University of Alberta

Ought Implies May Not

by

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Abstract

This thesis examines a deontic logic of contingent obligations. It is motivated by views in freewill and theological philosophy. Most common principles of standard deontic logic are examined. Furthermore, iterated deontic principles are considered as well as those pertinent to moral dilemmas. Finally, consequences to substantive ethics are considered.

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List of Symbols

RPL $A_1, \ldots, A_n$ / A ($n \geq 0$), where the inference from $A_1, \ldots, A_n$ to A is propositionally correct.

PL Tautologies of propositional logic (also used in proof abbreviations to indicate inference by RPL, or combinations of PL and RPL)

Mixed Deontic-Alethic Principles

$MD_{d\text{-}a}$ $OA \wedge OB \wedge \neg\Diamond'(A \wedge B)$

$NAT_{d\text{-}a}$ $OA \rightarrow \Diamond\neg OA$

$FAT_{d\text{-}a}$ $FA \rightarrow \Diamond\neg FA$

$OAP_{d\text{-}a}$ $OA \rightarrow \Diamond\neg A$

$OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$

$OI\nabla_{d\text{-}a}$ $OA \rightarrow \Diamond A \wedge \Diamond\neg A$

$ME_{d\text{-}a}$ $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$

$MIO_{d\text{-}a}$ $\Box A \rightarrow OA$

$OB_{d\text{-}a}$ $OA \rightarrow \Diamond(OA \wedge \neg A)$

$OF_{d\text{-}a}$ $OA \rightarrow \Diamond(OA \wedge A)$

$REP_{d\text{-}a}$ $B \leftrightarrow B'$ / $A \leftrightarrow A[B/B']$

Deontic Principles

AG_d $O(A \vee B) \wedge O\neg B \rightarrow O((A \vee B) \wedge \neg B)$

C_d $OA \wedge OB \rightarrow O(A \wedge B)$

Con_d $\neg O\bot$

D_d $OA \rightarrow PA$

M_d $O(A \wedge B) \rightarrow OA \wedge OB$

N_d $O\top$

$O4_d$ $OA \rightarrow OOA$

$O4C_d$ $OOA \rightarrow OA$

$OOIC_d$ $O(OA \rightarrow \Diamond A)$

OU_d $O(OA \rightarrow A)$

RE_d $A \leftrightarrow B$ / $OA \leftrightarrow OB$

RM_d $A \rightarrow B$ / $OA \rightarrow OB$

$RM|_d$ $A \rightarrow B$ / $O(A \mid C) \rightarrow O(B \mid C)$

RR_d $A \wedge B \rightarrow C$ / $OA \wedge OB \rightarrow OC$

RN_d A / OA

REP_d $B \leftrightarrow B'$ / $A \leftrightarrow A[B/B']$

RP_d A / PA

Df. O_d $OA \leftrightarrow \neg P\neg A$

Df. F_d $FA \leftrightarrow \neg PA$

Df. P_d $PA \leftrightarrow \neg O\neg A$

Alethic Principles

RE_a	$A \leftrightarrow B / \Box A \leftrightarrow \Box B$
RN_a	$A / \Box A$
RM_a	$A \rightarrow B / \Box A \rightarrow \Box B$
Df. $\Box_a$	$\Box A \leftrightarrow \neg\Diamond\neg A$
Df. $\Diamond_a$	$\Diamond A \leftrightarrow \neg\Box\neg A$
D_a	$\Box A \rightarrow \Diamond A$
T_a	$\Box A \rightarrow A$
B_a	$A \rightarrow \Box\Diamond A$
4_a	$\Box A \rightarrow \Box\Box A$
5_a	$\Diamond A \rightarrow \Box\Diamond A$
$4C_a$	$\Box\Box A \rightarrow \Box A$
$4\Diamond C_a$	$\Diamond\Diamond A \rightarrow \Diamond A$
$\Diamond M_a$	$\Diamond(A \wedge B) \rightarrow \Diamond A \wedge \Diamond B$

Chapter 1: Introduction and Preliminaries

1.1 Introduction

This thesis investigates a mixed deontic-alethic logic founded on two principles, 'ought' implies 'can' and 'ought' implies 'can not'. Together, they say that all obligations are contingent. There is an old tradition arguing for the former, while the latter principle has received less attention. Nevertheless, since the inception of deontic logic, several prominent authors have expressed intuitions supporting this view. More importantly, both of these principles are central in contemporary free will discussions, where deontic judgments have received increasing attention. The purpose here is to examine the deontic logic where obligations are contingent. This task is two-fold: a logical demonstration of common principles and their relationships within the proposed logic (or simply 'E∇') and the de facto "standard deontic logic," and an exposition of the philosophical ramifications of both logics. The proposed logic and standard deontic logic are compared, and finally, E∇ is evaluated based on the comparison.

The first chapter introduces basic principles and the formal apparatus of the logics involved. 'Ought' implies 'can' and 'ought' implies 'can not' are expounded and motivated in sections 1.2 and 1.3. The specific deontic and alethic concepts are explained in 1.4. In 1.4.5 the notation used in the formal parts of the thesis is given. The syntax, axiomatic systems, and semantics for KD are described in section 1.5 for KD and in 1.6 for E∇. Chapter 2 is divided into three parts: a discussion on typical KD principles (2.1), iterated and nested principles

(2.2) and principles pertinent to moral dilemmas (2.3). In each section, formal work is done (with references to the Appendix), as well as a discussion on correctness. Some remarks will be made on the ramifications of E∇ to common substantive ethical views. Finally, a comparison will be made and a preliminary remark on the worth of the proposed logic.

1.2 'Ought' implies 'Can' and 'Ought' implies 'Can Not'

The 'ought' implies 'can' principle states that whatever is obligatory can be done. For example, suppose Jane is obligated to pay her rent. This obligation holds only if it is possible for Jane to pay her rent. For example, if she is tied up by a burglar on rent day then she cannot pay her rent. In this case, the obligation is absolved in virtue of the fact that she could not have done the very thing asked of her. This principle is significant since agents are often placed in situations where they cannot fulfill obligations. Something must be appealed to in order to show that in fact there is no obligation. The 'ought' implies 'can' principle serves this function. As intuitive as this principle is, a number of authors have challenged it in the mid to late twentieth century.[1] Though there is controversy surrounding this principle, it is still defended by many and it will be taken as fundamental (Zimmerman 1996, Feldman 1986).

For the purposes of this preliminary discussion, the principle, abbreviated $OIC_{d\text{-}a}$, is symbolized as: $OA \rightarrow \Diamond A$, where 'O' is read "it is obligatory that" and '$\Diamond$' is read "it is possible that." For example, if it is obligatory that Jane pay her

[1] For critiques of 'ought' implies 'can' see Frankena (1963), White (1975) and Sinnott-Armstrong (1988).

rent then it is possible that Jane pays her rent.

In the example above, being incapable of fulfilling her obligation, Jane was absolved of her obligation. Typically, Jane can not only be prevented from paying her rent, she can also, of her own accord, not pay rent by spending her rent money on a trip to Cuba or simply forgetting. In such cases, Jane has done something other than what she was obligated to do. If this were not always so, there would be cases where Jane necessarily must do the very thing obligated of her. Although necessities in our lives abound, we rarely describe them colloquially as obligatory. It is unusual to describe an arithmetic truth, such as "2 + 2 = 4" as obligatory, or a fact about human nature, such as "all humans are animals." The general principle behind this intuitive idea is that whatever ought to be, could have been otherwise, or what ought to be implies alternative possibilities.

$$OAP_{d\text{-}a}\ OA \rightarrow \Diamond\neg A$$

Together, the two principles, $OAP_{d\text{-}a}$ and $OIC_{d\text{-}a}$ state that obligations both can be done and not, or whatever ought to be is contingent. A contingent principle is one that can be true and can be false. Thus, impossible, and necessary sentences are not contingent.

$$OI\nabla_{d\text{-}a}\ OA \rightarrow \Diamond A \wedge \Diamond\neg A$$

$OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ are similar in form, however, they are logically independent indicating that there is a significant difference between them. Suppose that 'A' expresses some necessary state of affairs, for example, that all bachelors are unmarried, and that this is obligatory. Since, whatever is necessary is possible, it is possible that all bachelors are unmarried, thereby establishing that $OIC_{d\text{-}a}$ is true for this 'A'. The same cannot be said of $OAP_{d\text{-}a}$. If indeed A is obligatory, it does not follow that the negation of A is possible, i.e. that not all bachelors are unmarried. By our assumption, this would be impossible. $OAP_{d\text{-}a}$ in this case is false, while $OIC_{d\text{-}a}$ is true, so $OIC_{d\text{-}a}$ is not logically dependent on $OAP_{d\text{-}a}$. Moreover, suppose that 'A' means to travel faster than light and that this is what ought to be. Then, $OIC_{d\text{-}a}$ is false, since faster than light travel is impossible, but $OAP_{d\text{-}a}$ is true because not traveling faster than light (e.g. at the speed of sound) is possible. Thus, the two principles are logically independent of one another.

1.3 Philosophical Motivations

1.3.1 Intuitions that Obligations are contingent

The contingency of obligation is explicitly intuitive to many philosophers and some logicians. The modern founder of deontic logic, von Wright, from his earliest to last paper defended the separation of the necessary and the obligatory (von Wright 1999). Peter Schotch in agreement states that "... a strongly held intuition to the effect that OP can only be true for P contingent. Everyone notices this when P is necessarily false (even when the sense of necessity is rather 'weak') but it also holds when P is necessarily true." (Schotch unpublished, pg. 134) Al-Hibri has the same intuitions (al-Hibri 1978, pg. 13).

In the next subsection an argument for contingent obligations is given, while the argument for $OAP_{d\text{-}a}$, the less discussed principle of the two, is provided now. Schotch conjectures that "... the pragmatic connection between terms of moral appraisal and praise (and blame) lies behind the intuition that only contingencies lie within the scope of such terms. We want to praise those who fulfill their oughts, which praise would be hollow indeed, for bringing about the truth of a theorem." (Schotch unpublished, pg. 134, footnote 6). From this conjecture, an argument can be surmised:

Where PR... means 'It is praiseworthy that ...'
O... means 'It is obligatory that ...'

1. $PR(A) \rightarrow \Diamond\neg A$
2. $OA \rightarrow PR(A)$
3. Therefore, $OA \rightarrow \Diamond\neg A$

The first premise says that what is praiseworthy could have been otherwise. To deny this principle would be to hold something praiseworthy yet necessary, and this does not seem plausible. It is highly unusual to praise something when it must happen. Mother Teresa's generosity is praiseworthy, at least in part, because she could have been less generous. Actions need not be extraordinary to deserve praise. A prompt return of rented videos deserves some praise, for they could be returned late. Even events are praised, such as the flooding of the Nile or the alleviation of human suffering or, perhaps, just a sunny day. Never are truths of arithmetic praised though one may applaud the discovery of one. The second principle is also plausible. Anything that is obligatory when done deserves some sort of praise. This does not mean that it is in fact praised, just that it is worthy of praise. The two principles yield, by propositional logic, the principle that obligations could have been otherwise.

1.3.2 OI∇ and Freewill

Prominent contemporary metaphysicians interested in freewill also hold that obligations are contingent. Roughly, the problem of free will is in reconciling determinism and freedom, two apparently conflicting positions. The prima facie consequence, troubling to many, is that determinism rules out choice. Moreover, since moral responsibility is intricately tied to choice, it seems that determinism also rules out all moral responsibility. Theorists often analyze freedom as the ability to do otherwise (or interchangeably, having alternative possibilities) and so the central principle debated is PAP:

PAP One is morally responsible only if one could have done otherwise (alternative possibilities)

Historically, much of the freewill debate has revolved around moral responsibility, particularly on PAP.[2] However, the debate has partly shifted to deontic grounds, that is, judgments of obligation, permission and prohibition. For example, David Widerker has shown interesting logical ties between PAP and deontic judgments (Widerker 1991)[3], while others have sought to consider the question of freewill from the deontic point of view alone (Haji 1998, 2000, Zimmerman 2003). The central free will problem in the deontic context is this: [4]

1. If determinism is true, one could not have done otherwise
2. <u>Determinism is true</u>
3. Therefore, one could not have done otherwise.
4. <u>It is morally obligatory only if one could have done otherwise</u>
5. Therefore, nothing is morally obligatory
6. <u>It is morally prohibited only if one could have done otherwise</u>
7. Therefore, nothing is morally prohibited[5]

Both deontic principles, $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$, are pertinent to freewill. The former implies that prohibitions imply could have done otherwise, while the latter implies that obligations do. The relevance of $OIC_{d\text{-}a}$ to the freewill debate was pointed out by (Haji 2003, pg. 29):

1. $O\neg A \rightarrow \Diamond\neg A$
2. $\underline{FA \leftrightarrow O\neg A}$
3. $FA \rightarrow \Diamond\neg A$

The first premise is an instance of 'ought' implies 'can'. The second

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2 The enthusiastic debate over PAP in recent decades is inspired by Frankfurt's seminal paper *Alternate Possibilities and Moral Responsibility* (1969).

3 Widerker argues that PAP follows from OIC.

4 Technically, $OAP_{d\text{-}a}$ and $OIC_{d\text{-}a}$ both imply logical possibility, whereas the sense of possibility relevant to freewill a stronger sense of personal possibility. The stronger sense implies the weaker sense of logical possibility, so the advocates of these stronger principles will accept the weaker principle. Moreover, the adversaries of these stronger principles will reject the weaker principles.

5 A more elaborate version, shortened here, is given in Haji's *Deontic Morality and Control* (2003, pg. 3)

principle, the forbidden ought not to be and vice versa, is uncontroversial. Therefore, if one accepts $OIC_{d\text{-}a}$, prohibitions imply alternative possibilities. Haji has argued and others have agreed (Zimmerman 2003, Kane 2000) that obligations also imply alternative possibilities i.e. that $OAP_{d\text{-}a}$ is valid. If freedom means that one could have done otherwise then why would only prohibitions require freedom but not obligations? Indeed, it is reasonable that freedom applies to all actions and not just prohibitions. "Why the asymmetry regarding metaphysical presuppositions of control? After all, wrongness and obligatoriness are normative appraisals of the same family; they are deontic normative statuses and thus it would seem that, barring cogent explanation to the contrary, the control-relevant [or freedom relevant] presuppositions of the one should also be those of the other." (Haji 2003, pg. 29)

There are three predominant views on freewill: the incompatibilist or the view that free will is not compatible with determinism and the compatibilist position, which views freedom and determinism as compatible. Incompatibilists are either hard-determinists or libertarian. Hard-determinists believe that determinism is true and so there is no freedom, while libertarians believe that determinism is false and that there is freedom. Compatibilists argue that freedom and determinism are compatible.

Like incompatibilists in general, hard determinists agree that freedom means one could have done otherwise. Persuaded by determinism, they accept the disastrous conclusion of the above argument; nothing is morally obligatory or forbidden. A libertarian accepts that obligation implies can and could have done otherwise. They save morality by rejecting determinism. This is Haji's stance.

Note that since this is a logical investigation, the examined sense of "could have done otherwise" is very weak. However, whatever stronger sense an incompatibilist wishes to argue for, may it be causal, ability or some other agential sense of "could have done otherwise" that sense will imply the logical sense, thereby committing incompatibilists in general to $OAP_{d\text{-}a}$, and $OIC_{d\text{-}a}$ and their logics.

On the other hand, compatibilists are generally skeptical of principles that require alternative possibilities. Persuaded by Frankfurt style cases, they deny PAP. Frankfurt cases are purported to show that agents sometimes could not have done otherwise yet are morally responsible (Frankfurt 1969). Thus, the compatibilist can avoid the disastrous conclusion of the above argument by denying that obligations and prohibitions imply could have done otherwise. John Fisher, a prominent compatibilist, has adopted this analogous response (Fischer 2000, pg. 361).

Whereas incompatibilists are committed to holding both $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ in the logical sense, it is not obvious what compatibilists are committed to. It is likely that compatibilists persuaded by Frankfurt cases would also accept logically necessary obligations, and thereby deny $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ but it does not follow that they must reject the logical sense of $OAP_{d\text{-}a}$. Consequently, the proposed logic E∇ (see section 1.6) is suitable for incompatibilists, while the standard deontic logic KD (see section 1.5) is suitable for compatibilists. However, some flavors of compatibilists i.e. those not persuaded by Frankfurt cases, could also adopt the E∇ logic. Either way, it is motivation for determining a deontic logic consistent with these views.

1.3.3 OI∇ in Theological Philosophy

Another motivation for developing the proposed logic is theologians and philosophers of religion often view contingency of obligations a conceptual truth. A contentious debate in this field of study is God's morality and freedom. Although the debate has a long history with proponents on either side, it has recently become active (Alston 1990). The status of God's morality plays a significant role in the central dilemma in philosophy of religion, namely Euthyphro's dilemma. Originating in Plato, the modern version of the dilemma is this. Is something right or wrong because God commands it, or does God command it because it is right or wrong? There are significant problems with both horns of this dilemma. If all there is to right and wrong is God's command then what is right and wrong seems arbitrary. Furthermore, it seems that God is omni-benevolent, only because he commands himself to do the right. On the other hand, if God commands something to be right, only because it is good (independently of his commands) then God's commands seem superfluous.

One answer to the first horn of the dilemma is that God is not good or evil; he is not the kind of thing that is moral. In other words, God has no obligations or prohibitions. Specifically relevant here is that commentators have argued for this position not based on peculiar theological principles but from conceptual truths of moral obligation. The basic moral claims have differed though what unites them is that nothing necessary can be obligatory.

One such principle is that obligation requires freedom i.e. the ability to do otherwise. Bruce R. Reichenbach in *Evil and a Good God*, argues that because God is necessarily good, in the strongest sense of the word, and since being good requires freedom i.e. the ability to do otherwise, God is not good or evil (1982). This is because God cannot do other than he does for he must do the good thing; he has no moral freedom. Alston gives the same argument, except he sees the relevant conceptual truth that obligations require the possibility of an opposition to what these obligations are (Alston 1990, pg. 308). Both views are committed to $OAP_{d\text{-}a}$.

1.4 Preliminaries for Deontic Logic

Three broad tasks are required to deontic logic, a consideration of the specific concepts involved, an examination of the kinds of sentences that will represent these concepts within the artificial language of logic, and finally a consideration into the nature of deontic logic itself. All three are large sub-fields of deontic logic, thus a brief overview of the issues and a statement of the position is all that is given. In 1.4.1, the relevant senses of obligation, permission and prohibition are outlined as well as necessity and possibility. Whether moral sentences are descriptive or prescriptive is considered in 1.4.2. The preliminaries of the formal work are in 1.4.3, i.e. the representation of deontic and alethic (logical) notions as operators. Section 1.4.4 examines a central issue concerned the nature of deontic logic, whether it is neutral to all substantive views or whether it is descriptive. Finally, in 1.4.5 the notation conventions used in the formal work are given.

1.4.1 Senses of 'Ought', 'May', 'Must' and 'Can'

Principles $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ express two distinct concepts: the concept of obligation (or moral necessity) and the concept of possibility. The English language contains many words and locutions that represent these notions, perhaps most frequently, 'must' and 'may'. Both express deontic concepts i.e. concepts of obligation and prohibition, as well as other forms of necessity, such as logical or physical necessity, possibility and impossibility. For example, one might say, "You must pay your fine," indicating that you are obligated (or morally necessitated) to pay the fine. Similarly, "you may park there" expresses a permission. On the other hand, 'must' in "light must travel at $3x10^8$ m/s in a vacuum" means physical necessity. Words like 'can', 'could', 'should' also have similar meanings and are just as ambiguous.

What is the problem with this ambiguity? First, given that the aim of deontic logic is a precise systematic study of the moral, it is essential that precise meanings of particular moral locutions are determined. This is a difficult task. Second, it is problematic to translate logical sentences, especially those that become more complex (e.g. iterated sentence such as $OOOA \rightarrow OO\Diamond A$) into meaningful sentences of English. These sentences are almost never uttered and even when they are, they are difficult to understand. This becomes particularly important when the logical principles discussed are being tested for correctness. Thus, it is doubtful that natural language is reliable is as a basis for judging putative principles, or as the source for precise moral terms (Hilpinen 1971). A better way to study deontic logic, understand its principles and judge their correctness is through modern semantic methods. The most well known semantic method is the model theoretic, specifically Kripke's standard models and Montague's minimal models. However, before discussing the semantics in sections 1.5.2 and 1.6.3, a brief informal explanation of the deontic and alethic notions is given.

The logical (or alethic) sense of possibility is the weakest sense of possibility, in that other senses of possibility imply the logical sense but not vice versa. Contrarily, the logical sense of necessity is the strongest. It is logically possible that "a turtle travels faster than the speed of light," and that "humans weigh over 1000 tons." Logical truths, such as "A → A," and analytic truths, such as "all bachelors are unmarried," are logically necessary. Furthermore, theorists often hold that mathematical truths are logically necessary. On the other hand, contradictions as in "it is raining and not raining," or sentences ascribing square-circle hood to rabbits are impossible. Intuitively, logical necessity can be thought of as truth in all possible worlds. At any possible world, "all bachelors are unmarried" will be true, for a bachelor cannot be married. A contradiction is false at all possible worlds while possible sentences are true at least one world.

