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PCAC AND π-π SCATTERING

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled PCAC AND π-π SCATTERING, submitted by Roy Geoffrey Levers in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

This thesis is concerned with two problems which are currently of interest; (a) investigation and understanding of the s-wave pion-pion interaction and (b) derivation of Current Algebra (CA) results without using CA. Problem (a) has recently received fairly intense theoretical and experimental study, as this interaction is obviously important due to the many processes, both virtual and real, in which it partakes. It also appears that the pion, at the present time, has a rather special rôle in particle physics resulting from its small mass (compared to other hadrons) and the Partially Conserved Axial-Vector Current Hypothesis (PCAC).

Experimentally this interaction is difficult to study as direct scattering experiments are, as yet, not feasible (quite unlike the π-N case); data has to be extracted, subject to some theoretical model, from interactions in which the π-π force plays a subsidiary rôle. Theoretically, much effort has been devoted to it since Weinberg carried out a CA calculation in 1966 and showed that the s-wave scattering lengths appeared to be much smaller than hitherto surmised. Results for the scattering lengths and phase shifts differ widely between authors.

Problem (b) has also been studied in the last year or so, in view of the general success of Gell-Man's "God-given"

(quark model) Chiral Algebra of Currents in many calculations.

In Chapter I consistency relations are derived between the amplitudes $\pi N \to 2\pi N$ and $\pi N \to \pi N$ from PCAC with a zero mass initial pion; from this the π-π interaction is extracted and compared to CA results. It seems that the latter can only be obtained on the assumption of peripheral (OPE) dominance of the non-nucleon pole part of the pion production amplitudes.

Chapter II investigates the effect of threshold unitarity and a crossed-channel isospin two (I_t=2) amplitude (which hitherto has been neglected, but must be present) on the CA π-π scattering lengths. Weinberg's CA calculation (modified with I_t=2) is used to constrain single pole approximations to the left-hand cut, from which the two s-wave phase shifts (for I=0,2) are determined, indicating that neither resonate.

Chapter III shows that the relation between anomalous nucleon magnetic moments and the off-shell photopion production amplitudes can be simply obtained from a first order e.m. modified version of PCAC. Such rules were originally obtained (1965-1966) by Fubini et al. from dispersive representations of CA commutators.

All of this work has appeared, or will appear, in the literature. Chapter I (with A.N. Kamal) in Phys. Rev. 162, 1543 (1967), Chapter II is in press (Nuovo Cimento, and Phys. Rev.) and Chapter III (with A.N. Kamal) in Nucl. Phys. B6, 32(1968).

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Here I would like to gratefully acknowledge the continuous help, advice and encouragement which has been given to me by my supervisor Dr. A.N. Kamal; it goes almost without saying that most of the problems investigated in this thesis have been initiated by him. In addition I would like to thank my wife, Christine, for solving special cases of equations (2.46) and arranging them in a form suitable for programming; in this regard I am also indebted to Mr. C.J. Elliott for initial help in my dialogue with the computer.

Finally, my thanks go to the Physics Department of the University of Alberta, as well as to the National Research Council of Canada, for their financial support over the last two and a half years.

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CHAPTER I

ABSTRACT: Reasons and methods for studying the π-π interaction are given. The concepts of Current Algebra (CA) and the Partially Conserved Axial-Vector Current Hypothesis (PCAC) are dealt with briefly. Using PCAC (only) in the single pion production process $\pi + N \rightarrow 2\pi + N$, consistency relations are derived between this process and the π-N elastic scattering amplitude for a zero mass initial pion. From this the amplitudes for the three isospin π-π scattering states at the unphysical point $s=0$, $t=u=m_\pi$ are extracted and compared to Weinberg's CA amplitudes and Iliopoulos' unitarily corrected version of this. It appears there will only be mutual agreement for the three isospin states if the peripheral process dominates the non-pole part of the pion production amplitude. Otherwise the I=0 amplitude is of the opposite sign and two orders of magnitude larger than the CA result. Means of reducing this to the CA value are studied, all without success.

1. INTRODUCTION

Lately much interest has centered upon the possible fundamental nature of the pion(1). From the viewpoint of nuclear physics its importance naturally arises due to its small mass, consequently generating the long range part of the inter-nucleon potential; in this sense it may be considered fundamental. It may possibly be fundamental from considerations involving PCAC where it appears to play a rather basic rôle at low energy. But at intermediate and high energies it does not appear to be particularly unusual, and SU(3) symmetry would suggest that it is just another member of a group of particles having no greater or less importance than (for example) the K-meson.

A question of importance, for believers in Regge Pole theory, is to decide whether or not the pion lies on a Regge trajectory-this is not settled at the present time. Thus, from this viewpoint the fundamental nature of the pion has not been decided. It is important to remember however that the pion plays a major rôle in many decay processes due to its small mass (for e.g. $K \to 2\pi$), as well as in virtual processes, such as the above mentioned nucleon-nucleon forces. It would seem that a study of the π-π interaction might throw a good deal of light on what one might consider to be a rather basic strong interaction "spinless billiard ball" problem.

Direct π-π scattering experiments are not presently feasible,[2] consequently there is a lack of the large amount of direct experimental information such as is obtained in the π-N case. Such indirect information as we do have comes from extracting data, <u>subject to some theoretical model</u>, from experiments in which the π-π interaction is involved in a subsidiary rôle, as in single pion-production in π-N collisions, backward π-N scattering and K decay or, more recently, in $p\bar{p} \to 3\pi$ scattering experiments and in angular correlations of the K_{ℓ_4} decay spectrum.

From the theoretical standpoint the situation is rather chaotic——nevertheless, certain extremely general trends are now in evidence; these will be studied in Chapter II. One of the first attempts to understand the π-π interaction was that of Chew and Mandelstam[3,4] in 1961 using the N/D technique in S-matrix theory and the then fairly new assumed Mandelstam representation. Given the <u>entire</u> left-hand cut the complete problem can be solved; in practice of course they had to use approximate methods such as s-wave dominance and iteration[3] or p-wave dominance and pole approximations[4]. This latter paper, where crossed-channel ρ exchange leads approximately to ρ-meson in the direct channel has led to the concept of the reciprocal bootstrap. This has been more successful in the π-N case where $N^*(1236)$ produces ample attraction in the P_{11} channel to bind the nucleon and N(938) also creates enough attraction in P_{33} channel to produce $N^*(1236)$.

Due to the success of a large number of calculations involving the notions of CA and PCAC, of which perhaps the best known is the Adler-Weisberger[5] sum rule for the axial-vector form factor g_A, Weinberg,[6] in 1966, with extra assumptions outside this framework, obtained s-wave π-π scattering lengths which were somewhat lower than those considered up to that time. Present attempts to calculate s-wave $\pi-\pi$ phase shifts and scattering lengths may simply be divided into two categories i.e. either one attempts to incorporate unitarity in Weinberg's approach, or one ignores CA and PCAC altogether. Much of this is considered in more detail in Chapter II. In this chapter we try to understand the meaning of Weinberg's calculation from a dynamical approach, in which CA is not used. As, however, we will always be referring to CA and PCAC it seems an appropriate point to briefly review what the former means, and what the latter might mean.

2. CURRENT ALGEBRA AND PCAC.

Let $V_\mu^\alpha(\underline{x},t)$, $A_\mu^\alpha(\underline{x},t)$ denote respectively the vector and axial-vector strangeness conserving weak hadronic currents, with α either an SU(2) or an SU(3) index. In the customary way one associates vector and axial-vector charges with these currents in the form $Q_\alpha^V = \int d^3x\ V_o^\alpha(\underline{x},t)$, and $Q_\alpha^A(t) = \int d^3x\ A_o^\alpha(\underline{x},t)$. The main difference between them is that Q_α^V is time independent whereas Q_α^A isn't, due to the conservation of V_μ^α and lack of conservation of A_μ^α (best noted by observing that charged pions decay into leptons).

Despite the fact that $\partial_\mu A_\mu^\alpha \neq 0$, Gell-Mann,(7) in 1961, postulated that the following group algebra holds.

$$[Q_\alpha^V, Q_\beta^V] = if_{\alpha\beta\gamma}Q_\gamma^V , \qquad (1.1)$$

$$[Q_\alpha^V, Q_\beta^A] = if_{\alpha\beta\gamma}Q_\gamma^A , \qquad (1.2)$$

$$[Q_\alpha^A, Q_\beta^A] = if_{\alpha\beta\gamma}Q_\gamma^V . \qquad (1.3)$$

Here $f_{\alpha\beta\gamma}$ are simple structure constants, and the commutators are equal time ones (our commutators will always be of this type, generally this will not explicitly be indicated). The real postulate of Gell-Mann is in the assumed closure property of eq.(1.3), so that we are dealing with the group $SU(2) \otimes SU(2)$ or $SU(3) \otimes SU(3)$; it is not generally satisfied in a field theory model of hadrons, although it is true in the quark model.

One may remove the integral signs in the above equations to obtain

$$[V_o^\alpha(x), V_\mu^\beta(y)] = if_{\alpha\beta\gamma}V_\mu^\gamma(x)\delta^3(\underline{x}-\underline{y}) + \text{S.T.}, \qquad (1.4)$$

$$[V_o^\alpha(x), A_\mu^\beta(y)] = [A_o^\alpha(x), V_\mu^\beta(y)]$$

$$= if_{\alpha\beta\gamma}A_\mu^\gamma(x)\ \delta^3(\underline{x}-\underline{y}) + \text{S.T.}, \qquad (1.5)$$

$$[A_o^\alpha(x), A_\mu^\beta(y)] = if_{\alpha\beta\gamma}V_\mu^\gamma(x)\delta^3(\underline{x}-\underline{y}) + \text{S.T.} \qquad (1.6)$$

where S. T. denotes Schwinger terms. They were first noted in the e.m. case by Gotô and Imanura[8] and rediscovered four years later by Schwinger[9]. These terms are proportional to spatial gradients of δ-distributions and may be unbounded,and arise when one considers rather carefully the limiting coincidence value of products of wave functions at non-coincident points. Okubo[10] has generalized the above proof of the existence of S. T. to currents which are not conserved, such as A_μ, obviously a rather important extension so as to include eqs. (1.5) and (1.6) as well as (1.4). In practical calculations equal time commutators of the above type often occur, and it is certainly rather important to have some method of dealing with the S. T. One possible way of circumventing the difficulty they pose is to ignore them,either pretending that they are c-numbers which do not contribute to matrix elements of interest, or that they disappear in the zero momentum limits in which most calculations become reasonably tractable. But perhaps the best way to overcome the whole problem is by not using CA; a problem considered in more detail in Chapter III.

As the pion decays into the hadronic vacuum state then

$$\langle o|A^\alpha_\mu(o)|\pi^\beta(k)\rangle = i\delta_{\alpha\beta}f_\pi k_\mu \quad , \qquad (1.7)$$

so that as $k^2 \neq 0$ and the pion decay form factor $f_\pi \neq 0$ then $\partial_\mu A^\alpha_\mu \neq 0$. Nambu, and Gell-Mann et al.[11] suggested that A^α_μ was conserved in the limit of zero pion mass, or infinite momentum transfer. The PCAC hypothesis states that $\partial_\mu A^\alpha_\mu$ is proportional to the renormalized pion field operator i.e.

$$\partial_\mu A^\alpha_\mu = C^\alpha \phi^\alpha \ . \tag{1.8}$$

Replacing the interpolating pion field by the divergence of a current is a very strange notion, however, field theory models exist where this is true. But before trying to understand its meaning let's use it to see what observable results it might imply.

From eq. (1.7) one immediately finds that

$$C^\alpha = m^2 f_\pi \ , \tag{1.9}$$

where m is the pion mass (which we will often denote the unit mass). Let us now insert A^α_μ between one-nucleon states of momentum p_1 and p_2; then the only invariant on which the form factors can depend is the squared momentum transfer $(p_2 - p_1)^2 = k^2$(say).

$$\langle p_2 | A^\alpha_\mu(o) | p_1 \rangle = iN_{12} \bar{u}(p_2) [g_A(k^2)\gamma_\mu \ + $$

$$+ f_A(k^2)\sigma_{\mu\nu}k_\nu + h_A(k^2)k_\mu]\gamma_5 \tau_\alpha u(p_1) \ , \tag{1.10}$$

where the spinor normalization factors are given by

$N_{12} = M/\sqrt{E_1 E_2}$ and γ, σ and τ matrices are defined in reference (12). Due to the Hermiticity of A_μ the three weak form factors $g_A(k^2)$, $f_A(k^2)$ and $h_A(k^2)$ are all real; if one believes that the axial-vector current has the same G-parity as the term proportional to g_A then f_A will be absent-in any case, upon taking the divergence of eq. (1.10) $k_\mu k_\nu \sigma_{\mu\nu} \equiv 0$ so our argument is unaffected by its inclusion. Thus

$$\langle p_2 | \partial_\mu A^\alpha_\mu(o) | p_1 \rangle = N_{12} \bar{u}(p_2)[2Mg_A(k^2) + k^2 h_A(k^2)]\gamma_5 \tau_\alpha u(p_1) \; , \tag{1.11}$$

with M the nucleon mass (= 6.7 m).

Now

$$\langle p_2 | \phi^\alpha | p_1 \rangle = \frac{1}{m^2-k^2} \langle p_2 | (\Box + m^2)\phi^\alpha | p_1 \rangle$$

$$= \frac{1}{m^2-k^2} \langle p_2 | J^\alpha_\pi | p_1 \rangle = N_{12} \frac{gK(k^2)}{m^2-k^2} \bar{u}(p_2)\gamma_5 \tau_\alpha u(p_1) \; , \tag{1.12}$$

$K(k^2)$ is the pionic form factor of the nucleon normalized so that $K(m^2) = 1$, and g is the rationalized renormalized pion-nucleon coupling constant ($g^2/4\pi \approx 14.6$).

From eqs. (1.8), (1.11) and (1.12)

$$C^\alpha(k^2) = \frac{(m^2-k^2)}{gK(k^2)} [2Mg_A(k^2) + k^2 h_A(k^2)] \; , \tag{1.13}$$

i.e. $$C^{\alpha}(o) = 2Mm^2 g_A(o)/gK(o) \; . \tag{1.14}$$

The assumption in PCAC is that $C^{\alpha}(k^2)$ is well approximated by $C^{\alpha}(o)$; from (1.9) and (1.14) one then obtains the famous Goldberger-Treiman[13] relationship

$$f_{\pi} = 2Mg_A(o)/gK(o) \; . \tag{1.15}$$

Experimentally f_{π} = 190 MeV. whereas from eq. (1.15) it would be 168.5 MeV. i.e. an error of 11%,rather good by the standards of particle physics. This, by itself, does not make eq. (1.8) any more meaningful - it can simply be regarded as a trick to use when inserting the divergence of A_{μ}^{α} between matrix elements (the possible usefulness of which has only been shown in one case so far!).

Weisberger,[14] Bernstein et al,[15] and others obtain eq. (1.15) from a different viewpoint. They regard the matrix elements of $\partial_{\mu}A_{\mu}^{\alpha}$ to be highly convergent operators obeying unsubtracted dispersion relations dominated by the pion-pole terms (the so-called pion-pole dominated divergence of the axial-vector current - PDDAC). This is certainly a very much weaker hypothesis than PCAC leading to the same result (1.15). But as noted by Adler,[16] Weisberger,[14] Meiere and Sugawara,[17] Iliopoulos[18] and others, PDDAC is not without ambiguity when there are more invariants than the one in which one disperses.

For example, consider the forward scattering of an axial-vector spurion of momentum k off a nucleon of momentum p; define our two independent invariants as $\nu = p.k/M$ and k^2 so that the scattering amplitude is $A(\nu,k^2)$. From PDDAC we neglect the integral of the continuum contribution from the branch cut and find $A(\nu,k^2) \simeq B(\nu)/(k^2-m^2)$ where $B(\nu)$ denotes the residue at $k^2=m^2$. But we could equally well use $\nu' = \nu + ak^2$ and k^2 as our independent variables - dispersing in k^2 again and using PDDAC. then $A(\nu',k^2) \simeq B(\nu'-am^2)/(k^2-m^2)$ which disagrees with the above result unless the residue is either a constant or a = o. It may be reasonable in this case to assume, as Weisberger does, that PDDAC means that one so chooses a that the residue is a maximum. This is somewhat difficult to do in practice, and appears an increasingly complicated manoevre to perform as the number of independent invariants increase, even assuming one has a definite prescription to maximize the residue at the expense of the continuum contribution. In the case we have considered Weisberger does it by choosing a so as to keep the pole as far away from the branch cut as possible; as he notes, a natural but somewhat arbitrary criterion of pole dominance.

Let us, throughout the remainder of this thesis, adopt the following attitude to PCAC i.e. a formal replacement the validity of which must ultimately be tested by experiment. This is essentially the viewpoint of Adler, and we write

$$\partial_\mu A_\mu^\alpha = C\phi^\alpha + R \ , \tag{1.16}$$

where R is *any* residual operator. Eq. (1.16) is then given real content for states $|a\rangle$ and $|b\rangle$ for which $\langle b|\phi^\alpha|a\rangle \neq 0$ by assuming that

$$|\langle b|R|a\rangle| \ll |\langle b|\phi^\alpha|a\rangle| \ , \tag{1.17}$$

for momentum transfer near the one pion-pole, with C a constant over this momentum transfer range. Thus eq. (1.8) now has meaning:- $\partial_\mu A_\mu^\alpha$ is assumed to be a good interpolating field for the pion and is a "smooth" operator over this momentum transfer range.

3. CONSISTENCY CONDITIONS FROM $\pi+N \rightarrow 2\pi+N$

PCAC is a remarkable hypothesis, involving as it does a relation between a weak hadronic current and a hadron; there is however, an even more remarkable relation derived by Adler(16) from PCAC. Consider an axial spurion $\partial_\mu A_\mu$ of momentum q_1 scattering off a nucleon (p_1) to produce a pion (q_2) and a nucleon (p_2) and define our two independent invariants as

$$\nu' = (p_1+p_2).q_1/2M$$

$$\nu'_B = -q_1.q_2/2M \; , \qquad (1.18)$$

i.e.

$$2M\nu' = s + \frac{t}{2} - M^2 - m^2 > 0$$

$$2M\nu'_B = -\frac{t}{2} - m^2 < 0 \; , \qquad (1.19)$$

where s and t are the usual Mandelstam invariants for this process.(19) Then at the unphysical point $\nu' = \nu'_B = q_1^2 = 0$ Adler found

$$g^2K = MA^+ \; , \qquad (1.20)$$

with A^+ defined by

$$A_{\alpha\beta} = A^+\delta_{\alpha\beta} + \tfrac{1}{2}\,[\tau_\alpha,\tau_\beta]A^- \; , \qquad (1.21)$$

where $A_{\alpha\beta}$ is the non-pole part of the π-N scattering amplitude.

Naturally one must have a plausible model to extend the physical $A^+(\nu'>o,\ \nu'_B<o,\ q_1^2=m^2)$ to the unphysical point $\nu' = \nu'_B = 0$ followed by a reasonable prescription to extrapolate from $q_1^2 = m^2$ to $q_1^2 = 0$. Adler does this (we will consider it in a little more detail later when we evaluate our constraints) and finds that $g^2K(q_1^2=0)/MA^+(\nu'=\nu'_B=q_1^2=0)$ is unity to within 10%, possibly 5%.

A very general constraint between the pion-nucleon elastic scattering amplitude and the pion production amplitude has been derived by Nambu et al.[20] from chirality invariance, and by Adler[16] from PCAC, although it has not been exploited in any detail. This will be the object of our considerations in the rest of this chapter.

The process we study is

$$\pi(k,\alpha) + N(p_1) = N(p_2) + \pi(k_1,\beta) + \pi(k_2,\gamma) \ . \qquad (1.22)$$

As we will always be concerned with operators sandwiched between an in state $|p_1>$ and an out state $<\pi(k_1,\beta),\pi(k_2,\gamma),p_2|$ it will prove very convenient to use the notation

$$<\beta\gamma|A^{\alpha}_{\mu}> \equiv <\pi(k_1,\beta),\ \pi(k_2,\gamma),\ p_2|A^{\alpha}_{\mu}(o)|p_1>/N'_{12} \ , \qquad (1.23)$$

where $N'_{12} = N_{12}/\sqrt{4k_{10}k_{20}}$. From the various momenta six independent variables can be formed (only five if particles are on their mass-shells).

$$\begin{aligned} s &= (k_1+k_2)^2 \ , & v &= (k_2+p_2)^2 \ , \\ t &= (k_1-k)^2 \ , & w &= (k+p_1)^2 \ , \\ u &= (k_2-k)^2 \ , & & k^2 \ , \end{aligned} \qquad (1.24)$$

with s, t and u the Mandelstam variables which we use later to describe π-π scattering. It will be often convenient to introduce the non-independent squared momentum transfer to the nucleon variable $\Delta^2 = (p_2-p_1)^2$ so that

$$s + t + u = \Delta^2 + k^2 + 2 . \qquad (1.25)$$

Then one finds

$$\langle\beta\gamma|A_\mu^\alpha\rangle = -ig_A(o)\bar{u}(p_2)\sum_{i=1}^{15} O_\mu^i A^i \gamma_5 u(p_1) , \qquad (1.26)$$

where the A^i are analytic functions of the variables of equation (1.24).

The O_μ^i have been chosen to be

$$O_\mu^1 = (p_2-p_1)_\mu , \qquad O_\mu^9 = \not{k}O_\mu^3 ,$$

$$O_\mu^2 = (p_1+k)_\mu , \qquad O_\mu^{10} = \not{k}O_\mu^4 ,$$

$$O_\mu^3 = (k_1+k_2)_\mu , \qquad O_\mu^{11} = (\not{k}_2-\not{k}_1)O_\mu^1 ,$$

$$O_\mu^4 = (k_2-k_1)_\mu , \qquad O_\mu^{12} = (\not{k}_2-\not{k}_1)O_\mu^2 ,$$

$$O_\mu^5 = \not{k}\gamma_\mu , \qquad O_\mu^{13} = (\not{k}_2-\not{k}_1)O_\mu^3 ,$$

$$O_\mu^6 = \gamma_\mu , \qquad O_\mu^{14} = (\not{k}_2-\not{k}_1)O_\mu^4 ,$$

$$O_\mu^7 = \not{k}O_\mu^1 , \qquad O_\mu^{15} = (\not{k}_2-\not{k}_1)O_\mu^6 .$$

$$O_\mu^8 = \not{k}O_\mu^2 , \qquad (1.27)$$

Our analysis only becomes tractable in the limit $k \to o$; as we will eventually be taking the divergence of eq. (1.26) (i.e. multiplication by $-ik_\mu$) then we need to know the singular contributions to the various A^i. These are the nucleon pole terms shown in figure (1) which behave like k^{-1} as $k \to o$; the blobs represent the amplitude $M_{\beta\gamma}$ for the process $N \to 2\pi N$, which is related to the amplitude $\pi N \to \pi N$ by crossing, with the conventional decomposition

$$M_{\beta\gamma} = A_{\beta\gamma} + \tfrac{1}{2}(\not{k}_2 - \not{k}_1)B_{\beta\gamma} \ , \qquad (1.28)$$

where $B_{\beta\gamma}$ contains the pole terms of the amplitude and satisfies the same relation as equation (1.21).

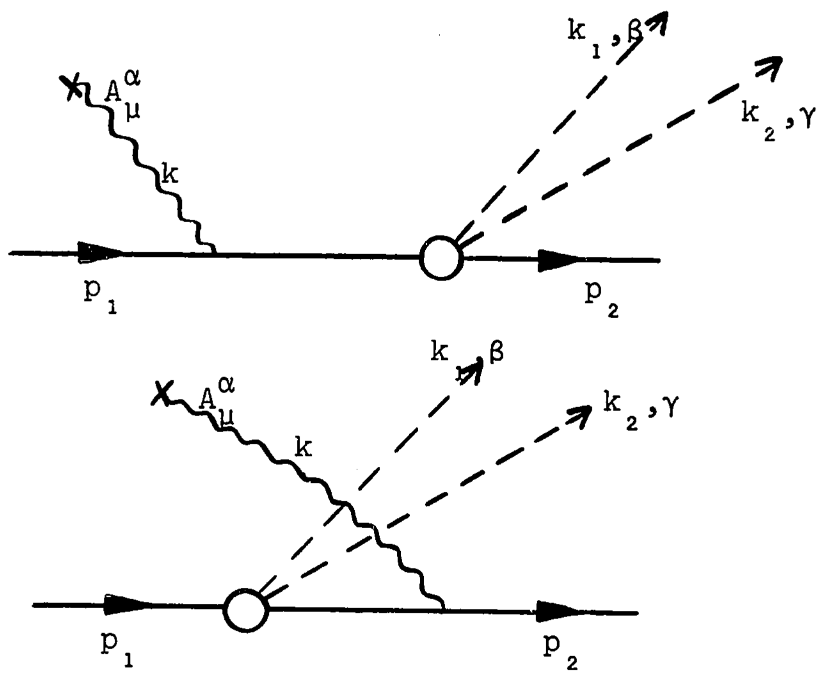


FIGURE (1). THE POLE DIAGRAMS. Wavy lines represent A_μ^α .

Evaluating these diagrams explicitly one finds, as $k \to o$

$$<\beta\gamma|A_\mu^\alpha>^{pole}_{k \to o} =$$

$$= ig_A(o)\bar{u}(p_2)[M_{\beta\gamma}\tau_\alpha\frac{(p_{1\mu}+M\gamma_\mu)}{p_1.k}\gamma_5 + \frac{(p_{2\mu}-M\gamma_\mu)}{p_2.k}\gamma_5\tau_\alpha M_{\beta\gamma}]u(p_1) \;, \tag{1.29}$$

i.e.

$$<\beta\gamma|\partial_\mu A_\mu^\alpha>^{pole}_{k \to o} =$$

$$= g_A(o)\bar{u}(p_2)[M_{\beta\gamma}\tau_\alpha(1 + \frac{M\not{k}}{p_1.k})\gamma_5 + (1-\frac{M\not{k}}{p_2.k})\gamma_5\tau_\alpha M_{\beta\gamma}]u(p_1) \;. \tag{1.30}$$

We now determine the divergence of the non-pole terms denoted by $\bar{A}^i = A^i - A^{i,pole}$; from equation (1.26), after a little tedious algebra one finds.

$$\bar{u}(p_2)\sum_{i=1}^{5} k_\mu O_\mu^i \bar{A}^i \gamma_5 u(p_1) = \tfrac{1}{2}(t+u-2)\bar{A}^1 +$$

$$+ \tfrac{1}{2}(w+k^2-M^2)\bar{A}^2 + \tfrac{1}{2}(2k^2+2-t-u)\bar{A}^3 +$$

$$+ \tfrac{1}{2}(t-u)\bar{A}^4 + k^2\bar{A}^5 \;. \tag{1.31}$$

In the limit $k \to o$ then $k^2 = o$, $w = M^2$ and $t = u = 1$ implying that the coefficients of the various $\bar{A}^i$ in equation

(1.31) vanish identically. As all our $\bar{A}^i$ are in fact kinematically analytic (this is not obvious, but is proved in the appendix to this chapter) the right hand side of equation (1.31) vanishes. By similar reasoning, in this same limit

$$\bar{u}(p_2)\sum_{i=6}^{10} k_\mu O^i_\mu \bar{A}^i \gamma_5 u(p_1) = \bar{u}(p_2) \not{k} \gamma_5 \bar{A}^6 u(p_1) \ , \qquad (1.32)$$

$$\bar{u}(p_2)\sum_{i=11}^{14} k_\mu O^i_\mu \bar{A}^i \gamma_5 u(p_1) = 0 \ , \qquad (1.33)$$

and $$\bar{u}(p_2) k_\mu O^{15}_\mu \bar{A}^{15} \gamma_5 u(p_1) = \bar{u}(p_2)(\not{K}_2 - \not{K}_1)\not{k}\gamma_5 \bar{A}^{15} u(p_1). \qquad (1.34)$$

Collecting all our expressions we have

$$\langle\beta\gamma|\partial_\mu A^\alpha_\mu\rangle_{k\to o} = g_A(o)\bar{u}(p_2)[M_{\beta\gamma}\tau_\alpha(1+M\not{k}/p_1.k)\gamma_5 +$$

$$+ (1-M\not{k}/p_2.k)\gamma_5\tau_\alpha M_{\beta\gamma} + \not{k}\bar{A}^6\gamma_5 + (\not{K}_2-\not{K}_1)\not{k}\bar{A}^{15}\gamma_5]u(p_1). \qquad (1.35)$$

From PCAC

$$\langle\beta\gamma|\partial_\mu A^\alpha_\mu\rangle = C\langle\beta\gamma|J^\alpha_\pi\rangle \ . \qquad (1.36)$$

Once again $\langle\beta\gamma|J^\alpha_\pi\rangle$ may be split up into pole terms (i.e. fig. (1) with A^α_μ replaced by a pion) and non-pole terms.

$$<\beta\gamma|J_\pi^\alpha>^{pole}_{k\to o} =$$

$$= g\frac{K(o)}{2}\,\bar{u}(p_2)[M_{\beta\gamma}\tau_\alpha\frac{\not k\gamma_5}{p_1.k} - \frac{\not k\gamma_5}{p_2.k}\tau_\alpha M_{\beta\gamma}]u(p_1)\ , \qquad (1.37)$$

Note that this implies that $C<\beta\gamma|J_\pi^\alpha>^{pole}_{k\to o}$ cancels exactly against a similar term in eq. (1.35) when inserted into eq. (1.36).

