University of Alberta

MODULATION OF THE HARMONIC SOLITON SOLUTIONS FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION

BY

SUNIL KUMAR BARRAN ©

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S.K. Barran

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: Dec. 1. 1998

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled MODULATION OF THE HARMONIC SOLITON SOLUTIONS FOR THE DEFOCUSING NON-LINEAR SCHRÖDINGER EQUATION submitted by SUNIL KUMAR BARRAN in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in APPLIED MATHEMATICS.

Mikhail Kovalyov

M. Kovalyov (Supervisor)

H.P. Künzle

H.P. Künzle

Martin Legaré

M. Legaré

Peter Schiavone

P. Schiavone

Date: Dec. 1. 1998

ABSTRACT

In this thesis we examine some of the properties of the slowly decaying solutions of the nonlinear Schrödinger equation. By the superpositioning slowly decaying solutions we get the so called harmonic soliton solutions. In particular we examine some of the modulating properties of the harmonic soliton solutions for the defocusing Schrödinger equation:

$$iu_t + u_{xx} - 2|u|^2 u = 0$$

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For

Mom and Dad

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Chapter 1

Beginnings

1.1 Introduction

Most wave motions are modeled by linear hyperbolic equations, which we can solve by means of the Fourier transform as we will discuss in Chapter 2. After some simplifications these models often lose their hyperbolicity and more important their linearity, [12]. As a result the Fourier transform method can no longer be applied.

In this thesis we investigate the possibility of a replacement of the Fourier transform for some nonlinear KdV-type equations, in particular the nonlinear defocusing Schrödinger equation. First we remark, that the Fourier transform method is a mathematical description of what is known in physics as linear modulation of waves. Linear modulation can be used to construct localized in space solutions for linear partial differential equations, so called wave packets. The natural question one can ask is whether the existence of nonlinear modulation yields exact solutions of a nonlinear partial differential equation. The analogy between Fourier analysis and the Inverse Scattering Theory used for solving the Cauchy problem for integrable equations suggests that if any nonlinear partial differential equations allow nonlinear modulation, then they would be the integrable equations. It will be shown that the nonlinear defocusing Schrödinger equation permits nonlinear modulation.

In Chapter 2 we tersely review the ideas behind the Fourier transform method for linear partial differential equations. We will review how to construct an arbitrary solution of the form $e^{i(kx-\omega(k)t)}$, which can be viewed as the simplest components. In Chapter 3 we use the representation of such solutions to derive the KdV and NLS equations. The solutions to the KdV and NLS can then be obtained by the inverse scattering method. Chapter 4 contains the main results of this thesis, namely the generation of the N-harmonic soliton solution for the defocusing Schrödinger and some time evolution profiles for the N-harmonic soliton case. These results are new and could possibly be applied to Soliton-Lasers and Soliton-Based Communication Systems, [13].

1.2 Background and Some History

Solitary waves were first observed by J. Scott Russel in 1834. After extensive experiments in the laboratory, he observed that:

1. solitary waves are long, shallow water waves of permanent form,

2. solitary waves propagate at speed $c = \sqrt{g(h+a)}$,

where g is the acceleration due to gravity, a is the amplitude and h is the uniform depth of the waveguide/channel. Further investigations into the existence of solitary waves were undertaken by Airy (1845), Stokes (1847), Boussinesq (1871-72) and Raleigh (1876). Korteweg and de Vries (1895) derived a nonlinear equation, named after them, governing long one dimensional, small amplitude, surface gravity waves in a shallow media, [14]. Further progress came when Fermi, Pasta and Ulam (1955) considered the problem of a one dimensional anharmonic lattice of equal masses coupled by nonlinear springs modeled by the KdV, [15]. This subsequently lead to the discovery of so called **solitary waves** i.e. a localized traveling wave solution, of the form *f(x-ct)*, of a nonlinear partial differential equation. Zabusky and Kruskal (1965) did extensive computer simulations for the initial value problem for the KdV:

$$u_t + uu_x + (0.022)^2\ u_{xxx} = 0$$

$$u(x,0) = \cos(\pi x), \qquad\qquad 0 \le x \le 2$$

where u, u_x, u_{xxx} are periodic functions on [0,2] for every t. They discovered some solitary wave-type solutions with some rather interesting properties, [16]. They called them **solitons**. They observed:

