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QUANTUM RADIATION OF RELATIVISTIC ACCELERATED POLARIZABLE BODIES

By

Dinesh Singh ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Quantum Radiation of Relativistic Accelerated Polarizable Bodies" submitted by Dinesh D. Singh in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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To the memory of my grandfather,

who did not quite make it to see this moment.

Abstract

It is well-known that a charged particle which is moving with a constant acceleration emits radiation. There exists a quantum analogue of this effect. A neutral polarizable body interacting with a quantum field and moving with an acceleration becomes a source of quantum radiation. The origin of this radiation are currents induced in the polarizable body by its interaction with zero-point fluctuations of the quantum field. When this body is moving with an acceleration, such currents generically produce a non-vanishing flux of energy-momentum. This effect is known as the dynamical Casimir effect. The thesis studies this effect for special models of a polarizable body and different types of its accelerated motion. The first example considered involves the constant accelerated expansion of semi-transparent spherical mirror-like surfaces, and specifically considers the cases of a single sphere and two concentric spheres. This expansion might model, for example, relativistic bubbles expected to have formed during cosmological phase transitions. The second example concerns the quantum radiation emitted from a small dielectric body, with small refractive index relative to the vacuum value. For comparison, the radiation due to a small ideal accelerated spherical mirror is also investigated. Special attention is paid to studying the quantum radiation in the relativistic regime, where earlier only a few results were obtained.

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Contents

Chapter 1

Introduction

1.1 An Overview of Quantum Field Theory in Curved Space-Time

Quantum mechanics and general relativity are undoubtedly two of the most influential scientific developments to emerge in the 20th century. In particular, the natural extension of quantum mechanics into quantum field theory has resulted with some of the most accurate predictions of any scientific theory. For example, quantum electrodynamics, the quantized description of the electromagnetic field, can successfully predict the value of the electron's anomalous magnetic moment to within one part in 10^{11} [36]. This leaves no doubt that quantum field theory, as described by a perturbation expansion, provides a description of Nature that agrees remarkably well with experiment. At the planetary and cosmological scales, general relativity as a theory of gravitation is also remarkably successful. For example, it accurately predicts the precession of Mercury's orbit previously unaccounted for by Newtonian gravitation, satisfies all known precision tests to date, such as the deflection of starlight passing near the Sun, and provides the basis for meaningful study of black holes and the evolution of the Universe as a whole.

Given the successes of quantum field theory and general relativity, many attempts to quantize the gravitational field have been proposed. Unfortunately, no one has yet found a self-consistent theory of quantum gravity. The primary reason why is because the basic perturbation approach of quantum field theory becomes *non-renormalizable* when applied to the gravitational field, since the gravitational fine coupling constant has dimensions of $(mass)^{-2}$. This has led many people to consider, as a first approximation, the study of quantum field theory within a classical gravitational background, where the quantized matter field is the source which induces the curvature of space-time. The general form of this approach is [22]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad = \quad 8\pi G \langle T_{\mu\nu} \rangle, \tag{1.1}$$

where the left-hand side of (1.1) is the classical Einstein field equation, and $\langle T_{\mu\nu} \rangle$, the quantum expectation value of the stress-energy-momentum tensor, also includes matter sources which are designated as classical fields.

The study of how the zero-point energy interacts with the gravitational field via (1.1) receives considerable attention since this theory's inception, and serves to challenge certain preconceived notions about space-time. Probably the most noteworthy of them all is the claim that conformal invariance is broken for the case of massless quantum fields, manifested by a non-zero trace in the quantum stress-energy tensor. Since the trace normally vanishes for massless classical fields, there exists what is known as a "trace" or "conformal" anomaly within the quantum field theory.

Probably the most celebrated single discovery in studying quantum field theory in curved space-time is the prediction by Hawking [26] that thermal radiation is emitted from a Schwarzschild black hole whose temperature is inversely proportional to its mass. This work not only solidifies a long-suspected connection between black holes and thermodynamics [5], it is widely accepted as a fundamental benchmark for confirmation of a successful quantum theory of gravity on theoretical grounds. Unfortunately, there is currently no observational evidence to support the existence of Hawking radiation.

Much more promising in terms of experimental confirmation is the prediction by Unruh [43, 48] that a uniformly accelerated particle detector through a Minkowski vacuum detects a thermal bath of photon radiation, whose temperature is proportional to the detector's acceleration. This is known as the *Unruh effect.* It is physically distinct from Hawking radiation because it concerns radiation due to the detector's acceleration through *flat* space-time, while Hawking radiation comes about due to black hole formation, which necessarily involves a strong gravitational field. However, there is a close relationship [48] between the two effects. This is because a continually accelerating observer asymptotically approaches the speed of light, which implies [5] that the corresponding plane of null rays acts like an event horizon, in the sense that there are events above it which are causally disconnected from the observer.

1.2 Moving Mirrors

The examples listed above suggest a wide number of avenues available for studying quantum field theory in curved space-time. In general, they involve considering how a quantum field couples to the classical gravitational background under certain configurations. An important field configuration is one which includes the presence of a potential barrier, whose strength may be very large relative to the propagating field. If this barrier is sharply localized in one spatial dimension, then it can be called a *boundary*. When the potential barrier becomes infinitesimally sharp and its magnitude grows infinitely large, the boundary becomes a totally reflecting surface, commonly known as a *mirror*.

A research area undertaken by many people is on the interaction between a quantum vacuum and a moving mirror. It is known that a stationary mirror in flat space-time creates a disturbance in the vacuum that forces the field modes to be constrained on its surface. The most noteworthy example is of two neutral parallel plate mirrors that feel an attractive force proportional to the inverse fourth power of their separation distance. This is the famous *Casimir effect*, which has been the subject of considerable study on how the vacuum energy is affected by non-trivial topologies. When considering moving mirrors, however, it is suggested that the time-dependent boundary conditions induce the emission of particles from the mirror surface in the form of quantum radiation. Considered originally by Moore [33] for the case of electromagnetic radiation, moving mirrors in quantum vacua described by quantum fields result in the *dynamical Casimir effect*, which forms the foundation behind this thesis work, and is discussed in much greater detail later.

1.3 Thesis Research

The purpose of this thesis is to examine the quantum radiation due to a massless scalar field emitted from polarizable bodies and moving mirrors in a number of situations. This basically entails finding the quantum vacuum expectation values for zero-point fluctuations of the field, as well as the quantum stress energy tensor. The motivation for this problem comes from a number of intriguing applications where processes involving mirrors are considered. In addition to using the (non-relativistic) dynamical Casimir effect to study sonoluminescence [39, 40], there are a number of open questions which require studying the dynamical Casimir effect in the relativistic domain. In particular, mirrors are intensively used in different gedanken experiments with black holes. The aim of these 'experiments' is to better understand the origin of quantum radiation from black holes, and the nature of different quantum states near black holes. More recently, systems requiring mirrors are used for studying the physical nature of generalized black hole physics [44, 45, 46] and the microscopic origin of black hole entropy [34]. In principle, quantum effects in systems with relativistic boundaries may have important cosmological applications. One example involves considering the creation and expansion of bubbles of a new phase during cosmological phase transitions. These applications are not considered here. Instead, attention is given to solving concrete problems with moving mirrors which allow for a complete analysis. The reason for this is that the results previously obtained for problems with four-dimensional relativistic mirrors are quite restricted, which is one of the problems when mirror modeling is used in the recent gedanken experiments.

Chapter 2 contains a concise account of the background material required to formulate the problems considered. This is followed, in Chapter 3, by a study of the quantum radiation due to one and two spherical semi-transparent expanding mirrors. Chapter 4 then analyzes the quantum radiation of an accelerated refractive body, and is followed in Chapter 5 by a similar study of an accelerated spherical mirror for comparison. The essential details of Chapters 3–5 are to find the field and stress-energy tensor vacuum expectation values, and especially describe the radiation fluxes emitted at distances very far away from the original disturbance due to the boundary conditions. A conclusion which draws together the results and points towards potential future developments from this thesis is then given in Chapter 6.

The space-time signature chosen for this thesis is $+(d-1)$, where d is the number of spatial dimensions, and it is assumed that $c = 1$ for all subsequent computations.

Chapter 2

Background Material

This chapter is intended to introduce the prerequisites for understanding how moving mirrors under various configurations and states of motion affect the vacuum energy associated with a quantum field. To motivate the general problem, a knowledge about the properties of *zero-point fluctuations* in quantum field theory is required. The tools to then describe these field fluctuations involve knowing how to quantize the scalar field, and how to develop the formalism of Green's functions to describe correlations between fields at different space-time events. This is contained in Section 2.1. Part of the analysis also requires knowing how quantum radiation appears at far distances from the source of emission. In particular, this is considered for polarized sources that are accelerated through space. These topics are contained in Section 2.2.

One of the conclusions determined in the formulation of quantum field theory in general space-times is that two observers may disagree about what constitutes a vacuum state. That is, one observer may perceive a vacuum described by a particular state, while another may detect particles according to the same state.

The inherent ambiguity of a particle concept which results from this conclusion renders it necessary to understand the notion of particle detection. Section 2.3 gives a description about Unruh detectors while under accelerated motion. Finally, Section 2.4 follows with a more detailed description of the physics behind the dynamical Casimir effect.

2.1 Essential Principles

One of the most significant concepts to emerge from quantum mechanics is *zero-point fluctuations.* That is, for a harmonic oscillator system with frequency ω in its ground state, there is a non-zero energy $E_0 = \hbar\omega/2$ that cannot be removed [35]. This has important implications in finding a sensible description of a vacuum, since the common-sense notion that a vacuum is absolute *nothingness* [1] is inconsistent with the previous statement.

Indeed, quantum field theory in flat space-time can be decomposed into modes. In this decomposition, the mode amplitudes are equivalent to harmonic oscillators. As is known from quantum mechanics, the creation and annihilation operators for an oscillator act on the given state by raising or lowering it by an integer, which corresponds to the creation or destruction of a quantum of excitation. Such a quantum is then interpreted as a *particle.* For example, for the electromagnetic field, [1] the elementary quantum of excitation is the photon. A vacuum in quantum field theory is, therefore, defined as a ground state in which none of the fields contain any excitation quanta.

However, this does not imply that the vacuum state has zero energy, since each vacuum state contributes a residual zero-point energy [5] of $\hbar\omega/2$. In fact, for a

Hamiltonian operator for a scalar field described by

$$\begin{aligned} H &= \frac{\hbar}{2}\sum_k \left(a_k^\dagger a_k + a_k a_k^\dagger\right)\omega_k \\ &= \sum_k \left(a_k^\dagger a_k + \frac{1}{2}\right)\hbar\omega_k, \end{aligned} \tag{2.1}$$

where $a_k^\dagger$ and a_k are the respective creation and annihilation operators, the Hamiltonian expectation value for a vacuum state $|0\rangle$ defined in terms of (2.1) becomes infinitely large, since

$$\langle 0|\, H \,|0\rangle = \frac{\hbar}{2}\sum_k \omega_k \to \infty. \tag{2.2}$$

In flat space-time, it is possible to remove the infinite zero-point energy by "normal-ordering" the Hamiltonian, whereby the annihilation operators appear to the right of the creation ones. Since energy measurements are only made in relative terms with respect to some arbitrary reference point, this procedure is perfectly reasonable. It follows that the zero-point energy density (on the average) is uniform over space and can be ignored. When considering a curved space-time scenario, however, this situation becomes much more complicated. This is because the zero-point energy makes a contribution to the stress-energy tensor [5] for the Einstein field equations (1.1), so it is necessary to first determine how to isolate the divergences in the stress-energy tensor. For this thesis, which involves flat space-time situations with non-trivial boundary conditions, the technical details are much simpler. This topic is discussed in greater detail later. For the moment, it is necessary to first discuss quantization of the scalar field.

2.1.1 Scalar Field Quantization

It is convenient to begin with the canonical quantization of a massless scalar field. For the sake of completeness, a brief overview is given for the case of a massive field in D-dimensional curved space-time. Consideration of the special case of flat space-time is given for specific examples in later chapters.

Starting [5] with an action S defined in terms of a Lagrangian density $\mathcal{L}(x)$

$$S = \int \mathcal{L}(x)\, d^D x, \tag{2.3}$$

the most standard form of the Lagrangian density $\mathcal{L}(x)$ for a free scalar field $\varphi(x)$ with mass m in the external field is

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g(x)}\left[g^{\mu\nu}(x)\nabla_\mu\varphi(x)\nabla_\nu\varphi(x) + \left(m^2 + \xi\, R(x)\right)\varphi^2(x)\right]. \tag{2.4}$$

Here, $g^{\mu\nu}(x)$ is the contravariant metric tensor, ∇_ν is the covariant derivative operator, and $g(x)$ is the determinant of $g_{\mu\nu}(x)$. In addition to the mass coupling with $\varphi^2(x)$, there is a first-order coupling with the Ricci curvature scalar $R(x)$. The constant scalar ξ often takes on the values $\xi = 0$ for *minimal coupling* and $\xi = (D-2)/(4(D-1))$ for *conformal coupling.* It is possible to add higher-order curvature invariants constructed from the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ [19], but (2.4) is the lowest-order form which involves gravitation.

Evaluating the Euler-Lagrange equations with respect to $\varphi(x)$ results in a generalized Klein-Gordon equation

$$\left[\Box - m^2 - \xi\, R(x)\right]\varphi(x) = 0, \tag{2.5}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. In defining a canonical quantization approach [5, 22], the space-time manifold must allow for the foliation of space-time into hypersurfaces Σ with dimension $(D-1)$. The remaining dimension, denoted by co-ordinate x_0

which acts as the time parameter, identifies each Σ by $x_0 = \text{constant}$ [22]. Then the normal derivative defined with respect to Σ is $n^\mu \nabla_\mu$, where n^μ is a future-directed normal vector [5] in the direction of x_0.

It is possible to define an inner product [5, 12] of two solutions $\varphi_1(x)$ and $\varphi_2(x)$ by

$$\begin{aligned}(\varphi_1, \varphi_2) &= -i \int_\Sigma [\varphi_1(x) n^\mu \nabla_\mu \varphi_2^*(x) - (n^\mu \nabla_\mu \varphi_1(x))\, \varphi_2^*(x)]\, \sqrt{-g_\Sigma(x)}\, d\Sigma \\ &= -i \int_\Sigma n^\mu \varphi_1(x) \overleftrightarrow{\nabla}_\mu\, \varphi_2^*(x) \sqrt{-g_\Sigma(x)}\, d\Sigma \\ &= -n^\mu \int_\Sigma J_\mu \sqrt{-g_\Sigma(x)}\, d\Sigma. \end{aligned} \tag{2.6}$$

where $*$ denotes complex conjugation and $g_\Sigma(x)$ is the determinant for the $(D-1)$-dimensional induced metric on the hypersurface. The special property of (2.6) is that it is independent of the choice of hypersurface Σ which can be proven using Gauss' law [12], and so integrand $J_\mu = i\,\varphi_1(x) \overleftrightarrow{\nabla}_\mu\, \varphi_2^*(x)$ is then a conserved current [22].

The basic postulate of canonical field quantization is to consider $\varphi(x)$ as an operator which satisfies the *equal time commutation relations*

$$\left[\varphi(x^0, \boldsymbol{x}), \varphi(x^0, \boldsymbol{x}')\right] = 0, \tag{2.7}$$

$$\left[\pi(x^0, \boldsymbol{x}), \pi(x^0, \boldsymbol{x}')\right] = 0, \tag{2.8}$$

$$\left[\varphi(x^0, \boldsymbol{x}), \pi(x^0, \boldsymbol{x}')\right] = i\,\delta^{D-1}(\boldsymbol{x} - \boldsymbol{x}'), \tag{2.9}$$

where $\boldsymbol{x}, \boldsymbol{x}'$ are points on Σ, and $\pi(x^0, \boldsymbol{x}) = \delta\mathcal{L}/\delta\dot{\varphi} = \sqrt{-g(x)} g^{\mu 0}(x) \nabla_\mu \varphi(x^0, \boldsymbol{x})$ is the canonical momentum conjugate to $\varphi(x^0, \boldsymbol{x})$, where $\dot{\varphi}(x^0, \boldsymbol{x}) \equiv \partial\varphi(x^0, \boldsymbol{x})/\partial x^0$.

The solution to (2.5) can be expanded in terms of a set of modes $u_k(x)$ in the form

$$\varphi(x) = \sum_k \left[a_k u_k(x) + a_k^\dagger u_k^*(x)\right], \tag{2.10}$$

where the modes satisfy normalization conditions

$$(u_k, u_{k'}) = \delta_{kk'}, \qquad (u_k^*, u_{k'}^*) = -\delta_{kk'}, \qquad (u_k, u_{k'}^*) = 0. \tag{2.11}$$

It then follows that $a_k, a_k^\dagger$ satisfy the commutation relations

$$[a_k, a_{k'}] = 0, \qquad \left[a_k^\dagger, a_{k'}^\dagger\right] = 0, \qquad \left[a_k, a_{k'}^\dagger\right] = \delta_{kk'}. \tag{2.12}$$

The creation and annihilation operators with label k are defined with respect to some Fock basis, and satisfy the conditions

$$a_k |0\rangle = 0, \tag{2.13}$$

$$a_k |n_k\rangle = \sqrt{n}\,|(n-1)_k\rangle, \tag{2.14}$$

$$a_k^\dagger |n_k\rangle = \sqrt{n+1}\,|(n+1)_k\rangle. \tag{2.15}$$

2.1.2 Zero-Point Fluctuations and Vacuum Polarization

It is mentioned that the zero-point energy density in flat space-time does not vary with location. Although this appears to be true on a statistical average, it is also true that there exist fluctuations in the vacuum which have significant measurable effects. The emergence of fluctuations in the vacuum can be understood in the following way. According to Heisenberg's energy-time uncertainty relation [1], it is known that

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \tag{2.16}$$

where ΔE and Δt are the uncertainties in the energy and time measurements, respectively. Although it does not hold the same status as the more familiar position-momentum uncertainty relation, within a relativistic field theory context (2.16) is perfectly acceptable, since both space and time co-ordinates can be treated as parameters [1]. For short enough time scales, this corresponds to large energy changes, where

$$\Delta t \sim 10^{-21} s, \qquad \Delta E \sim 1 \text{ MeV}, \tag{2.17}$$

$$\Delta t \sim 10^{-24} s, \qquad \Delta E \sim 1000 \text{ MeV}. \tag{2.18}$$

Clearly, this suggests that particle-antiparticle pairs can spontaneously emerge and annihilate with a lifetime in accord with (2.16), and that a non-localization effect in space-time exists for space-time due to this uncertainty relation.

It is generally not known to what extent gravitation affects the nature of zero-point fluctuations [32], particularly when considering uncertainties on length scales comparable to the Planck length $l_P \sim 10^{-35}$ m. Since general relativity is not renormalizable by the standard techniques of field quantization, a precise understanding of the problem is not solvable until a successful theory of quantum gravity emerges.

From classical electrodynamics [24, 30], it is an experimental fact that, when some dielectric object is placed in an external electric field, the bound charges induce an electric field to counteract it. This implies that the dielectric has a net polarization $\boldsymbol{P} \sim \chi_e \boldsymbol{E}$, where χ_e is the *electric suspectibility* of the medium. It follows that Gauss' law in differential form is modified to become

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 4\pi\rho, \tag{2.19}$$

where $\boldsymbol{D} = \epsilon \boldsymbol{E}$ is the dielectric displacement vector, $\epsilon = (1 + 4\pi\chi_e)\epsilon_0$, and ϵ_0 is the permittivity in vacuum. It is evident from (2.19) that the polarization has the effect of screening the otherwise "bare" charge density.

In a similar way, the vacuum also becomes polarized [1] due to the zero-point fluctuations. For example, a charged particle in the vacuum becomes exposed to virtual charged particles surrounding it, which serve to screen its "bare" or true charge, effectively reducing its strength. The lifetime of such particles of mass m is then on the order of $\Delta t \sim \hbar/mc^2$. Clearly, it follows that particles with small m serve to screen the charge more effectively than ones with large m, so electron-positron pair contributions dominate over muon pairs in the overall vacuum polarization, since the latter are 200 times as massive. In quantum field theory, it is the "screened" charge which is relevant to measurement, since vacuum polarization effects prevent the "bare" charge from ever being detected.

2.1.3 Green's Functions

Green's functions play an important technical role in quantum physics, and this formalism is used extensively in this thesis. It is, therefore, instructive to give a basic overview of this topic and how it relates the central issues of this thesis in studying the moving mirror problem.

It is known that, in mathematical physics, Green's functions provide a very powerful technique for solving inhomogeneous ordinary and partial differential equations, subject to either initial or boundary conditions.

To illustrate, consider in schematic form the differential equation [3]

$$L_x \psi(x) = f(x), \tag{2.20}$$

where L_x is a linear differential operator and $f(x)$ is an arbitrary function. Then the solution $\psi(x)$ can be written as the sum of a complementary function and a particular function

$$\psi(x) = \psi_c(x) + \psi_p(x), \tag{2.21}$$

where $\psi_c(x)$ satisfies the homogeneous equation $L_x \psi_c(x) = 0$, and

$$\psi_p(x) = -\int G(x, x') f(x') dx' \tag{2.22}$$

for some two-point function $G(x, x')$. If it can be shown that

$$L_x G(x, x') = -\delta(x - x'), \tag{2.23}$$

then $G(x, x')$ is a *Green's function.* This method of solving differential equations is very powerful because it provides a systematic approach towards solving inhomogeneous equations for suitably behaved functions $f(x)$, so long as the Green's function and the associated integral in (2.22) exist.

For this thesis, the problems under consideration involve the wave propagation of massless scalar fields through space in the presence of a source generating the field disturbance. In this case, the operator L_x can be identified with the d'Alembertian $\Box_x = \eta^{\mu\nu}\partial_\mu\partial_\nu$ for wave motion in Minkowski space-time, and $f(x)$ with some current source $J(x)$. Then the corresponding equation (2.20) for the massless field $\varphi(x)$ is

$$\Box_x \varphi(x) = J(x), \tag{2.24}$$

where the corresponding condition for (2.23) in D-dimensional space-time is

$$\Box_x G(x, x') = -\delta^D(x - x'). \tag{2.25}$$

Then the solution to (2.24) can be written as

$$\varphi(x) = \varphi_0(x) - \int G(x, x') J(x') d^D x', \tag{2.26}$$

where $\varphi_0(x)$ is a solution for the homogeneous equation $\Box_x \varphi_0(x) = 0$.

There is an immediate physical interpretation which comes forth from (2.26). Physical fields propagate inside null cones, and in order to preserve causality, it is assumed that this property occurs in future-directed null cones. For this reason, $G(x, x')$ must be chosen as the *retarded* or *causal Green's function* [30]. It should be noted that there is also a corresponding *advanced Green's function* by symmetry where $x'^0 > x^0$, which is usually discarded in classical problems because it violates causality.

The Green's functions and other objects which possess a similar singular structure play an important role in quantum field theory. For quantum scalar fields defined according to (2.7)–(2.9), the simplest form of Green's functions are the *Wightman functions*

$$\begin{aligned} G^+(x,x') &= G^-(x',x) \equiv \langle 0|\, \varphi(x)\, \varphi(x')\, |0\rangle \\ &= \sum_k u_k(x)\, u_k^*(x') \end{aligned} \tag{2.27}$$

when evaluated using (2.10) and (2.12). The $+(-)$ signifies that (2.27) is a positive (negative) frequency Wightman function.

It is from (2.27) that more complex forms of Green's functions can be constructed. For example, the *Pauli-Jordan* or *Schwinger function*, defined as the average of a field commutator relation, is

$$i\, G_0(x,x') \equiv G^+(x,x') - G^-(x,x') = \langle 0|\, [\varphi(x), \varphi(x')]\, |0\rangle\,, \tag{2.28}$$

while the corresponding average of field anticommutator relations, called *Hadamard's elementary function*, is

$$G^{(1)}(x,x') \equiv G^+(x,x') + G^-(x,x') = \langle 0|\, \{\varphi(x), \varphi(x')\}\, |0\rangle\,. \tag{2.29}$$

Strictly speaking, though (2.27)–(2.29) are called Green's functions, they only satisfy the homogeneous equation

$$\Box_x \mathcal{G}(x,x') \;=\; 0. \tag{2.30}$$

The retarded and advanced Green's functions introduced earlier are defined as

$$G_R(x,x') \;\equiv\; -\theta(x^0 - x'^0)\, G_0(x,x') \tag{2.31}$$

$$G_A(x,x') \;\equiv\; \theta(x'^0 - x^0)\, G_0(x,x'), \tag{2.32}$$

where the *Heaviside step function* θ is

$$\theta(x^0) \;=\; \begin{cases} 1 & x^0 \;>\; 0 \\ 0 & x^0 \;<\; 0. \end{cases} \tag{2.33}$$

They, and their average

$$\bar{G}(x,x') \;\equiv\; \frac{1}{2}\left[G_R(x,x') + G_A(x,x')\right], \tag{2.34}$$

satisfy (2.25).