The most general form for the non-pole terms is

$$<\beta\gamma|J_\pi^\alpha>^{non-pole} = \bar{u}(p_2)[F_{\alpha,\beta\gamma} + (\not k_2-\not k_1)G_{\alpha,\beta\gamma} +$$

$$+ \not k H_{\alpha,\beta\gamma} + (\not k_2-\not k_1)\not k L_{\alpha,\beta\gamma}]\gamma_5 u(p_1)\ . \qquad (1.38)$$

Substituting (1.35), (1.37) and (1.38) into (1.36) one finds

$$\bar{u}(p_2)[M_{\beta\gamma}\tau_\alpha\gamma_5 + \gamma_5\tau_\alpha M_{\beta\gamma} + \not k\gamma_5\bar{A}^6 + (\not k_2-\not k_1)\not k\gamma_5\bar{A}^{15}]u(p_1) =$$

$$= \frac{2M}{gK(o)}[F_{\alpha,\beta\gamma} + (\not k_2-\not k_1)G_{\alpha,\beta\gamma} + \not k H_{\alpha,\beta\gamma} + (\not k_2-\not k_1)\not k L_{\alpha,\beta\gamma}]\gamma_5 u(p_1)\ .$$

(1.39)

Now

$$M_{\beta\gamma}\tau_\alpha\gamma_5 + \gamma_5\tau_\alpha M_{\beta\gamma} = (A_{\beta\gamma}\tau_\alpha+\tau_\alpha A_{\beta\gamma})\gamma_5 +$$

$$+ \tfrac{1}{2}(\not k_2-\not k_1)(B_{\beta\gamma}\tau_\alpha-\tau_\alpha B_{\beta\gamma})\gamma_5\ . \qquad (1.40)$$

Thus, one finds

$$(2M/gK(o))F_{\alpha,\beta\gamma} = A_{\beta\gamma}\tau_{\alpha} + \tau_{\alpha}A_{\beta\gamma} , \qquad (1.41)$$

$$(2M/gK(o))G_{\alpha,\beta\gamma} = \tfrac{1}{2}[B_{\beta\gamma},\tau_{\alpha}] , \qquad (1.42)$$

$$(2M/gK(o))H_{\alpha,\beta\gamma} = \bar{A}^{6} , \qquad (1.43)$$

and $$(2M/gK(o))L_{\alpha,\beta\gamma} = \bar{A}^{15} . \qquad (1.44)$$

The last two equations are quite useless, as both $\bar{A}^6$ and $\bar{A}^{15}$ are unknown. As $k \to o$ however the intermediate nucleon in figure (1) is on the mass-shell so that eqs. (1.41) and (1.42) are directly related to the on-mass-shell π-N scattering amplitude.

$A_{\beta\gamma}$ and $B_{\beta\gamma}$ are (when $k \to o$) analytic functions of two invariants which we choose to be

$$\nu = -(p_1+p_2).k_1/2M ,$$

$$\nu_B = -(p_2-p_1).k_1/2M . \qquad (1.45)$$

(note the difference between equation (1.45) and Adler's definitions, eq. (1.18)). When $k = o$ we can imagine that we are dealing with the scattering process $p_1 + (-k_1) = p_2 + k_2$ with Mandelstam variables $s' = (p_1-k_1)^2$ and $t' = (p_2-p_1)^2 = \Delta^2$.

Then

$$2M\nu = s' + \frac{t'}{2} - M^2 - m^2 < o \ ,$$

and $$2M\nu_B = \frac{t'}{2} \leq o \ ; \qquad (1.46)$$

these should be compared to eq. (1.19). F and G are analytic functions of the six invariants defined in eq. (1.24), but when $k \to o$ they are naturally only functions of ν and ν_B

$$s = 4M\nu_B \ ,$$
$$t = u = 1 \ ,$$
$$v = 1 + M^2 + 2M(\nu-\nu_B) \ ,$$
$$w = M^2 \ ,$$
$$k^2 = o \ , \qquad (1.47)$$

It still remains for the isospin decomposition of eqs. (1.41) and (1.42) to be carried out. The right hand sides may be so decomposed by using eq. (1.21) with a similar decomposition for $B_{\beta\gamma}$. Denoting the isospin projection operators for the pion-production amplitude by $Q^{(2I,I')}_{\alpha,\beta\gamma}$, where I denotes the total initial (final)isospin and I' the isospin of the two pion subsystem——the final state then

$$Q^{(1,0)}_{\alpha,\beta\gamma} = \frac{1}{3}\,\delta_{\beta\gamma}\tau_\alpha \;,$$

$$Q^{(1,1)}_{\alpha,\beta\gamma} = \frac{1}{3\sqrt{2}}\,\tfrac{1}{2}[\tau_\beta,\tau_\gamma]\tau_\alpha \;,$$

$$Q^{(3,1)}_{\alpha,\beta\gamma} = \frac{1}{\sqrt{2}}\,[\delta_{\alpha\gamma}\tau_\beta - \delta_{\alpha\beta}\tau_\gamma - \frac{1}{3}\,[\tau_\beta,\tau_\gamma]\tau_\alpha] \;,$$

$$Q^{(3,2)}_{\alpha,\beta\gamma} = \frac{1}{\sqrt{10}}\,[\delta_{\alpha\gamma}\tau_\beta + \delta_{\alpha\beta}\tau_\gamma - \frac{2}{3}\,\delta_{\beta\gamma}\tau_\alpha] \;. \tag{1.48}$$

Therefore

$$A_{\beta\gamma}\tau_\alpha + \tau_\alpha A_{\beta\gamma} = 2A^+\tau_\alpha\delta_{\beta\gamma} + \tfrac{1}{2}\{\tau_\alpha[\tau_\beta,\tau_\gamma] + [\tau_\beta,\tau_\gamma]\tau_\alpha\}A^-$$

$$= 6A^+Q^{(1,0)}_{\alpha,\beta\gamma} + 2\sqrt{2}\,A^-[Q^{(1,1)}_{\alpha,\beta\gamma} - Q^{(3,1)}_{\alpha,\beta\gamma}] \;, \tag{1.49}$$

and

$$\tfrac{1}{2}[B_{\beta\gamma},\tau_\alpha] = \sqrt{2}\,B^-[Q^{(3,1)}_{\alpha,\beta\gamma} + 2Q^{(1,1)}_{\alpha,\beta\gamma}] \;. \tag{1.50}$$

Finally, one obtains the consistency conditions.

$$\frac{2M}{g\overline{K}(o)}\,F^{(1,0)} = 6A^+ \;, \tag{1.51}$$

$$\frac{2M}{g\overline{K}(o)}\,F^{(1,1)} = -\frac{2M}{g\overline{K}(o)}\,F^{(3,1)} = 2\sqrt{2}\,A^- \;, \tag{1.52}$$

$$F^{(3,2)} = G^{(3,2)} = G^{(1,0)} = 0 \;, \tag{1.53}$$

$$\frac{2M}{g\overline{K}(o)}\,G^{(1,1)} = \frac{4M}{g\overline{K}(o)}\,G^{(3,1)} = 2\sqrt{2}\,B^- \;, \tag{1.54}$$

with all F's and G's functions of ν and ν_B (only).
Note that under the exchange $k_1 \rightleftarrows k_2$ $(\nu\to-\nu,\ \nu_B\to\nu_B)$ $A^-(\nu,\nu_B)$ is odd i.e.

$$F^{(1,1)}(\nu=o,\nu_B) = F^{(3,1)}(\nu=o,\nu_B) = 0 \ , \qquad (1.55)$$

—a rather obvious condition as both $F^{(3,1)}$ and $F^{(1,1)}$ must be odd under the exchange of two final state pions.

4. EXTRACTION OF π-π AMPLITUDES FROM CONSTRAINTS

For the π-π scattering process $q_1(\alpha) + q_2(\beta) \to - q_3(\gamma) - q_4(\delta)$ we define our S, T and M matrices as

$$(S-1)_{\alpha\beta,\gamma\delta} = i(2\pi)^4\delta^4(\sum_{i=1}^{4} q_i)T_{\alpha\beta,\gamma\delta} \ ,$$

$$T_{\alpha\beta,\gamma\delta} = M_{\alpha\beta,\gamma\delta}/(2\pi)^6(16q_{10}q_{20}q_{30}q_{40})^{\frac{1}{2}} \ . \qquad (1.56)$$

Then

$$M_{\alpha\beta,\gamma\delta} = \sum_{I=o,1,2} M^I P^I_{\alpha\beta,\gamma\delta} \ , \qquad (1.57)$$

where $P^I_{\alpha\beta,\gamma\delta}$ are the usual isospin projection operators

$$P^{o} = \frac{1}{3}\,\delta_{\alpha\beta}\delta_{\gamma\delta} \ ,$$

$$P^{1} = \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \ ,$$

$$P^{2} = \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{2}{3}\,\delta_{\alpha\beta}\delta_{\gamma\delta}) \ , \qquad (1.58)$$

Weinberg's[(6)] CA calculation gave, at the unphysical point $s = o,\ t = u = 1$ the result (in pion mass units)

$$M^{o} = -\ 8.3 \ ,$$

$$M^{1} = M^{2} = o \ , \qquad (1.59)$$

whereas a CA calculation of Iliopoulos,[(18)] including unitarity, led to the four solutions.

$$M^{o} = +4.1, -1.3, -8.3 \text{ and } -14.5 \ ,$$

$$M^{1} = M^{2} = o \ . \qquad (1.60)$$

In the Chew-Low[(21)] extrapolation method one assumes that the amplitude for $\pi+N \rightarrow 2\pi+N$ is dominated by the one pion exchange (peripheral) process at low momentum transfer squared (Δ^{2}). One may write the M matrix for the whole process as

$$M = -\ igK(\Delta^{2})\frac{i\bar{u}(p_{2})\gamma_{5}u(p_{1})}{\Delta^{2} - m^{2}}M_{\pi}(k,q,k_{1},k_{2}) \ , \qquad (1.61)$$

with M_π the M matrix for the off-mass-shell π-π scattering amplitude ($q = k_1 + k_2 - k$); it would describe on-mass-shell scattering if $\Delta^2 \to m^2$. Now $\Delta^2 \leq o$ for the whole physical process to occur, but as m is small one may consider the point $\Delta^2 = m^2$ closest to the physical region for M, which from equation (1.61) will then be dominated by M_π.

It will transpire that our calculations simplify enormously when $\nu = \nu_B = o$, $k^2 = o$ i.e. $s = o$, $t = u = 1$; from eq. (1.25) $\Delta^2 = o$ also. In this limit therefore, as a first attempt at calculating M^I let us assume that the peripheral process dominates the *non-pole* part of the pion production amplitude i.e. that $F \simeq F_\pi$ in the constraint equations

$$2MF^{(1,o)}/gK(o) = 6A^+(\nu=\nu_B=k^2=o) \;,$$
$$F^{(1,1)} = F^{(3,1)} = F^{(3,2)} = o \;. \qquad (1.62)$$

Now

$$\bar{u}(p_2)F_{\pi\alpha,\beta\gamma}\gamma_5 u(p_1) = \lim_{\Delta^2\to o} M_{\alpha\delta,\beta\gamma}\frac{1}{\Delta^2-m^2}\{-igK(\Delta^2)\}\bar{u}(p_2)\gamma_5\tau_\delta u(p_1) \;,$$

i.e.

$$F_{\pi\alpha,\beta\gamma} = -\,gK(o)M_{\alpha\delta,\beta\gamma}\tau_\delta \;. \qquad (1.63)$$

But
$$P^{o}_{\alpha\delta,\beta\gamma}\tau_{\delta} = Q^{(1,o)}_{\alpha,\beta\gamma} ,$$

$$-\sqrt{2}\, P^{1}_{\alpha\delta,\beta\gamma}\tau_{\delta} = 2Q^{(1,1)}_{\alpha,\beta\gamma} + Q^{(3,1)}_{\alpha,\beta\gamma} ,$$

$$2P^{2}_{\alpha\delta,\beta\gamma}\tau_{\delta} = \sqrt{10}\, Q^{(3,2)}_{\alpha,\beta\gamma} , \qquad (1.64)$$

and so
$$F_{\pi}^{(1,o)} = -\, gK(o)M^{o} ,$$

$$F_{\pi}^{(3,1)} = gK(o)M^{1}/\sqrt{2} ,$$

$$F_{\pi}^{(1,1)} = \sqrt{2}gK(o)M^{1} ,$$

$$F_{\pi}^{(3,2)} = -\sqrt{10}gK(o)M^{2}/2 . \qquad (1.65)$$

Therefore, from eqs. (1.62) and (1.65)

$$M^{1} = M^{2} = o ,$$

$$M^{o} = -\,(3/M)A^{+}(o,o,o) . \qquad (1.66)$$

If we could show that $A^{+}(o,o,o) = 8.3M/3 = 18.6$ then eqs. (1.66) are exactly the same as Weinberg's CA calculations.

Let us note, to begin with, that $\nu_B = o$ corresponds to forward scattering (see eq. (1.46)); this implies in Adler's notation that $\nu_B' = -\,1/2M$ from eq. (1.19). Rather fortunately Adler has evaluated $A^{+}(\nu'=o,\ \nu_B'=-1/2M, q_1^2=o)$ $= A^{+}(\nu=o,\ \nu_B=o,\ k^2=o)$ by two independent methods with remarkable consistency; it will be rather convenient to examine both briefly.

The problem may be considered in two stages viz.

(i) Extrapolation of the on-mass-shell physical amplitude. $A^{+}(\nu'>o, \nu_B'= - 1/2M, q_1^2=1)$ to the unphysical amplitude $A^{+}(\nu'=o, \nu_B'=-1/2M, q_1^2=1)$

(ii) The use of a plausible model to continue this amplitude from the on-mass-shell point $q_1^2=1$ to $q_1^2=o$

In Adler's original problem (eq. (1.20)) he needed to evaluate the amplitude $A^{+}(\nu'=\nu_B'=q_1^2=o)$, this was done by using a once subtracted (at threshold) fixed momentum transfer dispersion relation. Using the phase shift analysis of Roper[22] with S, P, D and F waves the integral over the right hand cut was determined up to a pion laboratory kinetic energy of 700 MeV, this was a sufficiently high energy to go to due to the rapid convergence of the integral. The threshold subtraction constant can be expressed in terms of the S, P, D and F scattering lengths——it turns out that it is this term which makes the major contribution to $A^{+}(\nu'=\nu_B'=o, q_1^2=1)$, mostly from the P-wave. Unfortunately there was a variation of about 20% in the threshold subtraction constant, depending on whether one used all of Roper's scattering lengths for S, P, D and F waves, or only Roper's D and F wave scattering lengths combined with the S and P wave scattering lengths of Woolcock.[23]

He concluded that too much emphasis was placed on this subtraction constant, and introduced a method whereby one could "smear" the subtraction over a finite section of the real axis——this should sample the experimental curves better.

Here we apply this smeared subtraction method to determine $A^+(\nu=\nu_B=o,\ k^2=1) = A^+(\nu'=o,\ \nu_B'=-1/2M,\ q_1^2=1)$.

Let
$$F(x) = \frac{A^+(x,\nu_B'=-1/2M,q_1^2=1)}{[(x-1)(x+1)(x-\nu_m)(x+\nu_m)]^{\frac{1}{2}}}\,, \qquad (1.67)$$

with $\nu_m > 1$ (threshold for physical scattering). Then, as $F(x) \to o$ as $x \to \infty$, one may write an unsubtracted dispersion relation for F(x) as

$$F(x) = \frac{1}{\pi}\int_1^\infty dy\ \frac{\Delta F(y)}{2i}\left(\frac{1}{y-x} + \frac{1}{y+x}\right), \qquad (1.68)$$

where $\Delta F(y)$ is the discontinuity of F across the cut. The denominator in eq. (1.67) has opposite signs on opposite sides of its cut, which runs from 1 to ν_m; eq. (1.68) therefore leads to the "smeared" dispersion relation at zero momentum transfer

$$A^+(o,-1/2M,1) = \frac{2}{\pi}\int_1^{\nu_m} \frac{dy}{y}\ \frac{\mathrm{Re}A^+(y,-1/2M,1)\nu_m}{[(y-1)(y+1)(\nu_m-y)(\nu_m+y)]^{\frac{1}{2}}} -$$

$$-\frac{2}{\pi}\int_{\nu_m}^\infty \frac{dy}{y}\ \frac{\mathrm{Im}A^+(y,-1/2M,1)\nu_m}{[(y-1)(y+1)(y-\nu_m)(y+\nu_m)]^{\frac{1}{2}}}\ . \qquad (1.69)$$

Define the pion-nucleon centre of mass energy $W_m = M + \omega_m$; then $\nu_m = \nu_B' + \omega_m + \omega_m^2/2M$. Adler found, using Roper's phase shifts, that the variation of $A^+(o,-1/2M,1)$ was extremely small as ω_m was varied from about 60 MeV below to 60 MeV above the (3,3) resonance——more precisely

$$A^+(o,-1/2M,1) = 26.05 \pm 0.69 \ . \qquad (1.70)$$

For an alternative method Adler used forward scattering dispersion relations on

$$F^+(x) = A^+(x,-1/2M,1) + xB^+(x,-1/2M,1) \ , \qquad (1.71)$$

with a smeared subtraction. Using Roper's phase shifts for pion laboratory kinetic energies below 700 MeV, and the total π^+P and π^-p cross sections of Amblard et al.[24] above this energy he concluded that

$$A^+(o,-1/2M,1) = 26.15 \pm 0.2. \qquad (1.72)$$

From eqs.(1.70) and (1.72) one may therefore conclude

$$A^+(o,-1/2M,1) = 26.1 \pm 0.7 \ . \qquad (1.73)$$

Evaluation of this in the static limit retaining only the (3,3) resonance in the narrow resonance approximation gives a result only about 20% less than eq. (1.73), suggesting that a suitable model for going off-mass-shell would be of this type. By assuming that the ratio of the off to on-mass-shell (3,3) resonance amplitude is the same as its Born projection he concludes that taking the pion off-mass-shell reduces eq. (1.73) by about 0.5 i.e. finally

$$A^{+}(\nu=\nu_B=k^2=o) = 25.6 \pm 0.7 \ . \qquad (1.74)$$

From eq. (1.66) we therefore find that $M^{o} = -11.4 \pm 0.3$ i.e. about 30% less than Weinberg's CA amplitude. Thus, if we could assume peripheral dominance of the non-pole part of the pion-production amplitude we would have a satisfactory explanation of Weinberg's (rather low compared to previous expectations) CA result.

We must now evaluate other terms which might be important in the momentum limits we use, to determine their "perturbation" on the assumed peripheral dominant one. There are, of course, an extremely large number of other possible terms, but it seems reasonable to assume that the most important contributions come from processes involving two nucleon poles (figure 2(a)) and those involving at least one nucleon pole (figures 2(b) and 2(c)) as the nucleon propagators become large in the unphysical region where the constraint is evaluated.

An evaluation of these diagrams gives, at the point $\nu=\nu_B=k^2=o$

$$\bar{F}^{(1,0)}(\text{fig.2}) = \{4MA^{+} - 4\bar{B}^{-} + 2g^2K(o)\}g \ ,$$

$$\bar{F}^{(3,2)}(\text{fig.2}) = \sqrt{10}\{2MA^{+} + \bar{B}^{-} - 2g^2K(o)\}g \ ,$$

$$\bar{F}^{(1,1)}(\text{fig.2}) = \bar{F}^{(3,1)}(\text{fig.2}) = 0 \ , \qquad (1.75)$$

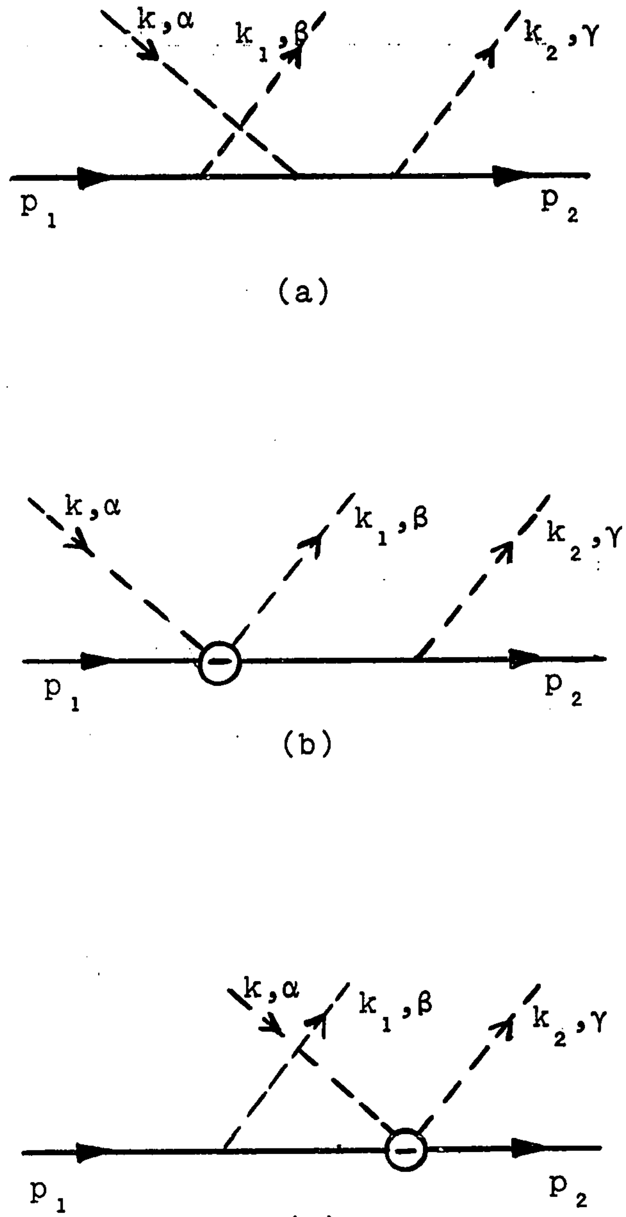


FIGURE (2). THE DOMINANT NON-POLE DIAGRAMS. ⊖ includes everything except the direct nucleon pole, which is accounted for in the pole diagrams. There are an equivalent set of diagrams with $k_1 \rightleftarrows k_2$.

where $\bar{B}^-$ denotes the non-pole part of B^-.

Then from this equation, together with eqs. (1.62), (1.63) and (1.66) one simply finds

$$M^o = 6g^2 - 3A^+/M - 4\bar{B}^-/K(o) \ ,$$

$$M^1 = o \ ,$$

$$M^2 = 2\bar{B}^-/K(o) \ . \tag{1.76}$$

Immediately one notes that $\bar{B}^-$ would have to be a rather large positive quantity $\approx 1.5\ g^2$ for M^o to be near the CA value; unfortunately this would make M^2 also rather large! As A^+ evaluated in the static limit with retention of only the (3,3) N^* resonance is close to the value evaluated by using the (presumably correct) fixed momentum transfer dispersion relations let us determine $\bar{B}^-$ in the same manner. Note that an approximation which is good for $\pi N \rightarrow \pi N$ need not be good for $N \rightarrow 2\pi N$ as the physical regions for the two processes are quite different. The approximation of using only the N^* resonance is expected to be reasonably good due to the small ratio of m/M and the fact that N^* is in the physical region for $N \rightarrow 2\pi N$.

$$\bar{B}^-(\nu'=o,\nu_B'=-1/2M) = \frac{2}{\pi}\int_1^\infty \frac{dx}{x}\,\mathrm{Im}\bar{B}^-(x,-1/2M) \ , \tag{1.77}$$

and

$$\mathrm{Im}\bar{B}^-(x,-1/2M) = 4\pi\left[\frac{3}{E+M} - \frac{1}{E-M}\right]\mathrm{Im}f_3^- \ , \qquad (1.78)$$

as given by Chew et al.[25]

$f_3^- = -f_{33}/3$, $E^2 = q^2 + M^2$ with q^2 the square of the c.m. momentum. In the static limit the right hand side of this last equation becomes $8\pi M\,\mathrm{Im}f_{33}/3q^2$, and in the narrow resonance approximation

$$\mathrm{Im}f_{33} = \frac{4\pi}{3}f^2q^2\delta(\omega-\omega_R) \ , \qquad (1.79)$$

where $W = M + \omega$, $x = (W^2-M^2-m^2)/2M$, ω_R is the position of the N^* resonance, and $f^2 = g^2/4M^2.4\pi \simeq 0.08$.

Finally

$$\bar{B}^-(\nu'=o,\nu_B'=-1/2M) = \frac{64}{9}\pi Mf^2\int_1^\infty \frac{dx}{x}\delta(\omega-\omega_R)$$

$$= 64\pi Mf^2/9x_R = 4.8 \ ,$$

i.e. $$\bar{B}^-(\nu=\nu_B=o,k^2=1) = 4.8 \ . \qquad (1.80)$$

Neglecting the off-mass-shell contribution and substituting into eqs. (1.76) one finds

$$M^o \simeq 1069.7 \ ,$$

$$M^1 = 0 \ ,$$

$$M^2 \simeq 9.6 \ . \qquad (1.81)$$

The most spectacular difference between this and the Weinberg CA result is that M° is not only 130 times larger, but of the opposite sign! As far as M° is concerned one could hardly regard this as a correction to the peripheral process; the question is, is there any process which we have neglected which might lead to a reduction of this M° to the CA result?