The senses of obligation, permission and prohibition require more explanation. In English, an obligation is expressed using the words 'should', 'ought' or 'must'. However, they often express notions not related to morality. The word 'ought' is used to express expectation, as in "The couple ought to be arriving soon." (Harman 2000, pg. 5) This sense lacks any normative character. It is simply an indication that some event is expected to happen. The 'ought' of prudence, as in "you ought to watch your pennies" and the 'ought' of aesthetic judgments, for example, "You ought to have used red" are both normative but neither is moral, and so, not pertinent to this discussion. There are interesting logical relationships between these senses, especially between the prudential and the moral 'ought', however this is beyond the current scope.[6]

There are two different senses of obligation in the literature, and which sense, the ought-to-be or the ought-to-do, is the subject of deontic logic proper is

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[6] For a discussion on the relationship between the prudential and moral sense of 'ought' see Feldman (1986, pg. 106-10).

a contentious issue. Furthermore, the logical relationships between these two senses are disputed.

The ought-to-be sense states that some state of affairs should obtain. For example, "John ought to be more honest," or "the world ought to be such that all children do not suffer." The former is an illustration of an agent's obligation while the latter is an impersonal obligation or ideal. Contrarily, the ought-to-do always involves an agent. It ascribes some action or state of affairs that the agent is obligated to do. For example, "Jones ought to pay his taxes."

Whether there is a logical relationship between the two senses is controversial. Some have argued that there is no relationship between the ought-to-be and the ought-to-do and opt for developing different but complementary logics (Castaneda 1970). Those that argue for a logical relationship between the senses see it either way, the ought-to-do defined in terms of the ought-to-be, or vice versa.

The more commonly held view is adopted here, namely that the ought-to-do is reducible to the ought-to-be. In other words, the ought-to-be sense is more basic than the ought-to-do, since the latter can be defined in terms of the former but not the other way around. This view was advanced in the 1930s, by Nicolai Hartmann but refined by Roderick Chisholm in 1964. Chisholm suggests that the ought-to-do sense, "S ought to bring it about that p," can be defined as "It ought to be that S brings it about that p." (Chisholm 1964, pg. 150) The relation between the agent and some state of affairs can be represented as an ought-to-be where the state of affairs involves that agent. For example, "John ought to bring it about that the payment of his rent is made," is an ought-to-do sentence relating John to a state of affairs. Chisholm analyzes this sentence as "it ought to be the case that Jones pays his rent." Thus, the ought-to-be sense of obligation can express both impersonal senses of 'ought' and those related to agents. Having one sense that is reducible to the other has the advantage that it requires one logic, which is more economical from a theoretical point of view. More importantly, the ought-to-be is considered the standard interpretation of obligation, and this is an advantage because a comparison of the proposed logic with the current norm is more useful.

1.4.2 Are moral statements prescriptive or descriptive?

Moral discourse is often expressed in terms of imperatives. Imperatives are directives, which do not express states of affairs and thus are neither true nor false. Consider the following examples:

I. You should drive on the left!
II. You ought to drive on the left.

Sentence (I) is an imperative expressing an order. It directs some agent to an action. Because it directs, it neither affirms nor denies a fact, and hence, prima facie, cannot be assigned a truth value. Sentence (II), an assertive or indicative sentence, is a different linguistic type altogether. This sentence describes a fact that ought to happen. The sentence can be ascribed a truth value; it is true in Britain and false in Canada.

Because imperatives cannot be ascribed truth values, the usual meta-logical concepts, such as validity, logical consequence and consistency are inappropriate. The latter are typically relationships between sentences (or propositions) that are truth bearers. For this reason, imperatives are not the proper subject matter of logic. On the other hand, 'ought' sentences are naturally thought of as imperatives and it is clear that imperatives can be inferred. The imperative, "You should drive on the left side of the road and pay attention!" entails, in some sense of imperative entailment, "You should drive on the left side of the road!" (Hilpinen 2001) These inferences suggest that there is a logic of imperatives. One way to resolve what has become know as Jorgensen's Dilemma is to deny that 'ought' statements are imperatives, in which case a voluminous portion of moral language needs to be accounted for (Hilpinen 2001). If these are not 'ought' statements, what are they? The standard solution to this problem, adopted here, is that imperatives can themselves be viewed as indicatives. Sentence (II) is the indicative description of the imperative sentence (I). The latter is either true or false. The same can be said for prohibitions. However, sentences expressing permissions seem to be indicative and do not require further representation. "Driving on the right side is permitted" is true in Canada, while false in Britain. Represented as indicatives, inferences of imperatives may use standard meta-logical concepts of truth, validity and consistency.

1.4.3 Logical Operators

Thus far, the meanings of deontic and alethic concepts were briefly described. Each of those senses requires an appropriate locution in the artificial logical language. Obligation, permission and prohibition are expressed using modal one-place operators in the artificial language of logic. Deontic concepts are expressed using 'O', 'P' and 'F', while the alethic concepts are expressed using '□', and '◊'. Informally, the operators are read as follows:

Figure 1.4.1 Deontic and Alethic Operators
Where ... is some sentence,
'O...' "It ought to be that ..."
'F...' "It is prohibited that ..."
'P...' "It is permitted that ..."
'□...' "It is logically necessary that ..."
'◊...' "It is logically possible that ..."

The domain and range of the operators are indicative sentences. For example, the range of 'O', in 'OA' "It is obligatory that A," is an indicative sentence and the domain, the sentence 'A', is also an indicative sentence. The domain and range of 'P' and 'F' are also indicative sentences. This is the typical syntactic representation in deontic logic.

Likewise, alethic concepts of necessity, possibility and impossibility are represented as one-place modal operators. Both operators have a domain and range of indicative sentences.

1.4.4 Logic as Topic-Neutral or Descriptive

It is sometimes remarked that every deontic logician has his own deontic logic. The lack of consensus is partly because a large number of paradoxes exist, the dispute over the kinds of sentences involved in deontic logic (whether imperatives or indicatives) is not settled, and the validity of central principles are disputed.

Deontic logicians often dismiss valid principles from deontic logic because they are contrary to tenable substantive views in ethics. On the other hand, they are quite content when they find a purported case of a principle that "cuts across all morality."[7] Those who argue in this vein are committed to the view that deontic logic is topic-neutral, that is "deontic logic ought to be neutral between competing moral theories." (Sayre-McCord 1986, pg. 179)

On this view, substantive views do not undercut principles of moral reasoning or moral discourse. Whether one is an ethical rationalist, divine command theorist or naturalist, one is still engaged in moral reasoning. In that case, some set of valid principles common to all substantive theories is expected. If some alleged theories or moral arguments lead to a denial of one of these principles, the alleged theory is discarded. Just as in propositional logic, if some particular, say, metaphysical view commits one to denying modus ponens, it is the theory that is rejected and its reasoning deemed flawed.

The difference between deontic logic, and propositional logic or modal logics of necessity is that in the former there is much more debate as to which principles really are valid. The vast variety of ethical views held over the course of history leave little doubt that in a topic-neutral deontic logic not a single principle remains. For example, Calvinists held that due to our evil nature certain obligations like being a good person could not be fulfilled. On the other hand, Leibniz argues that all obligations in the actual world are fulfilled, for our world

[7] Schotch and Jennings (1981) seem to be implicitly committed to the topic-neutrality thesis. See comment on $Con_d\ \neg O\bot$ which is valid on their account because it "cuts across all moral theories."

is the best of all possible worlds.[8] The theories could not be more distinct, and it is unlikely that a deontic logic can be neutral between them. Nor is this peculiar to idiosyncratic philosophers of the past; it is true of contemporary meta-ethics as well. A major point of contention is the issue of moral dilemmas (as we shall see in Chapter 2.3). The existence of moral dilemmas results in a score of principles excluded from deontic logic, and puts the entire enterprise of a topic-neutral deontic logic into question.

If this is correct, topic-neutrality is not a tenable requirement for any foreseeable deontic logic. Furthermore, the ideal of one deontic logic capturing the "underlying structure, of our moral discourse" is not tenable either. Thus, there is no neutral battleground for moral debate (Sayre-McCord 1986, pg. 179).

What then is the purpose of a deontic logic? An alternative view provides a more descriptive function. As Hansson (1971) and Sayre-McCord (1986) argue, deontic logic is a tool for meta-ethics and ethics proper. It is a rigorous framework for showing inferences in moral discourse but it does not claim to capture inferences for any substantive view. Therefore, deontic logic cannot reject an ethical view, just show whether it follows from it or not. Nevertheless, we can evaluate deontic logics in a limited way by comparing them. A logic is advantageous over another if it is broader in scope, i.e. it encompasses a greater number of ethical theories, especially those prominent in the literature. Moreover,

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[8] Ruth Marcus in *Iterated Deontic Obligations* briefly mentions this consequence of Leibniz' theory (Marcus 1966). Leibniz' view makes the actual world the ideal world, i.e. the best of all possible worlds, and hence the world which is the standard of obligation for all other worlds. In the actual world the principle $OA \rightarrow A$ is true because our world is the ideal world and so whatever is true at our world is ideal.

given equal substantive claims, a logic is advantageous if it is stronger than another. A deontic logic also reveals subtle consequences of substantive views perhaps not apparent at first sight. This thesis examines these issues, by comparing the proposed system E∇ with the established system KD.

1.4.5 Notation / Nomenclature

In summary, in section 1.4, the relationship between the logical artificial language and the pertinent normative and alethic discourse was established. The broad proto-semantic meanings of obligation, permission and prohibition were specified as well as those of logical (or alethic) necessity, possibility and impossibility. The logical operators that represented these concepts were shown to be modal one-place operators with a domain and range of assertive sentences. Before describing established systems of deontic logic, this section describes the general logical notation.

The notation adopted here is an extension of Chellas' *Modal Logic* text. The following examples illustrate the notation:

Figure 1.4.2 Examples Illustrating Notation Style

Type	*Label*	*Principle*
deontic schema	M_d	$O(A \wedge B) \rightarrow OA \wedge OB$
alethic schema	4_a	$\Box A \rightarrow \Box\Box A$
deontic-alethic schema	$NAT_{d\text{-}a}$	$OA \rightarrow \Diamond\neg OA$
deontic definition	Df. P_d	$PA \leftrightarrow \neg O \neg A$
deontic rule of inference	RE_d	$A \leftrightarrow B / OA \leftrightarrow OB$
semantic constraints	m_d)	If $\| A \cap B \| \in N^d_\alpha$ then $\| A \| \in N^d_\alpha$ and $\| B \| \in N^d_\alpha$

Syntactic principles (i.e. schemas and rules of inference) are labeled with a capital letter. The labels are from Chellas with a few exceptions, most notably principles that have mixed modalities. Definitions are indicated by the acronym 'Df.' followed by the principle label, while schemas and instances of schemas are indicated by the principle label alone. Similarly, semantic constraints are indicated by the small case letters of the syntactic principle they validate. In Figure 1.4.2, the semantic constraint m_d) validates the principle M_d. For both semantics and syntax, the subscripts indicate whether the definition, schema or schema instance is one with strictly deontic modalities (subscript 'd'), alethic modalities (subscript 'a'), or both (subscript 'd-a').

The names of the logical systems correspond to the schemas and/or rules of inference that constitute the systems. For example,

ED_d
$E_dOIC_{d\text{-}a}KT5_a$

The first system contains the schema D_d and the rule of inference RE_d. The system is defined on a language of only deontic modalities hence the subscript 'd'. The second indicates a system with the rule of inference RE_d, a deontic-alethic schema $OIC_{d\text{-}a}$, and alethic K_a, T_a and 5_a schemas.

1.5 KD and KD+

The modern conception of deontic logic began with von Wright's famous paper *Deontic Logic* in 1951. Leibniz in the 17^{th} century developed an informal system, and late medieval philosophers discussed principles of deontic logic as early as the 14^{th} century, but von Wright's paper created an outpouring of criticisms and improvements, establishing a legitimate field of study. Prior, Anderson, Kanger and others were among those who immediately contributed to what is now a vast literature on deontic logic. A system emerged, similar to von Wright's, called "standard deontic logic," often just 'SDL', which served as a benchmark for new deontic logics.

Before beginning the exposition of SDL, misleading terminology should be cleared up. In this thesis, following Chellas (1980, ch. 4), the "standard deontic logic" system will be referred to as the "normal deontic system," abbreviated as 'KD_d' (without the subscript, where obvious). This naming convention is somewhat misleading as Chellas' semantics are called "standard models," but this should not be confused with 'SDL', the standard deontic logic system. To avoid confusion, the label 'SDL' is abandoned and only Chellas' terminology used. 'KD_d' or "the normal deontic system" refers to the axiomatic framework, while 'SDM' stands for the standard deontic model. An extension of KD_d, $KD+_{d\text{-}a}$, is also discussed. KD_d expresses only deontic concepts, while $KD+_{d\text{-}a}$ expresses alethic concepts as well. Both will serve as benchmarks for several proposed logics, the weakest of which is named '$E\nabla_{d\text{-}a}$'. The following section will present the syntax and the semantics of KD_d and $KD+_{d\text{-}a}$. In section 1.6, $E\nabla_{d\text{-}a}$ is presented. It is the weakest basis for the proposed systems. An exposition of its syntax, some principles and semantics ends chapter 1.

1.5.1 Syntax

A language is defined using a vocabulary and a recursive definition of a sentence. Two different languages, $\mathcal{L}$ and $\mathcal{L}'$, are needed. The former expresses propositional logic sentences and deontic sentences, while the latter also expresses alethic sentences.

Definition 1.5.1 $\mathcal{L}$ Vocabulary

1. A denumerable set of atomic sentences: $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$, ...[9]
2. Zero-place operators: $\top$, $\bot$
3. One-place operators: $\neg$, O, P, F
4. Two-place operators: $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
5. Auxiliary symbols: (,)

Using this vocabulary all the sentences of language $\mathcal{L}$ are recursively defined:

Definition 1.5.2 $\mathcal{L}$ Language

1. $\mathbb{P}_n$ is a sentence, for n = 0, 1, ...
2. $\bot$, $\top$ are sentences
3. If A is a sentence then $\neg$A, OA, PA, and FA are sentences
4. If A and B are sentences then (A $\wedge$ B), (A $\vee$ B), (A $\rightarrow$ B), and (A $\leftrightarrow$ B) are sentences
5. Nothing else is a sentence

[9] 'Denumerable' means that there is a one to one correspondence from some set (in this case the set of atomic sentences) to the natural numbers

The second language, $\mathcal{L}'$ has the same vocabulary as $\mathcal{L}$, with the addition of '$\Box$' and '$\Diamond$'.

Definition 1.5.3 $\mathcal{L}'$ Vocabulary

1. All conditions of $\mathcal{L}$ Vocabulary - Definition 1.5.1
2. One-place operators: $\Box$, $\Diamond$,

Every sentence of $\mathcal{L}'$ can be defined as follows:

Definition 1.5.4 $\mathcal{L}'$ Language

1. All conditions on $\mathcal{L}$ Language – Definition 1.5.2
2. If A is a sentence then $\Box$A, $\Diamond$A are sentences

1.5.2 Systems

A *system* is a set of sentences. In particular, a set of sentences is a *system of modal logic* if and only if it is closed under the rule of inference RPL. 'RPL' states that whenever a set of sentences contains A_1... A_n then it contains A where A is a tautological consequence of A_1, ..., A_n. Thus, the smallest modal logic is the system PL, the set of all tautologies of propositional logic. Note that in annotating proofs 'PL' will indicate either some member of the system PL, i.e. some tautology, or an inference using the rule RPL. A *schema* is simply a set of sentences that share a certain syntactic form and a *schema instance* is a member of that set.

Definition 1.5.5 Modal Logic System

A set of sentences is a *system of modal logic* iff it is closed under RPL.

The KD_d system is defined on the language $\mathcal{L}$ (Definition 1.5.2). It consists of several schemas and rules of inference.

Definition 1.5.6 Normal Deontic System (KD_d)

1. PL All tautologies of propositional logic
2. MP $A \rightarrow B, A / B$
3. Df. O_d $OA \leftrightarrow \neg P \neg A$
4. Df. F_d $FA \leftrightarrow \neg PA$
5. D_d $OA \rightarrow PA$
6. K_d $O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$
7. RN_d $\top / O\top$

This system yields several theorems and derived rules of inference.

Theorems 1.5.1 KD_d Theorems

1. RE_d $A \leftrightarrow B / OA \leftrightarrow OB$
2. REP_d $B \leftrightarrow B' / A \leftrightarrow A[B/B']$
3. C_d $OA \wedge OB \rightarrow O(A \wedge B)$
4. M_d $O(A \wedge B) \rightarrow OA \wedge OB$
5. RM_d $A \rightarrow B / OA \rightarrow OB$
6. Con_d $\neg O\bot$
7. N_d $O\top$
8. $OA \rightarrow O(A \vee B)$
9. $FA \rightarrow F(A \wedge B)$
10. $PA \rightarrow P(A \vee B)$
11. $OA \vee OB \rightarrow O(A \vee B)$
12. $P(A \wedge B) \rightarrow PA \wedge PB$
13. $A \rightarrow B / FB \rightarrow FA$
14. $O(A \wedge B) \rightarrow OA$
15. RP_d A / PA

These principles will appear at some point during this thesis. They are listed here for reference purposes and their proofs given in the appendix, in order, 1.5.1-App – 1.5.15-App. Their explanation is deferred for the appropriate time, with the exception of REP_d. The rule of replacement allows for the substitution of logically equivalent principles within other principles. This is true of both substitutions inside propositional logic sentences as well as within modal sentences. Thus, the sentence '$\neg A$' can be replaced within '$OA \rightarrow \neg O \neg A$', resulting in '$OA \rightarrow \neg O(A \rightarrow \neg A)$,' because '$\neg A$' is logically equivalent with 'A

$\rightarrow \neg A$.' It is because of RE_d that replacement is possible within modalities (see modal case of REP_d proof 1.5.2-App).[10]

KD_d expresses only deontic concepts of obligation, permission and prohibition. It is not rich enough to express concepts of logical or alethic necessity. It is adequate neither for 'ought' implies 'can' nor 'ought' implies 'can not', because these principles express both deontic and alethic notions. However, these principles can be expressed in the language $\mathcal{L}'$ (Definition 1.5.4). The weakest proposed system, $E\nabla_{d\text{-}a}$, and its extensions are defined on $\mathcal{L}'$, as is $KD+_{d\text{-}a}$, an extension of KD_d.

The system $KD+_{d\text{-}a}$ is significant for two reasons. First, it is contained within Anderson's reduction of deontic logic to alethic logic (Proof 1.5.16-App); a system that has endured since the 50's and continues to be discussed in recent literature (Anderson 1966, Aqvist 2002). Although the intricacies of this particular logic are beyond the scope of this thesis, its longevity gives force to the principles contained therein, and so to $KD+_{d\text{-}a}$ principles. Second, a system that can express mixed principles can serve as a better system of comparison to $E\nabla_{d\text{-}a}$ and its extensions. KD+ is chosen in such a way that someone who held KD_d would likely hold $KD+_{d\text{-}a}$. For example, in KD_d principle N_d $O\top$ is valid, stating that tautologies are obligatory. In $KD+_{d\text{-}a}$ the principle $MIO_{d\text{-}a}$ $\Box A \rightarrow OA$ is valid,

[10] It is also the reason that obligations are not propositional attitudes. For propositional attitudes the rule of extensionality fails, as in "Lois Lane believes that Superman can leap over tall building, but she does not believe that Clark Kent can leap over tall buildings, even though the two are the same persons, and the propositions, Clark can leap over tall buildings is logically equivalent with Superman can leap over tall building."

which says that not only are tautologies obligatory but also all logically necessary sentences. Another instance is Con_d $\neg O\bot$, which is valid in KD_d. Con_d is a weak version of $OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$, a theorem of $KD+_{d\text{-}a}$.

Schemas for the logical sense of necessity are given in Definition 1.5.7. The system is equivalent with Lewis' famous S5 system.

Definition 1.5.7 $KT5_a$ System

1. Df. $\Box_a$ $\Box A \leftrightarrow \neg\Diamond\neg A$
2. K_a $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
3. T_a $\Box A \rightarrow A$
4. 5_a $\Diamond A \rightarrow \Box\Diamond A$
5. RN_a $\top / \Box\top$

The $KT5_a$ system yields the following theorems.

Theorems 1.5.2 $KT5_a$ Theorems

1. C_a $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$
2. M_a $\Box(A \wedge B) \rightarrow \Box A \wedge \Box B$
3. RM_a $A \rightarrow B / \Box A \rightarrow \Box B$
4. RE_a $A \leftrightarrow B / \Box A \leftrightarrow \Box B$
5. $4\Diamond C_a$ $\Diamond\Diamond A \rightarrow \Diamond A$
6. 5_a $\Diamond A \rightarrow \Box\Diamond A$
7. $\Box A \leftrightarrow \Box\Box A$
8. $\Box A \leftrightarrow \Diamond\Box A$
9. $\Diamond A \leftrightarrow \Diamond\Diamond A$
10. $\Diamond A \leftrightarrow \Box\Diamond A$
11. $\Diamond M_a$ $\Diamond(A \wedge B) \rightarrow \Diamond A \wedge \Diamond B$

Alethic schemas and theorems are not contested in this thesis. They are listed for reference purposes. The extension $KD+_{d\text{-}a}$ are now defined as:

Definition 1.5.8 Normal Deontic-Alethic System ($KD+_{d\text{-}a}$)

1. All schemas, definitions and rules of KD_d (Definition 1.5.6)
2. All schemas, definitions and rules of $KT5_a$ (Definition 1.5.7)
3. $MIO_{d\text{-}a}$ $\Box A \rightarrow OA$

Some additional theorems in KD+ are the following:

Theorems 1.5.3

1. Theorems of KD_d (Theorems 1.5.1)
2. Theorems of $KT5_a$ (Theorems 1.5.2)
3. $REP_{d\text{-}a}$ $B \leftrightarrow B' / A \leftrightarrow A[B/B']$
4. $OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$
5. $ME_{d\text{-}a}$ $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$
6. $\Box A \wedge OA \rightarrow \Box OA$
7. $PA \rightarrow \Diamond A$
8. $\Box A \rightarrow PA$

Further explanation of these theorems is delayed until chapter 2. Their proofs are in the appendix, Theorem (3) at 1.5.17-App – Theorem (9) at 1.5.22-App. The rule $REP_{d\text{-}a}$ is the same as REP_d except that the replacements can also be done within alethic modalities.

