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BANACH ALGEBRA STRUCTURE AND AMENABILITY OF A CLASS OF MATRIX ALGEBRAS WITH APPLICATIONS

BY

GHOLAM HOSSEIN ESSLAMZADEH ©

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Banach Algebra Structure and Amenability of a Class of Matrix Algebras with Applications** submitted by **Gholam Hossein Esslamzadeh** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Mathematics**.

A. T. Lau (Supervisor)

L. Marcoux (Chairman)

J. E. Lewis

N. Tomczak-Jaegermann

M. Razavy

W. Bade (University of California at Berkeley)

Date: October 31, 1997

To my Parents, Sisters and Brothers that I love the most

ABSTRACT

In this thesis we consider the structure and applications of a new class of matrix algebras that we call them ℓ^1-Munn algebras. Some functional analytic properties as well as the relations between certain members of this category and the algebra of compact operators on a separable Hilbert space are described. Then some characterizations of amenable ℓ^1-Munn algebras are proved. Semisimple ones in this category are also considered and semisimple ℓ^1-Munn algebras with bounded approximate identity are characterized.

ℓ^1-Munn algebras and semigroup algebras are connected by a generalization of the main result of Munn. Amenability and semisimplicity of semigroup algebras in terms of the ideal structure of the underlying semigroups are studied here and some characterizations of amenable semigroup algebras are obtained by using the corresponding results on ℓ^1-Munn algebras. Also a counter example to a conjecture of Duncan and Paterson is provided here.

The topological center of the second dual of ℓ^1-Munn algebras are considered and they are fully described in terms of the algebras that they were based on. Application of this result to semigroup algebras, gives a generalization of Young's theorem to semigroups. A variety of examples and open problems at the end, suggests some directions for future research.

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Table of contents

Chapter 1

Introduction

After Johnson introduced the notion of amenable Banach algebra [21], in [7, 8, 13, 14, 26] the authors studied the amenability of semigroup algebras for certain weighted cases. In the process of characterizing amenable weighted semigroup algebras we discovered a new category of Banach algebras which we call them ℓ^1-Munn algebras. Indeed a very special finite dimensional case of these algebras, without any topological structure, was introduced by Munn [30]. He used this algebra to interpret the algebraic semigroup algebra of a finite Rees matrix semigroup in terms of a matrix algebra over a finite group, and since then his technique has been used in the study of algebraic semigroup rings (for example see [35]). Quite recently, Duncan and Paterson [8, page 145] observed that Munn's technique can be used in the study of semigroup algebras of completely 0-simple semigroups with a finite number of idempotents. Besides, some special ℓ^1-Munn algebras have certain relations and interactions with well known algebras. These applications provide strong reasons to study these algebras as abstract objects and then apply the results to the concrete cases like semigroup algebras. We will follow this direction in this thesis.

This thesis consists of six chapters and one appendix. In chapter 2 we define our notations. In chapter 3 we introduce ℓ^1-Munn algebras and show some basic facts about their structure. In particular a ℓ^1-Munn algebra $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity if and only if it is unital, if and only if its sandwich matrix P is invertible and its index sets are finite. Then we obtain the following characterization of amenable ℓ^1-Munn algebras, by explicit construction of approximate diagonals: A ℓ^1-Munn algebra $\mathcal{LM}(\mathcal{A}, P)$ is amenable if and only if it has a bounded approximate identitiy and $\mathcal{A}$ is amenable. Also we consider the semisimplicity of these algebras and this is used to show the interaction of semisimplicity and amenability in the concrete case of semigroup algebras, in chapter 5. In private communications with Professor N. Gronbaek, he suggested that some of the results of this chapter can have an alternate proof by using Morita equivalence.

The second dual of a Banach algebra can be made into a Banach algebra in two ways as was first shown by Arens [1] and their study was continued by different authors for the case of group algebras and Fourier algebras [4, 11, 12 and 24]. One of the main problems that has been considered in the study of the second dual of a Banach algebra is the coincidence of two multiplications or more generally the topological center. For a recent and abstract approach see [29]. Isik, Pym and Ulger [20] showed that for any compact group the topological center of $L^1(G)^{**}$ is

$L^1(G)$. This was extended to any locally compact group by Lau and Losert [28]. Also Young [39] showed that for a locally compact group G, $L^1(G)$ is Arens regular if and only if G is finite. Chapter 4 starts with the study of the first and the second duals of ℓ^1-Munn algebras. One of the major results of this thesis is Theorem 4.3.2 in which we consider the topological center of the ℓ^1-Munn algebras. Part (ii) of this Theorem is indeed an analog of the Young's Theorem [39] for the ℓ^1-Munn algebras, as it shows that Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$ implies that at least two of the three cardinal numbers $|I|$, $|J|$ and $\dim \mathcal{A}$ are finite. The last section of chapter 4 is devoted to the study of involutive ℓ^1-Munn algebras, their positive elements, positive functionals and representations. In this section we characterize their positive functionals. Then we construct some of the representations of ℓ^1-Munn algebras from the representations of the underlying algebras and show their relations.

Most of the results of chapter 5 are based on the ideal structure and the structure of principal factors of semigroups. Since amenability of $\ell^1(S, \omega)$, ω a weight on S, implies that S is a regular semigroup with a finite number of idempotents [8, Theorem 2], in most of the results we consider without loss of generality this type of semigroups only. In this chapter we provide some characterizations of amenable weighted semigroup algebras which show that the amenability problem of the semigroup algebras is reduced to the completely [0-]simple case. Previously this was done only for inverse semigroups in [8, page 145]. We also give a counter example to the conjecture of Duncan and Paterson [8. page 145]. In [7. Theorem 8] the authors showed that for an inverse semigroup S. $\ell^1(S)$ is amenable if and only if every maximal subgroup of S is amenable. We extend this result by showing that for a regular semigroup with a finite number of idempotents. $\ell^1(S)$ is amenable if and only if every maximal subgroup of S is amenable and all of the principal factors of S have semisimple semigroup algebras. In particular if $\ell^1(S)$ is amenable. then it is semisimple. Consequently it is the semisimplicity of the semigroup algebras of the principal factors i.e. completely [0-]simple semigroups that build the above result. By applying Theorem 4.3.2 to the semigroup algebras and also giving some counter examples in section 6.1. we show that up to certain extent Young's Theorem [39] can be generalized to semigroups. In the last section we show that the involution which we defined on ℓ^1-Munn algebras naturally arises in some cases. for example in the case of semigroup algebras of inverse semigroups.

The last chapter is devoted to examples and open problems. In section 6.1 we give some examples of the algebras that appeared earlier in this thesis plus some interesting counter examples that were promised in the previous chapters. In section 6.2 we discuss some of the open problems that arise naturally from the chapters 3. 4 and 5.

Finally for the convenience of the reader, we have included an index of terms and symbols at the end.

Chapter 2

Notations and Preliminaries

2.1. Banach algebras

Let $\mathcal{A}$ be a Banach algebra. Throughout by $\mathcal{A}$ module we mean *Banach $\mathcal{A}$ module*, left, right or two-sided, whichever specified. We denote the *projective tensor product* of two $\mathcal{A}$ modules X and Y by $X \hat{\otimes} Y$. A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of $\mathcal{A}$ modules and bounded $\mathcal{A}$ module homomorphisms is called *admissible* [*split*] if f has a bounded linear [$\mathcal{A}$ module homomorphism] left inverse. A bounded net $\{e_\alpha\}$ in $\mathcal{A} \hat{\otimes} \mathcal{A}$ with $e_\alpha a - a e_\alpha \to 0$ and $\pi(e_\alpha a) \to a$ is called an ***approximate diagonal***. Here $\pi : \mathcal{A} \hat{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$ is the canonical projection. Let X be a Banach $\mathcal{A}$ bimodule. We will denote the set of all bounded [inner] derivations from $\mathcal{A}$ into X by $Z^1(\mathcal{A}, X)$ [$B^1(\mathcal{A}, X)$]. Also $\mathcal{H}^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/B^1(\mathcal{A}, X)$. $\mathcal{A}$ is called *amenable* if $\mathcal{H}^1(\mathcal{A}, X^*) = 0$ for every dual $\mathcal{A}$ bimodule X^* or equivalently if $\mathcal{A}$ has an approximate diagonal [20. Lemma 1.2 and Theorem 1.3]. $\mathcal{A}$ is called *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = 0$.

Suppose $\mathcal{A}$ is unital. $D(\mathcal{A}, 1)$ denotes the set of *normalized states on* $\mathcal{A}$. i.e. all $f \in \mathcal{A}^*$ such that $\|f\| = 1$ and $f(1) = 1$. Let $x \in \mathcal{A}$. $V(\mathcal{A}, x) = \{f(a) \ : \ f \in D(\mathcal{A}, 1)\}$ is called the *numerical range of* x. x is called *positive* if $V(\mathcal{A}, x) \subseteq \mathbb{R}^+ \cup \{0\}$.

An *involution* $*$ on $\mathcal{A}$ is a map $* \ : \ \mathcal{A} \longrightarrow \mathcal{A}$ that satisfies the following conditions: For every $a, b \in \mathcal{A}$. (i) $(a^*)^* = a$ (ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$ (iii) $(ab)^* = b^* a^*$. $\mathcal{A}$ is called *involutive* if it has an isometric involution.

Let $\mathcal{A}$ be an arbitrary Banach algebra. The *first* and *second Arens multiplications on* $\mathcal{A}^{**}$ that we denote by "$\circ$" and "." respectively are defined in three steps. For $a, b \in \mathcal{A}$. $f \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$, the elements $f \circ a$. $a.f$. $m \circ f$. $f.m$ of $\mathcal{A}^*$ and $m \circ n$. $m.n$ of $\mathcal{A}^{**}$ are defined in the following way:

$$\left\langle f \circ a \, , \, b \right\rangle = \left\langle f \, , \, ab \right\rangle \qquad \left\langle a.f \, , \, b \right\rangle = \left\langle f \, , \, ba \right\rangle$$

$$\left\langle m \circ f \, , \, a \right\rangle = \left\langle m \, , \, f \circ a \right\rangle \qquad \left\langle f.m \, , \, b \right\rangle = \left\langle m \, , \, b.f \right\rangle$$

$$\left\langle m \circ n \, , \, f \right\rangle = \left\langle m \, , \, n \circ f \right\rangle \qquad \left\langle m.n \, , \, f \right\rangle = \left\langle n \, , \, f.m \right\rangle.$$

Throughout we assume $\mathcal{A}^{**}$ has the first Arens multiplication. unless stated otherwise.

For fixed $n \in \mathcal{A}^{**}$ the map $m \mapsto m \circ n$ $[m \mapsto n.m]$ is weak* – weak* continuous, but the map $m \mapsto n \circ m$ $[m \mapsto m.n]$ is not necessarily weak* – weak* continuous. unless m is in $\mathcal{A}$. The *topological center* $\mathcal{Z}(\mathcal{A}^{**})$ *of* $\mathcal{A}^{**}$ is defined by

$$\mathcal{Z}(\mathcal{A}^{**}) = \{n \in \mathcal{A}^{**} \ : \ \text{The map } m \mapsto n \circ m \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

It can be shown that

$$\mathcal{Z}(\mathcal{A}^{**}) = \{n \in \mathcal{A}^{**} \ : \ n \circ m = n.m \text{ for all } m \in \mathcal{A}^{**}\}.$$

If $\mathcal{A}$ is commutative, then $\mathcal{Z}(\mathcal{A}^{**})$ is precisely the algebraic center of $\mathcal{A}^{**}$.

Throughout by $\mathcal{K}(\mathcal{H})$ we mean the algebra of compact operators on the Hilbert space $\mathcal{H}$.

2.2. Semigroups and semigroup algebras

In the algebraic notations for semigroups mainly we follow [5]. Throughout S $[G]$ is a semigroup [group] and E_S is the set of idempotent elements of S. If T is an ideal of S, then the *Rees factor semigroup* S/T is the result of collapsing T into a single element 0 and retaining the identity of elements of $S \setminus T$. We make the convention that $S/o = S$. If S has an identity, then $S^1 = S$; otherwise $S^1 = S \cup \{1\}$ where 1 is the identity joined to S. For $a \in S$, $J(a)$ is the principal ideal $S^1 a S^1$ and J_a is the set of elements $b \in J(a)$ such that $J(b) = J(a)$. The inclusion among the principal ideals induces the following order among the equivalence classes J_as: $J_a \le J_b$ if $J(a) \subseteq J(b)$ $[J_a < J_b$ if $J(a) \subset J(b)]$. By $I(a)$ we mean the ideal $\{b \in J(a) : J_b < J_a\}$ i.e. $I(a) = J(a) \setminus J_a$. On E_S we have a usual order: $e, f \in E_S$, $e \le f$ if $ef = fe = e$. An idempotent $e \in E_S$ is called *primitive* if it is nonzero and is minimal in the set of nonzero idempotents. A semigroup S with zero is *0-simple* if $\{0\}$ and S are the only ideals of S. S is called *completely* [0-] *simple* if it is [0-] simple and contains a nonzero primitive idempotent. The factors $J(a)/I(a)$, $a \in S$ are called the *principal factors* of S. Each principal factor of S is either 0-simple, simple, or null i.e. the product of any two element is zero [5, Lemma 2.39]. If every principal factor of S is 0-simple or simple, we say that S is *semisimple.*

A [relative] ideal series $S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = o$ that has no proper refinement is called a *principal* [*composition*] *series.* If S has a principal series as above, then the factors of this series are isomorphic in some order to the principal factors of S [5, Theorem 2.40].

A semigroup S is called *regular* if for every $a \in S$ there is a $b \in S$ such that $a = aba$. S is an *inverse semigroup* if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$.

Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i, j)$ is called a *Rees* $I \times J$ *matrix over* G^0 and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G. $S = G \times I \times J$ with the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$, $(a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $\mathcal{M}(G, P)$ [18, page 68]. Similarly if P is a $J \times I$ matrix over G^0, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation:

$$(a, i, j) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k) & if\ P_{jl} \neq 0 \\ 0 & if\ P_{jl} = 0 \end{cases}$$

$$(a, i, j) \circ 0 = 0 \circ (a, i, j) = 0 \circ 0 = 0.$$

This semigroup which is denoted by $\mathcal{M}^0(G, P)$ also can be described in the following way: The set of all Rees $I \times J$ matrices over G^0 form a semigroup under the binary operation $A \circ B = APB$, which is called the *Rees $I \times J$ matrix semigroup over G^0 with the sandwich matrix P* and is isomorphic to $\mathcal{M}^0(G, P)$ [18, pages 61-63]. An $I \times J$ matrix P over G^0 is called *regular* [*invertible*] if every row and every column of P contains at least [exactly] one nonzero entry.

For $f \in \ell^\infty(S)$ the left and right translations of f by $s \in S$ will be denoted by fs and sf respectively. S is called *amenable* if there exists $m \in \ell^\infty(S)^*$ such that $m \geq 0$, $m(1_S) = 1$ and $m(xf) = m(f) = m(fx)$, for every $x \in S$, $f \in \ell^\infty(S)$. As usual *semigroup algebra of S* is $\ell^1(S)$ with the convolution product:

$$(f * g)(x) = \sum_{uv=x} f(u)g(v), \quad f, g \in \ell^1(S), \quad x \in S.$$

If S has a zero, then we call the algebra $\ell^1(S)/\ell^1(0)$ the *contracted semigroup algebra of S*, where $\ell^1(0) = \ell^1(\{0\})$.

Chapter 3

ℓ^1-Munn algebras, their amenability and semisimplicity

3.1. Introduction

In this chapter we introduce the ℓ^1-Munn algebras and compare them with the algebras $\mathcal{K}(\mathcal{H})$ and the l^1-algebra that was defined in [37, page 710]. These will be done in section 3.2. The rest of section 3.2 is devoted to investigating some of the basic structural properties of ℓ^1- Munn algebras. Amenability and simplicity of these algebras will be studied in sections 3.3 and 3.4 respectively.

3.2. ℓ^1-Munn algebras, definition and basic properties

Definition 3.2.1. Let $\mathcal{A}$ be a unital Banach algebra, I and J be arbitrary index sets and P be a $J \times I$ nonzero matrix over $\mathcal{A}$ such that $sup\{\|P_{ij}\| \ : \ i \in I, j \in$

$J\} \leq 1$. Let $\mathcal{LM}(\mathcal{A}, P)$ be the vector space of all $I \times J$ matrices A over $\mathcal{A}$ such that $\sum_{i\in I}^{j\in J} \|A_{ij}\| < \infty$. Then it is easy to check that $\mathcal{LM}(\mathcal{A}, P)$ with the product $A \circ B = APB$, $A, B \in \mathcal{LM}(\mathcal{A}, P)$ and the ℓ^1-norm is a Banach algebra that we call it *ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P* or briefly *ℓ^1-Munn algebra*. When $I = J$ and P is the identity $J \times J$ matrix over $\mathcal{A}$, we denote $\mathcal{LM}(\mathcal{A}, P)$ by $\mathcal{LM}_J(\mathcal{A})$. In addition we denote $\mathcal{LM}_J(\mathbb{C})$ simply by $\mathcal{LM}_J$.

Convention. (i) From now on we use $\mathcal{A}$ for an arbitrary unital Banach algebra, I and J for index sets and P for the sandwich matrix exclusively.

(ii) Throughout $\{\varepsilon_{ij} : i \in I, j \in J\}$ is the standard matrix unit system of the matrix algebra under discussion.

(iii) As we will see in the applications, nonzero entries of P are invertible. Also P has no zero row or column i.e. P is *regular*. So from now on we assume P satisfies these conditions unless stated otherwise.

(iv) If we assume $1 < \|P\|_\infty < \infty$, then ℓ^1-norm is not an algebra norm but still is a complete norm. So by [31, Proposition 1.1.9] $\mathcal{LM}(\mathcal{A}, P)$ with a norm equivalent to ℓ^1-norm, is a Banach algebra. Obviously duals of $\mathcal{LM}(\mathcal{A}, P)$ in both norms are the same. So in this case for simplicity we don't refer to the equivalent norm explicity.