In figures 2(b) and 2(c) the $\ominus$ denotes the complete pion-nucleon amplitude except the direct nucleon pole. In our later evaluation of $\ominus$ in A^{+} we have used the two lowest mass resonances i.e. the $N^{*}P_{33}$ resonance at 1236 MeV. and the P_{11} (Roper) resonance at 1400 MeV., as well as S, D and F resonances. We have been unable to take into account processes similar to figure (2) where the intermediate propagator is not a nucleon, but P_{33} or P_{11}; this is due to our lack of knowlege of the amplitudes $\pi+N \rightarrow \pi+P_{33}$ (or P_{11}). But we would expect that their corrections to figures (2) would be of the order of $M/(M^{*2}-M^{2}+1)$ i.e. corrections of less than 20%, if these channels do not couple strongly.

Another possible term would be of the contact type, shown in figure (3).

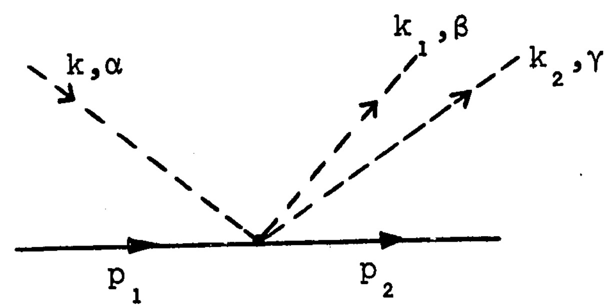


FIGURE (3). THE CONTACT TERM.

Its Lagrangian would be

$$L_{contact} = G \bar{N} \gamma_5 \vec{\tau}.\vec{\pi} N \vec{\pi}^2 . \qquad (1.82)$$

This gives the contributions

$$M^o(\text{contact}) = -5G/gK(o) ,$$

$$M^1(\text{contact}) = o ,$$

$$M^2(\text{contact}) = -2G/gK(o) . \qquad (1.83)$$

G would have to be extremely large, $\approx 1.2\ g^2$ to reduce M^o in equation (1.76) to the CA value, and once again this would make M^2 quite different from the CA. result.

Finally, we show that a diagram such as figure (4) may be neglected.

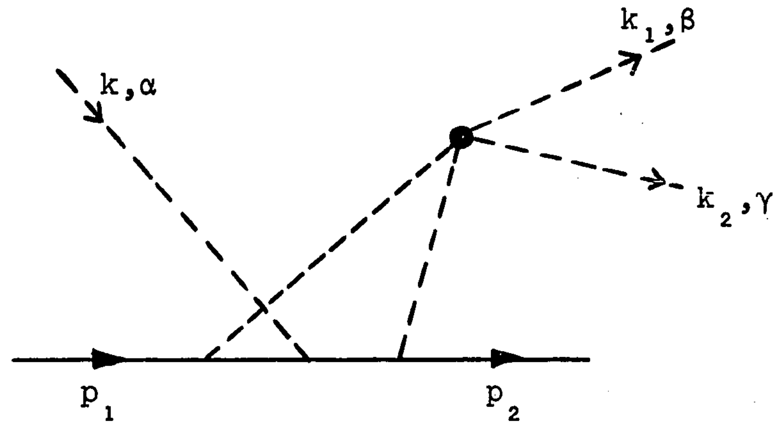


FIGURE (4). A PROCESS WHICH HAS BEEN NEGLECTED.

Let M_T^I be the total contribution to the π-π M matrix in the isospin state I (i.e. the very M matrix which we are expending our effort in obtaining!). Then the contributions of figure (4) to the various isospin states are.

$$M^0(\text{fig. }(4)) = -g[\log M - M\tan^{-1}2M]M_T^0/16\pi^2M^2K^2(o) \simeq$$

$$\simeq 0.02\ M_T^0$$

$$M^1(\text{fig.}(4)) = 0$$

$$M^2(\text{fig.}(4)) = g[\log M - M\tan^{-1}2M]M_T^2/8\pi^2M^2K^2(o] \simeq$$

$$\simeq -\ 0.04\ M_T^2 \qquad (1.84)$$

i.e. M^0 and M^2 from figure (4) make only contributions of 2% and 4% respectively to the total amplitudes.

5. CONCLUSION

From PCAC alone we have tried to extract information on the three isospin amplitudes at the unphysical point $s = o$, $t = u = 1$; these were then compared to Weinberg's CA calculation (eq. 1.59) and Iliopoulos' unitarily corrected version of this (eq. 1.60). By *assuming* peripheral dominance of non-pole amplitudes we obtain results similar to those given by CA; in this regard one notes that in the zero momentum pion limit N^* does not contribute to this amplitude in a phenomenological Lagrangian with derivative pion coupling to N and N^*.(26) If we don't assume peripheral dominance a much larger contribution to the $I = o$ amplitude is obtained when the non-pole diagrams of figure (2) are taken into account. A variety of processes were studied (in particular note eqs. (1.83) and (1.84)) with the hope of reducing M^o to the CA size; this, however, does not appear to be possible as those processes which could be of importance seem to have one common feature viz. their contributions to M^o and M^2 are of the same order of magnitude. It becomes apparent that to obtain the CA result a process is needed which contributes only to M^o and not to M^2, as only M^1 is exactly the same as the CA value. Such a mechanism is rather difficult to construct!

6. APPENDIX

To show that the amplitudes A^i do not have any kinematical singularities we follow the method due to Ball.[27]

$$T_\mu = \sum_{i=1}^{15} O^i_\mu \gamma_5 A^i \quad . \tag{A1}$$

Fifteen scalar quantities can be constructed from T_μ as follows:

$$T^i = \text{Trace}\,[\gamma_5 O^{\mu,i} \Lambda_+(p_2) T_\mu \Lambda_+(p_1)] \quad , \tag{A2}$$

where Λ_+ are the positive energy projection operators. By the Hall-Wightman theorem T^i will be analytic functions of the scalar invariants involved in the problem. We can write Eq.(A2) as

$$T^i = D^{ij} A^j \quad . \tag{A3}$$

The kinematical singularities of A^i would appear as zeroes of det D. The problem in general would be a complex one since D is a 15 x 15 matrix. Some simplifications can however be made as follows. Define the new quantities

$$X_1 = [(p_2-p_1)^2\, A^1 + (p_2-p_1)\cdot(p_1+k)\, A^2 + (p_2-p_1)\cdot(k_2-k_1)\, A^3 +$$

$$(p_2-p_1)\cdot(k_2-k_1) A^4] \quad ,$$

$$Y_1 = \text{In } X_1 \text{ replace } (A^1 \to A^7),\ (A^2 \to A^8),\ (A^3 \to A^9), \text{ and } (A^4 \to A^{10}),$$

$$Z_1 = \text{In } X_1 \text{ replace } (A^1 \to A^{11}),\ (A^2 \to A^{12}),\ (A^3 \to A^{13}), \text{ and } (A^4 \to A^{14}). \quad \text{(A4)}$$

$$X_2 = [(p_1+k)\cdot(p_2-p_1)\ A^1 + (p_1+k)^2\ A^2 + (p_1+k)\cdot(k_2+k_1)\ A^3 + (p_1+k)\cdot(k_2-k_1)\ A^4],$$

$$Y_2 = \text{In } X_2 \text{ replace } (A^1 \to A^7) \text{ etc.},$$

$$Z_2 = \text{In } X_2 \text{ replace } (A^1 \to A^{11}) \text{ etc.} \quad \text{(A5)}$$

$$X_3 = [(k_2+k_1)\cdot(p_2-p_1)\ A^1 + (k_2+k_1)\cdot(p_1+k)\ A^2 + (k_2+k_1)^2\ A^3],$$

$$Y_3 = \text{In } X_3 \text{ replace } (A^1 \to A^7) \text{ etc.},$$

$$Z_3 = \text{In } X_3 \text{ replace } (A^1 \to A^{11}) \text{ etc.} \quad \text{(A6)}$$

$$X_4 = [(k_2-k_1)\cdot(p_2-p_1)\ A^1 + (k_2-k_1)\cdot(p_1+k)\ A^2 + (k_2-k_1)^2\ A^4],$$

$$Y_4 = \text{In } X_4 \text{ replace } (A^1 \to A^7) \text{ etc.},$$

$$Z_4 = \text{In } X_4 \text{ replace } (A^1 \to A^{11}) \text{ etc.} \quad \text{(A7)}$$

$$a_1 = \text{Trace } [(\not{p}_2 - M)\ (\not{p}_1 + M)],$$

$$b_1 = -\text{Trace } [(\not{p}_2 - M)\ \not{k}\ (\not{p}_1 + M)],$$

$$c_1 = -\text{Trace } [(\not{p}_2 - M)\ (\not{k}_2 - \not{k}_1)\ (\not{p}_1 + M)],$$

$$d_1 = \text{Trace}\ [(\not{p}_2-M)\ \not{k}\ (\not{p}_2-\not{p}_1)\ (\not{p}_1+M)]\ ,$$

$$e_1 = \text{Trace}\ [(\not{p}_2-M)\ (\not{k}_2-\not{k}_1)\ (\not{p}_2-\not{p}_1)\ (\not{p}_1+M)]\ . \qquad \text{(A8)}$$

$$a_2 = -\text{Trace}\ [\not{k}(\not{p}_2-M)\ (\not{p}_1+M)]\ ,$$

$$b_2 = \text{Trace}\ [\not{k}(\not{p}_2-M)\ \not{k}\ (\not{p}_1+M)]\ ,$$

$$c_2 = \text{Trace}\ [\not{k}(\not{p}_1-M)(\not{k}_2-\not{k}_1)(\not{p}_1+M)]\ ,$$

$$d_2 = -\text{Trace}\ [\not{k}(\not{p}_2-M)\ \not{k}\ (\not{p}_2-\not{p}_1)(\not{p}_1+M)]\ ,$$

$$e_2 = -\text{Trace}\ [\not{k}(p_2-M)(\not{k}_2-\not{k}_1)(\not{p}_2-\not{p}_1)(\not{p}_1+M)]\ . \qquad \text{(A9)}$$

$$a_3 = -\text{Trace}\ [(\not{k}_2-\not{k}_1)(\not{p}_2-M)(\not{p}_2+M)]\ ,$$

$$b_3 = \text{Trace}\ [(\not{k}_2-\not{k}_1)(\not{p}_2-M)\ \not{k}\ (\not{p}_1+M)]\ ,$$

$$c_3 = \text{Trace}\ [(\not{k}_2-\not{k}_1)(\not{p}_2-M)(\not{k}_2-\not{k}_1)(\not{p}_1+M)]\ ,$$

$$d_3 = -\text{Trace}\ [(\not{k}_2-\not{k}_1)(\not{p}_2-M)\ \not{k}\ (\not{p}_2-\not{p}_1)(\not{p}_1+M)]\ ,$$

$$e_3 = -\text{Trace}\ [(\not{k}_2-\not{k}_1)(\not{p}_2-M)(\not{k}_2-\not{k}_1)(\not{p}_2-p_1)(\not{p}_1+M)]\ . \qquad \text{(A10)}$$

All the scalar products of four-vectors and the traces can be expressed in terms of the invariants of Eq.(1.24) of the text. We are interested in the possible kinematical singularities in these invariants. Using Eq.(A1) to (A10) one obtains a set of simultaneous equations

$$T^1 - d_1A^5 - e_1A^{15} = a_1X_1 + b_1Y_1 + c_1Z_1 ,$$

$$T^7 - d_2A^5 - e_2A^{15} = a_2X_1 + b_2Y_1 + c_2Z_1 ,$$

$$T^{11} - d_3A^5 - e_3A^{15} = a_3X_1 + b_3Y_1 + c_3Z_1 . \qquad \text{(A11)}$$

Three more sets of simultaneous equations result on using (T^2,T^8,T^{12}), (T^3,T^9,T^{13}) and (T^4,T^{10},T^{14}). The simplification brought about by our definition of X's, Y's, and Z's is that the transformation matrix that relates (T^1,T^7,T^{11}) to (X_1,Y_1,Y_1) also relates (T^2,T^8,T^{12}) to (X_2,Y_2,Z_2) (T^3,T^9,T^{13}) to (X_3,Y_3,Z_3) and (T^4,T^{10},T^{14}) to (X_4,Y_4,Z_4).

In the first step we solve for (X_1,Y_1,Z_1) in terms of the combinations on the left hand side of Eq.(A11). The zeroes of the determinant of the transformation matrix of Eq. (A11) will result if (i) $k_\mu = \alpha(p_2-p_1)_\mu + \beta(p_2+p_1)_\mu$, (ii) $(k_2-k_1)_\mu = \xi(p_2-p_1)_\mu + \eta(p_2+p_1)_\mu$, (iii) $k_\mu = \gamma(k_2-k_1)_\mu$ (iv) conditions (i) and (ii) are met simultaneously. The parameters α,β etc. can be expressed very easily in terms of the scalar invariants of Eq.(1.24) of the text. If condition (i) is met then the second row of the transformation matrix becomes $2M\alpha$ times the first. Similarly conditions (ii) and (iii) make the third row proportional to the first and the third row proportional to the second respectively. Condition (iv) makes both the second and the third rows proportional to the first. At any of these four conditions we get simple relations between appropriate T's. For example a direct substitution of condition (i) in Eq. (1.26) and (1.27) of the text, with the use of Dirac equation implies $O^7_\mu = 2M\alpha\, O^1_\mu$ which in turn implies $T^7 = 2M\alpha T^1$. At the same time $(a_2, \ldots\ldots, e_2) = 2M\alpha(a_1, \ldots\ldots, e_1)$. We then find that the numerator function in the solution of X_1, Y_1 and Z_1 vanishes simultaneously with the determinant in the denominator. Consequently no kinematical singularity results. Similar treatment confirms that X's, Y's and Z's are free from kinematical singularities under all the conditions (i) to (iv).

In the second step we invert equations (A4) - (A7) together with three more equations to complete the set of fifteen equations,

$$X_5 \equiv T^5 \;,$$

$$X_6 \equiv T^6 \;,$$

$$X_7 \equiv T^{15} \;, \tag{A12}$$

and solve for the original A^i. To illustrate the method adopted to show that A^i are free from kinematical singularities we discuss the solution for A^1. The equations to be solved are

$$X_1 = (p_2-p_1)^2\ A^1 + (p_2-p_1).(p_1+k)\ A^2 + (p_2-p_1).(k_2+k_1)\ A^3 +$$

$$(p_2-p_1).(k_2-k_1)\ A^4\ ,$$

$$X_2 = (p_1+k).(p_2-p_1)\ A^1 + (p_1+k)^2\ A^2 + (p_1+k).(k_2+k_1)\ A^3 +$$

$$(p_1 + k).(k_2-k_1)\ A^4\ ,$$

$$X_3 = (k_2+k_1).(p_2-p_1)\ A^1 + (k_2+k_1).(p_1+k)\ A^2 + (k_2+k_1)^2\ A^3 + 0\ ,$$

$$X_4 = (k_2-k_1).(p_2-p_1)\ A^1 + (k_2-k_1).(p_1+k)\ A^3 + 0 + (k_2-k_1)^2\ A^4 .$$

(A13)

The zeroes of the determinant of the transformation matrix will appear if the following conditions are satisfied: (i) $(p_1+k)_\mu = \alpha'(p_2-p_1)_\mu$ (ii) $(p_2-p_1)_\mu = \beta'(k_2+k_1)_\mu + \gamma'(k_2-k_1)$ (iii) $(p_1+k)_\mu = \xi'(k_2+k_1)_\mu + \eta'(k_2-k_1)_\mu$ (iv) conditions (ii) and (iii) are met simultaneously. Condition (i) makes the second row of the transformation matrix of (A13) proportional to the first. Condition (ii) makes the first row a linear combination of the third and the fourth rows. Condition (iii) makes the second row a linear combination of the third and fourth rows. Condition (iv) makes both the first and the second rows linear combinations of the third and fourth. Under condition (i) the determinant of the transformation matrix vanishes, but one also gets the relations $T^2 = \alpha' T^1$, $T^8 = \alpha' T^7$ and $T^{12} = \alpha' T^{11}$. The only solution of these equations is the trivial one, $X_2 = \alpha' X_1$, $Y_2 = \alpha' Y_1$, $Z_2 = \alpha' Z_1$. Under these conditions the first two rows of the determinant in the numerator of the solution for A^1 become proportional to each other and consequently no kinematical singularities result. Under condition (ii) one finds that the first row of the transformation matrix in Eq. (A13) is β' (third row) + γ'(fourth row). At the same time one also finds that $X_1 = \beta' X_3 + \gamma' X_4$, $Y_1 = \beta' Y_3 + \gamma' Y_4$ and $Z_1 = \beta' Z_3 + \gamma' Z_4$. The determinant in the numerator of the solution for A^1 now vanishes by virtue of the fact that the first row is a linear combination of the third and the fourth rows. By a similar method one can show that no kinematical singularities result when conditions (iii) and (iv) are met. By the above procedure it can be shown that all A^i except A^5, A^6 and A^{15} are free from kinematical singularities. Finally one can write A^5, A^6 and A^{15} in terms of T^5, T^6 and T^{15} and all other A^i which have been shown to be analytic. The problem reduces to that of solving a set of three simultaneous equations. Using the type of procedure adopted above one can also show that A^5, A^6 and A^{15} are free from kinematical singularities.

CHAPTER II

ABSTRACT: Weinberg's CA calculation of the s-wave π-π scattering lengths a_0 and a_2 incorporates neither $I = 2$ amplitude in the crossed channel (t-channel) nor the unitarity condition, and the below elastic threshold conditions on the π^0-π^0 amplitude as determined by Martin are not satisfied. We modify Weinberg's amplitude to include this t-channel contribution and use it to fix the position and residue of a single pole approximation to the unphysical cut, so as to obtain the phase shifts δ_0 and δ_2. For no value of the π-π coupling constant λ does δ_0 pass through 90^0, as indicated for example by Walker et al. and Malamud and Schlein.

Generalizing the usual isoscalar σ term in Iliopoulos' unitarily corrected CA calculation we are led to (generally) four sets of solutions for a given $I = 2$ amplitude in the t-channel, all of which satisfy the below threshold conditions. By considering the continuity of the coefficients of the scattering amplitude's power series parametrization in terms of the imaginary momentum variables k_s, k_t, k_u from first to second order we conclude that probably only one solution is meaningful. Surprisingly this gives Weinberg's results almost exactly in the limit of no $I_t = 2$ amplitude. It is this branch which falls within the experimental data of Pickup et al, giving two possible sets of a_0 and a_2, both with non-zero $I_t = 2$. Using the phenomenological bounds of Goebel and Shaw, and Chu and Desai on a_0 one finds $a_0 \approx (-0.5, -0.3)$, $a_2 \approx -0.3$, $a_2' \approx 0.1$ (phase convention, $\delta = ak$, $k \to 0$).

1. INCORPORATION OF $I_t = 2$ AMPLITUDE INTO THE WEINBERG RESULT.

As presently used, CA constraints are imposed at the point $s = u = 1$, $t = 0$ (i.e. the external masses of two of the pions are zero), and assumes that the scalar σ term is also an isoscalar. Here we remove this restriction and calculate the effect of the $\sigma^{(2)}\pi\pi$ vertex on the Weinberg amplitude, for a reasonable range of values of the vertex function f_2^{σ}.

Consider the scattering process

$$\pi(k_1,\alpha) + \pi(k_2,\beta) \to \pi(k_3,\gamma) + \pi(k_4,\delta) ,$$

where $\alpha,\beta,\gamma,\delta$ are isospin labels. The Mandelstam variables are defined in terms of the centre of mass variables $\nu = q^2$ and θ by $s = (k_1+k_2)^2 = 4(\nu+1)$, $t = (k_1-k_3)^2 = -2\nu(1-\cos\theta)$ and $u = (k_1-k_4)^2 = -2\nu(1+\cos\theta)$.

The S-matrix for the scattering amplitude is written as

$$S_{\alpha\beta,\gamma\delta} = \langle k_3,\gamma;k_4,\delta;\text{out}|k_1,\alpha;k_2,\beta;\text{in}\rangle \ , \tag{2.1}$$

with

$$(S-1)_{\alpha\beta,\gamma\delta} = -i(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)T_{\alpha\beta,\gamma\delta}$$

$$= -i(2\pi)^4 \frac{\delta^4(k_1+k_2-k_3-k_4)M_{\alpha\beta,\gamma\delta}}{(2\pi)^6(16k_{10}k_{20}k_{30}k_{40})^{\frac{1}{2}}} \ . \tag{2.2}$$

Following Chew and Mandelstam(3) $M_{\alpha\beta,\gamma\delta}$ may be written

$$M_{\alpha\beta,\gamma\delta} = A(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta} + B(s,t,u)\delta_{\alpha\gamma}\delta_{\beta\delta} + C(s,t,u)\delta_{\alpha\delta}\delta_{\beta\gamma} \ . \tag{2.3}$$

By using the LSZ. formalism(28) one contracts out a pion from the initial and final state in the usual manner to obtain

$$i(2\pi)^4 \frac{\delta^4(k_1+k_2-k_3-k_4)M_{\alpha\beta,\gamma\delta}}{(2\pi)^6(16k_{10}k_{20}k_{30}k_{40})^{\frac{1}{2}}} =$$

$$= \frac{(k_1^2-1)(k_3^2-1)}{(2\pi)^3(4k_{10}k_{30})^{\frac{1}{2}}f_\pi^2}\int d^4x d^4y \exp\{-i(k_1x-k_3z)\} \times$$

$$\times \ \langle k_4,\delta|T\{\partial_\mu A_\mu^\alpha(x),\partial_\rho A_\rho^\gamma(z)\}|k_2,\beta\rangle \ . \tag{2.4}$$

In this equation we have already used the PCAC hypothesis-substituting $\partial_\mu A^\alpha_\mu(x)/f_\pi$ for the pion isospin α field.

Now

$$T\{\partial_\mu A^\alpha_\mu(x),\partial_\rho A^\gamma_\rho(z)\} = \partial_\mu\partial_\rho T\{A^\alpha_\mu(x),A^\gamma_\rho(z)\} + $$
$$+ \partial_\mu\delta(x_o-z_o)[A^\gamma_o(z),\ A^\alpha_\mu(x)] - \delta(x_o-z_o)[A^\alpha_o(x),\partial_\rho A^\gamma_\rho(z)] \ . \tag{2.5}$$

Omitting possible Schwinger terms the equal time commutators may be written as

$$\delta(x_o-z_o)[A^\gamma_o(z),A^\alpha_\mu(x)] = 2i\varepsilon_{\gamma\alpha\beta}V^\beta_\mu(x)\delta^4(x-z) \ , \tag{2.6}$$

$$\delta(x_o-z_o)[A^\alpha_o(x),\partial_\rho A^\gamma_\rho(z)] = i\sigma_{\alpha\gamma}(x)\delta^4(x-z) \ . \tag{2.7}$$

Throughout the remainder of this chapter we will refer to eqs. (2.6) and (2.7) as the V-commutator and σ term respectively, or more simply as V and σ .

Evaluation of $\langle k_4,\delta|T\{A^\alpha_\mu(x),A^\alpha_\rho(z)\}|k_2,\beta\rangle$ would require various dynamical assumptions and approximations,(29) but in the soft pion limit $k_1 = k_3 \to 0$ this term disappears, as it has no poles in these four momenta. It is, of course, implicit in the idea of PCAC that this soft pion approximation is not greatly different from the hard pion result.