1.5.3 Semantics

The semantics for KD_d are given by a model constituted by a set W, a binary relation R, and a function P.

Definition 1.5.9 Standard Deontic Model (SDM)

M = <W, R, P> is a *standard deontic model* iff

1. W is a non-empty set
2. P(n) is a function from natural numbers to subsets of W (i.e. $P: \mathbb{N} \rightarrow W$)
3. R is a *serial* binary relation on W, i.e. for every $\alpha \in W$ in M, there is at least one $\beta \in W$, such that $\alpha R \beta$

The set W can be thought of as a set of possible worlds. A sentence is said to be true (or valid in the lowest degree) in a particular model M at a particular world α, abbreviated $\models^M_\alpha A$, where A is any sentence, and α is a member of W. Validity (or highest degree of validity) is expressed as truth at all worlds at every model.

Definition 1.5.10 Truth of a Sentence

$\models^M_\alpha$ A, abbr. sentence A is true at world α in model M

Definition 1.5.11 Validity

$\models$ A iff for every model M and every world α, $\models^M_\alpha A$

Another element of the model, P(n) is a function from the natural numbers to subsets of W. It delineates the atomic sentences in the model. Intuitively, the atomic sentences are the set of worlds at which they are true.

The truth conditions for propositional logic sentences are as follows:

Definition 1.5.12 Truth Conditions for Propositional Sentences

Let A be a sentence

Let $\alpha \in$ W in M

The truth conditions are stated as follows:

1. $\models^M_\alpha \mathbb{P}_n$ iff $\alpha \in P(n)$, for n = 0, 1, ...
2. $\models^M_\alpha \top$
3. Not $\models^M_\alpha \bot$
4. $\models^M_\alpha \neg A$ iff not $\models^M_\alpha A$
5. $\models^M_\alpha A \wedge B$ iff both $\models^M_\alpha A$ and $\models^M_\alpha B$
6. $\models^M_\alpha A \vee B$ iff either $\models^M_\alpha A$ or $\models^M_\alpha B$, or both
7. $\models^M_\alpha A \rightarrow B$ iff if $\models^M_\alpha A$ then $\models^M_\alpha B$
8. $\models^M_\alpha A \leftrightarrow B$ iff $\models^M_\alpha A$ iff $\models^M_\alpha B$

Condition 1 states that the atomic sentence in language $\mathcal{L}$ at a particular world α is true if and only if the particular world α belongs to the corresponding subset of W, i.e. P(n). Thus, the function P(n) is all that is needed in the model to determine the truth values of the atomic sentences. With the meanings of the logical connectives, the truth of all other complex non-modal sentences of $\mathcal{L}$ in M can be determined (Conditions 2-8). This is the familiar notion of truth-functionality, that all sentences of propositional logic can be determined strictly from the meanings of the logical connectives ($\wedge, \vee, \leftrightarrow, \rightarrow, \neg$) and their atomic sentences. On the other hand, the truth of modal sentences cannot be determined by the truth of atomic sentences and the meanings of logical connectives alone. The relation R between possible worlds is used to determine the truth value of modal sentences. The deontic sentence 'OA' is true at world α in a particular model M if and only if in every related world, A is true at that world.

Definition 1.5.13 Truth Conditions for Deontic Sentences

1. $\models^M_\alpha$ OA iff for every β in M such that $\alpha R\beta$, $\models^M_\beta$ A
2. $\models^M_\alpha$ PA iff for some β in M such that $\alpha R\beta$, $\models^M_\beta$ A

The relation R has several names: "alternativeness", "accessibility" or "sees it" relation. One kind of necessity is truth in all possible worlds, however there are other senses of necessity, such as physical necessity, deontic necessity, and temporal necessity. The specific sense is given by the relation R. In the deontic case (SDM – Definition 1.5.9), it is a serial relation. A serial relation is one where for every world α in M there is at least one world β such that $\alpha R\beta$. Every world has at least one deontic alternative. To pay your taxes, for example, is true in all deontic alternatives to this world, making tax payments obligatory at the current world. Parking ones car in a particular spot is permitted because it occurs as some morally accessible worlds.

Jaakko Hintikka developed an intuitive interpretation of SDM to aid in thinking about the models and the principles therein. He gives the following: OA is true at a possible world α iff A is true in all ideal worlds relative to α (Hintikka 1969). PA is true at a possible world α iff A is true some ideal worlds relative to

α. The intuitive description has the following properties. As an example, the principle OA → ¬O¬A is valid in standard deontic models. It states that whenever it ought to be that something is the case, the opposite cannot be obligatory. Suppose that A is obligatory at some world α then at all worlds deontically accessible from α, A is true. Whatever these worlds are A is true there, and hence ¬A cannot be true, since A and ¬A is not true in any world. However, ¬A is required at the deontic alternative worlds for O¬A to be true at α. Thus, a counter-example is not possible, i.e. OA ∧ O¬A. Therefore, OA → ¬O¬A is valid.

1.6 Proposed System E∇

1.6.1 Principles

The syntax of the weakest proposed system, E∇, is the same as that of KD+. The language $\mathcal{L}'$ (Definition 1.5.4) is defined using the vocabulary of both alethic and deontic operators (Definition 1.5.3). Instead of giving a long list of invalid deontic principles, it is easier to suggest obviously valid principles. As a starting point, some straightforward consequences of the principles motivated in section 1.3.2 will be shown, postponing the philosophical implication of these and other principles for chapter 2.

First, deontic logic is an extension of propositional logic. In $E\nabla_{d\text{-}a}$ and all its extensions, all tautologies of propositional logic are valid and all sentences are closed under propositional rules of inference. In other words, it is a modal logic system (Definition 1.5.5). The most basic and most common set of principles in deontic logic are those stating the relationship between obligation, permission and forbiddance.

Figure 1.6.1 Principles of Obligation, Permission and Prohibition

1. Df. O_d $OA \leftrightarrow \neg P \neg A$
2. Df. F_d $FA \leftrightarrow \neg PA$
3. Df. P_d $PA \leftrightarrow \neg O \neg A$
4. $FA \leftrightarrow O \neg A$
5. $PA \leftrightarrow \neg FA$

The first principle captures this equivalence. John ought to pay tax means the same as for him not to pay tax is impermissible. Principle (2) says that forbiddance (or interchangeably prohibition) is equivalent to the impermissible. An example of principle (3) is "if gardening is permissible then not gardening is not obligatory, and vise versa." Notice that if principles (1) and (2) are schemas, (3)-(5) can be derived from them (Appendix Proofs 1.6.1-App – 1.6.3-App). Principle (4) says that what is forbidden (or interchangeably, prohibited) is what ought not to be, and vice versa. The last principle says that what is permissible is not forbidden and vice versa.

The principles outlined at the very beginning of this essay are:

1. $OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$
2. $OAP_{d\text{-}a} OA \rightarrow \Diamond \neg A$
3. $OI\nabla_{d\text{-}a}$ $OA \rightarrow \Diamond A \wedge \Diamond \neg A$
4. $FIC_{d\text{-}a}$ $FA \rightarrow \Diamond A$
5. $FAP_{d\text{-}a}$ $FA \rightarrow \Diamond \neg A$
6. $FI\nabla_{d\text{-}a}$ $FA \rightarrow \Diamond A \wedge \Diamond \neg A$

Principles $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ are 'ought' implies 'can' and that 'ought' implies 'can not', and $OI\nabla_{d\text{-}a}$ the combination of the two, 'ought' implies 'contingency'. In $E\nabla$ systems, $FIC_{d\text{-}a}$ and $FAP_{d\text{-}a}$ follow from $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$

and the schema $FA \leftrightarrow \neg PA$, and together imply $FI\nabla_{d\text{-}a}$. A contingent sentence is one that can be true and can be false. Note that the contingency of obligations should not be confused with the validity of the principles.

1.6.2 System

The axiomatization of the weakest basis for the systems proposed is the system $E_dOI\nabla_{d\text{-}a}KT5_a$, abbreviated '$E\nabla$'.

Definition 1.6.1 System $E_dOI\nabla_{d\text{-}a}KT5_a$ (abbr. $E\nabla$)

1. All schemas, definitions and rules of $KT5_a$ (Definition 1.5.7)
2. Df. O_d $OA \leftrightarrow \neg P \neg A$
3. Df. F_d $FA \leftrightarrow \neg PA$
4. RE_d $A \leftrightarrow B \,/\, OA \leftrightarrow OB$
5. $OI\nabla_{d\text{-}a}$ $OA \rightarrow \Diamond A \wedge \Diamond \neg A$

It contains basic notions such as the inter-definability of obligation, permission and forbiddance, alethic principles for $KT5_a$, and mixed deontic-alethic principle $OI\nabla_{d\text{-}a}$. In this very weak system, the following theorems are derivable.

Theorems 1.6.1 System $E\nabla$

1. All theorems of $KT5_a$ (Theorems 1.5.2)
2. $OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$
3. $OAP_{d\text{-}a}$$OA \rightarrow \Diamond \neg A$
4. $FIC_{d\text{-}a}$ $FA \rightarrow \Diamond A$
5. $FAP_{d\text{-}a}$ $FA \rightarrow \Diamond \neg A$
6. Con_d $\neg O \bot$
7. $\neg O \top$
8. $\neg F \bot$
9. $\neg \Diamond A \rightarrow PA$
10. $P \bot$
11. $R \neg O_d$ $A \,/\, \neg OA$
12. $REP_{d\text{-}a}$ $B \leftrightarrow B' \,/\, A \leftrightarrow A[B/B']$

Principles (2) and (3) follow from $OI\nabla_{d\text{-}a}$. Principles (4) and (5) say the same thing for prohibitions. Con_d follows from $OIC_{d\text{-}a}$ and PL. Principles (7) – (11) are proven in the appendix, Proof 1.6.4-App – Proof 1.6.8-App and the general rule of replacement for both deontic and alethic modalities, $REP_{d\text{-}a}$, is proven in 1.5.17-App. This very weak system will be strengthened as further principles are considered in Chapter 2. It is offered as a common basis for other stronger systems.

There are principles, however, that cannot be valid even in this system. Since prohibitions and obligations are both contingent, it might be thought that permissions, too, are contingent, i.e. that the principle, $PA \rightarrow \Diamond A \wedge \Diamond\neg A$ is valid. However, this is not the case. Consider one part of the principle that permissions are contingent, i.e. that permissions imply can.

Proof 1.6.9 Inconsistency { $\Box A$, $PA \rightarrow \Diamond A$, $OAP_{d\text{-}a}$ }

1.	$\Box A$	Assumption
2.	$P\neg A \rightarrow \Diamond\neg A$	$PA \rightarrow \Diamond A$ instance
3.	$\neg\Diamond\neg A \rightarrow \neg P\neg A$	2, PL
4.	$\Box A \rightarrow \neg P\neg A$	3, Df. $\Box_a$
5.	$\Box A \rightarrow OA$	4, Df. O_d
6.	$OA \rightarrow \Diamond\neg A$	$OAP_{d\text{-}a}$
7.	$\Box A \rightarrow \Diamond\neg A$	5, 6 PL
8.	$\Diamond\neg A$	1, 7 MP
9.	$\neg\Diamond\neg A$	1, Df. $\Box_a$

The contradiction on lines 8 and 9 shows that $OAP_{d\text{-}a}$ and necessary sentences are not consistent with $PA \rightarrow \Diamond A$. The reason for the inconsistency is not that no permission is possible, but just that the principle $PA \rightarrow \Diamond A$ is not valid, that is not every permission is possible, some are impossible. This is clear from $P\bot$, a theorem of $E\nabla$. The plausibility of this result is taken up in chapter 2.1.2.

Further, permissions do not imply cannot, i.e. $PA \rightarrow \lozenge\neg A$, since $OIC_{d\text{-}a}$ is valid:

Proof 1.6.10: Inconsistency { $\square A$, $PA \rightarrow \lozenge\neg A$, $OIC_{d\text{-}a}$}

1.	$\square A$	Assumption
2.	$PA \rightarrow \lozenge\neg A$	$PA \rightarrow \lozenge\neg A$
3.	$\neg\lozenge\neg A \rightarrow \neg PA$	2, PL
4.	$\neg\lozenge\neg A \rightarrow \neg\neg O\neg A$	3, Df. P_d
5.	$\neg\lozenge\neg A \rightarrow O\neg A$	4, PL
6.	$\square A \rightarrow O\neg A$	Df. $\square_a$
7.	$O\neg A \rightarrow \lozenge\neg A$	$OIC_{d\text{-}a}$ instance
8.	$\square A \rightarrow \lozenge\neg A$	6, 7 PL
9.	$\lozenge\neg A$	1, 8 MP
10.	$\neg\lozenge\neg A$	1, Df. $\square_a$

Although it may initially seem plausible to have symmetry between obligation, permission and prohibition i.e. for all three to imply contingency, under closer inspection it is undesirable. The principle $PA \rightarrow \lozenge\neg A$ is clearly false, for it is equivalent with the principle $\square A \rightarrow \neg PA$, which must be false. This principle is counter-intuitive and it is enough to see that permissions do not imply contingency.

$MIO_{d\text{-}a}$ $\square A \rightarrow OA$ is another invalid principle if obligations are contingent. The principle says that all necessary sentences are obligatory. $MIO_{d\text{-}a}$ is a schema in KD+ and it is a theorem of Anderson's reduction of deontic logic to alethic.

Proof 1.6.11 Inconsistency {$MIO_{d\text{-}a}$, $OAP_{d\text{-}a}$}

1.	$\square A$	Assumption
2.	$\square A \rightarrow OA$	$MIO_{d\text{-}a}$ instance
3.	$OA \rightarrow \lozenge\neg A$	$OAP_{d\text{-}a}$ instance
4.	$\square A \rightarrow \lozenge\neg A$	2, 3 PL
5.	$\lozenge\neg A$	1, 4 MP
6.	$\neg\lozenge\neg A$	1, Df. $\square_a$

The inconsistency can be shown for the rule of O-necessitation as well, which states that all tautologies are obligatory, i.e. A / OA (Proof 1.6.12-App). These remarks are meant to be preliminary, showing basic valid and invalid principles in the proposed system.

1.6.3 Semantics

Standard models are appropriate for KD but since they must validate principles invalid in E∇, they are not suitable for E∇. In all standard models, the rule RN_d A / OA is closed under all tautologies. A sentence is obligatory at some world only if that sentence is at all deontic alternative worlds. However, since tautologies and more broadly necessary sentences are true at all possible worlds, they are true at all the deontic alternative worlds. Thus, tautologies will always be obligatory. Since standard models are inadequate, the description offered in terms of ideal worlds is also inadequate. Recall, that an obligation is true if and only if it is true at all ideal worlds. The right-to-left conditional fails because what is true at an ideal world, namely a necessary sentence, is not obligatory. Thus, when describing the semantics of $E\nabla_{d\text{-}a}$ ideal world descriptions are not used.

Minimal models (sometimes called neighborhood semantics or Montague-Scott semantics) are a generalization of normal models. Minimal models still rely on possible worlds and while the truth conditions for propositional logic sentences are determined in the same way, the truth conditions for necessity (i.e. deontic necessity) are determined differently. Instead of an accessibility relation between worlds, minimal models use a set of propositions (or sets of worlds) at each world to indicate that those propositions are, in whatever sense, necessary at that world.

The result is that the weakest model validates fewer principles compared to normal models.

The set of possible worlds, W, and the function P remains the same as for standard models. The relation R is replaced with a set of propositions (sets of worlds) N^d and N^a. N^d is the set of obligatory propositions, while N^a is the set of alethic propositions at a particular world α.

Definition 1.6.2 Minimal Deontic-Alethic Model

Where $\wp$ is the power set,

$M = \langle W, N^d, N^a, P \rangle$

1. W is a set
2. P(n) is a function from natural numbers to subsets of W
   $P: \mathbb{N} \rightarrow W$
3. N^d is a mapping from the natural numbers to sets of subsets of W i.e. $N^d{}_\alpha \subseteq \wp(W)$, for each world α in W
4. N^a is a mapping from the natural numbers to sets of subsets of W i.e. $N^a{}_\alpha \subseteq \wp(W)$, for each world α in W

To explain how N^d and N^a facilitate deontic and alethic necessity, the notion of a truth set is required. A truth set, $\| A \|$, is the set of worlds where A is true (Chellas 1980, ch. 2). Intuitively, a truth set is a proposition expressed by the sentence A. This is in the nominalist tradition that views propositions as sets of possible worlds rather than abstract entities envisioned by realists. Although nominalist, the current account is neutral between nominalists that view propositions as sets of real possible worlds, a la David Lewis, and those that view possible worlds as consistent representations.

Definition 1.6.3 Truth Sets

1. $\| \mathbb{P}_n \|^M = P(n)$, for n = 0, 1, 2, …
2. $\| \top \|^M = W$
3. $\| \bot \|^M = \varnothing$
4. $\| \neg A \|^M = - \| A \|^M$
5. $\| A \wedge B \|^M = \| A \|^M \cap \| B \|^M$
6. $\| A \vee B \|^M = \| A \|^M \cup \| B \|^M$
7. $\| A \rightarrow B \|^M = - \| A \|^M \cup \| B \|^M$
8. $\| A \leftrightarrow B \|^M = (- \| A \|^M \cup \| B \|^M) \cap (- \| B \|^M \cup \| A \|^M)$

The truth conditions for both deontic and alethic sentences are:

Definition 1.6.4 Semantics for $E_dOI\nabla_{d\text{-}a}KT5_a$ or $E\nabla$ System
Where M is a minimal model (Definition 1.6.2), the semantics for $E\nabla$ are:

1. df. o_d) $\models^M_\alpha OA$ iff $\| A \| \in N^d_\alpha$
2. df. p_d) $\models^M_\alpha PA$ iff $- \| A \| \notin N^d_\alpha$
3. df. f_d) $\models^M_\alpha FA$ iff $- \| A \| \in N^d_\alpha$
4. $oi\nabla_{d\text{-}a}$) if $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^a_\alpha$ and $\| A \| \notin N^a_\alpha$
5. df. $\Box_a$) $\models^M_\alpha \Box A$ iff $\| A \| \in N^a_\alpha$
6. df. $\Diamond_a$) $\models^M_\alpha \Diamond A$ iff $- \| A \| \notin N^a_\alpha$
7. t_a) if $\| A \| \in N^a_\alpha$ then $\alpha \in \| A \|$
8. v_a) if $\| A \| \notin N^a_\alpha$ then $\{\beta \text{ in } M: \| A \| \notin N^a_\beta\} \in N^a_\alpha$
9. n_a) $W \in N^a_\alpha$

The truth conditions for deontic sentences are defined in terms of sets of propositions at a particular world. The set of obligatory propositions at some world α is given by N^d_α. Permission is defined by the complement of the propositions that are not obligatory. Similarly, a prohibited proposition is one the complement of which belongs to the set of obligatory propositions. The truth conditions of alethic sentences are defined in the same way except that N^a_α is the set of necessary propositions. With conditions, t_a, v_a, and n_a the alethic portion of the model corresponds to the $KT5_a$ system or Lewis' S5 system. Thus, N^a

contains propositions expressing sentences true at all possible worlds, i.e. those that belong to every N^a at every possible world in the model M. The semantic conditions for bridge principles, i.e. those that contain two different kinds of modalities, such as $OI\nabla_{d\text{-}a}$, require both sets of sentences, N^d and N^a. The semantics in definition 1.6.4 correspond to the system $E_dO\nabla_{d\text{-}a}KT5_a$, abbreviated '$E\nabla$', which is the weakest system considered here. Several theorems are listed and the proofs given in the appendix Proof 1.6.13-App – 1.6.19-App.

Theorem 1.6.2 Semantics Theorems for $E\nabla$

1. $\models^M_\alpha OA$ iff $\neg P \neg A$
2. $oap_{d\text{-}a}$) If $\| A \| \in N^d_\alpha$ then $\| A \| \notin N^a_\alpha$
3. $oic_{d\text{-}a}$) If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^d_\alpha$
4. If $\| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$
5. If $- \| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$
6. n_a) $W \notin N^d$
7. con_d) $\varnothing \notin N^a$

Theorems 1.6.2-1 and 1.6.2-2 are the conditions corresponding to $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$. Theorem 1.6.2-3 shows that tautologies and necessary sentences do not belong to the set of obligatory propositions at any world. Theorem 1.6.2-4 indicates that contradictions do not belong to that set either.

Chapter 2: Deontic Principles and Substantive Ethics

In sections 1.5 and 1.6 three logical systems were described, KD and KD+, and E∇. In the former systems, necessary sentences are obligatory while in the latter they are not. The freewill motivation specified at the outset of the thesis described two major views in freewill, the incompatibilists and the compatibilists. The incompatibilists, who hold stronger versions of $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$, are committed to E∇ logics, while the compatibilists, especially those persuaded by Frankfurt cases are likely to adopt KD and KD+ systems.

