University of Alberta

Hidden Symmetries of Higher-Dimensional Rotating Black Holes

by

David Kubizňák ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Department of Physics

Edmonton, Alberta
Fall 2008

Your file Votre référence
ISBN: 978-0-494-46351-2
Our file Notre référence
ISBN: 978-0-494-46351-2

And if anyone knows anything about anything—it's Owl who knows something about something, said Bear to himself—or my name's not Winnie-the-Pooh—he said—which it is, he added—so there you are.

Winnie-the-Pooh, A. A. Milne (1926)

To

Hanka,
my grandfather, Josef,
and my parents, Jana and Jiří

Abstract

In this thesis we study higher-dimensional rotating black holes. Such black holes are widely discussed in string theory and brane-world models at present. We demonstrate that even the most general known Kerr-NUT-(A)dS spacetime, describing the general rotating higher-dimensional asymptotically (anti) de Sitter black hole with NUT parameters, is in many aspects similar to its four-dimensional counterpart. Namely, we show that it admits a fundamental hidden symmetry associated with the principal conformal Killing–Yano tensor. Such a tensor generates towers of hidden and explicit symmetries. The tower of Killing tensors is responsible for the existence of irreducible, quadratic in momenta, conserved integrals of geodesic motion. These integrals, together with the integrals corresponding to the tower of explicit symmetries, make geodesic equations in the Kerr-NUT-(A)dS spacetime completely integrable. We further demonstrate that in this spacetime the Hamilton–Jacobi, Klein–Gordon, and stationary string equations allow complete separation of variables and the problem of finding parallel-propagated frames reduces to the set of the first order ordinary differential equations. Moreover, we show that the Kerr-NUT-(A)dS spacetime is the most general Einstein space which possesses all these properties. We also explicitly derive the most general (off-shell) canonical metric admitting the principal conformal Killing–Yano tensor and demonstrate that such a metric is necessarily of the special algebraic type D of the higher-dimensional algebraic classification. The results presented in this thesis describe the new and complete picture of the relationship of hidden symmetries and rotating black holes in higher dimensions.

Preface

When in 1963 Kerr discovered an astrophysically relevant but relatively complicated metric describing the gravitational field of a rotating black hole, it seemed that no analytical predictions were possible even for the simplest particle geodesic motion. However, a 'miracle' happened, and it turned out that not only the geodesic motion can be analytically solved, but also the equations describing various perturbations of this background can be 'drastically' simplified. This opened a way for studying astrophysical processes, such as the plasma accretion around black holes, the radiation produced by infalling matter, the origin of jets, the production and propagation of waves produced in the vicinity of black holes, and it even led to estimates of the gravitational wave production in star collisions and galaxy merges. It also facilitated the study of more theoretical problems, such as the problem of stability of the Kerr solution, the calculation of the quasinormal modes, or the study of the Hawking radiation. A hidden symmetry responsible for this miracle can be mathematically described by a simple antisymmetric object, called the Killing–Yano tensor.

Recently, higher-dimensional rotating black hole spacetimes have become of high interest due to various developments in gravity and high energy physics. One of the reasons is the popularity of the scenarios with large extra dimensions. In these so-called brane world models, our Universe is represented by a four-dimensional brane floating in the higher-dimensional bulk space—where only gravity can propagate. One of the main predictions of these models is the possibility to produce higher-dimensional mini black holes in particle colliders. Such black holes may provide a window into higher dimensions as well as into non-perturbative gravitational physics which may already appear on the TeV scale. Naturally, one wants to study the properties of these black holes which are the higher-dimensional generalizations of the Kerr geometry.

Another motivation for studying higher-dimensional black hole spacetimes comes from string theory. When quantizing a string action, a conformal anomaly appears unless the spacetime dimension is $D = 26$ for bosonic strings or $D = 10$ for superstrings. Black holes in string theory are widely discussed in connection with the problem of microscopical explanation of the black hole entropy. The study of black holes in an anti-de Sitter background can improve the understanding of the AdS/CFT correspondence. For this reason, it is useful to understand the particle and field propagation in these backgrounds. For example, recently the structure of the black hole singularity was probed using geodesics and correlators on the dual CFT on the boundary.

This thesis is devoted to the study of hidden symmetries of higher-dimensional rotating black holes (with spherical horizon topology). We shall demonstrate that, despite their complexity, the higher-dimensional rotating black holes admit a similar hidden symmetry as their four-dimensional 'cousins'. In consequence, these black holes possess the following properties: the geodesic motion in these backgrounds is completely integrable, the Hamilton–Jacobi, Klein–Gordon, and Dirac equations allow the separation of variables, the metric element is of the algebraic type D, it allows a generalized Kerr–Schild form, and it is uniquely determined by the presence of the hidden symmetry. In this context, four dimensions are not exceptional, and four-dimensional miraculous properties of rotating black holes are, in some sense, even more miraculous in higher dimensions.

In **Part I** of the thesis we focus on hidden symmetries. The next chapter introduces the concept of hidden symmetries and outline their importance for understanding the properties of four-dimensional rotating black holes. It also overviews higher-dimensional black hole spacetimes and recapitulates the results on hidden symmetries which laid the groundwork for new developments described in this thesis. The basic definitions and properties of the Killing and Killing–Yano tensors, the objects responsible for hidden symmetries, are summarized in Chapter 2.

In Chapter 3, we introduce the central notion of this thesis, the notion of the *principal conformal Killing–Yano* (PCKY) tensor. We demonstrate how, based on a simple property of this object, one can generate a whole tower of Killing–Yano tensors and a corresponding tower of rank-2 Killing tensors. We also discuss another, in some sense more physical, method for generating various Killing tensors. This method is based on the construction of integrals of motion for geodesic motion which are of higher order in geodesic momenta and therefore associated with the corresponding Killing tensors. Finally, we demonstrate that the PCKY tensor also generates a tower of Killing vector fields. One of these Killing vectors plays an exceptional role and we reserve for it the notion *primary*.

In **Part II**, applying the previous results, we study the remarkable properties of higher-dimensional rotating black holes. In Chapter 4, we demonstrate that the general Kerr–NUT–(A)dS spacetime, describing the higher-dimensional arbitrarily rotating black hole with NUT parameters and the cosmological constant, possesses the PCKY tensor. Moreover, this tensor determines uniquely preferred (canonical) coordinates for this metric and hence its canonical form. This form is especially useful for the subsequent calculations. We also learn that it is possible to consider a broader class of the (off-shell) metrics which possess the (same) PCKY tensor. It will be shown later, that this class describes the most general metric element admitting the PCKY tensor. We call it the *canonical metric* element.

In Chapter 5, we demonstrate the *complete integrability* of geodesic motion in the canonical background. Namely, we prove that the constants of geodesic motion corresponding to the extended tower of Killing tensors and the constants of geodesic motion corresponding to the tower of Killing vectors are all functionally independent and that they all mutually Poisson commute of one another. The latter property is closely related to the fact that the corresponding Killing tensors and/or Killing vectors Schouten–Nijenhuis commute. We use the opportunity to briefly review the theory of the Schouten–Nijenhuis brackets and to remind that with respect to these brackets Killing tensors form a Lie algebra.

The *separability* of the Hamilton–Jacobi and Klein–Gordon equations in the background of the canonical metric is demonstrated in Chapter 6. Such a separability provides an independent proof of complete integrability of geodesic motion. It also allows to study the contribution of scalar field to the Hawking evaporation of these black holes. Several related results, directly connected with these developments, are mentioned. Namely, we recapitulate the theory of the separability structures and describe a recent achievement on the symmetry operators which underly the separability of the Hamilton–Jacobi and Klein–Gordon equations. Some open questions, primarily connected with the separability problem for higher spin equations in this background, are also briefly discussed.

In Chapter 7, we address the question of *uniqueness* and generality of these developments. This leads us to the study of metric elements admitting the PCKY tensor. In particular, we demonstrate the following two important results: First, we establish that the Kerr-NUT-AdS spacetime is the most general solution of the vacuum Einstein equations with the cosmological constant which possesses the PCKY tensor. Second, without imposing the Einstein equations, we explicitly derive the most general metric admitting such a tensor and show that it coincides with the canonical metric element. These results naturally generalize the results obtained earlier in four dimensions.

Part III of the thesis is devoted to further developments connected with the PCKY tensor. In Chapter 8, we demonstrate the separability of the Nambu–Goto equations for a stationary string in the background of the canonical spacetime. Such a string is generated by a 1-parameter family of Killing trajectories and the problem of finding its configuration reduces to a problem of finding a geodesic line in a (one dimension lower) effective background. The resulting integrability of this geodesic problem is connected with the existence of hidden symmetries which are inherited from the black hole background. More generally, we introduce the concept of ξ-branes, that is more dimensional objects with the worldvolume aligned along the set of Killing vector fields, and discuss their integrability in the Kerr-NUT-(A)dS spacetime.

In Chapter 9, we study the equations describing the parallel transport of orthonormal frames along timelike geodesics in the spacetime admitting the PCKY tensor. It is demonstrated how, in the presence of this tensor, these equations can be reduced to a set of the first order ordinary differential equations. Concrete examples of parallel-propagated frames in $D = 3, 4, 5$ canonical spacetimes are constructed and it is shown that the obtained set of equations can be solved by the separation of variables. In the last chapter we summarize the overall picture of the obtained results, link these results to the related achievements, and discuss possible future directions.

To keep the main text concise and fluent, we have moved the complementary material of various character to the **appendices**. In Appendix A some four-dimensional aspects of hidden symmetries are discussed. The first section plays the role of an introduction for newcomers to the problematic of hidden symmetries. On a simple four-dimensional example we describe the main ideas of the, much more complicated, higher-dimensional theory. In the second section we study hidden symmetries of the Plebański–Demiański class of solutions. Some physically important subcases are discussed in more detail. An account of historical developments leading to the discovery of the PCKY tensor in the general Kerr-NUT-(A)dS spacetimes is recorded in Appendix B. Miscellaneous results are gathered in Appendix C.

The results presented in this **doctoral thesis** were obtained during the course of the author's Ph.D. program at the University of Alberta between years 2005 and 2008. The thesis is based on the following published papers in peer reviewed journals: [Frolov & Kubizňák, 2007], [Kubizňák & Frolov, 2007], [Page *et al.*, 2007], [Frolov *et al.*, 2007], [Krtouš *et al.*, 2007a], [Krtouš *et al.*, 2007b], [Kubizňák & Krtouš, 2007], [Kubizňák & Frolov, 2008], [Frolov & Kubizňák, 2008], [Connell *et al.*, 2008b], [Krtouš *et al.*, 2008b].

Notations and conventions

Throughout the thesis, we use a mixture of invariant, tensorial, and matrix notations. The tensors are typed in boldface and their components (with due indices) in normal letters. We consider a D-dimensional manifold M^D equipped with a metric $\boldsymbol{g}$. Except Chapter 9 and the appendices the metric is symbolically of the Euclidean signature; it is related to the physical metric by a simple Wick rotation (see Chapter 4). The coordinate indices are denoted by the Latin letters from the beginning of the alphabet, $a, b, c, \dots, = 1, \dots, D$; we use the Einstein summation convention for them. These indices may be also understood as abstract, in the sense of [Wald, 1984]. A dot above tensors denotes the differentiation along the vector field; $\dot{\boldsymbol{T}} \equiv \nabla_u \boldsymbol{T} \equiv u^a \nabla_a \boldsymbol{T}$. In Chapter 7, we use the symbol $\check{}$ to distinguish an operator from the corresponding 2-form. For example, having a 2-form $\boldsymbol{F}$ with the (abstract) indices F_{ab}, $\check{\boldsymbol{F}}$ denotes the operator $F^a{}_b$. Where it cannot lead to confusion, a dot between tensors indicates contraction, e.g., $\boldsymbol{a} \cdot \boldsymbol{b} \equiv a^c b_c$. Similarly, $\check{\boldsymbol{h}} \cdot \check{\boldsymbol{h}} \cdot \boldsymbol{v}$ denotes a vector with components $h^a{}_b\, h^b{}_c\, v^c$. The symbols $\boldsymbol{d}x^a$,

$\boldsymbol{\partial}_{x^a}$, denote the coordinate 1-form, vector, associated with the coordinate x^a. In several place we also use matrix notations. Matrices are typed in normal letters and the standard matrix notations are used for them. For example, having a rank-2 tensor $\boldsymbol{A}$, the symbol A stands for the matrix of its components $A^a{}_b$, A^T denotes the transposed matrix, and $\mathrm{Tr}A$ stands for its trace.

The central object of the thesis is a *principal conformal Killing–Yano tensor*, that is a non-degenerate, closed, conformal Killing–Yano 2-form. We reserve for it an abbreviation PCKY and (except Chapter 8) the symbol $\boldsymbol{h}$. The associated primary (Killing) vector is denoted by $\boldsymbol{\xi}$; $\xi^b \equiv 1/(D-1)\nabla_a h^{ab}$. The (orthonormal) Darboux bases of 1-forms and vectors determined by the PCKY tensor are called the *canonical* bases and denoted by $\{\boldsymbol{\omega}\}$ and $\{\boldsymbol{e}\}$ (see Chapter 3). To distinguish the basis indices from the coordinate indices we use $\hat{}$. For example, $u^{\hat{a}}$ denotes the basis components of the velocity $\boldsymbol{u}$. The same symbol is also used to denote (differential) operators, for example, $\hat{\xi} \equiv i\xi^a\nabla_a$. Further conventions are introduced later in the text (see, e.g., Chapter 2).

Acknowledgements

I am grateful to Prof. Valeri P. Frolov, my supervisor, for providing me with an excellent research project, for inspiration, discussions, and encouragement during my whole Ph.D. program. It has been a great privilege to work with him and with such collaborators as Prof. Don N. Page, Prof. Pavel Krtouš, and Dr. Muraari Vasudevan from whom I have learnt many aspects of physics. I would like also to thank to the members of the Theoretical Physics Institute, especially to Prof. Dmitri Pogosyan and Dr. Andrei Zelnikov for numerous discussions, not only about physics, and to my colleagues and friends, Shohreh Abdolrahimi, Patrick Connell, Dan Gorbonos, Rituparno Goswami, Eun Ah Lee, Debolina Guha Majumdar, Aditya Saha, Andrey Shoom, Jan Páral, and especially to Stepan Grinek. Not least, I would like to thank Prof. Jiří Bičák for his kindness and continuous support and Prof. Alan A. Coley for his hospitality during my visit at the Dalhousie University.

I am grateful to the Golden Bell Jar Graduate Scholarship in Physics at the University of Alberta for financial support during my whole Ph.D. program.

Table of Contents

Part I

Hidden Symmetries

Chapter 1

Introduction

1.1 Symmetries

In modern theoretical physics one can hardly overestimate the role of symmetries. They comprise the most fundamental laws of nature, they allow us to classify solutions, in their presence complicated physical problems become tractable. The value of symmetries is especially high in nonlinear theories, such as general relativity.

In curved spacetime continuous symmetries (isometries) are generated by Killing vector fields. Such symmetries have clear geometrical meaning. Let us assume that in a given manifold we have a 1-parameter family of diffeomorphisms generated by a vector field $\boldsymbol{\xi}$. Such a vector field determines the dragging of tensors by the diffeomorphism transformation. If a tensor field $\boldsymbol{T}$ is invariant with respect to this dragging, that is, its Lie derivative along $\boldsymbol{\xi}$ vanishes, $\mathcal{L}_\xi \boldsymbol{T} = 0$, we have a symmetry. A vector field which generates transformations preserving the metric is called a *Killing vector field*, and the corresponding diffeomorphism—an *isometry*. According to the first Noether theorem continuous symmetries of the theory imply the existence of conserved quantities. For a covariant theory in an external gravitational field for each of the Killing vectors there exists a conserved quantity. For example, for a particle geodesic motion this conserved quantity is a projection of the particle momentum on the Killing vector.

Besides isometries the spacetime may also possess *hidden symmetries*, generated by either symmetric or antisymmetric tensor fields. Such symmetries are not directly related to the metric invariance under diffeomorphisms. They represent the genuine symmetries of the phase space rather than the configuration space. For example, the symmetric Killing tensors give rise to the conserved quantities of higher order in particle momenta, and underline the separability of the scalar field equations. Less known but even more fundamental are the antisymmetric Killing–Yano tensors which are related to the separability of field equations with spin, the existence of 'quantum' symmetry operators, and the presence of conserved charges.

1.2 Miraculous properties of the Kerr geometry

To illustrate the role of hidden symmetries in general relativity let us recapitulate the "miraculous" properties [Chandrasekhar, 1983] of the Kerr geometry. This astrophysical important solution was obtained in 1963 by Kerr [Kerr, 1963]. The metric is stationary and axially symmetric; it possesses two Killing vectors, ∂_t and ∂_ϕ, generating the time translation and the rotation. The Kerr solution is of the special algebraic type D of Petrov's classification [Petrov, 1954], [Petrov, 1969], it belongs to the special class of solutions which can be presented in the Kerr–Schild form [Kerr & Schild, 1965], [Kerr & Schild, 1969], [Debney *et al.*, 1969],

$$g_{ab} = \eta_{ab} + 2H l_a l_b \,. \tag{1.1}$$

Here, η is a flat metric and l is a null vector, in both metrics g and η.[1]

Although the Killing vector fields ∂_t and ∂_ϕ are not enough to provide a sufficient number of integrals of motion[2] in 1968 Carter [Carter, 1968a], [Carter, 1968b] demonstrated that both—the Hamilton–Jacobi and scalar field equations—can be separated. This proved, apart from other things, that there exists an additional integral of motion, 'mysterious' Carter's constant, which makes the particle geodesic motion completely integrable. In 1970, Walker and Penrose [Walker & Penrose, 1970] pointed out that Carter's constant is quadratic in particle momenta and its existence is directly connected with the symmetric Killing tensor [Stackel, 1895]

$$K_{ab} = K_{(ab)}\,, \quad \nabla_{(c}K_{ab)} = 0\,. \tag{1.2}$$

During the several following years it was discovered that it is not only the Klein–Gordon equation which allows the separation of variables in the Kerr geometry. In 1972, Teukolsky decoupled the equations for electromagnetic and gravitational perturbations, and separated variables in the resulting master equa-

[1] If the ansatz (1.1) is inserted into the Einstein equations, one effectively reduces the problem to a linear one (see, e.g., [Gürses & Gürsey, 1975]). This gives a powerful tool for the study of special solutions of the Einstein equations. This method works in higher dimensions as well. For example, the Kerr–Schild ansatz was used by Myers and Perry to obtain their higher-dimensional black hole solutions [Myers & Perry, 1986].

[2] For example, for a particle motion these isometries generate the conserved energy and azimuthal component of the angular momentum, which, together with the conservation of p^2, gives only three integrals of motion. For separability of the Hamilton–Jacobi equation in the Kerr spacetime the fourth integral of motion is required.

tions [Teukolsky, 1972]. One year later the massless neutrino equation by Teukolsky and Unruh [Teukolsky, 1973], [Unruh, 1973], and in 1976 the massive Dirac equation by Chandrasekhar and Page [Chandrasekhar, 1976], [Page, 1976] were separated.

Meanwhile a new breakthrough was achieved in the field of hidden symmetries when in 1973 Penrose and Floyd [Penrose, 1973], [Floyd, 1973] discovered that the Killing tensor for the Kerr metric can be written in the form

$$K_{ab} = f_{ac} f_b{}^c \,, \tag{1.3}$$

where the antisymmetric tensor $\boldsymbol{f}$ is the Killing–Yano (KY) tensor [Yano, 1952]

$$f_{ab} = f_{[ab]} \,, \quad \nabla_{(c} f_{a)b} = 0 \,. \tag{1.4}$$

A Killing–Yano tensor is in many aspects more fundamental than a Killing tensor. In particular, having a Killing–Yano tensor one can always construct the corresponding Killing tensor using Eq. (1.3). On the other hand, not every Killing tensor can be decomposed in terms of a Killing–Yano tensor (for necessary conditions see [Collinson, 1976], [Ferrando & Saez, 2002]).

Many of the remarkable properties of the Kerr spacetime are consequences of the existence of the Killing–Yano tensor. In particular, in 1974 Collinson demonstrated that the integrability conditions for a non-degenerate Killing–Yano tensor imply that the spacetime is necessary of the Petrov type D [Collinson, 1974].[3] In 1975, Hughston and Sommers showed that in the Kerr geometry the Killing–Yano tensor $\boldsymbol{f}$ generates both of its isometries [Hughston & Sommers, 1973]. Namely, the Killing vectors $\boldsymbol{\partial}_t$ and $\boldsymbol{\partial}_\phi$, can be written as follows:

$$\xi^a \equiv \frac{1}{3} \nabla_b (*f)^{ba} = (\partial_t)^a \,, \quad \eta^a \equiv -K^a_{\ b} \xi^b = (\partial_\phi)^b \,. \tag{1.5}$$

This means that, in fact, all the symmetries necessary for complete integrability of geodesic motion in the Kerr spacetime are 'derivable' from the existence of a single Killing–Yano tensor.

In 1977, Carter demonstrated [Carter, 1977] that given an isometry $\boldsymbol{\xi}$ and/or a Killing tensor $\boldsymbol{K}$ one can construct the operators

$$\hat{\xi} \equiv i \xi^a \nabla_a \,, \quad \hat{K} \equiv \nabla_a K^{ab} \nabla_b \,, \tag{1.6}$$

[3] All the vacuum type D solutions were obtained by Kinnersley [Kinnersley, 1969]. Demiański and Francaviglia showed that in the absence of acceleration these solutions admit Killing and Killing–Yano tensors [Demianski & Francaviglia, 1980]. A general (off-shell) metric element admitting a Killing–Yano tensor in four dimensions was obtained by Dietz and Rüdiger [Dietz & Rüdiger, 1981], [Dietz & Rüdiger, 1982], see also [Taxiarchis, 1985].

which commute with the scalar Laplacian[4]

$$\left[\Box\,,\hat{\xi}\right] = 0 = \left[\Box\,,\hat{K}\right]\,, \quad \Box \equiv \nabla_a g^{ab} \nabla_b\,. \tag{1.7}$$

Moreover, in the Kerr geometry these operators commute also between themselves and provide therefore good 'quantum' numbers for scalar fields. In 1979 Carter and McLenaghan found that an operator

$$\hat{f} \equiv i\gamma_5\gamma^a\Big(f_a{}^b\nabla_b - \frac{1}{6}\,\gamma^b\gamma^c\nabla_c f_{ab}\Big) \tag{1.8}$$

commutes with the Dirac operator $\gamma^a\nabla_a$ [Carter & McLenaghan, 1979]. This gives a new quantum number for the spinor wavefunction and explains why separation of the Dirac equation can be achieved. Similar symmetry operators for other equations with spin, including electromagnetic and gravitational perturbations, were constructed later [Kamran & McLenaghan, 1984], [Kamran, 1985], [del Castillo, 1988], [Kalnins & Miller, 1989],[Kalnins *et al.*, 1996].

In 1983, Marck solved equations for the parallel transport of an orthonormal frame along geodesics in the Kerr spacetime [Marck, 1983b], [Marck, 1983a] and used this result for the study of tidal forces. For this construction he used a simple fact that the vector

$$L_a \equiv f_{ab}p^b\,, \quad L_a p^a = 0\,, \tag{1.9}$$

is parallel-propagated along a geodesic $\boldsymbol{p}$.

In 1987, Carter [Carter, 1987] pointed out that the Killing–Yano tensor itself is derivable from a 1-form $\boldsymbol{b}$,

$$\boldsymbol{f} = *\boldsymbol{db}\,. \tag{1.10}$$

We call such a form $\boldsymbol{b}$ a (KY) *potential*. It satisfies the Maxwell equations and can be interpreted as a 4-potential of an electromagnetic field with the source current proportional to the *primary* Killing vector field $\boldsymbol{\partial}_t$, cf. Eq. (1.5). In 1989, Frolov *et al.* [Frolov *et al.*, 1989], [Carter & Frolov, 1989], [Carter *et al.*, 1991], separated equations for an equilibrium configuration of a cosmic string near the Kerr black hole. In 1993, Gibbons *et al.* demonstrated that due to the presence of Killing–Yano tensor the classical spinning particles in this background possess enhanced worldline supersymmetry [Gibbons *et al.*, 1993]. Conserved quantities in the

[4]In fact, the operator $\hat{K}$ defined by (1.6) commutes with $\Box$ provided that the background metric satisfies the vacuum Einstein or source-free Einstein–Maxwell equations. In more general spacetimes, however, a quantum anomaly proportional to a contraction of $\boldsymbol{K}$ with the Ricci tensor may appear. Such anomaly is not present if an additional condition (1.3) is satisfied [Cariglia, 2004].

Kerr geometry generated by f were discussed in 2006 by Jezierski and Łukasik [Jezierski & Lukasik, 2006].

To conclude this section we mention that many of the above statements and results, which we have formulated for the Kerr geometry, are in fact more general. Their validity can be extended to more general spacetimes, or even to an arbitrary number of spacetime dimensions. For example, the whole Carter's class of solutions [Carter, 1968c], [Carter, 1968b] (see also [Debever, 1971], [Plebański, 1975]) admits a KY tensor and possesses many of the discussed properties (see Appendix A). General results on Killing–Yano tensors and algebraic properties were gathered by Hall [Hall, 1987]. A relationship among the existence of Killing tensors and separability structures for the Hamilton–Jacobi equation in an arbitrary number of spacetime dimensions was discussed in [Woodhouse, 1975], [Benenti & Francaviglia, 1979], [Kalnins & Miller, 1981] (see also Section 6.3.1). We refer to Appendix A for further details on hidden symmetries in 4D.

1.3 Higher-dimensional black holes

Higher-dimensional black hole solutions have been studied for a long time. Already in 1963, Tangherlini [Tangherlini, 1963] obtained a higher-dimensional generalization of the Schwarzschild metric [Schwarzschild, 1916]. The charged version of the Tangherlini metric was found in 1986 by Myers and Perry [Myers & Perry, 1986]. In the same paper a general vacuum rotating black hole in higher dimensions was obtained. This solution, often called the Myers–Perry (MP) metric, generalizes the four-dimensional Kerr solution.[5] Main new feature of the MP metrics in D dimensions is that, instead of one rotation parameter, they have $m \equiv [(D-1)/2]$ rotation parameters, corresponding to m independent 2-planes of rotation. Later, in 1998, Hawking, Hunter, and Taylor-Robinson [Hawking *et al.*, 1999] found a 5D generalization of the 4D rotating black hole in asymptotically (anti) de Sitter space (Kerr-(A)dS metric [Carter, 1968b]). In 2004 Gibbons, Lü, Page, and Pope [Gibbons *et al.*, 2004], [Gibbons *et al.*, 2005] discovered the general Kerr-(A)dS metrics in arbitrary number of dimensions. After several attempts to include NUT [Newman *et al.*, 1963] parameters [Chong *et al.*, 2005],

[5]To be more precise, it is well known that physics of higher-dimensional black holes can be substantially different, and much richer, than in four dimensions. Whereas in four dimensions only the black holes with spherical horizon topology are allowed, it is expected that non-spherical horizon topologies are a generic feature of higher-dimensional gravity. Besides the class of rotating black holes solutions with spherical horizon, such as the Myers–Perry metrics and their generalizations, there exist other rotating 'black objects', for example black rings and their generalizations. In this thesis we concentrate only on the class of rotating black holes with spherical horizon topology. Such black holes can be considered as natural higher-dimensional generalizations of the Kerr metric.

[Chen *et al.*, 2007], in 2006 Chen, Lü, and Pope [Chen *et al.*, 2006a] found a general Kerr-NUT-(A)dS solution of the Einstein equations for all D. These metrics were obtained in special coordinates which are the natural higher-dimensional generalization of the Carter's 4D *canonical coordinates* [Carter, 1968b], [Debever, 1971], [Plebański, 1975]. So far, they remain the most general black-hole-type solutions of the Einstein equations with the cosmological constant (and spherical horizon topology) which are known analytically.[6] For a recent extended review on higher-dimensional black holes see, e.g., [Emparan & Reall, 2008].

In connection with these black holes the following natural questions arise: To what extent are the remarkable properties described for the four-dimensional black holes innate to four dimensions? Do some of them transfer to higher dimensions as well? And in particular, do some of the higher-dimensional black holes possess hidden symmetries?

1.4 Hidden symmetries in higher dimensions

The hidden symmetries of higher-dimensional rotating black holes were first discovered for the 5D Myers–Perry metrics [Frolov & Stojković, 2003b] ,[Frolov & Stojković, 2003a]. It was demonstrated that both, the Hamilton–Jacobi and scalar field equations, allow the separation of variables and the corresponding Killing tensor was obtained. Later it was shown that 5D results can be extended to an arbitrary number of dimensions, provided that rotation parameters of the MP metric can be divided into two classes, and within each of the classes these parameters are equal of one another [Vasudevan *et al.*, 2005a]. Similar results were found in the presence of the cosmological constant and NUT parameters [Lopez-Ortega, 2003], [Vasudevan *et al.*, 2005b], [Kunduri & Lucietti, 2005], [Vasudevan & Stevens, 2005], [Vasudevan, 2006a], [Chen *et al.*, 2006b], [Davis, 2006], [Vasudevan, 2006b]. It was also demonstrated that a stationary string configuration in the 5D Myers–Perry spacetime is completely integrable [Frolov & Stevens, 2004].

[6]Besides the brane-world scenarios, these black holes find their applications for the ADS/CFT correspondence. In the BPS limit the odd-dimensional metrics lead to the Sasaki–Einstein metrics [Hashimoto *et al.*, 2004], [Cvetic *et al.*, 2005b], [Cvetic *et al.*, 2005a] whereas the even-dimensional metrics lead to the Calabi–Yau spaces [Oota & Yasui, 2006], [Lü & Pope, 2007]. There have been also several attempts to generalize these solutions. For example, to find a similar solution of the Einstein–Maxwell equations either in an analytical form [Aliev & Frolov, 2004], [Aliev, 2006a], [Aliev, 2006b],[Aliev, 2007], [Kunz *et al.*, 2006b], [Chen & Lü, 2008], [Krtouš, 2007] or numerically [Kunz *et al.*, 2005],[Kunz *et al.*, 2006a],[Kunz *et al.*, 2007], [Brihaye & Delsate, 2007], [Kleihaus *et al.*, 2008]. See also [Charmousis & Gregory, 2004], [Podolský & Ortaggio, 2006], [Pravda *et al.*, 2007],[Ortaggio *et al.*, 2008], or [Houri *et al.*, 2008c], [Lü *et al.*, 2008a].

Were these results the most general possible? Or, were there somewhere hidden other symmetries which would allow a further progress? In particular, do the higher-dimensional rotating black holes admit the fundamental symmetry of a Killing–Yano tensor? These were the questions which stimulated further research.

The outcome of this research can be briefly summarized as follows:[7] Even the most general known higher-dimensional Kerr-NUT-(A)dS black holes possess many of the remarkable properties of their four-dimensional 'cousins' . Namely, the geodesic motion in these backgrounds is completely integrable, the Hamilton–Jacobi, Klein–Gordon, Dirac, and stationary string equations are completely separable. The metrics are of the type D of higher-dimensional algebraic classification and allow a generalized Kerr–Schild form. Many of these properties follow directly from the existence of a fundamental hidden symmetry associated with the *principal conformal Killing–Yano* (PCKY) tensor.

The PCKY tensor was first discovered for the Myers–Perry metrics [Frolov & Kubizňák, 2007], and soon after that for the completely general Kerr-NUT-(A)dS spacetimes [Kubizňák & Frolov, 2007]. Starting with this tensor, one can generate the whole *tower* of hidden symmetries [Krtouš *et al.*, 2007b] which are responsible for complete integrability of geodesic motion in these spacetimes [Page *et al.*, 2007], [Krtouš *et al.*, 2007a]. Such an integrability was independently proved by separating the Hamilton–Jacobi equation [Frolov *et al.*, 2007]. Due to the presence of hidden symmetries, also the Klein–Gordon [Frolov *et al.*, 2007] and Dirac [Oota & Yasui, 2008] equations allow the separation of variables in these backgrounds. Recently, extending the work of [Houri *et al.*, 2007], [Houri *et al.*, 2008a], the uniqueness of these results was demonstrated [Krtouš *et al.*, 2008b]. In particular, it was proved that, similar to the 4D case, the most general Einstein space admitting the PCKY tensor is the Kerr-NUT-(A)dS spacetime. Meanwhile, it was shown that the Kerr-NUT-(A)dS spacetime is of the algebraic type D of higher-dimensional algebraic classification [Hamamoto *et al.*, 2007] and that it allows a generalized Kerr–Schild form [Chen & Lü, 2008]. By relaxing some of the requirements imposed on the PCKY tensor more general spacetimes were recently discovered [Houri *et al.*, 2008c], [Houri *et al.*, 2008b]. Directly related to the PCKY tensor is also a proved complete integrability of a stationary string configuration in the vicinity of the Kerr-NUT-(A)dS black hole [Kubizňák & Frolov, 2008] and the possibility of constructing a parallel propagated frame in such a background [Connell *et al.*, 2008b]. (For other related works see Chapter 10.)

[7] Let us emphasize that not all of the results were obtained by the author of this thesis and/or his collaborators. What we summarize here is the overall progress which has been recently achieved in this direction.

Chapter 2

Killing–Yano and Killing tensors

In this chapter we review the basic objects responsible for symmetries of the spacetime and briefly discuss their properties. In particular, we introduce the Killing and Killing–Yano tensors, and their conformal generalizations. We also exploit the opportunity to establish some notations used later in the text.

2.1 Definitions

Killing vectors

Let us consider a D-dimensional spacetime with a metric g. A spacetime possesses an *isometry* generated by the *Killing vector* field $\boldsymbol{\xi}$ if this vector obeys the Killing equation

$$\nabla_{(a}\xi_{b)} = 0\,. \tag{2.1}$$

For a geodesic motion of a particle in such a curved spacetime the quantity $p^a\xi_a$, where $\boldsymbol{p}$ is the momentum of the particle, remains constant along the particle's trajectory. Similarly, for a null geodesic, $p^a\xi_a$ is conserved provided that $\boldsymbol{\xi}$ is a *conformal Killing vector* obeying the equation

$$\nabla_{(a}\xi_{b)} = \tilde{\xi} g_{ab}\,, \quad \tilde{\xi} = D^{-1}\nabla_b\xi^b\,. \tag{2.2}$$

An equivalent defining equation for the conformal Killing vector is

$$\nabla_a\xi_b = \nabla_{[a}\xi_{b]} + g_{ab}\tilde{\xi}\,. \tag{2.3}$$

That is, the conformal Killing vector is an object, the covariant derivative of which splits into the *'exterior'* and *'divergence'* parts.[1] When the first term vanishes the conformal Killing vector is *closed*. The vanishing of the second term means that we are dealing with the Killing vector. In the case when both parts are zero we have a *covariantly constant* (Killing) vector.

There exist two natural (symmetric and antisymmetric) generalizations of a (conformal) Killing vector. (For a 'hybrid' proposal, see, e.g., [Collinson & Howarth, 2000].)

Killing tensors

A symmetric (rank-p) *conformal Killing tensor* [Walker & Penrose, 1970] $\boldsymbol{Q}$ obeys the equations

$$Q_{a_1a_2\dots a_p} = Q_{(a_1a_2\dots a_p)}\,, \quad \nabla_{(b}Q_{a_1a_2\dots a_p)} = g_{(ba_1}\tilde{Q}_{a_2\dots a_p)}\,. \tag{2.4}$$

As in the case of a conformal Killing vector, the tensor $\tilde{\boldsymbol{Q}}$ is determined by tracing both sides of Eq. (2.4). In particular, a rank-2 conformal Killing tensor obeys the equations

$$\nabla_{(a}Q_{bc)} = g_{(ab}\tilde{Q}_{c)}\,, \quad \tilde{Q}_a = \frac{1}{D+2}\left(2\nabla_c Q^c{}_a + \nabla_a Q^c{}_c\right). \tag{2.5}$$

If $\tilde{\boldsymbol{Q}}$ vanishes, the tensor $\boldsymbol{Q}$ is called a *Killing tensor* [Stackel, 1895] and it is usually denoted by $\boldsymbol{K}$. So we have

$$K_{a_1a_2\dots a_p} = K_{(a_1a_2\dots a_p)}\,, \quad \nabla_{(b}K_{a_1a_2\dots a_p)} = 0\,. \tag{2.6}$$

Obviously, the metric is a (trivial) rank-2 Killing tensor. In the presence of the Killing tensor $\boldsymbol{K}$ the conserved quantity for a geodesic motion is

$$K = K_{a_1a_2\dots a_p}p^{a_1}p^{a_2}\dots p^{a_p}\,. \tag{2.7}$$

For null geodesics this quantity is conserved not only for a Killing tensor, but also for a conformal Killing tensor. Let us finally mention that a symmetrized tensor product of Killing vectors is a (reducible) Killing tensor. More generally, we call the Killing tensor *reducible* if it is a linear combination of the products of Killing tensors of a lower rank.

[1]For a general vector an additional term, the *'harmonic'* part, is present. It is the lack of this term what makes conformal Killing vectors 'special'.

Killing–Yano tensors

The *conformal Killing–Yano* (CKY) tensors were first proposed in 1968 by Kashiwada and Tachibana [Kashiwada, 1968], [Tachibana, 1969] as a generalization of the Killing–Yano (KY) tensors introduced by Yano in 1952 [Yano, 1952].

One of the simplest approaches to the CKY tensors is based on a natural generalization of the definition (2.3) of conformal Killing vectors. The CKY tensor $\boldsymbol{k}$ of rank-p is a p-form the covariant derivative of which has vanishing harmonic part, that is it splits into the exterior and divergence parts as follows:

$$\nabla_a k_{b_1 \ldots b_p} = \nabla_{[a} k_{b_1 \ldots b_p]} + p\, g_{a[b_1} \tilde{k}_{b_2 \ldots b_p]}\,, \tag{2.8}$$

$$\tilde{k}_{b_2 \ldots b_p} = \frac{1}{D-p+1} \nabla_c k^c{}_{b_2 \ldots b_p}\,. \tag{2.9}$$

[The tensor $\tilde{\boldsymbol{k}}$, (2.9), is determined by tracing both sides of the first equation.] The defining equation (2.8) is invariant under the Hodge duality; the exterior part transforms into the divergence part and vice versa. This implies that the dual $*\boldsymbol{k}$ is a CKY tensor whenever $\boldsymbol{k}$ is a CKY tensor (see also the next section).

Three special subclasses of CKY tensors are of particular interest: (a) *Killing–Yano* tensors with zero divergence part in (2.8) (b) *closed CKY* tensors with vanishing exterior part in (2.8) and (c) *covariantly constant* (KY) tensors with both parts vanishing. The subclasses (a) and (b) transform into each other under the Hodge duality.

In particular, a rank-2 CKY tensor $\boldsymbol{k}$ obeys the equations

$$\nabla_a k_{bc} = \nabla_{[a} k_{bc]} + 2\, g_{a[b}\xi_{c]}\,, \quad \xi_a = \frac{1}{D-1} \nabla_c k^c{}_a\,. \tag{2.10}$$

The vector $\boldsymbol{\xi}$, defined by the last equation, is called *primary*. It satisfies [Tachibana, 1969] (see also Appendix C.2)

$$\nabla_{(a}\xi_{b)} = \frac{1}{D-2} R_{c(a} k_{b)}{}^c\,. \tag{2.11}$$

Thus, in an Einstein space, that is when $R_{ab} = \Lambda g_{ab}$, $\boldsymbol{\xi}$ is the Killing vector.

An alternative (equivalent) definition of a rank-p CKY tensor naturally generalizes the definition (2.2) [Tachibana, 1969], [Jezierski, 1997], [Cariglia, 2004]. It reads

$$\nabla_{(a} k_{b_1)b_2 \ldots b_p} = g_{ab_1} \tilde{k}_{b_2 \ldots b_p} - (p-1)\, g_{[b_2(a} \tilde{k}_{b_1)\ldots b_p]}\,, \tag{2.12}$$

where $\tilde{\boldsymbol{k}}$ (obtained again by tracing procedure) is given by (2.9).

Let us finally mention two additional important properties of KY tensors. Having a KY tensor $\boldsymbol{f}$:

1. The tensor
$$L_{a_1 a_2 \dots a_{p-1}} \equiv f_{a_1 a_2 \dots a_p} p^{a_p} \tag{2.13}$$
is parallel-propagated along the geodesic $\boldsymbol{p}$.

2. The object
$$K_{ab} = \frac{c_p}{(p-1)!} f_{a a_2 \dots a_p} f_b{}^{a_2 \dots a_p} \tag{2.14}$$
is an associated Killing tensor. Here c_p is an arbitrary constant, which is often taken to be one. For a different convenient choice see Section 3.2.

Similarly, for a rank-2 CKY tensor $\boldsymbol{k}$ the object
$$Q_{ab} = k_{ac} k_b{}^c\,, \tag{2.15}$$
is an associated conformal Killing tensor.

Let us also mention that the exterior product of Killing vectors does not generally produce a KY tensor.[2] However, in Section 3.2.1 we shall prove the important fact that the exterior product of two closed CKY tensors is again a closed CKY tensor.

2.2 CKY tensors as differential forms

The CKY tensors are forms and operations with them are greatly simplified if one uses the 'invariant' language of differential forms. In this section we establish some of these notations. We also recast the CKY equation (2.8) into this language and prove its invariance under the Hodge duality.

If $\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_q$ are p- and q-forms, respectively, the external derivative $\boldsymbol{d}$ of their exterior (wedge) product $\wedge$ obeys a relation
$$\boldsymbol{d}(\boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q) = \boldsymbol{d}\boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q + (-1)^p \boldsymbol{\alpha}_p \wedge \boldsymbol{d}\boldsymbol{\beta}_q\,. \tag{2.16}$$
For an arbitrary form $\boldsymbol{\alpha}$ we denote
$$\boldsymbol{\alpha}^{\wedge m} \equiv \underbrace{\boldsymbol{\alpha} \wedge \dots \wedge \boldsymbol{\alpha}}_{\text{total of m factors}}\,. \tag{2.17}$$
A Hodge dual $*\boldsymbol{\alpha}_p$ of a p-form $\boldsymbol{\alpha}_p$ is a $(D-p)$-form defined as
$$(*\alpha_p)_{a_1 \dots a_{D-p}} = \frac{1}{p!} \alpha^{b_1 \dots b_p} e_{b_1 \dots b_p a_1 \dots a_{D-p}}\,, \tag{2.18}$$

[2] A trivial example when this works is , for example, the case of maximally symmetric spacetimes (see also Footnote 1 in Appendix A).

where $e_{a_1 \ldots a_D}$ is a totally antisymmetric tensor. The *co-derivative* $\boldsymbol{\delta}$ is defined as follows:

$$\boldsymbol{\delta}\boldsymbol{\alpha}_p = (-1)^p \epsilon_p * \boldsymbol{d} * \boldsymbol{\alpha}_p\,, \quad \epsilon_p = (-1)^{p(D-p)} \frac{\det(g)}{|\det(g)|}\,. \tag{2.19}$$

One also has $** \boldsymbol{\alpha}_p = \epsilon_p \boldsymbol{\alpha}_p$.

If $\{\boldsymbol{e}_{\hat{a}}\}$ is a basis of vectors, then the dual basis of 1-forms $\{\boldsymbol{\omega}^{\hat{a}}\}$ is defined by the relations $\boldsymbol{\omega}^{\hat{a}}(\boldsymbol{e}_{\hat{b}}) = \delta^{a}_{\hat{b}}$. We denote $g_{\hat{a}\hat{b}} = \boldsymbol{g}(\boldsymbol{e}_{\hat{a}}, \boldsymbol{e}_{\hat{b}})$ and by $g^{\hat{a}\hat{b}}$ the inverse matrix. The operations with the indices enumerating the basic vectors and forms are performed by using these matrices. In particular, $\boldsymbol{e}^{\hat{a}} = g^{\hat{a}\hat{b}} \boldsymbol{e}_{\hat{b}}$, and so on. We denote a covariant derivative along the vector $\boldsymbol{e}_{\hat{a}}$ by $\nabla_{\hat{a}}\,;\, \nabla_{\hat{a}} \equiv \nabla_{\boldsymbol{e}_{\hat{a}}}$. One has

$$\boldsymbol{d} = \boldsymbol{\omega}^{\hat{a}} \wedge \nabla_{\hat{a}}\,, \quad \boldsymbol{\delta} = -\boldsymbol{e}^{\hat{a}} \lrcorner \nabla_{\hat{a}}\,. \tag{2.20}$$

In tensor notations the 'hook' operator (inner derivative) along a vector $\boldsymbol{X}$, applied to a p-form $\boldsymbol{\alpha}_p$, $\boldsymbol{X} \lrcorner \boldsymbol{\alpha}_p$, corresponds to a contraction

$$(X \lrcorner \alpha_p)_{a_2 \ldots a_p} = X^{a_1} (\alpha_p)_{a_1 a_2 \ldots a_p}\,. \tag{2.21}$$

It satisfies the properties

$$\boldsymbol{e}^{\hat{a}} \lrcorner (\boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q) = (\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\alpha}_p) \wedge \boldsymbol{\beta}_q + (-1)^p \boldsymbol{\alpha}_p \wedge (\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\beta}_q)\,, \tag{2.22}$$

$$\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\omega}_{\hat{a}} = D\,, \quad \boldsymbol{\omega}_{\hat{a}} \wedge (\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\alpha}_p) = p\, \boldsymbol{\alpha}_p\,. \tag{2.23}$$

For a given vector $\boldsymbol{X}$ one defines $\boldsymbol{X}^\flat$ as a corresponding 1-form with the components $(X^\flat)_a = g_{ab} X^b$. In particular, one has $(\boldsymbol{e}_{\hat{a}})^\flat = g_{\hat{a}\hat{b}} \boldsymbol{\omega}^{\hat{b}}$. An inverse to $\flat$ operation is denoted by $\sharp$. Namely if $\boldsymbol{\alpha}$ is a 1-form then $\boldsymbol{\alpha}^\sharp$ denotes a vector with components $(\alpha^\sharp)^a = g^{ab} \alpha_b$. We refer to [Sternberg, S., 1964], [Kress, 1997] where these and many other useful relations can be found.

The definition (2.8) of the (rank-p) CKY tensor $\boldsymbol{k}$ reads [Benn *et al.*, 1997], [Benn & Charlton, 1997], [Kress, 1997]:

$$\nabla_X \boldsymbol{k} = \frac{1}{p+1} \boldsymbol{X} \lrcorner \boldsymbol{d}\boldsymbol{k} - \frac{1}{D-p+1} \boldsymbol{X}^\flat \wedge \boldsymbol{\delta}\boldsymbol{k}\,. \tag{2.24}$$

Here, the first term on the right-hand-side denotes the exterior part, the second term denotes the divergence part, and $\boldsymbol{X}$ is an arbitrary vector.

Using the relation

$$\boldsymbol{X} \lrcorner * \boldsymbol{\omega} = *(\boldsymbol{\omega} \wedge \boldsymbol{X}^\flat)\,, \tag{2.25}$$

it is easy to show that under the Hodge duality the exterior part transforms into the divergence part and vice versa. Indeed, we find

$$*(\boldsymbol{X} \lrcorner \boldsymbol{dk}) = -\boldsymbol{X}^\flat \wedge \boldsymbol{\delta}(*\boldsymbol{k}) \,, \quad -*(\boldsymbol{X}^\flat \wedge \boldsymbol{\delta k}) = \boldsymbol{X} \lrcorner \boldsymbol{d}(*\boldsymbol{k}) \,, \tag{2.26}$$

where we have used the definition (2.19). In particular, (2.24) implies

$$\nabla_X(*\boldsymbol{k}) = \frac{1}{p_*+1}\boldsymbol{X} \lrcorner \boldsymbol{d}(*\boldsymbol{k}) - \frac{1}{D-p_*+1}\boldsymbol{X}^\flat \wedge \boldsymbol{\delta}(*\boldsymbol{k}) \,, \quad p_* = D - p \,. \tag{2.27}$$

That is, the Hodge dual $*\boldsymbol{k}$ of a CKY tensor $\boldsymbol{k}$ is again a CKY tensor. Moreover, the Hodge dual of a closed CKY tensor is a KY tensor and vice versa.

For a (rank-p) closed CKY tensor $\boldsymbol{h}$, characterized by vanishing of the exterior part, $\boldsymbol{dh} = 0$, there exists locally a (KY) *potential* $\boldsymbol{b}$, which is a $(p-1)$-form, such that

$$\boldsymbol{h} = \boldsymbol{db} \,. \tag{2.28}$$

The Hodge dual of such a tensor $\boldsymbol{h}$,

$$\boldsymbol{f} = *\boldsymbol{h} = *\boldsymbol{db} \,, \tag{2.29}$$

is a Killing–Yano tensor ($\boldsymbol{\delta f} = 0$).

Chapter 3

Principal conformal Killing–Yano tensor and towers of hidden symmetries

In this chapter we introduce a notion of a *principal conformal Killing–Yano* (PCKY) tensor—the central object of this thesis. Starting with the PCKY tensor and the metric in any D-dimensional spacetime we show how to generate a tower of $n-1=[D/2]-1$ Killing–Yano tensors, of rank $D-2j$ for all $1\leq j\leq n-1$, and an extended tower of n rank-2 Killing tensors, giving n quadratic in momenta constants of geodesic motion that are in involution. We also discuss another, more physical, method for generating Killing tensors and outline a construction of $D-n$ vectors which turn out to be the independent commuting Killing vectors. Based on these results, we shall prove in Part II many of the remarkable properties of higher-dimensional rotating black hole spacetimes. This chapter is based on [Page *et al.*, 2007], [Krtouš *et al.*, 2007b], [Krtouš *et al.*, 2007a], and [Frolov & Kubizňák, 2008].

3.1 Principal conformal Killing–Yano tensor

3.1.1 Definition

In what follows we consider a D-dimensional spacetime M^D, equipped with the metric

$$\boldsymbol{g}=g_{ab}\boldsymbol{d}x^a\boldsymbol{d}x^b\,. \tag{3.1}$$

To treat both cases of even and odd dimensions simultaneously we denote

$$D=2n+\varepsilon\,, \tag{3.2}$$

where $\varepsilon = 0$ and $\varepsilon = 1$ for even and odd number of dimensions, respectively.

Definition ([Krtouš *et al.*, 2007b]). *A principal conformal Killing–Yano tensor $\boldsymbol{h}$ is a closed non-degenerate CKY 2-form, $\boldsymbol{h} = \frac{1}{2} h_{ab}\, \boldsymbol{dx}^a \wedge \boldsymbol{dx}^b$, obeying the following equation:*

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}^\flat\,, \tag{3.3}$$

where $\boldsymbol{X}$ is an arbitrary vector field.

The condition of *non-degeneracy* means that the skew symmetric matrix h_{ab} has the (matrix) rank $2n$ and that the eigenvalues of $\boldsymbol{h}$ are functionally independent in some spacetime domain. So, we exclude the possibility that $\boldsymbol{h}$ possesses the constant eigenvalues, and in particular, that it is covariantly constant; $\boldsymbol{\xi} \neq 0$. The equation (3.3) implies

$$\boldsymbol{dh} = 0\,, \quad \boldsymbol{\xi}^\flat = -\frac{1}{D-1}\boldsymbol{\delta h}\,. \tag{3.4}$$

In particular, this means that there exists a 1-form (KY) potential $\boldsymbol{b}$ such that

$$\boldsymbol{h} = \boldsymbol{db}\,. \tag{3.5}$$

The dual tensor

$$\boldsymbol{f} = *\boldsymbol{h} \tag{3.6}$$

is a principal Killing–Yano tensor [$(D-2)$-form]. In tensor notations the definition (3.3) of the PCKY tensor $\boldsymbol{h}$ reads

$$\nabla_c h_{ab} = 2 g_{c[a}\xi_{b]}\,, \quad \xi_b = \frac{1}{D-1}\nabla_d h^d{}_b\,. \tag{3.7}$$

3.1.2 Canonical basis and canonical coordinates

Let us consider an eigenvalue problem for a conformal Killing tensor $\boldsymbol{Q}$ associated with $\boldsymbol{h}$ [cf. Eq. (2.15)],

$$Q_{ab} \equiv h_{ac} h_b{}^c\,. \tag{3.8}$$

It is easy to show that in the Euclidean domain its eigenvalues x^2,

$$Q^a{}_b v^b = x^2 v^a\,, \tag{3.9}$$

are real and non-negative. Using a modified Gram–Schmidt procedure it is possible to show that there exists such an orthonormal basis in which the operator $\boldsymbol{h}$ has the following structure:

$$\mathrm{diag}(0, \ldots, 0, \Lambda_1, \ldots, \Lambda_p)\,, \tag{3.10}$$

where Λ_i are matrices of the form

$$\Lambda_i = \begin{pmatrix} 0 & -x_i I_i \\ x_i I_i & 0 \end{pmatrix}, \tag{3.11}$$

and I_i are unit matrices. We call such a basis an orthonormal *Darboux basis* (see also Section 9.2). Its elements are unit eigenvectors of the problem (3.9).

For a non-degenerate 2-form $\boldsymbol{h}$ the number of zeros in the Darboux decomposition (3.10) coincides with ε. Since all the eigenvalues x in (3.9) are different (we denote them x_μ, $\mu = 1, \ldots, n$), the matrices Λ_i are 2-dimensional. We denote the vectors of the Darboux basis by $\boldsymbol{e}_{\hat{\mu}}$ and $\tilde{\boldsymbol{e}}_{\hat{\mu}} \equiv \boldsymbol{e}_{\hat{n}+\hat{\mu}}$, where $\mu = 1, \ldots, n$, and enumerate them so that the orthonormal vectors $\boldsymbol{e}_{\hat{\mu}}$ and $\tilde{\boldsymbol{e}}_{\hat{\mu}}$ span a 2-dimensional plane of eigenvectors of (3.9) with the same eigenvalue x_μ. In an odd-dimensional spacetime we also have an additional basis vector $\boldsymbol{e}_{\hat{0}}$ (the eigenvector of (3.9) with $x = 0$). We further denote by $\boldsymbol{\omega}^{\hat{\mu}}$ and $\tilde{\boldsymbol{\omega}}^{\hat{\mu}} \equiv \boldsymbol{\omega}^{\hat{n}+\hat{\mu}}$ (and $\boldsymbol{\omega}^{\hat{0}}$ if $\varepsilon = 1$) the dual basis of 1-forms. The metric $\boldsymbol{g}$ and the PCKY tensor $\boldsymbol{h}$ in this basis take the form

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}}, \tag{3.12}$$

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}}. \tag{3.13}$$

In what follows we shall refer to bases $\{\boldsymbol{\omega}\}$ and $\{\boldsymbol{e}\}$ as the *cononical bases* of 1-forms and vectors associated with the PCKY tensor. These bases are fixed uniquely by the PCKY tensor up to a 2D rotation in each of the (KY) 2-planes $\boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}}$.

Since the 'eigenvalues' x_μ are functionally independent in some spacetime domain, we may use them as 'natural' coordinates. In fact, we shall demonstrate in Chapter 7 that these n coordinates can be 'upgraded' by adding $n + \varepsilon$ new coordinates ψ_i, determined completely by the PCKY tensor. Therefore, the PCKY tensor 'determines' in D dimensions D preferred coordinates. We call such preferred coordinates (x_μ, ψ_i) the *canonical coordinates*. The most general (off-shell) *canonical metric* element admitting the PCKY tensor is derived in Chapter 7. When it is written in the canonical coordinates, many of the coefficients of rotation vanish. We call the corresponding (special) canonical basis, the *principal canonical basis*. Such a basis is fixed uniquely; the freedom of 2D rotations was already exploited. (For more details see Chapter 7.)

To summarize, the PCKY tensor determines uniquely the class of canonical spacetimes, together with the preferred canonical coordinates and the preferred principal canonical basis, in which these spacetimes take a 'simple form'.

3.2 Towers of hidden symmetries

In this section we present a simple way how, from a single PCKY tensor, one can generate the whole towers of hidden symmetries. Our approach is based on the lemma of the following subsection. We also derive an explicit form of the tower of Killing tensors in the canonical basis.