From eq. (2.6) one sees that $[Q_A^\gamma(t), Q_A^\alpha(t)] = 2i\varepsilon_{\gamma\alpha\beta}Q_V^\beta$ and as (from the Conserved Vector Current Hypothesis) $\frac{d}{dt}Q_V^\beta = 0$ then,

$$[Q_A^\gamma(t), \frac{d}{dt} Q_A^\alpha(t)] = [Q_A^\alpha(t), \frac{d}{dt} Q_A^\gamma(t)] \ , \qquad (2.8)$$

which implies that in eq. (2.7) $\sigma_{\alpha\gamma} = \sigma_{\gamma\alpha}$. As the I = 2 s-wave interaction is known to be smaller than that in the I = 0 state the usual assumption made is that the σ term is not only scalar, but isoscalar also i.e.

$$\sigma_{\alpha\gamma}(x) = \delta_{\alpha\gamma}\sigma(x) = \delta_{\alpha\gamma}\delta_{\alpha'\gamma'}\sigma_{\alpha'\gamma'}(x) \ . \qquad (2.9)$$

This restriction may be removed by using the form suggested by Kamal.

$$\sigma_{\alpha\gamma}(x) = \tfrac{1}{2}(\delta_{\alpha\gamma'}\delta_{\gamma\alpha'} + \delta_{\alpha\alpha'}\delta_{\gamma\gamma'})\sigma_{\alpha'\gamma'}(x) \ , \qquad (2.10)$$

which is still symmetric in α and γ, as well as in α' and γ'. From now on we will use the form given by eq. (2.10) rather than that of eq. (2.9), as it has both an I = 0 and I = 2 component. One may write, quite generally

$$\langle k_4, \delta | V_\mu^i(o) | k_2, \beta \rangle = i\varepsilon_{i\beta\delta}(k_4+k_2)_\mu/(2\pi)^3(4k_{20}k_{40})^{\frac{1}{2}} \ ,$$

(2.11)

and

$$<k_4,\delta|\sigma_{\alpha\gamma}(o)|k_2,\beta> = f^{\sigma}_{\alpha\gamma,\beta\delta}\{k_2^2=k_4^2=1,(k_4-k_2)^2\}/(2\pi)^3(4k_{20}k_{40})^{\frac{1}{2}},$$

(2.12)

where the vertex function $f^{\sigma}_{\alpha\gamma,\beta\delta}$ may be written as

$$f^{\sigma}_{\alpha\gamma,\beta\delta} = f^{\sigma}_{o}\, P^{t(o)}_{\alpha\beta\gamma\delta} + f^{\sigma}_{2}\, P^{t(2)}_{\alpha\beta\gamma\delta} , \qquad (2.13)$$

with $P^{t(o)}_{\alpha\beta\gamma\delta}$ and $P^{t(2)}_{\alpha\beta\gamma\delta}$ the t-channel isospin projection operators for the I = 0 and I = 2 states, and $f^{\sigma}_{o}, f^{\sigma}_{2}$ their respective vertex functions. In the limit $k_1 = k_3 \to 0$ (i.e. $k_2 = k_4$) eqs. (2.11) and (2.12) may be rewritten in the form

$$<k_2,\delta|V^{i}_{\mu}(o)|k_2,\beta> = i\varepsilon_{i\beta\delta}k_{2\mu}/\ (2\pi)^3 k_{20} , \qquad (2.14)$$

and

$$<k_2,\delta|\sigma_{\alpha\gamma}(o)|k_2,\beta> = \frac{f^{\sigma}_{o}(1,1,o)P^{t(o)}_{\alpha\beta\gamma\delta}}{(2\pi)^3 2k_{20}} + \frac{f^{\sigma}_{2}(1,1,o)P^{t(2)}_{\alpha\beta\gamma\delta}}{(2\pi)^3 2k_{20}} .$$

(2.15)

From eqs. (2.4), (2.5), (2.14) and (2.15) one obtains, in the limit s = u = 1, t = o, three equations in A, B, C.

$$f^2_{\pi}M_{\alpha\beta,\gamma\delta}\Big|_{k_1=k_3\to 0} = (8k_1.k_2 + f^{\sigma}_{2})\delta_{\alpha\beta}\delta_{\gamma\delta} \quad +$$

$$+ (f_o^\sigma - \tfrac{2}{3}f_2^\sigma)\delta_{\alpha\gamma}\delta_{\beta\delta} + (f_2^\sigma - 8k_1.k_2)\delta_{\alpha\delta}\delta_{\beta\gamma} ,$$

i.e.

$$f_\pi^2 A = -8k_1.k_2 - f_2^\sigma ,$$

$$f_\pi^2 B = -f_o^\sigma + \tfrac{2}{3} f_2^\sigma ,$$

$$f_\pi^2 C = 8k_1.k_2 - f_2^\sigma . \qquad (2.16)$$

These take their simplest form when written as t-channel isospin amplitudes $M_t^{(I)}$.

$$f_\pi^2(3B+A+C) = f_\pi^2 M_t^{(o)} = -\ 3f_o^\sigma ,$$

$$f_\pi^2(A-C) = f_\pi^2 M_t^{(1)} = -\ 16k_1.k_2 ,$$

$$f_\pi^2(A+C) = f_\pi^2 M_t^{(2)} = -\ 2f_2^\sigma \qquad (2.17)$$

Weinberg's parametrization of M, taking account of isospin invariance, crossing symmetry and Bose statistics is

$$A = a + b(t+u) + cs + \ldots.$$

$$B = A + b(u+s) + ct + \ldots.$$

$$C = a + b(s+t) + cu + \ldots. , \qquad (2.18)$$

where +.... denotes omitted higher order terms in s, t, u and m whose coefficients are hopefully small (see Khuri[30]) From eqs. (2.17) and (2.18) one therefore finds

$$a + b + c = - f_2^{\sigma}/f_{\pi}^2$$

$$b - c = 4/f_{\pi}^2$$

$$5a + 8b + 2c = - 3f_0^{\sigma}/f_{\pi}^2 \ . \qquad (2.19)$$

Only two of these equations are linearly independent i.e. they do not give unique solutions for a, b, c but merely define a relation between f_0^{σ} and f_2^{σ}, which must be satisfied with the parametrization of eq. (2.18) viz.

$$5f_2^{\sigma} - 3f_0^{\sigma} = 12 \qquad \text{at } s = u = 1,\ t = o \ . \qquad (2.20)$$

It is well known that one may obtain another relation between a, b, c which is linearly independent of eqs. (2.19) by exploiting the Adler self-consistency condition (ASC). By contracting out one pion (say k_1, α) and taking the soft pion limit $k_1 \to 0$ this condition is

$$A = B = C = 0 \qquad \text{at } s = t = u = 1 \ , \qquad (2.21)$$

i.e.

$$a + 2b + c = 0 \ . \qquad (2.22)$$

We have a choice of any two of the eqs. (2.19) to combine with (2.22); in order to exhibit explicitly the dependence of our amplitudes on f_2^{σ} we choose the first two to give:-

$$a = (4-3f_2^\sigma)/f_\pi^2 \,,$$

$$b = f_2^\sigma/f_\pi^2 \,,$$

$$c = (f_2^\sigma-4)/f_\pi^2 \,, \qquad (2.23)$$

i.e.

$$A = \frac{4}{f_\pi^2}\{1-s + (s+t+u-3)f_2^\sigma/4\} \,,$$

$$B = A(s \leftrightarrow t), \quad C = A(s \leftrightarrow u) \,. \qquad (2.24)$$

From these one immediately determines the scattering lengths a_0 and a_2 as

$$a_0 = (7-5f_2^\sigma/4)L/4 = 0.19 - 2.5\, a_2' \,,$$

$$a_2 = -(1 + f_2^\sigma/4)L/2 = -0.06 - a_2' \,, \qquad (2.25)$$

with $L = 1/2\pi f_\pi^2$ and f_2^σ has been made to look like a multiple of a scattering length a_2' by rewriting it in the form $f_2^\sigma = 16\pi f_\pi^2 a_2'$. Naturally, when $a_2' = 0$ we recover the results of Weinberg.[6]

a_2' is an unknown parameter in our theory, and we must use experiments to determine what range of values are reasonable for it; from eq. (2.25) it is obvious that knowing any one of the set a_0, a_2, a_2' determines the other two uniquely. A little later we will study this equation in more detail, and compare it with a theory which includes the unitarity condition, for the moment however we concentrate on the theoretical implications of it for the s-wave phase shifts δ_0 and δ_2 .

2. DETERMINATION OF δ_0 AND δ_2 FROM CA AND N/D TECHNIQUE

In 1961, Desai[31] had shown, following the original work of Chew and Mandelstam, how one may determine δ_0 and δ_2 incorporating elastic unitarity in terms of a single real parameter λ (the π-π coupling constant) taking into account the then recently discovered 2π p-wave resonance (now called the ρ meson). Briefly his work may be described as follows. By definition the π-π coupling constant λ is (see Chew[3,4])

$$-\lambda \equiv A(\nu_0,\cos\theta=o) = B(\nu_0,\cos\theta=o) = C(\nu_0,\cos\theta=o) \quad , \qquad (2.26)$$

where $(\nu_0=-2/3,\cos\theta=o)$ is the symmetry point $s = t = u = 4/3$. Then the crossing relations at the symmetry point, for the amplitudes A^I and their derivatives with respect to ν (indicated by a prime) and $\cos\theta$ are,

(a) $A^0 = 5A^2/2 = -5\lambda$

(b) $A^{0'} = -2A^{2'} = \dfrac{2\partial(A^1/\nu)}{\partial\cos\theta}$

(c) $A^{0''} - 5A^{2''}/2 = -\frac{9}{2}\frac{\partial(A^1/\nu)'}{\partial\cos\theta}$

(d) $\frac{\partial^2(A^0/\nu^2)}{\partial\cos^2\theta} - \frac{5}{2}\frac{\partial^2(A^2/\nu^2)}{\partial\cos^2\theta} = \frac{3}{2}\frac{\partial(A^1/\nu)'}{\partial\cos\theta}$. (2.27)

Neglecting d and higher partial waves in eqs.(a) and (b), and f and higher partial waves in eqs.(c) and (d) one obtains the following approximate crossing relations.

(a′) $A^0_0 \simeq 5A^2_0/2 \simeq -5\lambda$

(b′) $A^{0\prime}_0 \simeq -2A^{2\prime}_0 \simeq 6a_1$

(c′) $A^{0''} - 5A^{2''}/2 \simeq -12a'_1$, (2.27′)

where $a_1 = A^1_1/\nu$ at ν_0. Eq. (c′) obtained from (c) takes into account the d-wave correction using (d).

Using the (assumed) Mandelstam representation, Chew and Mandelstam(3) showed that the $A^I_\ell(\nu)$ had singularities in the complex ν plane from $\nu = 0$ (threshold) along the real axis to infinity (the so called right-hand or physical cut R) and from $-\infty$ to -1 (the left-hand, unphysical or potential cut L). The partial wave dispersion relations are then

$$A^I_\ell(\nu) = \frac{1}{\pi}\int_L d\nu' \frac{\mathrm{Im}A^I_\ell(\nu')}{\nu'-\nu} + \frac{1}{\pi}\int_R d\nu' \frac{\mathrm{Im}A^I_\ell(\nu')}{\nu'-\nu} . \qquad (2.28)$$

In order to ensure the convergence of this equation for s-waves it seems necessary to introduce at least one subtraction, which in this particular case we make at ν_o viz.

$$A_o^I(\nu) = A_o^I(\nu_o) + \frac{\nu-\nu_o}{\pi} \int_{L,R} d\nu' \frac{\text{Im}A_o^I(\nu')}{(\nu'-\nu)(\nu'-\nu_o)} . \qquad (2.29)$$

In the usual manner, substituting $A_o^I(\nu) = N_I/D_I$ with N_I containing L, and D_I containing R, and using elastic unitarity in the form $\text{Im}\left[A_o^I(\nu)\right]^{-1} = -[\nu/(\nu+1)]^{\frac{1}{2}}$, then one finds

$$N_I(\nu) = A_o^I(\nu_o) + \frac{(\nu-\nu_o)}{\pi} \int_L d\nu' \frac{\text{Im}A_o^I(\nu')D_I(\nu')}{(\nu'-\nu)(\nu'-\nu_o)} , \qquad (2.30)$$

$$D_I(\nu) = 1 - \frac{(\nu-\nu_o)}{\pi} \int_R d\nu' \left(\frac{\nu'}{\nu'+1}\right)^{\frac{1}{2}} \frac{N_I(\nu')}{(\nu'-\nu)(\nu'-\nu_o)} . \qquad (2.31)$$

In the simplest approximation of these equations the left hand cut is replaced by a pole, with residue b_I and position $\omega_I = -\nu_I$; these equations then simplify to

$$N_I(\nu) = A_o^I(\nu_o) + (\nu-\nu_o)\frac{(\omega_I+\nu_o)}{(\omega_I+\nu)} B_I , \qquad (2.32)$$

$$D_I(\nu) = 1 - (\nu-\nu_o)[K(-\nu,-\nu_o)A_o^I(\nu_o) +$$

$$+ (\omega_I+\nu_o)K(\omega_I,-\nu)B_I] , \qquad (2.33)$$

where B_I is proportional to b_I and

$$K(x,y) = \frac{1}{\pi}\int \frac{d\nu\sqrt{\nu/(\nu+1)}}{(\nu+x)(\nu+y)} \quad . \tag{2.34}$$

As we are generally dealing with s-waves let's drop the ℓ suffix on A_ℓ^I and write as A_I, unless $\ell \neq 0$.
Desai had to determine the six s-wave parameters $A_0(\nu_0)$, $A_2(\nu_0)$, ω_0, ω_2, B_0 and B_2 in order to find $\cot\delta_I = \sqrt{(\nu+1)/\nu}\ \mathrm{Re}D_I(\nu)/N_I(\nu)$. This was done by using (a known) two parameter resonance form for $A_1^{(1)}(\nu)/\nu$ given by Frazer and Fulco[32] so that $A_1^{(1)}(\nu_0)$ and $A_1^{(1)\prime}(\nu_0)$ were known. Thus eq. (2.27) gave five conditions to determine the six unknowns; by fixing a priori ω_0 at some value guessed at from physical considerations the problem became determinate and phase shift curves could be plotted for various λ. λ is, of course, unknown in this theory, he did find however that $\lambda \gtrsim 0.03$ for solutions to exist.

The point I wish to make here is that eqs. (2.24) satisfy the crossing relations (2.27′), the last one being satisfied trivally as $A_0'' = A_2'' = a_1' \equiv 0$, the other two being

$$\lambda = -A_0(\nu_0)/5 = -A_2(\nu_0)/2 = L(3f_2^\sigma-4)/48, \tag{2.35}$$

and

$$A_0'(\nu_0)/2 = -A_2'(\nu_0) = L \ . \qquad (2.36)$$

Thus, in eqs. (2.32) and (2.33) we already know two of Desai's unknowns viz $A_0(\nu_0)$ and $A_2(\nu_0)$ either as functions of f_2^σ, or equivalently λ .

Now eqs. (2.24) may be rewritten as

$$A_0(\nu) = (8\nu+7-5f_2^\sigma/4)L/4 \ ,$$

$$A_2(\nu) = -(4\nu+2+f_2^\sigma/2)L/4 \ , \qquad (2.37)$$

or alternatively, using eq. (2.35) as

$$A_0(\nu) = 2(\nu-\nu_0)L - 5\lambda \ ,$$

$$A_2(\nu) = -(\nu-\nu_0)L - 2\lambda \ . \qquad (2.38)$$

These CA derived amplitudes were obtained in an off-mass-shell calculation and are assumed to be valid a little below and hopefully up to the elastic threshold; if $\partial_\mu A_\mu$ is really a smooth operator it may be expected that these latter amplitudes hold in the region $-1 \leq \nu \leq 0$ between the left and right branch cuts. To obtain the two unknowns ω_I and B_I one may then choose any two points in this region and substitute the relevant amplitudes into eqs. (2.32) and (2.33).

For a given isospin state these two points have been chosen to be at threshold, and at the zero of the amplitudes when $f_2^\sigma = 0$ (i.e. at $\nu_{I=o} = -7/8$ and $\nu_{I=2} = -\frac{1}{2}$).

A typical equation to be solved for the I = 2 case (the simplest!) is, with $\omega_2 = x$.

$$\frac{1}{x} + \frac{1}{\pi}\left[\frac{L}{2}(2x-1)(2+f_2^\sigma/2) - \frac{g(f_2^\sigma)a_2'x}{h(f_2^\sigma)}\right]\frac{\log(\sqrt{x}+\sqrt{x-1})}{\sqrt{x(x-1)}} =$$

$$= 2 + (a_2'/2-1)g(f_2^\sigma)/2h(f_2^\sigma) \ , \qquad (2.39)$$

where

$$g(f_2^\sigma) = \frac{2L}{3}\left[1+(2+f_2^\sigma/2)(-2/3+f_2^\sigma/2)\ \frac{3\sqrt{2}L}{16\pi}\tan^{-1}(1\sqrt{2})\right] \ , \qquad (2.40)$$

and

$$h(f_2^\sigma) = a_2' - \frac{L}{4}(f_2^\sigma/2-2/3)\left[1+\frac{a_2'}{2}\left(\frac{4\sqrt{2}}{\pi}\tan^{-1}(1/\sqrt{2})-3\right)\right]. \qquad (2.41)$$

There is a similar, although somewhat more complicated, equation to solve for the I=0 case, again for various f_2^σ(or λ) .

The most striking point to note in the solution of eq. (2.38) for ω_2 is that solutions only exist on L when $\lambda \gtrsim 0.029$ (i.e. $f_2^\sigma \gtrsim 5.5$), whereas solutions for ω_0 exist for a much larger range of λ; these are shown in figure (5). There appears to be little point in carrying the study beyond $\lambda \simeq 0.06$ as this would imply, from eq. (2.35), that the $I = 2$ state interaction would become unduly large, contradicting presently available experimental data. In figure (6) the resulting phase shifts are shown for the two values of λ, 0.03 and 0.06, corresponding to scattering lengths of $a_0 = 0.003$, $a_2 = -0.13$ and $a_0 = -0.15$, $a_2 = -0.19$ respectively. For $\lambda = 0.03$ δ_0 becomes very slightly negative (less than a degree) for very low values of ν; this is more accentuated and occurs at higher ν values as λ increases. In figure (7) we give the "experimental" δ_2 obtained by Walker et al.[33] and Baton et al.[34]; for δ_0 we clearly do not get a resonance, as many experimental analysis indicate- this situation is considered in greater detail in the next section. For the moment let us analyse the possible sources of error in our calculations, assuming from the outset the validity of dispersion relations and CA techniques. These errors may come from three sources viz:

(1) Solutions may be strongly dependent upon the choice of "matching up" points between the cuts.

(2) Approximation of the left-hand cut by a single pole.

(3) The CA amplitudes of eq. (2.24) do not incorporate elastic unitarity.

The dependence of the solution on the "matching up" points is relatively simple to check; it was found that although pole positions and residues can change by as much as 15% the effect on the phase shifts is very small indeed (2-3%) even for the larger values of ν.

The single pole approximation (2) represents the average effect of the crossed s and p-wave interactions on the right hand cut, and is justifiable only if no strong s-wave interaction exists at low energies. The ρ-meson will then give a peak on the left-hand cut at $\nu = -m_\rho^2/4$, which may be considered to generate a force which isn't too long a range for the single pole approximation to be valid. Presumably the error in δ_2 is larger than in δ_0, due to its pole being closer to the physical region, although it appears to reproduce the "experimental" data rather better than δ_0 does.

Interestingly enough Shaw and Morgan[35] using forward dispersion relations have obtained results for δ_0 and δ_2 somewhat similar to mine. My results are also very close indeed to what one may consider to be the much more sophisticated (from the phenomenological view-point!) calculation of Fulco and Wong.[36] Here they use forward dispersion relations (once-subtracted for the s-wave) incorporating

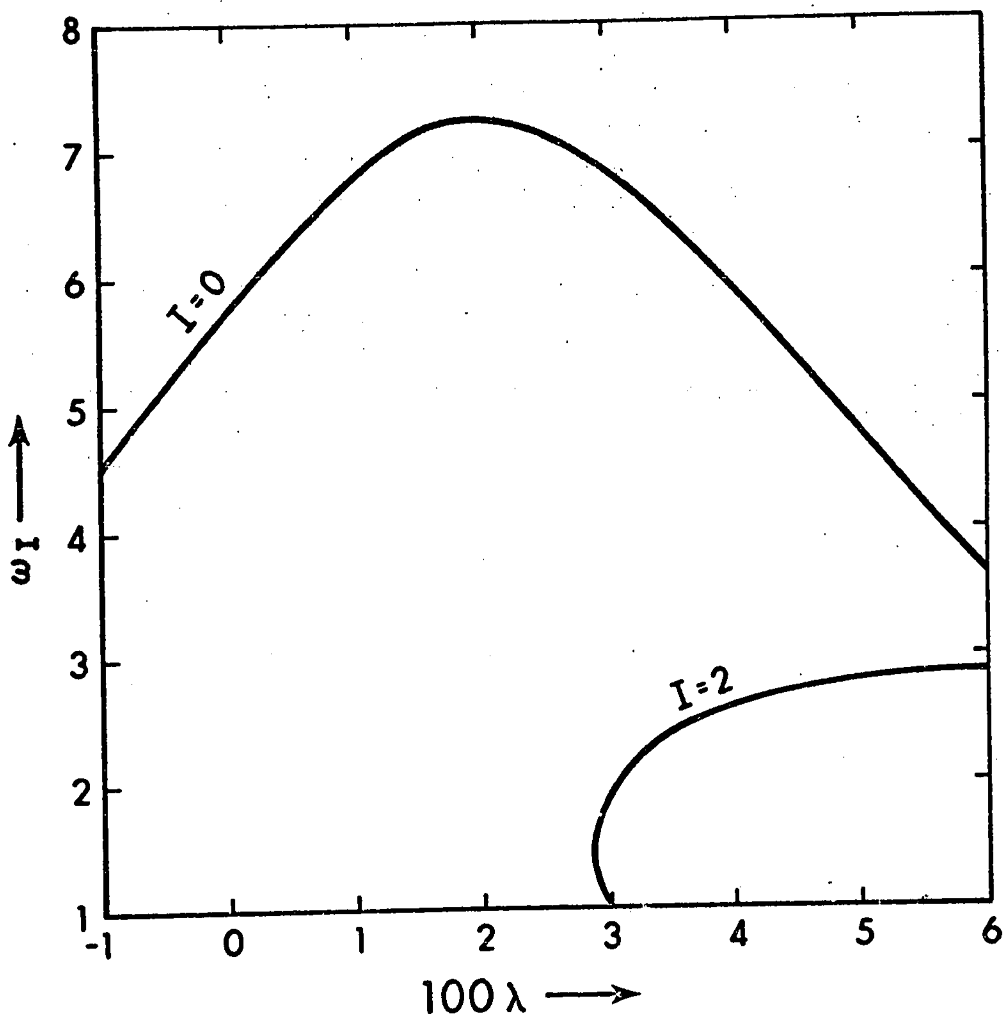


FIGURE 5. Pole position ω_I vs. λ for I = o,2.

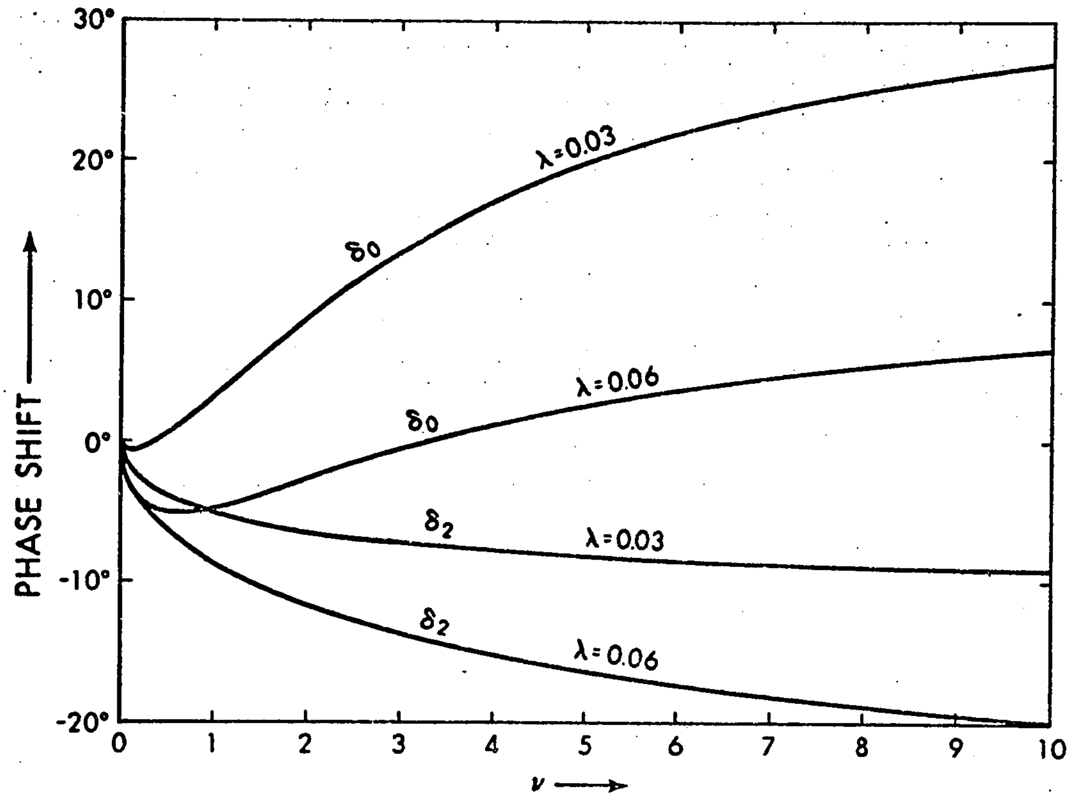


FIGURE 6. Phase shifts δ_0 and δ_2 vs. ν for two values of λ.

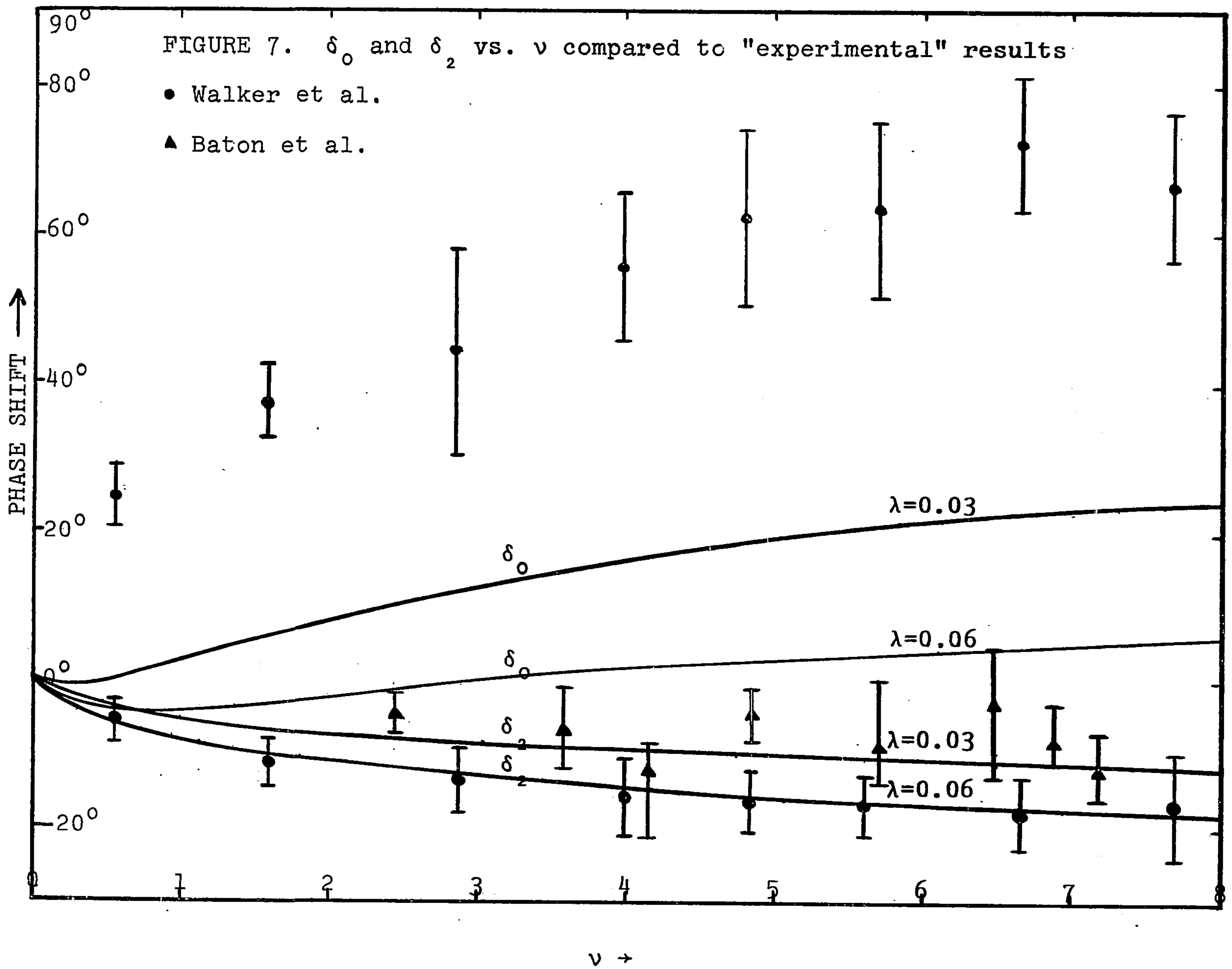


FIGURE 7. δ_0 and δ_2 vs. ν compared to "experimental" results

exact crossing symmetry but with a parametrization of the right-hand cut. For the absorptive parts they use the ρ and f mesons, and exact elastic unitarity for the s-waves below $\nu = 24$, and assume that above this value the absorptive part is proportional to s for all I (Pomeranchuk Theorem). Their conclusions are the same as mine viz., use of the rather low CA amplitudes appears not to give large phase shifts. This conclusion is supported by the pure CA effective range calculations of Brown and Goble.[37] One should mention a recent calculation of Tryon[38] which is similar in many ways to that of Fulco and Wong, using twice subtracted partial wave dispersion relations, unitarity, and crossing symmetry. He parametrizes the absorptive part by using Breit-Wigner forms for the ρ, f_o and f_o' mesons and various reasonable assumptions about the s-wave. His results fit almost exactly (except at very low energy) the results of Walker et al.; personally I would like to be convinced that his s-wave parametrization is unique. Tryon does note however that he is rather straining the validity of the Legendre expansion (useful when $\nu > -9$) as he obtains poles on the left hand cut which are well into the region of the double spectral function (i.e. $\nu = -11, -56$ and -79).

It is still far from clear as to what effect incorporation of elastic unitarity into the CA amplitudes is, although (or perhaps because!) it has received fairly intensive study recently by a number of authors. A very general trend seems to indicate that unitarity changes the Weinberg amplitude by only a small amount—a result which appears not to be particularly obvious. One particular approach, which we study and generalize in Section 4, is that of Iliopoulos; this incorporates elastic threshold unitarity in a Weinberg type CA calculation and leads to interesting multiple-valued scattering lengths.

3. "EXPERIMENTAL" PHASE SHIFTS.

