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Bianchi - IX Universe in Euclidean Quantum Cosmology

by

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ABSTRACT

The path-integral approximation of quantum cosmology is applied to a particular homogeneous and anisotropic model (Bianchi type-IX). The dynamical equations governing the system are examined for different equivalent sets of dynamical variables and time parameters. The imposition of specific boundary conditions, when the Universe shrinks to zero volume, changes drastically the whole solution (Instanton). Close to the boundaries, approximate solutions are written down explicitly, so the correct initial data for the differential equations are extracted and numerical solutions are derived. Using the first-order approximation for the wave function of the Universe, according to the Hartle-Hawking no-boundary proposal, the maximality of the wavefunction for an isotropic configuration is verified. The addition of a cosmological constant or some types of scalar fields is examined. A comparison with the Ashtekar-theoretical version of the model for non-zero value of the cosmological constant is attempted.

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1. Path - Integral quantum cosmology

In any attempt to apply quantum mechanics to the Universe as a whole the specification of the possible quantum mechanical states which the Universe can occupy is of central importance. This specification determines the possible dynamical behavior of the Universe. Moreover, if the uniqueness of the present Universe is to find any explanation in quantum gravity, it can only come from a restriction on the possible states available.

In quantum mechanics the state of a system is specified by giving its wave function on an appropriate configuration space. The possible wave functions can be constructed from the fundamental quantum mechanical amplitude for a complete history of the system which may be regarded as the starting point for quantum theory. In the case of a single particle a history is a path $x(t)$ and the amplitude for a particular path is proportional to $e^{iS[x(t)]}$, where $S[x(t)]$ is the classical action. From this basic amplitude, the amplitude for more restricted observations can be constructed by superposition. In particular, the amplitude that the particle, having been prepared in a certain way, is located at position x and nowhere else at time t is : $\Psi(x,t) = N \int_C \delta x(t) e^{iS[x(t)]}$. Here, N is a normalizing factor and the sum is over a class of paths which intersect x at time t and which are weighted in a way that reflects the preparation of the system. A state of particular interest in any quantum-mechanical theory is the ground state, or state of minimum excitation. This is naturally defined by the path integral, made definite by Wick rotating to Euclidean time, over the class of paths which have vanishing action in the past. Thus, for a ground state at $t = 0$ one would write $\Psi_0(x,0) = N \int \delta x(\tau) e^{\,I[x(\tau)]}$, where $I[\,x(\tau)]$ is the Euclidean action obtained from S by sending $t \rightarrow -i\tau$

and adjusting the sign so that it is positive. When the time is well-defined and the corresponding Hamiltonian is time-independent then this definition coincides with the lowest eigenfunction of the Hamiltonian.

In the case of flat space quantum field theory, the wave function is a functional of the field configuration of the spacelike surface of constant time, $\Psi = \Psi[\Phi(\vec{x}),t]$. The functional Ψ gives the amplitude that a particular field distribution $\Phi(\vec{x})$, occurs on this spacelike surface. The ground-state wave functional is $\Psi_0[\Phi(\vec{x}),0] = N\int\delta\Phi(x)e^{I[\Phi(x)]}$, where the integral is over all Euclidean field configurations for $\tau < 0$ which match $\Phi(\vec{x})$, on the surface $\tau = 0$ and leave the action finite at Euclidean infinity.

To evaluate the functional integral one first looks the non-singular stationary points of the action functional (classical solutions) and expands about them. Such critical points are called "Instantons".

In the case of quantum gravity new features enter. For definiteness and simplicity we restrict ourselves to spatially closed universes. There is no well-defined intrinsic measure of the location of a spacelike surface in the spacetime beyond that contained in the intrinsic or extrinsic geometry of the surface itself. One cannot move a given spacelike surface back and forth in time. It is easy to understand what is meant by fixing a field on a given spacelike surface. What is meant by fixing the four-geometry is less obvious.

In quantum mechanics, one fixes the coordinates or the momenta (or some combination) as the argument of the wave function, but not both. This is why for gravity one fixes h_{ij} or K_{ij} but not both, even thought a definite 4-geometry fixes both on a spacelike surface. Similarly, for a scalar field one fixes $\Phi(\vec{x})$, or $\dot{\Phi}(\vec{x})$ but not both, and for the electromagnetic field one fixes $\vec{B}(\vec{x})$ or $\vec{E}(\vec{x})$, but not both. For classical physics one would need to give both on a Cauchy surface to fix the classical evolution, but in quantum physics the argument of the wave function is only half of the phase-space variables.

Quantum dynamics is supplied′ by the functional integral $\Psi[h_{ij}] = N\int\delta g_{\mu\nu}(x)\ e^{iS_E[g_{\mu\nu}]}$, where S_E is the classical action for a given metric

$g_{\mu\nu}$ and the functional integral is over all four-geometries with a compact spacelike boundary on which the induced metric is h_{ij} and which to the past of that surface satisfy some appropriate condition to define the state. In particular the amplitude to go from a three-geometry h'_{ij} on an initial spacelike surface to a three-geometry h''_{ij} on a final spacelike surface is $\langle h''_{ij} | h'_{ij} \rangle = \int \delta g_{\mu\nu} \; e^{iS[g_{\mu\nu}]}$, where the sum is over all four-geometries which match h'_{ij} on the initial surface and h''_{ij} on the final surface. In these states, time cannot be specified. The proper time between the surfaces depends on the four-geometry in the sum.

Unfortunately, such an integral is not well-defined. First, the integral oscillates rapidly. We can try to remedy this problem by going to Euclidean spacetime, where $\Psi[h_{ij}]$ becomes $\Psi[h_{ij}] = N \int_C \delta g_{\mu\nu} \; e^{I[g_{\mu\nu}]}$, where I is the Euclidean version of the Einstein-Hilbert action appropriate to keeping the three-geometry fixed on the boundary, i.e.

$$I = -\frac{1}{16\pi}\int_M d^4x \;\sqrt{g}\;\; ({}^{(4)}R - 2\Lambda) \;-\frac{1}{8\pi}\int_{\partial M} d^3x \;\sqrt{h}\;\; K \quad , (G=1) \qquad (1.1)$$

The first term is integrated over spacetime and the second over its boundary. K is the trace of the extrinsic curvature K_{ij} of the boundary three-surface and $h = \det(h_{ij})$. In the case of matter fields included

$$\Psi[h_{ij}, \Phi] = \int \delta g_{\mu\nu} \delta\Phi \;\; e^{-(I_g[g_{\mu\nu}] + I_m[g_{\mu\nu}, \Phi])} \quad , \qquad (1.2),$$

where $I_m[g_{\mu\nu}, \Phi]$ is a matter action and $I_m[g_{\mu\nu}]$ is the Euclidean Einstein-Hilbert action. When suitably defined, the path-integral expression generates solutions to the Wheeler-DeWitt equation and momentum constraints :
$\hat{\mathcal{H}}\,\Psi = \hat{\mathcal{H}}_i \Psi = 0$.

However, this method does not solve the problem completely, because the Einstein-Hilbert action has a special feature that distinguishes it from most matter-field actions - it is not positive definite. As a consequence, the path-integral will not converge if taken over real Euclidean metrics. For example, if we make the conformal transformation $g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)$ keeping h_{ij} fixed, then the action contains the term $-\int(\nabla\Omega)^2 d^4x$ and if Ω changes rapidly the functional integral blows up. Since we observed that it is the conformal part of the Euclidean four-metric which is responsible for the Euclidean action becoming negative, it was suggested that the integral over all metrics be split up into a sum over conformal equivalence classes and a sum over conformal factors within each class, where the representant is chosen in such a way that $^{(4)}R = 4\Lambda$. If the integration over conformal factors was rotated to lie parallel to the imaginary axis ($\Omega \rightarrow i\Omega$), then the integral can be calculated and the result analytically continued.

This procedure does not include large classes of metrics and also it breaks down when the metric is coupled to non-conformally invariant manner - the positivity of the Euclidean matter action is not preserved by the conformal rotation. Rather than trying to rotate certain components of the four-metrics while leaving others alone, one could allow the four-metric to become fully complex and then integrate along some contour in the space of complex four-metrics.

There are a few more things needed to be specified before the Euclidean path-integral above may be regarded as properly and uniquely defined. These are gauge-fixing terms, a regularization scheme, a measure and boundary conditions. Actually, the class C may contain metrics of an open spacetime and in such a case we need boundary conditions. In the quantum mechanics of closed Universes there is no natural definition of energy for a closed Universe just as there is no independent standard of time. In a certain sense the total energy for a closed Universe is always zero - the gravitational energy cancelling the matter energy. It is logical to define a state of minimum excitation corresponding to the classical notion of a geometry of high symmetry. Such a ground-state wave function is given by the Hartle-Hawking "no boundary" proposal, as a Euclidean sum-over-histories of the form

$$\Psi_{NB}[h_{ij}, \Phi, \Sigma] = \sum_{M} \int \delta g_{\mu\nu}\, \delta\Phi\, e^{-I[g_{\mu\nu}, \Phi]} , \tag{1.3}$$

where I is the Euclidean action of the gravitational field $g_{\mu\nu}$ and the matter field Φ, on a four-manifold M. The sum over manifolds is over a class of compact manifolds which have as their only boundary the three-surface Σ, on which the arguments of the wave function are specified.

One of the major successes of the Hartle - Hawking proposal was to show how the classical Universes corresponding to the HH quantum state tend to have a long inflationary period, developing into large, homogeneous and isotropic Universes containing small-scale inhomogeneity and thus resembling our own Universe.

Since a geometry is a manifold with a metric, in a sum-over-histories quantum mechanics of spacetime it is, therefore, as natural to sum over manifolds as it is over metrics. Admitting different manifolds to the sums over geometries defining quantum amplitudes means allowing different possibilities for the topology of physical spacetime and allowing also for the possibility of quantum transitions between the topology of space at one time and another. We do not really know how to perform the sum over manifolds, because we do not know what measure to use. In practice, one can therefore do little more than consider each term in the sum over four-manifolds separately. Naively, the functional integral for a given M is taken to extend over real Euclidean four-metrics $g_{\mu\nu}$ and matter fields Φ on M. When defined in this way, however, the integral is divergent, and so a viable definition should include a prescription for the contour of integration in the space of complex metrics and matter-field configurations.

The quantum state of the Universe is exceedingly difficult to calculate. It is possible to gain some insight into its behavior by studying simplifying models, with a restricted number of degrees of freedom. When applying the no-boundary proposal in spatially homogeneous minisuperspace models, it is possible to consider two qualitatively different situations. The first is to take the boundary to consist of two separate components so that the interpolating four-geometries are bounded by an "initial" and a "final" three-surface. In this case the arguments of the wave function are two copies of the set of the minisuperspace coordinates. It is often argued that the appropriate object to study is then the density matrix obtained by tracing over states on one of the three-surfaces. The second possibility is to take the boundary to consist of a single connected component. In this case the minisuperspace geometries to be integrated over should "begin" at an initial value of the Euclidean time coordinate, where this time coordinate becomes singular, but the manifold closes in a regular way and "end" at some final value of the Euclidean time coordinate with the three-geometry specified by the arguments of the wave function. Although from a four-dimensional geometrical point of view, one would expect to have to specify only the boundary data at the actual three-surface boundary, the fact that we are working with a 3+1 formalism obliges us to specify, in addition, conditions for the metric variables at the "bottom" of the four-manifold. Then, the no-boundary proposal which means that the Universe does not have any boundaries in space or time (at least in the Euclidean regime), can be interpreted as giving the amplitude for the specific three-geometry to arise from a zero three-geometry, i.e. a single point (the Universe appears from nothing). Also, note that if the wave function is finite and nonzero at the zero-volume three-geometry, corresponding to the big-bang singularity, it allows the possibility of topological fluctuations of the three-geometry.

Since we cannot perform such an integration directly, except for some simple minisuperspace models, some prescriptions have been given to obtain the wave function approximately. In the case of simple systems for which we can explicitly write down the classical Euclidean action I_{cl} , we may approximate the wave function as $\Psi \approx e^{-I_{cl}}$.

When we sum over all compact geometries, an additional problem may appear. There may be more than one surface with the same h_{ij} , eg. in a four-sphere there are two spacelike surfaces with the same radii. Therefore, it is convenient to change the variables from h_{ij} to the "K representation" in which we use $h_{ij} = h_{ij}(\det h)^{-1/3}$ and the K trace of the external curvature K^i_j . The transformation formulas are

$$\Psi[\tilde{h}_{ij}, K] = \int_0^{\infty} dh \quad \exp[-\frac{4}{3l^3}\int d^3x \quad \sqrt{h}K] \quad \Psi[h_{ij}]$$

$$\Psi[h_{ij}] = -\frac{1}{2\pi i}\int_{\Gamma} dK \quad \exp[\frac{4}{3l^3}\int d^3x \quad \sqrt{h}K] \quad \Psi[\tilde{h}_{ij}, K] \, , \tag{1.4}$$

where Γ goes from $-i\infty$ to $+i\infty$ at the right of all singularities of $\Psi[\tilde{h}_{ij}, K]$ and l is the Planck length.

2. The position of relativistic Bianchi-IX Universe in Cosmology

General-relativistic cosmology was for many years concerned almost entirely with the simplest possible models. These are the models which are both isotropic, i.e. in which all spatial directions are equivalent, and spatially homogeneous, i.e. all points in space at a given time are equivalent. The condition of isotropy at every point leads uniquely to a certain metric form (Friedmann-Robertson-Walker). This observed large-scale isotropy, homogeneity and also flatness of our Universe described by a FRW spacetime filled with energy nearly equal to the critical density, constitutes an important cosmological problem. Actually, the conventional big-bang cosmology has no answer - it merely resorts to a very unnatural fine-tuning of the initial condition. For various reasons, attempts have been made to compare FRW models with other, less symmetrical models. We have the freedom to consider various models since the governing equations of general relativity must be supplemented by initial conditions, boundary conditions, symmetry conditions and/or other restrictions in order to yield definite solutions. The easiest models to consider are those which share with the FRW model the property of spatial-homogeneity. The philosophical reasons for considering non-FRW models are essentially that the Friedmann models offer no explanation for the observed symmetry, and in particular that regions now observable (by the microwave radiation) could not have been in causal contact at time of emission, so that the symmetry really seems to be imposed, rather than natural. A second aspect is that although FRW models start from a big-bang, thus satisfying the singularity theorems which strongly indicate such an origin for our Universe, they do not exhibit the most general types of singularity. In particular, small perturbations of FRW models exist, i.e. small at some time t after the big-bang, such that they grow as the singularity is approached. The singularity structure is therefore unstable and the FRW initial conditions are far from general, being, in some ill defined sense, isolated in the space of solutions. A further impetus to the study of non-FRW models came from the fact that small perturbations caused by random statistical fluctuations in FRW models do not appear to grow fast enough for this to provide a satisfactory account of galaxy formation.

Belief in homogeneity is really the outcome of a long series of reverses from a geocentric point of view. However, it is almost impossible to test homogeneity, because we see distant regions as they were a long time ago, and in order to compare them with the present-day we must find the appropriate evolution to obtain the present-day parameters of those distant regions, something that may well lead us into a circular argument. The only attempts at direct testing of homogeneity use the distribution of galaxies, and since the galaxies appear to be clustered on scales which may be very large indeed, the outcome of these tests is disputed. The principal advantage of the spatially-homogeneous models is that the physical variables depend only on time. Thus all equations reduce to ordinary differential equations.

We can directly test, in a number of ways, the isotropy of the Universe about us (e.g. the distribution of galaxies, distribution of radio sources, cosmic x-ray background, cosmic microwave background, cosmic magnetic field). There is no strong case for anisotropic models, but some of the data, if confirmed, could provide such evidence.

Bianchi type-IX or "Mixmaster" cosmologies generalize the closed ($k = +1$) FRW model by allowing the constant-time surfaces to be distorted three-spheres. This spatially homogeneous mixmaster gravitational collapse is a very famous gravitational collapse (a "big crunch") which gives us a "hint" of the sort of complexity one should expect for gravitational collapses with more than one degree of freedom. The Bianchi IX Universe has been much studied by cosmologists for various reasons:

(i) It is an anisotropically expanding Universe with closed space sections. It begins expanding in a strongly irregular fashion but can, in the course of time, isotropize and provide a good description of the large scale Universe today. When the anisotropy level is small it resembles an anisotropic perturbation of the closed Friedmann Universe.

(ii) It has been shown [3] that for general (inhomogeneous) vacuum closed Universes away from initial data with symmetries (Killing vectors) the initial data for the Einstein equations can be completely defined by four independent functions of three variables. That is, there is locally a diffeomorphism from the space of pairs of three-tensors describing the induced metric on a spacelike hypersurface and its extrinsic curvature onto the Hilbert space consisting of four arbitrary functions of three variables. The general Bianchi IX (as well as VI_h ,VII_h ,VIII), Cauchy data is specified by four arbitrary constants in vacuum and so in some sense, might be locally near a general vacuum solution to the Einstein equations. It is worth remarking that the type VIII and IX models have no Newtonian analogues, unlike type I-VII. Their unusual dynamics is a consequence of the non-Newtonian, or "magnetic" portion of the Weyl conformal curvature. They are intrinsically general relativistic phenomena. The oscillatory mode of approach to the singularity, described below, exists in the vacuum case only in models VIII and IX, and it is this circumstance that attaches to these models the special role of prototypes for constructing the general inhomogeneous solution of the Einstein equations in the neighborhood of the singularity.

(iii) Misner originally discovered that the mixmaster Universe has the unusual property that periodically, close to the initial singularity, light can circumnavigate the Universe. This is not possible in other simpler homogeneous models and probably also not in inhomogeneous models either. Misner hoped that this discovery might go some way towards providing an explanation for the remarkable degree of regularity displayed by the present-day Universe; regularity that extends over regions, which in the more conventional cosmological models like Friedmann's, could never have been causally connected during the expansion history of the Universe. How then did they manage to coordinate their structure to within one part in ten thousand today? Misner's suggestion was that if the Universe began in a manner resembling the Bianchi type IX model, then causal communication could, in principle, be established over the entire Universe in its earliest stages. Irregularities associated with the "big-bang" would be efficiently ironed-out by viscous transport processes and diffusive mixing. Unfortunately, subsequent studies revealed that the mixmaster model very rarely visited configurations conducive to "mixing" over very large regions. It has been proved that without an extremely specialized choice of the parameters of the model, the light will never have time to circle around the Universe in any direction during the first period of expansion. The mixmaster model could not guarantee the present structrure of the Universe independently of the initial data-the goal of the so called "chaotic cosmology" program.

We also note that Mach's idea that the inertial frame is determined by the distant stars is the ground of Wheeler's persuasive philosophical statement that the Machian problem is more precisely stated and understood when posed on a compact manifold.

We imagine that a 3+1 split has been performed, splitting the spacetime manifold into the topological product of a line (the "time" axis) and the three-dimensional spacelike hypersurfaces Σ_t (the dynamical degrees of freedom are the spatial components of the metric, the induced metric h_{ij} on Σ_t , which evolves in the "time" parameter "t"). In fact, we operate in a "synchronous reference frame" which brings the spacetime metric on the very simple form $ds^2 = -dt^2 + h_{ij}dx^i dx^j$. By definition, the general Bianchi IX spacetime has topology $\mathbf{R}\times S^3$ (product of a time axis and the compact three-sphere), with a simply transitive action of the isometry group $SU(2)$ on the S^3 spatial slices. The metric of a general Bianchi IX model can be put in the form $ds^2 = -dt^2 + \gamma_{ij}(t)\sigma^i(x)\sigma^j(x)$, $i,\ j = 1,2,3$. Here, σ^1, σ^2, and σ^3 are isometry invariant one-forms on the three sphere satisfying $\mathcal{L}_t\ \sigma^i = 0$ (where t^a is the unit normal to the homogeneous hypersurfaces) and γ_{ij} is a symmetric 3×3 matrix. The basis $(\sigma^1, \sigma^2, \sigma^3)$ obeys the Cartan structure equations in the exterior calculus $d\sigma^i = -\frac{1}{2}\varepsilon^i_{jk}\sigma^j \wedge \sigma^k$, where ε^i_{jk} is the completely antisymmetric tensor of rank 3. (Explicitly $d\sigma^1 = -\sigma^2 \wedge \sigma^3$, $d\sigma^2 = -\sigma^3 \wedge \sigma^1$, $d\sigma^3 = -\sigma^1 \wedge \sigma^2$). The diagonal Bianchi IX spacetimes are those for which the σ^i can be chosen so that γ_{ij} is a diagonal matrix for all time. This requirement is equivalent to demanding that on each homogeneous slice the eigenvectors of the extrinsic curvature tensor K^i_j coincide with the eigenvectors of the three-dimensional Ricci tensor $^{(3)}R^i_j$. For vacuum solutions, this condition is implied by the field equations, so the diagonal case encompasses all vacuum Bianchi IX spacetimes. Then, let $\gamma_{ij}(t)=\mathrm{diag}(a^2(t),\ \ b^2(t),\ \ c^2(t))$ and so

$$ds^2 = -dt^2 + a^2(\sigma^1)^2 + b^2(\sigma^2)^2 + c^2(\sigma^3)^2 .$$

Let ψ, θ, φ are the classical angle coordinates on $SO(3)$. Since we consider the simply connected covering space S^3 instead of $SO(3)$, we allow ψ to have fundamental domain $0 \le \psi \le 4\pi$, while θ and φ have their usual ranges $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. We let $(\sigma^1, \sigma^2, \sigma^3)$ be a basis for the space of left-invariant differential one-forms on the three-sphere $S^3 = SU(2)$. In local parametric coordinates we have :

$$\sigma^1 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi \ \ , \ \ \sigma^2 = -\sin\psi d\theta + \cos\psi \sin\theta d\varphi \ \ , \ \ \sigma^3 = d\psi + \cos\theta d\varphi \ .$$

The dual basis of vectors $\{K_i\}$ is: $K_1 = \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cot\theta \sin\psi \frac{\partial}{\partial\psi}$,

$$K_2 = \sin\psi \frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} + \cot\theta\cos\psi\frac{\partial}{\partial\psi},\ K_3 = \frac{\partial}{\partial\psi} \ .$$

Written out in terms of the coordinate differentials $d\psi$, $d\theta$, $d\varphi$ we get for the line element of the spacetime

$$ds^2 = -dt^2 + c^2 d\psi^2 + (a^2\cos^2\psi + b^2\sin^2\psi)d\theta^2 + \{\sin^2\theta(a^2\sin^2\psi + b^2\cos^2\psi) + c^2\cos^2\theta\}d\varphi^2 + (a^2 - b^2)\sin 2\psi \sin\theta d\theta d\varphi + 2c^2\cos\theta d\psi d\varphi \ , \qquad (2.1)$$

This is a toy-model spacetime metric (with $a(t), b(t), c(t)$, the three scale-functions, as degrees of freedom) which we can also evolve on approach to the "big crunch" spacetime singularity where the three-volume of the metric collapses to zero.

The space is closed and the three-volume of the compact space is given by

$V=\int\sqrt{\gamma}\,d\psi d\theta d\varphi=\int abc\sigma^1\wedge\sigma^2\wedge\sigma^3$, where $\gamma=\det(\gamma_{ij})=a^2b^2c^2\sin^2\theta$ is the determinant of the three-metric in the frame of ψ,θ,φ. Then

$$V=abc\int_0^{\pi}\sin\theta d\theta\int_0^{4\pi}d\psi\int_0^{2\pi}d\varphi=16\pi^2abc\ .$$

When $a=b=c=R/2$, the space reduces to the space of constant positive curvature with radius $R=2a$, which is the metric of highest symmetry on the group space $SU(2)$. The volume then reduces to the three-volume $V=2\pi^2R^3(t)$ of the compact (isotropic) Robertson-Walker space. If we couple the gravitational field to perfect fluid matter, the cosmological model with the ansatz for the metric is an anisotropic generalization of the well-known compact FRW model : It has different "Hubble constants" along different directions in the three-space. One may also interpret the metric as a closed FRW Universe on which is superposed circularly polarized gravitational waves with the longest wavelengths that will fit into a closed Universe [15].

Since the metric is spatially homogeneous, the full non-linear Einstein equations for this are a set of ordinary (non-linear) differential cquations. To see this more explicitly, introduce in place of the quantities a,b,c their logarithms $\alpha=\ln a$, $\beta=\ln b$, $\gamma=\ln c$, and a new time variable $\tau=\int dt/abc$ in place of the proper (synchronous) time t. With the inclusion of a perfect fluid matter source, the space-space components of Einstein's equations read:

$$2\alpha_{\tau\tau}=\frac{d^2}{d\tau^2}(\ln a^2)=(b^2-c^2)^2-a^4+8\pi(\rho-p)a^2b^2c^2 \qquad (2.2)$$

$$2\beta_{\tau\tau}=\frac{d^2}{d\tau^2}(\ln b^2)=(c^2-a^2)^2-b^4+8\pi(\rho-p)a^2b^2c^2 \qquad (2.3)$$

$$2\gamma_{\tau\tau}=\frac{d^2}{d\tau^2}(\ln c^2)=(a^2-b^2)^2-c^4+8\pi(\rho-p)a^2b^2c^2 \qquad (2.4)$$

and the time-time component reads

$$(\alpha+\beta+\gamma)_{\tau\tau}-2(\alpha_\tau\beta_\tau+\alpha_\tau\gamma_\tau+\beta_\tau\gamma_\tau)=-4\pi(\rho+3p)a^2b^2c^2 \qquad (2.5)$$

The quantities p and ρ denote the pressure and the energy density of the fluid. One may easily combine the equations (2.2), (2.3), (2.4), (2.5) to get the first integral $\tilde{I}=I-8\pi\rho a^2b^2c^2=0$, where

$$I = \alpha_\tau \beta_\tau + \alpha_\tau \gamma_\tau + \beta_\tau \gamma_\tau - \tfrac{1}{4}(a^4 + b^4 + c^4) + \tfrac{1}{2}(a^2 b^2 + a^2 c^2 + b^2 c^2) \qquad (2.6)$$

To be in accordance with the vacuum set of Einstein equations the solution should have $I = 0$. The dynamical equations for the compact FRW cosmology is recovered in the case of $a=b=c=R/2$.