3.2.1 Important property of closed CKY tensors

Lemma ([Krtouš *et al.*, 2007b]). *Let $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ be two closed CKY tensors. Then their exterior product $\boldsymbol{k} \equiv \boldsymbol{k}^{(1)} \wedge \boldsymbol{k}^{(2)}$ is also a closed CKY tensor.*

We shall prove this lemma in two steps. The fact that $\boldsymbol{k}$ is closed is trivial, it follows from Eq. (2.16). Let us first show that for a p-form $\boldsymbol{\alpha}_p$ obeying the equation

$$\nabla_X \boldsymbol{\alpha}_p = \boldsymbol{X}^\flat \wedge \boldsymbol{\gamma}_{p-1}\,, \tag{3.14}$$

one has

$$\boldsymbol{\gamma}_{p-1} = -\frac{1}{D-p+1}\,\boldsymbol{\delta\alpha}_p\,. \tag{3.15}$$

Indeed, using Eqs. (2.20), and relations (2.22), (2.23), we find

$$\begin{aligned} -\boldsymbol{\delta\alpha}_p &= \boldsymbol{e}^{\hat{a}} \lrcorner \nabla_{\hat{a}} \boldsymbol{\alpha}_p = \boldsymbol{e}^{\hat{a}} \lrcorner (\boldsymbol{\omega}_{\hat{a}} \wedge \boldsymbol{\gamma}_{p-1}) \\ &= (\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\omega}_{\hat{a}}) \boldsymbol{\gamma}_{p-1} - \boldsymbol{\omega}_{\hat{a}} \wedge (\boldsymbol{e}^{\hat{a}} \lrcorner \boldsymbol{\gamma}_{p-1}) = (D-p+1)\boldsymbol{\gamma}_{p-1}\,. \end{aligned}$$

The second step in the proof of the lemma is to show that if $\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_q$ are two closed CKY tensors then

$$\nabla_X (\boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q) = \boldsymbol{X}^\flat \wedge \boldsymbol{\gamma}_{p+q-1}\,. \tag{3.16}$$

Really, one has

$$\begin{aligned} \nabla_X (\boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q) &= \nabla_X \boldsymbol{\alpha}_p \wedge \boldsymbol{\beta}_q + \boldsymbol{\alpha}_p \wedge \nabla_X \boldsymbol{\beta}_q \\ &= -\frac{1}{D-p+1}(\boldsymbol{X}^\flat \wedge \boldsymbol{\delta\alpha}_p) \wedge \boldsymbol{\beta}_q - \frac{1}{D-q+1}\boldsymbol{\alpha}_p \wedge (\boldsymbol{X}^\flat \wedge \boldsymbol{\delta\beta}_q) \\ &= \boldsymbol{X}^\flat \wedge \boldsymbol{\gamma}_{p+q-1}\,, \end{aligned} \tag{3.17}$$

where

$$\gamma_{p+q-1} = -\frac{1}{D-p+1}\delta\alpha_p \wedge \beta_q - \frac{(-1)^p}{D-q+1}\alpha_p \wedge \delta\beta_q \,.$$

Combining (3.17) with (3.15) we arrive at the statement of the lemma. ♡

3.2.2 Towers of hidden symmetries

According to the lemma of the previous subsection, the PCKY tensor generates a set (tower) of new closed CKY tensors

$$h^{(j)} \equiv h^{\wedge j} = \underbrace{h \wedge \ldots \wedge h}_{\text{total of } j \text{ factors}} \,. \tag{3.18}$$

$h^{(j)}$ is a $(2j)$-form, and in particular $h^{(1)} = h$. Since h is non-degenerate, one has a set of n non-vanishing closed CKY tensors. In an even dimensional spacetime $h^{(n)}$ is proportional to the totally antisymmetric tensor, whereas it is dual to a Killing vector in odd dimensions. In both cases such a CKY tensor is trivial and can be excluded from the tower of hidden symmetries. Therefore we take $j = 1, \ldots, n-1$. The CKY tensors (3.18) can be generated from the potentials $b^{(j)}$ [cf. Eq. (2.28)]

$$b^{(j)} \equiv b \wedge h^{\wedge(j-1)}\,, \quad h^{(j)} = db^{(j)} \,. \tag{3.19}$$

Each $(2j)$-form $h^{(j)}$ determines a $(D-2j)$-form of the Killing–Yano tensor [cf. Eq. (2.29)]

$$f^{(j)} \equiv *h^{(j)} \,. \tag{3.20}$$

In their turn, these tensors give rise to the Killing tensors $K^{(j)}$

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1\ldots c_{D-2j-1}} f^{(j)}{}_b{}^{c_1\ldots c_{D-2j-1}} \,. \tag{3.21}$$

A choice of the coefficient in the definition (3.21) is adjusted to the canonical basis (see the next subsection). It is also convenient to include the metric g, which is a trivial Killing tensor, as an element $K^{(0)}$ of the tower of the Killing tensors. The total number of irreducible elements of this *extended tower* is n.

3.2.3 Explicit form of Killing tensors

Let us now explicitly evaluate the Killing tensors (3.21) in the canonical basis. Using identities

$$\varepsilon^{a_1\ldots a_r c_{r+1}\ldots c_D}\varepsilon_{b_1\ldots b_r c_{r+1}\ldots c_D} = r!(D-r)!\delta^{[a_1}_{b_1}\ldots\delta^{a_r]}_{b_r} \,, \tag{3.22}$$

$$(r+1)\,\delta^{[a}_{[b}\delta^{a_1}_{b_1}\dots\delta^{a_r]}_{b_r]} = \delta^{a}_{b}\delta^{[a_1}_{[b_1}\dots\delta^{a_r]}_{b_r]} - r\,\delta^{a}_{[b_1}\delta^{[a_1}_{|b|}\dots\delta^{a_r]}_{b_r]}\,, \tag{3.23}$$

we can rewrite (3.21) as

$$\begin{aligned} K^{(j)\,a}{}_{b} &= \frac{(2j+1)!}{(2^j j!)^2}\,\delta^{[a}_{[b}h^{a_1b_1}\dots h^{a_jb_j]}h_{a_1b_2}\dots h_{a_jb_j]} \\ &= \frac{(2j)!}{(2^j j!)^2}\Big(\delta^a_b h^{[a_1b_1}\dots h^{a_jb_j]}h_{[a_1b_1}\dots h_{a_jb_j]} - 2j\,h^{a[b_1}\dots h^{a_jb_j]}h_{b[b_1}\dots h_{a_jb_j]}\Big) \\ &= A^{(j)}\delta^a_b - \tilde{K}^{(j)\,a}{}_{b}\,. \end{aligned} \tag{3.24}$$

Here we have introduced

$$\begin{aligned} A^{(j)} &\equiv \frac{(2j)!}{(2^j j!)^2}h^{[a_1b_1}\dots h^{a_jb_j]}h_{[a_1b_1}\dots h_{a_jb_j]}\,, \\ \tilde{K}^{(j)\,a}{}_{b} &\equiv \frac{2j(2j)!}{(2^j j!)^2}h^{a[b_1}\dots h^{a_jb_j]}h_{b[b_1}\dots h_{a_jb_j]}\,. \end{aligned}$$

In the canonical basis, using (3.12) and (3.13), we find

$$\boldsymbol{h}^{(j)} = j!\sum_{\nu_1<\dots<\nu_j} x_{\nu_1}\dots x_{\nu_j}\boldsymbol{\omega}^{\hat{\nu}_1}\wedge\tilde{\boldsymbol{\omega}}^{\hat{\nu}_1}\wedge\dots\wedge\boldsymbol{\omega}^{\hat{\nu}_j}\wedge\tilde{\boldsymbol{\omega}}^{\hat{\nu}_j}\,. \tag{3.25}$$

$$\tilde{\boldsymbol{K}}^{(j)} = \sum_{\mu=1}^{n} x_\mu^2 A_\mu^{(j-1)}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}})\,, \tag{3.26}$$

$$A^{(j)} = \sum_{\nu_1<\dots<\nu_j} x_{\nu_1}^2\dots x_{\nu_j}^2\,,\quad A_\mu^{(j)} = \sum_{\substack{\nu_1<\dots<\nu_j\\ \nu_i\neq\mu}} x_{\nu_1}^2\dots x_{\nu_j}^2\,, \tag{3.27}$$

From the obvious fact that quantities (3.27) obey, $A^{(j)} = A_\mu^{(j)} + x_\mu^2 A_\mu^{(j-1)}$, we obtain the following form of the Killing tensors in the canonical basis:

$$\boldsymbol{K}^{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon A^{(j)}\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}}\,. \tag{3.28}$$

Let us finally mention that it was shown in [Houri *et al.*, 2008a] that $\tilde{K}^{(j)\,a}{}_{b} = Q^a{}_c K^{(j-1)\,c}{}_b$. Here $\boldsymbol{Q}$ is the conformal Killing tensor introduced in Section 3.1.2. Eq. (3.24) therefore gives the following recursive relation for $\boldsymbol{K}^{(j)}$:

$$\boldsymbol{K}^{(j)} = A^{(j)}\boldsymbol{g} - \boldsymbol{Q}\cdot\boldsymbol{K}^{(j-1)}\,,\quad \boldsymbol{K}^{(0)} = \boldsymbol{g}\,. \tag{3.29}$$

3.3 Other method for generating Killing tensors

In this section we describe, in some sense a more physical method, how to generate from the PCKY tensor various towers of Killing tensors. This method is based on the fact that Killing tensors are in one to one correspondence with conserved quantities for a geodesic motion which are of higher order in geodesic momenta [Walker & Penrose, 1970]. In particular, we extract these constants as invariants of the parallel-transported 2-form $\boldsymbol{F}$—obtained as a projection of the PCKY tensor to a subspace orthogonal to the velocity of a geodesic motion [Page *et al.*, 2007]. Depending on how these invariants are extracted one obtains different (related) sets of constants of motion and corresponding towers of Killing tensors. For example, the traces of powers of the operator $\boldsymbol{F}^2$ lead to the set of Killing tensors of increasing rank [Page *et al.*, 2007]. The advantage of this approach is that one can generate constants of geodesic motion with the help of *generating functions*. This gives a powerful tool how to study properties of these constants, such as their independence or Poisson commutativity (see Chapter 5). In particular, we introduce two generating functions: the first one generates constants given by the traces of powers of the operator $\boldsymbol{F}^2$, the second one leads to the earlier described tower of Killing-tensors [Krtouš *et al.*, 2007b].

3.3.1 2-form $\boldsymbol{F}$

Let γ be a geodesic affine parametrized by τ, $u^a = dx^a/d\tau$ be its 'velocity' tangent vector, and $w \equiv u^a u_a$ be its norm. We denote the covariant derivative of a tensor $\boldsymbol{T}$ along γ by

$$\dot{\boldsymbol{T}} \equiv \nabla_u \boldsymbol{T} = u^a \nabla_a \boldsymbol{T} \,. \tag{3.30}$$

In particular $\dot{\boldsymbol{u}} = 0$. Let us now consider the following 2-form $\boldsymbol{F}$ [Page *et al.*, 2007]:

$$\boldsymbol{F} \equiv \boldsymbol{u} \lrcorner (\boldsymbol{u}^\flat \wedge \boldsymbol{h}) = w\boldsymbol{h} - \boldsymbol{u}^\flat \wedge \boldsymbol{s} \,, \quad \boldsymbol{s} \equiv \boldsymbol{u} \lrcorner \boldsymbol{h} \,. \tag{3.31}$$

From the construction, such a form is automatically parallel-transported along γ. Indeed, we have

$$\nabla_u (\boldsymbol{u}^\flat \wedge \boldsymbol{h}) = \boldsymbol{u}^\flat \wedge \nabla_u \boldsymbol{h} = \boldsymbol{u}^\flat \wedge \boldsymbol{u}^\flat \wedge \boldsymbol{\xi}^\flat = 0 \,. \tag{3.32}$$

So, already $\boldsymbol{u}^\flat \wedge \boldsymbol{h} \propto \boldsymbol{u} \lrcorner \boldsymbol{f}$ is parallel-transported [cf. Eq. (2.13)], and the last contraction with $\boldsymbol{u}$ in (3.31) is just to obtain a 2-form with which it is easier to work. Since $\boldsymbol{F}$ is parallel-propagated along γ, any object constructed from $\boldsymbol{F}$ and the metric $\boldsymbol{g}$ is also parallel-propagated. In particular, this is true for the invariants constructed from $\boldsymbol{F}$, such as its eigenvalues. These are therefore constants of motion.[1]

Let us notice that $\boldsymbol{F}$ can be also written as

$$F_{ab} = wP^c_a h_{cd} P^d_b \,. \tag{3.33}$$

Here, $P^b_a = \delta^b_a - w^{-1}u^b u_a$ is the projector to the $(D-1)$-dimensional subspace V orthogonal to the 1-dimensional space U generated by $\boldsymbol{u}$. $P_{ab} = g_{ab} - w^{-1}u_a u_b$ can be also understood as a metric in V induced by its embedding into the tangent space T; $T = U \oplus V$. This means that $\boldsymbol{F}$ has a clear geometrical meaning: *it is the projection of the PCKY tensor* $\boldsymbol{h}$ *along the tangent vector* $\boldsymbol{u}$ *of geodesic* γ. F_{ab} and $F^a{}_b = g^{ac}F_{cb}$ can be considered as a 2-form and an operator, respectively, either in the subspace V or in the complete tangent space T. Since $F^a{}_b u^b = 0$, the vector $\boldsymbol{u} \in T$ is an eigenvector of $\boldsymbol{F}$ with a zero eigenvalue. One also immediately has

$$\dot{F}_{ab} = wP^c_a \dot{h}_{cd} P^d_b = wP^c_a u_{[c}\xi_{d]} P^d_b = 0 \,, \tag{3.34}$$

where we used the defining equation (3.7).

3.3.2 Killing tensors of increasing rank

One of the convenient ways [Page *et al.*, 2007] how to extract the invariants of $\boldsymbol{F}$ is to consider the traces of powers of the operator $\boldsymbol{F}^2$:

$$C_j \equiv w^{-j}\mathrm{Tr}[(-\boldsymbol{F}^2)^j] \,. \tag{3.35}$$

(The traces of odd powers of $\boldsymbol{F}$ are zero, because of the antisymmetry of $\boldsymbol{F}$.)

In what follows we shall use matrix notation in which F is the antisymmetric matrix with orthonormal components $F^{\hat{a}}{}_{\hat{b}}$, H is the antisymmetric matrix with components $h^{\hat{a}}{}_{\hat{b}}$, $Q \equiv -H^2$ is the symmetric matrix with components $Q^{\hat{a}}{}_{\hat{b}} = -h^{\hat{a}}{}_{\hat{c}} h^{\hat{c}}{}_{\hat{b}}$, W is the symmetric matrix with components $u^{\hat{a}}u_{\hat{b}}$, $w \equiv \mathrm{Tr}(W) = u^{\hat{c}}u_{\hat{c}}$, $P \equiv I - p$ is the projection onto the hyperplane V orthogonal to the velocity, and $p = W/w$. These matrices have the properties that $P^2 = P$ and $WH^{2j+1}W = 0$ for all nonnegative integers j. The component Eq. (3.33) becomes the matrix equation

$$F = wPHP \tag{3.36}$$

whose square is the symmetric matrix $F^2 = w^2 P(HP)^2$. So, we get the constants of motion

$$C_j = (-w)^j \mathrm{Tr}[(HP)^{2j}] \,. \tag{3.37}$$

The trace of the matrix product can be viewed diagrammatically as a loop formed

[1] Actually, from the fact that $\boldsymbol{F}$ is parallel-transported along γ one can obtain slightly more, see Chapter 9.

by joined vertices (each with two 'legs') corresponding to matrices in the product. In our case the loop is formed by alternating H and P vertices. Substituting $P = I - p$ we get a sum over all possible loops in which P is replaced either by I or by $-p$. In the case of the identity I the corresponding vertex is effectively eliminated, in the case of one dimensional projector p the loop splits into disconnected pieces. Namely, we can use identity

$$\mathrm{Tr}\big(H^{k_1}p\, H^{k_2}p \cdots H^{k_c}p\big) = \mathrm{Tr}\big(H^{k_1}p\big)\mathrm{Tr}\big(H^{k_2}p\big)\cdots\mathrm{Tr}\big(H^{k_c}p\big)\,. \tag{3.38}$$

The trace in (3.37) thus leads to

$$\mathrm{Tr}\big[(HP)^{2j}\big] = \mathrm{Tr}\big(H^{2j}\big) + \sum_{c=1}^{2j} \sum_{\substack{k_1\leq\cdots\leq k_c\\ k_1+\cdots+k_c=2j}} (-1)^c N^{2j}_{k_1\dots k_j} \prod_{i=1}^{c} \mathrm{Tr}\big(H^{k_i}p\big)\,. \tag{3.39}$$

The sum over c is the sum over number of 'splits' of the loop, the indices k_i are the 'lengths' of the splitted pieces, and the combinatorial factor $N^{2j}_{k_1\dots k_c}$ gives a number of ways in which the loop of the length $2j$ can be split to c pieces of lengths $k_1,\dots,k_c$. From the antisymmetry of H it follows that traces of odd powers of H (optionally multiplied by a projector) are zero. Setting $k_i = 2l_i$ and introducing Q as earlier we have

$$C_j = w^j\mathrm{Tr}\big(Q^j\big) + \sum_{c=1}^{j} \sum_{\substack{l_1\leq\cdots\leq l_c\\ l_1+\cdots+l_c=j}} (-1)^c\, 2\, N^{j}_{l_1\dots l_j} w^j \prod_{i=1}^{c} \mathrm{Tr}\big(Q^{l_i}p\big)\,, \tag{3.40}$$

where we have used $N^{2j}_{2l_1\dots 2l_c} = 2N^{j}_{l_1\dots l_c}$ which follows from the definition of N's. Let us define the following quantities:

$$w_j \equiv w\mathrm{Tr}(Q^j p) = \mathrm{Tr}(Q^j W) \equiv Q^{(j)}_{ab} u^a u^b\,. \tag{3.41}$$

Here, $Q^{(j)}_{ab}$ denote covariant components that form the tensor $\boldsymbol{Q}^{(j)}$ corresponding to the j-th power of the matrix Q. We also denote $Q^{(j)} \equiv \mathrm{Tr}\big(Q^j\big)$. For example, $Q^{(1)} = Q_c{}^c$, $Q^{(1)}_{ab} = Q_{ab}$, $Q^{(2)} = Q_c{}^d Q_d{}^c$, $Q^{(2)}_{ab} = Q_a{}^c Q_{cb}$, $Q^{(3)} = Q_c{}^d Q_d{}^e Q_e{}^c$, and $Q^{(3)}_{ab} = Q_a{}^c Q_c{}^d Q_{db}$. Then we finally obtain

$$C_j = w^j Q^{(j)} - 2j w^{j-1} w_j + 2\sum_{c=2}^{j} \sum_{\substack{l_1\leq\cdots\leq l_c\\ l_1+\cdots+l_c=j}} (-1)^c\, N^{j}_{l_1\dots l_j} w^{j-c} \prod_{i=1}^{c} w_{l_i}\,. \tag{3.42}$$

We can easily see that the C_j's have the form

$$C_j = K_{a_1 \dots a_{2j}} u^{a_1} \dots u^{a_{2j}} , \tag{3.43}$$

where $K_{a_1 \dots a_{2j}}$, formed from combinations of the metric g_{ab}, $\mathrm{Tr}(Q^j)$, and the $Q^{(i)}_{ab}$'s for $i \le j$, are Killing tensors [Walker & Penrose, 1970] in the sense of Eq. (2.6).

In particular, we get the first four constants of motion

$$\begin{aligned}
C_1 &= wQ^{(1)} - 2w_1 , \\
C_2 &= w^2 Q^{(2)} - 4w\, w_2 + 2w_1^2 , \\
C_3 &= w^3 Q^{(3)} - 6w^2 w_3 + 6w\, w_1 w_2 - 2w_1^2 , \\
C_4 &= w^4 Q^{(4)} - 8w^3 w_4 + 4w^2 w_2^2 + 8w^2 w_1 w_3 - 8w\, w_1^2 w_2 + 2w_1^4 .
\end{aligned} \tag{3.44}$$

Comparing with (3.43) we obtain the corresponding tower of (reducible) Killing tensors of increasing rank:

$$\begin{aligned}
K_{ab} &= g_{ab} Q^{(1)} - 2Q^{(1)}_{ab} , \\
K_{abcd} &= g_{(ab} g_{cd)} Q^{(2)} - 4 g_{(ab} Q^{(2)}_{cd)} + 2 Q^{(1)}_{(ab} Q^{(1)}_{cd)} , \\
K_{abcdef} &= g_{(ab} g_{cd} g_{ef)} Q^{(3)} - 6 g_{(ab} g_{cd} Q^{(3)}_{ef)} + 6 g_{(ab} Q^{(1)}_{cd} Q^{(2)}_{ef)} - 2 Q^{(1)}_{(ab} Q^{(1)}_{cd} Q^{(1)}_{ef)} , \\
K_{abcdefgh} &= g_{(ab} g_{cd} g_{ef} g_{gh)} Q^{(4)} - 8 g_{(ab} g_{cd} g_{ef} Q^{(4)}_{gh)} + 4 g_{(ab} g_{cd} Q^{(2)}_{ef} Q^{(2)}_{gh)} \\
&+ 8 g_{(ab} g_{cd} Q^{(1)}_{ef} Q^{(3)}_{gh)} - 8 g_{(ab} Q^{(1)}_{cd} Q^{(1)}_{ef} Q^{(2)}_{gh)} + 2 Q^{(1)}_{(ab} Q^{(1)}_{cd} Q^{(1)}_{ef} Q^{(1)}_{gh)} .
\end{aligned} \tag{3.45}$$

To write an explicit form of the constants of motion or the Killing tensors obtained we can use the canonical basis. There we have

$$\boldsymbol{Q}^{(j)} = Q^{(j)}_{\hat{a}\hat{b}} \boldsymbol{\omega}^{\hat{a}} \boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n} x_\mu^{2j} (\boldsymbol{\omega}^{\hat{\mu}} \boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}} \tilde{\boldsymbol{\omega}}^{\hat{\mu}}) , \tag{3.46}$$

and also

$$Q^{(j)} = 2 \sum_{\mu=1}^{n} x_\mu^{2j} , \qquad w_j = \sum_{\mu=1}^{n} x_\mu^{2j} (u_{\hat{\mu}}^2 + \tilde{u}_{\hat{\mu}}^2) , \tag{3.47}$$

where $u_{\hat{a}}$ denotes the basis components of the velocity

$$\boldsymbol{u}^\flat = \sum_{\mu=1}^{n} \left(u_{\hat{\mu}} \boldsymbol{\omega}^{\hat{\mu}} + \tilde{u}_{\hat{\mu}} \tilde{\boldsymbol{\omega}}^{\hat{\mu}} \right) + \varepsilon\, u_{\hat{0}} \boldsymbol{\omega}^{\hat{0}} . \tag{3.48}$$

3.3.3 Generating functions

Although formula (3.42) gives the constants of motion explicitly, the presence of combinatoric factors makes calculations difficult in practice. In this subsection we introduce generating functions [Krtouš *et al.*, 2007b] which allow to, aside from other things, write down a more useful formula how to evaluate these constants.

We introduce the *generating function* $W(\beta)$,

$$W(\beta) \equiv \det\bigl(I + \sqrt{\beta} w^{-1} F\bigr) \ . \tag{3.49}$$

Due to the antisymmetry of F and properties of the determinant, $W(\beta)$ can be rewritten as a function of β instead of $\sqrt{\beta}$, and in terms of H and P instead of F,

$$W(\beta) = \det{}^{1/2}\bigl(I - \beta w^{-2} F^2\bigr) = \det\bigl(I - \sqrt{\beta}\, H P\bigr) \ . \tag{3.50}$$

Because it is constructed only in terms of covariantly conserved quantities F and w, the generating function is conserved along γ, and the same is true for its derivatives with respect to β. We can thus define constants of motion κ_j as the coefficients in the β-expansion of $W(\beta)$:

$$W(\beta) \equiv \frac{1}{w} \sum_{j=0}^{\infty} \kappa_j \, \beta^j \ . \tag{3.51}$$

It turns out that all terms with $j > n$ are zero. To evaluate the observables κ_j, we can split $W(\beta)$ in the following way:

$$W(\beta) = W_0(\beta)\ \Sigma(\beta) \ , \tag{3.52}$$

where

$$\begin{aligned} W_0(\beta) &= \det\bigl(I - \sqrt{\beta} H\bigr) = \det{}^{1/2}\bigl(I + \beta Q\bigr) \ , \\ \Sigma(\beta) &= \det\Bigl(I + \frac{\sqrt{\beta} H}{I - \sqrt{\beta} H}\, p\Bigr) = \mathrm{Tr}\Bigl[\sum_{j=0}^{\infty} \bigl(\sqrt{\beta} H\bigr)^j p\Bigr] = \sum_{j=0}^{\infty} \mathrm{Tr}(H^{2j} p) \beta^j \\ &= \sum_{j=0}^{\infty} (-1)^j \mathrm{Tr}(Q^j p) \beta^j = \mathrm{Tr}\bigl[(I + \beta Q)^{-1} p\bigr] = \frac{1}{w} \sum_{j=0}^{\infty} (-1)^j w_j \beta^j \ . \end{aligned} \tag{3.53}$$

Here we have used the fact that the matrix in the determinant in the expression for $\Sigma(\beta)$ differs from I only in the one-dimensional subspace U given by $\boldsymbol{u}$. The generating function $W(\beta)$ thus splits into a part $W_0(\beta)$ independent of $\boldsymbol{u}$ and a part $\Sigma(\beta)$ linear in p. Such generating function therefore leads to the tower of

2nd-rank Killing tensors; these will be described in the next section.

Let us now consider a different generating function $Z(\beta)$,

$$Z(\beta) \equiv \log W(\beta)\,. \tag{3.54}$$

Using the relation

$$\log\Big[\det(I - A)\Big] = -\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{Tr}(A^n)\,, \tag{3.55}$$

we find

$$\begin{aligned} Z(\beta) &= \log W_0(\beta) + \log \Sigma(\beta) = \log\Big[\det\big(I - \sqrt{\beta}\,HP\big)\Big] \\ &= -\sum_{j=1}^{\infty} \frac{1}{2j}\mathrm{Tr}\big[(HP)^{2j}\big]\beta^j = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2j}\frac{C_j}{w^j}\beta^j\,. \end{aligned} \tag{3.56}$$

Constants C_j, given by (3.42), therefore correspond to terms proportional to β^j in the power expansion of $Z(\beta)$. The first term of (3.42) is obtained from $\log W_0(\beta)$, and the sum over all possible splittings of the loop corresponds to the β^j term of $\log\Sigma(\beta)$. Clearly, the j-th derivative of $\log\Sigma(\beta)$ (evaluated at $\beta = 0$) contains the sum over all possible products of l-th derivatives $\Sigma^{(l)}(0)$ which are proportional to w_l defined in (3.41). The factors $N^j_{l_1\dots l_2}$ can thus be obtained by the explicit computation of the derivatives of the generating function $\log\Sigma(\beta)$:

$$C_j = w^j Q^{(j)} - \frac{2(-w)^j}{(j-1)!}\frac{d^j}{d\beta^j}\left[\log\big(1 + \sum_{k=1}^{j}(-1)^k \frac{w_k}{w}\beta^k\big)\right]_{\beta=0}\,. \tag{3.57}$$

With the help of this formula and a software for algebraic manipulation one can easily generate constants of motion C_j.

The relation between $W(\beta)$ and $Z(\beta)$ implies that constants C_j and constants κ_j [introduced in (3.51)] are related as follows:

$$C_j = -\frac{2(-w)^j}{(j-1)!}\frac{d^j}{d\beta^j}\left[\log\big(w + \sum_{k=1}^{j}\kappa_k\beta^k\big)\right]_{\beta=0}\,. \tag{3.58}$$

So, C_j's are polynomial combinations of κ_j's and w with constant coefficients.

In particular, we get

$$\begin{aligned}
C_1 &= 2\kappa_1 \,, \\
C_2 &= -4w\kappa_2 + 2\kappa_1^2 \,, \\
C_3 &= 6w^2\kappa_3 - 6w\kappa_1\kappa_2 + 2\kappa_1^3 \,, \\
C_4 &= -8w^3\kappa_4 + 8w^2\kappa_1\kappa_3 + 4w^2\kappa_2^2 - 8w\kappa_1^2\kappa_2 + 2\kappa_1^4 \,.
\end{aligned} \tag{3.59}$$

3.3.4 Rank-2 Killing tensors

Using the canonical basis, let us now explicitly evaluate the 2nd-rank Killing tensors generated by function $W(\beta)$. We again introduce the quantities

$$A^{(j)} = \sum_{\nu_1<\dots<\nu_j} x_{\nu_1}^2 \dots x_{\nu_j}^2 \,, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1<\dots<\nu_j \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_j}^2 \,. \tag{3.60}$$

Then, with the help of relations (3.46) and (3.47), we find

$$W_0(\beta) = \det{}^{1/2}(I + \beta Q) = \prod_{\mu=1}^{n} (1 + \beta x_\mu^2) = \sum_{j=0}^{n} A^{(j)} \beta^j \,, \tag{3.61}$$

$$\Sigma(\beta) = \frac{1}{w} \sum_{j=0}^{\infty} (-1)^j w_j \beta^j = \frac{1}{w} \Big(\varepsilon u_{\hat{0}}^2 + \sum_{j=0}^{\infty} (-1)^j \beta^j \sum_{\mu=1}^{n} (u_{\hat{\mu}}^2 + \tilde{u}_{\hat{\mu}}^2)\, x_\mu^{2j} \Big). \tag{3.62}$$

The original generating function (3.52) reads

$$W(\beta) = \frac{1}{w} \sum_{j=0}^{n} \Big(\sum_{l=0}^{j} (-1)^l A^{(j-l)} w_l \Big) \beta^j = \frac{1}{w} \sum_{j=0}^{n} \Big[\varepsilon A^{(j)} u_{\hat{0}}^2 + \sum_{\mu=1}^{n} A_\mu^{(j)} \big(u_{\hat{\mu}}^2 + \tilde{u}_{\hat{\mu}}^2 \big) \Big] \beta^j \,. \tag{3.63}$$

Comparing Eq. (3.63) with Eq. (3.51), we can identify $n + \varepsilon$ conserved quantities κ_j (constants of geodesic motion, $j = 0, \dots, n + \varepsilon - 1$),

$$\kappa_j = \sum_{l=0}^{j} (-1)^l A^{(j-l)} w_l = \varepsilon A^{(j)} u_{\hat{0}}^2 + \sum_{\mu=1}^{n} A_\mu^{(j)} \big(u_{\hat{\mu}}^2 + \tilde{u}_{\hat{\mu}}^2 \big) \,, \tag{3.64}$$

which are quadratic in velocities. They are generated [Walker & Penrose, 1970] by the 2nd-rank Killing tensors $\boldsymbol{K}^{(j)}$

$$\kappa_j = K_{ab}^{(j)} u^a u^b \,, \quad \nabla_{(a} K_{bc)}^{(j)} = 0 \,, \tag{3.65}$$

where

$$\boldsymbol{K}^{(j)} = \sum_{l=0}^{j}(-1)^l A^{(j-l)}\boldsymbol{Q}^{(l)} = \sum_{\mu=1}^{n} A_\mu^{(j)}(\boldsymbol{\omega}^{\hat\mu}\boldsymbol{\omega}^{\hat\mu} + \tilde{\boldsymbol{\omega}}^{\hat\mu}\tilde{\boldsymbol{\omega}}^{\hat\mu}) + \varepsilon A^{(j)}\boldsymbol{\omega}^{\hat 0}\boldsymbol{\omega}^{\hat 0}\,. \tag{3.66}$$

The Killing tensor $\boldsymbol{K}^{(n)}$ is present only in an odd number of spacetime dimensions and it is reducible. Similar to Section 3.2.2 we exclude it from the set. The remaining tensors (3.66) coincide with those of the extended tower introduced in Section 3.2.2.

The first expression for $\boldsymbol{K}^{(j)}$ in (3.66) can be easily derived from the recursive relation (3.29) [Houri *et al.*, 2008a]. It immediately implies that

$$\boldsymbol{K}^{(i)} \cdot \boldsymbol{K}^{(j)} = \boldsymbol{K}^{(j)} \cdot \boldsymbol{K}^{(i)}\,, \tag{3.67}$$

and therefore $\boldsymbol{K}^{(j)}$'s have common eigenvectors (see also Section 6.3.1).

We shall prove in Chapter 5, that observables κ_j are in *involution*, that is, they mutually Poisson commute:

$$\{\kappa_i, \kappa_j\} = 0\,. \tag{3.68}$$

This is equivalent to vanishing of the Schouten–Nijenhuis (SN) brackets [Schouten, 1940], [Schouten, 1954], [Nijenhuis, 1955] for the corresponding Killing tensors (see Section 5.5):

$$\left[K^{(j)}, K^{(l)}\right]_{abc} \equiv K^{(j)}_{e(a}\nabla^e K^{(l)}_{bc)} - K^{(l)}_{e(a}\nabla^e K^{(j)}_{bc)} = 0\,. \tag{3.69}$$

Once (3.68) is proved for κ_j's, the relation (3.58) shows that also observables C_j are in involution, and vice versa. An independent proof of mutual Poisson commutativity of observables κ_j was later demonstrated in [Houri *et al.*, 2008a], using the method of generating functions.

3.4 Tower of Killing vectors

In the previous two sections we have seen that the PCKY tensor determines the whole set of hidden symmetries. In this section we demonstrate that it also naturally generates $n+\varepsilon$ vectors $\boldsymbol{\xi}^{(k)}$ $(k = 0, \ldots, n-1+\varepsilon)$ which turn out to be the independent commuting Killing vector fields [Krtouš *et al.*, 2007b], [Krtouš *et al.*, 2008b].

The *primary* (Killing) vector $\boldsymbol{\xi}^{(0)} \equiv \boldsymbol{\xi}$ is defined by (3.4),

$$\xi^{(0)}_b \equiv \xi_b = \frac{1}{D-1}\nabla_d h^d{}_b\,. \tag{3.70}$$

The *secondary* (Killing) vectors $\boldsymbol{\xi}^{(j)} \equiv \boldsymbol{\eta}^{(j)}$ ($j = 1, \dots, n-1$) can be constructed as

$$\xi^{(j)\,a} \equiv \eta^{(j)\,a} \equiv K^{(j)a}{}_{b}\xi^{b}\,. \tag{3.71}$$

In odd dimensions the last Killing vector is given by the n-th Killing–Yano tensor (see Section 3.2.2)

$$\boldsymbol{\xi}^{(n)} \equiv \boldsymbol{f}^{(n)}\,. \tag{3.72}$$

The proof that all these vectors are the mutually commuting Killing fields which also (Schauten-Nijenhuis) commute with the Killing tensors constructed in Section 3.3.4,

$$[\boldsymbol{\xi}^{(i)}, \boldsymbol{K}^{(j)}] = 0\,, \quad [\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}^{(j)}] = 0\,, \tag{3.73}$$

is demonstrated in Chapter 7.[2]

[2]Historically, this fact was first proved [Houri *et al.*, 2008a] under the additional conditions

$$\pounds_{\xi}\boldsymbol{g} = 0\,, \quad \pounds_{\xi}\boldsymbol{h} = 0\,. \tag{3.74}$$

The first condition requires that $\boldsymbol{\xi}$ is a Killing vector. [This condition is trivially satisfied in any Einstein space, cf. Eq. (2.11).] It is easy to see, that from the second condition it follows that also $\eta^{(j)}$'s are the Killing vectors. Indeed, from (3.3) we have $\nabla_{\xi}\boldsymbol{h} = 0$. Using (3.74), we find

$$\pounds_{\xi}\boldsymbol{K}^{(j)} = 0\,, \quad \nabla_{\xi}\boldsymbol{K}^{(j)} = 0\,, \tag{3.75}$$

and therefore

$$\nabla_{(a}\eta^{(j)}_{b)} = \frac{1}{2}\pounds_{\xi}K^{(j)}_{ab} - \nabla_{\xi}K^{(j)}_{ab} = 0\,. \tag{3.76}$$

It is shown in Chapter 7 that both conditions (3.74) follow from the existence of the PCKY tensor.

Part II

Remarkable Properties of Higher-Dimensional Rotating Black Holes

Chapter 4

PCKY tensor in the Kerr-NUT-(A)dS spacetimes

In this chapter, based on [Kubizňák & Frolov, 2007], we demonstrate that the general Kerr–NUT–(A)dS spacetime, describing the higher-dimensional arbitrarily rotating black hole with NUT parameters and the cosmological constant, possesses the PCKY tensor. We write the Kerr–NUT–(A)dS metric in canonical coordinates, completely determined by the PCKY tensor. In this (canonical) form, the metric can be considered as a natural higher-dimensional generalization of the Carter's canonical form for the 4D Kerr-NUT-(A)dS solution. The invariant (geometrical) definition of canonical coordinates makes the canonical form convenient for calculations. For example, it is these coordinates in which the Hamilton–Jacobi equation separates (see Chapter 6). We also introduce, a more general, (off-shell) canonical metric and its principal canonical basis.

4.1 Overview of the Kerr-NUT-(A)dS metrics

The most general known higher-dimensional ($D > 2$) solution describing rotating black holes with NUT parameters in an asymptotically (Anti) de Sitter spacetime (Kerr-NUT-(A)dS metric) was found by Chen, Lü, and Pope [Chen *et al.*, 2006a]. We write it in the following symmetric (analytically continued) form:

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{\boldsymbol{d}x_\mu^2}{Q_\mu} + Q_\mu \Big(\sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \Big(\sum_{j=0}^{n} A^{(j)} \boldsymbol{d}\psi_j \Big)^2. \tag{4.1}$$

Here, functions $A^{(j)}$, $A^{(j)}_\mu$ are given by (3.60), and

$$Q_\mu = \frac{X_\mu}{U_\mu}\,, \quad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2)\,. \tag{4.2}$$

Metric functions X_μ are functions of x_μ only, and for the Kerr-NUT-(A)dS solution take the form

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}\,. \tag{4.3}$$

Time is denoted by ψ_0, azimuthal coordinates by ψ_j, $j = 1, \ldots, m \equiv D - n - 1$, x_n is an analytical continuation of the Boyer–Lindquist type radial coordinate, and x_μ, $\mu = 1, \ldots, n-1$, stand for latitude coordinates.[1] The parameter c_n is proportional to the cosmological constant [Hamamoto *et al.*, 2007]

$$R_{ab} = (-1)^n (D-1) c_n\, g_{ab}\,, \tag{4.4}$$

and the remaining constants c_k, $c > 0$, and b_μ are related to rotation parameters, mass, and NUT parameters. One of these constants may be eliminated due to the scaling symmetry. The metric therefore constitutes the $(D-1-\varepsilon)$–parametric Einstein space (see [Chen *et al.*, 2006a] for more details). The limit of flat spacetime is recovered when $c_n = 0$ and all of the parameters b_μ are zero (equal of one another) in the even (odd) dimensional case.

The Kerr-NUT-(A)dS spacetime (4.1)–(4.3) may be understood as a higher-dimensional generalization of the four-dimensional Kerr-NUT-(A)dS solution obtained by Carter [Carter, 1968c], [Carter, 1968b]. Moreover, the coordinates (x_μ, ψ_j) used in the metric are the higher-dimensional analogue of the canonical coordinates [Carter, 1968b], [Carter, 1968c], [Debever, 1971], [Plebański, 1975]. As discussed in the next section, they have a well defined geometrical meaning. More generally, it is possible to consider a broader class of metrics (4.1) where X_μ's are arbitrary functions of one variable; $X_\mu = X_\mu(x_\mu)$. To stress that such metrics do not necessarily satisfy the Einstein equations we call them *off-shell* metrics. It will be shown in Chapter 7, that the most general metric element admitting the PCKY tensor, the *canonical metric* element, coincides with the off-shell spacetime (4.1). Therefore, from now on we refer to the off-shell spacetime (4.1), without imposing (4.3), as to the canonical metric. The canonical metric is of the special algebraic type D [Hamamoto *et al.*, 2007] of the higher-dimensional

[1] Similar to the 4D case (see Appendix A.1.3), the signature of the symmetric form of the metric depends on the domain of x_μ's and signs of X_μ's. The physical Kerr-NUT-(A)dS spacetime is recovered when standard radial coordinate $r = -ix_n$, and new parameter $M = (-i)^{1+\epsilon} b_n$, are introduced, that is, by a simple Wick rotation. See also Section 9.4.1.

algebraic classification [Milson *et al.*, 2005], [Coley *et al.*, 2004], [Coley, 2008]. Let us finally remark that formulas (4.1)–(4.3) are applicable also in $D = 3$ where one recovers the 2–parametric BTZ black hole [Banados *et al.*, 1992].

In what follows we shall also use the orthonormal form of the metric

$$g = \delta_{ab}\omega^{\hat{a}}\omega^{\hat{b}} = \sum_{\mu=1}^{n}(\omega^{\hat{\mu}}\omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}}\tilde{\omega}^{\hat{\mu}}) + \varepsilon\omega^{\hat{0}}\omega^{\hat{0}}\,, \tag{4.5}$$

$$\omega^{\hat{\mu}} = \frac{dx_\mu}{\sqrt{Q_\mu}}\,, \quad \tilde{\omega}^{\hat{\mu}} = \sqrt{Q_\mu}\sum_{j=0}^{n-1}A_\mu^{(j)}d\psi_j\,, \quad \omega^{\hat{0}} = \sqrt{\frac{-c}{A^{(n)}}}\sum_{j=0}^{n}A^{(j)}d\psi_j\,. \tag{4.6}$$

The inverse metric reads

$$g^{-1} = \sum_{\mu=1}^{n}\Big[Q_\mu(\partial_{x_\mu})^2 + \frac{1}{Q_\mu U_\mu^2}\Big(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\partial_{\psi_k}\Big)^2\Big] - \frac{\varepsilon}{cA^{(n)}}(\partial_{\psi_n})^2\,, \tag{4.7}$$

or, in the orthonormal form

$$g^{-1} = \delta^{ab}e_{\hat{a}}e_{\hat{b}} = \sum_{\mu=1}^{n}(e_{\hat{\mu}}e_{\hat{\mu}} + \tilde{e}_{\hat{\mu}}\tilde{e}_{\hat{\mu}}) + \varepsilon e_{\hat{0}}e_{\hat{0}}, \tag{4.8}$$

$$e_{\hat{\mu}} = \sqrt{Q_\mu}\partial_{x_\mu}\,, \quad \tilde{e}_{\hat{\mu}} = \frac{1}{\sqrt{Q_\mu}U_\mu}\sum_{j=0}^{m}(-x_\mu^2)^{n-1-j}\partial_{\psi_j}\,, \quad e_{\hat{0}} = \frac{\partial_{\psi_n}}{\sqrt{-cA^{(n)}}}\,. \tag{4.9}$$

The inverse relations to (4.9) are

$$\partial_{x_\mu} = \frac{e_{\hat{\mu}}}{\sqrt{Q_\mu}}\,, \quad \partial_{\psi_j} = \sum_{\mu=1}^{n}\sqrt{Q_\mu}A_\mu^{(j)}\tilde{e}_{\hat{\mu}} + \varepsilon A^{(j)}\sqrt{\frac{-c}{A^{(n)}}}e_{\hat{0}}\,, \quad \partial_{\psi_n} = \sqrt{-cA^{(n)}}e_{\hat{0}}\,. \tag{4.10}$$

The determinant of the metric g reads

$$g = \det(g) = \big(-cA^{(n)}\big)^{\varepsilon}U^2\,, \quad U \equiv \det\big[A_\mu^{(j)}\big] = \prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n}(x_\mu^2 - x_\nu^2)\,. \tag{4.11}$$

In the last expression, $A_\mu^{(j)}$, given by (3.60), is understood as the $n \times n$ matrix. Some algebraic identities regarding these functions or other properties of the canonical metric are gathered in Appendix C.4.

4.2 Principal conformal Killing–Yano tensor

The general canonical metric (4.1) described in the previous section, and in particular the Kerr-NUT-(A)dS spacetime (4.1)–(4.3), possesses a PCKY tensor [Kubizňák & Frolov, 2007].[2] The corresponding 1-form (KY) potential $\boldsymbol{b}$ reads

$$\boldsymbol{b} = \frac{1}{2}\sum_{j=0}^{n-1} A^{(j+1)} \boldsymbol{d}\psi_j \,. \tag{4.12}$$

The PCKY tensor, $\boldsymbol{h} = \boldsymbol{db}$, takes the following forms:

$$\boldsymbol{h} = \frac{1}{2}\sum_{j=0}^{n-1} \boldsymbol{d}A^{(j+1)} \wedge \boldsymbol{d}\psi_j = \frac{1}{2}\sum_{\mu=1}^{n}\Big[\boldsymbol{d}x_\mu^2 \wedge \sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\psi_j\Big] = \sum_{\mu=1}^{n} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}} \,. \tag{4.13}$$

The last expression shows that the basis $\{\boldsymbol{\omega}\}$, introduced in (4.6), is a canonical basis associated with the PCKY tensor $\boldsymbol{h}$ (see Section 3.1.2). In fact, this canonical basis has an additional nice property that many of the Ricci coefficients of rotation vanish [Hamamoto *et al.*, 2007], [Krtouš *et al.*, 2008b]; it is a *principal canonical basis* (see also Chapter 7).

Having a PCKY tensor and its canonical basis, we may employ the machinery of Chapter 3. In particular, we obtain the following extended tower of the 2nd-rank irreducible Killing tensors ($j = 0, \dots, n-1$):

$$\boldsymbol{K}^{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} (\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon A^{(j)} \boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}} \,. \tag{4.14}$$

The Killing fields (3.70)–(3.72) become ($i = 1, \dots, n-1$)

$$\boldsymbol{\xi}^{(0)} = \boldsymbol{\partial}_{\psi_0} \,, \quad \boldsymbol{\xi}^{(i)} = \boldsymbol{\partial}_{\psi_i} \,, \quad \boldsymbol{\xi}^{(n)} = \boldsymbol{\partial}_{\psi_n} \,. \tag{4.15}$$

This means that coordinates (x_μ, ψ_j) are *canonical coordinates*. All of them are completely determined by the PCKY tensor: 'essential' coordinates x_μ are connected with its eigenvalues (see Section 3.1.2), Killing coordinates ψ_j ($j = 0, \dots, m$) are defined by the tower of Killing vectors generated from this tensor. It is this invariant definition of coordinates what makes the form (4.1) of the canonical metric so convenient for calculations. For example, we shall see in Chapter 6 that these coordinates are the normal separable coordinates for which the

[2] In fact, the PCKY tensor in higher dimensions was first discovered for the Myers–Perry metrics [Frolov & Kubizňák, 2007], and only after that for the general Kerr-NUT-(A)dS spacetimes. For an account of these historical developments we refer the reader to Appendix B.

Hamilton–Jacobi and Klein–Gordon equations allow the separation of variables.

Chapter 5

Complete integrability of geodesic motion

In this chapter we demonstrate that in the canonical spacetime (4.1), the n constants of geodesic motion corresponding to the extended tower of Killing tensors and the $D-n$ constants of geodesic motion corresponding to the tower of Killing vectors are functionally independent of one another, making a total of D independent constants of motion in all dimensions D. The Poisson brackets of all pairs of these D constants are zero, so, the geodesic motion in these spacetimes is completely integrable [Page *et al.*, 2007], [Krtouš *et al.*, 2007a].

5.1 Constants of motion

In the previous chapter we have seen that the (off-shell) canonical spacetime (4.1) admits the PCKY tensor $\boldsymbol{h}$, (4.13), which in its turn generates the extended tower of n Killing tensors $\boldsymbol{K}^{(j)}$, (4.14), and the tower of $D-n$ Killing vectors $\boldsymbol{\partial}_{\psi_k}$, (4.15). Together, these objects give D constants of geodesic motion,[1]

$$\Psi_k = \xi^{(k)}_{\ a} u^a = \boldsymbol{u}\cdot\boldsymbol{\partial}_{\psi_k}\,,\quad \kappa_j = K^{(j)}_{\ ab} u^a u^b = \boldsymbol{u}\cdot\boldsymbol{K}^{(j)}\cdot\boldsymbol{u}\,. \tag{5.2}$$

Here, we have denoted the momentum of the geodesic motion $\boldsymbol{u}$, $u^a = dx^a/d\tau$, and we understand all mentioned quantities as observables (i.e. functions) on the phase space $\boldsymbol{\Gamma} \equiv \mathbf{T}^*M$. (For a review of the canonical mechanics on the

[1] Instead of constants κ_j, one can consider a different set of n constants corresponding to various invariants of the form $\boldsymbol{F}$, (3.31). For example, we may consider [cf. Eq. (3.37)]

$$\tilde{C}_0 \equiv w = \kappa_0 = \boldsymbol{u}\cdot\boldsymbol{u}\,,\quad \tilde{C}_j \equiv \mathrm{Tr}\big[(H\tilde{P})^{2j}\big] = (-w)^j C_j\,,\quad \tilde{P} \equiv wP = wI - W\,. \tag{5.1}$$

In Section 5.4, we shall use this choice to prove the Poisson commutativity of κ_j's.

phase space Γ see, e.g., Appendix of [Krtouš *et al.*, 2007a].)

Let us now explicitly evaluate constants (5.2) in the orthonormal basis (4.5). There we have

$$\boldsymbol{u}^\flat = \sum_{\mu=1}^{n} \left(u_{\hat\mu}\boldsymbol{\omega}^{\hat\mu} + \tilde u_{\hat\mu}\tilde{\boldsymbol{\omega}}^{\hat\mu}\right) + \varepsilon\, u_{\hat 0}\boldsymbol{\omega}^{\hat 0} . \tag{5.3}$$

Using (4.10) and (3.64) we find

$$\Psi_k = \sum_{\mu=1}^{n} \sqrt{Q_\mu} A^{(k)}_\mu \tilde u_{\hat\mu} + \varepsilon A^{(k)} \sqrt{\frac{-c}{A^{(n)}}}\, u_{\hat 0} , \quad \Psi_n = \sqrt{-cA^{(n)}}\, u_{\hat 0} , \tag{5.4}$$

$$\kappa_j = \sum_{l=0}^{j} (-1)^l A^{(j-l)} w_l = \sum_{\mu=1}^{n} A^{(j)}_\mu \left(u_{\hat\mu}^2 + \tilde u_{\hat\mu}^2\right) + \varepsilon A^{(j)} u_{\hat 0}^2 . \tag{5.5}$$

These formulas may be easily inverted using relation (4.10) and identities

$$\sum_{j=0}^{n-1} \frac{(-x_\nu^2)^{n-1-j}}{U_\nu} A^{(j)}_\mu = \delta^\mu_\nu , \quad \sum_{\mu=1}^{n} \frac{A^{(k)}_\mu}{x_\mu^2 U_\mu} = \frac{A^{(k)}}{A^{(n)}} , \tag{5.6}$$

proved in Appendix C.4. The result is given by formula (5.7) below.

5.2 Complete integrability

Definition. A motion in M^D is *completely integrable* if there exist D functionally independent integrals of motion which are in *involution*, that is, they mutually Poisson commute of one another [Arnol'd, 1989], [Kozlov, V. V., 1983].

Proposition. *The geodesic motion in the canonical spacetime* (4.1) *is completely integrable. The geodesic momentum* $\boldsymbol{u}$ *can be written in the form* (5.3), *where the basis components (expressed in terms of integrals of motion* Ψ_k *and* κ_j*) are:*

$$u_{\hat\mu} = \frac{\sigma_\mu}{(X_\mu U_\mu)^{1/2}} \left(X_\mu V_\mu - W_\mu^2\right)^{1/2} , \quad \tilde u_{\hat\mu} = \frac{1}{\sqrt{Q_\mu}} \frac{W_\mu}{U_\mu} , \quad u_{\hat 0} = \frac{\Psi_n}{A^{(n)}} \sqrt{\frac{A^{(n)}}{-c}} . \tag{5.7}$$

Constants $\sigma_\mu = \pm 1$ *are independent of one another, and*

$$V_\mu \equiv \sum_{j=0}^{m} (-x_\mu^2)^{n-1-j} \kappa_j , \quad W_\mu \equiv \sum_{k=0}^{m} (-x_\mu^2)^{n-1-k} \Psi_k , \quad \kappa_n \equiv -\frac{\Psi_n^2}{c} . \tag{5.8}$$

In order to prove this proposition, we need to establish the functional independence and Poisson commutativity of integrals of motion κ_j and Ψ_k. This is done in the following two sections.

The coordinate components of the velocity are

$$\dot{x}_\mu = \frac{\sigma_\mu}{|U_\mu|}\left(X_\mu V_\mu - W_\mu^2\right)^{1/2}, \quad \dot{\psi}_k = \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu X_\mu} W_\mu - \varepsilon \frac{\Psi_n}{cA^{(n)}}. \tag{5.9}$$

To obtain these expressions we have used (4.6), and the explicit form of the inverse metric (4.7). Using formula (C.48) proved in Appendix C.5 we can symbolically integrate equations for ψ_k:

$$\psi_k = \sum_{\mu=1}^{n} \int \frac{\sigma_\mu \mathrm{sign}(U_\mu) f_\mu^{(k)} dx_\mu}{\sqrt{X_\mu V_\mu - W_\mu^2}}, \quad f_\mu^{(k)} \equiv \frac{W_\mu}{X_\mu}(-x_\mu^2)^{n-1-k} - \varepsilon \frac{\Psi_n}{c x_\mu^2}. \tag{5.10}$$

Similarly, we can express the affine parameter τ as [cf. Eq. (C.52)]

$$\tau = \sum_{\mu=1}^{n} \int \frac{\sigma_\mu \mathrm{sign}(U_\mu)(-x_\mu^2)^{n-1} dx_\mu}{\sqrt{X_\mu V_\mu - W_\mu^2}}. \tag{5.11}$$

5.3 Independence of constants of motion

In this section we want to demonstrate that quantities κ_j and Ψ_k are independent at a generic point of the phase space $\mathbf{\Gamma} = \mathbf{T}^*M$. This means that their gradients on the phase space are linearly independent. To prove that it is sufficient to show that these gradients are independent in the vertical direction of the cotangent bundle $\mathbf{T}^*M$, i.e., that the derivatives of these quantities with respect to the momentum $\boldsymbol{u}$, are linearly independent. To achieve this we will study the wedge product of the 'vertical' derivatives. We denote the vertical derivative by $\boldsymbol{\partial}$. For observable f, $\boldsymbol{\partial} f \equiv \boldsymbol{\partial} f / \boldsymbol{\partial u}$ denotes a *vector* field on the manifold M^D, with components $\partial f / \partial u_a$.

Let us, instead of κ_j consider the equivalent set of observables

$$2\tilde{\kappa}_j \equiv (-1)^j \kappa_j = w_j + \dots . \tag{5.12}$$

Here we have used the first relation (5.5). 'Dots' denote terms which contain w_k with $k < j$. We are interested in the quantity[2]

$$J = \boldsymbol{\partial}\tilde{\kappa}_0 \wedge \dots \wedge \boldsymbol{\partial}\tilde{\kappa}_{n-1} \wedge \boldsymbol{\partial}\Psi_0 \wedge \dots \wedge \boldsymbol{\partial}\Psi_m . \tag{5.13}$$

[2]The wedge product is, strictly speaking, defined for (antisymmetric) forms. However, we can easily define the wedge product also for the vectors or lower the vector indices with the help of the metric to get forms.

Due to (5.2), (5.12), and the definition of w_j, (3.41), we have

$$\boldsymbol{\partial}\Psi_j = \boldsymbol{\partial}_{\psi_j}\,, \quad \boldsymbol{\partial}\tilde{\kappa}_j = \boldsymbol{Q}^j \cdot \boldsymbol{u} + \dots\,, \tag{5.14}$$

where 'dots' denote linear combinations of $\boldsymbol{Q}^k \cdot \boldsymbol{u}$ with $k < j$; $\boldsymbol{Q}^l \cdot \boldsymbol{u}$ represents the vector with components $Q^a_{a_1} Q^{a_1}_{a_2} \cdots Q^{a_{l-1}}_{a_j} u^{a_j}$. From the antisymmetry of the wedge product it follows that

$$\boldsymbol{J} = \boldsymbol{u} \wedge (\boldsymbol{Q} \cdot \boldsymbol{u}) \wedge \dots \wedge (\boldsymbol{Q}^{n-1} \cdot \boldsymbol{u}) \wedge \boldsymbol{\partial}_{\psi_0} \wedge \dots \wedge \boldsymbol{\partial}_{\psi_m}\,. \tag{5.15}$$

Let us now use the explicit form of $\boldsymbol{Q}^j$ [cf. Eq. (3.46)]

$$\boldsymbol{Q}^j = \sum_{\mu=1}^{n} x_\mu^{2j} \boldsymbol{e}_{\hat{\mu}}\, \boldsymbol{\omega}^{\hat{\mu}} + \sum_{\mu=1}^{n} x_\mu^{2j} \tilde{\boldsymbol{e}}_{\hat{\mu}}\, \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\,. \tag{5.16}$$

The second term acts on the subspace of vectors spanned on $\boldsymbol{\partial}_{\psi_j}$. Thus, thanks to the term $\boldsymbol{\partial}_{\psi_0} \wedge \dots \wedge \boldsymbol{\partial}_{\psi_m}$ in the wedge product, this part can be ignored in (5.15). Moreover, taking into account that $\boldsymbol{e}_{\hat{\mu}}\, \boldsymbol{\omega}^{\hat{\mu}} = \boldsymbol{\partial}_{x_\mu} \boldsymbol{d}x_\mu$, and $u^\mu = \boldsymbol{d}x_\mu \cdot \boldsymbol{u}$, the substitution of (5.16) into (5.15) leads to

$$J = u^1 \dots u^n\, U\, \boldsymbol{\partial}_{x_1} \wedge \dots \wedge \boldsymbol{\partial}_{x_n} \wedge \boldsymbol{\partial}_{\psi_0} \wedge \dots \wedge \boldsymbol{\partial}_{\psi_{D-n-1}}\,, \tag{5.17}$$

where

$$U = \sum_{\substack{\text{permutations}\\ \sigma \text{ of } 1\dots n}} \operatorname{sign}\sigma\; x_1^{2\sigma_1} \dots x_n^{2\sigma_n} = \prod_{\substack{\mu,\nu=1\dots n\\ \mu<\nu}} (x_\mu^2 - x_\nu^2)\,. \tag{5.18}$$

In a generic point of the phase space we have $u^\mu \neq 0$ and $x_\mu^2 \neq x_\nu^2$ (for $\mu \neq \nu$) and therefore $\boldsymbol{J} \neq 0$ there. Thus we have shown that the constants of motion are independent.