A particularly useful definition of vacuum correlation functions is the vacuum average of the time-ordered product of fields called the *Feynman Green's function* or *Feynman propagator*

$$\begin{aligned} i\, G_F(x,x') \;&\equiv\; \langle 0 |\, T\left(\varphi(x)\,\varphi(x')\right) |0\rangle \\ &=\; \theta(x^0 - x'^0)\, G^+(x,x') + \theta(x'^0 - x^0)\, G^-(x,x'), \end{aligned} \tag{2.35}$$

with the property that it takes on the same value when x and x' are interchanged, and also that it satisfies

$$\Box_x\, G_F(x,x') \;=\; \delta^D(x - x'). \tag{2.36}$$

This Green's function propagates positive frequencies to the future and negative frequencies to the past.

For quantum field theory in flat space-time (and more generally in a static space-time), it is often convenient to start with a so-called *Euclidean* formulation. By making a *Wick rotation* $x^0 \to i\,\tau$, the space becomes described by a Euclidean metric. This has the effect of converting the d'Alembertian $\Box_x$ into a positive-definite Euclidean form $\Box_E$, where $\eta_{\mu\nu} \to \delta_{\mu\nu}$. For many cases, it is possible to perform calculations first in the Euclidean formulation and then return the result to Minkowski space-time. In these calculations, the *Euclidean Green's function* becomes important, since its analytic continuation to Minkowski space-time coincides with the Feynman propagator.

In addition, there are boundary conditions to consider, which serve to restrict the set of solutions of interest. To discuss this, consider the Green's identity

$$\int [\phi(x')\,\Box'\psi(x') - \psi(x')\,\Box'\,\phi(x')]\,\sqrt{-g(x')}\,d^Dx'$$

$$= \int_\Sigma \left[\phi(x')\,\nabla'_\mu\psi(x') - \psi(x')\,\nabla'_\mu\,\phi(x')\right]\sqrt{-g_\Sigma(x')}\,d\Sigma^\mu. \qquad (2.37)$$

Letting $\phi(x') = \varphi(x')$ and $\psi(x') = G(x,x')$, it is shown that

$$\varphi(x') \;=\; -\int G(x,x')\,J(x')\,\sqrt{-g(x')}\,d^Dx'$$

$$+\int_\Sigma \left[G(x,x')\,\nabla'_\mu\varphi(x') - \varphi(x')\,\nabla'_\mu\,G(x,x')\right]\sqrt{-g_\Sigma(x')}\,d\Sigma^\mu. \,(2.38)$$

Quite often, two types of boundary conditions are considered. The first type is a *Dirichlet boundary condition*, where $G_D(x,x')|_{x\in\Sigma} = G_D(x,x')|_{x'\in\Sigma} = 0$. Then the first integral on the second line of (2.38) vanishes. The second type is the *Neumann boundary condition*, for which the normal derivative $n^\mu\nabla_\mu G_N(x,x')|_{x'\in\Sigma} =$

$n^\mu \nabla'_\mu G_N(x, x')\Big|_{x\in\Sigma} = 0$, where the prime denotes differentiation on x'.

2.2 Null Asymptotic Infinities and the Wave Zone Region

Often, it is important to understand the radiation falloff rate at far distances from the localized charge distributions causing the radiation emission. It is possible to study this for the case of electromagnetism by performing a multipole expansion [47] of the field. For the radiation emitted by a set of accelerating charges, the field's asymptotic nature is then described by an infinite set of multipole coefficients.

This is one way of considering the problem. There is, however, another way to accomplish this goal, which involves a transformation of the space-time itself. This is done by introducing a metric *conformal transformation* in the form

$$\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}, \tag{2.39}$$

where the conformal factor $\Omega^2(x)$ relates the physical metric $g_{\mu\nu}(x)$ to a non-physical one $\tilde{g}_{\mu\nu}(x)$ by stretching or shrinking the space-time. This effectively adjusts the length scale between respective space-time points. By choosing an appropriate conformal factor such that the length scale increases with increased distance from the source, it is possible to describe infinity as a *finite* location on $\tilde{g}_{\mu\nu}$.

To demonstrate this [47], consider Minkowski space-time in spherical co-ordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2.40}$$

where the co-ordinates hold their usual meaning, and introduce advanced and retarded null co-ordinates

$$v = t + r, \qquad u = t - r. \tag{2.41}$$

The new metric then becomes

$$ds^2 = -du\,dv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{2.42}$$

The objective here is to consider the *wave zone* region where u, θ, and φ are fixed while letting $v \to \infty$. Here, it is possible to apply a conformal transformation to (2.42) by letting

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)\,\eta_{\mu\nu}, \qquad \Omega^2 = 4(1+v^2)^{-1}(1+u^2)^{-1}. \tag{2.43}$$

By then defining a new set of co-ordinates $\tilde{T}, \tilde{R}$ as

$$\tilde{T} = \tan^{-1} v + \tan^{-1} u, \tag{2.44}$$

$$\tilde{R} = \tan^{-1} v - \tan^{-1} u, \tag{2.45}$$

the new co-ordinates have ranges

$$-\pi \leq \tilde{T} + \tilde{R} \leq \pi, \tag{2.46}$$

$$-\pi \leq \tilde{T} - \tilde{R} \leq \pi, \qquad 0 \leq R, \tag{2.47}$$

and the conformally-transformed metric in terms of these co-ordinates becomes

$$d\tilde{s}^2 = -d\tilde{T}^2 + d\tilde{R}^2 + \sin^2\tilde{R}\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{2.48}$$

This result (2.48) is precisely the metric for the Einstein static universe with the co-ordinate restrictions (2.46)–(2.47). It now becomes possible to identify different regions of infinity according to the boundaries defined by (2.48) in the following

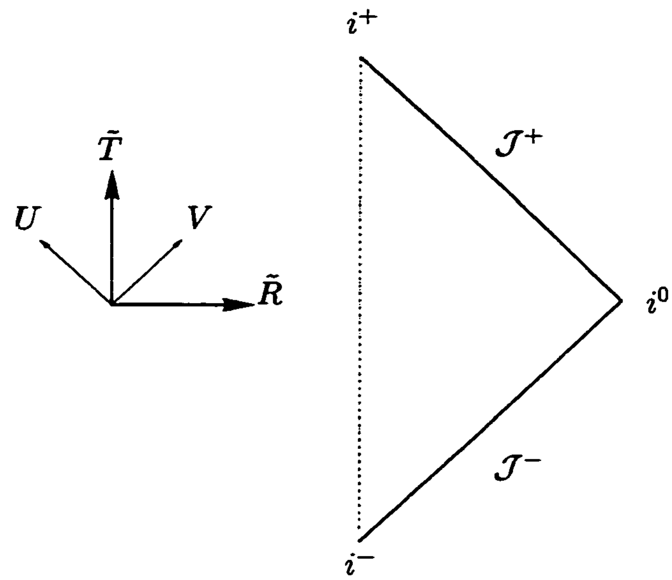

Figure 2.1: Conformal diagram for Minkowski space-time in two-dimensional representation (θ and φ suppressed).

way. First, there is the *past time-like infinity* i^- at the bottom described by $\tilde{R} = 0, \tilde{T} = -\pi$, along with the *future time-like infinity* i^+ at the bottom described by $\tilde{R} = 0, \tilde{T} = \pi$, where all the time-like geodesics run from i^- to i^+. Second, there is a *spatial infinity* i^0 at $\tilde{R} = \pi, \tilde{T} = 0$, where all the spacelike geodesics meet. Third, the outer edges of the manifold are described by three-dimensional null surfaces called the *past null infinity* $\mathscr{I}^-$ and *future null infinity* $\mathscr{I}^+$. These are described by $\tilde{T} = -\pi + \tilde{R}$ and $\tilde{T} = \pi - \tilde{R}$, respectively for $0 < \tilde{R} < \pi$, where the null geodesics follow 45° lines from $\mathscr{I}^-$ to $\mathscr{I}^+$.

It now becomes clear from (2.48) that, by a suitable choice of conformal factor and making the final co-ordinate change

$$V = \tilde{T} + \tilde{R}, \qquad U = \tilde{T} - \tilde{R}, \tag{2.49}$$

that the wave zone region for fixed U has a well-defined boundary on $\mathscr{I}^+$, and allows for a precise formulation of asymptotic behaviour of physical fields by performing a conformal transformation leading to the metric $d\tilde{s}^2$. To illustrate, it can be shown for the Klein-Gordon equation (2.5) with ξ defined for conformal coupling that, under a conformal transformation [5],

$$\left[\Box - m^2 - \{(D-2)/4(D-1)\}\, R(x)\right] \varphi(x)$$

$$\to \left[\tilde{\Box} - m^2 - \{(D-2)/4(D-1)\}\, \tilde{R}(x)\right] \tilde{\varphi}(x)$$

$$= \Omega^{-(D+2)/2}(x) \left[\Box - m^2 - \{(D-2)/4(D-1)\}\, R(x)\right] \varphi(x), \quad (2.50)$$

where

$$\tilde{\varphi}(x) \equiv \Omega^{(2-D)/2}(x)\, \varphi(x). \quad (2.51)$$

Classical and quantum scattering problems for massless fields can be described using this approach. Consider the field $\tilde{\varphi}$ in the conformal space, $\mathcal{J}^-$ is a characteristic (null) surface in this space. For given initial data on this surface, which are given by $\Phi^-(V, \omega_i)$, where ω_i are angle variables on a unit sphere, it is possible to solve the evolution equation and determine the field $\tilde{\varphi}$ everywhere in space-time. This includes its future null infinity boundary $\mathcal{J}^+$, where the field $\tilde{\varphi}$ takes the value $\Phi^+(U, \omega_i)$. Finding $\Phi^+(U, \omega_i)$ for given $\Phi^-(V, \omega_i)$ can be considered as a formulation of the classical scattering problem.

From (2.51), it follows that the 'physical' fields near $\mathcal{J}^\pm$ behave as $\varphi \sim \Phi^\pm / r^{(D-2)/2}$. This is the standard behaviour of a radiated field in the wave zone. Studying the characteristics of radiated fields in the wave zone can then be performed in two equivalent ways: either by direct series expansion of objects in the inverse powers of the radial distance r along future directed null rays, or in terms of special limits of these objects on $\mathcal{J}^+$. These approaches will not be distinguished further. In the following chapters, the objects of interest (that is, energy density fluxes and fluctuations) are calculated directly in physical space, but often use the notion of $\mathcal{J}^{\pm}$ null infinities.

2.3 Accelerated Unruh-DeWitt Detectors

In the thesis, the main discussion is on local observables without considering the particle content of the theory. In other words, we shall analyze how much energy is emitted during some interval of time, rather than the number of particles observed in this radiation. These two problems, being closely related nevertheless require different types of calculations. To calculate particle production requires a knowledge of Bogoliubov coefficients, while flux calculations are obtainable from Green's functions. In principle, by knowing a specially chosen Green's function (Wightman positive frequency function), it is possible to also answer questions about the particle aspect of the radiation. To illustrate this, a brief discussion of the Unruh effect is in order.

It is generally accepted that matter field quantization in a generic curved space-time background does not admit a meaningful particle interpretation. To understand why first requires a brief analysis of particles in Minkowski space-time [5]. For an inertial observer, the associated vacuum state is invariant under the Poincaré group of transformations, which also leave invariant the positive frequency field modes corresponding to it. Therefore, an agreed-upon notion of a vacuum state appears, and so observation of a particle by one detector can be conveyed unambiguously to all other detectors related by Lorentz transformations.

For detectors in curved space-time [5], however, the Poincaré symmetry is not available, though the space-time may have enough symmetry to allow for a definition of "in" and "out" regions, where the field interactions tend to zero. These regions can then be defined to have respective vacuum states, each described in terms of *distinct* sets of field modes. If the space-time is not stationary, the definitions of positive and negative frequencies in the past and future regions are not equivalent. Therefore, if a particle detector is constructed so that it does not register particles in the "in" region's vacuum state, it will detect quanta in the "out" region. It appears that particle detection is observer-dependent, and the detector can be excited even in the Minkowski vacuum in flat space-time, but only if it does not move inertially.

To illustrate this, consider a model for particle detection proposed by Unruh [43] and DeWitt [10]. It detects a massless scalar field φ. The interaction Lagrangian is chosen

$$L_{\text{int}} = m(\tau)\varphi(x(\tau)), \tag{2.52}$$

where $m(\tau)$, the monopole moment of the detector at proper time τ, is treated as a point object, and $x^\mu(\tau)$ is the world-line of the detector along a Killing trajectory. Let the detector have a discrete set of internal energy levels denoted by eigenstates $|E\rangle$, where the ground state corresponds to $E = 0$. With respect to the world-line, it is possible to describe $m(\tau)$ in terms of the Heisenberg equation or motion

$$m(\tau) = e^{iH\tau}\, m(0)\, e^{-iH\tau}, \tag{2.53}$$

where $H\,|E\rangle = E\,|E\rangle$. Then the matrix elements corresponding to a particle detection is given by

$$\langle E|\, m(\tau)\, |E'\rangle = e^{i(E-E')\tau}\, \langle E|\, m(0)\, |E'\rangle\,. \tag{2.54}$$

Consider the detector when initially prepared in the ground state, and the scalar field is in state ψ. Then, from (2.54), the transition amplitude from $|0, \psi\rangle$

to $|E, \psi'\rangle$ is

$$A(E, \psi'|0, \psi) \;=\; \langle E, \psi'| T \left(\exp \left(i \int_{-\infty}^{\infty} L_{\text{int}}\, d\tau \right) \right) |0, \psi\rangle \,, \tag{2.55}$$

where T is again the time-ordering operator. This expression (2.55) can then be treated as a first-order perturbation if the monopole moment is small enough that radiative corrections to L_{int} are negligible. Therefore,

$$\begin{aligned} A(E, \psi'|0, \psi) \;&\approx\; i\,\langle E, \psi'| \int_{-\infty}^{\infty} m(\tau)\varphi(x(\tau))\, d\tau\, |0, \psi\rangle \\ &=\; i\,\langle E|\, m(0)\, |0\rangle \int_{-\infty}^{\infty} e^{iE\tau}\, \langle \psi'|\, \varphi(x(\tau))\, |\psi\rangle\, d\tau, \end{aligned} \tag{2.56}$$

and the total probability of the detector reaching the excited state E is

$$\begin{aligned} P(E) \;&=\; \sum_{\psi'} |A(E, \psi'|0, \psi)|^2 \\ &=\; |\,\langle E|\, m(0)\, |0\rangle\, |^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau'\, e^{-iE(\tau-\tau')}\, \langle \psi|\, \varphi(x(\tau))\, \varphi(x(\tau'))\, |\psi\rangle \,. \end{aligned} \tag{2.57}$$

From (2.57), the double integral in the total probability is then known as the *response function*

$$\mathcal{F}(E) \;=\; \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau'\, e^{-iE(\tau-\tau')} G^{+} \left(\varphi(x(\tau))\, \varphi(x(\tau')) \right), \tag{2.58}$$

where the positive frequency Wightman function [5] is the correlation function that senses the "particle bath" once the detector is in motion. The matrix element in (2.57) indicates the *selectivity* of transition, which is determined by the internal properties of the detector.

2.4 Dynamical Casimir Effect

After this brief introduction, it is possible to discuss the essential ideas concerning this thesis, that of moving mirrors in flat space-time. It is, however, instructive to first briefly discuss some basic principles about stationary mirrors in a quantum vacuum, and then discuss the dynamical case.

2.4.1 Essential Principles

A *mirror* is a potential barrier sharply localized in space that is often (though not always, as shown in the next chapter) treated as a perfectly reflecting surface [17]. When in the presence of a quantum vacuum, the mirror acts like a potential barrier which forces the field modes to become constrained on its surface. This is the essence of the *Casimir effect*, and much literature has been produced on this subject since the first discovery. (For discussion, see [37] and references therein.) Originally developed as a theory to explain van der Waals attraction between atoms, it became re-interpreted as interactions with the zero-point fluctuations of a given field. The resultant Casimir force can be either attractive or repulsive, depending upon the mirror's geometry. For example, the Casimir force is repulsive for the case of a single spherical shell, originally considered [6] to stabilize a classical model of the electron by balancing out the Coulomb force.

Formally, the Casimir energy is a *divergent* quantity because all the field modes contribute in the infinite mode sum. However, the boundary only significantly affects only those modes which occupy a low number density of states, so those in a high number density have to be scaled out to determine the mirror's effect on the energy. In flat space-time, this is possible by subtracting the vacuum energy without the mirror. This process of *renormalization* then renders the Casimir energy finite.

The *dynamical Casimir effect* is a generalization of this phenomenon for a moving mirror system, in which photon radiation is emitted from the mirror's surface [17] while undergoing non-uniform motion in the presence a massless quantum field. The physical explanation [8] is that motion of the mirror induces motion of the surrounding field modes; since the mirror's position is continually changing with time, the field modes must accordingly change frequency, causing excitations [8] in the form of photon radiation for the case of electromagnetic radiation. The actual emission comes from fluctuating currents present on the surface of the mirror [8, 9]. It must be stressed here that the dynamical Casimir effect appears only when the mirror undergoes *accelerated* motion. Clearly, this is because of the equivalence between rest frames and uniformly moving frames due to Lorentz invariance.

Much of the established work on the dynamical Casimir effect involves mirror motion in two-dimensional space-time. This is because quantum field theory for a massless field in such a space-time is conformally flat, and so the computations simplify considerably, since it then becomes possible to map the original problem [17] into an equivalent problem with static boundaries. This results in a very general expression for the radiation flux at far distances which is proportional to the second derivative of the mirror's velocity [20, 12]. For space-times of higher dimensions, however, the situation is less easily solvable, since they often lack the symmetries to simplify the problem.

2.4.2 Vacuum Field Fluctuations, Stress-Energy Tensor, and Renormalization

The main objective in this thesis is to calculate the vacuum stress energy tensor due to moving mirrors in various situations. These objects are formally divergent and need to be renormalized in order to extract any physical meaning from them. Fortunately, there are some well-known methods available to remove these divergences. The one used in this thesis is called *point splitting*, in which the object of interest is first evaluated using a function defined at x, x', and then evaluated in the limit as $x' \to x$. The Green's functions defined above provide a natural means to evaluate these vacuum expectation values. The divergences can then be removed by renormalization before the limit is taken, leaving a finite result. This is reduced to changing $G(x, x')$ by

$$G^{\text{ren}}(x, x') = G(x, x') - G_0(x, x'), \tag{2.59}$$

where $G_0(x, x')$ is a free Green's function.

In flat space-time, $G_0^{(1)}(x, x')$ is a free Green's function, while for more general space-times [5], it can have divergent terms associated with curvature dependences.

The zero-point fluctuation expectation value is then

$$\begin{aligned} \left\langle \varphi^2(x) \right\rangle^{\text{ren}} &\equiv \langle 0|\, \varphi^2(x)\, |0\rangle^{\text{ren}} \\ &= \lim_{x' \to x} G^{\pm}_{\text{ren}}(x, x') = \frac{1}{2} \lim_{x' \to x} G^{(1)}_{\text{ren}}(x, x'). \end{aligned} \tag{2.60}$$

For the vacuum stress-energy tensor, it is formally obtained by first varying the action (2.3) with respect to the metric such that

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

$$
\begin{aligned}
&= (1-2\xi)\,\nabla_\mu\varphi\,\nabla_\nu\varphi + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\, g^{\alpha\beta}\,\nabla_\alpha\varphi\,\nabla_\beta\varphi \\
&\quad - 2\xi\,\varphi\,(\nabla_\mu\nabla_\nu\,\varphi - g_{\mu\nu}\,\Box\,\varphi) + \xi\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right)\varphi^2. \qquad (2.61)
\end{aligned}
$$

When considering conformal coupling for ξ, it follows that $T^\mu{}_\mu = 0$. In terms of the Hadamard function, it is shown that

$$
\begin{aligned}
\langle T_{\mu\nu}(x)\rangle^{\text{ren}} &\equiv \langle 0|\,T_{\mu\nu}(x)\,|0\rangle^{\text{ren}} \\
&= \lim_{x'\to x} D_{\mu\nu'} G^{(1)}_{\text{ren}}(x,x') \qquad (2.62)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{\mu\nu'} &= \left(\frac{1}{2} - \xi\right)\nabla_\mu\,\nabla_{\nu'} + \left(\xi - \frac{1}{4}\right) g_{\mu\nu}\, g^{\alpha\beta'}\,\nabla_\alpha\,\nabla_{\beta'} \\
&\quad - \xi\,\left[\nabla_\mu\nabla_\nu + \nabla_{\mu'}\nabla_{\nu'} - g_{\mu\nu'}\,(\Box + \Box')\right] + \frac{\xi}{2}\left(R_{\mu\nu'} - \frac{1}{2}\,g_{\mu\nu'}R\right).
\end{aligned}
$$

(2.63)

The unprimed operator acts on the first argument and the primed one acts on the second.

Chapter 3

Quantum Radiation from Spherical Semi-Transparent Expanding Mirrors

Most moving mirror problems considered were in two-dimensional space-time to take advantage of certain inherent symmetries that simplify computations. Among the relatively few in four-dimensional space-time is one [15, 16] which considers an expanding spherical mirror system where quantum radiation is emitted off the surface. The approach taken is to solve it as a static boundary value problem in Euclidean space-time by the method of images to obtain the Green's function, then Wick rotate the time co-ordinate to obtain the solution.

This chapter describes a variation on this original problem by considering semi-transparent boundaries instead of a true mirror surface. It has relevance in modelling quantum effects of relativistic bubbles likely to have been generated from cosmological phase transitions. Because the problem can be solved in arbitrary dimensions, the results are presented in D-dimensional space time. The steps in this chapter [17] are to first formulate the problem in Section 3.1, which establishes the underlying principles. In keeping with the previous work on this topic, [15] two models are considered. The first one (Model A) considers the expansion of one spherical mirror, while the second one (Model B) considers two concentric mirrors. In Section 3.2, an outline of the Green's function in D-dimensions is formally given, with specific calculations for both models. This is followed by calculations of the renormalized zero-point fluctuations and stress-energy tensor expectation values in Sections 3.3 and 3.4, respectively. After considering the emission of quantum radiation in Section 3.5, the chapter concludes with a brief discussion in Section 3.6.

3.1 Formulation of the Problem

Consider first a flat space-time metric in Cartesian co-ordinates We write the metric of the flat space-time metric in Cartesian co-ordinates as

$$ds^2 = -\,dT^2 + \sum_{i=1}^{d}(dX_i)^2\,. \tag{3.1}$$

where $d = D-1$ refers to the number of spatial dimensions. By Wick rotating the time $T \to iX_0$, the Euclidean form of (3.1) is

$$ds_E^2 = \sum_{\mu=0}^{d}(dX_\mu)^2\,. \tag{3.2}$$

From (2.25), the Euclidean Green's function in flat space-time then satisfies

$$\Box_E G(x, x') = -\delta^D(x, x'), \tag{3.3}$$

which satisfy the necessary boundary conditions, and where

$$\Box_E = \sum_{\mu=0}^{d}\frac{\partial^2}{\partial X_\mu^2}\,. \tag{3.4}$$

The boundary-free Euclidean Green's function $G_0(x, x')$ in Cartesian co-ordinates, by definition, vanishes at infinity, and so from (3.2) it takes the form

$$G_0(x, x') = \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2}} \frac{1}{|X - X'|^{D-2}}, \tag{3.5}$$

where

$$|X - X'|^2 = \sum_{\mu=0}^{d} (X_\mu - X'_\mu)^2 . \tag{3.6}$$

The zero-point fluctuations expectation value $\langle\hat{\varphi}^2(x)\rangle^{\text{ren}}$ is then (2.60) in terms of the renormalized Hadamard function

$$G^{(1)}_{\text{ren}}(x, x') = G^1(x, x') - 2G_0(x, x') . \tag{3.7}$$

and the stress-energy tensor is (2.62), where (2.63) in flat space-time is

$$\begin{aligned} D_{\mu\nu'} &= \left(\frac{1}{2} - \xi\right) \nabla_\mu \nabla_{\nu'} + \left(\xi - \frac{1}{4}\right) \eta_{\mu\nu'} \eta^{\alpha\beta'} \nabla_\alpha \nabla_{\beta'} \\ &\quad - \frac{1}{2} \xi \left(\nabla_\mu \nabla_\nu + \nabla_{\mu'} \nabla_{\nu'}\right) . \end{aligned} \tag{3.8}$$

The semi-transparent mirror can be introduced by including in (3.3) a potential in the form

$$V_\Sigma = V_0\, \delta^d(\Sigma) , \tag{3.9}$$

where Σ is a time-like d-dimensional hypersurface representing the motion of a $(d-1)$-dimensional mirror surface. The delta function in the potential allows for partial field transmission. When $V_0 \to \infty$, the transmission coefficient vanishes, leaving a perfectly reflecting mirror.