One can show that sufficiently near the singularity the perfect fluid matter terms may be neglected if the equation of state gives $p \le (2/3)\rho$. Thus, it is sufficient to investigate the empty space equations there. Then, the volume $\mathbf{V}$ cannot oscillate. Statements of monotonicity of the three-volume are equivalent whether given in t or in τ - time, since $dt = abc d\tau$ and $abc > 0$. The property of $\ln \mathbf{V}$ being a concave function (negative second derivative) does not translate from t to τ - time. Below we show that $\ln \mathbf{V}$ is a concave function in the t - time variable. Neglecting, for notational convenience, the factor $16\pi^2$ in the expression for the three-volume, we have $\ln \mathbf{V} = \ln a + \ln b + \ln c = \alpha + \beta + \gamma$ and the R_{00} equation for the mixmaster metric reads

$$\tfrac{1}{2}(\alpha + \beta + \gamma)_{\tau\tau} = \tfrac{1}{2}(\ln \mathbf{V})_{\tau\tau} = \alpha_\tau \beta_\tau + \alpha_\tau \gamma_\tau + \beta_\tau \gamma_\tau \ .$$

From the definition $\partial_t = (abc)^{-1} \partial_\tau = \mathbf{V}^{-1} \partial_\tau$ one arrives at

$\partial_t^2 = \mathbf{V}^{-2} \{ \partial_\tau^2 - (\ln \mathbf{V})_\tau \partial_\tau \}$ and hence

$$\begin{aligned} \mathbf{V}^2 \partial_t^2 (\ln \mathbf{V}) &= \mathbf{V} \partial_t^2 \mathbf{V} - (\partial_t \mathbf{V})^2 = (\ln \mathbf{V})_{\tau\tau} - (\ln \mathbf{V})_\tau^2 \\ &= 2(\alpha_\tau \beta_\tau + \alpha_\tau \gamma_\tau + \beta_\tau \gamma_\tau) - (\alpha_\tau + \beta_\tau + \gamma_\tau)^2 \\ &= -\alpha_\tau^2 - \beta_\tau^2 - \gamma_\tau^2 \le 0 \ . \end{aligned}$$

It follows that $\ln \mathbf{V}$ and therefore the volume $\mathbf{V}$ itself, can have no local minimum (where we should have $\mathbf{V}_t = 0$, $\mathbf{V}_{tt} > 0$). As a corollary it follows that volume oscillations are not possible.

A stronger result has recently been proved [18]. There do not exist any vacuum Bianchi IX solutions which expand for an infinite amount of proper time as measured by observers moving orthogonally to the homogeneous hypersurfaces. Furthermore, since every expanding vacuum Bianchi IX solution on a finite proper time interval is extendible into the future, every initially expanding, inextendible, vacuum Bianchi IX must recollapse. In the non-vacuum case, there do not exist any Bianchi IX Universes which expand for an infinite time, provided only that the matter satisfies the dominant energy condition and has non-negative average pressure (i.e. a non-negative trace of the spatial projection of the stress-energy tensor).

When the right-hand sides of equations (2.2), (2.3), (2.4) are set equal to zero and equation (2.5) becomes $\alpha_\tau \beta_\tau + \alpha_\tau \gamma_\tau + \beta_\tau \gamma_\tau = 0$, the field equations reduce exactly to those for the Bianchi I or Kasner Universe $ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_i^2$, where the field equations place two algebraic constraints on the three Kasner indices $\{p_i\}$: $\sum_{i}^{3} p_i = \sum_{i}^{3} p_i^2 = 1$. Approaching the singularity, two metric components go to zero while the third goes to infinity. Thus, a small comoving volume of fluid is squashed to zero in two directions and stretched to infinity in the third, while its volume shrinks to zero. This is to be contrasted with the Friedmann-Robertson-Walker singularity where the volume shrinks uniformly in every direction. If one thinks of equations (2.2), (2.3), (2.4) as a Hamiltonian description, then the left-hand side of these equations represent the kinetic terms whilst the right-hand side describes the potential. Whilst the motion is far from the potential walls (the shape of the potential appears in section 3), the right-hand sides are negligible and the Kasner solution is obtained. After a momentary collision with the potential wall, the model is perturbed into a different Kasner model.

To establish the mixmaster behavior as $t \to 0$ $(\tau \to -\infty)$, suppose it begins to evolve with $a >> b >> c$. Then $a^2 = Au\, sech\theta$, $b^2 = \frac{Au}{B} csch\theta\, exp(-\frac{\theta}{u})$, $c^2 = \frac{Au}{C} csch\theta\, exp(-\theta u)$, where A and u are constants and $\theta = Au(\tau - \tau_0)$. The two remaining constants B and C are defined as $C = (a^2\ c^2)_{\dot{a}=0}$, $B = (a^2\ b^2)_{\dot{a}=0}$ and they give the amplitude of the relative scales at the maximum of $a(t)$.

$(\cdot \equiv d/dt)$. The evolution towards the singularity proceeds through an infinite number of oscillations of the scale factors $a(t)$, $b(t)$ and $c(t)$ on any open interval of time t . Physically speaking one is following the evolution of a ball of gravitational wave energy as it collapses to zero volume. The collapse follows a series of cycles during which two of the scale factors ("radii") execute small oscillations whilst the third collapses monotonically. The change of behavior indicating the onset of a new cycle is the attainment of a local minimum by the monotonically falling function. During the new cycle the monotonic scale of the old cycle executes small oscillations whilst the scale factor it replaced now undertakes monotonic behavior until the next cycle commences. During any interval of time in which none of a,b or c have local maxima or minima the expansion is described to a good approximation by the Kasner model. As the collapse occurs the amplitude of both the large and the small oscillations increases steadily even though the overall volume decreases. So, briefly, the solution of the Einstein equations for a Bianchi IX metric in vacuum has the property that the evolution proceeds towards an initial Weyl curvature singularity via a chaotically unpredictable sequence of oscillations which ergodically pass close to a sequence of Kasner eras. It has also been shown that under certain conditions the introduction of a cosmological constant does not change the alternation of Kasner eras on approach to the initial singularity. Generally, most of the spatially homogeneous cosmologies have Weyl tensor singularities. An isotropic model has vanishing Weyl tensor. E.g., for the Kasner Universe the curvature invariant $C^{\mu\nu\kappa\lambda}C_{\mu\nu\kappa\lambda}$ turns out to be proportional to the inverse fourth power of the proper time.

3. The Euclidean Bianchi - IX spacetime

There has been considerable interest in "Instantons" in Yang-Mills theory. As we have stated, they may be defined as non-singular solutions of the classical equations in four-dimensional Euclidean space. They provide stationary phase points in the path-integral for the amplitude to tunnel between two topologically distinct vacua. Because gravity is a gauge theory like Yang-Mills, it seems reasonable to suppose that gravitational instantons may play a similar important role. We define a gravitational instanton to be a non-singular complete positive-definite metric which satisfies the classical vacuum Einstein equations or the Einstein equations with a Λ term. The Λ term can be regarded as a Lagrange multiplier for the four-volume. Also, solutions of Euclidean general relativity are important to the study of "spacetime foam" and other Euclidean quantum gravity theories. As stationary phase points, in the path-integral approach to quantum gravity, the dominant contribution is expected to occur near such metrics, so they may be considered the "atoms" out of which a quantum spacetime is built.

We are going to derive the action for the diagonal Euclidean Bianchi IX Universe, with three scale factors *a*,*b* and *c*. Let ξ be the proper distance, then the metric takes the form $ds^2 = d\xi^2 + a^2(\sigma^1)^2 + b^2(\sigma^2)^2 + c^2(\sigma^3)^2$ .The full action for the Euclidean equations has the form

$I = I_G + I_S + I_\Phi$, where $I_G = -\frac{1}{16\pi}\int_M \sqrt{g}\,({}^{(4)}R - 2\Lambda)\; d^4x$ is the Hilbert-Einstein action , $g = \det({}^{(4)}g_{\mu\nu})$, $I_S = -\frac{1}{8\pi}\int_{\partial M} \sqrt{h}K \; d^3x$ is the surface term, and I_Φ is the matter action : $I_\Phi = \frac{1}{16\pi}\int_M \sqrt{g}L_\Phi \; d^4x$.

The full action I supplies the field equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$. $T_{\mu\nu}$ is the energy-momentum tensor. If there is no matter field then ${}^{(4)}R = 4\Lambda \Rightarrow I_G = -\frac{\Lambda}{8\pi}\int_M \sqrt{g}\, d^4x$, and for $\Lambda=0$, the numerical value of I_G is zero. The Cartan method gives the Ricci tensor components for the above metric.

Define the one-forms $\theta^0 = d\xi,\ \theta^1 = a\sigma^1,\ \theta^2 = b\sigma^2,\ \theta^3 = c\sigma^3$. Then $ds^2 = (\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 = \mu_{\kappa\lambda}\ \theta^\kappa\theta^\lambda$; $\mu_{\kappa\lambda} = diag(1,1,1,1)$. Also define the curvature one and two-forms $\omega^\mu_\nu \equiv \Gamma^\mu_{\nu\lambda}\ \theta^\lambda$, $\Omega^\mu_\nu \equiv d\omega^\mu_\nu + \omega^\mu_\lambda \wedge \omega^\lambda_\nu$, which satisfy the structure equations $d\theta^\mu = -\omega^\mu_\nu \wedge \theta^\nu$, $\Omega^\mu_\nu = \frac{1}{2}R^\mu_{\nu\kappa\lambda}\ \theta^\kappa \wedge \theta^\lambda$. Also, from $\omega^{(\mu}_\nu \mu^{\kappa)\nu} = 0$ it follows that $\omega^0_0 = 0$, $\omega^0_i = -\omega^i_0$, $\omega^i_j = -\omega^j_i$. So,

$$d\theta^0 = 0 = -\omega^0_i \wedge \theta^i \ ,$$

$$d\theta^1 = \frac{\tilde{a}}{a}\theta^0 \wedge \theta^1 + \frac{a}{bc}\theta^2 \wedge \theta^3 = -\omega^1_0 \wedge \theta^0 - \omega^1_i \wedge \theta^i \ , \qquad (\sim \equiv \frac{d}{d\xi})$$

$$d\theta^2 = \frac{\tilde{b}}{b}\theta^0 \wedge \theta^2 + \frac{b}{ac}\theta^3 \wedge \theta^1 = -\omega^2_0 \wedge \theta^0 - \omega^2_i \wedge \theta^i \ ,$$

$$d\theta^3 = \frac{\tilde{c}}{c}\theta^0 \wedge \theta^3 + \frac{c}{ab}\theta^1 \wedge \theta^2 = -\omega^3_0 \wedge \theta^0 - \omega^3_i \wedge \theta^i \ .$$

From these we can put $\omega^1_0 = \frac{\tilde{a}}{a}\theta^1 = -\omega^0_1$, $\omega^2_0 = \frac{\tilde{b}}{b}\theta^2 = -\omega^0_2$, $\omega^3_0 = \frac{\tilde{c}}{c}\theta^3 = -\omega^0_3$, and $\omega^1_2 = s\theta^3$, $\omega^1_3 = v\theta^2$, $\omega^2_3 = u\theta^1$ and then obtain the algebraic system for s,v,u: $s - v = \frac{a}{bc}$, $s + u = \frac{b}{ac}$, $u - v = \frac{c}{ab}$. Solving this we get :

$$\omega^1_2 = \frac{1}{2}(\frac{b}{ac} + \frac{a}{bc} - \frac{c}{ab})\theta^3 = -\omega^2_1 \ , \qquad \omega^1_3 = \frac{1}{2}(\frac{b}{ac} - \frac{a}{bc} - \frac{c}{ab})\theta^2 = -\omega^3_1$$

$$\omega^2_3 = \frac{1}{2}(\frac{b}{ac} + \frac{c}{ab} - \frac{a}{bc})\theta^1 = -\omega^3_2 \ .$$

For the curvature 2-forms we have

$$\Omega^0_0 = \omega^0_i \wedge \omega^i_0 \ , \quad \Omega^1_1 = \Omega^2_2 = \Omega^3_3 = 0$$

$$\Omega_1^0 = -\Omega_0^1 = -d(\frac{\tilde{a}}{a}\theta^1) + \omega_i^0 \wedge \omega_1^i = \frac{\tilde{\tilde{a}}}{a}\theta^0 \wedge \theta^1 + \frac{1}{abc}[-a\tilde{a} + \frac{\tilde{b}}{2b}(a^2 + b^2 - c^2) - \frac{\tilde{c}}{2c}(b^2 - a^2 - c^2)]\theta^2 \wedge \theta^3$$

$$\Omega_2^0 = -\Omega_0^2 = -d(\frac{\tilde{b}}{b}\theta^2) + \omega_i^0 \wedge \omega_2^i = \frac{\tilde{\tilde{b}}}{b}\theta^0 \wedge \theta^2 + \frac{1}{abc}[b\tilde{b} - \frac{\tilde{a}}{2a}(a^2 + b^2 - c^2) - \frac{\tilde{c}}{2c}(b^2 + c^2 - a^2)]\theta^1 \wedge \theta^3$$

$$\Omega_3^0 = -\Omega_0^3 = -d(\frac{\tilde{c}}{c}\theta^3) + \omega_i^0 \wedge \omega_3^i = \frac{\tilde{\tilde{c}}}{c}\theta^0 \wedge \theta^3 + \frac{1}{abc}[-c\tilde{c} - \frac{\tilde{a}}{2a}(b^2 - a^2 - c^2) + \frac{\tilde{b}}{2b}(b^2 + c^2 - a^2)]\theta^1 \wedge \theta^2$$

$$\Omega_2^1 = -\Omega_1^2 = \frac{1}{2}d(\frac{a^2 + b^2 - c^2}{abc}\theta^3) + \omega_\mu^1 \wedge \omega_2^\mu =$$

$$= \frac{1}{2}[(\frac{a^2 + b^2 - c^2}{abc})^\sim + \frac{\tilde{c}}{abc^2}(a^2 + b^2 - c^2)]\theta^0 \wedge \theta^3 +$$

$$\frac{1}{2}[c^2(a^2 + b^2 - c^2) - 2abc^2\tilde{a}\tilde{b} + \frac{1}{2}(b^2 - a^2 - c^2)(b^2 + c^2 - a^2)]\frac{\theta^1 \wedge \theta^2}{(abc)^2}$$

$$\Omega_3^1 = -\Omega_1^3 = \frac{1}{2}d(\frac{b^2 - a^2 - c^2}{abc}\theta^2) + \omega_\mu^1 \wedge \omega_3^\mu =$$

$$= \frac{1}{2}[(\frac{b^2 - a^2 - c^2}{abc})^\sim + \frac{\tilde{b}}{ab^2c}(b^2 - a^2 - c^2)]\theta^0 \wedge \theta^2 -$$

$$\frac{1}{2}[b^2(b^2 - a^2 - c^2) + 2ab^2c\tilde{a}\tilde{c} + \frac{1}{2}(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)]\frac{\theta^1 \wedge \theta^3}{(abc)^2}$$

$$\Omega_3^2 = -\Omega_2^3 = \frac{1}{2}d(\frac{b^2 + c^2 - a^2}{abc}\theta^3) + \omega_\mu^2 \wedge \omega_3^\mu =$$

$$= \frac{1}{2}[(\frac{b^2 + c^2 - c^2}{abc})^\sim + \frac{\tilde{a}}{a^2bc}(b^2 + c^2 - a^2)]\theta^0 \wedge \theta^1 +$$

$$\frac{1}{2}[a^2(b^2 + c^2 - a^2) - 2a^2bc\tilde{b}\tilde{c} + \frac{1}{2}(a^2 + b^2 - c^2)(b^2 - a^2 - c^2)]\frac{\theta^2 \wedge \theta^3}{(abc)^2}\ .$$

Now, very easily we can get the curvature tensor components

$$R_{101}^0 = -\frac{\tilde{\tilde{a}}}{a}, \quad R_{102}^0 = R_{103}^0 = R_{112}^0 = R_{113}^0 = 0\,,$$

$$R_{123}^0 = \frac{1}{abc}[-a\tilde{a} + \frac{\tilde{b}}{2b}(a^2 + b^2 - c^2) - \frac{\tilde{c}}{2c}(b^2 - a^2 - c^2)],$$

$$R_{202}^0 = -\frac{\tilde{\tilde{b}}}{b}, \quad R_{201}^0 = R_{203}^0 = R_{212}^0 = R_{223}^0 = 0,$$

$$R_{213}^0 = \frac{1}{abc}[b\tilde{b} - \frac{\tilde{a}}{2a}(a^2 + b^2 - c^2) - \frac{\tilde{c}}{2c}(b^2 + c^2 - a^2)],$$

$$R^{0}_{303} = -\frac{\tilde{\tilde{c}}}{c}, \qquad R^{0}_{301} = R^{0}_{302} = R^{0}_{313} = R^{0}_{323} = 0,$$

$$R^{0}_{312} = \frac{1}{abc}[-c\tilde{c} - \frac{\tilde{a}}{2a}(b^2 - a^2 - c^2) + \frac{\tilde{b}}{2b}(b^2 + c^2 - a^2)]$$

$$R^{1}_{203} = \frac{1}{2}[(\frac{a^2 + b^2 - c^2}{abc})^{\sim} + \frac{\tilde{c}}{abc^2}(a^2 + b^2 - c^2)], \qquad R^{1}_{201} = R^{1}_{202} = R^{1}_{213} = R^{1}_{323} = 0$$

$$R^{1}_{212} = -\frac{\tilde{a}\tilde{b}}{ab} + \frac{1}{4(abc)^2}(a^4 + b^4 - 3c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2),$$

$$R^{1}_{302} = \frac{1}{2}[(\frac{b^2 - a^2 - c^2}{abc})^{\sim} + \frac{\tilde{b}}{ab^2c}(b^2 - a^2 - c^2)], \qquad R^{1}_{301} = R^{1}_{302} = R^{1}_{312} = 0,$$

$$R^{1}_{313} = -\frac{\tilde{a}\tilde{c}}{ac} + \frac{1}{4(abc)^2}(a^4 - 3b^4 + c^4 + 2a^2b^2 - 2a^2c^2 + 2b^2c^2),$$

$$R^{2}_{301} = \frac{1}{2}[(\frac{b^2 + c^2 - a^2}{abc})^{\sim} + \frac{\tilde{b}}{a^2bc}(b^2 + c^2 - a^2)], \qquad R^{2}_{302} = R^{2}_{303} = R^{2}_{312} = R^{2}_{313} = 0,$$

$$R^{2}_{323} = -\frac{\tilde{b}\tilde{c}}{bc} + \frac{1}{4(abc)^2}(-3a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 - 2b^2c^2),$$

From $\quad \Omega^{i}_{0} = -\Omega^{0}_{i} \;\Rightarrow\; R^{i}_{0\mu\nu} = -R^{0}_{i\mu\nu}$,

$\Omega^{i}_{i} = 0 \;\Rightarrow\; R^{i}_{i\mu\nu} = 0$ (no sum) ,

$\Omega^{2}_{1} = -\Omega^{1}_{2} \;\Rightarrow\; R^{2}_{1\mu\nu} = -R^{1}_{2\mu\nu}$ and also $R^{3}_{2\mu\nu} = -R^{2}_{3\mu\nu}$, $R^{3}_{1\mu\nu} = -R^{1}_{3\mu\nu}$.

The Ricci components are

$$R_{00} = R^{\mu}_{0\mu0} = R^{0}_{101} + R^{0}_{202} + R^{0}_{303} = -(\frac{\tilde{\tilde{a}}}{a} + \frac{\tilde{\tilde{b}}}{b} + \frac{\tilde{\tilde{c}}}{c}) ,$$

$$R_{11} = R^{\mu}_{1\mu1} = R^{0}_{101} + R^{1}_{212} + R^{1}_{313} = -\frac{\tilde{\tilde{a}}}{a} - \frac{\tilde{a}\tilde{b}}{ab} - \frac{\tilde{a}\tilde{c}}{ac} + \frac{1}{a^2} + \frac{1}{2(abc)^2}(a^4 - b^4 - c^4) ,$$

$$R_{22} = R^{\mu}_{2\mu2} = R^{0}_{202} + R^{2}_{121} + R^{3}_{232} = -\frac{\tilde{\tilde{b}}}{b} - \frac{\tilde{a}\tilde{b}}{ab} - \frac{\tilde{b}\tilde{c}}{bc} + \frac{1}{b^2} + \frac{1}{2(abc)^2}(b^4 - c^4 - a^4) ,$$

$$R_{33} = R^{\mu}_{3\mu3} = R^{0}_{303} + R^{1}_{313} + R^{2}_{323} = -\frac{\tilde{\tilde{c}}}{c} - \frac{\tilde{a}\tilde{c}}{ac} - \frac{\tilde{b}\tilde{c}}{bc} + \frac{1}{c^2} + \frac{1}{2(abc)^2}(c^4 - b^4 - a^4) .$$

Finally, the Ricci scalar is

$$^{(4)}R = \mu^{\kappa\lambda} R_{\kappa\lambda} = R_{00} + R_{11} + R_{22} + R_{33} \tag{3.1}$$
$$= -2\left(\frac{\tilde{\tilde{a}}}{a} + \frac{\tilde{\tilde{b}}}{b} + \frac{\tilde{\tilde{c}}}{c}\right) - 2\left(\frac{\tilde{a}\tilde{b}}{ab} + \frac{\tilde{a}\tilde{c}}{ac} + \frac{\tilde{b}\tilde{c}}{bc}\right) + \frac{1}{2(abc)^2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)$$

If we repeat the Cartan calculus only for the three-dimensional part of the metric we find for all the 1-forms, the 2-forms, and the curvature components the same results as above, except for all the terms including derivatives of a,b, or c which now disappear completely. So, the three-dimensional Ricci scalar is

$$^{(3)}R = \frac{1}{2(abc)^2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) \tag{3.1'}$$

The gravitational part of the action becomes

$$I_G = -\frac{1}{16\pi}\int(^{(4)}R - 2\Lambda)\sqrt{g}\, d^4x = -\frac{1}{16\pi}\int(^{(4)}R - 2\Lambda)abc\sin\theta\, d\psi\, d\varphi\, d\theta\, d\xi = -\pi\int(^{(4)}R - 2\Lambda)abc\, d\xi$$
$$= 2\pi\int abc\left(\frac{\tilde{\tilde{a}}}{a} + \frac{\tilde{\tilde{b}}}{b} + \frac{\tilde{\tilde{c}}}{c} + \frac{\tilde{a}\tilde{b}}{ab} + \frac{\tilde{a}\tilde{c}}{ac} + \frac{\tilde{b}\tilde{c}}{bc}\right)d\xi - \pi\int\frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{2abc}d\xi + 2\pi\Lambda\int abc\, d\xi$$

If there is no matter field then $I_G = -2\pi\Lambda\int abcd\xi$ is the numerical value. One has $\int\tilde{\tilde{a}}bcd\xi = \int(\tilde{a}bc)^{\sim}d\xi - \int(\tilde{a}\tilde{b}c + \tilde{a}\tilde{c}b)d\xi$ and similarly for the terms containing the second derivatives of b and c. But the integral of the total derivative gives a contribution which is canceled by the surface term I_S, as will be shown below. Finally, we obtain the following action (first order as usual in classical mechanics) :

$$I_g = -\pi\int\left[2abc\left(\frac{\tilde{a}\tilde{b}}{ab} + \frac{\tilde{a}\tilde{c}}{ac} + \frac{\tilde{b}\tilde{c}}{bc}\right) + \frac{1}{2abc}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)\right]d\xi + 2\pi\Lambda\int abcd\xi \tag{3.2}$$

where we have set $I_g = I_G + I_S$.

Now it is very simple to obtain the Euler-Lagrange equations of this action, which coincide with the Einstein equations and read :

$$(\tilde{a}bc)^{\sim} = \frac{1}{2abc}[a^4 - (b^2 - c^2)^2] - \Lambda abc \tag{3.3}$$

$$(a\tilde{b}c)^{\sim} = \frac{1}{2abc}[b^4 - (a^2 - c^2)^2] - \Lambda abc \tag{3.4}$$

$$(ab\tilde{c})^{\sim} = \frac{1}{2abc}[c^4 - (a^2 - b^2)^2] - \Lambda abc \quad (3.5)$$

This system is a constrained system due to the time reparametrization -the remnant of the full coordinate covariance of general relativity. The constraint may be derived from the action I_g if one imposes the condition that the integral is stationary under the replacement $d\xi = N(\xi)d\xi$, where $N(\xi)$ is an arbitrary function of ξ. The constraint says that the "Hamiltonian" corresponding to the Lagrangian of the action vanishes. Note that because the time is imaginary the roles of the physical Lagrangian and minus the Hamiltonian are interchanged. The constraint has the form

$$\frac{\tilde{a}\tilde{b}}{ab} + \frac{\tilde{a}\tilde{c}}{ac} + \frac{\tilde{b}\tilde{c}}{bc} = \frac{1}{4(abc)^2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4) - \Lambda \quad (3.6)$$

and puts a restriction on the initial values of $a,b,c,\tilde{a},\tilde{b},\tilde{c}$ which is preserved by the evolution equations. Adding the three dynamical equations and replacing in the constraint, we obtain another equivalent expression for that, namely

$$\frac{\tilde{\tilde{a}}}{a} + \frac{\tilde{\tilde{b}}}{b} + \frac{\tilde{\tilde{c}}}{c} + \Lambda = 0 \quad (3.7)$$

We define two more useful "time" parameters (actually it would be better to say "distance" parameters). One is the logarithmic one τ such that $d\xi = abcd\tau$ ($\cdot \equiv d/d\tau$). Now the lapse function of the metric is proportional to the spatial volume **V**. Then, the expression for the Ricci scalar and the action are :

$$^{(4)}R = -\frac{2}{(abc)^2}[\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} - (\frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + \frac{\dot{c}^2}{c^2}) - (\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc})] + \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{2(abc)^2} \quad (3.8)$$

$$I_g = -\pi\int[2(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}) + \frac{1}{2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)]d\tau + 2\pi\Lambda\int a^2b^2c^2 d\tau \quad (3.9)$$

The field equations and the constraint get the form:

$$(\ln a^2)^{\cdot\cdot} = a^4 - (b^2 - c^2)^2 - 2\Lambda a^2 b^2 c^2 \quad (3.10)$$

$$(\ln b^2)^{\cdot\cdot} = b^4 - (a^2 - c^2)^2 - 2\Lambda a^2 b^2 c^2 \quad (3.11)$$

$$(\ln c^2)^{\cdot\cdot} = c^4 - (a^2 - b^2)^2 - 2\Lambda a^2 b^2 c^2 \quad (3.12)$$

$$\begin{aligned}[\ln(a^2b^2c^2)]^{\cdot\cdot} &= 4\left(\frac{\dot a\dot b}{ab} + \frac{\dot a\dot c}{ac} + \frac{\dot b\dot c}{bc}\right) - 2\Lambda a^2 b^2 c^2 \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 - 6\Lambda a^2b^2c^2\end{aligned} \quad (3.13)$$

Second, we have the "time" η such that $d\xi = 2(abc)^{1/3} d\eta$, where now the lapse function is proportional to the linear dimension corresponding to the spatial volume ($' \equiv d/d\eta$). Both τ and η go to $-\infty$ when $\xi \to 0$ and to $+\infty$ for $\xi \to +\infty$, for some specific boundary condition we discuss latter.