5.4 Poisson brackets

Finally, we need to show that observables κ_j and Ψ_j Poisson commute. The Poisson bracket of two functions on the phase space Γ can be written as

$$\{A, B\} = \boldsymbol{\nabla} A \cdot \boldsymbol{\partial} B - \boldsymbol{\partial} A \cdot \boldsymbol{\nabla} B\,, \tag{5.19}$$

where $\boldsymbol{\nabla} F$ represents an arbitrary (torsion-free) covariant derivative which ignores the dependence of F on the momentum $\boldsymbol{u}$, and $\boldsymbol{\partial} B$ is the derivative of B with respect to the momentum $\boldsymbol{u}$. $\boldsymbol{\nabla} F$ and $\boldsymbol{\partial} F$ is a 1-form and a vector field on the spacetime M^D, respectively; the dot indicates a contraction between them.

Naturally, we use the covariant derivative $\boldsymbol{\nabla}$ generated by the metric connection on M^D.

Clearly, the commutation of any observable with the Hamiltonian

$$H = \frac{1}{2}w = \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} = \frac{1}{2}\kappa_0 \tag{5.20}$$

of the geodesic motion is equivalent to the conservation of the observable. So we have

$$\{\kappa_0, \kappa_j\} = 0\ , \quad \{\kappa_0, \Psi_j\} = 0\ . \tag{5.21}$$

The Poisson bracket between observables $\Psi_j = \boldsymbol{u}\cdot\boldsymbol{\partial}_{\psi_j}$ reduces to the Lie bracket of the Killing vector fields $\boldsymbol{\partial}_{\psi_j}$, which vanishes because $\boldsymbol{\partial}_{\psi_j}$ are coordinate vector fields:

$$\{\Psi_i, \Psi_j\} = \boldsymbol{\partial}_{\psi_j}\cdot(\boldsymbol{\nabla}\boldsymbol{\partial}_{\psi_i})\cdot\boldsymbol{u} - \boldsymbol{\partial}_{\psi_i}\cdot(\boldsymbol{\nabla}\boldsymbol{\partial}_{\psi_j})\cdot\boldsymbol{u} = \big[\boldsymbol{\partial}_{\psi_j}, \boldsymbol{\partial}_{\psi_i}\big]\cdot\boldsymbol{u} = 0\,. \tag{5.22}$$

The Poisson bracket of κ_i with the observable $\Psi_j = \boldsymbol{\partial}_{\psi_j}\cdot\boldsymbol{u}$, associated with the isometry $\boldsymbol{\partial}_{\psi_j}$, leads to the Lie derivative along this isometry,

$$\{\kappa_i, \Psi_j\} = \boldsymbol{\partial}_{\psi_j}\cdot\boldsymbol{\nabla}\kappa_i - \boldsymbol{\partial}\kappa_i\cdot\boldsymbol{\nabla}\boldsymbol{\partial}_{\psi_j}\cdot\boldsymbol{u} = \pounds_{\partial_{\psi_j}}\!\big(K^{ab}_{(i)}\big)u_a u_b \equiv \pounds_{\partial_{\psi_j}}\kappa_i = 0\ . \tag{5.23}$$

Here, the 'generalized' Lie derivative $\pounds_{\partial_{\psi_j}}\kappa_i$ ignores the dependence of κ_i on the momentum $\boldsymbol{u}$. It vanishes because $\boldsymbol{\partial}_{\psi_j}$ are Killing vectors and Killing tensors $\boldsymbol{K}^{(i)}$ respect the symmetry of the spacetime.

Finally, it remains to evaluate the brackets $\{\kappa_i, \kappa_j\}$. We shall do it in two steps: first, we prove that an equivalent set of observables $\tilde{C}_j$, given by (5.1), Poisson commute and, second, by relating these constants to κ_j we obtain the desired result. So, let us consider the observables $\tilde{C}_j$. Using the cyclic property of the trace, the derivative of $\tilde{C}_j$ in the spacetime direction is

$$\nabla_a\tilde{C}_j = 2j\,\mathrm{Tr}\big[(\nabla_a H)\tilde{P}(H\tilde{P})^{2j-1}\big]\ . \tag{5.24}$$

Here, $\nabla_a H$ is the matrix of components $\nabla_a h^b{}_c$ of the covariant derivative $\boldsymbol{\nabla h}$. Substituting for $\nabla_a h^b{}_c$ from Eq. (3.7) and using the antisymmetry of $\boldsymbol{h}$, we obtain

$$\begin{aligned}\tfrac{D-1}{2j}\nabla_e\tilde{C}_j &= \xi_{a_0}\tilde{P}^{a_0}_{b_1}h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_2}\dots h^{b_{2j-2}}{}_{a_{2j-1}}\tilde{P}^{a_{2j-1}}_{e} - g_{ea_{2j}}\tilde{P}^{a_{2j}}_{b_{2j-1}}h^{b_{2j-1}}{}_{a_{2j-1}}\dots h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_0}\xi^{b_0}\\ &= 2\,\xi_{a_0}\tilde{P}^{a_0}_{b_1}h^{b_1}{}_{a_1}\tilde{P}^{a_1}_{b_2}\dots h^{b_{2j-2}}{}_{a_{2j-1}}\tilde{P}^{a_{2j-1}}_{e}\ .\end{aligned} \tag{5.25}$$

For the derivative with respect to the momentum $\boldsymbol{u}$ we get

$$\tfrac{1}{4j}\partial^e\tilde{C}_j = u^e\big(h^{d_1}{}_{c_1}\tilde{P}^{c_1}_{d_2}h^{d_2}{}_{c_2}\tilde{P}^{c_2}_{d_3}\dots\tilde{P}^{c_{2j-1}}_{d_{2j}}h^{d_{2j}}{}_{d_1}\big) + h^e{}_{c_1}\tilde{P}^{c_1}_{d_2}h^{d_2}{}_{c_2}\tilde{P}^{c_2}_{d_3}\dots\tilde{P}^{c_{2j-1}}_{d_{2j}}h^{d_{2j}}{}_{c_{2j}}u^{c_{2j}}\ . \tag{5.26}$$

Substituting (5.25) and (5.26) into (5.19) for $\{\tilde{C}_i, \tilde{C}_j\}$ and using $\tilde{P}^a_b u^b = 0$, we find

$$\begin{aligned}\tfrac{D-1}{16ij}\{\tilde{C}_i, \tilde{C}_j\} =\ & \xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1}\dots \tilde{P}^{a_{2i-1}}_{b_{2i-1}} h^{b_{2i-1}}{}_{c_1}\dots \tilde{P}^{c_{2j-1}}_{d_{2j}} h^{d_{2j}}{}_{c_{2j}} u^{c_{2j}} \\ & -\xi_{a_0}\tilde{P}^{a_0}_{b_1} h^{b_1}{}_{a_1}\dots \tilde{P}^{a_{2j-1}}_{b_{2j-1}} h^{b_{2j-1}}{}_{c_1}\dots \tilde{P}^{c_{2i-1}}_{d_{2i}} h^{d_{2i}}{}_{c_{2i}} u^{c_{2i}} = 0\,. \qquad (5.27)\end{aligned}$$

We thus proved that constants $\tilde{C}_j$ mutually Poisson commute. The same is, of course, true for constants C_j, (3.37), which differ from $\tilde{C}_j$ only by rescaling (5.1). The generating function $Z(\beta)$, (3.54), is given by power series in β with coefficients given (up to constant factors) by constants C_j , cf. Eq. (3.56). Therefore this function, and similarly $W(\beta) = \exp Z(\beta)$, Poisson commute with κ_0 and Ψ_j, as well as with itself for different choices of β:

$$\big\{Z(\beta_1), Z(\beta_2)\big\} = 0\,, \quad \big\{W(\beta_1), W(\beta_2)\big\} = 0\,. \qquad (5.28)$$

This means, that also quantities κ_j generated from $W(\beta)$ mutually Poisson commute. Therefore all the constants of motion are in involution and the geodesic motion is completely integrable.

5.5 Lie algebra of Killing tensors

Let us use the opportunity to remind here that Killing tensors, as proper symmetry objects, form an appropriate Lie algebra. This will give us another point of view on the above calculations.

We start with an observation that the Poisson commutativity of constants corresponding to the isometries is equivalent to the vanishing of the Lie brackets of these isometries [cf. Eq. (5.22)]. Similarly, the Poisson bracket of a quantity corresponding to the Killing tensor and a quantity associated with the isometry leads to the Lie bracket of the Killing tensor along the isometry [cf. Eq. (5.23)]. More generally, it is well known that Killing tensors form a Lie subalgebra of a Lie algebra of all totally symmetric contravariant tensor fields on the manifold with respect to the *symmetric Schouten–Nijenhuis* (SN) brackets [Schouten, 1940], [Schouten, 1954], [Nijenhuis, 1955]. The vanishing of these brackets is equivalent to the Poisson commutativity of the corresponding constants of geodesic motion (see, e.g., [Benenti & Francaviglia, 1980]). In particular, we have the following equations:

$$\big[K_{(i)}, K_{(j)}\big]^{abc}_{\mathrm{SN}} \equiv K^{e(a}_{(i)}\nabla_e K^{bc)}_{(j)} - K^{e(a}_{(j)}\nabla_e K^{bc)}_{(i)} = 0\,, \qquad (5.29)$$

$$\big[\partial_{\psi_j}, K_{(i)}\big]^{ab}_{\mathrm{SN}} \equiv \mathcal{L}_{\partial_{\psi_j}} K^{ab}_{(i)} = 0\,. \qquad (5.30)$$

Taking the metric g as one of the Killing tensors in (5.29), we obtain the Killing tensor equation (2.6). [Such an equation simply states that an observable corresponding to the Killing tensor commutes with the Hamiltonian (5.20), and therefore constitutes a constant of geodesic motion.] Using the Schouten–Nijenhuis brackets and the method of generating functions, an independent proof of the Poisson commutativity of constants κ_j and Ψ_k generated by a PCKY tensor obeying (3.74) was recently demonstrated [Houri *et al.*, 2008a].

Finally, we would like to mention that an interesting question whether also Killing–Yano tensors form a closed Lie algebra was recently addressed in [Kastor *et al.*, 2007]. It is well known that forms on the manifold form a (graded) Lie algebra with respect to the *antisymmetric Schouten–Nijenhuis* (aSN) brackets. For a p-form α and a q-form β these are defined as

$$[\alpha,\beta]_{\text{aSN}}^{a_1\dots a_{p+q-1}} \equiv p\,\alpha^{b[a_1\dots a_{p-1}}\nabla_b\beta^{a_p\dots a_{p+q-1}]} + (-1)^{pq}q\,\beta^{b[a_1\dots a_{q-1}}\nabla_b\alpha^{a_q\dots a_{p+q-1}]}\,. \tag{5.31}$$

The definition is connection independent; covariant derivatives may be replaced with partial derivatives. When one of the forms is a vector, the bracket reduces to the Lie derivative.

One might expect that if Killing–Yano tensors are associated with symmetries in some 'appropriately generalized' sense they would form a closed subalgebra with respect to these brackets. The (aSN) bracket of a Killing vector and a rank-2 Killing-Yano tensor is indeed a rank-2 Killing–Yano tensor [Kastor *et al.*, 2007]. Unfortunately, for two Killing–Yano tensors this is not, except the special case of a constant curvature spacetime, generally true [Kastor *et al.*, 2007]. The KY tensor (as well as the PCKY tensor) in the Kerr spacetime are the counter examples. The geometrical meaning of the Killing–Yano symmetry therefore still remains veiled.

Chapter 6

Separation of variables

In this chapter, based on [Frolov *et al.*, 2007], we demonstrate the separability of the Hamilton–Jacobi and Klein–Gordon equations in the canonical spacetime (4.1). Such a separability provides an independent proof of complete integrability of geodesic motion. We also review some related results and briefly discuss an open problem of separability of equations with spin.

6.1 Hamilton–Jacobi equation

The Hamilton–Jacobi equation for geodesic motion on a manifold with metric g has the form

$$\frac{\partial S}{\partial \lambda} + g^{ab}\,\partial_a S\;\partial_b S = 0\,. \tag{6.1}$$

Here λ denotes an 'external' time which turns out to be an affine parameter of the corresponding geodesic motion. We want to demonstrate that in the background (4.1) the classical action S allows an additive separation of variables

$$S = -w\lambda + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{m} \Psi_k \psi_k\,, \tag{6.2}$$

with functions $S_\mu(x_\mu)$ of a single argument x_μ.

Substituting (6.2) into the Hamilton–Jacobi equation (6.1) and using the form (4.7) of the inverse metric, we obtain

$$\sum_{\mu=1}^{n}\Big[\frac{X_\mu S_\mu'^{\,2}}{U_\mu} + \frac{1}{X_\mu U_\mu}\Big(\sum_{j=0}^{m}(-x_\mu^2)^{n-1-k}\Psi_j\Big)^2\Big] - \varepsilon\frac{\Psi_n^2}{cA^{(n)}} - w = 0\,. \tag{6.3}$$

Here, S_μ' denotes the derivative of function S_μ with respect to its single argument x_μ. Using identities (proved in Appendix C.4)

$$1=\sum_{\mu=1}^{n}\frac{(-x_\mu^2)^{n-1}}{U_\mu}\,,\qquad \frac{1}{A^{(n)}}=\sum_{\mu=1}^{n}\frac{1}{x_\mu^2U_\mu}\,, \tag{6.4}$$

and the definition of W_μ, (5.8), we can rewrite the last equation in the form

$$\sum_{\mu=1}^{n}\frac{F_\mu}{U_\mu}=0\,, \tag{6.5}$$

where F_μ are functions of x_μ only:

$$F_\mu=\frac{W_\mu^2}{X_\mu}+X_\mu S_\mu'^{\,2}-w(-x_\mu^2)^{n-1}-\varepsilon\frac{\Psi_n^2}{cx_\mu^2}\,. \tag{6.6}$$

Applying Lemma 2 of Appendix C.4, we realize that the general solution of (6.5) is

$$F_\mu=\sum_{j=1}^{n-1}\kappa_j(-x_\mu^2)^{n-1-j}\,, \tag{6.7}$$

where κ_j are arbitrary constants. Denoting by

$$\kappa_0\equiv w\,,\quad \kappa_n\equiv-\frac{\Psi_n^2}{c}\,, \tag{6.8}$$

and using the definition of V_μ, (5.8), we combine (6.6) and (6.7) to obtain equations for S_μ',

$$S_\mu'^2=-\frac{W_\mu^2}{X_\mu^2}+\frac{V_\mu}{X_\mu}\,, \tag{6.9}$$

which can be solved by quadratures. Thus we have shown that the Hamilton–Jacobi equation (6.1) in the off-shell gravitational background (4.1) can be solved by the classical action S in the separated form (6.2), with S_μ satisfying (6.9). The separated solution contains D arbitrary constants. Namely, it contains $m+1=D-n$ constants Ψ_j ($j=0,\dots,m$) and n constants κ_k ($k=0,\dots,n-1$).

The gradient of S gives the momentum $p_a=\partial_a S$. Substituting our expression for S we obtain p_a in terms of the constants κ_k and Ψ_j:

$$p_j=\Psi_j\,,\quad p_\mu^2=-\frac{W_\mu^2}{X_\mu^2}+\frac{V_\mu}{X_\mu}\,. \tag{6.10}$$

These relations can be inverted. Clearly, $\Psi_j=p_j$ are constants linear in the momentum generated by Killing vectors $\boldsymbol{\partial}_{\psi_j}$. The constants κ_k are quadratic in momenta. They are connected with n (irreducible) Killing tensors $\boldsymbol{K}^{(k)}$, $(k = 0, \dots, n-1)$,

$$\kappa_k = K^{(k)}_{ab} p^a p^b \,, \quad \nabla_{(c} K^{(k)}_{ab)} = 0 \,. \tag{6.11}$$

One can easily find the explicit form of $K^{ab}_{(k)}$ by inverting (6.6). Let us multiply it by $A^{(k)}_\mu / U_\mu$, sum over μ, and use identities (see Appendix C.4)

$$\sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} A^{(k)}_\mu = \delta^j_k \,, \quad \sum_{\mu=1}^{n} \frac{A^{(k)}_\mu}{x_\mu^2 U_\mu} = \frac{A^{(k)}}{A^{(n)}} \,, \tag{6.12}$$

which are valid for $j, k = 0, \dots, n-1$. Then we obtain

$$\boldsymbol{K}_{(k)} = \sum_{\mu=1}^{n} \left[\frac{A^{(k)}_\mu}{X_\mu U_\mu} \Big(\sum_{j=0}^{m} (-x_\mu^2)^{n-1-j} \boldsymbol{\partial}_{\psi_j} \Big)^2 + A^{(k)}_\mu Q_\mu (\boldsymbol{\partial}_{x_\mu})^2 \right] - \frac{\varepsilon A^{(k)}}{c A^{(n)}} (\boldsymbol{\partial}_{\psi_n})^2 \,, \tag{6.13}$$

which are Killing tensors (4.14), written in the coordinate basis.

6.2 Klein–Gordon equation

The behavior of a massive scalar field Φ in the gravitational background $\boldsymbol{g}$ is governed by the Klein–Gordon equation

$$\Box \Phi = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} g^{ab} \partial_b \Phi) = \mu^2 \Phi . \tag{6.14}$$

In an Einstein space, this equation remains valid for the non-minimal coupling case as well. (The term ξR is constant and can be included into the definition of μ^2.)

Now, we demonstrate that the Klein–Gordon equation (6.14) in the canonical background (4.1) allows a multiplicative separation of variables

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{j=0}^{m} e^{i \Psi_j \psi_j} . \tag{6.15}$$

This equation has the following explicit form:

$$\sum_{\mu=1}^{n} \left[\partial_{x_\mu} \Big(\frac{\sqrt{|g|}}{U_\mu} X_\mu \partial_{x_\mu} \Phi \Big) + \frac{\sqrt{|g|}}{U_\mu X_\mu} \Big(\sum_{j=1}^{m} (-x_\mu^2)^{n-1-j} \partial_{\psi_j} \Big)^2 \Phi \right] - \varepsilon \frac{\sqrt{|g|}}{c A^{(n)}} \partial^2_{\psi_n} \Phi = \sqrt{|g|} \mu^2 \Phi \,. \tag{6.16}$$

Here, we have used the quasi-diagonal property of the inverse metric g^{ab}, (4.7), and the fact that ∂_{ψ_j} are Killing vectors. We further notice that [cf. Eq. (4.11)]

$$\sqrt{|g|} \propto U P^{\varepsilon}\,, \quad P \equiv \prod_{\mu=1}^{n} x_\mu\,, \tag{6.17}$$

where "$\propto$" means equality up to a constant factor [which can be ignored in Eq. (6.16)]. Using identities (6.4) and the obvious fact that $\partial_{x_\mu}(U/U_\mu) = 0$, we realize that (6.16) is equivalent to

$$\sum_{\mu=1}^{n} \frac{1}{U_\mu}\left[\frac{\partial_{x_\mu}(P^{\varepsilon} X_\mu \partial_{x_\mu}\Phi)}{P^{\varepsilon}} + \frac{1}{X_\mu}\Big(\sum_{j=1}^{m}(-x_\mu^2)^{n-1-j}\partial_{\psi_k}\Big)^2\Phi + \varepsilon\frac{\partial^2_{\psi_n}\Phi}{c x_\mu^2} - \mu^2(-x_\mu^2)^{n-1}\Phi\right] = 0\,. \tag{6.18}$$

Employing the ansatz (6.15), we have

$$\partial_{\psi_j}\Phi = i\Psi_j\Phi\,, \quad \partial_{x_\mu}\Phi = \frac{R'_\mu}{R_\mu}\Phi\,, \quad \partial^2_{x_\mu}\Phi = \frac{R''_\mu}{R_\mu}\Phi\,, \tag{6.19}$$

and the Klein–Gordon equation (6.18) takes the form

$$\sum_{\mu=1}^{n} \frac{G_\mu}{U_\mu}\Phi = 0\,, \tag{6.20}$$

where G_μ is function of x_μ only,

$$G_\mu = X_\mu\frac{R''_\mu}{R_\mu} + \frac{R'_\mu}{R_\mu}\Big(X'_\mu + \varepsilon\frac{X_\mu}{x_\mu}\Big) - \frac{W_\mu^2}{X_\mu} + \frac{\varepsilon\Psi_n^2}{c x_\mu^2} - \mu^2(-x_\mu^2)^{n-1}. \tag{6.21}$$

As earlier, the prime means the derivative of functions R_μ and X_μ with respect to their single argument x_μ, and we have used the definition (5.8) for W_μ. Employing again Lemma 2 of Appendix C.4, we realize that the general solution of (6.20) is

$$G_\mu = -\sum_{j=1}^{n-1}\kappa_j(-x_\mu^2)^{n-1-j}\,, \tag{6.22}$$

where κ_j are arbitrary constants.

Therefore, we have demonstrated that the Klein–Gordon equation (6.14) in the background (4.1) allows a multiplicative separation of variables (6.15), where functions $R_\mu(x_\mu)$ satisfy the ordinary second order differential equations

$$(X_\mu R'_\mu)' + \varepsilon \frac{X_\mu}{x_\mu} R'_\mu + \Big(V_\mu - \frac{W_\mu^2}{X_\mu}\Big) R_\mu = 0\,. \tag{6.23}$$

Here, functions V_μ and W_μ are defined in (5.8). They contain

$$\kappa_0 = -\mu^2\,, \quad \kappa_n = -\frac{\Psi_n^2}{c}\,, \tag{6.24}$$

and arbitrary separation constants Ψ_j ($j = 0, \dots, m$) and κ_k ($k = 1, \dots, n-1$). These constants are related to the constants obtained by the separation of the Hamilton–Jacobi equation by the *geometric optics approximation*. This connection is briefly discussed in the next section.

It should be emphasized that in the symmetric form of the metric (4.1) all equations (6.23) 'look the same'. However, in order to use the proved separability for concrete calculations in the physical Kerr-NUT-(A)dS spacetime one needs to specify metric functions X_μ to have the form (4.3) and perform a Wick rotation to the 'physical space' (see Footnote 1 in Chapter 4, and also Section 9.4.1). Such a transformation 'spoils' the symmetry between essential coordinates but the separability property remains. The equation for R_n then plays the role of an equation for propagating radial modes, whereas the other equations (with imposed regularity conditions) represent the eigenvalue problems. For a discussion of special sub-cases of these equations see, e.g, [Berti *et al.*, 2006] and reference therein.

6.3 Understanding connections

To obtain a more complete picture about the above described separability, let us in this section review two closely related results. Namely, we review the theory of separability structures, and briefly describe a recent result [Sergyeyev & Krtouš, 2008] on symmetry operators for the Klein–Gordon equation in the canonical background.

6.3.1 Separability structures

The separation of variables for the Hamilton–Jacobi equation in any number of spacetime dimensions allows a geometric characterization described by the theory of *separability structures*, see, e.g., [Benenti & Francaviglia, 1979], [Benenti & Francaviglia, 1980], [Demianski & Francaviglia, 1980], [Kalnins & Miller, 1981]. Let us briefly recall the main results of this theory.

Separability structures are classes of separable charts for which the Hamilton–Jacobi equation allows an additive separation of variables. For each separability structure there exists such a family of separable coordinates which admits a maximal number of, let us say r, ignorable coordinates. Each system in this family is called a *normal separable system* of coordinates. The corresponding separability structure is denoted by δ_r. Its existence is governed by the following central theorem:

Theorem. A manifold $(M^D, \boldsymbol{g})$ admits a δ_r-separability structure if and only if it admits r commuting Killing vectors $\boldsymbol{X}_{(k)}$ $(k = 0, \dots, r-1)$ and $D-r$ Killing tensors $\boldsymbol{K}_{(\alpha)}$ $(\alpha = 0, \dots, D-r-1)$, all of them independent, which satisfy:
(i) in the Lie algebra of Killing tensors with Schouten–Nijenhuis brackets the commutation relations

$$\begin{aligned} \left[K_{(\alpha)}, K_{(\beta)}\right]_{\mathrm{SN}}^{abc} &\equiv K_{(\alpha)}^{e(a} \nabla_e K_{(\beta)}^{bc)} - K_{(\beta)}^{e(a} \nabla_e K_{(\alpha)}^{bc)} = 0\,, \qquad (6.25)\\ \left[X_{(k)}, K_{(\beta)}\right]_{\mathrm{SN}}^{ab} &\equiv \mathcal{L}_{X_{(k)}} K_{(\alpha)}^{ab} = 0\,, \qquad (6.26)\end{aligned}$$

(ii) the Killing tensors $\boldsymbol{K}_{(\alpha)}$ have in common $D-r$ eigenvectors $\boldsymbol{X}_{(\alpha)}$ such that

$$[\boldsymbol{X}_{(\alpha)}, \boldsymbol{X}_{(\beta)}] = 0\,, \quad [\boldsymbol{X}_{(\alpha)}, \boldsymbol{X}_{(k)}] = 0\,, \quad \boldsymbol{g}(\boldsymbol{X}_{(\alpha)}, \boldsymbol{X}_{(k)}) = 0\,. \qquad (6.27)$$

Let us mention two implications of this theory (we refer to the original publications for more details).

(1) *The existence of separability structure implies complete integrability of geodesic motion.* Indeed, the requirement of independence means that r linear in momenta constants of motion $c_{(k)}$ associated with Killing vectors $\boldsymbol{X}_{(k)}$ and $(D-r)$ quadratic in momenta constants of motion $c_{(\alpha)}$ corresponding to Killing tensors $\boldsymbol{K}_{(\alpha)}$ are functionally independent. Moreover, equations (6.25), (6.26), are equivalent to (see also Section 5.5)

$$\{c_{(\alpha)}, c_{(\beta)}\} = 0\,, \quad \{c_{(k)}, c_{(\alpha)}\} = 0\,, \qquad (6.28)$$

which, together with $\{c_{(k)}, c_{(l)}\} = 0$ (following from the commutativity of Killing vectors), implies that all these D constants are in involution and hence the motion is completely integrable. In particular, this means that the proved separability of the Hamilton–Jacobi equation establishes an independent proof of complete integrability of geodesic motion, demonstrated in Chapter 5.

(2) D vectors $\{\boldsymbol{X}_{(\alpha)}, \boldsymbol{X}_{(k)}\}$ form a natural basis $\{\boldsymbol{\partial}_x\}$ associated with normal separable coordinates x^a. This allows to write down the *canonical metric element* of the 'separable' spacetime, e.g., [Benenti & Francaviglia, 1980]. The authors of [Houri *et al.*, 2008a], [Houri *et al.*, 2007] used this fact to prove that the existence of a PCKY tensor, obeying the assumptions (3.74), restricts the metric of

the spacetime to the canonical form (4.1). Their proof consists of showing that the tower of Killing tensors $\boldsymbol{K}^{(j)}$ together with the tower of Killing vectors $\boldsymbol{\xi}^{(k)}$ generated by a PCKY tensor (see Chapter 3) obey all the requirements of the above theorem. Hence, the Hamilton–Jacobi equation in a spacetime admitting this tensor is separable. The corresponding canonical element turns out to be the off-shell spacetime (4.1). In Chapter 7, we show that (4.1) follows from the existence of a PCKY tensor directly, without referring to the theory of separability structures, and even without imposing conditions (3.74).

Let us finally mention another theorem which relates the (additive) separability of the Hamilton–Jacobi equation with the (multiplicative) separability of the Klein–Gordon equation (see, e.g., [Benenti & Francaviglia, 1979]).

Theorem. The Klein–Gordon equation allows a multiplicative separation of variables if and only if the manifold $(M^D, \boldsymbol{g})$ possesses a separability structure in which the vectors $\boldsymbol{X}_{(\alpha)}$ are eigenvectors of the Ricci tensor.

Corollary. If the manifold is an Einstein space, the Hamilton–Jacobi equation is separable if and only if the same holds for the Klein–Gordon equation.

This corollary explains why, after separating the Hamilton–Jacobi equation, we were able to separate also the Klein–Gordon equation. Another explanation is in the following subsection.

6.3.2 Symmetry operators

After the Hamilton–Jacobi and Klein–Gordon equations in the canonical background were separated [Frolov *et al.*, 2007], it turned out that one can construct symmetry operators for these equations [Sergyeyev & Krtouš, 2008], which invariantly characterize such a separability. Following closely the latter paper, let us recapitulate this connection.

Following the 'first quantization rule', $p_a \to -i\alpha\nabla_a$, where α is some scaling constant, one can consider the operator counterparts of the conserved quantities (5.2). So, one can introduce the operators [cf. Eqs. (1.6)]

$$\hat{\xi}_{(k)} = -i\alpha\xi^{(k)a}\partial_a\,, \quad k=0,\dots,m\,, \tag{6.29}$$

$$\hat{K}_{(j)} = -\frac{\alpha^2}{\sqrt{|g|}}\,\partial_a\Big(\sqrt{|g|}K^{ab}_{(j)}\partial_b\Big)\,, \quad j=0,\dots,n-1\,. \tag{6.30}$$

It was proved in [Sergyeyev & Krtouš, 2008] that all these operators mutually commute in the canonical background (4.1). This means that there exist joint eigenfunctions, modes Φ, specified by the eigenvalues Ψ_k and κ_j, so that

$$\hat{\xi}_{(k)}\Phi = \Psi_k\Phi\,, \quad \hat{K}_{(j)}\Phi = \kappa_j\Phi\,. \tag{6.31}$$

In the background (4.1) these equations allow a separated solution

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{m} e^{\frac{i}{\alpha}\Psi_k \psi_k}, \tag{6.32}$$

provided that functions $R_\mu(x_\mu)$ obey the ordinary differential equations [Sergyeyev & Krtouš, 2008]

$$\alpha^2 \big(X_\mu R_\mu'\big)' + \varepsilon \alpha^2 \frac{X_\mu}{x_\mu} R_\mu' + \Big(V_\mu - \frac{W_\mu^2}{X_\mu}\Big) R_\mu = 0\,. \tag{6.33}$$

In particular, since $\hat{K}_{(0)} = -\alpha^2 \Box$, modes Φ (6.32) are solutions of the Klein–Gordon equation

$$\big(\alpha^2 \Box + \kappa_0\big)\Phi = 0\,, \tag{6.34}$$

and $\{\hat{\xi}_{(k)}, \hat{K}_{(j)}\}$ form a complete set of commuting *symmetry operators* for this equation (see, e.g., [Miller, 1977], [Fushchich & Nikitin, 1994] and references therein).

The constants of separation for the Klein–Gordon equation are related to the constants obtained for the Hamilton–Jacobi equation. Writing the solution of Eqs. (6.31) in the form

$$\Phi = A \exp\Big(\frac{i}{\alpha} S\Big)\,, \tag{6.35}$$

one obtains in the *geometric optics approximation*, $\alpha \to 0$, a new set of equations

$$\xi^a_{(k)} \partial_a S = \Psi_k\,, \quad K^{ab}_{(j)} \partial_a S \partial_b S = \kappa_j\,. \tag{6.36}$$

One can easily recognize the Hamilton–Jacobi equation (6.1) and, upon identifying $\partial_a S$ with p_a, the separation constants (6.10) and (6.11). Moreover, substituting $R_\mu = \exp\big(\frac{i}{\alpha} S_\mu\big)$ into (6.32) yields an additive separation ansatz (6.2), and in geometric optics approximation Eq. (6.33) gives directly Eq. (6.9).

6.4 Discussion

In this chapter we have demonstrated the separability of the Hamilton–Jacobi and the scalar field equations in the general canonical spacetime (4.1). This allows one to study the particle and light propagation in completely general higher-dimensional rotating black hole spacetimes, or to calculate the contribution of a scalar field to the bulk Hawking radiation of these black holes.

These results are very promising and might suggest that also the equations with spin can be in this background decoupled and separated. In fact, the separability of the massive Dirac equation was already demonstrated [Oota & Yasui, 2008]. We expect that, similar to the 4-dimensional case [Carter & McLenaghan, 1979], [Kamran & McLenaghan, 1984], in higher dimensions also the separability of the Dirac equation can be characterized by the corresponding symmetry operators. These operators are well known [Benn & Charlton, 1997], [Cariglia, 2004].

An important open question is a separability problem for the electromagnetic and the gravitational perturbations in higher-dimensional black hole spacetimes. A certain progress in this direction was achieved recently (see, e.g., [Kodama & Ishibashi, 2003], [Kunduri *et al.*, 2006], [Murata & Soda, 2008a]). These results are very important for the study of the stability of such black holes and different aspect of the Hawking radiation produced by them. Another important direction of research is to study the quasinormal modes in higher dimensions. The results obtained in these directions so far restricts mainly to non-rotating (or not generally rotating) black holes (see, e.g., [Ida *et al.*, 2003], [Cardoso *et al.*, 2003a], [Cardoso *et al.*, 2003b], [Cardoso *et al.*, 2003c], [Cardoso *et al.*, 2004], [Konoplya, 2003a], [Konoplya, 2003b], [Cardoso *et al.*, 2004], [Cardoso *et al.*, 2006], [Zhidenko, 2006], [Kanti *et al.*, 2006], [Lopez-Ortega, 2006b], [Lopez-Ortega, 2006a], [Lopez-Ortega, 2007], [Konoplya & Zhidenko, 2007], [Lopez-Ortega, 2008], [Kodama, 2008], [Kodama, 2007], [Casals *et al.*, 2008], [Murata & Soda, 2008b], and references therein).

At a first glance it seems that to attack these problems in full generality, for example in the way of Teukolsky [Teukolsky, 1972], [Teukolsky, 1973], may not be possible. On the other hand, it might be useful to study the invariant structures determined by the PCKY tensor. For example, in 4D the method of the Debye potentials [Benn *et al.*, 1997] allows one to decouple the electromagnetic perturbations. Unfortunately, this method seems to lie heavily on the self duality property of electromagnetic fields in four dimensions. Another starting point could be to search for the analogues of the 4-dimensional symmetry operators (see, e.g., [Kamran, 1985], [Kalnins *et al.*, 1986], [Kalnins & Miller, 1989], [Kalnins & Williams, 1990], [Kalnins *et al.*, 1992], [Kalnins *et al.*, 1996]) which invariantly characterize the separability of field equations with spin in the Kerr background.

It is an important open question to ask whether the existing symmetry connected with the towers of hidden symmetries generated by the PCKY tensor is enough to enable the decoupling and separation of the higher spin fields equations.

Chapter 7

Canonical metric and Kerr-NUT-(A)dS uniqueness

In previous chapters we have seen that the general off-shell metric element (4.1) admits the PCKY tensor $\boldsymbol{h}$, (4.13), from which complete integrability of geodesic motion and separability of the Hamilton–Jacobi and Klein–Gordon equations can be derived. In this chapter we want to address the question of uniqueness and generality of these results. This leads us to the study of metric elements admitting a PCKY tensor. In particular, we demonstrate the following two important results: First, we establish that the Kerr-NUT-AdS spacetime (4.1)–(4.3) is the most general solution of the vacuum Einstein equations with the cosmological constant which possesses the PCKY tensor. Second, without imposing the Einstein equations, we explicitly derive the canonical form of the metric admitting such a tensor and show that it coincides with the off-shell metric (4.1). These results naturally generalize the results obtained earlier in four dimensions. This chapter is based on [Krtouš *et al.*, 2008b].

7.1 Uniqueness of the Kerr-NUT-(A)dS spacetime

In this section we prove that the most general solution of the Einstein equations with the cosmological constant which admits a PCKY tensor is the Kerr-NUT-(A)dS spacetime (4.1)–(4.3). Instead of giving a formally organized proof we shall deduce this statement from filling two missing pieces in the mosaic of already known facts. Namely, it was demonstrated in [Houri *et al.*, 2008a], [Houri *et al.*, 2007] that the Kerr-NUT-(A)dS spacetime is the only Einstein space admitting the PCKY tensor obeying the additional restrictions

$$\pounds_{\xi}\boldsymbol{h}=0\,,\quad \pounds_{\xi}\boldsymbol{g}=0\,. \tag{7.1}$$

Here we prove that both these conditions already follow from the existence of the PCKY tensor, together with the restriction on a vacuum solution of the Einstein equations with the cosmological constant.

7.1.1 Condition on the PCKY tensor

In this subsection we concentrate on the first condition in (7.1). At first we consider the case of an even dimension, $D = 2n$, and then briefly discuss what happens in the odd-dimensional case. Besides the canonical basis (Section 3.1.2), it is convenient to introduce also a basis of complex null eigenvectors, $\{\boldsymbol{m}_{\hat\mu}, \bar{\boldsymbol{m}}_{\hat\mu}\}$, defined by the relations[1]

$$\check{\boldsymbol{h}} \cdot \boldsymbol{m}_{\hat\mu} \equiv -ix_\mu \boldsymbol{m}_{\hat\mu}\,, \quad \check{\boldsymbol{h}} \cdot \bar{\boldsymbol{m}}_{\hat\mu} \equiv ix_\mu \bar{\boldsymbol{m}}_{\hat\mu}\,. \tag{7.2}$$

Here, $\check{\boldsymbol{h}}$ is the operator with components $h^a{}_b$, bar denotes the complex conjugation, and x_μ's describe the eigenvalues of $\check{\boldsymbol{h}}$ [cf. Eq. (3.13)].[2] The complex null vectors satisfy the normalization

$$\boldsymbol{m}_{\hat\mu} \cdot \boldsymbol{m}_{\hat\nu} = \bar{\boldsymbol{m}}_{\hat\mu} \cdot \bar{\boldsymbol{m}}_{\hat\nu} = 0\,, \quad \boldsymbol{m}_{\hat\mu} \cdot \bar{\boldsymbol{m}}_{\hat\nu} = \delta_{\mu\nu}\,. \tag{7.3}$$

They are connected with the canonical vectors $\{\boldsymbol{e}_{\hat\mu}, \tilde{\boldsymbol{e}}_{\hat\mu}\}$ as follows:

$$\boldsymbol{m}_{\hat\mu} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat\mu} + i\boldsymbol{e}_{\hat\mu})\,, \quad \bar{\boldsymbol{m}}_{\hat\mu} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat\mu} - i\boldsymbol{e}_{\hat\mu})\,. \tag{7.4}$$

Let us further denote $D_{\hat\mu} \equiv \nabla_{m_\mu}$ and $\bar{D}_{\hat\mu} \equiv \nabla_{\bar{m}_{\hat\mu}}$. Using the PCKY Eq. (3.3) one has

$$(D_{\hat\mu}\check{\boldsymbol{h}}) \cdot \boldsymbol{m}_{\hat\nu} = (\boldsymbol{m}_{\hat\nu} \cdot \boldsymbol{\xi})\, \boldsymbol{m}_{\hat\mu}\,. \tag{7.5}$$

Applying $D_{\hat\mu}$ to (7.2) and using (7.5) one obtains

$$(\check{\boldsymbol{h}} + ix_\nu \boldsymbol{\delta}) \cdot D_{\hat\mu}\boldsymbol{m}_{\hat\nu} + i(D_{\hat\mu}x_\nu)\, \boldsymbol{m}_{\hat\nu} + (\boldsymbol{m}_{\hat\nu} \cdot \boldsymbol{\xi})\, \boldsymbol{m}_{\hat\mu} = 0\,. \tag{7.6}$$

By taking a scalar product of (7.6) with $\bar{\boldsymbol{m}}_{\hat\nu}$, using antisymmetry of $\boldsymbol{h}$ and Eq. (7.2) again, the first term cancels out. Considering two cases when $\nu = \mu$ and when $\nu \neq \mu$ one gets

$$D_{\hat\mu}x_\nu = 0 \quad \text{for } \nu \neq \mu\,, \quad D_{\hat\mu}x_\mu = i\boldsymbol{m}_{\hat\mu} \cdot \boldsymbol{\xi}\,. \tag{7.7}$$

[1]The eigenvectors of the PCKY tensor play a special role. One can prove that they coincide with the *principal null directions* (see Appendix C.1).

[2]In this chapter we do not have a fixed background; we are constructing the metric. It is therefore necessary to distinguish various positions of indices. In particular, we denote by $\boldsymbol{h}$ a PCKY 2-form, whereas by $\check{\boldsymbol{h}}$ a PCKY operator $h^a{}_b$.

Let us define functions Q_μ in terms of magnitudes of complex quantities $D_{\hat{\mu}} x_\mu$:

$$Q_\mu \equiv 2|D_{\hat{\mu}} x_\mu|^2 \,, \quad D_{\hat{\mu}} x_\mu = \frac{1}{\sqrt{2}} \sqrt{Q_\mu}\, e^{i\alpha} \,. \tag{7.8}$$

As mentioned in Section 3.1.2, the canonical basis is not fixed by Eqs. (3.12) and (3.13) uniquely. There remains a freedom of a rotation in each KY 2-plane, $\boldsymbol{\omega}^\mu \wedge \tilde{\boldsymbol{\omega}}^\mu$, which in terms of the null basis (7.4) reads $\boldsymbol{m}_{\hat{\mu}} \to \exp(i\varphi_\mu)\boldsymbol{m}_{\hat{\mu}}$. We uniquely fix the canonical basis by setting the phase factor $\alpha = \pi/2$. Then, we have

$$D_{\hat{\mu}} x_\mu = \frac{i}{\sqrt{2}} \sqrt{Q_\mu} \,. \tag{7.9}$$

Using (7.7) and (7.9) we find

$$\boldsymbol{\xi} = \frac{1}{\sqrt{2}} \sum_\mu \sqrt{Q_\mu}\, (\boldsymbol{m}_{\hat{\mu}} + \bar{\boldsymbol{m}}_{\hat{\mu}}) = \sum_\mu \sqrt{Q_\mu}\, \tilde{\boldsymbol{e}}_{\hat{\mu}}. \tag{7.10}$$

Eqs. (7.7) and (7.9) also give us that the gradient $\boldsymbol{d}x_\mu$ of the eigenvalue x_μ is proportional to $\boldsymbol{\omega}^{\hat{\mu}}$,

$$\boldsymbol{d}x_\mu = \sqrt{Q_\mu}\, \boldsymbol{\omega}^{\hat{\mu}} \,. \tag{7.11}$$

A simple calculation employing Eqs. (3.12), (7.10) and (7.11) shows that

$$\boldsymbol{\xi} \cdot \boldsymbol{h} = -\sum_\mu x_\mu \sqrt{Q_\mu}\, \boldsymbol{\omega}^{\hat{\mu}} = \boldsymbol{d}\Big(-\frac{1}{2} \sum_\mu x_\mu^2\Big) \,. \tag{7.12}$$

With the help of the fact that this 1-form is exact and using the closeness of $\boldsymbol{h}$ we immediately obtain the desired relation

$$\pounds_{\xi} \boldsymbol{h} = \boldsymbol{\xi} \cdot \boldsymbol{d}\boldsymbol{h} + \boldsymbol{d}(\boldsymbol{\xi} \cdot \boldsymbol{h}) = 0 \,. \tag{7.13}$$

In an odd-dimensional case, equipped with the extra direction $\boldsymbol{e}_{\hat{0}}$, we have besides (7.2) also an additional equation

$$\check{\boldsymbol{h}} \cdot \boldsymbol{e}_{\hat{0}} = 0 \,. \tag{7.14}$$

Let us denote $D_{\hat{0}} \equiv \nabla_{\boldsymbol{e}_{\hat{0}}}$, and apply this operator on (7.2). Proceeding analogously to the derivation of (7.7) we obtain

$$D_{\hat{0}} x_\nu = 0 \,, \tag{7.15}$$

and therefore Eq. (7.11) still holds. Moreover, denoting by

$$e_{\hat{0}} \cdot \boldsymbol{\xi} \equiv \sqrt{-\frac{c}{A^{(n)}}}\,, \tag{7.16}$$

we obtain the expression for $\boldsymbol{\xi}$, valid in any dimension D,

$$\boldsymbol{\xi} = \sum_{\mu} \sqrt{Q_\mu}\, \tilde{e}_{\hat{\mu}} + \varepsilon \sqrt{-\frac{c}{A^{(n)}}} e_{\hat{0}}\,. \tag{7.17}$$

Using (7.14), we finally find that Eqs. (7.12) and (7.13) remain unchanged.

7.1.2 Killing vector condition

The second condition in Eq. (7.1), which states that $\boldsymbol{\xi}$ is a Killing vector, is automatically satisfied in any Einstein space. Indeed, it was demonstrated in [Tachibana, 1969] (see also Appendix C.2) that

$$\nabla_{(a}\xi_{b)} = \frac{1}{D-2}\, R_{n(a} h_{b)}{}^{n}\,. \tag{7.18}$$

For spaces obeying the vacuum Einstein equations with the cosmological constant we have the Ricci tensor proportional to the metric and thanks to the antisymmetry of $\boldsymbol{h}$ we immediately get $\nabla_{(a}\xi_{b)} = 0$, that is $\pounds_{\xi} \boldsymbol{g} = 0$.

Thus, when the vacuum Einstein equations with the cosmological constant are imposed both conditions (7.1) are valid and using the results of [Houri *et al.*, 2008a], [Houri *et al.*, 2007] one can derive that the metric has to be the Kerr-NUT-(A)dS spacetime (4.1)–(4.3).

7.2 Canonical metric element

In this section we explicitly construct the *canonical metric* admitting the PCKY tensor. Namely, we show that the most general metric element admitting this tensor is the off-shell metric (4.1). Our demonstration extends the result of [Houri *et al.*, 2008a], [Houri *et al.*, 2007] where it was proved provided the additional conditions (7.1) and with the help of the theory of separability structures (see Section 6.3.1). Let us emphasize that, contrary to the previous section, we work off-shell, that is without imposing the Einstein equations.

It might seem that an obvious path to follow is to prove that (yet off-shell) both conditions (7.1) can still be derived from the very existence of the PCKY tensor and then use the result of [Houri *et al.*, 2008a], [Houri *et al.*, 2007]. In fact, going through the proof of the previous section, we realize that the first condition is indeed satisfied off-shell. However, it is not a straightforward task to prove the second condition (7.1) without imposing the Einstein equations. Therefore, instead we proceed in a different way. We explicitly demonstrate that besides n natural coordinates[3] x_μ, associated with the eigenvalues of $\boldsymbol{h}$, it is possible to introduce $n+\varepsilon$ additional coordinates ψ_k, associated with the tower of vectors generated from $\boldsymbol{h}$, (3.70)–(3.72), so that the metric and the PCKY tensor take the form (4.1) and (4.13), respectively. Here we sketch only main steps of the derivation and for simplicity restrict to an even dimension $D = 2n$. Technical details of this construction, including the odd-dimensional case, are in preparation [Krtouš *et al.*, 2008a].

First, taking all projections of equation (7.6), we collect a partial information about the Ricci coefficients. For example, we obtain that only those Ricci coefficients with at least two indices equal are nonvanishing. Next, using $\boldsymbol{\xi}\cdot \boldsymbol{d}x_\mu = 0$ we can calculate the Lie derivative of $\boldsymbol{e}_{\hat\mu}$ in terms of function q_μ,

$$q_\mu \equiv \boldsymbol{\xi}\cdot \boldsymbol{d}\big[\ln(\sqrt{Q_\mu})\big]\,. \tag{7.19}$$

Using duality relations and action of the PCKY tensor we find

$$\pounds_\xi \boldsymbol{e}_{\hat\mu} = q_\mu \boldsymbol{e}_{\hat\mu} + \sum_\nu E^\nu_\mu\, \tilde{\boldsymbol{e}}_{\hat\nu}\,, \quad \pounds_\xi \tilde{\boldsymbol{e}}_{\hat\mu} = -q_\mu \tilde{\boldsymbol{e}}_{\hat\mu}\,, \tag{7.20}$$

where E^ν_μ are yet unspecified. Expressing these Lie derivatives using covariant derivatives gives an additional information about the Ricci coefficients and determines E^ν_μ in terms of the Ricci coefficients and derivatives of Q_μ. It also guarantees that $\tilde{\boldsymbol{e}}_{\hat\nu}\cdot \boldsymbol{d}Q_\mu = 0$ for $\mu\neq\nu$ and $q_\mu = \tilde{\boldsymbol{e}}_{\hat\mu}\cdot \boldsymbol{d}\sqrt{Q_\mu}$. These facts allow us to calculate the Lie brackets among all the vectors $\boldsymbol{e}_{\hat\mu}, \tilde{\boldsymbol{e}}_{\hat\mu}$ of the canonical basis. They do not commute, with the exception: $[\tilde{\boldsymbol{e}}_{\hat\mu}, \tilde{\boldsymbol{e}}_{\hat\nu}] = 0$.

Now, we introduce a new basis $\{\boldsymbol{\epsilon}_\mu, \tilde{\boldsymbol{\epsilon}}_k\}$, $\mu = 1,\dots,n$, $k = 0,\dots,n-1$,

$$\boldsymbol{\epsilon}_\mu = \frac{1}{\sqrt{Q_\mu}}\,\boldsymbol{e}_{\hat\mu}\,, \quad \tilde{\boldsymbol{\epsilon}}_k = \sum_\mu A^{(k)}_\mu \sqrt{Q_\mu}\,\tilde{\boldsymbol{e}}_{\hat\mu}\,, \tag{7.21}$$

with $A^{(k)}_\mu$ given by (3.60). The meaning of the basis vectors $\tilde{\boldsymbol{\epsilon}}_k$ is elucidated by observing that they coincide with the vector fields $\boldsymbol{\xi}_{(k)}$, (3.70)–(3.71), generated from the PCKY tensor.

[3] Let us remind that it is a part of the definition of the PCKY tensor that its eigenvalues x_μ are functionally independent in some spacetime domain. This means that x_μ's are non-constant, independent, scalar functions with different values at a generic point of the manifold and one can use them as natural coordinates.

Using the known Ricci coefficients and the Jacobi identity we can prove that vectors of this frame do commute,

$$[\epsilon_\mu, \epsilon_\nu] = [\epsilon_\mu, \tilde{\epsilon}_k] = [\tilde{\epsilon}_k, \tilde{\epsilon}_l] = 0\,. \tag{7.22}$$

Moreover, for the dual frame

$$\epsilon^\mu = \sqrt{Q_\mu}\,\omega^{\hat{\mu}} = dx_\mu\,, \quad \tilde{\epsilon}^k = \sum_\mu \frac{(-x_\mu^2)^{n-1-k}}{U_\mu\sqrt{Q_\mu}}\,\tilde{\omega}^{\hat{\mu}} \tag{7.23}$$

we show

$$d\epsilon^\mu = 0\,, \quad d\tilde{\epsilon}^k = 0\,. \tag{7.24}$$

Both conditions (7.22) and (7.24) ensure that additionally to x_μ, $\mu = 1, \ldots, n$, it is possible to introduce coordinates ψ_k, $k = 0, \ldots, n-1$, such that

$$\epsilon_\mu = \partial_{x_\mu}\,, \quad \tilde{\epsilon}_k = \partial_{\psi_k}\,, \qquad \epsilon^\mu = dx_\mu\,, \quad \tilde{\epsilon}^k = d\psi_k\,. \tag{7.25}$$

Taking into account the inverse of Eqs. (7.23) we get

$$\omega^{\hat{\mu}} = \frac{1}{\sqrt{Q_\mu}}\,dx_\mu\,, \quad \tilde{\omega}^{\hat{\mu}} = \sqrt{Q_\mu}\sum_{k=0}^{n-1} A_\mu^{(k)}\,d\psi_k\,, \tag{7.26}$$

which coincides with the basis 1-forms of the orthonormal form of the metric (4.6), with unspecified metric functions Q_μ. However, in the process, we also learn that metric functions Q_μ must take the form (4.2), particularly that $q_\mu = 0$ and $E_\mu^\nu = 0$. This finishes the proof of our main result: we have constructed a coordinate system in which the canonical metric element admitting the PCKY tensor takes the off-shell form (4.1). Let us emphasize that this result was achieved without imposing the Einstein equations, starting only from the quantities determined by the PCKY tensor. As a corollary of this construction, we have established that all the vectors $\boldsymbol{\xi}_{(k)}$ are Killing vectors.

Let us finally mention, that very recently the authors of [Houri *et al.*, 2008c], [Houri *et al.*, 2008b] were able to construct the most general metric element admitting a closed CKY 2-form. Such a 2-form, besides the functionally independent eigenvalues, may also admit the constant eigenvalues.[4] The key observation for the construction is the fact that Eq. (3.3) for a closed CKY 2-form forbids the possibility of degenerate non-constant eigenvalues, that is, the Darboux subspaces corresponding to the non-constant eigenvalues are always

[4]This, in particular, incorporates the case of a covariantly constant PCKY tensor, that is a PCKY tensor for which the primary vector $\boldsymbol{\xi}$ vanishes. Such a tensor possesses only the constant eigenvalues.

2-dimensional. This means, that with respect to the functionally independent eigenvalues the metric 'behaves' as the canonical spacetime for the PCKY tensor, and one has to find only the 'trivial' part, corresponding to the constant eigenvalues. The resulting canonical element turns out to be the 'generalized Kerr-NUT-(A)dS spacetime' [Houri *et al.*, 2008c], [Houri *et al.*, 2008b], or more precisely, the Kaluza–Klein metric on the bundle over Kähler manifold whose fibres are canonical metric elements described above. These results complete the classification of all spacetimes admitting a closed CKY 2-form.

Part III

Further Developments

Chapter 8

Stationary strings and branes

In this chapter we demonstrate complete integrability of the Nambu–Goto equations for a stationary string in the canonical spacetime (4.1). The stationary string in D dimensions is generated by a 1-parameter family of Killing trajectories and the problem of finding a string configuration reduces to a problem of finding a geodesic line in an effective $(D-1)$-dimensional space. Resulting integrability of this geodesic problem is connected with the existence of hidden symmetries which are inherited from the black hole background. More generally, in a spacetime with p mutually commuting Killing vectors it is possible to introduce a concept of a ξ-brane, that is a p-brane with the worldvolume generated by these fields and a 1-dimensional curve. We discuss conditions of the integrability of such ξ-branes in the Kerr-NUT-(A)dS spacetime (4.1)–(4.3). This chapter is based on [Kubizňák & Frolov, 2008].

8.1 Introduction

There are several reasons why the problem of interaction of strings and branes with black holes attracted interest recently. Fundamental strings and branes are basic objects in string theory [Polchinski, 1998], and black holes (as well as other black objects) form an important class of solutions of the low-energy effective action in this theory (see, e.g., [Ortin, 2004]). On the other hand, cosmic strings and domain walls are topological defects which can be naturally created during phase transitions in the early Universe (see, e.g., [Vilenkin & Shellard, 1994], [Polchinski, 2004], [Davis & Kibble, 2005]). Their interaction with astrophysical black holes may result in interesting observational effects. In both cases we are dealing with a problem when the interacting objects are non-local and relativistic. An important example is an interaction of a bulk black hole with a brane representing our world in the brane world models (see, e.g., [Emparan *et al.*, 2000], [Frolov *et al.*, 2003], [Frolov *et al.*, 2004a], [Frolov *et al.*, 2004b], [Rodrigo, 2006], [Majumdar & Mukherjee, 2005]). A stationary test brane interacting with a bulk black hole can be used as a toy model for the study of (Euclidean) topology change transitions [Frolov, 2006]. This model demonstrates interesting scaling and self-similarity properties during such phase transitions, similar to the Choptuik critical collapse [Choptuik, 1993] and merger black hole transitions [Kol, 2006], [Asnin *et al.*, 2006]. These models may also have far going interesting consequences for the study of phase transitions in quantum chromodynamics (see, e.g., [Mateos *et al.*, 2006], [Kobayashi *et al.*, 2007], [Albash *et al.*, 2008], [Hoyos-Badajoz *et al.*, 2007]).