The precise form of (3.9) is dependent upon the configuration of interest. Given that

$$R^2 = \sum_{i=1}^{d} X_i^2 . \tag{3.10}$$

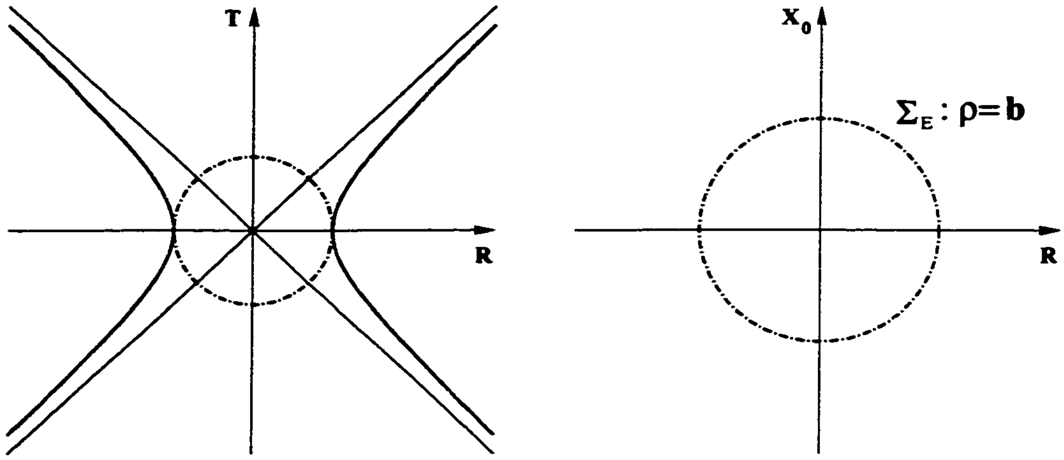

Figure 3.1: The geometry of Model A. Solid lines in the left figure represent motion of spherical semi-transparent mirror Σ in the space-time. A dash-dotted line in the right figure corresponds to the surface Σ_E in the Euclidean space obtained by the Wick's rotation $T \to iX_0$.

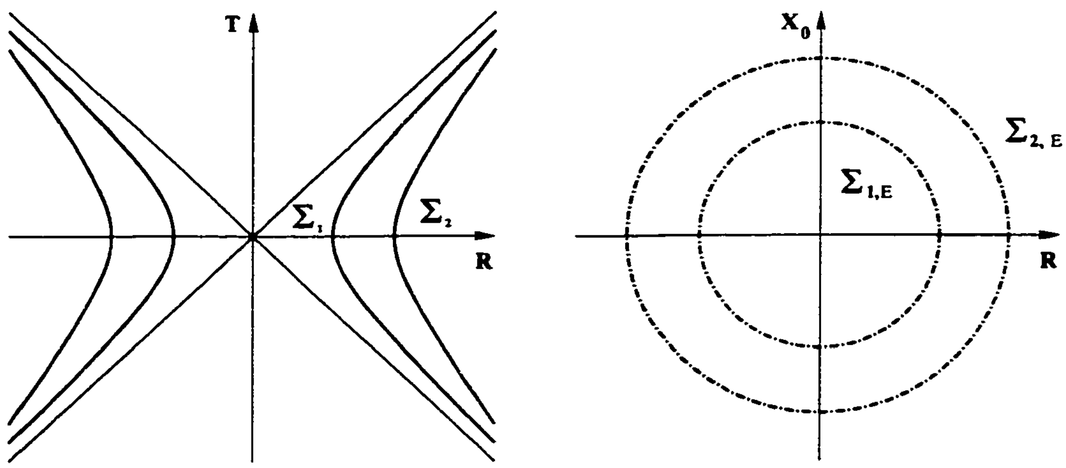

Figure 3.2: The geometry of Model B. Solid lines in the left figure represent motion of spherical semi-transparent mirrors Σ_1 and Σ_2 in the space-time. Dash-dotted lines in the right figure correspond to the surfaces $\Sigma_{1,E}$ and $\Sigma_{2,E}$ in the Euclidean space obtained by the Wick's rotation $T \to iX_0$.

the single mirror motion of Model A described by the surface world-sheet Σ is

$$R^2 - T^2 = b^2, \tag{3.11}$$

while for the concentric mirror system of Model B described by internal and external mirror world-sheets Σ_1 and Σ_2, respectively,

$$R^2 - T^2 = b_1^2, \tag{3.12}$$

$$R^2 - T^2 = b_2^2, \tag{3.13}$$

where $b_1 < b_2$ by assumption. It is evident from (3.10) (3.13) that the mirrors are spherical and are moving with constant acceleration $a_i = b_i^{-1}$ orthogonal to the mirrors' surfaces, since their motion describe hyperboloids in space-time. By a Wick rotation $T \to iX_0$, the Σ_i become $\Sigma_{E,i}$, a d-dimensional sphere S^d of radius b embedded in a D-dimensional Euclidean space. Therefore, the Euclidean forms of (3.11) (3.13) for Σ_E , $\Sigma_{1,E}$, and $\Sigma_{2,E}$, respectively, are

$$R^2 + X_0^2 = b^2, \tag{3.14}$$

$$R^2 + X_0^2 = b_1^2, \tag{3.15}$$

$$R^2 + X_0^2 = b_2^2. \tag{3.16}$$

3.2 Calculation of Green's Functions

3.2.1 Green's Functions in Spherical Co-ordinates

Having re-formulated the problem in terms of Euclidean space-time, the next step is to obtain the D-dimensional Green's function. Since the boundary now describes a d-dimensional sphere, it is appropriate to re-define the problem by parametrizing the line element $d\Omega_d^2$ on a unit d-dimensional sphere. At this point, it should be noted that the description below applies to $D > 2$ only, and the two-dimensional is considered separately as a special case later.

Starting with

$$d\Omega_1^2 = d\vartheta_1^2 , \tag{3.17}$$

$$d\Omega_n^2 = d\vartheta_n^2 + \sin^2\vartheta_n \, d\Omega_{n-1}^2 , \quad (n = 2, \ldots, d) . \tag{3.18}$$

the line element $d\Omega_d^2$ can be written in the form

$$d\Omega_d^2 = \sum_{i,j=1}^{d} \Omega_{ij}^{(d)} \, d\vartheta_i \, d\vartheta_j , \tag{3.19}$$

where

$$\Omega_{ij}^{(d)} = \mathrm{diag}\,(1, \sin^2\vartheta_d(1, \sin^2\vartheta_{d-1}(\ldots))) ,$$

$$\sqrt{\Omega_d} \equiv \sqrt{\det\left(\Omega_{ij}^{(d)}\right)} = \sin^{d-1}\vartheta_d \, \sin^{d-2}\vartheta_{d-1} \ldots \sin\vartheta_2 . \tag{3.20}$$

Then the line element (3.2) can be re-written as

$$ds_E^2 = d\rho^2 + \rho^2 \, d\Omega_d^2 , \tag{3.21}$$

where

$$R = \rho \sin\vartheta_d , \quad X_0 = \rho\cos\vartheta_d . \tag{3.22}$$

Using this formalism, the Euclidean d'Alembertian operator for (3.4) is

$$\Box_E = \frac{1}{\rho^d}\frac{\partial}{\partial\rho}\left(\rho^d \frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\Delta_d , \tag{3.23}$$

where Δ_d is the Laplace-Beltrami operator [7, 38] on S^d. The eigenfunctions corresponding to S^d are generalized spherical harmonics $Y(\Omega)$ which satisfy the eigenvalue equation

$$\Delta_d Y(\Omega) = \lambda Y(\Omega) \tag{3.24}$$

It can be shown [27] that the spherical harmonics on S^d are

$$Y_{l_d \ldots l_1}(\vartheta_d, \ldots, \vartheta_1) = \left[\prod_{n=2}^{d} {}_n\hat{P}_{l_n}^{l_{d-1}}(\vartheta_n)\right] \frac{e^{il_1\vartheta_1}}{\sqrt{2\pi}}, \tag{3.25}$$

where

$$ {}_n\hat{P}_\lambda^l(\vartheta) = \left[\frac{2\lambda+n-1}{2} \frac{(\lambda+l+n-2)!}{(\lambda-l)!}\right]^{1/2} (\sin\vartheta)^{-(n-2)/2}$$

$$\times P_{\lambda+(n-2)/2}^{-(l+(n-2)/2)}(\cos\vartheta), \tag{3.26}$$

and where $P_\nu^\mu(x)$ is the associated Legendre function of the first kind. The integers $l_1, l_2, \ldots, l_d$ in (3.25)–(3.26) which satisfy

$$|l_1| \leq l_2 \leq \ldots \leq l_{d-1} \leq l_d, \tag{3.27}$$

are equivalent to the magnetic quantum numbers in three-dimensional space. It follows that the generalized spherical harmonics satisfy the normalization

$$\int d\vartheta_1 \ldots d\vartheta_d \sqrt{\Omega_d}\, Y_{l_d \ldots l_1} \cdot Y^*{}_{l'_d \ldots l'_1} = \delta_{l_d l'_d} \cdots \delta_{l_1 l'_1}, \tag{3.28}$$

and form a complete set of functions on S^d. Therefore,

$$\Delta_d Y_{l_d \ldots l_1} = -l_d(l_d + d - 1) Y_{l_d \ldots l_1}, \tag{3.29}$$

where, to simplify notation, $L = l_d$, $W = \{l_{d-1}, \ldots, l_1\}$. Then

$$\Delta_d Y_{LW} = -L(L+d-1)Y_{LW}, \quad (L \geq 0), \tag{3.30}$$

$$\int d\Omega_d\, Y_{LW}(\Omega)\, Y^*{}_{L'W'}(\Omega') = \delta_{LL'}\, \delta_{WW'}, \tag{3.31}$$

where

$$d\Omega_d = d\vartheta_1 \ldots d\vartheta_d \sqrt{\Omega_d}, \tag{3.32}$$

$$\delta_{WW'} = \delta_{l_{d-1} l'_{d-1}} \cdots \delta_{l_1 l'_1}, \tag{3.33}$$

and

$$\sum_{L=0}^{\infty}\sum_{W} Y_{LW}(\Omega)\, Y^{*}{}_{LW}(\Omega') \;=\; \delta^{d}(\Omega,\Omega')\,, \tag{3.34}$$

is the completeness relation.

The Euclidean Green's function can now be written in the form

$$G_E(x,x') \;=\; \sum_{L=0}^{\infty}\sum_{W} \mathcal{G}_L(\rho,\rho')\, Y_{LW}(\Omega)\, Y^{*}{}_{LW}(\Omega')\,, \tag{3.35}$$

where (3.35) satisfies the equation

$$(\Box_E - V_{\Sigma_E})G_E(x,x') = -\delta^{d}(x,x')\,. \tag{3.36}$$

Because the external potential V_{Σ_E} is dependent only on ρ, it remains invariant under rotations. Therefore, the radial Green's function $\mathcal{G}_L(\rho,\rho')$ obeys the equation

$$\left[\frac{d}{d\rho}\left(\rho^{d}\,\frac{d}{d\rho}\right) - \rho^{d-2}\,L(L+d-1) - \rho^{d}\,V_{\Sigma}(\rho)\right]\mathcal{G}_L(\rho,\rho') \;=\; -\delta(\rho-\rho'), \tag{3.37}$$

which is self-adjoint, so that $\mathcal{G}_L(\rho,\rho') = \mathcal{G}_L(\rho',\rho)$.

Since the radial Green's function $\mathcal{G}_L$ has no dependence on the collective index W, it is possible to sum over W in (3.35) [38], which leads to

$$\sum_{W} Y_{LW}(\Omega)\, Y^{*}{}_{LW}(\Omega') \;=\; \eta_d\,(2L+d-1)\,C_L^{(d-1)/2}(\cos\gamma)\,, \tag{3.38}$$

where γ is the angle between Ω and Ω' on the unit sphere S^d, $C_L^{(d-1)/2}(x)$ is the Gegenbauer polynomial, and

$$\eta_d \;=\; \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}}$$

$$\;=\; \frac{1}{(d-1)\,\mathcal{V}_d}\,, \tag{3.39}$$

where

$$\mathcal{V}_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}, \tag{3.40}$$

is the volume for the d-dimensional unit sphere.

By writing $C_L^{(d-1)/2}(x)$ in terms of the hypergeometric function $F(a, b; c; z)$

$$C_L^{(d-1)/2}(x) = \frac{(L+d-2)!}{L!(d-2)!} F\left(-L, L+d-1; \frac{d}{2}; \frac{(1-x)}{2}\right), \tag{3.41}$$

and noting the normalization condition $F(a, b, c; z = 0) = 1$, it follows that

$$C_L^{(d-1)/2}(1) = \frac{(L+d-2)!}{L!(d-2)!}. \tag{3.42}$$

Then the Euclidean Green's function is

$$G_E(x, x') = \eta_d \sum_{L=0}^{\infty} (2L+d-1) \mathcal{G}_L(\rho, \rho')\, C_L^{(d-1)/2}(\cos\gamma), \tag{3.43}$$

where γ is a geodesic distance between two points $(\vartheta_1, \ldots \vartheta_d)$ and $(\vartheta_1^{'}, \ldots \vartheta_d^{'})$ on the unit sphere. It is defined as $\gamma = \gamma_d$ by the following relations

$$\gamma_1 = \vartheta_1 - \vartheta_1^{'}, \tag{3.44}$$

$$\cos\gamma_n = \cos\vartheta_n \cos\vartheta_n^{'} + \sin\vartheta_n \sin\vartheta_n^{'} \cos\gamma_{n-1}, \quad n = 2, \ldots, d. \tag{3.45}$$

An ambiguity in the choice of the spherical co-ordinates connected with rigid rotations of space can be used to put $\vartheta_1 = \ldots = \vartheta_{d-1} = 0$ and $\vartheta_1^{'} = \ldots = \vartheta_{d-1}^{'} = 0)$ for any chosen pair of points on the unit sphere.

3.2.2 Model A

To obtain the radial Green's function for the single mirror problem, it is necessary to solve the homogeneous equation

$$\left[\frac{d}{d\rho}\left(\rho^d \frac{d}{d\rho}\right) - \rho^{d-2} L(L+d-1)\right] R_L = 0. \tag{3.46}$$

For the single mirror, the potential in (3.37) is

$$V_\Sigma(\rho) = V_0\,\delta(\rho - b)\,. \tag{3.47}$$

Clearly, there are two linearly independent solutions, an increasing one ρ^{L+} which is regular at the origin, and a decreasing one ρ^{L-} which vanishes at infinity, where

$$L_+ = L\,, \quad L_- = -(L + d - 1)\,. \tag{3.48}$$

From here, the radial Green's function can be constructed for the

$$\mathcal{G}_L(\rho,\rho') = \begin{cases} (A_-\,\rho^{L-} + A_+\,\rho^{L+})(\rho')^{L+}\,, & 0 \le \rho' < \rho < b, \\ B\rho^{L-}\,(\rho')^{L+}\,, & 0 \le \rho' < b < \rho, \\ \rho^{L-}\left(C_-\,(\rho')^{L-} + C_+\,(\rho')^{L+}\right), & 0 \le b < \rho' < \rho. \end{cases} \tag{3.49}$$

The inherent symmetry in the Green's function determines $\mathcal{G}_L(\rho,\rho')$ for other possible relative positions of ρ and ρ'.

The radial Green's function $\mathcal{G}_L(\rho,\rho')$ is continuous throughout the region, while its first derivatives have jump discontinuities. The form and location of the jumps directly follow from equation (3.37) with potential (3.47). The jump at $\rho = \rho'$ then takes the form

$$\left[\frac{d\mathcal{G}_L(\rho,\rho')}{d\rho}\right]_{\rho=\rho'+0} - \left[\frac{d\mathcal{G}_L(\rho,\rho')}{d\rho}\right]_{\rho=\rho'-0} = -\frac{1}{(\rho')^d}\,. \tag{3.50}$$

For $\rho' \neq b$, the jump at $\rho = b$ is

$$\left[\frac{d\mathcal{G}_L(\rho,\rho')}{d\rho}\right]_{\rho=b+0} - \left[\frac{d\mathcal{G}_L(\rho,\rho')}{d\rho}\right]_{\rho=b-0} = V_0\,\mathcal{G}_L(b,\rho')\,. \tag{3.51}$$

By using (3.49)–(3.51), the coefficients in (3.49) are determined to be

$$A_- = C_+ = \frac{1}{2(L+\beta_d)}\,, \quad B = \frac{1}{2(L+\beta_d+U_0)}\,,$$

$$A_+ \;=\; -\frac{U_0}{2(L+\beta_d)(L+\beta_d+U_0)}\frac{1}{b^{2L+d-1}}\,,$$

$$C_- \;=\; -\frac{U_0}{2(L+\beta_d)(L+\beta_d+U_0)}b^{2L+d-1}\,, \tag{3.52}$$

where

$$\beta_d \;=\; \frac{d-1}{2}\,, \quad U_0 \;=\; \frac{bV_0}{2}\,. \tag{3.53}$$

For the case when $V_0 = 0$, the relations among (3.49) simplify to

$$A_- \;=\; C_+ \;=\; B \;=\; \frac{1}{2L+d-1}\,, \quad A_+ \;=\; C_- \;=\; 0\,. \tag{3.54}$$

In this limit, the Green's function $G_E(x,x')$ coincides with the free Green's function (3.5). This can be easily verified by using the relation [38]

$$\frac{1}{|X-X'|^{d-1}} = \frac{1}{\eta_d}\sum_{L=0}^{\infty}\sum_{W}\frac{\rho_>^{L_-}\,\rho_<^{L_+}}{2L+d-1}\,Y_{LW}(\Omega)\,Y^*{}_{LW}(\Omega')\,, \tag{3.55}$$

where $\rho_> = \max(\rho,\rho')$ and $\rho_< = \min(\rho,\rho')$.

By subtracting the free Green's function $G_0(x,x')$ from $G_E(x,x')$, the renormalized Green's function is

$$G_E^{\rm ren}(x,x') \;=\; \sum_{L=0}^{\infty}\sum_{W}\mathcal{G}_L^{\rm ren}(\rho,\rho')\,Y_{LW}(\Omega)\,Y^*{}_{LW}(\Omega')\,,$$

$$=\; \eta_d\sum_{L}(2L+d-1)\,\mathcal{G}_L^{\rm ren}(\rho,\rho')\,C_L^{(d-1)/2}(\cos(\gamma))\,, \tag{3.56}$$

where

$$\mathcal{G}_L^{\rm ren}(\rho,\rho') \;=\; -\frac{U_0}{2(L+\beta_d)(L+\beta_d+U_0)}\frac{1}{b^{d-1}}\begin{cases}\left(\dfrac{\rho\rho'}{b^2}\right)^{L_+}, & 0\le\rho,\ \rho'<b,\\[2ex] \left(\dfrac{\rho\rho'}{b^2}\right)^{L_-}, & b\le\rho,\ \rho'.\end{cases} \tag{3.57}$$

3.2.3 Model B

Calculating the Green's function for Model B is similar to that of Model A, except that

$$V_\Sigma(\rho) = V_1\,\delta(\rho - b_1) + V_2\,\delta(\rho - b_2)\,. \tag{3.58}$$

It follows that

$$\mathcal{G}_L(\rho,\rho') = \begin{cases} \left(A_-\,\rho^{L-} + A_+\,\rho^{L+}\right)(\rho')^{L+}, & 0 \le \rho' < \rho < b_1, \\ \left(B_-\,\rho^{L-} + B_+\,\rho^{L+}\right)(\rho')^{L+}, & 0 \le \rho' < b_1 < \rho < b_2, \\ C\,\rho^{L-}(\rho')^{L+}\,, & 0 \le \rho' < b_1 < b_2 < \rho, \\ C^{-1}\left(D_-\,\rho^{L-} + D_+\,\rho^{L+}\right)\left(E_-(\rho')^{L-} + E_+(\rho')^{L+}\right), & 0 < b_1 < \rho' < \rho < b_2, \\ \rho^{L-}\left(F_-(\rho')^{L-} + F_+(\rho')^{L+}\right), & 0 < b_1 < \rho' < b_2 < \rho, \\ \rho^{L-}\left(G_-(\rho')^{L-} + G_+(\rho')^{L+}\right), & 0 < b_1 < b_2 < \rho' < \rho. \end{cases} \tag{3.59}$$

After straightforward but long calculations, it is shown that

$$A_+ = -\frac{1}{2\Omega_L\,b_1^{2L+d-1}}\left[U_1 + U_2\,\gamma_L + \frac{U_1\,U_2}{L+\beta_d}\,(1-\gamma_L)\right],$$

$$A_- = G_+ = \frac{1}{2(L+\beta_d)}, \qquad C = \frac{L+\beta_d}{2\Omega_L},$$

$$B_+ = D_+ = -\frac{U_2}{2\Omega_L}\frac{1}{b_2^{2L+d-1}}, \qquad B_- = D_- = \frac{L+\beta_d+U_2}{2\Omega_L},$$

$$E_+ \;=\; F_+ \;=\; \frac{L+\beta_d+U_1}{2\Omega_L}\,, \qquad E_- \;=\; F_- \;=\; -\frac{U_1}{2\Omega_L} b_1^{2L+d-1}\,,$$

$$G_- \;=\; -\frac{b_2^{2L+d-1}}{2\Omega_L}\left[U_1\,\gamma_L + U_2 + \frac{U_1\,U_2}{L+\beta_d}\,(1-\gamma_L)\right]. \tag{3.60}$$

As with Model A, the notational simplifications are

$$U_1 \;=\; \frac{b_1 V_1}{2}, \qquad U_2 \;=\; \frac{b_2 V_2}{2}, \qquad \gamma_L \;=\; \left(\frac{b_1}{b_2}\right)^{2L+d-1}, \qquad \beta_d \;=\; \frac{d-1}{2},$$

$$\Omega_L \;=\; (L+\beta_d+U_1)(L+\beta_d+U_2) - U_1 U_2 \gamma_L\,. \tag{3.61}$$

Having now obtained expressions for the radial Green's functions, it is necessary to renormalize them by subtracting off the vacuum radial Green's function $\mathcal{G}_L^0$, in order to calculate $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ and $\langle\hat{T}^{\nu}_{\mu}\rangle^{\text{ren}}$. To find $\mathcal{G}_L^0$ requires setting $U_1 = U_2 = 0$ from (3.59). As expected, the vacuum radial Green's function has the same value in the *inner* region ($\rho' < \rho < b_1$), *outer* region ($b_2 < \rho' < \rho$), and *intermediate* region ($b_1 < \rho' < \rho < b_2$), so that

$$\mathcal{G}_L^0(\rho,\rho') \;=\; \frac{1}{2(L+\beta_d)}\,\rho^{L_-}\,\rho'^{L_+}\,. \tag{3.62}$$

Therefore, it follows that

$$\mathcal{G}_L^{\text{ren}}(\rho,\rho') \;=\; A_+(\rho\rho')^L\,, \tag{3.63}$$

for the inner region ($\rho' < \rho < b_1$),

$$\mathcal{G}_L^{\text{ren}}(\rho,\rho') \;=\; \frac{G_-}{(\rho\rho')^{L+d-1}}\,, \tag{3.64}$$

for the outer region ($b_2 < \rho' < \rho$), and

$$\mathcal{G}_L^{\text{ren}}(\rho,\rho') \;=\; -\frac{1}{2\Omega_L\,(L+\beta_d)}\left[U_1\,(L+\beta_d+U_2)\,\frac{b_1^{2L+d-1}}{(\rho\rho')^{L+d-1}}\right.$$

$$+ U_2 \,(L + \beta_d + U_1)\, \frac{(\rho\rho')^L}{b_2^{2L+d-1}} - U_1\, U_2\, \gamma_L \left(\frac{\rho^L}{(\rho')^{L+d-1}} + \frac{(\rho')^L}{\rho^{L+d-1}} \right) \Bigg] \tag{3.65}$$

in the intermediate region ($b_1 < \rho' < \rho < b_2$).