We come back to the surface term I_s. For the extrinsic curvature we have the general formula $K_{ij} = \frac{1}{2N}(\dot g_{ij} - 2N_{(i,j)})$, where $ds^2 = (N^iN_i + N^2)d\tau^2 + 2N_i d\tau dx^i + g_{ij}dx^i dx^j$. From the ansatz we use, it is $N^i = N_i = 0, \quad N = abc$. Thus :

$K_{ij} = \frac{1}{2abc}\dot g_{ij}$. So, if the frame is $\{d\tau, d\psi, d\theta, d\varphi\}$ it will be

$$K_{\psi\psi} = \frac{1}{2abc}\dot g_{\psi\psi} = \frac{1}{2abc}(c^2)^{\cdot} = \frac{\dot c}{ab}$$

$$K_{\psi\theta} = K_{\theta\psi} = \frac{1}{2abc}\dot g_{\psi\theta} = 0$$

$$K_{\psi\varphi} = K_{\varphi\psi} = \frac{1}{2abc}\dot g_{\psi\varphi} = \frac{1}{2abc}(c^2\cos\theta)^{\cdot} = \frac{\dot c}{ab}\cos\theta$$

$$K_{\theta\theta} = \frac{1}{2abc}\dot g_{\theta\theta} = \frac{1}{2abc}(a^2\cos^2\psi + b^2\sin^2\psi)^{\cdot} = \frac{\dot a}{bc}\cos^2\psi + \frac{\dot b}{ac}\sin^2\psi$$

$$K_{\theta\varphi} = K_{\varphi\theta} = \frac{1}{2abc}\dot g_{\theta\varphi} = \frac{1}{2abc}\left(\frac{a^2-b^2}{2}\sin2\psi\sin\theta\right)^{\cdot} = \frac{1}{2}\left(\frac{\dot a}{bc} - \frac{\dot b}{ac}\right)\sin2\psi\sin\theta$$

$$K_{\varphi\varphi} = \frac{1}{2abc}\dot g_{\varphi\varphi} = \sin^2\theta\left(\frac{\dot a}{bc}\sin^2\psi + \frac{\dot b}{ac}\cos^2\psi\right) + \frac{\dot c}{ab}\cos^2\theta\ .$$

A lengthy but straightforward calculation gives for the scalar K :

$K = g^{ij} K_{ij} = \frac{1}{abc}(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c})$. Replacing in I_s we get

$$I_s = -\frac{1}{8\pi}(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}) \int sin\theta d\psi d\theta d\varphi = -2\pi(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}) \tag{3.14}$$

The integral of the total derivative appearing in I_G is :

$$2\pi\int(\tilde{a}bc + a\tilde{b}c + ab\tilde{c})^{\tilde{}} d\xi = 2\pi\int(abc\tilde{)}^{\tilde{}} d\xi = 2\pi\int[\frac{(abc)^{\cdot}}{abc}]^{\cdot} d\tau$$

$$= 2\pi\int(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c})^{\cdot} d\tau \tag{3.14'}$$

and cancels the surface term (3.14) if the term coming from the upper limit of the integral (3.14') is zero.

At this point we introduce the Misner variables α, β_+, β_- , writing the metric as follows : $ds^2 = d\xi^2 + e^{2\alpha} e^{2\beta}{}_{ij}\sigma^i\sigma^j$, where $(\beta_{ij}(\xi))$ is a 3×3 traceless matrix (e^β denotes matrix exponentiation). For the diagonal Bianchi IX Universe, only two independent parameters must be contained in the matrix (β_{ij}), namely $(\beta_{ij}) = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$. The relation of Misner variables with the scale factors a, b, c is the following :

$$a = e^{\alpha+\beta_+ +\sqrt{3}\beta_-} \quad , \quad b = e^{\alpha+\beta_+ -\sqrt{3}\beta_-} \quad , \quad c = e^{\alpha-2\beta_+} \qquad \text{or}$$

$$\alpha = \frac{1}{3}\ln(abc) \quad , \quad \beta_+ = \frac{1}{6}\ln(\frac{ab}{c^2}) \quad , \quad \beta_- = \frac{1}{2\sqrt{3}}\ln(\frac{a}{b}) \ .$$

Then $V = 16\pi^2 e^{3\alpha}$. This form separates the expansion (volume change) and the anisotropy (shape change). If put $\beta_+ = \beta_- = 0$, we obtain the FRW model with $R = 2e^\alpha$, R being the radius of the Universe. As $\alpha \to -\infty$ we approach the singularity at $R = 0$. We derive the various preceding results in terms of α, β_+, β_-.

Since $\tilde{a} = (\tilde{\alpha} + \tilde{\beta}_+ + \sqrt{3}\tilde{\beta}_-)a$, $\tilde{b} = (\tilde{\alpha} + \tilde{\beta}_+ - \sqrt{3}\tilde{\beta}_-)b$, $\tilde{c} = (\tilde{\alpha} - 2\tilde{\beta}_+)c$, and thus $\frac{\tilde{a}\tilde{b}}{ab} + \frac{\tilde{a}\tilde{c}}{ac} + \frac{\tilde{b}\tilde{c}}{bc} = 3(\tilde{\alpha}^2 - \tilde{\beta}_+^2 - \tilde{\beta}_-^2)$, the Euclidean action (3.2) takes the form

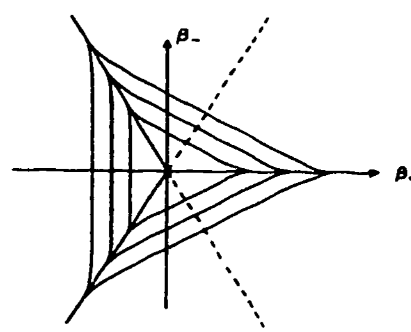
$$I_g = 6\pi \int e^{3\alpha} (\tilde{\beta}_+^2 + \tilde{\beta}_-^2 - \tilde{\alpha}^2 - \frac{\rho}{3} e^{-2\alpha} + \frac{\Lambda}{3}) d\xi \tag{3.15}$$

where

$$\begin{aligned} \rho &= 2e^{4\beta_+} + 2e^{-2\beta_+ + 2\sqrt{3}\beta_-} + 2e^{-2\beta_+ - 2\sqrt{3}\beta_-} - e^{4\beta_+ + 4\sqrt{3}\beta_-} - e^{4\beta_+ - 4\sqrt{3}\beta_-} - e^{-8\beta_+} \\ &= 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) - 4e^{4\beta_+} \sinh^2(2\sqrt{3}\beta_-) - e^{-8\beta_+} \\ &= 3 - 3V . \end{aligned} \tag{3.16}$$

The potential $V(\beta_+, \beta_-)$ is $V \geq 0$ and the diagram showing the contours of this is the following, where the three-fold rotational symmetry and the two-fold reflection symmetry is obvious.

The potential has exponentially steep walls with equipotentials forming equilaterals triangles in the (β_+, β_-) plane. However, the corners of the triangular contours are not closed, but rather have thin channels leading off to infinity. The potential is time-independent, since it has no dependence on ξ or even on α .



The dynamical equations arising from the action (3.15) can be thought of as those of a particle moving in the time-dependent potential $e^{3\alpha}(-\frac{\rho}{12}e^{-2\alpha} + \frac{\Lambda}{3})$. At large negative values of α this potential is flat. As α increases so does this potential, becoming infinitely steep as $\alpha \to +\infty$. Care must be taken however since α is not always a monotonic function of ξ or η.

Going back to the Lorentzian case, the walls expand as the Universe collapses to zero volume at zero proper time. Encounters with walls become increasingly rare as the proper time goes to zero. In all models (except Bianchi VIII & IX) the potential is open in at least one direction. The closed potentials allow recurrence to occur and the motion explores all of the phase-space available to it, constrained only by the conservation laws of energy and momentum [3]. The shape of the VIII & IX potentials in the corner channels supplies the instability necessary for chaotic behavior. Chaotic behavior requires that neighboring trajectories of Universe points diverge as they are followed both forwards and backwards in time. Approximate "bounce laws" may be obtained to describe the interaction of the system point with the potential walls (where the Kasner exponents change during this interaction). It is worthwhile to say that chaotic behavior does not appear in the Euclidean model. The following argument is due to [5]. If H^2 is the square of the Euclidean Arnowitt-Deser-Misner Hamiltonian, then

$$H^2 = p_+^2 + p_-^2 + 24\pi^2 h^{(3)}R = p_+^2 + p_-^2 + 36\pi^2 e^{4\alpha}(1-V) \text{ , since } {}^{(3)}R = \frac{1}{2}e^{-2\alpha}\rho,$$

and $h={}^{(3)}g$ is the determinant of the three-dimensional metric. A positive sign in front of the third term on the right-hand of the above equation appears, instead of a negative one in the space-time case. The shape of the equipotentials is unchanged in passing to the Euclidean case. The potential minimum at the origin becomes a maximum. The walls decrease steeply to $-\infty$ as $|\beta_\pm|$ increases. For convenience, let a single (triangular) equipotential represents for some fixed α and H^2 (with $H^2 < 0$ allowed) the curve described by the last equation with $p_+ = p_- = 0$. Since the region inside the curve for the same fixed α corresponds to greater H^2, it is forbidden to the system point. Thus the system point is either above or outside the potential (now a barrier) in the Euclidean case. Even if $H^2 < 0$ is possible, there is no Euclidean analog of the system point repeatedly bouncing off the corners of the potential. The presence of at most one bounce for the system point in the Euclidean case means that the separation between trajectories with infinitesimally different initial points will be linear rather than exponential in α. In the Euclidean case there is no qualitative change in the dynamics analogous to the change from expanding to contracting directions in the spacetime case which occurs when the point enters the "corners" of the potential. Because the system point must remain outside the corners of the potentials, qualitative dynamical changes which cause the chaotic behavior of spacetime type-IX cosmologies cannot occur in the Euclidean analog solutions.

In the logarithmic time the action is

$$I_g = 6\pi\int(\dot\beta_+^2 + \dot\beta_-^2 - \dot\alpha^2 - \frac{\rho}{12}e^{4\alpha} + \frac{\Lambda}{3}e^{6\alpha})d\tau \qquad (3.17)$$

The constraint equation is $\dot\beta_+^2 + \dot\beta_-^2 - \dot\alpha^2 = -\frac{\rho}{12}e^{4\alpha} + \frac{\Lambda}{3}e^{6\alpha}$ (3.18). Varying with respect to the dynamical variables α, β_+, β_- gives rise to the second-order evolution equations :

$$\ddot\alpha = \frac{1}{6}e^{4\alpha}\rho - \Lambda e^{6\alpha} \qquad (3.19)$$

$$\ddot\beta_\pm = \frac{1}{24}e^{4\alpha}\frac{\partial\rho}{\partial\beta_\pm} \qquad (3.20)$$

With respect to the time η, where $d\eta = \frac{1}{2}e^{2\alpha}d\tau = \frac{1}{2}e^{-\alpha}d\xi$, the action is

$$I_g = 3\pi\int(\beta_+'^2 + \beta_-'^2 - \alpha'^2 - \frac{\rho}{3} + \frac{4\Lambda}{3}e^{2\alpha})e^{2\alpha}d\eta \qquad (3.21)$$

The constraint is $\beta_+'^2 + \beta_-'^2 - \alpha'^2 = -\frac{\rho}{3} + \frac{4\Lambda}{3}e^{2\alpha}$ (3.22), and the field equations are :

$$\alpha'' = -2(\beta_+'^2 + \beta_-'^2) - \frac{4}{3}\Lambda e^{2\alpha} \qquad (3.23)$$

$$\beta_\pm'' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} - 2\alpha'\beta_\pm' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} \mp 2\beta_\pm'\sqrt{\beta_+'^2 + \beta_-'^2 + \frac{1}{3}\rho - \frac{4}{3}\Lambda e^{2\alpha}} \qquad (3.24).$$

For $\Lambda=0$, this system has a special feature. The (β_+, β_-) motion decouples from the α motion, after using the constraint equation to eliminate α. The evolution in the β_+, β_- plane is governed by a system of two coupled equations.

Finally, we note for completeness that the surface action I_s is $I_S = -6\pi\dot\alpha = -3\pi e^{2\alpha}\alpha'$.

4. Nut and Bolt boundaries

Nearly all known gravitational instantons possess continuous symmetry groups of at least two parameters. There is a classification scheme [8] based on the existence of at least a one parameter group.

We consider an oriented manifold M with a positive definite metric $g_{\mu\nu}$ which admits at least a one-parameter isometry group G. Denote by $\mu_\tau : M \rightarrow M$ the action of the group, where τ is the group parameter and by $K = K^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tau}$ the Killing vector. The isometry group G is said to have a fixed point where $K=0$. At a fixed point p the action of μ_τ on the manifold M gives rise to an isometry $\mu_{\tau *} : T_p(M) \rightarrow T_p(M)$ where $T_p(M)$ is the tangent space at p. $\mu_{\tau *}$ is generated by the antisymmetric matrix $K_{\mu ;\nu}$. Antisymmetric 4x4 matrices can have rank 0, 2, or 4. The zero case is not interesting because it would imply that the Killing vector K was zero everywhere. Since then $\mu_{\tau *}$ is the identity, and μ commutes with the exponential map at p, i.e. $\mu_\tau \circ \exp X = \exp(\mu_{\tau *}(X))$, $\forall X \in T_p(M)$, it follows that the action of the group G is trivial. In the case that $K_{\mu ;\nu}$ has rank 2, there will be a two-dimensional subspace T_1 of $T_p(M)$ which is left invariant by $\mu_{\tau *}$. The action of $\mu_{\tau *}$ will rotate T_2, the two-dimensional orthogonal complement of T_1, into itself. Thus the canonical form in this case is $\mu_{\tau *} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\kappa\tau & \sin\kappa\tau \\ 0 & 0 & -\sin\kappa\tau & \cos\kappa\tau \end{pmatrix}$, where κ is the surface gravity and is given by the non-zero skew eigenvalue of $K_{\mu ;\nu}$ in an orthonormal frame. From this one can see that $\mu_{\tau *}$, and hence μ_τ, must be periodic with a period $2\pi\kappa^{-1}$. The image of T_1 under the exponential map will not be moved by μ_τ, and so will contribute a 2-dimensional oriented totally geodesic submanifold of fixed points. Gibbons and Hawking [8] named such a 2-dimensional fixed point set a Bolt. A simple example is provided by the horizon 2-sphere of the Euclidean Schwarzschild solution, with G being the periodic group of imaginary time translations. In the case that $K_{\mu;\nu}$ has the maximal rank 4 there can be no directions at p which are left invariant under $\mu_{\tau *}$. Thus p must be an isolated fixed point. Gibbons and Hawking called it a Nut after the fixed point at the centre of the Euclidean self-dual Taub-NUT solution. In this case there will be two orthogonal 2-dimensional subspaces T_1 and T_2 which are mapped into themselves by $\mu_{\tau *}$. The canonical form is

$$\mu_{\tau *} = \begin{pmatrix} \cos\kappa_1\tau & \sin\kappa_1\tau & 0 & 0 \\ -\sin\kappa_1\tau & \cos\kappa_1\tau & 0 & 0 \\ 0 & 0 & \cos\kappa_2\tau & \sin\kappa_2\tau \\ 0 & 0 & -\sin\kappa_2\tau & \cos\kappa_2\tau \end{pmatrix},$$

where κ_1, κ_2 are the skew-eigenvalues of $K_{\mu;\nu}$ in an orthonormal frame. For some purposes it is convenient to subdivide nuts into 2 classes - "Nuts" and "anti-Nuts" - depending on whether the sign of $\kappa_1.\kappa_2$ is positive or negative respectively.

We express Nut and Bolt boundary conditions in terms of the more convenient objects a, b, c, ξ. Let one of $\{a, b, c\}$ vanishes, for example c, at a point p. Then the corresponding vector will have zero length at p. This means that the orbit of G through p can no longer be 3-dimensional. In fact the orbit through p corresponds to a subgroup H of G and hence must be one-dimensional or the entire group G. In the first case, which is a Bolt, only c vanishes and the orbit through p corresponds to G/G_3. This is a homogeneous 2-space whose second fundamental form vanishes. This means that a tends to b on the Bolt with vanishing derivative with respect to ξ. By considering the limiting form of the metric on a 2-surface orthogonal to the Bolt -i.e. the (τ, ψ) plane in the coordinates $(\tau, \psi, \theta, \varphi)$ - one can readily see that c must vanish as $\frac{1}{2}\xi$, if ψ has range 4π, whereas c vanishes as ξ if ψ has range 2π. Strictly speaking, the word Bolt, introduced in [8], applies only when $a = b, \forall \xi$. In this case there is an additional Killing vector K_3 (i.e. the group is extended to $U(2)$) and $c = 0$ is the locus of its fixed point set - i.e. its Bolt. In the second case the orbit through p is just p itself and is refered as a Nut. In this case all the $\{a, b, c\}$ must vanish as $\frac{1}{2}\xi$ as $\xi \to 0$, in order that the orbits be a nested sequence of 3-spheres near p. Below we give the behavior of the various quantities close to the Nut or the Bolt point.

Nut case

$\xi \to 0, \quad a, b, c \sim \frac{1}{2}\xi \to 0$

$\alpha \sim \frac{1}{3}\ln(\frac{1}{8}\xi^3) = \ln(\frac{1}{2}\xi) \to -\infty$

$\beta_+ \sim 0 \ , \ \beta_- \sim 0$

$\tilde{a}, \tilde{b}, \tilde{c} \sim \frac{1}{2} \ , \ \tilde{\alpha} \sim \dfrac{1}{\xi} \to +\infty, \ \tilde{\beta}_+, \tilde{\beta}_- \sim 0$

$d\xi = abc\,d\tau \sim \frac{1}{8}\xi^3 d\tau$

$\dot{a}, \dot{b}, \dot{c} \sim \tilde{a}\dfrac{d\xi}{d\tau} \sim \frac{1}{2}\frac{1}{8}\xi^3 \to 0 \ , \ \dot{\alpha} \sim \frac{1}{8}\xi^2 \to 0 \ , \ \dot{\beta}_\pm \sim 0$

$d\xi = 2(abc)^{1/3} d\eta \sim \xi \ d\eta$

$a', b', c' \sim \tilde{a}\dfrac{d\xi}{d\eta} \sim \dfrac{\xi}{2} \to 0 \ , \ \alpha' \sim 1 \ , \ \beta'_\pm \sim 0$

Bolt case

$\xi \to 0, \quad a, b \to a_0, \quad c \to \frac{1}{2}\xi, \quad \tilde{a}, \tilde{b} \to 0$

$\alpha \sim \frac{1}{3}\ln(a_0^2 \frac{1}{2}\xi) \to -\infty, \quad \beta_+ \sim \frac{1}{3} \ln(\dfrac{2a_0}{\xi}) \to +\infty \ , \ \beta_- \sim 0$

$\tilde{c} \sim \dfrac{1}{2}, \ \tilde{\alpha} \sim \dfrac{1}{3\xi} \to +\infty, \ \tilde{\beta}_+ \sim -\dfrac{1}{3\xi} \to -\infty, \ \tilde{\beta}_- \sim 0$

$d\xi \sim \frac{1}{2}a_0^2 \xi \ d\tau \ , \ \dot{a}, \dot{b} \sim \tilde{a}\frac{1}{2}a_0^2\xi \to 0 \ , \ \dot{c} \sim \frac{1}{4}a_0^2\xi \to 0$

$\dot{\alpha} \sim \frac{1}{6}a_0^2 \ , \ \dot{\beta}_+ \sim -\frac{1}{6}a_0^2 \ , \ \dot{\beta}_- \sim 0$

$d\xi \sim (4a_0^2\xi)^{1/3} d\eta \ , \ a', b' \to 0 \ , \ c' \sim (\frac{1}{2}a_0^2\xi)^{1/3} \to 0$

$\alpha' \sim \frac{1}{3}(\frac{2a_0}{\xi})^{2/3} \to +\infty$, $\beta'_+ \sim -\frac{1}{3}(\frac{2a_0}{\xi})^{2/3} \to -\infty$, $\beta' \sim 0$

Definition : If, as a function of τ, no two of a, b, and c are equal, we call the solution triaxial. Otherwise, it is called biaxial. (In analogous problems with quadratic forms in optics the terms biaxial and triaxial are often used).

The action until now has the form $I = \int_{\tau}^{\tau'} L d\tau + I_S$, where the Lagrangian $L = L(a,b,c,\dot{a},\dot{b},\dot{c})$. Then

$$\delta I = \int_{-\infty}^{+\infty} d\tau (\frac{\delta L}{\delta a}\delta a + \frac{\delta L}{\delta b}\delta b + \frac{\delta L}{\delta c}\delta c) + (\frac{\partial L}{\partial \dot{a}}\delta a + \frac{\partial L}{\partial \dot{b}}\delta b + \frac{\partial L}{\partial \dot{c}}\delta c)\Big|_{\tau=-\infty}^{\tau=+\infty} + \delta I_S \ ,$$

where $\frac{\delta L}{\delta a} = \frac{\partial L}{\partial a} - \frac{d}{d\tau}(\frac{\partial L}{\partial \dot{a}})$ the usual Euler-Lagrange variational derivative. At the boundary $\tau = +\infty$ we fix the intrinsic metric, so $\delta a = \delta b = \delta c|_{\tau=+\infty} = 0$. Then

$$\delta I = \int_{-\infty}^{\tau'} d\tau (\frac{\delta L}{\delta a}\delta a + \frac{\delta L}{\delta b}\delta b + \frac{\delta L}{\delta c}\delta c) - (\frac{\partial L}{\partial \dot{a}}\delta a + \frac{\partial L}{\partial \dot{b}}\delta b + \frac{\partial L}{\partial \dot{c}}\delta c)\Big|_{\tau=-\infty} + \delta I_S \ ,$$

where I_S is the expression (3.14) and

$$L = -\pi[2(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}) + \frac{1}{2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)] + 2\pi\Lambda a^2b^2c^2 \, .$$

We must arrive at the field equations $\frac{\delta L}{\delta a} = \frac{\delta L}{\delta b} = \frac{\delta L}{\delta c} = 0$ and thus the following equation must hold

$$-(\frac{\partial L}{\partial \dot{a}}\delta a + \frac{\partial L}{\partial \dot{b}}\delta b + \frac{\partial L}{\partial \dot{c}}\delta c)\Big|_{\tau=-\infty} + \delta I_S = 0 \ .$$

It is $\frac{\partial L}{\partial \dot{a}} = -\frac{2\pi}{\dot{a}}(\frac{\dot{b}}{b} + \frac{\dot{c}}{c})$ and similarly for $\frac{\partial L}{\partial \dot{b}}$, $\frac{\partial L}{\partial \dot{c}}$. Also

$$\delta I_S = 2\pi(\frac{\delta \dot{a}}{a} + \frac{\delta \dot{b}}{b} + \frac{\delta \dot{c}}{c} - \frac{\dot{a}}{a}\frac{\delta a}{a} - \frac{\dot{b}}{b}\frac{\delta b}{b} - \frac{\dot{c}}{c}\frac{\delta c}{c})\Big|_{\tau=-\infty} \ .$$

Thus

$$(\frac{\dot{b}}{b} + \frac{\dot{c}}{c} - \frac{\dot{a}}{a})\frac{\delta a}{a} + (\frac{\dot{a}}{a} + \frac{\dot{c}}{c} - \frac{\dot{b}}{b})\frac{\delta b}{b} + (\frac{\dot{a}}{a} + \frac{\dot{b}}{b} - \frac{\dot{c}}{c})\frac{\delta c}{c} + \frac{\delta \dot{a}}{a} + \frac{\delta \dot{b}}{b} + \frac{\delta \dot{c}}{c}\Big|_{\tau=-\infty} = 0 \ .$$

This is the initial condition that must be satisfied. One sufficient but not necessary choice is to take $\dot{a} = \dot{b} = \dot{c}\big|_{\tau=-\infty} = 0$.

Proposition : The only solutions satisfying $\dot{a} = \dot{b} = \dot{c}\big|_{\tau_0} = 0$ on some spacelike surface Σ_{τ_0}, $(\tau_0 > -\infty)$, which give a compact Einstein metric are biaxial.

Proof : To have a compact metric, a, b, and c cannot all remain non-zero : we must "close off the space". This can be done, as we already said, if one or three of a, b, c vanish. This is because the orbits of *SU(2)* must collapse in dimension. The collapsed orbits are still homogeneous spaces of *SU(2)*. The only possibilities are zero-dimensional points (Nuts, $a = b = c = 0$) or two-dimensional subspaces (Bolts, one of a, b, c vanishes and the other two become equal). The resulting Bolt is either an S^2 or an RP^2. Now, let us suppose that a, b, and c are all unequal on the surface Σ_{τ_0}. With no loss of generality we may then assume $a > b > c$ on this surface. From the equations (3.10), (3.11), (3.12), for $(\ln a)^{\cdot\cdot}$, $(\ln b)^{\cdot\cdot}$, $(\ln c)^{\cdot\cdot}$ we obtain by subtraction :

$$(\ln \frac{a}{b})^{\cdot\cdot} = (a^2 - b^2)(a^2 + b^2 - c^2) \quad , \quad (\ln \frac{a}{c})^{\cdot\cdot} = (a^2 - c^2)(a^2 + c^2 - b^2).$$

At $\tau = \tau_0$, one has $\ln \frac{a}{b} > 0$, $(\ln \frac{a}{b})^{\cdot} = 0$, $(\ln \frac{a}{b})^{\cdot\cdot} > 0$, $\ln \frac{a}{c} > 0$, $(\ln \frac{a}{c})^{\cdot} = 0$, $(\ln \frac{a}{c})^{\cdot\cdot} > 0$ and thus $\ln \frac{a}{b}$ and $\ln \frac{a}{c}$ for all preceding instants is strictly positive. Thus, if $a > b > c$ at $\tau = \tau_0$, then $a > b$ and $a > c$ for all times at which $a > 0$. One cannot have a Bolt, requiring one of the three, which could only be b or c, to go to zero, while the remaining pair, which must include a , become equal (non-zero). Neither can one have a Nut for the following reason : Since $(\ln \frac{a}{b})^{\cdot\cdot} > 0$ for each time, one has $(\ln \frac{a}{b})^{\cdot} < 0$, $\forall\, \tau < \tau_0$. At this point, one should have a look at the beginning of section 7, where the relations (7.3) and (7.4) are stated for the behavior near a Nut. So,

$\frac{d\xi}{d\tau}(\ln\frac{a}{b})^{\sim} < 0 \Rightarrow (\ln\frac{a}{b})^{\sim} < 0, \forall \tau < \tau_0$. From (7.3) one has $\tau_1 < \tau_2 \Rightarrow \xi_1 < \xi_2$, so

$(\ln\frac{a}{b})^{\sim} < 0, \forall \xi < \xi_0$ (ξ_0 corresponds to τ_0). From (7.5) ,

$$\ln\frac{a}{b} \approx \ln\frac{1 + 2a_1\xi^2}{1 + 2b_1\xi^2} \approx \ln[(1 + 2a_1\xi^2)(1 - 2b_1\xi^2) \approx \ln[1 + 2(a_1 - b_1)\xi^2] \approx 2(a_1 - b_1)\xi^2$$

where $a_1 > b_1$, since $a > b$. Thus, close to $\xi = 0$ it is $(\ln\frac{a}{b})^{\sim} \approx 4(a_1 - b_1)\xi > 0$ .This contradiction means that the existence of a Nut is also impossible. The result is that $a = b$ or $a = c$ (or both) will be true for all time and the metric is biaxial. (q.e.d.)