The following Lemma in a sense is a generalization of the Lemma 4, page 231 [3] which can be proved with a similar argument. The Lemma 3.2.3 is well known for the case that J is finite, see [33, page 4]. The general case can be proved with the same technique and using Lemma 3.2.2. We will present the proofs of these two Lemmas in the appendix.

Lemma 3.2.2. Every $u \in \mathcal{LM}_J \widehat{\otimes} \mathcal{A}$ has a unique expression in the form $u = \sum_{i,j \in J} \varepsilon_{ij} \otimes a_{ij}$.

Lemma 3.2.3. $\mathcal{LM}_J(\mathcal{A})$ is isometrically algebra isomorphic to $\mathcal{LM}_J \widehat{\otimes} \mathcal{A}$.

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\epsilon_n : n \in \mathbb{N}\}$. Using Tanbays's notation [37, page 710], let A be a $\mathbb{N} \times \mathbb{N}$ matrix with complex entries for which there is a $m \in \mathbb{R}^+$ such that $\sum_{i \in \mathbb{N}} |A_{ij}| \leq m$ for all $j \in \mathbb{N}$ and $\sum_{j \in \mathbb{N}} |A_{ij}| \leq m$ for all $i \in \mathbb{N}$. Let $\mathcal{M}_0$ be the collection of all such matrices. Then $\mathcal{M}_0$ is a self adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Define the norm $|||.|||$ on $\mathcal{M}_0$ by

$$|||A||| = \inf\left\{m \ : \ \sum_{i \in \mathbb{N}} |A_{ij}| \leq m, \ \ \sum_{j \in \mathbb{N}} |A_{ij}| \leq m\right\}.$$

Then for every $A \in \mathcal{M}_0$ we have $\|A\| \leq |||A|||$, where $\|.\|$ is the operator norm [37, page 710]. Let $\mathcal{M} = \overline{\mathcal{M}_0}^{\|.\|}$. With these notations we have:

Proposition 3.2.4. (i) $\mathcal{LM}_{\mathbb{N}}$ is a proper ideal in $\mathcal{M}$ and for every $T, U \in \mathcal{M}_0$, $A \in \mathcal{LM}_{\mathbb{N}}$ we have: $\|TAU\|_1 \leq |||T||| \ \|A\|_1 \ |||U|||$.

(ii) $\mathcal{LM}_{\mathbb{N}} \subsetneqq \overline{\mathcal{LM}_{\mathbb{N}}}^{\|.\|} = \mathcal{K}(\mathcal{H}) \subsetneqq \mathcal{M}$.

Proof.(i) Let $A \in \mathcal{LM}_{\mathbb{N}}$, $T, U \in \mathcal{M}_0$. Then it is easy to check that $\|TA\|_1 \leq |||T||| \ \|A\|_1$ and $\|AU\|_1 \leq \|A\|_1 \ |||U|||$. Combination of these two relations with the fact that identity matrix is in $\mathcal{M} \setminus \mathcal{LM}_{\mathbb{N}}$, proves the first part.

(ii) Let $T \in \mathcal{LM}_{\mathbb{N}}$. There is a sequence T_n of matrices, each with a finite number of nonzero entries such that $T_n \xrightarrow{\|.\|_1} T$. Now by part (i) and the fact preceding the

Proposition. $\|.\| \leq |||.||| \leq \|.\|_1$ which implies that $T_n \xrightarrow{\|.\|} T$ and since $T_n \in \mathcal{K}(\mathcal{H})$, then $T \in \mathcal{K}(\mathcal{H})$. Also the diagonal matrix $diag(1/n)$ is in $\mathcal{K}(\mathcal{H}) \setminus \mathcal{LM}_1$; which shows the first inclusion is proper. On the other hand every finite rank operator is the (operator norm) limit of matrices with a finite number of nonzero entries. So the middle equality holds.

The last inclusion was shown by Tanbay [37, page710].///

Lemma 3.2.5. Suppose I and J are finite and V $[W]$ is an invertible $J \times J$ $[I \times I]$ matrix over $\mathcal{A}$. Let $\mathcal{B} = \mathcal{LM}(\mathcal{A}, P)$ and $\mathcal{C} = \mathcal{LM}(\mathcal{A}, VP)$ $[\mathcal{C} = \mathcal{LM}(\mathcal{A}, PW)]$. Then $\mathcal{B}$ and $\mathcal{C}$ are topologically algebra isomorphic.

Proof. Define the map $\phi : \mathcal{B} \longrightarrow \mathcal{C}$ by $\phi(A) = AV^{-1}$ $[\phi(A) = W^{-1}A]$. It is easy to check that ϕ is an onto algebra isomorphism. Let $A \in \mathcal{B}$. Then,

$$\|\phi(A)\| \leq \sum_{i \in I, j \in J} \sum_{k \in J} \|A_{ik}\| \|V_{kj}^{-1}\| \leq \|A\| \|V^{-1}\|_1.$$

So by the open mapping theorem ϕ is a topological algebra isomorphism.///

Lemma 3.2.6. Let I and J be finite of orders m and n respectively. Then there is an invertible $n \times n$ matrix V over $\mathcal{A}$, an invertible $m \times m$ matrix W over $\mathcal{A}$, a natural number k, $k \leq \min(m, n)$, and a $(n-k) \times (m-k)$ matrix E over $\mathcal{A}$ such that

$$VPW = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.$$

Proof. It is easy to show that each of the following linear algebraic operations is equivalent to multiplying P on the left [right] by an (invertible) elementary matrix:

(i) Multiplying a row [column] of P by an invertible element a of $\mathcal{A}$.

(ii) Adding a row [column] of P to another row [column] of P.

(iii) Interchanging two rows [columns] of P.

Since every nonzero $a \in \mathcal{A}$ can be written as $a = 2\|a\|((\frac{a}{2\|a\|} - 1) + 1)$ which is a difference of two invertible elements, then we can combine parts (i) and (ii) to get:

(iv) Adding a nonzero multiple of one row [column] of P to another row [column].

Now we can do a finite sequence of the above operations to get:

$$\begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}$$

which is the result of multiplying P on the left and right by appropriate invertible matrices V and W respectively.///

Lemma 3.2.7. The following conditions are equivalent:

(i) $\mathcal{LM}(\mathcal{A}, P)$ has an identity,

(ii) $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity,

(iii) I and J are finite and $\mathcal{LM}(\mathcal{A}, P)$ has a left and a right approximate identity.

(iv) I and J are finite and P is invertible.

Proof. (ii)$\Longrightarrow$(iii) We need only to show that the index sets are finite. Let $\{E^\gamma : \gamma \in \Gamma\}$ be a bounded approximate identity for $\mathcal{LM}(\mathcal{A}, P)$, $E^\gamma = [c^\gamma_{ij}]$ and $\|E^\gamma\| \leq M$ for all $\gamma \in \Gamma$. Then for given $k \in I$, $l \in J$ we have:

$$0 = \lim_\gamma \|E^\gamma \circ \varepsilon_{kl} - \varepsilon_{kl}\| = \lim_\gamma \|\sum_{i,j} c^\gamma_{ij}\varepsilon_{ij} \circ \varepsilon_{kl} - \varepsilon_{kl}\| = \lim_\gamma \|\sum_{i,j} c^\gamma_{ij} P_{jk}\varepsilon_{il} - \varepsilon_{kl}\|$$
$$= \lim_\gamma \left(\sum_{i \neq k} \|\sum_j c^\gamma_{ij} P_{jk}\| + \|\sum_j c^\gamma_{kj} P_{jk} - 1\| \right).$$

So $\lim_\gamma \|\sum_j c^\gamma_{kj} P_{jk} - 1\| = 0$. Let $\varepsilon > 0$ be given and every $k \in I$, $\gamma_k \in \Gamma$ be such that for every $\gamma \geq \gamma_k$, $1 - \varepsilon < \sum_j \|c^\gamma_{kj}\|$. Now if I is infinite, choose $N \in \mathbb{N}$ such that $(1-\varepsilon)N > M$, then choose distinct $k_1, \ldots k_N \in I$ and $\gamma \geq \max\{\gamma_{k_1}, \ldots \gamma_{k_N}\}$. We have:

$$M < (1-\varepsilon)N < \sum_{i=1}^{N} \sum_j \|c^\gamma_{k_i j}\| \leq \sum_{i,j} \|c^\gamma_{ij}\| \leq M$$

which is a contradiction. So I is finite. Similarly if we apply E^γ to the right, we conclude that J must be finite.

(iii)$\Longrightarrow$(iv) Suppose P is not invertible and $\{E^\gamma : \gamma \in \Gamma\}$ be a left approximate identity for $\mathcal{LM}(\mathcal{A}, P)$. By Lemma 3.2.6 there are invertible matrices V and W, a non invertible matrix E and a positive integer k such that:

$$Q = VPW = \begin{bmatrix} I_k & 0 \\ 0 & E \end{bmatrix}.$$

Assume $n = |J| \leq |I| = m$. By induction on m we can show that there is a nonzero column matrix Y in $\mathcal{A}^m$ such that $QY = 0$ and hence there is a nonzero column matrix X in $\mathcal{A}^m$ such that $PX = 0$. Now if $B \in \mathcal{LM}(\mathcal{A}, P)$ is the matrix that all of its columns are equal to X, then $B = \lim E_\gamma PB = 0$ which is a contradiction. So P must be invertible. If $n > m$, by using a right approximate identity and a similar argument we conclude that P must be invertible.

(iv)$\Longrightarrow$(i) By Lemma 3.2.5, $\mathcal{LM}(\mathcal{A}, P) \cong \mathcal{LM}_J(\mathcal{A})$ and since $\mathcal{LM}_J(\mathcal{A})$ is unital, then so is $\mathcal{LM}(\mathcal{A}, P)$.///

3.3. Amenability of the ℓ^1 -Munn algebras

Theorem 3.3.1. The following conditions are equivalent:

(i) $\mathcal{LM}(\mathcal{A}, P)$ is amenable.

(ii) $\mathcal{A}$ is amenable, I and$J$ are finite and P is invertible.

Proof. (i)$\Longrightarrow$(ii) Since $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity, then by Lemma 3.2.7, I and J are finite and P is invertible. So by using Lemma 3.2.5 and then Lemma 3.2.3 we conclude that $\mathcal{LM}(\mathcal{A}, P)$ is topologically algebra isomorphic to $\mathcal{M}_m \widehat{\otimes} \mathcal{A}$ where $m = |I| = |J|$. Since $\mathcal{M}_m \widehat{\otimes} \mathcal{A}$ is amenable, then it has an approximate diagonal which by Lemma 3.2.2 can be represented in the form:

$$\left\{ \sum_{k=1}^{\infty} \left(\sum_{i,j=1}^{m} \varepsilon_{ij} \otimes a_{ij}^{\alpha k} \right) \otimes \left(\sum_{r,l=1}^{m} \varepsilon_{rl} \otimes b_{rl}^{\alpha k} \right) \quad : \quad \alpha \in I \right\}.$$

Let $\sum_{s,t=1}^{m} \varepsilon_{st} \otimes x_{st} \in \mathcal{M}_m \widehat{\otimes} \mathcal{A}$. Then,

$$\begin{aligned}
\sum_{s,t=1}^{m} \varepsilon_{st} \otimes x_{st} &= \lim_{\alpha} \pi \left(\sum_{k=1}^{\infty} \left(\sum_{i,j=1}^{m} \varepsilon_{ij} \otimes a_{ij}^{\alpha k} \right) \otimes \left(\sum_{r,l=1}^{m} \varepsilon_{rl} \otimes b_{rl}^{\alpha k} \right) \right) \left(\sum_{s,t=1}^{m} \varepsilon_{st} \otimes x_{st} \right) \\
&= \lim_{\alpha} \sum_{k=1}^{\infty} \left(\sum_{i,j,l,t=1}^{m} \varepsilon_{it} \otimes a_{ij}^{\alpha k} b_{jl}^{\alpha k} x_{lt} \right) \\
&= \sum_{i,t=1}^{m} \left(\varepsilon_{it} \otimes \lim_{\alpha} \sum_{k=1}^{\infty} \sum_{j,l=1}^{m} a_{ij}^{\alpha k} b_{jl}^{\alpha k} x_{lt} \right).
\end{aligned}$$

Therefore we have:

$$\sum_{i,t=1}^{m} \varepsilon_{it} \otimes \left(x_{it} - \lim_{\alpha} \sum_{k=1}^{\infty} \sum_{j,l=1}^{m} a_{ij}^{\alpha k} b_{jl}^{\alpha k} x_{lt} \right) = 0. \tag{1}$$

On the other hand,

$$0=\lim_{\alpha}\left(\left(\sum_{k=1}^{\infty}\left(\sum_{i,j=1}^{m}\varepsilon_{ij}\otimes a_{ij}^{\alpha k}\right)\otimes\left(\sum_{r,l=1}^{m}\varepsilon_{rl}\otimes b_{rl}^{\alpha k}\right)\right)\left(\sum_{s,t=1}^{m}\varepsilon_{st}\otimes x_{st}\right)\right.$$
$$\left.-\left(\sum_{s,t=1}^{m}\varepsilon_{st}\otimes x_{st}\right)\left(\sum_{k=1}^{\infty}\left(\sum_{i,j=1}^{m}\varepsilon_{ij}\otimes a_{ij}^{\alpha k}\right)\otimes\left(\sum_{r,l}^{m}\varepsilon_{rl}\otimes b_{rl}^{\alpha k}\right)\right)\right)$$
$$=\lim_{\alpha}\sum_{k=1}^{\infty}\left(\sum_{i,j,r,l,t=1}^{m}\varepsilon_{ij}\otimes a_{ij}^{\alpha k}\otimes\varepsilon_{rt}\otimes b_{rl}^{\alpha k}x_{lt}-\sum_{s,t,j,r,l=1}^{m}\varepsilon_{sj}\otimes x_{st}a_{tj}^{\alpha k}\otimes\varepsilon_{rl}\otimes b_{rl}^{\alpha k}\right).$$

Let ψ be the onto linear isometry:

$$\psi\ :\ (\mathcal{M}_m\widehat{\otimes}\ \mathcal{A})\widehat{\otimes}(\mathcal{M}_m\widehat{\otimes}\ \mathcal{A})\longrightarrow(\mathcal{M}_m\widehat{\otimes}\ \mathcal{M}_m)\widehat{\otimes}(\mathcal{A}\ \widehat{\otimes}\ \mathcal{A})$$

$$\psi((c\otimes x)\otimes(d\otimes y))=(c\otimes d)\otimes(x\otimes y),\ \ c,d\in\mathcal{M}_m,\ \ x,y\in\mathcal{A}.$$

If we apply ψ to the above identity, we get:

$$0=\lim_{\alpha}\sum_{k=1}^{\infty}\left(\sum_{i,j,r,l,t=1}^{m}\varepsilon_{ij}\otimes\varepsilon_{rt}\otimes a_{ij}^{\alpha k}\otimes b_{rl}^{\alpha k}x_{lt}-\sum_{i,j,r,l,t=1}^{m}\varepsilon_{ij}\otimes\varepsilon_{rt}\otimes x_{il}a_{lj}^{\alpha k}\otimes b_{rt}^{\alpha k}\right)$$
$$=\sum_{i,j,r,t=1}^{m}\left((\varepsilon_{ij}\otimes\varepsilon_{rt})\otimes\lim_{\alpha}\sum_{k=1}^{\infty}\sum_{l=1}^{m}(a_{ij}^{\alpha k}\otimes b_{rl}^{\alpha k}x_{lt}-x_{il}a_{lj}^{\alpha k}\otimes b_{rt}^{\alpha k})\right).$$

So for every $i,j,r,t\le m$ we have:

$$\lim_{\alpha}\sum_{k=1}^{\infty}\sum_{l=1}^{m}(a_{ij}^{\alpha k}\otimes b_{rl}^{\alpha k}x_{lt}-x_{il}a_{lj}^{\alpha k}\otimes b_{rt}^{\alpha k})=0. \tag{2}$$

Suppose $x\in\mathcal{A}$, $x_{11}=x$ and $x_{it}=0$ if $i\neq 1$ or $t\neq 1$. By the relation (1) we have:

$$x=\lim_{\alpha}\sum_{k=1}^{\infty}\sum_{j=1}^{m}a_{1j}^{\alpha k}b_{j1}^{\alpha k}x=\lim_{\alpha}\pi\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m}a_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k}\right)x. \tag{3}$$

Suppose $j=r$ and $i=t=1$. Under the assumptions preceding relation (3), we conclude from the relation (2) that:

$$\lim_{\alpha}\sum_{k=1}^{\infty}(a_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k}x-xa_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k})=0.$$

Taking sum over j, we get:

$$\lim_{\alpha}\left(\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m} a_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k}\right)x - x\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m} a_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k}\right)\right) = 0. \tag{4}$$

Relations (3) and (4) together with the boundedness of the given approximate diagonal of $\mathcal{M}_m \hat{\otimes} \mathcal{A}$ imply that:

$$\left\{\sum_{k=1}^{\infty}\sum_{j=1}^{m} a_{1j}^{\alpha k}\otimes b_{j1}^{\alpha k} \;:\; \alpha\in I\right\}$$

is an approximate diagonal for $\mathcal{A}$. Therefore $\mathcal{A}$ is amenable.

(ii)$\Longrightarrow$(i) As in the previous part $\mathcal{LM}(\mathcal{A}, P)$ is topologically algebra isomorphic to $\mathcal{M}_m \hat{\otimes} \mathcal{A}$ for some $m \in \mathbb{N}$. So by [21, Proposition 5.4] $\mathcal{LM}(\mathcal{A}, P)$ is amenable.///

Remark 3.3.2. In the proof of the above Theorem we constructed an approximate diagonal for $\mathcal{A}$ from an approximate diagonal for $\mathcal{M}_m \hat{\otimes} \mathcal{A}$ which is the converse of [21, Proposition 5.4] for this particular algebra. This is the only partial converse for that result, known to the author. Besides this constructive method can be used to provide an alternate proof for [21, Proposition 5.4] without involvement of derivations and their extensions. This will be presented in the appendix.

