University of Alberta

Landsberg Spaces in Finsler Geometry

by

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Abstract

Finsler geometry studies the length of paths in a space. Two important specializations of Finsler spaces are Landsberg spaces and Berwald spaces. All Berwald spaces are Landsberg spaces. An example of a Landsberg space which is not a Berwald space is not known. This thesis consists of three sections. The first introduces the basic results and notation to discuss these spaces. The second examines theorems in the mathematical literature which consider conditions under which Landsberg spaces become Berwald spaces. The third section contains original work which introduces new classes of metrics: these may contain Landsberg spaces which are not Berwald.

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Joseph Modayil

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Introduction

The study of Finsler geometry concerns the lengths of paths in a space. Within Finsler spaces, Landsberg spaces and Berwald spaces are of particular importance. Berwald spaces include the set of Riemannian and Minkowski spaces. The problem with Berwald spaces is that they are too benign—their geodesics (the "natural" paths) can often be given by a Riemannian metric. Yet, many important applications have been derived from their theory [1, 2, 3]. Landsberg spaces are a generalization of Berwald spaces. They are spaces where the Chern/Rund connection coincides with the Berwald connection. Also, the horizontal part of the Cartan connection coincides with that of the Berwald connection for this class of Finsler spaces. This is significant as the Cartan connection is a metrical and h-torsion free connection, while the Berwald connection comes from the theory of sprays. Both types of connections are generalizations of the Levi-Civita connection of Riemannian geometry. The first section covers the background information necessary to discuss Landsberg and Berwald spaces.

There are several theorems regarding Landsberg spaces—however, *a specific example of a Landsberg space which is not a Berwald space is not yet known.* The second section presents several reduction theorems and proofs, taken from the mathematical literature, which are of the following form:

> with property $\mathcal{P}$, a Landsberg space is a Berwald space.

In the third section, the author introduces some new types of metrics. These may contain Landsberg spaces which are not Berwald.

1.1 Preliminaries from differentiable manifold theory

1.1.1 Definition A **differentiable manifold** $\mathcal{M}^n$ is a separable Hausdorff space $\mathcal{M}$ with a maximal collection of charts $\{U_\lambda, h_\lambda\}_{\lambda\in\Lambda}$ such that

- $h_\lambda : U_\lambda \to V_\lambda \subset \mathbb{R}^n$ is a homeomorphism where V_λ is an open set;
- for all $\alpha, \beta \in \Lambda$ the composition $h_\alpha \circ h_\beta^{-1}$ is C^∞ when restricted to the domain $h_\beta(U_\alpha \cap U_\beta)$.

1.1.2 Definition Let $\mathcal{M}^n$ be a manifold, $p \in \mathcal{M}^n$. A **tangent vector** ξ **at p** is an assignment of a n-tuple of numbers for every $\alpha \in \Lambda$, denoted ξ^i_α with $i = 1, \dots, n$, which obeys the relation

$$\xi_\beta = D(h_\beta \circ h_\alpha^{-1})|_{h_\alpha(p)} \cdot \xi_\alpha.$$

D denotes the derivative operator; this is evaluated at $h_\alpha(p)$. The **tangent space at a point** $p \in \mathcal{M}^n$, denoted $T_p\mathcal{M}^n$, is a vector space formed by the disjoint union of all tangent vectors at p. The **tangent space** of $\mathcal{M}^n$, denoted by $T\mathcal{M}^n$, is the disjoint union of all $T_p\mathcal{M}^n$. The tangent space can also be considered as a tangent manifold, and as a fiber bundle over $\mathcal{M}^n$.

1.1.3 Notation Often a particular coordinate chart λ is implicitly assumed and coordinates are imposed from V_λ. Namely, a point $p \in U_\lambda$ is identified with $h_\lambda(p)$, so the standard Euclidean coordinates $(x^1, \dots, x^n)$ of $\mathbb{R}^n$ when restricted to V_λ are interpreted as coordinates on $\mathcal{M}^n$. Furthermore, a basis for the tangent space at $p \in \mathcal{M}^n$ is given by $\frac{\partial}{\partial x^i}$. Vectors in this tangent space can be expressed as $y = \sum_{i=1}^n y^i \frac{\partial}{\partial x^i}$. Hence coordinates in $T\mathcal{M}^n$ are given by $(x^1, \dots, x^n, y^1, \dots, y^n)$. This local coordinate description is abbreviated by the notation (x, y).

1.1.4 Definition The term **conical tangent manifold** is used to denote an open region of the tangent manifold that

i. does not contain $(x, 0)$,

ii. contains $(x, \lambda y)$ for $\lambda > 0$ if it contains (x, y).

The conical tangent manifold is represented by the symbol $\widetilde{TM}$.

1.1.5 Definition Consider the map $\pi : T\mathcal{M}^n \to \mathcal{M}^n$ where $\pi(x, y) = x$. Taking the differential of this map yields $D\pi : TT\mathcal{M}^n \to T\mathcal{M}^n$. Let $\mathcal{V}T\mathcal{M}$ denote the kernel of $D\pi$; this space is a sub-bundle of $TT\mathcal{M}^n$ over $T\mathcal{M}^n$. A **(spray) vector field on** $T\mathcal{M}^n$ is a C^∞ section of $\mathcal{V}T\mathcal{M}$. A (spray) vector field is just a second order ordinary differential equation; it does not have to be homogeneous of degree 2. Define $(\mathcal{V}T\mathcal{M})^*$ as the dual vector bundle to $\mathcal{V}T\mathcal{M}$ over $T\mathcal{M}^n$. A **(spray) one-form field on** $T\mathcal{M}^n$ is a C^∞ section of $(\mathcal{V}T\mathcal{M})^*$. A C^∞ section of

$$\underbrace{\mathcal{V}T\mathcal{M} \otimes \cdots \otimes \mathcal{V}T\mathcal{M}}_{r \text{ copies}} \otimes \underbrace{(\mathcal{V}T\mathcal{M})^* \otimes \cdots \otimes (\mathcal{V}T\mathcal{M})^*}_{s \text{ copies}}$$

is called a **(spray)** (r, s) **tensor field on** $\mathcal{M}^n$. The term tensor field will refer to spray tensor fields in this paper.

1.1.6 Notation **Einstein summation** is used to simplify the notation of objects in coordinate form. Namely, Latin indices which are used once in subscript and once in the superscript of a term are assumed to be summed in an expression over the values $1, \ldots, n$. For example if A is a $(0,3)$ tensor, then $A_{ijk}y^j = \sum_{j=1}^{n} A_{ijk}y^j$ and the result is a $(0,2)$ tensor. Summing in this manner is also called **transvection**. An index with the number 0 is used as a shorthand notation to indicate that the index has been transvected with y^i. For example, $A_{i0k} = A_{ijk}y^j$.

1.1.7 Notation In local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ denote partial differentiation as follows.

$$\partial_k = \frac{\partial}{\partial x^k} \qquad \dot{\partial}_k = \frac{\partial}{\partial y^k}$$

1.1.8 Definition Let $\widetilde{TM}$ denote a conical region in a tangent manifold. A **fundamental function**

$$L : \widetilde{TM} \to \mathbb{R}$$

is used to define lengths of parameterized paths in the manifold. Namely, if $\alpha : [0,1] \to \mathcal{M}^n$ is a path, then it has length $Length(\alpha) = \int_0^1 L(\alpha(t), \alpha'(t))dt$. Note that in general, this definition of length depends on parameterization of the path.

If $x = (x^1, \dots, x^n)$ denotes local coordinates on $\mathcal{M}^n$ and y denotes the induced tangent vector coordinates on $T\mathcal{M}^n$ then the fundamental function is often written in the form $L(x, y)$.

1.1.9 Definition A **metric tensor** associated to a fundamental function L is given by

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2.$$

1.1.10 Definition Let V be a vector space. A function $f : V \to \mathbb{R}$ is **positively homogeneous** (or p-homogeneous) of degree $d \in \mathbb{R}$ if for each $y \in V$, $f(\lambda y) = \lambda^d f(y)$ for all $\lambda > 0$.

1.1.11 Proposition **(Euler)** A smooth function $f : V \to \mathbb{R}$, with V open in $\mathbb{R}^n$, is p-homogeneous of degree d if and only if

$$\frac{\partial f}{\partial v^i} v^i = d \cdot f(v) \tag{1.1}$$

where v^i are an orthogonal basis of V.

Proof: We follow [2] for the proof. Suppose f is p-homogeneous of degree d. Then $f(pv) = p^d f(v)$ for $p \in \mathbb{R}_{>0}$. Differentiate this equation with respect to p to obtain

$$\frac{\partial f}{\partial pv^i} v^i = d \cdot p^{d-1} f(v)$$

and set $p = 1$ to get

$$\frac{\partial f}{\partial v^i} v^i = d \cdot f(v).$$

For the reverse implication, assume the above equation holds. Evaluate f at pv to obtain

$$d \cdot f(pv) = \left.\frac{\partial f}{\partial v^i}\right|_{pv} pv^i = p\frac{\partial f(pv)}{\partial p}.$$

Now for fixed v, define $g(p) = f(pv)$. The above equation can then be written as

$$\frac{d \cdot g(p)}{p} = \frac{\partial g(p)}{\partial p}.$$

This is a separable differential equation which yields the solution $g(p) = p^d g(1)$. This implies $f(pv) = p^d f(v)$ so f is p-homogeneous of degree d. $\square$

1.1.12 Definition A **Finsler space** is a pair $(\widetilde{TM}, L)$ where $\mathcal{M}^n$ is a manifold, and L is a fundamental function which satisfies properties below.

1. $L \geq 0$, and L is C^∞ on $\widetilde{TM}$.

2. L is positively homogeneous of degree one in y. This implies that the length of a path does not depend on parameterization, though it may depend on orientation.

3. The induced metric tensor $g_{ij}(x, y)$ is **positive definite**. This means that if $(x, y) \in \widetilde{TM}$ and $\zeta \in T_x\mathcal{M}^n$ then $g_{ij}(x, y)\zeta^i\zeta^j > 0$.

The last requirement is sometimes replaced with regularity ($\det(g_{ij}) \neq 0$) to include more general spaces.

1.1.13 Definition The **indicatrix** at a point $p \in \mathcal{M}^n$ is defined to be the set $I_p = \{(p, y) \in \widetilde{TM} | L(p, y) = 1\}$.

1.2 Metric properties

Although Finsler spaces are commonly called Finsler metric spaces, they are not necessarily metric spaces in the traditional sense. A metric on a topological space $\mathcal{T}$ is well known as a function

$$d : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$$

which satisfies the following properties:

i. $d(x, y) \geq 0 \ \forall x, y \in \mathcal{T}$,

ii. $d(x, y) = 0 \iff x = y$,

iii. $d(x, y) = d(y, x)$ (symmetry),

iv. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Originally Finsler spaces satisfied all of these condition and the terminology of metric space was well justified. The theory of Finsler spaces under the original restricted assumptions was pursued by Busemann [8]. He showed that on a one or two dimensional space X, defining a metric on X which satisfies certain assumptions will imply that X is a manifold. Moreover, a metric defined on a manifold (of arbitrary dimension) which satisfied a few additional assumptions had to be Finsler.

Later, the requirement of symmetry was dropped for many applications. One striking example of the strength of this is the discovery of a non-symmetric Finsler structure of constant curvature on the sphere [7]. No symmetric example is known which is not Riemannian.

On reduction to a Finsler metric defined only on a conical region, positive definiteness no longer ensures the triangle inequality. The example indicates how it can fail.

1.2.1 Example Let $\mathcal{M}^n = \mathbb{R}^2$, and let

$$\widetilde{TM} = \{(x^1, x^2, y^1, y^2) \in T\mathcal{M}^n |\ y^2 \neq 0\}.$$

Define the fundamental function

$$L = \begin{cases} \sqrt{(y^1 + 0.5)^2 + (y^2)^2} & \text{if } y^2 > 0. \\ \sqrt{(y^1 - 0.5)^2 + (y^2)^2} & \text{if } y^2 < 0. \end{cases}$$

The fundamental function is clearly positive definite in $\widetilde{TM}$. Let

$$p_1 = (0, 0, 1.5, -0.5), \quad p_2 = (0, 0, 0.5, 1).$$

Then

$$L(p_1) + L(p_2) = \sqrt{1 + .25} + \sqrt{1 + 1} < \sqrt{(2.5)^2 + .25} = L(p_1 + p_2).$$

Hence, the triangle inequality does not hold in general.

Note in Figure 1.1 that the line connecting p_1 and p_2 crosses the line $y^2 = 0$ and hence does not lie entirely inside $\widetilde{TM}$. This is essentially the only way that the triangle inequality can be broken. The triangle inequality does hold in a weaker form, as is shown in the following theorem.

1.2.2 Lemma **(Rund [19])** Let L be the fundamental function of a Finsler space $\widetilde{TM}$. Let $(x, y) \in \widetilde{TM}$ and $(x, \xi) \in T\mathcal{M}^n$. Then

$$(\dot{\partial}_i \dot{\partial}_j \frac{1}{2} L(x, y))\xi^i \xi^j \geq 0,$$

with equality holding if and only if $\xi = sy$ for some s.