Biaxial Bianchi type-IX metrics have an extra Killing vector, i.e. they are invariant under a group homomorphic to $U(1) \times U(2)$. If $a = b$ and c vanishes at a Bolt with $a = b \neq 0$ and $\frac{da}{d\xi} = 0 = \frac{db}{d\xi}$ the generator of the $U(1)$ factor has a fixed point set at the Bolt.

Finally, we find the limiting value of the surface action at the Nut and Bolt point. It is $I_S = -2\pi(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c})$.

At the Nut limit , $I_S \to -2\pi(\frac{\frac{1}{2}\frac{1}{8}\xi^3}{\frac{1}{2}\xi}) \cdot 3 = -\frac{3\pi}{4}\xi^2 \to 0,\ \xi \to 0$ (4.1)

At the Bolt limit ,

$$I_S \to -2\pi(\frac{\tilde{a}\frac{a_0^2}{2}\xi}{a_0} + \frac{\tilde{b}\frac{a_0^2}{2}\xi}{a_0} + \frac{\frac{a_0^2}{4}\xi}{\frac{1}{2}\xi}) \to -2\pi(0 + 0 + \frac{a_0^2}{2}) = -\pi a_0^2,\ \xi \to 0 \qquad (4.2)$$

Obviously, the same results arise if one works in a different time-parameter.

Since Nuts and Bolts are not boundaries of the manifold, one does not have a contribution from I_S there. However, the integration by parts of I_G gives a surface term which is canceled by I_s at the true boundary but would not be at a Bolt (at a Nut the contribution is zero).

5. A few known solutions

If in the dynamical equations for $\Lambda = 0$ we put the condition that two of the invariant directions have equal magnitude, e.g., $a = b$, this leads to the general Taub–NUT family of solutions which is invariant under a 4-parameter group with the Lie algebra of $U(2)$:

$$a^2 = b^2 = \tfrac{1}{4} q \sinh q(\tau - \tau_2) \operatorname{csch}^2 \tfrac{1}{2} q(\tau - \tau_1) \quad , \quad c^2 = q \operatorname{csch} q(\tau - \tau_2) \tag{5.1}$$

where (q, τ_1, τ_2) are constants of integration. The most familiar form

$$ds^2 = (r^2 - n^2)(r^2 - 2mr + n^2)^{-1} dr^2 + 4n(r^2 - n^2)^{-1}(r^2 - 2mr + n^2)(\sigma^3)^2 + (r^2 - n^2)((\sigma^1)^2 + (\sigma^2)^2) \tag{5.2}$$

is obtained using the transformation :

$$n^2 = -\tfrac{1}{4} q \operatorname{csch} q(\tau - \tau_1) \quad , \quad m = n \cosh q(\tau_2 - \tau_1) ,$$

$$r = [\frac{q}{4n} \coth \tfrac{1}{2} q(\tau - \tau_1) - \coth q(\tau_2 - \tau_1)] \tag{5.3}$$

Two special cases are of note :

A) $\tau_1 = \tau_2$. These are the Eguchi-Hanson metrics [4,7]. Real q corresponds to their type II. Imaginary q corresponds to their type I. One can obtain their metric by the following transformation : $m = n + \dfrac{g^4}{128 n^3}$, $r = m + \dfrac{\rho^2}{8n}$, where g is Eguchi and Hanson's parameter. If one lets $n \to \infty$, this gives the metric

$$ds^2 = (1 - \frac{g^4}{\rho^4})^{-1} d\rho + \frac{1}{4}\rho^4 (1 - \frac{g^4}{\rho^4})(\sigma^3)^2 + \frac{1}{4}\rho^2((\sigma^1)^2 + (\sigma^2)^2) \tag{5.4}$$

B) $q = 0$. This corresponds to the self-dual solution discussed by Hawking [13]. All Bianchi IX solutions with self-dual curvature may be obtained systematically. The self-dual conditions lead, after a single integration, to the following equations :

$$2\frac{\dot{a}}{a} = b^2 + c^2 - a^2 - 2\lambda_1 bc \ ,$$

$$2\frac{\dot{b}}{b} = a^2 + c^2 - b^2 - 2\lambda_2 ac \ ,$$

$$2\frac{\dot{c}}{c} = a^2 + b^2 - c^2 - 2\lambda_3 ab \qquad (5.5)$$

The $\{\lambda_i\}$ are constants obeying $\lambda_1 = \lambda_2\lambda_3$, $\lambda_3 = \lambda_1\lambda_2$, $\lambda_2 = \lambda_3\lambda_1$. Then the possible solutions for λ_i are : S1) $(\lambda_1,\lambda_2,\lambda_3) = (0,0,0)$, S2) $(\lambda_1,\lambda_2,\lambda_3) = (1,1,1)$, S3) $(\lambda_1,\lambda_2,\lambda_3) = (-1,-1,1)$ and cyclic permutations. In fact case S3) is not distinct from case S2) since it may be obtained by the substitution $c \rightarrow -c$. Both cases A) and B) have self-dual curvature. Case S1) may be obtained directly without integration by requiring that the connection forms ω^i_j in the basis $(abcd\tau,\ a\sigma^1,\ b\sigma^2,\ c\sigma^3)$ be self-dual.

If define $\omega_1 \equiv bc,\ \omega_2 \equiv ac,\ \omega_3 \equiv ab$ or $a^2 = \frac{\omega_2\omega_3}{\omega_1},\ b^2 = \frac{\omega_1\omega_3}{\omega_2},\ c^2 = \frac{\omega_1\omega_2}{\omega_3}$, then the system of the self-dual equations becomes :

$$\dot{\omega}_1 = \omega_2\omega_3 - \omega_1(\lambda_2\omega_2 + \lambda_3\omega_3)$$

$$\dot{\omega}_2 = \omega_1\omega_3 - \omega_2(\lambda_1\omega_1 + \lambda_3\omega_3)$$

$$\dot{\omega}_3 = \omega_1\omega_2 - \omega_3(\lambda_1\omega_1 + \lambda_2\omega_2) \qquad (5.6)$$

For $\lambda_1 = \lambda_2 = \lambda_3 = 0$ the Euler system is defined, describing the motion of a rigid body around its centre of gravity : $\dot{\omega}_1 = \omega_2\omega_3$, and cyclically.

For $\lambda_1 = \lambda_2 = \lambda_3 = 1$ we get the Darboux system describing a problem of geometry of second degree surfaces : $\dot{\omega}_1 = \omega_2\omega_3 - \omega_1\omega_2 - \omega_1\omega_3$, and cyclically.

The Euler system has been integrated by Abel and Jacobi and in the Bianchi IX model by Belinski, Gibbons, Page, and Pope (BGPP) with elliptic functions [4]. If $u = -c_1\tau + c_2$ then the general solution of the Euler system is :

$$a^2 = c_1\frac{cnudnu}{snu} \ ,\quad b^2 = c_1\frac{cnu}{snudnu}\ ,\quad c^2 = c_1\frac{dnu}{snucnu} \qquad (5.7)$$

where sn, cn, dn are the standard Jacobian elliptic functions with modulus k. An alternative form is $ds^2 = \frac{1}{4}P^{-1/2}dx^2 + P^{1/2}(\frac{(\sigma^1)^2}{x-x_1} + \frac{(\sigma^2)^2}{x-x_2} + \frac{(\sigma^3)^2}{x-x_3})$, where $dx = 2a^2b^2c^2d\tau = 2P^{1/2}d\tau$, $x - x_1 = b^2c^2$, $x - x_2 = a^2c^2$, $x - x_3 = a^2b^2$, $P = (x-x_1)(x-x_2)(x-x_3) = (abc)^4$. The only singularities of the general solution of the Euler system are movable simple poles. In the Misner variables and η- time the Euler system has the following form :

$$\beta_+' = \tfrac{2}{3}[-e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) + e^{-4\beta_+}]$$

$$\beta_-' = -\tfrac{2}{\sqrt{3}}e^{2\beta_+}\sinh(2\sqrt{3}\beta_-)$$

$$\alpha' = \tfrac{1}{3}[2e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) + e^{-4\beta_+}] \qquad (5.7')$$

The Darboux system has been integrated by Halphen and Bureau with Hermite modular elliptic functions. The general solution of the Darboux system is only defined inside or outside a movable disk. It is holomorphic in its domain of definition, and its only singularity is a movable natural boundary defined by the circle. Then the self–dual system takes the form :

$$(\omega_1 + \omega_2)^{\cdot} = -2\omega_1\omega_2 \ , \ (\omega_1 + \omega_3)^{\cdot} = -2\omega_1\omega_3 \ , \ (\omega_2 + \omega_3)^{\cdot} = -2\omega_2\omega_3 \qquad (5.8)$$

These equations were explicitly solved (or linearized) by Atiyah and Hitchin (AH) [2] as follows. Consider any solution of the linear differential equation $\frac{d^2u}{d\theta^2} + \frac{u}{4}\csc^2\theta = 0$ (5.9), where θ and τ are related by the linear equation $d\theta = u^2 d\tau$. Then

$$\omega_1 = -u\frac{du}{d\theta} - \frac{1}{2}u^2\csc\theta \ , \ \omega_2 = -u\frac{du}{d\theta} + \frac{1}{2}u^2\cot\theta \ , \ \omega_3 = -u\frac{du}{d\theta} + \frac{1}{2}u^2\csc\theta \qquad (5.10)$$

is a three-parameter family of solutions. A solution for u is $u(\theta) = \sqrt{2\sin\theta}\,K(\sin\theta/2)$, $0 \le \theta < \pi$ (5.11), where $K(k)$ is the complete elliptic integral $K(k) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$. At ∞ ,, i.e. as $\theta \to \pi$, this metric (Atiyah-Hitchin) is exponentially close to the Taub–NUT metric (with a negative mass parameter), while at the origin (the Nut) it is regular. In the Misner variables and in η-time the Darboux system has the following form :

$$\beta_+' = \frac{2}{3}[e^{2\beta_+}(1-\cosh(2\sqrt{3}\beta_-)) - e^{-\beta_+}\cosh(\sqrt{3}\beta_-) + e^{-4\beta_+}]$$

$$\beta_-' = \frac{2}{\sqrt{3}}[-e^{2\beta_+}\sinh(2\sqrt{3}\beta_-) + e^{-\beta_+}\sinh(\sqrt{3}\beta_-)]$$

$$\alpha' = \frac{1}{3}[2e^{2\beta_+}(\cosh(2\sqrt{3}\beta_-)-1) - 4e^{-\beta_+}\cosh(\sqrt{3}\beta_-) + e^{-4\beta_+}] \qquad (5.12).$$

From the self-dual system (5.8) we obtain by integration

$$2\left(\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}\right) = a^2+b^2+c^2-2(\lambda_1 bc+\lambda_2 ac+\lambda_3 ab) \qquad (5.13)$$

and the surface action becomes

$$I_S = \pi[a^2+b^2+c^2-2(\lambda_1 bc+\lambda_2 ac+\lambda_3 ab)] \qquad (5.14)$$

Then at the Nut limit $a,b,c \to 0 \Rightarrow I_S \to 0$. If we got the limit of this expression for the Bolt boundary, we would obtain $I_S \to 2\pi(1-\lambda_3)a_0^2$ and so for $\lambda_3 = 1$, $I_S \to 0$ while for $\lambda_3 = 0$, $I_S \to 2\pi a_0^2$. Both results do not agree with the limit we have found in (4.2) for a Bolt. This confirms that the self-dual action corresponds to a Nut boundary condition and agrees with that (non-round case only if $\lambda_1 = \lambda_2 = \lambda_3 = 1$).

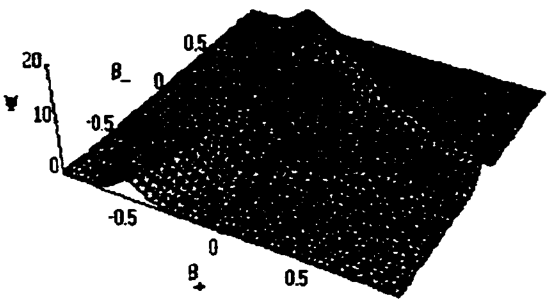
In the BGPP case we have

$$I_S = \pi(a^2+b^2+c^2) = \pi e^{2\alpha}[e^{-4\beta_+} + 2e^{2\beta_+}\cosh(2\sqrt{3}\beta_-)] = e^{2\alpha} f_{BGPP}(\beta_+,\beta_-) \qquad (5.15)$$

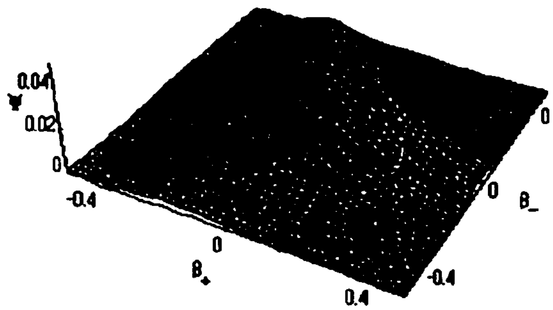
while in the AH case it is

$$I_S = \pi(a^2+b^2+c^2-2ab-2ac-2bc) \qquad (5.16)$$
$$= \pi e^{2\alpha}[e^{-4\beta_+} + 2e^{2\beta_+}(\cosh(2\sqrt{3}\beta_-)-1) - 4e^{-\beta_+}\cosh(\sqrt{3}\beta_-)] = e^{2\alpha} f_{AH}(\beta_+,\beta_-)$$

and the approximate wave function $\Psi = e^{-I_S}$ for both cases and for some constant α is shown by the following diagrams



$\Psi = \Psi_{AH}(\beta_+, \beta_-)$



$\Psi = \Psi_{BGPP}(\beta_+, \beta\ )$

with the only differences being that the AH wave function is much higher, sharper and more extended.

A point with the BGPP action is that it is appropriate for matching the boundary to an asymptotically Euclidean self-dual solution and so gives a "wormhole" wavefunction rather than a HH "no-boundary" wavefunction as one would get from compact 4-metrics with no other boundary. At the origin the BGPP solution is singular.

The three local solutions discussed are valid so long as $\{a, b, c\}$ are finite and non-zero. If any of $\{a, b, c\}$ cease to be finite and non-zero in a finite proper distance interval, the manifold may be incomplete. If G is *SU(2)* and all three of $\{a, b, c\}$ diverge as $\frac{1}{2}\times$(proper distance), we have a Euclidean infinity. The Taub-NUT infinity corresponds to $a \to \xi, b \to \xi, c \to constant$ (or any permutation of $\{a, b, c\}$) as the proper distance ξ tends to infinity. In the Eguchi-Hanson metric, $\{a, b, c\}$ diverge as $\frac{1}{2}\xi$ but G is *SO(3)*, giving a sort of "conical" Euclidean infinity. Also, we have the Nut and Bolt boundary conditions we discussed before. Among the various ways the five regular boundary conditions can be combined to give regular manifolds, most of them are rejected due to topological statements [9].

6. A new set of dynamical variables

In this section we introduce a new set of dynamical variables x, y instead of β_+, β_-. Let

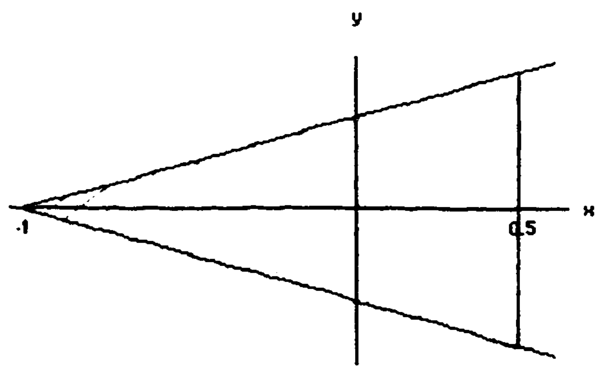
$$x + iy = -\frac{a\overline{w} + bw + c}{a+b+c}, \text{ where } w = e^{2\pi i/3}.$$

Then

$$x = \frac{1}{2}\frac{a+b-2c}{a+b+c} \quad , \quad y = \frac{\sqrt{3}}{2}\frac{a-b}{a+b+c} \tag{6.1}$$

$$\text{or } x = \frac{\cosh(\sqrt{3}\beta_-) - e^{-3\beta_+}}{2\cosh(\sqrt{3}\beta_-) + e^{-3\beta_+}} \quad , \quad y = \frac{\sqrt{3}\sinh(\sqrt{3}\beta_-)}{2\cosh(\sqrt{3}\beta_-) + e^{-3\beta_+}} \tag{6.2}$$

Then $e^{\sqrt{3}\beta_- - 3\beta_+} = \dfrac{2x-1}{\sqrt{3}y - x - 1} > 0$, $e^{2\sqrt{3}\beta_-} = \dfrac{x+1+\sqrt{3}y}{x+1-\sqrt{3}y} > 0$.



These two inequalities hold only in the region above, the interior of the triangle, defined by the straight lines $y = \dfrac{x+1}{\sqrt{3}}, y = -\dfrac{x+1}{\sqrt{3}}$ and $x = \dfrac{1}{2}$. The segment of the line $x = \frac{1}{2}$ is not included for β_+ finite, while for $\beta_+ \to +\infty \Rightarrow x \to \frac{1}{2}$.

The inverse transformation is

$$\beta_+ = \frac{1}{6}\ln\frac{(x+1)^2 - 3y^2}{(1-2x)^2} \quad , \quad \beta_- = \frac{1}{2\sqrt{3}}\ln\frac{x+1+\sqrt{3}y}{x+1-\sqrt{3}y} \tag{6.3}$$

The Nut point $(\beta_+, \beta_-) = (0{,}0)$ is mapped to $(x, y) = (0{,}0)$, while the Bolt $(\beta_+, \beta_-) = (+\infty, 0)$ to the point $(x, y) = (\frac{1}{2}, 0)$. For $c = 0 \Rightarrow x = \frac{1}{2},\ y = \frac{\sqrt{3}}{2}\frac{a-b}{a+b}$ and the three-dimensional Ricci scalar ${}^{(3)}R$ given by (3.1') becomes singular on this edge.

The advantage of this description is that the region in which the problem is focused is finite, while the disadvantage is that the kinetic energy is not diagonal and the resulting equations are more complicated.

The quantity ρ becomes

$$\rho = \frac{3}{(1-2x)^{4/3}}\frac{(1-2x)^2(4y^2+4x+1) - 16y^2(x+1)^2}{((x+1)^2 - 3y^2)^{4/3}} \tag{6.4}$$

For $\Lambda = 0$ the constraint and the equation for α, in η-time, are :

$$\alpha'^2 = \alpha_1 x'^2 + \alpha_2 y'^2 + \alpha_3 x'y' + \frac{\rho}{3} \tag{6.5}$$

$$\alpha'' = -2(\alpha_1 x'^2 + \alpha_2 y'^2 + \alpha_3 x'y') \tag{6.6}$$

where

$$\alpha_1 \equiv \frac{(x+1-4y^2)(x+1-y^2) + y^2(4x^2-3x+2)}{(1-2x)^2[(x+1)^2 - 3y^2]^2} \ , \ \alpha_2 \equiv \frac{y^2 + (x+1)^2}{[(x+1)^2 - 3y^2]^2}$$

$$\alpha_3 \equiv \frac{4y(x^2+y^2-1)}{(1-2x)[(x+1)^2 - 3y^2]^2} \ . \tag{6.7}$$

The second order equations for β_+, β_- are

$$\beta''_\pm = \beta_1^\pm x'' + \beta_2^\pm y'' + \beta_3^\pm x'^2 + \beta_4^\pm y'^2 + \beta_5^\pm x'y' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} - 2\alpha'(\beta_6^\pm x' + \beta_7^\pm y') \tag{6.8}$$

or after solving for x'', y'' :

$$x'' = \frac{1}{\beta_1^+\beta_2^- - \beta_2^+\beta_1^-}[(\beta_2^+\beta_3^- - \beta_3^+\beta_2^-)x'^2 + (\beta_2^+\beta_4^- - \beta_4^+\beta_2^-)y'^2 + (\beta_2^+\beta_5^- - \beta_5^+\beta_2^-)x'y'$$
$$+ 2\alpha'(\beta_2^+\beta_6^- - \beta_6^+\beta_2^-)x' + 2\alpha'(\beta_2^+\beta_7^- - \beta_7^+\beta_2^-)y' - \frac{1}{6}(\frac{\partial\rho}{\partial\beta_+}\beta_2^- - \frac{\partial\rho}{\partial\beta_-}\beta_2^+)] \tag{6.9}$$

$$y'' = \frac{1}{\beta_1^+ \beta_2^- - \beta_2^+ \beta_1^-}[(-\beta_1^+ \beta_3^- + \beta_3^+ \beta_1^-)x'^2 - (\beta_1^+ \beta_4^- - \beta_4^+ \beta_1^-)y'^2 - (\beta_1^+ \beta_5^- - \beta_5^+ \beta_1^-)x'y'$$

$$- 2\alpha'(\beta_1^+ \beta_6^- - \beta_6^+ \beta_1^-)x' - 2\alpha'(\beta_1^+ \beta_7^- - \beta_7^+ \beta_1^-)y' + \frac{1}{6}(\frac{\partial \rho}{\partial \beta_+}\beta_1^- - \frac{\partial \rho}{\partial \beta_-}\beta_1^+)] \qquad (6.10),$$

where

$$\beta_1^+ = \frac{x+1-2y^2}{(1-2x)[(x+1)^2 - 3y^2]}, \ \beta_2^+ = \frac{-y}{(x+1)^2 - 3y^2},$$

$$\beta_3^+ = \frac{(x+1)^2(4x+1) - 12y^2x(x+1) - 9y^2 + 12y^4}{(1-2x)^2[(x+1)^2 - 3y^2]^2}, \ \beta_4^+ = -\frac{(x+1)^2 + 3y^2}{[(x+1)^2 - 3y^2]^2},$$

$$\beta_5^+ = \frac{4y(x+1)}{[(x+1)^2 - 3y^2]^2}, \ \beta_6^+ = \beta_1^+, \ \beta_7^+ = \beta_2^+, \ \beta_1^- = \beta_2^+, \ \beta_2^- = \frac{x+1}{(x+1)^2 - 3y^2},$$

$$\beta_3^- = \tfrac{1}{2}\beta_5^+, \ \beta_4^- = \tfrac{3}{2}\beta_5^+, \ \beta_5^- = 2\beta_4^+, \ \beta_6^- = \beta_2^+, \ \beta_7^- = \beta_2^-,$$

$$\frac{\partial \rho}{\partial \beta_+} = \frac{24}{(1-2x)^{4/3}} \frac{(1-2x)^2(x^2 - 2x - y^2) - 8y^2(x+1)^2}{[(x+1)^2 - 3y^2]^{4/3}},$$

$$\frac{\partial \rho}{\partial \beta_-} = \frac{48y(x+1)}{(1-2x)^{4/3}} \frac{2x^2 - 8x - 1 - 6y^2}{[(x+1)^2 - 3y^2]^{4/3}}. \qquad (6.11)$$

The coefficients $\beta_2^+ \beta_7^- - \beta_7^+ \beta_2^-$ and $\beta_1^+ \beta_6^- - \beta_6^+ \beta_1^-$ of $\alpha' y'$ and $\alpha' x'$ of equations (6.9) and (6.10) respectively are zero, while the coefficients $(\beta_2^+ \beta_6^- - \beta_6^+ \beta_2^-)/(\beta_1^+ \beta_2^- - \beta_2^+ \beta_1^-)$ and $(\beta_1^+ \beta_7^- - \beta_7^+ \beta_1^-)/(\beta_1^+ \beta_2^- - \beta_2^+ \beta_1^-)$ are equal to -1 and +1 respectively , so these equations are a little more simplified.

The Atiyah-Hitchin action in x-y variables takes the following form after a few calculations

$$I_S = 3\pi e^{2\alpha} \frac{4x^2 + 4y^2 - 1}{(1-2x)^{2/3}[(x+1)^2 - 3y^2]^{2/3}} = e^{2\alpha} f(x,y) \qquad (6.12)$$

and both I_S and $\Psi = e^{-I_S}$ have exactly the same form with the relative ones of section 5.

7. Nut - solution in β_+,β_- for $\Lambda = 0$

Close to the Nut point ($\beta_+ \approx \beta_- \approx \beta'_+ \approx \beta'_- \approx 0$), the constraint (3.22) is approximated by $\alpha'^2 = \beta_+'^2 + \beta_-'^2 + \frac{\rho}{3} \approx \frac{\rho}{3} \approx 1$, so $\alpha \approx \eta$. The relations between the times defined before are :

$$d\xi = 2e^{\alpha} d\eta \approx 2e^{\eta} d\eta \;\Rightarrow\; \xi \approx 2e^{\eta} \;\Rightarrow\; \eta \approx \ln(\tfrac{1}{2}\xi) \tag{7.1}$$

Then for $\xi \to 0 \Rightarrow \eta \to -\infty \Rightarrow \alpha \to -\infty$. From equation (3.18) we have

$$\dot{\alpha}^2 = \dot{\beta}_+^2 + \dot{\beta}_-^2 - \frac{\rho}{12} e^{4\alpha} \approx -\frac{\rho}{12} e^{4\alpha} \approx \frac{1}{4} e^{4\alpha} \Rightarrow \dot{\alpha} \approx \frac{1}{2} e^{2\alpha} \Rightarrow \alpha \approx -\frac{1}{2} \ln(-\tau). \tag{7.2}$$

So $\tau < 0$ close to the Nut, and

$$d\xi = e^{3\alpha} d\tau \approx (-\tau)^{-3/2} d\tau \Rightarrow \xi \approx 2(-\tau)^{-1/2} \tag{7.3}$$

Then for $\xi \to 0 \Rightarrow \tau \to -\infty \Rightarrow \alpha \to -\infty$. We approximate ρ near $(\beta_+,\beta_-) = (0,0)$ by $\rho \approx 3 - 24(\beta_+^2 + \beta_-^2)$ and the equations (3.24) for $\beta_\pm$ become $\beta_\pm'' \approx 8\beta_\pm - 2\beta_\pm' \Rightarrow \beta_\pm'' + 2\beta_\pm' - 8\beta_\pm \approx 0$. If $\beta_\pm \sim e^{\lambda\eta}$, then the characteristic equation is $\lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda = -4,\ +2$. The negative value is rejected since it does not satisfy the boundary condition and then $\beta_\pm \approx b_\pm e^{2\eta} \approx \frac{1}{4} b_\pm \xi^2$. (7.4)

Thus

$$a = e^{\alpha} e^{\beta_+} e^{\sqrt{3}\beta_-} \approx \tfrac{1}{2}\xi\,(1 + \tfrac{1}{4} b_+ \xi^2)(1 + \tfrac{\sqrt{3}}{4} b_- \xi^2) \approx \tfrac{1}{2}\xi + \tfrac{1}{8}(b_+ + \sqrt{3} b_-)\xi^3 \approx \tfrac{1}{2}\xi + a_1\xi^3 + a_2\xi^5 + \cdots$$

$$b = e^{\alpha} e^{\beta_+} e^{-\sqrt{3}\beta_-} \approx \tfrac{1}{2}\xi\,(1 + \tfrac{1}{4} b_+ \xi^2)(1 - \tfrac{\sqrt{3}}{4} b_- \xi^2) \approx \tfrac{1}{2}\xi + \tfrac{1}{8}(b_+ - \sqrt{3} b_-)\xi^3 \approx \tfrac{1}{2}\xi + b_1\xi^3 + b_2\xi^5 + \cdots$$

$$c = e^{\alpha} e^{-2\beta_+} \approx \tfrac{1}{2}\xi\,(1 - \tfrac{2}{4} b_+ \xi^2) \approx \tfrac{1}{2}\xi - \tfrac{1}{4} b_+ \xi^3 \approx \tfrac{1}{2}\xi + c_1\xi^3 + c_2\xi^5 + \cdots \tag{7.5}$$

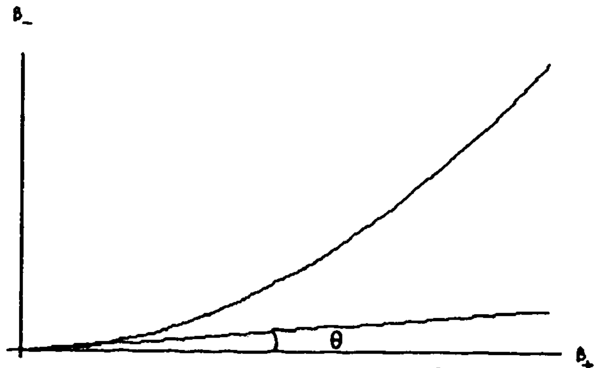
where $a_1 = \frac{1}{8}(b_+ + \sqrt{3} b_-)$, $b_1 = \frac{1}{8}(b_+ - \sqrt{3} b_-)$, $c_1 = -\frac{1}{4} b_+$.