Even in an idealized case, when one neglects the effects connected with the thickness of the strings and branes and their tension, this problem is quite complicated. The reason is evident: the Dirac–Nambu–Goto action for these objects in an external gravitational field is very nonlinear. In a general case numerical calculations are required (see, e.g., [Snajdr *et al.*, 2002], [Snajdr & Frolov, 2003], [Dubath *et al.*, 2007]). When the effects of thickness and tension are taken into account these numerical calculations become even more involved (see, e.g., [Morisawa *et al.*, 2000], [Morisawa *et al.*, 2003], [Flachi & Tanaka, 2005], [Flachi *et al.*, 2006], [Flachi & Tanaka, 2007]).

Study of *stationary* configurations of strings and branes in a background of a stationary black hole is simpler problem which in several cases allows complete solution. One of the examples is a stationary string in the Kerr spacetime. It was shown [Frolov *et al.*, 1989] that the Hamilton–Jacobi equation for such a string allows a complete separation of variables. It was also demonstrated [Carter & Frolov, 1989], [Carter *et al.*, 1991] that this property is directly connected with the hidden symmetry of the Kerr metric generated by the Killing tensor [Walker & Penrose, 1970] discovered by Carter in 1968 [Carter, 1968b]. More recently, Carters's method was applied to 5-dimensional rotating black holes and the Killing tensor was found in these spacetimes [Frolov & Stojković, 2003b], [Frolov & Stojković, 2003a]. This result was used to show that the equations for a stationary string in the 5-dimensional Myers–Perry metric are completely integrable [Frolov & Stevens, 2004].

Here we demonstrate that this result allows a generalization to higher dimensional rotating black holes in an arbitrary number of spacetime dimensions. Namely, we show that a stationary string configuration is completely integrable in the canonical spacetime (4.1). We use the fact that after performing a dimensional reduction along the Killing trajectories, the stationary string equation in a D-dimensional stationary spacetime can be reduced to a geodesic equation in a $(D-1)$-dimensional space with a metric conformal to the reduced metric. The separability of the Hamilton–Jacobi equation in this effective metric follows from the separability of the Hamilton–Jacobi equation in the original D-

dimensional canonical spacetime proved in Chapter 6 and a special property of the primary (timelike) Killing vector.

There is a natural generalization of the concept of a stationary string in the case when there exist several mutually commuting Killing vectors. If p is a number of these fields one may consider a $(p+1)$-hypersurface generated by the Killing vectors passing through a 1-dimensional line. We call a *ξ-brane* an extended object, a p-brane, with the worldvolume associated with this hypersurface. We discuss integrability conditions for ξ-branes in the Kerr-NUT-(A)dS spacetime (4.1)–(4.3) and give some examples of integrable systems.

8.2 Stationary strings

Consider a string in a stationary D-dimensional spacetime M^D. Let x^a ($a = 0, \ldots, D-1$) be coordinates in it and

$$ds^2 = g_{ab} dx^a dx^b \tag{8.1}$$

be its metric. We denote by ξ^a the corresponding Killing vector which is timelike at least in some domain of M^D. We call the string *stationary* if ξ^a is tangent to the 2-dimensional worldsheet Σ_ξ of the string in this domain. In other words, the surface Σ_ξ is generated by a 1-parameter family of the Killing trajectories (the integral lines of ξ^a).

A general formalism for studying a stationary spacetime based on its foliation by Killing trajectories was developed by Geroch [Geroch, 1971]. In this approach, one considers a set S of the Killing trajectories as a quotient space and introduce the structure of the differential Riemannian manifold on it. The projector h_{ab} onto S is related to the metric g_{ab} as follows:

$$g_{ab} = h_{ab} + \xi_a \xi_b / \xi^2 \,. \tag{8.2}$$

In this formalism, a stationary string is uniquely determined by a curve in S.

The equation for this curve follows from the Nambu–Goto action

$$I = -\mu \int d^2\zeta \, |\gamma|^{1/2} \,. \tag{8.3}$$

Here μ is the string tension. As it enters the Nambu–Goto action as a common factor, its value is not important and one can always put $\mu = 1$. The string worldsheet can be parametrized by $x^a = x^a(\zeta^A)$, where ζ^A are coordinates on

Σ_ξ, $(A = 0, 1)$. We denote by γ_{AB} the induced metric on Σ_ξ

$$\gamma_{AB} = \frac{\partial x^a}{\partial \zeta^A}\frac{\partial x^a}{\partial \zeta^B} g_{ab}\,, \tag{8.4}$$

and by γ its determinant.

Let Killing time parameter be t, so that $\xi^a \partial_a = \partial_t$, and let y^i be coordinates which are constant along the Killing trajectories (coordinates in S). Then, the non-vanishing components of the projection operator h_{ab} are h_{ij} (reduced metric) and the metric (8.1)-(8.2) takes the form

$$\begin{aligned} ds^2 &= -F(dt + A_i dy^i)^2 + h_{ij}dy^i dy^j\,, && (8.5)\\ F &= g_{tt} = -\xi_a\xi^a\,, \quad A_i = g_{ti}/g_{tt}\,. && (8.6)\end{aligned}$$

From (8.2) it also follows that in these coordinates $h^{ij} = g^{ij}$.

We choose $\zeta^0 = t$ and denote $\zeta^1 = \sigma$. Then the string configuration is determined by $y^i = y^i(\sigma)$. The induced metric is

$$d\gamma^2 = \gamma_{AB} d\zeta^A d\zeta^B = -F(dt + A d\sigma)^2 + dl^2\,, \tag{8.7}$$

where

$$dl^2 = h\, d\sigma^2\,, \quad A = A_i \frac{dy^i}{d\sigma}\,, \quad h = h_{ij}\frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}\,, \tag{8.8}$$

and it has the following determinant

$$\gamma = \det(\gamma_{AB}) = -Fh\,. \tag{8.9}$$

So, the Nambu–Goto action is

$$\begin{aligned} I &= -\Delta t E\,, && (8.10)\\ E &= \mu\int \sqrt{F}dl = \mu\int d\sigma\sqrt{F h_{ij}\frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}}\,. && (8.11)\end{aligned}$$

In a static spacetime the equation (8.11) has a very simple meaning: The energy density of a string is proportional to its proper length dl multiplied by the redshift factor $\sqrt{F}$.

The problem of a stationary string configuration therefore reduces to that of a geodesic in the $(D-1)$-dimensional effective background

$$dH^2 = H_{ij} dy^i dy^j = F h_{ij} dy^i dy^j\,. \tag{8.12}$$

In order to solve this geodesic problem we shall use the Hamilton-Jacobi method. That is, we shall attempt for the additive separation of the Hamilton-

Jacobi equation

$$\frac{\partial S}{\partial \sigma} + H^{ij}\, \partial_i S\, \partial_j S = 0\,, \tag{8.13}$$

where H^{ij} is the inverse of the effective metric (8.12) with the components given by

$$F H^{ij} = h^{ij} = g^{ij}\,. \tag{8.14}$$

If the Hamilton–Jacobi equation can be separated, the effective geodesic motion and hence also the stationary string configuration are completely integrable, (see Section 6.3.1).

8.3 Stationary strings in canonical spacetime

In this section we prove complete integrability of a stationary string configuration in the canonical spacetime (4.1). As explained earlier, in such a spacetime the primary Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_{\psi_0}$ plays a special role. This vector is (after the analytical continuation to the physical domain) timelike in the black hole exterior. It is also the one most 'directly connected' with the PCKY tensor.[1] We call a string stationary if it is tangent to the primary Killing vector. For such string one has the following form of the effective metric:

$$\begin{aligned} F H^{ij}\boldsymbol{\partial}_i\boldsymbol{\partial}_j &= \sum_{\mu=1}^{n}\Big[Q_\mu(\boldsymbol{\partial}_{x_\mu})^2 + \frac{1}{Q_\mu U_\mu^2}\Big(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}\boldsymbol{\partial}_{\psi_k}\Big)^2\Big] - \frac{\varepsilon}{cA^{(n)}}(\boldsymbol{\partial}_{\psi_n})^2\,, \\ F &= \sum_{\mu=1}^{n} Q_\mu - \frac{\varepsilon c}{A^{(n)}}\,. \end{aligned} \tag{8.15}$$

The expression is similar to (4.7), with the only difference that in the sum over k the term $k = 0$ is omitted. This corresponds to the natural projection given by (8.14).

In the background of the metric H_{ij} the Hamilton–Jacobi equation (8.13) allows the additive separation of variables

$$S = w\sigma + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=1}^{m} L_k \psi_k \tag{8.16}$$

[1]Let us remark here that the asymmetry among the primary Killing vector $\boldsymbol{\partial}_{\psi_0}$ and the secondary Killing vectors $\boldsymbol{\partial}_{\psi_j}$ can be also viewed as arising from the requirement that, in addition to the Hamilton–Jacobi, also the Klein–Gordon equation is separable (see, e.g., [Carter & Frolov, 1989] and references therein).

with functions $S_\mu(x_\mu)$ of a single argument x_μ. Substituting (8.16) into (8.13) we obtain

$$Fw + \sum_{\mu=1}^{n} \Big[Q_\mu S_\mu'^2 + \frac{1}{X_\mu U_\mu} \Big(\sum_{k=1}^{m} (-x_\mu^2)^{n-1-k} L_k \Big)^2 \Big] - \frac{\varepsilon L_n^2}{c A^{(n)}} = 0\,, \tag{8.17}$$

where $S_\mu{}'$ denotes the derivative of function S_μ with respect to its single argument x_μ. Using the explicit form of F and algebraic identity

$$\frac{1}{A^{(n)}} = \sum_{\mu=1}^{n} \frac{1}{x_\mu^2 U_\mu}\,, \tag{8.18}$$

we can rewrite the last equation in the form

$$\sum_{\mu=1}^{n} \frac{G_\mu}{U_\mu} = 0\,, \tag{8.19}$$

where G_μ are functions of x_μ only:

$$G_\mu = X_\mu \left(S_\mu'^2 + w \right) + \frac{1}{X_\mu} \Big(\sum_{k=1}^{m} (-x_\mu^2)^{n-1-k} L_k \Big)^2 - \varepsilon \, \frac{L_n^2/c + wc}{x_\mu^2}\,. \tag{8.20}$$

Applying Lemma 2 of Appendix C.4, we write the general solution of (8.19) as

$$G_\mu = \sum_{k=1}^{n-1} c_k (-x_\mu^2)^{n-1-k}\,, \tag{8.21}$$

where c_k are arbitrary constants. So, we have obtained the equations for S_μ',

$$S_\mu'^2 = \frac{1}{X_\mu} \Big[\sum_{k=1}^{n-1} c_k \, (-x_\mu^2)^{n-1-k} + \varepsilon \, \frac{L_n^2/c + wc}{x_\mu^2} \Big] - \frac{1}{X_\mu^2} \Big(\sum_{k=1}^{m} (-x_\mu^2)^{n-1-k} L_k \Big)^2 - w\,, \tag{8.22}$$

which can be solved by quadratures.

This completes the demonstration that in the canonical background (4.1) the reduced $(D-1)$-dimensional geodesic problem (8.11) allows the separation of the Hamilton–Jacobi equation (8.13) and therefore the stationary string configuration is completely integrable.

8.4 Inherited hidden symmetries

The resulting complete integrability of the stationary string configuration in the canonical spacetime (4.1) is connected with the existence of hidden symmetries of the $(D-1)$-dimensional effective metric H_{ij}. Namely, there exist $(n-1)$ irreducible Killing tensors $C^{ij}_{(k)}$, $(k=1,\dots,n-1)$, which give the constants of motion

$$c_k = C^{ij}_{(k)} p_i p_j\,, \quad D_{(m} C^{(k)}_{ij)} = 0\,, \tag{8.23}$$

and allow the separation of the Hamilton–Jacobi equation (8.13) in the background H_{ij}. In the last formula $p_i = \partial_i S$ are the 'momenta of geodesic motion' and D_i denotes the covariant derivative with respect to H_{ij}.

Similar to Chapter 6, one can easily find the explicit form of $C^{ij}_{(k)}$ by inverting (8.20). Let us multiply it by $A^{(l)}_\mu / U_\mu$, sum over μ, and use identities (6.12). Then we obtain

$$C^{ij}_{(k)} = {}^{(\psi_0)}K^{ij}_{(k)} - F_{(k)} H^{ij}\,, \quad F_{(k)} \equiv \sum_{\mu=1}^{n} Q_\mu A^{(k)}_\mu - \varepsilon\, \frac{cA^{(k)}}{A^{(n)}}\,. \tag{8.24}$$

Here ${}^{(\psi_0)}K^{ij}_{(k)}$ are natural projections of the Killing tensors (6.13) for the D-dimensional canonical spacetime,

$${}^{(\psi_0)}K^{ij}_{(k)} \partial_i \partial_j = \sum_{\mu=1}^{n} \Big[A^{(k)}_\mu Q_\mu (\partial_{x_\mu})^2 + \frac{A^{(k)}_\mu}{Q_\mu U_\mu^2} \Big(\sum_{l=1}^{m} (-x_\mu^2)^{n-1-l} \partial_{\psi_l} \Big)^2 \Big] - \frac{\varepsilon A^{(k)}}{cA^{(n)}} (\partial_{\psi_n})^2\,. \tag{8.25}$$

Similar to (8.15), the direction ∂_{ψ_0} is projected out. Therefore, one can say that the hidden symmetries of the $(D-1)$-dimensional effective metric H_{ij} are 'inherited' from the hidden symmetries of g_{ab}.

A nontrivial property which follows from the separability of the Hamilton–Jacobi equation (see Chapter 6) is that the constants c_k mutually Poisson commute, or equivalently, the Schouten–Nijenhuis brackets, in the background H_{ij}, of the corresponding Killing tensors vanish:

$$\big[C_{(k)}, C_{(l)}\big]^{ijm}_{H} = C^{n(i}_{(k)} D_n C^{jm)}_{(l)} - C^{n(i}_{(l)} D_n C^{jm)}_{(k)} = 0\,. \tag{8.26}$$

Let us also mention that the objects ${}^{(\psi_0)}K^{ij}_{(k)}$ are the Killing tensors for the reduced metric h_{ij} and obey

$$\big[{}^{(\psi_0)}K_{(k)}, {}^{(\psi_0)}K_{(l)}\big]^{ijm}_{h} = 0\,. \tag{8.27}$$

These results can be easily obtained by separating the Hamilton–Jacobi equation in the background of the reduced metric h_{ij}. We expect them to be more general. (For a discussion and necessary conditions regarding the projection of a single Killing tensor see [Carter & Frolov, 1989].)

We have seen that the existence of the Killing tensors $C^{ij}_{(k)}$ for the metric H_{ij} is the property inherited from the canonical metric g_{ab}. As we have learned in Part II, the latter possesses even more fundamental symmetry—connected with the PCKY tensor from which all the Killing tensors (6.13) are derivable. A natural question arises whether H_{ij} also 'inherits' any Killing–Yano tensor. In a general case the answer is negative. The necessary conditions for a Killing tensor in 4D to be the 'square' of a Killing–Yano tensor were given by Collinson [Collinson, 1976] (see also [Ferrando & Saez, 2002]). One can easily check that they are not satisfied and hence the 4D metric H_{ij} does not admit any Killing–Yano tensor. In higher dimensions we can exclude the existence of a PCKY tensor. Indeed, as demonstrated in Chapter 7, the higher-dimensional metric element admitting the PCKY tensor is the canonical spacetime (4.1), i.e., the spacetime different from H_{ij}.

8.5 ξ-branes

In the above consideration we have focused on stationary strings, that is strings generated by a 1-parameter family of timelike Killing trajectories. There are two natural ways how one may try to generalize this construction. First, one may consider other Killing vector fields, and/or second, in the case when there exist more than one Killing vector, one may consider hypersurfaces formed by the set of Killing trajectories passing through the same 1-dimensional curve. Let us discuss these generalizations in more detail.

For simplicity we assume that the spacetime M^D allows p mutually commuting Killing vectors which we denote by $\xi^a_{(M)}$, $(M, N = 1, \ldots, p)$. The Frobenius theorem implies that for each point of the spacetime M^D there exists (at least locally) a submanifold of dimension p generated by the Killing vectors $\xi^a_{(M)}$ passing through this point. In other words, the set $\xi = \{\boldsymbol{\xi}_{(M)}\}$ defines a foliation of M^D. Similar to what was done in the Geroch formalism for one Killing vector field, one can define a quotient space S of M^D determined by the action of the isometry group generated by ξ. This generalization of the Geroch's formalism was developed in [Mansouri & Witten, 1984]. The metric g_{ab} of the spacetime M^D can be written as

$$g_{ab} = h_{ab} + \Xi_{ab}\,, \quad h_{ab}\xi^a_{(M)} = 0\,, \quad \Xi_{ab} = \sum_{M,N=1}^{p} a^{MN}\xi_{(M)a}\xi_{(N)b}\,. \tag{8.28}$$

Here a^{MN} is the $(p \times p)$ matrix which is inverse to the $(p \times p)$ matrix $a_{MN} = \xi_{(M)a}\xi^a_{(N)}$: $a^{MN}a_{NK} = \delta^M_K$. A tensor h_{ab} is a projection operator onto S.

Let us denote by y^i $(D-p)$ coordinates which are constant along the Killing surfaces generated by the set ξ, and by ψ^M the Killing parameters defined by the conditions

$$\xi^a_{(M)}\boldsymbol{\partial}_a = \boldsymbol{\partial}_{\psi^M} . \tag{8.29}$$

The metric g_{ab} in these coordinates $(x^a) = (y^i, \psi^M)$ takes the form

$$ds^2 = h_{ij}dy^i dy^j + \sum_{M,N=1}^{p} a^{MN}(\xi_{(M)a}dx^a)(\xi_{(N)b}dx^b) . \tag{8.30}$$

In these coordinates we also have

$$a_{MN} = \xi_{(M)a}\xi^a_{(N)} = \xi_{(N)M} = \xi_{(M)N} . \tag{8.31}$$

A natural generalization of stationary strings Σ_ξ are $(p+1)$-dimensional objects Σ^p_ξ which are formed by a 1-parameter family of Killing surfaces. We call them *ξ-branes*. In (y^i, ψ^M)-coordinates the equation of Σ^p_ξ is $y^i = y^i(\sigma)$. For this parametrization coordinates on Σ^p_ξ are $(\zeta^A) = (\psi^M, \sigma)$ $(A, B = 1, \dots, p+1)$. The induced metric on the ξ-brane takes the form

$$d\gamma^2 = \gamma_{AB}d\zeta^A d\zeta^B = (h+u)d\sigma^2 + 2\, d\sigma \sum_{M=1}^{p} \xi_{(M)\sigma}d\psi^M + \sum_{M,N=1}^{p} a_{MN}d\psi^M d\psi^N . \tag{8.32}$$

Here we have defined

$$h \equiv h_{ij}\frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma} , \quad \xi_{(M)\sigma} \equiv \xi_{(M)i}\frac{dy^i}{d\sigma} , \quad u \equiv \sum_{M,N=1}^{p} a^{MN}\xi_{(M)\sigma}\xi_{(N)\sigma} . \tag{8.33}$$

In order to derive (8.32) we used (8.31).

The metric γ_{AB} can be considered as a block matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8.34}$$

where A is a 1-dimensional matrix and D is a matrix $(p \times p)$. If $|Z|$ is a determinant of a matrix Z, then one has the following relation for the determinant of a block matrix (see, e.g., [Gantmacher, 1959])

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C| . \tag{8.35}$$

Using this equation one obtains

$$\gamma = \det(\gamma_{AB}) = \begin{vmatrix} h+u & \xi_{(M)\sigma} \\ \xi_{(N)\sigma} & a_{MN} \end{vmatrix} = h\,\mathcal{F}_{\xi}\,, \tag{8.36}$$

where

$$\mathcal{F}_{\xi} = \det(a_{MN}) = \det(\xi^{a}_{(M)}\xi_{(N)a}) \tag{8.37}$$

is the Gram determinant for the set $\xi = \{\boldsymbol{\xi}_{(M)}\}$ of the Killing vectors.

The Dirac–Nambu–Goto action for a $(p+1)$-dimensional brane is

$$I = -\mu \int d^{p+1}\zeta \sqrt{|\gamma|}\,, \tag{8.38}$$

where γ is the determinant of the induced metric on the brane γ_{AB}. For a ξ-brane this action reduces to the following expression[2]

$$I = -\mu V \mathcal{E}\,, \quad dl^2 = h\, d\sigma^2\,, \quad V = \int d^p\psi^N\,, \quad \mathcal{E} = \int \sqrt{\mathcal{F}_{\xi}} dl\,. \tag{8.41}$$

Thus after the dimensional reduction the problem of finding a configuration of a ξ-brane reduces to a problem of solving a geodesic equation in the reduced $(D-p)$-dimensional space with the metric

$$dH^2 = H_{ij}dy^i dy^j = \mathcal{F}_{\xi} h_{ij} dy^i dy^j\,. \tag{8.42}$$

If the original metric g_{ab} admits a Killing tensor K^{ab} then, since $h^{ij} = g^{ij}$, the natural projection ${}^{\{\xi\}}K^{ij}$ is a Killing tensor for the metric h_{ij}. However, the full effective metric H_{ij} does not inherit this symmetry unless the 'red-shift' factor $\mathcal{F}_{\xi}$ is of the special 'separable form'. Only then, the Hamilton–Jacobi equation (8.13) for the geodesic motion in the metric (8.42) allows complete separation of variables.

[2]In our derivation we have focused on a 1-dimensional line in S generating ξ-branes. The same construction remains valid for, let us say, q-dimensional hyperspace in S in the case of a $(p+q)$-dimensional brane. Then, denoting coordinates on the worldvolume of such brane by $(\zeta^A) = (\psi^M, \sigma^\alpha)$, $(\alpha, \beta = 1, \dots, q)$, and repeating the same steps one would obtain

$$\gamma = \det(h_{\alpha\beta})F_{\xi} = hF_{\xi}\,, \quad h_{\alpha\beta} = h_{ij}\frac{dy^i}{d\sigma^\alpha}\frac{dy^j}{d\sigma^\beta}\,, \tag{8.39}$$

$$I = -\mu V \mathcal{E}, \quad \mathcal{E} = \int \sqrt{\mathcal{F}_{\xi}} dv, \quad dv = \sqrt{h}\, d^q\sigma\,. \tag{8.40}$$

8.6 ξ-branes in Kerr-NUT-AdS spacetime

8.6.1 Separability condition

Let us discuss now the problem of integrability of ξ-branes in the Kerr-NUT-(A)dS background (4.1)–(4.3). There we have $m+1$ Killing fields $\boldsymbol{\partial}_{\psi_k}$, $k = 0, \dots, m$, and we may choose any arbitrary subset of them as the set ξ. In general, however, the corresponding red-shift factor $\mathcal{F}_\xi$ will not be of the separable form.

More specifically, one requires that the red-shift factor can be written as

$$\mathcal{F}_\xi = \sum_{\mu=1}^{n} \frac{f_\mu(x_\mu)}{U_\mu}\,, \tag{8.43}$$

with f_μ functions of x_μ only, in order to allow the separation of variables for the Hamilton–Jacobi equation in the effective background H_{ij}. The corresponding Killing tensors ($k = 1, \dots, n-1$) would be then

$${}^{\{\xi\}}C^{ij}_{(k)} = {}^{\{\xi\}}K^{ij}_{(k)} - f_{(k)} H^{ij}\,, \tag{8.44}$$

where ${}^{\{\xi\}}K^{ij}_{(k)}$ are due natural projections of the 'primordial' Killing tensors (6.13), with directions from the set ξ projected out, and

$$f_{(k)} \equiv \sum_{\mu=1}^{n} \frac{f_\mu A^{(k)}_\mu}{U_\mu}\,. \tag{8.45}$$

In the case of a stationary string, i.e., for $\xi = \{\boldsymbol{\partial}_{\psi_0}\}$, the red-shift factor (8.15), the norm of the primary Killing field $\boldsymbol{\partial}_{\psi_0}$, possesses the property (8.43), with

$$f_\mu = X_\mu - \varepsilon \frac{c}{x_\mu^2}\,, \tag{8.46}$$

and the integrability proved in Section 8.3 is justified.

8.6.2 ξ-branes in 4D

In 4D a stationary string is the only nontrivial example of a ξ-brane for which (in these coordinates) integrability can be proved. Indeed, as discussed in [Carter & Frolov, 1989] only in the exceptionally symmetric case of the de Sitter space itself one can obtain the integrability of the axially symmetric ξ-string with $\xi = \{\boldsymbol{\partial}_{\psi_1}\}$.

The last possibility of a ξ-brane in 4D Kerr-NUT-(A)dS spacetime is the axially symmetric stationary domain wall, $\xi = \{\partial_{\psi_0}, \partial_{\psi_1}\}$. Let us consider this important example in more detail. The action takes the form

$$I = -\mu \Delta\psi_0 \Delta\psi_1 \mathcal{E}\,, \quad \mathcal{E} = \int d\sigma \sqrt{H_{ij}\frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}}\,, \tag{8.47}$$

where the effective 2-dimensional metric is

$$dH^2 = H_{ij}dy^i dy^j = \mathcal{F}_\xi \left(\frac{dx_1^2}{Q_1} + \frac{dx_2^2}{Q_2}\right). \tag{8.48}$$

The red-shift factor reads

$$\mathcal{F}_\xi = \begin{vmatrix} g_{\psi_0\psi_0} & g_{\psi_0\psi_1} \\ g_{\psi_0\psi_1} & g_{\psi_1\psi_1} \end{vmatrix} = \sum_{\mu=1}^{2} \frac{f_\mu}{U_\mu}\,, \tag{8.49}$$

where

$$f_\mu = x_\mu^2 X_\mu (X_1 + X_2). \tag{8.50}$$

Evidently, f_μ becomes function of x_μ only in the case when all parameters in metric functions X_μ, (4.3), but c_0, vanish. Only in that trivial case the Hamilton–Jacobi equation for the axially symmetric stationary domain wall in 4D can be separated.

8.6.3 ξ-branes in 5D

In 5D the situation is more interesting. There we can prove integrability of the axisymmetric ξ-string, $\xi = \{\partial_{\psi_1}\}$, under the condition that parameter $c_1 = 0$. Indeed, then the red-shift factor takes the separable form (8.43) with

$$f_1(x_1) = 2b_2x_1^4 + cx_1^2\,, \quad f_2(x_2) = 2b_1x_2^4 + cx_2^2\,. \tag{8.51}$$

Also, the axially symmetric stationary ξ-brane, $\xi = \{\partial_{\psi_0}, \partial_{\psi_1}\}$ is completely integrable in the case of a vacuum ($c_2 = 0$) 5D spacetime (4.1)–(4.3) with $c_1 = 0$. In that case,

$$f_1(x_1) = 4b_1b_2x_1^2 + 2cb_1\,, \quad f_2(x_2) = 4b_1b_2x_2^2 + 2cb_2\,. \tag{8.52}$$

In both cases the nontrivial Killing tensor responsible for the integrability is given by (8.44).

However restrictive and unlikely to be generally satisfied the condition (8.43) seems, the above examples illustrate the special cases where complete integrability of ξ-branes can be analytically proved. We postpone the discussion of the existence of other nontrivial examples elsewhere.

8.7 Summary

We have studied integrability of the Nambu–Goto equations for a stationary string configuration near a higher-dimensional rotating black hole. In a general stationary spacetime this problem reduces to finding a geodesic in the effective $(D-1)$-dimensional background H_{ij}. In the canonical spacetime (4.1) the geodesic equation can be integrated by a separation of variables of the corresponding Hamilton–Jacobi equation. This separability is a consequence of the fact that H_{ij} inherits some of the hidden symmetries of the black hole. Namely, it inherits $(n-1)$ irreducible mutually commuting Killing tensors which correspond to natural projections of the Killing tensors present in g_{ab}. In a general case there are no Killing–Yano tensors generating these Killing tensors.

The problem of integrating the equations for ξ-branes is more complicated. We have given some examples where these equations are completely integrable, but in the general case complete integrability is not possible. It would be interesting to find other, physically interesting, examples of completely integrable ξ-branes in higher dimensional black hole spacetimes. It is also interesting to study cases where there exist non-complete but non-trivial sets of (quadratic in momenta) integrals of motion for ξ-branes related to the hidden symmetries of the black hole background.

Chapter 9

Parallel transport of frames

In this chapter, based on [Connell *et al.*, 2008b], we obtain and study the equations describing the parallel transport of orthonormal frames along timelike geodesics in a spacetime admitting the PCKY tensor $\boldsymbol{h}$. We demonstrate that the operator $\boldsymbol{F}$, obtained by a projection of $\boldsymbol{h}$ to a subspace orthogonal to the velocity, has in a generic case eigenspaces of dimension not greater than 2. Each of these eigenspaces is independently parallel-propagated. This allows one to reduce the parallel transport equations to a set of the first order ordinary differential equations for the angles of rotation in the 2D eigenspaces. Examples of $D = 3, 4, 5$ canonical spacetimes, (4.1), are considered and it is shown that the obtained first order equations can be solved by a separation of variables. This chapter is based on [Connell *et al.*, 2008b].

9.1 Introduction

One of the remarkable properties of the 4D Kerr metric, discovered by Marck in 1983, is that the equations of parallel transport can be integrated [Marck, 1983b], [Marck, 1983a]. Even more generally, a parallel-propagated frame along a geodesic can be constructed explicitly in any 4D spacetime admitting the rank-2 Killing–Yano tensor [Kamran & Marck, 1986]. The purpose of the present chapter is to extend these results to the case of a spacetime with an arbitrary number of dimensions admitting the PCKY tensor $\boldsymbol{h}$. It was demonstrated in Chapter 7 that such a spacetime is necessary described by the canonical metric (4.1), and in Chapter 5 that the particle geodesic motion is there completely integrable.

Solving the parallel transport equations in curved spacetime is useful for many physical problems. In the case of timelike geodesics it can be used for studying the behavior of extended objects moving in the Kerr and more general geometries. In particular, it facilitated the study of tidal forces acting on a moving body, for example a star, in the background of a massive black hole (see, e.g., [Luminet & Marck, 1985], [Laguna *et al.*, 1993], [Frolov *et al.*, 1994], [Diener *et al.*, 1997], [Shibata, 1996], [Ishii *et al.*, 2005]). Even more useful is to solve the parallel transport along null geodesics. For example, in geometric optics approximation linearly polarized photons and gravitons propagate along null geodesics while the corresponding polarization vectors are parallel-transported along the worldline [Misner *et al.*, 1973]. This property was used to study the scattering of a polarized radiation by black holes (see, e.g., [Stark & Connors, 1977], [Connors & Stark, 1977], [Connors *et al.*, 1980] and references therein). The parallel-propagated frames are very convenient for investigating the form and shape of a thin 'pencil of light' propagating in an external gravitational field. In the derivation of the equations for optical scalars such parallel propagating frames play an important technical role (see, e.g., [Pirani, 1965], [Frolov, 1977]). Another problem where such frames are useful is the, so called, peeling-off property of the gravitational radiation in an asymptotically flat spacetime (see, e.g., [Krtouš & Podolský, 2004] and references therein). In quantum physics the parallel transport of frames is an important technical element of the point splitting method which is used for calculation of renormalized values of local observables in a curved spacetime (such as vacuum expectation values of currents, stress-energy tensor, etc.). Solving of the parallel transport equations is especially useful when fields with spin are considered (see, e.g., [Christensen, 1978]).

Here, we describe how to construct a parallel-propagated frame along timelike (spacelike) geodesics. The case of null geodesics requires an additional consideration and is under preparation [Connell *et al.*, 2008a]. Let us outline the main idea of our construction. Any 2-form determines what is called a *Darboux basis*, that is a basis in which it has a simple standard form. We have already encountered the Darboux basis of $\boldsymbol{h}$ which we called a *canonical* basis (see Section 3.1.2). Since $\boldsymbol{h}$ is non-degenerate its Darboux subspaces are two-dimensional.[1] This means that the 'local' Darboux basis, defined in the tangent space of any spacetime point, is determined up to 2D rotations in the Darboux subspaces. The union of local Darboux bases of $\boldsymbol{h}$ forms a global canonical basis in the tangent bundle of the spacetime manifold. In the case of the canonical metric (4.1), there exists a special global canonical basis in which the Ricci rotation coefficients are simplified; the *principal canonical* basis. This basis is completely determined by the PCKY tensor (see Chapter 7).

Consider now a timelike geodesic describing the motion of a particle with velocity $\boldsymbol{u}$. We focus our attention on the 2-form $\boldsymbol{F}$, (3.31), obtained as a pro-

[1] In an odd number of spacetime dimensions there exists an additional one-dimensional zero-eigenvalue Darboux subspace of $\boldsymbol{h}$.

jection of the PCKY tensor $\boldsymbol{h}$ to a subspace orthogonal to the velocity $\boldsymbol{u}$. $\boldsymbol{F}$ has its own Darboux basis, which we call *comoving*. For any chosen geodesic the comoving basis is determined along its trajectory. We have seen in Section 3.3.1 that $\boldsymbol{F}$ is parallel-transported along the geodesic. In particular, this means that its eigenvalues and its Darboux subspaces, which we call the *eigenspaces* of $\boldsymbol{F}$, are parallel-transported. We shall show that for *generic* geodesics the eigenspaces of $\boldsymbol{F}$ are at most 2-dimensional. In fact, the eigenspaces with non-zero eigenvalues are 2-dimensional, and the zero-value eigenspace is 1-dimensional for an odd number of spacetime dimensions and 2-dimensional for even. So, the comoving basis is defined up to rotations in each of the 2D eigenspaces. The *parallel-propagated* basis is a special comoving basis. It can be found by solving a set of the first order ordinary differential equations for the angles of rotation in the 2D eigenspaces.

For special geodesic trajectories the 2-form $\boldsymbol{F}$ may become *degenerate*, that is at least one of its eigenspaces will have more than 2 dimensions. We shall demonstrate that the eigenspaces with non-vanishing eigenvalues in such a degenerate case may be 4-dimensional. In the odd number of spacetime dimensions one may also have a 3-dimensional eigenspace with a zero eigenvalue. Nevertheless, in these degenerate cases one can also obtain the parallel-transported basis by (now rather more complicated) time dependent rotations of the comoving basis.

9.2 Comoving basis

In this section, we shall construct a comoving basis, that is a Darboux basis of the operator $\boldsymbol{F}$, and briefly describe its properties.

9.2.1 Operator $\boldsymbol{F}$ for timelike geodesics

Let γ be a timelike geodesic affine parameterized by τ, and $u^a = dx^a/d\tau$ be its unit tangent vector (velocity), with the norm $w = \boldsymbol{u} \cdot \boldsymbol{u} = -1$. Then the parallel-transported 2-form $\boldsymbol{F}$ can be written as [cf. Eq. (3.31)]

$$\boldsymbol{F} = \boldsymbol{h} + \boldsymbol{u}^\flat \wedge \boldsymbol{s}\,, \quad \boldsymbol{s} = \boldsymbol{u} \lrcorner\, \boldsymbol{h}\,. \tag{9.1}$$

This form is obtained by a projection of the PCKY tensor $\boldsymbol{h}$ to a subspace orthogonal to the velocity $\boldsymbol{u}$ [cf. Eq. (3.33)]. Consequently, $\boldsymbol{u}$ is an eigenvector of the operator $\boldsymbol{F}$ with a zero eigenvalue.

At a chosen point of the spacetime the tangent space T splits into a 1-dimensional space U generated by $\boldsymbol{u}$, and a $(D-1)$-dimensional subspace V orthogonal to

u;

$$T = U \oplus V\,. \tag{9.2}$$

F_{ab} and $F^a{}_b$ can be considered as a 2-form and an operator, respectively, either in the subspace V or in the complete tangent space T.

9.2.2 Comoving basis

We demonstrate now that there exists such an orthonormal basis in V in which the operator $\boldsymbol{F}$ has the (matrix) form (see, e.g., [Prasolov, 1994])

$$\mathrm{diag}(0,\ldots,0,\Lambda_1,\ldots,\Lambda_p)\,, \tag{9.3}$$

where Λ_μ are matrices of the form

$$\Lambda_\mu = \begin{pmatrix} 0 & \lambda_\mu I_\mu \\ -\lambda_\mu I_\mu & 0 \end{pmatrix}\,, \tag{9.4}$$

and I_μ are unit matrices.

The operator $\boldsymbol{F}$ maps a linear space V into itself. If $(\boldsymbol{v},\boldsymbol{w}) = P_{ab}v^a w^b$ is a scalar product in V, then an adjoint operator $\boldsymbol{F}^+$ defined by the relation

$$(\boldsymbol{v},\boldsymbol{F}\boldsymbol{w}) = (\boldsymbol{F}^+\boldsymbol{v},\boldsymbol{w}) \tag{9.5}$$

obeys the relation $\boldsymbol{F}^+ = -\boldsymbol{F}$, and $\boldsymbol{F}^+\boldsymbol{F} = -\boldsymbol{F}^2$ is a positive self-adjoint operator. Its spectrum is

$$\mathrm{Spec}(-\boldsymbol{F}^2) = \{0,\lambda_1^2,\ldots,\lambda_p^2\}\,. \tag{9.6}$$

We choose λ_μ to be non-negative and order them so that

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_p\,. \tag{9.7}$$

(If $-\boldsymbol{F}^2$ does not have a zero eigenvalue, the first term λ_0 in (9.6) is omitted.) The spectrum of $\boldsymbol{F}$ is

$$\mathrm{Spec}(\boldsymbol{F}) = \{0, i\lambda_1, -i\lambda_1,\ldots,i\lambda_p,-i\lambda_p\}\,. \tag{9.8}$$

Consider a non-zero λ_μ. We denote

$$V_\mu^\pm = \mathrm{Ker}(\boldsymbol{F}\pm i\lambda_\mu\boldsymbol{I})\,,\quad q_\mu = \dim(V_\mu^\pm)\,. \tag{9.9}$$

Thus the eigenvalues and the eigenspaces of $\boldsymbol{F}$ are well defined but they are not real. In order to obtain Darboux form (9.3) it is sufficient to consider a full space

V_μ, which is a pair of eigenspaces for complex conjugate eigenvalues

$$V_\mu = V_\mu^+ + V_\mu^- \,, \quad \dim(V_\mu) = 2q_\mu \,. \tag{9.10}$$

Using a modified version of the Gram–Schmidt process one can construct a real orthonormal basis in V_μ

$$\{{}^1\boldsymbol{n}_{\hat\mu}, {}^1\tilde{\boldsymbol{n}}_{\hat\mu} \ldots, {}^{q_\mu}\boldsymbol{n}_{\hat\mu}, {}^{q_\mu}\tilde{\boldsymbol{n}}_{\hat\mu}\}\,, \tag{9.11}$$

which has the property (see, e.g., [Prasolov, 1994])

$$\boldsymbol{F}\,{}^j\boldsymbol{n}_{\hat\mu} = -\lambda_\mu\,{}^j\tilde{\boldsymbol{n}}_{\hat\mu}\,, \quad \boldsymbol{F}\,{}^j\tilde{\boldsymbol{n}}_{\hat\mu} = \lambda_\mu{}^j\boldsymbol{n}_{\hat\mu}\,. \tag{9.12}$$

Obviously, the space V_μ is the eigenspace of $\boldsymbol{F}^2$ and the vectors (9.11) form the complete set of orthonormal eigenvectors[2] of $\boldsymbol{F}^2$ corresponding to $-\lambda_\mu^2$:

$$\boldsymbol{F}^2\boldsymbol{v} = -\lambda_\mu^2\boldsymbol{v}\,, \quad \boldsymbol{v} \in V_\mu\,. \tag{9.14}$$

If $\lambda = 0$ and the corresponding subspace V_0 has q_0 dimensions, we denote an orthonormal basis in V_0 by

$$\{{}^1\boldsymbol{n}_{\hat 0}, \ldots, {}^{q_0}\boldsymbol{n}_{\hat 0}\}\,. \tag{9.15}$$

The subspaces V_μ are mutually orthogonal and their direct sum forms the space V:

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_p\,. \tag{9.16}$$

We further denote by

$$\{{}_1\boldsymbol{\varsigma}^{\hat 0}, \ldots, {}_{q_0}\boldsymbol{\varsigma}^{\hat 0}\}\,, \quad \{{}_1\boldsymbol{\varsigma}^{\hat\mu}, {}_1\tilde{\boldsymbol{\varsigma}}^{\hat\mu} \ldots, {}_{q_\mu}\boldsymbol{\varsigma}^{\hat\mu}, {}_{q_\mu}\tilde{\boldsymbol{\varsigma}}^{\hat\mu}\}\,, \tag{9.17}$$

bases of forms dual to the constructed orthonormal vector bases (9.15), (9.11). These forms give bases in the cotangent spaces V_0^* and V_μ^*. We combine the bases (9.15), (9.11), and (9.17) with $\mu = 0, \ldots, p$ to obtain a complete orthonormal basis of vectors (forms) in the space V (V^*). The duality conditions read

$${}_s\boldsymbol{\varsigma}^{\hat\mu}({}^{s'}\boldsymbol{n}_{\hat\mu'}) = {}_s\tilde{\boldsymbol{\varsigma}}^{\hat\mu}({}^{s'}\tilde{\boldsymbol{n}}_{\hat\mu'}) = \delta^\mu_{\mu'}\delta^{s'}_s\,, \quad {}_s\boldsymbol{\varsigma}^{\hat\mu}({}^{s'}\tilde{\boldsymbol{n}}_{\hat\mu'}) = {}_s\tilde{\boldsymbol{\varsigma}}^{\hat\mu}({}^{s'}\boldsymbol{n}_{\hat\mu'}) = 0\,. \tag{9.18}$$

Here, for a given $\mu = 0, \ldots, p$ index s takes the values $s = 1, \ldots, q_\mu$. It is evident

---

[2] One can also introduce the complex eigenvectors of $\boldsymbol{F}$:

$${}^j\boldsymbol{n}^\pm_{\hat\mu} = \frac{1}{\sqrt{2}}({}^j\boldsymbol{n}_{\hat\mu} \pm i\,{}^j\tilde{\boldsymbol{n}}_{\hat\mu})\,, \quad \boldsymbol{F}\,{}^j\boldsymbol{n}^\pm_{\hat\mu} = \pm i\lambda_\mu{}^j\boldsymbol{n}^\pm_{\hat\mu}\,, \tag{9.13}$$

which form the bases in $V_\mu^\pm$ [cf. Eqs. (7.2), (7.4)]. However, we shall not do so here and consider the real basis of V_μ, (9.11), instead.

from the orthonormality of the constructed basis that we also have

$$({}^{s}\boldsymbol{n}_{\hat{\mu}})^{\flat} = {}_{s}\boldsymbol{\varsigma}^{\hat{\mu}}\,, \quad ({}^{s}\tilde{\boldsymbol{n}}_{\hat{\mu}})^{\flat} = {}_{s}\tilde{\boldsymbol{\varsigma}}^{\hat{\mu}}\,, \qquad ({}_{s}\boldsymbol{\varsigma}^{\hat{\mu}})^{\sharp} = {}^{s}\boldsymbol{n}_{\hat{\mu}}\,, \quad ({}_{s}\tilde{\boldsymbol{\varsigma}}^{\hat{\mu}})^{\sharp} = {}^{s}\tilde{\boldsymbol{n}}_{\hat{\mu}}\,. \tag{9.19}$$

In this basis the antisymmetric operator $\boldsymbol{F}$, (9.1), takes the form (9.3).

For briefness in what follows we shall use the following terminology. We call V_μ an *eigenspace* of $\boldsymbol{F}$ corresponding to its *eigenvalue* λ_μ. We call the basis $\{\boldsymbol{n}\}$ ($\{\boldsymbol{\varsigma}\}$), in which the operator $\boldsymbol{F}$ takes the Darboux form (9.3), an *orthonormal Darboux basis*, or simply the *Darboux basis*.[3]

If we consider $\boldsymbol{F}$ as an operator in the complete tangent space T, the corresponding orthonormal Darboux basis is enlarged by adding the vector $\boldsymbol{u}$ to it. In this enlarged basis the operator $\boldsymbol{F}$ has the same form (9.3), with the only difference that now the total number of zeros is not q_0, but $q_0 + 1$. To remind that the constructed basis depends on the velocity $\boldsymbol{u}$ of a particle and $\boldsymbol{u}$ is one of its elements we call this basis *comoving*. The characteristic property of the comoving frame is that all spatial components of the velocity vanish.

Although so far our construction was local (we considered a chosen spacetime point), one can naturally extend the comoving basis along the whole geodesic trajectory. In a general case, however, the constructed comoving frame is not parallel-propagated. The parallel-propagated frame can be obtained by performing additional rotations in each of the parallel-propagated eigenspaces of $\boldsymbol{F}$. The equations for the corresponding rotation angles will be derived in the next section. Before we do that we demonstrate that due to the fact that the PCKY tensor $\boldsymbol{h}$ is non-degenerate the structure of the eigenspaces of $\boldsymbol{F}$, and hence the comoving basis, significantly simplifies.

[3] In a symplectic vector space with a non-degenerate 2-form ω the Darboux basis is defined as a basis in which ω takes the (matrix) form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{9.20}$$

where I is the unit matrix. When the symplectic space possesses also a positive definite scalar product, in general it is impossible to find a basis in which the metric takes the standard diagonal form and simultaneously transform ω into (9.20). However, one can put ω into the form similar to (9.3). This is why we call the above described modification of the Darboux basis an *orthonormal Darboux basis*.

9.2.3 Eigenspaces of F

In the comoving frame constructed above the 2-form $\boldsymbol{F}$ reads

$$\boldsymbol{F} = \sum_{\mu=1}^{p} \lambda_\mu \Big(\sum_{j=1}^{q_\mu} \boldsymbol{\mathfrak{s}}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\mathfrak{s}}}^{\hat{\mu}}\Big) . \tag{9.21}$$

We shall also use the following notation

$$S(\boldsymbol{F}) = \{0^{(q_0+1)}, \lambda_1^{(q_1)}, \dots, \lambda_p^{(q_p)}\} \tag{9.22}$$

to encode the complete information about the eigenvalues of $\boldsymbol{F}$ and the dimensionality of the corresponding subspaces. The extra zero eigenvalue corresponds to the 1-dimensional subspace U spanned by $\boldsymbol{u}$. One also has

$$D = 2n + \varepsilon = 1 + q_0 + 2k\,, \quad k = \sum_{\mu=1}^{p} q_\mu\,. \tag{9.23}$$

Structure of V_0

Let us now exploit the condition that $\boldsymbol{h}$ is *non-degenerate*, that is, its (matrix) rank is $2n$. Then one has

$$q_0 = \begin{cases} 1\,, & \text{for } \varepsilon = 0\,, \\ 0 \text{ or } 2\,, & \text{for } \varepsilon = 1\,. \end{cases} \tag{9.24}$$

Let us prove this assertion. From the definition (9.1) of $\boldsymbol{F}$ we find

$$\boldsymbol{h}^{\wedge m} = \boldsymbol{F}^{\wedge m} - m\boldsymbol{F}^{\wedge(m-1)} \wedge \boldsymbol{u}^\flat \wedge \boldsymbol{s}\,, \tag{9.25}$$

where we have used the property of the exterior product (2.16). It is obvious from (9.21) that the (matrix) rank of $\boldsymbol{F}$ is $2k$, that is $\boldsymbol{F}^{\wedge(k+1)} = 0$. So, using (9.25) we have $\boldsymbol{h}^{\wedge(k+2)} = 0$. It means that for a non-degenerate (matrix rank $2n$) $\boldsymbol{h}$ we have $k + 2 \geq n + 1$. Employing (9.23) this is equivalent to $q_0 \leq 1 + \varepsilon$ which, together with the fact that q_0 has to be even for D odd and vice versa, proves (9.24).

Let us now consider a nontrivial V_0, that is V_0 with $q_0 = 1 + \varepsilon$, $n - 1 = k$. The vectors spanning it can be found as the eigenvectors of the operator $\boldsymbol{F}^2$ with zero eigenvalue, not belonging to U. There is, however, a more direct way which was already used by Marck in 4D. Let us consider a Killing–Yano $(2 + \varepsilon)$-form [cf. Eq. (3.20)]

$$\boldsymbol{f} = *\boldsymbol{h}^{\wedge k}\,, \tag{9.26}$$

and use it to define a $(1+\varepsilon)$-form

$$z \equiv u \lrcorner f \,. \tag{9.27}$$

Using relation (2.25) and Eq. (9.25) one obtains

$$z = u \lrcorner * h^{\wedge k} = *(h^{\wedge k} \wedge u^\flat) = *(F^{\wedge k} \wedge u^\flat) \,. \tag{9.28}$$

Employing (9.21) we have

$$F^{\wedge k} = B \, {}_1\varsigma^{\hat{1}} \wedge {}_1\bar{\varsigma}^{\hat{1}} \wedge \ldots \wedge {}_{q_p}\bar{\varsigma}^{\hat{p}} \,, \quad B \equiv k! \prod_{\mu=1}^{p} \lambda_\mu^{q_\mu} \,. \tag{9.29}$$

This means that z spans V_0^*. In an even number of spacetime dimensions the space V_0^* is 1-dimensional and $\varsigma^{\hat{0}} = z/|z|$. Hence, using (9.19), $n_{\hat{0}} = z^\sharp/|z|$ spans V_0. In the odd number of spacetime dimensions

$$z = \text{const} \; {}_1\varsigma^{\hat{0}} \wedge {}_2\varsigma^{\hat{0}} \,. \tag{9.30}$$

Hence, the 2-form z determines the orthonormal basis $\{{}^1n_{\hat{0}}, {}^2n_{\hat{0}}\}$ in V_0 up to a 2D rotation.

Let us finally consider the odd-dimensional case in more detail. Expanding the characteristic equation for the operator F one has

$$0 = \det(F - \lambda I) = a(u) + b(u)\lambda^2 + \ldots \tag{9.31}$$

The condition that $q_0 = 2$ implies that $a(u) = \det(F) = 0$. This imposes a constraint on u. It means that $q_0 = 2$ is a *degenerate* case which happens only for special trajectories u. For a *generic* (not special) u one has trivial V_0 with $q_0 = 0$.

Eigenspaces V_μ

Using the requirement that the eigenvalues of a PCKY tensor h are functionally independent, or in other words, that in a generic point of the manifold the Darboux subspaces of h have no more than 2 dimensions, it is possible to show (see Appendix C.7) that the dimensionalities of the eigenspaces of F with non-zero eigenvalues obey the inequalities $q_\mu \leq 2$. The case of $q_\mu = 2$ is possible only in a degenerate case when the vector u obeys a special condition.

9.3 Equations of parallel transport

In this section we describe how to obtain the parallel-transported basis from the comoving basis constructed above. The crucial fact for the construction is that the 2-form $\boldsymbol{F}$ is parallel-transported along $\boldsymbol{u}$ (see Section 3.3.1)

$$\dot{\boldsymbol{F}} = \nabla_u \boldsymbol{F} = 0\,. \tag{9.32}$$

This means that any object constructed from $\boldsymbol{F}$ and the metric $\boldsymbol{g}$ is also parallel-transported. In particular, this is true for the operator $\boldsymbol{F}^2$ and its eigenvalues $-\lambda_\mu^2$. We have used this property in Section 3.3 to construct the tower of Killing tensors which, in their turn, imply complete integrability of particle geodesic motion (see Chapter 5). Here, we go a little bit further. Namely, we prove that Darboux subspaces of $\boldsymbol{F}$, the eigenspaces V_μ, are independently parallel-transported, that is

$$\dot{\boldsymbol{v}} \in V_\mu \quad \text{for}\, \forall\, \boldsymbol{v} \in V_\mu\,. \tag{9.33}$$

Indeed, using (9.14), we find

$$\boldsymbol{F}^2 \dot{\boldsymbol{v}} = \nabla_u\big(\boldsymbol{F}^2 \boldsymbol{v}\big) = \nabla_u\big(-\lambda_\mu^2 \boldsymbol{v}\big) = -\lambda_\mu^2 \dot{\boldsymbol{v}}\,, \tag{9.34}$$

which proves (9.33).[4]

This means, that the parallel-propagated basis can be obtained from the comoving basis by time dependent rotations in the eigenspaces of $\boldsymbol{F}$. We denote the corresponding matrix of rotations by $O(\tau)$. Similar to $\boldsymbol{F}$ it has the following structure

$$O = \mathrm{diag}(O_{\hat{0}}, O_{\hat{1}}, \ldots, O_{\hat{p}})\,. \tag{9.35}$$

For $\lambda_\mu > 0$, $O_{\hat{\mu}}$ are $2q_\mu \times 2q_\mu$ orthogonal matrices. Let $\{{}^s\boldsymbol{p}_{\hat{\mu}}, {}^s\tilde{\boldsymbol{p}}_{\hat{\mu}}\}$ be a parallel-propagated basis in the eigenspace V_μ and $\{{}^s\boldsymbol{n}_{\hat{\mu}}, {}^s\tilde{\boldsymbol{n}}_{\hat{\mu}}\}$ be the 'original' comoving basis. Then

$$\begin{pmatrix} {}^s\boldsymbol{p}_{\hat{\mu}} \\ {}^s\tilde{\boldsymbol{p}}_{\hat{\mu}} \end{pmatrix} = \sum_{s'=1}^{q_\mu} O_{\hat{\mu}}{}^{s}{}_{s'} \begin{pmatrix} {}^{s'}\boldsymbol{n}_{\hat{\mu}} \\ {}^{s'}\tilde{\boldsymbol{n}}_{\hat{\mu}} \end{pmatrix}\,. \tag{9.36}$$

Here, for fixed values $\{\hat{\mu}, s, s'\}$, $O_{\hat{\mu}}{}^{s}{}_{s'}$ are 2×2 matrices. Differentiating (9.36) along the geodesic and using the fact that $\{{}^s\boldsymbol{p}_{\hat{\mu}}, {}^s\tilde{\boldsymbol{p}}_{\hat{\mu}}\}$ are parallel-propagated one

[4] Let us emphasize that the dimension of an eigenspace of $\boldsymbol{F}$ is also constant along γ. For generic geodesics the eigenspaces of $\boldsymbol{F}$ with non-zero eigenvalues are always 2-dimensional, while the subspace with zero eigenvalue ($U \oplus V_0$) has $2 - \varepsilon$ dimensions. There might also exist a zero measure set of special geodesics for which either an eigenspace of $\boldsymbol{F}$ with non-zero eigenvalue has not 2 but 4 dimensions or (in the odd dimensional case) the eigenspace of $\boldsymbol{F}$ with zero eigenvalue has 3 dimensions. (See previous section.)

gets

$$\sum_{s'=1}^{q_\mu} \dot{O}_{\hat{\mu}}{}^{s}{}_{s'} \begin{pmatrix} {}^{s'}\boldsymbol{n}_{\hat{\mu}} \\ {}^{s'}\tilde{\boldsymbol{n}}_{\hat{\mu}} \end{pmatrix} = -\sum_{s'=1}^{q_\mu} O_{\hat{\mu}}{}^{s}{}_{s'} \begin{pmatrix} {}^{s'}\dot{\boldsymbol{n}}_{\hat{\mu}} \\ {}^{s'}\dot{\tilde{\boldsymbol{n}}}_{\hat{\mu}} \end{pmatrix} . \tag{9.37}$$

This gives the following set of the first order differential equations for $O_{\hat{\mu}}{}^{s}{}_{s'}$

$$\dot{O}_{\hat{\mu}}{}^{s}{}_{s'} = -\sum_{s''=1}^{q_\mu} O_{\hat{\mu}}{}^{s}{}_{s''} N_{\hat{\mu}}{}^{s''}_{s'} , \tag{9.38}$$

where

$$N_{\hat{\mu}}{}^{s''}_{s'} = \begin{pmatrix} ({}^{s'}\dot{\boldsymbol{n}}_{\hat{\mu}}, {}^{s''}\boldsymbol{n}_{\hat{\mu}}) & ({}^{s'}\dot{\boldsymbol{n}}_{\hat{\mu}}, {}^{s''}\tilde{\boldsymbol{n}}_{\hat{\mu}}) \\ ({}^{s'}\dot{\tilde{\boldsymbol{n}}}_{\hat{\mu}}, {}^{s''}\boldsymbol{n}_{\hat{\mu}}) & ({}^{s'}\dot{\tilde{\boldsymbol{n}}}_{\hat{\mu}}, {}^{s''}\tilde{\boldsymbol{n}}_{\hat{\mu}}) \end{pmatrix} . \tag{9.39}$$

For generic geodesics the parallel transport equations are greatly simplified. In this case each of the eigenspaces V_μ is two-dimensional. The equations (9.36) take the form

$$\boldsymbol{p}_{\hat{\mu}} = \cos\beta_\mu \boldsymbol{n}_{\hat{\mu}} - \sin\beta_\mu \tilde{\boldsymbol{n}}_{\hat{\mu}} , \quad \tilde{\boldsymbol{p}}_{\hat{\mu}} = \sin\beta_\mu \boldsymbol{n}_{\hat{\mu}} + \cos\beta_\mu \tilde{\boldsymbol{n}}_{\hat{\mu}} . \tag{9.40}$$

It is easy to check that Eqs. (9.38) reduce to the following first order equations

$$\dot{\beta}_\mu = (\dot{\tilde{\boldsymbol{n}}}_{\hat{\mu}}, \boldsymbol{n}_{\hat{\mu}}) = -(\boldsymbol{n}_{\hat{\mu}}, \dot{\tilde{\boldsymbol{n}}}_{\hat{\mu}}) . \tag{9.41}$$

If at the initial point $\tau = 0$ bases $\{\boldsymbol{p}\}$ and $\{\boldsymbol{n}\}$ coincide, the initial conditions for Eqs. (9.41) are

$$\beta_\mu(\tau = 0) = 0 . \tag{9.42}$$

For $\lambda_0 = 0$, $O_{\hat{0}}$ is a $q_0 \times q_0$ matrix. In even number of spacetime dimensions, $q_0 = 1$, and V_0 is spanned by $\boldsymbol{n}_{\hat{0}}$ which is already parallel-propagated. Therefore we have $O_{\hat{0}} = 1$. For odd number of spacetime dimensions, $O_{\hat{0}}$ is present only in the degenerate case, $q_0 = 2$, that is when V_0 is spanned by $\{{}^1\boldsymbol{n}_{\hat{0}}, {}^2\boldsymbol{n}_{\hat{0}}\}$. The parallel-propagated vectors $\{{}^1\boldsymbol{p}_{\hat{0}}, {}^2\boldsymbol{p}_{\hat{0}}\}$ are then given by the analogue of the equations (9.40)–(9.42).