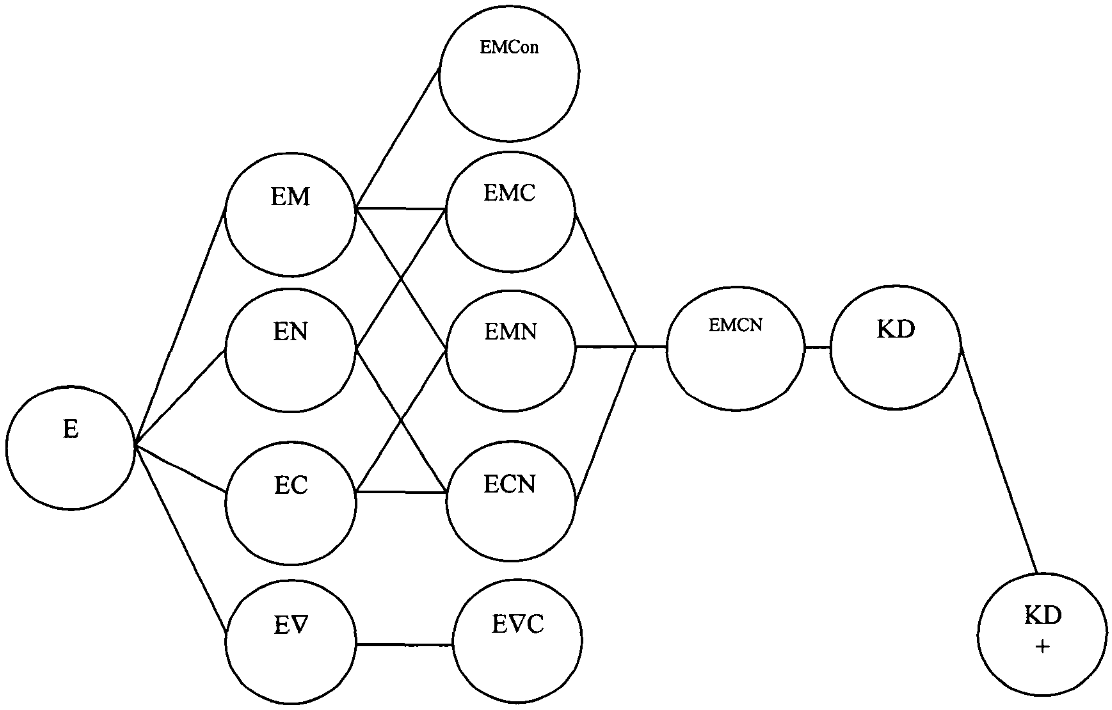
In this chapter, deontic and mixed deontic-alethic principles are taken up and their ramifications to substantive ethical theories examined. The principles are examined in three sections: those valid in KD (section 2.1), iterated and nested principles (section 2.2) and principles related to moral dilemmas (section 2.3). Section 2.4 examines consequences of E∇ to substantive ethical views.

The discussion of KD principles is set in the context of a group of systems called classical modal systems.[11] Chellas' classification is supplemented by alethic schemas and mixed bridge principles (1980, ch. 8). Normal systems contain the schemas M_d, C_d, N_d and are closed under the rule $RE_{d\text{-}a}$. The normal deontic system KD_d is one such system, as well as the extension $KD+_{d\text{-}a}$. Regular systems, the weakest of which is EMC_d, are closed under RR_d $A \wedge B \rightarrow C$ / $OA \wedge$

[11] The name 'classical modal system' refers to extensions of PL with the rule RE. The name should not be confused with classical logic in general.

$OB \rightarrow OC$ and contain the schema Df O_d $OA \leftrightarrow \neg P \neg A$. Along with the same schema, the rule RM_d is characteristic of monotonic systems, the weakest of which is EM_d. Finally, the weakest classical system, E_d, contains Df. O_d and is closed under the rule of inference RE_d $A \leftrightarrow B$ / $OA \leftrightarrow OB$. Every monotonic system is a classical system and every regular system is monotonic. Normal systems are regular systems with the schema N_d $O\top$. Every $E\nabla_{d\text{-}a}$ system or its extension is a classical system. Figure 2.0.1 illustrates the relationships between these systems. Their horizontal ordering indicates their strength. Note that all the systems are deontic systems, with the exceptions of $KD+_{d\text{-}a}$, $E\nabla_{d\text{-}a}$ and its extensions, which contain the alethic system $KT5_a$, as well as their respective bridge principles, $MIO_{d\text{-}a}$, and $OI\nabla_{d\text{-}a}$. For brevity, the systems in Figure 2.0.1 are labeled without indicating the alethic schemas and the proper subscripts.

Figure 2.0.1 Classification of Classical Modal Systems



2.1 Principles of KD

2.1.1 Interrelationship of Obligation, Permission and Prohibition

As was mentioned in 1.6.1, deontic logicians almost universally accept the inter-definability (theorems 1.6.1) of obligation, permission and prohibition. Keeping these in mind, the more complicated Maupassant's "La Parure" example is examined.[12]

Suppose that John is obliged to pay his taxes, and he appears on a public list of very famous people who have been ignoring the law and their public

12 Example and name given by Adam Morton. The name refers to a short story written by Maupassant in 1893 entitled the "La Pararure" or "the Necklace."

obligations by not filing any tax returns, so everyone says to him "John, you should pay your taxes." In fact, because of a quirk in the Canadian tax system, given the precise combination of circumstances, the amount of taxes John owes is exactly $0. So, it seems that no matter what John does, he will have paid his taxes; it is inevitable that he will, moreover, he cannot do anything to avoid paying his taxes. Nevertheless, he is obligated to. This example challenges the initial claim that obligations require alternative possibilities, for allegedly John has no alternative possibilities, yet he is still obligated to pay.

Though this appears to be an obligation, especially given that everyone tells John that he is obligated to pay his tax, in fact it is not an obligation. Suppose that it is an obligation. Then it is also true that $\neg P \neg A$, which is to say that it is not permissible to not pay the tax, or it is forbidden for John not to pay his tax. This seems to be clearly false, since John owes $0. Not paying his taxes is a completely acceptable or permissible action. It is more appropriate to describe either action as permissible, that is hold that where 'A' means paying tax, PA and $P\neg A$. It is permissible for John to pay his taxes and it is permissible for John not to pay his taxes.

2.1.2 N_d $O\top$ and $P\bot$

Using Chellas' classification, all tautologies are closed under RN_d A / OA in all normal systems (ECMN). The principle N_d is valid in all systems closed under RN_d, and where all tautologies are closed under RN_d, N_d is valid. Further, N_d is a straightforward consequence of $MIO_{d\text{-}a}$ $\Box A \rightarrow OA$, which is a schema of

$KD+_{d\text{-}a}$. On the other hand, the principle that whatever is impossible is permissible, $\neg\Diamond A \rightarrow PA$, is a consequence of $E\nabla_{d\text{-}a}$ (Proof 1.6.6-App). It follows directly from $OAP_{d\text{-}a}$. A consequence of this principle is that contradictions are permissible. So, established systems view tautologies and necessary sentences as obligatory, while $E\nabla_{d\text{-}a}$ holds contradictions and impossible sentences as obligatory. The two cannot be valid together as the inconsistencies between N_d and $OAP_{d\text{-}a}$ (Proof 2.1.1-App), and $MIO_{d\text{-}a}$ and $OAP_{d\text{-}a}$ (Proof – 2.1.2-App) show. Semantically, N_d states tautologies are true in all deontically accessible worlds, i.e. at all the ideal worlds. Of course, tautologies are true at every world and so they will be true at all ideal worlds. Similarly, $OAP_{d\text{-}a}$ states that what ought to be must not be true at all possible worlds. Are necessary obligations problematic? Alternatively, is it problematic is all necessary sentences are not obligatory? This is the issue in the current sub-section.

The rule RN_d (or equivalently N_d) is often accepted on solely pragmatic grounds.[13] Introducing RN_d simplifies derivations in general but most importantly, the semantics of deontic logic becomes a special kind of normal model for which completeness proofs are easier (Hansson 1971, pg. 135). Although important, the pragmatic reason is secondary to the issue of whether it is counter-intuitive. After all, principles that are immediately removed from any deontic logic, such as T_d $OA \rightarrow A$, are removed because they are counter-intuitive, whether convenient for simplifying completeness proofs or not. RN_d

[13] Van Fraassen introduces for pragmatic reasons the rule O(A/A), from which the conditional analogue of RN_d can be derived (van Fraassen 1971, pg. 421).

should stand the same test.

Prominent critics have rejected RN_d. Von Wright writes, "This always seemed to me highly counterintuitive, sheer nonsense." (von Wright 1981, pg. 8) Peter Schotch and al-Hibri also repudiate the rule (al-Hibri 1978, pg. 13). However, not everyone agrees. Those that argue for it say that it is harmless (even though, perhaps, unintuitive) and it can be accepted on pragmatic grounds. Hilpinen's position is paradigmatic. "The denial of $O\top$ excludes only those cases in which nothing whatsoever is obligatory, in other words, it excludes *empty* normative systems. In a sense, '$O(p \vee \neg p)$' does not exclude even this, since an obligation of this form is an 'empty' obligation, that is an obligation that is impossible not to fulfill."(Hilpinen 1971, pg. 13) Prior agrees. He writes of the stronger principle $MIO_{d\text{-}a}$ $\Box A \rightarrow OA$, "But surely this proposition is harmless (this obligation, if it be one, is one that is always met and need not worry us)."(Prior 1958, pg. 138) The reason that RN_d and $MIO_{d\text{-}a}$ are valid, despite the fact that they are odd, is that they are harmless.

What distinguishes counter-intuitive cases such as T_d $OA \rightarrow A$ from RN_d? It seems that contrary cases to T_d are obvious, i.e. cases where an obligation is not fulfilled but not with $O\top$. This is not surprising as conversation about necessary obligations rarely occurs. Therefore, it may be said that T_d is harmful in the sense that immediate counter examples exist for it, while not for $O\top$, though both may be counter-intuitive.

In contrast, $E\nabla$ is committed to the very negation of N_d i.e. $\neg O\top$.

Equivalently, this says that contradictions are permissible, i.e. $P\bot$. It is unlikely that the latter is intuitive to anyone, but as with N_d, intuitions against it may not be as strong as they are against T_d $OA \rightarrow A$. However, is it harmless? Contradictions and impossible sentences are never true; what they express never happens. It seems just as harmless to have permissible sentences that never happen as obligations that are always fulfilled.

At this point, the issue is at an impasse. Both $O\top$ and $P\bot$ seem counter-intuitive but not quite like T_d. Neither can be granted on grounds that it is harmless because the other seems just as harmless.

The issue can be pursued further by examining the consequences of rejecting each principle and giving advantage to the principle that retains more principles. As will be shown later, accepting $\neg O\top$ requires the rejection of M_d $O(A \wedge B) \rightarrow OA \wedge OB$ and $ME_{d\text{-}a}$ $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$. On the other hand, accepting N_d $O\top$ requires rejecting $OAP_{d\text{-}a}$ along with the main idea that obligations are contingent. The latter principles were argued for in section 1.3, but M_d and $ME_{d\text{-}a}$ are considered in the sections to come; if these principles cannot stand on independent reasons than there can be no advantage to having them as consequences.

It should be also said that taking both as contingent, that contradictions are sometimes permissible and sometimes not, and that necessary obligations are sometimes true and sometimes false, yields none of the benefits of rejecting one principle and keeping the other, while retains the drawbacks of both. There will still be obligatory tautologies, and contradictions that are permissible, and whatever principles have to be rejected from the removal of one, will also have to rejected from the removal of both, thus the deontic logic will be much weaker if this is done. It is better to take one harmless principle and strengthen the logic rather than reject both harmless principles and make the logic much weaker.

2.1.3 M_d

In the current section, M_d is considered. It is valid in normal, regular, and monotonic systems (see Figure 2.0.1). Most importantly, it is valid in normal systems KD and KD+. This principle is closely related to RM_d. In all classical systems where M_d is valid, the rule RM_d holds. Likewise, in all systems where obligation is closed under logical consequence, the principle M_d is valid. Thus, the two will be used interchangeably. The weakest classical system containing RM_d is the monotonic system EM_d.

Definition 2.1.1 Monotonic System (EM_d)

1. Df. O_d $OA \leftrightarrow \neg P \neg A$
2. RM_d $A \rightarrow B / OA \rightarrow OB$

The model that corresponds to the monotonic system is Chellas' supplemented minimal model (Chellas 1980, ch. 7).

Definition 2.1.2 Supplemented Minimal Model

Where M is a minimal model and for every α in M and every sentence A and B in M:

1. df. o_d) $\models^M_\alpha OA$ iff $\| A \| \in N^d_\alpha$
2. df. p_d) $\models^M_\alpha PA$ iff $- \| A \| \notin N^d_\alpha$
3. m_d) if $\{ \| A \| \cap \| B \| \} \in N^d_\alpha$ then $\| A \| \in N^d_\alpha$ and $\| B \| \in N^d_\alpha$

A consequence of this model is theorem 2.1.1 (Proof 2.1.3-App). It states that if tautologies are obligatory ($W \in N^d_\alpha$) then there is at least one obligation ($N^d_\alpha \neq \varnothing$). Moreover, if there is at least one obligation ($N^d_\alpha \neq \varnothing$) then necessary sentences are obligatory ($W \in N^d_\alpha$).

Theorem 2.1.1 $N^d_\alpha \neq \varnothing \leftrightarrow W \in N^d_\alpha$

According to Theorem 2.1.1, the reason that N_d (corresponding to $W \in N^d_\alpha$) is invalid in supplemented models is that there are worlds where nothing is obligatory. However, this is hardly what was intended in rejecting $O\top$. Further, there is no way to retain M_d and hold $OAP_{d\text{-}a}$ as valid. The significance of this is that removing N_d from classical systems does not aid the goal of not having tautologies that are obligatory, since it does so at the cost of not having any obligations at all.

For although, tautologies in system EM do not have to be obligatory, this is only at worlds where there are no obligations at all. This is also evident syntactically, with the result that necessary sentences are also inconsistent with RM_d, as long as there is at least one obligation (proof 2.1.4-App). The rest of the section will consider M_d independent of $OAP_{d\text{-}a}$.

The removal of M_d is costly as it permeates our most basic moral reasoning. When someone says, "John should obey his parents" it can be safely inferred by M_d that "John should obey his mother." It is by means of this principle that moral agents are persuaded of the logical consequences of their obligations (Schotch and Jennings 1981, pg. 151). These inferences are important and common in our moral reasoning.

The strength of this principle can be seen from various intuitive theorems:

Theorems 2.1.2 EM_d Theorems

1. $OA \vee OB \rightarrow O(A \vee B)$
2. $P(A \wedge B) \rightarrow PA \wedge PB$
3. $A \rightarrow B \,/\, FB \rightarrow FA$
4. $O(A \wedge B) \rightarrow OA$
5. $K_d\ O(A \rightarrow B) \rightarrow OA \rightarrow OB$

Each proof is given in the appendix, proofs 1.5.11-App to 1.5.14-App, except K_d, which is at 2.1.5-App. The first principle says, for example, if either it ought to be that John is losing weight or it ought to be that he is gaining weight then it ought to be that he is either losing or gaining weight. Principle (2) says that if two states of affairs are permissible together then they are each permissible. If it is permissible to be at a picnic and at the water park then it is permissible to be at the park and it is permissible to be at the water park. An example of (3) is that if robbing a bank implies committing a crime then if it is forbidden to commit a crime, so it is forbidden to rob a bank. An instance of principle (4) might be, "If it ought to be the case that we sing and dance at a friends' party, then it certainly ought to be the case that we sing at the party." Principles I – IV are valid in all EM systems, i.e. supplemented models. In systems as strong as EMC, K_d follows. Principle K_d is also a schema in KD.

On the other hand, RM_d also allows for theorems that have stirred a debate as old as deontic logic itself. The three oldest paradoxes are Alf Ross' theorem (Theorems 2.1.3-1), the penitent paradox (Theorems 2.1.3-2) and the free choice paradox (Theorems 2.1.3-3).

Theorems 2.1.3 EM_d Theorems

1. $OA \rightarrow O(A \vee B)$
2. $FA \rightarrow F(A \wedge B)$
3. $PA \rightarrow P(A \vee B)$

Alf Ross was the first to point out problematic instances of theorem 2.1.3-1 in the early 40's. As he points out, the theorem seems to allow for counter-intuitive obligations. The penitent's paradox appears to make anything forbidden as long as some prohibition is true. For, 'B' in Theorem 2.1.3-(2) can be any sentence whatsoever. Similarly, the free choice paradox seems to allow for too many permissions as 'B' can be something forbidden.

Although these paradoxes do not lead to contradictions, deontic logicians often argue that they lead to results that do not agree with our moral intuitions. A different objection claims that these principles debilitate deontic logic in guiding action. Both of these objections are flawed, however, there are serious counterexamples to M_d that are decisive.

The proofs for these theorems are given in the appendix, 1.5.8-App – 1.5.10-App. RM_d is the sole principle used in these theorems. As such, the three paradoxes can be addressed simultaneously. In the following section Alf Ross' paradox is considered but the discussion will equally apply to other paradoxes.

Alf Ross suggests the following instance (Ross 1941):

1. $O\neg B$
   Peter should not burn the letter
2. $OA \rightarrow O(A \vee B)$
   Peter should send the letter implies that Peter should send the letter or burn it.
3. OA
   Peter should send the letter
4. $\therefore O(A \vee B)$
   Therefore, Peter should send the letter or burn it.

Suppose Peter does not in fact send the letter but burns it. Not only that, he decides to go for a walk instead. Peter is not an angel, and he admits he burnt the letter but as consolation, he points to the infinite number of obligations that he has fulfilled. Since, $O(A \vee B)$ follows from OA, for any B, he has fulfilled the obligation that he ought to send the letter or go for a walk. He has done a number of other things that can all be substituted for 'B'. Ross' example is counter-intuitive because it seems that the obligation $O(A \vee B)$ can be fulfilled by doing any B, even when B ought not occur.

However, can this really occur according to deontic logic? Consider the semantic model. In SDM Ross' example says that if A is true in all deontically perfect worlds then $A \vee B$ is true. However, why should this be a problem? If in the ideal world Peter sends the letter then it will be true that he has sent the letter or burnt it. In this particular case, the disjunction is true because only one of the disjuncts is true, i.e. that he has sent the letter. Since, if Peter should not burn the letter then in the ideal world $\neg B$ is true, the letter is not burnt and the disjunction still holds. The semantics clearly show that Peter cannot burn the letter and fulfill an obligation. Thus, Alf Ross' paradox and analogously the penitent and free choice paradoxes are not counter intuitive once the precise meaning of obligation is kept in mind.

Nevertheless, so a second objection goes, given these paradoxes deontic logic leaves agents perplexed as to what act to follow. An infinite number of disjunctive obligations are derivable and no deontic resource to point out exactly which of these obligations to fulfill (Hansson 1971, pg. 132). Deontic logic fails to guide action.

In fact, a formal result can be derived that points agents in a unique direction. Al-Hibri emphasizes an instance of C_d: AG_d $O(A \vee B) \wedge O\neg B \rightarrow O((A \vee B) \wedge \neg B)$ (al Hibri, 1978, pg. 40). With it, a specific action can be derived. Take Peter's example.

Proof 2.1.6 $O\neg B \wedge O(A \vee B) \rightarrow OA$

1.	$O(A \vee B) \wedge O\neg B \rightarrow O((A \vee B) \wedge \neg B)$	AG_d
2.	$O(A \vee B) \wedge O\neg B \rightarrow O(A \wedge \neg B)$	1, PL, REP_d
3.	$O(A \wedge \neg B) \rightarrow OA \wedge O\neg B$	M_d instance
4.	$O(A \vee B) \wedge O\neg B \rightarrow OA \wedge O\neg B$	2, 3 PL
5.	$O(A \vee B) \wedge O\neg B \rightarrow OA$	4 PL

For disjunctive obligations where one of the disjuncts is forbidden, one cannot fulfill the disjunctive obligation by doing the forbidden act, but must fulfill it by doing the other act. It is not enough to derive some result from deontic logic, like a disjunctive obligation, and then proceed to action from that alone. Although it is true that there are an infinite number of obligations at a particular world, to get a specific prescription "...We must look not only on the obligation uttered or asserted, but on the deontic system as a whole."(Hansson 1971, pg. 132) To put the matter succinctly, "the simplest fact to remember about disjunctive obligations is that they present us with the choice on how to fulfill our obligations. But at no time should we choose a way of fulfilling one's obligation by violating another..." (al-Hibri 1978, pg. 41) The objection that M_d leads to an infinite number of disjunctive obligations, and thus fails to guide action is incorrect.

Finally, a highly problematic consequence of M_d is that it allows for only perfect obligations, i.e. those that do not imply anything that ought not to be.

Consider the following:

1. If poverty is decreasing in Canada then Canada is poor poverty.
2. If it is obligatory that poverty is decreasing in Canada then it is obligatory that Canada have poverty.