Then $b_- = \frac{4}{\sqrt{3}}(a_1 - b_1)$ and $b_+ = 4(a_1 + b_1) = -4c_1 \;\Rightarrow\; a_1 + b_1 + c_1 = 0$. This is an accurate relation which also arises if one substitutes into the dynamical equations the complete form for *a*, *b*, *c* as expanded in powers of ξ.

The solution should scale when a_1, b_1, c_1 scale, i.e. when $b_\pm$ scales ($b_\pm \rightarrow \lambda b_\pm$) or $(a_1, b_1, c_1) \rightarrow \lambda(a_1, b_1, c_1)$. Then if $\xi \rightarrow \lambda^{-1/2}\xi$, one has $(a,b,c) \rightarrow \lambda^{-1/2}(a,b,c)$.
If θ is the angle shown in the figure then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{b_+} \text{ and from } \beta_+ = \tfrac{1}{4} b_\pm \xi^2 + \cdots \Rightarrow \tilde{\beta}_\pm = \tfrac{1}{2} b_\pm \xi + \cdots$$

$$\Rightarrow \tilde{\tilde{\beta}}_+ = \tfrac{1}{2} b_+ + \cdots \Rightarrow \tilde{\tilde{\beta}}_+(0) = \tfrac{1}{2} b_+ \text{, we get } \tan\theta = \frac{\tilde{\tilde{\beta}}_-(0)}{\tilde{\tilde{\beta}}_+(0)}.$$



But $\beta_+(\xi) = \beta_+(0) + \tilde{\beta}_+(0)\xi + \tfrac{1}{2}\tilde{\tilde{\beta}}_\pm(0)\xi^2 + \cdots$ and applying De l' Hospital's rule twice, we obtain

$$\frac{\tilde{\tilde{\beta}}_-(0)}{\tilde{\tilde{\beta}}_+(0)} = \lim_{\xi \to 0} \frac{\beta_-(\xi)}{\beta_+(\xi)} \Rightarrow \tan\theta = \lim_{\xi \to 0} \frac{\beta_-(\xi)}{\beta_+(\xi)} = \left(\frac{\beta_-}{\beta_+}\right)_0 .$$ One has

$$\beta_+ \approx \tfrac{1}{6}\ln\left(\frac{a_1 b_1}{c_1^2}\right) + \tfrac{1}{6}\ln\left(\frac{a_2 b_2}{c_2^2}\right) + \cdots \text{ and } \beta_- \approx \frac{1}{2\sqrt{3}}\ln\left(\frac{a_1}{b_1}\right) + \frac{1}{2\sqrt{3}}\ln\left(\frac{a_2}{b_2}\right) + \cdots .$$

Thus, up to first order one has

$$\tan\theta = \sqrt{3}\ln\left(\frac{a_1}{b_1}\right)\left[\ln\left(\frac{a_1 b_1}{c_1^2}\right)\right]^{-1} \Rightarrow \tan\theta = \sqrt{3}\ln\left(\frac{a_1}{b_1}\right) \Big/ \left[\ln\left(\frac{a_1 b_1}{(a_1 + b_1)^2}\right)\right], \; a_1 . b_1 < 0.$$

We can go further and find the second order approximation for α:

$$\alpha'^2 \approx 4(b_+^2 + b_-^2)e^{4\eta} + 1 - 8(b_+^2 + b_-^2)e^{4\eta} \Rightarrow \alpha' \approx 1 - 2(b_+^2 + b_-^2)e^{4\eta} \approx 1 - 2(\beta_+^2 + \beta_-^2)$$

$$\Rightarrow \qquad \alpha \approx \eta - \tfrac{1}{2}(b_+^2 + b_-^2)e^{4\eta} \approx \eta - \tfrac{1}{2}(\beta_+^2 + \beta_-^2) \qquad (7.6)$$

This suggests that to get a time t such that the motion in the (β_+, β_-) plane starts off at unit velocity, giving $\beta_+^{2'} + \beta_-^2 \approx t^2$, we have to choose

$t = \sqrt{\frac{1}{2}(1 - \alpha'^2)}$. The quantity $\alpha' = \frac{d\alpha}{d\eta}$ is a time-parameter which decreases monotonically with η since $\alpha'' < 0$. Then one should get $\frac{d^2\beta_\pm}{dt} = f_\pm(\beta_\pm, \frac{d\beta_\pm}{dt}, t)$, which one can integrate from $\beta_\pm(t = 0) = 0$.

After a few calculations we find that :

$$\frac{d}{d\eta} = \frac{2t}{\beta_{+t}^2 + \beta_{-t}^2} \frac{d}{dt} \;\&$$

$$\frac{d^2}{d\eta^2} = \frac{4t^2}{(\beta_{+t}^2 + \beta_{-t}^2)^2} [\frac{d^2}{dt^2} - \frac{2}{\beta_{+t}^2 + \beta_{-t}^2}(\beta_{+t}\beta_{+tt} + \beta_{-t}\beta_{-tt})\frac{d}{dt} + \frac{1}{t}\frac{d}{dt}] ,$$

where the index t means differentiation with respect to t. Then :

$(\beta_+'^2 + \beta_-'^2)(\beta_{+t}^2 + \beta_{-t}^2) = 4t^2$ and the constraint equation takes the form :

$\beta_{+t}^2 + \beta_{-t}^2 - \alpha_t^2 + \frac{1}{12t^2}\rho(\beta_{+t}^2 + \beta_{-t}^2)^2 = 0$, while the equation for the expansion parameter is

$$\alpha_{tt} = -2(\beta_{+t}^2 + \beta_{-t}^2) - \frac{1}{t}\alpha_t + \frac{2}{\beta_{+t}^2 + \beta_{-t}^2}(\beta_{+t}\beta_{+tt} + \beta_{-t}\beta_{-tt})\alpha_t \; .$$

The action takes the form

$$I_g = 3\pi\int \frac{2t}{\beta_{+t}^2 + \beta_{-t}^2}[\beta_{+t}^2 + \beta_{-t}^2 - \alpha_t^2 - \frac{\rho}{3}\frac{(\beta_{+t}^2 + \beta_{-t}^2)^2}{4t^2}]e^{2\alpha}dt \; .$$

The equations for β_+, β_- are mixed and after solving for β_{+tt}, β_{-tt} we find

$$\beta_{\pm tt} = \frac{1}{t}\beta_{\pm t}[1 + (1 - 2t^2)(\beta_{+t}^2 + \beta_{-t}^2)] + \frac{(\beta_{+t}^2 + \beta_{-t}^2)^2}{24t^2}\frac{\partial\rho}{\partial\beta_\pm} + \frac{\beta_{+t}^2 + \beta_{-t}^2}{12t^2}\beta_{\mp t}(\beta_{+t}\frac{\partial\rho}{\partial\beta_-} - \beta_{-t}\frac{\partial\rho}{\partial\beta_+})$$

These equations are quite complicated, and we do not go any further with these.

We come back to the system (3.23), (3.24) ($\Lambda = 0$). To make it first-order we set $\alpha' = H$, $\beta_+' = p_+$, $\beta_-' = p_-$. The constraint takes the form

$$H^2 = p_+^2 + p_-^2 + \frac{\rho}{3},$$

and the dynamical equations become

$$H' = -2(p_+^2 + p_-^2) \tag{7.7}$$

$$p_+' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_+} - 2Hp_+ \qquad (7.8)$$

So, we have a system for $(H, \beta_+, \beta_-, p_+, p_-)$ of five first-order equations. In order to find the initial conditions, we take a highly negative initial value for η, say η_0. From the approximate solution (7.4) we have

$$\beta_+' \approx 2b_+e^{2\eta} \Rightarrow \beta_+' \approx 2\beta_+ \Rightarrow \frac{\beta_+'}{\beta_+} \to 2, \quad \xi \to 0 .$$

We could make a better approximation and add cubic terms, but this is not necessary. If we set $\varepsilon = e^{2\eta_0} \Leftrightarrow \eta_0 = \frac{1}{2}\ln\varepsilon$, $b_+ = \cos\theta$, $b_- = \sin\theta$, then

$$(\beta_+)_0 = \varepsilon\cos\theta \;,\; (\beta_-)_0 = \varepsilon\sin\theta \;,\; (p_+)_0 = 2\varepsilon\cos\theta \;,\; (p_-)_0 = 2\varepsilon\sin\theta. \qquad (7.9)$$

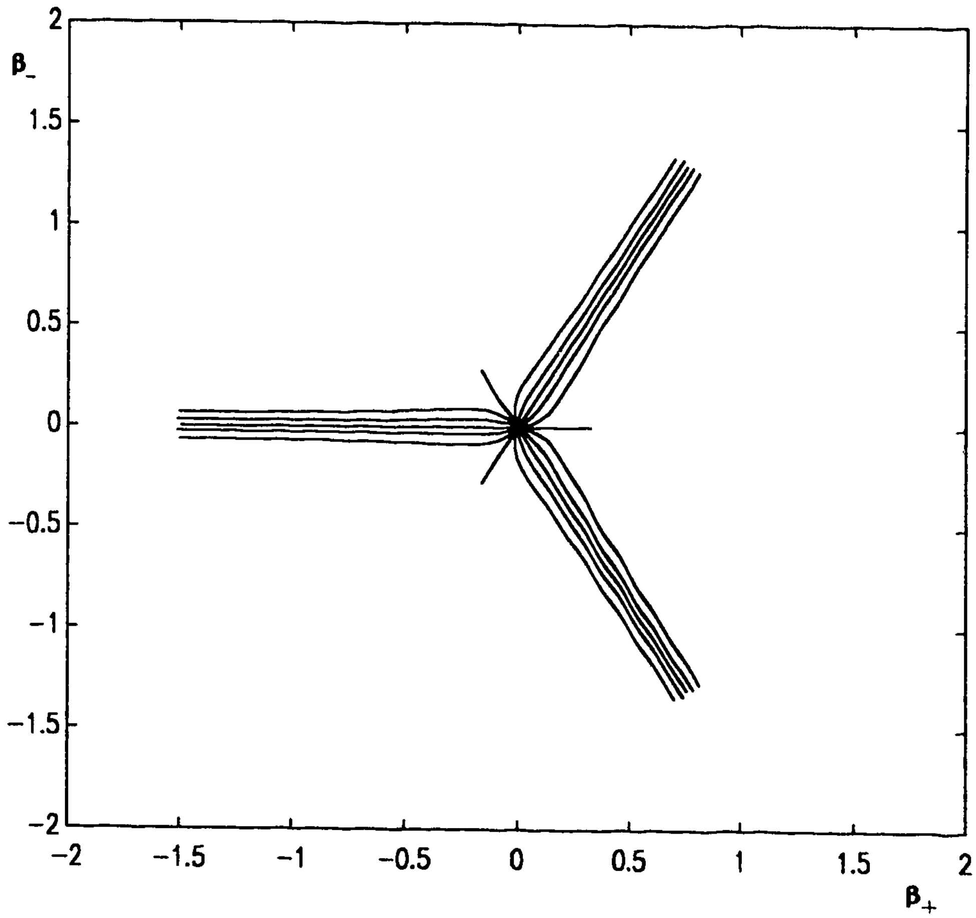
There is only one parameter in this problem - the angle θ determining the curve in the β_+, β_- plane. Let's try to explain this. There are three 2nd-order equations plus one 1st-order constraint equation reducing the number of parameters to 5, namely : $H, \beta_+, \beta_-, p_+, p_-$. But one can take out one parameter each for the origin of the time (or η), and for the scale (since there are no dimensional constants in the equations). These leave 3 parameters. Regularity at the origin presumably takes out 2 more, leaving 1. This is the angle θ.

If we use the constraint we can eliminate the primed quantities from the action and this takes the form :

$$I = -\frac{\pi}{4}\int \rho e^{2\alpha} d\eta \, , \qquad (7.10)$$

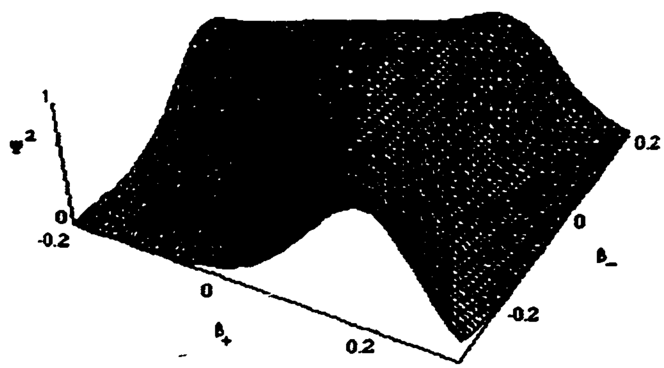
which is a simpler one for computational work.

Using the Runge-Kutta method the computer gives the following picture for the solutions for various values of the angle-parameter θ:

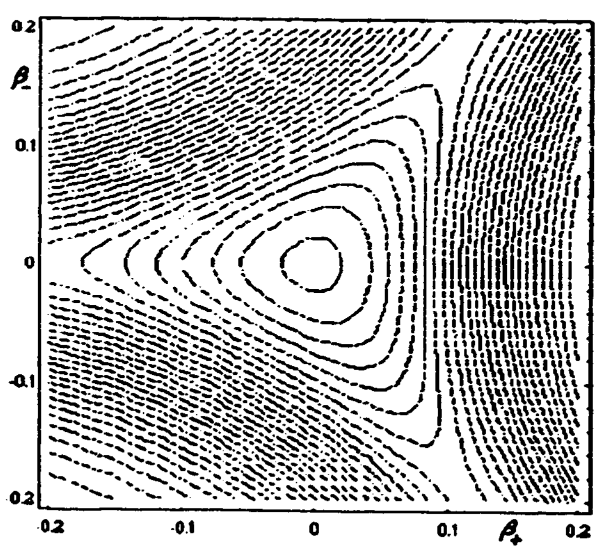


The figure has the expected symmetry under rotation of 120^0 as well as the reflection symmetry with respect to the β_+ axis, the 60^0 and the 300^0 straight lines (as the potential has). We observe that whatever the initial ratio of expansion of the two anisotropic scale factors β_+, β_- is, after some period of time the solution approaches a specific way of expansion : that of the 60^0 , the 180^0, and the 300^0 solution. The curves do not approach asymptotically the 60^0 solution, but asymptotically become parallel to it. Also, after some search we find that there are curves somewhere between $\theta = 108^0$ and $\theta = 112^0$ which do cross each other. This happens only very close to the origin and this must be a mistake due to numerical approximations made by the computer. An explanation of the fact that solutions do not cross is given in section 9. As we approach the solution of 120^0 we have to make the value of θ extremely close to 120^0, in order to find a nearby curve. This means that the "density" of solutions is much bigger in the region of 60^0 and decreases considerably as we approach the 120^0 solution. Certainly, all the above happen also for the symmetric regions after rotation.

We can plot the first order approximation of the wavefunction $\Psi = e^{-I}$ or the probability distribution $|\Psi|^2$, with α fixed at the boundary, as functions of β_+, β_-. The resulting distribution has the six-fold symmetry, due to the symmetry of the potential.



We can also get the contour plot of Ψ^2 as follows:



The wavefunction Ψ decreases for increasing anisotropy $\left(\beta_+^2+\beta_-^2\right)^{1/2}$. So the maximum of the probability corresponds to an isotropic universe. One can use the symmetry of the wavefunction to see that $\Psi\left(\beta_+<0,\beta_-=0\right)=\Psi\left(\beta_+=\frac{\beta_-}{\sqrt{3}}:\beta_+>0\right)$. Also the probability of the points (β_+,β_-) near the three directions of symmetry is remarkably bigger relative to the other directions in (β_+,β_-) plane.

8. The Hamilton - Jacobi - equation BGPP solution

Another approach is to use the Hamilton-Jacobi equation. In canonical formulation the action for the diagonal Bianchi IX model has the form $I = \int (p_+ \frac{d\beta_+}{dt} + p_- \frac{d\beta_-}{dt} + p_\alpha \frac{d\alpha}{dt} - NH)dt$, where $d\xi = Ndt$ and the Hamiltonian $H = 0 = -p_\alpha^{\ 2} + p_+^{\ 2} + p_-^{\ 2} + 24\pi^{2\,(3)}g^{(3)}R$. With ${}^{(3)}g = e^{6\alpha}$ and ${}^{(3)}R = \frac{\rho}{2}e^{-2\alpha}$ we obtain $24\pi^{2\,(3)}g^{(3)}R = 12\pi^2 e^{4\alpha}\rho$. The Hamilton-Jacobi equation is

$$H(q^i, p_i = \frac{\partial I}{\partial q^i}) = 0 \text{ is } -(\frac{\partial I}{\partial \alpha})^2 + (\frac{\partial I}{\partial \beta_+})^2 + (\frac{\partial I}{\partial \beta_-})^2 + 12\pi^2 e^{4\alpha}\rho = 0 \tag{8.1}$$

Rescaling the action $I \to I' = \frac{1}{36\pi^2} I$ and calling I' again I, we obtain

$$-(\frac{\partial I}{\partial \alpha})^2 + (\frac{\partial I}{\partial \beta_+})^2 + (\frac{\partial I}{\partial \beta_-})^2 + \frac{1}{3}e^{4\alpha}\rho = 0 \tag{8.2}$$

Let $I = e^{2\alpha} f(\beta_+, \beta_-)$. Then

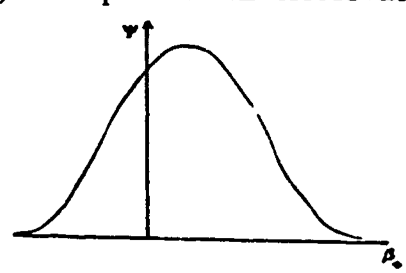
$$(\frac{\partial f}{\partial \beta_+})^2 + (\frac{\partial f}{\partial \beta_-})^2 - 4f^2 + \frac{1}{3}\rho = 0 \tag{8.3}$$

One solution of this equation is

$$f(\beta_+, \beta_-) = \tfrac{1}{6}[e^{-4\beta_+} + 2e^{2\beta_+} \cosh(2\sqrt{3}\beta_-)], \tag{8.4}$$

as we can verify by direct computation.

Then the action is exactly the BGPP action (5.15) except for an irrelevant multiplying factor in front.



The wave function $\Psi \approx e^{-I}$ falls sharply away from a maximum at $\beta_+ = \beta_- = 0$ (the $k = +1$ FRW Universe). The special case, where $\beta_- = 0$, $\Psi(\beta_+, \alpha = const.)$ is shown in the figure beside. The figure may also be taken as a roughly representative cross-section of the contour plot. One feature of this is that near the singularity ($\alpha \to -\infty$) it is spread out over the β_+, β_- plane, since the coefficient $e^{2\alpha}$ of f in I is tiny and so e^{-I} does not go to zero very rapidly. As the Universe moves away from the singularity, since at $\beta_+ = \beta_- = 0$, $f = \frac{1}{2}$, then $e^{-I} = e^{-\frac{1}{2}e^{2\alpha}}$, which shrinks rapidly with α. But for fixed α, the relative shrinkage of e^{-I} with $\beta_\pm$ as they move away from 0 is faster for larger α, so e^{-I} is more sharply peaked near the $k=+1$ FRW metric. If one regards $\Psi^*\Psi$ as representing the probability of finding the Universe in some state, it is possible to interpret this behavior as showing that as the Universe evolves it becomes more and more probable to find it in the FRW state. While it is tempting to regard the square of these functions in the Hartle-Hawking interpretation as the probability of finding the Universe in a corresponding state, there are a number of different probability interpretations which would not agree with such an assignment. Actually, the BGPP action does not lead to classical solutions, regular at the center, so it does not give the HH wavefunction. In the canonical quantization of the Bianchi IX model, if we select [21] a factor ordering and write $\Psi = We^{-S}$ and then replace this form in the Wheeler-DeWitt equation for Ψ, then a very complicated equation arises, containing both W, S. If we set

$$\left(\frac{\partial S}{\partial \beta_+}\right)^2 + \left(\frac{\partial S}{\partial \beta_-}\right)^2 - 4S^2 + \frac{1}{3}\rho = 0 \ ,$$

which is exactly the Hamilton-Jacobi equation of our case, with $ds^2 = d\xi^2 + g_{ij}\sigma^i\sigma^j$ then the Wheeler-DeWitt equation is somehow simplified. I think this agreement between the canonical quantization method and the path-integral approximation method is something positive towards the direction of convergence of these two.

9. The AH solution

For the system (5.12) we have obtained in the case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in η-time, we can find an approximate solution with a regular centre.

For small β_+, β_- it is up to first order in β_+, β_- :

$$3\beta_+' \approx -(1+2\beta_+ + 2\sqrt{3}\beta_-) - (1+2\beta_+ - 2\sqrt{3}\beta_-) + 2(1-4\beta_+) - (1-\beta_+ - \sqrt{3}\beta_-) - (1-\beta_+ + \sqrt{3}\beta_-) + 2(1+2\beta_+)$$
$$= -6\beta_+ \Rightarrow \beta_+' \approx -2\beta_+ \Rightarrow \beta_+ \approx b_+ e^{-2\eta} \tag{9.1}$$

$$\sqrt{3}\beta_-' \approx -(1+2\beta_+ + 2\sqrt{3}\beta_-) + (1+2\beta_+ - 2\sqrt{3}\beta_-) - (1-\beta_+ - \sqrt{3}\beta_-) + (1-\beta_+ + \sqrt{3}\beta_-)$$
$$= -2\sqrt{3}\beta_- \Rightarrow \beta_-' \approx -2\beta_- \Rightarrow \beta_- \approx b_- e^{-2\eta} \tag{9.2}$$

$$3\alpha' \approx (1+2\beta_+ + 2\sqrt{3}\beta_-) + (1+2\beta_+ - 2\sqrt{3}\beta_-) + (1-4\beta_+) - 2(1-\beta_+ - \sqrt{3}\beta_-) - 2(1-\beta_+ + \sqrt{3}\beta_-) - 2(1+2\beta_+)$$
$$= -3 \Rightarrow \alpha' \approx -1 \Rightarrow \alpha \approx -\eta \tag{9.3}$$

For $\xi \to 0$, it is $\beta_+ \to 0 \Rightarrow e^{-2\eta} \to 0 \Rightarrow \eta \to +\infty$. After coordinate change $\eta \to -\eta$, AH should be same as the full Nut case. Also there $\alpha \approx \eta$. But the behavior of α is the same, i.e. $\alpha \to -\infty$.

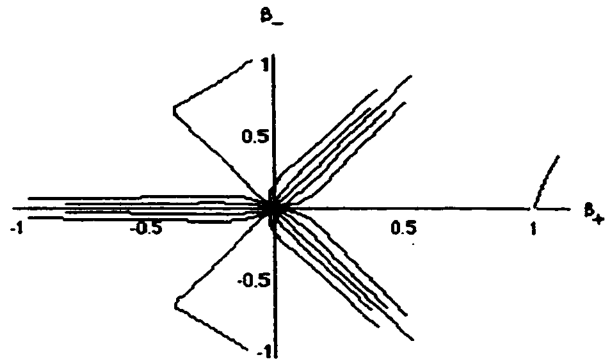
For the proper distance ξ, it is

$$d\xi = 2e^{\alpha} d\eta \approx 2e^{-\eta} d\eta \Rightarrow \xi \approx -2e^{-\eta} + c \tag{9.4}$$

But, it is required from the Nut condition that when $\eta \to +\infty \Rightarrow \xi \to 0$, thus $c = 0$ and $\xi \approx -2e^{-\eta}$. The only difference from before is that now ξ approaches the center from negative values. We notice something more : The self-dual equations satisfy the constraint equation, which yield the expression $\alpha'^2 \approx 1$ near the origin. In the full Nut case, we kept the positive value for α', now the self-dual equations force us to accept the minus sign.

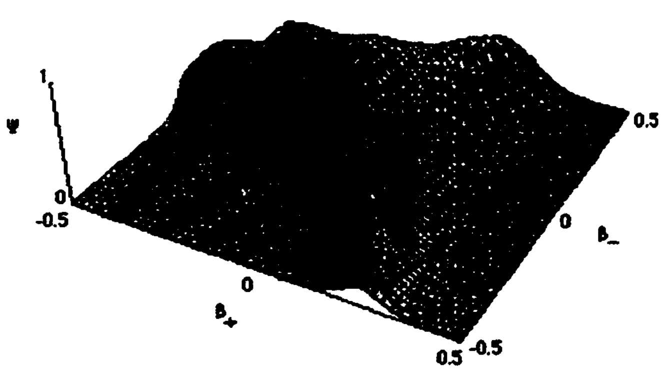
For a very negative value of η, say η_0, let $\varepsilon = e^{-2\eta_0} \Leftrightarrow \eta_0 = -\frac{1}{2}\ln\varepsilon$ and then $(\beta_+)_0 = \varepsilon\cos\theta$, $(\beta_-)_0 = \varepsilon\sin\theta$ are the initial conditions for solving the system of the two 1st-order differential equations of β_+, β_-. The unique parameter of the problem is again the angle θ.