As expected, $\mathcal{G}_L^{\mathrm{ren}}(\rho, \rho')$ is a symmetric function of its arguments. It is easy to verify that when one of the potentials, U_1 or U_2, vanishes, expressions (3.63)–(3.65) for $\mathcal{G}_L^{\mathrm{ren}}$ reduce to (3.57). In the other limit of perfectly reflecting mirrors, when $U_1 = U_2 = \infty$, relations (3.63)–(3.65) correctly reproduce the result of [15].

3.3 Calculation of $\langle \hat{\varphi}^2 \rangle^{\mathrm{ren}}$

3.3.1 Model A

Finding $\langle \hat{\varphi}^2 \rangle^{\mathrm{ren}}$ amounts to determining $G_E^{\mathrm{ren}}(x, x')$ in the coincidence limit $x' \to x$

$$\langle \hat{\varphi}^2 \rangle^{\mathrm{ren}} \;=\; G_E^{\mathrm{ren}}(x, x)\,, \tag{3.66}$$

which implies that $\rho' = \rho$ and angle $\gamma = 0$. For Model A, this means that, for (3.42), (3.43), and (3.57), the result is

$$\langle \hat{\varphi}^2(\rho) \rangle^{\mathrm{ren}} \;=\; -\frac{U_0\, \eta_d}{b^{d-1}} \begin{cases} F^{(d)}((\rho/b)^2, U_0 + (d-1)/2), & \rho < b, \\ \left(\frac{b}{\rho}\right)^{2(d-1)} F^{(d)}((b/\rho)^2, U_0 + (d-1)/2), & b < \rho, \end{cases} \tag{3.67}$$

where the function $F^{(d)}$ is

$$F^{(d)}(z, \beta) \;=\; \sum_{L=0}^{\infty} \frac{(L+d-2)!}{L!\,(d-2)!\,(L+\beta)}\, z^L\,. \tag{3.68}$$

It can be shown that at the centre $\rho = 0$,

$$\langle \hat{\varphi}^2(\rho = 0) \rangle^{\text{ren}} = -\frac{\eta_d}{b^{d-1}} \frac{U_0}{(d-1)/2 + U_0}, \tag{3.69}$$

and from the properties of hypergeometric functions, that (3.67) becomes divergent as $\rho \to b$.

Because the divergence involves the behaviour of the series at large L, it is possible to estimate the leading divergence by setting $\beta = 1$ in (3.68). Then it follows that the leading divergence of $F^{(d)}$ near $z = 1$ has the form

$$F^{(d)}(z, \beta) \sim \frac{1}{(d-2)(1-z)^{d-2}}. \tag{3.70}$$

Therefore, the divergent part of $\langle \hat{\varphi}^2(\rho = 0) \rangle^{\text{ren}}$ at $\rho = b$ is

$$\langle \hat{\varphi}^2(\rho) \rangle^{\text{ren}} \sim -\frac{U_0}{b^{d-1}} \frac{\eta_d}{(d-2)} \frac{1}{(1-x^2)^{d-2}}, \tag{3.71}$$

where

$$x = \begin{cases} \rho/b, & \rho \leq b, \\ \\ b/\rho, & b > \rho. \end{cases} \tag{3.72}$$

In the limit as $U_0 \to \infty$, elementary functions can be used to express $\langle \hat{\varphi}^2(x) \rangle^{\text{ren}}$, so that

$$\langle \hat{\varphi}^2(x) \rangle^{\text{ren}} = -\frac{\eta_d}{b^{d-1}} \frac{1}{|1 - (\rho/b)^2|^{d-1}}. \tag{3.73}$$

For $d = 3$, this result directly follows from expression (3.5) of Ref. [15] for the Green's function of a scalar massless field in the presence of an ideal mirror with the same choice of the surface Σ as for Model A. For $d \neq 3$ the result (3.73) can be verified by the method of images found in Ref. [15].

4-Dimensional Special Case

A specific consideration is given for the four-dimensional form of $\langle\hat{\varphi}^2\rangle^{\text{ren}}$, where $d = 3$. In dealing with the divergent part of the zero-point fluctuations, it is evident that a relationship exists between it and the singularity of the potential. Then, the function $F^{(3)}$ can be written in the form

$$F^{(3)}(z,\beta) \;=\; h(z,\beta) + g(z,\beta)\,, \tag{3.74}$$

where $h(z,\beta)$ is defined to be the divergent part. It can be shown that

$$\begin{aligned} h(z,\beta) \;&=\; \frac{1}{1-z} + (1-\beta)\ln(1-z) + \frac{1-\beta}{\beta} \\ &\quad + \left[\frac{2}{1+\beta} - (2-\beta)\right] z + \left[\frac{3}{2+\beta} - \frac{\beta-3}{2}\right] z^2\,, \end{aligned} \tag{3.75}$$

$$g(z,\beta) \;=\; -(1-\beta)\tilde{g}(z,\beta)\,, \tag{3.76}$$

$$\tilde{g}(z,\beta) \;=\; \beta\sum_{L=1}^{\infty}\frac{z^L}{L(L+\beta)} + \frac{1}{1-\beta}\left\{\left[\frac{2}{1+\beta} - (2-\beta)\right] z + \left[\frac{3}{2+\beta} - \frac{\beta-3}{2}\right] z^2\right\}. \tag{3.77}$$

The first line of (3.75) shows the divergent components of $F^{(3)}$ when $z = 1$, while the second line is included by construction so that the first three terms of its Taylor series expansion coincides with those of $F^{(3)}$ directly from (3.68). As for (3.77), it is convergent at $z = 1$. Therefore, it is possible to write $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ in the form

$$-\frac{4\pi^2 b^2}{U_0}\langle\hat{\varphi}(\rho)^2\rangle^{\text{ren}} \;=\; H(\rho,U_0) + U_0 G(\rho,U_0)\,, \tag{3.78}$$

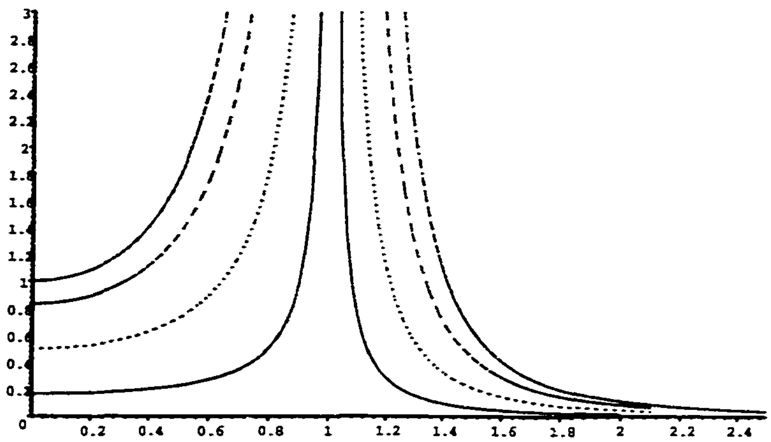

Figure 3.3: $-4\pi^2 b^2 \langle\hat{\varphi}^2\rangle^{\text{ren}}$ in 4-dimensional space-time as the function of ρ/b for different values of the parameter U_0: solid line – $U_0 = 0.2$, dotted line – $U_0 = 1.0$, and dashed line – $U_0 = 5.0$. The dashed and dotted line corresponds to an ideally reflecting mirror ($U_0 = \infty$).

where

$$H(\rho, U_0) = \begin{cases} h((\rho/b)^2, 1 + U_0), & \rho \le b, \\ \\ (b/\rho)^4 h((b/\rho)^2, 1 + U_0), & b > \rho, \end{cases} \tag{3.79}$$

$$G(\rho, U_0) = \begin{cases} \tilde{g}((\rho/b)^2, 1 + U_0), & \rho \le b, \\ \\ (b/\rho)^4 \tilde{g}((b/\rho)^2, 1 + U_0), & b > \rho. \end{cases} \tag{3.80}$$

The plots of functions H and G for different values of U_0 are shown in Figures 3.4 and 3.5.

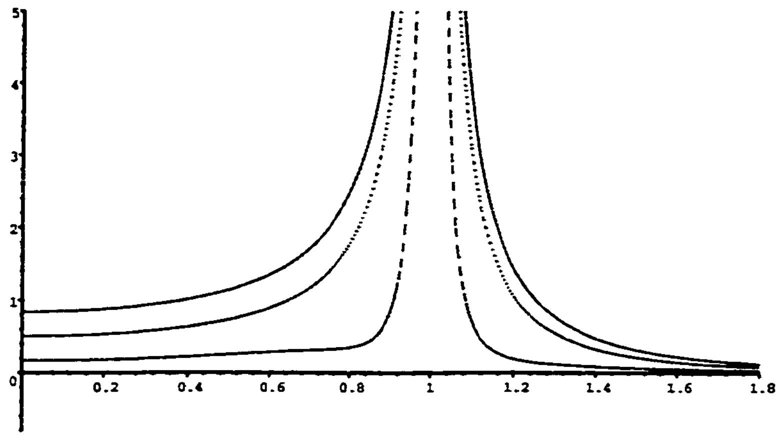

Figure 3.4: Function $H(\rho, U_0)$ for different values of the parameter U_0: solid line – $U_0 = 0.2$, dotted line – $U_0 = 1.0$, and dashed line – $U_0 = 5.0$.

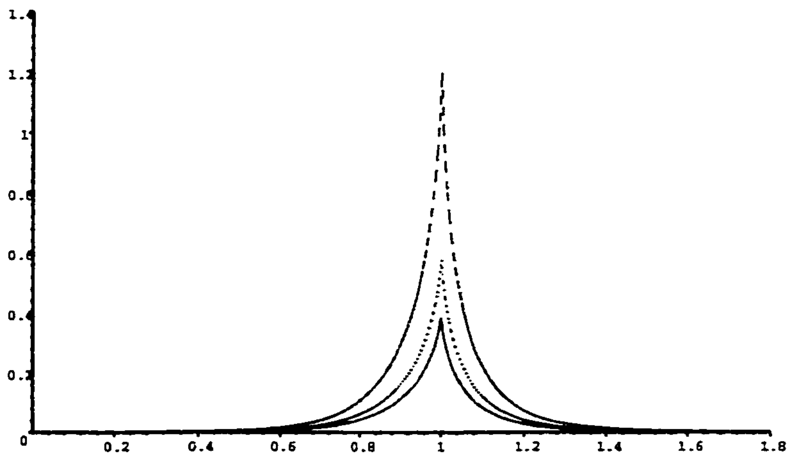

Figure 3.5: Function $G(\rho, U_0)$ for different values of the parameter U_0: solid line – $U_0 = 0.2$, dotted line – $U_0 = 1.0$, and dashed line – $U_0 = 5.0$.

3.3.2 Model B

The zero-point fluctuations for the concentric mirror problem can be described using (3.66), (3.42), and (3.63)–(3.65), resulting in

$$\langle\hat{\varphi}^2\rangle^{\text{ren}} = \frac{\eta_d}{(d-2)!}\sum_{L=0}^{\infty}\frac{(L+\beta_d)(L+d-2)!}{L!}\mathcal{R}_L(\rho)\,, \tag{3.81}$$

where

$$\mathcal{R}_L(\rho) = \begin{cases} -\dfrac{1}{\Omega_L\, b_1^{d-1}}\left(\dfrac{\rho}{b_1}\right)^{2L}\left[U_1 + U_2\,\gamma_L + \dfrac{U_1\,U_2}{L+\beta_d}\,(1-\gamma_L)\right], & \rho \le b_1 \\[2ex] -\dfrac{1}{\Omega_L(L+\beta_d)}\dfrac{1}{\rho^{d-1}}\left[U_1(L+\beta_d+U_2)\left(\dfrac{b_1}{\rho}\right)^{2L+d-1}\right. & \\[2ex] \left. +U_2(L+\beta_d+U_1)\left(\dfrac{\rho}{b_2}\right)^{2L+d-1} - 2U_1U_2\gamma_L\right], & b_1 \le \rho \le b_2\,, \\[2ex] -\dfrac{1}{\Omega_L\,\rho^{d-1}}\left(\dfrac{b_2}{\rho}\right)^{2L+d-1}\left[U_2 + U_1\,\gamma_L + \dfrac{U_1\,U_2}{L+\beta_d}\,(1-\gamma_L)\right], & \rho \ge b_2. \end{cases} \tag{3.82}$$

where $\gamma_L = (b_1/b_2)^{2L+d-1}$.

From (3.81) and (3.82), it is evident that only the $L = 0$ modes contribute to $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ at the origin. Therefore, it follows that

$$\langle\hat{\varphi}^2(\rho=0)\rangle^{\text{ren}} = -\frac{\eta_d}{b_1^{d-1}}\frac{\beta_d(U_1+\gamma_0)+U_1U_2(1-\gamma_0)}{(\beta_d+U_1)(\beta_d+U_2)-U_1U_2\gamma_0}\,, \tag{3.83}$$

where $\gamma_0 = (b_1/b_2)^{d-1}$.

To illustrate a typical behavior of $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ for the two-mirror problem, Figure 6 is plotted $-4\pi^2b^2\langle\hat{\varphi}^2\rangle^{\text{ren}}$ in 4-dimensional space-time as a function of ρ for different values of the parameter U_1 and U_2, and for $b_1 = 1$ and $b_2 = 3$.

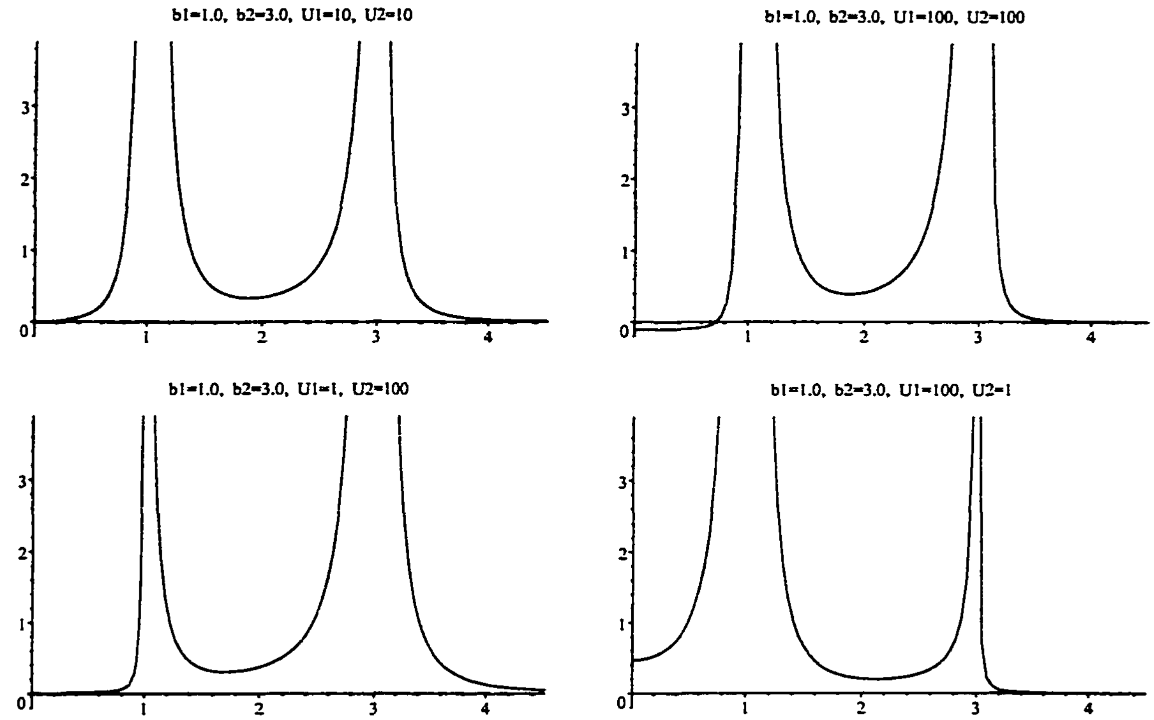

Figure 3.6: $-4\pi^2 b^2 \langle\hat{\varphi}^2\rangle^{\text{ren}}$ in 4-dimensional space-time as the function of ρ/b for different values of the parameter U_1 and U_2.

3.4 Stress-Energy Tensor

It is evident that, because the boundary conditions are only radially dependent, they are invariant under the $O(d+1)$ symmetry group. This suggests that the renormalized stress-energy tensor has the form

$$\langle \hat{T}_\mu^\nu \rangle^{\rm ren} = \operatorname{diag}(\epsilon, p, \ldots p)\,, \tag{3.84}$$

where ϵ and p are functions of ρ. Then the conservation law

$$\nabla_\nu \langle \hat{T}_\mu^\nu \rangle^{\rm ren} = 0 \tag{3.85}$$

implies

$$\frac{1}{\rho^d}\frac{d}{d\rho}\left(\rho^d\,\epsilon\right) - \frac{d}{\rho}p = 0\,, \tag{3.86}$$

or equivalently

$$\frac{d\epsilon}{d\rho} = -\frac{d}{\rho}(\epsilon - p)\,. \tag{3.87}$$

When considering the conformally invariant case, $\langle \hat{T}_\mu^\mu \rangle^{\rm ren} = \epsilon + d\cdot p = 0$, while the conservation law (3.86), (3.87) results in

$$\epsilon = \frac{\epsilon_0}{\rho^{d+1}}\,, \quad p = -\frac{\epsilon_0}{d\,\rho^{d+1}}\,. \tag{3.88}$$

From (3.88), it becomes known that the renormalized stress-energy tensor for the conformal invariant theory is uniquely determined by the symmetry and the conservation law up to one constant. In a more general case, it is sufficient to determine only one function of one variable ρ, such as $\epsilon - p$.

Using equations (2.62) and (3.8) results in

$$\epsilon - p = \lim_{x'\to x}\left[\mathcal{R} + \frac{1}{\rho^2}\mathcal{N}\right] G^{\rm ren}(x,x')\,, \tag{3.89}$$

where

$$\mathcal{R} = (1-2\xi)\partial_\rho \partial_{\rho'} - \xi\left(\partial_\rho^2 + \partial_{\rho'}^2\right) + \frac{\xi}{\rho}(\partial_\rho + \partial_{\rho'})\,, \tag{3.90}$$

$$\mathcal{N} = -(1-2\xi)\,\partial_\eta\,\partial_{\eta'} + \xi\left[\partial_\eta^2 + \partial_{\eta'}^2\right], \tag{3.91}$$

and the η are the generalized angle co-ordinates. (The last term in the right-hand-side of (3.90) arises because in the spherical co-ordinates $\Gamma^{\rho}_{\eta\eta} = -\rho \neq 0$.) From this formalism, it is possible to obtain the renormalized stress-energy tensors for the models considered.

3.4.1 Model A

From using (3.56), (3.57), and (3.89), it is shown that

$$\epsilon - p = -\frac{U_0\,\eta_d}{b^{d-1}}\left[\left(\mathcal{R} + \frac{1}{\rho^2}\mathcal{N}\right) Q_\pm(\rho,\rho',\gamma)\right]_{\substack{\rho'=\rho\\ \gamma=0}}, \tag{3.92}$$

where

$$Q_\pm(\rho,\rho',\gamma) = \sum_{L=0}^{\infty} \frac{1}{(L+\beta_d+U_0)}\left(\frac{\rho\rho'}{b^2}\right)^{L_\pm} C_L^{(d-1)/2}(\cos\gamma)\,. \tag{3.93}$$

When acting on $Q_\pm$, it is straightforward to show that

$$\left[\mathcal{R}\,(\rho\rho')^{L_\pm}\right]_{\rho=\rho'} = \left[(1-4\xi)\,L_\pm^2 + 4\xi L_\pm\right]\rho^{2(L_\pm-1)}\,, \tag{3.94}$$

$$\left[\mathcal{N}\,C_L^{(d-1)/2}(\cos\gamma)\right]_{\gamma=0} = \frac{L_+\,L_-}{d}\,C_L^{(d-1)/2}(1)\,, \tag{3.95}$$

where (3.95) is obtained from

$$\left.\frac{d}{dz}F(a,b;c;z)\right|_{z=0} = \frac{ab}{c}\,, \tag{3.96}$$

a property of hypergeometric functions.

Therefore, from (3.94), (3.95), and (3.41), the relationship obtained is

$$\epsilon - p = -\frac{U_0\,\eta_d}{b^{d-1}\,\rho^2}\sum_{L=0}^{\infty}\frac{\mathcal{A}_{\pm}^{\xi}}{(L+\beta_d+U_0)}\,\frac{(L+d-2)!}{L!\,(d-2)!}\left(\frac{\rho^2}{b^2}\right)^{L_\pm}, \tag{3.97}$$

where

$$\mathcal{A}_{L,\xi}^{\pm} = (1-4\xi)\,L_\pm^2 + 4\xi L_\pm + \frac{L_+\,L_-}{d} = L_\pm\,(L_\pm - 1)\left(\frac{d-1}{d} - 4\xi\right), \tag{3.98}$$

where the signs + and − correspond to the internal ($\rho < b$) and external ($\rho > b$) problems, respectively. Using (3.87), the following expression for ϵ is

$$\epsilon = \frac{U_0\,\eta_d\,d}{2\,b^{d-1}\,\rho^2}\sum_{L=0}^{\infty}\frac{\mathcal{A}_{\pm}^{\xi}}{(L_\pm-1)\,(L+\beta_d+U_0)}\,\frac{(L+d-2)!}{L!\,(d-2)!}\left(\frac{\rho^2}{b^2}\right)^{L_\pm}, \tag{3.99}$$

It is evident that the first two terms (with $L = 0$ and $L = 1$) (3.97) for the inner problem vanish. Then, in the general case, $\epsilon - p \sim \rho^2$ for small ρ, while ϵ remains finite at $\rho = 0$.

For the special case of conformal invariant theory $\xi = \xi_d = (d-1)/4d$, (3.98) shows that the coefficients $\mathcal{A}_{L,\xi_d}^{\pm}$ vanish. It then follows that $\epsilon = p$, while (3.88) shows that $\epsilon = p = 0$. The result (3.99) is similar to what happens in the case of an ideally reflecting mirror considered in [15].

For the minimal coupling consideration $\xi = 0$,

$$\mathcal{A}_{L,\xi=0}^{\pm} = \frac{d-1}{d}\,L_\pm(L_\pm - 1)\,. \tag{3.100}$$

In considering the internal ($\rho < b$) problem first. In this case,

$$\mathcal{A}_{L,\xi=0}^{+} = \frac{(d-1)L(L-1)}{d}\,. \tag{3.101}$$

Substituting $\mathcal{A}_{L,\xi=0}^{+}$ into (3.97), it is found that for $\rho < b$

$$\epsilon - p = -\mathcal{B}\,U_0\,\frac{\rho^2}{b^{d+3}}\,F^{(d+2)}\left((\rho/b)^2,\, U_0 + (d+3)/2\right), \tag{3.102}$$

where $F^{(d)}$ is the function determined by (3.68) and

$$\mathcal{B} = \eta_d (d-1)^2 . \tag{3.103}$$

Similarly, for the external ($\rho > b$) problem,

$$\mathcal{A}^-_{L,\xi=0} = \frac{(d-1)(L+d)(L+d-1)}{d} . \tag{3.104}$$

Substituting (3.104) into (3.97), it follows that

$$\epsilon - p = -\mathcal{B}\, U_0 \frac{b^{d-1}}{\rho^{2d}} F^{d+2}\left((b/\rho)^2,\, U_0 + (d-1)/2\right) . \tag{3.105}$$

The stress-energy conservation equation (3.87) allows for a way to obtain ϵ, and then p). To do this requires making use of the integral equations for hypergeometric functions

$$\int_0^z dz\, F^{(a)}(z,\beta) = \frac{1}{a-2} F^{(a-1)}(z,\beta-1) , \tag{3.106}$$

$$\int_0^z dz\, z^{a-2}\, F^{(a+1)}(z,\beta) = \frac{1}{a-1} z^{a-1}\, F^{(a)}(z,\beta) , \tag{3.107}$$

which lead to

$$\epsilon = \frac{\mathcal{B}\, U_0}{2b^{d+1}} \begin{cases} F^{(d+1)}\left(\left((\rho/b)^2,\, U_0 + (d+1)/2\right)\right) , & \rho < b , \\ \\ -\left(\dfrac{b}{\rho}\right)^{2d} F^{(d+1)}\left(\left((b/\rho)^2,\, U_0 + (d-1)/2\right)\right) , & \rho > b . \end{cases} \tag{3.108}$$

For an ideal reflecting mirror, $U_0 = \infty$, the expressions for $\epsilon - p$ and ϵ are greatly simplified. By considering the following property of functions $F^{(d)}(z,\beta)$

$$\lim_{\beta\to\infty} (\beta F^{(d)}(z,\beta)) = \frac{1}{(1-z)^{d-1}} , \tag{3.109}$$

it can be shown that

$$\epsilon = \pm\frac{\mathcal{B}}{2} \frac{b^{d-1}}{|b^2-\rho^2|^d} , \tag{3.110}$$

$$\epsilon - p = -\mathcal{B}\, b^{d-1} \frac{\rho^2}{|b^2-\rho^2|^{d-1}} . \tag{3.111}$$

The signs $\pm$ in (3.110) correspond to the inner and outer problem, respectively. For $d = 3$, expressions (3.110) – (3.111) reproduce the results obtained in [15].