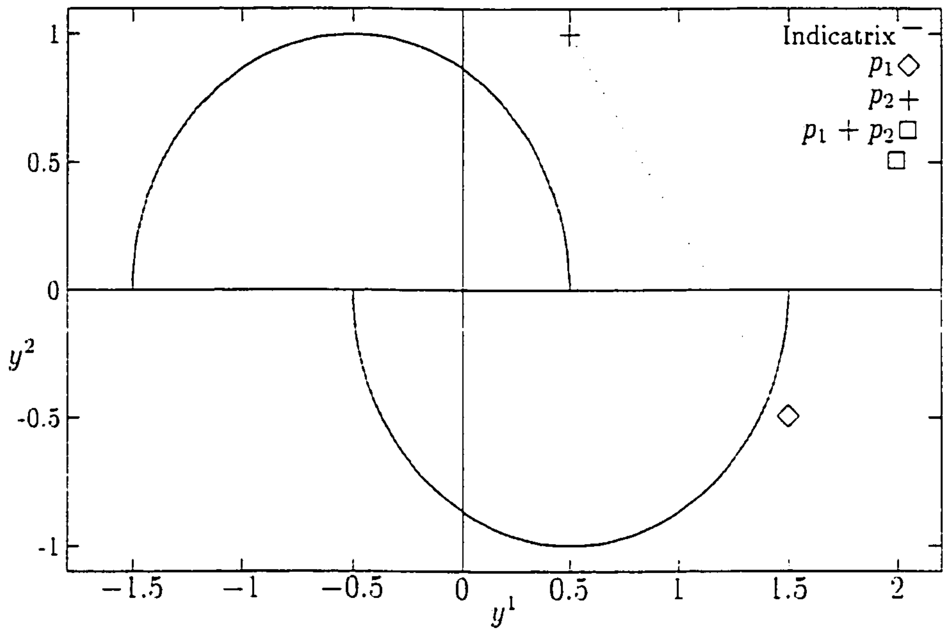

Figure 1.1: The indicatrix of the metric L, with the points p_1, p_2, $p_1 + p_2$.

Proof: Let $(x, y) \in \widetilde{TM}$, and let $\xi, \nu \in T_x M^n$. Then

$$g_{ij}(x,y)(\xi^i + s\nu^i)(\xi^j + s\nu^j) = g_{ij}(x,y)\xi^i\xi^j + 2sg_{ij}(x,y)\xi^i\nu^j + s^2 g_{ij}(x,y)\nu^i\nu^j,$$

where $s \in \mathbb{R}$. The equation can be viewed as a quadratic equation in s. Since g_{ij} is positive definite by assumption, the left hand term can only be zero if $\xi + s\nu = 0$. Hence, the quadratic equation in s has at most one real solution. Then the discriminant of the equation must be negative, so

$$(g_{ij}\xi^i\nu^j)^2 \leq (g_{ij}\xi^i\xi^j)(g_{ij}\nu^i\nu^j). \tag{1.2}$$

The definition of the metric tensor can be evaluated explicitly.

$$g_{ij}(x,y) = \dot{\partial}_i\dot{\partial}_j\frac{1}{2}L^2 = \dot{\partial}_i(L\dot{\partial}_j L) = (\dot{\partial}_i L)(\dot{\partial}_j L) + L(\dot{\partial}_i\dot{\partial}_j L) \tag{1.3}$$

Euler's theorem yields

$$(\dot{\partial}_i\dot{\partial}_j\frac{1}{2}L^2)y^j = (\dot{\partial}_i L)(\dot{\partial}_j L)y^j + L(\dot{\partial}_i\dot{\partial}_j L)y^j = (\dot{\partial}_i L)1L + L0 = (\dot{\partial}_i L)L, \text{ and}$$

$$(\dot{\partial}_i\dot{\partial}_j\frac{1}{2}L^2)y^j y^i = L(\dot{\partial}_i L)y^i = L^2$$

by the one and zero degree p-homogeneity of $\dot{\partial}_j L$ and $\dot{\partial}_i\dot{\partial}_j L$ respectively. The first of the above two equations can be solved $\dot{\partial}_i L = \frac{1}{2}L^{-1}(\dot{\partial}_i\dot{\partial}_j L^2)y^j$ and this can be substituted into Equation (1.3) to get

$$\frac{1}{2}(\dot{\partial}_i\dot{\partial}_j L^2)(x,y) = \frac{1}{4}L^{-2}(\dot{\partial}_i\dot{\partial}_h L^2)y^h(\dot{\partial}_k\dot{\partial}_j L^2)y^k + L(\dot{\partial}_i\dot{\partial}_j L).$$

Solving for the term $(\dot{\partial}_i\dot{\partial}_j L)$ in the above and contracting by $\xi^i\xi^j$ yields

$$(\dot{\partial}_i\dot{\partial}_j L)\xi^i\xi^j = \frac{1}{2}L^{-1}\{(\dot{\partial}_i\dot{\partial}_j L^2)\xi^i\xi^j - \frac{1}{2}L^{-2}\left((\dot{\partial}_i\dot{\partial}_j L^2)y^i\xi^j\right)^2\}.$$

Substitute the inequality 1.2 into the above with $\nu = y$.

$$\begin{aligned}(\dot{\partial}_i\dot{\partial}_j L)\xi^i\xi^j &\geq \frac{1}{2}L^{-1}\{(\dot{\partial}_i\dot{\partial}_j L^2)\xi^i\xi^j - \frac{1}{2}L^{-2}(\dot{\partial}_i\dot{\partial}_j L^2)\xi^i\xi^j(\dot{\partial}_i\dot{\partial}_j L^2)y^iy^j\} \\ &\geq \frac{1}{2}L^{-1}\{(\dot{\partial}_i\dot{\partial}_j L^2)\xi^i\xi^j - L^{-2}(\dot{\partial}_i\dot{\partial}_j L^2)\xi^i\xi^j L^2\} \\ &\geq 0\end{aligned}$$

□

1.2.3 Theorem **(Rund [19])** Let L be the fundamental function of a Finsler manifold $\widetilde{TM}$. Let $x \in \mathcal{M}^n$. Let $(x,\xi),(x,\nu) \in \widetilde{TM}$. Further, assume $(x, r\xi + (1-r)\nu) \in \widetilde{TM}$ for all $r \in [0,1]$. Then $L(x,\xi+\nu) \leq L(x,\xi) + L(x,\nu)$, and equality holds $\iff \xi = r\nu$ with $r > 0$.

Proof: Let x^i, ν^i, ξ^i be given as in the theorem statement in coordinate form. The mean value theorem yields the equation

$$L(x,\xi) = L(x,\nu) + (\dot{\partial}_i L(x,\nu))(\xi^i - \nu^i) + \Phi_{ij}(\xi^i - \nu^i)(\xi^j - \nu^j).$$

The remainder term in the above is given by

$$\Phi_{ij} = \dot{\partial}_i\dot{\partial}_j L(x, s\nu + (1-s)\xi),$$

where $s \in (0,1)$ is some unknown number given by the mean value theorem. Since L is p-homogeneous of degree 1, $L(x,\nu) = \dot{\partial}_i L(x,\nu)\nu^i$. Hence the first equation reduces to

$$L(x,\xi) = (\dot{\partial}_i L(x,\nu))\xi^i + \Phi_{ij}(\xi^i - \nu^i)(\xi^j - \nu^j).$$

Note that $(x, 0) \notin \widetilde{TM}$ so $\xi \neq s\nu$ for $s < 0$. This ensures that Φ_{ij} is well -defined, as L is smooth at $(x, s\xi + (1-s)\nu) \in \widetilde{TM}$ by assumption. Define

$$\Phi = \Phi_{ij}(\xi^i - \nu^i)(\xi^j - \nu^j).$$

Now by Lemma 1.2.2, $\Phi \geq 0$ with equality if and only if there is a number r such that

$$\xi^i - \nu^i = r(s\nu + (1-s)\xi),$$

which is equivalent to

$$\xi^i(1 + rs - r) = \nu^i(1 + rs).$$

Now $1 + rs = 0 \implies r = -1/s \implies (1 + rs - r) = 1/s \neq 0$, hence both (1+rs) and (1+rs-r) can not vanish simultaneously. This implies that neither can vanish at all, for it would imply that one of ξ or ν was the zero vector.

Hence Φ is non-negative and vanishes if and only if $\xi = t\nu$ for some $t > 0$. This implies

$$L(x, \xi) \geq \dot{\partial}_i L(x, \nu)\xi^i,$$

with equality only in the aforementioned case. Applying this result to the points (x, ξ) and $(x, \xi + \nu)$ yields

$$L(x, \xi) \geq \dot{\partial}_i L(x, \xi + \nu)\xi^i.$$

Applying the result to the points (x, ν) and $(x, \xi + \nu)$ yields

$$L(x, \nu) \geq \dot{\partial}_i L(x, \xi + \nu)\nu^i.$$

This implies

$$L(x, \nu) + L(x, \xi) \geq \dot{\partial}_i L(x, \xi + \nu)(\nu^i + \xi^i).$$

Since L is p-homogeneous of degree 1, the right hand side equals $(L, x, \xi + \nu)$ by Euler's theorem. This is the desired result. □

The theorem shows that the Finsler function is locally convex in $\widetilde{TM}$. The above theorem can be useful when visually examining an indicatrix; if the indicatrix of a fundamental function is not locally convex, then it does not give rise to a positive definite Finsler space. Strict convexity is not sufficient to ensure that the metric is positive definite—the requirement is that the quadratic form osculating to the indicatrix must be non-degenerate.

1.3 Finsler connections

1.3.1 Definition A **frame** at a point $p \in \mathcal{M}^n$ is an ordered set of n linearly independent tangent vectors. The set of all frames above a point p forms the **structure group** $GL(n)$. The collection of all frames over all points forms the principal bundle called the **frame bundle** over $\mathcal{M}^n$, and is denoted by $L(\mathcal{M}^n)$.

$$\begin{array}{ccc} F(\mathcal{M}^n) & \xrightarrow{\pi_T^*} & L(\mathcal{M}^n) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_L} \\ \widetilde{TM} & \xrightarrow[\pi_T]{} & \mathcal{M}^n \end{array}$$

Define the **spray bundle** over $\mathcal{M}^n$ as the pullback of the frame bundle over the conical tangent bundle. The spray bundle is denoted by $F(\mathcal{M}^n)$, and it is a principal bundle with structure group $GL(n)$.

1.3.2 Definition There is a map $\beta_g : F(\mathcal{M}^n) \to F(\mathcal{M}^n)$ called **right translation**. It is understood that $g \in GL(n)$, and it maps $(x, y, z) \mapsto (x, y, zg)$, where $(x, y, z) \in \pi_1^{-1}((x, y))$.

1.3.3 Definition Let $\mathcal{M}^n$ be a manifold which is a base space of the vector bundle E with a projection map $\pi : E \to \mathcal{M}^n$. A **distribution** in E over $\mathcal{M}^n$ is a C^∞ map Δ, which maps $p \in \mathcal{M}^n$ to a subspace in $\pi^{-1}(p)$. The **vertical sub-bundle** VE of TE is the kernel of $D\pi : TE \to T\mathcal{M}^n$. The fiber of VE above a point $y \in T\mathcal{M}^n$ is denoted $E_y^v = (D\pi)^{-1}(y)$.

1.3.4 Definition A **spray connection** N on a manifold M is a distribution in $E = TM$ over M, where for all $y \in M$, $T_yE = E_y^v \oplus N(y)$. A **linear connection** in $F(\mathcal{M}^n)$ is a spray connection on $F(\mathcal{M}^n)$ which also satisfies $(D\beta_g)(N(y)) = N(yg)$.

1.3.5 Definition A **pre-Finsler connection** is a pair (Γ, N), where Γ is a linear connection in $F\mathcal{M}^n$, and N is a spray connection on $\widetilde{TM}$. In coordinate notation, such a connection is represented by the triad $(F^i_{jk}, N^i_j, V^i_{jk})$. A pre-Finsler connection is called a **Finsler connection** if it defined using a fundamental function. The Levi-Civita connection γ^i_{jk} of Riemannian geometry corresponds to F^i_{jk}.

1.3.6 Remark Let $(F^i_{jk}, N^i_j, V^i_{jk})$ be a pre-Finsler connection on $\mathcal{M}^n$. In general F, and V are p-homogeneous of degrees 0 and -1 respectively, while N^i_j has no p-homogeneous constraint. For the connections which are examined in this work, it is assumed that N^i_j is p-homogeneous of degree 1.

1.3.7 Definition The **Cartan torsion tensor** is defined as

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}.$$

1.3.8 Definition The **angular metric tensor** is defined as $h_{ij} = g_{ij} - L^{-2}y_i y_j$.

1.3.9 Definition A space is said to be **C-Reducible** if the Cartan torsion tensor is of the special form

$$C_{ijk} = \frac{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j}{n+1}.$$

The term C_i is referred to as the **contracted torsion tensor**, and is defined by $C_i = g^{jk}C_{ijk}$. All two dimensional spaces are of this form, as are Riemannian spaces, so they are excluded for convenience.[2][p. 52]

1.3.10 Notation The following symbols are defined. The degree listed is the degree of p-homogeneity .

Name	Degree	Definition
Levi-Civita Symbol (first kind)	0	$\gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})$
Levi-Civita Symbol (second kind)	0	$\gamma^i_{jk} = g^{is}\gamma_{sjk}$
Spray functions	2	$G^i = \frac{1}{2}\gamma^i_{jk}y^j y^k$
Nonlinear connection of Cartan	1	$G^i_j = \dot{\partial}_j G^i$
Spray connection coefficients	0	$G^i_{jk} = \dot{\partial}_k G^i_j$

The spray functions come from the calculus of variations. Namely, if s is a path between two points p and q on the manifold, then s is called a geodesic if the length of s is a local extremum. This condition, $\delta s = 0$, is converted into the differential equations

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}y^j y^k = 0.$$

The term $\gamma^i_{jk}y^j y^k$ is set to equal $2G^i$. This system will be *a classical Douglas spray if and only if G^i is p-homogeneous of degree 2.*

1.3.11 Notation Berwald's nonlinear operator is defined to be $\delta_k = \partial_k - N_k^r(\dot{\partial}_r)$.

1.3.12 Definition The **h-covariant derivative** relative to $(F^i_{jk}, N^i_j, V^i_{jk})$ of a (1,1) tensor field A^i_j is given by

$$A^i_{j|k} = \delta_k A^i_j + A^s_j F^i_{sk} - A^i_s F^s_{jk}.$$

The short bar is used to denote the h-covariant derivative in coordinate form. In non-coordinate form, this operator is represented by the symbol ∇^h.