This example is a variant on a series of problems called the "Good Samaritan" problems.[14] The inference follows by the rule RM_d. Line 1 is a logical consequence; there must be poverty if poverty is decreasing. An evidently plausible obligation is that poverty ought to decrease in Canada, and so by RM_d, it follows that Canada ought to have poverty. The inference is entirely flawed and the model reveals why. Let 'A' stand for "poverty is decreasing in Canada" and 'B' for "Canada has poverty." That 'B' follows from 'A' is true in any world; it is a logical consequence. Suppose that OA is true in the actual world then A will be true in all deontic alternatives, in all the ideal worlds. However, if A is true in the ideal world, B will be true there as well, because it is a logical consequence of A. In all ideal worlds, the deontic alternatives to the actual world, A is true.

[14] A similar example was considered by Schotch and Jennings (1981). It states that if we feed the starving poor implies that there are starving poor. Even though we ought to feed the starving poor that does not imply that there ought to be the case that there are starving poor.

Therefore, it is obligatory that A in the actual world. It ought to be that Canada has poverty. This is an entirely implausible result.

On the other hand, that poverty ought to be decreasing might be thought false and the inference justifiable. Though an intuitive obligation, SDM semantics requires that A be obligatory if and only if A is true at *all* the ideal deontic alternatives. Now, one such ideal world is one where there is no poverty at all, i.e. ¬B. However, if this is so, by the converse of the logical consequence, ¬A, poverty is not decreasing in Canada, and so, it is obligatory that poverty is not decreasing in Canada or it is forbidden for poverty to decrease in Canada.

In the first case, A is an imperfect ought because it involves poverty or otherwise something bad. That bad thing became obligatory by RM_d. In the second case, the deontic alternatives were taken to be truly ideal, i.e. without any "imperfect" obligations. In this case, it turned out that imperfect obligations are forbidden. Both results are serious reasons to reject RM_d. On the one hand, obligations that clearly ought not to be are added to morality (i.e. that Canada ought to be poor), while on the other hand, imperfect obligations, much needed in daily life, become forbidden.

That the above example is not concerned with agents is irrelevant, as similar examples can be constructed with agents involved. If John is reducing to normal weight then John is overweight. John ought to reduce to normal weight implies that John ought to be overweight.

The general problem is that ideal worlds are standards for obligation and in such worlds perfect and imperfect obligations cannot coexist. In minimal models where M_d is not valid, perfect and imperfect obligations can coexist as standards for obligations. Perfect ideal obligations and imperfect ideals can both belong to set of sentences obligatory at some world. It can be obligatory that Canada not have any poverty, indicated by membership of the set of all worlds where Canada has no poverty in N^d_α. In addition, the set of all worlds where Canadian poverty is decreasing can also belong to N^d_α.

Conditional Obligation

An alleged solution is that the counter-example is an instance of a conditional obligation (Schotch and Jennings 1981, pg. 156). A distinction needs to be made where obligations hold given certain conditions and where they hold irrespective of any conditions. It is true that poverty should be decreasing but only when there is poverty. Poverty should not occur, but given that it does, it ought to be decreasing.

Further syntactic resources are required, namely a dyadic operator for obligation. The operator 'O(A | B)' means it is obligatory that A given that B has happened.[15] One such system is defined in Chellas (1980, ch. 10.2). To represent the counter-example the $RCOM_d$ principle is allegedly required. Several other systems have agreed on at least these principles.[16]

$RCOM_d \qquad B \rightarrow B' \,/\, O(B \mid A) \rightarrow O(B' \mid A)$

[15] The dyadic operator can be defined in terms of unconditional obligations and suitable conditionals. See Chellas chapter 10.3.

[16] See Bass van Fraassen (1972) and Georg von Wright (1971b)

A minimal model for conditional obligation is given as:

Definition 2.1.3 Minimal Conditional Model (MCM)
$M = \langle W, f, P \rangle$

1. W is a set of worlds
2. f is a mapping that selects a collection of proposition (set of worlds) $f(\alpha, X)$ for each world α and proposition, or condition, X. Formally, f: $W \times \wp(W) \rightarrow \wp(\wp(W))$
3. P(n) is a function from natural numbers to subsets of W

   $P: \mathbb{N} \rightarrow W$

The semantics are the same as minimal models used for E∇ except that a function is used to represent conditional obligation, whereas N_d, a set of deontically necessary propositions was used to model monadic obligation. The function f determines the set of propositions obligatory at some world, given the condition (or proposition) X. The truth conditions for conditional obligation are as follows:

Definition 2.1.4 Truth Conditions for Conditional Obligation
$\models^M_\alpha O(B \mid A)$ iff $\| B \| \in f(\alpha, \| A \|)$

With these resources added the problematic sentences are represented formally as: "Poverty is decreasing in Canada" as '$O(D \mid P)$' meaning it ought to be that poverty is decreasing in Canada given that there is poverty in Canada."

To retain the conditional obligation counter-part to M_d, namely the rule of inference $RCOM_d$ $A \rightarrow B \,/\, O(A \mid C) \rightarrow O(B \mid C)$, the following condition has to hold in minimal conditional models.

cm) if $Y \cap Y' \in f(\alpha, X)$ then $Y \in f(\alpha, X)$ and $Y' \in f(\alpha, X)$

The intersection of two propositions given some condition are obligatory imply that each of the propositions are obligatory given the condition.

However, the analogous rule has an analogous problem. It ought to be that poverty is decreasing given that there is poverty implies that there ought to be poverty given that there is poverty. Although, the implication is valid and it is certainly true that poverty ought to be decreasing given that there is poverty, why is it true that poverty ought to be, given that there is poverty? O(B | B) is clearly false.

Semantically, the intersection of the set of worlds where poverty is decreasing in Canada and the set of worlds where Canada is poor, i.e. the set of worlds where poverty is decreasing in Canada (since worlds where poverty is decreasing are a subset of the worlds where there is poverty) are worlds that ought to be. By cm) this implies that the sets of worlds where Canada is poor given that it is poor are worlds that ought to be.

In most typical conditional obligation systems, i.e. Chellas' (1980), van Fraassen's (1972), and al-Hibri's (1978) this result occurs. Therefore, conditional obligation cannot resolve this matter. If one believes that it can then the burden is on them to provide such a logic.

Alf Ross' paradox, the penitent's paradox and the free choice paradox put RM_d in question. It was argued that these are not paradoxes. A closer look at the semantics revealed that the intended meanings are not problematic. A different objection stated that the same paradoxes prevented deontic logic from guiding action. However, this problem can be addressed by considering the deontic system as a whole, and using other principles to derive unique obligations, where there are disjunctive obligations. Nevertheless, cases of the Good Samaritan paradox precluded the possibility of having obligations whose content may be imperfect, such as that it ought to be that poverty is decreasing. Alternatively, only perfect obligations could be held making deontic logic useless in our imperfect worlds. Since popular conditional obligation approaches do not resolve the problem, it follows that M_d is not valid and the set of all sentences be is not closed under RM_d. Note that the reasons given for the rejection of M_d are those entirely independent of the proposed system.

2.1.4 C_d and D_d

The agglomeration principle C_d says that if it ought to be that something occurs and it ought to be that something else occurs then they both ought to occur. If it ought to be that John cooks dinner and it ought to be that he buys wine then he ought to cook dinner and buy wine. The principle is valid in KD and KD+, and in E∇C systems. However, if the existence of genuine moral dilemmas is asserted, C_d is controversial. This will be discussed in section 2.3.2. Moral dilemmas aside, this principle is a valid principle.

The characteristic deontic principle in normal deontic systems is D_d $OA \rightarrow PA$. This principle states that whatever is obligatory is permissible. If it ought to be that John hikes up the mountain then it is permissible that he does so. The principle is equivalent with $OA \rightarrow \neg O \neg A$, which is the same as $\neg(OA \wedge O\neg A)$. This principle clearly precludes moral dilemmas, that is, if someone ought to do something then they should not be obligated to do the opposite. However, if genuine moral dilemmas are denied and cases of moral conflict are always apparent then D_d is a plausible principle. More on the relationship between D_d and moral dilemmas will be said in section 2.3.

The principle D_d is valid in standard systems, KD and KD+. It need not be added as an axiom to E∇C systems, since it follows from OIC_{d-a} and C_d (proof 2.1.6-App). Thus, D_d is a perfectly sound principles given that there no genuine moral dilemmas.

2.1.5 Conclusion on KD_d Principles

In considering the fundamental principles of KD_d, the proposed system and established were shown to be at variance. N_d and M_d are valid in KD_d but are inconsistent with $E\nabla_{d-a}$ systems. It was argued that there is no defensible independent reason for the preference of $O\top$ over $P\bot$. The outstanding question was how drastically a logic has to be weakened in order to keep or remove N_d. It was shown that M_d implies N_d if there is at least one obligation and so rejecting N_d practically means rejecting M_d. This is an important principle whose inferences make up much of our moral reasoning. However, M_d has serious objections from Good Samaritan type paradoxes. Thus, M_d has to be rejected independently of the fact that it is inconsistent with the principles of $E\nabla_{d-a}$. If these reasons are correct, KD_{d-a} and $E\nabla_{d-a}$, at least up to this point are roughly comparable. Without convincing reasons for adopting N_d, and the fact that M_d has to be rejected, C_d and D_d are left, both of which are valid in extension of $E\nabla_{d-a}$, namely $EC\nabla_{d-a}$. However, there are more principles to consider. Iterated principles are next, followed by those related to moral dilemmas.

The tentative strengthened system $EC\nabla_{d\text{-}a}$ and its semantics arising from the discussion in 2.1 is stated:

Definition 2.1.3 $EC_d\nabla_{d\text{-}a}KT5_a$ ($EC\nabla_{d\text{-}a}$)

1. All schemas and rules of inference of $KT5_a$ (Definition 1.5.7)
2. Df. O_d $\quad OA \leftrightarrow \neg P \neg A$
3. $OI\nabla_{d\text{-}a}$ $\quad OA \rightarrow \Diamond A \wedge \Diamond \neg A$
4. C_d $\quad OA \wedge OB \rightarrow O(A \wedge B)$

Theorem 2.1.3 $EC\nabla_{d\text{-}a}$ Theorems

1. D_d $OA \rightarrow PA$
2. $OA \rightarrow \neg O \neg A$
3. $\neg(OA \vee O\neg A)$

The proofs are in the Appendix 2.1.6-App. Theorems 2.1.3(2)-(3) follow from D_d by propositional logic (PL).

Definition 2.1.4 Minimal Models for $E\nabla C$

Where M is a minimal model and for every α in M and every sentence A and B in M,
M is a minimal model for $E\nabla C$ iff:

1. df. o_d) $\models^M_\alpha OA$ iff $\| A \| \in N^d_\alpha$
2. df. p_d) $\models^M_\alpha PA$ iff $- \| A \| \notin N^d_\alpha$
3. $oi\nabla_d$) if $\| A \| \in N^d_\alpha$ then $\| A \| \notin N^a_\alpha$ and $\| A \| \notin N^d_\alpha$
4. c_d) if $\| A \| \in N^d_\alpha$ and $\| B \| \in N^d_\alpha$ then $\| A \| \cap \| B \| \in N^d_\alpha$

2.2 Iterated / Nested Principles

In this section, iterated and nested principles are examined. A modality is any sequence of operators '$\neg$', '$\Box$', '$\Diamond$', including '$\cdot$', the empty sequence. Two modalities 'ψ' and 'φ' are equivalent if and only if for every A the sentence $\varphi A \leftrightarrow \psi A$ is a theorem. For example, in any system with the theorem $\Box A \leftrightarrow \Box\Box A$, modalities $\Box$ and $\Box\Box$ are equivalent. The mentioned theorem is called a reduction law because it reduces one modality to another. An iterated modal principle is one that contains a sequence of two or more modal operators, while a nested modal principle has multiple modalities it but it need not involve a sequence. For example, $\Box\Box A$, $OOA \leftrightarrow OA$, $\Box\Diamond A \leftrightarrow \Diamond A$ are iterated modal principles while $\Diamond(A \rightarrow (A \rightarrow \Box A)$ and $O(OA \rightarrow \Diamond A)$ are nested principles.

The system for alethic necessity is $KT5_a$ (Definition 1.5.7). This system contains alethic iterated and nested principles but they will be omitted in the following discussion. Schema 5_a $\Diamond A \rightarrow \Box\Diamond A$ allows for iterated alethic principles. The iterated principles together yield four reduction laws.

Theorem 2.2.1 $KT5_a$ (alethic) Reduction Laws

1. $\Box A \leftrightarrow \Box\Box A$
2. $\Diamond A \leftrightarrow \Diamond\Diamond A$
3. $\Box A \leftrightarrow \Diamond\Box A$
4. $\Diamond A \leftrightarrow \Box\Diamond A$

It can also be shown that there are no further reduction laws (for example, that $\Box A \leftrightarrow \Diamond A$, is not a theorem), establishing that there are exactly four. The upshot of this is that all alethic modalities can be reduced to six modalities, '$\Box$', '$\Diamond$', '$\cdot$', '$\neg$', '$\neg\Box$', and '$\neg\Diamond$'.

Several deontic principles, both iterated and nested are considered. They are selected partly for their prima facie value to the proposed system as well as their prominence in the literature.

1. $OO\top$
2. $O4_d$ $OA \rightarrow OOA$
3. $O4C_d$ $OOA \rightarrow OA$
4. OU_d $O(OA \rightarrow A)$

The subject of iterated and nested modalities is a controversial issue. It seems that consensus has not been reach on any of these principles.

2.2.1 OO$\top$

The simplest of the iterated deontic modalities are those where the 'O' or 'F' operator are repeated, as in OOA and FFA, where the sentence A is a tautology. The following semantic condition makes OO$\top$ valid: $\{\beta \text{ in } M: W \in N^d_\beta\} \in N^d_\alpha$. All the worlds where O$\top$ ($W \in N^d_\beta$) is true belong to the set of morally necessary sentences at α. However, in a E∇C model (Definition 2.1.2), since $W \notin N^d_\alpha$ is a theorem (see Theorem 1.6.2), the set of worlds where tautologies are obligatory will be the empty set. However, the empty set does not belong to the set of morally necessary propositions at any world, for $\varnothing \notin N^d_\alpha$ (See Theorem 1.6.2). The same result holds for longer iterations, and for theorems. For instance, O(OA $\rightarrow \Diamond$A) is invalid, because OA $\rightarrow \Diamond$A is a theorem, true at all worlds. Nevertheless, permissions may be iterated. For example, PP$\top$ (and further iterations) are valid sentences. Thus, KD allows for iterations of both modalities, since the rule A / PA is valid as well as A / OA (Proof 1.5.15-App, Definition 1.5.6). The evident oddity is that in E∇ iterations of obligations and prohibitions (where the atomic sentence is a tautology or theorem) are invalid, whereas iterations of permissions (where the atomic sentence is a tautology or theorem) are valid.

2.2.2 $O4_d$

Another often-discussed iterated principle is $O4_d$ OA $\rightarrow$ OOA. In standard models, the principle is valid when the deontic alternativeness relation is transitive. Added to SDM, the relation R becomes serial and transitive. It states that whatever is obligatory at some world will be obligatory at all the deontic alternatives, at all the ideal worlds. Obeying the law is obligatory at our world, and by $O4_d$, it continues to be obligatory in worlds that are deontically perfect relative to ours.

Chellas thinks the principle is plausible, appealing to the favorable result that the deontic alternatives lead to worlds that are in some way better (Chellas 1980, ch. 6). Prior agrees; he finds $O4_d$ quite reasonable (Prior 1955). From the formal point of view, this principle together with $O4C_d$ OOA $\rightarrow$ OA, yields the reduction law OOA $\leftrightarrow$ OA, which is highly advantageous for simplifying the deontic system.

For minimal models, the semantic condition on M where α is a possible world is:

$o4_d$) if $\| A \| \in N^d_\alpha$ then $\{ \beta \text{ in } M\text{: } \| A \| \in N^d_\beta \} \in N^d_\alpha$

If some proposition belongs to the set of obligatory sentences at α, so do all the worlds at which that proposition is obligatory. However, the foregoing account is problematic with iterated obligations when there are absolute obligations.

Proof 2.2.1 Inconsistency $\{\Box OA, OOA, OAP_{d\text{-}a}\}$

1.	OOA	Assumption
2.	$\Box OA$	Assumption
3.	$OOA \rightarrow \Diamond\neg OA$	$OAP_{d\text{-}a}$ instance
4.	$\Diamond\neg OA$	1, 3 PL
5.	$\neg\Diamond\neg OA$	2, Df. $\Box_a$

A similar proof can be shown for prohibitions. The E∇ deontic-alethic system will not allow for absolute obligations and iterated obligations. One can hold iterated obligations, but then none of those obligations iterated can be absolutely obligatory. On the other hand, if one holds absolute obligations, those iterations cannot be iterated. If one holds an ethical theory where some obligations are absolute and others are not, then those that are not absolute may be iterated.

2.2.3 OU_d

Next, the nested principle OU_d $O(OA \rightarrow A)$ is considered. To validate OU_d in SDM, the relation R is secondarily reflexive. A relation R is secondarily reflexive if and only if for any possible world α and β, if $\alpha R\beta$ then $\beta R\beta$. It ought to be the case that whatever ought to be the case is the case. At all ideal worlds, all obligations are fulfilled.

In minimal models the semantic condition that validates OU_d is:

ou_d) $\{ - \{\beta \text{ in } M: \| A \| \in N^d_\beta\} \cup \{\beta \text{ in } M: \beta \in \| A \|\}\} \in N^d_\alpha$

More informally, the principle states that obligations ought to be fulfilled. The spirit of this principle is intuitive enough. Obligations are sometimes fulfilled

(OA $\wedge$ A) and sometimes not (OA $\wedge$ $\neg$A), but they always ought to be fulfilled. This intuitive principle cannot be held with E∇ either. The instance O(O$\top$ $\rightarrow$ $\top$) is false because on the foregoing account, obligations of tautologies are always false, and the conditional of this instance is always true because the consequent is never false.

2.2.4 $O4C_d$

The semantic condition for OOA $\rightarrow$ OA is:

$o4c_d$) if { β in M: $\| A \| \in N^d_\beta$ } $\in N^d_\alpha$ then $\| A \| \in N^d_\alpha$

If some occurrence is ideal in ideal worlds relative to α then it is obligatory in α. This appears plausible. For example, if it ought to be the case that walking the dog is obligatory then it ought to be the case that dogs are walked. An obligation that ought to hold naturally implies that it ought to be. On the scenario described, principle $O4C_d$ OOA $\rightarrow$ OA will turn out to be vacuously true, since the iterated obligations will never be true. This does not show an inconsistency with $O4C_d$ but shows the principle to be vacuous only when one grants absolute obligations. Thus, $O4C_d$ should not be rejected on these grounds.

2.2.5 Iterated Alethic within Deontic Modalities

More problems arise when alethic modalities are embedded in deontic modalities. For example, it ought to be the case that free speech is possible, or it ought to be the case that criminals are necessarily punished. Contingent obligations do not allow for alethic modal sentences, whether necessary or possible, to occur within the deontic operators.

Proof 2.2.3 Inconsistency $\{OI\nabla_{d\text{-}a}, O\Diamond A, 4\Diamond C_a, 5_a\}$

1.	$O\Diamond A$	Assumption
2.	$O\Diamond A \rightarrow \Diamond\Diamond A$	$OIC_{d\text{-}a}$ instance
3.	$\Diamond\Diamond A$	1, 2 PL
4.	$\Diamond\Diamond A \rightarrow \Diamond A$	$4\Diamond C_a$
5.	$\Diamond A$	3, 4 PL
6.	$\Diamond A \rightarrow \Box\Diamond A$	5_a instance
7.	$\Box\Diamond A$	5, 6 PL
8.	$O\Diamond A \rightarrow \Diamond\neg\Diamond A$	$OAP_{d\text{-}a}$ instance
9.	$\Diamond\neg\Diamond A$	1, 8 PL
10.	$\neg\Box\neg\neg\Diamond A$	9, Df. $\Diamond_a$
11.	$\neg\Box\Diamond A$	10, $REP_{d\text{-}a}$, PL

Lines 7 and 11 contradict each other. Thus, it is only $OIC_{d\text{-}a}$ and $OAP_{d\text{-}a}$ that together disallow possible sentences within the obligatory operator. A similar proof can be shown for obligatory necessary statements.

Proof 2.2.4 Inconsistency $\{ OI\nabla_{d\text{-}a}, O\Box A, 4\Diamond_a, 5_a \}$

1.	$O\Box A$	Assumption
2.	$O\Box A \rightarrow \Diamond\Box A$	$OIC_{d\text{-}a}$ instance
3.	$\Diamond\Box A$	1, 2 PL
4.	$O\Box A \rightarrow \Diamond\neg\Box A$	$OAP_{d\text{-}a}$ instance
5.	$\Diamond\neg\Box A$	1, 4 PL
6.	$\Diamond\Box A \rightarrow \Box A$	5_a instance
7.	$\Box A$	3, 6 PL
8.	$\Diamond\neg\neg\Diamond\neg A$	5, Df. $\Box_a$, $REP_{d\text{-}a}$
9.	$\Diamond\Diamond\neg A$	8, PL
10.	$\Diamond\Diamond\neg A \rightarrow \Diamond\neg A$	$4\Diamond_a$ instance
11.	$\Diamond\neg A$	9, 10 PL
12.	$\neg\Diamond\neg A$	7, Df. $\Box_a$

Thus, no alethic sentence, whether it is necessary or possible, can be true inside the deontic operator 'O'. The proofs can be repeated with the $FAP_{d\text{-}a}$ and $FIC_{d\text{-}a}$ principles to establish that $F\Box A$ and $F\Diamond A$ cannot occur either. Similar proofs can be given for sentences (3) – (4).