1. solitary wave is a stable formation;

2. when solitary waves move with different velocities, the faster one will overtake the slower one and, after a complicated nonlinear interaction, the solitons emerge in their original form with possible delays (phase shifts) due to interaction (Fig. 1).

3. every sufficiently smooth and exponentially decaying solution of the KdV with initial condition $u(x,0) = \varphi(x)$, decomposes as $t \longrightarrow \infty$ into a finite number of solitary waves of various speeds and a dispersive tail (Fig. 2).

This kind of behavior is expected for linear problems, for example for the wave equation $u_{tt} = c^2\, u_{xx}$ with $u(x,0) = \varphi(x)$ on $0 < x < l$, $u(0,t) = u(l,t)$ and $u_t(x,0) = \psi(x)$, since each eigenfunction evolves separately, but that this could happen for a nonlinear problem was not expected. Later on, a number of other equations with solutions possessing properties similar to that of solitons were discovered, e.g.

- Nonlinear Schrödinger: $iu_t + u_{xx} + 2\nu|u|^2 u = 0, \qquad \nu = \pm 1$
- Sine-Gordon: $u_{tt} - u_{xx} + \sin(u) = 0$
- Kadomstev-Petviashvili: $(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$

We start by reviewing the ideas behind Fourier transform which will subsequently lead us to the KdV and the NLS.

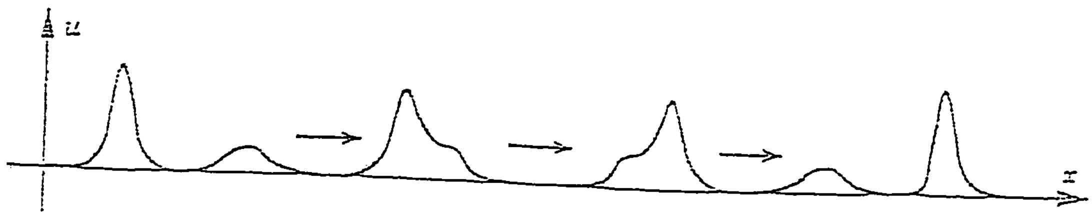

Figure 1

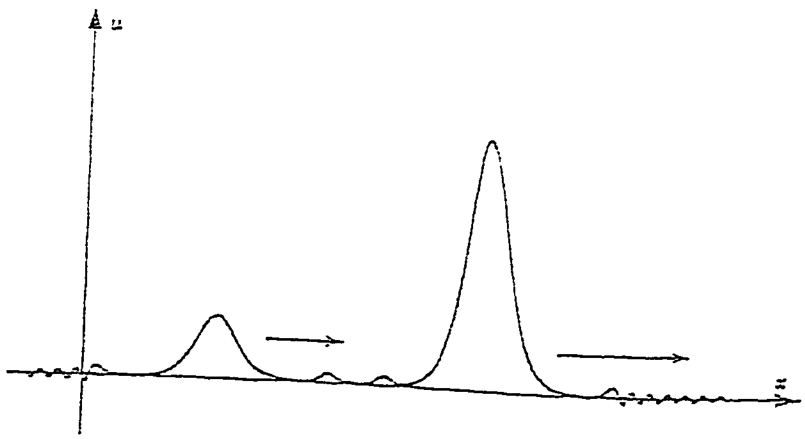

Figure 2

Chapter 2

Fourier Transform for linear equations

The most general Cauchy problem solvable by the Fourier transform is the one reducible to the form, [7]