1.3.13 Definition The **v-covariant derivative** relative to $(F^i_{jk}, N^i_j, V^i_{jk})$ of a (1,1) tensor field A^i_j is given by

$$A^i_j\big|_k = \dot{\partial}_k A^i_j + A^s_j V^i_{sk} - A^i_s V^s_{jk}.$$

The long bar denotes the operation when the tensor is in coordinate form, and the symbol ∇^v is used when the tensor is given in non-coordinate form.

1.3.14 Notation The symbol $(j|k)$, where j and k are indices, is used to denote that all the terms in front of the symbol should be added with the symbols j and k interchanged. In other words

$$A^i_{jk} = B^i_{jk} - D^i_{jk} - (j|k) \implies A^i_{jk} = B^i_{jk} - D^i_{jk} - \{B^i_{kj} - D^i_{kj}\}.$$

1.3.15 Definition Given a pre-Finsler connection $(F^i_{jk}, N^i_j, V^i_{jk})$ the following tensors are defined. The degree of their p-homogeneity is listed in brackets.

- **Deflection tensor** $D= D^i_j = F^i_{rj}y^r - N^i_j$ (1)
- K **curvature** $= K^i_{hjk} = \delta_k F^i_{hj} + F^r_{hj}F^i_{rk} - (j|k)$ (0)
- F **tensor** $= F^i_{hjk} = \dot{\partial}_k F^i_{hj}$ (-1)

1.3.16 Definition Given a pre-Finsler connection $(F^i_{jk}, N^i_j, V^i_{jk})$ the following curvatures and torsions are defined as in [2].

- **(h) h -torsion** $T = T^i_{jk} = F^i_{jk} - (j|k)$ (0)
- **(v) h-torsion** $R^1= R^i_{jk} = \delta_k N^i_j - (j|k)$ (1)

- (h) **hv-torsion** $V = V^i_{jk}$ (-1)
- (v) **hv-torsion** $P^1 = P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{jk}$ (0)
- (v) **v-torsion** $S^1 = S^i_{jk} = V^i_{jk} - (j|k)$ (-1)
- **h-curvature** $R^2 = R^i_{hjk} = K^i_{hjk} + V^i_{hr} R^r_{jk}$ (0)
- **hv-curvature** $P^2 = P^i_{hjk} = F^i_{hjk} - V^i_{hk|j} + V^i_{kr} P^r_{jk}$ (-1)
- **v-curvature** $S^2 = S^i_{hjk} = \dot{\partial}_k V^i_{hj} + V^r_{hj} V^i_{rk} - (j|k)$ (-2)

1.3.17 Definition The **Cartan connection** is a Finsler connection given by the triad $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ where

$$\Gamma^{*i}_{jk} = g^{is}\Gamma^{*}_{sjk} = g^{is}(\gamma_{sjk} - C_{sjr}G^r_k - C_{jkr}G^r_s + C_{skr}G^r_j) = \frac{1}{2}g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk})$$

and $C^i_{jk} = g^{is}C_{sjk}$. The symbol Γ^{*}_{ijk} is obtained from the definition of γ_{ijk} by replacing ∂_k with δ_k. By construction, C^i_{jk} is the Christoffel symbol of the second kind in the tangent space of a fixed value of x.

M. Matsumoto showed that this connection is uniquely determined by $D = 0, T = 0, S^1 = 0, \nabla^h g = 0, \nabla^v g = 0$. The last two imply that raising and lowering indices is compatible with the covariant derivatives. [2]

1.3.18 Definition The **Rund connection** is defined to be $R\Gamma = (\Gamma^{*i}_{jk}, G^i_j, 0)$.

1.3.19 Definition The **Berwald connection** is $B\Gamma = (G^i_{jk}, G^i_j, 0)$. Covariant derivatives taken with this connection are denoted by (; , ||) for the h and v directions respectively.

T. Okada showed the Berwald connection is uniquely determined by $\nabla^h L = 0, T = 0, D = 0, P^1 = 0, V = 0$. [18] Raising and lowering coefficients are not necessarily metric compatible, but they can be computed using Proposition 1.4.7.

1.3.20 Notation The **h-curvature** tensor R^2 of the **Berwald connection** is written as H, and is given by

$$H^i_{hjk} = \delta_k G^i_{hj} + G^r_{hj} G^i_{rk} - (j|k).$$

The **hv-curvature** tensor P^2 of the **Berwald connection** is written as G and is given by $G^i_{hjk} = \dot{\partial}_k G^i_{hj}$.

1.3.21 Lemma The Berwald, Cartan, and Rund connections all give the same value for a horizontal covariant derivative followed by a transvection.

Proof: All that needs to be shown is $(\Gamma^{*i}_{jk} - G^i_{jk})y^k = 0$ or equivalently $\Gamma^{*i}_{jk}y^k = G^i_j$. This last equation is clear since $D = 0$ for the Cartan connection. □

1.4 Finsler identities

A number of useful identities hold for homogeneous functions.

1.4.1 Proposition Let R, S, and T be functions on $\widetilde{TM}$ which are p-homogeneous of degree s, s, and t respectively. Then the following combinations are p-homogeneous of the given degree.

Function	Degree
ST	s+t
S/T	s-t
$R+S$	s
$\partial_i S$, $\nabla^h S$	s
$\dot{\partial}_i S$, $\nabla^v S$	s-1

Proof: The proof of the first three is trivial.

$$\begin{aligned}\partial_i S(x,\lambda y) &= \frac{\partial S(x,\lambda y)}{\partial x^i} = \lim_{h\to 0} \frac{S(x^j + \delta^j_i h, \lambda y^j) - S(x^j, \lambda y^j)}{h} \\ &= \lim_{h\to 0} \lambda^s \frac{S(x^j + \delta^j_i h, y^j) - S(x^j, y^j)}{h} = \lambda^s \frac{\partial S(x,y)}{\partial x^i} = \lambda^s \partial_i S(x,y)\end{aligned}$$

The covariant horizontal derivative of S is a partial derivative with respect to x summed with the products of S and functions which are p-homogeneous of degree zero. Hence the horizontal covariant derivative does not affect homogeneity. Euler's theorem gives the result for $\dot{\partial}_i S$ since differentiation of the result in Euler's theorem yields

$$\begin{aligned}
\dot\partial_i\dot\partial_j S(x,y)y^j &= \dot\partial_i s S(x,y)\\
\dot\partial_j(\dot\partial_i S(x,y))y^j + \dot\partial_j S(x,y)\delta_i^j &= s\dot\partial_i S(x,y)\\
\dot\partial_j(\dot\partial_i S(x,y))y^j &= (s-1)\dot\partial_i S(x,y),
\end{aligned}$$

where by Euler's theorem, the last line implies $\dot\partial_i S(x,y)$ is p-homogeneous of degree s-1. Finally the vertical covariant derivative consists of a partial derivative with respect to y summed with products of S and functions which are p-homogeneous of degree minus one. □

1.4.2 Definition The **inverse metric tensor** g^{ij} is defined by the relationship $g^{is}g_{sj} = \delta^i_j$ where

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

If g_{ij} is considered as a matrix of entries, then g^{ij} will be the matrix inverse.

1.4.3 Notation The indices of a tensor can be raised or lowered by contraction with the metric. If S^i and T_i are defined tensors, then $S_i := g_{is}S^s$ and $T^i := g^{is}T_s$. If there are indices at both the top and bottom of a tensor, then conventions dictate the new position of the index.

1.4.4 Definition The **length** of a vector ξ at a point (x,y) is given by the formula $\sqrt{g_{ij}(x,y)\xi^i\xi^j}$. Similarly, the length of a one-form w_i at a point (x,y) is given by $\sqrt{g^{ij}(x,y)w_iw_j}$. It should be noted that $L(x,y) = \sqrt{g_{ij}(x,y)y^iy^j}$.

1.4.5 Notation All functions, tensors and connections are defined on $\widetilde{TM}$: everything in Finsler geometry has angular dependence. This dependence is often suppressed for notational clarity. For example g_{ij} is written instead of $g_{ij}(x,y)$. Note that although g_{ij} is 0 degree p-homogeneous it still depends on the ratios y^k/y^l.

1.4.6 Definition A **canonical vector field** is defined on $\widetilde{TM}$ by $l^i = y^i/L$. The covariant form of this vector field is given by $l_i = L^{-1}g_{ij}y^j = L^{-1}\dot\partial_i\frac{1}{2}L^2 = \dot\partial_i L$. Both l^i and l_i have unit length.

1.4.7 Proposition

$$g_{ij}g^{ij} = n \qquad g^{ij}h_{ij} = (n-1)$$
$$g_{ij}y^iy^j = L^2 \qquad h_{ij}y^i = 0$$
$$g^{ih}_{lk} = -g^{jh}g^{is}g_{sjlk}, l \in \{|, |, ;, ||\}$$

Proof: The first four are immediate.

$$\begin{aligned}
g_{ij}g^{ij} &= \delta^i_i = n \\
g^{ij}h_{ij} &= h^i_i = \delta^i_i - l^il_i = n-1 \\
g_{ij}y^iy^j &= \frac{1}{2}(\dot\partial_i\dot\partial_j L^2)y^iy^j = \frac{1}{2}(\dot\partial_j L^2)y^j = \frac{1}{2}2L^2 = L^2 \text{ by Euler's theorem.} \\
h_{ij}y^i &= (g_{ij} - l_il_j)y^i = y_j - Ll_j = 0
\end{aligned}$$

The last relationship follows from

$$0 = (\delta^i_j)_{lk} = (g^{is}g_{sj})_{lk} = g^{is}_{lk}g_{sj} + g^{is}g_{sjlk} \text{ so}$$

$$g^{is}_{lk}g_{sj} = -g^{is}g_{sjlk} \implies g^{is}_{lk}g_{sj}g^{jh} = -g^{jh}g^{is}g_{sjlk} \implies g^{is}_{lk}\delta^h_s = -g^{jh}g^{is}g_{sjlk}.$$

This implies that the covariant derivative of the inverse metric tensor is zero in $C\Gamma$, and in $B\Gamma$ we have

$$g^{ih}_{;k} = -g^{jh}g^{is}g_{sj;k} = 2g^{jh}g^{is}P_{sjk} = 2P^{ih}_{k}, \text{ and}$$

$$g^{ih}_{||k} = -g^{jh}g^{is}g_{sj||k} = 2g^{jh}g^{is}C_{sjk} = 2C^{ih}_{k}.$$

□

The following proposition lists many identities used in Theorem 2.5.1

1.4.8 Proposition Berwald identities for h and l and y.

Proof:

$$
\begin{aligned}
y^i_{||a} &= \delta^i_a \\
y_{i||a} &= (L\dot\partial_i L)_{||a} = \dot\partial_a(L\dot\partial_i L) = \dot\partial_a\dot\partial_i(\frac{1}{2}L^2) = g_{ia} \\
l^i_{||a} &= (y^i L^{-1})_{||a} = \delta^i_a L^{-1} - L^{-2} l_a y^i = L^{-1}(\delta^i_a - y_a y^i L^{-2}) = L^{-1} h^i_a \\
l_{i||a} &= (L^{-1} y_i)_{||a} = -L^{-2} l_a y_i + l^{-1} y_{i||a} = L^{-1}(g_{ia} - L^{-2} y_a y_i) = L^{-1} h_{ia} \\
h_{ab||c} &= (g_{ab} - l_a l_b)_{||c} = 2C_{abc} - l_{a||c} l_b - l_a l_{b||c} = 2C_{abc} - L^{-1} h_{ac} l_b - l_a L^{-1} h_{bc} \\
h^i_{a||b} &= (\delta^i_a - l^I l_a)_{||b} = -(l^i_{||b} l_a + l^i l_{a||b}) \\
&= -(L^{-1} h^i_b l_a + l^i h_{ab} L^{-1}) = -L^{-1}(h^i_b l_a + l^i h_{ab}) \\
y_i h^i_{a||b} &= -L^{-1}(L h_{ab}) = -h_{ab} \\
h^i_{a||b||c} &= -(L^{-1}(h^i_b l_a + l^i h_{ab}))_{||c} \\
&= L^{-1}_{||c}(h^i_b l_a + l^i h_{ab}) - L^{-1}(h^i_{b||c} l_a + h^i_b l_{a||c} + l^i_{||c} h_{ab} + l^i h_{ab||c}) \\
&= L^{-2} l_c (h^i_b l_a + l^i h_{ab}) - L^{-1}((-L^{-1}(h^i_c l_b + l^i h_{bc}))l_a + h^i_b l_{a||c} + (L^{-1} h^i_c) h_{ab} \\
&\quad + l^i(2C_{abc} - L^{-1} h_{ac} l_b - l_a L^{-1} h_{bc})) \\
y_i h^i_{a||b||c} &= L^{-2} l_c (h^i_b l_a + l^i h_{ab}) - L^{-1}((-L^{-1}(L h_{bc}))l_a + 0 + 0 \\
&\quad + L(2C_{abc} - L^{-1} h_{ac} l_b - L^{-1} l_a h_{bc})) \\
&= L^{-1} l_c h_{ab} - L^{-1}(-h_{bc} l_a + 2LC_{abc} - h_{ac} l_b - l_a h_{bc}) \\
&= L^{-1}(l_c h_{ab} + l_b h_{ac} - 2LC_{abc} + 2h_{bc} l_a)
\end{aligned}
$$

□

1.5 Landsberg spaces

1.5.1 Definition A **Landsberg space** is a Finsler space where $\Gamma^{*i}_{jk} = G^i_{jk}$ [2]. If this condition holds in one coordinate system, it will hold in any coordinate system. The geometric invariance of this definition lies in the fact that the difference of these two connections is zero.