9.4 Parallel transport in Kerr-NUT-(A)dS spacetimes

In this section we shall concretize the above procedure for the particular form of the canonical spacetime (4.1). As it is somewhat unnatural to construct a parallel-propagated frame in the unphysical (Wick rotated) space, we use the opportunity to recast these metrics into the physical signature.

9.4.1 Kerr-NUT-(A)dS spacetimes

In the physical signature, the Kerr-NUT-(A)dS spacetime (4.1)–(4.3) can be written as

$$g = \sum_{\mu=1}^{n-1} (\omega^{\hat\mu}\omega^{\hat\mu} + \tilde\omega^{\hat\mu}\tilde\omega^{\hat\mu}) + \omega^{\hat n}\omega^{\hat n} - \tilde\omega^{\hat n}\tilde\omega^{\hat n} + \varepsilon\, \omega^{\hat\epsilon}\omega^{\hat\epsilon}, \tag{9.43}$$

where the basis 1-forms are ($\mu = 1, \dots, n-1$)

$$\begin{aligned} \omega^{\hat n} &= \frac{dr}{\sqrt{Q_n}}, \quad \omega^{\hat\mu} = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad \omega^{\hat\epsilon} = \sqrt{\frac{-c}{A^{(n)}}} \sum_{j=0}^{n} A^{(j)} d\psi_j, \\ \tilde\omega^{\hat n} &= \sqrt{Q_n} \sum_{j=0}^{n-1} A_n^{(j)} d\psi_j, \quad \tilde\omega^{\hat\mu} = \sqrt{Q_\mu} \sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j . \end{aligned} \tag{9.44}$$

Notice that we enumerate the basis $\{\omega\}$ so that $\tilde\omega^{\hat n}$ is (the only one) timelike 1-form. Here, quantities $A_\mu^{(j)}, A^{(j)}, Q_\mu, U_\mu$ are defined exactly as before with the only difference that we now understand $x_n^2 = -r^2$, and

$$X_n = -\sum_{k=\varepsilon}^{n} c_k(-r^2)^k - 2Mr^{1-\varepsilon} + \frac{\varepsilon c}{r^2}, \quad X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}. \tag{9.45}$$

Time is denoted by ψ_0, azimuthal coordinates by ψ_j, $j = 1, \dots, m$, r is the Boyer-Lindquist type radial coordinate, and x_μ, $\mu = 1, \dots, n-1$, stand for latitude coordinates. Again, it is possible to consider a broader class of the *off-shell* metrics (9.43) where $X_n(r), X_\mu(x_\mu)$, are arbitrary functions.

The PCKY tensor reads [cf. Eq. (4.13)]

$$h = \sum_{\mu=1}^{n-1} x_\mu \omega^{\hat\mu} \wedge \tilde\omega^{\hat\mu} - r\omega^{\hat n} \wedge \tilde\omega^{\hat n}. \tag{9.46}$$

Obviously, the basis $\{\omega\}$ remains a principal canonical basis. The second-rank irreducible Killing tensors are [cf. Eq. (4.14)]

$$K^{(j)} = \sum_{\mu=1}^{n-1} A_\mu^{(j)} (\omega^{\hat\mu}\omega^{\hat\mu} + \tilde\omega^{\hat\mu}\tilde\omega^{\hat\mu}) + A_n^{(j)} (\omega^{\hat n}\omega^{\hat n} - \tilde\omega^{\hat n}\tilde\omega^{\hat n}) + \varepsilon A^{(j)} \omega^{\hat\epsilon}\omega^{\hat\epsilon}. \tag{9.47}$$

The velocity of a (timelike) geodesic reads

$$\boldsymbol{u}^\flat = \sum_{\mu=1}^{n}\left(u_{\hat\mu}\boldsymbol{\omega}^{\hat\mu} + \tilde{u}_{\hat\mu}\tilde{\boldsymbol{\omega}}^{\hat\mu}\right) + \varepsilon\, u_{\hat\epsilon}\boldsymbol{\omega}^{\hat\epsilon}\,, \tag{9.48}$$

where [cf. Eqs. (5.7), (5.8)]

$$\begin{aligned} u_{\hat n} &= \frac{\sigma_n}{(X_n U_n)^{1/2}}\left(W_n^2 - X_n V_n\right)^{1/2}, \quad u_{\hat\mu} = \frac{\sigma_\mu}{(X_\mu U_\mu)^{1/2}}\left(X_\mu V_\mu - W_\mu^2\right)^{1/2}, \\ \tilde{u}_{\hat n} &= \frac{W_n}{(X_n U_n)^{1/2}}\,, \quad \tilde{u}_{\hat\mu} = \frac{\mathrm{sign}(U_\mu) W_\mu}{(X_\mu U_\mu)^{1/2}}\,, \quad u_{\hat\epsilon} = -\frac{\Psi_n}{\sqrt{-cA^{(n)}}}\,. \end{aligned} \tag{9.49}$$

The constants $\sigma_\mu = \pm 1$ ($\mu = 1, \dots, n$) are independent of one another, and

$$\begin{aligned} V_n &\equiv -\sum_{j=0}^{m} r^{2(n-1-j)}\kappa_j\,, \quad V_\mu \equiv \sum_{j=0}^{m}(-x_\mu^2)^{n-1-j}\kappa_j\,, \quad \kappa_n = -\frac{\Psi_n^2}{c}\,, \\ W_n &\equiv \sum_{j=0}^{m} r^{2(n-1-j)}\Psi_j\,, \quad W_\mu \equiv \sum_{j=0}^{m}(-x_\mu^2)^{n-1-j}\Psi_j\,. \end{aligned} \tag{9.50}$$

The constant κ_0 denotes the normalization of the velocity, which for a timelike geodesic is $\kappa_0 = -1$.

We shall construct a parallel-propagated frame for geodesic motion in three steps. At first we use the freedom of local rotations in the 2D Darboux spaces of $\boldsymbol{h}$ to introduce the *velocity adapted* canonical basis in which n components of the velocity vanish. As the second step, by studying the eigenvalue problem for the operator $\boldsymbol{F}^2$ we find a transformation connecting the velocity adapted basis to a comoving basis. And finally, we derive the equations for the rotation angles in the eigenspaces of $\boldsymbol{F}$ which transform the obtained comoving basis into the parallel-propagated one.

9.4.2 Velocity adapted canonical basis

To construct the velocity adapted canonical basis we perform the boost transformation in the $\{\tilde{\omega}^{\hat n}, \omega^{\hat n}\}$ 2-plane and the rotation transformations in each of the $\{\tilde{\omega}^{\hat\mu}, \omega^{\hat\mu}\}$, $\mu < n$, 2-planes:

$$\begin{aligned} &\tilde{\boldsymbol{o}}^{\hat n} \equiv \cosh\alpha_n\tilde{\boldsymbol{\omega}}^{\hat n} + \sinh\alpha_n\boldsymbol{\omega}^{\hat n}\,, \quad \boldsymbol{o}^{\hat n} \equiv \sinh\alpha_n\tilde{\boldsymbol{\omega}}^{\hat n} + \cosh\alpha_n\boldsymbol{\omega}^{\hat n}\,, \\ &\tilde{\boldsymbol{o}}^{\hat\mu} \equiv \cos\alpha_\mu\tilde{\boldsymbol{\omega}}^{\hat\mu} + \sin\alpha_\mu\boldsymbol{\omega}^{\hat\mu}\,, \quad \boldsymbol{o}^{\hat\mu} \equiv -\sin\alpha_\mu\tilde{\boldsymbol{\omega}}^{\hat\mu} + \cos\alpha_\mu\boldsymbol{\omega}^{\hat\mu}\,, \quad \boldsymbol{o}^{\hat\epsilon} \equiv \boldsymbol{\omega}^{\hat\epsilon}\,. \end{aligned} \tag{9.51}$$

For arbitrary angles α_μ ($\mu = 1, \dots, n$) this transformation preserves the form of the metric and of the PCKY tensor:

$$
\begin{aligned}
\boldsymbol{g} &= \sum_{\mu=1}^{n-1} (\boldsymbol{o}^{\hat{\mu}} \boldsymbol{o}^{\hat{\mu}} + \tilde{\boldsymbol{o}}^{\hat{\mu}} \tilde{\boldsymbol{o}}^{\hat{\mu}}) + \boldsymbol{o}^{\hat{n}} \boldsymbol{o}^{\hat{n}} - \tilde{\boldsymbol{o}}^{\hat{n}} \tilde{\boldsymbol{o}}^{\hat{n}} + \varepsilon\, \boldsymbol{o}^{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}} , \\
\boldsymbol{h} &= \sum_{\mu=1}^{n-1} x_\mu \boldsymbol{o}^{\hat{\mu}} \wedge \tilde{\boldsymbol{o}}^{\hat{\mu}} - r \boldsymbol{o}^{\hat{n}} \wedge \tilde{\boldsymbol{o}}^{\hat{n}} .
\end{aligned}
\tag{9.52}
$$

Let us define

$$
\tilde{v}_{\hat{n}} \equiv -\sqrt{\tilde{u}_{\hat{n}}^2 - u_{\hat{n}}^2} = -\sqrt{\frac{V_n}{U_n}} , \quad \tilde{v}_{\hat{\mu}} \equiv \sqrt{\tilde{u}_{\hat{\mu}}^2 + u_{\hat{\mu}}^2} = \sqrt{\frac{V_\mu}{U_\mu}} . \tag{9.53}
$$

Then, specifying the values of α_μ to be

$$
\cosh \alpha_n = \frac{\tilde{u}_{\hat{n}}}{\tilde{v}_{\hat{n}}} , \quad \sinh \alpha_n = \frac{u_{\hat{n}}}{\tilde{v}_{\hat{n}}} , \qquad \cos \alpha_\mu = \frac{\tilde{u}_{\hat{\mu}}}{\tilde{v}_{\hat{\mu}}} , \quad \sin \alpha_\mu = \frac{u_{\hat{\mu}}}{\tilde{v}_{\hat{\mu}}} , \tag{9.54}
$$

one obtains the following form of the velocity:

$$
\boldsymbol{u}^\flat = \sum_{\mu=1}^{n} \tilde{v}_{\hat{\mu}} \tilde{\boldsymbol{o}}^{\hat{\mu}} + \varepsilon u_{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}} . \tag{9.55}
$$

It means that after this transformation the velocity vector $\boldsymbol{u}$ has only $(n + \varepsilon)$ non-vanishing components. This simplifies considerably the construction of the comoving and the parallel-propagated bases. Notice also that the boost in the $\{\tilde{\boldsymbol{\omega}}^{\hat{n}}, \boldsymbol{\omega}^{\hat{n}}\}$ 2-plane is function of r only and the rotation in each $\{\tilde{\boldsymbol{\omega}}^{\hat{\mu}}, \boldsymbol{\omega}^{\hat{\mu}}\}$ 2-plane is function of x_μ only. The components of the velocity in the adapted basis $\{\boldsymbol{o}\}$ depend on constants κ_j only; constants Ψ_j and σ_μ are absorbed in the definition of the new frame.

9.4.3 Parallel-propagated frame

At this point we have constructed the velocity adapted basis $\{\boldsymbol{o}\}$. Such a basis is still canonical [$\boldsymbol{h}$ and $\boldsymbol{g}$ take the form (9.52)]; in this basis the particle's velocity takes the significantly simplified form (9.55). The next step is to solve the eigenvalue problem for $\boldsymbol{F}^2$ and find the comoving basis. We further consider only the case of generic geodesics.[5] For such geodesics the operator $\boldsymbol{F}^2$ possesses twice

[5] A degenerate case which requires a special consideration arises when initially different elements of $S(\boldsymbol{F})$, (9.22), coincide of one another. It happens for special values of integrals of

degenerate non-zero eigenvalues. The nontrivial eigenspace V_0 is present only in even dimensions, where it is spanned by a properly normalized $\boldsymbol{z}^\sharp$. Therefore the problem of finding the parallel-propagated frame in the off-shell spacetimes (9.43)–(9.44) reduces to finding the eigenvectors $\{\boldsymbol{n}_{\hat\mu}, \tilde{\boldsymbol{n}}_{\hat\mu}\}$ spanning the 2-plane eigenspaces V_μ and subsequent 2D rotations (9.40)–(9.42) in these spaces.

In our setup it is somewhat more natural to construct, instead of the vector basis $\{\boldsymbol{p}\}$, the parallel-propagated basis of forms $\{\boldsymbol{\pi}\}$. In the generic case it consists of

$$\{\boldsymbol{u}^\flat, \boldsymbol{z}, \boldsymbol{\pi}^{\hat 1}, \tilde{\boldsymbol{\pi}}^{\hat 1}, \dots, \boldsymbol{\pi}^{\hat n}, \tilde{\boldsymbol{\pi}}^{\hat n}\}\,. \tag{9.56}$$

(The element $\boldsymbol{z}$ is present only in even dimensions.) If $\{\boldsymbol{\varsigma}^{\hat\mu}, \tilde{\boldsymbol{\varsigma}}^{\hat\mu}\}$ are comoving basis forms spanning V_μ^*, then [cf. Eqs. (9.40)–(9.42)]

$$\boldsymbol{\pi}^{\hat\mu} = \boldsymbol{\varsigma}^{\hat\mu}\cos\beta_\mu - \tilde{\boldsymbol{\varsigma}}^{\hat\mu}\sin\beta_\mu\,, \quad \tilde{\boldsymbol{\pi}}^{\hat\mu} = \boldsymbol{\varsigma}^{\hat\mu}\sin\beta_\mu + \tilde{\boldsymbol{\varsigma}}^{\hat\mu}\cos\beta_\mu\,, \tag{9.57}$$

where

$$\dot\beta_\mu = (\tilde{\boldsymbol{\varsigma}}^{\hat\mu}, \dot{\boldsymbol{\varsigma}}^{\hat\mu}) = -(\boldsymbol{\varsigma}^{\hat\mu}, \dot{\tilde{\boldsymbol{\varsigma}}}^{\hat\mu})\,, \quad \beta_\mu(\tau=0) = 0\,. \tag{9.58}$$

The rotation angles $\dot\beta_\mu$, as given by (9.58), are functions of r and x_μ. In the case when $\dot\beta_\mu$ can be brought into the form

$$\dot\beta_\mu = \frac{f_n^{(\mu)}(r)}{U_n} + \sum_{\nu=1}^{n-1}\frac{f_\nu^{(\mu)}(x_\nu)}{U_\nu}\,, \tag{9.59}$$

the problem (9.58) is *separable* and the particular solution is given by (see Appendix C.5)

$$\beta_\mu = \int\frac{\sigma_n f_n^{(\mu)}dr}{\sqrt{W_n^2 - X_nV_n}} + \sum_{\nu=1}^{n-1}\int\frac{\sigma_\nu \mathrm{sign}(U_\nu) f_\nu^{(\mu)}dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}\,. \tag{9.60}$$

9.5 Examples

We shall now illustrate the above described formalism by considering $D = 3, 4, 5$ off-shell spacetimes (9.43). We take the normalization of the velocity -1 and normalize other vectors of the parallel-transported frame to $+1$. In the derivation of the equations for $\dot\beta_\mu$ we used the Maple program.

motion characterizing the geodesic trajectories. The larger is the number of spacetime dimensions the larger is the number of different degenerate cases. Some of them will be discussed in the next section.

9.5.1 3D spacetime: BTZ black holes

Generic case

As the first example we consider the case when $D = 3$, that is when the metric (9.43) describes a BTZ black hole [Banados *et al.*, 1992]. We first discuss the generic case, $q_0 = 0$, and then briefly mention what happens for the degenerate geodesics with $q_0 = 2$. Since in three dimensions $n = 1$ we drop everywhere index μ.

So, we have the metric

$$g = -\tilde{\omega}\tilde{\omega} + \omega\omega + \omega^{\hat{\epsilon}}\omega^{\hat{\epsilon}}\,, \tag{9.61}$$

where

$$\tilde{\omega} = \sqrt{X}\boldsymbol{d}\psi_0\,,\quad \boldsymbol{\omega}=\frac{\boldsymbol{d}r}{\sqrt{X}}\,,\quad \omega^{\hat{\epsilon}}=\frac{\sqrt{c}}{r}(\boldsymbol{d}\psi_0- r^2\boldsymbol{d}\psi_1)\,,\quad X = c_1r^2 - 2M + \frac{c}{r^2}\,. \tag{9.62}$$

The parameter c_1 is proportional to the cosmological constant and parameters M and $c > 0$ are related to mass and rotation parameter.

The PCKY tensor and the Killing tensor are

$$\boldsymbol{h} = -r\boldsymbol{\omega}\wedge\tilde{\boldsymbol{\omega}}\,,\quad \boldsymbol{K} = -r^2\boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}\,. \tag{9.63}$$

The geodesic velocity reads

$$\boldsymbol{u}^\flat = \tilde{u}\tilde{\boldsymbol{\omega}} + u\boldsymbol{\omega} + u_{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}\,, \tag{9.64}$$

$$\tilde{u} = \frac{W}{\sqrt{X}}\,,\quad u=\sigma\sqrt{\frac{W^2}{X}-V}\,,\quad u_{\hat{\epsilon}} = -\frac{\Psi_1}{\sqrt{c}r}\,, \tag{9.65}$$

$$W \equiv \Psi_0 + \frac{\Psi_1}{r^2}\,,\quad V\equiv 1+\frac{\Psi_1^2}{cr^2}\,. \tag{9.66}$$

In the velocity adapted frame $\{\tilde{\boldsymbol{o}}, \boldsymbol{o}, \boldsymbol{o}^{\hat{\epsilon}}\}$, given by (9.51), we have

$$\boldsymbol{u}^\flat = \tilde{v}\tilde{\boldsymbol{o}} + u_{\hat{\epsilon}}\boldsymbol{o}^{\hat{\epsilon}}\,,\quad \tilde{v} = -\sqrt{V}\,, \tag{9.67}$$

$$\boldsymbol{F} = ru_{\hat{0}}\boldsymbol{o}\wedge(u_{\hat{0}}\tilde{\boldsymbol{o}} + \tilde{v}\boldsymbol{o}^{\hat{\epsilon}})\,. \tag{9.68}$$

We find

$$S(\boldsymbol{F}) = \{0^{(1)},\lambda^{(2)}\}\,,\quad \lambda = \frac{|\Psi_1|}{\sqrt{c}}\,. \tag{9.69}$$

The zero eigenvalue corresponds to the space U^* spanned by $\boldsymbol{u}^\flat$. In the non-degenerate case, that is when $\Psi_1 \neq 0$, the eigenspace V_0^* is trivial. The orthonormal forms spanning V_λ^* are

$$\varsigma = \boldsymbol{o}\,, \quad \tilde{\varsigma} = u_{\hat{\epsilon}}\tilde{\boldsymbol{o}} + \tilde{v}\boldsymbol{o}^{\hat{\epsilon}}\,. \tag{9.70}$$

Using (9.58) one finds

$$\dot{\beta} = \frac{C}{r^2+\lambda^2}\,, \quad C \equiv \frac{c - \Psi_0\Psi_1}{\sqrt{c}}\,. \tag{9.71}$$

The parallel-transported forms $\{\boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$ are given by (9.57), where

$$\beta = \int \frac{\sigma C dr}{(r^2+\lambda^2)\sqrt{W^2 - XV}}\,. \tag{9.72}$$

Degenerate case

Let us now consider special geodesic trajectories with $\Psi_1 = 0$ for which $q_0 = 2$. For such trajectories one has

$$\tilde{u} = \frac{\Psi_0}{\sqrt{X}}\,, \quad u = \sigma\sqrt{\frac{\Psi_0^2}{X} - 1}\,, \quad u_{\hat{\epsilon}} = 0\,. \tag{9.73}$$

In the adapted basis the velocity is $\boldsymbol{u}^\flat = -\tilde{\boldsymbol{o}}$. Operators $\boldsymbol{F}$ and $\boldsymbol{F}^2$ become trivial. The space V_0^* is spanned by $\{{}_1\varsigma^{\hat{0}}, {}_2\varsigma^{\hat{0}}\}$, where

$${}_1\varsigma^{\hat{0}} = \boldsymbol{o}, \quad {}_2\varsigma^{\hat{0}} = -\boldsymbol{o}^{\hat{\epsilon}}\,. \tag{9.74}$$

Similar to (9.57) and (9.58) parallel-transported forms can be written as follows:

$$\begin{aligned} \boldsymbol{\pi} &= {}_1\varsigma^{\hat{0}}\cos\beta - {}_2\varsigma^{\hat{0}}\sin\beta\,, \quad \tilde{\boldsymbol{\pi}} = {}_1\varsigma^{\hat{0}}\sin\beta + {}_2\varsigma^{\hat{0}}\cos\beta\,, \\ \dot{\beta} &= ({}_2\varsigma^{\hat{0}}, {}_1\dot{\varsigma}^{\hat{0}}) = -({}_1\varsigma^{\hat{0}}, {}_2\dot{\varsigma}^{\hat{0}})\,, \quad \beta(\tau = 0) = 0\,. \end{aligned} \tag{9.75}$$

Using these equations we find $\dot{\beta} = \sqrt{c}/r^2$ and hence

$$\beta = \int \frac{\sigma\sqrt{c}dr}{r^2\sqrt{\Psi_0^2 - X}}\,. \tag{9.76}$$

Notice that this relation can be obtained from (9.72) by taking the limit $\Psi_1 \to 0$.

To conclude, the parallel-propagated orthonormal frame around a BTZ black hole is $\{\boldsymbol{u}^\flat, \boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$. This frame remains parallel-propagated also off-shell, when X given by (9.62) becomes an arbitrary function of r.

9.5.2 4D spacetime: Carter's family of solutions

Let us now consider the case of $D = 4$. We have

$$g = -\tilde{\omega}^{\hat{2}}\tilde{\omega}^{\hat{2}} + \omega^{\hat{2}}\omega^{\hat{2}} + \tilde{\omega}^{\hat{1}}\tilde{\omega}^{\hat{1}} + \omega^{\hat{1}}\omega^{\hat{1}}\,, \tag{9.77}$$

where

$$\begin{aligned} \tilde{\omega}^{\hat{2}} &= \sqrt{\frac{X_2}{U_2}}(d\psi_0 + x_1^2 d\psi_1)\,, \quad & \omega^{\hat{2}} &= \sqrt{\frac{U_2}{X_2}}\,dr\,, \\ \tilde{\omega}^{\hat{1}} &= \sqrt{\frac{X_1}{U_1}}(d\psi_0 - r^2 d\psi_1)\,, \quad & \omega^{\hat{1}} &= \sqrt{\frac{U_1}{X_1}}\,dx_1\,. \end{aligned} \tag{9.78}$$

Here, $U_2 = -U_1 = x_1^2 + r^2$, and we shall not be specifying functions $X_1(x_1), X_2(r)$ at this point.

The PCKY tensor and the Killing tensor are:

$$h = x_1\omega^{\hat{1}} \wedge \tilde{\omega}^{\hat{1}} - r\omega^{\hat{2}} \wedge \tilde{\omega}^{\hat{2}}\,, \tag{9.79}$$

$$K = x_1^2(\omega^{\hat{2}}\omega^{\hat{2}} - \tilde{\omega}^{\hat{2}}\tilde{\omega}^{\hat{2}}) - r^2(\tilde{\omega}^{\hat{1}}\tilde{\omega}^{\hat{1}} + \omega^{\hat{1}}\omega^{\hat{1}})\,. \tag{9.80}$$

The components of the velocity are

$$\begin{aligned} \tilde{u}_{\hat{2}} &= \frac{W_2}{\sqrt{X_2U_2}}\,, \quad & u_{\hat{2}} &= \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{W_2^2 - X_2V_2}\,, \\ \tilde{u}_{\hat{1}} &= \frac{-W_1}{\sqrt{X_1U_1}}\,, \quad & u_{\hat{1}} &= \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}\,, \end{aligned} \tag{9.81}$$

where

$$\begin{aligned} W_2 &= r^2\Psi_0 + \Psi_1\,, \quad & V_2 &= r^2 - \kappa_1\,, \\ W_1 &= -\,x_1^2\Psi_0 + \Psi_1\,, \quad & V_1 &= x_1^2 + \kappa_1\,. \end{aligned} \tag{9.82}$$

The constants of geodesic motion Ψ_0 and Ψ_1 are associated with isometries and $\kappa_1 < 0$ corresponds to the Killing tensor (9.80). In the velocity adapted frame $\{\tilde{o}^{\hat{2}}, o^{\hat{2}}, \tilde{o}^{\hat{1}}, o^{\hat{1}}\}$ given by (9.51) we have

$$u^\flat = \tilde{v}_{\hat{2}}\tilde{o}^{\hat{2}} + \tilde{v}_{\hat{1}}\tilde{o}^{\hat{1}}\,, \quad \tilde{v}_{\hat{2}} = -\sqrt{\frac{V_2}{U_2}}\,, \quad \tilde{v}_{\hat{1}} = \sqrt{\frac{V_1}{U_1}}\,, \tag{9.83}$$

$$F = (r\tilde{v}_{\hat{1}}o^{\hat{2}} + x_1\tilde{v}_{\hat{2}}o^{\hat{1}}) \wedge (\tilde{v}_{\hat{1}}\tilde{o}^{\hat{2}} + \tilde{v}_{\hat{2}}\tilde{o}^{\hat{1}})\,. \tag{9.84}$$

We find

$$S(F) = \{0^{(2)}, \lambda^{(2)}\}\,, \quad \lambda = \sqrt{-\kappa_1}\,. \tag{9.85}$$

The first zero eigenvalue corresponds to U^*, while the second one corresponds to the eigenspace V_0^*, spanned by 1-form $\boldsymbol{z}$ (9.27). When normalized $\boldsymbol{z}$ reads

$$\boldsymbol{z} = \lambda^{-1}(-x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{2}} + r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{1}})\,. \tag{9.86}$$

The orthonormal forms spanning V_λ^* are:

$$\boldsymbol{\varsigma} = \tilde{v}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{v}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{1}}\,, \quad \tilde{\boldsymbol{\varsigma}} = \lambda^{-1}(r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{2}} + x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{1}})\,. \tag{9.87}$$

Using (9.58) one finds

$$\dot{\beta} = \frac{C}{(x_1^2-\lambda^2)(r^2+\lambda^2)} = \frac{f_1}{U_1} + \frac{f_2}{U_2}\,, \quad f_1 \equiv -\frac{C}{x_1^2-\lambda^2}\,, \quad f_2 \equiv \frac{C}{r^2+\lambda^2}\,, \tag{9.88}$$

where $C \equiv \lambda\big(\lambda^2\Psi_0 - \Psi_1\big)$. Therefore, β allows a separation of variables and can be written as

$$\beta = \int \frac{\sigma_2 f_2 dr}{\sqrt{W_2^2 - X_2V_2}} - \int \frac{\sigma_1 f_1 dx_1}{\sqrt{X_1V_1 - W_1^2}}\,, \tag{9.89}$$

where functions f_1, f_2, are defined in (9.88). The parallel-transported forms $\{\boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$ are given by (9.57).

To summarize, the parallel-propagated orthonormal frame in the spacetime (9.77)–(9.78) is $\{\boldsymbol{u}^\flat, \boldsymbol{z}, \boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$. This parallel-propagated basis is constructed for arbitrary functions $X_1(x_1), X_2(r)$, and in particular for the Carter's class of solutions [Carter, 1968c], [Carter, 1968b]—describing among others a rotating charged black hole in the cosmological background (see Appendix A). So, we have re-derived the results obtained earlier in [Marck, 1983b], [Kamran & Marck, 1986].

9.5.3 5D Kerr-NUT-(A)dS spacetime

Generic case

As the last example we consider the 5D canonical spacetime. Similar to the 3D case we shall first obtain the parallel-propagated frame for generic geodesics and then briefly discuss what happens for the special trajectories characterized by $q_0 = 2$, or $q_1 = 2$. The metric reads

$$\boldsymbol{g} = -\tilde{\boldsymbol{\omega}}^{\hat{2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \boldsymbol{\omega}^{\hat{2}}\boldsymbol{\omega}^{\hat{2}} + \tilde{\boldsymbol{\omega}}^{\hat{1}}\tilde{\boldsymbol{\omega}}^{\hat{1}} + \boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{1}} + \boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}\,, \tag{9.90}$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^{\hat{2}} &= \sqrt{\frac{X_2}{U_2}}(\boldsymbol{d}\psi_0 + x_1^2\boldsymbol{d}\psi_1)\,,\ \boldsymbol{\omega}^{\hat{2}} = \sqrt{\frac{U_2}{X_2}}\,\boldsymbol{d}r\,,\\
\tilde{\boldsymbol{\omega}}^{\hat{1}} &= \sqrt{\frac{X_1}{U_1}}(\boldsymbol{d}\psi_0 - r^2\boldsymbol{d}\psi_1)\,,\ \boldsymbol{\omega}^{\hat{1}} = \sqrt{\frac{U_1}{X_1}}\,\boldsymbol{d}x_1\,,\\
\boldsymbol{\omega}^{\hat{\epsilon}} &= \frac{\sqrt{c}}{rx_1}\,\left[\boldsymbol{d}\psi_0 + (x_1^2 - r^2)\boldsymbol{d}\psi_1 - x_1^2r^2\boldsymbol{d}\psi_2\right]\,,
\end{aligned}
\tag{9.91}
$$

and $U_2 = -U_1 = x_1^2 + r^2$. The PCKY tensor and the Killing tensor for this metric are

$$
\boldsymbol{h} \;=\; x_1\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}} - r\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}}\,, \tag{9.92}
$$

$$
\boldsymbol{K} \;=\; x_1^2(-\tilde{\boldsymbol{\omega}}^{\hat{2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \boldsymbol{\omega}^{\hat{2}}\boldsymbol{\omega}^{\hat{2}} + \boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}) - r^2(\tilde{\boldsymbol{\omega}}^{\hat{1}}\tilde{\boldsymbol{\omega}}^{\hat{1}} + \boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{1}} + \boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}})\,. \tag{9.93}
$$

The components of the velocity are

$$
\begin{aligned}
\tilde{u}_{\hat{2}} &= \frac{W_2}{\sqrt{X_2U_2}}\,, \quad u_{\hat{2}} = \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{W_2^2 - X_2V_2}\,,\\
\tilde{u}_{\hat{1}} &= \frac{-W_1}{\sqrt{X_1U_1}}\,, \quad u_{\hat{1}} = \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}\,, \quad u_{\hat{\epsilon}} = -\frac{\Psi_2}{r\sqrt{cx_1^2}}\,,
\end{aligned}
\tag{9.94}
$$

where

$$
\begin{aligned}
W_1 &= -\,x_1^2\Psi_0 + \Psi_1 - \frac{\Psi_2}{x_1^2}\,, \quad V_1 = x_1^2 + \kappa_1 + \frac{\Psi_2^2}{cx_1^2}\,,\\
W_2 &= r^2\Psi_0 + \Psi_1 + \frac{\Psi_2}{r^2}\,, \quad V_2 = r^2 - \kappa_1 + \frac{\Psi_2^2}{cr^2}\,.
\end{aligned}
\tag{9.95}
$$

In the velocity adapted frame $\{\boldsymbol{o}\}$ given by (9.51) we have

$$
\boldsymbol{u}^\flat = \tilde{v}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{v}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{1}} + u_{\hat{\epsilon}}\boldsymbol{o}^{\hat{\epsilon}}\,, \quad \tilde{v}_{\hat{2}} = -\sqrt{\frac{V_2}{U_2}}\,, \quad \tilde{v}_{\hat{1}} = \sqrt{\frac{V_1}{U_1}}\,. \tag{9.96}
$$

The 2-form $\boldsymbol{F}$ and the 2-form $\boldsymbol{z}$ are

$$
\begin{aligned}
\boldsymbol{F} \;=\;& (r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{2}} + x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{1}}) \wedge (\tilde{v}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{v}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{1}})\\
&+ru_{\hat{\epsilon}}\boldsymbol{o}^{\hat{2}} \wedge (\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{\epsilon}} + u_{\hat{\epsilon}}\tilde{\boldsymbol{o}}^{\hat{2}}) + x_1u_{\hat{\epsilon}}\boldsymbol{o}^{\hat{1}} \wedge (\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{\epsilon}} - u_{\hat{\epsilon}}\tilde{\boldsymbol{o}}^{\hat{1}})\,,
\end{aligned}
\tag{9.97}
$$

$$
\boldsymbol{z} \;=\; \boldsymbol{o}^{\hat{\epsilon}} \wedge (r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{1}} - x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{2}}) + u_{\hat{\epsilon}}(x_1\boldsymbol{o}^{\hat{2}} \wedge \tilde{\boldsymbol{o}}^{\hat{2}} + r\boldsymbol{o}^{\hat{1}} \wedge \tilde{\boldsymbol{o}}^{\hat{1}})\,. \tag{9.98}
$$

One has

$$
S(\boldsymbol{F}) = \{0^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}\}\,, \tag{9.99}
$$

where

$$\lambda_1 = \sqrt{-\frac{\kappa_1 + d}{2}}\,, \quad \lambda_2 = \sqrt{-\frac{\kappa_1 - d}{2}}\,, \quad d \equiv \sqrt{\kappa_1^2 - 4\frac{\Psi_2^2}{c}}\,. \tag{9.100}$$

The zero eigenvalue corresponds to U^*. The eigenspace V_1^* is spanned by

$$\begin{aligned} \varsigma^{\hat{1}} &= N_1\Big(-\frac{x_1 F_r(\lambda_2)}{\sqrt{U_2}}\,\tilde{o}^{\hat{2}} + \frac{r F_{x_1}(\lambda_2)}{\sqrt{U_2}}\,\tilde{o}^{\hat{1}} + o^{\hat{\epsilon}}\Big)\,, \\ \tilde{\varsigma}^{\hat{1}} &= \tilde{N}_1\Big(F_r(\lambda_2)F_{x_1}(\lambda_1)o^{\hat{2}} + o^{\hat{1}}\Big)\,. \end{aligned} \tag{9.101}$$

Here N_1 and $\tilde{N}_1$ stand for normalizations,

$$\begin{aligned} N_1 &= \sqrt{U_2}/\sqrt{U_2 + r^2 F_{x_1}(\lambda_2)^2 - x_1^2 F_r(\lambda_2)^2}\,, \\ \tilde{N}_1 &= 1/\sqrt{1 + F_{x_1}(\lambda_1)^2 F_r(\lambda_2)^2}\,, \end{aligned} \tag{9.102}$$

and we have introduced

$$F_r(\lambda) \equiv \frac{x_1 u_{\hat{\epsilon}}\sqrt{V_2}}{x_1^2 u_{\hat{\epsilon}}^2 + \lambda^2}\,, \quad F_{x_1}(\lambda) \equiv \frac{r u_{\hat{\epsilon}}\sqrt{-V_1}}{r^2 u_{\hat{\epsilon}}^2 - \lambda^2}\,, \tag{9.103}$$

which are functions of r, x_1, respectively. Using (9.58) we find

$$\begin{aligned} \dot{\beta}_1 &= \frac{C^{(1)}}{(x_1^2 - \lambda_1^2)(r^2 + \lambda_1^2)} = \frac{f_1^{(1)}}{U_1} + \frac{f_2^{(1)}}{U_2}\,, \\ f_1^{(1)} &\equiv -\frac{C^{(1)}}{x_1^2 - \lambda_1^2}\,, \quad f_2^{(1)} \equiv \frac{C^{(1)}}{r^2 + \lambda_1^2}\,, \quad C^{(1)} \equiv -\frac{\Psi_2\Psi_1 - \lambda_1^2\Psi_2\Psi_0 - c\lambda_2^2}{\lambda_2\sqrt{c}}\,. \end{aligned} \tag{9.104}$$

This means that β_1 can be separated as follows:

$$\beta_1 = \int \frac{\sigma_2 f_2^{(1)} dr}{\sqrt{W_2^2 - X_2 V_2}} - \int \frac{\sigma_1 f_1^{(1)} dx_1}{\sqrt{X_1 V_1 - W_1^2}}\,. \tag{9.105}$$

The parallel-propagated forms $\{\boldsymbol{\pi}^{\hat{1}}, \tilde{\boldsymbol{\pi}}^{\hat{1}}\}$ are given by rotation (9.57). Similarly one finds that

$$\begin{aligned}
\boldsymbol{\varsigma}^{\hat{2}} &= N_2\Big(-\frac{x_1 F_r(\lambda_1)}{\sqrt{U_2}}\,\tilde{\boldsymbol{o}}^{\hat{2}} + \frac{r F_{x_1}(\lambda_1)}{\sqrt{U_2}}\,\tilde{\boldsymbol{o}}^{\hat{1}} + \boldsymbol{o}^{\hat{\epsilon}}\Big)\,,\\
N_2 &= \sqrt{U_2}/\sqrt{U_2 + r^2 F_{x_1}(\lambda_1)^2 - x_1^2 F_r(\lambda_1)^2}\,,\\
\tilde{\boldsymbol{\varsigma}}^{\hat{2}} &= \tilde{N}_2\Big(F_r(\lambda_1)F_{x_1}(\lambda_2)\boldsymbol{o}^{\hat{2}} + \boldsymbol{o}^{\hat{1}}\Big)\,,\\
\tilde{N}_2 &= 1/\sqrt{1 + F_{x_1}(\lambda_2)^2 F_r(\lambda_1)^2}\,,
\end{aligned} \tag{9.106}$$

span the eigenspace V_2^*. Using (9.58) we have

$$\begin{aligned}
\dot{\beta}_2 &= \frac{C^{(2)}}{(x_1^2 - \lambda_2^2)(r^2 + \lambda_2^2)} = \frac{f_1^{(2)}}{U_1} + \frac{f_2^{(2)}}{U_2}\,,\\
f_1^{(2)} &\equiv -\frac{C^{(2)}}{x_1^2 - \lambda_2^2}\,,\quad f_2^{(2)} \equiv \frac{C^{(2)}}{r^2 + \lambda_2^2}\,,\quad C^{(2)} \equiv \frac{\Psi_2\Psi_1 - \lambda_2^2\Psi_2\Psi_0 - c\lambda_1^2}{\lambda_1\sqrt{c}}\,.
\end{aligned} \tag{9.107}$$

The parallel-propagated forms $\{\boldsymbol{\pi}^{\hat{2}}, \tilde{\boldsymbol{\pi}}^{\hat{2}}\}$ are given by rotation (9.57), where

$$\beta_2 = \int \frac{\sigma_2 f_2^{(2)}\,dr}{\sqrt{W_2^2 - X_2 V_2}} - \int \frac{\sigma_1 f_1^{(2)}\,dx_1}{\sqrt{X_1 V_1 - W_1^2}}\,. \tag{9.108}$$

Degenerate case

In $D = 5$ two different degenerate cases are possible. One can have either a 2-dimensional V_0 ($q_0 = 2$) which happens for the special geodesics characterized by $\Psi_2 = 0$, or a 4-dimensional V_λ ($q_1 = 2$) which happens when $\kappa_1^2 = 4\Psi_2^2/c$. The latter case is more complicated and the general formulas (9.36)–(9.39) have to be used. We shall not do this here and rather concentrate on the first degeneracy which has an interesting consequence.

So, we consider the special geodesics characterized by $\Psi_2 = 0$. It can be checked by direct calculations that in this case the results can be obtained by taking the limit $\Psi_2 \to 0$ of previous formulas. In particular, one has $u_{\hat{\epsilon}} = 0$, and functions W_1, W_2, V_1, V_2, are the same as in (9.82). The velocity $\boldsymbol{u}$ and the 2-form $\boldsymbol{F}$ become effectively 4-dimensional—equal to (9.83) and (9.84), respectively. The 2-form $\boldsymbol{z}$ (9.98) reduces to

$$\boldsymbol{z} = \boldsymbol{o}^{\hat{\epsilon}} \wedge (r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{1}} - x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{2}})\,, \tag{9.109}$$

and

$$S(\boldsymbol{F}) = \{0^{(3)}, \lambda^{(2)}\}, \quad \lambda = \sqrt{-\kappa_1}. \tag{9.110}$$

The eigenspace V_0^* is spread by $\{{}_1\varsigma^{\hat{0}}, {}_2\varsigma^{\hat{0}}\}$, where

$${}_1\varsigma^{\hat{0}} = \boldsymbol{o}^{\hat{\epsilon}}, \quad {}_2\varsigma^{\hat{0}} = \lambda^{-1}(r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{1}} - x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{2}}). \tag{9.111}$$

[Notice that ${}_2\varsigma^{\hat{0}}$ is identical to the normalized 4-dimensional 1-form $\boldsymbol{z}$ given by (9.86).] The angle of rotation in the $\{{}_1\varsigma^{\hat{0}}, {}_2\varsigma^{\hat{0}}\}$ 2-plane obeys the equation

$$\dot{\beta}_1 = \frac{C^{(1)}}{x_1^2 r^2} = \frac{f_1^{(1)}}{U_1} + \frac{f_2^{(1)}}{U_2}, \quad f_1^{(1)} \equiv -\frac{C^{(1)}}{x_1^2}, \quad f_2^{(1)} \equiv \frac{C^{(1)}}{r^2}, \quad C^{(1)} \equiv \lambda\sqrt{c}. \tag{9.112}$$

Thus β_1 is given by (9.105) with functions $f_1^{(1)}, f_2^{(1)}$, defined in (9.112). The eigenspace V_λ^* is spread by

$$\varsigma = \tilde{v}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{v}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{1}}, \quad \tilde{\varsigma} = \lambda^{-1}(r\tilde{v}_{\hat{1}}\boldsymbol{o}^{\hat{2}} + x_1\tilde{v}_{\hat{2}}\boldsymbol{o}^{\hat{1}}), \tag{9.113}$$

which is identical to the 4D case. Thus the rotation angle β_2 coincides with β given by (9.89).

Summary of 5D

To summarize, we have demonstrated that also in $D = 5$ Kerr-NUT-(A)dS spacetime the rotation angles in 2D eigenspaces can be separated and the parallel-transported frame $\{\boldsymbol{\pi}\}$ explicitly constructed. This result is again valid off-shell, that is for arbitrary functions $X_2(r)$, $X_1(x_1)$.

The special degenerate case characterized by $\Psi_2 = 0$ has the following interesting feature. The zero-value eigenspace is spanned by the 4-dimensional $\boldsymbol{u}^\flat$, by the 4-dimensional 1-form $\boldsymbol{z}$, and $\boldsymbol{o}^{\hat{\epsilon}}$. The structure of V_λ^* is identical to the 4D case and the equation of parallel transport in this plane is identical to the equation in 4D. Therefore this 5D degenerate problem effectively reduces to the generic 4D problem plus the rotation in the 2D $\{\boldsymbol{z}, \boldsymbol{o}^{\hat{\epsilon}}\}$ plane.

This indicates that a similar reduction might be valid also in higher dimensions. Namely, one may expect that the degenerate odd-dimensional problem, $\Psi_n = 0$, effectively reduces to the problem in a spacetime of one dimension lower plus the rotation in the 2D $\{\boldsymbol{z}, \boldsymbol{o}^{\hat{\epsilon}}\}$ plane. If this is so, one can use this odd-dimensional degenerate case to generate the solution for the generic (one dimension lower) even-dimensional problem.

9.6 Conclusions

In this chapter we have described the construction of a parallel-transported frame in a spacetime admitting the PCKY tensor $\boldsymbol{h}$. This tensor determines a canonical (Darboux) basis at each point of the spacetime. The geodesic motion of a particle in such a space can be characterized by the components of its velocity $\boldsymbol{u}$ with respect to this basis. For a moving particle it is also natural to introduce a comoving basis, which is just a Darboux basis of $\boldsymbol{F}$, where $\boldsymbol{F}$ is a projection of $\boldsymbol{h}$ along the velocity $\boldsymbol{u}$. Since $\boldsymbol{F}$ is parallel-propagated along $\boldsymbol{u}$, its eigenvalues are constant along the geodesic and its eigenspaces are parallel-propagated. We have demonstrated that for a generic motion the parallel-propagated basis can be obtained from the comoving basis by simple two-dimensional rotations in the 2D eigenspaces of $\boldsymbol{F}$.

To illustrate the general theory we have considered the parallel transport in the Kerr-NUT-(A)dS spacetimes. Namely, we have newly constructed the parallel-propagated frames in three and five dimensions and re-derived the results [Marck, 1983b], [Kamran & Marck, 1986] in 4D. One of the interesting features of the 4D construction, observed already by Marck, is that the equation for the rotation angle allows a separation of variables. Remarkably, we have shown that also in five dimensions equations for the rotation angles can be solved by a separation of variables. Moreover, the 4D result can be understood as a special degenerate case of the 5D construction. Is this a general property valid in the Kerr-NUT-(A)dS spacetime in any number of dimensions? What underlines the separability of the rotation angles? These are interesting open questions.

The present analysis was restricted to the problem of parallel transport along timelike geodesics. The generalization to the case of spatial geodesics is straightforward. The case of null geodesics requires additional consideration and is under preparation [Connell *et al.*, 2008a]. To conclude this chapter we would like to mention that the above described possibility of solving the parallel transport equations in the Kerr-NUT-(A)dS spacetime is one more evidence of the miraculous properties of these metrics connected with their hidden symmetries.

Chapter 10

Summary of results

In this thesis we have described recent developments of the theory of higher-dimensional black holes. We focused mainly on the problem of hidden symmetries and separation of variables. (For more general discussion of the modern status of the theory of higher-dimensional black holes see, e.g., a recent review [Emparan & Reall, 2008].) By studying the hidden symmetries we have demonstrated that higher-dimensional black holes are in many aspects similar to their four-dimensional counterparts.

Explicit spacetime symmetries are represented by Killing vectors. Hidden symmetries are related to generalizations of this concept. One of the most important of these generalizations is the hidden symmetry encoded in the principal conformal Killing–Yano tensor. We have demonstrated that the Myers–Perry metric, describing the higher-dimensional general rotating black hole, as well as its generalization, the Kerr-NUT-(A)dS metric which includes the NUT parameters and the cosmological constant, admit such a tensor.

The PCKY tensor generates towers of hidden and explicit symmetries. The tower of Killing tensors is responsible for the existence of irreducible, quadratic in momenta, conserved integrals of geodesic motion. These integrals, together with the integrals corresponding to the tower of explicit symmetries, make geodesic equations in the Kerr-NUT-(A)dS spacetime completely integrable. We have further demonstrated that the Hamilton–Jacobi and Klein–Gordon equations allow complete separation of variables in this spacetime. The separability of the Dirac equation was proved in [Oota & Yasui, 2008]. We have also shown that the Nambu–Goto equations for a stationary test string in the Kerr-NUT-(A)dS background can be completely separated, and that the problem of finding parallel-propagated frames in these backgrounds reduces to the set of the first order ordinary differential equations. It was also demonstrated [Chen & Lü, 2008] that the Kerr-NUT-(A)dS solution can be presented in the generalized Kerr–Schild form and that it belongs to the class of spacetimes of the special algebraic type D [Hamamoto *et al.*, 2007], [Pravda *et al.*, 2007] of the higher-dimensional algebraic classification. All these remarkable properties make higher-dimensional black hole solutions very similar to the 4D black holes.

To complete this analogy we have addressed the question of generality and uniqueness of these developments. Namely, we have studied the most general metric elements admitting the PCKY tensor—with and without imposing the Einstein equations. The result can be summarized by the following theorem:
Theorem. *The most general spacetime admitting the PCKY tensor is the canonical metric* (4.1). *It possesses the following properties:*

1. *It is of the algebraic type D.*

2. *It allows a separation of variables for the Hamilton–Jacobi, Klein–Gordon, Dirac, and stationary string equations.*

3. *The geodesic motion in such a spacetime is completely integrable. The problem of finding parallel-propagated frames reduces to the set of the first order ordinary differential equations.*

4. *When the Einstein equations with the cosmological constant are imposed the canonical metric becomes the Kerr-NUT-(A)dS spacetime* (4.1)–(4.3).

This theorem naturally generalizes the results obtained earlier in four dimensions (see Appendix A).

Our work motivates further developments in this field. For example, the proved separability of the scalar field equations has led to the construction of the corresponding symmetry operators [Sergyeyev & Krtouš, 2008]. It also opens the possibility that the equations with spin can be decoupled and separated. As described above, the separation of the Dirac equation was already demonstrated [Oota & Yasui, 2008]. An important open question is a separability problem for the electromagnetic and gravitational perturbations in higher-dimensional black hole spacetimes. Such a separability would provide us the important tools for studying the stability, quasinormal modes, and different aspects of the Hawking radiation of higher-dimensional black holes. A certain progress in this direction was already achieved (see, e.g., [Kodama & Ishibashi, 2003], [Kunduri *et al.*, 2006], [Murata & Soda, 2008a]). However, most of the results obtained in these directions so far (see also references in Chapter 6) assumed some additional restrictions on the parameters characterizing black hole solutions. This reminds a situation for the Klein–Gordon and Dirac equations before the general results on their separability were proved.

The similarity of higher-dimensional black holes with their four-dimensional cousins also inspires the search for new higher-dimensional solutions. For example, by relaxing some of the conditions on the PCKY tensor the authors of

[Houri *et al.*, 2008c], [Houri *et al.*, 2008b] were able to find 'generalized Kerr-NUT-(A)dS spacetimes'. Similarly, by generalizing our (unsuccessful) procedure of rescaling the Kerr-NUT-(A)dS metric [Kubizňák & Krtouš, 2007] (see also Appendix C.6), new five-dimensional black hole solutions were recently obtained [Lü *et al.*, 2008a], [Lü *et al.*, 2008b].

The curvature of the Kerr-NUT-(A)dS spacetime and its algebraic type were studied in [Hamamoto *et al.*, 2007]. The relationship between the existence of the PCKY tensor and the uniqueness of this spacetime was first addressed in [Houri *et al.*, 2007], [Houri *et al.*, 2008a]. Other recent papers which deal with hidden symmetries or the subjects addressed in this thesis are, for example, [Acik *et al.*, 2008b], [Acik *et al.*, 2008c], [Acik *et al.*, 2008a], [Ahmedov & Aliev, 2008], [Connell & Frolov, 2008], [Gooding & Frolov, 2008], [Hackmann & Lammerzahl, 2008], [Hioki & Miyamoto, 2008], [Kagramanova *et al.*, 2008], [Papadopoulos, 2008], [Wu, 2008a], [Wu, 2008b].

The results presented in this thesis can be used for studying the particle and light propagation in higher-dimensional rotating black hole spacetimes. They allow us to calculate the contribution of the scalar and Dirac fields to the bulk Hawking radiation, without any restrictions on black hole parameters. The important open questions are: Is it possible to decouple the higher spin massless field equations in the background of the general Kerr-NUT-(A)dS metric? And do they allow separation of variables? Recent result on the separability of the massive Dirac equation is quite promising. Separability of the higher spin equations, and especially the equations for the gravitational perturbations, would provide one with powerful tools, important, for example, for studying the stability of higher-dimensional black hole solutions. One might hope that it will not take too long before this and other interesting open questions connected with the existence of hidden symmetries in higher-dimensional black hole spacetimes will find their answers.

Bibliography

[Acik *et al.*, 2008a] ACIK, O., ERTEM, U., ONDER, M., & VERCIN, A. 2008a. First-order symmetries of Dirac equation in curved background: a unified dynamical symmetry condition, arXiv:0806.1328.

[Acik *et al.*, 2008b] ACIK, O., ERTEM, U., ONDER, M., & VERCIN, A. 2008b. Killing-Yano Forms of a Class of Spherically Symmetric Space-Times I: A Unified Generation of Killing Vector Fields, 0803.3327.

[Acik *et al.*, 2008c] ACIK, O., ERTEM, U., ONDER, M., & VERCIN, A. 2008c. Killing-Yano Forms of a Class of Spherically Symmetric Space-Times II: A Unified Generation of Higher Forms, arXiv:0803.3328.

[Ahmedov & Aliev, 2008] AHMEDOV, H., & ALIEV, A. N. 2008. Stationary Strings in the Spacetime of Rotating Black Holes in Five-Dimensional Minimal Gauged Supergravity, arXiv:0805.1594.

[Albash *et al.*, 2008] ALBASH, T., FILEV, V. G., JOHNSON, C. V., & KUNDU, A. 2008. A topology-changing phase transition and the dynamics of flavour. *Phys. Rev.*, **D77**, 066004, hep-th/0605088.

[Aliev, 2006a] ALIEV, A. N. 2006a. Charged Slowly Rotating Black Holes in Five Dimensions. *Mod. Phys. Lett.*, **A21**, 751–758, gr-qc/0505003.

[Aliev, 2006b] ALIEV, A. N. 2006b. Rotating black holes in higher dimensional Einstein- Maxwell gravity. *Phys. Rev.*, **D74**, 024011, hep-th/0604207.

[Aliev, 2007] ALIEV, A. N. 2007. Electromagnetic Properties of Kerr-Anti-de Sitter Black Holes. *Phys. Rev.*, **D75**, 084041, hep-th/0702129.

[Aliev & Frolov, 2004] ALIEV, A. N., & FROLOV, V. P. 2004. Five dimensional rotating black hole in a uniform magnetic field: The gyromagnetic ratio. *Phys. Rev.*, **D69**, 084022, hep-th/0401095.

[Alonso-Alberca *et al.*, 2000] ALONSO-ALBERCA, N., MEESSEN, P., & ORTIN, T. 2000. Supersymmetry of topological Kerr-Newman-Taub-NUT-adS spacetimes. *Class. Quantum Grav.*, **17**, 2783–2798, hep-th/0003071.

[Arnol'd, 1989] ARNOL'D, V. I. 1989. *Mathematical methods of classical mechanics.* New York: Springer.

[Asnin *et al.*, 2006] ASNIN, V., KOL, B., & SMOLKIN, M. 2006. Analytic evidence for continuous self similarity of the critical merger solution. *Class. Quantum Grav.*, **23**, 6805–6827, hep-th/0607129.

[Banados *et al.*, 1992] BANADOS, M., TEITELBOIM, C., & ZANELLI, J. 1992. The Black hole in three-dimensional space-time. *Phys. Rev. Lett.*, **69**, 1849–1851, hep-th/9204099.

[Benenti & Francaviglia, 1979] BENENTI, S., & FRANCAVIGLIA, M. 1979. Remarks on certain separability structures and their applications to general relativity. *Gen. Rel. Grav.*, **10**, 79–92.