$$P(D,\tau)u = 0 \qquad \text{for } t \geq 0 \tag{2.1}$$

$$\tau^k\, u = 0 \quad \text{for } k = 0,\ 1, \ldots,\ m-2 \ \ \textit{and}\ t = 0 \tag{2.2}$$

$$\tau^{m-1}\, u = f(x) \qquad \text{for } t = 0 \tag{2.3}$$

where

$$D = (D_1, \ldots, D_n) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \quad \tau = \frac{\partial}{\partial t} \tag{2.4}$$

and $P(D,\tau) = P(D_1, \ldots, D_n, \tau)$ is a polynomial of degree m in its $n+1$ arguments.

Consider the simplest case of (2.1)-(2.3) that appears in the theory of waves, [2]:

$$u_t = \frac{-\omega(k)}{k}\, u_x \tag{2.5}$$

$$u(x,0) = f(x) \tag{2.6}$$

A formal solution of the standard problem (2.5)-(2.6) can be obtained using the Fourier transformation

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{ikx} dk \tag{2.7}$$

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx} dx \tag{2.8}$$

Assuming the validity of the interchange of derivative and integral, by taking the Fourier transform of (2.5), we get:

$$\frac{d\hat{u}}{dt} = -i\omega(k)\hat{u} \tag{2.9}$$

which is a linear differential equation with solution:

$$\hat{u}(k,t) = \hat{u}(k,0)\, e^{-i\omega(k)t} \tag{2.10}$$

where

$$\hat{u}(k,0) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx \tag{2.11}$$

The solution of (2.5)-(2.6) is:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{i(kx-\omega t)} dk \qquad (2.12)$$

If the system (2.10) is conservative, i.e. $\omega(k)$ is a real function and dispersive ($\omega''(k) \neq 0$), then the solution may decay into wavepackets which move with group velocities $\omega'(k)$ and decays as $t \longrightarrow \infty$. So an arbitrary solution can be decomposed into $e^{i(kx-\omega(k)t)}$, which can be viewed as the simplest components of that solution.

Chapter 3

Origins

3.1 Derivation of KdV and NLS equations

We first begin by deriving the well known KdV equation and then the NLS equation since the analysis of the NLS equation parallels that of the KdV equation. We showed in the previous chapter that an arbitrary wave can be decomposed into harmonics $e^{i(kx-\omega(k)t)}$. Consider the plane wave corresponding to one of these harmonics:

$$v(x,t) = e^{i(kx-\omega t)} \tag{3.1}$$

Assume the dispersion relation is of the form:

$$\omega(k) = \alpha k - \beta k^3 \tag{3.2}$$

A partial differential equation for $v(x,t)$ with this dispersion relation is:

$$v_t + \alpha v_x + \beta v_{xxx} = 0, \tag{3.3}$$

which is often called the linearized KdV equation, [6]. In shallow water theory, the following conservation law must also hold, [2]:

$$\partial_t v + \partial_x \mathbf{j} = 0 \tag{3.4}$$

For the linearized version of the KdV equation, $\mathbf{j}$ is given by:

$$\mathbf{j} = \alpha v + \beta v_{xx} + \frac{1}{2}\gamma v^2 \tag{3.5}$$

Combining (3.2),(3.3),(3.4) together we obtain, [2]:

$$v_t + \alpha v_x + \beta v_{xxx} + \gamma v v_x = 0 \tag{3.6}$$

Using the rescaling,

$$x = \sqrt{\beta}\xi + \alpha\sqrt{\beta}\tau, \tag{3.7}$$

$$t = \sqrt{\beta}\tau \tag{3.8}$$

and

$$w(\xi, \tau) = \frac{\gamma}{6} v(x, t) = \frac{\gamma}{6} v(\sqrt{\beta}\xi + \alpha\sqrt{\beta}\tau, \sqrt{\beta}\tau) \tag{3.9}$$

we can rewrite (3.6) in the standard form

$$w_\tau + w_{\xi\xi\xi} + 6ww_\xi = 0 \tag{3.10}$$

In a similar fashion we derive the nonlinear Schrödinger equation (3.11):

$$iu_t + u_{xx} + 2\nu|u|^2 u = 0 \tag{3.11}$$

For a modulate wavetrain with most of the energy in wave numbers close to some values k_0, $\hat{f}(k)$ is concentrated near $k = k_0$, and one can approximate (2.12) by:

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\ e^{i[kx - t(\omega_0 + (k-k_0)\omega_0' + (k-k_0)^2 \omega_0''/2)]} dk \tag{3.12}$$

where $\omega_0 = \omega(k_0),\ \omega_0' = \omega'(k_0),\ \omega_0'' = \omega''(k_0)$.

Let $k - k_0 = \kappa$, then:

$$\Phi = \frac{1}{2\pi}\ e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \hat{f}(k_0 + \kappa)\ e^{i[\kappa x - t(\kappa\omega_0' + \kappa^2 \omega_0''/2)]} dx =$$

$$= \frac{1}{2\pi}\ e^{i(k_0 x - \omega_0 t)} \cdot \varphi(t, x), \tag{3.13}$$

where $\varphi(t, x)$ describes the modulation in (3.13) and satisfies

$$i(\varphi_t + \omega_0' \varphi_x) + \frac{1}{2}\omega_0'' \varphi_{xx} = 0 \tag{3.14}$$

with the corresponding dispersion relation:

$$W(\kappa) = \kappa\omega_0' + \frac{1}{2}\kappa^2\ \omega_0'' \tag{3.15}$$

The equation for Φ with the original dispersion relation

$$\omega(k) = \omega_0 + (k - k_0)\omega_0' + \frac{(k-k_0)^2}{2}\omega_0'' \tag{3.16}$$

is:

$$i\Phi_t - \left(\omega_0 - k_0\omega_0' + \frac{k_0^2 \omega_0''}{2}\right)\Phi + i(\omega_0' - k_0\omega_0'')\Phi_x +$$

$$+ \frac{\omega_0''}{2}\Phi_{xx} = 0 \tag{3.17}$$

If the approximation (3.14) to the linear dispersion is combined with a cubic nonlinearity, [2], we have:

$$i(\varphi_t + \omega_0'\varphi_x) + \frac{1}{2}\omega_0''\varphi_{xx} + q|\varphi|^2\varphi = 0 \tag{3.18}$$

Since $\varphi(t,x) = a \cdot e^{i(\kappa x - Wt)}$ is still a solution to (3.18), we get that the nonlinear correction to the dispersion relation modifies W to:

$$W = \kappa\omega_0' + \frac{\kappa^2}{2}\omega_0'' - q|\varphi|^2 \tag{3.19}$$

Equation (3.18) can be normalized by choosing a frame of reference moving with linear group velocity ω_0' to eliminate the φ_x term in (3.18) and then, after rescaling the variables, [2], we get the nonlinear Schrödinger equation (3.11).

There are actually two NLS's, one with $\nu = 1$, the other with $\nu = -1$. The NLS can be considered as the Hartree-Fock equation for a one-dimensional quantum Boson gas with $\delta-$point interaction. Then ν plays the role of a coupling constant: the case $\nu > 0$ corresponds to attractive interaction between particles and $\nu < 0$ is the repulsive case, [6]. The two cases are essentially different in optical applications, describing selffocusing or defocusing of the light rays in nonlinear waveguides, [10].