3.4. Semisimplicity of the ℓ^1-Munn algebras

Theorem 3.4.1. The following conditions are equivalent:

(i) I and J are finite and $\mathcal{LM}(\mathcal{A}, P)$ is semisimple,

(ii) $\mathcal{A}$ is semisimple and $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity.

Proof.(i) $\Longrightarrow$(ii) First we show that semisimplicity of $\mathcal{LM}(\mathcal{A}, P)$ implies that P is invertible. Suppose P is not invertible. Then as in the proof of (iii) $\Longrightarrow$(iv) of Lemma 3.2.7, there is a nonzero matrix B in $\mathcal{LM}(\mathcal{A}, P)$ such that $PB = 0$ or $BP = 0$. So $B \in Rad(\mathcal{LM}(\mathcal{A}, P))$ which is a contradiction. Therefore P is invertible and by Lemma 3.2.7, $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity. Now let $|I| = m$ and ϕ be the topological algebra isomorphism from $\mathcal{LM}(\mathcal{A}, P)$ onto $\mathcal{M}_m(\mathcal{A})$, as in Lemma 3.2.5. By [31, Proposition 4.3.12],

$$\mathcal{M}_m(Rad(\mathcal{A})) = Rad(\mathcal{M}_m(\mathcal{A})) = \phi(Rad(\mathcal{LM}(\mathcal{A}, P))) = 0.$$

Therefore $\mathcal{A}$ is semisimple.

(ii)$\Longrightarrow$(i) By Lemma 3.2.7 and Lemma 3.2.5 we need only to show that $\mathcal{M}_m(\mathcal{A})$ is semisimple which can be done similarly to the previous part, by applying [31, Proposition 4.3.12]./ / /

Chapter 4

Duality of ℓ^1 -Munn algebras

4.1. Introduction

In this Chapter we study the duals of the ℓ^1-Munn algebras and characterize their topological center. These will be done in the sections 4.2 and 4.3. In the section 4.4 we consider involutive ℓ^1-Munn algebras, their positive linear functionals and representations.

Let X be an arbitrary Banach space. It is well known that $\ell^1(I, X)^*$ is isometrically isomorphic to $\ell^\infty(I, X^*)$. In particular $\mathcal{LM}(\mathcal{A}, P)^* = \ell^\infty(I \times J, \mathcal{A}^*)$. Note that if I is a finite index set, then $\ell^1(I, X) = X^I = \ell^\infty(I, X)$ as vector spaces and equivalence of the two norms in this case implies that these two are topologically isomorphic. So in this case all of the Banach spaces $\ell^\infty(I, X)^*$, $\ell^\infty(I, X^*)$, $\ell^1(I, X)^*$ and $\ell^1(I, X^*)$ are topologically isomorphic. Besides these facts, we can embed $\ell^\infty(I, X^*)$ into $\mathcal{B}\left(\ell^1(I, X), \ell^1(I)\right)$. This can be done simply by defining an injective contraction.

$$o : \ell^\infty(I, X^*) \longrightarrow \mathcal{B}\left(\ell^1(I, X), \ell^1(I)\right), \quad o(F)((a_i)_{i\in I}) = \left(F(i)(a_i)\right)_{i\in I}.$$

4.2. First and second duals of ℓ^1-Munn algebras

Convention. If $F \in \ell^\infty(I \times J, \mathcal{A}^*)$, then we denote $F(i,j)$ by F_{ij}. Also by $f\varepsilon_{ij} \in \ell^\infty(I \times J, \mathcal{A}^*)$ we mean an $I \times J$ matrix over $\mathcal{A}^*$ that has f as its (i,j)th entry and 0 elsewhere. From now on by υ we mean the map $\upsilon : \mathcal{LM}(\mathcal{A}, P)^* \longrightarrow \ell^\infty(I \times J, \mathcal{A}^*)$, $\left\langle \upsilon(F)(i,j) \, , \, a\right\rangle = \left\langle F \, , \, a\varepsilon_{ij}\right\rangle$. We will denote $\upsilon(F)$ by $\widetilde{F}$. We will use similar notations for $\ell^\infty(I \times J, \mathcal{A}^{**})$ and its elements. Also when I and J are finite we will use the same notations to identify the two Banach spaces $\mathcal{LM}(\mathcal{A}, P)^{**}$ and $\mathcal{LM}\left(\mathcal{A}^{**}, P\right)$.

Remark 4.2.1. In Lemma 4.2.2, Proposition 4.3.1 and Lemma 4.2.5 we do not use the invertiblity of nonzero entries of P. So we can drop this condition in those results. Indeed we will need this case in one of the steps in the proof of Theorem 4.3.2 (i).

Lemma 4.2.2. If the index sets are finite, then $\mathcal{LM}(\mathcal{A}, P)^{**}$ is topologically algebra isomorphic to $\mathcal{LM}\left(\mathcal{A}^{**}, P\right)$, when both of $\mathcal{A}^{**}$ and $\mathcal{LM}(\mathcal{A}, P)^{**}$ are equipped with the first [second] Arens product.

Proof. Using the facts that were mentioned at the beginning of this section, we need only to show that the linear isomorphism $\upsilon \; : \mathcal{LM}(\mathcal{A}, P)^{**} \longrightarrow \mathcal{LM}\left(\mathcal{A}^{**}, P\right)$

is multiplicative. Throughout we will use the fact that restriction of Arens product of $\mathcal{A}^{**}$ to $\mathcal{A}$ agrees with the multiplication of $\mathcal{A}$. Let $A, X \in \mathcal{LM}(\mathcal{A}, P)$, $F \in \mathcal{LM}(\mathcal{A}, P)^*$ and $M, N \in \mathcal{LM}(\mathcal{A}, P)^{**}$. Then,

$$\left\langle (\widetilde{F \circ A})_{ij} \ , \ X_{ij} \right\rangle = \left\langle F \ , \ A \circ X_{ij}\varepsilon_{ij} \right\rangle = \sum_{k,l} \left\langle F \ , \ A_{kl}P_{li}X_{ij}\varepsilon_{kj} \right\rangle$$
$$= \sum_{k,l} \left\langle \widetilde{F}_{kj} \ , \ A_{kl}P_{li}X_{ij} \right\rangle = \left\langle \sum_{k,l} \widetilde{F}_{kj} \circ (A_{kl}P_{li}) \ , \ X_{ij} \right\rangle.$$

So

$$(\widetilde{F \circ A})_{ij} = \sum_{k,l} \widetilde{F}_{kj} \circ (A_{kl}P_{li}).$$

Applying this relation to $\widetilde{M \circ F}$, we get:

$$\left\langle (\widetilde{M \circ F})_{ij} \ , \ A_{ij} \right\rangle = \left\langle M \ , \ F \circ (A_{ij}\varepsilon_{ij}) \right\rangle = \sum_{r,s} \left\langle \widetilde{M}_{rs} \ , \ \widetilde{F}_{is} \circ (A_{ij}P_{jr}) \right\rangle$$
$$= \sum_{r,s} \left\langle A_{ij} \circ P_{jr} \ , \ \widetilde{M}_{rs} \circ \widetilde{F}_{is} \right\rangle = \left\langle \sum_{r,s} P_{jr} \circ (\widetilde{M}_{rs} \circ \widetilde{F}_{is}) \ , \ A_{ij} \right\rangle.$$

So

$$(\widetilde{M \circ F})_{ij} = \sum_{r,s} P_{jr} \circ (\widetilde{M}_{rs} \circ \widetilde{F}_{is}).$$

Now

$$\left\langle (\widetilde{N \circ M})_{ij} \ , \ \widetilde{F}_{ij} \right\rangle = \left\langle N \ , \ M \circ (\widetilde{F}_{ij}\varepsilon_{ij}) \right\rangle = \sum_{r,l} \left\langle \widetilde{N}_{il} \ , \ P_{lr} \circ (\widetilde{M}_{rj} \circ \widetilde{F}_{ij}) \right\rangle$$
$$= \left\langle \sum_{r,l} \widetilde{N}_{il} \circ (P_{lr} \circ \widetilde{M}_{rj}) \ , \ \widetilde{F}_{ij} \right\rangle.$$

Thus

$$(\widetilde{N \circ M})_{ij} = \sum_{r,l} \widetilde{N}_{il} \circ (P_{lr} \circ \widetilde{M}_{rj}). \tag{1}$$

Similarly for the second Arens product we can prove the following identities:

$$(\widetilde{A.F})_{ij} = \sum_{k,l}(P_{jl}A_{lk}).\tilde{F}_{ik}$$
$$(\widetilde{F.M})_{ij} = \sum_{r,s}(\tilde{F}_{rj}.\widetilde{M}_{rs}).P_{si}$$
$$(\widetilde{N.M})_{ij} = \sum_{r,l}(\tilde{N}_{il}.P_{lr}).\widetilde{M}_{rj}. \qquad (2)$$

Therefore $\psi(N \circ M) = \psi(N) \circ \psi(M)$ and $\psi(N.M) = \psi(N).\psi(M)$.///

Corollary 4.2.3. Suppose the index sets are finite. Then $\mathcal{LM}(\mathcal{A}, P)$ is an ideal in $\mathcal{LM}(\mathcal{A}, P)^{**}$ if and only if $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$.

Proof. It is easy to check that $\psi(\mathcal{LM}(\mathcal{A}, P)) = \mathcal{LM}(\mathcal{A}, P)$.

Suppose $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$. Let $A \in \mathcal{LM}(\mathcal{A}, P)$ and $M \in \mathcal{LM}(\mathcal{A}, P)^{**}$. By Lemma 4.2.2 and relation (1) in its proof we get:

$$(\widetilde{A \circ M})_{ij} = \sum_{r,l}(A_{il}P_{lr}) \circ \widetilde{M}_{rj} \in \mathcal{A} \text{ for all } i \in I,\ j \in J.$$

Thus $\widetilde{A \circ M} \in \mathcal{LM}(\mathcal{A}, P)$. Similarly $\widetilde{M \circ A} \in \mathcal{LM}(\mathcal{A}, P)$. Therefore $\mathcal{LM}(\mathcal{A}, P)$ is an ideal in $\mathcal{LM}(\mathcal{A}, P)^{**}$.

Conversely suppose $\mathcal{LM}(\mathcal{A}, P)$ is an ideal in $\mathcal{LM}(\mathcal{A}, P)^{**}$ and fix $i \in I,\ j \in J$ such that $P_{ji} \neq 0$. Let $m \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$. Let $M = \psi^{-1}(m\varepsilon_{ij})$ and $A = P_{ji}^{-1}a\varepsilon_{ij}$. By relation (1) in the proof of Lemma 4.2.2 $(\widetilde{M \circ A})_{ij} = m \circ a$. Now by assumption $M \circ A \in \mathcal{LM}(\mathcal{A}, P)$ and hence $\widetilde{M \circ A} \in \mathcal{LM}(\mathcal{A}, P)$. Therefore $m \circ a \in \mathcal{A}$.///

Definition 4.2.4. We will call the following relations *mixed associativity relations*. They can be proved easily just by using the definition of Arens product.

For every $a, x \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $m \in \mathcal{A}^{**}$,

$$(f\circ x).a = f\circ(xa) \qquad a\circ(x.f) = (ax).f \qquad (x.f)\circ a = x.(f\circ a)$$

$$(a.m)\circ f = a.(m\circ f) \qquad f.(m\circ a) = (f.m)\circ a$$

$$(m\circ a).x = m\circ(ax) \qquad a\circ(x.m) = (ax).m.$$

Lemma 4.2.5. Suppose the index sets are finite and let $\theta : \mathcal{LM}(\mathcal{A}^{**}, Q) \longrightarrow \mathcal{LM}(\mathcal{A}^{**}, P)$ be the topological algebra isomorphism that was defined in Lemma 3.2.5. Then the restriction of θ to $\mathcal{LM}(\mathcal{A}, P)$ which we will denote by θ again, maps $\mathcal{LM}(\mathcal{A}, Q)$ onto $\mathcal{LM}(\mathcal{A}, P)$ and makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{LM}(\mathcal{A},Q)^{**} & \xrightarrow{\theta^{**}} & \mathcal{LM}(\mathcal{A},P)^{**} \\ \big\downarrow \psi & & \big\downarrow \psi \\ \mathcal{LM}(\mathcal{A}^{**},Q) & \xrightarrow{\theta} & \mathcal{LM}(\mathcal{A}^{**},P). \end{array}$$

Proof. The first statement of the Lemma follows from the argument of Lemma 3.2.5. Let $\lambda : \mathcal{LM}(\mathcal{A}, Q) \hookrightarrow \mathcal{LM}(\mathcal{A}^{**}, Q)$ and $\mu : \mathcal{LM}(\mathcal{A}^{**}, Q) \hookrightarrow \mathcal{LM}(\mathcal{A}^{**}, Q)^{**}$ be the natural embeddings. It is easy to check that the following diagram is commutative i.e. $\lambda^{**} = \mu\psi$

$$\begin{array}{ccc} \mathcal{LM}(\mathcal{A},Q)^{**} & & \\ \big\downarrow \psi & \searrow^{\lambda^{**}} & \\ \mathcal{LM}(\mathcal{A}^{**},Q) & \xrightarrow{\mu} & \mathcal{LM}(\mathcal{A}^{**},Q)^{**}. \end{array}$$

Let $M \in \mathcal{LM}(\mathcal{A}, Q)^{**}$ and $f \in \mathcal{A}^{*}$. Whenever necessary, we will assume $f\varepsilon_{ij} \in \ell^{\infty}(I\times J, \mathcal{A}^{***}) = \mathcal{LM}(\mathcal{A}^{**}, P)^{*}$. By using the above commutative diagram we get:

$$\Big\langle (\psi\theta^{**}(M))_{ij}\ ,\ f\Big\rangle = \Big\langle M\ ,\ \theta^{*}(f\varepsilon_{ij})\Big\rangle = \Big\langle \lambda^{**}(M)\ .\ \theta^{*}(f\varepsilon_{ij})\Big\rangle$$
$$= \Big\langle \theta^{*}(f\varepsilon_{ij})\ .\ \psi(M)\Big\rangle = \Big\langle (\theta\psi(M))_{ij}\ .\ f\Big\rangle.$$

Therefore $\psi\theta^{**} = \theta\psi$ as required.///

4.3. Topological center of the second dual of ℓ^1-Munn algebras

Proposition 4.3.1. If the index sets I and J are finite, then

$$\mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right) \subseteq \psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right)).$$

In particular if $\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right) = \mathcal{LM}(\mathcal{A}, P)$, then $\mathcal{Z}(\mathcal{A}^{**}) = \mathcal{A}$.

Proof. Let $\widetilde{M} \in \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right)$ and $M = \psi^{-1}(\widetilde{M}) \in \mathcal{LM}(\mathcal{A}, P)^{**}$. As $\mathcal{A} \subseteq \mathcal{Z}(\mathcal{A}^{**})$, by relation (1) in the proof of Lemma 4.2.2 for every $N \in \mathcal{LM}(\mathcal{A}, P)^{**}$ we have:

$$(\widetilde{M \circ N})_{ij} = \sum_{r,l} \widetilde{M}_{il} \circ (P_{lr} \circ \widetilde{N}_{rj}) = \sum_{r,l} \widetilde{M}_{il}.(P_{lr}.\widetilde{N}_{rj}) = (\widetilde{M.N})_{ij}.$$

So $M \circ N = M.N$ and hence $\widetilde{M} \in \psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right))$. The second statement follows from the first part and the fact that $\psi(\mathcal{LM}(\mathcal{A}, P)) = \mathcal{LM}(\mathcal{A}, P)$.///

Now we prove the main result of this section which explains the relation between the topological center of the second dual of the ℓ^1-Munn algebras and some other finiteness conditions, in particular existence of bounded approximate identities in the ℓ^1-Munn algebras. Throughout the proof we will use the following fact without any specific reference: If $\theta : \mathcal{B} \longrightarrow \mathcal{C}$ is a topological algebra isomorphism between two arbitrary Banach algebras $\mathcal{B}$ and $\mathcal{C}$, then $\theta^{**}\left(\mathcal{Z}(\mathcal{B}^{**})\right) = \mathcal{Z}(\mathcal{C}^{**})$.

Theorem 4.3.2. (i) $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity if and only if the index sets are finite and $\psi(\mathcal{Z}\left(\mathcal{LM}(\mathcal{A}, P)^{**}\right)) = \mathcal{LM}\left(\mathcal{Z}(\mathcal{A}^{**}), P\right)$.

(ii) Suppose $\mathcal{A}$ admits a nonzero multiplicative linear functional. Then $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular i.e. $\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**}) = \mathcal{LM}(\mathcal{A}, P)^{**}$ if and only if one of the following conditions hold:

(a) $\mathcal{A}$ is Arens regular and both of the index sets are finite.

(b) $\mathcal{A}$ is finite dimensional and one of the index sets is finite.

Proof. (i)($\Longrightarrow$) Suppose $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity. By Lemma 3.2.7, I and J are finite and P is invertible. Assume $\theta : \mathcal{LM}(\mathcal{A}^{**}, P) \longrightarrow \mathcal{LM}_J(\mathcal{A}^{**})$ is the topological algebra isomorphism that was constructed in Lemma 3.2.5. By Proposition 4.3.1,

$$\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), P) \subseteq \psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})).$$

Let $M \in \mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})$, $n \in \mathcal{A}^{**}$ and $i, j \in J$. Let $\tilde{N} = n\varepsilon_{ij}$ and $N = \psi^{-1}(\tilde{N})$. Then by relations (1) and (2) in the proof of Lemma 4.2.2 we have:

$$\widetilde{M}_{ij} \circ n = (\widetilde{M \circ N})_{ij} = (\widetilde{M.N})_{ij} = \widetilde{M}_{ij}.n.$$

So $\psi(M) \in \mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**}))$ and for this special case the equality $\mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**}))$ $= \psi(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}))$ holds.