In Landsberg spaces the Berwald connection will be h-metrical and symmetric. This is a desirable property since this is the condition that distinguishes the Levi-Civita connection in Riemannian geometry. Landsberg spaces can also be characterized by a result of Ichijyō:

In order that a connected Finsler manifold M be a Landsberg space, it is necessary and sufficient that, for arbitrary two points p and q in M and for any piecewise differentiable curve l joining p and q, the holonomy mapping from $T_p(M)$ to $T_q(M)$ along l with respect to Cartan's non-linear connection $G^i_j = \Gamma^{*i}_{mj} y^m$ is always an affine mapping from the tangent Riemannian space $T_p(M)$ to the tangent Riemannian space $T_q(M)$. [10]

1.5.2 Proposition. In a Landsberg space, h-covariant differentiation is the same in $B\Gamma$, $C\Gamma$, and $R\Gamma$.

Proof: This follows directly from the definition of the h-covariant derivative since $\Gamma^{*i}_{jk} = G^i_{jk}$. □

Several other conditions are equivalent to the given definition.

1.5.3 Theorem The following are all equivalent [2].

(1) $\Gamma^{*i}_{jk} = G^i_{jk}$ (2) $C_{ijk;0} = 0$ (3) $g_{ij;k} = 0$ (4) $P^i_{jk} = 0$

(5) $P^i_{jkl} = 0$ (6) $G^h_{ijk} = C^h_{ik|j}$ (7) $G^h_{ijk} y_h = 0$

Proof: The result for (2) and (3) follow from a Ricci identity with the Berwald connection. In particular it yields

$$\begin{aligned} g_{ij;h\|k} - g_{ij\|k;h} &= -g_{rj} G^r_{ihk} - g_{ir} G^r_{jhk} \text{ by definition.} \\ g_{ij;h\|k} - 2C_{ijk;h} &= -G_{ijhk} - G_{jihk} \\ g_{ij;0\|k} - 2C_{ijk;0} &= -G_{ij0k} - G_{ji0k} \text{ contraction by } y^h \\ g_{ij;0\|k} - 2C_{ijk;0} &= 0 \end{aligned}$$

since by Euler's theorem $G_{abhk} y^h = 0$. Now, $g_{ij;h\|k} y^h = (g_{ij;h} y^h)_{\|k} - g_{ij;k}$ so $g_{ij;0\|k} = -g_{ij;k}$ as $g_{ij;0} = 0$. Hence,

$$g_{ij;k} = -2C_{ijk;0}.$$

Define $D^i_{jk} = \Gamma^{*i}_{jk} - G^i_{jk}$, and consider

$$\begin{aligned} g_{ij|k} - g_{ij;k} &= 0 + 2C_{ijk;0} \text{ by above and h-metricity of } C\Gamma, && (1.4) \\ &= -g_{rj} D^r_{ik} - g_{ri} D^r_{jk} \text{ by the definitions.} && (1.5) \end{aligned}$$

Symmetry implies $C_{ijk;0} = C_{ikj;0}$, so the above yields $g_{rj} D^r_{ik} = g_{rk} D^r_{ij}$ since D^r_{jk} is symmetric in the lower two indices. Also $C_{ijk;0} = C_{kji;0}$ implies $g_{ri} D^r_{jk} = g_{rk} D^r_{ji}$.

Hence Equation (1.4) implies $2C_{ijk;0} = -2g_{rk}D^r_{ij}$ which by the symmetry the Cartan torsion tensor implies $C_{ijk;0} = g_{ir}D^r_{jk}$. Now $C^i_{jk;0} = -D^i_{jk}$, so (1),(2) and (3) are equivalent.

For (4) and (5), identities for $C\Gamma$ [2] are used. The identity is

$$p_{hijk} := C_{jhk|i} + C^r_{ih}P_{rjk} + P_{hijk} - (i|j) = 0.$$

Then $p_{hijk} + p_{ijhk} - p_{jhik} = 0$ will imply that

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + C^r_{hj}P_{rjk} - C^r_{ij}P_{rhk}. \tag{1.6}$$

Transvect by y^h in the above and use Euler's theorem to find

$$P_{ijk} = P_{0ijk} = C_{ijk|0} - C^r_{ij}P_{r0k},$$

where first equality follows from a Ricci identity. Moreover, if this expression is transvected by y^j it shows $P_{i0k} = 0$. Hence $P_{ijk} = C_{ijk|0} = C_{ijk;0}$ so (4) is equivalent to (3). Next the Bianchi identity

$$q_{hijk} := P_{hri}C^r_{kj} - P_{hkj}\big|_i + P_{ihkj} - (i|j) = 0$$

implies that $q_{hijk} + q_{ijhk} - q_{jhik} = 0$. This in turn yields

$$P_{hijk} = P_{jki}\big|_h + P_{kir}C^r_{jh} - (h|i).$$

Hence (4) implies (5). From the Ricci identity mentioned earlier, $P_{0ijk} = P_{ijk}$, (5) implies (4).

For (6), first assume the space is Landsberg. Then the P^1 and P^2 tensors (of $C\Gamma$) are zero, so substituting into the definition of the P^2 tensor yields $G^i_{hjk} = C^i_{hk|j}$. Conversely, (6) implies $C^h_{ik|j}y^j = \dot{\partial}_j G^h_{ik}y^j = 0$ since G^h_{ik} is p-homogeneous of degree zero. It is clear that $C^h_{ik|0} = 0$ is equivalent to (2) since the connection is metrical.

For (7), first note that in a Landsberg space the tensor $C_{ijk|l}$ is fully symmetric in all indices. This occurs since the P^2 and P^1 tensors are zero in Equation (1.6). Hence (6) implies (7) as

$$y_r G^r_{ijk} = y^h g_{hr} C^r_{ik|j} = y^h C_{ihk|j} = 0.$$

Conversely, assuming (7) a Ricci identity can be applied [2, 2.5.2.5].

$$\begin{aligned} y_{i;j\|k} - y_{i\|k;j} &= -y_r G^r_{ijk} \\ 0 - g_{ik;j} &= -y_r G^r_{ijk} \end{aligned}$$

Hence (7) implies (2). □

In a Landsberg space, the Cartan connection has a zero hv-curvature tensor. Indeed, in $C\Gamma$ the following are zero: T, P^1, S^1, P^2, D for Landsberg spaces.

1.6 Berwald spaces

1.6.1 Definition A Finsler space is a Berwald space if the connection $G^i_{jk}(x, y) = G^i_{jk}(x)$ is a function of x alone. This is a geometric definition because if it holds in one coordinate chart, it will hold in all coordinate charts.

1.6.2 Example A Riemannian space is one where the metric tensor is a function of x alone. It is easy to see that a Riemannian space is Berwald. A space is said to be locally Minkowski if every point has a neighborhood in which the metric tensor is a function of y alone. These spaces are also Berwald.

1.6.3 Theorem The following are equivalent.
(1) L is Berwald (2) $G^i_{jkl} = 0$
(3) $C_{ijk|l} = 0$ (4) $C_{ijk;l} = 0$

Proof: (1) $\iff$ (2) is clear from the definition. Moreover (2) will imply that a Berwald space is a Landsberg space by a later result. In a Landsberg space, the tensor $G^i_{jkl} = C^i_{jk|l} = C^i_{jk;l}$ so (2) implies (3) and (4). Finally (3) or (4) implies the space is Landsberg, so the above equation implies (2). □

1.6.4 Remark A Berwald space has several additional properties. The Berwald connection has a zero hv-curvature tensor. Normal coordinates are smoothly defined at the origin, in contrast to the general theory where normal coordinates are C^0 on the zero section of the tangent bundle. Many applications are known [1, 2, 3]. Due to close ties with Riemannian geometry, the theory of Laplacians and Finslerian diffusions is well developed on Berwald spaces [3].

Chapter 2

Landsberg spaces which are Berwald

2.1 One-form metrics in two dimensions

2.1.1 Definition Let $\mathcal{M}^n$ be a manifold with n linearly independent one-forms $a_i^\alpha(x)$ with $\alpha \in \{1, \ldots, n\}$, and the functions $a^\alpha(x, y) := a_i^\alpha(x) y^i$ are defined. A Finsler space is said to have a **one-form Finsler metric** if the fundamental function is a function of a^α, namely $L(x, y) = L(a^1, \ldots, a^n)$.

Two objects play a fundamental role in almost all theory involving two-dimensional Finsler spaces: the Berwald frame, and the main scalar.

2.1.2 Definition Locally define the unit length one-form m_i by $m_i l^i = 0$. This one-form is uniquely determined up to an orientation. Raising the index with the inverse metric, the **Berwald frame** is defined as the pair $\{l^i, m^i\}$. The metric tensor can be written $g_{ij} = l_i l_j + \epsilon m_i m_j$ where $\epsilon = \pm 1$. When $\epsilon = -1$, the metric is not positive definite.

2.1.3 Definition The Cartan torsion tensor goes to zero when transvected with y on any index. This implies $LC_{ijk} = I m_i m_j m_k$ for some function $I : \widetilde{TM} \to \mathbb{R}$: I is called the **main scalar** of a two dimensional Finsler space.

2.1.4 Proposition A two dimensional Finsler space is Landsberg if and only if $I_{|0} = 0$.

Proof: Since $g_{ij|k} = 0 = l_{i|k}$, we must have $m_{i|k}m_j = -m_i m_{j|k}$. This implies $m_{i|k} = 0$. By the definition of the main scalar

$$C_{ijk|0} = (I/L)_{|0} m_i m_j m_k = I_{|0} m_i m_j m_k / L,$$

since $L_{|0} = 0$. Hence $C_{ijk|0} = 0 \iff I_{|0} = 0$. □

2.1.5 Proposition A two dimensional Finsler space is a Berwald space if the main scalar is constant.

Proof: If the main scalar is a constant, then the definition of the main scalar implies $(LC_{ijk})_{|l} = 0$. This in turn implies $L_{|l}C_{ijk} = -LC_{ijk|l}$, so $C_{ijk|l} = 0$. Hence the space is Berwald. □

The previous statement can be strengthened to state that a Berwald space in two dimensions is either locally Minkowski, or the main scalar is constant [2].

2.1.6 Theorem **(Matsumoto and Shimada [16])** If a one-form metric in two dimensions is a Landsberg space, then it is also a Berwald space.

Proof: For a scalar function $F : \widetilde{TM} \to \mathbb{R}$ on a one-form space, where F can be written in terms of the one-forms, define $F_\alpha := \frac{\partial F}{\partial a^\alpha}$. Consider the h-covariant derivative of a scalar F.

$$F_{|i} = \partial_i F - G_i^r \dot{\partial}_r F = F_\alpha \partial_i a^\alpha - G_i^r F_\alpha \dot{\partial}_r a^\alpha = F_\alpha a^\alpha_{|i}$$

Since $L_{|i} = 0$, the above formula implies that $L_\alpha a^\alpha_{|i} = 0$ which can be weakened to state $L_\alpha a^\alpha_{|0} = 0$. Since I can be written as a function of the one-forms, the assumption that the space is Landsberg, $I_{|0} = 0$, implies $I_\alpha a^\alpha_{|0} = 0$. Because the space is two-dimensional, the above equations imply one of two possibilities.

(i) I_α and L_α are proportional ($I_\alpha = \lambda L_\alpha$ for some scalar λ). Since I is p-homogeneous of degree 0 and L is p-homogeneous of degree 1, and L_α is not zero for every α , we must have $\lambda = 0 = I_\alpha$. Hence I is a constant, and this implies the space is Berwald.

(ii) $a^\alpha_{|0} = 0$. Expanding the h-covariant derivative yields

$$\begin{aligned} \partial_i a^\alpha y^i &= y^i G_i^r \dot{\partial}_r a^\alpha, \\ \partial_i a^\alpha_j(x) y^j y^i &= 2G^r \dot{\partial}_r a^\alpha_j(x) y^j = 2a^\alpha_j(x) G^j. \end{aligned}$$

Multiply the above equation with the inverse matrix a^k_α to get

$$2G^k = a^k_\alpha(x)\partial_i a^\alpha_j(x) y^i y^j.$$

Differentiating this expression twice shows that G^i_{jk} is a function of x alone which implies the space is Berwald.

□

2.2 Randers and Kropina spaces in two dimensions

C-reducible spaces in dimension three or greater are all Randers or Kropina spaces. Both are metrics of (α, β) type: this means that the fundamental function can be specified in terms of a Riemannian metric $a_{ij}(x)$ with $\alpha = \sqrt{a_{ij}y^i y^j}$, and a one-form $b_i(x)$ with $\beta = b_i y^i$. A Randers space is given by the fundamental function $L = \alpha + \beta$. The metric is positive definite as long as the length of the one-form $b_i(x)$ is less than one when measured by the Riemannian metric a_{ij}. A Kropina space is given by $L = \alpha^2/\beta$ and these spaces are never positive definite. These two spaces have been studied extensively by Finsler geometers as there are some physical applications for these metrics [2].

In [9], Hashiguchi, Hōjō and Matsumoto showed that two dimensional Randers and Kropina spaces which are Landsberg are Berwald spaces. Their proof, which follows, does not require the positive definiteness of the metric a_{ij}; a_{ij} is only assumed to be regular. First some notation is required.

2.2.1 Notation Given an (α, β) metric, let the connection associated to the Riemannian metric a_{ij} be denoted by $\Gamma^i_{jk}(x)$. Let covariant differentiation with respect to this connection be denoted with the symbol $/$: $\nabla^h A = A_{/i}$. Define the following.

$$\begin{aligned} b^i &:= a^{ir}b_r & (b)^2 &:= a^{rs}b_r b_s \\ r_{ij} &:= \tfrac{1}{2}(b_{i/j} + b_{j/i}) & s_{ij} &:= \tfrac{1}{2}(b_{i/j} - b_{j/i}) \\ r^i_j &:= a^{ir}r_{rj} & s^i_j &:= a^{ir}s_{rj} \\ r_i &:= b_r r^r_i & s_i &:= b_r s^r_i \end{aligned}$$

It should be evident from the above definitions that a_{ij} is used as a metric. Define the difference tensor $B^i_{jk} := G^i_{jk} - \Gamma^i_{jk}$. Define the difference vector $B^i := G^i - \frac{1}{2}\Gamma^i_{00}$. Define $L_\alpha = \frac{\partial L}{\partial \alpha}$ and $L_\beta := \frac{\partial L}{\partial \beta}$.