1. $O\Box A$
2. $O\Diamond A$
3. $O\neg\Box A$
4. $O\neg\Diamond A$

But not for,

5. OA
6. $O\neg A$

As is well known, in $KT5_a$ iterated modalities can be reduced to six different modalities, using the reduction laws specified in (Definition 1.5.7). So, any sentence where the deontic 'O' or 'F' is followed by any alethic sentence (except those where the iterated alethic proposition is equivalent to OA or $O\neg A$), will be false. This result is not true for permission. It is perfectly true to have $P\Box A$ or $P\Diamond A$.

This is problematic for principles that might seem quite intuitive. For example, $OA \rightarrow O\Diamond A$, if it is obligatory to pay tax then it ought to be possible to pay taxes. The defense of this is that what just is, whether necessary or possible is not obligatory because it just is and cannot be otherwise. Yet principles such as these might be thought perfectly plausible.

2.2.6 Conclusion for Iterated Principles

Four principles were taken up in 2.2. All four except $O4C_d$ had to be rejected because they are inconsistent with $E\nabla$. The problem with iterated obligations of tautologies was that all such obligations are invalid, but all iterated permissions of tautologies are valid. On there other hand $O4_d$ $OA \rightarrow OOA$ and OU_d $O(OA \rightarrow A)$ are intuitive and the rejection of these is a clear disadvantage.

It was also a problem because ideal worlds are typically thought to take us to better worlds. OU_d is invalid, and this prima facia is counter-intuitive of all – all obligations should be fulfilled. This rejection of $O4_d$ alone made the reduction laws $OA \leftrightarrow OOA$ invalid, which is a drawback from the formal standpoint. Most troubling is the conclusion that possible sentences (in addition to necessary) cannot occur within obligations or prohibitions. Some further consequences of these results to substantive ethics will be taken in section 2.4. In conclusion, a major drawback of E∇ is that it cannot handle iterated deontic modalities nor certain nested principles. KD and KD+ are advantageous in this regard.

2.3 Moral Dilemmas

Moral agents can be conflicted in a variety of ways. A general may be in conflict over the outcome of two military strategies, a monk over his religious duties or a citizen about the laws of his country. These do not concern morality per se but prudence, religious convictions and law, respectively. Moreover, any combination of these different kinds of conflicts can bewilder an agent, however, this discussion is limited to conflicts arising in the moral domain only.

In the context of deontic logic, if one does accept moral dilemmas, several intuitive deontic principles are questioned. Moreover, it is particularly interesting for this thesis because moral dilemmas introduce some sort of inconsistency, for if moral dilemmas can exist then one thing can be required by morality and the very opposite. Prima facie, this seems counter to the principle that obligations are contingent, that is, that they are consistent (OI∇).

To begin, a moral dilemma is a conflict between at least two moral obligations. Not every conflict between moral obligations is a moral dilemma. For, there are two kinds of obligations, prima facie and all-things-considered obligations.[17] A prima facie obligation is one where it is morally wrong not to adopt that alternative if there were no moral justification for not adopting it. An all-things-considered obligation is what one should do after taking into account all prima-facie obligations. In cases of moral conflict, the all-things-considered obligation will be the strongest prima-facie obligation and it is said to "override" all other prima-facie obligations. Thus, moral conflict in general is conflict between prima-facie obligations, but a moral dilemma is a conflict between all-things-considered obligations. Take Plato's Dilemma as an example. Suppose Socrates borrows Plato's rifle and promises to return it to him on Monday. On that day, Socrates comes asking for it with a look of rage in his eyes. Plato knows that if he gives it to him he will use it to kill his unfaithful wife. In this case, though Plato has a prima facie moral obligation to give him the rifle, he does not have an all-things-considered moral obligation to do so. The moral considerations against giving Socrates the rifle – the fact that he will kill his wife – outweigh the moral considerations in favor of giving it to him. In this case, Plato's all-thing-considered obligation overrides the prima facia obligation to return the rifle.

A paradigmatic example of a moral dilemma is Sophie's choice. For example, Sophie has to choose between the deaths of one of her two children who happen to be identical twins. If she does not choose, they will both die. Sophie

[17] The contemporary distinction between prima-facie and all-things-considered obligations are contemporary products of David Ross' prima-facie duties and duties proper.

should save each of her children but she cannot save both. She has two equally strong prima facie obligations to save each of her children, and thus two all-things-considered obligations.

As was stated at the outset, two conflicting obligations seem to introduce some sort of inconsistency; one must do something and must also do something else but cannot do both, especially in cases where one is obligated to do one thing and also not to do that very same thing. Marcus shows that there is nothing logically contradictory about this (Marcus 1980, pg. 59). Formally, OA and O¬A are not contradictory, rather OA and ¬OA. Both Sophie's moral obligations can be fulfilled and the dilemma avoided by teleporting both of her children away from danger. Although this is highly unlikely and not within the powers of Sophie, it shows that there is no logical inconsistency. A stronger sense of possibility is required to formulate a dilemma. The impossibility is not of the logical kind but one that involves the agent. Call this the personal sense. This does not alter the prima facie case for examining moral dilemmas, since the inconsistency is felt within the deontic operators and that is precisely what OI∇ denies. A definition of a moral dilemma can be given as:

Definition 2.3.1 Moral Dilemma

1. There is an all-things-considered obligation that A.
2. There is an all-things-considered obligation that B.
3. It is not personally possible to do both A and B.

Formally this can be simply stated as:

$MD_{d\text{-}a}\ OA \wedge OB \wedge \neg\Diamond'(A \wedge B)$

where $\Diamond'$ is personal sense of possibility.

The central issue is whether there are any genuine moral dilemmas. Contending that there are no moral dilemmas, theorists show that they lead to contradictions in conjunction with common deontic principles. Others point to phenomenal experience such as guilt, regret and indecision, as evidence for moral dilemmas but in so doing deny some common deontic principles. The impact that the proposed deontic logic has on the issue of moral dilemmas will be shown.

2.3.1 Moral Dilemmas and Inconsistency: D_d

It is uncontroversial that D_d $OA \rightarrow PA$ must be rejected if there are moral dilemmas. It is equivalent with $\neg(OA \wedge O\neg A)$, which states that there cannot be an obligation and obligation of its opposite (proof 2.5.2). John made a promise to a host that he will be at dinner by seven and having forgotten the original promise, he promises the hostess that he will not be arriving to dinner at seven. John ought to be at dinner by seven, but he also ought not to be there by seven.

Proof 2.3.1 Inconsistency $\{MD_{d\text{-}a}, D_d\}$

1.	$OA \wedge O\neg A \wedge \neg\Diamond'(A \wedge B)$	$MD_{d\text{-}a}$ instance
2.	$OA \rightarrow PA$	D_d
3.	$OA \rightarrow \neg O\neg A$	2, Df. P_d
4.	$\neg(OA \wedge O\neg A)$	3, PL
5.	$OA \wedge O\neg A$	1 PL

The deontic accessibility relation is serial, stating that at every world there is at least one alternative. In standard models, moral dilemmas cannot arise because at the deontic alternatives a sentence and its negation, that is a contradiction, cannot be true because contradictions are not true at any world. Deontic logicians concerned with moral dilemmas have routinely rejected this principle, Chellas (1980, ch. 6), Schotch and Jennings (1981) to name a few.

2.3.2 Moral Dilemmas and Inconsistency: C_d and $OIC_{d\text{-}a}$

A much more contentious issue with moral dilemmas is over the agglomeration principle C_d $OA \wedge OB \rightarrow O(A \wedge B)$ and $OIC_{d\text{-}a}$ $OA \rightarrow \Diamond A$. The two principles are inconsistent with moral dilemmas (proof 2.3.2), and the issue is which principle one should reject.

Proof 2.3.2 $OA \rightarrow PA$

1.	$O(A \wedge \neg A) \rightarrow \Diamond(A \wedge \neg A)$	$OIC_{d\text{-}a}$ instance
2.	$\neg\Diamond(A \wedge \neg A) \rightarrow \neg O(A \wedge \neg A)$	1, PL
3.	$\neg\Diamond\neg\top \rightarrow \neg O(A \wedge \neg A)$	2, PL
4.	$\Box\top \rightarrow \neg O(A \wedge \neg A)$	3, Df. $\Box_a$
5.	$\Box\top$	PL
6.	$\neg O(A \wedge \neg A)$	4, 5 PL
7.	$OA \wedge O\neg A \rightarrow O(A \wedge \neg A)$	C_d instance
8.	$\neg O(A \wedge \neg A) \rightarrow \neg(OA \wedge O\neg A)$	7, PL
9.	$\neg(OA \wedge O\neg A)$	8, PL
10.	$OA \rightarrow \neg O\neg A$	9, PL
11.	$OA \rightarrow PA$	10, Df. P_d

Because, D_d was shown to be invalid, either $OIC_{d\text{-}a}$ or C_d must be rejected. $OIC_{d\text{-}a}$ was taken as fundamental so on the proposed logic $E\nabla_{d\text{-}a}$ it is C_d that must be rejected.

There is a further reason independent of the above to reject C_d as suggested by Chellas (1980, ch. 6). Accepting C_d conflates two principles, Con_d $\neg O\bot$ and D_d $OA \rightarrow PA$, which should be distinguished, because in semantics where moral dilemmas are possible the former is valid while the latter not. The two principles are logically equivalent in systems with C_d and M_d, KD and KD+ being two examples.

Proof 2.3.3 $\neg O(A \wedge \neg A) \leftrightarrow OA \rightarrow PA$

1.	$OA \wedge O\neg A \rightarrow O(A \wedge \neg A)$	C_d instance
2.	$\neg O(A \wedge \neg A) \rightarrow \neg (OA \wedge O\neg A)$	1, PL
3.	$\neg O(A \wedge \neg A) \rightarrow OA \rightarrow \neg O\neg A$	2, PL
4.	$\neg O(A \wedge \neg A) \rightarrow (OA \rightarrow PA)$	3, Df. P_d, $REP_{d\text{-}a}$
5.	$O(A \wedge \neg A) \rightarrow OA \wedge O\neg A$	M_d instance
6.	$\neg(OA \wedge O\neg A) \rightarrow \neg O(A \wedge \neg A)$	5, PL
7.	$(OA \rightarrow \neg O\neg A) \rightarrow \neg O(A \wedge \neg A)$	6, PL, $REP_{d\text{-}a}$
8.	$(OA \rightarrow PA) \rightarrow \neg O(A \wedge \neg A)$	7, Df. P_d, $REP_{d\text{-}a}$
9.	$\neg O(A \wedge \neg A) \leftrightarrow OA \rightarrow \neg O\neg A$	4, 8 PL
10.	$\neg O(A \wedge \neg A) \leftrightarrow OA \rightarrow PA$	9, Df. P_d

As was argued in 2.3.1, D_d has to be rejected, and so by the biconditional just proven Con_d $\neg O\bot$ has to go as well. The problem is that moral dilemmas put into question D_d but not Con_d, for the latter seems quite plausible.

The principles that lie behind the conflation are more easily seen with the equivalent biconditional $O(A \wedge \neg A) \leftrightarrow OA \wedge O\neg A$. In the left to right direction this is an instance of M_d, if a contradiction is obligatory then moral conflict ensues. There does not seem to be anything problematic about this direction, especially if Con_d is true, the antecedent is false and truth of the principle is vacuous. Thus, M_d is not to be blamed for the conflation of D_d and Con_d. On the other hand, the right to left direction of the biconditional is problematic for it says that whenever there is a moral conflict, a contradiction will be obligatory. Though an ethicist might claim that moral conflicts exist, it would be difficult to see why they would want to hold that contradictions are obligatory. These obligations can never be fulfilled and thus, cannot be part of morality proper. Thus, C_d, the only principle relevant in the right-to-left conditional, is to be rejected if one wants to hold that there are moral dilemmas.

In summary, if there are genuine moral dilemmas then D_d has to be rejected, but then either OIC_{d-a} or C_d will also have be rejected; the former follows from the latter. On the proposed logic, C_d has to be rejected because OIC_{d-a} is fundamental. A further reason that C_d should be rejected instead of OIC_{d-a} is that C_d conflates two principles, D_d and Con_d, the former which needs to be rejected in systems with moral dilemmas whereas the latter is clearly valid.

2.3.3 Moral Dilemmas and Inconsistency: ME_{d-a}

The principle ME_{d-a} $\Box(A \rightarrow B) \rightarrow OA \rightarrow OB$ states that if one ought to do A and it is logically necessary that if one does A, one does B then one ought to do B as well. The principle has other forms

I. $\Box(A \rightarrow B) \wedge OA \rightarrow OB$
II. $\neg\Diamond(A \wedge B) \rightarrow (OA \rightarrow O\neg B)$
III. $OA \wedge \neg\Diamond(A \wedge B) \rightarrow O\neg B$

ME_{d-a} is quite weak given that it concerns logical necessity. Nevertheless, it is inconsistent with OAP_{d-a} and simple necessary sentences, $\Box B$ and $\Box(A \rightarrow B)$.

Proof 2.3.4 Inconsistency $\{\Box(A \rightarrow B), \Box B, OA, ME_{d-a}, OAP_{d-a}\}$

1.	$\Box(A \rightarrow B)$	Assumption
2.	$\Box B$	Assumption
3.	OA	Assumption
4.	$\Box(A \rightarrow B) \wedge OA$	1, 3 PL
5.	$\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$	ME_{d-a}
6.	$(\Box(A \rightarrow B) \wedge OA) \rightarrow OB$	5 PL
7.	$OB \rightarrow \Diamond\neg B$	OAP_{d-a} instance
8.	$(\Box(A \rightarrow B) \wedge OA) \rightarrow \Diamond\neg B$	6, 7 PL
9.	$\Diamond\neg B$	4, 8 MP
10.	$\neg\Diamond\neg B$	2, Df. N_a

In its stronger form, with the personal sense of possibility, ME_{d-a} is intuitive in many cases. If I ought to cook you dinner and I cannot cook a dinner without opening the stove then I ought to open the stove. The principle tells us that if you ought to do something and the only way to do it is by doing something else then you ought to do that. With respect to moral dilemmas, the stronger principle is problematic. In conjunction with D_d it leads to an inconsistency.

Proof 2.3.5 Inconsistency $\{MD_{d\text{-}a}, ME'_{d\text{-}a}, D_d\}$ (Brink 1994)

1. $OA \wedge OB \wedge \neg\Diamond'(A \wedge B)$	$MD_{d\text{-}a}$
2. OA	1, PL
3. $\neg\Diamond'(A \wedge B)$	1, PL
4. $\neg\neg\Box'\neg(A \wedge B)$	3, Df. $\Diamond'$
5. $\Box'\neg(A \wedge B)$	4, PL
6. $\Box'\neg(A \wedge B) \wedge OA$	2, 5 PL
7. $\Box'(A \rightarrow \neg B) \rightarrow (OA \rightarrow O\neg B)$	$ME'_{d\text{-}a}$ instance
8. $\Box'\neg(A \wedge B) \rightarrow (OA \rightarrow O\neg B)$	7, PL, $REP_{d\text{-}a}$
9. $(\Box'\neg(A \wedge B) \wedge OA) \rightarrow O\neg B$	8, PL
10. $O\neg B$	6, 9 MP
11. OB	1, PL
12. $OB \wedge O\neg B$	10, 11 PL
13. $OB \rightarrow PB$	D_d
14. $OB \rightarrow \neg O\neg B$	13, Df. P_d
15. $\neg(OB \wedge O\neg B)$	14, PL

Lines 12 and 15 contradict and so, either D_d or $ME_{d\text{-}a}$ are invalid. It may seem that just as D_d is the only culprit in this special case of moral dilemmas (where $OA \wedge O\neg A$), so it is in the more general case (in Proof 2.3.5), indicating that $ME'_{d\text{-}a}$ is of no concern. Although D_d must be discarded if there are moral dilemmas, $ME'_{d\text{-}a}$ is also problematic and should be discarded despite the fact that it alone does not lead to a proper contradiction in (Proof 2.3.5). To see the problem it is enough to examine lines 1 – 12, since up to that point ME' is the only relevant principle. Line 12, $OB \wedge O\neg B$ is not a contradiction, however, it is problematic in several ways. Unease with this consequence is expressed by

Lemmon (1965), Brink (1994) but an explanation of the unease is wanting. Sinnott-Armstrong states the concern: "If we assume ME, $OA \wedge OB \wedge \neg\Diamond'(A \wedge B)$ formulation [of a moral dilemma] entails the $OA \wedge O\neg A$ formulation." (Sinnott-Armstrong 1988, pg. 123) Sophie's dilemma illustrates the problem. She has an obligation to save both of her children. Suppose the distraught mother, does the unthinkable, she decides to watch TV instead. It does not seem that she has done the right thing, no matter how difficult the choice was for her. Yet, $ME'_{d\text{-}a}$ states that from a moral dilemma, $OA \wedge O\neg A$ follows. The mother in not choosing either child has fulfilled one obligation, namely $O\neg A$. But this seems to be plainly wrong. This is not an obligation and it should not be allowed into morality. This does not preclude moral dilemmas of the sort where one is obligated to do something and the negation of that very thing. These dilemmas do arise, but they are not the only ones. Sophie's choice, a paradigmatic case of a moral dilemma is not such a case.

Independently of moral dilemmas, $ME_{d\text{-}a}$ inherits the difficulties of M_d, for the latter can be derived from the former.

Proof 2.3.6

1.	$A \rightarrow B$	Assumption
2.	$\Box(A \rightarrow B)$	1, RN_a
3.	$\Box(A \rightarrow B) \rightarrow OA \rightarrow OB$	$ME_{d\text{-}a}$
4.	$OA \rightarrow OB$	2, 3 MP

Thus, the problematic counter-examples and the general problem of M_d requiring only perfect obligations is a reason to reject $ME_{d\text{-}a}$ as well.

2.3.4 Conclusion on Moral Dilemmas

The debate about moral dilemmas is complex. The existence of moral dilemmas was not defended but the discussion assumed that they exist. If there are genuine moral dilemmas then D_d is straightforwardly rejected. Also, either C_d or $OIC_{d\text{-}a}$ has to be given up. $OIC_{d\text{-}a}$ is taken as fundamental and therefore, C_d has to be rejected, but also for the independent reason that C_d is the culprit in the conflation of D_d and Con_d. The principle $ME_{d\text{-}a}$ has to be rejected with the adoption of $OAP_{d\text{-}a}$, but it was shown that this principle is not valid because it makes every moral dilemma into a special case of a moral dilemma where OA and $O\neg A$. This was shown to be problematic because obligations could be fulfilled by failing to fulfill either obligation, and this is clearly wrong. If the previous reasoning is cogent, then there are few principles left in deontic systems with the acceptance of moral dilemmas. Moreover, $ME_{d\text{-}a}$ inherits the problems of M_d.

To the weakest proposed system, $E\nabla$ nothing can be added. In the section above, it was argued that Con_d $\neg O\bot$ is valid, but this is a consequence of $OIC_{d\text{-}a}$.

However, this is not an idiosyncrasy of $E\nabla_{d\text{-}a}$. Independent reasons were given for the rejection of C_d, $ME_{d\text{-}a}$, and D_d that require these principles to be invalid in KD_d and $KD+_{d\text{-}a}$. All that remains is Con_d and M_d, which is consistent with moral dilemmas. A weak monotonic system (EMCon) is left with Con_d (See Figure 2.0.1). Given that there are moral dilemmas, normal systems are better in that they allow for M_d whereas $E\nabla_{d\text{-}a}$ systems do not. Chellas' deontic system is one example (1980, ch. 6). However, if the reasoning for the rejection of M_d is correct, the two systems remain comparable. Therefore, an advantage for either the E∇ systems or KD systems has not been gained here.

2.4 Substantive Ethics

2.4.1 Error Theorists

Error theorists such as J. Mackie argue that all moral judgments are false (Mackie 1977, ch. 1). Roughly, he holds two theses: the conceptual or cognitivist thesis that our moral judgments are beliefs that are truth apt and the ontological thesis that there are no moral facts in the world (Miller 2003, pg. 112). From this Mackie concludes that all our moral judgments are false because there are no moral facts to make the beliefs true. To be an error theorist is to hold the following valid deontic principle: $\neg OA$, where A is any sentence whatsoever. This thesis is inconsistent with RN_d and thus KD and KD+ systems. On the other hand, it is consistent with E∇ systems. The remaining principles of deontic logic are entirely consistent because they are conditional, making for example principle D_d $OA \rightarrow PA$, etc. vacuously valid because the antecedent in each case is false. Error theorists hold that all permissions are false i.e. $\neg PA$. A theorem of E∇ is that all contradictions and necessary sentences are permissible, thus, this account is committed to some moral judgments and inconsistent with incompatibilist logics like E∇.

2.4.2 Absolutism

This section will examine the consistency of E∇ with basic tenets of absolutist ethical views, in comparison to KD deontic logics. It may be thought that contingent obligations preclude the *possibility* of absolute obligation common in absolutist ethical theories. This is a confusion of two different but similar principles. Formally, an absolute obligation is written as '$\Box OA$'. A contingent obligation is OA, where $\Diamond A$ and $\Diamond \neg A$. Whatever is obligatory may be contingent, while the obligation itself may be necessary. It may be absolutely obligatory to honour your parents, i.e. $\Box OA$, where 'A' means honouring your parents but it is certainly contingent that parents are honoured. There are those who honour them and those that do not.

Here are several principles absolute theorists might find attractive.