The present non-feasibility of direct π-π scattering experiments has already been noted in the introduction to Chapter I; data must therefore be extracted, *subject to some theoretical model*, from experiments in which pions play a rôle. This has the very obvious disadvantage of resulting in conclusions which can be highly model dependent, even if the experimental data is rather "clean". One particular theoretical model, which has undergone a number of refinements, is of the peripheral type in which a one pion exchange (OPE) interaction is assumed to dominate a quasi two body collision.[39] This, by itself, fails to yield the rather striking forward production angle peaking experimentally observed and violates the unitarity limit for low partial waves. Ad hoc form factor modifications initiated by Ferrari and Selleri[40] appeared to have a rather drastic momentum dependence, and in the case of NNρ the calculated nucleon e.m. form factor was quite inaccurate. Furthermore accurate experimental polarization data of the produced particle showed significant derivations from the form factor modified OPE.

It was soon realized[41] that the existence of a large number of inelastic competing channels modified the low partial wave OPE quasi two body interactions (absorption). These form factor and absorption modifications are the ones used at the present time; here we indicate briefly some of the basic ideas and results.

Walker et al.[33] have managed to obtain both δ_o and δ_2 by collecting, from various experimental groups, data on the reactions $\pi^- p \rightarrow \rho^- p$ and $\pi^- p \rightarrow \rho^o n$ at energies of 2, 3 and 7 Gev. The pionic decays of ρ^- and ρ^o both exhibit a forward-backward asymmetry due to interference between s and p-wave π-π amplitudes. This leads to an energy dependent minimum in the angular distribution; by using these experimentally determined minimum points and the p-wave cross section data one may extract the s-wave phase shifts, taking absorption and spin flip effects into account, assuming that the asymmetry does not exhibit a strong dependence on the momentum transfer to the nucleon. Bander et al.[42] suggest however that the latter effect is quite pronounced, and that s-wave phase shifts depend quite strongly on the p-wave ones, which cannot unambiguously be determined by using a Breit-Wigner form. For what it is worth we have already given the results of Walker et al. for δ_2 in figure (7), their result for δ_0 is on an average three times greater than my conclusions and indicate a possible broad isoscalar resonance (the ε-meson) around 850-900 Mev. Walker's results for δ_0 in the range 300-500 Mev are similar to Jones et al.[43] (who do not consider absorption or form factor modifications), and for δ_2 similar to Baton et al.[34] (who consider rather carefully the extrapolation in the momentum transfer to the nucleon variable and obtain $a_2 = -0.07$).

The foregoing peripheral type calculations, as well as many others, are either dependent upon a detailed theoretical treatment of absorption, or ignore it partially or completely. Schlein,[44] utilizing the reaction $\pi^- p \to \pi^- \pi^+ n$, has given a method of extracting the π-π phase shifts in the vicinity of the ρ-meson resonance which does not involve a detailed knowledge of the helicity amplitudes; although some of them need to be parametrized enough constraints remain to test his model. With this method Malamud and Schlein[44] extract δ_0 (the details need not concern us here) and find three solutions, all of which suggest an ε-meson exists-the preferred set indicating that δ_0 passes through 90° at about 730 Mev.

All of the calculations mentioned so far are really restricted to the vicinity of the ρ-mass region, as they ultimately depend for their validity on the interference between p and s-waves. Biswas et al.[45] have used a different method for evaluating δ_0 and δ_2 at dipion masses lower than that of the ρ-meson by utilizing interference between the nuclear and coulomb contributions, which are comparable in the forward direction; unfortunately their statistics are so bad it is impossible to sensibly extract any detailed information.

An alternative and independent method was begun in 1962 by Hamilton et al.[46] and Atkinson,[47] and has been studied again recently by Lovelace et al.[48] viz. backward π-N scattering. It consists in the observation that the backward π-N scattering amplitude is analytic in the cut ν-plane (cuts from $(-\infty,-1)$ and $(0,\infty)$) except for the nucleon pole, and that the left-hand cut describes the process $N\bar{N} \to \pi\pi$, physical only for $\nu < -16M^2$. By extrapolating the positive isotopic spin backward π-N data from its physical region $\nu > 0$ to $\nu < -1$ the π-π phases can be deduced. This extrapolation is, of course, the weakest part of this whole method, as amply demonstrated by Lovelace, who obtains a vast number of solutions for δ_0 with only one thing in common i.e. sooner or later they pass through 90^o! This contrasts with the earlier work where δ_0 rose fairly rapidly to about 30^o-40^o before slowly falling off. The method has the obvious advantage of using the considerable amount of experimental knowledge on π-N scattering.

Recently a great deal of interest has been focussed on K_{ℓ_4} decays; we refer to the paper of Pais and Treiman[49] for an extensive bibliography. The form factors of this decay carry direct information about the π-π interaction, in particular that in the isospin zero state (if one believes in the semileptonic $\Delta I = \frac{1}{2}$ rule). These influence the intensity and polarization spectra which are receiving increasingly accurate measurements—it would be premature to use these results at the present time, although Kacser et al.[50] exclude the possibility of a σ—resonance on rather limited statistics. The Glasgow group[51] indicate in their analysis of K_{ℓ_4} decay that δ_o is negative at low energies, compatible with my calculation.

4. ELASTIC THRESHOLD UNITARITY.

Define $f^{oo}(s,t) = <\pi^o,\pi^o,\text{out}|\pi^o,\pi^o,\text{in}> = A+B+C$ with $f_o^{oo}(s)$ its s-wave projection. Then Jin, Martin and Common[52] have shown from axiomatic field theory (positivity of the absorptive part and crossing symmetry) that the following rigorous results must be satisfied.

$$\text{(a)} \quad \left.\frac{\partial^2 f^{00}}{\partial s^2}(s,t)\right|_{s=t=u=4/3} > 0$$

$$\text{(b)} \quad f_0^{00}(4) > f_0^{00}(s) \qquad \text{for} \quad 4 > s \geq 0$$

$$\text{(c)} \quad f_0^{00}(0) \geq f_0^{00}(2+2/\sqrt{3})$$

$$\text{(d)} \quad f_0^{00}(3.205) > f_0^{00}(0.2134) > f_0^{00}(2.9863).$$

(2.42)

Furthermore, $f_0^{00}(s)$ has *at least* one minimum for 1.02 > s > 1.7. It is immediately obvious, as other authors have noted, that Weinberg's amplitudes do not satisfy these conditions— neither do the modified ones when $f_2^{\sigma} \neq 0$. By incorporation of elastic threshold unitarity (only) in a special way we will see that Martin's axiomatic conditions will be satisfied.

Write $M_{\alpha\beta,\gamma\delta}$ as

$$M_{\alpha\beta,\gamma\delta} = -16\pi \sum_{\substack{\ell,I \\ \ell+I \text{ even}}} t_\ell^I(\nu)(2\ell+1)P_\ell(\cos\theta)P_{\alpha\beta\gamma\delta}^{s(I)} .$$

(2.43)

Then the elastic unitarity condition written in the form most suitable for our purposes is

$$\text{Im}\, t_\ell^I = \frac{q}{\sqrt{s}} |t_\ell^I|^2 \, . \tag{2.44}$$

Following Iliopoulos' generalization of Weinberg's expansion (eqs. (2.18)) $M_{\alpha\beta,\gamma\delta}$ is expanded in a power series in $k_s = \sqrt{4-s}/2$, $k_t = \sqrt{4-t}/2$ and $k_u = \sqrt{4-u}/2$ so that the domain of analyticity of $M_{\alpha\beta,\gamma\delta}$ is expanded so as to include the physical threshold.

$$A = a - 2e + b(k_t+k_u) + ck_s + (d+e)k_s^2 + e(k_t^2+k_u^2)+\ldots,$$

$$B = A(k_s \rightleftarrows k_t) \, ,$$

$$C = A(k_s \rightleftarrows k_u) \, . \tag{2.45}$$

The on-mass-shell relation s+t+u = 4 is written in the form $k_s^2+k_t^2+k_u^2 = 2$. Eq. (2.45) is so arranged as to illustrate its off-mass-shell nature, characterized by e(=0 for an on-mass-shell amplitude).

Unitarity (U) in both I = 0 and I = 2 s-wave states, the Adler self-consistency condition, the V-commutator and σ-term give the following five equations in five unknowns:

$$-32\pi(2b+3c) = (5a+8b+2c+2d+8e)^2 \;, \qquad U(I=0)$$

$$-16\pi b = (a+b+c+d+e)^2 \;, \qquad U(I=2)$$

$$a + \sqrt{3}b + \frac{\sqrt{3}}{2}c + \frac{3d}{4} + \frac{1e}{4} = 0 \;, \qquad ASC$$

$$b - c - \sqrt{3}\,d = \frac{-16\sqrt{3}}{f_\pi^2} \;, \qquad V$$

$$2a + (\sqrt{3}+2)b + \sqrt{3}c + \frac{3d}{2} + e = -32a_2'\pi\,\sigma \qquad (2.46)$$

Before solving these equations let us look at the below threshold analyticity requirements on $f^{oo}(s,t)$ and $f_o^{oo}(s)$. From eq. (2.45) one finds

$$f^{oo}(s,t) = 3a + 2d + (2b+c)(k_s+k_t+k_u) \;, \qquad (2.47)$$

$$\left.\frac{\partial^2 f^{oo}(s,t)}{\partial s^2}\right|_{s=t=u=4/3} = -\frac{3\sqrt{3}}{64\sqrt{2}}(2b+c) \;. \qquad (2.48)$$

Note that in order to satisfy eq. (2.42a), 2b+c must be negative.

$$f_o^{oo}(s) = 3a + 2d + (2b+c)h(s) \;, \qquad (2.49)$$

with

$$h(s) = \sqrt{4-s}\,/2 + 2(s^{3/2}-8)/3(s-4) \;. \qquad (2.50)$$

This latter function h(s), shown in figure (8) has several interesting properties i.e. it has a maximum variation of 18% in the range 4 > s > 0 with a single *maximum* at s = 1.65 (which is in the range 1.7 > s > 1.02) and obeys the inequalities of eqs. (2.42(b),(c) and (d)) with the inequality signs reversed. Put more succinctly,-h(s) remarkably has all the properties required for $f_0^{00}(s)$. From eq. (2.49) this again indicates that we need (2b+c) to be negative. Thus, in order to satisfy *all* the below threshold inequalities we need 2b+c < 0. Although from eqs. (2.46){u(I=0)} and {u(I=2)} it is obvious that 2b + 3c < 0 and b < 0 it is certainly not clear that 2b + c < 0. Before leaving this section note that it is the linear expansion terms of eq. (2.45) which contribute to satisfy Martin's conditions for 2b + c < 0, not the quadratic ones.

5. NUMERICAL SOLUTIONS.

Eqs. (2.46) were solved on the APL/360 system in operation here. It was found that no real solutions existed for $a_2' > 0.15$. The equations were therefore solved for $-0.4 < a_2' < 0.15$ i.e. $-29.1 < f_2^{\sigma} < 10.9$. Results are shown in figures (9) and (10) for a_0 and a_2 respectively; for a given a_2' there are two solutions for $0.11 < a_2' < 0.15$ labelled 1(a) and 1(b), and four solutions for $a_2' < 0.11$ labelled 1(a), 1(b), 2(a) and 2(b). The solutions for a_0

are almost parabolic in shape.

For comparison purposes the modified Weinberg scattering lengths (eqs. (2.25)) are shown; they do not exhibit the cut-off behaviour due to incorporation of threshold unitarity, and diverge rapidly from the unitarily corrected scattering lengths for $a_2' > 0$. When $a_2' = 0$ we recover the four-fold nature of the solutions as obtained by Iliopoulos, although his values are incorrect. A very important point to note is that $2b+c < 0$ for all a_2' for all four solutions i.e. Martin's conditions are automatically satisfied. Selected numerical results are shown in table (1)

a_2'	Weinberg	1(a)	1(b)	2(a)	2(b)	
-0.1	0.44	0.56	-2.83	-2.59	0.31	a_0
0	0.19	0.18	-2.45	-2.18	-0.10	a_0
+0.1	-0.06	-0.37	-1.90	-1.42	-0.86	a_0
-0.1	0.04	0.13	0.09	-2.33	-2.37	a_2
0	-0.06	-0.06	-0.10	-2.15	-2.18	a_2
+0.1	-0.16	-0.29	-0.33	-1.93	-1.94	a_2

TABLE 1

Comparison of second order scattering lengths incorporating threshold unitarity with those of Weinberg for three values of a_2'.

Naturally, the fact that we have incorporated elastic threshold unitarity and isospin two amplitude in the t-channel, as well as satisfying below threshold conditions, is very gratifying; however, we do appear to be over-endowed with solutions. The question is, how can one discriminate between them? One possible way would be to contract out three or four pions and obtain the soft pion limit of all of them, so as to obtain constraints on $M_{\alpha\beta,\gamma\delta}$ at the unphysical points s+t+u=1 and s=t=u=o. Upon expansion of $M_{\alpha\beta,\gamma\delta}$ to third order in k_s, k_t, k_u the extra five coefficients could be determined, and an attempt made to determine the smoothest solution in transition from second to third order. Such a procedure unfortunately involves a knowledge of "σ-type" terms i.e.

(a) $\delta(x_o-y_o)[A_o^{\alpha}(x),\sigma_{\beta\gamma}(y)]$ at $s + t + u = 1$,

(b) $\delta(x_o-y_o)\delta(x_o-z_o)[A_o^{\alpha}(x),[A_o^{\beta}(y), \sigma_{\gamma\delta}(z)]]$, at $s = t = u = o$,

(2.51)

which are not known without introducing specific models.

Here we suggest a possible approach to remove the ambiguity caused by the multiplicity of solutions. $M_{\alpha\beta,\gamma\delta}$ may simply be determined to first order in k_s,k_t,k_u(i.e. solve eqs.(2.46)with $d = e \equiv o$); one may then try to decide whether a first order solution merges *smoothly* with any of the four second order ones. Obviously, if the change is rather violent from first to second order the parametrization of eq.(2.45) is meaningless.

At this stage a further ambiguity arises, namely, there is no unique first order solution due to the existence of five conditions to determine the three unknowns a, b, c; we have a choice of conditions to impose. As it is not particularly obvious which choice to make, all possibilities have been examined viz. the CA conditions ASC+V+σ or U(I=o) and U(I=2) combined with any one of ASC, V or σ. Unitarity combined with the V-commutator does not give real values for a, b, c (for any f_2^σ) whereas unitarity with the σ-model gives four distinct possibilities; altogether we therefore have six sets of solutions. They are exhibited as broken curves in figures (11)-(14), the full curves being the second order solutions. Figures (15) and (16) show the variation of d and e with a_2'. It is immediately obvious that the curves 1(a) merge most smoothly into one another, also the value of d is the smallest, as it should be. Note that scattering amplitudes are determined at the on-mass-shell point where $k_s^2+k_t^2+k_u^2 = 2$ so that the coefficient of e vanishes (which is equivalent to e = o for on-mass-shell considerations). Figure (17) and (18) show first order scattering lengths a_o and a_2 vs. a_2' - on comparison with figures (9) and (10) respectively one notes the vast change from first to second order in the scattering lengths except for solution 1(a). Figures (19) and (20) compare first and second order

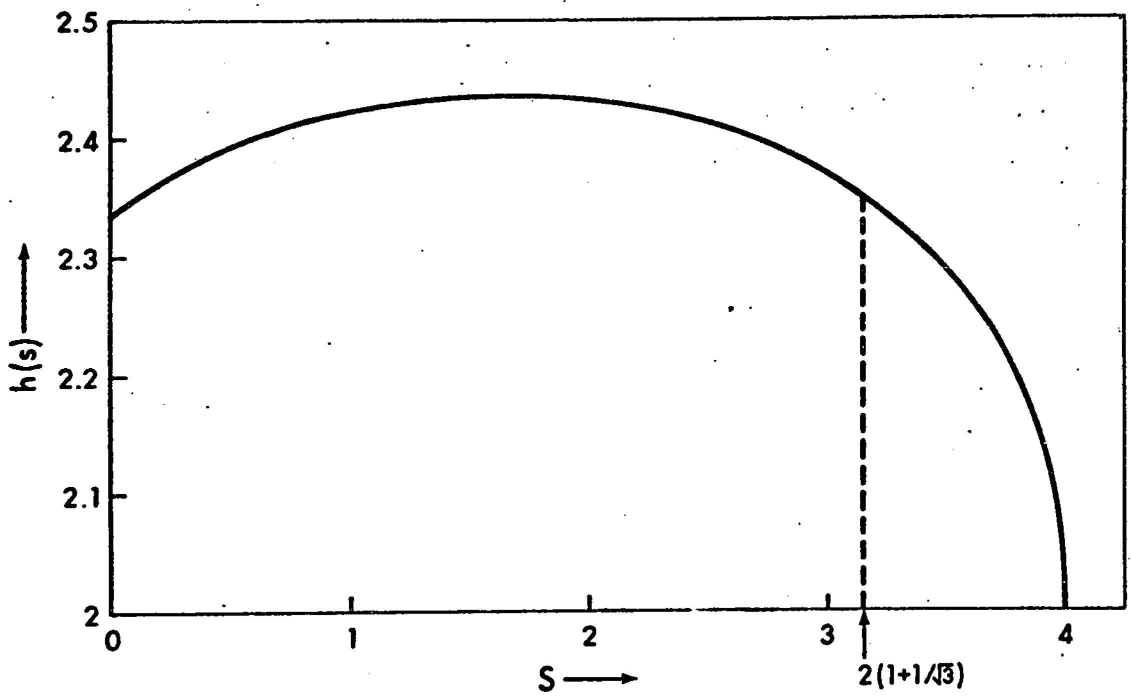


FIGURE 8. h(s) vs. s in the range $4 \geq s \geq o$.

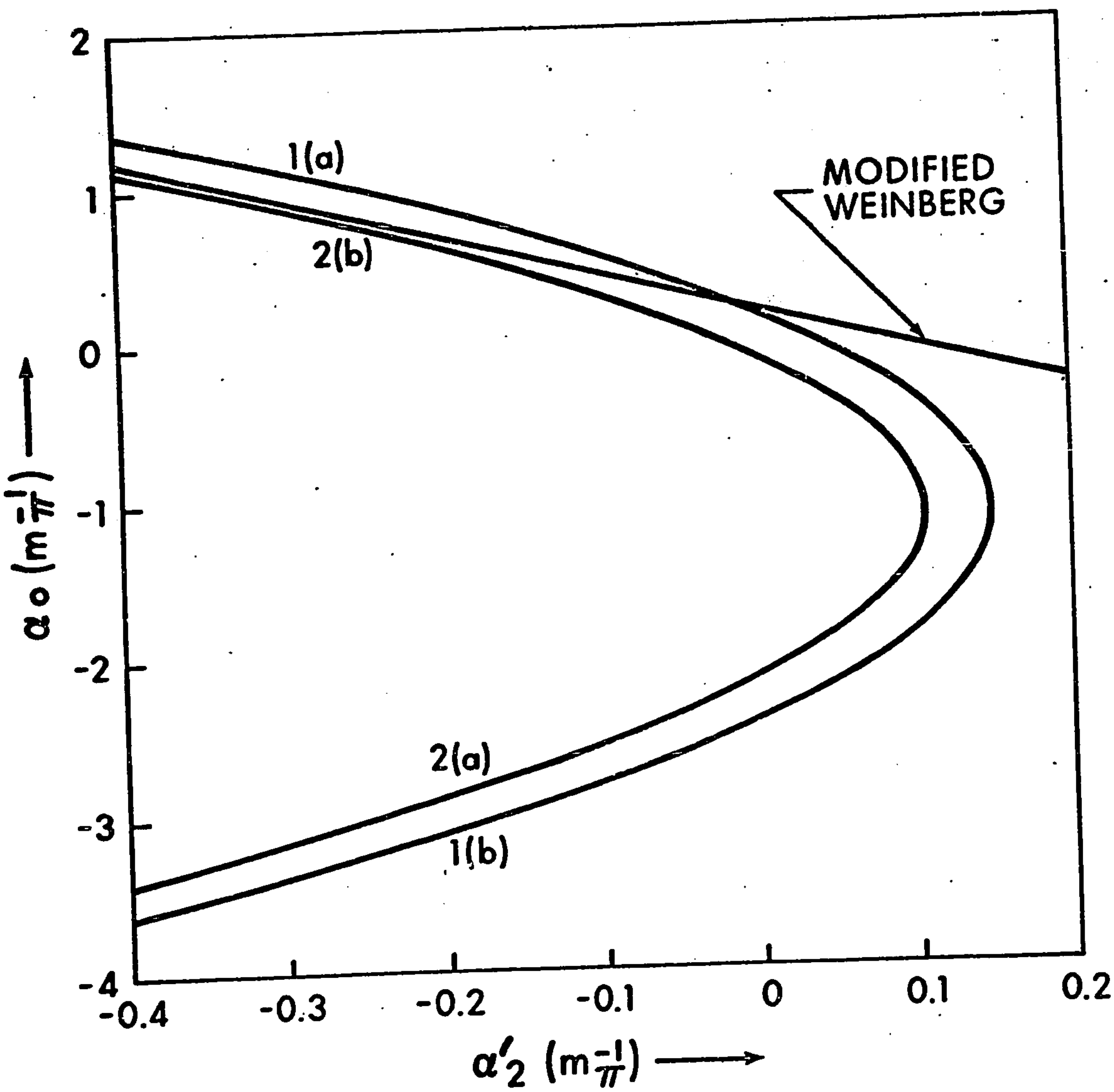


FIGURE 9. a_o vs. a'_2 compared to modified Weinberg a_o.

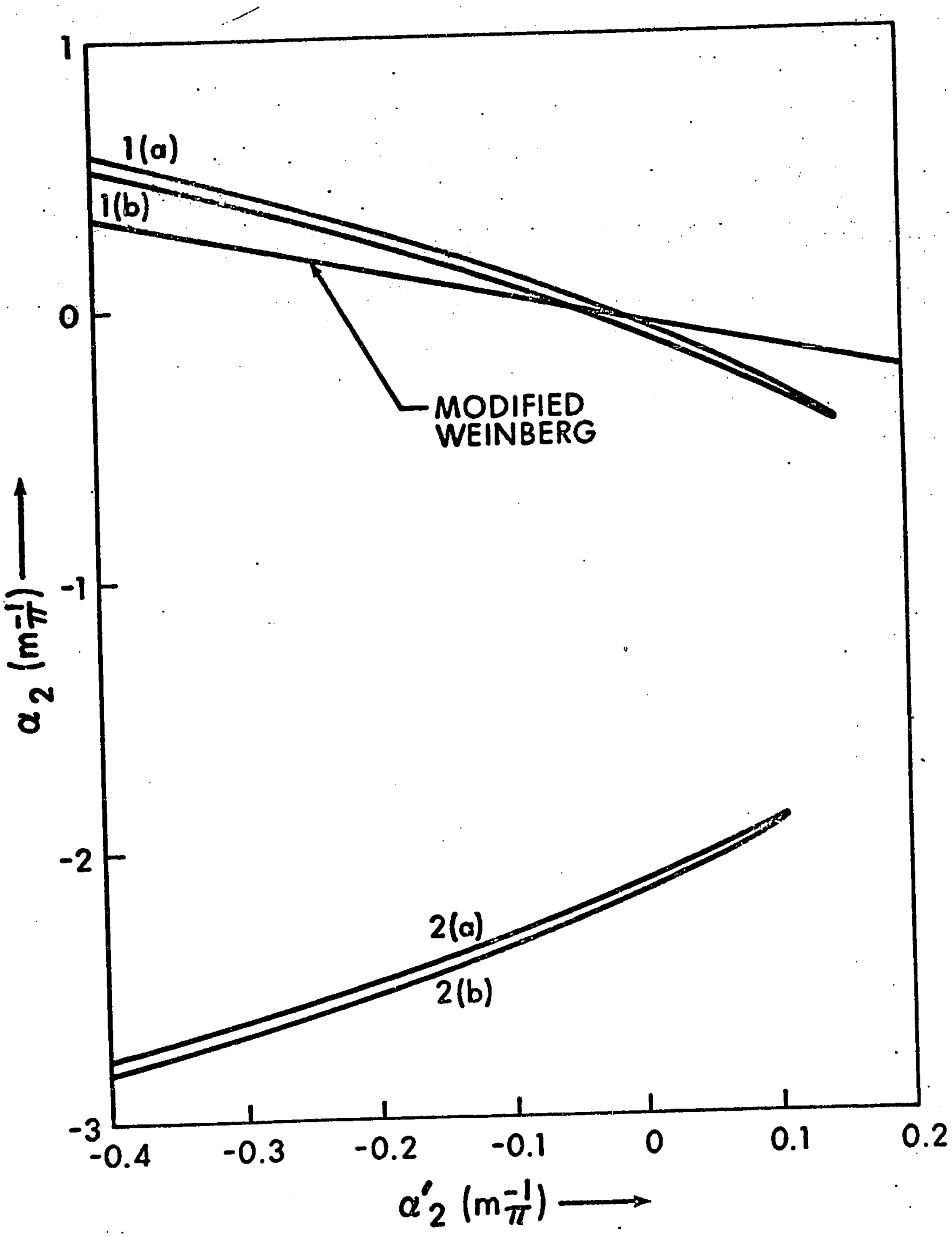


FIGURE 10. a_2 vs. a'_2 compared to modified Weinberg a_2.

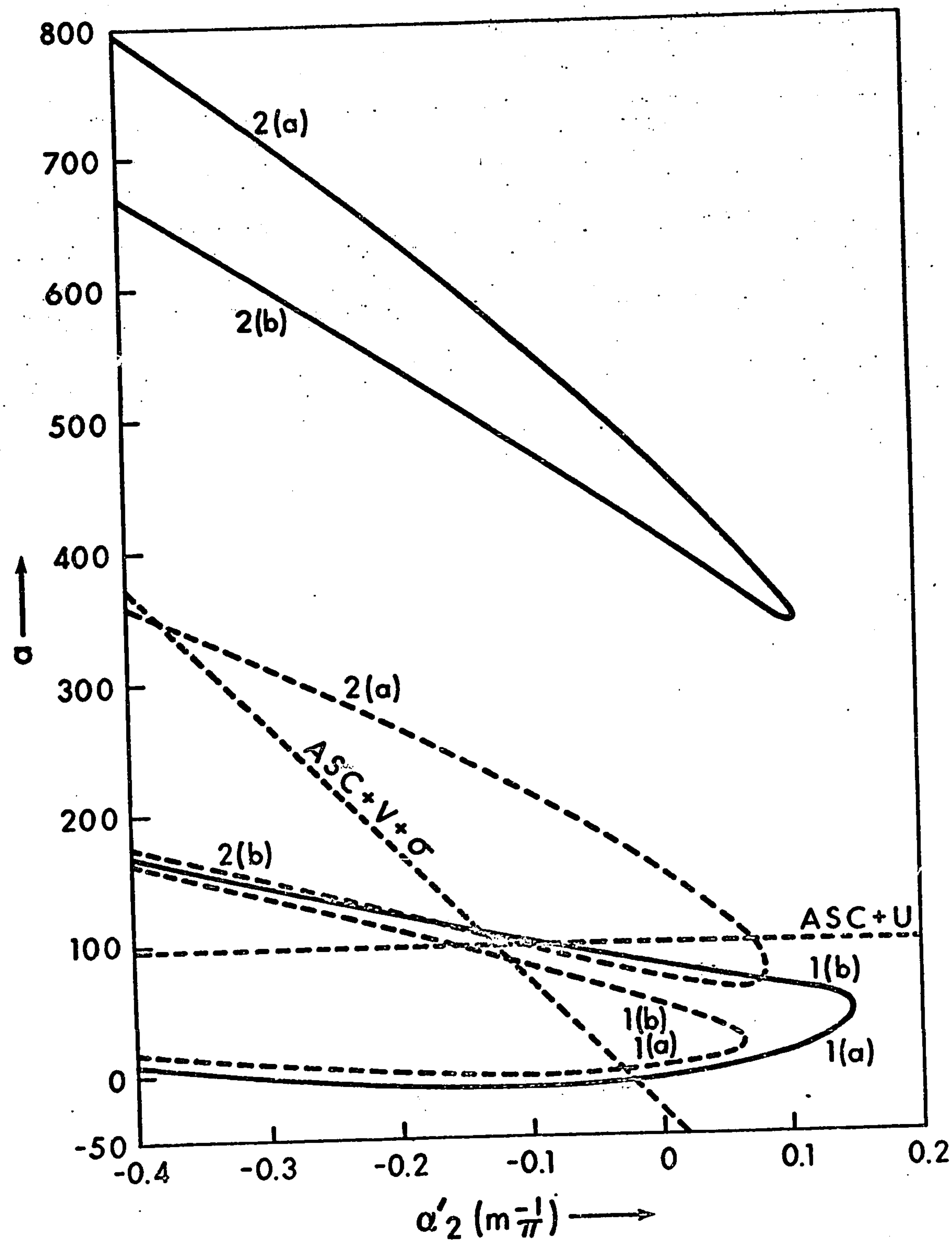


FIGURE 11. a vs. a_2' (1st and 2nd order expansions).