Now $L_{|i} = 0 = L_\alpha \alpha_{|i} + L_\beta \beta_{|i}$ implies

$$\alpha_{|i} = -\frac{L_\beta}{L_\alpha}\beta_{|i}. \tag{2.1}$$

Now consider the h-covariant derivative of β.

$$\begin{aligned}
\beta_{|i} y^i &= (b_r y^r)_{|i} y^i = b_{r|i} y^i y^r = (\delta_i b_r - G^s_{ri} b_s) y^r y^i \\
&= (\delta_i b_r - \Gamma^s_{ri} b_s + \Gamma^s_{ri} b_s - G^s_{ri} b_s) y^r y^i = (b_{r/i} - B^s_{ri} b_s) y^r y^i \\
&= b_{0/0} - B^s_{00} b_s = r_{00} - 2B^s b_s
\end{aligned} \tag{2.2}$$

A similar procedure applied to the $(b)^2$ yields another relation.

$$\begin{aligned}
(b)^2_{|i} y^i &= (b)^2_{/i} y^i = (a^{rs} b_r b_s)_{/i} y^i = y^i a^{rs} (b_{r/i} b_s + b_r b_{s/i}) \\
&= 2y^i a^{rs} b_s (b_{r/i}) = 2y^i a^{rs} b_s (r_{ri} + s_{ri}) = 2y^i b_s (r^s_i + s^s_i) \\
&= 2y^i (r_i + s_i) = 2(r_0 + s_0)
\end{aligned} \tag{2.3}$$

Defining the scalar $\rho = (b)^2\alpha^2 - \beta^2$, the above equations imply

$$\begin{aligned}
\rho_{|i} y^i &= (b)^2_{|i}\alpha^2 + 2(b)^2 \alpha \alpha_{|i} y^i - 2\beta\beta_{|i} y^i \\
&= 2(r_0 + s_0)\alpha^2 + 2(b)^2\alpha(-\frac{L_\beta}{L_\alpha}\beta_{|i}) y^i - 2\beta\beta_{|i} y^i \\
&= 2(r_0 + s_0)\alpha^2 - 2((b)^2\alpha\frac{L_\beta}{L_\alpha} + \beta)\beta_{|i} y^i \\
&= 2(r_0 + s_0)\alpha^2 - 2((b)^2\alpha\frac{L_\beta}{L_\alpha} + \beta)(r_{00} - 2b_s B^s).
\end{aligned} \tag{2.4}$$

2.2.2 Lemma In [5], Báscó and Matsumoto showed that if $b_i(x)y^i$ is a factor of $a_{ij}(x)y^i y^j$ then the dimension of the space is two, $(b)^2 = 0$, and there exists $\delta = d_i(x)y^i$ which satisfies $d_i b^i = 0$.

2.2.3 Lemma In two dimensional (α, β) spaces with $\beta \neq 0$

- $(b)^2 \neq 0 \implies \exists\epsilon = \pm 1, \exists\delta = d_i(x)y^i$ such that $\alpha^2 = \beta^2/(b)^2 + \epsilon\delta^2$.
- $(b)^2 = 0 \implies \exists\delta = d_i(x)y^i$ where $\alpha^2 = \beta\delta$.

Proof: If $(b)^2$ is not zero then defining $c_{ij} = a_{ij} - b_i b_j/(b)^2$ yields

$$c_{ij}b^j = (a_{ij} - \frac{b_i b_j}{(b)^2})b^j = b_i - b_i = 0.$$

Let d_i be defined by $d^i b_i = 0$ with $d_j d^j = \epsilon$. This implies $c_{ij}d^j = a_{ij}d^j = d_i$. Let $y^i = \lambda d^i + \mu b^i$.

$$c_{ij}y^i y^j = c_{ij}(\lambda d^i + \mu b^i)(\lambda d^j + \mu b^j) = \lambda^2 a_{ij} d^i d^j = \lambda^2 \epsilon$$

Now consider $\delta^2 = d_i y^i d_j y^j = d_i(\lambda d^i + \mu b^i)d_j(\lambda d^j + \mu b^j) = \lambda^2 d_i d^i d_j d^j = \lambda^2$. Hence $\alpha^2 - \beta^2/(b)^2 = c_{00} = \epsilon\delta^2$, which completes the first part.

If $(b)^2 = 0$, there are two cases to consider.

- If $b_1, b_2 \neq 0$, define the one-form d by the equations $a_{11} = b_1 d_1$ and $a_{22} = b_2 d_2$. Let $a = \det(a_{ij}) = a_{11}a_{22} - a_{12}a_{21}$. To show $\alpha^2 = \beta\delta$, it remains to show that $a_{12} = (b_1 d_2 + d_1 b_2)/2$. Consideration of the last term yields

$$\begin{aligned}\frac{1}{2}(b_1 d_2 + b_2 d_1) &= \frac{1}{2}\frac{(b_1)^2 b_2 d_2 + (b_2)^2 b_1 d_1}{b_1 b_2} \\ &= \frac{1}{2b_1 b_2}\frac{(b_1)^2 a_{22} + b_2^2 a_{11}}{b_1 b_2} \\ &= \frac{a}{2b_1 b_2}(a^{11}(b_1)^2 + a^{22}(b_2)^2.\end{aligned}$$

  since $a^{11} = a_{22}/a$, $a^{22} = a_{11}/a$. The assumption that $(b)^2 = 0$ now implies

$$0 = a^{ij}b_i b_j = a^{11}(b_1)^2 + a^{22}(b_2)^2 + 2a^{12}b_1 b_2 = \frac{b_1 b_2}{a}(b_1 d_2 + b_2 d_1) + 2a^{12}b_1 b_2.$$

  As $b_1 b_2 \neq 0$, we have $a^{12} = -\frac{b_1 d_2 + b_2 d_1}{2a}$ so $a_{12} = (b_1 d_2 + b_2 d_1)/2$.

- Now suppose $b_1 \neq 0$ and $b_2 = 0$ (the case where the indices are transposed is similar). We now have

$$0 = (b)^2 = b^1 b_1 = a^{11} b_1 \implies a^{11} = 0 \implies a_{22} = 0.$$

  Define (d_1, d_2) by $a_{11} = b_1 d_1$ and $a_{12} = b_1 d_2/2$. This yields $a_{ij} = (b_i d_j + b_j d_i)/2$. □

2.2.4 Lemma Consider a two dimensional (α,β) space. If there are two functions $f(x), g(x)$ which satisfy

i) $f\alpha^2 + g\beta^2 = 0$ then $f = g = 0$; or if

ii) $f\beta^2 + g\rho = 0$ then $f = g = 0$ or $f = g, (b)^2 = 0$.

iii) Similarly, if there are two one-forms λ, μ which satisfy $\lambda\alpha^2 + \mu\beta^2 = 0$, then either $(b)^2 \neq 0, \lambda = \mu = 0$ or $(b)^2 = 0, \lambda = f(x)\beta, \mu = -f(x)\delta$, where δ is from the above lemma.

Proof:

i) $f\alpha^2 + g\beta^2 = 0$ implies $\alpha^2 = -(g/f)\beta^2$ if $f \neq 0$. This is not possible since $a_{ij} = -(g/f)b_ib_j$ would imply $\det(a_{ij}) = 0$. Hence $f = 0 = g$.

ii) Assume $f\beta^2 + g\rho = 0$. If $(b)^2 = 0$ then $\rho = -\beta^2$ and $f = g$. Suppose $(b)^2 \neq 0$. Then $f\beta^2 + g\beta^2 + g(b)^2\alpha^2 = 0$. By the first case, this implies $f = g = 0$.

iii) If $(b)^2 \neq 0$ then using Lemma 2.2.3 implies

$$\lambda_i\left(\frac{b_jb_k}{(b)^2} + \epsilon d_jd_k\right) + \mu_ib_jb_k = 0.$$

Then $\lambda_i = \mu_i = 0$ since β and δ are independent. If $(b)^2 = 0$, then Lemma 2.2.3 implies $\lambda\beta\delta + \mu\beta^2 = 0$ which implies $\lambda\delta = -\mu\beta$. Since these terms are linear functions in y with β and δ being independent, $\lambda = f(x)\beta$ for some $f(x)$. This choice will also imply $\mu = -f(x)\delta$.

□

2.2.5 Theorem A two dimensional Randers space which is Landsberg is Berwald.

Proof: The difference vector in a Randers space is computed using a formula from [14] as

$$2B^i = \frac{1}{L}(r_{00} - 2\alpha s_0)y^i 2\alpha s^i_0.$$

The main scalar of a Randers space of signature $\epsilon = \pm 1$ is given by [2]

$$I^2 = \epsilon\frac{9\rho}{4L\alpha}.$$

Taking a h-covariant derivative of $9\rho = 4L\alpha\epsilon I^2$ yields

$$\begin{aligned} 9\rho_{|i} &= 4L_{|i}\alpha\epsilon I^2 + 4L\alpha_{|i}\epsilon I^2 + 4L\alpha\epsilon I^2_{|i} \\ &= 0 + 4L(-\frac{L_\beta}{L_\alpha}\beta_{|i})\epsilon\frac{9\rho}{4L\alpha\epsilon} + 4L\alpha\epsilon I^2_{|i} \\ &= -\frac{9\rho\beta_{|i}}{\alpha} + 4L\alpha\epsilon I^2_{|i}. \end{aligned}$$

Rearranging and a transvection yields the following.

$$\begin{aligned} \frac{4}{9}L\alpha\epsilon I^2_{|i}y^i &= \{\rho_{|i} + \frac{\rho\beta_{|i}}{\alpha}\}y^i \\ &= 2(r_0+s_0)\alpha^2 - 2((b)^2\alpha+\beta)(r_{00}-2b_rB^r) + \rho(r_{00}-2b_rB^r)(1/\alpha) \\ &= 2(r_0+s_0)\alpha^2 - (2(b)^2\alpha + 2\beta - \frac{(b)^2\alpha^2-\beta^2}{\alpha})(r_{00}-b_rB^r) \\ &= 2(r_0+s_0)\alpha^2 - \frac{1}{\alpha}((b)^2\alpha^2+2\alpha\beta+\beta^2)(r_{00}-2b_rB^r) \end{aligned}$$

Using the above formula for the difference vector allows the last term to be rewritten.

$$\begin{aligned} r_{00} - 2B_rB^r &= r_{00} - \frac{1}{L}(r_{00}-2\alpha s_0)y^ib_i - 2\alpha s_0^ib_i \\ &= \frac{L}{L}r_{00} - \frac{1}{L}(r_{00}\beta - 2\alpha s_0\beta) - 2\alpha s_0 \\ &= \frac{1}{L}(r_{00}(L-\beta) + 2\alpha s_0(\beta - L)) \\ &= \frac{\alpha}{L}(r_{00}-2\alpha s_0) \end{aligned}$$

This now implies

$$\frac{4}{9}L\alpha\epsilon I^2_{|i}y^i = 2(r_0+s_0)\alpha^2 - \frac{1}{L}((b)^2\alpha^2+2\alpha\beta+\beta^2)(r_{00}-2\alpha s_0).$$

Now $I^2_{|0} = 0 \iff I_{|0} = 0$, so the space is Landsberg $\iff$

$$\begin{aligned} 0 &= 2(r_0+s_0)L\alpha^2 - ((b)^2\alpha^2+2\alpha\beta+\beta^2)(r_{00}-2\alpha s_0) \\ &= 2(r_0+s_0)(\alpha+\beta)\alpha^2 - ((b)^2\alpha^2+\beta^2)(r_{00}-2\alpha s_0) - 2\alpha\beta(r_{00}-2\alpha s_0) \\ &= 2(r_0+s_0)(\beta)\alpha^2 - ((b)^2\alpha^2+\beta^2)r_{00} + 2\alpha\beta(2\alpha s_0) \\ &\quad +2\alpha\{(r_0+s_0)\alpha^2 + ((b)^2\alpha^2+\beta^2)s_0 - \beta r_{00}\} \\ &= 2(r_0+3s_0)(\beta)\alpha^2 - ((b)^2\alpha^2+\beta^2)r_{00} + \\ &\quad +2\alpha\{(r_0+s_0)\alpha^2 + ((b)^2\alpha^2+\beta^2)s_0 - \beta r_{00}\}. \end{aligned}$$

The above equation splits the terms into two pieces: one is polynomial in y^i and the other is an irrational in y^i. Hence the pieces must individually be equal to zero.

$$0 = 2\alpha^2\beta(r_0 + 3s_0) - ((b)^2\alpha^2 + \beta^2)r_{00} \tag{2.5}$$

$$0 = \alpha^2(r_0 + s_0) - \beta r_{00} + ((b)^2\alpha^2 + \beta^2)s_0 \tag{2.6}$$

Now suppose that $(b)^2 \neq 0$. By Equation (2.5) this implies $\alpha^2 r_{00}$ has β as a factor. Since $(b)^2 \neq 0$, by the first lemma above $r_{00} = \beta\omega$ where ω is a one-form. This implies one factor of β can be canceled from Equation (2.5), leaving

$$\alpha^2(2r_0 + 6s_0 - (b)^2\omega) - \beta^2\omega = 0.$$

Applying Lemma 2.2.4 implies that $\omega = 0$. This implies that $r_{00} = 0$. Making this substitution into Equation (2.6) and applying the lemma again implies $s_0 = 0$.

Suppose that $(b)^2 = 0$. Equation (2.5) can be simplified to $\alpha^2(2r_0+6s_0)-\beta r_{00} = 0$, by removing an excess factor of β. Subtracting Equation (2.6) from the above yields

$$\alpha^2((r_0 + 5s_0) - \beta^2 s_0 = 0.$$

Now Lemma 2.2.4 yields $s_0 = f\delta$ and $r_0 = f(\beta - 5\delta)$. The second lemma gives $\alpha^2 = \beta\delta$ which implies $r_{00} = 2f\delta(\beta - 2\delta)$. Moreover $(b)^2 = 0$ and Equation (2.3) imply $f(\beta - 4\delta) = 0$. Since $\beta = 4\delta \implies \alpha^2 = 4\delta^2$ which means that α is not regular, f must be zero. Again, this implies $r_{00} = 0$ and $s_0 = 0$.