3.2 Inverse Scattering Theory for KdV and NLS equations

To illustrate the method of solving KdV, NLS-type equations using the method of inverse scattering developed by Gardner-Green-Kruskal-Miura, [17, 18], we start by considering the simplest of the equations, [5]. Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.20}$$

with

$$u(x, 0) = f(x) \tag{3.21}$$

where $f(x)$ is sufficiently smooth and decays rapidly as $|x| \longrightarrow \infty$. The basic idea is to relate the KdV equation to the time-independent Schrödinger scattering problem:

$$(L - \lambda)\psi := \psi_{xx} + u\psi = 0 \tag{3.22}$$

The time-dependence of the eigenfunctions (3.22) is given by

$$\psi_t = M\psi := (\gamma + u_x)\psi - (4\lambda - 2u)\psi_x, \tag{3.23}$$

where γ is an arbitrary constant parameter and λ is the spectral parameter.

The KdV can then be written using the Lax pair [M,L], as:

$$L_t = [M, L] \tag{3.24}$$

where the operators M and L are given by:

$$L := -\frac{\partial^2}{\partial x^2} + u \tag{3.25}$$

$$M := 4\frac{\partial^3}{\partial x^3} - 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) \tag{3.26}$$

Equation (3.24) is satisfied if and only if $u(x, t)$ satisfies the KdV.

The solution of the KdV with initial condition (3.21) is as follows:

At $t = 0$, we need to determine the spectrum of the Schrödinger equation (3.22) which consists of a finite number of discrete eigenvalues, $\lambda_n = \xi_n^2 > 0$, $n = 1, \ldots, N$ and continuous spectrum, $\lambda = -k^2 < 0$. The corresponding eigenfunctions are:

1. For $\lambda = \xi_n^2$: $\psi_n(x,t) \approx c_n(t)\ e^{-\xi_n x}$ as $x \longrightarrow \infty$, with $\int_{-\infty}^{\infty} \psi_n^2(x,t)dx = 1$

2. For $\lambda = -k^2$:

   (a) $\psi(x,t) \approx e^{-ikx} + r(k,t)\ e^{ikx}$ as $x \longrightarrow \infty$

   (b) $\psi(x,t) \approx a(k,t)\ e^{-ikx}$ as $x \longrightarrow -\infty$

where $r(k,t)$ and $a(k,t)$ are the reflection and transmission coefficients respectively.

At $t = 0$ we define the scattering data to be:

$$S(\lambda, 0) = \left(\{\xi_n, c_n(0)\}_{n=1}^{N},\ r(k,0),\ a(k,0)\right)$$

The time evolution of the scattering data

$$S(\lambda, t) = \left(\{\xi_n, c_n(t)\}_{n=1}^{N},\ r(k,t),\ a(k,t)\right)$$

is given by the formulae:

- ξ_n=constant
- $c_n(t) = c_n(0)\ e^{4\xi_n^2 t}$, $\qquad n = 1, \ldots, N$
- $a(k,t) = a(k,0)$

- $r(k,t) = r(k,0)\ e^{8ik^3 t}$

Using the scattering data define

$$F(x;t) = \sum_{n=1}^{N} c_n^2(t)\ e^{-\xi_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k,t)\ e^{ikx} dk \tag{3.27}$$

Then we recover the potential $u(x,t)$ via:

$$u(x,t) = 2\frac{\partial}{\partial x}[K(x,x;t)] \tag{3.28}$$

where $K(x,x;t)$ is the solution of the Gel'fand-Levitan-Marchenko equation

$$K(x,y;t) + F(x+y;t) + \int_x^{\infty} K(x,z;t)F(z+y;t)dz = 0 \tag{3.29}$$

The method just described is a concise version of the method of inverse scattering developed by Gardner-Green-Kruskal-Miura.

Zakharov and Shabat (1972) showed that the inverse scattering approach may also be applied to solve the NLS, [1]. In particular they showed that the nontrivial operators

$$L := i\begin{pmatrix} 1+\kappa & 0 \\ 0 & 1-\kappa \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & \bar{u}^1 \\ u & 0 \end{pmatrix}$$

and

$$M := i\kappa\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} -\frac{i|u|^2}{1+\kappa} & \bar{u}_x \\ -u_x & \frac{i|u|^2}{1-\kappa} \end{pmatrix}, \nu = \frac{2}{1-\kappa^2}$$

satisfy Lax's equation (3.24) if and only if $u(x,t)$ is a solution of the NLS equation.