1. $\Box A \wedge OA \rightarrow \Box OA$
2. $OA \rightarrow \Box OA$
3. $\Box OA \rightarrow O\Box A$
4. $\Box OA \rightarrow O\Diamond A$

The first principles $OA \wedge \Box A \rightarrow \Box OA$ says that if the obligatory is necessary then it is necessarily or absolutely obligatory. It states that necessary obligations cannot be obligatory in one world but not in another. This principle follows from $MIO_{d\text{-}a}$ but not vice versa (Proof 1.5.20-App). It is also a consequence of $E\nabla$ but it is entirely vacuous in those systems. If the proposition expressed by A belongs to the set of necessary propositions at some world then it belongs to the set of necessary propositions at every other world. Since, A is necessary at every world then it is not obligatory at any world, and so, the antecedent of $\Box A \wedge OA \rightarrow \Box OA$ will be false at every world. This principle does involve absolute obligations but it does not seem to be significant.

Absolutists of various sorts will accept principle (2) $OA \rightarrow \Box OA$. This principle is consistent with $OI\nabla$ as well as KD+. It says that whatever is obligatory is obligatory in all possible worlds. For instance, refraining from indiscriminate killing is sometimes held to be an obligation that is true in all possible worlds.

A further problem with iterated modalities and absolute obligations concerns principles (3) and (4). It seems appropriate to say that if all obligations are absolute then it ought to be necessarily true that A. Since all obligations are true at all worlds then it ought to be that what the obligation prescribes is true at all worlds, or succinctly it ought to be that all obligations are fulfilled. It was shown in 2.2.5 that obligations of alethic sentences are false, but then $\Box OA \rightarrow O\Box A$ is invalid. The same is true for $\Box OA \rightarrow O\Diamond A$. If an obligation holds in every world then it seem reasonable to say that it ought to be possible that A occurs, or it ought to be possible that the obligation is fulfilled.

Another consequence for absolutist views is that they cannot hold iterated obligations or prohibitions. Proof 2.2.1 revealed that iterated obligations are inconsistent with absolute obligations. So, whatever is absolutely obligatory, say OA, once iterated i.e. OOA, is false.

In summary, it is perfectly consistent to hold absolute obligations with the forgoing account. Furthermore, the stronger principle, $OA \rightarrow \Box OA$ that requires that all obligations are necessary is also consistent with E∇ systems. Thus far, KD+ and E∇ systems agree. KD+ is superior in the sense that an absolutist theorist might prefer principles (3) and (4), and iterated obligations, whereas in E∇ this is not possible.

2.4.3 Hypothetical Imperative

An example of the hypothetical imperative is "If I want to learn how to drive then I ought to take a driver's ed." The form of this sentence is $A \rightarrow OB$. This form does not pick out only hypothetical imperatives because, for instance, the categorical imperative can be an instance of the same form. What is essential to a hypothetical syllogism is that the condition is some end, and the ought is some means of doing that. (Darwall 1998, pg. 156) According to Kant, this principle is entailed by the idea of a rational agent pursuing an end. If one wills an end, then one wills the means of attaining that end. There may be more than one means of attaining that end, but certain means are indispensable. A Kant says, "whoever wills an end, ... wills also the means which are indispensably necessary and in his power." (Darwall 1998, pg. 157) It would be incoherent to want to learn how to drive and to believe that the only way to learn is to take driver's ed, and then *not* will to take driver's ed. Kant is referring to the aforementioned principle $ME_{d\text{-}a}$: $\Box(A \rightarrow B) \rightarrow OA \rightarrow OB$, where A is some end willed, and B is the means to that end.

This principle is not particular to Kant – others have adopted it. It is quite common in the natural law tradition. For instance, Pufendorf takes natural laws teach how a man should conduct himself, to become a good member of human society. The fundamental natural law is that every man must cherish and maintain sociability, so far as in him lies. Pufendorf uses the $ME_{d\text{-}a}$ principle to argue for specific obligations from his fundamental natural law.

A = maintains sociability
B = doing something that maintains sociability

1.	OA	Assumption
2.	$\Box(A \rightarrow B)$	Assumption
3.	$\Box(A \rightarrow B) \rightarrow OA \rightarrow OB$	$ME_{d\text{-}a}$
4.	Therefore, OB	2, 3 MP

Puffendorf's fundamental obligation is that sociability ought to be maintained (OA). Suppose that keeping promises is necessary for maintaining sociability. By the $ME_{d\text{-}a}$ principle it follows that promises ought to be kept. Thus, although absolutes are consistent with E∇ account, they cannot be held by principles that argue from means to ends.

2.4.4 Naturalism

The basic ethical naturalist claim is that moral judgments are rendered true or false by a natural state of affairs and it is the natural state of affairs to which a true moral judgment affords us access. (Miller 2003, pg. 4) A natural state of affairs is a property, which figures in one of the natural science or in psychology. A naturalist might define the good in terms of the following natural states of affairs: tending towards individual well-being, or producing the greatest happiness, or adhering to the conventions of society. The definition of naturalism does not preclude a priori access to the good. Take Michael Smith's view. He holds that "our judgments about what we are morally required to do are simply judgments about what the categorical requirements of rationality or reason demand of us." (Smith 1994) This definition is known a priori while it is naturalist. The specifics of what reason or rationality requires of us is a question for psychology and that is investigated through empirical means.

Keeping the a priori naturalist in mind, the following principles are considered:

1. $NAT_{d\text{-}a}\ OA \rightarrow \Diamond\neg OA$
2. $FAT_{d\text{-}a}\ FA \rightarrow \Diamond\neg FA$

Since, a moral judgment is rendered true or false by a natural state of affairs, whatever natural properties those obligations have, they could turn out to be false as with all natural states of affairs.

However, the KD+ account is inconsistent with these principles.

Proof 2.4.1 Inconsistency { $\Box A$, $MIO_{d\text{-}a}$, $NAT_{d\text{-}a}$}

1. $\Box A$	Assumption
2. OA	$MIO_{d\text{-}a}$
3. $\Box A \wedge OA$	1, 2 PL
4. $\Box A \rightarrow OA$	$MIO_{d\text{-}a}$
5. $\Box\Box A \rightarrow \Box OA$	4, RM_a
6. $\Box\Box A \leftrightarrow \Box A$	Theorem KD+
7. $\Box A \rightarrow \Box OA$	5, 6 PL
8. $\Box OA$	1, 7 MP
9. $OA \rightarrow \Diamond\neg OA$	$NAT_{d\text{-}a}$
10. $\Diamond\neg OA$	2, 9, MP
11. $\neg\Diamond\neg OA$	8, Df. $\Box_a$

Having at least one necessary sentence is inconsistent with holding $MIO_{d\text{-}a}$ and the naturalist principles $NAT_{d\text{-}a}$. A similar proof can be shown for $FAT_{d\text{-}a}$. Note that the principle on line 6 is a reduction law that follows from 4_a and 5_a. Therefore, a philosophical consequence of having $MIO_{d\text{-}a}$ is that one cannot hold these principles of naturalism. On the other hand, these two principles are consistent with $E\nabla$ systems.

However, the apparent progress is stymied by the need of naturalists to extend the principles to permissions. For the same reasons as for obligations and prohibition, one would expect that for whatever is permissible, it is possible that it is not permissible i.e. PA → ◊¬PA. It is hard to see why the first two would be held without the third.

2.4.5 Ideal Agents

A peculiar consequence of this logic relates to ideal agents. Consider the argument:

X is a some morally good act

1. □"(God does X)
2. O(God does X) → ◊"¬(God does X) OAP$_{d\text{-}a}$ instance
3. □"(God does X) → ¬O(God does X) 2, PL
4. Therefore, ¬O(God does X), for any X 1, 3 MP

The sense of necessity, indicated by "□"", is the metaphysical sense of necessity. In terms of possible worlds, metaphysical necessity is sometimes equated with logical necessity, in that metaphysically necessary truths are true in all possible worlds. Since, God is impeccable he cannot act in an evil way in any possible world, for that would not be God. But, since doing the obligatory requires that something other than that can be done, God does not have any obligations. The same argument can be made for prohibition with the same conclusion. Bringing about war, pestilence, famine and drought are actions not prohibited to God.

Whether God is amoral or whether God has obligations is a topic in the philosophy of religion. In recent commentary, the issue has been resurrected with a similar argument (Alston 1990):

1. An agent has a moral obligation to do an action only if there is a general

moral principle that plays a governing or regulative role

2. A general moral principle plays a governing or regulative role for agents only if it is possible that they violate that principles
3. It isn't possible that God violate any general moral principles
4. Therefore, God does not have any moral obligations

This argument is similar to the first as it is does not involve, in premise 3, the principle $OA \rightarrow \Diamond\neg A$ but rather $OB_{d\text{-}a}$ $OA \rightarrow \Diamond(OA \wedge \neg A)$. An obligation can be violated only if it is possible for that obligation to occur and it not be satisfied. This principle is consistent with E∇ and not with KD+ systems (Proof 2.4.2). Thus, the proposed system is fruitful because accommodates such reason whereas KD and KD+ do not.

2.5 Conclusion

In conclusion, several principles were examined in roughly three groups. The KD principles were examined and it was concluded that the two systems are comparable, given that the independent reasons for the rejection of M_d are cogent. The reason for its rejection was a variant of the Good Samaritan paradox that showed M_d requiring obligations that were clearly false. This yielded system E∇C, which contained D_d as theorem but not M_d, and N_d.

Iterated deontic and nested principles were taken up in 2.2. The principles OU_d $O(OA \rightarrow A)$ and $O4_d$ $OA \rightarrow OOA$ could not be added to E∇. More importantly, it was shown that sentences that contained alethic sentences within a deontic operator where all false. This was problematic on its own and because several intuitive principles had to be rejected. These principles worked perfectly fine with KD. The only iterated principle that could be added was $O4C_d$ $OOA \rightarrow OA$. Despite this addition, iterated principles are a significant problem for E∇ systems.

With respect to moral dilemmas, the situation was equally grim. Virtually every principle had to be thrown out given the existence of moral dilemmas and no advantage was seen from either. It was shown that C_d had to go, because it conflicted with OIC_d. D_d could not be held as well. Principle $ME_{d\text{-}a}$ $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$, though it need not be removed from KD from any inconsistency, had to be for other reasons. The weakest system E∇ was left, while the KD system was reduced to EMCon. Again, if the reasoning for the rejection of M_d is correct the two systems are comparable (Con_d is a consequences of EY).

Certain issues with substantive ethics were examined. It was shown that absolutist ethics is compatible with the proposed account in a limited way. Iterated obligations could not be held with absolute obligations. Also, hypothetical imperatives had to be rejected. A priori naturalist ethics showed no advantage to the proposed account or to the established one. God was shown to be amoral on the proposed account, a consequence favored by some in theological philosophy. An incompatibilist is committed to these results. In general, those that accept contingent obligations are also committed to these consequences.

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Appendix: Proofs

Proof 1.5.1-App RE_d $A \leftrightarrow B$ / $OA \leftrightarrow OB$

1.	$A \leftrightarrow B$	Assumption
2.	$A \rightarrow B$	1, PL
3.	$O(A \rightarrow B)$	2, RN_d
4.	$O(A \rightarrow B) \rightarrow OA \rightarrow OB$	K_d
5.	$OA \rightarrow OB$	3, 4 MP
6.	$B \rightarrow A$	1, PL
7.	$O(B \rightarrow A)$	6, RN_d
8.	$O(B \rightarrow A) \rightarrow OB \rightarrow OA$	K_d instance
9.	$OB \rightarrow OA$	7, 8 MP
10.	$OA \leftrightarrow OB$	5, 9 PL

Proof 1.5.2-App: REP_d $B \leftrightarrow B'$ / $A \leftrightarrow A[B/B']$

- Hypothesis: $B \leftrightarrow B'$
- Prove: $A \leftrightarrow A[B/B']$
- $A[B/B']$ means any sentence that results from A by replacing *zero or more* occurrences of B, in A, by B'

- Suppose A = B
    - Either 0 replacements or more than 0 replacements of occurrences
    - For 0 replacements
        - $A[B/B'] = A$
        - Therefore, $A \leftrightarrow A[B/B']$ since, $A \leftrightarrow A$
    - For more than zero replacements
        - $A[B/B'] = B'$, since $A = B$
        - Therefore, $A \leftrightarrow A[B/B']$ since, $B \leftrightarrow B'$, $A = B$
- Suppose A and B are distinct
- Prove: the result holds for A
- Base Case:
    - $A = \mathbb{P}_n$, i.e. A is atomic
        1. $\mathbb{P}_n[B/B'] = \mathbb{P}_n$, since $\mathbb{P}_n$ and B are distinct
        2. Therefore, $A \leftrightarrow A[B/B']$, since $\mathbb{P}_n \leftrightarrow \mathbb{P}_n$
    - $A = \bot$
        1. $\bot[B/B'] = \bot$, since $\bot$ and B are distinct
        2. Therefore, $A \leftrightarrow A[B/B']$, since $\bot \leftrightarrow \bot$

- $A = \top$
  1. $\top[B/B'] = \top$, since $\top$ and B are distinct
  2. Therefore, $A \leftrightarrow A[B/B']$, since $\top \leftrightarrow \top$

Inductive Hypothesis (ih): complexity of A, i.e. the result holds out for all sentences shorted than A

- $A = \neg C$
  1. $C \leftrightarrow C[B/B']$ (ih)
  2. $\neg C \leftrightarrow \neg(C[B/B'])$ 1, PL
  3. $\neg C \leftrightarrow \neg C[B/B']$ 2, PL i.e. $\neg(C[B/B']) = \neg C[B/B']$
  4. $A \leftrightarrow A[B/B']$ 3, $A = \neg C$
- $A = C \vee D$
  1. $C \leftrightarrow C[B/B'] \wedge D \leftrightarrow D[B/B']$ (ih)
  2. $C \vee D \leftrightarrow (C \vee D)[B/B']$ 1, PL
  3. $A \leftrightarrow A[B/B']$ 2, $A = C \vee D$
- $A = C \wedge D$
  1. $C \leftrightarrow C[B/B'] \wedge D \leftrightarrow D[B/B']$ (ih)
  2. $C \wedge D \leftrightarrow (C \wedge D)[B/B']$ 1, PL
  3. $A \leftrightarrow A[B/B']$ 2, $A = C \vee D$
- $A = C \rightarrow D$
  1. $C \leftrightarrow C[B/B'] \wedge D \leftrightarrow D[B/B']$ (ih)
  2. $C \leftrightarrow C[B/B'] \leftrightarrow D \leftrightarrow D[B/B']$ 1, PL
  3. $C \rightarrow D \leftrightarrow (C[B/B'] \rightarrow D[B/B'])$ 2, PL
  4. $C \rightarrow D \leftrightarrow (C \rightarrow D)[B/B']$
     since, $(C \rightarrow D)[B/B'] = (C[B/B'] \rightarrow D[B/B'])$
  5. $A \leftrightarrow A[B/B']$ 4, $A = C \rightarrow D$
- $A = OC$
  1. $C \leftrightarrow C[B/B']$ (ih)
  2. $OC \leftrightarrow O(C[B/B'])$ 1, RE_d
  3. $OC \leftrightarrow OC[B/B']$ since, $O(C[B/B']) = OC[B/B']$
  4. $A \leftrightarrow A[B/B']$ 3, $A = OC$
- $A = PC$
  1. $C \leftrightarrow C[B/B']$ (ih)
  2. $\neg C \leftrightarrow \neg(C[B/B'])$ 1, PL
  3. $\neg C \leftrightarrow \neg C[B/B']$ 2, PL
  4. $O\neg C \leftrightarrow O\neg C[B/B']$ 3, RE_d
  5. $\neg O\neg C \leftrightarrow \neg O\neg C[B/B']$ 4, PL
  6. $PC \leftrightarrow PC[B/B']$ 5, Df. P_d
  7. $A \leftrightarrow A[B/B']$ 6, $A = PC$

- $A = FC$

1.	$C \leftrightarrow C[B/B']$	(ih)
2.	$\neg C \leftrightarrow \neg C[B/B']$	1, PL
3.	$O\neg C \leftrightarrow O\neg(C[B/B'])$	1, RE_d
4.	$O\neg C \leftrightarrow O\neg C[B/B']$	since, $O(C[B/B']) = OC[B/B']$
5.	$FC \leftrightarrow FC[B/B']$	4, Df. F_d
6.	$A \leftrightarrow A[B/B']$	5, $A = FC$

Proof 1.5.3-App C_d $OA \wedge OB \rightarrow O(A \wedge B)$

1.	$(A \wedge B) \rightarrow (A \wedge B)$	PL
2.	$A \rightarrow (B \rightarrow (A \wedge B))$	1, PL
3.	$OA \rightarrow O(B \rightarrow (A \wedge B))$	2, RM_d
4.	$O(B \rightarrow (A \wedge B)) \rightarrow (OB \rightarrow O(A \wedge B))$	K_d instance
5.	$OA \rightarrow (OB \rightarrow O(A \wedge B))$	3, 4, PL
6.	$(OA \wedge OB) \rightarrow O(A \wedge B)$	5, PL

Proof 1.5.4-App M_d $O(A \wedge B) \rightarrow OA \wedge OB$

1.	$A \wedge B \rightarrow B$	PL
2.	$O(A \wedge B) \rightarrow OB$	1, RM_d
3.	$A \wedge B \rightarrow A$	PL
4.	$O(A \wedge B) \rightarrow OA$	3, RM_d
5.	$O(A \wedge B) \rightarrow OA \wedge OB$	2, 4 PL

Proof 1.5.5-App RM_d $A \rightarrow B$ / $OA \rightarrow OB$

1.	$A \rightarrow B$	Assumption
2.	$O(A \rightarrow B)$	1, RN_d
3.	$O(A \rightarrow B) \rightarrow OA \rightarrow OB$	K_d
4.	$OA \rightarrow OB$	2, 3 MP

Proof 1.5.6-App Con_d $\neg O\bot$

1.	$OA \rightarrow PA$	D_d
2.	$OA \rightarrow \neg O\neg A$	1, Df. P_d
3.	$\neg(OA \wedge O\neg A)$	2, PL
4.	$O(A \wedge \neg A) \rightarrow (OA \wedge O\neg A)$	M_d instance
5.	$\neg(OA \wedge O\neg A) \rightarrow \neg O(A \wedge \neg A)$	4, PL
6.	$\neg O(A \wedge \neg A)$	3, 5 MP

Proof 1.5.7-App $N_d\ O\top$

1. $\top$ PL
2. $O\top$ 1, RN_d

Proof 1.5.8-App $OA \rightarrow O(A \vee B)$

1. $A \rightarrow A \vee B$ PL
2. $OA \rightarrow O(A \vee B)$ 1, RM_d

Proof 1.5.9-App $FA \rightarrow F(A \wedge B)$

1. $\neg A \rightarrow \neg A \vee \neg B$ PL
2. $O\neg A \rightarrow O(\neg A \vee \neg B)$ 1, RM_d
3. $FA \rightarrow O(\neg A \vee \neg B)$ 2, Df. F_d
4. $FA \rightarrow O\neg(A \wedge B)$ 3, PL, REP_d
5. $FA \rightarrow F(A \wedge B)$ 4, Df. F_d

Proof 1.5.10-App $PA \rightarrow P(A \vee B)$

1. $\neg A \wedge \neg B \rightarrow \neg A$ PL
2. $O(\neg A \wedge \neg B) \rightarrow O\neg A$ 1, RM_d
3. $\neg O\neg A \rightarrow \neg O(\neg A \wedge \neg B)$ 2, PL
4. $PA \rightarrow \neg O(\neg A \wedge \neg B)$ 3, Df. P_d
5. $PA \rightarrow \neg O\neg(A \vee B)$ 4, PL, REP_d
6. $PA \rightarrow P(A \vee B)$ 5, Df. P_d

Proof 1.5.11-App $OA \vee OB \rightarrow O(A \vee B)$

1. $A \rightarrow A \vee B$ PL
2. $OA \rightarrow O(A \vee B)$ 1, RM_d
3. $B \rightarrow (A \vee B)$ PL
4. $OB \rightarrow O(A \vee B)$ 3, RM_d
5. $OA \vee OB \rightarrow O(A \vee B)$ 2, 4 PL

Proof 1.5.12-App $P(A \wedge B) \rightarrow PA \wedge PB$

1.	$\neg A \rightarrow \neg A \vee \neg B$	PL
2.	$O\neg A \rightarrow O(\neg A \vee \neg B)$	1, RM_d
3.	$\neg O(\neg A \vee \neg B) \rightarrow \neg O\neg A$	2, PL
4.	$\neg\neg P\neg (\neg A \vee \neg B) \rightarrow \neg O\neg A$	3, Df. O_d, REP_d
5.	$P\neg(\neg A \vee \neg B) \rightarrow \neg O\neg A$	4, PL
6.	$P\neg(\neg A \vee \neg B) \rightarrow PA$	5, Df. P_d
7.	$P(A \wedge B) \rightarrow PA$	6, PL, REP_d
8.	$\neg B \rightarrow \neg A \vee \neg B$	PL
9.	$O\neg B \rightarrow O(\neg A \vee \neg B)$	8, RM_d
10.	$\neg O(\neg A \vee \neg B) \rightarrow \neg O\neg B$	9, PL
11.	$\neg\neg P\neg(\neg A \vee \neg B) \rightarrow \neg O\neg B$	10, Df. O_d, REP_d
12.	$P\neg(\neg A \vee \neg B) \rightarrow \neg O\neg B$	11, PL
13.	$P\neg(\neg A \vee \neg B) \rightarrow PB$	12, Df. P_d
14.	$P(A \wedge B) \rightarrow PB$	13, PL, REP_d
15.	$P(A \wedge B) \rightarrow PA \wedge PB$	7, 14 PL

Proof 1.5.13-App $A \rightarrow B$ / $FB \rightarrow FA$

1.	$A \rightarrow B$	Assumption
2.	$\neg B \rightarrow \neg A$	1, PL
3.	$O\neg B \rightarrow O\neg A$	2, RM_d
4.	$FB \rightarrow FA$	3, Df. F_d

Proof 1.5.14-App $O(A \wedge B) \rightarrow OA$

1.	$A \wedge B \rightarrow A$	PL
2.	$O(A \wedge B) \rightarrow OA$	1, RM_d

Proof 1.5.15-App A / PA

1.	A	Assumption
2.	$\Box A$	1, RN_a
3.	$O\neg A \rightarrow \Diamond\neg A$	$OIC_{d\text{-}a}$ instance
4.	$\neg\Diamond\neg A \rightarrow \neg O\neg A$	3, PL
5.	$\Box A \rightarrow \neg O\neg A$	4, Df. $\Box_a$
6.	$\Box A \rightarrow PA$	5, Df. P_d
7.	PA	2, 6 MP

Proof 1.5.16-App Containment Proof: KD+ within Anderson's

Anderson's System (AS)

1. PL All tautologies of propositional logic
2. K_a $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
3. Q_a $\Diamond Q$
4. Df. O_d $OA = \Box(Q \rightarrow A)$
5. Df. P_d $PA = \Diamond(Q \wedge A)$
6. RN_a $A / \Box A$
7. MP $A \rightarrow B, A / B$

'Q' is a propositional constant meaning "what morality prescribes"

KD+ System

1. PL All tautologies of propositional logic
2. K_d $O(A \rightarrow B) \rightarrow OA \rightarrow OB$
3. D_d $OA \rightarrow PA$
4. MP $A \rightarrow B, A / B$
5. $MIO_{d\text{-}a}$ $\Box A \rightarrow OA$

Containment Proof: All schemas and rules of inference of KD+ are derivable in AS.