For rest of this part we follow the terminology of Lemma 4.2.5. The equality $\theta^{**}(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})) = \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})$ implies that if we restrict the maps ψ and θ^{**} to the $\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**})$ in the commutative diagram of Lemma 4.2.5, we will get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}) & \xrightarrow{\theta^{**}} & \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**}) \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\psi} \\ \mathcal{LM}_J(\mathcal{A}^{**}) & \xrightarrow{\quad\theta\quad} & \mathcal{LM}\,(\mathcal{A}^{**}, P)\,. \end{array}$$

So we have:

$$\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})) = \theta\psi(\theta^{**})^{-1}(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})) = \theta\psi(\mathcal{Z}(\mathcal{LM}_J(\mathcal{A})^{**}))$$

$$= \theta(\mathcal{LM}_J(\mathcal{Z}(\mathcal{A}^{**}))) = \mathcal{LM}\,(\mathcal{Z}(\mathcal{A}^{**}), P)\,.$$

($\Longleftarrow$) Suppose I and J are finite and $\psi(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})) = \mathcal{LM}\,(\mathcal{Z}(\mathcal{A}^{**}), P)$. By Lemma 3.2.7. it is enough to show that P is invertible. We will do this by induction on $k = \max\{|I|, |J|\}$. If $k = 1$. trivially P is invertible. So we assume $k > 1$. If P is not invertible. then by Lemma 3.2.6. there is a $r < k$ such that P is equivalent to $Q = \begin{bmatrix} I_r & 0 \\ 0 & E \end{bmatrix}$ and E is non invertible. By induction assumption there is a $M \in \mathcal{Z}(\mathcal{LM}(\mathcal{A}, E)^{**})$ such that $\psi(M) \notin \mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), E)$. So $\widetilde{M}_{ij} \notin \mathcal{Z}(\mathcal{A}^{**})$ for some $i \in I$, $j \in J$. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{M} \end{bmatrix} \notin \mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), Q). \tag{1}$$

On the other hand $M \in \mathcal{Z}(\mathcal{LM}(\mathcal{A}, E)^{**})$ together with relations (1) and (2) in the proof of Lemma 4.2.2 imply that for every $N \in \mathcal{LM}(\mathcal{A}, E)^{**}$ we have: $\widetilde{M} \circ E \circ \widetilde{N} = \widetilde{M}.E.\widetilde{N}$ where these products are ordinary matrix products when $\mathcal{A}^{**}$ is equipped with the first and the second Arens products respectively. By applying this identity for the special case that $\widetilde{N}$ has only one nonzero column. we conclude that it is true for every finite matrix $\widetilde{N}$ of appropriate size on $\mathcal{A}^{**}$ that the above matrix product is defined. So for every $\begin{bmatrix} \widetilde{N}_1 & \widetilde{N}_2 \\ \widetilde{N}_3 & \widetilde{N}_4 \end{bmatrix} \in \mathcal{LM}(\mathcal{A}^{**}, Q)$,

$$\begin{aligned} \upsilon^{-1}\left(\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{M} \end{bmatrix}\right) \circ \upsilon^{-1}\left(\begin{bmatrix} \widetilde{N}_1 & \widetilde{N}_2 \\ \widetilde{N}_3 & \widetilde{N}_4 \end{bmatrix}\right) &= \upsilon^{-1}\left(\begin{bmatrix} 0 & 0 \\ \widetilde{M} \circ E \circ \widetilde{N}_3 & \widetilde{M} \circ E \circ \widetilde{N}_4 \end{bmatrix}\right) \\ &= \upsilon^{-1}\left(\begin{bmatrix} 0 & 0 \\ \widetilde{M}.E.\widetilde{N}_3 & \widetilde{M}.E.\widetilde{N}_4 \end{bmatrix}\right) \\ &= \upsilon^{-1}\left(\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{M} \end{bmatrix}\right).\upsilon^{-1}\left(\begin{bmatrix} \widetilde{N}_1 & \widetilde{N}_2 \\ \widetilde{N}_3 & \widetilde{N}_4 \end{bmatrix}\right). \end{aligned}$$

Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & \widetilde{M} \end{bmatrix} \in \upsilon(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, Q)^{**})).$$

Comparing this with relation (1), we conclude that

$$\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), Q) \neq \upsilon(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, Q)^{**})). \tag{2}$$

On the other hand by using the commutative diagram of Lemma 4.2.5 and our assumption we get:

$$\begin{aligned} \upsilon(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, Q)^{**})) = \theta^{-1}\upsilon\theta^{**}(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, Q)^{**})) &= \theta^{-1}\upsilon(\mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})) \\ = \theta^{-1}(\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), P)) &= \mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), Q). \end{aligned}$$

which contradicts (2). Therefore P is invertible.

(ii)($\Longleftarrow$) Suppose $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular. First assume both of I and J are infinite and choose countable infinite subsets $\{i_n\}$ and $\{j_m\}$ from them respectively. Choose $k \in J$, $l \in I$ such that $P_{kl} \neq 0$ and let V be a closed subspace of $\mathcal{A}$ such that $\mathcal{A} = V \oplus \left\langle P_{kl} \right\rangle$. Suppose $h \in \mathcal{A}^*$ is such that $h(P_{kl}) = 1$ and $h = 0$ on V.

Define $\widetilde{F} : I \times J \longrightarrow \mathcal{A}^*$ by $\widetilde{F}_{i_n j_m} = h$ if $n > m$ and $\widetilde{F}_{ij} = 0$ otherwise. Let $X_n = \varepsilon_{i_n k}$, $Y_m = \varepsilon_{l j_m} \in \mathcal{LM}(\mathcal{A}, P)$, $m, n \in \mathbb{N}$. Then

$$\left\langle F \, , \, X_n \circ Y_m \right\rangle = \sum_{i,j} \left\langle \widetilde{F}_{ij} \, , \, (X_n \circ Y_m)_{ij} \right\rangle = \left\langle \widetilde{F}_{i_n j_m} \, , \, P_{kl} \right\rangle.$$

So $\lim_m \lim_n \left\langle F \, , \, X_n \circ Y_m \right\rangle = h(P_{kl}) = 1$ and $\lim_n \lim_m \left\langle F \, , \, X_n \circ Y_m \right\rangle = 0$ which contradicts Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$ by [31, Theorem 1.4.11]. Therefore at least one of the I and J must be finite.

If $\dim \mathcal{A} < \infty$, we have condition (b). If $\mathcal{A}$ is infinite dimensional, then we will show that the other index set is finite too.

We may assume that J is finite but I is infinite. By regularity of P there is at least one row of P that has infinitely many nonzero entries, say row k. Let $\{i_n : n \in \mathbb{N}\} \subseteq I$ be such that $P_{k i_n} \neq 0$, $n \in \mathbb{N}$. Choose a sequence $\{a_m : m \in \mathbb{N}\}$ in $\mathcal{A}$ such that $\{a_m : m \in \mathbb{N}\} \cup \{1\}$ is linearly independent and $\|a_m\| = \frac{1}{2}$. Then by [34, Theorem 10.7] for every $m \in \mathbb{N}$, $b_m = 1 - a_m$ is invertible and $\|b_m^{-1}\| \leq \frac{13}{8}$. Moreover $\{b_m : m \in \mathbb{N}\}$ is linearly independent. For every $n > 1$ let $V_n = \left\langle b_1, \dots, b_{n-1} \right\rangle$ and W_n be a closed subspace of $\mathcal{A}$ such that $\mathcal{A} = V_n \dotplus W_n$. Then $V_1 \subsetneqq V_2 \subsetneqq \dots$ and $W_1 \supsetneqq W_2 \supsetneqq \dots$.

Now let h be a nonzero multiplicative linear functional on $\mathcal{A}$, $X_n = \varepsilon_{i_n k}$ and $Y_m = (P_{k i_m})^{-1} b_m \varepsilon_{i_m k} \in \mathcal{LM}(\mathcal{A}, P)$, $\quad n, m \in \mathbb{N}$.

Define the map $\widetilde{F} : I \times J \longrightarrow \mathcal{A}^*$ in the following way: For every $n > 1$, $\widetilde{F}_{i_n k} = h$ on V_n and $\widetilde{F}_{i_n k} = 0$ on W_n and $\widetilde{F}_{ij} = 0$ otherwise. Then $F(X_n \circ Y_m) = F(b_m \varepsilon_{i_n k}) =$

$\widetilde{F}_{i_n k}(b_m)$. So $\lim_n \lim_m F(X_n \circ Y_m) = 0$ but since h is multiplicative we have:

$$|\lim_m \lim_n F(X_n \circ Y_m)| = \lim_m |h(b_m)| = \lim_m \frac{1}{|h(b_m^{-1})|} \geq \frac{8}{13}.$$

Again by [31. Theorem 1.4.11] this contradicts Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$. So I is also finite. Now we need only to show that $\mathcal{A}$ is Arens regular. Let $i \in I$, $j \in J$ be such that $P_{ji} \neq 0$. Suppose $m, n \in \mathcal{A}^{**}$, $\widetilde{M} = m\varepsilon_{ij}$ and $\widetilde{N} = (n \circ (P_{ji})^{-1})\varepsilon_{ij}$. Arens regularity of $\mathcal{LM}(\mathcal{A}, P)$ implies that $\widetilde{N \circ M} = \widetilde{N.M}$. So by using relations (1) and (2) in the proof of Lemma 4.2.2 and mixed associativity identities we get:

$$n \circ m = (\widetilde{N \circ M})_{ij} = (\widetilde{N.M})_{ij} = n.m.$$

($\Longrightarrow$) Suppose condition (a) holds. By Proposition 4.3.1.

$$\mathcal{LM}(\mathcal{A}, P)^{**} = \upsilon^{-1}(\mathcal{LM}(\mathcal{A}^{**}, P)) = \upsilon^{-1}(\mathcal{LM}(\mathcal{Z}(\mathcal{A}^{**}), P)) \subseteq \mathcal{Z}(\mathcal{LM}(\mathcal{A}, P)^{**})$$

So in this case $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular.

Now suppose condition (b) holds. We may assume that J is finite. Let $\{X^n : n \in \mathbb{N}\}$ and $\{Y^m : m \in \mathbb{N}\}$ be sequences in $\mathcal{LM}(\mathcal{A}, P)$, bounded by $M \in \mathbb{R}^+$ and $F \in \mathcal{LM}(\mathcal{A}, P)^*$ be such that both of the limits $\lim_n \lim_m F(X^n \circ Y^m)$ and $\lim_m \lim_n F(X^n \circ Y^m)$ exist.

Define the sequence $\{g_m : m \in \mathbb{N}\}$ of functions on $\mathbb{N} \times J \times J$ by $g_m(n, j, k) = \sum_{i \in I} \widetilde{F}_{ij}\left(X^n_{ik} \sum_{l \in I} P_{kl} Y^m_{lj}\right)$. Since $\{g_m : m \in \mathbb{N}\}$ is a uniformly bounded sequence of complex functions (with the uniform bound $\|F\|M^2$) on a countable set, then it has a point wise convergent subsequence, say $\{g_{m_r} : r \in \mathbb{N}\}$.

With a similar discussion we can find a sequence $\{n_s\}$ in $\mathbb{N}$ such that $\{\lim_r \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right) : \quad s \in \mathbb{N}\}$ is convergent for every $j, k \in J$. So:

$$\lim_n \lim_m F(X^n \circ Y^m) = \lim_s \lim_r \sum_{j,k\in J} \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right)$$
$$= \sum_{j,k\in J} \lim_s \lim_r \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n_s} \sum_{l\in I} P_{kl} Y_{lj}^{m_r}\right).$$

Now by doing the same process on the $\lim_r \lim_s F(X_{n_s} \circ Y_{m_r})$ and denoting the new subsequences by $\{X^n : \quad n \in \mathbb{N}\}$ and $\{Y^m : \quad m \in \mathbb{N}\}$ again, we will get:

$$\begin{aligned} \lim_n \lim_m F(X^n \circ Y^m) &= \sum_{j,k\in J} \lim_n \lim_m \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right) \\ \lim_m \lim_n F(X^n \circ Y^m) &= \sum_{j,k\in J} \lim_m \lim_n \sum_{i\in I} \widetilde{F}_{ij}\left(X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right). \end{aligned} \tag{3}$$

Suppose $\{f_t\}$ be a (finite) basis for $\mathcal{A}^*$ and $\widetilde{F}_{ij} = \sum_{t=1}^{q} \alpha_{ij}^t f_t$, $i \in I$, $j \in J$. Since $\mathcal{A}$ is finite dimensional, like the previous step we can pass to subsequences iteratively and rename the subsequences by $\{X^n : \quad n \in \mathbb{N}\}$ and $\{Y^m : \quad m \in \mathbb{N}\}$ again, to get:

$$\begin{aligned} \lim_n \lim_m F(X^n \circ Y^m) &= \sum_{j,k\in J} \sum_{t} \lim_n \lim_m f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right) \\ \lim_m \lim_n F(X^n \circ Y^m) &= \sum_{j,k\in J} \sum_{t} \lim_m \lim_n f_t\left(\sum_{i\in I} \alpha_{ij}^t X_{ik}^{n} \sum_{l\in I} P_{kl} Y_{lj}^{m}\right). \end{aligned} \tag{4}$$

Now fix j, k, t and define:

$$a_n = \sum_{i\in I} \alpha_{ij}^t X_{ik}^n, \quad b_m = \sum_{l\in I} P_{kl} Y_{lj}^m, \quad m, n \in \mathbb{N}.$$

Then by Arens regularity of $\mathcal{A}$ (as it is finite dimensional). we have:

$$\lim_n \lim_m f_t\left(\sum_{i\in I} \alpha^t_{ij} X^n_{ik} \sum_{l\in I} P_{kl} Y^m_{lj}\right) = \lim_n \lim_m f_t(a_n b_m) = \lim_m \lim_n f_t(a_n b_m)$$
$$= \lim_m \lim_n f_t\left(\sum_{i\in I} \alpha^t_{ij} X^n_{ik} \sum_{l\in I} P_{kl} Y^m_{lj}\right).$$

So the right hands of the relations (4) are equal and hence by [31. Theorem 1.4.11] $\mathcal{LM}(\mathcal{A}, P)$ is Arens regular.///

4.4. Involutive ℓ^1 -Munn algebras, positive functionals and representations

Throughout this section we assume $\mathcal{A}$ is a unital involutive Banach algebra and P is $J \times J$. Define a map $*\ : \mathcal{LM}(\mathcal{A}, P) \longrightarrow \mathcal{LM}(\mathcal{A}, P)$ by $[a_{ij}]^* = [a^*_{ji}]$. Throughout by A^*_{ij} we mean $(A_{ij})^*$

Lemma 4.4.1. The above map $*$ is an isometric involution on $\mathcal{LM}(\mathcal{A}, P)$ if and only if P is self adjoint in the sense that $P^*_{ij} = P_{ji}$. $i, j \in J$.

Proof. Suppose $*$ is an isometric involution. Let $A = \varepsilon_{ij}$ and $B = \varepsilon_{rs}$. Since $(A \circ B)^* = B^* \circ A^*$, then $P^*_{jr}\varepsilon_{si} = P_{rj}\varepsilon_{si}$ which implies $P^*_{jr} = P_{rj}$. Therefore P is self adjoint.

Conversely suppose P is self adjoint. We need only to check that for every $A, B \in \mathcal{LM}(\mathcal{A}, P)$, $(A \circ B)^* = (B^* \circ A^*)$ as the other properties of involution follow from those of $\mathcal{A}$.

$$((A \circ B)^*)_{ij} = (A \circ B)^*_{ji} = (\sum_{k,l} A_{jk} P_{kl} B_{li})^* = \sum_{k,l} B^*_{li} P^*_{kl} A^*_{jk}$$
$$= \sum_{k,l} (B^*)_{il} P_{lk} (A^*)_{kj} = (B^* \circ A^*)_{ij}.$$

So $(A \circ B)^* = (B^* \circ A^*)$.///

Convention. For the rest of this section we assume P is self adjoint. Moreover since involutive Banach algebras with bounded approximate identities are of special interest. except in Lemma 4.4.4 and Proposition 4.4.5. we assume $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity. Consequently J is finite. $\mathcal{LM}(\mathcal{A}, P)$ is unital (Lemma 3.2.7) and $P \in \mathcal{LM}(\mathcal{A}, P)$. For simplicity, for every $X \in \ell^2(J)$, $j \in J$ we denote $X(j)$ by X_j.

Lemma 4.4.2. (i) If $X \in \mathcal{LM}(\mathcal{A}, P)$ is positive. then $X_{ij} \geq 0$ for all $i, j \in J$.

(ii) If $\mathcal{LM}(\mathcal{A}, P)$ is a C^*-algebra, then $\|P_{ii}\| = 1$, $\quad i \in J$.

Proof.(i) Recall that $D(\mathcal{A}, 1)$ and $V(\mathcal{A}, x)$ denote the set of normalized states on $\mathcal{A}$ annd numerical range of x respectively. Since $\mathcal{LM}(\mathcal{A}, P)^* = \ell^\infty(J \times J, \mathcal{A}^*)$, then for every $i, j \in J$

$$D(\mathcal{A}, 1)\varepsilon_{ij} = \{f\varepsilon_{ij} \in \mathcal{A}^* \ : \quad \|f\| = 1,\ f(1) = 1\} \subseteq D(\mathcal{LM}(\mathcal{A}, P), 1).$$

So

$$V(\mathcal{A}, X_{ij}) = \{f(X_{ij}) \ : \ f \in D(\mathcal{A}, 1)\}$$

$$= \{(f\varepsilon_{ij})(X) \ : \ f\varepsilon_{ij} \in D(\mathcal{A}, 1)\varepsilon_{ij}\}$$

$$\subseteq \{F(X) \ : \ F \in D(\mathcal{LM}(\mathcal{A}, P), 1)\}$$

$$= V(\mathcal{LM}(\mathcal{A}, P), X) \subseteq \mathbb{R}^+.$$

(ii) Let $j \in J$ and $A = \varepsilon_{jj}$. Then $\|P_{jj}\| = \|A \circ A^*\| = \|A\|^2 = 1.///$

Naturality of involution. The involution that we defined on the ℓ^1-Munn algebras coincides with the natural involution not only in the case of $\mathcal{M}_m$ but also in the concrete case of the contracted semigroup algebra of completely 0-simple semigroups as we will see in the section 5.6.