The figure below shows the solutions in the β_+, β plane.

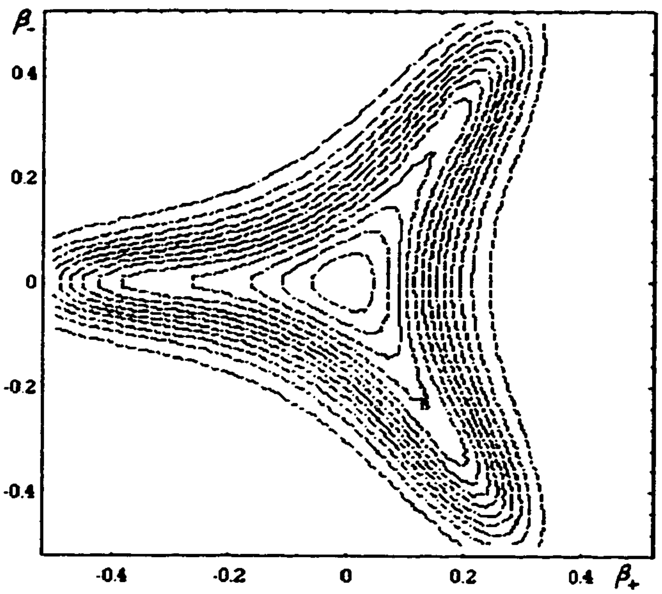


This figure is in complete agreement with the full Nut case, except for the three solutions for $\theta = 0^\circ$, $\theta = 120^0$ and , $\theta = 240^0$ whose decline from the straight lines is due to the fact that the sines cannot be represented exactly by a finite number of digits in the computer and this leads to round-off errors making the numerical results stray from the straight lines. This last figure has been derived using the "Mathematica" package, while for the similar figure of section 7, the "Matlab" package has been used. Actually, this last package is basically made for numerical manipulations, while the former one is also available for symbolic manipulations. We note this fact and keep the figure with the limited accuracy (at least for the three curves), to indicate that the choice of a package may sometimes influence the final results slightly.

For the wavefunction we have got, using the formula (7.10) for calculating the action I, with α fixed at the boundary, the following picture :



while the contour plot is shown below



The same features arise as in the full-Nut case of section 7. However, the peak of the self-dual wavefunction is much more distinct than that of the full-Nut Ψ. But as the distance from (0,0) increases, the full-Nut Ψ obtains values smaller than the corresponding ones of the self-dual Ψ. This difference arises from using different rules for the scale at the boundary. More accurate results can be obtained if we plot $(I/e^{2\alpha})(\beta_+,\beta_-)$ instead of $I(\beta_+,\beta_-)$.

One more remarkable thing : The polynomial single-valued form of the AH action for the configuration variables implies that there must be precisely one trajectory solution through each point in the (β_+,β_-) space, so there should be solutions everywhere, and, furthermore, the solutions should not cross (only one solution through each point).

Remark : If we define a time parameter $t = -\eta$ increasing instead of decreasing, then the new equations' figure in the (β_+,β_-) plane is the same.

10. Bolt - solution in β_+, β_- for $\Lambda = 0$

In the case of a Bolt boundary, we have found in the table of section 4, that $\dot{\alpha} \approx \frac{a_0^2}{6}$, $\xi \to 0$. This is also verified from the constraint equation (3.18), if we set $\dot{\beta}_- \sim 0$, $\dot{\beta}_+ \sim -\frac{a_0^2}{6}$ and also use the asymptotic behavior of ρ for $\beta_- = 0$, $\beta_+ \to +\infty$ $(\rho \sim 4e^{-2\beta_+})$. Thus $\rho e^{4\alpha} \to 0$, $\xi \to 0$.

In the η-time it is $\alpha' \approx \frac{1}{3}(\frac{2a_0}{\xi})^{2/3} \approx \frac{a_0^2}{3} e^{-2\alpha}$, while $\beta_+' \approx -\frac{1}{3}(\frac{2a_0}{\xi})^{2/3}$.

Thus $\alpha' \approx -\beta_+'$, and the equation (3.24) for β_+ becomes

$$\beta_+'' \approx -\frac{1}{6}\frac{\partial \rho}{\partial \beta_+} + 2\beta_+'^2 \text{ , where } \frac{\partial \rho}{\partial \beta_+} \approx -8e^{-2\beta_+} \to 0.$$

So, $\beta_+'' \approx 2\beta_+'^2$ is the approximate equation for β_+ close to the Bolt. The equation for α, (3.23), becomes $\alpha'' \approx -2\alpha'^2$. If $y \equiv \alpha'$ then

$$y' \approx -2y^2 \;\Rightarrow\; y(\eta) = \alpha'(\eta) \approx \frac{1}{2\eta} \;\Rightarrow\; \alpha(\eta) \approx \frac{1}{2}\ln|\eta| .$$

For $\alpha \to -\infty \Rightarrow |\eta| \to 0$. Then $\beta_+ \approx -\frac{1}{2}\ln|\eta| \to +\infty$.

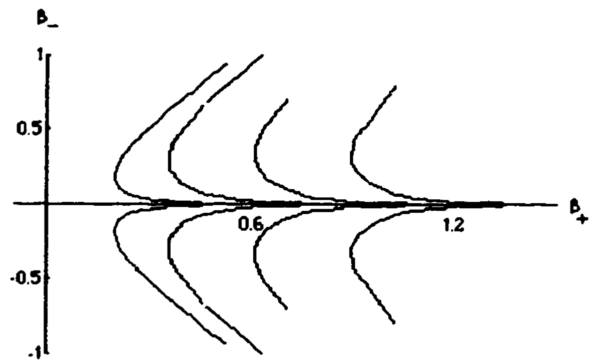
The relation of ξ and η is found by

$d\xi = 2e^{\alpha}d\eta \approx 2\sqrt{|\eta|}d\eta \;\Rightarrow\; \xi \approx \pm\frac{4}{3}|-\eta|^{3/2}$, where the positive sign corresponds to $\eta > 0$, while the negative to $\eta < 0$. If we want $\xi \to 0^+$ at the Bolt, then $\xi \approx \frac{4}{3}\eta^{3/2}$, $\eta \to 0^+$ and then $\beta_+ \approx -\alpha \approx -\frac{1}{2}\ln\eta$ and $\beta_+' \approx -\frac{1}{2\eta}$.

The unique arbitrary parameter parametrizing the paths is

$$\beta' = \dot{\beta} \frac{d\tau}{d\eta} = 2\dot{\beta}\, e^{-2\alpha} \text{ , where } \dot{\beta}_- \to 0, \;\; e^{-2\alpha} \to +\infty \text{ .}$$

For some $\eta_0 \to 0^+$ we put the initial data for $(\alpha', \beta_+, \beta_-, \beta_+', \beta_-')$ according to the relations above and obtain the following figure



From the table of section 4 one has $\beta_-' \sim 0$, so the values of β_-' for this plot are very small.

11. Nut - solution in x, y for $\Lambda = 0$

The system of differential equations governing this problem consists of the equations (6.5), (6.6), (6.9), (6.10). We have noted that close to the Nut point it is $(x, y) \to (0,0)$ and we approximate the quantities x, y by

$$x \approx \frac{1}{2}\frac{(1+\sqrt{3}\beta_-)+(1-\sqrt{3}\beta_-)-2(1-3\beta_+)}{(1+\sqrt{3}\beta_-)+(1-\sqrt{3}\beta_-)+(1-3\beta_+)} = \frac{\beta_+}{1-\beta_+} \approx \beta_+ \tag{11.1}$$

$$y \approx \frac{\sqrt{3}}{2}\frac{(1+\sqrt{3}\beta_-)-(1-\sqrt{3}\beta_-)}{(1+\sqrt{3}\beta_-)+(1-\sqrt{3}\beta_-)+(1-3\beta_+)} = \frac{\beta_-}{1-\beta_+} \approx \beta_- \tag{11.2}$$

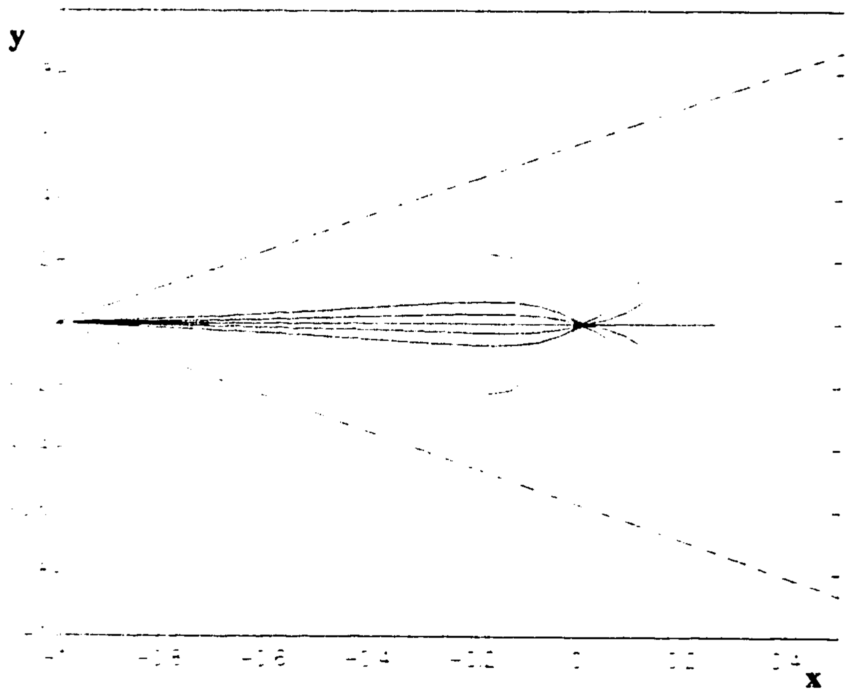
Then

$$x' \approx \frac{\beta_+'(1-\beta_+)-\beta_+(-\beta_+')}{(1-\beta_+)^2} = \frac{\beta_+'}{(1-\beta_+)^2} \approx \beta_+' \tag{11.3}$$

$$y' \approx \frac{\beta_-'(1-\beta_+)-\beta_-(-\beta_+')}{(1-\beta_+)^2} = \frac{\beta_-' - \beta_-'\beta_+ + \beta_-\beta_+'}{(1-\beta_+)^2} \approx \beta_-' \tag{11.4}$$

All the relations about times are the same with those at the Nut case in β_+, β_- variables. If $\varepsilon = e^{2\eta_0}$ then $(\beta_+)_0 = \varepsilon\cos\theta$, $(\beta_-)_0 = \varepsilon\sin\theta$ and $x_0 = \varepsilon\cos\theta$, $y_0 = \varepsilon\sin\theta$, $x_0' = 2\varepsilon\cos\theta$, $y_0' = 2\varepsilon\sin\theta$ (where the second-order terms with respect to ε in the expression for y' have been neglected). So, the initial data for the system (α', x, y, x', y') are the same with those for $(\alpha', \beta_+, \beta_-, \beta_+', \beta_-')$.

For various values of the angle-parameter θ, we get the following figure for the solutions in the x-y plane.



Certainly, everything is restricted in the allowed domain of definition of x,y. The expected symmetry is also present here. The point (-1,0) is a pole to which all solutions between 120° and 240° converge. Due to computational difficulties (the algorithm did not manage to pass over all values of time), the remaining solutions are partially plotted. Observing the plots, it seems that all of them are directed (if extended) towards the corner points $(0.5,\sqrt{3}/2)$ or $(0.5,-\sqrt{3}/2)$. Even for very small values of the angle θ, or for θ approaching 120° from below, this feature seems to be preserved. In section 6, we have stated that for $\beta_+ \to +\infty \Rightarrow x \to 1/2$. At this limit, one has from (6.2) that $y \to \frac{\sqrt{3}}{2}\frac{1-e^{-2\sqrt{3}\beta_-}}{1+e^{-2\sqrt{3}\beta_-}}$. So if β_- goes to a finite number, y will also go to a finite number, smaller than $\sqrt{3}/2$, and for $\beta_- \to +\infty \Rightarrow y \to \sqrt{3}/2$. From this statement, we can conclude that all the β_+,β_- Nut-solutions (section 7) with $0<\theta<120^0$, will be mapped to $x\,,y$ solutions which approach the corner point $(0.5,\sqrt{3}/2)$, since all these β_+,β_- solutions have $\beta_\pm \to +\infty$ for $\xi \to +\infty$. The same argument also holds for the other two symmetry directions.

12. Bolt - solution in x, y for $\Lambda = 0$

Close to the Bolt $(\beta_+, \beta_-) \to (+\infty, 0)$, one has $(x, y) \to (\frac{1}{2}, 0)$. We have already seen that as a parameter for the paths, one can take the variable β' . We have seen that $\beta_+ \approx -\frac{1}{2} \ln \eta$, as $\eta \to 0$

$$\beta'_+ \approx -\frac{1}{2\eta}, \quad \eta \to 0 \text{ and } \beta_- \sim 0 , \ \eta \to 0 \tag{12.1}$$

Then $e^{\pm\sqrt{3}\beta_-}$ can be approximated by $1 \pm \sqrt{3}\beta_-$ and then

$$x \approx \frac{1 - \eta^{3/2}}{2 + \eta^{3/2}} \approx \frac{1}{2} , \ \eta \to 0 \tag{12.2}$$

$$\text{and } y \approx \frac{3\beta_-}{2 + \eta^{3/2}} \approx \frac{3\beta_-}{2} \sim 0 , \ \eta \to 0 \tag{12.3}$$

Differentiating these formulas, we obtain the velocities

$$x' \approx -\frac{9}{2} \frac{\eta^{1/2}}{(2 + \eta^{3/2})^2} \approx -\frac{9}{8}\eta^{1/2} \tag{12.4}$$

$$y' \approx \frac{3}{(2 + \eta^{3/2})^2} [\beta'_-(2 + \eta^{3/2}) - \frac{3}{2}\beta_- \eta^{1/2}] \approx \frac{3\beta'}{2} \tag{12.5}$$

$$\text{Also} \quad \alpha' \approx -\beta'_+ \approx \frac{1}{2\eta} . \tag{12.6}$$

From these relations we can choose initial data if we select some very small value $\eta_0 > 0$ for the time and parametrize the paths by very small positive values of β'_- . Unfortunately, when we put these initial data in the computer program, the algorithm does not start ; it stops from the beginning. This could probably mean that this problem, as it is stated, contains some sort of anomaly and needs to be reformulated. The root of this anomaly may be due to the following fact: close to the Bolt $(\frac{1}{2},0)$ the slope of a solution is

$$\frac{dy}{dx}=\frac{y'}{x'}=-\frac{4}{3}\frac{\beta_-'}{\sqrt{\eta}}.$$

So, the limiting value of the slope is $-\infty$. The curvature of the trajectory in the *(x,y)* plane is :

$$[1+(\frac{d^2y}{dx^2})^2]^{\frac{3}{2}}\frac{d^2y}{d^2x}=\frac{x'^3}{(x'^2+y'^2)^{\frac{3}{2}}}(\frac{y''}{x'^2}+\frac{y'}{x''})\approx\frac{9}{8\beta_-'^2}\left(\eta^2-\frac{4}{9}\eta\frac{\beta_-''}{\beta_-'}\right)\to 0\ ,\ \eta\to 0\ .$$

One might ask whether there is some other choice of variables instead of (β_+,β_-) or (x, y), let (u,v) such that (1) a Bolt is at a finite point (u, v), (2) a curve into that point is uniquely determined by the slope there, and the curve has finite curvature there (as happens in either (β_+,β_-) or (x, y) coordinates for a Nut).

13. Nut - solution with cosmological constant in β_+,β_-

We define a new parameter $\alpha \equiv \Lambda e^{2\alpha}$ and then

$$\alpha' = \frac{1}{2}\frac{\alpha'}{\alpha} \quad , \quad \alpha'' = \frac{1}{2}\left[\frac{\alpha''}{\alpha} - \left(\frac{\alpha'}{\alpha}\right)^2\right]$$

The system of equations (3.22), (3.23), (3.24) takes the form :

$$\tfrac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 = \beta_+'^2 + \beta_-'^2 + \frac{\rho}{3} - \frac{4}{3}\alpha \tag{13.1}$$

$$\frac{\alpha''}{\alpha} - \left(\frac{\alpha'}{\alpha}\right)^2 = -4(\beta_+'^2 + \beta_-'^2) - \tfrac{8}{3}\alpha \tag{13.2}$$

$$\beta_\pm'' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} - \frac{\alpha'}{\alpha}\beta_\pm' \tag{13.3}$$

If we set $\alpha' = H$, $\beta_+' = p_+$, $\beta_-' = p_-$, then

$$\tfrac{1}{4}\left(\frac{H}{\alpha}\right)^2 = p_+^2 + p_-^2 + \frac{\rho}{3} - \frac{4}{3}\alpha \tag{13.4}$$

$$H' = \frac{H^2}{\alpha} - 4\alpha(p_+^2 + p_-^2) - \tfrac{8}{3}\alpha^2 \tag{13.5}$$

$$p_\pm' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} - \frac{H}{\alpha}p_\pm \tag{13.6}$$

Now, we have 3 second-order equations and 1 first-order constraint equation, giving 5 independent parameters. If we take out one parameter for the origin of time and two more for the regularity conditions then we are left with two independent parameters. The solution does not scale any more.

From the constraint (3.22) it is $\alpha'^2 \approx \frac{\rho}{3} \approx 1 \;\Rightarrow\; \alpha' \approx 1 \;\Rightarrow\; \alpha \approx \eta + k$. This is the crucial difference from the case in which Λ=0, namely, that now a second parameter k appears (besides θ). Then from $\xi \approx 2\int e^{\alpha} d\eta \Rightarrow \xi \approx 2e^{k}e^{\eta} \approx 2e^{\alpha}$

(13.7) , close to $(\beta_+, \beta_-) = (0,0)$. From the equations for $\beta_\pm$ we have the usual approximation

$$\beta_+'' + 2\beta_+' - 8\beta_+ \approx 0 \;\Rightarrow\; \beta_+ \approx b_+ e^{2\eta} \approx \tfrac{1}{4} b_+ \xi^2 e^{-2k} \qquad (13.8)$$

Then $a \approx \frac{1}{2}\xi(1 + \frac{1}{4}b_+ \xi^2 e^{-2k})(1 + \frac{\sqrt{3}}{4} b_- \xi^2 e^{-2k}) \approx \frac{1}{2}\xi + \frac{1}{8}(b_+ + \sqrt{3}b_-)\xi^3 e^{-2k}$.

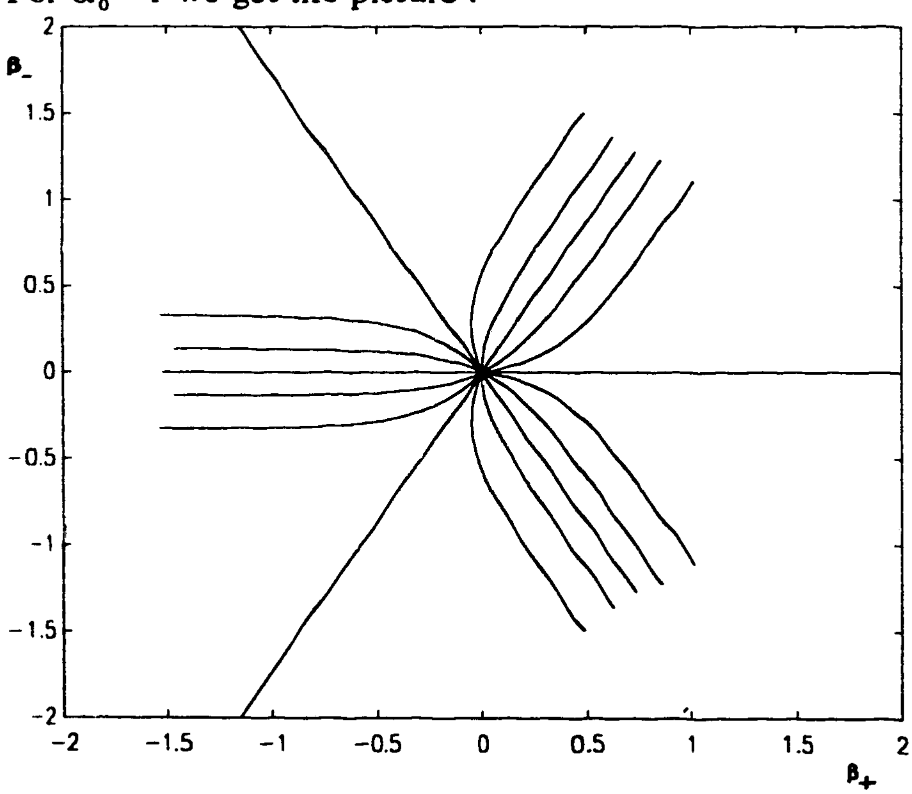
Now, if we scale ξ, $b_\pm$, there is no way to cancel e^{-2k} such that the parameter a scales. Many paths in $(\alpha, \beta_+, \beta_-)$ space, parametrized by k, give the same projection on the (β_+, β_-) plane.

The parameter α is approximated by $\alpha \approx \Lambda e^{2k} e^{2\eta} \;\Rightarrow\; \alpha' \approx 2\Lambda e^{2k} e^{2\eta} \approx 2\alpha$.

So, we can consider as the second parameter of the problem the quantity $\sigma \equiv \Lambda e^{2k}$ and then $\alpha \approx \sigma e^{2\eta}$, $\alpha' \approx 2\sigma e^{2\eta} \approx 2\alpha$.

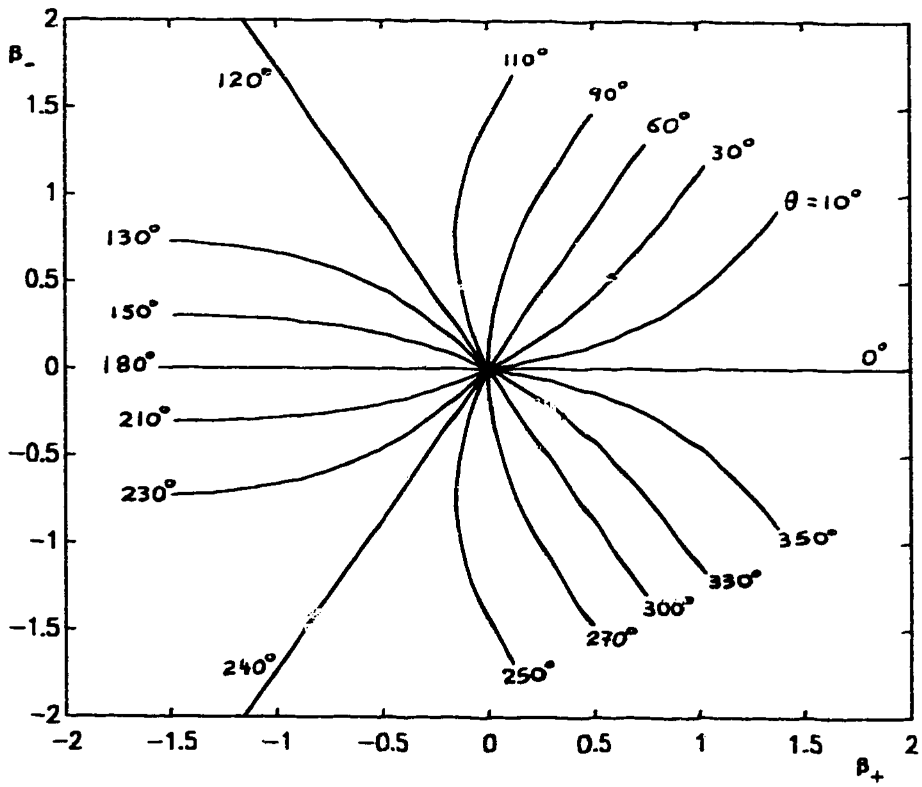
The initial data for the system $(\alpha, H, \beta_+, \beta_-, p_+, p_-)$ are set if $\varepsilon = e^{2\eta_0}$ and then $\alpha_0 = \sigma\varepsilon$, $H_0 = 2\sigma\varepsilon$, $(\beta_+)_0 = \varepsilon\cos\theta$, $(\beta_-)_0 = \varepsilon\sin\theta$, $(p_+)_0 = 2\varepsilon\cos\theta$, $(p_-)_0 = 2\varepsilon\sin\theta$.

For $\alpha_0 = 1$ we get the picture :

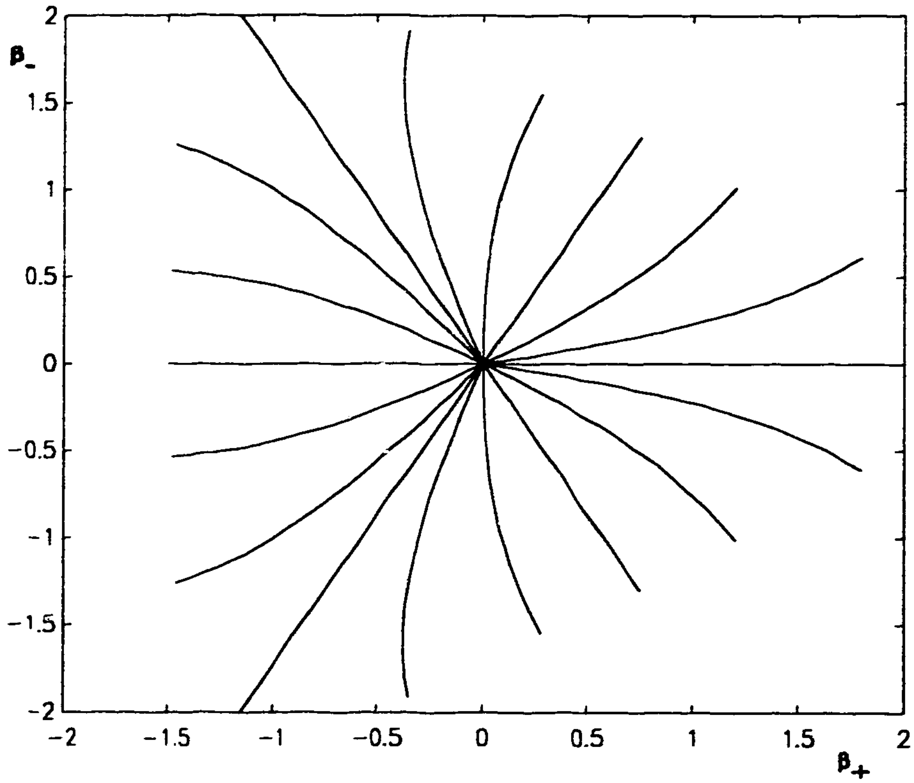


If α_0 becomes smaller, the solutions become closer to the $\theta = 60^0$ solution, but even for $\alpha_0 = 0.001$, the difference is not so significant.