3.4.2 Model B

Though much more involved in detail, the calculations of $\epsilon - p$ and ϵ for the two mirror problem are similar to those of Model A. As well as requiring (3.94) and (3.95), it is necessary to know

$$\left[\mathcal{R}\left(\rho^{L\pm}\rho'^{L\mp}\right)\right]_{\rho'=\rho} = \frac{1}{\rho^{d+1}}\left[L_+L_- - \xi(d^2-1)\right]. \tag{3.112}$$

For this section, only the final results are listed below. For the *inner region* ($\rho < b_1$),

$$\epsilon - p = -\frac{\eta_d}{\rho^2}\sum_{L=0}^{\infty}\frac{(L+\beta_d)\,(L+d-2)!}{L!\,(d-2)!}\mathcal{A}^+_{L,\xi}\,\mathcal{R}^+_L(\rho)\,, \tag{3.113}$$

$$\epsilon = \frac{\eta_d\, d}{2\rho^2}\sum_{L=0}^{\infty}\frac{(L+\beta_d)\,(L+d-2)!}{L!\,(d-2)!\,(L-1)}\mathcal{A}^+_{L,\xi}\,\mathcal{R}^+_L(\rho)\,, \tag{3.114}$$

$$\mathcal{R}^+_L(\rho) = \frac{1}{b_1^{d-1}}\left(\frac{\rho}{b_1}\right)^{2L}\frac{U_1 + U_2\gamma_L + \dfrac{U_1U_2}{L+\beta_d}(1-\gamma_L)}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1U_2\gamma_L}. \tag{3.115}$$

For the *outer region* ($\rho > b_2$),

$$\epsilon - p = -\frac{\eta_d b_2^{d-1}}{\rho^{2d}}\sum_{L=0}^{\infty}\frac{(L+\beta_d)\,(L+d-2)!}{L!\,(d-2)!}\mathcal{A}^-_{L,\xi}\,\mathcal{R}^-_L(\rho)\,, \tag{3.116}$$

$$\epsilon = -\frac{\eta_d b_2^{d-1}\, d}{2\rho^{2d}}\sum_{L=0}^{\infty}\frac{(L+\beta_d)\,(L+d-2)!}{L!\,(d-2)!\,(L+d)}\mathcal{A}^-_{L,\xi}\,\mathcal{R}^-_L(\rho)\,, \tag{3.117}$$

$$\mathcal{R}^-_L(\rho) = \left(\frac{b_2}{\rho}\right)^{2L}\frac{U_1\gamma_L + U_2 + \dfrac{U_1U_2}{L+\beta_d}(1-\gamma_L)}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1U_2\gamma_L}. \tag{3.118}$$

For the *intermediate region* ($b_2 > \rho > b_1$),

$$\epsilon - p = -\frac{\eta_d}{\rho^{d+1}} \sum_{L=0}^{\infty} \frac{(L+d-2)!}{L!\,(d-2)!} \left[\mathcal{A}^-_{L,\xi}\,\tilde{\mathcal{R}}^-_L(\rho) + \mathcal{A}^+_{L,\xi}\,\tilde{\mathcal{R}}^+_L(\rho) + \mathcal{A}^0_{L,\xi}\,\tilde{\mathcal{R}}^0_L\right], \tag{3.119}$$

$$\epsilon = \frac{\eta_d\, d}{2\rho^{d+1}} \sum_{L=0}^{\infty} \frac{(L+d-2)!}{L!\,(d-2)!} \times \left[\frac{\mathcal{A}^-_{L,\xi}}{(L_- - 1)}\,\tilde{\mathcal{R}}^-_L(\rho) + \frac{\mathcal{A}^+_{L,\xi}}{(L_+ - 1)}\,\tilde{\mathcal{R}}^+_L(\rho) - \frac{2}{d+1}\,\mathcal{A}^0_{L,\xi}\,\tilde{\mathcal{R}}^0_L\right]. \tag{3.120}$$

The coefficients $\mathcal{A}^{\pm}_{L,\xi}$ in relations (3.113), (3.116) and (3.119) are given by (3.98), and

$$\begin{aligned}\mathcal{A}^0_{L,\xi} &= \frac{d+1}{d}\,L_+\,L_- - \xi(d^2-1)\\ &= -\left[\frac{d+1}{d}\,L\,(L+d-1) + \xi\,(d^2-1)\right].\end{aligned} \tag{3.121}$$

The notations below are also introduced, whereby

$$\tilde{\mathcal{R}}^-_L(\rho) = \frac{U_1\,(L+\beta_d+U_2)}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1 U_2 \gamma_L}\left(\frac{b_1}{\rho}\right)^{2L+d-1}, \tag{3.122}$$

$$\tilde{\mathcal{R}}^+_L(\rho) = \frac{U_2\,(L+\beta_d+U_1)}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1 U_2 \gamma_L}\left(\frac{\rho}{b_2}\right)^{2L+d-1}, \tag{3.123}$$

$$\tilde{\mathcal{R}}^0_L = \frac{2\,U_1\,U_2\,\gamma_L}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1 U_2 \gamma_L}. \tag{3.124}$$

Expressions for ϵ in each of the three regions are obtainable by integrating relation (3.87).

In considering the conformally invariant theory for this problem, (3.98) indicates that $\mathcal{A}^{\pm}_{L,\xi} = 0$ when $\xi = \xi_d = (d-1)/4d$. From (3.114), (3.115), (3.116), and

(3.117), it becomes evident that $\epsilon = p = 0$ in both in the inner and outer regions. For the intermediate region, it can be shown that

$$\epsilon - p \;=\; \frac{C}{\rho^{d+1}}, \tag{3.125}$$

where

$$\begin{aligned} C \;&=\; 2U_1\, U_2\, \eta_d\, \frac{d+1}{d} \sum_{L=0}^{\infty} \frac{(L+d-2)!}{L!\,(d-2)!} \\ &\times \frac{\gamma_L \left[L(L+d-1) + \dfrac{(d-1)^2}{4} \right]}{(L+\beta_d+U_1)(L+\beta_d+U_2) - U_1 U_2 \gamma_L}. \end{aligned} \tag{3.126}$$

Using relation $\epsilon = -d \cdot p$, it follows that

$$\epsilon = \frac{d}{d+1} \frac{C}{\rho^{d+1}}. \tag{3.127}$$

It is straightforward to verify that, in the limit of ideally reflecting mirrors ($U_1, U_2 \to \infty$) in 4-dimensional space ($d = 3$), relations (3.125) – (3.127) correctly reproduce the results of [15].

3.5 Radiation From Mirrors

At this point, it is possible to consider the quantum radiation from expanding spherical mirrors with constant acceleration. The main goal here is to obtain an expression for the radiation at infinity due to such a mirror. The approach taken here is to start with $\langle \hat{\varphi}^2 \rangle^{\mathrm{ren}}$ and $\langle \hat{T}^{\mu}_{\nu} \rangle^{\mathrm{ren}}$ as defined in Euclidean space-time and make an analytical continuation into Lorentzian space-time.

3.5.1 $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ at $\mathcal{J}^+$

For this problem, it is sufficient to consider Model B, since Model A occurs in the limit when $U_1 \to 0$. Therefore, the exterior region is

$$\langle\hat{\varphi}^2\rangle^{\text{ren}}(\rho) = -\frac{\eta_d b_2^{d-1}}{\rho^{2(d-1)}} \sum_{L=0}^{\infty} \frac{(L+\beta_d)\,(L+d-2)!}{L!\,(d-2)!} \mathcal{R}_L^-(\rho)\,. \tag{3.128}$$

Using (3.22), it follows that

$$\rho^2 = R^2 + X_0^2\,. \tag{3.129}$$

By performing a Wick rotation $X_0 = iT$, this quantity becomes

$$\rho^2 = R^2 - T^2\,. \tag{3.130}$$

By letting $u = T - R$ be the retarded time, it follows that at large R

$$\rho^2 \approx -2uR\,. \tag{3.131}$$

For this definition of u, the value $u = 0$ corresponds to the moment of the retarded time when both expanding mirrors reach $\mathcal{J}^+$. For a part of $\mathcal{J}^+$ lying to the past of this moment (that is outside the mirror), this corresponds to $u < 0$. It happens that the leading contribution at $\mathcal{J}^+$ is given by the term $L = 0$ in (3.128). Therefore,

$$\langle\hat{\varphi}^2\rangle^{\text{ren}}(u) \approx -\frac{\eta_d}{R^{d-1}} \frac{b_2^{d-1}}{(-2u)^{d-1}} \frac{d-1}{2} \mathcal{R}_0^-\,, \tag{3.132}$$

$$\mathcal{R}_0^- = \frac{U_1\gamma_0 + U_2 + \dfrac{U_1U_2}{\beta_d}(1-\gamma_0)}{(\beta_d+U_1)(\beta_d+U_2) - U_1U_2\gamma_0}\,, \tag{3.133}$$

where $\gamma_0 = (b_1/b_2)^{d-1}$.

3.5.2 $\langle \hat{T}^{\mu}_{\nu} \rangle^{\rm ren}$ at $\mathcal{J}^+$

For Euclidean space-time, the starting point in the general expression of $\langle \hat{T}^{\mu}_{\nu} \rangle^{\rm ren}$, is

$$T^{\mu}_{\nu} \equiv \langle \hat{T}^{\mu}_{\nu} \rangle^{\rm ren} = (\epsilon - p)\, \delta^{(\rho)}_{\nu}\, \delta^{\mu}_{(\rho)} + p\delta^{\mu}_{\nu}\,, \tag{3.134}$$

where $\delta^{(\rho)}_{\nu}$ is the Kronecker delta which is non-vanishing only when ν corresponds to the co-ordinate ρ. The parentheses around ρ indicate that it is a *fixed* co-ordinate, for which no summation occurs. From (3.134), it becomes evident that, in (X_0, R) co-ordinates

$$\begin{aligned} T_{X_0X_0} &= \frac{1}{\rho^2}\left(X_0^2\epsilon + R^2 p\right)\,, \\ T_{X_0R} &= \frac{1}{\rho^2} X_0\, R\,(\epsilon - p)\,, \\ T_{RR} &= \frac{1}{\rho^2}\left(R^2\epsilon + X_0^2 p\right)\,. \end{aligned} \tag{3.135}$$

By making a Wick's rotation $X_0 = iT$, we get

$$\begin{aligned} T_{TT} &= \frac{1}{\rho^2}\left(T^2\epsilon - R^2 p\right)\,, \\ T_{TR} &= -\frac{1}{\rho^2} T\, R\,(\epsilon - p)\,, \\ T_{RR} &= \frac{1}{\rho^2}\left(R^2\epsilon - T^2 p\right)\,. \end{aligned} \tag{3.136}$$

For constant $u = T - R$ and $R \to \infty$, it follows that

$$T_{TT} \approx -T_{TR} \approx T_{RR} \approx \frac{R^2}{\rho^2}(\epsilon - p)\,. \tag{3.137}$$

For large R, the leading contribution to $\epsilon - p$ is given by the $L = 0$ term in (3.116). Therefore,

$$\epsilon - p \approx -\frac{\eta_d b_2^{d-1}}{\rho^{2d}} \frac{d-1}{2} \mathcal{A}^{-}_{0,\xi} \mathcal{R}^{-}_{0}\,, \tag{3.138}$$

where $\mathcal{R}_0^-$ is given by (3.133) and

$$\mathcal{A}_{0,\xi}^- = (d-1)\,(d-1-4\xi\,d)\;. \tag{3.139}$$

Combining these results, it follows that, for $T_{\mu\nu}$ in the asymptotic region,

$$T_{\mu\nu} \;=\; \frac{\dot{E}}{\mathcal{S}_{d-1}}\, l_\mu\, l_\nu\,, \tag{3.140}$$

where $l_\mu = u_{,\mu}$, $u = T - R$ is the retarded time,

$$\dot{E} \equiv \frac{dE}{du} \;=\; -\frac{1}{4\pi}\, B\left(\frac{d-1}{2}, \frac{1}{2}\right) \frac{b_2^{d-1}}{(-2u)^{d+1}}\,(d-1)^2\,(d-1-4\xi\,d)\,\mathcal{R}_0^-\,, \tag{3.141}$$

and

$$\mathcal{S}_{d-1} = R^{d-1}\,\mathcal{V}_{d-1} \tag{3.142}$$

is the surface area of a $(d-1)$-dimensional sphere of radius R. The function $B(z,w)$ in relation (3.141) is the Beta function $B(z,w) = \Gamma(z)\Gamma(w)/\Gamma(z+w)$.

For an ideally reflecting external mirror, $U_2 = \infty$, $\mathcal{R}_0^- = 2/(d-1)$, so that for the radiation of such a mirror in 4-dimensional space-time ($d = 3$) there is

$$\frac{dE}{du} = -\frac{b_2^2}{4\pi\, u^4}(1-6\xi)\,. \tag{3.143}$$

For $\xi = 0$, this result reproduces the result obtained in [15].

By making the substitution $u \to v = T + R$ in relations (3.132) and (3.140)–(3.141) the expressions for $\langle \hat{\varphi}^2 \rangle^{\text{ren}}$ and $T_{\mu\nu}$ at $\mathcal{J}^-$ are obtained.

3.6 Discussion

To re-iterate, this chapter investigates the quantum effects generated by spherical partially transparent mirrors expanding with a constant acceleration in a D-dimensional flat space-time. Considering a scalar massless field with an arbitrary parameter of non-minimal coupling ξ, it is demonstrated that the choice of parameter does not affect the field equation and the expectation value of $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ but results in different expressions for the stress-energy tensor. The partially transparent mirror can be modelled with a δ-like potential in the field equation.

It is demonstrated that the leading terms of $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ and $\langle\hat{T}^{\mu}_{\nu}\rangle^{\text{ren}}$ at $\mathcal{J}^+$ has quite a simple form, (3.132) and (3.140)–(3.141), respectively. Both expressions infinitely grow at $u = 0$, the moment of the retarded time when the mirrors reach $\mathcal{J}^+$. The same divergence takes place at the moment $v = 0$ of the advanced time when the mirrors start their motion from $\mathcal{J}^-$. Both of these divergences are evidently connected with the adopted idealization of the problem: an infinite time of the accelerated motion. It should be also emphasized that since the Euclidean approach is used and $\langle\hat{\varphi}^2\rangle^{\text{ren}}$ and $\langle\hat{T}^{\mu}_{\nu}\rangle^{\text{ren}}$ are obtained by a Wick rotation from their Euclidean values, the corresponding quantities in the physical space-time are given for a special choice of state of the quantum field, namely the state which is invariant under time reflection $T \to -T$

It is important to note that in Model B for the conformally invariant case (that is, for a special choice of the coupling $\xi = (d-1)/(4d)$), the stress-energy tensor identically vanishes both inside the inner mirror and outside the external mirror. For the conformal field in Model A this occurs everywhere. As indicated earlier (see [15] and [16]), this is a direct consequence of the conformal invariance of the models. A more general discussion of the properties of the vacuum in Minkowski space-time under conformal transformations can be found in [31] and [32].

A number of remarks can be made about $\langle \hat{T}^{\mu}_{\nu} \rangle^{\text{ren}}$ computations in a two-dimensional space-time.[1] Because $\xi_{d=1} = 0$ in two dimensions, the conformal and canonical stress-energy tensors are identical. Therefore, there is no loss of generality by considering $\xi = 0$ in this special case. The spherical harmonics (which enter for example into the expression (3.35) for the Green's function) are simply functions $\exp(im\eta)/\sqrt{2\pi}$, and $(L, W) \equiv m = 0, \pm 1, \pm 2, \ldots$. For all modes with $m \neq 0$, the general relation (3.65) can be easily verified for the 2-dimensional case by setting $\beta_d = 0$. The only concerns the mode $m = 0$ because it becomes logarithmically divergent, as shown by the general solution to the radial equation (3.37) either at $\rho = 0$ or $\rho = \infty$. For this reason, in the general case there does not exist a radial Green's function $\mathcal{G}_{m=0}(\rho, \rho')$ which remains finite at both boundaries. Fortunately, these zero modes do not contribute to $\langle \hat{T}^{\mu}_{\nu} \rangle^{\text{ren}}$ and the corresponding ambiguity has no bearing on the problem. The calculations for Model B give $\epsilon = p = 0$ inside the inner mirror and outside the outer mirror, while between the mirrors there is

$$
\begin{aligned}
\epsilon - p &= -\frac{2}{\pi}\frac{U_1 U_2}{\rho^2}\sum_{m=1}^{\infty}\frac{m\,\gamma_m}{\Omega_m}, \\
\epsilon &= -\frac{U_1 U_2}{\pi\rho^2}\sum_{m=1}^{\infty}\frac{m\,\gamma_m}{\Omega_m},
\end{aligned}
\tag{3.144}
$$

where $\gamma_m = (b_1/b_2)^{2m}$, and $\Omega_m = (m + U_1)(m + U_2) - U_1 U_2\,\gamma_m$. Both quantities ϵ and p evidently vanish if one of the potentials U_i vanishes. This is exactly the result which must be expected for Model A.

[1]We shall not consider $\langle \hat{\varphi}^2 \rangle^{\text{ren}}$ because of the infrared problem, and so is not well-behaved quantity. In particular, $\langle \hat{T}^{\mu}_{\nu} \rangle^{\text{ren}}$ is logarithmically divergent at infinity for any value of the mirror potential.

Chapter 4

Quantum Radiation from a Uniformly Accelerated Refractive Body

A second example of the moving mirror problem considered in this thesis concerns the quantum radiation emitted due to a uniformly accelerated refractive body. In electromagnetism, the presence of a dielectric has a significant influence on the propagation of waves in a medium. For example, given a fast-moving object in such a medium, it is possible to observe Cerenkov radiation [30] when the object moves faster than the speed of light in the medium.

The degree that an uncharged object becomes polarized is also directly proportional to the dielectric properties of its internal degrees of freedom. When at rest, this body interacts with the electromagnetic field in a way that induces a change in the total energy of the surrounding vacuum, leading to the Casimir effect. For a polarized body under accelerated motion, the dynamical Casimir effect occurs due to interactions between the zero-point and fluctuations within its dipole moment. This problem has been studied considerably in two-dimensional space-time, especially when dealing with bodies of very high polarizability so that their surfaces can be approximated by a reflecting mirror-like boundary. A four-dimensional treatement is much less accessible, except in situations where the geometry simplifies the problem sufficiently.

This chapter considers the effect of quantum radiation for a small polarizable body undergoing uniform acceleration. To simplify calculations, the internal degrees of freedom interact with a massless scalar field instead of an electromagnetic field. As well, the *refractive index* n differs only slightly from the vacuum value of $n = 1$, so that a perturbation expansion in terms of $n - 1$ becomes permissible. This leads to a correction of the vacuum Hadamard function due to the object's acceleration in space.

The organization of this chapter begins with a formulation of the problem in Section 4.1. This is followed in Section 4.2 by a calculation of the perturbed Hadamard function in the presence of a refractive object with small dimensions relative to the surrounding vacuum. Calculations of $\langle\varphi^2\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}\rangle^{\text{ren}}$ in the wave zone limit are then obtained in Section 4.3, followed by a brief discussion in Section 4.4.

4.1 Formulation of the Problem

4.1.1 A Refractive Body in Static Space-Time

The first step is to establish how to describe a refractive body in the presence of a vacuum region of space-time. To do this, consider a space-time described by a static metric

$$ds^2 = -A^2\,dt^2 + \gamma_{ij}\,dx^i\,dx^j\,, \tag{4.1}$$

where by definition A and γ_{ij} are time-independent. In relation to (4.1), consider another metric ds_n^2 with a time-independent function $n = n(x^i)$, where

$$ds_n^2 = -\frac{A^2}{n}\,dt^2 + n\gamma_{ij}\,dx^i\,dx^j\,. \tag{4.2}$$

It is important to note that (4.2) is merely a construction designed to describe a space-time corresponding to a refractive medium with index of refraction n. In so doing, the ultimate purpose of this construction is to generate a d'Alembertian operator $\Box_n$ from (4.2) which can be decomposed into an operator $\Box$ defined by the metric (4.1) and some operator $D(n)$ left over that acts like a corresponding source operator for an inhomogeneous wave equation.

To verify that n is indeed a refractive index, consider the case where (4.2) describes null rays, implying

$$ds_n^2 = 0\,. \tag{4.3}$$

Then, for a Killing observer moving with proper velocity $u^\mu = \xi^\mu/A$, with ξ^μ the Killing vector associated with (4.1) and $A^2 = -\xi^\mu\,\xi_\mu$, (4.3) leads to

$$\frac{dl}{d\tau} = \frac{1}{n}\,, \tag{4.4}$$

where $d\tau = A\,dt$ is the propagation time for the null ray, and $dl = \sqrt{\gamma_{ij}\,dx^i\,dx^j}$ is the proper distance for the metric (4.1). The condition (4.4) verifies that null rays move with speed $1/n < 1$ with respect to a static observer, which is a property of a refractive index.

The next step is to describe field modes in the refractive body and somehow relate them to the static space-time. For a massless scalar field

$$\Box_n\,\varphi = 0, \tag{4.5}$$

where $\Box_n$ is the d'Alembertian operator for the metric (4.2). In terms of (4.1), it is shown that

$$\Box_n = \frac{1}{n}\left[\Box - D(n)\right], \tag{4.6}$$

where

$$D(n) = \frac{n^2 - 1}{A^2}\,\partial_t^2. \tag{4.7}$$

It follows that

$$\Box\varphi = D(n)\,\varphi, \tag{4.8}$$

where $D(n)\,\varphi$ is a source term for the $\Box$ operator, which is perturbative[1], according to the condition $|n-1| \ll 1$.

It is possible to generalize (4.8)-(4.7) to include dispersive media. By letting $n = n(\omega)$ for a monochromatic wave of frequency ω, it follows that $\partial_t^2 \to -\omega^2$, which takes the dispersion into account.

Finally, it is necessary to find a way to localize the region containing the refractive medium. Assuming that the body is static and rigid, a world-tube described by a three-dimensional surface Σ can be formed by Killing trajectories passing through the body's surface. The four-dimensional region inside is then Γ, where

[1]There is an ambiguity in the form of the metric (4.2). It is possible to multiply the metric by any function $f(n)$ for which $f(1) = 1$. This operation modifies the form of the operator $D(n)$. In particular, a term proportional $\nabla n \nabla$ would be generated. For a wave of characteristic frequency ω it gives a contribution $\sim \omega\Delta n/b$ where b is a size of the body. It can be considered as a perturbation only if it is much smaller than the leading derivative terms of the unperturbed operator which are of the order ω^2. For our problem $\omega \sim a$ where a is the acceleration of the body, and $ab \ll 1$. In order to escape problems connected with the applicability of the perturbation approach and to simplify the calculations, the special case $f = 1$ is chosen.

$\Sigma = \partial\Gamma$. It follows that the operator (4.7) can be written as

$$D(n) = \frac{n^2-1}{A^2}\,\vartheta(\Gamma)\,\partial_t^2\,, \tag{4.9}$$

where ϑ is the Heaviside step function.