1. Proof $\vdash_{AS}$ PL
   1. PL is an Axiom of AS
2. Proof $\vdash_{AS}$ MP
   1. MP is a rule of inference of AS
3. Proof $\vdash_{AS}$ $MIO_{d\text{-}a}$
   1. $A \rightarrow Q \rightarrow A$ — PL
   2. $\Box(A \rightarrow (Q \rightarrow A))$ — 1, RN_a
   3. $\Box(A \rightarrow (Q \rightarrow A)) \rightarrow \Box A \rightarrow \Box(Q \rightarrow A)$ — K_d instance
   4. $\Box A \rightarrow \Box(Q \rightarrow A)$ — 2, 3 MP
   5. $\Box A \rightarrow OA$ — 4, Df. O_d
4. Proof $\vdash_{AS}$ D_d
   1. $OA \wedge O\neg A \rightarrow O(A \wedge \neg A)$ — C_d instance
   2. $\neg O(A \wedge \neg A) \rightarrow \neg(OA \wedge O\neg A)$ — 1, PL
   3. $\neg\Box((Q \rightarrow A) \wedge (Q \rightarrow \neg A)) \rightarrow \neg(OA \wedge O\neg A)$ — 2, Df. O_d
   4. $\neg\Box(Q \rightarrow (A \wedge \neg A)) \rightarrow \neg(OA \wedge O\neg A)$ — 3, PL, REP
   5. $\neg\Box(\neg(A \wedge \neg A) \rightarrow \neg Q) \rightarrow \neg(OA \wedge O\neg A)$ — 4, PL, REP
   6. $\neg\Box(\top \rightarrow \neg Q) \rightarrow \neg(OA \wedge O\neg A)$ — 5, PL, REP
   7. $\neg\Box\neg Q \rightarrow \neg(OA \wedge O\neg A)$ — 6, PL, REP
   8. $\Diamond Q$ — Q_a
   9. $\neg\Box\neg Q$ — 8, Df. $\Box_a$
   10. $\neg(OA \wedge O\neg A)$ — 7, 9 MP

11. $OA \to \neg O \neg A$ 10, PL

12. $OA \to PA$ 11, Df. P_d

5. Proof $\vdash_{AS} K_d$
   1. $(Q \to (A \to B)) \to [(Q \to A) \to (Q \to B)]$ PL
   2. $\Box(Q \to (A \to B)) \to \Box[(Q \to A) \to (Q \to B)]$ 1, RM_a
   3. $O(A \to B) \to \Box[(Q \to A) \to (Q \to B)]$ 2, Df. O_d
   4. $\Box[(Q \to A) \to (Q \to B)] \to \Box(Q \to A) \to \Box(Q \to B)$ K_a
   5. $O(A \to B) \to \Box(Q \to A) \to \Box(Q \to B)$ 4, Df. O_d
   6. $O(A \to B) \to OA \to OB$ 5, Df. O_d
6. Proof $\vdash_{AS} RN_d$
   1. A Assumption
   2. $Q \to A$ 1, PL
   3. $\Box(Q \to A)$ 2, RN_a
   4. OA 3, Df. O_d

Proof 1.5.17-App $REP_{d\text{-}a}$ $B \leftrightarrow B' / A \leftrightarrow A[B/B']$

- REP_d (Proof 1.5.2-App) with the two additional conditions:
- $A = \Box C$
  1. $C \leftrightarrow C[B/B']$ (ih)
  2. $\Box C \leftrightarrow \Box(C[B/B'])$ 1, RE_a
  3. $\Box C \leftrightarrow \Box C[B/B']$ 2, $\Box(C[B/B']) = \Box C[B/B']$
  4. $A \leftrightarrow A[B/B']$ 3, $A = \Box C$
- $A = \Diamond C$
  1. $C \leftrightarrow C[B/B']$ (ih)
  2. $\neg C \leftrightarrow \neg(C[B/B'])$ 1, PL
  3. $\neg C \leftrightarrow \neg C[B/B']$ 2, PL
  4. $\Box\neg C \leftrightarrow \Box\neg C[B/B']$ 3, RE_a
  5. $\neg\Box\neg C \leftrightarrow \neg\Box\neg C[B/B']$ 4, PL
  6. $\Diamond C \leftrightarrow \Diamond C[B/B']$ 5, Df. $\Diamond_a$
  7. $A \leftrightarrow A[B/B']$ 6, $A = \Diamond C$

Proof 1.5.18-App $OIC_{d\text{-}a}$ $OA \to \Diamond A$

1. $\Box\neg A \to O\neg A$ $MIO_{d\text{-}a}$ instance
2. $O\neg A \to P\neg A$ D_d instance
3. $\Box\neg A \to P\neg A$ 1, 2 PL
4. $\neg P\neg A \to \neg\Box\neg A$ 3, PL
5. $OA \to \Diamond A$ 4, Df. O_d

Proof 1.5.19-App $ME_{d\text{-}a}$ $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$

1. $\Box(A \rightarrow B) \rightarrow O(A \rightarrow B)$ — $MIO_{d\text{-}a}$ instance
2. $O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$ — K_d
3. $\Box(A \rightarrow B) \rightarrow (OA \rightarrow OB)$ — 1, 2 PL

Proof 1.5.20-App $\Box A \wedge OA \rightarrow \Box OA$

1. $\Box A \rightarrow OA$ — $MIO_{d\text{-}a}$
2. $\Box\Box A \rightarrow \Box OA$ — 1, RM_a
3. $\Box\Box A \leftrightarrow \Box A$ — Theorem 1.5.2 (7)
4. $\Box A \rightarrow \Box OA$ — 2, 3 PL
5. $\Box A \wedge OA \rightarrow \Box OA$ — 4, PL

Proof 1.5.21-App $PA \rightarrow \Diamond A$

1. $\Box\neg A \rightarrow O\neg A$ — MIO_d instance
2. $\neg O\neg A \rightarrow \neg\Box\neg A$ — 1, PL
3. $PA \rightarrow \neg\Box\neg A$ — 2, Df. P_d
4. $PA \rightarrow \Diamond A$ — 3, Df. $\Diamond_a$

Proof 1.5.22-App $\Box A \rightarrow PA$

1. $\Box A \rightarrow OA$ — $MIO_{d\text{-}a}$ instance
2. $OA \rightarrow PA$ — D_d instance
3. $\Box A \rightarrow PA$ — 1, 2 PL

Proof 1.6.1-App $FA \leftrightarrow O\neg A$

1. $O\neg A \leftrightarrow \neg P\neg\neg A$ — Df. O_d instance
2. $O\neg A \leftrightarrow \neg PA$ — 1, PL, REP_d
3. $\neg PA \leftrightarrow O\neg A$ — 1, PL
4. $FA \leftrightarrow O\neg A$ — 3, Df. F_d

Proof 1.6.2-App $PA \leftrightarrow \neg O\neg A$

1. $O\neg A \leftrightarrow \neg P\neg\neg A$ — Df. O_d instance
2. $\neg O\neg A \leftrightarrow P\neg\neg A$ — 1, PL
3. $\neg O\neg A \leftrightarrow PA$ — 2, PL, REP_d
4. $PA \leftrightarrow \neg O\neg A$ — 3. PL

Proof 1.6.3-App $PA \leftrightarrow \neg FA$

1.	$PA \leftrightarrow \neg\neg PA$	PL
2.	$PA \leftrightarrow \neg FA$	1, Df. F_d

Proof 1.6.4-App $\neg O\top$

1.	$\top$	PL
2.	$\Box\top$	1, RN_a
3.	$O\top \rightarrow \Diamond\neg\top$	$OAP_{d\text{-}a}$ instance
4.	$\neg\Diamond\neg\top \rightarrow \neg O\top$	3, PL
5.	$\Box\top \rightarrow \neg O\top$	4, Df. $\Box_a$
6.	$\neg O\top$	2, 5 MP

Theorem 1.6.5-App $\neg F\bot$

1.	$\neg O\top$	Theorem 1.6.4
2.	$\neg\neg P\neg\top$	1, Df. P_d
3.	$\neg\neg P\bot$	2, PL
4.	$\neg F\bot$	3, Df. F_d

Theorem 1.6.6-App $\neg\Diamond A \rightarrow PA$

1.	$O\neg A \rightarrow \Diamond\neg\neg A$	$OAP_{d\text{-}a}$ instance
2.	$O\neg A \rightarrow \Diamond A$	1, PL, $REP_{d\text{-}a}$
3.	$\neg\Diamond A \rightarrow \neg O\neg A$	2, PL
4.	$\neg\Diamond A \rightarrow PA$	3, Df. P_d

Theorem 1.6.7-App $P\bot$

1.	$\neg O\top$	Theorem 1.6.4
2.	$\neg\neg P\neg\top$	1, Df. P_d
3.	$P\neg\top$	2, PL
4.	$P\bot$	3, PL

Theorem 1.6.8-App $R\neg O_d$ A / $\neg$OA

1.	A	Assumption
2.	$\Box A$	1, RN_a
3.	$OA \rightarrow \Diamond\neg A$	$OAP_{d\text{-}a}$
4.	$\neg\Diamond\neg A \rightarrow \neg OA$	3, PL
5.	$\Box A \rightarrow \neg OA$	4, Df. $\Box_a$
6.	$\neg OA$	2, 5 MP

Proof 1.6.12-App Inconsistency $\{RN_d, OAP_{d\text{-}a}\}$

1.	$\top$	PL
2.	$O\top$	1, RN_d
3.	$O\top \rightarrow \Diamond\neg\top$	$OAP_{d\text{-}a}$ instance
4.	$\Box\top$	1, RN_a
5.	$\Diamond\neg\top$	2, 3 MP
6.	$\neg\Diamond\neg\top$	5, Df. $\Box_a$

Theorem 1.6.13-App $\models^M_\alpha OA$ iff $\neg P\neg A$

1.	$\models^M_\alpha$ OA iff $\| A \| \in N^d_\alpha$	df. o_d)
2.	$\models^M_\alpha$ OA iff $- \| \neg A \| \in N^d_\alpha$	1, Definition 1.6.3
3.	$\models^M_\alpha$ OA iff not $- \| \neg A \| \notin N^d_\alpha$	2, PL
4.	$\models^M_\alpha$ OA iff not $P\neg A$	3, df. p_d)
5.	$\models^M_\alpha$ OA iff $\neg P\neg A$	4, PL

Theorem 1.6.14-App $oic_{d\text{-}a}$) If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^a_\alpha$

1.	If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^a_\alpha$ and $\| A \| \notin N^a_\alpha$	$oi\nabla_{d\text{-}a}$)
2.	If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^a_\alpha$	1, PL

Theorem 1.6.15-App $oap_{d\text{-}a}$) If $\| A \| \in N^d_\alpha$ then $\| A \| \notin N^a_\alpha$

1.	If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^a_\alpha$ and $\| A \| \notin N^a_\alpha$	$oi\nabla_{d\text{-}a}$)
2.	If $\| A \| \in N^d_\alpha$ then $\| A \| \notin N^a_\alpha$	1, PL

Theorem 1.6.16-App If $\| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$

1. If $\| A \| \in N^d_\alpha$ then $\| A \| \notin N^a_\alpha$ — $oap_{d\text{-}a}$)
2. If not $\| A \| \notin N^a_\alpha$ then not $\| A \| \in N^d_\alpha$ — 1, PL
3. If $\| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$ — 1, PL

Theorem 1.6.17-App If $- \| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$

1. If $\| A \| \in N^d_\alpha$ then $- \| A \| \notin N^d_\alpha$ — $oic_{d\text{-}a}$)
2. If not $- \| A \| \notin N^a_\alpha$ then not $\| A \| \in N^d_\alpha$ — 1, PL
3. If $- \| A \| \in N^a_\alpha$ then $\| A \| \notin N^d_\alpha$ — 2, PL

Theorem 1.6.18-App $W \notin N^d_\alpha$

1. If $\| \top \| \in N^d_\alpha$ then $- \| \top \| \notin N^a_\alpha$ and $\| \top \| \notin N^a_\alpha$ — $oi\nabla_{d\text{-}a}$)
2. If $W \in N^d_\alpha$ then $- W \notin N^a_\alpha$ and $W \notin N^a_\alpha$ — Definition 1.6.3
3. If $W \in N^d_\alpha$ then $W \notin N^a_\alpha$ — 2, PL
4. If not $W \notin N^a_\alpha$ then not $W \in N^d_\alpha$ — 3, PL
5. If $W \in N^a_\alpha$ then $W \notin N^d_\alpha$ — 4, PL
6. $W \in N^a_\alpha$ — n_a
7. $W \notin N^d_\alpha$ — 5, 6 MP

Theorem 1.6.19-App $\varnothing \notin N^d_\alpha$

1. If $\| \bot \| \in N^d_\alpha$ then $- \| \bot \| \notin N^a_\alpha$ and $\| \bot \| \notin N^a_\alpha$ — $oi\nabla_{d\text{-}a}$)
2. If $\varnothing \in N^d_\alpha$ then $W \notin N^a_\alpha$ and $\varnothing \notin N^a_\alpha$ — Definition 1.6.3
3. If $\varnothing \in N^d_\alpha$ then $W \notin N^a_\alpha$ — 2, PL
4. If not $W \notin N^a_\alpha$ then not $\varnothing \in N^d_\alpha$ — 3, PL
5. If $W \in N^a_\alpha$ then $\varnothing \notin N^d_\alpha$ — 4, PL
6. $W \in N^a_\alpha$ — n_d)
7. $\varnothing \notin N^d_\alpha$ — 5, 6 MP

Proof 2.1.1-App Inconsistency $\{N_d, OAP_{d\text{-}a}\}$

1. $O\top$ — N_d
2. $O\top \rightarrow \Diamond\neg\top$ — $OAP_{d\text{-}a}$ instance
3. $\Diamond\neg\top$ — 1, 2 MP
4. $\neg\Diamond\neg\top$ — 2, Df. $\Box_a$

Proof 2.1.2-App Inconsistency {$MIO_{d\text{-}a}$, $OAP_{d\text{-}a}$}

1. $\Box\top \rightarrow O\top$ — $MIO_{d\text{-}a}$ instance
2. $O\top \rightarrow \Diamond\neg\top$ — $OAP_{d\text{-}a}$ instance
3. $\Box\top \rightarrow \Diamond\neg\top$ — 1, 2 PL
4. $\Box\top \rightarrow \neg\Box\top$ — 1, 2 PL
5. $\top$ — PL
6. $\Box\top$ — 5, RN_d
7. $\neg\Box\top$ — 4, PL

Proof 2.1.3-App $N^d_\alpha \neq \varnothing \leftrightarrow W \in N^d_\alpha$

Suppose $N^d_\alpha \neq \varnothing$ Then

Either (1) $N^d_\alpha = \{ \varnothing \}$

OR (2) $A \in N^d_\alpha$, where A is some arbitrary sentence

In case (1), i.e. $N^d_\alpha = \{ \varnothing \}$

1. $\varnothing \in N^d_\alpha$ — by assumption
2. $W \cap \varnothing \in N^d_\alpha$ — by set theory
3. If $W \cap \varnothing \in N^d_\alpha$ then $W \in N^d_\alpha$ and $\varnothing \in N^d_\alpha$ — m)
4. $W \in N^d_\alpha$ and $\varnothing \in N^d_\alpha$ — 2, 3 PL
5. $W \in N^d_\alpha$ — 4, PL

In case (2), i.e. $A \in N^d_\alpha$, where A is some arbitrary sentence

1. $A \in N^d_\alpha$ — Assumption
2. $W \cap A \in N^d_\alpha$ — set theory
3. If $W \cap A \in N^d_\alpha$ then $W \in N^d_\alpha$ and $A \in N^d_\alpha$ — m)
4. $W \in N^d_\alpha$ and $\varnothing \in N^d_\alpha$ — 2, 3 PL
5. $W \in N^d_\alpha$ — 4, PL

So, $W \in N^d_\alpha$ in either case

So, the left to right conditional holds

Suppose $W \in N^d_\alpha$ Then

Plainly, $N^d_\alpha \neq \varnothing$

So, the right to left condition holds.

Therefore, the conditional holds in both directions.

Therefore, $N^d_\alpha \neq \varnothing \leftrightarrow W \in N^d_\alpha$.

Proof 2.1.4-App Inconsistency {$\Box A$, $OAP_{d\text{-}a}$, RM_d}

1. $\Box A$
2. $OA \rightarrow \Diamond\neg A$
3. $OOA \rightarrow O\Diamond\neg A$

Proof 2.1.5-App K_d $O(A \rightarrow B) \rightarrow OA \rightarrow OB$

1.	$(A \rightarrow B) \wedge A \rightarrow A$	PL
2.	$O((A \rightarrow B) \wedge A) \rightarrow OA$	1, RM_d
3.	$O(A \rightarrow B) \wedge OA \rightarrow O((A \rightarrow B) \wedge A)$	C_d instance
4.	$O(A \rightarrow B) \wedge OA \rightarrow OB$	2, 3 PL
5.	$O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$	4, PL

Proof 2.1.6-App D_d $OA \rightarrow PA$

1.	$\neg(A \wedge \neg A)$	PL
2.	$\Box\neg (A \wedge \neg A)$	1, RN_a
3.	$O(A \wedge \neg A) \rightarrow \Diamond(A \wedge \neg A)$	$OIC_{d\text{-}a}$ instance
4.	$\neg\Diamond(A \wedge \neg A) \rightarrow \neg O(A \wedge \neg A)$	3, PL
5.	$\neg\neg\Box\neg(A \wedge \neg A) \rightarrow \neg O(A \wedge \neg A)$	4, Df. $\Box_a$
6.	$\Box\neg(A \wedge \neg A) \rightarrow \neg O(A \wedge \neg A)$	5, PL
7.	$\neg O(A \wedge \neg A)$	2, 6 MP
8.	$OA \wedge O\neg A \rightarrow O(A \wedge \neg A)$	C_d instance
9.	$\neg(OA \wedge O\neg A)$	7, 9 MP
10.	$\neg O(A \wedge \neg A) \rightarrow \neg(OA \wedge O\neg A)$	8, PL
11.	$OA \rightarrow \neg O\neg A$	10, PL
12.	$OA \rightarrow PA$	11, Df. P_d

Proof 2.2.2-App $O4C_d$

1.	$O(OA \rightarrow A)$	OU_d
2.	$O(OA \rightarrow A) \rightarrow OOA \rightarrow OA$	K_d instance
3.	$OA \rightarrow OOA$	1, 2 MP

Proof 2.4.2-App Inconsistency { $\Box A$, $OB_{d\text{-}a}$, $MIO_{d\text{-}a}$}

1.	$\Box A$	Assumption
2.	$\Box A \rightarrow OA$	$MIO_{d\text{-}a}$
3.	$OA \rightarrow \Diamond(OA \wedge \neg A)$	$OB_{d\text{-}a}$
4.	$\Box A \rightarrow \Diamond(OA \wedge \neg A)$	2, 3 PL
5.	$\Diamond(OA \wedge \neg A)$	1, 4 MP
6.	$\Diamond(OA \wedge \neg A) \rightarrow OA \wedge \neg A$	$\Diamond M_a$ (Theorems 1.5.2)
7.	$\Diamond OA \wedge \Diamond\neg A$	5, 6 MP
8.	$\Diamond\neg A$	6, PL
9.	$\neg\Diamond\neg A$	1, Df. $\Box_a$