In two dimensions the conditions $r_{00} = 0$, or equivalently $b_{i/j} + b_{j/i} = 0$, becomes

$$b_{1/1} = b_{2/2} = 0 \text{ and } b_{1/2} = -b_{2/1}$$

which implies that the condition $s_0 = 0$ is reduced to

$$b^1 b_{1/1} + b^2 b_{2/1} = b^2 b_{2/1} = 0 \text{ and } b^1 b_{1/2} + b^2 b_{2/2} = -b^1 b_{2/1} = 0.$$

Hence $b_{i/j} = 0$, a condition that implies a space is Berwald [11]. □

2.2.6 Theorem If a Kropina space of two dimensions is Landsberg with $(b)^2 \neq 0$, then it is Berwald.

Proof: $L = \alpha^2/\beta$ in a Kropina space and the difference vector is computed from [14] to be

$$B^i = \frac{1}{2(b)^2}(r_{00} + Ls_0)(b^i - \frac{2}{L}y^i) - Ls_0^i.$$

This equation yields the relation

$$\begin{aligned} r_{00} - 2b_i B^i &= r_{00} - \frac{1}{(b)^2}(r_{00} + Ls_0)((b)^2 - \frac{2}{L}\beta) + Ls_0 \\ &= \frac{1}{(b)^2}(r_{00} + Ls_0)(\frac{2}{L}\beta). \end{aligned}$$

The main scalar is given by [12]

$$\epsilon I^2 = \frac{9\rho}{2L(b)^2\beta}.$$

Bringing the denominator to the left side and taking an h-covariant derivative yields

$$\begin{aligned} 2L(b)^2\beta\epsilon I^2_{|i} + 2L(b)^2\beta_{|i}\epsilon I^2 + 2L(b)^2_{|i}\beta\epsilon I^2 &= 9\rho_{|i} \\ 2L(b)^2\beta\epsilon I^2_{|i} + \frac{9\rho}{\beta}\beta_{|i} + \frac{9\rho}{(b)^2}(b)^2_{|i} &= 9\rho_{|i} \\ 2L(b)^2\beta\epsilon I^2_{|i} &= -\frac{9\rho}{\beta}\beta_{|i} - \frac{9\rho}{(b)^2}(b)^2_{|i} + 9\rho_{|i}. \end{aligned}$$

Contracting the above equation with y^i yields

$$\begin{aligned} 2L(b)^2\beta\epsilon I^2_{|0} &= -\frac{9\rho}{\beta}\beta_{|0} - \frac{9\rho}{(b)^2}(b)^2_{|0} + 9\rho_{|0} \\ &= -\frac{9\rho}{\beta}(r_{00} - 2b_iB^i) - \frac{9\rho}{(b)^2}2(r_0 + s_0) \\ &\quad +9(2(r_0 + s_0)\alpha^2 - 2(\frac{L_\beta}{L_\alpha}(b)^2\alpha + \beta)(r_{00} - 2b_iB^i)) \\ &= -\frac{9\rho}{\beta}\frac{2\beta(r_{00} + Ls_0)}{L(b)^2} - \frac{9\rho}{(b)^2}2(r_0 + s_0) + 9\frac{\alpha^2(b)^2}{(b)^2}2(r_0 + s_0) \\ &\quad -9\cdot 2(-\frac{\alpha}{2\beta}(b)^2\alpha + \beta)\frac{2\beta(r_{00} + Ls_0)}{L(b)^2}) \\ &= -9\rho\frac{2(r_{00} + Ls_0)}{L(b)^2} + \frac{9\beta^2}{(b)^2}2(r_0 + s_0) + 9\cdot 2(\alpha^2(b)^2 - 2\beta^2)\frac{(r_{00} + Ls_0)}{L(b)^2}) \\ &= -9\cdot 2\rho\frac{(r_{00} + Ls_0)}{L(b)^2} + \frac{9\beta^2}{(b)^2}2(r_0 + s_0) + 9\cdot 2(\rho - \beta^2)\frac{(r_{00} + Ls_0)}{L(b)^2}) \\ &= +18\frac{\beta^2(Lr_0 + Ls_0)}{L(b)^2} - 18\beta^2\frac{(r_{00} + Ls_0)}{L(b)^2}) \\ &= +18\beta\frac{L\beta r_0 + L\beta s_0 - \beta r_{00} - \beta Ls_0}{L(b)^2} \\ &= +18\beta\frac{\alpha^2 r_0 - \beta r_{00}}{L(b)^2}. \end{aligned}$$

Now the first lemma implies $r_0 = f\beta$ with $f = f(x)$. Hence the above equation in a Landsberg space ($I^2_{|0} = 0$) implies $r_{00} = f\alpha^2$, which yields the equation

$$b_{i/j} + b_{j/i} = 2fa_{ij}. \tag{2.7}$$

Writing the terms out in two dimensions yields

$$b_{1/1} = fa_{11}, \qquad b_{2/2} = fa_{2/2}, \qquad b_{1/2} + b_{2/1} = 2fa_{12}.$$

Define the functions (f_1, f_2) by the following relations

$$b^1 f_1 + b^2 f_2 = f, \qquad -b_2 f_1 + b_1 f_2 = \frac{1}{2}(b_{1/2} - b_{2/1}).$$

These two functions are well defined since $(b)^2 \neq 0$. In conjunction with Equation (2.7), this implies $b_{i/j} = (b^r f_r)a_{ij} + b_i f_j - b_j f_i$. This last relation implies the space is Berwald [12]. □

2.3 C-reducible in more than two dimensions

Matsumoto and Hōjō showed that all C-reducible spaces for $n > 2$ are of Randers or Kropina type. [15]

2.3.1 Theorem **(Matsumoto [13])** If a C-reducible space is Landsberg with $n > 2$, then the space is Berwald.

Proof: In a Landsberg space, the hv-curvature $P_{ijkl} = g_{js}P^s_{ikl}$ of Cartan's connection is zero. Now there is a Bianchi identity [2, eqn 2.4.3.13] which states

$$P_{ijkl} = C_{jkl|l} - C_{ikl|j} + C_{ikr}C^r_{jl|0} - C_{jkr}C^r_{il|0}.$$

The last two terms vanish in a Landsberg space, so the vanishing of the P^2 tensor implies that

$$C_{jkl|i} = C_{ikl|j}. \tag{2.8}$$

Since h-covariant differentiation is metrical,

$$C_{j|i} = g^{kl}C_{jkl|i} = g^{kl}C_{ikl|j} = C_{i|j}.$$

Now substitution of the definition of a C-reducible space into Equation (2.8) yields

$$h_{jk}C_{l|i} + h_{jl}C_{k|i} - h_{ik}C_{l|j} - h_{il}C_{k|j} = 0 \tag{2.9}$$

using $h_{ij|k} = 0$.

The contraction of the middle two terms by g^{li} yields

$$\begin{aligned} g^{li}h_{jl}C_{k|i} - g^{li}h_{ik}C_{l|j} &= g^{li}(g_{jl} - l_jl_l)C_{k|i} - g^{li}(g_{ik} - l_il_k)C_{l|j} \\ &= (g^i_j - l_jl^i)C_{k|i} - (g^l_k - l^l l_k)C_{l|j} \\ &= \delta^i_j C_{k|i} - l_jl^iC_{k|i} - \delta^l_k C_{l|j} + l^l l_k C_{l|j} \\ &= \delta^i_j C_{k|i} - \delta^l_k C_{l|j} + l^l l_k C_{l|j} - l_j l^i C_{k|i} \\ &= l^l l_k C_{l|j} - l_j l^i C_{k|i} \\ &= y^l L^{-1} l_k C_{j|l} - l_j L^{-1} y^i C_{k|i} \\ &= L^{-1} l_k C_{j|0} - l_j L^{-1} C_{k|0}. \end{aligned}$$

Moreover, $C_{k|0} = \{g^{rs}C_{krs}\}_{|l}y^l = g^{rs}C_{krs|l}y^l = 0$ since $g^{rs}{}_{|l} = 0$. Hence contraction of Equation (2.9) by g^{li} yields

$$h_{jk}g^{li}C_{l|i} - (n-1)C_{k|j} = 0$$

by Proposition 1.4.7. This is rewritten as

$$C_{i|j} = \mu h_{ij} \tag{2.10}$$

where μ is a scalar on $\widetilde{TM}$.

Now consider the v-curvature tensor. The definition of the Cartan connection yields

$$\begin{aligned} S_{hijk} &= -g_{is}S^s_{hjk} \\ &= -g_{is}\{C^r_{hj}C^s_{rk} - C^r_{hk}C^s_{rj}\} \\ &= C^r_{hk}C_{rij} - C^r_{hj}C_{rik}. \end{aligned}$$

Using the definition of C-reducibility, the above can be expanded using $C^r_{st} = g^{ra}C_{sat}$.

$$\begin{aligned} S_{ijkl} = &\frac{1}{n+1}(h_{il}C_r + h_{ir}C_l + h_{lr}C_i)\frac{1}{n+1}(h^r_jC_k + h_{jk}C^r + h^r_kC_j) \\ &-\frac{1}{(n+1)^2}(h_{ik}C_r + h_{ir}C_k + h_{kr}C_i)(h^r_jC_l + h_{jl}C^r + h^r_lC_j) \end{aligned}$$

Using the identity $h_{ir}C^r = g_{ir}C^r - l_i l_r C^r = C_i$ where appropriate the above is expanded.

$$\begin{aligned}
S_{ijkl} &= \tfrac{1}{(n+1)^2} \{(h_{il}C_jC_k + h_{il}h_{jk}C^rC_r + \underbrace{h_{il}C_kC_j}_{6} + \underbrace{h_{ij}C_lC_k}_{3} + h_{jk}C_iC_l + \underbrace{h_{ik}C_lC_j}_{2} \\
&\quad + \underbrace{h_{lj}C_iC_k}_{5} + \underbrace{h_{jk}C_lC_i}_{4} + \underbrace{h_{lk}C_iC_j}_{1}) - (h_{ik}C_jC_l + h_{ik}h_{jl}C_rC^r + \underbrace{h_{ik}C_lC_j}_{2} \\
&\quad + \underbrace{h_{ij}C_kC_l}_{3} + h_{ji}C_iC_k + \underbrace{h_{il}C_kC_j}_{6} + \underbrace{h_{kj}C_iC_l}_{4} + \underbrace{h_{jl}C_kC_i}_{4} + \underbrace{h_{kl}C_iC_j}_{1})\} \\
&= \tfrac{1}{(n+1)^2} (h_{il}C_jC_k + h_{il}h_{jk}C_rC^r + h_{jk}C_iC_l - h_{ik}C_jC_l - h_{ik}h_{jl}C^rC_r - h_{jl}C_iC_k) \\
&= \tfrac{1}{(n+1)^2} (h_{il}(\frac{1}{2}h_{jk}C_rC^r + C_jC_k) + h_{jk}(\frac{1}{2}h_{il}C_rC^r + C_iC_l) \\
&\quad - h_{ik}(\frac{1}{2}h_{jl}C_rC^r + C_jC_l) + h_{jl}(\frac{1}{2}h_{ik}C^rC_r + C_iC_k)) \\
&= \tfrac{1}{(n+1)^2} (h_{il}C_{jk} + h_{jk}C_{il} - h_{ik}C_{jl} - h_{jl}C_{ik})
\end{aligned}$$

The last line of the above follows from the definition $C_{ij} := \frac{1}{2}C^rC_rh_{ij} + C_iC_j$. One Bianchi identity [2, eqn 2.4.3.4] in a purely covariant form yields

$$S_{mhij|k} + \{P_{mhri}C^r_{kj} - S_{mhir}P^r_{kj} - P_{mhkj}\,|_i - (i|j)\} = 0.$$

Since the P^1 and P^2 tensors are zero in a Landsberg space, this implies that $S_{hijk|l} = 0$. Now observe,

$$\begin{aligned}
C_{ij|k} &= \frac{1}{2}h_{ij}(C^r_{|k}C_r + C^rC_{r|k}) + C_{i|k}C_j + C_iC_{j|k} \\
&= \frac{1}{2}h_{ij}(g^{rs}C_{s|k}C_r + C^rC_{r|k}) + C_{i|k}C_j + C_iC_{j|k} \\
&= \frac{1}{2}h_{ij}(C_{s|k}C_s + C^rC_{r|k}) + C_{i|k}C_j + C_iC_{j|k} \\
&= h_{ij}C^rC_{r|k} + C_{i|k}C_j + C_iC_{j|k}.
\end{aligned}$$

Using Equation (2.10) the above becomes

$$\begin{aligned}
C_{ij|k} &= h_{ij}C^r\mu h_{rk} + \mu h_{ik}C_j + C_i\mu h_{jk} \\
&= \mu(h_{ij}C^k + h_{ik}C_j + C_ih_{jk}) \\
&= \mu(n+1)C_{ijk}.
\end{aligned}$$

Hence,

$$\begin{aligned} 0 &= S_{hijk|l} \\ &= \frac{1}{(n+1)^2}(h_{hk}C_{ij|l} + h_{ij}C_{hk|l} - h_{hj}C_{ik|l} - h_{ik}C_{hj|l}) \\ &= \mu(h_{hk}C_{ijl} + h_{ij}C_{hkl} - h_{hj}C_{ikl} - h_{ik}C_{hjl}). \end{aligned}$$

Contract the above equation by g^{hk} followed by g^{ij}.

$$\begin{aligned} 0 &= g^{ij}\mu g^{hk}(h_{hk}C_{ijl} + h_{ij}C_{hkl} - h_{hj}C_{ikl} - h_{ik}C_{hjl}) \\ &= g^{ij}\mu((n-1)C_{ijl} + h_{ij}C_l - h_{hj}C^h_{il} - h_{ik}C^k_{jl}) \\ &= g^{ij}\mu((n-1)C_{ijl} + h_{ij}C_l - C_{ijl} - C_{jil}) \\ &= g^{ij}\mu((n-3)C_{ijl} + h_{ij}C_l) \\ &= \mu((n-3)C_l + (n-1)C_l) \\ &= \mu(2n-4)C_l \end{aligned}$$

Since $n \neq 2$ either μ or C_l must be zero. If C_l is zero, then the definition of C-reducible implies that $C_{ijk} = 0$ so the space is Riemannian and Berwald. If $\mu = 0$ then $C_{i|j} = 0$. The definition of C-reducible shows that $C_{ijk|l} = 0$ so the space is Berwald. □

2.4 Landsberg spaces with vanishing projective Douglas tensor

Define the tensor $G_{ij} = G^k_{ijk}$.