4.1.2 Uniformly Accelerated Body

It is now possible to develop this problem for a uniformly accelerating body moving in flat space-time. To begin, let the metric ds^2 be described by the usual Cartesian co-ordinates in the form

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2\,. \tag{4.10}$$

For an accelerating observer moving in the $X-$direction with acceleration a, the world-line γ for the motion is

$$X^2 - T^2 = l^2 \equiv a^{-2}\,, \tag{4.11}$$

where l is a characteristic length parameter associated with γ. For the sake of convenience, introduce dimensionless co-ordinates

$$t = \frac{T}{l}\,, \quad x = \frac{X}{l}\,, \quad y = \frac{Y}{l}\,, \quad z = \frac{Z}{l}\,. \tag{4.12}$$

By a re-definition of x and t in terms of dimensionless Rindler co-ordinates ξ, η in the form

$$x = (1+\xi)\cosh\eta\,, \quad t = (1+\xi)\sinh\eta\,, \tag{4.13}$$

the metric becomes

$$ds^2 = l^2\left[-(1+\xi)^2\,d\eta^2 + d\xi^2 + dy^2 + dz^2\right]\,, \tag{4.14}$$

which is valid in the wedge $x > |t|$. Then (4.11) for γ takes the form

$$\xi = 0, \tag{4.15}$$

while $l\eta$ is the proper time along γ. The surface $\eta = \eta_0$ is a plane with a three-dimensional flat metric.

It is convenient to parametrize (ξ, y, z) in terms of spherical co-ordinates (β, θ, ϕ) by

$$(\xi, y, z) = \beta n^i, \tag{4.16}$$

where

$$n^i = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi) \tag{4.17}$$

is a unit vector directed from the origin $\xi = y = z = 0$ to the point (ξ, y, z).

For a small uniformly accelerated body with a size much smaller than l, such a body is at rest in the reference frame (4.14), and the surface of the body is described by the equation

$$\beta = \beta_0(\theta, \varphi). \tag{4.18}$$

The combination of $\eta = \eta_0$ and (4.18), which defines the surface Σ identifies the position on Σ at time η_0. For the special case when the body is a sphere of radius b, $\beta_0 = b/l$ is a constant.

4.2 Quantum Radiation of an Accelerated Refractive Body

4.2.1 Hadamard Functions and Stress-Energy Tensor

It is shown below how to obtain the solution to (4.8) as a perturbation from the source-free solution, which is defined inside the world-tube Γ. Given that $D(n)$ is treated as a perturbation to solve this problem, the unperturbed field equation is

$$\Box\varphi = 0\,. \tag{4.19}$$

The Hadamard function for the Minkowski space-time vacuum is

$$\begin{aligned} G_0^{(1)}(x,x') &\equiv \langle 0|\hat{\varphi}(x)\,\hat{\varphi}(x') + \hat{\varphi}(x')\,\hat{\varphi}(x)|0\rangle \\ &= \frac{1}{2\pi^2 l^2}\,\frac{1}{s^2(x,x')}\,, \end{aligned} \tag{4.20}$$

where

$$s^2(x,x') = -(t-t')^2 + (x-x')^2 + (y-y')^2 + (z-z')^2 \tag{4.21}$$

is the invariant point separation in terms of dimensionless Cartesian co-ordinates x and x'.

Consider now the inhomogeneous equation

$$\Box\varphi = j\,, \tag{4.22}$$

where j in schematic form corresponds to $D(n)\,\varphi$ from (4.8). Its solution is

$$\varphi(x) = \varphi_0(x) - l^4 \int G_0^{\mathrm{ret}}(x,x')\,j(x')\,d^4x'\,, \tag{4.23}$$

where the retarded Green function G_0^{ret} is

$$G_0^{\text{ret}}(x,x') = \frac{1}{2\pi l^2}\,\delta(s^2(x,x'))\,\vartheta(t-t')\,. \tag{4.24}$$

By considering the right-hand side of (4.8) as a perturbation and using (4.23), the solution is

$$G^{(1)}(x,x') = G_0^{(1)}(x,x') + G_{\text{ren}}^{(1)}(x,x')\,, \tag{4.25}$$

where

$$G_{\text{ren}}^{(1)}(x,x') = -l^4\int_\Gamma d^4x''\left[G_0^{\text{ret}}(x,x'')\,D''\,G_0^{(1)}(x',x'')\right.$$

$$\left.+\,G_0^{\text{ret}}(x',x'')\,D''\,G_0^{(1)}(x,x'')\right] \tag{4.26}$$

where the notation D'' indicates that the operator D acts on the argument x''.

The notation $G_{\text{ren}}^{(1)}$ is also used for $G^{(1)} - G_0^{(1)}$, suggesting that this object is required for the calculation of physically observable quantities after subtracting the contribution of zero-point fluctuations in a boundary-free space-time. It becomes self-evident that the integration in (4.26) is performed over the interior of the world-tube Γ, and that $\langle\varphi^2\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}\rangle^{\text{ren}}$ can be found for $G_{\text{ren}}^{(1)}(x,x')$.

4.3 $\langle\varphi^2(x)\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}(x)\rangle^{\text{ren}}$ in the Wave Zone

4.3.1 $\langle\varphi^2\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}\rangle^{\text{ren}}$ on $\mathscr{J}^+$

Having now established the form of the renormalized Hadamard function, it is possible to proceed in obtaining $\langle\varphi^2\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}\rangle^{\text{ren}}$, where the observation point x is in the wave zone limit. The form for $\langle\varphi^2\rangle^{\text{ren}}$ is defined from (2.60), while $\langle T_{\mu\nu}\rangle^{\text{ren}}$ follows from (2.62) and (3.8).

By the same reasoning as shown in (4.16)–(4.18) to use spherical co-ordinates in Γ, it is convenient to use spherical co-ordinates to describe points in the wave zone region. Therefore, let

$$r = \sqrt{x^2+y^2+z^2}\,, \quad u = t - r\,, \quad (x,y,z) = rN^i\,,$$

$$N^i = (\cos\Theta, \sin\Theta\cos\Phi, \sin\Theta\sin\Phi)\,, \tag{4.27}$$

where u is the dimensionless retarded time, and (r,Θ,Φ) are spherical coordinates in the inertial reference frame. The wave-zone corresponds to taking the limit $r \to \infty$ with u and N^i fixed.

For the choice of point splitting, it is desirable to consider separations in the t-direction by letting

$$r = r'\,, \quad N^i = N^{i'} \tag{4.28}$$

in $G^{(1)}_{\text{ren}}$. Then it is possible to define the separation in terms of a parameter h so that $u' - u = t' - t = h$, or for fixed u_0,

$$u = u_0 - \frac{h}{2}\,, \quad u' = u_0 + \frac{h}{2}\,. \tag{4.29}$$

Then $G^{(1)}_{\text{ren}}(x,x')$ becomes a function of r, N^i, u_0 and h, in the form

$$G^{(1)}_{\text{ren}}(x,x')\Big|_{\substack{\text{point}\\ \text{splitted}}} = G(r,u_0,N^i,h)\,, \tag{4.30}$$

and by the symmetry in its co-ordinates, G is an *even* function of h.

It is straightforward to simply set $h = 0$ in (4.30) to obtain $\langle\varphi^2\rangle^{\text{ren}}$. It is considerably more complicated to find the corresponding energy density flux. By definition,

$$\frac{dE}{dUd\Omega} \equiv R^2\langle T_{UU}\rangle^{\text{ren}} \equiv r^2\langle T_{uu}\rangle^{\text{ren}} = 4\pi r^2 \lim_{h\to 0}\left[D_{uu}\,G(r,u_0,N^i,h)\right]\,, \tag{4.31}$$

where

$$D_{uu} = \frac{1}{8}(1-4\xi)\,\partial_{u_0}^2 - \frac{1}{2}\,\partial_h^2 \tag{4.32}$$

is the differential operator in the new co-ordinates. Since the leading-order term of G is proportional to r^{-2}, (4.30) is written in the form

$$G(r, u_0, N^i, h) = \frac{1}{r^2 l^2}\left[\mathcal{G}_1(u_0, \Theta) + h^2\,\mathcal{G}_2(u_0, \Theta) + O(h^4)\right], \tag{4.33}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are dimensionless, and l^2 is included to restore the correct dimensions. Because the system is invariant under rotations in the (y, z)−plane, (4.33) cannot have any dependence on Φ. It follows that

$$\langle\varphi^2\rangle^{\rm ren} \sim \frac{\mathcal{G}_1(u_0, \Theta)}{2r^2 l^2}, \tag{4.34}$$

$$\frac{dE}{dU d\Omega} \sim \frac{1}{l^2}\left[\frac{1}{8}(1-4\xi)\,\partial_{u_0}^2 \mathcal{G}_1(u_0, \Theta) - \mathcal{G}_2(u_0, \Theta)\right] \tag{4.35}$$

are the formal results for the vacuum fluctuation and energy density flux, respectively.

4.3.2 Boost Invariance

To obtain a more precise description of $\mathcal{G}_1$ and $\mathcal{G}_2$, it is possible to take advantage of the boost invariance symmetry in flat space-time. Given that a uniformly accelerating body is invariant under boost transformations

$$t \to \tilde{t} = \gamma(t + vx),$$

$$x \to \tilde{x} = \gamma(x + vt),$$

$$\tilde{y} = y, \quad \tilde{z} = z, \quad \gamma = (1 - v^2)^{-1/2}, \tag{4.36}$$

it is necessary to find the relationship between retarded spherical co-ordinates (u, r, Θ, Φ) and $(\tilde{u}, \tilde{r}, \tilde{\Theta}, \tilde{\Phi})$. Then in the wave zone limit, where $r \to \infty$ and u, Θ, Φ are constant, it follows that

$$\tilde{r} \;\approx\; \gamma r(1 + v\cos\Theta) + \gamma\frac{vu\cos\Theta}{1 + v\cos\Theta}, \tag{4.37}$$

$$\tilde{u} \;\approx\; \frac{u}{\gamma(1 + v\cos\Theta)}, \tag{4.38}$$

$$\tan\tilde{\Theta} \;=\; \frac{\sin\Theta}{\gamma(\cos\Theta + v)}, \quad \tilde{\Phi} \;=\; \Phi. \tag{4.39}$$

Because $\langle\varphi^2\rangle^{\text{ren}}$ is a Lorentz scalar, its invariance under the transformation (4.36) implies that

$$\mathcal{G}_1(\tilde{u}, \tilde{\Theta}) \;=\; \gamma^2(1 + v\cos\Theta)^2\,\mathcal{G}_1(u, \Theta), \tag{4.40}$$

where the second term in (4.37) is absorbed into $\tilde{r}$.

Having now (4.37)–(4.40), the invariance condition can be established by a variation with respect to v, such that

$$\left[\left(\frac{\partial\tilde{u}}{\partial v}\right)_{v=0}\partial_{\tilde{u}} + \left(\frac{\partial\tilde{\Theta}}{\partial v}\right)_{v=0}\partial_{\tilde{\Theta}}\right]\mathcal{G}_1(\tilde{u}, \tilde{\Theta}) \;=\; \frac{\partial}{\partial v}\left[\gamma^2(1 + v\cos\Theta)^2\mathcal{G}_1(u, \Theta)\right]$$

$$= \; 2\cos\Theta\,\mathcal{G}_1(u, \Theta)\,. \tag{4.41}$$

From (4.38) and (4.39), where

$$\left(\frac{\partial\tilde{u}}{\partial v}\right)_{v=0} = -u\cos\Theta\,, \quad \left(\frac{\partial\tilde{\Theta}}{\partial v}\right)_{v=0} \;=\; -\sin\Theta\,, \tag{4.42}$$

it follows that

$$\frac{\partial\mathcal{G}_1}{\partial(\ln u)} + \frac{\partial\mathcal{G}_1}{\partial\ln(\sin\Theta)} \;=\; -2\mathcal{G}_1\,. \tag{4.43}$$

The general solution of (4.43) is then

$$\mathcal{G}_1(u,\Theta) = \frac{1}{u^2}\mathcal{H}_1\left(\frac{\sin\Theta}{u}\right). \tag{4.44}$$

By following the same procedure as for $\mathcal{G}_1(u,\Theta)$, it can be shown that

$$\mathcal{G}_2(u,\Theta) = \frac{1}{u^4}\mathcal{H}_2\left(\frac{\sin\Theta}{u}\right). \tag{4.45}$$

4.3.3 Wave Zone Approximation

As noted earlier, the field modes in the dielectric are defined within the world-tube Γ. Therefore,

$$\int_\Gamma d^4x'' \ldots = \int d\eta \int d\Omega \int_0^{\beta_0(\theta,\phi)} \beta^2 A(\beta,\theta)\, d\beta \ldots \equiv \int d^4v \ldots, \tag{4.46}$$

where $A(\beta,\theta) = 1 + \beta\cos\theta$, $d\Omega = \sin\theta\, d\theta\, d\phi$ and β_0 defines the boundary of the body, as described by (4.18). Then the perturbation operator (4.9) in these co-ordinates is

$$D = \frac{(n^2-1)\mathcal{D}}{l^2}, \tag{4.47}$$

where

$$\mathcal{D} = A^{-2}\,\vartheta(\beta_0(\theta,\phi) - \beta)\;\partial_\eta^2. \tag{4.48}$$

Both Green's functions $G_0^{\rm ret}$ and $G_0^{(1)}$ depend only on the distance s between a point in the wave zone and a point inside or on the boundary of the world tube Γ. From (4.13), (4.16), (4.17), and the definition of the four-vector (t,x,y,z), it can be shown that

$$s^2(x,x'') = -2r\,w - u^2 + 2uA\sinh\eta + 1 + 2\beta n^1 + \beta^2, \tag{4.49}$$

where

$$w = u + \beta\, \boldsymbol{n}_\perp \boldsymbol{N}_\perp - F(\eta)\, A\,, \quad F(\eta) = \sinh\eta - N^1 \cosh\eta\,. \tag{4.50}$$

The vectors $\boldsymbol{n}$ and $\boldsymbol{N}$ are defined by equations (4.17) and (4.27), respectively, and $\boldsymbol{n}_\perp = (0, n^2, n^3)$, $\boldsymbol{N}_\perp = (0, N^2, N^3)$. Recall that (u, r, Θ, Φ) are retarded spherical coordinates of the point x in the wave zone, and $(\eta, \beta, \theta, \phi)$ are Rindler spherical coordinates of the point x'' in the tube Γ.

Since the leading term in $G^{(1)}_{\text{ren}}$ in the wave zone is $1/r^2$, all terms in (4.49) independent of r are small. Therefore,

$$s^2(x, x'') \approx -2rw_-\,, \quad s^2(x', x'') \approx -2rw_+\,, \tag{4.51}$$

where

$$w_\pm \;=\; w_0 \pm \frac{h}{2}\,, \quad w_0 \;=\; u_0 + \beta\, \boldsymbol{n}_\perp \boldsymbol{N}_\perp - F(\eta)\, A\,. \tag{4.52}$$

By combining all these results and using (4.33), it follows that

$$\mathcal{G}_1(u_0, \Theta) + h^2 \mathcal{G}_2(u_0, \Theta) + \ldots \;=$$

$$\frac{n^2 - 1}{16\pi^3} \int d^4 v \left[\delta(w_-)\, \mathcal{D}\left(\frac{1}{w_+}\right) + \delta(w_+)\, \mathcal{D}\left(\frac{1}{w_-}\right) \right]. \tag{4.53}$$

The ϑ-function which enters the definition (4.24) of G_0^{ret} can be omitted, since a future-directed null cone emitted from a point in the wave zone never crosses the tube Γ.

To explicitly evaluate (4.53), it is useful to evaluate the integral

$$I(\eta_0, h) \;=\; \int_{-\infty}^{\infty} d\eta\, \delta(w_-)\, \partial_\eta^2 \left(\frac{1}{w_+}\right) \tag{4.54}$$

over the proper time η. From (4.52), it can be shown that

$$\partial_\eta^2 \left(\frac{1}{w_+}\right) \;=\; \frac{F(\eta)\, A}{w_\pm^2} + \frac{2(F'(\eta))^2 A^2}{w_\pm^3}\,, \tag{4.55}$$

where $(\ )' = \partial_\eta(\ )$, and the substitution

$$F''(\eta) = F(\eta) \tag{4.56}$$

is made, given (4.50). Because $F(\eta)$ is a monotonically increasing function where $F(\pm\infty) = \pm\infty$ for $\Theta \neq 0$, for any number c there is a unique η_i such that $w_0(\eta) = c$. has a unique solution for any c. For the purpose of evaluating the integral, it is possible to find an η_0 and $\eta_\pm$ such that they satisfy

$$w_0(\eta_0) = 0, \quad w_0(\eta_\pm) = \mp\frac{h}{2}. \tag{4.57}$$

The next step is to determine $\delta(w_-)$ in terms of $\delta(\eta_-)$. Therefore,

$$\delta(w_-) = \frac{\delta(\eta_-)}{F'(\eta_-)\,A}. \tag{4.58}$$

By substitution of (4.55) and (4.58) into (4.54), it follows that

$$I(\eta_0, h) = \frac{1}{F'(\eta_-)}\left[\frac{1}{h^2}F(\eta_-) + \frac{2}{h^3}(F'(\eta_-))^2\,A\right]. \tag{4.59}$$

In order to evaluate

$$J(\eta_0, h) = \int_{-\infty}^{\infty} d\eta\left[\delta(w_-)\,\partial_\eta^2\left(\frac{1}{w_+}\right) + \delta(w_+)\,\partial_\eta^2\left(\frac{1}{w_-}\right)\right] \tag{4.60}$$

in (4.53), it is sufficient by symmetry to evaluate (4.59) for the first term of (4.60) and let $h \to -h$ for the second term. That is,

$$J(\eta_0, h) = I(\eta_0, h) + I(\eta_0, -h). \tag{4.61}$$

From (4.61), it becomes evident that only terms with even powers of h contribute to the integral.

The key point in evaluating $I(\eta_0, h)$ is to describe $F(\eta_-), F'(\eta_-)$ in terms of a Taylor expansion about η_0, using h as an order parameter. To ensure that (4.59) can be evaluated up to order h^2 then requires that terms up to order h^5 are kept for $F(\eta_-), F'(\eta_-)$. Therefore,

$$\begin{aligned} F(\eta_-) &= F(\eta_0 + (\eta_- - \eta_0)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} F_0^{(n)}(\eta_0)\,(\eta_- - \eta_0)^n \\ &= F_0 \cosh(\eta_- - \eta_0) + F_0' \sinh(\eta_- - \eta_0)\,, \end{aligned} \tag{4.62}$$

where $F_0 \equiv F(\eta_0)$, $F_0' \equiv F'(\eta_0)$, using (4.56).

From (4.62), it becomes evident that an expression for $\eta_- - \eta_0$ in terms of h is required. To do this, the equation $w_0(\eta_-) = h/2$ can be solved where $w_0(\eta_-)$ is written in terms of a Taylor expansion

$$\begin{aligned} w_0(\eta_-) &= -A \sum_{n=1}^{\infty} \frac{1}{n!} F_0^{(n)}\,(\eta_- - \eta_0)^n \\ &= \frac{h}{2}, \qquad F_0^{(n)} = F^{(n)}(\eta_0) \end{aligned} \tag{4.63}$$

and

$$\eta_- = \eta_0 + \sum_{k=1}^{\infty} \frac{h^k}{k!} z_k\,. \tag{4.64}$$

The coefficients z_k are defined in terms of F_0, F_0', which need to be evaluated. By substituting (4.64) into (4.63), it becomes evident that the terms of order h sum to $h/2$, while higher order terms equal zero identically. In terms of z_k, it follows that

$$z_1 = -\frac{1}{2AF_0'}, \quad z_2 = -\frac{F_0}{4A^2(F_0')^3}, \quad z_3 = -\frac{-3(F_0)^2 + (F_0')^2}{8A^3(F_0')^5},$$

$$z_4 = \frac{3F_0[-5(F_0)^2 + 3(F_0')^2]}{16A^4(F_0')^7},$$

$$z_5 = -\frac{3[35(F_0)^4 - 30(F_0)^2(F_0')^2 + 3(F_0')^4]}{32A^5(F_0')^9}, \tag{4.65}$$

computed using Maple.

By substituting the expansions into (4.59), it can be shown that the final expression for the integral is

$$J(\eta_0, h) = -\frac{F_0 \sin^2\Theta}{2A^2(F_0')^5}\left[1 + \frac{h^2}{8A^2(F_0')^4}[(4\cosh^2\eta_0 + 3)\cos^2\Theta\right.$$

$$\left. - 8\sinh\eta_0\cosh\eta_0\cos\Theta - 7 + 4\cosh^2\eta_0]\right]. \tag{4.66}$$

Therefore, the right-hand side of (4.53) is

$$\frac{n^2-1}{16\pi^3}\int d\Omega \int_0^{\beta_0(\theta,\phi)} d\beta\, \beta^2 A^{-1} J(\eta_0, h). \tag{4.67}$$

This integral can be solved perturbatively, since $\beta \ll 1$. This can be accomplished by solving $w_0(\eta_0) = 0$, or

$$F(\eta_0) = A^{-1}(\beta, \theta)(u_0 + \beta\, \boldsymbol{n}_\perp \boldsymbol{N}_\perp). \tag{4.68}$$

By now defining a solution $\tilde{\eta}_0$ for the equation

$$F(\tilde{\eta}_0) = u_0, \tag{4.69}$$

then

$$\eta_0 - \tilde{\eta}_0 \approx \frac{\beta}{F'_0}(\boldsymbol{n}_\perp \boldsymbol{N}_\perp - u_0\cos\theta), \tag{4.70}$$

where $F'_0 = F'(\tilde{\eta}_0)$. The result (4.70) shows that a linear perturbation about η_0 is proportional to β, so it is sufficient to neglect this correction and similar ones in $A(\beta, \theta)$, leading to

$$\frac{(n^2-1)\,V}{16\pi^3 l^3} J_0(\tilde{\eta}_0, h), \tag{4.71}$$

where V is the volume of the body. Comparison with (4.53) then leads to

$$\mathcal{G}_1(u_0,\Theta) = -\mathcal{B}\,\frac{F_0\sin^2\Theta}{(F_0')^5}\,, \tag{4.72}$$

$$\mathcal{G}_2(u_0,\Theta) = -\mathcal{B}\,\frac{F_0\sin^2\Theta}{8(F_0')^9}\left(4F_0^2-3\sin^2\Theta\right)\,, \tag{4.73}$$

where

$$\mathcal{B} = \frac{(n^2-1)\,V}{32\pi^3 l^3}\,. \tag{4.74}$$

Finally, to evaluate F_0 and F_0' in terms of u_0 and Θ, first note that $F_0 = u_0$ to zeroth order in β. To then obtain F_0', it is necessary to first solve the equation

$$F_0 \equiv \sinh\eta_0 - \sin\Theta\cosh\eta_0 = u_0\,, \tag{4.75}$$

and determine $\eta_0 = \eta_0(u_0,\cos\Theta)$, by substituting this value into the definition of F_0', where

$$F_0' \equiv \cosh\eta_0 - \sin\Theta\sinh\eta_0\,. \tag{4.76}$$

4.3.4 $\langle\varphi^2\rangle^{\text{ren}}$ and Energy Density Flux

By using relations (4.75) and (4.76), it can be shown that $\mathcal{G}_{1,2}$ obey the symmetry relations (4.44)–(4.45), and $\mathcal{G}_{1,2}$ can be rewritten in the form

$$\mathcal{G}_1(u,\Theta) = -\mathcal{B}\,\frac{g_1(z)}{u^2}\,, \quad \mathcal{G}_2(u,\Theta) = -\mathcal{B}\,\frac{g_2(z)}{u^4}\,, \tag{4.77}$$

where $z = \sin\Theta/u$, and

$$g_1(z) = \frac{z^2}{(1+z^2)^{5/2}}\,, \quad g_2(z) = \frac{z^2(4-3z^2)}{8(1+z^2)^{9/2}}\,. \tag{4.78}$$

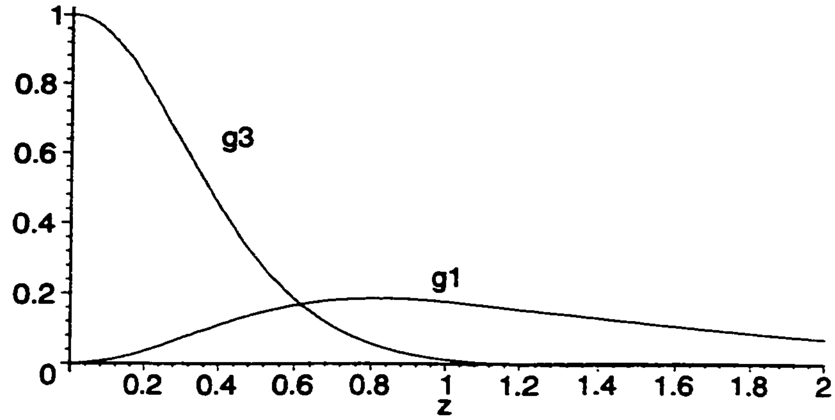

Figure 4.1: Plots of functions $g_1(z)$ $(g1)$ and $g_3(z)$ $(g3)$.