Remark 4.4.3. Let V be an invertible $J \times J$ matrix on $\mathcal{A}$ and $o \ : \ \mathcal{LM}(\mathcal{A}, P) \longrightarrow \mathcal{LM}(\mathcal{A}, VP)$, $A \mapsto AV^{-1}$ be the topological isomorphism as in the Lemma 3.2.5. o is not a $*-$homomorphism unless $A^*V^{-1} = (V^{-1})^*A^*$ which is not true in general. Suppose this condition holds. Even if we assume $\mathcal{A} = \mathbb{C}$ and $|J| = m < \infty$, then $V^{-1} = (V^{-1})^*$. So V^{-1} commutes with every $A \in \mathcal{M}_m$ and hence by Shur's Lemma it is a scalar multiple of identity. Therefore o is just a scalar multiple of the identity map.

Lemma 4.4.4. Let $F \in \mathcal{LM}(\mathcal{A}, P)^*$. If $F \geq 0$, then for every $a, b \in \mathcal{A}$, $i, j, k, l \in J$ we have:

(i) $\widetilde{F}_{ij}(a) = \overline{\widetilde{F}_{ji}(a^*)}$.

(ii) $\widetilde{F}_{ii}(aP_{jj}a^*) \geq 0$ and if P has a nonzero positive diagonal entry, then $\widetilde{F}_{ii} \geq 0$.

(iii) $|\widetilde{F}_{kj}(b^*P_{li}a)|^2 \le \widetilde{F}_{jj}(a^*P_{ii}a)\widetilde{F}_{kk}(b^*P_{ll}b)$. In particular if all diagonal entries of P are zero, then there is no nonzero positive functional on $\mathcal{LM}(\mathcal{A}, P)$.

Conversely if F satisfies (i) and (ii), then $F(A \circ A^*) \in \mathbb{R}$ for every $A \in \mathcal{LM}(\mathcal{A}, P)$.

Proof. (i) Let $r \in J$ be such that $P_{ir} \neq 0$, $A = a\varepsilon_{ij}$ and $B = P_{ir}^{-1}\varepsilon_{ri}$. Since $F \geq 0$, then $F(A^* \circ B) = \overline{F(B^* \circ A)}$. So

$$\widetilde{F}_{ji}(a^*) = F(a^*\varepsilon_{ji}) = F(A^* \circ B) = \overline{F(B^* \circ A)} = \overline{F(a\varepsilon_{ij})} = \overline{\widetilde{F}_{ij}(a)}.$$

(ii) Let $A = a\varepsilon_{ji}$. Then

$$\widetilde{F}_{ii}(aP_{jj}a^*) = F(aP_{jj}a^*\varepsilon_{ii}) = F(A \circ A^*) \geq 0.$$

Now suppose $P_{jj} > 0$. By [3, Lemma 7 page 207] P_{jj} has a positive square root $u \in \mathcal{A}$. Since $u(uP_{jj}^{-1}) = 1 = (P_{jj}^{-1}u)u$, then u has a two sided inverse. Let $c = au^{-1}$. Then $cP_{jj}c^* = aa^*$. So $\widetilde{F}_{ii}(aa^*) = \widetilde{F}_{ii}(cP_{jj}c^*) \geq 0$ i.e. $\widetilde{F}_{ii} \geq 0$

(iii) Let $A = a\varepsilon_{ij}$ and $B = b\varepsilon_{lk}$. Then

$$\begin{aligned}|\widetilde{F}_{kj}(b^*P_{li}a)|^2 &= |F(b^*P_{li}a\varepsilon_{kj})|^2 = |F(B^* \circ A)|^2 \\ &\leq F(A^* \circ A)F(B^* \circ B) = F(a^*P_{ii}a\varepsilon_{jj})F(b^*P_{ll}b\varepsilon_{kk}) \\ &= \widetilde{F}_{jj}(a^*P_{ii}a)\widetilde{F}_{kk}(b^*P_{ll}b).\end{aligned}$$

So the inequality in part (iii) holds. Now suppose all diagonal entries of P are zero and $i, l \in J$ are such that $P_{li} \neq 0$. Let $k, j \in J$, $b \in \mathcal{A}$ and $a = P_{li}^{-1}$. Then

$$|\widetilde{F}_{kj}(b)|^2 = \widetilde{F}_{kj}(bP_{li}a)|^2 \leq \widetilde{F}_{jj}(a^*P_{ii}a)\widetilde{F}_{kk}(b^*P_{ll}b) = 0.$$

Therefore $F = 0$.

Conversely assume F satisfies (i) and (ii). Then for every $A \in \mathcal{LM}(\mathcal{A}, P)$,

$$F(A \circ A^*) = \sum_{i,j} \sum_{k,l} \widetilde{F}_{ij}(A_{ik}P_{kl}A^*_{jl}) = \sum_{i} \sum_{k,l} \widetilde{F}_{ii}(A_{ik}P_{kl}A^*_{il})$$
$$+ \sum_{i<j} \sum_{k,l} \widetilde{F}_{ij}(A_{ik}P_{kl}A^*_{jl}) + \sum_{i>j} \sum_{k,l} \overline{\widetilde{F}_{ji}(A_{jl}P_{lk}A^*_{ik})}$$
$$= \sum_{i} \left(\sum_{k} \widetilde{F}_{ii}(A_{ik}P_{kk}A^*_{ik}) + \sum_{k<l} \widetilde{F}_{ii}(A_{ik}P_{kl}A^*_{il}) + \sum_{k>l} \overline{\widetilde{F}_{ii}(A_{il}P_{lk}A^*_{ik})} \right)$$
$$+ \sum_{i<j} \sum_{k,l} \widetilde{F}_{ij}(A_{ik}P_{kl}A^*_{jl}) + \sum_{i<j} \sum_{k,l} \overline{\widetilde{F}_{ij}(A_{ik}P_{kl}A^*_{jl})}$$
$$= \sum_{i,k} \widetilde{F}_{ii}(A_{ik}P_{kk}A^*_{ik}) + \sum_{i} \sum_{k<l} 2Re\left(\widetilde{F}_{ii}(A_{ik}P_{kl}A^*_{il}\right)$$
$$+ \sum_{i<j} \sum_{k,l} 2Re\left(\widetilde{F}_{ij}(A_{ik}P_{kl}A^*_{jl})\right).$$

Therefore $F(A \circ A^*) \in \mathbb{R}$, by part (ii).///

Proposition 4.4.5. If $F \in \mathcal{LM}(\mathcal{A}, P)^*$ is positive, then $[\widetilde{F}_{ij}(A \circ A^*)_{ij}]$ is a positive operator on $\ell^2(J)$ for every $A \in \mathcal{LM}(\mathcal{A}, P)$. If J is finite, then this condition is sufficient for F to be positive.

Proof. Suppose $F \geq 0$, $X \in \ell^2(J)$ and $A \in \mathcal{LM}(\mathcal{A}, P)$. Then,

$$\left\langle [\widetilde{F}_{ij}(A \circ A^*)_{ij}](X)\ ,\ X \right\rangle = \sum_{i} \left\langle \sum_{j} \widetilde{F}_{ij}(A \circ A^*)_{ij}X_j\ .\ X_i \right\rangle$$
$$= \sum_{i,j} \widetilde{F}_{ij}\left(\sum_{k,l} A_{ik}P_{kl}(A_{jl})^* \right) X_j\overline{X}_i$$
$$= \sum_{i,j} \widetilde{F}_{ij}\left(\sum_{k,l} (\overline{X}_i A_{ik})P_{kl}(\overline{X}_j A_{jl})^* \right). \qquad (1)$$

Let B be the $J \times J$ matrix on $\mathcal{A}$ defined by $B_{ij} = \overline{X}_i A_{ij}$. Since

$$\|B\|_1 = \sum_{i,j} \|\overline{X}_i A_{ij}\| \leq \sum_{i,j} \|A_{ij}\|\|X\| = \|A\|\|X\| < \infty,$$

then $B \in \mathcal{LM}(\mathcal{A}, P)$. Now by relation (1),

$$\left\langle [\widetilde{F}_{ij}(A \circ A^*)_{ij}]X \ , \ X \right\rangle = \sum_{i,j} \widetilde{F}_{ij} \left(\sum_{k,l} B_{ik} P_{kl} (B_{jl})^* \right)$$
$$= \sum_{i,j} \widetilde{F}_{ij} (B \circ B^*)_{ij} = F(B \circ B^*) \geq 0.$$

Therefore $[\widetilde{F}_{ij}(A \circ A^*)_{ij}]$ is a positive operator on $\ell^2(J)$.

Conversely suppose J is finite and $[\widetilde{F}_{ij}(A \circ A^*)_{ij}]$ is a positive operator for every $A \in \mathcal{LM}(\mathcal{A}, P)$. Let $X = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Then

$$F(A \circ A^*) = \sum_{i,j} \widetilde{F}_{ij}(A \circ A^*)_{ij} = \left\langle [\widetilde{F}_{ij}(A \circ A^*)_{ij}]X \ , \ X \right\rangle \geq 0.$$

Therefore $F \geq 0$.///

Notation. Let $(\pi, \mathcal{H})$ be a $*-$representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ and $A \in \mathcal{LM}(\mathcal{A}, P)$. The matrix $[\pi(A_{ij})]_{i,j \in J}$ can be considered as a bounded operator on $\mathcal{H}^J$, the Hilbert space direct sum of J copies of $\mathcal{H}$. From now on we denote $[\pi(A_{ij})]_{i,j \in J}$ simply by $\pi(A)$. If moreover P as an element of $\mathcal{LM}_J(\mathcal{A})$ is positive, then by [3, Lemma 7 page 207] P has a positive square root $U \in \mathcal{LM}_J(\mathcal{A})$. Since U is ℓ^1-bounded, then $U \in \mathcal{LM}(\mathcal{A}, P)$ as well. So we can relate the $*-$representations of $\mathcal{LM}(\mathcal{A}, P)$ and those of $\mathcal{A}$ by defining a map $\widetilde{\pi} : \mathcal{LM}(\mathcal{A}, P) \longrightarrow \mathcal{B}(\mathcal{H}^J)$ in the following way: $\widetilde{\pi}(A) = \pi(U)\pi(A)\pi(U)$. The following Proposition shows that this is a $*-$representation. Throughout $h\delta_j \in \mathcal{H}^J$ denotes the vector that has h as its jth component and zero elsewhere.

Proposition 4.4.6. (i) $\widetilde{\pi}$ is a $*-$representation of $\mathcal{LM}(\mathcal{A}, P)$ on $\mathcal{H}^J$.

(ii) $(\tilde{\pi}, \mathcal{H}^J)$ is irreducible [resp. faithful, nondegenerate] if and only if $(\pi, \mathcal{H})$ is irreducible [resp. faithful, nondegenerate].

(iii) If $(\pi, \mathcal{H})$ is cyclic, then $(\tilde{\pi}, \mathcal{H}^J)$ is cyclic.

Proof. (i) Since π is multiplicative, then for every $A, B \in \mathcal{LM}(\mathcal{A}, P)$, $\pi(AB) = \pi(A)\pi(B)$. So

$$\tilde{\pi}(A \circ B) = \tilde{\pi}(AU^2 B) = \pi(U)\pi(A)\pi(U)\pi(U)\pi(B)\pi(U) = \tilde{\pi}(A)\tilde{\pi}(B).$$

Also $\pi(A^*) = \pi(A)^*$ implies that

$$\tilde{\pi}(A^*) = \pi(U)\pi(A)^*\pi(U) = (\pi(U)\pi(A)\pi(U))^* = \tilde{\pi}(A)^*.$$

Linearity of $\tilde{\pi}$ is obvious. Therefore $\tilde{\pi}$ is a $*-$representation.

(ii) suppose $\tilde{\pi}$ is reducible and $\mathcal{V}$ is a non trivial invariant subspace of $\mathcal{H}^J$ under $\tilde{\pi}(\mathcal{LM}(\mathcal{A}, P))$. Let $j \in J$ be such that $\mathcal{T}_j(\mathcal{V}) \neq 0$ where $\mathcal{T}_j : \mathcal{H}^J \longrightarrow \mathcal{H}$ is the canonical projection. Let $a \in \mathcal{A}$ and $A = U^{-1}(a\varepsilon_{ij})U^{-1}$. Then by assumption for every $X \in \mathcal{V}$,

$$\begin{aligned}(\pi(a)\mathcal{T}_j(X))\delta_j = (\pi(a)\varepsilon_{jj})(X) = \pi(a\varepsilon_{jj})(X) &= \pi\left(UU^{-1}(a\varepsilon_{jj})U^{-1}U\right)(X) \\ &= \left(\pi(U)\pi(A)\pi(U)\right)(X) = \tilde{\pi}(A)(X) \in \mathcal{V}.\end{aligned}$$

So $\pi(a)\mathcal{T}_j(X) \in \mathcal{T}_j(\mathcal{V})$ and hence $\mathcal{T}_j(\mathcal{V})$ is invariant under $\pi(\mathcal{A})$ i.e. π is reducible.

Conversely suppose $\mathcal{W}$ is a nontrivial invariant subspace of $\mathcal{H}$ under $\pi(\mathcal{A})$. Then it is easy to see that $\mathcal{W}^J$ is invariant under $\tilde{\pi}(\mathcal{LM}(\mathcal{A}, P))$. So $\tilde{\pi}$ is reducible.

Suppose π is nondegenerate. Let $X \in \mathcal{H}^J$ and $j \in J$ be such that $X_j \neq 0$. Choose $a \in \mathcal{A}$ such that $\pi(a)X_j \neq 0$. If $A = U^{-1}(a\varepsilon_{jj})U^{-1}$, then similar to the previous part one can check that $\tilde{\pi}(A)(X) = \pi(a)X_j\delta_j$ which is nonzero by assumption. Therefore $\tilde{\pi}$ is nondegenerate.

Conversely suppose $\tilde{\pi}$ is nondegenerate and $x \in \mathcal{H}$. Choose $A \in \mathcal{LM}(\mathcal{A}, P)$ such that $Y = \tilde{\pi}(A)(x\delta_1) \neq 0$. Let $j \in J$ be such that $Y_j \neq 0$ and $a = (U^*AU)_{j1}$. Then

$$\pi(a)(x) = (\pi(U^*AU))_{j1}(x) = (\tilde{\pi}(A))_{j1}(x) = Y_j \neq 0.$$

So π is nondegenerate.

The last equivalence is fairly easy to check.

(iii) Suppose π is cyclic with cyclic vector ξ. Let X be the vector in $\mathcal{H}^J$ that all of its components are ξ. Since $\overline{\left\langle \pi(\mathcal{A})(\xi) \right\rangle} = \mathcal{H}$, then $\overline{\left\langle (\pi(\mathcal{A})(\xi))\, \delta_i \right\rangle} = \mathcal{H}\delta_i$. But

$$\begin{aligned}(\pi(\mathcal{A})(\xi))\,\delta_i = (\pi(\mathcal{A})(\varepsilon_{i1}))\,(X) &= \tilde{\pi}\,(U^{*-1}(\mathcal{A}\varepsilon_{i1})U^{*-1})\,(X) \\ &\subseteq \tilde{\pi}\,(\mathcal{LM}(\mathcal{A}, P))\,(X).\end{aligned}$$

So

$$\mathcal{H}^J = \oplus_{j\in J}\mathcal{H}\delta_j = \oplus_{j\in J}\overline{\left\langle (\pi(\mathcal{A})(\xi))\,\delta_j \right\rangle} \subseteq \overline{\left\langle \tilde{\pi}\,(\mathcal{LM}(\mathcal{A}, P))\,(X) \right\rangle}.$$

Therefore $\tilde{\pi}$ is cyclic with cyclic vector X.///

Chapter 5

Applications to semigroup algebras

5.1. Introduction

In this chapter we apply the results of chapters 3 and 4 to the semigroup algebras. We characterize the amenable semigroup algebras and semisimple ones in sections 5.2 and 5.3 respectively. Then in section 5.4 we prove a generalized version of Young's Theorem for the semigroups. Most of the results of sections 5.2 and 5.3 are valid for certain weighted semigroups. But since the proofs are basically the same, we state and prove the results for the nonweighted case, then we state the weighted version. This is done in section 5.5. In the last section we consider involutive semigroups and we show that the involution which we defined in section 4.4, naturally arises from the inverse semigroup algebras.

5.2. Amenable semigroup algebras

Without any topological assumptions, for finite semigroups part (iii) of the following Lemma is due to Munn [30, 3.1].

Lemma 5.2.1. Let T be an ideal of S.

(i) $\ell^1(T)$ is isometrically algebra isomorphic to a closed complemented ideal of $\ell^1(S)$.

(ii) If S has a zero element, then $\ell^1(S)$ is topologically algebra isomorphic to $\ell^1(S)/\ell^1(0) \dotplus \ell^1(0)$.

(iii) $\ell^1(S)/\ell^1(T)$ is isometrically algebra isomorphic to $\ell^1(S/T)/\ell^1(0)$.

Proof. (i) Straightforward.