If α_0 becomes larger then, for example, for the value $\alpha_0 = 10$ we get :



If $\alpha_0 = 50$ then the figure becomes nearly "radial", and as α_0 increases the curves tend to become straighter, as seen from the following diagram :



For negative value of α_0 (not plotted here), the range of the solutions in the (β_+ , β_-) plane is extremely small compared to that of an equal positive value of α_0.

14. Bolt - solution with cosmological constant in β_+, β

As in section 10 concerning the Bolt-solution in β_+, β for $\Lambda = 0$, we have also here the approximate equations:

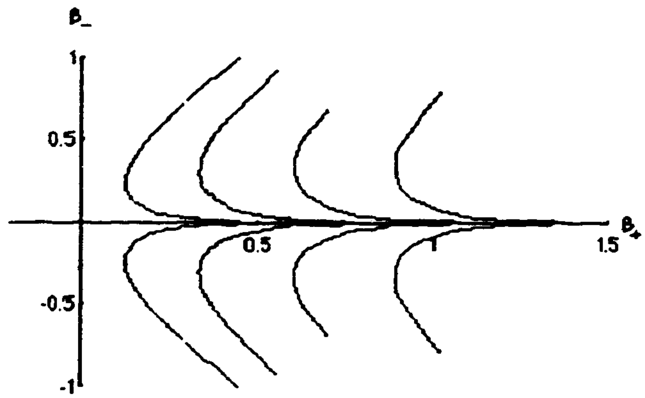
$$\beta_+'' \approx 2\beta_+'^2 \quad , \quad \alpha'' \approx -2\alpha'^2$$

Thus, $\alpha' \approx \dfrac{1}{2\eta} \Rightarrow \alpha(\eta) \approx \dfrac{1}{2} \ln|\eta| + k \Rightarrow \alpha(\eta) \to -\infty, \qquad |\eta| \to 0$

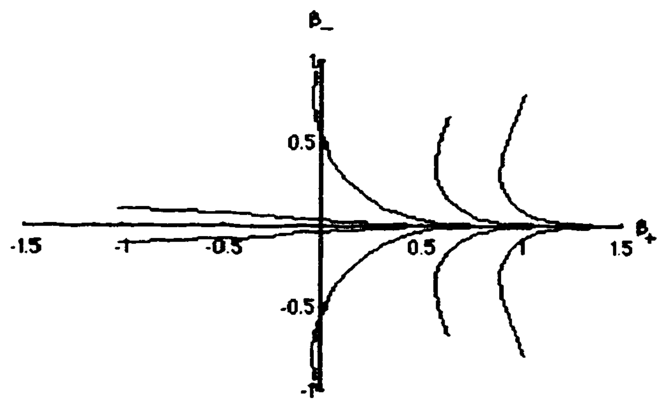
and $\quad \beta_+' \approx -\alpha' \approx -\dfrac{1}{2\eta} \Rightarrow \beta_+ \approx -\dfrac{1}{2} \ln|\eta| \Rightarrow \beta_+ \to +\infty, \qquad |\eta| \to 0$.

The governing equations are those of the preceding section and so only two independent parameters should remain.

From $d\xi = 2e^{\alpha} d\eta \approx 2e^{k} |\eta|^{1/2} d\eta \Rightarrow \xi \approx \pm \dfrac{4}{3} e^{k} |-\eta|^{3/2} + (const.)$, where the plus sign is for $\eta > 0$, while the minus for $\eta < 0$. If $\xi \to 0^+$ at the Bolt, then $\eta \to 0^+$ and $\xi \approx \dfrac{4}{3} e^{k} \eta^{3/2}$. The parameter α, close to the Bolt is approximated by: $\alpha \approx \Lambda e^{2k} |\eta| = \Lambda e^{2k} \eta \Rightarrow \alpha' \approx \Lambda e^{2k}$. Since β_-' is one of the two parameters of the problem, we can consider as the second parameter the quantity $\sigma \equiv \Lambda e^{2k}$ and then $\alpha \approx \sigma\eta$, $\alpha' \approx \sigma$. For some $\eta_0 \to 0^+$, we put the initial data for $(\alpha, \alpha', \beta_+, \beta_-, \beta_+', \beta_-')$ in agreement with the relations above for some fixed value of σ. For σ=1 the solutions in the $\beta_+ - \beta$ space are shown in the following diagram.



When the cosmological constant Λ increases we get the following picture for a bigger σ (say σ=10) and for the same values of β' as before.



We observe that as in the Nut case, when the cosmological constant increases, the solutions tend to become "straighter" lines. This means that the effect of Λ on the solutions is similar to that of increasing the initial velocity β'_-.

15. Nut - solution with cosmological constant in x, y

We consider the system of equations (3.22), (3.23), (3.24) again. In $x-y$ variables the quantity $\beta_+'^2+\beta_-'^2+\frac{\rho}{3}$ has been evaluated in section 6, and this is what we called there α'^2. So, the constraint becomes

$$\frac{H^2}{4\alpha^2}+\frac{4}{3}\alpha=\alpha_1 p_+^{\,2}+\alpha_2 p_-^{\,2}+\alpha_3 p_+ p_- +\frac{\rho}{3}, \qquad (15.1)$$

where $p_+\equiv x'$, $p_-\equiv y'$, $\alpha\equiv \Lambda e^{2a}$, $H\equiv\alpha'$.

Also $-2(\beta_+'^2+\beta_-'^2)$ is the old α'', thus

$$\frac{H'}{2\alpha}-\frac{H^2}{2\alpha^2}+\frac{4}{3}\alpha=-2(\alpha_1 p_+^{\,2}+\alpha_2 p_-^{\,2}+\alpha_3 p_+ p_-)\ \Leftrightarrow$$

$$H'=-4\alpha(\alpha_1 p_+^{\,2}+\alpha_2 p_-^{\,2}+\alpha_3 p_+ p_-)+\frac{H^2}{\alpha}-\frac{8}{3}\alpha^2 \qquad (15.2)$$

From the remaining two equations (6.9), (6.10) just replacing $2\alpha'$ by $\frac{H}{\alpha}$, we obtain

$$p_+'=\frac{1}{\beta_1^+\beta_2^- - \beta_1^-\beta_2^+}[(\beta_2^+\beta_3^- - \beta_2^-\beta_3^+)p_+^{\,2}+(\beta_2^+\beta_4^- - \beta_2^-\beta_4^+)p_-^{\,2}+(\beta_2^+\beta_5^- - \beta_2^-\beta_5^+)p_+p_-$$
$$+\frac{H}{\alpha}(\beta_2^+\beta_6^- - \beta_2^-\beta_6^+)p_+ +\frac{H}{\alpha}(\beta_2^+\beta_7^- - \beta_2^-\beta_7^+)p_- -\frac{1}{6}(\frac{\partial\rho}{\partial\beta_+}\beta_2^- -\frac{\partial\rho}{\partial\beta_-}\beta_2^+)] \qquad (15.3)$$

$$p_-'=\frac{-1}{\beta_1^+\beta_2^- - \beta_1^-\beta_2^+}[(\beta_1^+\beta_3^- - \beta_1^-\beta_3^+)p_+^{\,2}+(\beta_1^+\beta_4^- - \beta_1^-\beta_4^+)p_-^{\,2}+(\beta_1^+\beta_5^- - \beta_1^-\beta_5^+)p_+p_-$$
$$+\frac{H}{\alpha}(\beta_1^+\beta_6^- - \beta_1^-\beta_6^+)p_+ +\frac{H}{\alpha}(\beta_1^+\beta_7^- - \beta_1^-\beta_7^+)p_- -\frac{1}{6}(\frac{\partial\rho}{\partial\beta_+}\beta_1^- -\frac{\partial\rho}{\partial\beta_-}\beta_1^+)] \qquad (15.4)$$

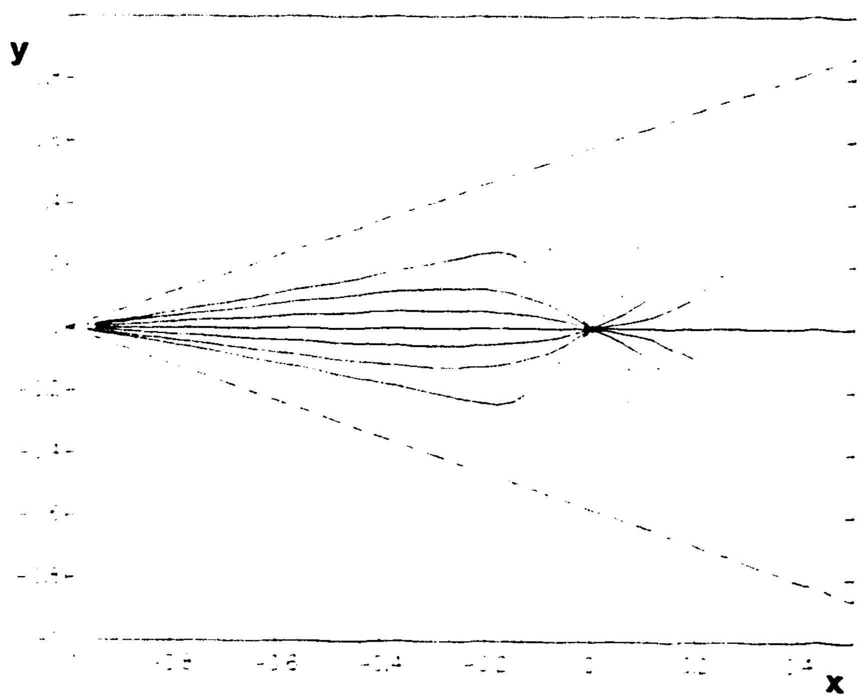
where the various $\alpha_i, \beta_i^{\pm}$ are the same functions of x, y as defined there. The system now is $(\alpha, H, x, y, p_+, p_-)$.

If $\varepsilon = e^{2\eta_0}$ then the initial data are :

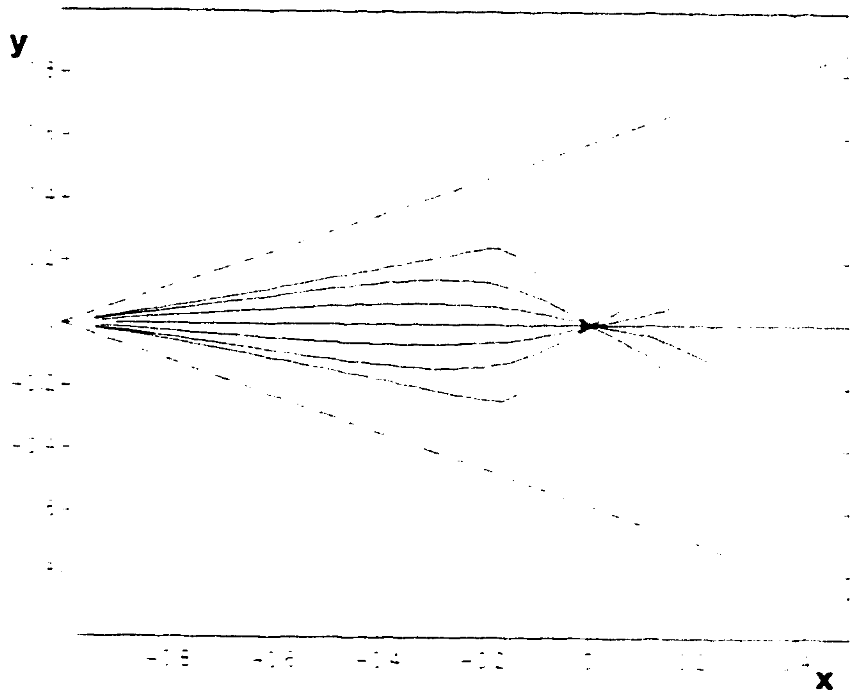
$\alpha_0 = \sigma\varepsilon$, $H_0 = 2\sigma\varepsilon$, $x_0 = \varepsilon\cos\theta$, $y_0 = \varepsilon\sin\theta$, $x_0' = 2\varepsilon\cos\theta$, $y_0' = 2\varepsilon\sin\theta$,
where σ is the second parameter of the problem, as in section 13.

For various positive values $(\Lambda > 0)$ of the parameter α_0 , we get various pictures for the solutions in the x-y space.

For α_0=0.1 it is

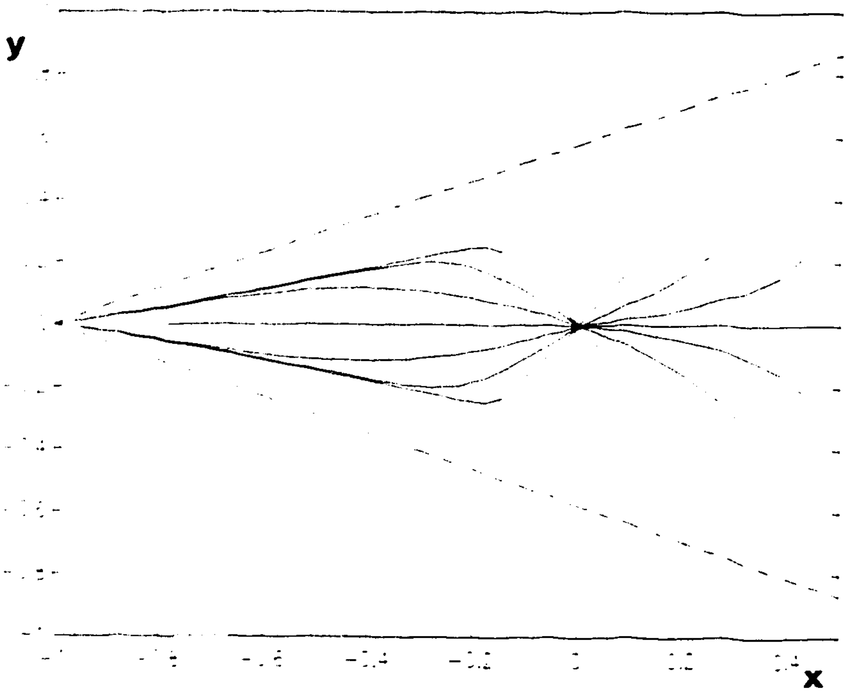


For α_0=1 we get the picture:



where now the distances between corresponding curves (curves with the same θ's as before) become bigger.

For α_o even bigger (α_o=100), the solutions are as follows:



where the phenomenon of spreading of solution becomes more obvious. This is an agreement with the β_+ , β_- - description (section 13) of the problem. By studying the preceding three pictures, it can be concluded that the three solutions for $\theta=0^\circ$, $\theta = 120^\circ$ and $\theta = 240^\circ$ remain practically fixed. This also agrees with the β_+ , β_- picture where these solutions are fixed straight lines at angles $\theta=0^\circ$, $\theta=120^\circ$ and $\theta=240^\circ$.

16. An approach through Ashtekar theory

Kodama has pointed out that the Chern-Simons functional S_{CS} provides an exact solution to the Ashtekar-Hamilton-Jacobi equation of general relativity with a non-zero cosmological constant. Louko [19] has investigated S_{CS} in the Bianchi IX spatially homogeneous cosmological model with S^3 spatial surfaces. Among the classical solutions generated by S_{CS}, there is a two parameter family of Euclidean spacetimes that have a regular closing of the Nut-type. This implies that, in this model, a wave function of the semiclassical form $e^{iS_{CS}}$ can be regarded as compatible with the no-boundary proposal of Hartle-Hawking.

The (Lorentzian) spacetime metric is

$$ds^2 = \frac{1}{32\pi^2}[-\sigma_1\sigma_2\sigma_3 \underset{\sim}{N}^2 dt^2 + \sigma_1^{-1}\sigma_2\sigma_3(\sigma^1)^2 + \sigma_2^{-1}\sigma_3\sigma_1(\sigma^2)^2 + \sigma_3^{-1}\sigma_1\sigma_2(\sigma^3)^2] \; ,$$

where σ^i are the left invariant one-forms on $SU(2) \cong S^3$. The rescaled lapse $\underset{\sim}{N}$ and the components σ_I of the inverse densitized triad are functions of t only and have nothing to do with σ^i. The Ashtekar action takes the form $S = \int dt(-\sigma_I \dot{A}_I - \underset{\sim}{N}h)$, where the Hamiltonian constraint $h(\sigma_I, A_I)$ is given by

$$h = -\sigma_1\sigma_2(A_1A_2 \mp iA_3) - \sigma_2\sigma_3(A_2A_3 \mp iA_1) - \sigma_3\sigma_1(A_3A_1 \mp iA_2) + 3\lambda\sigma_1\sigma_2\sigma_3$$

Here $\lambda \equiv \frac{\Lambda}{96\pi^2}$. The upper and lower signs correspond to the two possible signs of i in the definition of the Ashtekar connection.

The fundamental Poisson brackets are $\{A_I, \sigma_J\} = -\delta_{IJ}$. Dirac quantization with A_I as the configuration variables leads to the Wheeler-DeWitt equation

$\hat{h}(i\frac{\partial}{\partial A_I}, A_I)\Psi = 0$, for the wave function $\Psi(A_I)$. Given such a solution with the (approximate) form e^{iS}, a semiclassical expansion shows that S (approximately) solves the Hamilton-Jacobi equation $h(-\frac{\partial S}{\partial A_I}, A_I) = 0$. The wave function is therefore associated with the spacetimes obtained by solving the equations

$$\sigma_I = -\frac{\partial S}{\partial A_I} \quad , \quad \dot{A}_I = \underset{\sim}{n}\{A_I, h\} = -\underset{\sim}{n}\frac{\partial h}{\partial \sigma_I} \, .$$

The Chern-Simons action takes the form $S_{CS} = -\frac{1}{\lambda}[A_1A_2A_3 \mp \frac{i}{2}(A_1^{\ 2} + A_2^{\ 2} + A_3^{\ 2})]$ and solves the Hamilton-Jacobi equation. Replacing this in the two equations above we obtain

$$\lambda\sigma_1 = A_2A_3 \mp iA_1 \quad , \quad \lambda\sigma_2 = A_3A_1 \mp iA_2 \quad , \quad \lambda\sigma_3 = A_1A_2 \mp iA_3$$

$$\dot{A}_1 = -\lambda\underset{\sim}{n}\sigma_2\sigma_3 \quad , \quad \dot{A}_2 = -\lambda\underset{\sim}{n}\sigma_3\sigma_1 \quad , \quad \dot{A}_3 = -\lambda\underset{\sim}{n}\sigma_1\sigma_2 \quad , \quad (\dot{\ } \equiv \frac{d}{dt}) \, .$$

Going to the Euclidean case, set $\underset{\sim}{n} = \pm i(\sigma_1\sigma_2\sigma_3)^{-1\,2}$ and take all σ_I positive without loss of generality. Define

$$\sigma_1^{\ -1}\sigma_2\sigma_3 = \tfrac{1}{4}a^2 \quad , \quad \sigma_2^{\ -1}\sigma_3\sigma_1 = \tfrac{1}{4}b^2 \quad , \quad \sigma_3^{\ -1}\sigma_1\sigma_2 = \tfrac{1}{4}c^2 \quad , \quad a,b,c > 0 \tag{16.1}$$

and $A_I = \pm iB_I$. Then

$$S_{CS} = \pm\frac{i}{\lambda}[B_1B_2B_3 - \frac{1}{2}(B_1^{\ 2} + B_2^{\ 2} + B_3^{\ 2})] \tag{16.2}$$

and $\tfrac{1}{4}\lambda bc = B_1 - B_2B_3 \quad , \quad \tfrac{1}{4}\lambda ca = B_2 - B_3B_1 \quad , \quad \tfrac{1}{4}\lambda ab = B_3 - B_1B_2 \qquad (16.3)$

$$\dot{B}_1 = -\tfrac{1}{2}\lambda a \quad , \quad \dot{B}_2 = -\tfrac{1}{2}\lambda b \quad , \quad \dot{B}_3 = -\tfrac{1}{2}\lambda c \tag{16.4}$$

Now the metric has become

$$ds^2 = \frac{1}{32\pi^2}dt^2 + \frac{1}{128\pi^2}[a^2(\sigma^1)^2 + b^2(\sigma^2)^2 + c^2(\sigma^3)^2] \tag{16.5}$$

The proper distance is $\xi = \frac{1}{\sqrt{32}\pi}t$. We will find the equations governing a, b, c and compare these with the standard form of Bianchi IX equations containing a, b, and c. Differentiating (16.3) and substituting from (16.4) we get

$$\left.\begin{aligned} \tfrac{1}{2}(\dot b c + b\dot c) + a &= bB_3 + cB_2 \\ \tfrac{1}{2}(\dot c a + c\dot a) + b &= cB_1 + aB_3 \\ \tfrac{1}{2}(\dot a b + a\dot b) + c &= aB_2 + bB_1 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} B_1 &= -\frac{a^2}{2bc} + \frac{b}{2c} + \frac{c}{2b} + \frac{\dot a}{2} \\ B_2 &= -\frac{b^2}{2ac} + \frac{a}{2c} + \frac{c}{2a} + \frac{\dot b}{2} \\ B_3 &= -\frac{c^2}{2ab} + \frac{a}{2c} + \frac{b}{2a} + \frac{\dot c}{2} \end{aligned}\right\} \tag{16.6}$$

Differentiating these once more and replacing $\dot B_i$ from (16.4) we find

$$\left.\begin{aligned} &b^2c^2\ddot a - 2abc\dot a + c(a^2+b^2-c^2)\dot b + b(a^2+c^2-b^2)\dot c + \lambda a b^2 c^2 = 0 \\ &a^2c^2\ddot b + c(a^2+b^2-c^2)\dot a - 2abc\dot b + a(b^2+c^2-a^2)\dot c + \lambda a^2 b c^2 = 0 \\ &a^2b^2\ddot c + b(a^2+c^2-b^2)\dot a + a(b^2+c^2-a^2)\dot b - 2abc\dot c + \lambda a^2 b^2 c = 0 \end{aligned}\right\} \tag{16.7}$$

Now we substitute the values of B_1, B_2, B_3 from (16.6) into (16.3) and have

$$\left.\begin{aligned} \lambda a^2b^2c^2 &= 2a^2bc\dot a - ac(a^2+b^2-c^2)\dot b - ab(a^2+c^2-b^2)\dot c - a^2bc\dot b\dot c \\ &\quad + (b^4+c^4-3a^4+2a^2b^2+2a^2c^2-2b^2c^2) \\ \lambda a^2b^2c^2 &= -bc(a^2+b^2-c^2)\dot a + 2ab^2c\dot b - ab(b^2+c^2-a^2)\dot c - ab^2c\dot a\dot c \\ &\quad + (a^4+c^4-3b^4+2a^2b^2+2b^2c^2-2a^2c^2) \\ \lambda a^2b^2c^2 &= -bc(a^2+c^2-b^2)\dot a - ac(b^2+c^2-a^2)\dot b + 2abc^2\dot c - abc^2\dot a\dot b \\ &\quad + (a^4+b^4-3c^4+2a^2c^2+2b^2c^2-2a^2b^2) \end{aligned}\right\} \tag{16.8}$$

Using (16.8) we eliminate in (16.7) the terms linear in $\dot a, \dot b, \dot c$ and thus obtain

$$\left.\begin{aligned} &\frac{\ddot a}{a} - \frac{\dot b\dot c}{bc} + \frac{1}{a^2b^2c^2}(b^4+c^4-3a^4+2a^2b^2+2a^2c^2-2b^2c^2) = 0 \\ &\frac{\ddot b}{b} - \frac{\dot a\dot c}{ac} + \frac{1}{a^2b^2c^2}(a^4+c^4-3b^4+2a^2b^2+2b^2c^2-2a^2c^2) = 0 \\ &\frac{\ddot c}{c} - \frac{\dot a\dot b}{ab} + \frac{1}{a^2b^2c^2}(a^4+b^4-3c^4+2a^2c^2+2b^2c^2-2a^2b^2) = 0 \end{aligned}\right\} \tag{16.9}$$

We can take the form of this system with the time variable being the proper distance ξ. The only difference is that the term $\frac{\ddot a}{a} - \frac{\dot b\dot c}{bc}$ (and similarly in the other two equations) will be multiplied by the factor $\frac{1}{32\pi^2}$, and the time differentiation will be with respect to ξ.

From system of equations (3.3), (3.4), (3.5) governing the three usual scale-factors a, b, c with $\Lambda \neq 0$, and using the constraint equation (3.6) we find exactly the system (16.9) for a, b, c instead of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ except that the differentiations are with respect to ξ. Then it is obvious that

$$\mathbf{a} = 8\sqrt{2}\pi a \quad , \quad \mathbf{b} = 8\sqrt{2}\pi b \quad , \quad \mathbf{c} = 8\sqrt{2}\pi c \tag{16.10}$$

and the metric takes the common form $ds^2 = d\xi^2 + a^2(\sigma^1)^2 + b^2(\sigma^2)^2 + c^2(\sigma^3)^2$.

The simple algebraic form of S_{CS} gives the correct equations. It is tempting though try to express S_{CS} with relation to a, b, c (in a closed form) instead of B_1, B_2, B_3 . After a few calculations, system (16.3) takes the equivalent form :

$$B_3^5 - \tfrac{1}{4}\lambda \mathbf{a}\mathbf{b}B_3^4 - 2B_3^3 + \tfrac{1}{2}\lambda \mathbf{a}\mathbf{b}(1 - \tfrac{1}{8}\lambda \mathbf{c}^2)B_3^2 + (1 - \tfrac{1}{16}\lambda^2\mathbf{c}^2(\mathbf{a}^2 + \mathbf{b}^2))B_3 = \tfrac{1}{4}\lambda \mathbf{a}\mathbf{b}(1 + \lambda \mathbf{c}^2)$$

$$B_2 = \tfrac{1}{4}\lambda \mathbf{c}\frac{\mathbf{a} + \mathbf{b}B_3}{1 - B_3^2}$$

$$B_1 = \tfrac{1}{4}\lambda \mathbf{c}\frac{\mathbf{b} + \mathbf{a}B_3}{1 - B_3^2} \tag{16.11}$$

For the fifth-order algebraic equation for B_3 it is not possible to find the solution explicitly.

Let us define the Misner variables as :

$$\mathbf{a} = e^{\alpha + \beta_+ + \sqrt{3}\beta_-} \quad , \quad \mathbf{b} = e^{\alpha + \beta_+ - \sqrt{3}\beta_-} \quad , \quad \mathbf{c} = e^{\alpha - 2\beta_+} \; , \tag{16.12}$$

where the scale factors of anisotropy β_+, β_- have exactly the same numerical value as the usual ones, while α differs from the usual α by the additional constant $\ln(8\sqrt{2}\pi)$.