Using equations (4.34) and (4.35) and restoring the dimensional R, $U = T - R$, acceleration a, and $z = \sin\Theta/(aU)$, it follows that

$$\langle\varphi^2\rangle^{\text{ren}} \sim -\frac{(n^2-1)ab^3 g_1(z)}{48\pi^2 R^2 U^2}, \tag{4.79}$$

$$\frac{dE}{dU d\Omega} = -\frac{(n^2-1)ab^3}{12\pi^2 U^4}(1-5\xi)g_3(z), \tag{4.80}$$

where

$$g_3(z) = \frac{1-3z^2/4}{(1+z^2)^{9/2}}. \tag{4.81}$$

Plots of functions g_1 and g_3 are shown in Figure 4.1.

By then integrating over angles, the total energy density flux is

$$\frac{dE}{dU} = -\frac{(n^2-1)(ab)^3}{30\pi U^2}(1-5\xi)f(aU), \tag{4.82}$$

where

$$f(u) = \frac{1+4u^2+5u^4+10u^6}{(1+u^2)^4}. \tag{4.83}$$

For the sake of comparison, the energy density flux for a uniformly accelerated point-like charge can be found in [21, 23].

4.4 Discussion

For a dielectric body with size $b \ll a^{-1}$, the expression for the zero-point fluctuations and the total energy density flux are described by (4.79) and (4.82), respectively. Both the functions $g_1(z)$ and $g_2(z)$ are regular at the origin and go to zero at $z \to \infty$. The most immediate features about $\langle\varphi^2\rangle^{\text{ren}}$ are that its value is independent of the coupling constant ξ, and that it is negative-valued. As for dE/dU, there is a dependence on ξ and is negative-valued only for $\xi < 1/5$. It is noteworthy that, for conformal coupling $\xi = 1/6$, there is a small non-zero flux off the dielectric which persists in the wave zone limit.

For both $\langle\varphi^2\rangle^{\text{ren}}$ and dE/dU, there is a divergence of $\sim U^{-2}$ as $U \sim 0$. This emerges due to the idealization of the problem by assumption that the constant acceleration occurs for infinite time, which is not realistic.

It is useful to repeat the calculations for the more realistic electromagnetic field. While some general features related to the symmetry of the problem should be common to both types of fields, certain details such as the angular distribution of energy flux may be dependent upon the field spin, which will then yield differences. It is especially important to realize that, while the scalar field model has an inherent ambiguity in definition, the dependence of the electromagnetic field equations on the dielectric and magnetic properties of the media is uniquely specified.

In consideration of this problem, it is assumed that a uniformly accelerated refractive body has zero temperature. It may be possible to consider a finite temperature extension in the situation when heat is applied to the dielectric body. Especially worthy of consideration is a case when the body's temperature coincides with the Unruh temperature corresponding to its acceleration.

Chapter 5

Quantum Radiation of a Uniformly Accelerated Spherical Mirror

A considerable degree of attention is placed on studying electromagnetic radiation from a uniformly accelerated charge. Indeed, such investigations form a major part of what is considered well-established knowledge of classical electrodynamics. The Larmor radiation power spectrum [30] due to non-relativistically accelerated charges is a significant example of this cornerstone of classical physics. Not surprisingly, considerable attention is also given to how quantum electromagnetic fields behave due to accelerated charges. The presence of bremsstrahlung radiation [36] due to the sudden acceleration of an electron along with other radiative corrections make for important examples in this regard.

A similar level of focus is currently given to how moving mirrors interact with quantum fields while under constant acceleration. For a point-like mirror, this is well-understood in two-dimensional space-time, where conformal flatness in the mirror leads to enormous simplifications in solving the problem. Because the dynamical Casimir effect is highly dependent on the geometry of the mirror, it is often difficult to solve this type of problem, since the global symmetry attained in a two-dimensional space-time no longer exists in higher dimensions. If, however, the mirror has a simple enough shape, then it may be possible to obtain a solution that allows for study of the geometrical effects on the quantum radiation for comparison with a strictly point-like mirror. Indeed, such a solution exists for a spherical mirror moving with constant acceleration.

This chapter begins with a formulation of the problem in Section 5.1 to understand the underlying geometrical considerations. This is followed by Section 5.2 with a calculation of the vacuum fluctuations present in the wave zone limit due to the spherical mirror, while Section 5.3 has a corresponding calculation of the quantum stress-energy tensor and energy density flux in the wave zone.

5.1 Formulation of the Problem

5.1.1 Geometrical Considerations

The purpose here is to establish the geometrical considerations for a spherical mirror moving with constant acceleration. To do this, begin with the Minkowski metric in Cartesian co-ordinates

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \,. \tag{5.1}$$

In order to describe accelerated motion in the Z-direction, introduce Rindler co-ordinates

$$T = \rho \sinh \tau , \quad Z = \rho \cosh \tau , \tag{5.2}$$

where ρ identifies the world-line and τ is the proper time. The new metric is then

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + dX^2 + dY^2 , \tag{5.3}$$

When $X = Y = 0$, $\rho = \rho_0$ becomes the world-line for a uniformly accelerated observer with constant acceleration $a = \rho_0^{-1}$, while the three-dimensional plane of constant τ is a set of events which are simultaneous from the point of view of the observer.

In terms of both Rindler and Cartesian co-ordinates, the boundary condition Σ_+ of a uniformly accelerated spherical mirror is described by

$$b^2 = X^2 + Y^2 + (\rho - \rho_0)^2 \tag{5.4}$$

$$= X^2 + Y^2 + (\sqrt{Z^2 - T^2} - a^{-1})^2, \tag{5.5}$$

where b is the mirror's radius which is smaller than the distance to the horizon ρ_0. In fact, since (5.5) is invariant under the reflection $Z \to -Z$, two surfaces Σ_+ and Σ_- are described (see Fig 5.1), where the former corresponds to $Z > 0$. As the mirror gets uniformly accelerated towards $\mathcal{J}^+$, it interacts with the field modes which propagate into the future along null surfaces emanating from the mirror. The solution is described in terms of the Hadamard function $G^{(1)}(x, x')$, subject to the boundary condition

$$G^{(1)}(x, x')|_{x \in \Sigma} = G^{(1)}(x, x')|_{x' \in \Sigma} = 0 . \tag{5.6}$$

It possesses a symmetry between arguments x and x' which corresponds to the time reversal symmetry $T \to -T$. It is possible to take advantage of this type

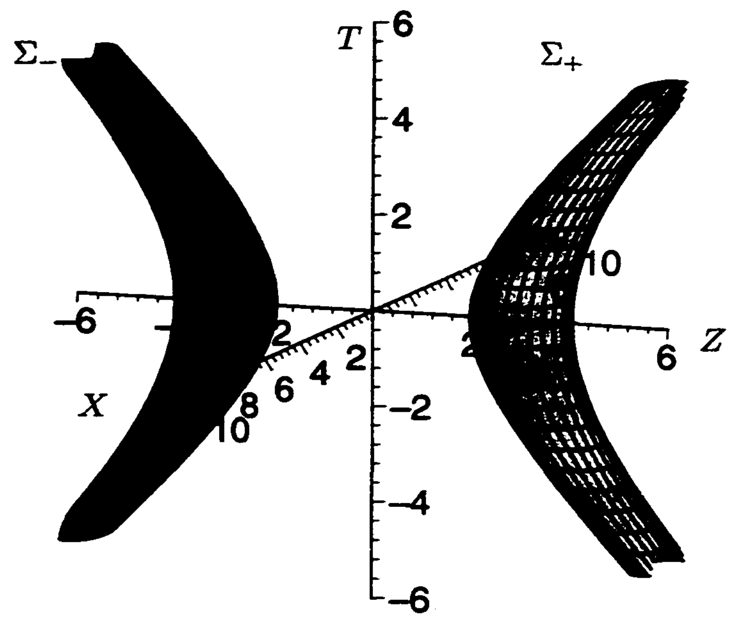

Figure 5.1: Co-ordinate surface describing hyperbolic trajectories corresponding to a spherical mirror under constant acceleration in Minkowski space-time. There is a reflection symmetry when $Z \to -Z$, which gives the second hyperbolic surface in the $-Z$ half of the co-ordinate space. The Y co-ordinate is suppressed.

of symmetry by finding the Euclidean Green's function and performing a Wick rotation, as described below. Suppose T is rotated in the complex plane, where

$$T \;\to\; i\mathcal{T}\,. \tag{5.7}$$

Then the Euclidean metric is

$$ds_E^2 \;=\; d\mathcal{T}^2 + dX^2 + dY^2 + dZ^2\,, \tag{5.8}$$

where the boundary condition is

$$X^2 + Y^2 + (\sqrt{Z^2 + \mathcal{T}^2} - a^{-1})^2 \;=\; b^2\,. \tag{5.9}$$

This surface Σ_E is a 4-dimensional torus $S^1 \times S^2$ obtained by the rotation of a sphere S^2 of the radius b around a circle S^1 with radius a^{-1} ($b < a^{-1}$). (See Fig 5.2.) It is required that $G_E(x, x')$ satisfies the equation

$$\Box_E G_E(x, x') \;=\; -\frac{1}{\sqrt{g}}\delta(x - x') \tag{5.10}$$

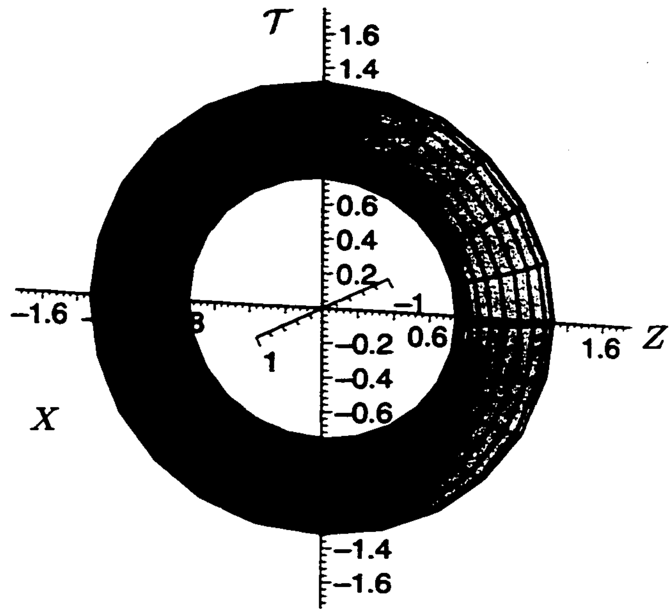

Figure 5.2: Co-ordinate surface describing a torus for a spherical mirror in Euclidean space-time, where the time co-ordinate T is analytically continued to $T \to i\mathcal{T}$. The Y co-ordinate is suppressed.

defined in the exterior of the torus Σ_E and satisfying the boundary conditions

$$G_E(x,x')|_{x\in\Sigma} \;=\; G_E(x,x')|_{x'\in\Sigma} \;=\; 0\,. \tag{5.11}$$

For space-like separation of the arguments, the Hadamard function $G^{(1)}$ can be obtained from G_E by the Wick rotation

$$G^{(1)}(x,x') \;=\; 2G_E(x,x')|_{\mathcal{T}\to -iT, \mathcal{T}'\to -iT'}\,. \tag{5.12}$$

5.1.2 Euclidean Green's Function

The Euclidean Green's function $G_E(x,x')$ coincides with an electric potential at a point x created by a point charge at x' in four-dimensional Euclidean space-time, and in the presence of a conducting surface Σ_E. This problem can be solved using the toroidal coordinates, for which the four-dimensional Laplace operator $\Box_E$ admits separation of variables. The toroidal coordinates $(\eta, \psi, \gamma, \phi)$ are then related to the Cartesian coordinates by

$$X = \frac{c\sin\gamma}{B(\eta,\gamma)}\cos\phi\,, \quad Y = \frac{c\sin\gamma}{B(\eta,\gamma)}\sin\phi\,, \tag{5.13}$$

$$Z = \frac{c\sinh\eta}{B(\eta,\gamma)}\cos\psi\,, \quad \mathcal{T} = \frac{c\sinh\eta}{B(\eta,\gamma)}\sin\psi\,, \tag{5.14}$$

where $B(\eta,\gamma) = \cosh\eta - \cos\gamma$, and c is a constant. The metric (5.8) in these coordinates takes the form

$$ds_E^2 = \Omega^2\, d\bar{s}^2\,, \quad \Omega = \frac{c}{B(\eta,\gamma)}\,, \tag{5.15}$$

$$d\bar{s}^2 = dH^2 + dS^2\,, \tag{5.16}$$

where

$$dH^2 = d\eta^2 + \sinh^2\eta\, d\psi^2 \tag{5.17}$$

is a metric on a hyperboloid H and

$$dS^2 = d\gamma^2 + \sin^2\gamma\, d\phi^2 \tag{5.18}$$

is a metric on the unit sphere S. The d'Alembertian operator corresponding to the metric (5.16) is of the form

$$\Box = \Delta_H + \Delta_S\,, \tag{5.19}$$

where

$$\Delta_H = \partial_\eta^2 + \coth\eta\,\partial_\eta + \frac{1}{\sinh^2\eta}\,\partial_\psi^2\,, \tag{5.20}$$

$$\Delta_S = \partial_\gamma^2 + \cot\gamma\,\partial_\gamma + \frac{1}{\sin^2\gamma}\,\partial_\phi^2 \tag{5.21}$$

are the Laplace operators on the unit hyperboloid and unit sphere, respectively. For this co-ordinate system, a surface of constant η describes a torus. By substituting (5.13)–(5.14) into (5.9), it is shown that

$$c = \sqrt{\frac{1}{a^2} - b^2}, \quad \tanh \eta_0 = \sqrt{1-(ab)^2}, \tag{5.22}$$

where η_0 is the value of η corresponding to Σ_E. Points with $\eta < \eta_0$ lie in the exterior of Σ_E.

It proves useful to develop the following expressions for the square of the distance $\mathcal{R}^2$ from the origin to the point $x = (X, Y, Z, \mathcal{T})$, and for the square of the distance $\mathcal{R}^2(x, x')$ between points x and x'. They are

$$\mathcal{R}^2 = c^2 \frac{\cosh \eta + \cos \gamma}{\cosh \eta - \cos \gamma}, \tag{5.23}$$

$$\mathcal{R}^2(x, x') = \frac{2\, c^2 \,(\cosh \Lambda - \cos \lambda)}{(\cosh \eta - \cos \gamma)(\cosh \eta' - \cos \gamma')}, \tag{5.24}$$

where λ and Λ are geodesic distances on a unit sphere S and a unit hyperboloid H, respectively. They are defined as

$$\cos \lambda = \cos \gamma \, \cos \gamma' + \cos(\phi - \phi') \, \sin \gamma \, \sin \gamma', \tag{5.25}$$

$$\cosh \Lambda = \cosh \eta \, \cosh \eta' - \cos(\psi - \psi') \, \sinh \eta \, \sinh \eta'. \tag{5.26}$$

Given that ds^2 and $d\tilde{s}^2$ are conformally related space-times by (5.15), it can be shown that the d'Alembertian operators are related by

$$\Box - \frac{1}{6} R = \Omega^{-3} \left(\tilde{\Box} - \frac{1}{6} \tilde{R} \right) \Omega. \tag{5.27}$$

For the current problem, R_E and $\tilde{R}$ for metrics ds_E^2 and $d\tilde{s}^2$ given by (5.15) – (5.16) vanish and using (5.27) it is shown that

$$\tilde{\Box}\, \tilde{G}(x, x') = -\frac{1}{\sqrt{\tilde{g}}} \delta(x - x'), \tag{5.28}$$

where

$$G_E(x,x') = \Omega^{-1}(x)\,\Omega^{-1}(x')\,\tilde{G}(x,x')\,. \tag{5.29}$$

Having now established the basic formalism, it is possible to now obtain the Green's function $\tilde{G}$ satisfying (5.28). Expanding over spherical harmonics $Y_{\ell m}$ which form a complete set on the unit sphere, $\tilde{G}$ is written as the mode sum

$$\begin{aligned} \tilde{G}(x,x') &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \tilde{G}_\ell(p,p')\,Y_{\ell m}(q)\,Y^*_{\ell m}(q') \\ &= \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\,\tilde{G}_\ell(p,p')\,P_\ell(\cos\lambda)\,, \end{aligned} \tag{5.30}$$

where $P_\ell(z)$ is the Legendre polynomial, p, p' are points on H, q and q' are points on S, for $x = (p,q)$, $x' = (p',q')$, and λ is the geodesic distance (angle) between $q = (\gamma,\phi)$ and $q' = (\gamma',\phi')$ on S defined by (5.25). The functions $\tilde{G}_\ell(p,p')$ are 2-dimensional Green functions of the operator $\Delta_H - \ell(\ell+1)$, which are regular inside the disc $0 \le \eta < \eta_0$ and obey the Dirichlet boundary conditions at the boundary of the disc. Using the Fourier decomposition with respect to the angle variable ψ it follows that

$$\tilde{G}_\ell(p,p') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-im(\psi-\psi')}\,\mathcal{G}_{\ell m}(\eta,\eta')\,, \tag{5.31}$$

where $\mathcal{G}_{\ell m}(\eta,\eta')$ obeys the equation

$$\left[\frac{d^2}{d\eta^2} + \coth\eta\frac{d}{d\eta} - \frac{m^2}{\sinh^2\eta} - \ell(\ell+1)\right]\mathcal{G}_{\ell m}(\eta,\eta') = -\frac{\delta(\eta-\eta')}{\sinh\eta}\,, \tag{5.32}$$

satisfying the boundary condition

$$\mathcal{G}_{\ell m}(\eta_0,\eta') = \mathcal{G}_{\ell m}(\eta,\eta_0) = 0\,. \tag{5.33}$$

The required Green's functions $\mathcal{G}_{\ell m}$ must also be regular at $\eta = 0$.

Linear independent solutions of the homogeneous version of the equation (5.32) are the associated Legendre functions P_ℓ^m and Q_ℓ^m. In terms of hypergeometric functions F, the Legendre functions are defined as (see [11], eq. 3.2.3 and 3.2.5)

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F(-\nu, \nu+1; 1-\mu; (1-z)/2) \qquad (5.34)$$

$$\begin{aligned} Q_\nu^\mu(z) &= e^{i\mu\pi} 2^{-\nu-1} \sqrt{\pi} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} z^{-\nu-\mu-1} (z^2-1)^{\mu/2} \\ &\times F\left(1+\frac{\nu}{2}+\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}; \nu+\frac{3}{2}; z^{-2}\right) \end{aligned} \qquad (5.35)$$

for arbitrary parameters ν, μ, and complex argument z.

Since ℓ and m are independent parameters in the equation, we shall need the Legendre functions for both $|m| \leq \ell$ and $|m| > \ell$ are required. For the latter case and for the standard definition of $P_\ell^m(z)$, these functions vanish, while $\Gamma(\ell - m + 1)P_\ell^m$ remain finite in the limit of integer ℓ and m. Therefore, instead of $P_\nu^\mu(z)$, it is more convenient to use the functions

$$\mathcal{P}_\nu^\mu(z) = \Gamma(\nu-\mu+1) P_\nu^\mu(z) \,, \qquad (5.36)$$

where it is understood that these functions are defined by continuity for integer ν and μ. As well,

$$\mathcal{Q}_\nu^\mu(z) = \frac{e^{-\mu\pi}}{\Gamma(\nu+\mu+1)} Q_\nu^\mu(z) \,. \qquad (5.37)$$

The $\mathcal{P}_\nu^\mu(z)$ and $\mathcal{Q}_\nu^\mu(z)$ with complex arguments ν, μ, and z are analytic functions defined in the complex plane z with a cut along the real axis lying to the left of $z = 1$. For integer value $\mu = m$, these functions obey the symmetry relations

$$\mathcal{P}_\nu^{-m}(z) = \mathcal{P}_\nu^m(z) \,, \quad \mathcal{Q}_\nu^{-m}(z) = \mathcal{Q}_\nu^m(z) \,. \qquad (5.38)$$

Using relation (3.2.13) of [11], the Wronskian of the functions $\mathcal{P}^{\mu}_{\nu}(z)$ and $\mathcal{Q}^{\mu}_{\nu}(z)$ is

$$W[\mathcal{P}^{\mu}_{\nu}(z), \mathcal{Q}^{\mu}_{\nu}(z)] \equiv \mathcal{P}^{\mu}_{\nu}(z)\frac{d}{dz}\mathcal{Q}^{\mu}_{\nu}(z) - \mathcal{Q}^{\mu}_{\nu}(z)\frac{d}{dz}\mathcal{P}^{\mu}_{\nu}(z) = \frac{1}{1-z^2}. \quad (5.39)$$

The functions $\mathcal{P}^{m}_{\ell}$ are regular at $z = 1$, while the functions

$$\mathcal{O}^{m}_{\ell}(z|z_0) = \mathcal{Q}^{m}_{\ell}(z) - \frac{\mathcal{Q}^{m}_{\ell}(z_0)}{\mathcal{P}^{m}_{\ell}(z_0)}\mathcal{P}^{m}_{\ell}(z) \quad (5.40)$$

are constructed so that they vanish at $z = z_0$. By letting $z_0 = \cosh\eta_0$, the one-dimensional Green function $\mathcal{G}_{\ell m}(\eta, \eta')$ is then

$$\mathcal{G}_{\ell m}(\eta, \eta') = \mathcal{P}^{m}_{\ell}(\cosh\eta_<)\,\mathcal{O}^{m}_{\ell}(\cosh\eta_>|z_0)\,. \quad (5.41)$$

Therefore, by combining the obtained results and using the symmetry properties of (5.38), it follows that the Euclidean Green function $\tilde{G}$ is

$$\tilde{G}(x, x') = \frac{1}{8\pi^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\lambda)$$

$$\times\sum_{m=0}^{\infty}\beta_m\,\cos[m(\psi-\psi')]\,\mathcal{P}^{m}_{\ell}(\cosh\eta_<)\,\mathcal{O}^{m}_{\ell}(\cosh\eta_>|z_0)\,, \quad (5.42)$$

where $\beta_0 = 1$ and $\beta_{m\geq 1} = 2$.

Since the Green's function must eventually become renormalized, it is necessary to find the boundary independent part of $\tilde{G}(x, x')$ and subtract it off. Using relation 3.11.4 from [11] there is

$$\sum_{m=0}^{\infty}\beta_m\,\cos[m(\psi-\psi')]\,\mathcal{P}^{m}_{\ell}(z_<)\,\mathcal{Q}^{m}_{\ell}(z_>) =$$

$$Q_\ell\left(z_< z_> - \cos(\psi-\psi')\sqrt{(z_>^2-1)(z_<^2-1)}\right)\,. \quad (5.43)$$

This relation, together with the Heine formula (see 3.11.10 [11])

$$\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(t')Q_\ell(t) = \frac{1}{t-t'}, \quad (5.44)$$

leads to

$$\tilde{G}^0(x,x') = \frac{(\cosh\eta - \cos\gamma)(\cosh\eta' - \cos\gamma')}{8\pi^2 a^2 (\cosh\Lambda - \cos\lambda)}, \tag{5.45}$$

where λ and Λ are defined by (5.25) and (5.26). This result implies that G_E^0 related to $\tilde{G}^0$ by (5.29) coincides with the *vacuum* Green function

$$G_E^0(x,x') = \frac{1}{4\pi^2\mathcal{R}^2(x,x')}. \tag{5.46}$$

The renormalized Euclidean Green's function defined as

$$G_E^{\text{ren}}(x,x') = G_E(x,x') - G_E^0(x,x') \tag{5.47}$$

then has the series representation

$$G_E^{\text{ren}}(x,x') = -\frac{BB'}{8\pi^2c^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\lambda)\sum_{m=0}^{\infty}\beta_m\cos[m(\psi-\psi')]$$

$$\times\,\mathcal{P}_\ell^m(\cosh\eta)\,\mathcal{P}_\ell^m(\cosh\eta')\,\frac{\mathcal{Q}_\ell^m(\cosh\eta_0)}{\mathcal{P}_\ell^m(\cosh\eta_0)}, \tag{5.48}$$

where $B = \cosh\eta - \cos\gamma$ and $B' = \cosh\eta' - \cos\gamma'$.