(ii) Consider the following short exact sequence of $\ell^1(S)$ modules and module homomorphisms:

$$0 \longrightarrow \ell^1(0) \xrightarrow{i} \ell^1(S) \xrightarrow{\tau} \ell^1(S)/\ell^1(0) \longrightarrow 0$$

where i is the inclusion map and τ is the canonical map. Define the map $\psi : \ell^1(S) \longrightarrow \ell^1(0)$ by $\psi(f) = f * \delta_0$, $f \in \ell^1(S)$. For every $f = \sum_{s\in S} f(s)\delta_s$, $h = \sum_{s\in S} h(s)\delta_s \in \ell^1(S)$ we have,

$$\psi(f * h) = f * \left(\sum_{s\in S} h(s)\delta_s * \delta_0\right) = f * \left(\delta_0 * \sum_{s\in S} h(s)\delta_s\right) = \psi(f) * h.$$

Similarly $\psi(f * h) = f * \psi(h)$. So ψ is a bounded $\ell^1(S)$ module homomorphism and since ψ is a left inverse for i, then the sequence splits. Now as in the argument of

[19. Theorem IV.1.18] the map

$$\phi: \ \ell^1(S) \longrightarrow (\ell^1(S)/\ell^1(0)) \oplus \ell^1(0), \quad \phi(f) = (\psi(f), \tau(f))$$

is an $\ell^1(S)$ module isomorphism. Moreover for every $f, h \in \ell^1(S)$ we have:

$$\phi(f * h) = ((f * \delta_0) * (h * \delta_0), \overline{f}\,\overline{h}) = \phi(f)\phi(h).$$

Thus ϕ is a bounded algebra isomorphism and hence it is a topological algebra isomorphism, by the open mapping theorem.

(iii) Define the map $\theta: \ \ell^1(S) \longrightarrow \ell^1(S/T)/\ell^1(0)$ by $\theta(f) = h + \ell^1(0)$ where

$$h(\bar{s}) = \begin{cases} f(s) & if \quad \bar{s} \neq 0 \\ 0 & if \quad \bar{s} = 0. \end{cases}$$

One can show that θ is an onto algebra homomorphism with kernel $\ell^1(T)$. So the map

$$\psi: \ell^1(S)/\ell^1(T) \longrightarrow \ell^1(S/T)/\ell^1(0)$$

$$\psi(f + \ell^1(T)) = \theta(f)$$

is an algebra isomorphism. Since $\|\psi(f + \ell^1(T))\| = \|\theta(f)\| = \|f + \ell^1(T)\|$, then ψ is an isometrical algebra isomorphism.///

The following Lemma is more or less known [8, page 141]. The finiteness of index sets is based on the observation that every nonzero entry $(P_k)_{ji}$ of the sandwich matrix P_k produces a nonzero idempotent $((P_k)_{ji}^{-1}, i, j)$ in the k-th principal factor and consequently in S. Also note that for regular semigroups there is no distinction between principal and composition series, since regular semigroups are semisimple.

Lemma 5.2.2. If S is a regular semigroup with E_S finite. then S has a principal series $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = o$. Moreover there are natural numbers n_k. l_k, a group G_k. $k = 1, \dots, m$. a regular $l_k \times n_k$ matrix P_k on G_k^0 such that $S_k/S_{k+1} = \mathcal{M}^0(G_k, P_k)$, $k = 1, \dots, m-1$ and a $l_m \times n_m$ matrix P_m on G_m such that $S_m = \mathcal{M}(G_m, P_m)$.

Proposition 5.2.3. For a semigroup S. $\ell^1(S)$ is amenable if and only if S has a principal series $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = o$ and $\ell^1(T)$ is amenable for every principal factor T of S.

Proof. Suppose $\ell^1(S)$ is amenable. [8. Theorem 2] and Lemma 5.2.2 imply that S has a principal series $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = o$. [21. Proposition 5.1]. and [6. Theorem 3.7] together with Lemma 5.2.1 imply that $\ell^1(S_{k+1}/S_k)/\ell^1(0)$. $k = 1, \dots m$ is amenable. Since the factors S_k/S_{k+1}. $k = 1, \dots, m$ are the principal factors of S [5. Theorem 2.40], then $\ell^1(T)$ is amenable for every principal factor T of S.

Conversely suppose S has a principal series $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = o$ and $\ell^1(T)$ is amenable for every principal factor T of S. By [5. Theorem 2.40] and inductive application of Lemma 5.2.1 and [21. Proposition 5.1] one can conclude that $\ell^1(S)$ is amenable.///

In order to provide a counter example to the conjecture of Duncan and Paterson [8. Page 145], we need the following Lemma.

Lemma 5.2.4. Suppose S is a semigroup which admits a principal series $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = \phi$ such that S_m is an inverse semigroup and every Rees factor semigroup S_k/S_{k+1}, $k = 1, \dots, m-1$ is completely [0-]simple with invertible sandwich matrix. Then S is an inverse semigroup.

Proof. First suppose $T = \mathcal{M}^0(G, P)$ where P is invertible; In particular $|I| = |J|$. Let Q be the identity $I \times I$ matrix on G^0 i.e. $Q_{ii} = \epsilon$ and $Q_{ij} = 0$ for $i \neq j$. [5, Corollary 3.12] with the identity map on G^0 as ω, $U = P$ and $V = Q$ implies that $\mathcal{M}^0(G, P) \cong \mathcal{M}^0(G, Q)$. Now by [5, Theorem 3.9] T is an inverse semigroup.

By the previous part, S_k/S_{k+1} is an inverse semigroup, $k = 1, \dots, m$. In the rest of the proof we use the fact that if $x, y \in S_k \setminus S_{k+1}$ are such that $\bar{x} = \bar{y}$, then $x = y$. Let $x \in S_k \setminus S_{k+1}$ for some $k \leq m$ and $\bar{x^*}$ be the inverse of $\bar{x}$ in S_k/S_{k+1}. Then $\bar{x^*} \neq 0$, because otherwise we would have $\bar{x} = \bar{x}\bar{x^*}\bar{x} = 0$. Thus $xx^*x = x$ and $x^*xx^* = x^*$ and hence x has an inverse x^* in S. If y is another inverse for x, then $\bar{x}\bar{y}\bar{x} = \bar{x}$ and $\bar{y}\bar{x}\bar{y} = \bar{y}$. So $\bar{y} = \bar{x^*}$ which implies $y = x^*$ as $\bar{y} = \bar{x^*} \neq 0$. Therefore x^* is the unique inverse of x in $S_k \setminus S_{k+1}$. If $k > 1$, then by a similar argument we can see that x has no inverse in $S \setminus S_k$. Therefore S is an inverse semigroup.///

Remarks 5.2.5. (i) With the notations of Lemma 5.2.2 , Duncan and Paterson [8, page 145] have conjectured that if $\ell^1(S)$ is amenable, then G_k is amenable for every $k \leq m$, $S_m = G_m$ and P_k is invertible for every $k = 1, \dots, m-1$. If this conjecture is true, then by Lemma 5.2.4, amenability of $\ell^1(S)$ implies that S is an inverse semigroup, which is not the case as we will see in Example 6.1.3.

(ii) Existence of a principal series is a crucial assumption in the second part of the Proposition 5.2.3 and can not be dropped as we will see in Example 6.1.2.

(iii) Let S be a semigroup such that $\ell^1(S)$ is amenable and $S = S_1 \supset S_2 \supset \dots \supset S_m \supset S_{m+1} = \phi$ be a composition series for S. Duncan and Paterson [8, page 141] have asked which ones of these ideals S_k have amenable semigroup algebra. Lemma 5.2.1 (i) provides a positive answer to this question.

(iv) It is well known that when G is a group and H is a normal subgroup of G, then $\ell^1(G)$ is amenable if and only if $\ell^1(H)$ and $\ell^1(G/H)$ are amenable. Proposition 5.2.3 is the analog of this result for semigroups.

(v) Proposition 5.2.3 reduces the amenability problem to the completely (0-)simple case.

(vi) Existence of an identity in $\ell^1(S)$ does not imply that S has an identity even in the special case of regular Rees matrix semigroups. One can check that a regular Rees matrix semigroup $S = \mathcal{M}^0(G, P)$ [$S = \mathcal{M}(G, P)$] has an identity if and only if $|I| = |J| = 1$.

Now we characterize those regular Rees matrix semigroups S for which $\ell^1(S)$ is amenable. In order to do this, we need the following Proposition. The algebraic version of the first part for finite semigroups without any analytical assumption is due to Munn [30, 3.8]. Also in [8, page 145] the authors have remarked it for the case of finite index sets, without proof. However the general case can be proved directly by showing that the following map is an isometrical algebra isomorphism:

$$\phi : \mathcal{LM}\left(\ell^1(G), P\right) \longrightarrow \ell^1(S)/\ell^1(0)$$
$$\phi\left([f_{ij}]\right) = \left(\sum_{i\in I}\sum_{j\in J}\sum_{g\in G} f_{ij}(g)\delta_{(g,i,j)}\right) + \ell^1(0).$$

Our proof is totally different from Munn's proof. Indeed Munn's proof is based on the finite dimensionality of $\ell^1(S)$.

Proposition 5.2.6. Suppose $S = \mathcal{M}^0(G, P)$. Identify the zero of G^0 with the zero of the ℓ^1-Munn algebra $\mathcal{LM}\left(\ell^1(G), P\right)$, where P is considered as a matrix over $\ell^1(G)$. Then $\ell^1(S)/\ell^1(0)$ is isometrically algebra isomorphic to $\mathcal{LM}\left(\ell^1(G), P\right)$. A similar statement holds for $S = \mathcal{M}(G, P)$.

Proof. It is easy to check that the above mentioned map ϕ is a linear isometry. Let $[f_{ij}], [h_{rt}] \in \mathcal{LM}\left(\ell^1(G), P\right)$. Then,

$$\phi([f_{ij}] \circ [h_{rt}]) = \left(\sum_{i\in I}\sum_{t\in J}\sum_{g\in G}\sum_{l\in I}\sum_{k\in J}(f_{ik} * P_{kl} * h_{lt})(g)\delta_{(g,i,t)}\right) + \ell^1(0)$$

$$= \left(\sum_{i,l\in I}\sum_{t,k\in J}\sum_{x,z\in G} f_{ik}(x)h_{lt}(z)\delta_{(xP_{kl}z,i,t)}\right) + \ell^1(0). \qquad (1)$$

On the other hand we have:

$$\phi([f_{ij}])\phi([h_{rt}]) = \left(\sum_{i\in I}\sum_{j\in J}\sum_{x\in G} f_{ij}(x)\delta_{(x,i,j)} * \sum_{r\in I}\sum_{t\in J}\sum_{z\in G} h_{rt}(z)\delta_{(z,r,t)}\right) + \ell^1(0)$$

$$= \left(\sum_{i,r\in I}\sum_{j,t\in J}\sum_{x,z\in G} f_{ij}(x)h_{rt}(z)\delta_{(xP_{jr}z,i,t)}\right) + \ell^1(0). \qquad (2)$$

From (1) and (2), we conclude that ϕ is an algebra homomorphism.

Let $f + \ell^1(0) \in \ell^1(S)/\ell^1(0)$, $f = \sum_{t\in S} f(t)\delta_t$. Define $f_{ij}(g) = f((g, i, j))$, $g \in G$, $i \in I$, $j \in J$. Then $[f_{ij}] \in \mathcal{LM}\left(\ell^1(G), P\right)$ and $\phi([f_{ij}]) = f + \ell^1(0)$. Therefore ϕ is onto.

In the case of $S = \mathcal{M}(G, P)$, similar to the previous case $\ell^1(S)$ is isometrically algebra isomorphic to $\mathcal{LM}\left(\ell^1(G), P\right)$ via the map

$$\phi: \mathcal{LM}\left(\ell^1(G), P\right) \longrightarrow \ell^1(S)$$

$$\phi([f_{ij}]) = \sum_{i\in I}\sum_{j\in J}\sum_{g\in G} f_{ij}(g)\delta_{(g,i,j)}. \qquad ///$$

Now we can have an alternate proof of the main result of [8] for a special case.

Corollary 5.2.7 [8, Theorem 2] Suppose S is a regular semigroup that admits a principal series. Then amenability of $\ell^1(S)$ implies that E_S is finite.

Proof. We use the notations of Lemma 5.2.2. By Proposition 5.2.3 and Proposition 5.2.6, $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ is amenable and hence by Theorem 3.3.1 the index sets of every principal factor are finite. Now using regularity of sandwich matrices of the principal factors and the fact that every nonzero idempotent corresponds to a nonzero entry of the sandwich matrices, we conclude that E_S is finite.///

Theorem 5.2.8. With the notations of Lemma 5.2.2, the following conditions are equivalent:

(i) $\ell^1(S)$ is amenable,

(ii) $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ has an identity and G_k is amenable, $k = 1, \ldots, m$.

Proof. (ii)$\Longrightarrow$(i) Similarly to the argument of Proposition 5.2.6, we can define a bounded algebra homomorphism:

$$\phi: \mathcal{LM}\left(\ell^1(G_k), P_k\right) \longrightarrow \ell^1(S_k/S_{k+1})/\ell^1(0)$$

$$\phi([f_{rt}]) = \left(\sum_{r=1}^{n_k}\sum_{t=1}^{l_k}\sum_{g\in G} f_{rt}(g)\delta_{(g,r,t)}\right) + \ell^1(0).$$

Since $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ is amenable (Lemma 3.2.7 and Theorem 4.1). and ϕ is a continuous algebra homomorphism with dense range. then $\ell^1(S_k/S_{k+1})/\ell^1(0)$ is amenable by [21, Proposition 5.3] $k = 1, \dots, m-1$. Similarly $\ell^1(S_m/S_{m+1}) = \ell^1(S_m)$ is amenable. Therefore $\ell^1(S)$ is amenable, by Proposition 5.2.3 .

(i)$\Longrightarrow$(ii) By Proposition 5.2.3, $\ell^1(S_k/S_{k+1})$. $k = 1, \dots, m$ is amenable. Since $\ell^1(S_k/S_{k+1})$ is isometrically algebra isomorphic to $\mathcal{LM}\left(\ell^1(G_k), P_k\right)$ (Proposition 5.2.6), then the later algebra is amenable. Now the result follows from Theorem 3.3.1 and Lemma 3.2.7.///

5.3. Semisimple semigroup algebras

A special case of the following Lemma for inverse semigroups has been proved in [8, Theorem 8], with a very technical method. Here we present an elementary proof for the general case.

Lemma 5.3.1. Under the assumptions and notations of the Lemma 5.2.2, the maximal subgroups of S (up to isomorphism) are precisely G_k, $k = 1, \dots, m$ and the trivial group $\{0\}$ (in the case that P_k has a zero entry for some $k \leq m$).

Proof. Let G be a maximal subgroup of S. Suppose $G \cap (S_k \setminus S_{k+1}) \neq \phi$ for some $k \leq m$. If $G \cap S_{k+1} \neq \phi$, then choose $x \in G \cap (S_k \setminus S_{k+1})$ and $y \in G \cap S_{k+1}$. We have $x = (xy^{-1})y \in S_{k+1}$ which is a contradiction. Therefore $G \subseteq (S_k \setminus S_{k+1})$ for some $k \leq m$. For simplicity in the rest of the proof we denote P_k by P.

Case I. Suppose $G \subseteq S_m$ and $(f, i, j) \in S_m = \mathcal{M}(G_m, P_m)$ be the identity of

G. Then for every $(h, r, t) \in G$ we have $r = i$ and $t = j$. Now define the map $\upsilon : G \longrightarrow G_m$ by $\upsilon((h, i, j)) = hP_{ji}$. Then υ is a group homomorphism. Moreover if $\upsilon((h, i, j)) = e$, then $(h, i, j) = (P_{ji}^{-1}, i, j) = (f, i, j)$. Therefore υ is a group monomorphism. On the other hand the set $H = \{(h, i, j) \mid h \in G_m\} \supseteq G$ forms a subgroup of S_m under the product of S which is isomorphic to G_m by a similar argument. Indeed this shows that S has at least one subgroup isomorphic to G_m. Now since G is maximal, then $H = G$. Therefore G is isomorphic to G_m.

Case II. Suppose $G \subseteq (S_k \setminus S_{k+1})$ for some $k < m$. $\overline{G} \simeq G$ is the image of G in S_k/S_{k+1} and (f, i, j) is the identity of $\overline{G}$. Then we can show that all of the elements of $\overline{G}$ are of the form (h, i, j), as in the previous case. Now if $P_{ij} \neq 0$, then similar to the first case we can show that $G \simeq G_k$ and S has at least one maximal subgroup of this kind. If $P_{ij} = 0$ then every subgroup H of S in $S_k \setminus S_{k+1}$ that $H \simeq \overline{H} \subseteq \{(h, i, j) \mid h \in G_k\}$, is the trivial group $\{0\}$, since the product of $\overline{H}$ is zero. Moreover any zero entry of P_k gives the trivial group $\{0\}$ as a maximal subgroup as we showed.///

It is well known that $\ell^1(S)$ is semisimple for every inverse semigroup [38, Theorem 2]. So the following theorem is the general form of [7, Theorem 8] which was proved for the special case of inverse semigroups.

Theorem 5.3.2. Let S be a regular semigroup with a finite number of idempotents. The following conditions are equivalent:

(i) $\ell^1(S)$ is amenable,

(ii) Every maximal subgroup of S is amenable and $\ell^1(T)$ is semisimple for every principal factor T of S.

In particular if $\ell^1(S)$ is amenable, then it is semisimple.

Proof. Throughout the proof we use the notations of the Lemma 5.2.2 .

(i)$\Longrightarrow$(ii) $\mathcal{LM}(\ell^1(G_k), P_k)$ is amenable by Proposition 5.2.3 and Proposition 5.2.6 $k = 1, ..., m$. Now amenability of maximal subgroups of S follows from Lemma 5.3.1. On the other hand Theorem 3.4.1 implies that $\mathcal{LM}(\ell^1(G_k), P_k)$ is semisimple, $k = 1, ..., m$. Thus by [31 Theorem 4.3.2(c)] and Lemma 5.2.1 (iii) $Rad\,(\ell^1(S_k/S_{k+1})) = Rad\,(\ell^1(0)) = 0$, $k = 1, ..., m-1$. Therefore $\ell^1(T)$ is semisimple for every principal factor T of S.