[Benenti & Francaviglia, 1980] BENENTI, S., & FRANCAVIGLIA, M. 1980. The theory of separability of the Hamilton-Jacobi equation and its applications to General Relativity. *In: General Relativity and Gravitation, vol. I.* New York: Plenum Press.

[Benn & Charlton, 1997] BENN, I. M., & CHARLTON, P. 1997. Dirac symmetry operators from conformal Killing-Yano tensors. *Class. Quantum Grav.*, **14**, 1037–1042, gr-qc/9612011.

[Benn *et al.*, 1997] BENN, I. M., CHARLTON, P., & KRESS, J. M. 1997. Debye potentials for Maxwell and Dirac fields from a generalisation of the Killing-Yano equation. *J. Math. Phys.*, **38**, 4504–4527, gr-qc/9610037.

[Berti *et al.*, 2006] BERTI, E., CARDOSO, V., & CASALS, M. 2006. Eigenvalues and eigenfunctions of spin-weighted spheroidal harmonics in four and higher dimensions. *Phys. Rev.*, **D73**, 024013, gr-qc/0511111.

[Brihaye & Delsate, 2007] BRIHAYE, Y., & DELSATE, T. 2007. Charged-rotating black holes and black strings in higher dimensional Einstein-Maxwell theory with a positive cosmological constant. *Class. Quantum Grav.*, **24**, 4691–4710, gr-qc/0703146.

[Cardoso *et al.*, 2003a] CARDOSO, V., DIAS, O. J. C., & LEMOS, J. P. S. 2003a. Gravitational radiation in D-dimensional spacetimes. *Phys. Rev.*, **D67**, 064026, hep-th/0212168.

[Cardoso *et al.*, 2003b] CARDOSO, V., YOSHIDA, S., DIAS, O. J. C., & LEMOS, J. P. S. 2003b. Late-time tails of wave propagation in higher dimensional spacetimes. *Phys. Rev.*, **D68**, 061503, hep-th/0307122.

[Cardoso *et al.*, 2003c] CARDOSO, V., LEMOS, J. P. S., & YOSHIDA, S. 2003c. Scalar gravitational perturbations and quasinormal modes in the five dimensional Schwarzschild black hole. *J. High Energy Phys.*, **12**, 041, hep-th/0311260.

[Cardoso *et al.*, 2004] CARDOSO, V., LEMOS, J. P. S., & YOSHIDA, S. 2004. Quasinormal modes of Schwarzschild black holes in four and higher dimensions. *Phys. Rev.*, **D69**, 044004, gr-qc/0309112.

[Cardoso *et al.*, 2006] CARDOSO, V., CAVAGLIA, M., & GUALTIERI, L. 2006. Black hole particle emission in higher-dimensional spacetimes. *Phys. Rev. Lett.*, **96**, 071301, hep-th/0512002.

[Cariglia, 2004] CARIGLIA, M. 2004. New quantum numbers for the Dirac equation in curved spacetime. *Class. Quantum Grav.*, **21**, 1051–1078, hep-th/0305153.

[Carter, 1968a] CARTER, B. 1968a. Global structure of the Kerr family of gravitational fields. *Phys. Rev.*, **174**, 1559–1571.

[Carter, 1968b] CARTER, B. 1968b. Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations. *Commun. Math. Phys.*, **10**, 280–310.

[Carter, 1968c] CARTER, B. 1968c. A new family of Einstein spaces. *Phys. Lett.*, **26A**, 399–400.

[Carter, 1977] CARTER, B. 1977. Killing Tensor Quantum Numbers and Conserved Currents in Curved Space. *Phys. Rev.*, **D16**, 3395–3414.

[Carter, 1987] CARTER, B. 1987. Separability of the Killing Maxwell system underlying the generalized angular momentum constant in the Kerr-Newman black hole metrics. *J. Math. Phys.*, **28**, 1535–1538.

[Carter & Frolov, 1989] CARTER, B., & FROLOV, V. P. 1989. Separability of string equilibrium equations in a generalized Kerr-de Sitter background. *Class. Quantum Grav.*, **6**, 569–580.

[Carter & McLenaghan, 1979] CARTER, B., & MCLENAGHAN, R. G. 1979. Generalized total angular momentum operator for the Dirac equation in curved space-time. *Phys. Rev.*, **D19**, 1093–1097.

[Carter *et al.*, 1991] CARTER, B., FROLOV, V. P., & HEINRICH, O. 1991. Mechanics of stationary strings: separability of non-dispersive models in a black hole background. *Class. Quantum Grav.*, **8**, 135.

[Casals *et al.*, 2008] CASALS, M., DOLAN, S. R., KANTI, P., & WINSTANLEY, E. 2008. Bulk Emission of Scalars by a Rotating Black Hole. *JHEP*, **06**, 071, arXiv:0801.4910.

[Chandrasekhar, 1976] CHANDRASEKHAR, S. 1976. The Solution of Dirac's Equation in Kerr Geometry. *Proc. Roy. Soc. Lond.*, **A349**, 571–575.

[Chandrasekhar, 1983] CHANDRASEKHAR, S. 1983. *The Mathematical Theory of Black Holes*. Oxford: Clarendon Press.

[Charmousis & Gregory, 2004] CHARMOUSIS, C., & GREGORY, R. 2004. Axisymmetric metrics in arbitrary dimensions. *Class. Quantum Grav.*, **21**, 527–554, gr-qc/0306069.

[Chen & Lü, 2008] CHEN, W., & LÜ, H. 2008. Kerr-Schild Structure and Harmonic 2-forms on (A)dS-Kerr-NUT Metrics. *Phys. Lett.*, **B658**, 158–163, arXiv:0705.4471.

[Chen *et al.*, 2006a] CHEN, W., LÜ, H., & POPE, C. N. 2006a. General Kerr-NUT-AdS metrics in all dimensions. *Class. Quantum Grav.*, **23**, 5323–5340, hep-th/0604125.

[Chen *et al.*, 2006b] CHEN, W., LÜ, H., & POPE, C. N. 2006b. Separability in cohomogeneity-2 Kerr-NUT-AdS metrics. *J. High Energy Phys.*, **04**, 008, hep-th/0602084.

[Chen *et al.*, 2007] CHEN, W., LÜ, H., & POPE, C. N. 2007. Kerr-de Sitter black holes with NUT charges. *Nucl. Phys.*, **B762**, 38–54, hep-th/0601002.

[Chong *et al.*, 2005] CHONG, Z. W., GIBBONS, G. W., LÜ, H., & POPE, C. N. 2005. Separability and killing tensors in Kerr-Taub-NUT-de Sitter metrics in higher dimensions. *Phys. Lett.*, **B609**, 124–132, hep-th/0405061.

[Choptuik, 1993] CHOPTUIK, M. W. 1993. Universality and scaling in gravitational collapse of a massless scalar field. *Phys. Rev. Lett.*, **70**, 9–12.

[Christensen, 1978] CHRISTENSEN, S. M. 1978. Regularization, Renormalization, and Covariant Geodesic Point Separation. *Phys. Rev.*, **D17**, 946–963.

[Coley, 2008] COLEY, A. A. 2008. Classification of the Weyl Tensor in Higher Dimensions and Applications. *Class. Quantum Grav.*, **25**, 033001, arXiv:0710.1598.

[Coley *et al.*, 2004] COLEY, A. A., MILSON, R., PRAVDA, V., & PRAVDOVÁ, A. 2004. Classification of the Weyl tensor in higher-dimensions. *Class. Quantum Grav.*, **21**, L35–L42, gr-qc/0401008.

[Coley *et al.*, 2008] COLEY, A. A., PAPADOPOULOS, G. O., & KUBIZŇÁK, D. 2008. A note on algebraic type of spacetimes admitting conformal Killing–Yano tensors in higher dimensions. *in preparation.*

[Collinson, 1974] COLLINSON, C. D. 1974. The existence of Killing tensors in empty space-times. *Tensor*, **28**, 173.

[Collinson, 1976] COLLINSON, C. D. 1976. On the relationship between Killing tensors and Killing–Yano tensors. *Int. J. Theor. Phys.*, **15**, 311.

[Collinson & Howarth, 2000] COLLINSON, C. D., & HOWARTH, L. 2000. Generalized Killing Tensors. *Gen. Rel. Grav.*, **32**, 1767.

[Connell *et al.*, 2008a] CONNELL, P., FROLOV, V. P., KRTOUŠ, P., & KUBIZŇÁK, D. 2008a. Parallel-propagated frame along null geodesics in higher-dimensional black hole spacetimes. *in preparation.*

[Connell *et al.*, 2008b] CONNELL, P., FROLOV, V. P., & KUBIZŇÁK, D. 2008b. Solving parallel transport equations in the higher- dimensional Kerr-NUT-(A)dS spacetimes. *Phys. Rev.*, **D78**, 024042, arXiv:0803.3259.

[Connell & Frolov, 2008] CONNELL, PATRICK, & FROLOV, VALERI P. 2008. Ray-tracing in four and higher dimensional black holes: An analytical approximation. *Phys. Rev. D*, **78**, 024032, arXiv:0804.4112.

[Connors & Stark, 1977] CONNORS, P. A., & STARK, R. F. 1977. Observable gravitational effects on polarised radiation coming from near a black hole. *Nature*, **269**, 128.

[Connors *et al.*, 1980] CONNORS, P. A., PIRAN, T., & STARK, R. F. 1980. Polarization features of X-ray radiation emitted near black holes. *Astrophys. J.*, **235**, 224.

[Cvetic *et al.*, 2005a] CVETIC, M., LÜ, H., PAGE, D. N., & POPE, C. N. 2005a. New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter, hep-th/0505223.

[Cvetic *et al.*, 2005b] CVETIC, M., LÜ, H., PAGE, D. N., & POPE, C. N. 2005b. New Einstein-Sasaki spaces in five and higher dimensions. *Phys. Rev. Lett.*, **95**, 071101, hep-th/0504225.

[Davis & Kibble, 2005] DAVIS, A.-C., & KIBBLE, T. W. B. 2005. Fundamental cosmic strings. *Contemp. Phys.*, **46**, 313–322, hep-th/0505050.

[Davis, 2006] DAVIS, P. 2006. A Killing tensor for higher dimensional Kerr-AdS black holes with NUT charge. *Class. Quantum Grav.*, **23**, 3607–3618, hep-th/0602118.

[Debever, 1971] DEBEVER, R. 1971. On type D expanding solutions of Einstein-Maxwell equations. *Bull. Soc. Math. Belg.*, **23**, 360–376.

[Debney *et al.*, 1969] DEBNEY, G. C., KERR, R. P., & SCHILD, A. 1969. Solutions of the Einstein and Einstein–Maxwell Equations. *J. Math. Phys.*, **10**, 1842.

[del Castillo, 1988] DEL CASTILLO, G. F. TORRES. 1988. The separability of Maxwell's equations in type-D backgrounds. *J. Math. Phys.*, **29**, 971.

[Demianski & Francaviglia, 1980] DEMIANSKI, M., & FRANCAVIGLIA, M. 1980. Separability structures and Killing–Yano tensors in vacuum type-D space-times without acceleration. *Int. J. Theor. Phys.*, **19**, 675.

[Diener *et al.*, 1997] DIENER, P., FROLOV, V. P., KHOKHLOV, A. M., NOVIKOV, I. D., & PETHICK, C. J. 1997. Relativistic tidal interaction of stars with a rotating black hole. *Astrophys. J.*, **479**, 164.

[Dietz & Rüdiger, 1981] DIETZ, W., & RÜDIGER, R. 1981. Spacetimes admitting Killing–Yano tensors I. *Proc. R. Soc. Lond., Ser A*, **375**, 361.

[Dietz & Rüdiger, 1982] DIETZ, W., & RÜDIGER, R. 1982. Spacetimes admitting Killing–Yano tensors II. *Proc. R. Soc. Lond., Ser A*, **381**, 315.

[Dubath *et al.*, 2007] DUBATH, F., SAKELLARIADOU, M., & VIALLET, C. M. 2007. Scattering of cosmic strings by black holes: Loop formation. *Int. J. Mod. Phys.*, **D16**, 1311–1325, gr-qc/0609089.

[Emparan *et al.*, 2000] EMPARAN, R., HOROWITZ, G. T., & MYERS, R. C. 2000. Exact description of black holes on branes. *J. High Energy Phys.*, **01**, 007, hep-th/9911043.

[Emparan & Reall, 2008] EMPARAN, ROBERTO, & REALL, HARVEY S. 2008. Black Holes in Higher Dimensions, arXiv:0801.3471.

[Ferrando & Saez, 2002] FERRANDO, J. J., & SAEZ, J. A. 2002. A type-Rainich approach to the Killing-Yano tensors. *Gen. Rel. Grav.*, **35**, 1191, gr-qc/0212085.

[Flachi & Tanaka, 2005] FLACHI, A., & TANAKA, T. 2005. Escape of black holes from the brane. *Phys. Rev. Lett.*, **95**, 161302, hep-th/0506145.

[Flachi & Tanaka, 2007] FLACHI, A., & TANAKA, T. 2007. Branes and black holes in collision. *Phys. Rev.*, **D76**, 025007, hep-th/0703019.

[Flachi *et al.*, 2006] FLACHI, A., PUJOLAS, O., SASAKI, M., & TANAKA, T. 2006. Black holes escaping from domain walls. *Phys. Rev.*, **D73**, 125017.

[Floyd, 1973] FLOYD, R. 1973. *The dynamics of Kerr fields*. Ph.D. thesis, London University, London, United Kingdom.

[Frolov, 1977] FROLOV, V. P. 1977. Newman–Penrose formalism in general relativity. *Akad. Nauk SSR*, **96**, 72.

[Frolov, 2006] FROLOV, V. P. 2006. Merger transitions in brane-black-hole systems: Criticality, scaling, and self-similarity. *Phys. Rev.*, **D74**, 044006, gr-qc/0604114.

[Frolov & Kubizňák, 2007] FROLOV, V. P., & KUBIZŇÁK, D. 2007. Hidden symmetries of higher-dimensional rotating black holes. *Phys. Rev. Lett.*, **98**, 011101, gr-qc/0605058.

[Frolov & Kubizňák, 2008] FROLOV, V. P., & KUBIZŇÁK, D. 2008. Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables. *Class. Quantum Grav.*, **25**, 154005, arXiv:0802.0322.

[Frolov & Stevens, 2004] FROLOV, V. P., & STEVENS, K. A. 2004. Stationary strings near a higher-dimensional rotating black hole. *Phys. Rev.*, **D70**, 044035, gr-qc/0404035.

[Frolov & Stojković, 2003a] FROLOV, V. P., & STOJKOVIĆ, D. 2003a. Particle and light motion in a space-time of a five- dimensional rotating black hole. *Phys. Rev.*, **D68**, 064011, gr-qc/0301016.

[Frolov & Stojković, 2003b] FROLOV, V. P., & STOJKOVIĆ, D. 2003b. Quantum radiation from a 5-dimensional rotating black hole. *Phys. Rev.*, **D67**, 084004, gr-qc/0211055.

[Frolov *et al.*, 2003] FROLOV, V. P., SNAJDR, M., & STOJKOVIĆ, D. 2003. Interaction of a brane with a moving bulk black hole. *Phys. Rev.*, **D68**, 044002, gr-qc/0304083.

[Frolov *et al.*, 2004a] FROLOV, V. P., FURSAEV, D. V., & STOJKOVIĆ, D. 2004a. Interaction of higher-dimensional rotating black holes with branes. *Class. Quantum Grav.*, **21**, 3483–3498, gr-qc/0403054.

[Frolov *et al.*, 2004b] FROLOV, V. P., FURSAEV, D. V., & STOJKOVIC, D. 2004b. Rotating black holes in brane worlds. *J. High Energy Phys.*, **06**, 057, gr-qc/0403002.

[Frolov *et al.*, 2007] FROLOV, V. P., KRTOUŠ, P., & KUBIZŇÁK, D. 2007. Separability of Hamilton-Jacobi and Klein-Gordon equations in general Kerr-NUT-AdS spacetimes. *J. High Energy Phys.*, **02**, 005, hep-th/0611245.

[Frolov *et al.*, 1989] FROLOV, VALERI P., SKARZHINSKY, V., ZELNIKOV, A., & HEINRICH, O. 1989. Equilibrium configurations of a cosmic string near a rotating black hole. *Phys. Lett.*, **B224**, 255.

[Frolov *et al.*, 1994] FROLOV, VALERI P., KHOKHLOV, A. M., NOVIKOV, I. D., & PETHICK, C. J. 1994. Relativistic tidal interaction of a white dwrf with a massive black hole. *Astroph. J.*, **432**, 680.

[Fushchich & Nikitin, 1994] FUSHCHICH, W. I., & NIKITIN, A. G. 1994. *Symmetry of Equations of Quantum Mechanics*. New York: Allerton Press.

[Gantmacher, 1959] GANTMACHER, F. R. 1959. *The theory of matrices*. Vol. I. New York: Chelsea.

[Geroch, 1971] GEROCH, R. 1971. A Method for generating solutions of Einstein's equations. *J. Math. Phys.*, **12**, 918–924.

[Gibbons *et al.*, 1993] GIBBONS, G. W., RIETDIJK, R. H., & VAN HOLTEN, J. W. 1993. SUSY in the sky. *Nucl. Phys.*, **B404**, 42–64, hep-th/9303112.

[Gibbons *et al.*, 2004] GIBBONS, G. W., LÜ, H., PAGE, D. N., & POPE, C. N. 2004. Rotating black holes in higher dimensions with a cosmological constant. *Phys. Rev. Lett.*, **93**, 171102, hep-th/0409155.

[Gibbons *et al.*, 2005] GIBBONS, G. W., LÜ, H., PAGE, D. N., & POPE, C. N. 2005. The general Kerr-de Sitter metrics in all dimensions. *J. Geom. Phys.*, **53**, 49–73, hep-th/0404008.

[Gooding & Frolov, 2008] GOODING, CISCO, & FROLOV, ANDREI V. 2008. Five-Dimensional Black Hole Capture Cross-Sections. *Phys. Rev.*, **D77**, 104026, arXiv:0803.1031.

[Griffiths & Podolský, 2005] GRIFFITHS, J. B., & PODOLSKÝ, J. 2005. Accelerating and rotating black holes. *Class. Quantum Grav.*, **22**, 3467–3480, gr-qc/0507021.

[Griffiths & Podolský, 2006a] GRIFFITHS, J. B., & PODOLSKÝ, J. 2006a. Global aspects of accelerating and rotating black hole space-times. *Class. Quantum Grav.*, **23**, 555–568, gr-qc/0511122.

[Griffiths & Podolský, 2006b] GRIFFITHS, J. B., & PODOLSKÝ, J. 2006b. A new look at the Plebanski-Demianski family of solutions. *Int. J. Mod. Phys.*, **D15**, 335–370, gr-qc/0511091.

[Griffiths & Podolský, 2007] GRIFFITHS, J. B., & PODOLSKÝ, J. 2007. On the parameters of the Kerr-NUT-(anti-)de Sitter space- time. *Class. Quantum Grav.*, **24**, 1687–1690, gr-qc/0702042.

[Gürses & Gürsey, 1975] GÜRSES, M., & GÜRSEY, F. 1975. Gravitational field of a spinning mass as an example of algebraically special metrics. *J. Math. Phys.*, **16**, 2385.

[Hackmann & Lammerzahl, 2008] HACKMANN, EVA, & LAMMERZAHL, CLAUS. 2008. Complete Analytic Solution of the Geodesic Equation in Schwarzschild- (Anti-) de Sitter Spacetimes. *Phys. Rev. Lett.*, **100**, 171101.

[Hall, 1987] HALL, G. S. 1987. Killing–Yano tensors in general relativity. *Int. J. Theor. Phys.*, **26**, 71–81.

[Hamamoto *et al.*, 2007] HAMAMOTO, N., HOURI, T., OOTA, T., & YASUI, Y. 2007. Kerr-NUT-de Sitter curvature in all dimensions. *J. Phys.*, **A40**, F177–F184, hep-th/0611285.

[Hashimoto *et al.*, 2004] HASHIMOTO, Y., SAKAGUCHI, M., & YASUI, Y. 2004. Sasaki-Einstein twist of Kerr-AdS black holes. *Phys. Lett.*, **B600**, 270–274, hep-th/0407114.

[Hawking *et al.*, 1999] HAWKING, S. W., HUNTER, C. J., & TAYLOR-ROBINSON, M. M. 1999. Rotation and the AdS/CFT correspondence. *Phys. Rev.*, **D59**, 064005, hep-th/9811056.

[Hioki & Miyamoto, 2008] HIOKI, K., & MIYAMOTO, U. 2008. Hidden symmetries, null geodesics, and photon capture in Sen black hole. *Phys. Rev.*, **D78**, 044007, arXiv:0805.3146.

[Houri *et al.*, 2007] HOURI, T., OOTA, T., & YASUI, Y. 2007. Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness. *Phys. Lett.*, **B656**, 214–216, arXiv:0708.1368.

[Houri *et al.*, 2008a] HOURI, T., OOTA, T., & YASUI, Y. 2008a. Closed conformal Killing-Yano tensor and geodesic integrability. *J. Phys.*, **A41**, 025204, arXiv:0707.4039.

[Houri *et al.*, 2008b] HOURI, T., OOTA, T., & YASUI, Y. 2008b. Closed conformal Killing-Yano tensor and uniqueness of generalized Kerr-NUT-de Sitter spacetime, arXiv:0805.3877.

[Houri *et al.*, 2008c] HOURI, T., OOTA, T., & YASUI, Y. 2008c. Generalized Kerr-NUT-de Sitter metrics in all dimensions. *Phys. Lett.*, **B666**, 391–394, arXiv:0805.0838.

[Hoyos-Badajoz *et al.*, 2007] HOYOS-BADAJOZ, C., LANDSTEINER, K., & MONTERO, S. 2007. Holographic Meson Melting. *J. High Energy Phys.*, **04**, 031, hep-th/0612169.

[Hughston & Sommers, 1973] HUGHSTON, L. P., & SOMMERS, P. 1973. The symmetries of Kerr black holes. *Commun. Math. Phys.*, **33**, 129–133.

[Ida *et al.*, 2003] IDA, D., UCHIDA, Y., & MORISAWA, Y. 2003. The scalar perturbation of the higher-dimensional rotating black holes. *Phys. Rev.*, **D67**, 084019, gr-qc/0212035.

[Ishii *et al.*, 2005] ISHII, M., SHIBATA, M., & MINO, Y. 2005. Black hole tidal problem in the Fermi normal coordinates. *Phys. Rev.*, **D71**, 044017, gr-qc/0501084.

[Jezierski, 1997] JEZIERSKI, J. 1997. Conformal Yano-Killing tensors and asymptotic CYK tensors for the Schwarzschild metric. *Class. Quantum Grav.*, **14**, 1679–1688.

[Jezierski & Lukasik, 2007] JEZIERSKI, J., & LUKASIK, M. 2007. Conformal Yano-Killing tensors for the Taub-NUT metric. *Class. Quantum Grav.*, **24**, 1331–1340, gr-qc/0610090.

[Jezierski & Lukasik, 2006] JEZIERSKI, JACEK, & LUKASIK, MACIEJ. 2006. Conformal Yano-Killing tensor for the Kerr metric and conserved quantities. *Class. Quant. Grav.*, **23**, 2895–2918, gr-qc/0510058.

[Kagramanova *et al.*, 2008] KAGRAMANOVA, V., KUNZ, J., & LAMMERZAHL, C. 2008. Charged particle interferometry in Plebanski-Demianski spacetimes. *Class. Quantum Grav.*, **25**, 105023, arXiv:0801.4514.

[Kalnins & Miller, 1981] KALNINS, E. G., & MILLER, JR, W. 1981. Killing tensors and nonorthogonal variable separation for Hamilton–Jacobi equations. *SIAM J. Math. Anal.*, **12**, 617.

[Kalnins & Miller, 1989] KALNINS, E. G., & MILLER, JR, W. 1989. Killing–Yano tensors and variable separation in Kerr geometry. *J. Math. Phys.*, **30**, 2630.

[Kalnins & Williams, 1990] KALNINS, E. G., & WILLIAMS, G. C. 1990. Symmetry operators and separation of variables for spin wave equations in oblate spheroidal coordinates. *J. Math. Phys.*, **31**, 1739.

[Kalnins *et al.*, 1986] KALNINS, E. G., MILLER, JR, W., & WILLIAMS, G. C. 1986. Electromagnetic Waves in Kerr geometry. *Proc. R. Soc. Lond., Ser A*, **408**, 23–30.

[Kalnins *et al.*, 1992] KALNINS, E. G., MCLENAGHAN, R. G., & WILLIAMS, G. C. 1992. Symmetry operators for Maxwell's equations on curved spacetime. *Proc. R. Soc. Lond., Ser A*, **439**, 103–113.

[Kalnins *et al.*, 1996] KALNINS, E. G., WILLIAMS, G. C., & MILLER, W. 1996. Intrinsic characterization of the separation constant for spin one and gravitational perturbations in Kerr geometry. *Proc. R. Soc. Lond., Ser A*, **452**, 997–1006.

[Kamran, 1985] KAMRAN, N. 1985. Separation of variables for the Rarita-Schwinger equation on all type D vacuum backgrounds. *J. Math. Phys.*, **26**, 1740–1742.

[Kamran & Marck, 1986] KAMRAN, N., & MARCK, J. A. 1986. Parallel-propagated frame along the geodesics of the metrics admitting a Killing–Yano tensor. *J. Math. Phys.*, **27**, 1589.

[Kamran & McLenaghan, 1984] KAMRAN, N., & MCLENAGHAN, R. G. 1984. Symmetry operators for neutrino and Dirac fields on curved space-time. *Phys. Rev.*, **D30**, 357–362.

[Kanti *et al.*, 2006] KANTI, P., KONOPLYA, R. A., & ZHIDENKO, A. 2006. Quasinormal modes of brane-localised standard model fields. II: Kerr black holes. *Phys. Rev.*, **D74**, 064008, gr-qc/0607048.

[Kashiwada, 1968] KASHIWADA, T. 1968. On conformal Killing tensors. *Nat. Sci. Rep. Ochanomizu Univ.*, **19**, 67–74.

[Kastor & Traschen, 2004] KASTOR, D., & TRASCHEN, J. 2004. Conserved gravitational charges from Yano tensors. *J. High Energy Phys.*, **08**, 045, hep-th/0406052.

[Kastor *et al.*, 2007] KASTOR, DAVID, RAY, SOURYA, & TRASCHEN, JENNIE. 2007. Do Killing-Yano tensors form a Lie algebra? *Class. Quant. Grav.*, **24**, 3759–3768, arXiv:0705.0535.

[Kerr, 1963] KERR, R. P. 1963. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, **11**, 237–238.

[Kerr & Schild, 1965] KERR, R. P., & SCHILD, A. 1965. Some algebraically degenerate solutions of Einstein's gravitational field equations. *Proc. Symp. Appl. Math.*, **17**, 199.

[Kerr & Schild, 1969] KERR, R. P., & SCHILD, A. 1969. Solutions of the Einstein and Einstein-Maxwell Equations. *J. Math. Phys.*, **10**, 1842.

[Kinnersley, 1969] KINNERSLEY, W. 1969. Type D Vacuum Metrics. *J. Math. Phys.*, **10**, 1195–1203.

[Kleihaus *et al.*, 2008] KLEIHAUS, B., KUNZ, J., & NAVARRO-LERIDA, F. 2008. Rotating Black Holes in Higher Dimensions. *AIP Conf. Proc.*, **977**, 94–115, arXiv:0710.2291.

[Klemm *et al.*, 1998] KLEMM, D., MORETTI, V., & VANZO, L. 1998. Rotating topological black holes. *Phys. Rev.*, **D57**, 6127–6137, gr-qc/9710123.

[Kobayashi *et al.*, 2007] KOBAYASHI, S., MATEOS, D., MATSUURA, S., MYERS, R. C., & THOMSON, R. M. 2007. Holographic phase transitions at finite baryon density. *J. High Energy Phys.*, **02**, 016, hep-th/0611099.

[Kodama, 2008] KODAMA, H. 2008. Superradiance and Instability of Black Holes. *Prog. Theor. Phys. Suppl.*, **172**, 11–20, arXiv:0711.4184.

[Kodama & Ishibashi, 2003] KODAMA, H., & ISHIBASHI, A. 2003. A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions. *Prog. Theor. Phys.*, **110**, 701–722, hep-th/0305147.

[Kodama, 2007] KODAMA, HIDEO. 2007. Perturbations and Stability of Higher-Dimensional Black Holes, arXiv:0712.2703.

[Kol, 2006] KOL, BARAK. 2006. Choptuik scaling and the merger transition. *J. High Energy Phys.*, **10**, 017, hep-th/0502033.

[Konoplya, 2003a] KONOPLYA, R. A. 2003a. Gravitational quasinormal radiation of higher-dimensional black holes. *Phys. Rev.*, **D68**, 124017, hep-th/0309030.

[Konoplya, 2003b] KONOPLYA, R. A. 2003b. Quasinormal behavior of the d-dimensional Schwarzschild black hole and higher order WKB approach. *Phys. Rev.*, **D68**, 024018, gr-qc/0303052.

[Konoplya & Zhidenko, 2007] KONOPLYA, R. A., & ZHIDENKO, A. 2007. Stability of multidimensional black holes: Complete numerical analysis. *Nucl. Phys.*, **B777**, 182, hep-th/0703231.

[Kozlov, V. V., 1983] KOZLOV, V. V. 1983. Integrability and non-integrability in Hamiltonian mechanics. *Russ. Math. Surv.*, **38:1**, 1–76.

[Kress, 1997] KRESS, J. 1997. *Generalized conformal Killing–Yano tensors: applications to electrodynamics*. Ph.D. thesis, University of Newcastle, Newcastle, England.

[Krtouš, 2007] KRTOUŠ, P. 2007. Electromagnetic Field in Higher-Dimensional Black-Hole Spacetimes. *Phys. Rev.*, **D76**, 084035, arXiv:0707.0002.

[Krtouš *et al.*, 2007a] KRTOUŠ, P., KUBIZŇÁK, D., PAGE, D. N., & VASUDEVAN, M. 2007a. Constants of Geodesic Motion in Higher-Dimensional Black- Hole Spacetimes. *Phys. Rev.*, **D76**, 084034, arXiv:0707.0001.

[Krtouš *et al.*, 2007b] KRTOUŠ, P., KUBIZŇÁK, D., PAGE, D. N., & FROLOV, V. P. 2007b. Killing-Yano tensors, rank-2 Killing tensors, and conserved quantities in higher dimensions. *J. High Energy Phys.*, **02**, 004, hep-th/0612029.

[Krtouš *et al.*, 2008a] KRTOUŠ, P., FROLOV, V. P., & KUBIZŇÁK, D. 2008a. *in preparation.*

[Krtouš *et al.*, 2008b] KRTOUŠ, P., FROLOV, V. P., & KUBIZŇÁK, D. 2008b. Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime, arXiv:0804.4705.

[Krtouš & Podolský, 2004] KRTOUŠ, PAVEL, & PODOLSKÝ, JIŘÍ. 2004. Asymptotic directional structure of radiative fields in spacetimes with a cosmological constant. *Class. Quantum Grav.*, **21**, R233–R273, gr-qc/0502095.

[Kubizňák & Frolov, 2007] KUBIZŇÁK, D., & FROLOV, V. P. 2007. Hidden symmetry of higher dimensional Kerr-NUT-AdS spacetimes. *Class. Quantum Grav.*, **24**, F1–F6, gr-qc/0610144.

[Kubizňák & Frolov, 2008] KUBIZŇÁK, D., & FROLOV, V. P. 2008. Stationary strings and branes in the higher-dimensional Kerr-NUT-(A)dS spacetimes. *J. High Energy Phys.*, **02**, 007, arXiv:0711.2300.

[Kubizňák & Krtouš, 2007] KUBIZŇÁK, D., & KRTOUŠ, P. 2007. On conformal Killing-Yano tensors for Plebanski-Demianski family of solutions. *Phys. Rev.*, **D76**, 084036, arXiv:0707.0409.

[Kunduri & Lucietti, 2005] KUNDURI, H. K., & LUCIETTI, J. 2005. Integrability and the Kerr-(A)dS black hole in five dimensions. *Phys. Rev.*, **D71**, 104021, hep-th/0502124.

[Kunduri *et al.*, 2006] KUNDURI, H. K., LUCIETTI, J., & REALL, H. S. 2006. Gravitational perturbations of higher dimensional rotating black holes: Tensor Perturbations. *Phys. Rev.*, **D74**, 084021, hep-th/0606076.

[Kunz *et al.*, 2005] KUNZ, J., NAVARRO-LERIDA, F., & PETERSEN, A. K. 2005. Five-dimensional charged rotating black holes. *Phys. Lett.*, **B614**, 104–112, gr-qc/0503010.

[Kunz *et al.*, 2006a] KUNZ, J., NAVARRO-LERIDA, F., & VIEBAHN, J. 2006a. Charged rotating black holes in odd dimensions. *Phys. Lett.*, **B639**, 362–367, hep-th/0605075.

[Kunz *et al.*, 2006b] KUNZ, J., MAISON, D., NAVARRO-LERIDA, F., & VIEBAHN, J. 2006b. Rotating Einstein-Maxwell-dilaton black holes in D dimensions. *Phys. Lett.*, **B639**, 95–100, hep-th/0606005.

[Kunz *et al.*, 2007] KUNZ, J., NAVARRO-LERIDA, F., & RADU, E. 2007. Higher dimensional rotating black holes in Einstein- Maxwell theory with negative cosmological constant. *Phys. Lett.*, **B649**, 463–471, gr-qc/0702086.

[Laguna *et al.*, 1993] LAGUNA, P., MILLER, W. A., ZUREK, W. H., & DAVIES, M. B. 1993. Tidal disruptions by supermassive black holes: Hydrodynamic evolution of stars on a Schwarzschild background. *Astrophys. J.*, **410**, L83.

[Lopez-Ortega, 2003] LOPEZ-ORTEGA, A. 2003. Klein-Gordon field in the rotating black branes wrapped on Einstein spaces. *Gen. Rel. Grav.*, **35**, 1785–1797.

[Lopez-Ortega, 2006a] LOPEZ-ORTEGA, A. 2006a. Electromagnetic quasinormal modes of D-dimensional black holes. *Gen. Rel. Grav.*, **38**, 1747–1770, gr-qc/0605034.

[Lopez-Ortega, 2006b] LOPEZ-ORTEGA, A. 2006b. Quasinormal modes of D-dimensional de Sitter spacetime. *Gen. Rel. Grav.*, **38**, 1565–1591, gr-qc/0605027.

[Lopez-Ortega, 2007] LOPEZ-ORTEGA, A. 2007. Dirac quasinormal modes of D-dimensional de Sitter spacetime. *Gen. Rel. Grav.*, **39**, 1011–1029, arXiv:0704.2468.

[Lopez-Ortega, 2008] LOPEZ-ORTEGA, A. 2008. Electromagnetic quasinormal modes of D-dimensional black holes II. *Gen. Rel. Grav.*, **40**, arXiv:0706.2933.

[Lü & Pope, 2007] LÜ, H., & POPE, C. N. 2007. Resolutions of cones over Einstein-Sasaki spaces. *Nucl. Phys.*, **B782**, 171–188, hep-th/0605222.

[Lü *et al.*, 2008a] LÜ, H., MEI, J., & POPE, C. N. 2008a. New Black Holes in Five Dimensions, arXiv:0804.1152.

[Lü *et al.*, 2008b] LÜ, H., MEI, J., & POPE, C. N. 2008b. New Charged Black Holes in Five Dimensions, arXiv:0806.2204.

[Luminet & Marck, 1985] LUMINET, J. P., & MARCK, J. A. 1985. Tidal squeezing of stars by Schwarzschild black holes. *Mon, Not. R. Astron. Soc.*, **212**, 57.

[Majumdar & Mukherjee, 2005] MAJUMDAR, A. S., & MUKHERJEE, N. 2005. Braneworld black holes in cosmology and astrophysics. *Int. J. Mod. Phys.*, **D14**, 1095, astro-ph/0503473.

[Mansouri & Witten, 1984] MANSOURI, F., & WITTEN, L. 1984. Isometries and dimensional reduction. *J. Math. Phys.*, **25**, 1991.

[Marck, 1983a] MARCK, J. A. 1983a. Parallel tetrad on null geodesics in Kerr and Kerr–Newman space–time. *Phys. Lett.*, **97A**, 140.

[Marck, 1983b] MARCK, J. A. 1983b. Solution to the equations of Parallel transport in Kerr geometry; tidal tensor. *Proc. R. Soc. Lond., Ser A*, **385**, 431.

[Mateos *et al.*, 2006] MATEOS, D., MYERS, R. C., & THOMSON, R.M. 2006. Holographic phase transitions with fundamental matter. *Phys. Rev. Lett.*, **97**, 091601, hep-th/0605046.

[Miller, 1977] MILLER, JR, WILLARD. 1977. *Symmetry and Separation of Variables.* Reading, Massachusetts: Addison-Wesley.

[Milson *et al.*, 2005] MILSON, R., COLEY, A. A., PRAVDA, V., & PRAVDOVÁ, A. 2005. Alignment and algebraically special tensors in Lorentzian geometry. *Int. J. Geom. Meth. Mod. Phys.*, **2**, 41–61, gr-qc/0401010.

[Milson, 2004] MILSON, ROBERT. 2004. Alignment and the classification of Lorentz-signature tensors, gr-qc/0411036.

[Misner *et al.*, 1973] MISNER, C. W., THORNE, K. S., & WHEELER, J. A. 1973. *Gravitation*. San Francisco: Freeman.

[Morisawa *et al.*, 2000] MORISAWA, Y., YAMAZAKI, R., IDA, D., ISHIBASHI, A., & NAKAO, K. 2000. Thick domain walls intersecting a black hole. *Phys. Rev.*, **D62**, 084022, gr-qc/0005022.

[Morisawa *et al.*, 2003] MORISAWA, Y., IDA, D., ISHIBASHI, A., & NAKAO, K. 2003. Thick domain walls around a black hole. *Phys. Rev.*, **D67**, 025017, gr-qc/0209070.

[Murata & Soda, 2008a] MURATA, K., & SODA, J. 2008a. A Note on Separability of Field Equations in Myers-Perry Spacetimes. *Class. Quantum Grav.*, **25**, 035006, arXiv:0710.0221.

[Murata & Soda, 2008b] MURATA, KEIJU, & SODA, JIRO. 2008b. Stability of Five-dimensional Myers-Perry Black Holes with Equal Angular Momenta, arXiv:0803.1371.

[Myers & Perry, 1986] MYERS, R. C., & PERRY, M. J. 1986. Black holes in higher dimensional space-times. *Ann. Phys. (N.Y.)*, **172**, 304–347.

[Newman *et al.*, 1963] NEWMAN, E., TAMUBRINO, L., & UNTI, T. 1963. Empty space generalization of the Schwarzschild metric. *J. Math. Phys.*, **4**, 915.

[Nijenhuis, 1955] NIJENHUIS, A. 1955. Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A*, **58**, 390–397, 398–403.

[Oota & Yasui, 2006] OOTA, T., & YASUI, Y. 2006. Explicit toric metric on resolved Calabi-Yau cone. *Phys. Lett.*, **B639**, 54–56, hep-th/0605129.

[Oota & Yasui, 2008] OOTA, T., & YASUI, Y. 2008. Separability of Dirac equation in higher dimensional Kerr- NUT-de Sitter spacetime. *Phys. Lett.*, **B659**, 688–693, arXiv:0711.0078.

[Ortaggio *et al.*, 2008] ORTAGGIO, M., PODOLSKÝ, J., & ŽOFKA, M. 2008. Robinson-Trautman spacetimes with an electromagnetic field in higher dimensions. *Class. Quantum Grav.*, **25**, 025006, arXiv:0708.4299.

[Ortin, 2004] ORTIN, T. 2004. *Gravity and Strings*. Cambridge, England: Cambridge Univ. Press.

[Page, 1976] PAGE, D. N. 1976. Dirac Equation Around a Charged, Rotating Black Hole. *Phys. Rev.*, **D14**, 1509–1510.

[Page *et al.*, 2007] PAGE, D. N., KUBIZŇÁK, D., VASUDEVAN, M., & KRTOUŠ, P. 2007. Complete Integrability of Geodesic Motion in General Higher-Dimensional Rotating Black Hole Spacetimes. *Phys. Rev. Lett.*, **98**, 061102, hep-th/0611083.

[Papadopoulos, 2008] PAPADOPOULOS, G. 2008. Killing-Yano equations and G-structures. *Class. Quant. Grav.*, **25**, 105016, arXiv:0712.0542.

[Penrose, 1973] PENROSE, R. 1973. Naked singularities. *Annals N. Y. Acad. Sci.*, **224**, 125–134.

[Petrov, 1954] PETROV, A. Z. 1954. Classification of spaces defined by gravitational fields. *Sci. Not. Kazan. State Univ.*, **114**, 55.

[Petrov, 1969] PETROV, A. Z. 1969. *Einstein spaces*. New York: Pergamon.

[Pirani, 1965] PIRANI, F. A. E. 1965. Introduction to gravitational radiation theory. *Pages 249–372 of:* DESER, S., & FORD, K. W. (eds), *Brandeis Lectures on General Relativity*. Englewood Cliffs, NJ: Prentice-Hall.

[Plebański, 1975] PLEBAŃSKI, J. F. 1975. A class of solutions of Einstein–Maxwell equations. *Ann. Phys., NY*, **90**, 196–255.

[Plebański & Demiański, 1976] PLEBAŃSKI, J. F., & DEMIAŃSKI, M. 1976. Rotating charged and uniformly accelerated mass in general relativity. *Ann. Phys. (N.Y.)*, **98**, 98–127.

[Podolský & Griffiths, 2006] PODOLSKÝ, J., & GRIFFITHS, J. B. 2006. Accelerating Kerr-Newman black holes in (anti-)de Sitter space-time. *Phys. Rev. D*, **73**, 044018, gr-qc/0601130.

[Podolský & Ortaggio, 2006] PODOLSKÝ, J., & ORTAGGIO, M. 2006. Robinson-Trautman spacetimes in higher dimensions. *Class. Quantum Grav.*, **23**, 5785–5797, gr-qc/0605136.

[Polchinski, 1998] POLCHINSKI, J. 1998. *String Theory*. Vol. I, II. Cambridge, England: Cambridge Univ. Press.

[Polchinski, 2004] POLCHINSKI, J. 2004. Introduction to cosmic F- and D-strings, hep-th/0412244.

[Prasolov, 1994] PRASOLOV, V. V. 1994. *Problems and Theorems in Linear Algebra*. Translations of Mathematical monographs, vol. 134. Providence, Rhode Island: American Math. Society.

[Pravda *et al.*, 2007] PRAVDA, V., PRAVDOVÁ, A., & ORTAGGIO, M. 2007. Type D Einstein spacetimes in higher dimensions. *Class. Quantum Grav.*, **24**, 4407–4428, arXiv:0704.0435.

[Rodrigo, 2006] RODRIGO, E. 2006. Higher-Dimensional Bulk Wormholes and their Manifestations in Brane Worlds. *Phys. Rev.*, **D74**, 104025, gr-qc/0701031.

[Rosquist *et al.*, 2007] ROSQUIST, K., BYLUND, T., & SAMUELSSON, L. 2007. Carter's constant revealed, arXiv:0710.4260.

[Schouten, 1940] SCHOUTEN, J. A. 1940. Ueber Differentialkomitanten zweier kontravarianter. *Nederl. Akad. Wetensch., Proc.*, **43**, 449–452.

[Schouten, 1954] SCHOUTEN, J. A. 1954. On the differential operators of first order in tensor calculus. *In: Convegno Internazionale di Geometria Differenziale.*

[Schwarzschild, 1916] SCHWARZSCHILD, K. 1916. Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Deutsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech.*, **1916**, 189–196.

[Sergyeyev & Krtouš, 2008] SERGYEYEV, A., & KRTOUŠ, P. 2008. Complete Set of Commuting Symmetry Operators for the Klein-Gordon Equation in Generalized Higher-Dimensional Kerr-NUT-(A)dS Spacetimes. *Phys. Rev.*, **D77**, 044033, arXiv:0711.4623.

[Shibata, 1996] SHIBATA, M. 1996. Relativistic Roche-Riemann problems around a black hole. *Prog. Theor. Phys.*, **96**, 917–932.

[Snajdr & Frolov, 2003] SNAJDR, M., & FROLOV, V. P. 2003. Capture and critical scattering of a long cosmic string by a rotating black hole. *Class. Quantum Grav.*, **20**, 1303–1320, gr-qc/0211018.

[Snajdr *et al.*, 2002] SNAJDR, M., FROLOV, V. P., & DEVILLIERS, J.-P. 2002. Scattering of a long cosmic string by a rotating black hole. *Class. Quantum Grav.*, **19**, 5987–6008, gr-qc/0208009.

[Stackel, 1895] STACKEL, P. 1895. Sur lintegration de lquation differentielle de Hamilton. *C. R. Acad. Sci. Paris Ser. IV*, **121**, 489.

[Stark & Connors, 1977] STARK, R. F., & CONNORS, P. A. 1977. Observational test for the existence of a rotating black hole in Cyg X-1. *Nature*, **266**, 429.

[Sternberg, S., 1964] STERNBERG, S. 1964. *Lectures on Differential Geometry*. Englwood Cliffs, NJ: Prentice Hall.

[Tachibana, 1969] TACHIBANA, S. 1969. On conformal Killing tensor in a Riemannian space. *Tôhoku Math. J.*, **21**, 56.

[Tangherlini, 1963] TANGHERLINI, F. R. 1963. Schwartzschild Field in N Dimensions and the Dimensionality of Space Problem. *Nuovo Cimento*, **27**, 636.

[Taxiarchis, 1985] TAXIARCHIS, P. 1985. Space-times admitting Penrose–Floyd tensors. *Gen. Rel. Grav.*, **17**, 149.

[Teukolsky, 1972] TEUKOLSKY, S. A. 1972. Rotating black holes - separable wave equations for gravitational and electromagnetic perturbations. *Phys. Rev. Lett.*, **29**, 1114–1118.

[Teukolsky, 1973] TEUKOLSKY, S. A. 1973. Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations. *Astrophys. J.*, **185**, 635–647.

[Unruh, 1973] UNRUH, W. G. 1973. Separability of the neutrino equations in a Kerr background. *Phys. Rev. Lett.*, **31**, 1265.

[Vasudevan, 2006a] VASUDEVAN, M. 2006a. A note on particles and scalar fields in higher dimensional nutty spacetimes. *Phys. Lett.*, **B632**, 532–536, gr-qc/0511028.

[Vasudevan, 2006b] VASUDEVAN, M. 2006b. *Symmetries of black holes and D-branes*. Ph.D. thesis, University of Alberta, Edmonton, Alberta, Canada.

[Vasudevan & Stevens, 2005] VASUDEVAN, M., & STEVENS, K. A. 2005. Integrability of particle motion and scalar field propagation in Kerr-(anti) de Sitter black hole spacetimes in all dimensions. *Phys. Rev.*, **D72**, 124008, gr-qc/0507096.

[Vasudevan *et al.*, 2005a] VASUDEVAN, M., STEVENS, K. A., & PAGE, D. N. 2005a. Particle motion and scalar field propagation in Myers- Perry black hole spacetimes in all dimensions. *Class. Quantum Grav.*, **22**, 1469–1482, gr-qc/0407030.

[Vasudevan *et al.*, 2005b] VASUDEVAN, M., STEVENS, K. A., & PAGE, D. N. 2005b. Separability of the Hamilton-Jacobi and Klein-Gordon equations in Kerr-de Sitter metrics. *Class. Quantum Grav.*, **22**, 339–352, gr-qc/0405125.

[Vilenkin & Shellard, 1994] VILENKIN, A., & SHELLARD, E. P. S. 1994. *Cosmic Strings and other Topological Defects*. Cambridge, England: Cambridge University Press.

[Wald, 1984] WALD, R. M. 1984. *General Relativity*. Chicago and London: The University of Chicago Press.

[Walker & Penrose, 1970] WALKER, M., & PENROSE, R. 1970. On Quadratic First Integrals of the Geodesic Equations for Type {22} Spacetimes. *Commun. Math. Phys.*, **18**, 265–274.

[Woodhouse, 1975] WOODHOUSE, N. M. J. 1975. Killing tensors and separation of the Hamilton–Jacobi equation. *Commun. Math. Phys.*, **44**, 9–38.

[Wu, 2008a] WU, S.-Q. 2008a. Separability of massive Dirac's equation in 5-dimensional Myers-Perry black hole geometry and its relation to a rank- three Killing-Yano tensor, arXiv:0807.2114.

[Wu, 2008b] WU, S.-Q. 2008b. Symmetry operators and separability of the massive Dirac's equation in the general 5-dimensional Kerr-(anti-)de Sitter black hole background, arXiv:0808.3435.

[Yano, 1952] YANO, K. 1952. Some Remarks on Tensor Fields and Curvature. *Ann. Math.*, **55**, 328–347.

[Zhidenko, 2006] ZHIDENKO, A. 2006. Massive scalar field quasi-normal modes of higher dimensional black holes. *Phys. Rev.*, **D74**, 064017, gr-qc/0607133.

Appendix A

On hidden symmetries in 4D

In this appendix we discuss some aspects of hidden symmetries in 4D. The first section plays the role of an introduction for newcomers to the field where, on the well known 4D case, we illustrate the main ideas of the more complicated higher-dimensional theory developed in Parts I and II of this thesis. In the second section, based on [Kubizňák & Krtouš, 2007], we discuss hidden symmetries for the Plebański–Demiański family of type D solutions.

A.1 Introduction for newcomers

The key object of the theory in higher dimensions is a *principal conformal Killing–Yano* (PCKY) tensor. We start discussing this object and its properties in a 4D flat spacetime and demonstrate how it generates other objects (Killing–Yano and Killing tensors) responsible for hidden symmetries. Then we show how this PCKY tensor allows one easily to 'generate' the 4D Kerr-NUT-(A)dS metric starting from the flat one—written in the *canonical coordinates* determined by this tensor. Finally, we discuss the separation of variables in the 4D Kerr-NUT-(A)dS spacetime in the canonical coordinates.

A.1.1 Principal conformal Killing–Yano tensor

Consider a 4D flat spacetime with the metric

$$dS^2 = \eta_{ab} dX^a dX^b = -dT^2 + dX^2 + dY^2 + dZ^2\,. \tag{A.1}$$

The PCKY tensor $\boldsymbol{h}$ is a (non-degenerate) rank-2 closed conformal Killing–Yano tensor. Therefore, there exists a 1-form potential $\boldsymbol{b}$, so that $\boldsymbol{h} = \boldsymbol{db}$. Let us consider the following ansatz:

$$b = \frac{1}{2}\left[-R^2 dT + a(Y dX - X dY)\right], \quad R^2 = X^2 + Y^2 + Z^2 . \tag{A.2}$$

Our choice of this special form for the potential b will become clear later, when it will be shown that this is a flat spacetime limit of the potential for the PCKY tensor in the Kerr-NUT-(A)dS spacetime. For a moment we just mention that the form (A.2) of the potential b singles out time coordinate T, a two-dimensional (X, Y) plane in space, and contains an arbitrary constant a.

It can be easily shown that

$$h = db = dT \wedge (X dX + Y dY + Z dZ) + a dY \wedge dX \tag{A.3}$$

is a closed conformal Killing–Yano tensor [cf. also (B.31), for the higher-dimensional case]. It means that its dual 2-form $f = *h$ is the Killing–Yano (KY) tensor.[1]

$$f = X dZ \wedge dY + Z dY \wedge dX + Y dX \wedge dZ + a dZ \wedge dT . \tag{A.5}$$

Let us put, for a moment, $a = 0$. Then the KY tensor f_{ab} has only spatial components f_{ik}, and the Killing tensor $K_{ab} = f_{ac} f_b{}^c$, (1.3), reads

$$K_{ij} = R^2 \delta_{ij} - X^i X^j = \sum_{k=X,Y,Z} \xi_{(k)\,i} \xi_{(k)\,j} , \quad \xi_{(k)\,i} = \epsilon_{kji} X^j . \tag{A.6}$$

Here $\xi_{(k)\,i}$ are the spatial rotational Killing vectors. Therefore, the Killing tensor K can be written as a sum of products of Killing vectors, and thus it is *reducible*. Parallel-propagated vector $L_a = f_{ab} p^b$, (1.9), with the nontrivial components

$$L_i = f_{ik} p^k = \epsilon_{ijk} X^j p^k = \xi_{(k)\,i} p^k , \tag{A.7}$$

[1] In D dimensions the maximum number of (linear independent) Killing–Yano tensors of a given rank-p is

$$N_p = \binom{D}{p} + \binom{D}{p+1} = \frac{(D+1)!}{(D-p)!(p+1)!} . \tag{A.4}$$

This reflects the fact that, similar to Killing vectors, Killing–Yano tensors are completely determined by the values of their components and the values of their (completely antisymmetric) first derivatives at a given point. Flat space has the maximum number of independent Killing–Yano tensors of each rank. Any KY tensor there can be written as a linear combination of 'translational' KY tensors (which are a simple wedge product of translational Killing vectors) and 'rotational' KY tensors (which are a wedge product of translations with a spacetime rotation, completely antisymmetrized) [Kastor & Traschen, 2004]. In particular case of $D = 4$ we have 10 rank-2 KY tensors (6 translational and 4 rotational).

has the meaning of the conserved angular momentum.[2] The conserved quantity $K = K_{ab}p^a p^b$,

$$K(a=0) = \sum_{k=X,Y,Z} L_k^2 = \vec{L}^2\,, \tag{A.8}$$

is the square of the total angular momentum.

For $a \neq 0$ the conserved quantity

$$K = \vec{L}^2 + 2aEL_Z + a^2(E^2 - p_Z^2) \tag{A.9}$$

is also reducible. Here $E = -p_T$ and p_Z are the conserved energy and the momentum in the Z-direction, respectively.

A.1.2 'Derivation' of the 4D Kerr-NUT-(A)dS metric

Consider a general case with $a \neq 0$. We first introduce the ellipsoidal coordinates[3]

$$X = \sqrt{r^2+a^2}\sin\theta\cos\phi, \quad Y = \sqrt{r^2+a^2}\sin\theta\sin\phi, \quad Z = r\cos\theta\,, \tag{A.10}$$

and rewrite the metric, the potential, the PCKY tensor, and the KY tensor as

$$\begin{aligned}
dS^2 =& -dT^2 + (r^2+a^2)\sin^2\theta d\phi^2 + (r^2+a^2\cos^2\theta)\Big(\frac{dr^2}{r^2+a^2} + d\theta^2\Big)\,,\\
\boldsymbol{b} =& \frac{1}{2}\big[-(r^2+a^2\sin^2\theta)\boldsymbol{dT} - a\sin^2\theta(r^2+a^2)\boldsymbol{d\phi}\big]\,,\\
\boldsymbol{h} =& -r\boldsymbol{dr}\wedge(\boldsymbol{dT} + a\sin^2\theta\boldsymbol{d\phi}) - a\sin\theta\cos\theta\boldsymbol{d\theta}\wedge\big[a\boldsymbol{dT} + (r^2+a^2)\boldsymbol{d\phi}\big]\,,\\
\boldsymbol{f} =& a\cos\theta\boldsymbol{dr}\wedge(\boldsymbol{dT} + a\sin^2\theta\boldsymbol{d\phi}) - r\sin\theta\boldsymbol{d\theta}\wedge\big[a\boldsymbol{dT} + (r^2+a^2)\boldsymbol{d\phi}\big]\,.
\end{aligned} \tag{A.11}$$

Second, we introduce the new coordinates

$$y = a\cos\theta, \quad t = T + a\phi, \quad \psi = -\phi/a\,, \tag{A.12}$$

in which the metric takes the 'algebraic' form

$$dS^2 = -\frac{\Delta_r(dt+y^2d\psi)^2}{r^2+y^2} + \frac{\Delta_y(dt-r^2d\psi)^2}{r^2+y^2} + \frac{(r^2+y^2)dr^2}{\Delta_r} + \frac{(r^2+y^2)dy^2}{\Delta_y}\,, \tag{A.13}$$

[2] In general, for a simple spacelike ($f_{ab}f^{ab} > 0$) Killing–Yano tensor $\boldsymbol{f}$, there exists a close analogy between the angular momentum of classical mechanics and the vector $L^a = f^{ab}p_b$ [Dietz & Rüdiger, 1981].