5.1.3 Wave Zone Region

After obtaining the Euclidean Green's function, it is necessary to analytically continue the final results from Euclidean to Minkowski space-time. To do this, it must first be noted that a Wick rotation of co-ordinate ψ in (5.14), $\psi \to -i\psi$, corresponds to $\mathcal{T} \to -iT$ in Cartesian co-ordinates. An immediate problem arises, however, in that this leads to $|T|/Z = |\tanh\psi| < 1$, and so only covers the right $R+$ and left R_- wedges of the total Minkowski spacetime.

To cover the wave zone region, located in the upper T_+ wedge, requires the following procedure. The first step is to shift ψ in the form

$$\psi \to \pi/2 - \psi. \tag{5.49}$$

After this, by making the following analytical continuation

$$\psi \to i\psi\,, \quad \gamma \to i\gamma\,, \quad c \to -ic\,, \tag{5.50}$$

and identifying $i\mathcal{T}$ with T, the result is

$$X = \frac{c\sinh\gamma}{\tilde{B}(\eta,\gamma)}\cos\phi\,, \quad Y = \frac{c\sinh\gamma}{\tilde{B}(\eta,\gamma)}\sin\phi\,, \tag{5.51}$$

$$Z = \frac{c\sinh\eta}{\tilde{B}(\eta,\gamma)}\sinh\psi\,, \quad T = \frac{c\sinh\eta}{\tilde{B}(\eta,\gamma)}\cosh\psi\,, \tag{5.52}$$

where $\tilde{B}(\eta,\gamma) = \cosh\eta - \cosh\gamma$. From (5.52), it follows that $|T|/|Z| = |\coth\psi| > 1$, and so the co-ordinates (X, Y, Z, T) cover the region located outside $R_\pm$.

The concluding steps are to re-parametrize the space-time in terms of more physically intuitive parameters. By letting $R^2 = X^2 + Y^2 + Z^2$, it is shown that

$$R = c\,\frac{\sqrt{\sinh^2\gamma + \sinh^2\eta\,\sinh^2\psi}}{\cosh\eta - \cosh\gamma}\,, \tag{5.53}$$

$$T^2 - R^2 = 2UR + U^2 = c^2\,\frac{\cosh\eta + \cosh\gamma}{\cosh\eta - \cosh\gamma}\,, \tag{5.54}$$

where $U = T - R$ is the retarded time coordinate. For $U > 0$, (5.54) is well-defined if $\eta > |\gamma|$. Then, in the limit as $R \to \infty$ with U a fixed parameter, $|\gamma| \to \eta$.

Dividing expression (5.54) by $2R$ and using (5.54) in the leading (zero) order, it is shown that

$$U = c\,\frac{\coth\eta}{\cosh\psi}\,. \tag{5.55}$$

By defining the spatial co-ordinates in the form

$$X = R\sin\Theta\cos\Phi\,, \quad Y = R\sin\Theta\sin\Phi\,, \quad Z = R\cos\Theta\,, \tag{5.56}$$

where the angles Θ and Φ specify a null ray (a generator) of the null cone for u a constant in the asymptotic region $R \to \infty$, then it can be shown that

$$\sin\Theta = \frac{1}{\cosh\psi}, \quad \Phi = \phi. \tag{5.57}$$

Equation (5.55) can be written as

$$\coth\eta = \frac{U}{c\sin\Theta}, \tag{5.58}$$

or equivalently

$$\cosh\eta = \frac{1}{\sqrt{1-(c\sin\Theta/U)^2}}. \tag{5.59}$$

As well, for $R \to \infty$, it follows that

$$\cosh\eta - \cosh\gamma = \frac{c^2\cosh\eta}{RU}. \tag{5.60}$$

5.2 $\langle\varphi^2\rangle^{\text{ren}}$ in the Wave Zone

In the coincidence limit the function $G_E^{\text{ren}}(x,x')$ is finite and it gives $\langle\varphi^2\rangle^{\text{ren}}$. Therefore, it follows that

$$\langle\varphi^2(x)\rangle^{\text{ren}} = N_E \sum_{\ell=0}^{\infty}(2\ell+1)\sum_{m=0}^{\infty}\beta_m\,[\mathcal{P}_\ell^m(\cosh\eta)]^2\,\frac{\mathcal{Q}_\ell^m(\cosh\eta_0)}{\mathcal{P}_\ell^m(\cosh\eta_0)}, \tag{5.61}$$

$$N_E = -\frac{(\cosh\eta-\cos\gamma)^2}{8\pi^2c^2}. \tag{5.62}$$

Under analytical continuation (5.49)–(5.50) the expressions which enter under summation remain unchanged, while the factor N_E is transformed into

$$N = \frac{(\cosh\eta-\cosh\gamma)^2}{8\pi^2c^2}. \tag{5.63}$$

Using (5.60) it is shown, in the wave-zone limit, this factor has the asymptotic value

$$N \sim \frac{c^2 \cosh^2 \eta}{8\pi^2 R^2 U^2}. \tag{5.64}$$

Finally, the representation for $\langle\varphi^2\rangle^{\text{ren}}$ in the wave zone is

$$\langle\varphi^2(x)\rangle^{\text{ren}} \sim \frac{c^2}{8\pi^2 R^2 U^2}\,\Phi(z, z_0)\,, \tag{5.65}$$

where

$$\Phi(z, z_0) = z^2 \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{m=0}^{\infty} \beta_m \,[\mathcal{P}_\ell^m(z)]^2 \,\frac{\mathcal{Q}_\ell^m(z_0)}{\mathcal{P}_\ell^m(z_0)}\,, \tag{5.66}$$

and

$$z = \frac{1}{\sqrt{1-(c\sin\Theta/U)^2}}\,, \quad z_0 = \cosh\eta_0 = \frac{1}{ab}\,, \quad c = \sqrt{\frac{1}{a^2} - b^2}\,. \tag{5.67}$$

It should be noted that functions of the type $F_1(z, z_0)$, apart from constants a and b which specify the problem, have dependence only on the combination $U/\sin\Theta$. Because of invariance under rotations in the X–Y-plane, there is no dependence on Φ. Moreover, the equation (5.5) for Σ is also invariant under a boost transformation in the T–Z-plane. It is shown in [18] that, as the result of this symmetry, $\langle\varphi^2\rangle^{\text{ren}}$ near $\mathcal{J}^+$ must have the form $\sim R^{-2}U^{-2}f(\sin\Theta/U)$. The fact that the result (5.65) has this form dictated by the symmetry of the problem gives an independent test of the correctness of the calculations.

It is also evident that all the dependence on U enters through the dimensionless time parameter $u = U/c$. If the size of the sphere is small, then $z_0 \to \infty$. In this approach, the asymptotic expansions of the Legendre functions are

$$\mathcal{P}_\nu^\mu(z_0) \sim \frac{(2z_0)^\nu}{\sqrt{\pi}}\,\Gamma\left(\nu + \frac{1}{2}\right), \quad Re(\nu) > -1/2\,, \tag{5.68}$$

$$\mathcal{Q}_\nu^\mu(z_0) \sim \frac{(2z_0)^{-\nu-1}\sqrt{\pi}}{\Gamma(\nu+\frac{3}{2})}. \tag{5.69}$$

Therefore,

$$\frac{\mathcal{Q}_\nu^\mu(z_0)}{\mathcal{P}_\nu^\mu(z_0)} \sim \frac{2\pi}{(2z_0)^{2\nu+1}(2\nu+1)\,[\Gamma(\nu+\frac{1}{2})]^2}, \quad Re(\nu) > -1/2\,. \tag{5.70}$$

Since the asymptotic of this ratio depends on m, $\Phi(z, z_0)$ can be written as an expansion with respect to b, in the form

$$\Phi(z, z_0) = 2\pi z^2 \sum_{\ell=0}^{\infty} \frac{1}{(2z_0)^{2\ell+1}\,[\Gamma(\ell+\frac{1}{2})]^2} F_\ell(z)\,, \tag{5.71}$$

where

$$F_\ell(z) = \sum_{m=0}^{\infty} \beta_m\,[\mathcal{P}_\ell^m(z)]^2\,. \tag{5.72}$$

The leading contribution for small b is proportional to b, and given by $\ell = 0$ term in the series (5.72). Notice that for integer m,

$$\mathcal{P}_0^m(z) = 2^{-m}(z^2-1)^{m/2}\,F\left(m+1, m; m+1; \frac{1-z}{2}\right) \tag{5.73}$$

(see formula 3.6.1.1 of [11]). Using the following property of the hypergeometric function

$$F(b, a; b; \xi) = (1-\xi)^{-a}\,, \tag{5.74}$$

it follows that

$$\mathcal{P}_0^m(z) = \left(\frac{z-1}{z+1}\right)^{m/2}. \tag{5.75}$$

To calculate $\mathcal{P}_\ell^m(z)$ for $\ell > 1$, the relation

$$\mathcal{P}_{\ell+1}^m(z) = (z^2-1)\frac{d\mathcal{P}_\ell^m(z)}{dz} + (\ell+1)\,z\,\mathcal{P}_\ell^m(z) \tag{5.76}$$

can be used. It is true that

$$\mathcal{P}_1^m(z) = (z+m)\left(\frac{z-1}{z+1}\right)^{m/2},$$

$$\mathcal{P}_2^m(z) = (3z^2+3zm+m^2-1)\left(\frac{z-1}{z+1}\right)^{m/2}. \tag{5.77}$$

By using Maple, the summation of (5.72) can be performed. It can be shown that $F_\ell(z)$ is a polynomial of z of the order $2\ell + 1$, where the first few harmonics of $F_\ell(z)$ are

$$F_0(z) = z\,, \tag{5.78}$$

$$F_1(z) = \frac{1}{2}(5z^3 - 2z^2 - 3z + 2)\,, \tag{5.79}$$

$$F_2(z) = \frac{z}{2}(63z^4 - 18z^3 - 70z^2 + 12z + 15)\,. \tag{5.80}$$

Using these results, it follows that the leading contribution to $\langle\varphi^2(x)\rangle^{\text{ren}}$ for the small radius of the mirror $b \ll a^{-1}$ is

$$\langle\varphi^2(x)\rangle^{\text{ren}} \sim \frac{abc^2}{8\pi^2 R^2 U^2\,[1 - (c\sin\Theta/U)^2]^{3/2}}\,. \tag{5.81}$$

By using relations (5.65), (5.71), and (5.79), higher order corrections to $\langle\varphi^2(x)\rangle^{\text{ren}}$ as powers of ab can be obtained.

5.3 Energy Density Flux in the Wave Zone

The calculation of the energy density flux in the wave zone

$$\frac{dE}{dU} = \lim_{R\to\infty} 4\pi R^2 \,\langle T_{UU}\rangle^{\text{ren}} \tag{5.82}$$

is similar to the calculation of $\langle\varphi^2(x)\rangle^{\text{ren}}$ but more involved. First, it is evident that the analytical continuation of G^{ren} given by (5.48) behaves as

$$G^{\text{ren}}(x, x') \sim \frac{g(U,\Theta,\Phi;U',\Theta',\Phi')}{RR'} \tag{5.83}$$

in the wave zone region. It is evident that, when calculating $\langle T_{UU}\rangle^{\text{ren}}$, it is sufficient to keep only derivatives with respect to U and U'. All other derivatives effectively introduce extra powers of R^{-1}, and do not contribute to the flux of the energy density at infinity. In the wave zone, it follows that

$$\frac{dE}{dU} = [(1-2\xi)\partial_U\partial_{U'} - \xi(\partial_U^2 + \partial_{U'}^2)]H(U,U';\Theta)\,, \tag{5.84}$$

where $H(U,U';\Theta) = g(U,\Theta,\Phi;U',\Theta,\Phi)$. ($H$ does not depend on Φ since g depends only on the difference $\Phi - \Phi'$.) Using the analytical continuation of (5.48), it is shown that

$$H(U,U';\Theta) = -\frac{c^2}{2\pi}\frac{zz'\,\mathcal{H}(z,z'|z_0)}{UU'}\,, \tag{5.85}$$

where

$$\mathcal{H}(z,z'|z_0) = \sum_{\ell=0}^{\infty}(2\ell+1)\mathcal{F}_\ell(z,z'|z_0)\,, \tag{5.86}$$

$$\mathcal{F}_\ell(z,z'|z_0) = \sum_{m=0}^{\infty}\beta_m\mathcal{P}_\ell^m(z)\,\mathcal{P}_\ell^m(z)\,\frac{\mathcal{Q}_\ell^m(z_0)}{\mathcal{P}_\ell^m(z_0)}\,. \tag{5.87}$$

For simplicity, consider $\xi = 0$ and calculate the canonical energy density flux. It can be shown that

$$\frac{\partial z}{\partial U} = \frac{z(1-z^2)}{U}\,. \tag{5.88}$$

Therefore,

$$\partial_U\left(\frac{zK(z)}{U}\right) = -\frac{z^2}{U^2}\mathcal{D}_z K(z)\,, \tag{5.89}$$

where

$$\mathcal{D}_z = z + (z^2-1)\partial_z\,. \tag{5.90}$$

The energy density flux can be written as

$$\frac{dE^{\text{can}}}{dU} = -\frac{c^2z^4}{2\pi U^4}\,[\mathcal{D}_z\mathcal{D}_{z'}\mathcal{H}(z,z'|z_0)]_{z'=z}\,. \tag{5.91}$$

For small value of the dimensionless parameter ab, the leading term is given by $\ell = 0$ modes and so

$$\mathcal{H}(z, z'|z_0) \sim \frac{1}{z_0} \sum_{m=0}^{\infty} \beta_m \mathcal{P}_0^m(z)\, \mathcal{P}_0^m(z') \,. \tag{5.92}$$

According to (5.76),

$$\mathcal{D}_z \mathcal{P}_0^m(z) = \mathcal{P}_1^m(z) \,, \tag{5.93}$$

and using (5.72), the result is

$$\frac{dE^{\text{can}}}{dU} = -\frac{c^2}{2\pi z_0} \frac{z^4}{U^4} F_1(z) \,, \tag{5.94}$$

where $F_1(z)$ is known from (5.79). Combining these results, the leading order contribution to the canonical energy density flux in the wave zone is

$$\frac{dE^{\text{can}}}{dU} = -\frac{c^2}{4\pi z_0} \frac{z^4}{U^4} (5z^3 - 2z^2 - 3z + 2) \,. \tag{5.95}$$

At this point, it is important to note that (5.95) has a well-defined physical meaning for only a certain region of Rindler space. This is evident from the definition of z in (5.67), where it can be shown that $z \to \infty$ when $U \to c \sin \Theta$. Then (5.95) becomes infinitely negative, suggesting that an infinite negative energy gets radiated away. Furthermore, when $U < c \sin \Theta$, z becomes complex, which leads to complex values for the energy density flux. This is clearly unphysical, which reflects the need to introduce the restriction that $|U| > |c \sin \Theta|$ for (5.95) to make any sense, where c corresponds to the sphere's radius by $c = b \sinh \eta_0$. It is also important to note that $z = 1$ when $\sin \Theta = 0$ or $c \to 0$, which implies that (5.95) is regular for all values of $U \neq 0$ under either condition. Obviously, $c \to 0$ corresponds to the condition that the mirrored surface appears like a point-like particle.

Determining the energy density flux for nonzero ξ is somewhat more involved, though still straightforward. Evaluating the second-order partial derivatives in (5.84), it is shown that

$$\partial_U^2 \left(\frac{zK(z)}{U} \right) = \frac{z^3}{U^3} \mathcal{E}_z K(z) , \tag{5.96}$$

where

$$\mathcal{E}_z = 3z^2 - 1 + 5z(z^2 - 1)\partial_z + (z^2 - 1)^2 \partial_z^2 . \tag{5.97}$$

It follows from (5.84) and (5.85) that

$$\frac{dE}{dU} = \frac{z^4}{U^4} \left[(1 - 2\xi) \mathcal{D}_z \mathcal{D}_{z'} - \xi \left(\mathcal{E}_z + \mathcal{E}_{z'} \right) \right] H(U, U'; \Theta) , \tag{5.98}$$

Because $\mathcal{E}_z$ acts only on $\mathcal{P}_0^m(z)$ in (5.92), the sum in (5.98) has to be evaluated before taking the coincidence limit. It can, therefore, be shown that

$$\sum_{m=0}^{\infty} \beta_m \mathcal{P}_0^m(z) \, \mathcal{P}_0^m(z') = 2 \left[1 - q(z) q(z') \right]^{-1} - 1, \tag{5.99}$$

where $q(z) = \sqrt{(z-1)/(z+1)}$. When $z' \to z$, the sum goes to $F_0(z) = z$ as expected. By evaluating (5.99) for $\mathcal{E}_z$ and taking the coincidence limit, it is shown that

$$\left[\mathcal{E}_z \mathcal{H}(z, z' | z_0) \right]_{z'=z} = \frac{1}{z_0} \left[5z^3 - 3z \right] . \tag{5.100}$$

Then the leading order contribution for the energy density flux in the wave zone for nonzero ξ is

$$\frac{dE(\xi)}{dU} = -\frac{c^2}{4\pi z_0} \frac{z^4}{U^4} \Big[(1 - 2\xi)(5z^3 - 2z^2 - 3z + 2)$$

$$- 2\xi \; (5z^3 - 3z) \Big] . \tag{5.101}$$

Chapter 6

Conclusion

In this thesis, a number of systems with moving mirror-like boundaries are studied. For the mirror configurations described in Chapters 3–5, vacuum expectation values for zero-point fluctuations $\langle\varphi^2(x)\rangle$ and the stress-energy tensor $\langle T_{\mu\nu}(x)\rangle$ are calculated. At this point, it is useful to review and compare the obtained final results.

In Chapter 3, the quantum radiation emitted from a single and two concentric spherical semi-transparent expanding mirrors is studied. The formulation is done in D-dimensional Euclidean space-time with static mirrors of $d = D - 1$ dimensions to describe the accelerated motion of $d-1$-dimensional mirrors. The semi-transparency of the mirrors is modelled by delta potentials of finite magnitude. The renormalized result (3.66), which for the mirror potential $U_0 \to \infty$ leads to (3.71), is an expression consistent with the result obtained by Frolov and Serebriany [15] for a perfectly reflecting mirror with dimension $d = 3$. A special consideration of the $d = 3$ case shows that, from (3.72)–(3.78), the convergent part of $\langle\varphi^2(x)\rangle^{\text{ren}}$ decays to zero by a power of four greater for the region exterior to the mirror compared to the interior. The divergent part of $\langle\varphi^2(x)\rangle^{\text{ren}}$ is most evident at the mirror surface, which is expected because the boundary introduces a first-derivative discontinuity in the first place. The concentric mirror boundary case for $\langle\varphi^2(x)\rangle^{\text{ren}}$ is a considerably more complicated expression given by (3.79)–(3.81). It is not obvious how to extract a convergent part from (3.79)–(3.80) as outlined for the single mirror. However, the associated graphs suggest that, for a mirror with magnitude considerably smaller than the other, the backscattering of the field fluctuations due to it is considerably less pronounced than that of the other mirror. Again, this is not surprising, since continuity at the mirrors' surfaces requires that the weaker potential allows for greater field transmission than for the stronger one.

In calculating the stress-energy tensor for these mirror configurations, it is shown by rotational invariance that $\langle T_{\mu\nu}(x)\rangle$ can be described in terms of the energy density ε, defined by (3.82)–(3.86). For the special case of conformal invariant theory, where $\xi = (d-1)/4d$, the stress-energy tensor for a single mirror vanishes, which agrees with the results of Frolov and Serebriany [15]. This is also true for the concentric mirror problem in the exterior and interior regions. As for the intermediate region, the results (3.123)–(3.125) are also consistent with the results of Frolov and Serebriany [15].

In Chapter 4, the vacuum zero-point fluctuations and stress-energy tensor for a uniformly accelerating refractive polarizable body is studied. By assuming that the refractive index n is only slightly larger than unity, and that the dielectric body of radius b is much less than the characteristic length scale l, it is possible to obtain expressions for $\langle\varphi^2(x)\rangle^{\text{ren}}$ and $\langle T_{\mu\nu}(x)\rangle$ as perturbation expansions relative to a static metric. The results here can be compared with the same quantities calculated in Chapter 5 due to a uniformly accelerating spherical mirror. What is found between the two sets of calculations is that, apart from differences in the coefficients, the values (4.79) and (5.79) for $\langle\varphi^2(x)\rangle^{\text{ren}}$ in the wave zone both vary as u^{-2}, where u is the retarded time, with corrections proportional to z^{-3}, where $z = \sin\Theta/u$. As for the energy density flux, the leading order term obtained in Chapter 5 for non-zero ξ (5.98) varies as u^{-4}, while the corresponding term in Chapter 4 (4.82) also has contributions which vary as u^{-6}, u^{-8}, u^{-10}, and higher order corrections in u^{-1}. It is expected that, because of the perturbation approximation required in finding (4.82), it is likely that more complicated expressions in Chapter 4 are expected when $n \gg 1$.

The examples considered here are chosen knowing that they possess certain symmetries to ensure a tractable solution. In particular, all of them involve mirror expansion (Chapter 3) or motion (Chapters 4-5) with uniform acceleration, in which case the respective boundary conditions are boost invariant. This symmetry, along with spherical symmetry, allow for a separation of variables to solve these problems. It is also significant that, after performing a Wick rotation into Euclidean space, the examples presented are reducible to equivalent electrostatic boundary value problems. In Chapters 3-5, it is found that the vacuum expectation of the energy density flux is negative. This result does not contradict with the general expectation that the total energy must be positive for a system in flat spacetime. It is expected that, for a more physically realistic problem when a mirror is accelerated for only a finite time, the total emitted energy must always be positive (see, for example, the discussion in [13] and references therein).

An important question emerges from studying quantum aspects of uniform accelerated motion. From the principle of equivalence, the properties of a system with respect to a uniformly accelerated observer are the same as properties of the same system in a homogeneous gravitational field with respect to an observer at rest, provided when the states are chosen with this correspondence. However, in a Rindler frame, which is equivalent to a static gravitational field, the boundaries are at rest with an invariant system under the discrete symmetry $t \to -t$, where t is the Rindler proper time. As a result, there are no energy fluxes in this frame, which seems contradictory.

A similar problem is discussed in detail for classical radiation of a uniformly accelerated charge, and can be found in [23] and the references therein. It is important to consider the relative acceleration between the emitter and observer in order to determine the observation of radiation. It is best to make these observations in the wave zone region far away from the radiative system. For uniformly accelerating observers, the wave zone is always located outside of the region of space-time covered by the Rindler frame. In order to pass to the wave zone from the Rindler wedge, it is necessary to cross the null horizon surface separating the regions.

It is noted that the energy density flux calculations in Chapters 3-5 come with a negative sign, which implies that the mirror surfaces radiate negative energy. However, this result is derived when considering the radiation flux for a particular *instant* of retarded time U, and does not necessarily suggest that the *total* radiated energy E is negative over a *finite* time interval. To show whether the total energy is positive requires the integration of the flux density over the given retarded time interval, excluding $U = 0$, where the formula becomes singular. It is not immediately obvious from the examples shown in Chapters 3-5 that E is generally positive, since the formalism is highly dependent upon the choice of quantum state, and perhaps also on the mirror's trajectory through space-time. A more careful analysis of this situation for the cases considered may be in order.

A particularly significant paper [28] discusses the classical radiation of a uniformly accelerated charge using a quantum approach. Here, it is shown that this radiation in the wave zone is due to the zero-frequency modes of the field in the Rindler frame. This approach is a departure from the chosen method here, where the information about the quantum field energy density fluxes in the wave zone are obtained by a simple analytical continuation of the observables from the wedge covered by the Rindler frame. A possible future direction is to repeat this approach for the problems considered in the thesis, which would make the analysis here much more involved due to the complexity of the required calculations.

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