2.4.1 Definition The **projective Douglas tensor** is defined by

$$D^i_{hjk} = G^i_{hjk} - \frac{1}{n+1}(y^i G_{hj||k} + \delta^i_j G_{hk} + \delta^i_h G_{jk} + \delta^i_k G_{hj}).$$

The Douglas tensor is invariant under projective transformations.

2.4.2 Theorem ([4, 6]) A Landsberg space with vanishing Douglas tensor is a Berwald space.

Proof: Assume that the Douglas tensor is zero. Then tranvecting the definition of the Douglas tensor by h^l_h yields

$$\begin{aligned} h^l_h G^h_{ijk} &= \tfrac{1}{n+1} \; h^l_h\{y^h G_{ij||k} + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij}\} \\ (\delta^l_h - l^l l_h) G^h_{ijk} &= \tfrac{1}{n+1} \; \{h^l_h y^h G_{ij||k} + h^l_i G_{jk} + h^l_j G_{ik} + h^l_k G_{ij}\} \\ G^l_{ijk} &= \tfrac{1}{n+1} \; \{h^l_i G_{jk} + h^l_j G_{ik} + h^l_k G_{ij}\}. \end{aligned}$$

With $G_{ihjk} = g_{hs} G^s_{ijk}$ the index l can be lowered in the above.

$$G_{iljk} = \frac{1}{n+1}\{h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij}\} \tag{2.11}$$

Next note that $G_{ihjk} = G_{hijk}$ since $G_{hijk} = C_{hij|k}$ by Theorem 1.5.3-(6) and the C tensor is fully symmetric. Now using Equation (2.11) in the above identity yields

$$\begin{aligned} h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij} &= h_{il} G_{jk} + h_{ji} G_{lk} + h_{ki} G_{lj} \\ g^{jl} h_{jl} G_{ik} &= g^{jl}\{h_{ji} G_{lk} - h_{kl} G_{ij} + h_{ki} G_{lj}\} \\ (n-1) G_{ik} &= h^l_i G_{lk} - h^j_k G_{ij} + h_{ki} G_{lj} g^{jl} \\ G_{ik} &= \frac{1}{n-1}\{(\delta^l_i - l^l l_i) G_{lk} - (\delta^j_k - l^j l_k) G_{ij} + h_{ki} G\} \\ &\quad \text{with } G := G_{ij} g^{ij} \\ G_{ik} &= \frac{1}{n-1} h_{ki} G \end{aligned}$$

since $G_{0k} = G_{i0} = 0$. Using the above evaluation of G_{ij}, Equation (2.11) reduces to

$$G_{ihjk} = \frac{G}{n^2-1}(h_{hi} h_{jk} + h_{hj} h_{ik} + h_{hk} h_{ij}). \tag{2.12}$$

Note $h_{ij}C^j_{kl} = g_{ij}C^j_{kl} - l_i l_j C^j_{kl} = C_{kil}$. Next, a Bianchi identity is applied [2, eqn 2.4.3.4].

$$
\begin{aligned}
0 &= S_{hijk;l} \\
&= (C_{hkr}C^r_{ij})_{;l} - (C_{hjr}C^r_{ik})_{;l} \\
&= C_{hkr;l}C^r_{ij} + C_{hkr}C^r_{ij;l} - C_{hjr;l}C^r_{ik} - C_{hjr}C^r_{ik;l} \\
&= G_{hkrl}C^r_{ij} + C_{hkr}g^{rs}C_{isj;l} - G_{hjrl}C^r_{ik} - C_{hjr}g^{rs}C_{isk;l} \\
&= G_{hkrl}C^r_{ij} + C_{hkr}g^{rs}G_{isjl} - G_{hjrl}C^r_{ik} - C_{hjr}g^{rs}G_{iskl} \\
&= \frac{G}{n^2-1}\{(h_{hk}h_{rl} + h_{hr}h_{kl} + h_{hl}h_{rk})C^r_{ij} + C^s_{hk}(h_{is}h_{jl} + h_{ij}h_{sl} + h_{il}h_{sj}) \\
&\qquad -(h_{hj}h_{rl} + h_{hr}h_{jl} + h_{hl}h_{jr})C^r_{ik} - C^s_{hj}(h_{is}h_{kl} + h_{ik}h_{sl} + h_{il}h_{sk})\} \\
&= G\{h_{hk}C_{ijl} + \underbrace{C_{ijh}h_{kl}}_{1} + \underbrace{h_{hl}C_{ijk}}_{2} + \underbrace{C_{hik}h_{jl}}_{3} + h_{ij}C_{hkl} + \underbrace{h_{il}C_{hjk}}_{4} \\
&\qquad -h_{hj}C_{ikl} - \underbrace{C_{hik}h_{jl}}_{3} - \underbrace{h_{hl}C_{ijk}}_{2} - \underbrace{C_{hij}h_{kl}}_{1} - h_{ik}C_{hjl} - \underbrace{h_{il}C_{hjk}}_{4}\} \\
&= G\{h_{hk}C_{ijl} + h_{ij}C_{hkl} - h_{hj}C_{ikl} - h_{ik}C_{hjl}\}
\end{aligned}
$$

Sum the last line by tranvecting with $g^{kl}g^{ij}$.

$$
\begin{aligned}
0 &= G\{g^{kl}g^{ij}h_{hk}C_{ijl} + g^{kl}g^{ij}h_{ij}C_{hkl} - g^{kl}g^{ij}h_{hj}C_{ikl} - g^{kl}g^{ij}h_{ik}C_{hjl}\} \\
&= G\{g^{ij}h_{hk}C^k_{ij} + g^{kl}(n-1)C_{hkl} - g^{kl}h_{hj}C^j_{kl} - g^{kl}h_{ik}C^i_{hl}\} \\
&= G\{g^{ij}C_{hij} + (n-1)C_h - g^{kl}C_{hkl} - g^{kl}C_{hkl}\} \\
&= G\{C_h + (n-1)C_h - C_h - C_h\} \\
&= G(n-2)C_h
\end{aligned}
$$

Now, $n > 2$ by assumption so $G = 0$, or $C_h = 0$. If $G = 0$, then by Equation (2.12) we have $G_{hijk} = 0$ so the space is Berwald. If $C_h = 0$ then note by Theorem 1.5.3

$$C_{h|k} = (g^{rs}C_{hrs})_{|k} = g^{rs}C_{hrs|k} = g^{rs}G_{hkrs} = G_{hk}.$$

This implies $C_{h|k} = 0 = G_{hk}$, so again $G_{hijk} = 0$. □

2.5 Constant curvature

2.5.1 Theorem (Numata [17]) A Landsberg space of scalar curvature K is a Berwald space if $K \neq 0$ and $n \geq 3$.

Numata went further to show that the space will be a Riemannian space of constant curvature. We need some definitions to make sense of the above.

2.5.2 Definition A Finsler space is of **scalar curvature** if the tensor $R_{i0k} = KL^2h_{ik}$ where $R_{i0k} := g_{ir}R^r_{0k}$ by definition. The space has **constant curvature** if K is a constant. Note the function K is p-homogeneous of degree 0

Proof: Assume $\mathcal{M}^n$ is a Landsberg space with non-zero scalar curvature. The Berwald connection has the Bianchi identity

$$G_l{}^i{}_{km;j} - G_l{}^i{}_{jm;k} + H_l^i{}_{jk;m} = 0.$$

Contract the above by y^k. Then $G^i_{lkm;j}$ can be written out as homogeneous terms which when contracted with y^k will give zero. Hence, the above equation reduces to

$$G^i_{ljm;0} - H^i_{ljk||m}y^k = 0.$$

The contraction of the above with $3y_i$ is quickly seen to be

$$0 = \{3G^i_{ljm;0}y_i - 3H^i_{ljk||m}\}y^ky_i = 3(G^i_{ljm}y_i)_{;0} - 3H^i_{ljk||m}y^ky_i = -3H^i_{ljk||m}y^ky_i. \quad (2.13)$$

The second equality sign in the above follows since the Berwald connection is h-metrical in a Landsberg space. The third equality follows from Theorem 1.5.3-(7).

In a Landsberg space, the Berwald h-curvature tensor has the special form $H^i_{ljk} = R^i_{jk||l}$ due to a Bianchi identity. Moreover,

$$\begin{aligned} R^i_{jk} &= h^i_kK_j - (j|k), \\ K_j &= \frac{1}{3}L^2K_{||j} + LKl_j. \end{aligned}$$

The remaining term of Equation (2.13) is computed using Proposition 1.4.8 as required.

$$
\begin{aligned}
0 = 3H^i_{ljk||m}y^k y_i &= 3R^i_{jk||l||m}y^k y_i \\
&= y^k y_i(h^i_{k||l||m}3\dot{K}_j - h^i_{j||l||m}3\dot{K}_k) \\
&\quad +\{y^k y_i h^i_{k||l}3\dot{K}_{j||m} - y^k y_i h^i_{j||l}3\dot{K}_{k||m} + (l|m)\} + \\
&\quad 3y^k(\underbrace{y_i h^i_k}_{0}\dot{K}_{j||l||m} - \underbrace{y_i h^i_j}_{0}\dot{K}_{k||l||m}) \\
&= (2h_{lm})3\dot{K}_j - L^{-1}(l_m h_{jl} + l_l h_{jm} - 2LC_{jlm} + 2h_{lm}l_j)(0 + 3L^2\dot{K}) \\
&\quad +\{0 - y^k(-h_{jl})(L^2\dot{K}_{||k} + 3\dot{K}y_k)_{||m} + (l|m)\} \\
&= 2h_{lm}(3\dot{K}_j - 3\dot{K}y_j) - 3L\dot{K}(l_m h_{jl} + l_l h_{jm} - 2LC_{jlm}) \\
&\quad +\{y^k h_{jl}(2Ll_m\dot{K}_{||k} + L^2\dot{K}_{||k||m} + 3\dot{K}_{||m}y_k + 3\dot{K}g_{km}) + (l|m)\} \\
&= 2h_{lm}(3\dot{K}_j - 3\dot{K}y_j) - 3L\dot{K}(l_m h_{jl} + l_l h_{jm} - 2LC_{jlm}) \\
&\quad +\{h_{jl}(0 - L^2\dot{K}_{||m} + 3\dot{K}_{||m}L^2 + 3\dot{K}y_m) + (l|m)\} \\
&= 2h_{lm}(L^2\dot{K}_{||j}) - 3(y_m\dot{K}h_{jl} + y_l\dot{K}h_{jm} - 2L^2\dot{K}C_{jlm}) \\
&\quad +h_{jl}(2\dot{K}_{||m}L^2 + 3\dot{K}y_m) + h_{jm}(2\dot{K}_{||l}L^2 + 3\dot{K}y_l) \\
&= 2L^2(h_{lm}\dot{K}_{||j} + h_{jl}\dot{K}_{||m} + h_{jm}\dot{K}_{||l} + 3\dot{K}C_{jlm})
\end{aligned}
$$

Since $L \neq 0$ on $\widetilde{TM}$, the above equation implies

$$C_{jlm} = -\frac{1}{3\dot{K}}(h_{jl}\dot{K}_{||m} + h_{jm}\dot{K}_{||l} + h_{lm}\dot{K}_{||j}). \tag{2.14}$$

provided that $\dot{K} \neq 0$. Contraction of the above equation by g^{lm} yields

$$C_j = -\frac{1}{3\dot{K}}(\dot{K}_{||j} + \dot{K}_{||j} + (n-1)\dot{K}_{||j}), \tag{2.15}$$

since $\dot{K}$ is p-homogeneous of degree 0 implies

$$g^{lm}h_{jl}\dot{K}_{||m} = h^m_j\dot{K}_{||m} = \delta^m_j\dot{K}_{||m} + l_j\underbrace{l^m\dot{K}_{||m}}_{\text{0 by Euler}} = \dot{K}_{||j}.$$

Now Equation (2.15) implies $\dot{K}_{||j} = -\frac{3}{n+1}\dot{K}C_j$. When this is substituted back into Equation (2.14), the Cartan torsion tensor is seen to be C-reducible. Since the dimension of the space is assumed to be greater than three, Theorem 2.3.1 implies that the space is Berwald. □

Chapter 3

New classes of Metrics

In the previous section, several important type of Finsler metrics were shown to be Berwald if they were Landsberg. There are other types of metrics which reduce to Berwald spaces. Often, these spaces are also Riemannian—for example Randers spaces and spaces of scalar curvature become Riemannian if they are Landsberg. In this section some different methods of creating Finsler metrics are introduced. The potential for Landsberg spaces to exist within these classes is then examined.

3.1 Geometrical mean metrics

Given two fundamental Finsler functions G and H, the fundamental function $L = \sqrt{GH}$ is defined and called the geometric mean of G and H. It is clearly positive and positive homogeneous. It is not however always positive definite. Positive definiteness can be obtained in some appropriately chosen region as the following example indicates.