[3] In this step, we associate constant a with 'rotation' parameter.

where

$$\Delta_r = r^2 + a^2, \quad \Delta_y = a^2 - y^2. \tag{A.14}$$

The hidden symmetries are

$$\begin{aligned} \boldsymbol{b} &= \frac{1}{2}\left[(y^2 - r^2 - a^2)\boldsymbol{dt} - r^2 y^2 \boldsymbol{d\psi}\right], \\ \boldsymbol{h} &= y\boldsymbol{dy} \wedge (\boldsymbol{dt} - r^2\boldsymbol{d\psi}) - r\boldsymbol{dr} \wedge (\boldsymbol{dt} + y^2\boldsymbol{d\psi}), \\ \boldsymbol{f} &= r\boldsymbol{dy} \wedge (\boldsymbol{dt} - r^2\boldsymbol{d\psi}) + y\boldsymbol{dr} \wedge (\boldsymbol{dt} + y^2\boldsymbol{d\psi}). \end{aligned} \tag{A.15}$$

In the potential $\boldsymbol{b}$ the term proportional to a^2 is constant and may be omitted. We remind that (A.13)–(A.14) is just a metric of the flat space written in special coordinates.

Let us consider now the metric (A.13) without imposing conditions (A.14) on functions Δ_r and Δ_y, but assuming that they are functions of r and y, respectively. Remarkably, we realize that the objects $\boldsymbol{b}$, $\boldsymbol{h}$, and $\boldsymbol{f}$ (A.15) are again the potential, the PCKY tensor, and the KY tensor. We call (A.13) with arbitrary $\Delta_r(r)$ and $\Delta_y(y)$, a *canonical (off-shell) metric.* It possesses the hidden symmetries (A.15).

In particular, let us impose the vacuum Einstein equations with the cosmological constant

$$R_{ab} = -3\lambda g_{ab}. \tag{A.16}$$

These equations are satisfied provided that

$$\frac{d^2\Delta_r}{dr^2} + \frac{d^2\Delta_y}{dy^2} = 12\lambda(r^2 + y^2). \tag{A.17}$$

The most general solution of this equation is

$$\Delta_r = (r^2 + a^2)(1 + \lambda r^2) - 2Mr, \quad \Delta_y = (a^2 - y^2)(1 - \lambda y^2) + 2Ny. \tag{A.18}$$

In other words, a simple replacement of functions (A.14) by more general polynomials (A.18) generates a non-trivial solution of the Einstein equations from a flat one. This solution is the Kerr-NUT-(A)dS metric written in the *canonical form* (see, e.g., [Chen *et al.*, 2006a]). It obeys the Einstein equations with the cosmological constant, cf. Eq. (A.16). M stands for mass, and parameters a and N are connected with rotation and NUT parameter [Griffiths & Podolský, 2006b].

Let us remark here that *the canonical metric* (A.13) *is the most general metric element admitting the (non-degenerate) PCKY tensor* [Dietz & Rüdiger, 1981], [Taxiarchis, 1985]. The derivation above therefore establishes that *the most general Einstein space admitting the PCKY tensor is the Kerr-NUT-(A)dS spacetime* (A.13), (A.18). We also remark that even a more general family of solutions–the Carter's spacetime (described in the next section) can be written in the form (A.13).

In the Kerr-NUT-(A)dS spacetime neither the square of the total angular momentum, $\vec{L}^2$, nor the projection of the momentum on the Z-axis, p_Z, which enter (A.9) have well defined meaning. However, the quadratic in momentum quantity, $K^{ab}p_a p_b$, is well defined and conserved. In the absence of the cosmological constant and NUT parameter, that is for the Kerr black hole, this quantity can be presented in the form (A.9) in the asymptotic region, where the spacetime is practically flat. The angular momentum and other quantities which enter (A.9) must be then understood as the corresponding asymptotically conserved quantities. Since the energy E and the angular momentum along the axis of symmetry L_Z are conserved exactly in any stationary axisymmetric spacetime they can be excluded from (A.9) and the asymptotically conserved quantity can be written as follows [Rosquist *et al.*, 2007]:

$$Q = L_X^2 + L_Y^2 - a^2 p_Z^2 \,. \tag{A.19}$$

For a scattering of particles in the Kerr metric, the presence of an exact integral of motion connected with the Carter's constant implies that the quantity Q calculated for the incoming from infinity particle must be the same as Q calculated at infinity for the outgoing particle. An interesting question is the following: suppose that such a conservation law is established for any scattering of particles by a localized object, can one conclude that the metric of this object possesses a hidden symmetry?

A.1.3 Symmetric form of the metric

Let us perform the 'Wick' rotation in radial coordinate r. This transforms the metric (A.13), (A.18) and its hidden symmetries into a *symmetric form* [Chen *et al.*, 2006a]. After the transformation

$$r = ix\,, \quad M = iN_x\,, \quad N = N_y\,, \tag{A.20}$$

the metric and the KY objects are

$$\begin{aligned}
ds^2 &= \frac{\Delta_x(dt+y^2d\psi)^2}{x^2-y^2} + \frac{\Delta_y(dt+x^2d\psi)^2}{y^2-x^2} + \frac{(x^2-y^2)dx^2}{\Delta_x} + \frac{(y^2-x^2)dy^2}{\Delta_y}\,, && \text{(A.21)}\\
\Delta_x &= (a^2-x^2)(1-\lambda x^2)+2N_x x\,,\ \Delta_y = (a^2-y^2)(1-\lambda y^2)+2N_y y\,, && \text{(A.22)}\\
\boldsymbol{b} &= \frac{1}{2}\big[(x^2+y^2)\boldsymbol{dt} + x^2y^2\boldsymbol{d\psi}\big]\,, && \text{(A.23)}\\
\boldsymbol{h} &= y\boldsymbol{dy}\wedge(\boldsymbol{dt}+x^2\boldsymbol{d\psi}) + x\boldsymbol{dx}\wedge(\boldsymbol{dt}+y^2\boldsymbol{d\psi})\,, && \text{(A.24)}\\
\boldsymbol{f} &= x\boldsymbol{dy}\wedge(\boldsymbol{dt}+x^2\boldsymbol{d\psi}) + y\boldsymbol{dx}\wedge(\boldsymbol{dt}+y^2\boldsymbol{d\psi})\,. && \text{(A.25)}
\end{aligned}$$

These forms of the Kerr-NUT-(A)dS spacetime and of the KY potential allow the natural generalizations to higher dimensions [Chen *et al.*, 2006a], [Kubizňák & Frolov, 2007], which are used throughout the thesis.[4]

A.1.4 PCKY tensor and canonical coordinates

We demonstrate now that coordinates (t, x, y, ψ) used in (A.21)–(A.25) have a deep invariant meaning. We start in a flat spacetime (A.1). Let us define

$$Q^a_{\ b} = -h^{ac}h_{cb}\,, \qquad \Delta^a_{\ b} = Q^a_{\ b} - q\delta^a_{\ b}\,, \tag{A.26}$$

then one has

$$\Delta^a_{\ b} = \begin{pmatrix} -R^2 - q & a\,Y & -a\,X & 0 \\ -a\,Y & a^2 - X^2 - q & -Y\,X & -Z\,X \\ a\,X & -Y\,X & a^2 - Y^2 - q & -Z\,Y \\ 0 & -Z\,X & -Z\,Y & -Z^2 - q \end{pmatrix}. \tag{A.27}$$

The condition $\det(\Delta) = 0$ which determines the eigenvalues q of the operator $\boldsymbol{Q}$ is equivalent to the following equation:[5]

$$q^2 + (R^2 - a^2)q - a^2\,Z^2 = 0\,. \tag{A.29}$$

Hence, the eigenvalues of $\boldsymbol{Q}$ are

$$q_\pm = \frac{1}{2}\Big[a^2 - R^2 \pm \sqrt{(R^2-a^2)^2 + 4a^2Z^2}\,\Big]\,. \tag{A.30}$$

Simple calculations using (A.10), (A.12), and (A.20), give

$$q_+ = a^2\cos^2\theta = y^2\,, \quad q_- = -r^2 = x^2\,. \tag{A.31}$$

Thus the coordinates x and y in (A.21) are uniquely determined as the eigenvalues of the operator $\boldsymbol{Q}$ constructed from the PCKY tensor $\boldsymbol{h}$. Let us show now

[4]It is obvious from the derivation that this symmetric form of the metric and of its hidden symmetries is an analytical continuation of the real physical quantities (A.13), (A.15), (A.18). The signature of the metric for this continuation depends on the domain of coordinates x and y and signs of Δ_x and Δ_y. For example, for $x^2 > y^2$ and $\Delta_x > 0$, $\Delta_y < 0$ it is of the Euclidean signature. The transition to the physical space is given by (A.20).

[5]The tensor $\boldsymbol{Q}$ is the conformal Killing tensor. It is related to $\boldsymbol{K}$ as

$$K_{ab} = Q_{ab} - \frac{1}{2}g_{ab}Q^c_{\ c}\,, \qquad Q_{ab} = K_{ab} - \frac{1}{D-2}\,g_{ab}K^c_{\ c}\,. \tag{A.28}$$

that the same tensor $\boldsymbol{h}$ uniquely determines the coordinates t and ψ. The *primary* Killing vector $\boldsymbol{\xi}$ and the the *secondary* Killing vector $\boldsymbol{\eta}$ are [cf. Eqs. (1.5)]

$$\xi^a = \frac{1}{3}\nabla_c h^{ca} = (\partial_T)^a\,, \tag{A.32}$$

$$\eta^a = -K^{ab}\xi_b = a^2(\partial_T)^a + aY(\partial_X)^a - aX(\partial_Y)^a\,. \tag{A.33}$$

In coordinates (A.12) one has

$$\boldsymbol{\xi} = \boldsymbol{\partial}_t\,, \quad \boldsymbol{\eta} = \boldsymbol{\partial}_\psi\,. \tag{A.34}$$

This means that the coordinates t and ψ are the affine parameters along the primary and secondary Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, determined by the tensor $\boldsymbol{h}$.

It can be checked that the same is true for (the symmetric form of) the canonical metric (A.21) with the PCKY tensor $\boldsymbol{h}$ given by (A.24). This underlines the exceptional role of the PCKY tensor. Remarkably, the existence of a similar object in higher dimensions generates the higher-dimensional Kerr-NUT-(A)dS spacetime and determines its canonical coordinates, in a way exactly analogous to four dimensions (see Chapter 7).

A.1.5 Separation of variables

The last subject we would like to discuss in this brief review of properties of the 4D Kerr-NUT-(A)dS metric is the separation of variables for the Hamilton–Jacobi and Klein–Gordon equations. More generally, we consider these equations in the off-shell spacetime (A.21), with $\Delta_x(x)$ and $\Delta_y(y)$ arbitrary functions.

Let us first discuss the Klein–Gordon equation

$$\Box\Phi - \mu^2\Phi = 0\,. \tag{A.35}$$

The separation of variables of equation (A.35) in canonical coordinates (τ, x, y, ψ) means that Φ can be decomposed into modes

$$\Phi = e^{i\varepsilon\tau + im\psi}X(x)Y(y)\,. \tag{A.36}$$

Indeed, substituting this expression in the Klein–Gordon equation (A.35) one obtains

$$(\Delta_x X')' + V_x X = 0\,, \quad V_x = \kappa + \mu^2 x^2 - \frac{(\varepsilon x^2 - m)^2}{\Delta_x}\,, \tag{A.37}$$

$$(\Delta_y Y')' + V_y Y = 0\,, \quad V_y = \kappa + \mu^2 y^2 - \frac{(\varepsilon y^2 - m)^2}{\Delta_y}\,. \tag{A.38}$$

Here, the prime stands for the derivative of function with respect to its single argument. The separation constants ε and m are connected with the symmetries generated by the Killing vectors $\boldsymbol{\xi} = \boldsymbol{\partial}_t$ and $\boldsymbol{\eta} = \boldsymbol{\partial}_\psi$. An additional separation constant κ is connected with the hidden symmetry generated by the Killing tensor $\boldsymbol{K}$. It should be emphasized, that in order to use the proved separability for concrete calculations in the physical Kerr-NUT-(A)dS spacetime (A.13), (A.18), one needs to specify functions Δ_x and Δ_y to have the form (A.22) and perform the Wick transformation inverse to (A.20). This transformation 'spoils' the symmetry between the essential coordinates but the separability property remains. In coordinates r and y in the 'physical' sector equations (A.37) and (A.38) play different roles. Eq. (A.38) with imposed regularity conditions serves as an eigenvalue problem which determines the spectrum of κ. Eq. (A.37) is a radial equation for propagating modes.

Similarly, the Hamilton–Jacobi equation for geodesic motion

$$\partial_\lambda S + g^{ab}\partial_a S \partial_b S = 0 \tag{A.39}$$

in the background (A.21) allows a separation of variables and S can be written in the form

$$S = \mu^2\lambda + \varepsilon\tau + m\psi + S_x(x) + S_y(y)\,. \tag{A.40}$$

The functions S_x and S_y obey the equations

$$(S'_x)^2 = \frac{V_x}{\Delta_x}\,, \quad (S'_y)^2 = \frac{V_y}{\Delta_y}\,. \tag{A.41}$$

The separability of the Hamilton–Jacobi and Klein–Gordon equations demonstrated above is directly connected with the existence of the Killing tensor and the corresponding symmetry operator [Carter, 1977] (see also Section 6.3.2). Let us mention that a similar intrinsic characterization of the separation constants (connected with the PCKY tensor) can be found in the case of the Dirac equation [Carter & McLenaghan, 1979], [Kamran & McLenaghan, 1984], as well as in the case of the massless equations with spin [Kamran, 1985], [Kalnins *et al.*, 1986], [Kalnins & Miller, 1989], [Kalnins & Williams, 1990], [Kalnins *et al.*, 1992], [Kalnins *et al.*, 1996]. For example, one of the very convenient ways how to prove that the Maxwell equations in the background (A.21) decouples and separate is the method of the Debye potentials, directly based on the existence of the CKY tensor [Benn *et al.*, 1997].

A.2 CKY tensors for the Plebański–Demiański class of solutions

In this section, based on [Kubizňák & Krtouš, 2007], we present explicit expressions for the conformal Killing–Yano tensors for the Plebański–Demiański family of type D solutions. Some physically important special cases are discussed in more detail. In particular, it is demonstrated how the conformal Killing–Yano tensor becomes the Killing–Yano tensor for the solutions without acceleration.

A.2.1 Plebański–Demiański metric

The important family of type D spacetimes in four dimensions, including the black-hole spacetimes like the Kerr metric, the metrics describing the accelerating sources as the C-metric, or the non-expanding Kundt's class type D solutions, can be represented by the general seven-parameter metric discovered by Plebański and Demiański [Plebański & Demiański, 1976] (cf. also [Debever, 1971]). Recently, Griffiths and Podolský [Griffiths & Podolský, 2005], [Griffiths & Podolský, 2006b], [Griffiths & Podolský, 2006a], [Podolský & Griffiths, 2006], [Griffiths & Podolský, 2007], put this metric into a new form which enables a better physical interpretation of parameters and simplifies a procedure how to derive all special cases. Among subclasses of this solution let us mention the six-parameter family of metrics without acceleration derived and studied already by Carter [Carter, 1968c], [Carter, 1968b] and later by Plebański [Plebański, 1975].

It turns out that the elegant form of the Plebański–Demiański metric not only yields new solutions in 4D (see, e.g., [Klemm *et al.*, 1998], [Alonso-Alberca *et al.*, 2000]), but also inspires for its generalizations into higher dimensions. For example, Chen, Lü, and Pope [Chen *et al.*, 2006a] were able to cast the Einstein space subclass of Carter's non-accelerating solutions into higher dimensions—thus constructing the general Kerr–NUT–(A)dS metrics in all dimensions. These are discussed in the main text. Recently, even more general solutions in 5D [Lü *et al.*, 2008a], [Lü *et al.*, 2008b], directly inspired by the Plebański–Demiański metric, were obtained (see also Appendix C.6).

One of the most remarkable properties of the Carter's subclass of non-accelerating solutions, which is also inherited by its higher dimensional generalization (see Chapter 4), is the existence of hidden symmetries associated with the Killing–Yano tensor [Penrose, 1973], [Carter, 1987]. In four dimensions the integrability conditions for the existence of a non-degenerate Killing–Yano tensor restricts the Petrov type of spacetime to type D (see, e.g., [Collinson, 1974], [Dietz & Rüdiger, 1981]). Demiański and Francaviglia [Demianski & Francaviglia, 1980] demonstrated that from the known type D solutions only spacetimes without acceleration of sources actually admit this tensor. The purpose of this section is to show that the general Plebański–Demiański metric admits the conformal generalization of the Killing–Yano tensor. We also explicitly demonstrate how in the absence of acceleration this tensor becomes the known Killing–Yano tensor of the Carter's metric. The explicit expressions for this tensor for the physically important cases are presented.

The original form of the Plebański–Demiański metric [Plebański & Demiański, 1976] is given by

$$\boldsymbol{g} = \Omega^2\Big[-\frac{Q(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{P(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{P}\,\boldsymbol{d}p^2 + \frac{r^2+p^2}{Q}\,\boldsymbol{d}r^2\Big]\,. \quad \text{(A.42)}$$

This metric obeys the Einstein–Maxwell equations with the electric and magnetic charges e and g and the cosmological constant Λ provided that functions $P = P(p)$ and $Q = Q(r)$ take the particular form

$$\begin{aligned} Q &= k+e^2+g^2-2mr+\epsilon r^2-2nr^3-(k+\Lambda/3)r^4\,, \\ P &= k+2np-\epsilon p^2+2mp^3-(k+e^2+g^2+\Lambda/3)p^4\,, \end{aligned} \quad \text{(A.43)}$$

the conformal factor is

$$\Omega^{-1} = 1 - pr\,, \quad \text{(A.44)}$$

and the vector potential reads

$$\boldsymbol{A} = -\frac{1}{r^2+p^2}\,\Big[e\,r\,\big(\boldsymbol{d}\tau - p^2\,\boldsymbol{d}\sigma\big) + g\,p\,\big(\boldsymbol{d}\tau + r^2\,\boldsymbol{d}\sigma\big)\Big]\,. \quad \text{(A.45)}$$

The general Plebański–Demiański metric (A.42) admits the CKY tensor [Kubizňák & Krtouš, 2007]

$$\boldsymbol{k} = \Omega^3\Big[p\,\boldsymbol{d}r \wedge (\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma) + r\,\boldsymbol{d}p \wedge (\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma)\Big]\,. \quad \text{(A.46)}$$

Using the Maple program, one can easily verify that Eqs. (2.10),

$$\nabla_a k_{bc} = \nabla_{[a}k_{bc]} + 2\,g_{a[b}\xi_{c]}\,, \quad \xi_a = \frac{1}{3}\,\nabla_c k^c{}_a\,, \quad \text{(A.47)}$$

are satisfied. An independent proof is given at the end of this appendix. The Hodge dual, $\boldsymbol{h} = *\boldsymbol{k}$, is also a CKY tensor. It reads

$$\boldsymbol{h} = \Omega^3\Big[r\,\boldsymbol{d}r \wedge (p^2\boldsymbol{d}\sigma - \boldsymbol{d}\tau) + p\,\boldsymbol{d}p \wedge (r^2\boldsymbol{d}\sigma + \boldsymbol{d}\tau)\Big]\,, \quad \text{(A.48)}$$

which is equivalent to

$$h = \Omega^3\, db\,, \tag{A.49}$$

where

$$b = \frac{1}{2}\big[(p^2 - r^2)d\tau + p^2 r^2 d\sigma\big]\ . \tag{A.50}$$

It is interesting to mention that k and h are CKY tensors for the metric (A.42), with an arbitrary conformal factor Ω and arbitrary functions $P(p)$, $Q(r)$, i.e., irrespectively of the fact whether the metric (A.42) solves the Einstein equations or not. We shall return to this remark later. We shall also see that in the absence of acceleration of sources, k becomes the KY tensor and h becomes the closed CKY tensor.

For Ω given by (A.44) and arbitrary functions $P(p)$ and $Q(r)$ both isometries of the spacetime follow from the existence of h and k as follows [cf. Eqs. (A.32), (A.33)]:

$$\xi_{(h)} \equiv -\frac{1}{3}\delta h = \partial_\tau\ , \qquad \xi_{(k)} \equiv -\frac{1}{3}\delta k = \partial_\sigma\ . \tag{A.51}$$

The conformal Killing tensor, (2.15), associated with k reads

$$Q_{(k)} = \Omega^4\Big[\frac{Qp^2(d\tau - p^2 d\sigma)^2}{r^2+p^2} + \frac{Pr^2(d\tau + r^2 d\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{P}\, r^2 dp^2 - \frac{r^2+p^2}{Q}\, p^2 dr^2\Big]\ . \tag{A.52}$$

It inherits the 'universality' of k, i.e., it is a conformal Killing tensor of the metric (A.42) with an arbitrary Ω, and arbitrary $Q(r)$ and $P(p)$. In the absence of acceleration $Q_{(k)}$ becomes a Killing tensor which generates the Carter's constant for a geodesic motion [Carter, 1968a]. The conformal Killing tensor associated with h is

$$Q_{(h)} = \Omega^4\Big[\frac{Qr^2(d\tau - p^2 d\sigma)^2}{r^2+p^2} + \frac{Pp^2(d\tau + r^2 d\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{P}\, p^2 dp^2 - \frac{r^2+p^2}{Q}\, r^2 dr^2\Big]\ . \tag{A.53}$$

Both tensors are related as

$$Q_{(h)} = Q_{(k)} + \Omega^2(p^2 - r^2)\, g\ . \tag{A.54}$$

Following [Griffiths & Podolský, 2006b] one can easily perform the transformations of coordinates and parameters to obtain the complete family of type D spacetimes and the corresponding particular forms of CKY tensors. In the next two sections we consider two special cases. First, we deal with the generalized black holes and, second, we demonstrate what happens when the acceleration of sources is removed.

A.2.2 Generalized black holes

Following [Griffiths & Podolský, 2006b], let us introduce two new continuous parameters α (the acceleration) and ω (the 'twist') by the rescaling

$$p \longrightarrow \sqrt{\alpha\omega}p\,, \quad r \longrightarrow \sqrt{\frac{\alpha}{\omega}}r\,, \quad \sigma \longrightarrow \sqrt{\frac{\omega}{\alpha^3}}\sigma\,, \quad \tau \longrightarrow \sqrt{\frac{\omega}{\alpha}}\tau\,, \tag{A.55}$$

and relabel the other parameters as

$$m \longrightarrow \Big(\frac{\alpha}{\omega}\Big)^{3/2}m\,, \quad n \longrightarrow \Big(\frac{\alpha}{\omega}\Big)^{3/2}n\,, \quad e \longrightarrow \frac{\alpha}{\omega}e\,, \quad g \longrightarrow \frac{\alpha}{\omega}g\,, \quad \epsilon \longrightarrow \frac{\alpha}{\omega}\epsilon\,, \quad k \longrightarrow \alpha^2 k\,. \tag{A.56}$$

Then the metric and the vector potential take the form

$$\begin{aligned} \boldsymbol{g} &= \Omega^2\Big[-\frac{Q(\boldsymbol{d}\tau-\omega p^2\boldsymbol{d}\sigma)^2}{r^2+\omega^2p^2}+\frac{P(\omega\boldsymbol{d}\tau+r^2\boldsymbol{d}\sigma)^2}{r^2+\omega^2p^2}+\frac{r^2+\omega^2p^2}{P}\,\boldsymbol{d}p^2+\frac{r^2+\omega^2p^2}{Q}\,\boldsymbol{d}r^2\Big]\,, \\ \boldsymbol{A} &= -\frac{1}{r^2+\omega^2p^2}\Big[e\,r\,(\boldsymbol{d}\tau-\omega p^2\,\boldsymbol{d}\sigma)+g\,p\,(\omega\boldsymbol{d}\tau+r^2\,\boldsymbol{d}\sigma)\Big]\,, \end{aligned} \tag{A.57}$$

with

$$\begin{aligned} \Omega^{-1} &= 1-\alpha\,pr\,, && \text{(A.58)} \\ Q &= \omega^2k+e^2+g^2-2mr+\epsilon r^2-\frac{2\alpha n}{\omega}r^3-\big(\alpha^2k+\frac{\Lambda}{3}\big)r^4\,, && \text{(A.59)} \\ P &= k+\frac{2n}{\omega}p-\epsilon p^2+2\alpha mp^3-\Big[\alpha^2(\omega^2k+e^2+g^2)+\omega^2\frac{\Lambda}{3}\Big]p^4\,. && \text{(A.60)} \end{aligned}$$

The CKY tensors are (up to trivial constant factors)

$$\Omega^{-3}\boldsymbol{k} = \omega p\,\boldsymbol{d}r\wedge(\boldsymbol{d}\tau-\omega p^2\boldsymbol{d}\sigma)+r\,\boldsymbol{d}p\wedge(\omega\boldsymbol{d}\tau+r^2\boldsymbol{d}\sigma)\,, \tag{A.61}$$

and, $\boldsymbol{h} = \Omega^3\,\boldsymbol{db}$, with Ω given in (A.58), and

$$\boldsymbol{b} = \frac{1}{2}\big[(\omega^2p^2-r^2)\boldsymbol{d}\tau+\omega p^2r^2\boldsymbol{d}\sigma\big]\,. \tag{A.62}$$

Let's consider two special cases. First, we relabel $\omega = a$, perform an additional coordinate transformation

$$p \longrightarrow \cos\theta\,, \quad \tau \longrightarrow \tau - a\phi\,, \quad \sigma \longrightarrow -\phi\,, \tag{A.63}$$

and set

$$k = 1\,, \quad \epsilon = 1-\alpha^2(a^2+e^2+g^2)-\frac{\Lambda}{3}a^2\,, \quad n = -\alpha am\,. \tag{A.64}$$

(One parameter—NUT charge—was set to zero and the scaling freedom was used to eliminate the other two.) We have obtained a six-parameter solution which describes the accelerating rotating charged black hole with the cosmological constant:

$$g = \Omega^2\Big\{-\frac{Q}{\Delta}\big[d\tau - a\sin^2\theta d\phi\big]^2 + \frac{\Delta}{Q}dr^2 + \frac{P}{\Delta}\big[a d\tau - (r^2+a^2)d\phi\big]^2 + \frac{\Delta}{P}\sin^2\theta d\theta^2\Big\}\,, \tag{A.65}$$

where

$$\begin{aligned}\Omega^{-1} &= 1-\alpha r\cos\theta\,,\quad \Delta = r^2+a^2\cos^2\theta\,,\\ Q &= (a^2+e^2+g^2-2mr+r^2)(1-\alpha^2r^2) - \frac{\Lambda}{3}(a^2+r^2)r^2\,,\\ \frac{P}{\sin^2\theta} &= 1-2\alpha m\cos\theta + \Big[\alpha^2(a^2+e^2+g^2)+\frac{\Lambda a^2}{3}\Big]\cos^2\theta\,.\end{aligned} \tag{A.66}$$

In the brackets in (A.65) we can easily recognize the familiar form of the Kerr solution. The conformal factor and the modification of metric functions correspond to the acceleration and the cosmological constant. The CKY tensor k takes the form

$$\Omega^{-3}k = a\cos\theta\, dr\wedge\big[d\tau - a\sin^2\theta\, d\phi\big] - r\sin\theta\, d\theta\wedge\big[a d\tau - (r^2+a^2)\, d\phi\big]\,, \tag{A.67}$$

where Ω is given in (A.66). Except the conformal factor we recovered the Killing–Yano tensor for the Kerr metric derived by Penrose and Floyd [Penrose, 1973], [Floyd, 1973].

The second interesting example is obtained if instead of (A.63) and (A.64) we perform

$$p\to\frac{l+a\cos\theta}{\omega}\,,\qquad \tau\to\tau-\frac{(l+a)^2}{a}\phi\,,\qquad \sigma\to-\frac{\omega}{a}\phi\,, \tag{A.68}$$

set the acceleration to zero, $\alpha=0$, and adjust

$$\epsilon = 1-\frac{\Lambda}{3}(a^2+6l^2)\,,\quad n = l+\frac{\Lambda l}{3}(a^2-4l^2)\,,\quad \omega^2k = (1-l^2\Lambda)(a^2-l^2)\,. \tag{A.69}$$

Then we have a non-accelerated rotating charged black hole with NUT parameter and the cosmological constant:

$$\begin{aligned} g = &-\frac{Q}{\Delta}\big[d\tau - (a\sin^2\theta + 4l\sin^2\tfrac{\theta}{2})\, d\phi\big]^2 + \frac{\Delta}{Q}\, dr^2\\ &+\frac{P}{\Delta}\Big\{a d\tau - \big[r^2+(a+l)^2\big]d\phi\Big\}^2 + \frac{\Delta}{P}\sin^2\theta\, d\theta^2\,,\end{aligned} \tag{A.70}$$

where

$$\frac{P}{\sin^2\theta} = 1+\frac{4\Lambda}{3}al\cos\theta+\frac{\Lambda}{3}a^2\cos^2\theta\,,\quad \Delta = r^2+(l+a\cos\theta)^2\,,$$
$$Q = a^2-l^2+e^2+g^2-2mr+r^2-\frac{\Lambda}{3}\Big[3(a^2-l^2)\,l^2+(a^2+6l^2)r^2+r^4\Big]\,. \quad \text{(A.71)}$$

The CKY tensor $\boldsymbol{k}$ becomes the KY tensor (see also the next subsection) and takes the form

$$\boldsymbol{k} = (l+a\cos\theta)\,\boldsymbol{d}r\wedge\big[\boldsymbol{d}\tau-(a\sin^2\theta+4l\sin^2\frac{\theta}{2})\,\boldsymbol{d}\phi\big]$$
$$-r\sin\theta\,\boldsymbol{d}\theta\wedge\Big\{a\boldsymbol{d}\tau-\big[r^2+(a+l)^2\big]\boldsymbol{d}\phi\Big\}\,. \quad \text{(A.72)}$$

The dual CKY tensor becomes closed, $\boldsymbol{h}=\boldsymbol{d}\boldsymbol{b}$, with

$$\boldsymbol{b} = \frac{1}{2}\Big\{\big[(l+a\cos\theta)^2-r^2\big]\big[a\boldsymbol{d}\tau-(l+a)^2\boldsymbol{d}\phi\big]-r^2(l+a\cos\theta)^2\boldsymbol{d}\phi\Big\}\,. \quad \text{(A.73)}$$

In particular, in vacuum ($e=g=\Lambda=0$) we recover the KY tensor for the Kerr metric ($l=0$), respectively for the NUT solution ($a=0$) studied recently in [Jezierski & Lukasik, 2006], respectively [Jezierski & Lukasik, 2007].

A.2.3 Carter's metric

Let us take the Plebański–Demiański metric in the form (A.57) and set the acceleration to zero, $\alpha=0$, and $\omega=1$. Then the conformal factor becomes $\Omega=1$ and we recover the Carter's family of non-accelerating solutions [Carter, 1968c], [Carter, 1968b] in the form used in [Plebański, 1975]:

$$\boldsymbol{g} = -\frac{Q(\boldsymbol{d}\tau-p^2\boldsymbol{d}\sigma)^2}{r^2+p^2}+\frac{P(\boldsymbol{d}\tau+r^2\boldsymbol{d}\sigma)^2}{r^2+p^2}+\frac{r^2+p^2}{P}\,\boldsymbol{d}p^2+\frac{r^2+p^2}{Q}\,\boldsymbol{d}r^2\,, \quad \text{(A.74)}$$

where

$$Q = k+e^2+g^2-2mr+\epsilon r^2-\frac{\Lambda}{3}r^4\,,$$
$$P = k+2np-\epsilon p^2-\frac{\Lambda}{3}p^4\,, \quad \text{(A.75)}$$

and the vector potential is given by (A.45). Notice that (A.74) coincides with the canonical form (A.13) discussed in the previous section.

We also get

$$k = p\, dr \wedge (d\tau - p^2 d\sigma) + r\, dp \wedge (d\tau + r^2 d\sigma)\ , \tag{A.76}$$

which is the Killing–Yano tensor given by Carter [Carter, 1987]. Its dual,

$$h = *k = db\ , \tag{A.77}$$

with b given by (A.50), becomes the closed CKY tensor. These are the hidden symmetries for the canonical metric [independent of the particular form of $P(p)$ and $Q(r)$]. The conformal Killing tensor (A.52) becomes the Killing tensor

$$K = \frac{Qp^2(d\tau - p^2 d\sigma)^2}{r^2+p^2} + \frac{Pr^2(d\tau + r^2 d\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{P} r^2 dp^2 - \frac{r^2+p^2}{Q} p^2 dr^2\ . \tag{A.78}$$

Both isometries of spacetime may be derived from the existence of k, but in a different manner than before. We have $\xi_{(h)} = \partial_\tau$ whereas $\xi_{(k)} = 0$ since k is now a KY tensor. Nevertheless, the second isometry is given by [cf. Eq. (A.33)]

$$(\partial_\sigma)^a = K^a_{\ b}\, \xi^b_{(h)}\ . \tag{A.79}$$

Let us observe that the full Plebański–Demiański metric with acceleration is related to the (non-accelerating) Carter's metric only by a conformal rescaling and a modification of metric functions $P(p)$ and $Q(r)$.[6] It allows us to use the theorem (see, e.g., [Tachibana, 1969], [Jezierski & Lukasik, 2006]) which states that whenever k is a CKY tensor for the metric g then $\Omega^3 k$ is a CKY tensor for the conformally rescaled metric $\Omega^2 g$. This would justify the transition from the known KY tensor (A.76) to the CKY tensor (A.46), up to the fact, that in the transition from (A.42) to (A.74) we also need to change metric functions $P(p)$ and $Q(r)$. Fortunately, as mentioned above, the 'universality' of k, i.e., the property that (A.76) remains KY tensor for the metric (A.74) with arbitrary function $P(p)$ and $Q(r)$, can be demonstrated. Indeed, the only nontrivial components of the covariant derivative ∇k, namely

$$\nabla_p k_{\sigma\tau} = \nabla_r k_{p\sigma} = \nabla_\sigma k_{rp} = r^2 + p^2\ , \tag{A.80}$$

are completely independent of the form of $Q(r)$ and $P(p)$. Therefore one can start with the metric g (A.74), with the KY tensor k (A.76), and with arbitrary functions $P(p)$ and $Q(r)$ so that, after performing the conformal scaling $g \to \Omega^2 g$ we obtain the metric (A.42). The theorem ensures that $\Omega^3 k$ is the universal

[6] One might hope that such a transition could work also in higher dimensions. For the demonstration that it is not so, see Appendix C.6.

CKY tensor for the new metric, and in particular for the Plebański–Demiański solution, where Ω is given by (A.44) and functions $P(p)$ and $Q(r)$ by (A.43).

Appendix B

PCKY tensor in the Myers–Perry spacetimes

Historically, the PCKY tensor in higher-dimensional black hole spacetimes was first discovered [Frolov & Kubizňák, 2007] for the Myers–Perry metrics [Myers & Perry, 1986], and its existence was verified with the help of the Maple program up to $D \leq 8$. A little bit later, an (unpublished) analytical calculation, using the Kerr–Schild form of the Myers–Perry metrics, proved its existence in an arbitrary number of spacetime dimensions. Soon after that, the PCKY tensor was discovered for the Gibbons–Lü–Page–Pope [Gibbons *et al.*, 2004], [Gibbons *et al.*, 2005] Kerr-(A)dS metrics (unpublished), and finally [Kubizňák & Frolov, 2007] for the general Kerr-NUT-(A)dS spacetimes [Chen *et al.*, 2006a]. In this appendix we give a brief account of these historical developments and present the sketch of the (unpublished) proof justifying the existence of the PCKY tensor in the Myers–Perry spacetimes.

B.1 Myers–Perry metrics and their symmetries

The Myers–Perry (MP) metrics [Myers & Perry, 1986] are the most general known vacuum solutions for the higher-dimensional rotating black holes. These solutions allow the Kerr–Schild form [Myers & Perry, 1986], they are of the type D of the higher-dimensional algebraic classification [Milson *et al.*, 2005], [Coley *et al.*, 2004], [Coley, 2008]. The metrics have slightly different form for the odd and even number of spacetime dimensions D. We can write them compactly as

$$\boldsymbol{g} = -\boldsymbol{d}t^2+\frac{U\boldsymbol{d}r^2}{V-2M}+\frac{2M}{U}\Big(\boldsymbol{d}t+\sum_{i=1}^{m}a_i\mu_i^2\boldsymbol{d}\phi_i\Big)^2+\sum_{i=1}^{m}(r^2+a_i^2)(\mu_i^2\boldsymbol{d}\phi_i^2+\boldsymbol{d}\mu_i^2)+\epsilon r^2\boldsymbol{d}\nu^2\,, \tag{B.1}$$

where

$$V \equiv r^{\epsilon-2}\prod_{i=1}^{m}(r^2+a_i^2)\,, \quad U \equiv V\Big(1-\sum_{i=1}^{m}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\Big)\,. \tag{B.2}$$

Here $m \equiv [(D-1)/2]$, where $[A]$ means the integer part of A. We define ϵ to be 1 for D even and 0 for odd. (This is different from the Kerr-NUT-(A)dS ε; $\epsilon = 1-\varepsilon$.) The coordinates μ_i are not independent. They obey the following constraint:

$$\sum_{i=1}^{m}\mu_i^2 + \epsilon\nu^2 = 1\,. \tag{B.3}$$

The MP metrics possess the PCKY tensor [Frolov & Kubizňák, 2007], and the, derivable from it, towers of hidden symmetries and tower of Killing vectors. The latter is related to the obvious $m+1$ isometries, $\boldsymbol{\partial}_t$, $\boldsymbol{\partial}_{\phi_i}$, $i=1,\dots,m$, present in the spacetime (see, e.g., Eq. (B.8) below). The KY potential $\boldsymbol{b}$ for the MP metric (B.1) reads [Frolov & Kubizňák, 2007]

$$\boldsymbol{b} = \frac{1}{2}\Big[\big(r^2+\sum_{i=1}^{m}a_i^2\mu_i^2\big)\boldsymbol{dt} + \sum_{i=1}^{m}a_i\mu_i^2(r^2+a_i^2)\boldsymbol{d\phi_i}\Big]\,. \tag{B.4}$$

The corresponding PCKY tensor $\boldsymbol{h}$ is

$$\boldsymbol{h} = \sum_{i=1}^{m}a_i\mu_i\boldsymbol{d\mu_i}\wedge\Big[a_i\boldsymbol{dt} + (r^2+a_i^2)\boldsymbol{d\phi_i}\Big] + r\boldsymbol{dr}\wedge\big(\boldsymbol{dt}+\sum_{i=1}^{m}a_i\mu_i^2\boldsymbol{d\phi_i}\big)\,. \tag{B.5}$$

Here and later on in similar formulas the summation over i is taken from 1 to m for both—even and odd number of spacetime dimensions D; the coordinates μ_i are independent when D is even whereas they obey the constraint (B.3) when D is odd.

Historically, the existence of the PCKY tensor was discovered with the help of the Maple Program. At the time of discovery it was known that the 5D MP metric possesses the Killing tensor and allows the separation of variables for the Hamilton–Jacobi and scalar field equations [Frolov & Stojković, 2003b], [Frolov & Stojković, 2003a]. It was also known that such a separation is possible in higher dimensions, provided that the rotation parameters a_i are divided into two classes, and within each of the classes they are equal of one another [Vasudevan *et al.*, 2005a]. One of the motivations to search for the hidden symmetry associated with Killing–Yano tensors was the task to solve the parallel transport of orthonormal frames. (This task was finally accomplished three years later, see Chapter 9.)

We further remark that, by the time of the discovery of the PCKY tensor (B.5) it was completely unknown that such a tensor allows one to generate other hidden symmetries or Killing vectors, except those of the dual Killing–Yano tensor $\boldsymbol{f} = *\boldsymbol{h}$,

$$f_{a_1 \ldots a_{D-2}} = (*h)_{a_1 \ldots a_{D-2}} = \frac{1}{2}\, e_{a_1 \ldots a_{D-2}}{}^{cd} h_{cd}\,, \tag{B.6}$$

the associated Killing tensor

$$K_{ab} = \frac{1}{(D-3)!}\, f_{a a_1 \ldots a_{D-3}} f_b{}^{a_1 \ldots a_{D-3}} = h_{ac} h_b{}^c - \frac{1}{2}\, g_{ab} h_{cd} h^{cd}\,, \tag{B.7}$$

and the corresponding two Killing vectors,

$$\xi^a = \frac{1}{D-1}\, \nabla_c h^{ca}\,, \quad \eta_a = K_{ac}\xi^c\,. \tag{B.8}$$

Specifically, the explicit expressions for $\boldsymbol{f}$ in four and five dimensions, and the following ('computer-empirical') formulas:

$$K^{ab}\boldsymbol{\partial}_a\boldsymbol{\partial}_b = \sum_{i=1}^{m}\Big[a_i^2(\mu_i^2-1)\,\boldsymbol{g}^{-1} + a_i^2\mu_i^2\boldsymbol{\partial}_t^2 + \frac{1}{\mu_i^2}\,\boldsymbol{\partial}_{\phi_i}^2\Big] + \sum_{i=1}^{m-1+\epsilon}\boldsymbol{\partial}_{\mu_i}^2 - 2\boldsymbol{Z}\boldsymbol{Z} - \boldsymbol{\xi}\boldsymbol{\zeta} - \boldsymbol{\zeta}\boldsymbol{\xi}\,, \tag{B.9}$$

$$\boldsymbol{\xi} = \boldsymbol{\partial}_t\,, \quad \boldsymbol{\eta} = \sum_{i=1}^{m} a_i^2\,\boldsymbol{\xi} - \boldsymbol{\zeta}\,, \quad \boldsymbol{\zeta} \equiv \sum_{i=1}^{m} a_i\boldsymbol{\partial}_{\phi_i}\,, \quad \boldsymbol{Z} \equiv \sum_{i=1}^{m-1+\epsilon} \mu_i\boldsymbol{\partial}_{\mu_i}\,, \tag{B.10}$$

were known. These expressions were verified with the help of the Maple program up to $D = 8$. In 4D, the expression for $\boldsymbol{f}$ coincided with the KY tensor discovered by Penrose and Floyd [Penrose, 1973], [Floyd, 1973], and the Killing tensor (B.9) reduced to the Killing tensor obtained in [Carter, 1968a], [Walker & Penrose, 1970]. In $D = 5$, the Killing tensor (B.9) gave the Killing tensor obtained in [Frolov & Stojković, 2003b], [Frolov & Stojković, 2003a], after the (constant) term $\boldsymbol{\xi}\boldsymbol{\zeta} + \boldsymbol{\zeta}\boldsymbol{\xi}$ was omitted. All these 'computer-empirical' results were put onto the solid ground by the (never published) analytical proof proving the existence of the PCKY tensor in an arbitrary number of spacetime dimensions (see Appendix B.3).

Soon after this proof was finished, the PCKY tensor for the general Kerr-NUT-(A)dS spacetimes was discovered [Kubizňák & Frolov, 2007]. Let us in the next section briefly mention the pre-stage of this development. The main impulse was the discovery of the PCKY tensor for the Kerr-(A)dS black holes, together with the transformation of this object into its 'universal' canonical form.

B.2 Kerr-(A)dS black holes and their symmetries

A generalization of the MP metric which includes the cosmological constant, the Kerr-(A)dS solution in all dimensions, was obtained in 2004 by Gibbons, Lü, Page, and Pope [Gibbons *et al.*, 2004], [Gibbons *et al.*, 2005]. The metric obeys the Einstein equations

$$R_{ab} = (D-1)\lambda g_{ab}\,. \tag{B.11}$$

Similar to the MP metric, the solution allows the Kerr–Schild form and it is of the algebraic type D. In the 'generalized' Boyer–Lindquist coordinates it takes the form

$$\begin{aligned} \boldsymbol{g} &= -W(1-\lambda r^2)\boldsymbol{d}t^2 + \frac{2M}{U}\Big(W\boldsymbol{d}t + \sum_{i=1}^{m}\frac{a_i\mu_i^2\boldsymbol{d}\phi_i}{1+\lambda a_i^2}\Big)^2 + \sum_{i=1}^{m}\frac{r^2+a_i^2}{1+\lambda a_i^2}\,(\mu_i^2\boldsymbol{d}\phi_i^2+\boldsymbol{d}\mu_i^2) \\ &+\frac{U\boldsymbol{d}r^2}{V-2M} + \frac{\lambda}{W(1-\lambda r^2)}\Big(\sum_{i=1}^{m}\frac{r^2+a_i^2}{1+\lambda a_i^2}\,\mu_i\boldsymbol{d}\mu_i + \epsilon r^2\nu\boldsymbol{d}\nu\Big)^2 + \epsilon r^2\boldsymbol{d}\nu^2\,, \end{aligned} \tag{B.12}$$

where

$$\begin{aligned} W &\equiv \sum_{i=1}^{m}\frac{\mu_i^2}{1+\lambda a_i^2} + \epsilon\nu^2\,, \quad V \equiv r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{m}(r^2+a_i^2)\,, \\ U &\equiv \frac{V}{1-\lambda r^2}\Big(1-\sum_{i=1}^{m}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\Big)\,. \end{aligned} \tag{B.13}$$

Here, we use the same notations and constraint (B.3) as for the MP metrics. The Kerr-(A)dS spacetime possesses the PCKY tensor, derivable from the KY potential

$$\boldsymbol{b} = \frac{1}{2}\Big\{\Big[r^2 + \sum_{i=1}^{m}a_i^2\mu_i^2(1-\lambda\frac{r^2+a_i^2}{1+\lambda a_i^2})\Big]\boldsymbol{d}t + \sum_{i=1}^{m}a_i\mu_i^2\frac{r^2+a_i^2}{1+\lambda a_i^2}\boldsymbol{d}\phi_i\Big\}\,. \tag{B.14}$$

The PCKY tensor, $\boldsymbol{h} = \boldsymbol{d}\boldsymbol{b}$, reads

$$\boldsymbol{h} = \sum_{i=1}^{m}a_i\mu_i\boldsymbol{d}\mu_i\wedge\Big[a_i\boldsymbol{d}t + \frac{r^2+a_i^2}{1+\lambda a_i^2}(\boldsymbol{d}\phi_i - \lambda a_i\boldsymbol{d}t)\Big] + r\boldsymbol{d}r\wedge\Big[\boldsymbol{d}t + \sum_{i=1}^{m}a_i\mu_i^2(\boldsymbol{d}\phi_i - \lambda a_i\boldsymbol{d}t)\Big]\,. \tag{B.15}$$

This formula was explicitly verified, using the Maple program, for $D \leq 7$. Moreover, when the cosmological constant λ is set to zero the KY potential (B.14) reduces to that of the Myers–Perry metric (B.4).

The crucial impulse for the discovery of the PCKY tensor in the general Kerr-

NUT-(A)dS spacetimes (4.1), presented in the main text, was the following observation. In $D = 4$, the metric (B.12) describes the Kerr black hole in the (A)dS background. Such a black hole possesses only one rotation parameter which, as usual, we denote by a. We also put $\mu_1 = \sin\theta$, $\phi_1 = \phi$, and perform the additional linear transformation of ϕ and t:

$$\boldsymbol{d}t \to \boldsymbol{d}t\,, \quad \boldsymbol{d}\phi \to -a\lambda\boldsymbol{d}t - (1+\lambda a^2)\boldsymbol{d}\phi\,. \tag{B.16}$$

Then one recovers the 'standard' form of the Kerr-(A)dS metric. Up to a constant factor, one also has

$$\boldsymbol{f} = r\sin\theta\boldsymbol{d}\theta\wedge\left[a\boldsymbol{d}t + (r^2+a^2)\boldsymbol{d}\phi\right] - a\cos\theta\boldsymbol{d}r\wedge\left[\boldsymbol{d}t + a\sin^2\theta\boldsymbol{d}\phi\right]\,, \tag{B.17}$$

which is the KY tensor discovered by Penrose and Floyd [Penrose, 1973], [Floyd, 1973] for the Kerr metric. In this form, however, it holds also for the nontrivial λ. Moreover, further transformation

$$t \to \tau - a^2\sigma\,, \quad \phi \to a\sigma\,, \quad \cos\theta \to -p/a\,, \tag{B.18}$$

brings the metric and the KY tensor into their canonical forms (A.13) and (A.15) described in Appendix A. In particular, we have

$$\boldsymbol{f} = r\boldsymbol{d}p\wedge(\boldsymbol{d}\tau + r^2\boldsymbol{d}\sigma) + p\boldsymbol{d}r\wedge(\boldsymbol{d}\tau - p^2\boldsymbol{d}\sigma)\,. \tag{B.19}$$

In this form $\boldsymbol{f}$ is completely 'universal'; it neither depends on λ nor on a. In fact, it is the KY tensor for the general Carter's solution described in Appendix A.2.3. This inspired the author of this thesis to search for a convenient coordinate transformation which would transform the known higher-dimensional PCKY tensor (B.5) or (B.15) into its 'universal' form. This is how the PCKY tensor (4.13) for the Kerr-NUT-(A)dS metric (4.1) was discovered.

B.3 PCKY tensor in the MP spacetime: the proof

In this section we prove that the tensor $\boldsymbol{h}$, given by (B.5), is indeed the PCKY tensor for the MP metric (B.1). Namely, we prove that it obeys the closed CKY equations (3.7). We proceed in two steps. First, we transform the metric and $\boldsymbol{h}$ into the Kerr–Schild coordinates. Second, following [Myers & Perry, 1986], we introduce the convenient orthonormal basis in which the verification of (3.7) is, although elaborate, rather straightforward.

B.3.1 Kerr–Schild form

Let us start with the transformation

$$dt = d\tau - \frac{2M}{V-2M}\,dr\,, \quad d\phi_i = d\varphi_i + \frac{V}{V-2M}\frac{a_i}{r^2+a_i^2}\,dr\,, \tag{B.20}$$

which transforms the metric element (B.1) into the 'Edington-like' form. We further introduce the Kerr–Schild coordinates

$$x_i \equiv \mu_i\sqrt{r^2+a_i^2}\cos\bigl(\varphi_i-\arctan\frac{a_i}{r}\bigr)\,,\ y_i \equiv \mu_i\sqrt{r^2+a_i^2}\sin\bigl(\varphi_i-\arctan\frac{a_i}{r}\bigr)\,,\ z\equiv\epsilon\nu r\,. \tag{B.21}$$

Here, index $i=1,\dots,m$, and the last coordinate z is introduced only in an even number of spacetime dimensions. The inverse transformation reads

$$\mu_i^2 = \frac{x_i^2+y_i^2}{r^2+a_i^2}\,, \quad \varphi_i = \arctan\frac{a_i}{r}+\arctan\frac{y_i}{x_i}\,, \quad \epsilon\nu = \frac{z}{r}\,. \tag{B.22}$$

These relations imply

$$\mu_i d\mu_i = \frac{x_i dx_i + y_i dy_i}{r^2+a_i^2} - \frac{(x_i^2+y_i^2)r\,dr}{(r^2+a_i^2)^2}\,, \quad d\varphi_i = \frac{x_i dy_i - y_i dx_i}{x_i^2+y_i^2} - \frac{a_i dr}{r^2+a_i^2}\,. \tag{B.23}$$

The constraint (B.3) reads[1]

$$\sum_{i=1}^{m}\frac{x_i^2+y_i^2}{r^2+a_i^2} + \epsilon\frac{z^2}{r^2} = 1\,. \tag{B.24}$$

Differentiating this expression we find

$$\partial_{x_i} r = \frac{r x_i}{F(r^2+a_i^2)}\,, \quad \partial_{y_i} r = \frac{r y_i}{F(r^2+a_i^2)}\,, \quad \partial_z r = \frac{\epsilon z}{Fr}\,, \tag{B.25}$$

$$F \equiv \frac{U}{V} = 1 - \sum_{i=1}^{m}\frac{a_i^2(x_i^2+y_i^2)}{(r^2+a_i^2)^2} = r^2\sum_{i=1}^{m}\frac{x_i^2+y_i^2}{(r^2+a_i^2)^2} + \epsilon\frac{z^2}{r^2}\,. \tag{B.26}$$

Therefore,

$$dr = \frac{r}{F}\sum_{i=1}^{m}\frac{x_i dx_i + y_i dy_i}{r^2+a_i^2} + \epsilon\frac{z\,dz}{Fr}\,. \tag{B.27}$$

[1]The 'Boyer–Lindquist' form (B.1) of the MP metric is in the original paper [Myers & Perry, 1986] derived from the Kerr–Schild ansatz (B.28). We are now going backwards. In the original derivation the constraint (B.3) is understood as a defining equation for the coordinate r. It also expresses the fact that the vector $\boldsymbol{k}$, (B.30), is null.

Using these relations we find that the metric takes the 'Kerr–Schild' form

$$g = \eta + hkk\,, \tag{B.28}$$

where

$$\eta = -d\tau^2 + \sum_{i=1}^{m}(dx_i^2 + dy_i^2)\,, \quad h = \frac{2M}{U}\,, \tag{B.29}$$

$$k = d\tau + \sum_{i=1}^{m} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2} + \epsilon \frac{z dz}{r}\,. \tag{B.30}$$

The PCKY tensor (B.5) reads

$$h = \sum_{i=1}^{m}\big[(x_i dx_i + y_i dy_i)\wedge d\tau + a_i dx_i \wedge dy_i\big] + \epsilon z dz \wedge d\tau\,. \tag{B.31}$$

The last expression is particularly useful for obtaining the flat space limit of this tensor [cf. Eq. (A.3)].

B.3.2 Basis forms

So far, we have covered both cases of the odd(even)-dimensional spacetime simultaneously. It is now natural to split the subsequent calculations. Here, we concentrate on the even-dimensional case (the proof in an odd number of dimensions is slightly modified but analogous).