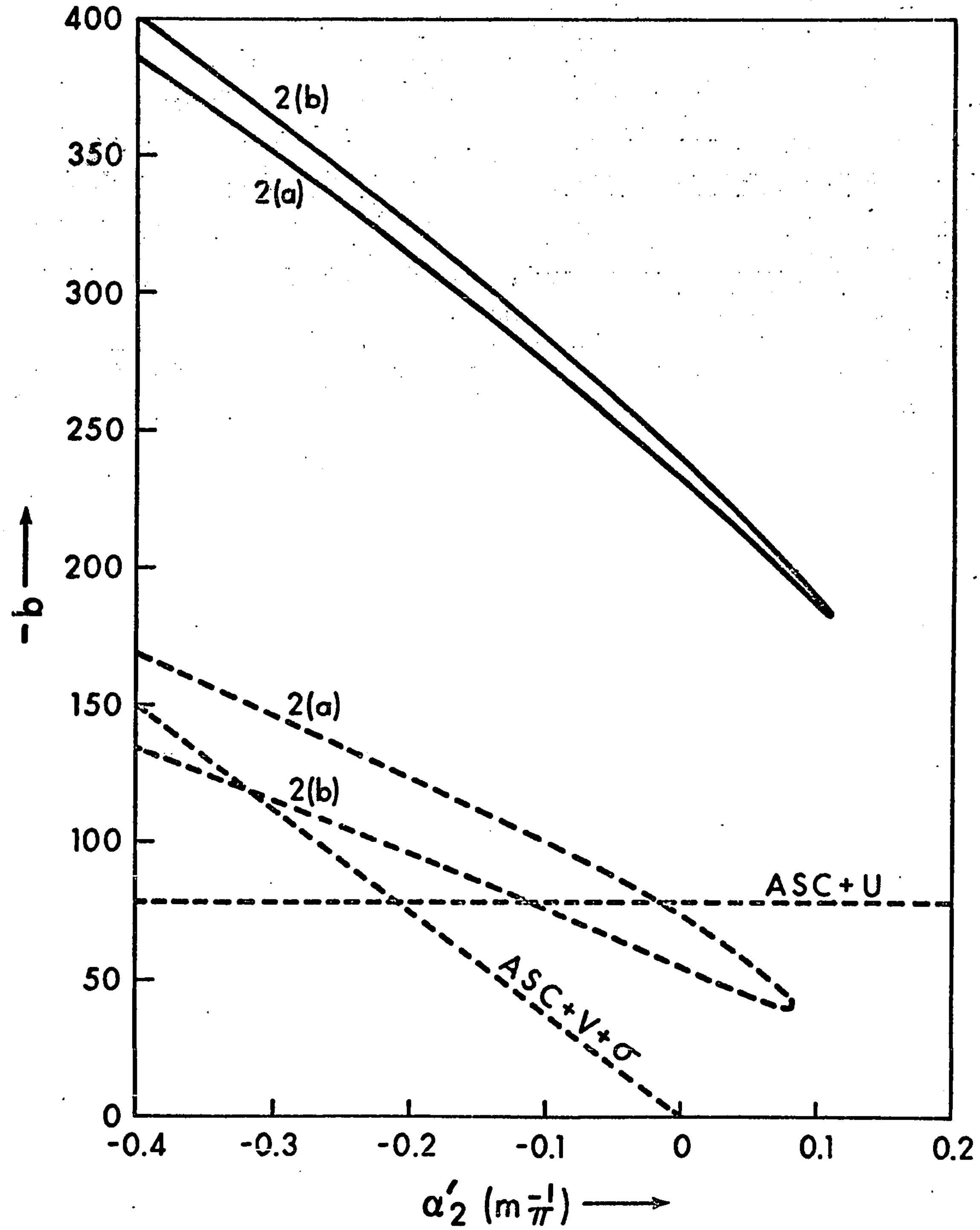


FIGURE 12. -b vs. a'_2 (1st and 2nd order expansions).

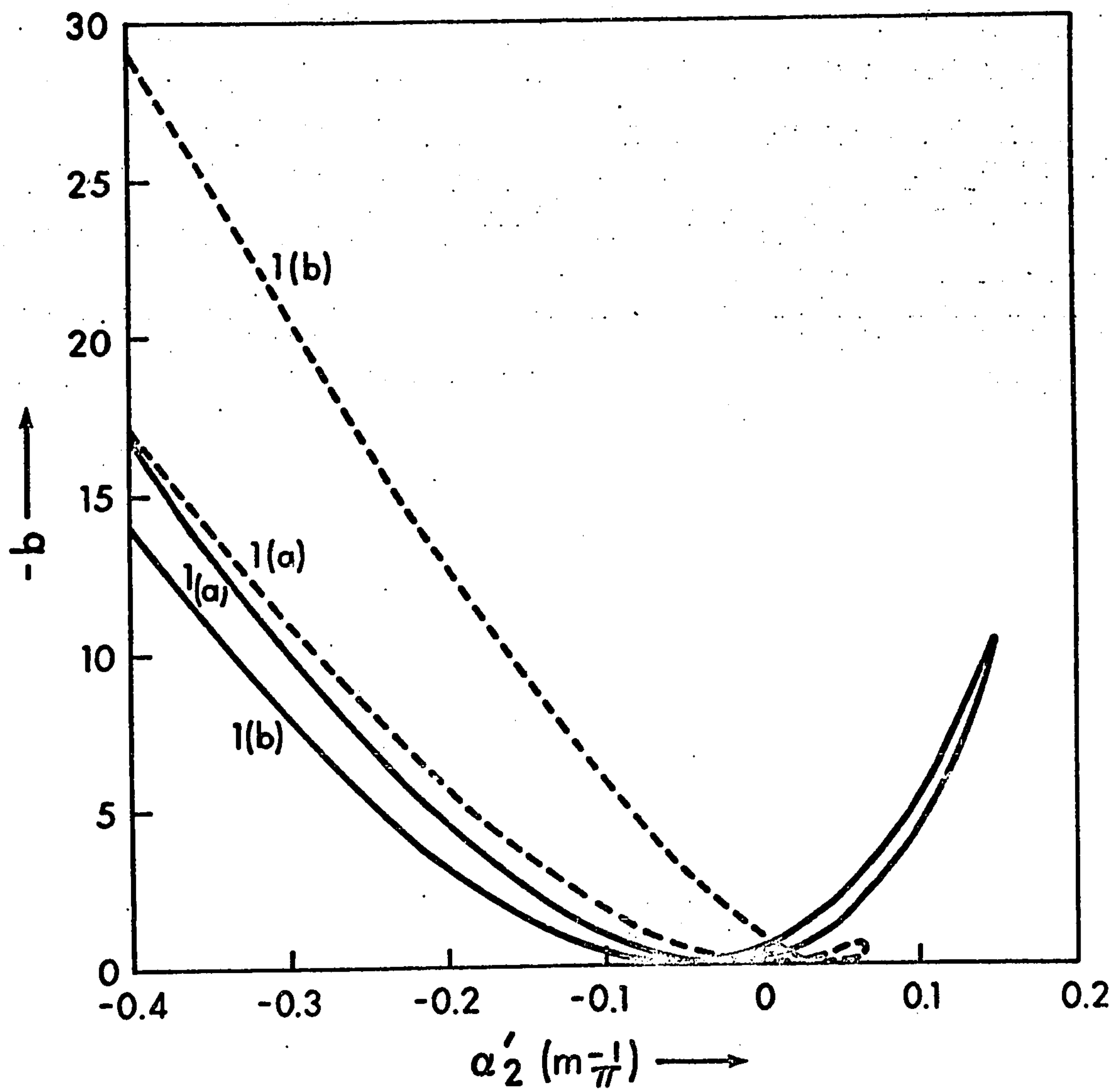


FIGURE 13. -b vs. a'_2 (1st and 2nd order expansions), solutions 1(a) and 1(b) on increased scale.

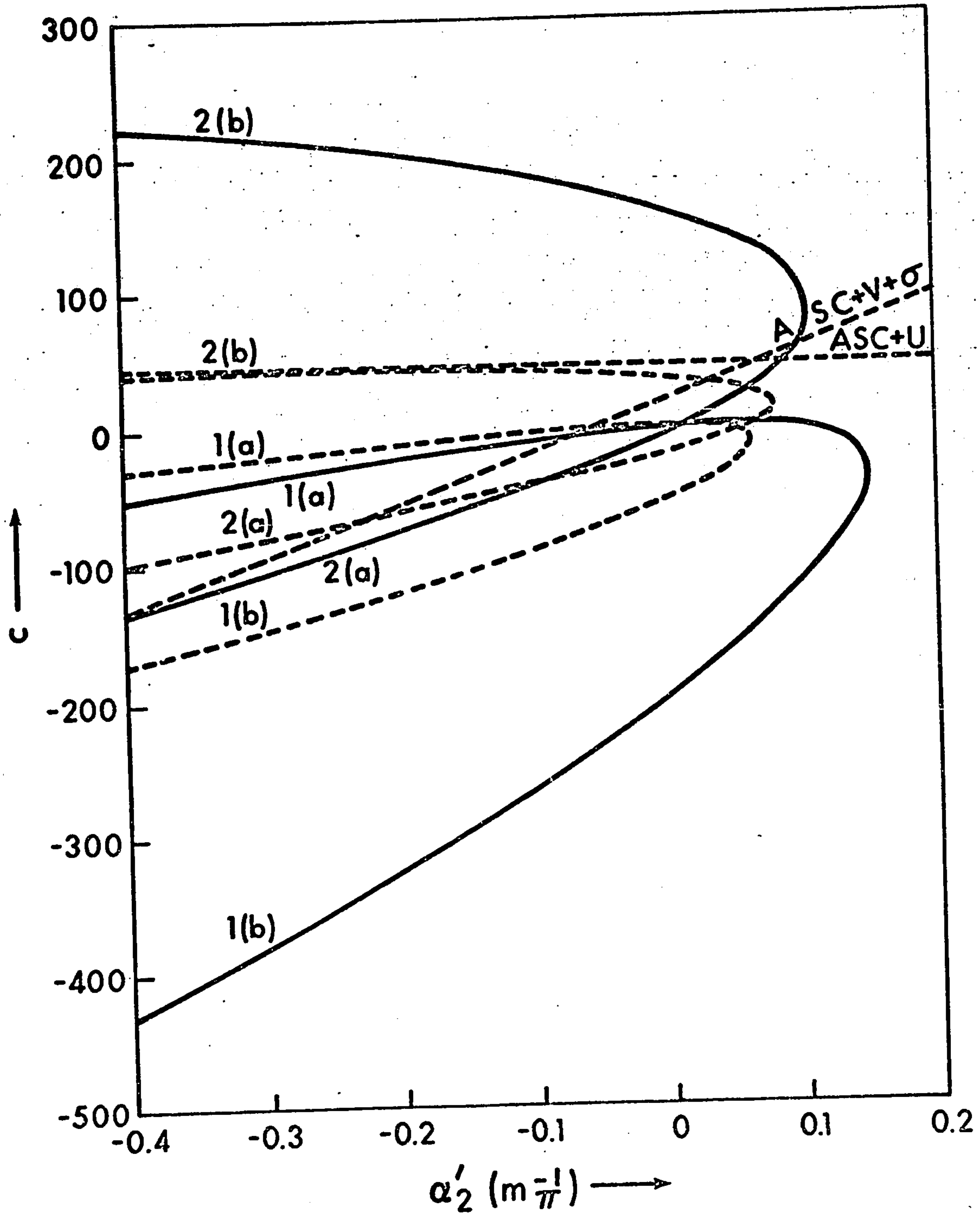


FIGURE 14. c vs. a_2' (1st and 2nd order expansions).

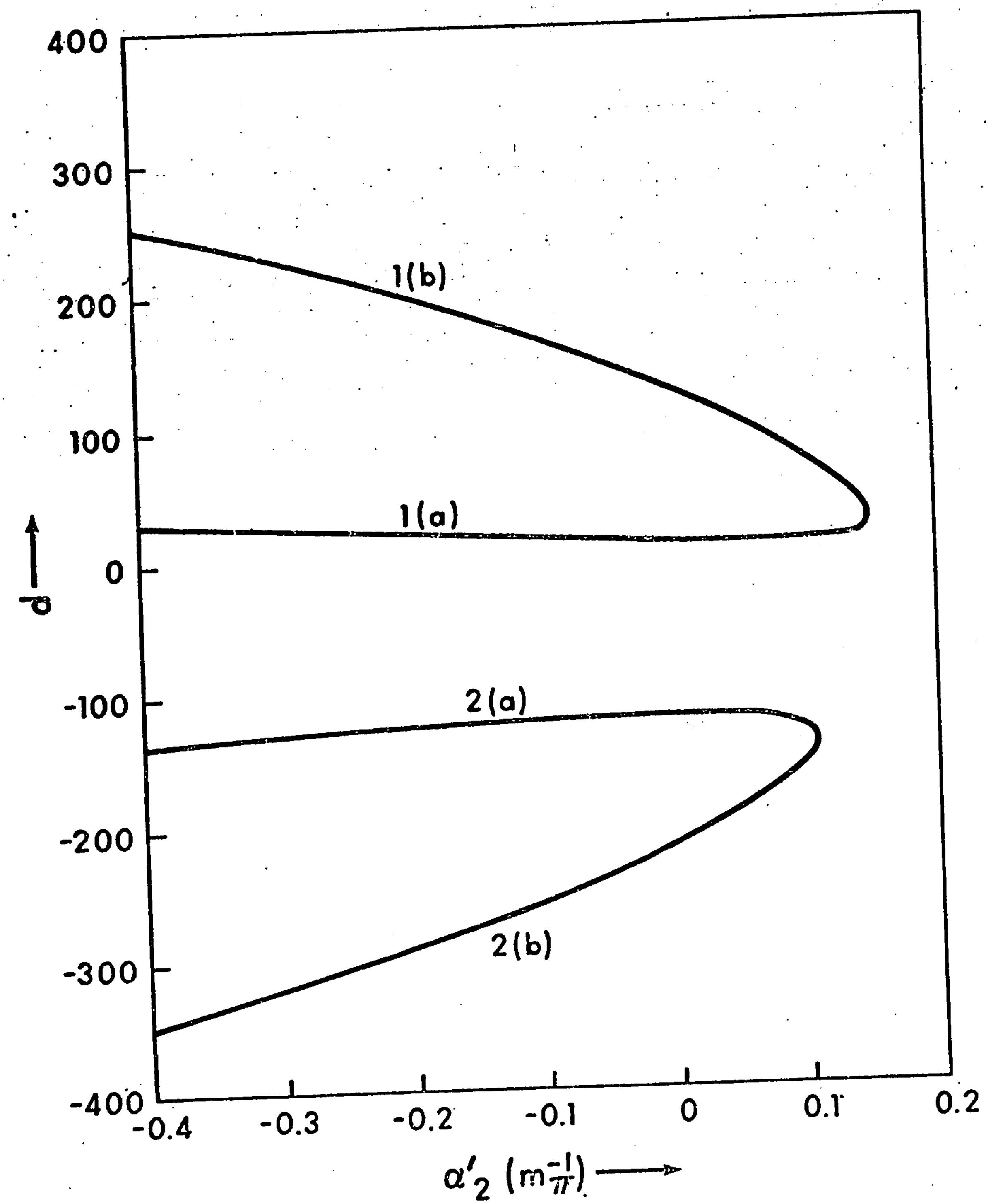


FIGURE 15. d vs. a_2' (2nd order).

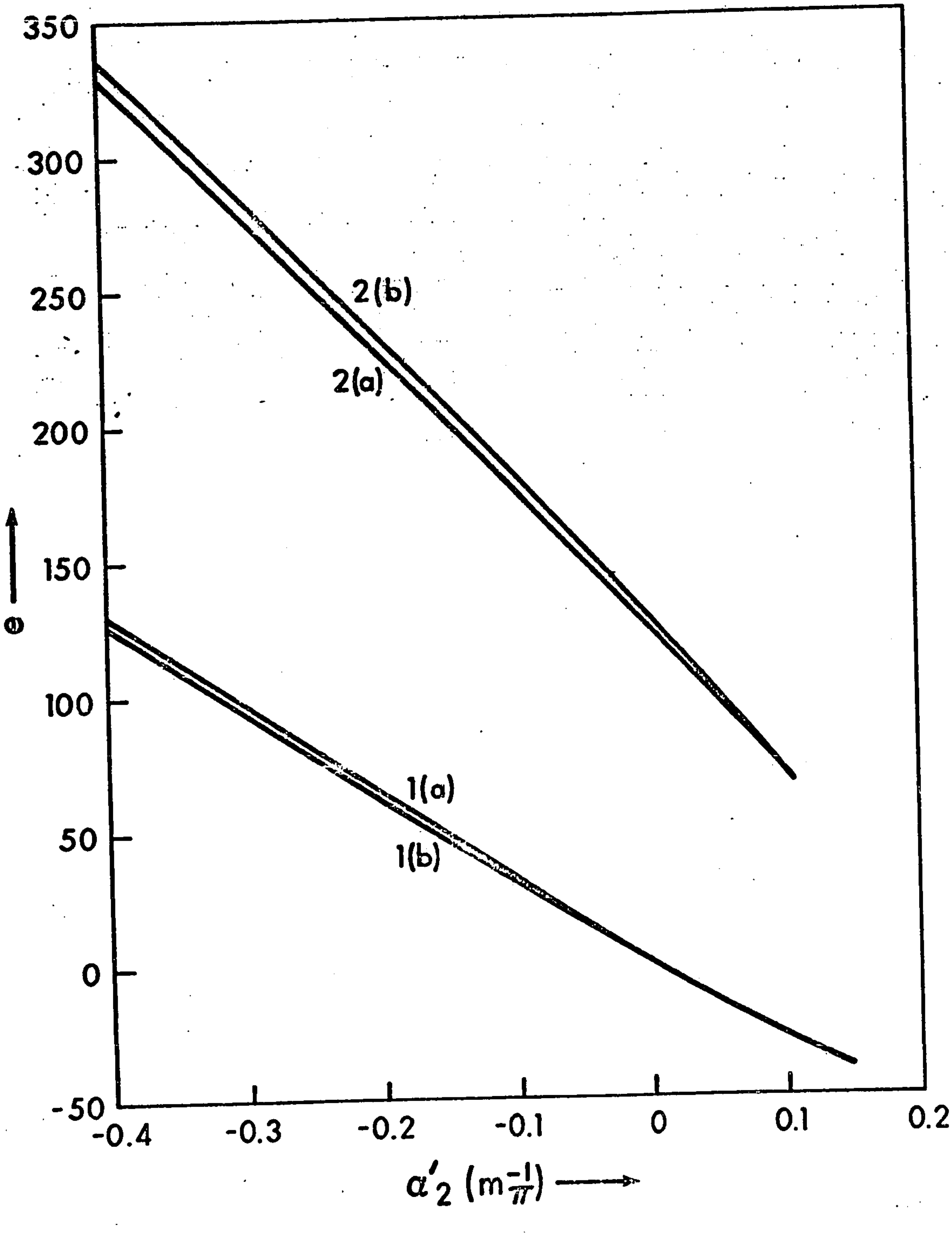


FIGURE 16. e vs. a_2' (2nd order).

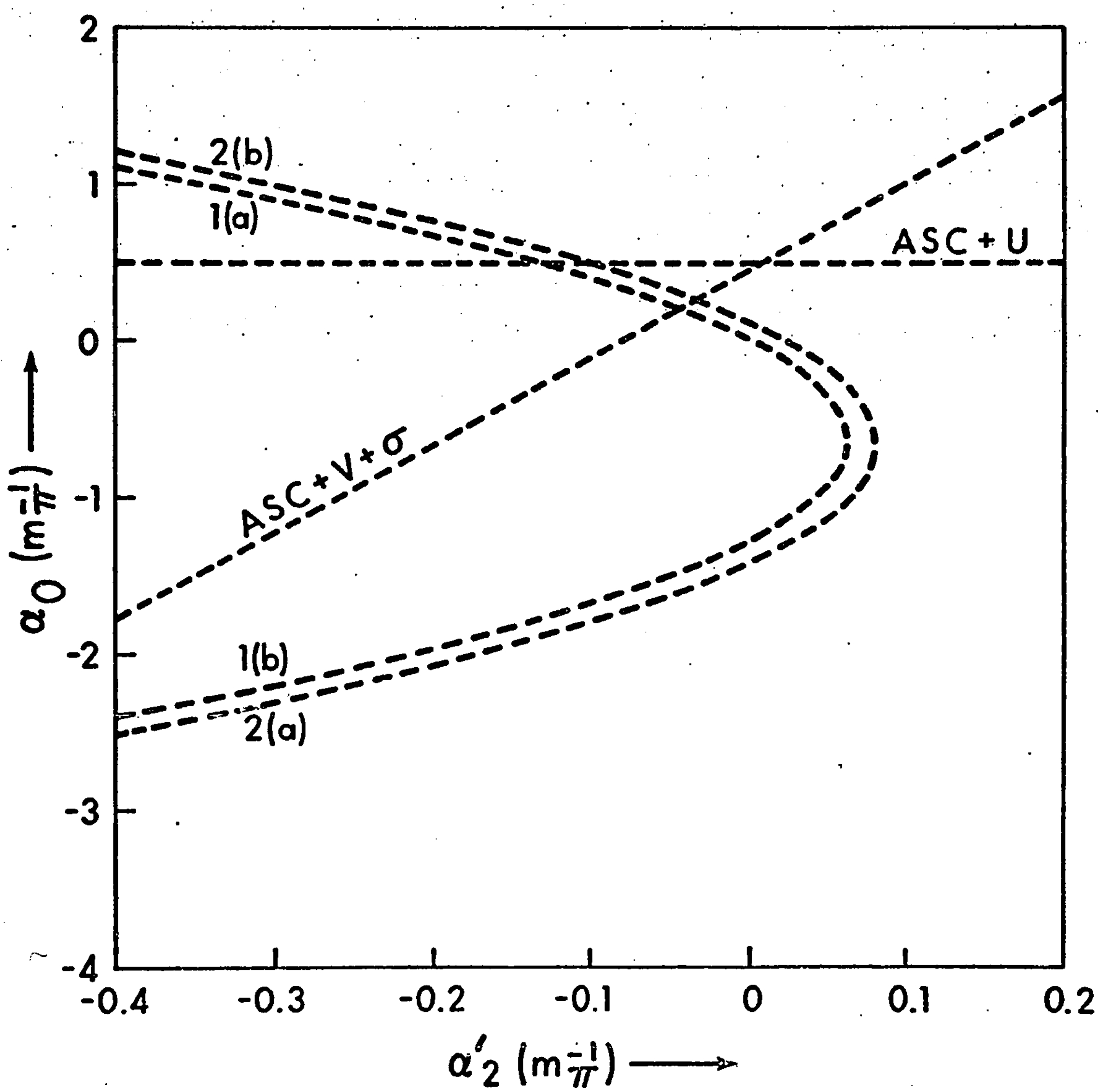


FIGURE 17. a_o vs. a_2' (1st order).

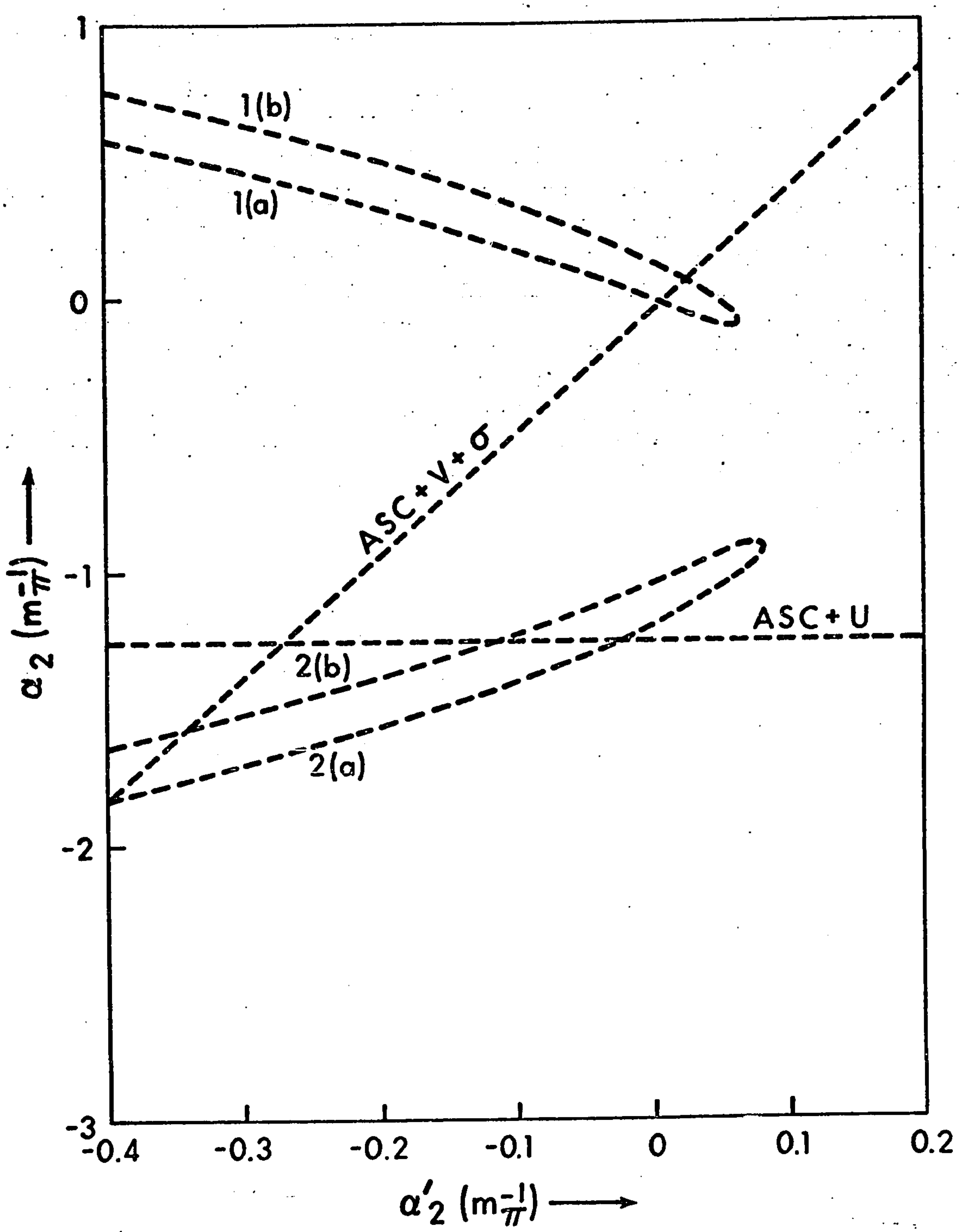


FIGURE 18. a_2 vs. a_2' (1st order).