(ii)$\Longrightarrow$(i) By Lemma 5.3.1 $\ell^1(G_k)$ is amenable, $k = 1, ..., m$. Also Lemma 5.2.2 (ii) implies that $\ell^1(S_k/S_{k+1})/\ell^1(0)$ is an ideal of $\ell^1(S_k/S_{k+1})$, $k = 1, ..., m-1$: So by [31, Theorem 4.3.2(a)],

$$Rad(\ell^1(S_k/S_{k+1})/\ell^1(0)) = (\ell^1(S_k/S_{k+1})/\ell^1(0)) \cap Rad(\ell^1(S_k/S_{k+1})) = 0.$$

Therefore $\mathcal{LM}(\ell^1(G_k), P_k)$ is semisimple. Similarly semisimplicity of $\ell^1(S_m)$ implies that $\mathcal{LM}(\ell^1(G_m), P_m)$ is semisimple. Now by Theorem 3.3.1 and Theorem 3.4.1 $\mathcal{LM}(\ell^1(G_k), P_k)$ is amenable, $k = 1, ..., m$. Therefore $\ell^1(S)$ is amenable by Lemma 3.2.7 and Proposition 5.2.3.

For the last statement it is enough to show that if $\ell^1(T)$ is semisimple for every principal factor T of S, then $\ell^1(S)$ is semisimple. As in the previous part we can check that $\ell^1(S_k)/\ell^1(S_{k+1})$ is semisimple, $k = 1, ..., m-1$. Now [31, Theorem

4.3.2(c)] implies that $Rad(\ell^1(S_{m-1})) = Rad(\ell^1(S_m)) = 0$. By doing this process repeatedly we conclude that $Rad(\ell^1(S)) = 0$.///

5.4. Young's theorem for semigroup algebras

Young [39] showed that for a locally compact group G, $L^1(G)$ is Arens regular if and only if G is finite. The following Theorem is an extension Young's theorem to semigroups.

Theorem 5.4.1. In any of the following two cases if $\ell^1(S)$ is Arens regular, then S is finite.

(i) S is a regular semigroup with a finite number of idempotents.

(ii) S is an inverse semigroup which admits a principal series.

Proof. (i) We use the notations of Lemma 5.2.2 . By Lemma 5.5.2, Proposition 5.2.6 and [31, Corollary 1.4.12], $\mathcal{LM}\ (\ell^1(G_k), P_k)$ is Arens regular. Now Theorem 4.3.2 (ii) implies $\ell^1(G_k)$ is Arens regular, $k = 1, ..., m$. So by Young's Theorem [39], G_k is finite, $k = 1, ..., m$. Therefore each principal factor of S and consequently S itself is finite.

(ii) Using the same argument of part (i) except the fact that the index sets of principal factors can be infinite initially, we conclude that $\mathcal{LM}\ (\ell^1(G_k), P_k)$ is Arens regular, $k = 1, ..., m$. Let $1 \leq k \leq m$. By Theorem 4.3.2 (ii) we have two possibilities:

Case 1: $\ell^1(G_k)$ is Arens regular and both index sets are finite. Then by Young's Theorem G_k and consequently S_k/S_{k+1} is finite.

Case 2: $\ell^1(G_k)$ is finite dimensional and one of the index sets is finite. In this case by [5, Theorem 3.9 page 102] both of the index sets are of the same cardinality and hence are finite. So S_k/S_{k+1} is finite.

Therefore in any case we conclude that principal factors of S are finite which implies that S is finite.///

Remark 5.4.2. Young's Theorem [39] is not true in general for regular semigroups even if they admit a principal series, as we will see in the Example 6.1.6. On the other hand Theorem 5.4.2 says that it is true for regular semigroups with a finite number of idempotents. These two facts together say that up to what extent Young's Theorem can be extended.

5.5. Weighted semigroup algebras

Definition 5.5.1 Suppose S is a semigroup. A function $\omega : S \longrightarrow \mathbb{R}^+ \cup \{0\}$ is called a ***weight*** if $\omega(xy) \leq \omega(x)\omega(y)$, $x, y \in S$ and $\omega(x) > 0$ for every nonzero $x \in S$ [in the case that S has a zero element]. If $\omega(x) > 0$ for all $x \in S$, then ω is called a ***positive weight***. If S has a zero element and ω is a weight on S, then $\ell^1(S,\omega)/\ell^1(0,\omega)$ is a Banach algebra that we denote it by $\ell^1(S,\widehat{\omega})$. A weight ω on the semigroup $\mathcal{M}(G,P)$ [$\mathcal{M}^0(G,P)$] is called *uniform* if ω is independent from the indices in the sense that for every $g \in G$, $\omega((g,i,j)) = \omega((g,l,k))$ $i, l \in I$, $j, k \in J$.

Note that a nonzero weight on a group is automatically positive. So in this case we don't mention the word positive.

The following two results are generalizations of Lemma 5.2.1 and Proposition 5.2.3 that can be proved with similar arguments.

Lemma 5.5.2. Let ω be a weight on the semigroup S and T be an ideal of S.

(i) If ω is positive, then $\ell^1(T,\omega)$ is isometrically algebra isomorphic to a closed complemented ideal of $\ell^1(S,\omega)$.

(ii) If S has a zero element, there is a continuous algebra isomorphism from $\ell^1(S,\omega)$ onto $\ell^1(S,\widehat{\omega}) \oplus \ell^1(0,\omega)$. In particular if ω is positive, then $\ell^1(S,\omega)$ is topologically isomorphic to $\ell^1(S,\widehat{\omega}) \oplus \ell^1(0,\omega)$.

(iii) The map $\overline{\omega}: \ S/T \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined by

$$\overline{\omega}(\bar{s}) = \begin{cases} \omega(s) & if \quad \bar{s} \neq 0 \\ 0 & if \quad \bar{s} = 0 \end{cases}$$

is a weight on S/T and $\ell^1(S,\omega)/\ell^1(T,\omega)$ is isometrically algebra isomorphic to $\ell^1(S/T,\widehat{\overline{\omega}})$.

Proposition 5.5.3. Let S be a semigroup and ω be a positive weight on S. Then $\ell^1(S,\omega)$ is amenable if and only if S has a principal series $S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \phi$ and $\ell^1(T,\widehat{\overline{\omega}})$ is amenable for every principal factor T of S.

Lemma 5.5.4. (i) Suppose ω is a weight on the semigroup $S = \mathcal{M}(G,P)$ $[\mathcal{M}^0(G,P)]$. If

$$\sup\{\omega(P_{ji}^{-1},i,j) |\ i \in I, j \in J\} < \infty \ \left[\sup\{\omega(P_{ji}^{-1},i,j) |\ i \in I, j \in J,\ P_{ji} \neq 0\} < \infty\right].$$

then the map

$$\omega_{max} : G \longrightarrow \mathbb{R}^+, \quad \omega_{max}(g) = \sup\{\omega(gP_{ji}^{-1}, i, j) | \quad i \in I, \ j \in J\}$$

$$[\omega_{max}(g) = \sup\{\omega(gP_{ji}^{-1}, i, j) | \quad i \in I, \ j \in J, \ P_{ji} \neq 0\}]$$

is a weight on G.

(ii) Suppose ω is a weight on the group G and all [nonzero] entries of P are equal to the identity e of G. Then the map $\widehat{\omega} : \mathcal{M}(G, P) \longrightarrow \mathbb{R}^+$ [$\widehat{\omega} : \mathcal{M}^0(G, P) \longrightarrow \mathbb{R}^+ \cup \{0\}$] defined by $\widehat{\omega}(g, i, j) = \omega(g)$ [and $\widehat{\omega}(0) = 0$] is a positive weight [weight] on S.

Proof. Straightforward.///

The following results are generalizations of Proposition 5.2.6, Corollary 5.2.7 and Theorem 5.2.8 repectively. They can be proved by using Lemma 5.2.4 and arguments similar to the nonweighted case.

Proposition 5.5.5. Suppose $S = \mathcal{M}^0(G, P)$ and ω is a uniform weight on S that satisfies the assumptions of Lemma 5.5.4(i) and $M = \sup\{\omega(P_{ji}^{-2}, i, j) | i \in I, \ j \in J, \ P_{ji} \neq 0\} < \infty$. Identify the zero of G^0 with the zero of the ℓ^1-Munn algebra $\mathcal{LM}\left(\ell^1(G, \omega_{max}), P\right)$, where P is considered as a matrix over $\ell^1(G, \omega_{max})$. Then $\ell^1(S, \widehat{\omega})$ is topologically algebra isomorphic to $\mathcal{LM}\left(\ell^1(G, \omega_{max}), P\right)$. In the case that $\omega = 1$, this isomorphism is isometrical.

A similar statement holds for $S = \mathcal{M}(G, P)$.

Corollary 5.5.6. [8, Theorem 2] Suppose S is a regular semigroup that admits a principal series and ω is a positive weight on S that satisfies the conditions of

Proposition 5.5.5 on every principal factor of S. Then amenability of $\ell^1(S,\omega)$ implies that E_S is finite.

Theorem 5.5.7. Let ω be a positive weight on the regular semigroup S with a finite number of idempotents. With the notations of Lemma 5.2.2, the following condition implies that $\ell^1(S,\omega)$ is amenable:

$\mathcal{LM}\ (\ell^1(G_k,\overline{\omega}_{max}), P_k)$ has an identity and $\ell^1(G_k,\overline{\omega}_{max})$ is amenable for every $k = 1,\dots,m$.

If moreover $\overline{\omega}$ is uniform on S_k/S_{k+1} for every $k \leq m$, then the converse is also true.

5.6. Involutive semigroup algebras

In this section we show the naturality of the involution that we defined on the ℓ^1-Munn algebras in section 4.4.

Let J be arbitrary and $S = \mathcal{M}^0(G,P)$ where P is a self adjoint $J \times J$ matrix on G i.e. if $P_{ij} \neq 0$, then $P_{ij}^{-1} = P_{ji}$. Note that P is self adjoint on $\ell^1(G)$, since $\delta_g^* = \delta_g^{-1}$. So $\mathcal{LM}\ (\ell^1(G),P)$ is involutive by Lemma 4.4.1.

On the other hand we can define an involution on $\mathcal{M}^0(G,P)$ by $(g,i,j)^* = (g^{-1},j,i)$, since P is self adjoint on G. This involution coincides with the natural involution i.e. inverse, in the case of inverse semigroups. Indeed by [5, Theorem 3.9, page 102] we can assume P is the identity matrix on G. Then the inverse of (g,i,j) is (g^{-1},j,i), i.e. $(g,i,j)^* = (g^{-1},j,i)$. This defines an involution on $\ell^1(S)/\ell^1(0)$ in the following way:

For every $f = \sum_{g\in G}\sum_{i,j} f((g,i,j))\delta_{(g,i,j)} + f(0)\delta_0 \in \ell^1(S)$ define f^* by $f^* = \sum_{g\in G}\sum_{i,j} \overline{f((g,i,j))}\delta_{(g^{-1},j,i)} + \overline{f(0)}\delta_0$ and $(f+\ell^1(0))^* = f^* + \ell^1(0)$.

We showed in Proposition 5.2.6 that the map $\phi : \mathcal{LM}\ (\ell^1(G), P) \longrightarrow \ell^1(S)/\ell^1(0)$ $\phi([f_{ij}]) = \left(\sum_{i\in I}\sum_{j\in J}\sum_{g\in G} f_{ij}(g)\delta_{(g,i,j)}\right) + \ell^1(0)$ is an isometrical algebra isomorphism. With these notations we have:

Proposition 5.6.1. The involution on $\ell^1(S)/\ell^1(0)$ that was defined before the Proposition coincides with the involution induced by ϕ i.e.

$(f+\ell^1(0))^* = \phi\left((\phi^{-1}(f+\ell^1(0)))^*\right), \quad f \in \ell^1(S).$

Proof. Let $f \in \ell^1(S)$. Then $\phi^{-1}(f+\ell^1(0)) = [f_{ij}]$ where $f_{ij}(g) = f(g,i,j)$. So $(\phi^{-1}(f+\ell^1(0)))^* = [f^*_{ji}]$. Since $f^*_{ji}(g) = \overline{f_{ji}(g^{-1})}\Delta(g^{-1}) = \overline{f(g^{-1},j,i)} \quad g \in G$. then

$$\begin{aligned}
\phi\left((\phi^{-1}(f+\ell^1(0)))^*\right) &= \sum_{g\in G}\sum_{i,j} f^*_{ji}(g)\delta_{(g,i,j)} + \ell^1(0) \\
&= \sum_{g\in G}\sum_{i,j} \overline{f(g^{-1},j,i)}\delta_{(g,i,j)} + \ell^1(0) \\
&= \sum_{g\in G}\sum_{i,j} \overline{f(g,i,j)}\delta_{(g^{-1},j,i)} + \ell^1(0) \\
&= f^* + \ell^1(0).
\end{aligned}$$

Therefore both definitions coincide.///

Chapter 6

Examples and open problems

6.1. Some examples

Example 6.1.1. Let $\mathcal{A}$ be an arbitrary unital Banach algebra, $I = J$ be finite of order n and $P = \begin{bmatrix} 1 & & 0 & \\ -1 & \ddots & & \\ o & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$

Let $\mathcal{D}$ be the set of all lower triangular elements $[a_{ij}]$ of $\mathcal{LM}(\mathcal{A}, P)$ that in every column all entries on and under the main diagonal are equal i.e. there is a subset $\{a_1, ...a_n\}$ of $\mathcal{A}$ such that $a_{ij} = a_j$ if $i \geq j$ and $a_{ij} = 0$ otherwise. Clearly $\mathcal{D}$ is a closed subalgebra of $\mathcal{LM}(\mathcal{A}, P)$ and one can check that multiplication of $\mathcal{D}$ which is inherited from $\mathcal{LM}(\mathcal{A}, P)$ coincides with Shur i.e. componentwise multiplication. Now suppose μ is a probability measure and $\mathcal{A} = L^1(\mu)$. Then $\mathcal{D}$ is a special algebra of triangular arrays of random variables that are of interest to the experimental scientists as the data of previous experiments are usually reused in the new (bigger) samples.

Example 6.1.2. Let $S = \mathbb{N}$ be the natural numbers with the binary operation $m.n = \min(m, n)$. Then,

(i) $\ell^1(S)$ is not amenable as E_S is infinite.

(ii) $\ell^1(T)$ is amenable for every principal factor T of S, since $J(a)/I(a) \simeq \{0, 1\}$ with the usual product, for every $a \in S$, which has amenable semigroup algebra.

(iii) S has no principal series. Indeed $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \ldots$ is a chain of ideals of S.

Example 6.1.3. Let $G = \{1\}$ be the trivial group, $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $S = \mathcal{M}^0(G, P)$. Then,

(i) $\ell^1(S)$ is amenable. Indeed Proposition 5.2.6 implies that $\ell^1(S)/\ell^1(0)$ is isometrically algebra isomorphic to $\mathcal{LM}(\ell^1(G), P)$ which is amenable.

(ii) S is not an inverse semigroup, since $\varepsilon_{11} = (1, 1, 1)$ has two different inverses ε_{11} and $\varepsilon_{12} = (1, 1, 2)$.

Example 6.1.4. Let m and n be natural numbers, $G_1 = \{1\}$, $G_2, \ldots, G_{m-1}$ be groups such that all of them but G_2 are amenable and $G_m = \{0\}$. Let $T_i = \mathcal{M}^0(G_i, I_n^i)$, $i = 1, \ldots, m-1$ where I_n^i is the identity $n \times n$ matrix over G_i^0 and $T_m = \{0\}$. We identify the zeros of all of these semigroups with $0 \in G_m$. Suppose S is the direct union of $T_1, \ldots, T_m$ i.e. $S = T_1 \cup \ldots \cup T_m$ with the product $a.b = ab$ if $a, b \in T_i$ for some $i \leq m$ and $a.b = 0$ otherwise. Then,

(i) S is regular as every T_i is an inverse semigroup [5, Theorem 3.9] and E_S is finite, since $E_S = E_{T_1} \cup E_{T_2} \cup \ldots \cup E_{T_m}$.

(ii) S is amenable as S has a zero element. But $\ell^1(S)$ is not amenable, since G_2 which is a maximal subgroup of S, is not amenable.

Example 6.1.5. Suppose S is the semigroup of Example 6.1.4. Then,

(i) $\ell^1(T)$ is semisimple for every principal factor T of S, since $T_1, \ldots, T_m$ are the principal factors of S and each T_i is an inverse semigroup [38, Theorem 2].

(ii) At least one maximal subgroup of S is not amenable and the same as $\ell^1(S)$.

Example 6.1.6. Let G be a finite group, $|I| = n$ for some $n \in \mathbb{N}$ and J be an arbitrary infinite index set. Let P be a regular matrix over G^0. Then $S = \mathcal{M}^0(G, P)$ is a regular semigroup with a principal series. By Theorem 4.3.2(ii) $\mathcal{LM}(\ell^1(G), P)$ is Arens regular and hence so is $\ell^1(S)/\ell^1(0)$ as it is topologically algebra isomorphic to $\mathcal{LM}(\ell^1(G), P)$ (Proposition 5.2.6). Now it is easy to check that $\ell^1(S)/\ell^1(0) \oplus \ell^1(0)$ is Arens regular. But by Lemma 5.2.1(ii) this algebra is topologically algebra isomorphic to $\ell^1(S)$. Therefore $\ell^1(S)$ is Arens regular, but S is infinite. Note that E_S is infinite in this example which shows the conditions of Theorem 5.4.1 can not be weakened.

6.2. Discussion and open problems

In section 3.2 we compared the algebra $\mathcal{LM}_{\mathbb{N}}$ and the matrix algebra $\mathcal{M}$ that was introduced in [37, page 710]. By Theorem 3.3.1, $\mathcal{LM}_{\mathbb{N}}$ is not amenable.

Problem 1. Is the algebra $\mathcal{M}$ amenable?