3.1.1 Example Since the dependence on the x^i coordinate does not affect the positive definiteness of the resulting metric, assume G and H are independent of x^i. It suffices to examine the situation in two dimensions as in higher dimensions similar structure appears. By appropriate choice of coordinates, assume that $G^2 = (y^1)^2+(y^2)^2$. With a rotation of this basis if required, H can now be expressed as $H^2 = a(b(y^1)^2+(1/b)(y^2)^2)$ where a and b are real constants. The positive definiteness of the resulting metric can be examined by ensuring that both the trace and determinant

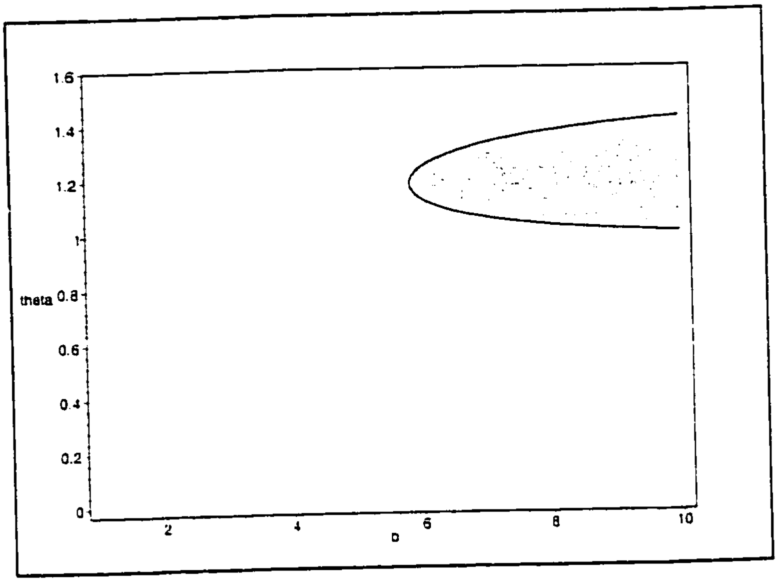

Figure 3.1: The geometric mean. It is not positive definite in the shaded region.

of g_{ij} are positive. The trace will be positive. The value of a will not affect the sign of the determinant since it can be factored out. Moreover, the determinant is a 0-degree homogeneous function in y^i, so its value only depends on $\theta = \arctan(y^1/y^2)$. Finally, the study of $b \geq 1$ will correspond to $b \leq 1$ with y^1 and y^2 interchanged in the formulas. Hence the determinant needs only be examined for $b \geq 1$ and $\theta \in [0, \pi/2]$. The region where the metric is not positive definite is shaded in Figure 3.1. The point where the determinant first becomes zero, in the figure, is $\theta = 3\pi/8$ and $b = 3 + 2\sqrt{2}$.

The above example proves the following proposition

3.1.2 Proposition Let G and H be (two dimensional) Riemannian metrics and let L be their geometric mean. Let I_x denote the indicatrix bundle G at $x \in \mathcal{M}^n$. Let $p_x, q_x \in I_x$ with

$$H(p_x) = \min_{v \in I_x} H(v), \qquad H(q_x) = \max_{v \in I_x} H(v).$$

Then L is positive definite everywhere $\Longleftrightarrow \forall x \in \mathcal{M}^n, \frac{H(q_x)}{H(p_x)} \leq (3 + 2\sqrt{2})$.

3.1.3 Remark The construction of a geometric mean can be used with metrics which are not Riemannian and it can also be used with more than two metrics. In general, define the geometric mean of p Finsler fundamental functions G_i as

$$L = \prod_{i=1}^{p} G_i^{q_i} \text{ with } q_i \in \mathbb{Q} \text{ and } \sum_{i=1}^{p} q_i = 1.$$

L may not be positive definite everywhere in general. The construction of the geometric mean in this manner is analogous to the construction of the m-th root metric $L = \sqrt[m]{y^1 y^2 \ldots y^m}$.

3.1.4 Proposition Let $G_1, \ldots, G_p$ be Riemannian metrics on a two dimensional manifold. Assume that L is a geometric mean of these metrics. If L is positive definite and Berwald on $T\mathcal{M}^n$ with the 0-section removed, then the space is locally Minkowski or $G_i = \sigma_i(x)L$ for all i.

Proof: Let L be Berwald and positive definite on

$$\widetilde{TM} = T\mathcal{M}^n \backslash \{(x, 0) | x \in \mathcal{M}^n\}.$$

A theorem of Szabó [20] asserts that L is either Riemannian or locally Minkowski. Assume the space is Riemannian. Choose $n \in \mathbb{N}$ such that $q_i n \in \mathbb{N}$ for all i. Now L is a quadratic polynomial in y^i and $G_i^{q_i n}$ divides L^n. Since L is a positive definite Riemannian space, L is an irreducible polynomial of degree 2. Now $R[y^1, \ldots, y^n]$ is a unique factorization domain since R is a unique factorization domain. This implies $G_i^{q_i n}$, a polynomial in y^i, must be a power of L. Since G is a irreducible quadratic □
polynomial in y^i, this implies that $G_i = c_i(x)L$.

3.2 Arithmetic mean metrics

Metrics of the fundamental function $L = \sqrt{\tilde{L}^2 + \hat{L}^2}$ have the property of being the arithmetic mean of two standard metric tensors. Moreover, the metric tensor is split into two components, one dependent on y and the other on x. This allows some simplifications when computing.

3.2.1 Proposition Let $\hat{L}$ and $\tilde{L}$ be Finsler metrics defined on $\widetilde{TM}$. Then $L = \sqrt{\hat{L}^2 + \tilde{L}^2}/2$ is a Finsler metric on $\widetilde{TM}$.

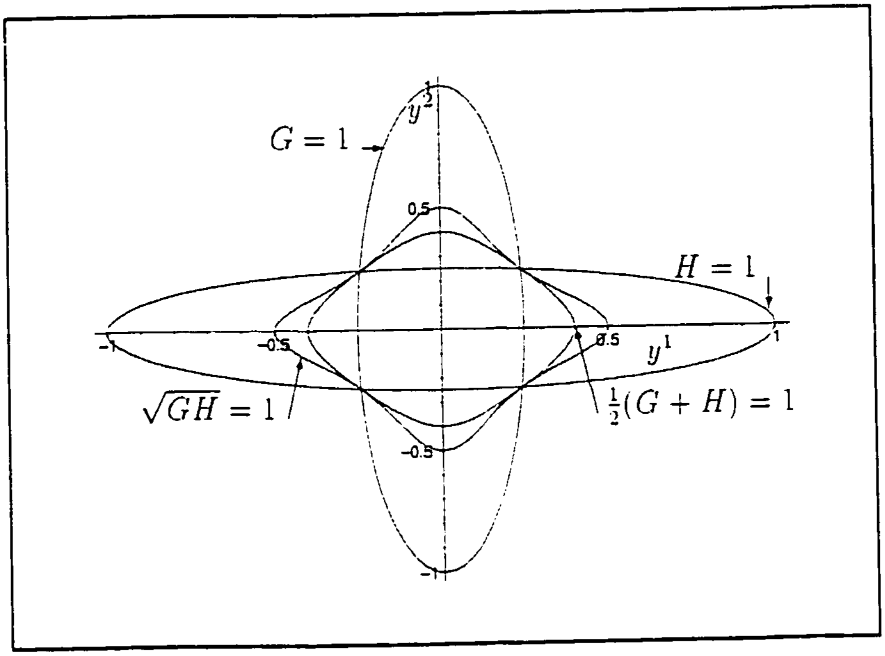

Figure 3.2: Indicatrix comparison of arithmetic and geometric means of G and H

Proof: It is clear that L is positive, and that it is p-homogeneous of degree one. Regularity is easily verified since for $(x, y) \in \widetilde{TM}$ and $\xi \in TM$

$$g_{ij}(x, y)\xi^i\xi^j = \hat{g}_{ij}\xi^i\xi^j + \tilde{g}_{ij}\xi^i\xi^j.$$

□

3.2.2 Example It is useful to compare the geometric and arithmetic means. Let $\mathcal{M}^n = \mathbb{R}^2$ with coordinates in $\widetilde{TM}$ of (x^1, x^2, y^1, y^2). Define the two metrics $G = \sqrt{16(y^1)^2 + (y^2)^2}$ and $H = \sqrt{(y^1)^2 + 16(y^2)^2}$. The indicatricies of these metrics and their means is given in Figure 3.2. It is seen that the geometric mean is not positive definite everywhere in $\widetilde{TM}$, since there is a region where the indicatrix is not convex. However, there is an open subset where $\sqrt{GH}$ is positive definite. In contrast, the arithmetic mean is positive definite everywhere as expected.

This type of metric is considered in [4], and they show that among Landsberg spaces, only Berwald spaces share common geodesics with Riemannian spaces.

3.3 Search for analytic existence

It is possible to examine the existence of Landsberg spaces in a manner analogous to the use of powers series in solving ordinary differential equations in one dimension. This can be used to examine the existence and construction of Landsberg spaces which are not Berwald. Unfortunately, the problem is computationally quite long, so definitive results have not yet been obtained.

The method is as follows. First assume that a fundamental function L is defined in $\widetilde{TM}$. An arbitrary point p is picked on the indicatrix of L. A special set of coordinates to ease computation is chosen, and a Taylor expansion of the function is performed about the point p. The function L is now represented by

$$L = \sum_i \sum_j \sum_k \sum_l a_{ijkl}(x^1)^i(x^2)^j(y^1)^k(y^2)^l.$$

Using this representation, all the tensors, connections, and constraints can be formulated in polynomial expressions of a_{ijkl}. Moreover, the conditions of homogeneity and positive-definiteness can also be enforced. The space will be Landsberg and not Berwald if $C_{ijk:0} = 0$ and $C_{ijk;l} \neq 0$.

To attempt to show that all Landsberg spaces are Berwald, reduces to showing that the constant terms of $C_{ijk;l}$ lie in the ideal of the polynomial constraints given by $C_{ijk:0}$. The theory of Gröbner basis allows these computations to be performed on the computer by symbolic computation. This is done using the constraint terms from $C_{ijk;0} = 0$ of total degree $\leq d$ for some $d \in \mathbb{N}$. Unfortunately, these computations can take double exponential time in the number of variables, which makes examining higher dimensional spaces much more difficult. Moreover, even if no Landsberg spaces exist, d could be arbitrarily large. This has resulted in the inability to resolve this issue using present computing resources.

Conversely, this theory can be used to attempt to construct Landsberg spaces which are not Berwald. This is done by choosing values for some a_{ijkl} to ensure that $C_{ijk;l}$ is not identically zero at p, and then attempting to find recurrence formulas for the remainder of a_{ijkl}. Elimination theory, which has also developed with Gröbner basis theory, can be used for the final step. Unfortunately, this process is not precise, and the computations can take even longer.

3.4 Conclusions

The number of reduction theorems that have been found is quite discouraging. The form of these theorems shows that finding a two-dimensional Landsberg space which is not Berwald will likely involve methods which are different from those used for higher dimensional cases. Since many common classes of metrics in Finsler geometry are Berwald when they are Landsberg, it seems that Landsberg spaces which are not Berwald are quite rare, and possibly do not exist.

Bibliography

[1] P.L. Antonelli and R.H. Bradbury. *Volterra-Hamilton models in the ecology and evolution of colonial organisms*, volume 2 of *Series in mathematical biology and medicine*. World Scientific, 1996.

[2] P.L. Antonelli, R.S. Ingarden, and M. Matsumoto. *The theory of Sprays and Finsler Spaces with applications in Physics and Biology*, volume 58 of *Fundamental Theories of Physics*. Kluwer Academic Publishers, 1993.

[3] P.L. Antonelli and T.J. Zastawniak. *Fundamentals of Finslerian diffusion with applications*, volume 101 of *Fundamental theories of physics*. Kluwer Academic, 1999.

[4] S. Bácsó, F. Hosvay, and B. Kis. Landsberg spaces with common geodesics. *Publ. Math. Debrecen*, 42(1–2):145–154, 1993.

[5] S. Bácsó and M. Matsumoto. Projective changes between Finsler spaces with (α, β) metric. *Tensor, N.S.*, 55:252–257, 1994.

[6] S. Bácsó and M. Matsumoto. Reduction theorems of certain Landsberg spaces to Berwald spaces. *Publ. Math. Debrecen*, 48(3–4):357–366, 1996.

[7] R.L. Bryant. Finsler spheres. In *Finsler Geometry*, volume 196 of *Contemporary Mathematics*, pages 27–41. AMS, July 1995.

[8] H. Busemann. *Metric methods in Finsler Spaces and in the Foundations of Geometry*. Number 8 in Annals of Mathematics Studies. Princeton University Press, 1942.

[9] M. Hashiguchi, S. Hōjō, and M. Matsumoto. Landsberg spaces of dimension two with (α, β) metric. *Tensor, N.S.*, 1996.

[10] Y. Ichijyō. On special Finsler Connections with the vanishing hv-Curvature tensor. *Tensor, N.S.*, 32:149–155, 1978.

[11] S. Kikuchi. On the condition that a space with (α, β)-metric be locally Minkowskian. *Tensor, N.S.*, 33:242–246. 1979.

[12] M. Kitayama, M. Azuma, and M. Matsumoto. On Finsler space with (α, β)-metric. Regularity, geodesics, and main scalars. *J. Hokkaido Univ. of Education*, 46:1–10, 1995.

[13] M. Matsumoto. On C-reducible Finsler spaces. *Tensor, N.S.*, 24:29–37, 1972.

[14] M. Matsumoto. The Berwald connection of a Finsler space with an (α, β)-metric. *Tensor, N.S.*, 50:18–21, 1991.

[15] M. Matsumoto and S. Hōjō. A conclusive theorem on C-reducible Finsler spaces. *Tensor, N.S.*, 32:225–230, 1978.

[16] M. Matsumoto and H. Shimada. On Finsler spaces with 1-form metric. *Tensor, N.S.*, 32:161–169, 1978.

[17] S. Numata. On Landsberg spaces of scalar curvature. *Tensor, N.S.*, 1975.

[18] T. Okada. Minkowskian product of Finsler spaces and the Berwald connection. *J. Math. Kyoto Univ.*, 22:323–332, 1982.

[19] H. Rund. *The Differential geometry of Finsler Spaces*. Springer-Verlag, 1959.

[20] Z. Szabó. Positive definite Berwald spaces. *Tensor, N.S.*, 35:25–39, 1981.