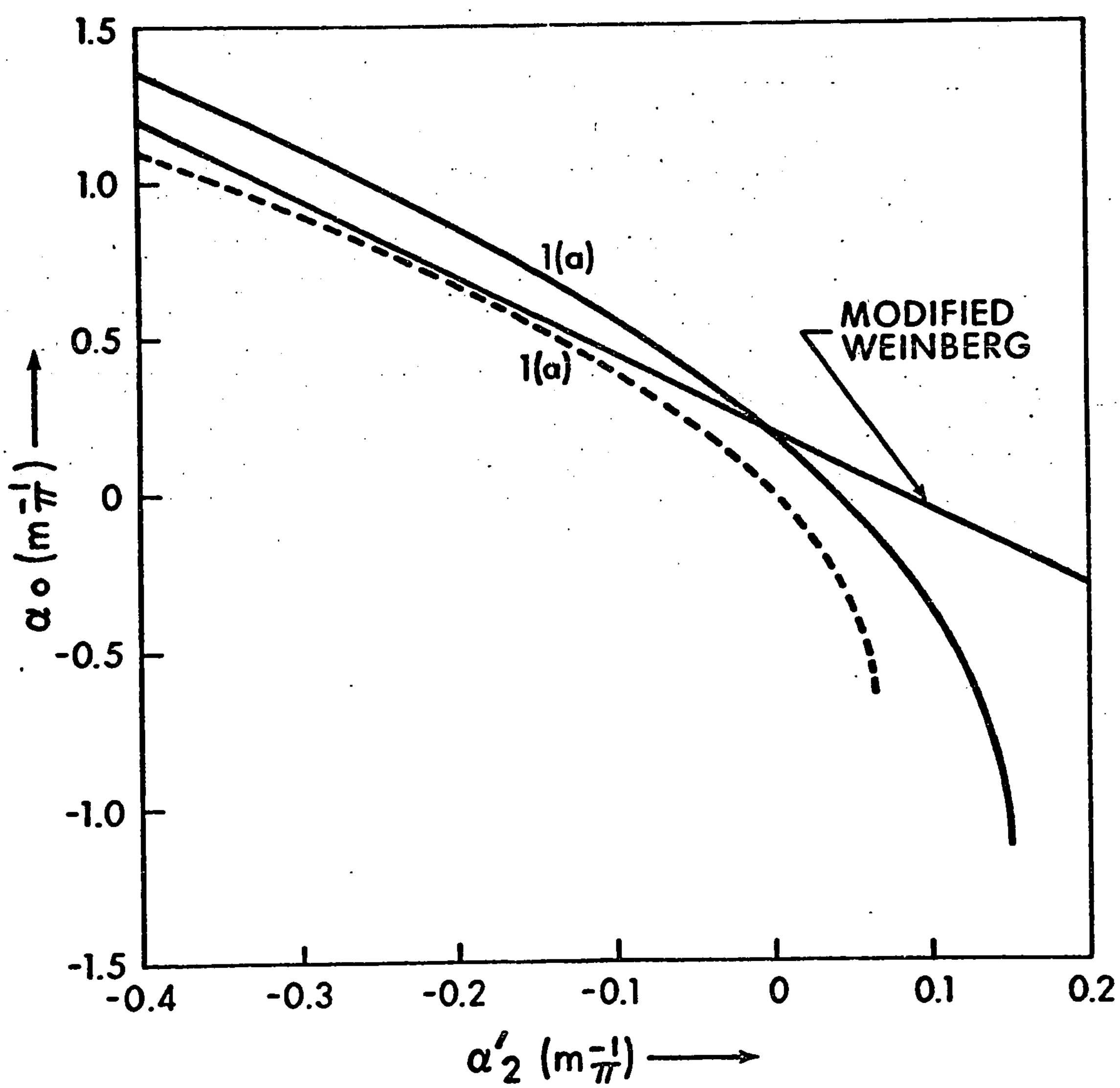


FIGURE 19. a_o vs. a_2' (1st and 2nd order) for solution 1(a) compared to modified Weinberg a_o.

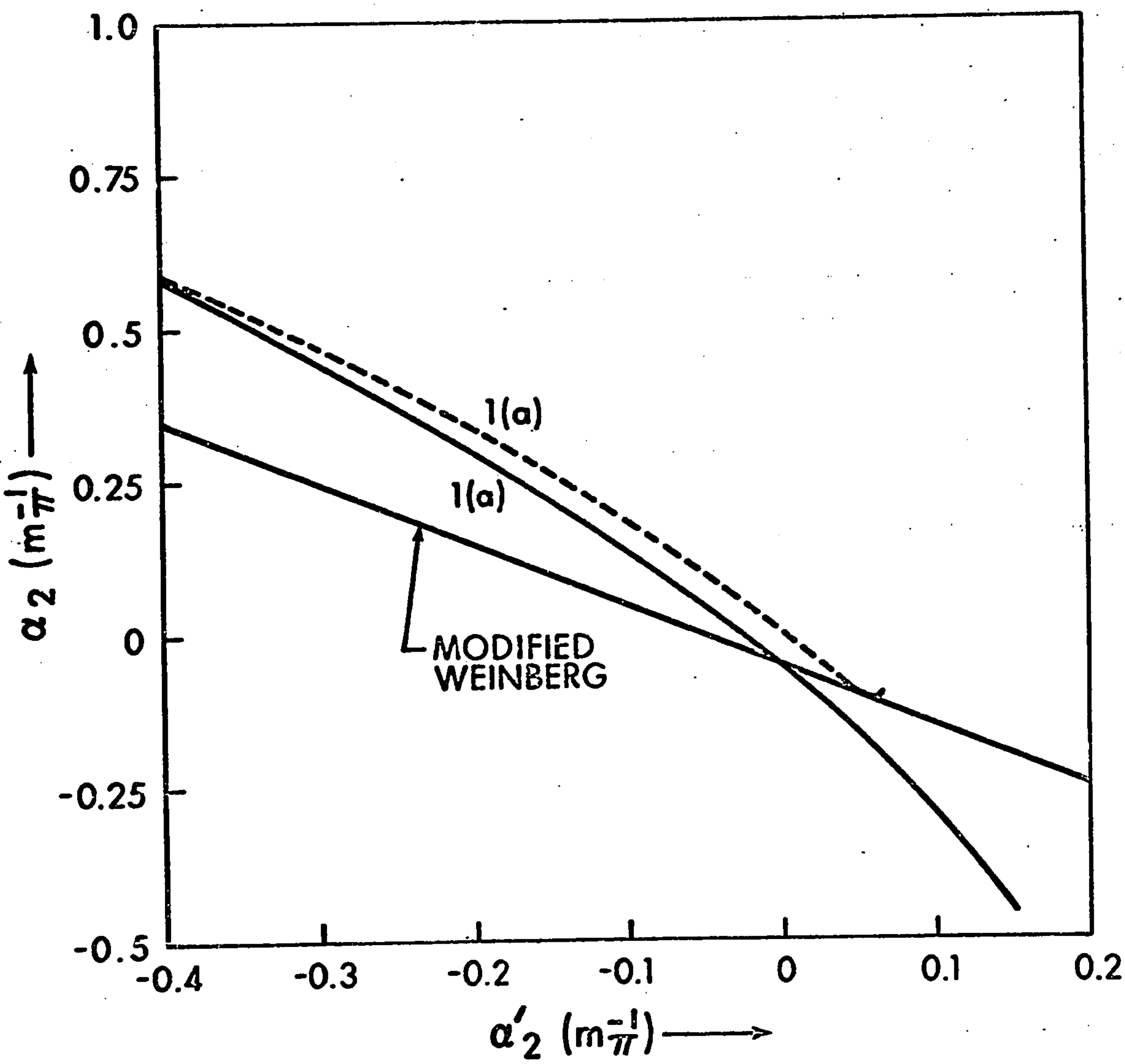


FIGURE 20. a_2 vs. a'_2 (1st and 2nd order) for solution 1(a) compared to modified Weinberg a_2 .

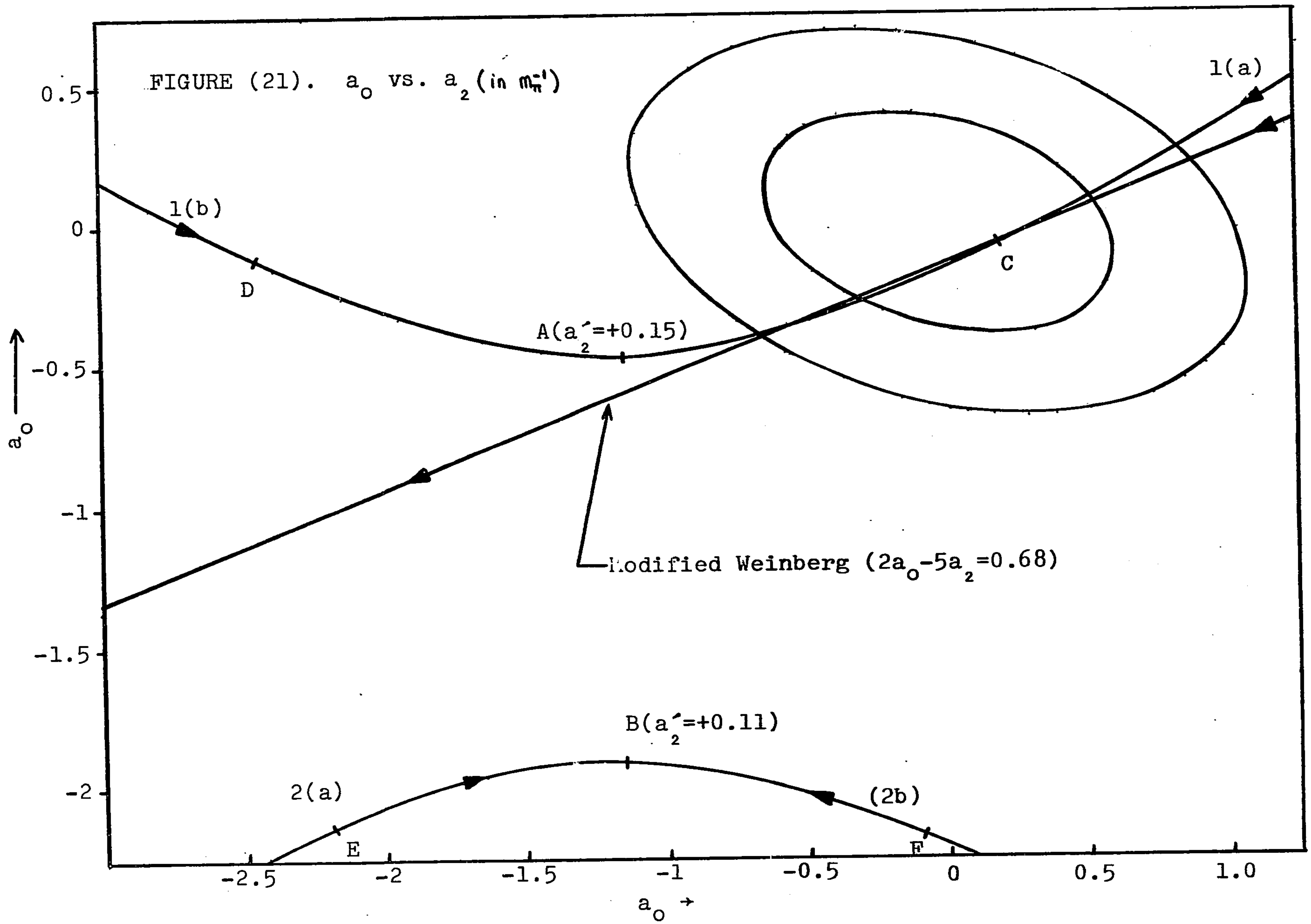


FIGURE (21). a_0 vs. a_2 (in m_π^{-1})

solutions with those of Weinberg; in the transition from first to second order a_2 decreases slightly whereas a_0 increases by a larger amount —— presumably if one could go to third order the change from second order would be smaller than the latter, but in the same relative directions.

Figures (19) and (20) give a definite relation between a_0 and a_2, if for example we know a_0 then a_2' is determined and a unique a_2 may be obtained. If $a_2' = o$ the Weinberg scattering lengths (see table (1)) coincide with the unitarily corrected ones —— this does appear a somewhat surprising conclusion.

6. SUMMARY OF SCATTERING LENGTH RESULTS.

In figure (21) we plot our four solutions 1(a), 1(b), 2(a) and 2(b) in the a_0-a_2 plane; points A and B are the points for maximum a_2' and indicate, of course, the dividing lines between these solutions. Arrows indicate increasing a_2'; the points C, D. E and F denote values at which $a_2' = o$.

Eliminating a_2' from eqs.(2.25) the modified Weinberg relation between a_0 and a_2 becomes

$$2a_0 - 5a_2 = 0.68 . \tag{2.52}$$

Notice (once again) how close this curve is to our favoured solution 1(a). In passing let us note that there is a well known approximate relation connecting the s and p-wave scattering lengths given by,

$$2a_0 - 5a_2 \simeq 18a_1 \ . \qquad (2.53)$$

This relation is simply obtained from eqs.(2.27) by continuing the amplitudes from ν_0 to threshold with retention of s and p-waves only. Comparing these two equations then $a_1 \simeq 0.038$, a result which relates very well to Olsson's (1967)[53] experimental result of 0.040 ± 0.002.

Now Pickup et al.[54] (see in this regard the paper by A.N. Kamal,[55]) using the Chew-Low extrapolation method on $\pi^- p \to \pi^- \pi^+ n$, $\pi^- \pi^0 p$ have plotted $\bar{\sigma} = \frac{1}{2}(\sigma_{\pi^- \pi^0} + \sigma_{\pi^+ \pi^-})$ in the range $o \leq \nu \leq 9$ and shown that its threshold value $\bar{\sigma}_0 = (40 \pm 15)$mb. But one also has

$$\bar{\sigma}_0 = \frac{4\pi}{9}(2a_0^2 + 5a_2^2 + 2a_0 a_2) \ . \qquad (2.54)$$

If we let $\bar{\sigma}_0$ take the values 20 and 60mb. and plot the relevant ellipses in the a_0-a_2 plane then the acutal results should certainly lie in between them, as in figure (21). It is quite obvious from this figure that the point $a_2' = o$ is *not* in this region i.e. Weinberg's original CA result is certainly not compatible with the Pickup data. When

Weinberg's calculation appeared many people were surprised as to the smallness of his results compared to previous calculations (this does not necessarily mean it is incorrect!). I merely suggest that one of the reasons for this is due to the neglect of a crossed channel isospin two interaction; this affects in particular a_0, which is more strongly dependent upon a_2' then a_2 is. From figure (21) this would appear to be even more important than unitarity. One must bear in mind however that figure (21) can be deceptive; although a_2' is zero at almost exactly the same point (i.e. at C) on both the modified Weinberg curve, and solution 1(a), the variation of a_2' along the former is much greater than the latter —— numerical values are noted below.

Using the Pickup ellipses as our scattering length boundaries one finds the following two ranges of possible solutions.

$$a_0 = (-0.67,-0.31) ,$$

$$a_2 = (-0.39,-0.26) ,$$

$$a_2' = (0.13, 0.09) ,$$

$$a_2'(\text{Weinberg}) = (0.35,0.22), \qquad (2.55)$$

$$a_o = (0.53,0.83) ,$$

$$a_2 = (0.10,0.27) ,$$

$$a_2' = (-0.09,-0.19) ,$$

$$a_2'(\text{Weinberg}) = (-0.14,-0.27). \qquad (2.56)$$

In both solutions $|a_o|>|a_2|$ which seems experimentally to be the case. Solution (2.56) corresponds to an attractive I = o (as well as attractive I = 2) interaction favoured in all pre-1963 calculations on backward π-N scattering,[56] and has not lost its popularity in the recent spate of interest in π-π scattering generated by Weinberg's calculation (for e.g. Fulco and Wong). There have, however, been many recent calculations[57] in which a_o is not only repulsive, but often quite strongly so. This is also experimentally indicated by some calculations on K_{ℓ_4} decay, and has led Goebel and Shaw[58] to try and obtain phenomenological bounds on a_o. With their assumptions and data (analyticity of a forward scattering amplitude plus ratio of total to forward elastic differential cross section) they obtain a negative lower bound of $a_o = -0.5$. Chu and Desai,[59] saturating a simply obtained symmetry point sum rule with ρ and f_o mesons conclude that a_o must be small and negative. These two results, if one accepts them at face value (often difficult to do!) would indicate that part of our solution (2.55) is the preferred one. As a reasonably happy consequence this has

$a_2 < o$, a result which appears to be gaining increasing acceptance. Thus, I find

$$a_0 = (-0.5,-0.31)$$

$$a_2 = (-0.32,-0.26)$$

$$a_2' = (0.115,0.09)$$

$$a_2' \text{ (Weinberg)} = (0.28,0.22) \qquad (2.57)$$

Finally, let us note that our Weinberg type phase shift calculations were only valid for a_2'(Weinberg) > 0.07, and were arbitrarily truncated at a_2'(Weinberg) = 0.13 in order not to make this interaction unduly large. This ensures that a_2' (Weinberg) is just inside the inner Pickup ellipse. From eq.(2.57) it would now seem reasonable to repeat this calculation with solution 1(a), as one can obviously have lower values of a_2' between the ellipses.

7. CONCLUSION.

It has been shown that Weinberg's CA π-π scattering length calculations may be generalized to include a crossed channel isospin two interaction (which surely must be present). Using dispersion relations, below threshold CA amplitudes, and a single pole approximation to the left hand cut (for each isospin state) the isospin zero s-wave phase shift δ_0 has been found to be slightly negative just above elastic threshold, before becoming positive and reaching a maximum of about 30° at $\nu = 10$. This occurs at the minimum allowed value of $\lambda = 0.03$ (the phenomenological π-π coupling constant), any increase in λ decreases δ_0 so that it will never resonate. δ_2 is always small and negative, and is compatible with, for example, the "experimental" results of Walker et al. and Baton et al.

This modified Weinberg calculation does not incorporate elastic unitarity, and furthermore does not satisfy the below threshold conditions of Martin on the π^{0}-π^{0} amplitude. A simple procedure is used to incorporate threshold unitarity in the above method, originally due to Iliopoulos. This forces the results to obey Martin's conditions but produces four sets of solutions, from continuity arguments only one of these is favoured, appearing also to be the one experimentally favoured. It seems that unitarity is important from the subsidiary rôle it plays in reducing the amount of the crossed channel isospin two scattering length a_2' which is necessary to bring the theoretical results within "experimental" bounds. With the experimental limits of Pickup et al., and the phenomenological bounds of Goebel and Shaw, and Chu and Desai one finds $a_0 \simeq (-0.5,-0.3)$, $a_2 \simeq -0.3$ and $a_2' \simeq 0.1$.

CHAPTER III

ABSTRACT: The off-shell gauge violation in the photoproduction of a soft pion or axial-vector spurion is shown to be completely contained in the nucleon pole terms of the amplitude. Using this one can show that the PCAC hypothesis, so modified as to take first order electromagnetic interactions into account, is sufficient to determine sum rules between the anomalous isoscalar and isovector nucleon magnetic moments and the soft-pion forward production amplitude. Such sum rules have been previously obtained using the techniques of current algebra and dispersion relations.

1. INTRODUCTION

In the last few years considerable interest has arisen in the possible significance and range of validity of various theoretical results presently ascribed to Gell-Mann's(7) chiral algebra of currents. One of the most important sum rules obtained from this current algebra is the Adler-Weisberger(5) result for the axial-vector β-decay coupling constant renormalization. However, Veltman(60) has indicated, in a model in which PCAC is modified to include minimal couplings of photon and intermediate weak vector and axial-vector bosons, that the Adler-Weisberger sum rule can merely be obtained by considering weak axial-vector boson scattering off nucleons, without the necessity for commutation relations. Even more simply, the Cabibbo-Radicati[61] sum rule for the difference between the proton and neutron total magnetic moments, originally obtained from the commutation relations of electric dipole moments, follows immediately from the minimal photon coupling to the vector hadronic weak current. Cordes and Moffat[62] have shown that reasonable results for g_A can be obtained by breaking the closure property of the $SU(3)\otimes SU(3)$ group in the axial-charge axial-charge commutation term. Fayyazuddin and Hussain[63] obtain the Adler-Weisberger result from weak amplitude superconvergent dispersion relations.

These results certainly indicate that the Adler-Weisberger and Cabibbo-Radicati sum rules are not a critical test of CA. Here we show further that the experimentally well satisfied CA derived sum rules of Fubini, Furlan and Rossetti[64] relating the anomalous isoscalar (μ^s) and isovector (μ^v) nucleon magnetic moments to the pion photoproduction amplitude, may simply be obtained from the PCAC hypothesis modified to take first order electromagnetic interactions into account, without using commutation relations.

Section 2 studies the modification of the usual gauge condition for a photoproduced off-mass-shell pion (or axial-vector spurion), and notes in the soft pion (axial-vector spurion) momentum limit $k\to 0$ that the entire gauge violation is contained in the nucleon pole terms. This non-CA method is compared with the earlier CA one and the similarities noted. Amplitudes for pion (axial-vector spurion) photoproduction are given, in Section 3 these amplitudes are substituted into the modified PCAC hypothesis sandwiched between a nucleon, and a nucleon photon state, in the soft limit $k^2=0$. The analysis simplifies considerably in the limit $k\to 0$, whence the sum rules are obtained. Finally we note the work which Nauenberg, Boulware and Brown, and Berman and Frishman have done concerning the interrelation between CA and modified PCAC, in the same period in which the aforementioned study was undertaken.

2. OFF-SHELL GAUGE CONDITIONS

We consider the photoproduction process $\gamma(q)+N(p_1)\to N(p_2)+\pi^i(k)$, and use the invariants $\nu=-(p_1+p_2).k/2M$, $\nu'=q.k/2M$, where M is the nucleon mass. The PCAC hypothesis, as modified to take account of first order e.m. effects may be written as[65]

$$\partial_\mu A^i_\mu - e\varepsilon^{3ij} a_\mu A^j_\mu = f_\pi \phi^i \ , \qquad (3.1)$$

where (i,j) are isospin labels, the pion mass is unity, a_μ the e.m. field, $\phi^\pm=\surd 2(\phi^1\pm i\phi^2)/2$, $\phi^0=\phi^3$ are renormalized pion field operators, and $A^\pm_\mu=(A^1_\mu \pm iA^2_\mu)/2$, $A^0_\mu=A^3_\mu$; f_π is given in eq. (1.15).

Sandwiching eq. (3.1) between states $\langle p_2|$ and $|p_1,q\rangle$, contracting out the photon, and using the convenient notation[66]

$$\langle p_2|A^j_\mu|p_1\rangle = N_{12}N^j_\mu \quad , \quad \langle p_2,q|A^i_\mu|p_1\rangle = eN'_{12}e_\nu M^i_{\nu\mu} \quad , \qquad (3.2)$$

(related to $\langle p_2|A^i_\mu|p_1,q\rangle$ by crossing)

$$\langle p_2|J^i_\pi|p_1,q\rangle = -(k^2-1)\langle p_2|\phi^i|p_1,q\rangle = eN'_{12}e_\nu T^i_\nu \quad , \qquad (3.3)$$

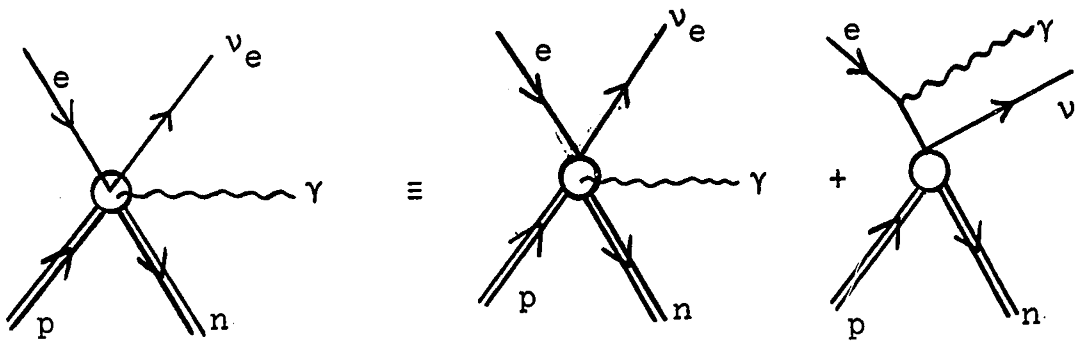
with e_ν the photon polarization vector, one obtains

$$f_\pi e_\nu T^i_\nu(\nu,\nu',k^2) = (k^2-1)e_\nu[ik_\mu M^i_{\nu\mu}(\nu,\nu',k^2) - \varepsilon^{3ij}N^j_\nu(\nu,\nu',k^2)], \qquad (3.4)$$

i.e. the gauge condition on T^i_ν becomes (substituting q_ν for e_ν)

$$f_\pi q_\nu T^i_\nu = (k^2-1)q_\nu(ik_\mu M^i_{\nu\mu}-\varepsilon^{3ij}N^j_\nu) \ . \qquad (3.5)$$

Eq. (3.5) is not in a very convenient form, as the right hand side involves radiative and non-radiative weak amplitudes. However, a separate constraint relating $M^i_{\nu\mu}$ to N^j_ν has been derived by Amati and Jengo[67] and Adler and Dothan[68] by utilizing gauge invariance on the radiative nucleon β-decay (or K-capture) amplitude; we consider this briefly here.



(a) (b) (c)

Figure 22. In the radiative K-capture process (a) the photon is emitted from either the hadrons (b) or the electron (c).