Existence of bounded approximate identity is necessary for amenability [21. Proposition 1.6]. Analog of this for weak amenability is the following:

If $\mathcal{A}$ is weakly amenable, then $\overline{\mathcal{A}^2} = \mathcal{A}$ [15, Proposition 2.4]. We characterized those ℓ^1-Munn algebras that have bounded approximate identity in Lemma 3.2.7. But the above step toward weak amenability has not been taken yet. So one can ask:

Problem 2. Characterize those ℓ^1-Munn algebras $\mathcal{LM}(\mathcal{A}, P)$ for which $\overline{\mathcal{LM}(\mathcal{A}, P)^2} = \mathcal{LM}(\mathcal{A}, P)$.

After characterizing those ℓ^1-Munn algebras that have bounded approximate identity, the next step is finding necessary and sufficient conditions for amenability which was done in Theorem 3.3.1. But the weakly amenable case is still open.

Problem 3. Characterize weakly amenable ℓ^1-Munn algebras.

In Proposition 3.2.4 we showed that $\mathcal{LM}_{\aleph}$ is an ideal in $\mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space. As $\mathcal{K}(\mathcal{H})$ is semisimple, $\mathcal{LM}_{\aleph}$ is also semisimple, by [31, Theorem 4.3.2], but in the general case of infinite index sets the problem is still open. More generally,

Problem 4. What is $Rad(\mathcal{LM}(\mathcal{A}, P))$, when at least one of the index sets is infinite?

As the positive cones of involutive Banach algebras are primary tools in the study of the structure of such algebras, we started studying positive elements of involutive ℓ^1-Munn algebras in Lemma 4.4.2. In particular we showed that if $B \in \mathcal{LM}(\mathcal{A}, P)_+$, then $B_{ij} \in \mathcal{A}_+$ for all $i, j \in J$.

Problem 5. Is the converse of this statement true? More generally characterize positive elements of involutive ℓ^1-Munn algebras.

As we observed in Lemma 4.4.2 and Lemma 4.4.4, diagonal entries of P play an important role in determining the structure of an involutive ℓ^1-Munn algebra. Being a C^*-algebra or more generally admiting a nonzero positive linear functional imposes certain necessities on the diagonal entries of P, but still no sufficient condition is known. However we conjecture that the sufficient conditions should be stronger than the necessary conditions obtained in 4.4.2 and 4.4.4. More generally we can ask:

Problem 6. Characterize ℓ^1-Munn algebras that admit a nonzero positive functional.

In Lemma 4.4.4 we showed that if $F \in \mathcal{LM}(\mathcal{A}, P)^*$ is positive, then for every $a, b \in \mathcal{A}$, $i, j, k, l \in J$ we have:

$\widetilde{F}_{ij}(a) = \overline{\widetilde{F}_{ji}(a^*)}$

$\widetilde{F}_{ii}(aP_{jj}a^*) \geq 0$

$|\widetilde{F}_{kj}(b^*P_{li}a)|^2 \leq \widetilde{F}_{jj}(a^*P_{ii}a)\widetilde{F}_{kk}(b^*P_{ll}b)$.

But still we don't know whether the converse is true or not.

Problem 7. Do these conditions imply that $F \geq 0$?

In section 4.4 we constructed a *-representation $\{\widetilde{\pi}, \mathcal{H}^J\}$ from a *-representation $\{\pi, \mathcal{H}\}$ of $\mathcal{A}$ and we showed that $\widetilde{\pi}$ is irreducible [resp. faithful, nondegenerate], if and only if π is. Besides we showed that if π is cyclic, then so is $\tilde{\tilde{\pi}}$.

Problem 8. Is the converse of the last statement true?

In Theorem 5.3.2 we obtained the relation between amenability and semisimplicity for semigroup algebras by showing that amenability of $\ell^1(S)$ is equivalent to the following statement:

Every maximal subgroup of S is amenable and $\ell^1(T)$ is semisimple for every principal factor T of S.

In the same Theorem we showed that the above semisimlicity statement implies that $\ell^1(S)$ is semisimple.

Problem 9. (Conjecture). In the Theorem 5.3.2. we can replace semisimplicity of $\ell^1(T)$ for every principal factor T of S by the weaker condition. semisimplicity of $\ell^1(S)$.

In the weighted case the result might be totally different and it is not known even for the groups. More clearly suppose ω is a weight on the group G. Then.

Problem 10. Does amenability of $\ell^1(G,\omega)$ implies that $\ell^1(G,\omega)$ is semisimple?

In the proof of Theorem 5.3.2 we used the semisimplicity of the group algebras. It is possible that the same argument works for the weighted semigroups. if problem 10 has a positive answer.

Problem 11. Suppose problem 10 has a positive answer. Is the weighted version of Theorem 5.3.2 true?

Appendix

Some generalizations and alternate proofs of some known results

In this appendix as it was mentioned in the Remark 3.3.2, we present an alternate proof of [21, Proposition 5.4] that does not depend on the extension of derivations. Also we present the proofs of Lemma 3.2.2 and Lemma 3.2.3.

[21, Proposition 5.4.] If $\mathcal{A}$ and $\mathcal{B}$ are amenable Banach algebras, then $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is amenable.

Proof. Suppose $\{e_\gamma : \gamma \in \Gamma\}$ and $\{f_\lambda : \lambda \in \Lambda\}$ are approximate diagonals for $\mathcal{A}$ and $\mathcal{B}$ respectively. We show that $\{\psi^{-1}(e_\gamma \otimes f_\lambda) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is an approximate diagonal for $\mathcal{A} \widehat{\otimes} \mathcal{B}$ where as in Theorem 3.3.1, ψ is the linear isometry defined by:

$$\psi : (\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B}) \longrightarrow (\mathcal{A} \widehat{\otimes} \mathcal{A}) \widehat{\otimes} (\mathcal{B} \widehat{\otimes} \mathcal{B})$$

$$\psi((c \otimes x) \otimes (d \otimes y)) = (c \otimes d) \otimes (x \otimes y), \ c, d \in \mathcal{A}, \ x, y \in \mathcal{B}.$$

For every $c, d, f \in \mathcal{A}$ $x, y, z \in \mathcal{B}$ we have:

$$\upsilon\Big((c\otimes x)\otimes(d\otimes y)\Big)(f\otimes z) = (c\otimes df)\otimes(x\otimes yz)$$
$$= \upsilon\Big((c\otimes x)\otimes(df\otimes yz)\Big)$$
$$= \upsilon\Big(((c\otimes x)\otimes(d\otimes y))(f\otimes z)\Big).$$

So υ is a right $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ module homomorphism. Similarly it is a left $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ module homomorphism. Hence both of υ and υ^{-1} are $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ bimodule homomorphisms. Moreover let

$$\pi : (\mathcal{A}\,\widehat{\otimes}\,\mathcal{B})\,\widehat{\otimes}\,(\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}) \longrightarrow (\mathcal{A}\,\widehat{\otimes}\,\mathcal{B})$$
$$\pi_1 : \mathcal{A}\,\widehat{\otimes}\,\mathcal{A} \longrightarrow \mathcal{A}$$
$$\pi_2 : \mathcal{B}\,\widehat{\otimes}\,\mathcal{B} \longrightarrow \mathcal{B}$$

be the canonical maps. Then one can show that $\pi_1 \otimes \pi_2 = \pi\upsilon^{-1}$. So for every $a\otimes x \in \mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ we have:

$$\lim_{(\gamma,\lambda)}\Big(\big(\upsilon^{-1}(e_\gamma\otimes f_\lambda)\big)(a\otimes x) - (a\otimes x)\big(\upsilon^{-1}(e_\gamma\otimes f_\lambda)\big)\Big) =$$
$$\upsilon^{-1}\Big(\lim_{(\gamma,\lambda)}\Big((e_\gamma\otimes f_\lambda)(a\otimes x) - (a\otimes x)(e_\gamma\otimes f_\lambda)\Big)\Big) =$$
$$\upsilon^{-1}\Big(\lim_{(\gamma,\lambda)}\Big(e_\gamma a\otimes(f_\lambda x - xf_\lambda) - (e_\gamma a - ae_\gamma)\otimes xf_\lambda\Big)\Big) = 0$$

and

$$\lim_{(\gamma,\lambda)}\pi\upsilon^{-1}\Big((e_\gamma\otimes f_\lambda)(a\otimes x)\Big) = \lim_{(\gamma,\lambda)}(\pi_1\otimes\pi_2)\Big((e_\gamma\otimes f_\lambda)(a\otimes x)\Big)$$
$$= \lim_{(\gamma,\lambda)}\Big((\pi_1(e_\gamma a)\otimes\pi_2(f_\lambda x)\Big) = a\otimes x.$$

Therefore $\{\psi^{-1}(e_\gamma \hat{\otimes} f_\lambda) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is an approximate diagonal for $\mathcal{A} \hat{\otimes} \mathcal{B}$.///

Lemma 3.2.2. Every $u \in \mathcal{LM}_J \hat{\otimes} \mathcal{A}$ has a unique expression in the form $u = \sum_{i,j \in J} \varepsilon_{ij} \otimes a_{ij}$.

Proof. We know that u has an expression $u = \sum_{n=1}^{\infty} B^n \otimes a_n$, $B^n \in \mathcal{LM}_J$, $a_n \in \mathcal{A}$. So if we assume $a_{ij} = \sum_{n=1}^{\infty} B^n_{ij} a_n$, then

$$u = \sum_{n=1}^{\infty} \left(\sum_{i,j \in J} B^n_{ij} \varepsilon_{ij} \right) \otimes a_n = \sum_{i,j \in J} \varepsilon_{ij} \otimes \left(\sum_{n=1}^{\infty} B^n_{ij} a_n \right) = \sum_{i,j \in J} \varepsilon_{ij} \otimes a_{ij}.$$

For the uniqueness we need only to show that if $\sum_{i,j \in J} \varepsilon_{ij} \otimes c_{ij} = 0$, then $c_{ij} = 0$, $i,j \in J$. Here we use the technique of [3, Lemma 4, page 231]. By [3, proposition 12, page 234], $\mathcal{LM}_J \hat{\otimes} \mathcal{A}$ can be represented as a linear subspace of $BL(\mathcal{LM}_J^*, \mathcal{A}^*; \mathbb{C})$, consisting of all elements of the form $\omega = \sum_{i=1}^{\infty} x_i \otimes y_i$ with $\sum_{i=1}^{\infty} \|x_i\| \|y_i\| < \infty$. Here $BL(\mathcal{LM}_J^*, \mathcal{A}^*; \mathbb{C})$ is the space of bounded bilinear functionals on $\mathcal{LM}_J^* \times \mathcal{A}^*$ and the action of $x \otimes y \in \mathcal{LM}_J \hat{\otimes} \mathcal{A}$ on $\mathcal{LM}_J^* \times \mathcal{A}^*$ is defined by $(x \otimes y)(f, g) = f(x)g(y)$.

Let $f \in \mathcal{LM}_J^*$ and $g \in \mathcal{A}^*$. Then $\sum_{i,j \in J} g(c_{ij}) \varepsilon_{ij} \in \mathcal{LM}_J$ and

$$f\left(\sum_{i,j \in J} g(c_{ij}) \varepsilon_{ij} \right) = \sum_{i,j \in J} g(c_{ij}) f(\varepsilon_{ij}) = \left(\sum_{i,j \in J} \varepsilon_{ij} \otimes c_{ij} \right)(f, g) = 0.$$

As f and g are arbitrary, then $\sum_{i,j \in J} g(c_{ij}) \varepsilon_{ij} = 0$ and hence $c_{ij} = 0$ for all $i, j \in J$.///

Lemma 3.2.3. $\mathcal{LM}_J(\mathcal{A})$ is isometrically algebra isomorphic to $\mathcal{LM}_J \hat{\otimes} \mathcal{A}$.

Proof. Define the map $\phi : \mathcal{LM}_J(\mathcal{A}) \longrightarrow \mathcal{LM}_J \hat{\otimes} \mathcal{A}$ by $\phi(A) = \sum_{i,j \in J} \varepsilon_{ij} \otimes A_{ij}$. It is easy to check that ϕ is an algebra homomorphism. Also by Lemma 3.2.2 it is onto.

Let $A \in \mathcal{LM}_J(\mathcal{A})$ and $u = \phi(A)$. Then $\|u\| \leq \sum_{i,j\in J} \|A_{ij}\| = \|A\|$. On the other hand if $u = \sum_{n=1}^{\infty} B^n \otimes a_n$, then $u = \sum_{i,j\in J} \varepsilon_{ij} \otimes (\sum_{n=1}^{\infty} B^n_{ij} a_n)$ and by the uniqueness part of Lemma 3.2.2, $A_{ij} = \sum_{n=1}^{\infty} B^n_{ij} a_n$. Thus

$$\|A\| = \sum_{i,j\in J} \|\sum_{n=1}^{\infty} B^n_{ij} a_n\| \leq \sum_{n=1}^{\infty} \|a_n\| \sum_{i,j\in J} \|B^n_{ij}\| = \sum_{n=1}^{\infty} \|a_n\| \|B^n\|.$$

Since this is true for every expression $\sum_{n=1}^{\infty} B^n \otimes a_n$ of u, then $\|A\| \leq \|u\|$ and hence $\|A\| = \|u\|$, i.e. ϕ is an isometry.///

References

1. R. Arens, *Operations induced in function classes*, Monatsh Math. **55** (1951), 1-19.

2. W .G. Bade, P. C. Curtis, Jr and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **55** (1987), 359–377.

3. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer Verlag, Berlin, 1973.

4. P. Civin and B. Yood, *The second conjugate space of a Banach algebra*, Pacific J. Math. **11** (1961), 847–870.

5. A. H. Clifford and J. B. Preston, *The algebraic theory of semigroups I*, Amer. Math. Soc. Surveys 7, 1961.

6. P. C. Curtis, Jr and R. J. Loy, *The structure of amenable Banach algebras*, J. London Math. Soc. **40** (1989), 89–104.

7. J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh **80A** (1978), 309–321.

8. J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand. **66** (1990), 141–146.

9. G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, preprint.

10. G. H. Esslamzadeh, *Representation theory and positive functionals of involutive l^1 Munn algebras*, in preparation.

11. F. Ghahramani and A.T. M. Lau, *Isometric isomorphisms between the second conjugate algebras of group algebras*, Bull. London Math. Soc. **(4)20** (1988), 342–344.

12. F. Ghahramani, A.T. M. Lau and V. Losert. *Isometric isomorphisms between Banach algebras related to locally compact groups*. Trans. Amer. Math. Soc. **321** (1990), 273–283.

13. N. Groenbaek, *Amenability of weighted discrete convolution algebras*, Proc. Edinburgh Math. Soc. **110A** (1988), 351–360.

14. ———. *Amenability of discrete convolution algebras, the commutative case*. Pasific J. Math. **143** (1990), 243–249.

15. ———. *Weak and cyclic amenability for non-commutative Banach algebras*. Proc. Edinburgh Math. Soc. **35** (1992), 315–328.

16. ———. *Morita equivalence for Banach algebras*. J. Pur Appl. Alg. **99** (1995). 183–219.

17. E. Hewitt and K. A. Ross. *Abstract harmonic analysis I*. Springer-Verlag. Berlin. 1987.

18. J. M. Howie. *An introduction to semigroup theory*. Academic Press. London. 1976.

19. T. W. Hungerford. *Algebra*. Springer-Verlag. New York. 1974.

20. N. Isik. J. Pym and A. Ulger. *The second dual of the group algebra of a compact group*. J. London Math. Soc. **(2)35** (1987). 135–148.

21. B. E. Johnson, *Cohomology in Banach algebras*, Memo. Amer. Math. Soc. **127** (1972).

22. ———. *Approximate diagonals and cohomology of certain annihilator Banach algebras*. Amer. J. Math. **94** (1972), 685–698.

23. A. Y. Khelemskii, *Flat Banach modules and amenable algebras*, Amer. Math. Soc. Translations of the Trans. Moscow Math. Soc. (1985), 199–244.

24. A. T. M. Lau, *The second conjugate algebra of the Fourier algebra of a locally compact group*. Trans. Amer. Math. Soc. **267** (1981), 53–63.

25. ______. *Characterizations of amenable Banach algebras*. Proc. Amer. Math. Soc. **70** (1978), 156–160.

26. A. T. M. Lau and R. J. Loy, *Amenability of convolution algebras*, Math. Scand **79** (1996), 283–296.

27. ______. *Weak amenability of Banach algebras on locally compact groups*. J. Fun. Anal. **145** (1997), 175–204.

28. A. T. M. Lau and V. Losert, *On the second conjugate algebra of the group algebra of a locally compact group*, J. London Math. Soc. **37** (1988), 464–470.

29. A. T. M. Lau and A. Ülger, *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. **(3)348** (1996), 1191–1212.

30. W. D. Munn, *On semigroup algebras*, Proc. Cambridge Phil. Soc. **51** (1955), 1–15.

31. T. Palmer, *Banach algebras and the general theory of *-algebras. Vol. I*, Cambridge University Press, 1994.

32. A. L. T. Paterson, *Amenability*, Math. Surveys and Monographs 29, Amer. Math. Soc. Providence, R. I., 1988.

33. V. I. Paulsen, *Completely bounded maps and dialations*, Pitman research notes in math. 146, Longman Sci. Tech., London, 1986.

34. W. Rudin, *Functional Analysis*, Tata McGraw Hill, New Delhi, 1989.

35. N. Sundhir R., *On classification of semigroup rings*, Semigroup Forum **(2)40** (1990), 159–179.

36. B. Tanbay, *Pure state extensions on algebras of operators*, University of California at Berkeley, PhD Thesis, 1989.

37. B. Tanbay. *Pure state extensions and compressability of the l^1-algebra*. Proc. Amer. Math. Soc. **(3)113** (1991). 707–713.

38. J. R. Wordingham. *The left *-representation of an inverse semigroup*. Proc. Amer. Math. Soc. **86** (1982). 55–58.

39. N. J. Young, *The irregularity of multiplication in group algebras*. Quart. J. Math. Oxford **(2)24** (1973). 59–62.

Symbol Index

Index

IMAGE EVALUATION TEST TARGET (QA-3)

150mm

6"

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