Let us introduce the basis of 1-forms $E^a = e^a_\mu dx^\mu$,

$$E^u \equiv du + A^k dx^k + \frac{1}{2}A^2 dv\,, \quad E^v \equiv dv - HE^u\,, \quad E^k \equiv dx^k + A^k dv\,, \tag{B.32}$$

$$dv = E^v + HE^u\,, \quad du = E^u + \frac{1}{2}A^2 dv - A^k E^k\,, \quad dx^k = E^k - A^k dv\,, \tag{B.33}$$

in which the (even-dimensional) metric (B.28) takes the form

$$g = -2E^{(u}E^{v)} + E^k E^k\,. \tag{B.34}$$

Here, we have defined $x^k \equiv (x^i, y^j)\,,\ A^k \equiv (qB^i, qC^j)\,,\ E^k \equiv (E^i_x, E^j_y)\,,$

$$B^i \equiv \frac{rx^i - a_i y^i}{r^2 + a_i^2}\,, \quad C^i \equiv \frac{ry^i + a_i x^i}{r^2 + a_i^2}\,, \quad q \equiv \frac{\sqrt{2}r}{r+z}\,, \quad H \equiv \frac{M}{2Uq^2}\,. \tag{B.35}$$

Indices i, j run over $1, \ldots, m$, whereas indices k, l, o through $1, \ldots, 2m$; due Einstein summation conventions are used. Also, $a_{i_x} = a_{i_y} = a_i$ whenever $i_x = i_y$. Using the fact that

$$\begin{aligned}
\frac{1}{2}A^2 &\equiv \frac{1}{2}A^kA^k = \frac{q^2}{2}(B^iB^i + C^iC^i) = \frac{q^2}{2}\frac{x_i^2+y_i^2}{r^2+a_i^2} = \frac{r-z}{r+z} = \sqrt{2}q - 1\,, && \text{(B.36)}\\
C^ia_i &= x_i - rB^i\,, \quad B^ia_i = rC^i - y_i\,, && \text{(B.37)}\\
X &\equiv x^kA^k = qr\frac{x_i^2+y_i^2}{r^2+a_i^2} = \frac{q}{r}(r^2-z^2) = \sqrt{2}(r-z)\,, && \text{(B.38)}
\end{aligned}$$

we find

$$\begin{aligned}
\boldsymbol{dv}\wedge\boldsymbol{du} &= -\boldsymbol{E}^u\wedge\boldsymbol{E}^v - \boldsymbol{dv}\wedge A^k\boldsymbol{E}^k\,,\\
a_i\boldsymbol{dx}_i\wedge\boldsymbol{dy}_i &= a_i\boldsymbol{E}^i_x\wedge\boldsymbol{E}^i_y + \boldsymbol{dv}\wedge(qx^k\boldsymbol{E}^k - rA^k\boldsymbol{E}^k)\,,\\
x^k\boldsymbol{dx}^k\wedge(\boldsymbol{du}+\boldsymbol{dv}) &= X\boldsymbol{E}^u\wedge\boldsymbol{E}^v + x^k\boldsymbol{E}^k\wedge(\boldsymbol{E}^u - A^l\boldsymbol{E}^l) + \boldsymbol{dv}\wedge(XA^l\boldsymbol{E}^l - \sqrt{2}qx^l\boldsymbol{E}^l)\,.
\end{aligned}$$

Plugging these expressions into (B.31), we find the following form of the PCKY tensor in the chosen basis:

$$\boldsymbol{h} = r\boldsymbol{E}^u\wedge\boldsymbol{E}^v + \frac{x^k}{\sqrt{2}}\boldsymbol{E}^k\wedge\boldsymbol{E}^u + (a_i\delta^{ki_x}\delta^{li_y} - \frac{x^kA^l}{\sqrt{2}})\boldsymbol{E}^k\wedge\boldsymbol{E}^l\,. \tag{B.39}$$

Let us conclude this subsection with introducing the dual basis operators $D_a = e^\mu_a\partial_\mu$,

$$\begin{aligned}
D &\equiv D_v = \partial_v - A^k\partial_k + \frac{1}{2}A^2\partial_u\,, \quad \Delta \equiv D_u = \partial_u + HD\,, \quad \delta^k \equiv D_k = \partial_k - A^k\partial_u\,,\\
\partial_u &= \Delta - HD, \quad \partial_v = D + \frac{1}{2}A^2\partial_u + A^k\delta^k, \quad \partial_k = \delta^k + A^k\partial_u\,. && \text{(B.40)}
\end{aligned}$$

B.3.3 Connection coefficients for the MP metric

The connection coefficients Γ_{abc} (antisymmetric in the first two indices) are obtained from relations

$$\boldsymbol{dE}^a = -\frac{1}{2}D^a{}_{bc}\boldsymbol{E}^b\wedge\boldsymbol{E}^c\,, \quad \Gamma_{abc} = \frac{1}{2}(D_{cab} + D_{bac} - D_{abc})\,. \tag{B.41}$$

When calculating these coefficients, we shall use the fact that the exterior derivative $\boldsymbol{d}$ can be expressed as

$$\boldsymbol{d} = \boldsymbol{E}^u\Delta + \boldsymbol{E}^vD + \boldsymbol{E}^l\delta^l\,. \tag{B.42}$$

Calculations are aided by the fact (proved in the next subsection) that

$$DA^k = 0\,. \tag{B.43}$$

So we find

$$\begin{aligned}
\boldsymbol{dE}^u &= \boldsymbol{d}A^k\wedge\boldsymbol{E}^k = -\Delta A^k\boldsymbol{E}^k\wedge\boldsymbol{E}^u + \delta^l A^k\boldsymbol{E}^l\wedge\boldsymbol{E}^k\,,\\
\boldsymbol{dE}^v &= DH\boldsymbol{E}^u\wedge\boldsymbol{E}^v + (H\Delta A^k - \delta^k H)\boldsymbol{E}^k\wedge\boldsymbol{E}^u - H\delta^l A^k\boldsymbol{E}^l\wedge\boldsymbol{E}^k\,,\\
\boldsymbol{dE}^l &= \Delta A^l\boldsymbol{E}^u\wedge\boldsymbol{E}^v + \delta^k A^l\boldsymbol{E}^k\wedge\boldsymbol{E}^v + H\delta^k A^l\boldsymbol{E}^k\wedge\boldsymbol{E}^u\,.
\end{aligned} \tag{B.44}$$

Comparing with (B.41) we identify

$$\begin{aligned}
D^u{}_{ku} &= \Delta A^k, \quad D^u{}_{lk} = -F^{lk}, \quad D^v{}_{uv} = -DH, \quad D^v{}_{ku} = \delta^k H - H\Delta A^k,\\
D^v{}_{lk} &= HF^{lk}, \quad D^l{}_{uv} = -\Delta A^l, \quad D^l{}_{kv} = -\delta^k A^l, \quad D^l{}_{ku} = -H\delta^k A^l,
\end{aligned} \tag{B.45}$$

where we introduced

$$F^{lk} \equiv \delta^l A^k - \delta^k A^l = -F^{kl} = F_{lk}\,. \tag{B.46}$$

This leads to the following coefficients:

$$\begin{aligned}
\Gamma^v{}_{vu} &= -DH\,, \;\; \Gamma^v{}_{ku} = H\Delta A^k - \delta^k H\,, \;\; \Gamma^v{}_{kl} = -H\delta^k A^l\,, \;\; \Gamma^u{}_{ku} = -\Delta A^k\,,\\
\Gamma^u{}_{kl} &= -\delta^l A^k\,, \;\; \Gamma^u{}_{uu} = DH\,, \;\; \Gamma^k{}_{uu} = H\Delta A^k - \delta^k H\,, \;\; \Gamma^k{}_{ul} = -H\delta^k A^l\,,\\
\Gamma^k{}_{vu} &= -\Delta A^k\,, \;\; \Gamma^k{}_{vl} = -\delta^l A^k\,, \;\; \Gamma^l{}_{ku} = -HF_{lk}\,.
\end{aligned} \tag{B.47}$$

B.3.4 Covariant derivatives of the PCKY tensor

In order to verify the closed CKY equation (3.7),

$$\nabla_c h_{ab} = 2g_{c[a}\xi_{b]}, \quad \xi_b = \frac{1}{D-1}\nabla_d h^d{}_b\,, \tag{B.48}$$

we need to calculate the covariant derivatives of $\boldsymbol{h}$,

$$h_{ab;\,c} = D_c h_{ab} - \Gamma^d{}_{ac}h_{db} - \Gamma^d{}_{bc}h_{ad}\,, \tag{B.49}$$

where [cf. Eq. (B.39)]

$$h_{uv} = r\,, \quad h_{ku} = \frac{1}{\sqrt{2}}x^k\,, \quad h_{kl} = -\frac{2}{\sqrt{2}}x^{[k}A^{l]} + 2a_i\delta^{i_x[k}\delta^{l]i_y}\,. \tag{B.50}$$

The most lengthy parts of the calculation (the details of which we moved to the next subsection) are summarized by the following lemmas:

Lemma 1 ('Orthogonality relations').

$$\text{a)}\quad r\delta^l A^k - h_{ok}\delta^l A^o = q\delta^{kl}\,, \quad \text{b)}\quad r\delta^k A^l - h_{ko}\delta^o A^l = q\delta^{kl}\,. \tag{B.51}$$

Lemma 2.

$$r\Delta A^k - h_{lk}\Delta A^l = -\frac{1}{\sqrt{2}}A^k\,. \tag{B.52}$$

Lemma 3.

$$h_{ku;u} = 0\,. \tag{B.53}$$

Summing a) and b) in (B.51), we immediately get

$$h_{ko}F^{lo} = 2q\delta^{kl} - 2r\delta^{(l}A^{k)}\,, \quad h_{[k|o|}F_{l]o} = 0\,. \tag{B.54}$$

We also need the following relations:

$$\Delta h_{kl} + \frac{2}{\sqrt{2}}x^{[k}\Delta A^{l]} = 0\,, \;\; Dh_{kl} = 0\,, \;\; \delta^o h_{kl} + \frac{2}{\sqrt{2}}x^{[k}\delta^{|o|}A^{l]} = \frac{2}{\sqrt{2}}\delta^{o[l}A^{k]}\,, \tag{B.55}$$

which follow from previous identities.

Applying all these lemmas and identities, we find that the only nontrivial covariant derivatives of $\boldsymbol{h}$ are

$$\begin{aligned} h_{uv;u} &= \frac{1}{\sqrt{2}} - qH\,, \;\; h_{uv;v} = -q\,, \;\; h_{ku;v} = -\frac{1}{\sqrt{2}}A^k\,, \;\; h_{ku;l} = \delta^{lk}(\frac{1}{\sqrt{2}} - qH)\,, \\ h_{kl;o} &= \frac{2}{\sqrt{2}}\delta^{o[l}A^{k]}\,, \;\; h_{vk;u} = \frac{1}{\sqrt{2}}A^k\,, \;\; h_{kv;l} = q\delta^{kl}\,. \end{aligned} \tag{B.56}$$

Specifically, we find

$$\boldsymbol{\xi}^\flat = -\frac{1}{D-1}\boldsymbol{\delta h} = \Big(\frac{1}{\sqrt{2}} - qH\Big)\boldsymbol{E}^u + q\boldsymbol{E}^v - \frac{1}{\sqrt{2}}A^k\boldsymbol{E}^k\,. \tag{B.57}$$

It is now straightforward to verify that Eq. (B.48) holds. ♡

B.3.5 Proofs of lemmas

In this subsection we gather the proofs of the above statements. Let us denote $c_k \equiv 1/(r^2 + a_k^2)$. For example, the constraint (B.24), and the definition of F, (B.26), are

$$c_k x_k^2 + \frac{z^2}{r^2} = 1\,, \quad F = 1 - a_k^2 x_k^2 c_k^2 = r^2 x_k^2 c_k^2 + \frac{z^2}{r^2}\,. \tag{B.58}$$

We also find

$$c_k x^k A_k = qrx_k^2 c_k^2 = \frac{q}{r}\big(F - \frac{z^2}{r^2}\big) = \frac{r}{q}\, c_k A_k^2\,, \quad a_k^2 x_k A_k c_k^2 = qra_k^2 x_k c_k^3\,. \qquad \text{(B.59)}$$

Let us first prove the relation (B.43). Using Eqs. (B.25), (B.36), (B.37), and (B.59), we find

$$\begin{aligned} Dr &= \partial_v r - A^k\partial_k r + \frac{1}{2}A^2\partial_u r = -\frac{z}{\sqrt{2}Fr} - \frac{rc_k x^k A^k}{F} + \frac{(\sqrt{2}q-1)z}{\sqrt{2}Fr} = -q\,, \\ Dz &= q - \sqrt{2}\,, \quad Dq = 0\,, \quad Dx^k = -A^k\,, \quad Dc_k = 2rqc_k^2\,. \end{aligned} \qquad \text{(B.60)}$$

Therefore one has [and similarly for $D(qC^i) = 0$]

$$D(qB^i) = qD(B^i) = q^2\Big[-x^i c_i - rc_i B^i + a_i c_i C^i + 2rc_i^2(rx^i - a_i y^i)\Big] = 0\,, \qquad \text{(B.61)}$$

where (B.37) and the definition of B^i were used. ♡

Proof of Lemma 1. Let us decompose h_{kl} into its constant part $\hat{h}_{kl}$ and the 'rest'

$$h_{kl} = \hat{h}_{kl} + \tilde{h}_{kl}\,, \quad \hat{h}_{kl} \equiv 2a_i\delta^{i_x[k}\delta^{l]i_y}\,, \quad \tilde{h}_{kl} \equiv -\frac{2}{\sqrt{2}}\,x^{[k}A^{l]}\,. \qquad \text{(B.62)}$$

We first notice that [see (B.36)]

$$\begin{aligned} \frac{1}{2}\,\delta^l A^2 &= \sqrt{2}\,\delta^l q\,, \quad \delta^l x^k = \delta^{lk}\,, \quad \delta^l z = -\frac{1}{\sqrt{2}}A^l\,, & \text{(B.63)} \\ x^k\delta^l A^k &= \delta^l X - A^l = \sqrt{2}\delta^l r\,, \quad \tilde{h}_{ok}\delta^l A^o = -A^k\delta^l r + x^k\delta^l q\,. & \text{(B.64)} \end{aligned}$$

Furthermore, using (B.37), we find

$$\begin{aligned} \hat{h}_{ok}\delta^l A^o &= \hat{h}_{i_x k}\delta^l A^{i_x} + \hat{h}_{i_y k}\delta^l A^{i_y} = \delta^l(qB^i a_i)\delta^{i_y k} - \delta^l(qC^i a_i)\delta^{i_x k} \\ &= \delta^l(qrC^i - qy_i)\delta^{i_y k} - \delta^l(qx_i - qrB^i)\delta^{i_x k} \\ &= A^k\delta^l r - x^k\delta^l q + r\delta^l A^k - q\delta^{lk}. \end{aligned} \qquad \text{(B.65)}$$

Combining (B.65) with (B.64) completes the proof of (B.51) a).

To prove (B.51) b), we successively find

$$\delta^l r = \frac{rx^l c_l}{F} - \frac{zA^l}{\sqrt{2}rF}\,,\ \ \delta^l q = \frac{q^2A^l}{2r} + \frac{zq^2\delta^l r}{\sqrt{2}r^2} = \frac{q^2 zx^l c_l}{\sqrt{2}rF} + \frac{q^2A^l}{2r}\Big(1-\frac{z^2}{Fr^2}\Big)\,, \tag{B.66}$$

$$\begin{aligned}\delta^k A^l =& A^kA^l\Big(\frac{q}{2r} - \frac{qz^2}{2r^3F} + \frac{\sqrt{2}zc_l}{F}\Big) + A^lx^k\Big(\frac{qzc_k}{\sqrt{2}rF} - \frac{2r^2c_kc_l}{F}\Big) - A^kx^l\frac{qzc_l}{\sqrt{2}rF}\\ &+x^lx^k\frac{qrc_lc_k}{F} + \delta^{kl}qrc_k + 2qa_ic_i\delta^{i_x[k}\delta^{l]i_y}\,.\end{aligned} \tag{B.67}$$

Moreover, using identities (B.59), we find

$$\begin{aligned}&rA^k - h_{kl}A^l = A^k(r+z) - \frac{qz}{r}x^k\,,\ \ rx^kc_k - h_{kl}x^lc_l = \frac{zA^k}{rq} + \frac{qx^k}{\sqrt{2}r}\Big(F-\frac{z^2}{r^2}\Big)\,,\\ &a_ic_i(r\delta^{i_x[k}\delta^{l]i_y} - h_{ko}\delta^{i_x[o}\delta^{l]i_y}) = -\frac{A^kA^l}{\sqrt{2}q} + \frac{rc_lx^lA^k}{\sqrt{2}} + \frac{rc_lx^kA^l}{\sqrt{2}} - \frac{qc_lx^lx^k}{\sqrt{2}} + a_l^2c_l\delta^{lk}\,.\end{aligned} \tag{B.68}$$

Using these relations one can express, $r\delta^kA^l - h_{ko}\delta^oA^l$, in terms of 'independent' coefficients $A^kA^l, A^lx^k, A^kx^l, x^lx^k, \delta^{lk}$. Finally, using

$$\frac{q}{2} + \frac{zq}{2r} = \frac{1}{\sqrt{2}}\,, \tag{B.69}$$

and the definition of q one can verify that each term, but $q\delta^{lk}$, vanishes which completes the proof of (B.51) b). ♡

Proof of Lemma 2. Let us decompose h_{kl} as in (B.62). Then we find

$$\frac{1}{2}\Delta A^2 = \sqrt{2}\Delta q\,,\ \ \Delta x^k = -A^kH\,,\ \ \Delta z = \frac{1}{\sqrt{2}} + H(q-\sqrt{2})\,, \tag{B.70}$$

$$x^k\Delta A^k = \sqrt{2}(\Delta r + qH) - 1\,,\ \ \tilde{h}_{lk}\Delta A^l = x^k\Delta q - A^k\Delta r - qHA^k + \frac{A_k}{\sqrt{2}}\,. \tag{B.71}$$

With the help of (B.37) we find

$$\hat{h}_{lk}\Delta A^l = -x^k\Delta q + A^k\Delta r + qHA^k + r\Delta A^k\,. \tag{B.72}$$

Combination of (B.71) and (B.72), gives (B.52). ♡

Proof of Lemma 3. This proof is the most difficult part of the whole calculation. We sketch only the main steps. Using (B.52) and (B.70), we find

$$h_{ku;\,u} = 2rH\Delta A^k - r\delta^kH - \frac{1}{\sqrt{2}}x^kDH + h_{kl}\delta^lH + \frac{1}{\sqrt{2}}Hx^lF^{lk}\,. \tag{B.73}$$

Our task is now to show that this expression is equal to zero. First of all, using the following identities:

$$\begin{aligned} x^l\delta^l r &= \frac{r-z}{F}\,, \quad A^l\delta^l r = n - \frac{nz}{rF}\,, \quad x^l\delta^l q = \frac{q^2X}{2r}\Big(1+\frac{z}{rF}\Big)\,, \\ x^l\delta^l A^k &= \frac{r-z}{F}\,qx^k c_k + A^k\Big[\sqrt{2}q + \frac{X}{2F}\big(\frac{zq}{r^2} - 2\sqrt{2}rc_k\big)\Big]\,, \end{aligned} \tag{B.74}$$

we find

$$\frac{1}{\sqrt{2}}\,x^l F^{lk} = \frac{x^l\delta^l A^k}{\sqrt{2}} - \delta^k r = -\frac{\sqrt{2}qzx^kc_k}{F} + A^k\Big(q + \frac{qz}{rF} - \frac{Xrc_k}{F}\Big)\,. \tag{B.75}$$

Next, from (B.25) it follows that

$$2r\Delta A^k = A^k\Big(-q + \frac{qz^2}{Fr^2} - \frac{2\sqrt{2}zrc_k}{F}\Big) + \frac{x^kc_k\sqrt{2}zq}{F}\,. \tag{B.76}$$

Contracting (B.67), we find

$$\delta^l A^l = 2qr\Big(\sum_{i=1}^m c_i + \frac{N_3}{F}\Big)\,, \quad N_3 \equiv a_k^2 x_k^2 c_k^3 = \frac{a_k^2 x_k A_k c_k^2}{qr}\,. \tag{B.77}$$

So we have

$$\begin{aligned} DF &= -2qrN_3\,, \quad D\lg V = \frac{q}{r} - 2qr\sum_{i=1}^m c_i\,, \\ \frac{DH}{H} &= \frac{1}{H}D\Big(\frac{M}{2q^2FV}\Big) = -D\lg(FV) = -\frac{q}{r} + 2qr\Big(\sum_{i=1}^m c_i + \frac{N_3}{F}\Big)\,, \end{aligned} \tag{B.78}$$

and therefore, using (B.77), we get

$$\frac{x^k}{\sqrt{2}}\frac{DH}{H} = \frac{x^k}{\sqrt{2}}\Big(\delta^l A^l - \frac{q}{r}\Big)\,. \tag{B.79}$$

Similarly, we obtain

$$\begin{aligned} \delta^k\lg q^2 &= \frac{qA^k}{r} + \frac{\sqrt{2}zq\delta^k r}{r^2}\,, \quad \delta^k\lg V = \frac{\delta^k r}{r}\big(2r^2\sum_{i=1}^m c_i - 1\big)\,, \\ \delta^k F &= -2a_k^2x^kc_k^2 + 4r\delta^k rN_3\,, \\ \frac{\delta^k H}{H} &= -\delta^k\lg(q^2FV) = \frac{2a_k^2x^kc_k^2}{F} - \frac{qA^k}{r} + \frac{\delta^k r}{r}\Big(1 - \frac{\sqrt{2}zq}{r} - \frac{2r^2N_3}{F} - \frac{r\delta^l A^l}{q}\Big)\,. \end{aligned} \tag{B.80}$$

Finally, using (B.68), and

$$r\delta^k r - h_{kl}\delta^l r = \frac{qx^k}{\sqrt{2}}\,, \quad ra_k^2 x^k c_k^2 - h_{kl}a_l^2 x^l c_l^2 = A^k\Big(\frac{F-1}{\sqrt{2}} + \frac{a_k^2 c_k}{q}\Big) + \frac{qrN_3 x^k}{\sqrt{2}}\,, \tag{B.81}$$

one has

$$\frac{1}{H}\big(r\delta^k H - h_{kl}\delta^l H\big) = A^k\Big(\frac{2a_k^2 c_k}{Fq} - \frac{\sqrt{2}}{F}\Big) + x^k\Big(\frac{q}{\sqrt{2}r} - \frac{\delta^l A^l}{\sqrt{2}}\Big)\,. \tag{B.82}$$

Combining the results (B.75), (B.76), (B.79), and (B.82), we find that

$$h_{ku;u} = \frac{1}{\sqrt{2}}\,x^l F^{lk} + 2r\Delta A^k - \frac{x^k}{\sqrt{2}}\frac{DH}{H} - \frac{1}{H}(r\delta^k H - h_{kl}\delta^l H) = 0\,. \quad \heartsuit \tag{B.83}$$

Appendix C

Miscellaneous results

In this appendix we gather various results concerning the PCKY tensor and other related topics. Namely, we prove that the eigenvectors of the PCKY tensor coincide with the principal null directions, we review the algebraic integrability conditions for the existence of a CKY 2-form and relate them with the algebraic type of the spacetime, we review some algebraic identities used throughout the text and comment on the separability of the first order differential equations in Kerr-NUT-(A)dS spacetimes, we outline the unsuccessful attempt to generalize these metrics to the Plebański–Demiański solution in higher dimensions, and finally briefly comment on the degeneracy of the eigenvalues of the operator $\boldsymbol{F}$ used in Chapter 9.

C.1 Principal null directions as eigenvectors of the PCKY tensor

Lemma. *Eigenvectors of the PCKY tensor* $\boldsymbol{h}$ *coincide with the 'principal null directions'. That is, the solution of the eigenvalue problem*

$$\boldsymbol{l} \lrcorner \boldsymbol{h} = \lambda \boldsymbol{l}^\flat \,, \tag{C.1}$$

is a geodesic WAND (Weyl aligned null direction).

Proof: Contracting (C.1) with $\boldsymbol{l}$ immediately implies that $\boldsymbol{l}$ is null. To prove that it is geodesic let us introduce the complex null Darboux basis for $\boldsymbol{h}$,

$$\boldsymbol{h} = \lambda\, \boldsymbol{l}^\flat \wedge \boldsymbol{n}^\flat + \sum_i \nu_i \boldsymbol{m}_i{}^\flat \wedge \bar{\boldsymbol{m}}_i{}^\flat \,, \tag{C.2}$$

with the only non-vanishing scalar products

$$(\boldsymbol{l},\boldsymbol{n})=-1\,,\quad(\boldsymbol{m}_i,\bar{\boldsymbol{m}}_i)=1\,.\tag{C.3}$$

Let us denote by $\dot{\boldsymbol{T}}\equiv\nabla_l\boldsymbol{T}$, and in particular $\boldsymbol{z}\equiv\dot{\boldsymbol{l}}$. Using the PCKY equation (3.3) and (C.1) we find

$$\nabla_l(\boldsymbol{l}\lrcorner\boldsymbol{h})=\boldsymbol{z}\lrcorner\boldsymbol{h}-\boldsymbol{l}^\flat(\boldsymbol{l}\lrcorner\boldsymbol{\xi})=\lambda\boldsymbol{z}^\flat+\dot{\lambda}\boldsymbol{l}^\flat\,.$$

Re-arranging the last equation we get

$$\boldsymbol{z}\lrcorner\boldsymbol{h}=\boldsymbol{l}^\flat\big(\dot{\lambda}+\boldsymbol{l}\lrcorner\boldsymbol{\xi}\big)+\lambda\boldsymbol{z}^\flat\,.\tag{C.4}$$

On the left-hand-side we plug the expression (C.2) for $\boldsymbol{h}$, and contract both sides with $\boldsymbol{n}$, to obtain

$$\boldsymbol{n}\lrcorner(\boldsymbol{z}\lrcorner\boldsymbol{h})=\lambda(\boldsymbol{n},\boldsymbol{z})=\lambda(\boldsymbol{n},\boldsymbol{z})-\big(\dot{\lambda}+\boldsymbol{l}\lrcorner\boldsymbol{\xi}\big)\,.$$

From here it follows that $\dot{\lambda}=-\boldsymbol{l}\lrcorner\boldsymbol{\xi}$. Plugging this expression into (C.4) we find that $\boldsymbol{z}\lrcorner\boldsymbol{h}=\lambda\boldsymbol{z}^\flat$. Comparing with (C.1) we conclude that

$$\boldsymbol{z}=\nabla_l\boldsymbol{l}=\alpha_l\boldsymbol{l}\,.\tag{C.5}$$

This means that $\boldsymbol{l}$ is a (non-affine parametrized) null geodesic. One can restore the affine parametrization by performing a proper boost in $\{\boldsymbol{l},\boldsymbol{n}\}$ 2-plane, so that afterwards

$$\nabla_l\boldsymbol{l}=0\,.\tag{C.6}$$

Similarly, one can consider null geodesics in other directions, such as

$$\boldsymbol{m}_i\lrcorner\boldsymbol{h}=-\nu_i\boldsymbol{m}_i^\flat\implies\nabla_{m_i}\boldsymbol{m}_i=\alpha_i\boldsymbol{m}_i\,.\tag{C.7}$$

It remains to prove that $\boldsymbol{l}$ is WAND. The most simple way to show this, is to use the explicit form of the eigenvector $\boldsymbol{l}$ in the most general spacetime admitting the PCKY tensor (see Chapter 7), and refer to the paper [Hamamoto *et al.*, 2007] where it was shown that such a vector is WAND. ♡

C.2 Integrability conditions for a CKY 2-form

In this section, following closely [Tachibana, 1969], we repeat the derivation of the integrability conditions for the existence of a CKY 2-form $\boldsymbol{k}$, (2.10), written as the algebraic relations between components of k_{cd} and the curvature tensor.

We use the conventions of [Wald, 1984]. For example, we have

$$\begin{aligned}(\nabla_a\nabla_b-\nabla_b\nabla_a)k_{cd} &= R_{abc}{}^e k_{ed}+R_{abd}{}^e k_{ce}\,, &\text{(C.8)}\\ R_{ac}=R_{ca} &= R_{abc}{}^b = R^b{}_{abc}\,, &\text{(C.9)}\end{aligned}$$

and the following definition of the Weyl tensor C_{abcd}:

$$R_{abcd}=C_{abcd}+\frac{2}{D-2}\big(g_{a[c}R_{d]b}-g_{b[c}R_{d]a}\big)-\frac{2}{(D-1)(D-2)}Rg_{a[c}g_{d]b}\,. \tag{C.10}$$

The defining equation for a CKY 2-form reads

$$\nabla_{(c}k_{a)b}=g_{ca}\xi_b-\xi_{(c}g_{a)b}\,,\quad \xi_b=\frac{1}{D-1}\nabla_d k^d{}_b\,, \tag{C.11}$$

or equivalently

$$\nabla_b k_{cd}=\nabla_{[b}k_{cd]}+2g_{b[c}\xi_{d]}\,. \tag{C.12}$$

What conditions follow from these equations? Differentiating Eq. (C.11) and shuffling the indices we have

$$\begin{aligned}\nabla_a\big(\nabla_{(b}k_{c)d}\big) &= g_{bc}\rho_{ad}-\rho_{a(b}g_{c)d}\,, &\text{(C.13)}\\ \nabla_b\big(\nabla_{(a}k_{c)d}\big) &= g_{ac}\rho_{bd}-\rho_{b(a}g_{c)d}\,, &\text{(C.14)}\\ \nabla_c\big(\nabla_{(a}k_{b)d}\big) &= g_{ab}\rho_{cd}-\rho_{c(a}g_{b)d}\,. &\text{(C.15)}\end{aligned}$$

Here and later we use the abbreviations

$$\rho_{ab}\equiv\nabla_a\xi_b\,,\quad S_{ab}\equiv\rho_{ab}+\rho_{ba}\,,\quad A_{ab}\equiv\rho_{ab}-\rho_{ba}\,.$$

Calculating (C.13)+(C.14)−(C.15), and using the Bianchi identity $R_{[abc]}{}^e=0$, we obtain

$$\begin{aligned}2\nabla_a\nabla_b k_{cd} &= k_{eb}R_{acd}{}^e+k_{ec}R_{bad}{}^e+k_{ea}R_{bcd}{}^e-g_{cd}S_{ab}+g_{bd}A_{ca}+g_{ad}A_{cb}\\ &\quad+2\big(k_{ed}R_{cba}{}^e+g_{bc}\rho_{ad}+g_{ac}\rho_{bd}-g_{ab}\rho_{cd}\big) &\text{(C.16)}\end{aligned}$$

Similarly, we have

$$\begin{aligned}2\nabla_a\nabla_c k_{db} &= k_{ec}R_{adb}{}^e+k_{ed}R_{cab}{}^e+k_{ea}R_{cdb}{}^e-g_{db}S_{ac}+g_{cb}A_{da}+g_{ab}A_{dc}\\ &\quad+2\big(k_{eb}R_{dca}{}^e+g_{cd}\rho_{ab}+g_{ad}\rho_{cb}-g_{ac}\rho_{db}\big)\,, &\text{(C.17)}\\ 2\nabla_a\nabla_d k_{bc} &= k_{ed}R_{abc}{}^e+k_{eb}R_{dac}{}^e+k_{ea}R_{dbc}{}^e-g_{bc}S_{ad}+g_{dc}A_{ba}+g_{ac}A_{bd}\\ &\quad+2\big(k_{ec}R_{bda}{}^e+g_{db}\rho_{ac}+g_{ab}\rho_{dc}-g_{ad}\rho_{bc}\big)\,. &\text{(C.18)}\end{aligned}$$

From here it follows that

$$\nabla_a\big(\nabla_{[b}k_{cd]}\big) = \nabla_a\nabla_b k_{cd} - 2\rho_{a[d}g_{c]b}\,. \tag{C.19}$$

Adding equations (C.16), (C.17), and (C.18), and using equation (C.19), we have

$$\begin{aligned} 2\nabla_a\nabla_b k_{cd} = \; & k_{ed}R_{cba}{}^{e} + k_{eb}R_{dca}{}^{e} + k_{ec}R_{bda}{}^{e} + 2g_{cb}\rho_{ad} \\ - \; & 2g_{db}\rho_{ac} + g_{ac}A_{bd} + g_{ab}A_{dc} + g_{ad}A_{cb}\,. \end{aligned}$$

Subtracting (C.16) from this equation we get

$$k_{ea}R_{bcd}{}^{e} + k_{eb}R_{adc}{}^{e} + k_{ec}R_{dab}{}^{e} + k_{ed}R_{cba}{}^{e} + g_{db}S_{ac} + g_{ac}S_{db} - g_{ab}S_{dc} - g_{cd}S_{ab} = 0\,. \tag{C.20}$$

Contracting indices a and b in this equation and using $k^{ea}(R_{adce} + R_{acde}) = 0$, we have

$$2k_{e(d}R_{c)}^{e} + (D-2)S_{dc} + S_a^a g_{cd} = 0\,. \tag{C.21}$$

Contracting the last equation once more we obtain $S_a^a = 0$. So we get[1]

$$\frac{1}{2}S_a^a = \nabla_a\xi^a = 0\,, \quad \frac{1}{2}S_{dc} = \nabla_{(d}\xi_{c)} = \frac{1}{D-2}R_{e(c}k_{d)}{}^{e}\,. \tag{C.22}$$

Denoting by

$$T_{bca}{}^{e} \equiv R_{bca}{}^{e} + \frac{2}{D-2}R^{e}_{[b}g_{c]a}\,, \tag{C.23}$$

and plugging (C.22) into (C.20) we finally obtain the following algebraic conditions for the existence of a rank-2 CKY tensor k_{ab}:

$$\big(T_{bcd}{}^{e}\delta_a^f + T_{adc}{}^{e}\delta_b^f + T_{dab}{}^{e}\delta_c^f + T_{cba}{}^{e}\delta_d^f\big)k_{fe} = 0\,. \tag{C.24}$$

C.3 On algebraic type and CKY tensors

Let us briefly comment on the relationship of the algebraic type of the spacetime and the existence of a CKY 2-form. It was proved by Collinson [Collinson, 1974], that the vacuum spacetime admitting a non-degenerate Killing–Yano 2-form is necessary of the algebraic type D. Here we outline the proof that the same remains true for the existence a non-degenerate CKY 2-form. This proof can be easily extended to higher dimensions [Coley *et al.*, 2008].

Our starting point are the algebraic relations between the components of k_{ab}

[1] In an Einstein space, that is when $R_{ec} \propto g_{ec}$, the last equation implies that ξ^a is a Killing vector.

and the curvature tensor (C.24). In a particular case of vacuum, we have

$$\left(C_{bcd}{}^{e}\delta_{a}^{f} + C_{adc}{}^{e}\delta_{b}^{f} + C_{dab}{}^{e}\delta_{c}^{f} + C_{cba}{}^{e}\delta_{d}^{f}\right)k_{fe} = 0\,, \tag{C.25}$$

where C_{abcd} denotes the Weyl tensor—related to the Riemann tensor by (C.10). In an arbitrary number of dimensions we have the following canonical forms of 2-forms (see, e.g., [Milson, 2004]):

$$k_{ab} = \lambda_0 n_{[a}l_{b]} + \sum_{p=1}^{[D/2]-1} \lambda_p m^{2p}_{[a} m^{2p+1}_{b]}\,, \tag{C.26}$$

$$k_{ab} = \lambda_0 n_{[a}m^{D-1}_{b]} + \sum_{p=1}^{[(D-3)/2]} \lambda_p m^{2p}_{[a} m^{2p+1}_{b]}\,, \tag{C.27}$$

$$k_{ab} = \lambda_0\left(n_{[a}m^{D-1}_{b]} + l_{[a}m^{D-1}_{b]}\right) + \sum_{p=1}^{(D-3)/2} \lambda_p m^{2p}_{[a} m^{2p+1}_{b]}\,. \tag{C.28}$$

Here, $\boldsymbol{n}$ has boost weight 1, $\boldsymbol{l}$ has boost weight -1, and the remaining spacelike forms are of boost weight 0.

To prove that the existence of CKY 2-form k_{ab} in a vacuum implies that the spacetime is of the algebraic type D, it is sufficient to prove that the algebraic relations (C.25) with these canonical forms eliminate all, but the boost weight zero, components of the Weyl tensor.

C.3.1 Non-degenerate CKY in four dimensions

In 4D, the non-degenerate 2-form takes the canonical form

$$k_{ab} = \lambda_0 n_{[a}l_{b]} + \lambda_1 m^2_{[a} m^3_{b]}\,. \tag{C.29}$$

Denoting by $F(a,b,c,d)$ the left-hand-side of (C.25), we successively find

$$
\begin{aligned}
F(1,1,2,2) &= 2\lambda_0 C_{1212} - 2\lambda_1 C_{1213} = 0\,,\\
F(1,1,2,3) &= -\lambda_1 C_{1313} = 0\,,\\
F(0,0,2,2) &= -2\lambda_0 C_{0202} - 2\lambda_1 C_{0203} = 0\,,\\
F(0,0,2,3) &= -\lambda_1 C_{0303} = 0\,,\\
F(0,1,3,1) &= \lambda_0 C_{0113} + \lambda_1 C_{0112} = 0\,,\\
F(1,2,3,2) &= \lambda_0 C_{1223} - \lambda_1 C_{1323} = 0\,,\\
F(0,0,1,2) &= -\lambda_0 C_{0102} - \lambda_1 C_{0103} = 0\,,\\
F(0,2,3,2) &= -\lambda_0 C_{0223} - \lambda_1 C_{0323} = 0\,,\\
C_{0212} &= C_{0313}\,, \quad C_{0312} = -C_{0213}\,.
\end{aligned}
\tag{C.30}
$$

This implies that only the following components of the Weyl:

$$
C_{0101}\,,\ C_{0123}\,,\ C_{0313}\,,\ C_{2323}\,,
\tag{C.31}
$$

and the components obtainable from them by symmetries of this tensor may be present in the spacetime admitting a CKY tensor (C.29). All of them are of boost zero and hence the spacetime is necessary of the algebraic type D.

C.4 Some algebraic identities

Throughout the thesis we use various algebraic identities. Most of them are directly related to the canonical form of the PCKY tensor or the canonical form of the Kerr-NUT-(A)dS spacetime. In this section we make a short overview of such identities.

The following functions are used throughout the text:

$$
U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2)\,, \quad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k,\,\nu_i\neq\mu}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2\,, \quad A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2\,.
\tag{C.32}
$$

These functions appear in the definition of the canonical metric (4.1), they appear in the expressions for the PCKY tensor (4.13), or the expressions for the Killing tensors (4.14). One can easily generate $A^{(k)}$, $A_\mu^{(k)}$ with the help of [Krtouš *et al.*, 2007b], [Oota & Yasui, 2008]

$$\prod_{\nu=1}^{n}(t-x_\nu^2) = A^{(0)}t^n - A^{(1)}t^{n-1} + \cdots + (-1)^n A^{(n)}\,, \tag{C.33}$$

$$\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(t-x_\nu^2) = A_\mu^{(0)}t^{n-1} - A_\mu^{(1)}t^{n-2} + \cdots + (-1)^{n-1} A_\mu^{(n-1)}\,. \tag{C.34}$$

One might also introduce quantity U, [Frolov *et al.*, 2007], [Kubizňák & Frolov, 2007]

$$U \equiv \det\left[A_\mu^{(j)}\right] = \prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n}(x_\mu^2 - x_\nu^2)\,, \tag{C.35}$$

which is simply related to the determinant of the canonical metric [cf. Eq. (4.11)]

$$g = \det(g_{ab}) = \left(-cA^{(n)}\right)^\varepsilon U^2\,. \tag{C.36}$$

In the first expression (C.35), $A_\mu^{(j)}$ $(j=0,\dots,n-1)$, is understood as the $n\times n$ matrix.

Lemma 1 ([Frolov *et al.*, 2007]). *The $n\times n$ matrix $B^\mu_{(k)} \equiv (-x_\mu^2)^{n-1-k}/U_\mu$ is an inverse of $A_\mu^{(k)}$. That is,*

$$\sum_{k=0}^{n-1}\frac{(-x_\mu^2)^{n-1-k}}{U_\mu}A_\nu^{(k)} = \delta_\mu^\nu\,, \quad \sum_{\mu=1}^{n}\frac{(-x_\mu^2)^{n-1-k}}{U_\mu}A_\mu^{(l)} = \delta_k^l\,. \tag{C.37}$$

In particular, we obtain the following important identities:

$$\sum_{\mu=1}^{n}\frac{(-x_\mu^2)^{n-1}}{U_\mu} = 1\,, \tag{C.38a}$$

$$\sum_{\mu=1}^{n}\frac{x_\mu^{2k}}{U_\mu} = 0 \quad \text{for} \quad k=0,\dots,n-2\,, \tag{C.38b}$$

$$\sum_{\mu=1}^{n}\frac{1}{x_\mu^2 U_\mu} = \frac{1}{A^{(n)}}\,, \tag{C.38c}$$

$$\sum_{\mu=1}^{n}\frac{A_\mu^{(k)}}{x_\mu^2 U_\mu} = \frac{A^{(k)}}{A^{(n)}} \quad \text{for} \quad k=0,\dots,n-1\,, \tag{C.38d}$$

The first two relations follow immediately from (C.37) (set $l=0$ in the latter expression). (C.38c) follows from (C.38a) by substituting $x_\mu \to 1/x_\mu$. (C.38d) can be verified using (C.38c), (C.37), and the fact that $A_\mu^{(k)} = A^{(k)} - x_\mu^2 A_\mu^{(k-1)}$.

The following lemma plays the central role for the separability in the canonical spacetimes:

Lemma 2. *The most general solution of the equation*

$$\sum_{\nu=1}^{n} \frac{f_\nu(x_\nu)}{U_\nu} = 0 \tag{C.39}$$

is given by

$$f_\nu(x_\nu) = \sum_{j=1}^{n-1} C_j(-x_\nu^2)^{n-1-j}\,, \tag{C.40}$$

where C_j are arbitrary constants.

This lemma was already used in [Hamamoto *et al.*, 2007], [Frolov *et al.*, 2007]. Its proper proof can be found in [Krtouš, 2007]. We finally mention the identity

$$\partial_{x_\mu}\left[\frac{U}{U_\mu}\right] = 0\,, \tag{C.41}$$

(used in the separation of the Klein–Gordon equation) which obviously follows from the definition of U and U_μ. Many more useful relations can be found, for example, in [Krtouš, 2007].

C.5 Integrability of some functions in Kerr-NUT-(A)dS spacetimes through separation of variables

In the main text we have encountered several situations where we have to solve an ordinary differential equation (or the set of equations, $j = 1, \dots, l$)

$$\dot{F}_j = G_j(x_\mu)\,. \tag{C.42}$$

Here the dot denotes the derivative with respect to an affine parameter and the right-hand-side is in general complicated function of x_μ's (or possibly r in the Lorentzian case). Such equations were, for example, obtained for the components ψ_j, (5.9), of the geodesic velocity in Chapter 5, or for the rotation angles β_μ in Chapter 9. It turns out that some of these equations may be 'symbolically' integrated as they allow an additive separation of variables. Let us probe this possibility in more detail. The separability means, that we seek the solution in the form

$$F_j = \sum_{\nu=1}^{n} F_j^{(\nu)}(x_\nu)\,. \tag{C.43}$$

Using the first relation (5.9), one finds

$$\dot{F}_j = \sum_{\nu=1}^{n} \big(F_j^{(\nu)}\big)' \dot{x}_\nu = \sum_{\nu=1}^{n} \frac{\sigma_\nu \mathrm{sign}(U_\nu)\big(F_j^{(\nu)}\big)' \sqrt{X_\nu V_\nu - W_\nu^2}}{U_\nu}\,. \tag{C.44}$$

Prime denotes the derivative with respect to a single argument. For each ν the numerator of the last expression is function of x_ν only. If $\dot{F}_j$ given by (C.42) can be brought into the form

$$\dot{F}_j = \sum_{\nu=1}^{n} \frac{f_j^{(\nu)}(x_\nu)}{U_\nu}\,, \tag{C.45}$$

the problem is separable. By comparing (C.44) with (C.45) we arrive at

$$\sum_{\nu=1}^{n} \frac{g_j^{(\nu)}(x_\nu)}{U_\nu} = 0\,, \quad g_j^{(\nu)} = \sigma_\nu \mathrm{sign}(U_\nu)\big(F_j^{(\nu)}\big)' \sqrt{X_\nu V_\nu - W_\nu^2} - f_j^{(\nu)}\,. \tag{C.46}$$

The general solution of (C.46) is (see Lemma 2 of the previous section)

$$g_j^{(\nu)} = \sum_{k=1}^{n-1} C_j^{(k)} (-x_\nu^2)^{n-1-k}\,. \tag{C.47}$$

However, what we need is a particular solution. For such a solution, we may choose all the constants $C_j^{(k)} = 0$. (In fact, a different choice of $C_j^{(k)}$ leads only to a different additive constant for F_j.) So we have

$$F_j = \sum_{\nu=1}^{n} \int \frac{\sigma_\nu \mathrm{sign}(U_\nu) f_j^{(\nu)} dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}\,. \tag{C.48}$$

The situation in the Lorentzian case is exactly analogous. If the right-hand-side of (C.42) can be brought into the form

$$\dot{F}_j = \frac{f_j^{(r)}(r)}{U_n} + \sum_{\nu=1}^{n-1} \frac{f_j^{(\nu)}(x_\nu)}{U_\nu}\,, \tag{C.49}$$

the separated solution reads

$$F_j = \int \frac{\sigma_n f_j^{(r)} dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\nu=1}^{n-1} \int \frac{\sigma_\nu \mathrm{sign}(U_\nu) f_j^{(\nu)} dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}} . \tag{C.50}$$

As a particular example let us consider the affine parameter itself. Due to the identity (C.38a),

$$1 = \sum_{\mu=1}^{n} \frac{f^{(\mu)}}{U_\mu} , \quad f^{(\mu)} = (-x_\mu^2)^{n-1} , \tag{C.51}$$

we find that

$$\tau = \sum_{\mu=1}^{n} \int \frac{\sigma_\mu \mathrm{sign}(U_\mu)(-x_\mu^2)^{n-1} dx_\mu}{\sqrt{X_\mu V_\mu - W_\mu^2}} . \tag{C.52}$$

C.6 Higher-dimensional Plebański–Demiański?

As we have mentioned in Chapter 4, the form (4.1) of the higher-dimensional Kerr-NUT-(A)dS spacetime can be considered as a 'natural' higher-dimensional generalization of the Carter's canonical form (A.74) of the 4D Kerr-NUT-(A)dS spacetime. In four dimensions, it is well known how even more general solutions can be 'generated' from the canonical form. The electromagnetic charge is added by a simple change of metric functions Q and P, the accelerated class of solutions is obtained by a conformal rescaling of the canonical element. All these classes are uniformly described by the Plebański–Demiański class of solutions, see Appendix A.2.

This inspires the search for more general higher-dimensional solutions. It is natural to ask, whether it is not possible to generalize the higher-dimensional Kerr-NUT-(A)dS spacetimes in a way exactly analogous to the 4D case. The problem of charging these solutions was addressed in [Krtouš, 2007]. It turned out that such a procedure is not sufficiently general.[2] Here we demonstrate that also the attempt to 'accelerate' Kerr-NUT-(A)dS spacetime in a way analogous to 4D, i.e., by a conformal rescaling (cf. Appendix A.2), generally fails [Kubizňák & Krtouš, 2007].

We consider the metric

$$\tilde{g} = \Omega^2 g , \tag{C.53}$$

[2]The higher-dimensional 'Kerr–Newman solution' is still analytically unknown. We conjecture that such a solution may not be of the algebraic type D. If this is so, its form might be quite different from the form of the Myers–Perry metric. The similarity of these solutions in four dimensions stems from the special properties of electromagnetism in 4D.

where g is a canonical metric (4.1) and Ω^2 is an unknown conformal factor. We ask the question, whether it is possible to adjust Ω and metric functions $X_\mu(x_\mu)$ so that the scaled metric $\tilde{g}$ satisfy the vacuum Einstein equations. Due to the same argument which we used in four dimensions (see Appendix A.2.3) such a metric would possess a conformal Killing–Yano tensor $\tilde{h} = \Omega^3 h$.

The Ricci tensor $\tilde{\mathbf{Ric}}$ of the rescaled metric $\tilde{g}$ is related to the Ricci tensor of the unscaled metric g by a well known expression (see, e.g., appendix in [Wald, 1984]), which can be written as

$$\tilde{\mathbf{Ric}} = \mathbf{Ric} + (D-2)\Omega\boldsymbol{\nabla}\boldsymbol{\nabla}\Omega^{-1} + \boldsymbol{g}\Big[\Omega\boldsymbol{\nabla}^2\Omega^{-1} - (D-1)\Omega^2\big(\boldsymbol{\nabla}\Omega^{-1}\big)^2\Big]\,. \tag{C.54}$$

Here, the 'square' of 1-forms is defined using the inverse unscaled metric g^{-1}. We require $\tilde{\mathbf{Ric}} = -\lambda\,\tilde{g}$ with λ proportional to the cosmological constant. The Ricci tensor $\tilde{\mathbf{Ric}}$ thus must be diagonal in the frame $\{\omega\}$. The conditions on off-diagonal terms give equations for the conformal factor Ω.

In a generic odd dimension these conditions are too strong—they admit only a constant conformal factor Ω. In even dimensions the conditions on off-diagonal terms lead to equations

$$\Omega^{-1}{}_{,\mu\nu} = \frac{x_\nu\,\Omega^{-1}{}_{,\mu}}{x_\nu^2 - x_\mu^2} + \frac{x_\mu\,\Omega^{-1}{}_{,\nu}}{x_\mu^2 - x_\nu^2}\,,\qquad 0 = \frac{x_\mu\,\Omega^{-1}{}_{,\mu}}{x_\nu^2 - x_\mu^2} + \frac{x_\nu\,\Omega^{-1}{}_{,\nu}}{x_\mu^2 - x_\nu^2}\,. \tag{C.55}$$

It gives the conformal factor depending on two constants c and a,

$$\Omega^{-1} = c + a\,x_1\ldots x_n\,, \tag{C.56}$$

which is obviously a generalization of the four-dimensional factor (A.44) (with $c=1$ and $a=i\alpha$).

Unfortunately, the conditions for diagonal terms of the Ricci tensor are in even dimensions $D>4$ rather restrictive. Analyzing first the condition for the scalar curvature and then checking all diagonal terms one finds that either[3]

$$\Omega^{-1} = x_1\ldots x_2\,,\qquad X_\mu = \bar{b}_\mu\,x_\mu^{2n-1} + \sum_{k=0}^{n} c_k\,x_\mu^{2k}\,, \tag{C.57}$$

with $\lambda = (D-1)\,c_0$, or

$$\Omega^{-1} = 1 + a\,x_1\ldots x_2\,,\qquad X_\mu = \sum_{k=0}^{n} c_k\,x_\mu^{2k}\,, \tag{C.58}$$

[3]The trivial global scaling was eliminated by setting $a=1$ in (C.57) and $c=1$ in (C.58).

with $\lambda = (D-1)\big[(-1)^{n-1}c_n + a^2 c_0\big]$. The first case is not a new solution: the substitution

$$x_\mu = 1/\bar{x}_\mu\,, \quad \psi_j = \bar{\psi}_{n-1-j}\,, \quad X_\mu = \bar{x}_\mu^{-n+1}\bar{X}_\mu\,, \tag{C.59}$$

transforms the rescaled metric $\tilde{g}$ back to the form (4.1) in 'barred' coordinates. In the second case metric functions X_μ depend on a smaller number of parameters and one has to expect that the metric describes only a subclass of the 'accelerated black hole solutions'. It is actually the trivial subclass—it was shown in [Hamamoto *et al.*, 2007] that the metric (4.1) with X_μ given by (C.58) represents the maximally symmetric spacetime; therefore the scaled metric $\tilde{g}$, being the Einstein space conformally related to the maximally symmetric spacetime, must describe also the maximally symmetric spacetime. In analogy with the four-dimensional case we expect that the metric (C.53) with metric functions (C.58) describes the Minkowski or (anti-)de Sitter space in some kind of 'rotating accelerated' coordinates. Such an interpretation, however, requires a further detailed study.[4]

The form of the Plebański–Demiański metric motivates the construction of new solutions in higher dimensions. Its non-accelerated subclass, the Carter's metric, has been already generalized into higher dimensions by Chen, Lü, and Pope [Chen *et al.*, 2006a]. However, it seems that further generalizations, although almost obvious at a first sight, cannot be easily obtained. For example, the attempts to 'naturally' charge these solutions failed so far (see, e.g., [Krtouš, 2007], [Chen & Lü, 2008]). Here we have demonstrated that also the generalization to accelerated solutions is not straightforward. In particular, we have shown that the direct analogue of the Plebański–Demiański family of solutions (with acceleration) cannot be in higher dimensions obtained in a manner similar to the four-dimensional case, that is, by a conformal scaling of the Chen–Lü–Pope metric, possibly with the 'natural' change of metric functions. The question about the existence of the C-metric in higher dimensions therefore still remains open.

[4] We have to conclude that a non-trivial generalization of the Plebański–Demiański metric into a generic higher dimension cannot be found by a conformal rescaling (C.53) of the canonical element (4.1). However, recently it was demonstrated that a nontrivial generalization of the 5D MP metric can be obtained with the help of two scaling factors [Lü *et al.*, 2008a]. The solution contains three independent and one adjustable parameter. It is obtained by gluing the rescaled 4D canonical metric with a (differently rescaled) part corresponding to the fifth dimension. This solution also gives rise to a new charged black hole of 5D minimal supergravity [Lü *et al.*, 2008b].

C.7 Remarks on the degeneracy of eigenvalues of $\boldsymbol{F}$

In this section we comment on the degeneracy of the eigenvalues of the 2-form $\boldsymbol{F}$ used in Chapter 9.[5] Namely, we prove that due to that fact that the PCKY tensor $\boldsymbol{h}$ is by definition non-degenerate, the corresponding 2-form $\boldsymbol{F}$, (9.1), possesses (q_0+1)-times degenerate zero eigenvalue, where $q_0 = 1$ in even number of spacetime dimensions and $q_0 = 0, 2$ for generic, special trajectories in an odd number of spacetime dimensions. Moreover, the multiplicity of non-zero eigenvalues is governed by $q_\mu \leq 2$, and the equality occurs only for special geodesic trajectories.

Consider a *non-degenerate* $\boldsymbol{h}$, then

$$\Delta(\lambda) = \det(h^a_{\ b} - \lambda\delta^a_b) = (-\lambda)^\varepsilon \prod_{k=1}^{n} (\lambda^2 + \nu_k^2)\,, \tag{C.60}$$

where all ν_k are different. Let us re-calculate this determinant in terms of $\boldsymbol{F}$ and compare the results. For this calculation we use the Darboux basis of $\boldsymbol{F}$ and corresponding matrix form of the objects. In particular, we have the expression (9.3) for $\boldsymbol{F}$, and

$$s = (s_{\hat{0}}, s_{\hat{1}}, \dots s_{\hat{p}})\,, \quad s_{\hat{0}} = ({}^1 s_{\hat{0}}, \dots, {}^{q_0} s_{\hat{0}})\,, \quad s_{\hat{\mu}} = ({}^1 s_{\hat{\mu}}, {}^1 \bar{s}_{\hat{\mu}}, \dots, {}^{q_\mu} s_{\hat{\mu}}, {}^{q_\mu} \bar{s}_{\hat{\mu}})\,, \tag{C.61}$$

for the 1-form $\boldsymbol{s}$ defined in (9.1). Using (9.1), (9.3), and (C.61), one can rewrite (C.60) as

$$\Delta(\lambda) = \det(F^a_{\ b} - u^a s_b + s^a u_b - \lambda\delta^a_b) = \begin{vmatrix} A & B \\ C & E \end{vmatrix}\,, \tag{C.62}$$

where $A = -\lambda$, $B = -s$, $C = -s^T$, and E is the $(D-1)$-dimensional matrix of the form

$$E = \mathrm{diag}(-\lambda I_{q_0}, {}^1 Z, \dots, {}^p Z)\,, \quad {}^\mu Z = \begin{pmatrix} -\lambda I_\mu & \lambda_\mu I_\mu \\ -\lambda_\mu I_\mu & -\lambda I_\mu \end{pmatrix}\,. \tag{C.63}$$

Here I_{q_0} is a unit $q_0 \times q_0$ matrix, and we use X^T to denote a matrix transposed to X. It is easy to check that

$$\begin{aligned} E^{-1} &= \mathrm{diag}(-\lambda^{-1} I_{q_0}, {}^1 Z^{-1}, \dots, {}^p Z^{-1})\,, \\ {}^\mu Z^{-1} &= Q_\mu^{-1}\, {}^\mu Z^T\,, \ \ Q_\mu = (\lambda^2 + \lambda_\mu^2)\,, \ \ \det({}^\mu Z) = Q_\mu^{q_\mu}\,. \end{aligned} \tag{C.64}$$

[5] As this appendix directly relates to Chapter 9, we consider the case of the Lorentzian signature and $\boldsymbol{F}$ for timelike geodesics, that is, $\boldsymbol{F}$ given by (9.1). Also other notations are the same as in Chapter 9.

One has the following relation for the determinant of a block matrix (see, e.g., [Gantmacher, 1959])

$$\begin{vmatrix} A & B \\ C & E \end{vmatrix} = \mathcal{A}\,|E|\,, \quad \mathcal{A} = |A - BE^{-1}C|\,. \tag{C.65}$$

Using (C.63) and (C.64), one finds

$$\det(E) = (-\lambda)^{q_0}\prod_{\mu=1}^{p} Q_\mu^{q_\mu}\,, \quad \mathcal{A} = -\lambda - sE^{-1}s^T. \tag{C.66}$$

Combining all these relations one obtains

$$\begin{aligned} \Delta(\lambda) &= (-\lambda)^{q_0-1}\prod_{\mu=1}^{p} Q_\mu^{q_\mu}\left[\lambda^2\Big(1 - \sum_{\mu=1}^{p}\frac{s_{\hat{\mu}}^2}{Q_\mu}\Big) - s_{\hat{0}}^2\right]\,, \\ s_{\hat{0}}^2 &= \sum_{i=1}^{q_0} {}^i s_{\hat{0}}^2\,, \quad s_{\hat{\mu}}^2 = \sum_{i=1}^{q_\mu}({}^i s_{\hat{\mu}}^2 + {}^{\bar{i}} s_{\hat{\mu}}^2)\,. \end{aligned} \tag{C.67}$$

Let us now compare (C.60) and (C.67). First of all, let us compare the powers of $(-\lambda)$. For $s_{\hat{0}}^2 \neq 0$ we have match for $q_0 - 1 = \varepsilon$, whereas the case $s_{\hat{0}}^2 = 0$ may happen only in odd dimensions and one must have $q_0 = 0$ [cf. (9.24)]. Another result of the comparison is that $q_\mu \leq 2$. Really, if $q_\mu > 2$, then at least 2 roots of $\Delta(\lambda)$ in (C.67) coincide. This contradicts the assumption previously stated, since for a non-degenerate operator $\boldsymbol{h}$ the characteristic polynomial has only single roots $\lambda^2 = -\nu_k^2$. The case when $q_\mu = 2$ is degenerate. It is valid only for a special value of the velocity $\boldsymbol{u}$. Really, in this case one of the eigenvalues, say ν_k, of $\boldsymbol{h}$ coincides with one of the eigenvalues of $\boldsymbol{F}$ so that one has $\det(\boldsymbol{F} - \nu_k\boldsymbol{I}) = 0$. The latter is an equation restricting the value of $\boldsymbol{u}$.