Denote the weak leptonic current by j^{ℓ}_{μ}, where

$$j^{\ell}_{\mu} = \bar{u}_{\nu_e}\gamma_{\mu}(1-i\gamma_5)u_e \tag{3.6}$$

Let's denote the matrix element of figure (22a) by $eN'_{12}e_{\nu}L^{-}_{\nu\mu}j^{\ell}_{\mu}$; then we have

$$eN'_{12}e_{\nu}L^{-}_{\nu\mu}j^{\ell}_{\mu} = eN'_{12}e_{\nu}M^{-}_{\nu\mu}j^{\ell}_{\mu} + N_{12}N^{-}_{\mu}\bar{u}_{\nu_e}\gamma_{\mu}(1-i\gamma_5) \times$$

$$\times \frac{i}{\not{K}_e-\not{q}-m_e}(-ie\gamma_{\nu}e_{\nu})u_e/\sqrt{2q_o} ,$$

i.e.

$$e_{\nu}L^{-}_{\nu\mu}j^{\ell}_{\mu} = e_{\nu}M^{-}_{\nu\mu}j^{\ell}_{\mu} + N^{-}_{\mu}\bar{u}_{\nu_e}\gamma_{\mu}(1-i\gamma_5)\frac{1}{\not{K}_e-\not{q}-m_e}\not{e}u_e . \tag{3.7}$$

Substituting q_{ν} for e_{ν} into this equation, and noting that $q_{\nu}L^{-}_{\nu\mu} \equiv 0$ then one immediately finds

$$q_\nu M^-_{\nu\mu} = N^-_\mu \ , \tag{3.8}$$

as $j^\ell_\mu \neq 0$. In the general isospin case one obtains

$$q_\nu M^i_{\nu\mu} = -i\varepsilon^{3ij} N^j_\mu \ . \tag{3.9}$$

Substituting eq. (3.9) into eq. (3.5) then

$$f_\pi q_\nu T^i_\nu = (k^2-1)\varepsilon^{3ij}(k_\nu - q_\nu)N^j_\nu$$

$$= \frac{if_\pi(k^2-1)\varepsilon^{3ij}}{(k-q)^2-1} \langle p_2 | J^i_\pi | p_1 \rangle \ , \quad \text{from PCAC.}$$

i.e.

$$q_\nu T^i_\nu(\nu,\nu',k^2) = \frac{-(k^2-1)}{(k-q)^2-1} \, gK\{(k-q)^2\}\bar{u}(p_2)Q^i_- \gamma_5 u(p_1) . \tag{3.10}$$

We use the isospin projection operators

$$Q^i_o = \tau_i$$

$$Q^i_+ = \tfrac{1}{2}\{\tau_i, \tau_3\} = \delta_{i3}$$

$$Q^i_- = \tfrac{1}{2}[\tau_i, \tau_3] = -i\varepsilon^{3ij}\tau_j \ . \tag{3.11}$$

It is instructive to compare the CA method for obtaining eq. (3.10), as given, for example, by Nauenberg[69]. Define a quantity $T^i_{\nu\mu}$ as

$$T^i_{\nu\mu} = -i\int d^4x e^{ikx} \langle p_2 | [A^i_\mu(x), V^3_\nu(0)] | p_1 \rangle \theta(x_o) . \tag{3.12}$$

Multiplying by k_μ, integrating by parts and dropping the surface term then

$$k_\mu T^i_{\nu\mu} = \int d^4x e^{ikx} \{ \langle p_2 | [\partial_\mu A^i_\mu(x) , V^3_\nu(0)] | p_1 \rangle \theta(x_o) + $$
$$+ \langle p_2 | [A^i_o(x) , V^3_\nu(0)] | p_1 \rangle \delta(x_o) \} . \tag{3.13}$$

But from CA.

$$[A^i_o(x), V^3_\nu(0)] \delta(x_o) = -i\varepsilon^{3ij} A^j_\nu(x) \delta^4(x) , \tag{3.14}$$

and substituting this result into eq. (3.13) one finds

$$\int d^4x e^{ikx} \langle p_2 | [\partial_\mu A^i_\mu(x), V^3_\nu(0)] | p_1 \rangle \theta(x_o) =$$
$$= i\varepsilon^{3ij} N^j_\nu + k_\mu T^i_{\nu\mu} . \tag{3.15}$$

Define $$S^i_\nu = \langle \pi^i(k), p_2 | V^3_\nu(0) | p_1 \rangle$$
$$= i\int d^4x e^{ikx} K_x \langle p_2 | [\phi^i(x), V^3_\nu(0)] | p_1 \rangle \theta(x_o) , \tag{3.16}$$

where the latter result is from the LSZ reduction formula with K_x the Klein-Gordon operator.

Denote the analytic continuation of S^i_ν away from $k^2=1$ as $\tilde{S}^i_\nu$ with

$$\tilde{S}^i_\nu = \frac{i(k^2-1)}{f_\pi}\int d^4x e^{ikx} \langle p_2|[\partial_\mu A^i_\mu(x), V^3_\nu(0)]|p_1\rangle \theta(x_0) \quad , \qquad (3.17)$$

where we have already used PCAC in substituting $\partial_\mu A^i_\mu(x)/f_\pi$ for the interpolating pion field $\phi^i(x)$.

From eqs. (3.15) and (3.17) eliminate the term containing $\theta(x_o)$ and multiply by q_ν to obtain

$$f_\pi q_\nu \tilde{S}^i_\nu = (k^2-1)q_\nu(ik_\mu T^i_{\nu\mu} - \varepsilon^{3ij}N^j_\nu) \quad . \qquad (3.18)$$

Notice the similarity of this equation to eq. (3.4).

Multiplying eq. (3.12) by q_ν and noting that $\partial_\nu V^3_\nu = 0$ one simply finds

$$q_\nu T^i_{\nu\mu} = -i\varepsilon^{3ij}N^j_\mu \quad , \qquad (3.19)$$

which should be compared to eq. (3.9). Combining these last two equations and using PCAC one again obtains eq. (3.10), with $\tilde{S}^i_\nu$ replacing T^i_ν .

It will transpire that our calculations become tractable in the soft pion limit $k \to 0$; it is convenient to arrive at this limit via the intermediate stage $k^2 \to 0$. In the limit of interest ($k \to 0$) we therefore see from eq. (3.10) that our soft pion gauge condition becomes

$$q_\nu T^i_\nu(0,0,0) = -gK(0)\bar{u}(p_2)Q^i_-\gamma_5 u(p_1) \quad , \qquad (3.20)$$

i.e. soft-pion gauge violation occurs only in the negative isospin projection states. It will be convenient therefore to divide the photopion production amplitude T^i_ν into three parts.

$$T^i_\nu(\nu,\nu',k^2) = P^i_\nu(\nu,\nu',k^2) + \bar{P}^i_\nu(\nu,\nu',k^2) + R^{i(-)}_\nu(\nu,\nu',k^2) \quad , \tag{3.21}$$

where $P^i_\nu(\nu,\nu',k^2)$ denotes nucleon pole terms, $\bar{P}^i_\nu(\nu,\nu',k^2)$ denotes the non-nucleon pole terms *which do not violate gauge invariance* for any value of k, and $R^{i(-)}_\nu(\nu,\nu',k^2)$ the remaining gauge violating non-nucleon pole terms $(R^{i(-)}_\nu(\nu,\nu',1) \neq 0) q_\nu(P^i_\nu+R^{i(-)}_\nu)|_{\nu,\nu',1} = 0$ i.e. the nucleon pole gauge violating terms (which must necessarily be the negative isospin projection ones) cancel out the non-nucleon pole gauge-violating ones[70]. The nucleon pole terms are given by

$$P^i_\nu(\nu,\nu',k^2)$$
$$= \tfrac{1}{2} gK(k^2)\bar{u}(p_2)\left(\gamma_5\tau_i \frac{1}{\not{p}_1+\not{q}-M}\Gamma_\nu(0) + \Gamma_\nu(0)\frac{1}{\not{p}_2-\not{q}-M}\gamma_5\tau_i\right)u(p_1), \tag{3.22}$$

where

$$\Gamma_\nu(0) = (1+\tau_3)\gamma_\nu + i(\sigma_{\nu\mu}/2M)q_\mu(\mu^S+\mu^V\tau_3) \tag{3.23}$$

and μ^S, μ^V are in units e/2M. As $q_\nu q_\mu \sigma_{\nu\mu} \equiv 0$ there will be no gauge violation in the anomalous moment part, and it immediately follows that

$$q_\nu P^i_\nu(\nu,\nu',k^2) = -gK(k^2)\bar{u}(p_2)Q^i_-\gamma_5 u(p_1) \ . \qquad (3.24)$$

From eqs. (9) and (16) one sees that for $k\to 0$, all the gauge violation comes from the nucleon pole terms i.e.

$$q_\nu R^{i(-)}_\nu(0,0,0) = 0 \ . \qquad (3.25)$$

It is therefore not necessary to evaluate $R^{i(-)}_\nu(\nu,\nu',k^2)$ as in the limit of interest it is not gauge violating i.e. it can be grouped with the non-nucleon pole terms $\bar{P}^i_\nu(\nu,\nu',k^2)$ which satisfy $q_\nu \bar{P}^i_\nu(\nu,\nu',k^2) = 0$ for all k.

From eq. (11) we find

$$P^i_\nu(\nu,\nu',0) = \frac{gK(0)}{4M}[F^i_\nu(\text{non-anom.}) - F^i_\nu(\text{anom.})] \ , \qquad (3.26)$$

with

$$F^i_\nu(\text{non-anom.}) = \bar{u}(p_2)\left(\frac{\not{k}\gamma_\nu\tau_i(1+\tau_3)}{\nu-\nu'} + \frac{(1+\tau_3)\tau_i\gamma_\nu\not{k}}{\nu+\nu'}\right)\gamma_5 u(p_1) \ ,$$

$$F^i_\nu(\text{anom.}) = \frac{iq_\mu\bar{u}(p_2)}{2M}\left(\frac{k\sigma_{\nu\mu}\tau_i(\mu^S+\mu^V\tau_3)}{\nu-\nu'} - \frac{(\mu^S+\mu^V\tau_3)\tau_i\sigma_{\nu\mu}\not{k}}{\nu+\nu'}\right)\gamma_5 u(p_1) . \qquad (3.27)$$

The most general gauge invariant terms which contribute to $\bar{F}^1_\nu$ are those given by Chew et al.[25] viz

$$\bar{F}^1_\nu(\nu,\nu';k^2) = \bar{u}(p_2) \sum_{r=1}^{4} O^r_\nu \bar{A}^1_r(\nu,\nu';k^2)\gamma_5 u(p_1) \ , \tag{3.28}$$

where

$$\bar{A}^1_r = A^1_r - A^{1(\text{pole})}_r , \tag{3.29}$$

and

$$\begin{aligned} O^1_\nu &= \tfrac{1}{2}[\gamma_\nu, \not{q}] \ , \\ O^2_\nu &= 2(q\cdot kP_\nu - P\cdot qk_\nu) \ , \\ O^3_\nu &= q\cdot k\gamma_\nu - \not{q}k_\nu \ , \\ O^4_\nu &= 2(P\cdot q\gamma_\nu - \not{q}P_\nu) + 2M(\gamma_\nu\not{q} - q_\nu) \ , \end{aligned} \tag{3.30}$$

with

$$P = \tfrac{1}{2}(p_1+p_2) \ .$$

As $q_\nu O^r_\nu = 0$ for all r, then $q_\nu \bar{F}^1_\nu(\nu,\nu';k^2) = 0$ for all k, which is a necessary condition for our above analysis to be valid.

We must now proceed to evaluate the right-hand side of eq. (3.4). In complete analogy with our treatment of T^i_ν we break $k_\mu M^i_{\nu\mu}$ into three parts, viz.

$$k_\mu M^i_{\nu\mu} = k_\mu P^i_{\nu\mu} + k_\mu \bar{P}^i_{\nu\mu} + k_\mu R^{i\,(-)}_{\nu\mu} \quad , \tag{3.31}$$

where again the non-nucleon pole terms satisfy $q_\nu k_\mu \bar{P}^i_{\nu\mu}(\nu,\nu',k^2) = 0$ for all k. Now,

$$k_\mu P^i_{\nu\mu}(\nu,\nu',k^2) = \tfrac{1}{2}i\bar{u}(p_2)[k_\mu F_\mu(k^2)\gamma_5\tau_i \frac{1}{\not p_1+\not q-M}\Gamma_\nu(0) + \Gamma_\nu(0)\frac{1}{\not p_2-\not q-M}k_\mu F_\mu(k^2)\gamma_5\tau_i]u(p_1) \quad , \tag{3.32}$$

where

$$F_\mu(k^2) = g_A(k^2)\gamma_\mu + h_A(k^2)k_\mu \quad . \tag{3.33}$$

This gives

$$q_\nu k_\mu P^i_{\nu\mu}(\nu,\nu',k^2) = i\bar{u}(p_2)k_\mu F_\mu(k^2)Q^i_-\gamma_5 u(p_1) \quad . \tag{3.34}$$

From eqs. (3.9), (3.31) and (3.34) one finds

$$q_\nu k_\mu R^{i(-)}_{\nu\mu}(\nu,\nu',k^2) = i\bar{u}(p_2)k_\mu(2F_\mu\{k-q)^2\}-F_\mu(k^2))Q^i_-\gamma_5 u(p_1) \quad , \tag{3.35}$$

i.e.

$$\lim_{k\to 0}\,[q_\nu k_\mu R^{i(-)}_{\nu\mu}(\nu,\nu',k^2)] = 0 \,. \qquad (3.36)$$

Thus, once again the off-shell gauge violation in $k_\mu M^i_{\nu\mu}$ is completely contained in $k_\mu P^i_{\nu\mu}$ for $k\to 0$. In this limit we may therefore regard $k_\mu R^{i(-)}_{\nu\mu}$ as contained in $k_\mu \bar{P}^i_{\nu\mu}$.

From eq. (3.32) our pole term for $k^2 = 0$ becomes

$$k_\mu P^i_{\nu\mu}(\nu,\nu',0) = \tfrac{1}{2} i g_A(0)\bar{u}(p_2)[\gamma_\nu 2Q^i_- - \frac{iq_\mu \sigma_{\nu\mu}}{M}(\mu^S Q^i_o + \mu^V Q^i_+]\gamma_5 u(p_1)$$
$$+ \tfrac{1}{2} i g_A(0)\bar{u}(p_2)[(\frac{\not{k}\gamma_\nu \tau_i(1+\tau_3)}{\nu-\nu'} + \frac{(1+\tau_3)\tau_i\gamma_\nu\not{k}}{\nu+\nu'})$$
$$- \frac{iq_\mu}{2M}(\frac{\not{k}\sigma_{\nu\mu}\tau_i(\mu^S+\mu^V\tau_3)}{\nu-\nu'} - \frac{(\mu^S+\mu^V\tau_3)\tau_i\sigma_{\nu\mu}\not{k}}{\nu+\nu'})]\gamma_5 u(p_1)\,.$$

(3.37)

We must now evaluate our non-nucleon pole terms, which may be written as

$$k_\mu \bar{P}^i_{\nu\mu}(\nu,\nu',k^2) = i g_A(0)\bar{u}(p_2)\sum_{r=1}^{32} k_\mu S^r_{\nu\mu}\bar{B}_r(\nu,\nu',k^2)\gamma_5 u(p_1)\,,$$

(3.38)

where the thirty-two possible $S^r_{\nu\mu}$ are formed from $P_\nu, k_\nu, \gamma_\nu, P_\mu, k_\mu, \gamma_\mu, q_\mu$ and $\not{k}$ (or equivalently $\not{q}$). Note that we choose $q\cdot e = 0$.

Thirty-two is a large number of invariants to deal with. In our limit of interest we will be concerned with evaluating terms such as

$$\lim_{k\to 0} [\bar{u}(p_2)k_\mu S^r_{\nu\mu}\bar{B}_r(\nu,\nu',k^2)\gamma_5 u(p_1)]$$

which are not necessarily zero due to possible kinematical singularities induced in $\bar{B}_r$ by our choice of $S^r_{\nu\mu}$. It would be extremely convenient if one could free oneself from having to study the kinematical analytic structure of $\bar{B}_r(\nu,\nu,k^2)$, as the relevant calculations are long and tedious (see, for example, the appendix to Chapter I).

Let $S^r_\nu = k_\mu S^r_{\nu\mu}$. Then S^r_ν can be considered to be formed from the available vectors P_ν, k_ν, γ_ν *subject to the constraint*

$$\lim_{k\to 0} S^r_\nu = 0 \;;$$

this does not imply that

$$\lim_{k\to 0} \{S^r_\nu \bar{C}_r(\nu,\nu',k^2)\} = 0 \;,$$

where $\bar{C}_r(\nu,\nu',k^2)$ denotes an invariant scattering amplitude. It turns out that there are six possible S^r_ν (without the constraint) which we choose to be:

$$S^1_\nu = Mk_\nu, \qquad S^4_\nu = \not{k}P_\nu,$$

$$S^2_\nu = \frac{M}{2}[\gamma_\nu,\not{k}], \qquad S^5_\nu = MP_\nu,$$

$$S^3_\nu = \not{k}k_\nu, \qquad S^6_\nu = M^2\gamma_\nu. \qquad (3.39)$$

Use of the above mentioned constraint shows that S^5_ν and S^6_ν must be eliminated, as they do not go to zero with k. Instead of eq. (3.38) for the non-nucleon pole terms we will have the much simpler one

$$k_\mu \bar{P}^i_{\nu\mu}(\nu,\nu',k^2) = ig_A(0)\bar{u}(p_2) \sum_{r=1}^{4} S^r_\nu \bar{C}^i_r(\nu,\nu',k^2)\gamma_5 u(p_1). \qquad (3.40)$$

3. DETERMINATION OF SUM RULES

Considering eq. (3.4) for the particular case $k^2 = 0$, we substitute eqs. (3.21) (3.26) and (3.28) into the left-hand side, and eqs. (3.31), (3.37) and (3.40) into the right-hand side to obtain

$$\frac{f_\pi g K(0)}{4M} (e\cdot F^i(\text{non-anom.}) - e\cdot F^i(\text{anom.}))$$

$$+ f_\pi \bar{u}(p_2) \sum_{r=1}^{4} e\cdot O^r \bar{A}^i_r(\nu,\nu',0)\gamma_5 u(p_1) + f_\pi e\cdot R^{i(-)}(\nu,\nu',0)$$

$$= \tfrac{1}{2}g_A(0)(e\cdot F^i(\text{non-anom.}) - e\cdot F^i(\text{anom.}))$$

$$+ g_A(0)\bar{u}(p_2)[\not{e}Q_-^i - ie_\nu q_\mu \frac{\sigma_{\nu\mu}}{2M}(\mu^S Q_o^i + \mu^V Q_+^i)]\gamma_5 u(p_1)$$

$$+ g_A(0)\bar{u}(p_2) \sum_{r=1}^{4} e\cdot S^r \bar{C}_r^i(\nu,\nu',0) - ik_\mu e_\nu R_{\nu\mu}^{i\,(-)}(\nu,\nu',0)$$

$$- g_A(0)\bar{u}(p_2)\not{e}\gamma_5 Q_-^i u(p_1). \qquad (3.41)$$

Noting that $f_\pi gK(0)/4M = \frac{1}{2}g_A(0)$ it can be immediately seen that terms proportional to $e\cdot F^i$ cancel; $\not{e}Q_-^i$ terms also disappear. If we now take limits of this equation as $k\to 0$ we simply obtain

$$f_\pi \bar{u}(p_2) \sum_{r=1}^{4} e\cdot O^r \bar{A}_r^i(\nu,\nu',0)\gamma_5 u(p_1)|_{k\to 0}$$

$$= -\frac{ie_\nu q_u}{2M} g_A(0)\bar{u}(p_2)\sigma_{\nu\mu}(\mu^S Q_o^i + \mu^V Q_+^i)\gamma_5 u(p_1)|_{k\to 0}$$

$$+ g_A(0)\bar{u}(p_2) \sum_{r=1}^{4} e\cdot S^r \bar{C}_r^i(\nu,\nu',0)\gamma_5 u(p_1)|_{k\to 0} .$$

$$(3.42)$$

In this limit we note that the terms $e\cdot R^{i(-)}$ and $k_\mu e_\nu R_{\nu\mu}^{i(-)}$ have been absorbed respectively into $e.O^r\bar{A}_r^i$ and $e\cdot S^r\bar{C}_r^i$ due to the previously proved gauge invariance for $k\to 0$. Furthermore, as $k\to 0$, $O_\nu^r \to 0$ for $r = 2,3,4$ and the corresponding $\bar{A}_r^i$ are kinematically analytic in k. Putting $e\cdot S^r$ in terms of independent invariants we find:

$$\bar{u}(p_2)[\not{e},\not{k}]\{\frac{2M\bar{A}_1{}^i}{gK(0)} - \frac{1}{2M}(\mu^S Q_o^i + \mu^V Q_+^i) - M\bar{C}_2^i\}\gamma_5 u(p_1)\Big|_{k\to 0}$$

(3.43)

$$= 2\bar{u}(p_2)\{Me\cdot k(\bar{C}_1^i - 2\bar{C}_3^i) + \not{k}e\cdot P\bar{C}_4^i - 2Me\cdot P(\bar{C}_2^i + \bar{C}_4^i) + \\ + \not{k}e.k\bar{C}_3^i\}\gamma_5 u(p_1)\Big|_{k\to o} \quad ,$$

where $e\cdot k$, $[\not{e},\not{k}]$, $\not{k}e\cdot k$, $\not{k}e\cdot P$ and $e\cdot P$ are independent invariants(71), so before taking the limit $k\to 0$ we see that

$$\bar{C}_4^i(\nu,\nu',0) = 0, \quad \text{and} \quad \bar{C}_2^i(\nu,\nu',0) + \bar{C}_4^i(\nu,\nu',0) = 0,$$

i.e.

$$\bar{C}_2^i(\nu,\nu',0) = 0 = \bar{C}_4^i(\nu,\nu',0). \qquad (3.44)$$

These invariant scattering amplitudes can be equated to zero because their relevant invariant matrices $\not{k}e\cdot P$ and $e\cdot P \neq 0$ as $k\to 0$. But although

$$\lim_{k\to o}(\bar{u}(p_2)e\cdot k\bar{C}_3^i(\nu,\nu',0)\gamma_5 u(p_1)) = 0,$$

as $e\cdot k$ is an independent invariant, we cannot conclude that

$\bar{C}^{i}_{3}(\nu,\nu',0) = 0$ as $e\cdot k\to 0$ when $k\to 0$. Similar considerations apply to $\bar{C}^{i}_{1}(\nu,\nu',0)$.

However, all that is necessary to complete our demonstration is that $\bar{C}^{i}_{2}(\nu,\nu',0) = 0$. Equating the coefficients of the independent invariant $[\not{e},\not{q}]$, which does not approach zero as k does, we obtain

$$2M\bar{A}^{i}_{1}(0,0,0)/gK(0) = (\mu^{S}Q^{i}_{O} + \mu^{V}Q^{i}_{+})/2M. \qquad (3.45)$$

Taking the various isospin projections this gives:

$$\mu^{S} = 4M^{2}\bar{A}^{O}_{1}(0,0,0)/gK(0), \qquad (3.46)$$

$$\mu^{V} = 4M^{2}\bar{A}^{(+)}_{1}(0,0,0)/gK(0), \qquad (3.47)$$

and

$$\bar{A}^{(-)}_{1}(0,0,0) = 0. \qquad (3.48)$$

Eqs. (3.46) and (3.47) are the Fubini, Furlan and Rossetti sum rules, derived by using a dispersive representation for the commutators of the isoscalar and isovector parts of the e.m. current and the corresponding axial charges sandwiched between nucleon states. Eq. (3.47) relates μ^{V} to the *uncharged* photoproduction amplitudes $\gamma+p\rightarrow\pi^0+p$ and $\gamma+n\rightarrow\pi^0+n$, whereas eq. (3.46) relates μ^s to either uncharged photoproduction amplitudes (as Fubini et al. have done) or to the charged ones $\gamma+p\rightarrow\pi^++n$, $\gamma+n\rightarrow\pi^-+p$. Eq. (3.48) is trivially satisfied, as $\bar{A}_1^{(-)}$ is an odd function of ν .

Adler and Dothan(68) have derived eq. (3.46) not using CA. Both eqs. (3.45) and (3.46) have been examined for consistency by Fubini et al. and quite good agreement found.(72) Adler and Gilman(73) have investigated eq. (3.47) by using photoproduction data to parametrize resonant photopion multipoles and conclude that the agreement is to within 15%. It is interesting to note that very recently Tapper(74) has rederived eqs. (3.46) - (3.48) in a similar approach to mine and has used eq. (3.46) to relate the $\rho\pi\gamma$ vertex to the $\rho\gamma$ vertex, g and μ^S/μ^V .

Veltman (60) originally postulated that modified divergence equations, such as eq. (3.1) were independent of CA. However Nauenberg(69) has shown (in the e.m. case) that the modified divergence equations imply the equal time CA commutation relations between vector charge densities and vector (or axial-vector) currents *with the Schwinger terms*(9) *included;* this result has been extended by Boulware and Brown(75) to include first order modifications due to the weak interaction.

Berman and Frishman(76) have now shown that the modified divergence equations are equivalent to equal time commutators of charges (i.e. *not* charge densities) with currents, ***which do not include Schwinger terms.*** Therefore, any low energy calculation using the modified divergence equations which reproduces the CA result with the Schwinger terms neglected naturally implies that the latter do not affect low energy conclusions (see reference (77) for a particular example of this).

4. CONCLUSION

It has been shown that the original results of Fubini et al. relating anomalous nucleon magnetic moments to off-shell photopion amplitudes, obtained by using dispersive representations for CA commutators, can be simply obtained by considering minimal e.m. modification of PCAC. The method is fairly simple, demanding only care in the manipulation of the off-shell gauge condition, which differs from the usual on-shell case.

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12. σ denotes Pauli spin matrices, and τ the Pauli isospin matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and satisfy the relations $[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$,

$$\{\sigma_i,\sigma_j\} = 2\delta_{ij} .$$

We use the following representation for the γ matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and satisfy

$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$, with $g_{\mu\nu}$ = diag. (1,-1,-1,-1).

Only γ_0 is hermitian, γ_i and γ_5 are anti-hermitian.

$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$.

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$$R_\nu^{i(-)} = R_{\pi\nu}^{i(-)} + 2gk_\nu K'(k^2)\bar{u}(p_2)Q_-^i\gamma_5 u(p_1) + O\{(q.k)k_\nu\}$$

with $$R_{\pi\nu}^{i(-)} = -g\,\frac{(2k-q)_\nu}{(k-q)^2-m^2}\,K\{(k-q)^2\}\bar{u}(p_2)Q_-^i\gamma_5 u(p_1).$$

71. q and k are not independent invariants. In particular the following relations hold when sandwiched between nucleon spinors $\bar{u}(p_2)$, and $\gamma_5 u(p_1)$.

$$\not{k} = \not{q} - 2M$$

$$[\not{e},\not{k}] = [\not{e},\not{q}] - 4\,P.e$$

72. By approximating the continuum contribution to $\bar{A}^o$ and $\bar{A}_1^{(+)}$ with the (3,3) and (3,1) (at 1515 MeV) resonances they find $\mu^V = 1.90$, $\mu^S = -0.088$ compared to the experimental values of 1.85 and - 0.06 respectively.

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77. Consider the three processes $K^+\rightarrow\pi^o + \ell^+ + \nu$, $K^+\rightarrow\ell^+ + \nu$, and $\pi^+\rightarrow\ell^+ + \nu$.

Let $\langle\pi^o(q)|V_\mu^-|K^+(k)\rangle = f_+(k+q)_\mu + f_-(k-q)_\mu$ where the form factors f_+ and f_- are functions of q^2 and q.k. Then C.G. Callan and S.B. Treiman (Phys. Rev. Letters 16, 153 (1966)) have shown that $f_+ + f_- = f_K/f_\pi$ at q=o in a CA calculation in which they neglected the Schwinger terms in the commutator $[A_3^o(x),V_\mu^-(o)]\delta(x_o)$. The same result is simply obtained by using modified PCAC (see reference 76).