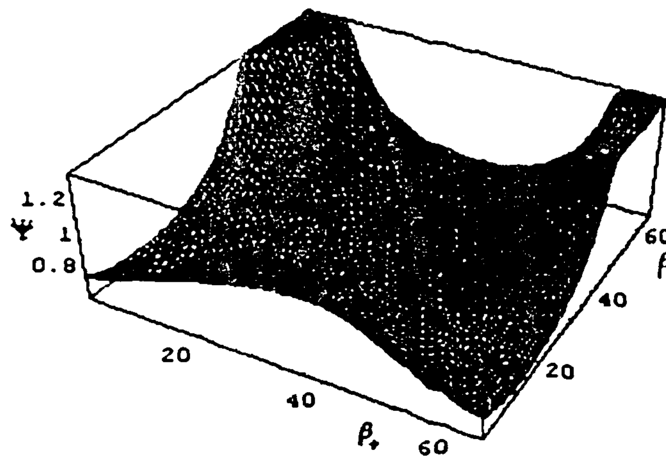
If we define the variable $\boldsymbol{\alpha} \equiv \lambda e^{2\alpha}$ instead of α, and express system (16.11) in $\boldsymbol{\alpha}, \beta_+, \beta_-$ variables we obtain

$$B_3^5 - \tfrac{1}{4}\boldsymbol{\alpha} e^{2\beta_+}B_3^4 - 2B_3^2 + \tfrac{1}{2}\boldsymbol{\alpha}(e^{2\beta_+} - \tfrac{1}{8}\boldsymbol{\alpha} e^{-2\beta_+})B_3^2 + (1 - \tfrac{1}{8}\boldsymbol{\alpha}^2 e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-))B_3$$
$$- \tfrac{1}{4}\boldsymbol{\alpha}(e^{2\beta_+} + \tfrac{1}{4}\boldsymbol{\alpha} e^{-2\beta_+}) = 0$$

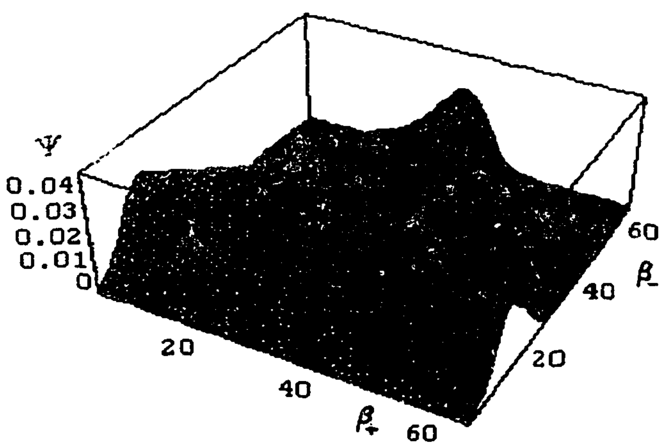
$$B_2 = \tfrac{1}{4}\boldsymbol{\alpha} e^{-\beta_+}\frac{e^{\sqrt{3}\beta_-} + e^{-\sqrt{3}\beta_-}B_3}{1 - B_3^2}$$

$$B_1 = \tfrac{1}{4}\boldsymbol{\alpha} e^{-\beta_+}\frac{e^{-\sqrt{3}\beta_-} + e^{\sqrt{3}\beta_-}B_3}{1 - B_3^2} \tag{16.13}$$

Now the cosmological constant has disappeared. For some constant α, we can plot $B_3 = B_3(\beta_+,\beta_-)$, $B_2 = B_2(\beta_+,\beta_-)$, $B_1 = B_1(\beta_+,\beta_-)$ and then $i\lambda S_{CS} = \mp[B_1 B_2 B_3 - \frac{1}{2}(B_1^2 + B_2^2 + B_3^2)]$ and $\psi = e^{\mp[B_1 B_2 B_3 - \frac{1}{2}(B_1^2 + B_2^2 + B_3^2)]}$ as functions of β_+,β_-, where $\Psi = \psi^{\frac{1}{\lambda}}$, and ψ is the wave function approximation. Doing this for $\alpha = -10$ $(\Lambda < 0)$ we get the picture



where the positive sign in the exponential has been used. Then for every $\lambda < 0$ the wave function $\Psi = \psi^{\frac{1}{\lambda}}$ will have a wave packet shape as follows



Varying the value of the cosmological constant, we observe that the peak is sharper when $|\lambda|$ is smaller. It is interesting to note that Ψ has the familiar form as before, but the method used now does not solve differential equations but only algebraic ones. This deduction is due to Ashtekar theory.

For $\lambda > 0$, Ψ does not appear to have a peak but changes monotonically with β_+, β (at least for $\alpha = 0.1, 1, 10$).

We come back to the systems (16.3), (16.4) to find a solution at the limit of small anisotropy. If we define furthermore F, g_+, g by

$$B_1 = \tfrac{1}{2} F e^{g_+ + \sqrt{3} g_-}, \quad B_2 = \tfrac{1}{2} F e^{g_+ - \sqrt{3} g_-}, \quad B_3 = \tfrac{1}{2} F e^{-2g_+} \tag{16.14}$$

and expand to linear order in $\beta_\pm, g_\pm$ then

$$\begin{cases} \dot{F} = -\lambda e^{\alpha} \\ \lambda e^{2\alpha} = F(2 - F) \\ (F g_\pm)\dot{} = -\lambda e^{\alpha} \beta_\pm \\ \lambda e^{\alpha} \beta_\pm = -F(2 + F) g_\pm \end{cases} \tag{16.15}$$

For $\lambda > 0$, the solution is

$$\begin{cases} e^{\alpha} = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} t) \\ \beta_\pm = (\beta_\pm)_0 (2 + \tan^2(\tfrac{1}{2}\sqrt{\lambda} t)) \tan^2(\tfrac{1}{2}\sqrt{\lambda} t) \\ F = 1 + \cos(\sqrt{\lambda} t) \\ g_\pm = -(\beta_\pm)_0 \tan^4(\tfrac{1}{2}\sqrt{\lambda} t) \end{cases} \tag{16.16}$$

where $(\beta_\pm)_0$ are constants of integration and $0 < t < \frac{\pi}{\sqrt{\lambda}}$.

For $\lambda < 0$, there are two solutions. One is given by (16.16) understood in the sense of analytic continuation in λ with $0 < t < \infty$. The other solution is

$$\begin{cases} e^{\alpha} = \frac{-1}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} t) \\ \beta_\pm = (\gamma_\pm)_0 (\coth^2(\tfrac{1}{2}\sqrt{-\lambda} t) - 2) \coth^2(\tfrac{1}{2}\sqrt{-\lambda} t) \\ F = 1 - \cosh(\sqrt{-\lambda} t) \\ g_\pm = -(\gamma_\pm)_0 \coth^4(\tfrac{1}{2}\sqrt{-\lambda} t) \end{cases} \tag{16.17}$$

where $(\gamma_\pm)_0$ constants and $-\infty < t < \infty$.

It can now be verified that inserting the linearized solution (16.16) into the initial metric ansatz gives a Euclidean metric that can be regularly extended to $t = 0$ by adding just one point to the manifold : one can view the new point as the coordinate singularity at the origin of a hyperspherical coordinate system in which t is the radial coordinate. Since $t \to 0$ is the limit where the linearized solution is accurate, we see that the corresponding exact metrics can be similarly extended to $t = 0$. Then, the closing of the geometry at $t = 0$ is of the Nut-type. This regular closing of the geometry is precisely the property characterizing the classical solutions that are relevant for the no-boundary proposal of Hartle and Hawking, in the sense that wave functions satisfying the no-boundary proposal are expected to get their dominant semiclassical contribution from one or more such regular classical solutions. Therefore in this model, a wave function of the semiclassical form $e^{iS_{cs}}$ is compatible with a semiclassical estimate to the no-boundary wave function.

17. Bianchi IX model coupled to a massless scalar field

If matter is included in a model for the Universe then the action of the system is $I = I_g + I_\Phi$, where

$$I_g = -\int\sqrt{g}\,({}^{(4)}R - 2\Lambda)d^4x + I_S \quad , \quad I_\Phi = \int\sqrt{g}\,L_\Phi d^4x \tag{17.1}$$

The numerical factor $\frac{1}{16\pi}$ has been omitted for simplicity. The field Φ is considered constant on the surfaces of homogeneity and so $\Phi = \Phi(\eta)$. The Lagrangian L_Φ is a function of Φ and its derivatives.

For a free scalar field it is by definition

$$I_\Phi = \int d^4x\sqrt{g}\,(-\Phi \,\Box\, \Phi + m^2 \frac{\Phi^2}{2}) \tag{17.2}$$

and since $\int d^4x\sqrt{g}\Phi \,\Box\, \Phi = \int d^4x \Phi\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi) = -\int d^4x\sqrt{g}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} \Rightarrow$

$$\begin{aligned} I_\Phi &= \int d^4x\sqrt{g}\,(g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + m^2 \frac{\Phi^2}{2}) \\ &= \int d^4x\sqrt{g}\,(\tfrac{1}{4}e^{-2\alpha}\Phi'^2 + m^2\frac{\Phi^2}{2}) \\ &= \int_0^\pi sin\theta d\theta \int_0^{2\pi} d\varphi \int_0^{4\pi} d\psi \int 2e^{-2\alpha}e^{6\alpha}(\tfrac{1}{4}e^{-2\alpha}\Phi'^2 + m^2\frac{\Phi^2}{2})d\eta \\ &= 8\pi^2 \int d\eta\, e^{2\alpha}(\Phi'^2 + 2m^2\Phi^2e^{2\alpha}) \end{aligned} \tag{17.3}$$

Equation (17.2) contains the surface term

$-\int d^4x\partial_\mu(\Phi\sqrt{g}g^{\mu\nu}\partial_\nu\Phi) = -16\pi^2\int d\xi(e^{3\alpha}\Phi\tilde{\Phi})^{\sim}$, so the surface action I_S, eq. (3.14), should be modified in order to cancel this new surface term of I_Φ.

So,

$$I = 6\pi^2\int(\beta_+'^2 + \beta_-'^2 - \alpha'^2 - \frac{\rho}{3} + \frac{4\Lambda}{3}e^{2\alpha})e^{2\alpha}d\eta + 8\pi^2\int(\Phi'^2 + 2m^2\Phi^2e^{2\alpha})e^{2\alpha}d\eta$$

$$= 2\pi^2 \int L d\eta \quad ; \quad \text{with}$$

$$L = 3(\beta_+'^2 + \beta_-'^2 - \alpha'^2 - \tfrac{\rho}{3} + \tfrac{4\Lambda}{3} e^{2\alpha})e^{2\alpha} + 4(\Phi'^2 + 2m^2\Phi^2 e^{2\alpha})e^{2\alpha} \tag{17.4}$$

This is the Lagrangian of the system and evaluating the various derivatives of L we get the Euler-Lagrange equations :

$$\beta_\pm'' = -\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} - 2\alpha'\beta_\pm' \tag{17.5}$$

$$\alpha'' = -2(\beta_+'^2 + \beta_-'^2) - \tfrac{4\Lambda}{3} e^{2\alpha} - \tfrac{8}{3}\Phi'^2 - \tfrac{8}{3} m^2\Phi^2 e^{2\alpha} \tag{17.6}$$

$$\Phi'' = -2\alpha'\Phi' + 2m^2 e^{2\alpha}\Phi \tag{17.7}$$

and the constraint

$$\beta_+'^2 + \beta_-'^2 - \alpha'^2 + \tfrac{4}{3}\Phi'^2 + \frac{\rho}{3} - \frac{4\Lambda}{3} e^{2\alpha} - \tfrac{8}{3} m^2\Phi^2 e^{2\alpha} = 0 \tag{17.8}.$$

We are only interested in the case with $\Lambda = 0, \quad m = 0$. Then the equation (17.5) remains unchanged while the others become :

$$\alpha'' = -2(\beta_+'^2 + \beta_-'^2) - \tfrac{8}{3}\Phi'^2 \tag{17.9}$$

$$\Phi'' = -2\alpha'\Phi' \tag{17.10}$$

$$\beta_+'^2 + \beta_-'^2 - \alpha'^2 + \tfrac{4}{3}\Phi'^2 + \frac{\rho}{3} = 0 \tag{17.11}$$

We have four 2nd-order equations plus one 1st-order constraint , giving 7 independent parameters : $\alpha, \beta_+, \beta_+', \beta_-, \beta_-', \Phi, \Phi'$. Again, we take out one parameter each for the origin of the time, for the scale, and for the origin of Φ (since only Φ' enters into the equations). This leaves 4, just as in the case of the cosmological constant. Regularity at the origin takes out 2 more, leaving 2 at last.

If set $p_\pm = \beta_\pm'$, $H = \alpha'$, $g = \Phi'$, then :

$$p_\pm' = -2Hp_\pm - \frac{1}{6}\frac{\partial\rho}{\partial p_\pm} \tag{17.12}$$

$$H' = -2(p_+^2 + p_-^2) - \tfrac{8}{3}g^2 \tag{17.13}$$

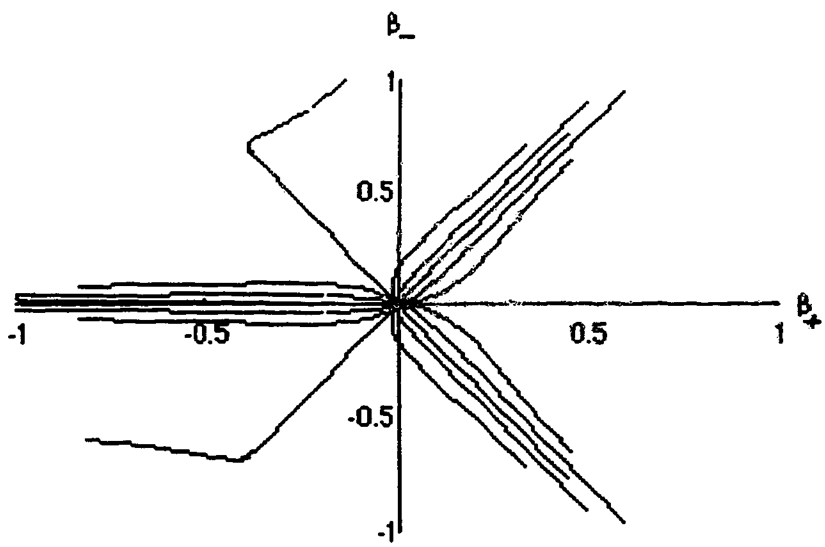
$$g' = -2Hg \tag{17.14}$$

$$H^2 = p_+^2 + p_-^2 + \frac{\rho}{3} + \frac{4}{3}g^2 \tag{17.15}$$

Equation (17.10) is equivalent to $(e^{2\alpha}\Phi')' = 0 \Rightarrow \Phi' = ce^{-2\alpha}$ (17.16)

Regularity at a Nut origin means $\beta_+' \approx 0$, near the center, and $\frac{d\Phi}{d\xi} \underset{\xi\to 0}{\to} 0$, thus

$\Phi' = \frac{d\Phi}{d\xi}\frac{d\xi}{d\eta} \underset{\xi\to 0}{\to} 0$. So, the constraint (17.11) becomes $\alpha'^2 \approx 1 \Rightarrow \alpha' \approx 1 \Rightarrow \alpha \approx \eta$, and from (17.16) one has $c=0$. Thus, $\Phi = \Phi_0 =$ constant, everywhere. The relation between ξ and η in this region, is found by : $d\xi = 2e^{\alpha}d\eta \Rightarrow d\xi \approx 2e^{\eta}d\eta \Rightarrow \xi \approx 2e^{\eta}$ and so for $\eta \to -\infty \Rightarrow \xi \to 0^+$. If $\varepsilon = e^{2\eta_0}$, the initial conditions are : $(\beta_+)_0 = \varepsilon\cos\theta$, $(\beta_-)_0 = \varepsilon\sin\theta$, $(p_+)_0 = 2\varepsilon\cos\theta$, $(p_-)_0 = 2\varepsilon\sin\theta$, $H_0 = 1$, $g_0 = 0$. The second parameter of the problem, namely the value Φ_0 of the scalar field, does not enter into the equations (17.12), (17.13), (17.14), so there is only one configuration of solutions, shown in the following picture



We observe that this picture is very similar (slightly different), even quantitatively, to the case where no field is present (section 7).

18. Bianchi IX model conformally coupled to a scalar field

For the action term I_Φ of equations (17.1) it is

$$I_\Phi = \int d^4x\sqrt{g}(\Phi \Box \Phi + \tfrac{1}{6}{}^{(4)}R\Phi^2 - \rho^2\Phi^4) \tag{18.1}$$

$$= -\int d^4x\sqrt{g}(g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} - \tfrac{1}{6}{}^{(4)}R\Phi^2 + \rho^2\Phi^2)$$

$$= -32\pi^2 \int e^{4\alpha}(\tfrac{1}{4}e^{-2\alpha}\Phi'^2 - \tfrac{1}{6}{}^{(4)}R\Phi^2 + \rho^2\Phi^4)d\eta \tag{18.2}$$

From equation (3.8) for the Ricci scalar we have

$${}^{(4)}R = -\frac{3}{2e^{2\alpha}}(\alpha'' + \alpha'^2 + \beta_+'^2 + \beta_-'^2) + {}^{(3)}R(\alpha, \beta_+, \beta_-)$$

So, $I_\Phi = -8\pi^2 \int [\frac{\Phi'^2}{\Phi^2} - 2\frac{\alpha'\Phi'}{\Phi} - \alpha'^2 + \beta_+'^2 + \beta_-'^2 - \frac{2}{3}{}^{(3)}Re^{2\alpha} + 4\rho^2\Phi^2 e^{2\alpha}]\Phi^2 e^{2\alpha} d\eta$

As in the case of section 17, I_S also here, takes a different form in order to cancel the two surface terms coming from $\Phi \Box \Phi$ and $(1/6){}^{(4)}R\Phi^2$ in (18.1).

The total action takes the form

$$I = 2\pi^2 \int [(3 - 4\Phi^2)(\beta_+'^2 + \beta_-'^2 - \alpha'^2) - 4\Phi'^2 + 8\Phi\alpha'\Phi' - \rho + 4\Lambda e^{2\alpha} + \frac{8}{3}{}^{(3)}R\Phi^2 e^{2\alpha} - 16\rho^2\Phi^4 e^{2\alpha}]e^{2\alpha} d\eta \tag{18.3}$$

It is $\frac{8}{3}{}^{(3)}R\Phi^2 e^{2\alpha} = \frac{4}{3}\rho\Phi^2 \Rightarrow -\rho + \frac{8}{3}{}^{(3)}R\Phi^2 e^{2\alpha} = -\frac{\rho}{3}(3 - 4\Phi^2)$

Varying the action, the Euler-Lagrange equations arise :

$$\beta_\pm'' = -2\alpha'\beta_\pm' - \frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm} + \frac{8\Phi}{3 - 4\Phi^2}\Phi'\beta_\pm' \tag{18.4}$$

$$\alpha'' = -2(\beta_+'^2 + \beta_-'^2) + \frac{12\Phi'^2}{3 - 4\Phi^2} + \frac{4\Phi\Phi''}{3 - 4\Phi^2} - \frac{4\Lambda e^{2\alpha}}{3 - 4\Phi^2} + 16\rho^2 e^{2\alpha}\frac{\Phi^4}{3 - 4\Phi^2} \tag{18.5}$$

where we have used the constraint equation

$$(3-4\Phi^2)(\beta_+'^2+\beta_-'^2-\alpha'^2)-4\Phi'^2+8\Phi\alpha'\Phi'+\rho-4\Lambda e^{2\alpha}-\frac{4}{3}\rho\Phi^2+16\rho^2\Phi^4e^{2\alpha}=0 \tag{18.6}$$

and the equation for Φ is :

$$\frac{\Phi''}{\Phi}-\alpha''+2\frac{\alpha'\Phi'}{\Phi}-\alpha'^2-\beta_+'^2-\beta_-'^2+\frac{1}{3}\rho-8\rho^2\Phi^2e^{2\alpha}=0 \tag{18.7}$$

Solving algebraically the system of equations (18.5), (18.7) for α'',Φ'' we obtain

$$\alpha''=-6\frac{1-2\Phi^2}{3-8\Phi^2}(\beta_+'^2+\beta_-'^2)+\frac{4\Phi'^2}{3-8\Phi^2}\alpha'^2+\frac{12}{3-8\Phi^2}\Phi'^2-\frac{8\Phi}{3-8\Phi^2}\alpha'\Phi'$$
$$-\frac{4\Lambda e^{2\alpha}}{3-8\Phi^2}+\frac{48\Phi^4}{3-8\Phi^2}\rho^2e^{2\alpha}-\frac{4\rho}{3}\frac{\Phi^2}{3-8\Phi^2} \tag{18.8}$$

and substituting α'^2 from the constraint, we get

$$\alpha''=-2(\beta_+'^2+\beta_-'^2)+4\frac{9-16\Phi^2}{(3-4\Phi^2)(3-8\Phi^2)}\Phi'^2-\frac{8\Phi}{3-4\Phi^2}\alpha'\Phi'$$
$$-\frac{12\Lambda e^{2\alpha}}{(3-4\Phi^2)(3-8\Phi^2)}+\frac{16\rho^2\Phi^4e^{2\alpha}(9-8\Phi^2)}{(3-4\Phi^2)(3-8\Phi^2)} \tag{18.9}$$

Similarly,

$$\Phi''=-2\alpha'\Phi'+\frac{8\Phi}{3-8\Phi^2}\Phi'^2-\frac{8\Phi}{3-8\Phi^2}\Lambda e^{2\alpha}+\frac{24\Phi^3}{3-8\Phi^2}\rho^2e^{2\alpha} \tag{18.10}$$

So, the full system of equations we have consists of (18.4), (18.6), (18.9), (18.10).

We are only interested in the case $\Lambda=0$, $\rho=0$. Then we have

$$\beta_\pm''=-2\alpha'\beta_\pm'-\frac{1}{6}\frac{\partial\rho}{\partial\beta_\pm}+\frac{8\Phi}{3-4\Phi^2}\Phi'\beta_\pm' \tag{18.11}$$

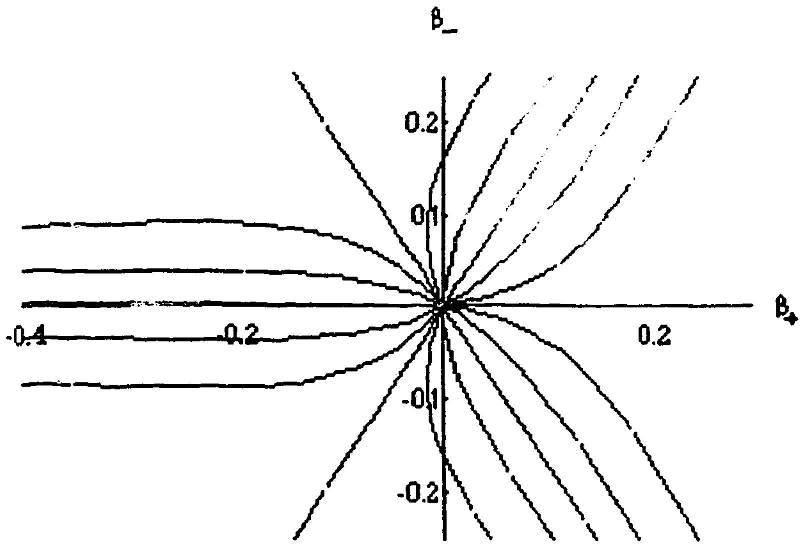
$$\alpha''=-2(\beta_+'^2+\beta_-'^2)+\frac{4(9-16\Phi^2)}{(3-4\Phi^2)(3-8\Phi^2)}\Phi'^2-\frac{8\Phi}{3-4\Phi^2}\alpha'\Phi' \tag{18.12}$$

$$\Phi''=-2\alpha'\Phi'+\frac{8\Phi}{3-8\Phi^2}\Phi'^2 \tag{18.13}$$

$$\alpha'^2=\beta_+'^2+\beta_-'^2+\frac{\rho}{3}-\frac{4}{3-4\Phi^2}\Phi'^2+\frac{8\Phi}{3-4\Phi^2}\alpha'\Phi' \tag{18.14}$$

The scaling, the time translation, the constraint and the two regularity conditions lead to three independent parameters.

As in the previous section, regularity at the origin gives $\Phi' \underset{\xi \to 0}{\to} 0$. Supposed that Φ does not approach the values $\pm\frac{\sqrt{3}}{2}$ close to the Nut, then we get from equation (18.14) that $\alpha' \approx 1$, and from (18.11) that $\beta_\pm \approx b_\pm e^{2\eta}$. If we set, as usually, $p_\pm = \beta'_\pm$, $H = \alpha'$, $g = \Phi'$, then we have a first-order differential system for $(H, \beta_+, \beta_-, p_+, p_-, \Phi, g)$. The initial values of $\beta_\pm, p_\pm$ are the usual ones. The initial value of the scalar field Φ cannot be determined elsewhere, and giving to that various values, the following figure arises (with non-remarkable differences observed between figures with different values of Φ_0).



Now, as it is easily seen, the solutions do not stay close to the three symmetry directions and the figure becomes nearly "radial", as in the case of the cosmological constant.

Conclusion

A lot of important work on the study of relativistic Bianchi-IX model has been done by various people. The significance of this model and its chaotic behavior, approaching the singularity, were revised. The action and the dynamical equations governing the Euclidean Bianchi-IX model were derived using various time parameters. When the Universe shrinks to zero volume, the Nut and the Bolt boundary conditions arise, and the asymptotic form of the various quantities was stated. Closed expressions of the self-dual actions corresponding to the (asymptotically Euclidean but singular) BGPP and/or the (regular at the center) AH solutions were extracted. A new set of dynamical variables was introduced, mapping the entire plane of the two anisotropy scale factors into a finite triangular region. Approximate solutions, close to the Nut, were obtained and they supplied the correct initial data for the full system of the differential equations. This way, the numerical solutions and the first-order approximation of the Hartle-Hawking no boundary wavefunction were obtained, with the expected properties (three-fold symmetry of the potential, peak at the center) verified. The Euclidean Hamilton-Jacobi equation contains the BGPP action as solution and its semiclassical wavefunction coincides with a canonical quantization solution, given in the bibliography. For the simplified first-order differential equations of the regular (AH) self-dual case, results were given, in agreement with the previous ones. In the Bolt case, with or without a cosmological constant, the time derivative of one of the two anisotropy scale factors was found to be able to serve as a parameter in the space of solutions and the figures of such solutions were given. The role of Λ is similar to that of increasing the initial value of that velocity. In the Nut case, the presence of Λ makes the solutions spreading, in either the infinite or the finite region parametrization of the anisotripy. The application of the Ashtekar theory to the Bianchi-IX model uses the Chern-Simmons action to provide a solution for a nonzero cosmological constant. Transforming the system of the Euclidean equations from the Ashtekar variables to the usual scale factors, the equivalence of this description with that of general relativity was confirmed for this model. The difficulty in solving the system of differential equations of general relativity is transformed into the difficulty in solving an algebraic system of equations for the Ashtekar variables in terms of the usual scale factors. Solving implicitly such a system, the figure of the CS action and of the wavefunction were found to agree with the previous figures. The coupling of the Bianchi IX model to a massless scalar field does not bring any significant difference to the configuration of the solutions, while the conformal coupling to a scalar field makes the solutions spreading.

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