If we let

$$F(x) = \sum_{n=1}^{N} c_n \ e^{ik_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\xi) \ e^{i\xi x} d\xi \qquad (3.30)$$

where

- $r(\xi, t) = r(\xi, 0) \ e^{6i\xi^2 t}$
- $c_j(t) = c_j(0) \ e^{4ik_j^2 t}$

Then we obtain the Gel'fand-Levitan-Marchenko system:

$$K_1(x,y) = \bar{F}(x+y) + \int_x^{\infty} \bar{K}_2(x,s)\bar{F}(s+y)ds \qquad (3.31)$$

$$\bar{K}_2(x,y) = -\int_x^{\infty} K_1(x,s)F(s+y)ds \qquad (3.32)$$

which allows us to recover the potential via

$$u(x) = 2iK_1(x,x), \ \ \int_x^{\infty} |u(s)|^2 ds = -2K_2(x,x).$$

In the reflectionless case the reflection coefficient is zero, i.e. $r(k,t) = r(k,0) = 0$, and we can solve the Gel'fand-Levitan-Marchenko equation using the method of separation of variables, [18], [19].

[1] $\bar{u}$ denotes the complex conjugate of u

Chapter 4

Soliton Theory

4.1 Derivation of the N-soliton solution

Besides the method described in Chapter 3, we can obtain the N-soliton solutions for the NLS following the methods of [1], [4], and [6]. We first observe that the NLS can be written as:

$$U_t - V_x = [V,\ U] \tag{4.1}$$

which is a compatibility condition for:

$$\frac{\partial \Phi}{\partial x} = U\Phi \tag{4.2}$$

$$\frac{\partial \Phi}{\partial t} = V\Phi \tag{4.3}$$

where

$$U = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + i\lambda \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

and

$$V = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix} \tag{4.5}$$

The field functions $q(x,t)$ and $r(x,t)$ are independent of λ. Note that (4.1) can also be written in Lax's form:

$$\left(U - \frac{\partial}{\partial x}\right)_t = \left[V,\, U - \frac{\partial}{\partial x}\right]$$

We can easily obtain the field functions for both NLS's with $r = \pm\bar{q} = 0$ and $\nu = \pm 1$.

For $r = \bar{q} = 0$ and $\nu = \pm 1$ equations (4.4) and (4.5) assumes the form:

$$U_0 = i\lambda \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

$$V_0 = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.7}$$

and equations (4.2), (4,3) degenerates into

$$\frac{\partial \Phi_0}{\partial x} = U_0 \Phi_0 \tag{4.8}$$

$$\frac{\partial \Phi_0}{\partial t} = V_0 \Phi_0 \tag{4.9}$$

Solving the system (4.8), (4.9) we get,

$$\Phi_0 = \begin{pmatrix} \varphi_{01}(x,t,\lambda) & \psi_{01}(x,t,\lambda) \\ \varphi_{02}(x,t,\lambda) & \psi_{02}(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda(x+2i\lambda t)} & 0 \\ 0 & e^{-i\lambda(x+2i\lambda t)} \end{pmatrix} \tag{4.10}$$

With Φ_0 given we can now obtain N-soliton solutions, $q(x,t)$, by means of the N-fold Bäcklund-Darboux-Matveev transformations [4], [6], in the form:

$$q(x,t) = -2i \frac{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \lambda_1^2 & \lambda_1^2\beta_1 & \cdots & \lambda_1^{N-1} & \lambda_1^N \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \lambda_2^2 & \lambda_2^2\beta_2 & \cdots & \lambda_2^{N-1} & \lambda_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_{2N} & \lambda_{2N} & \lambda_{2N}\beta_{2N} & \lambda_{2N}^2 & \lambda_{2N}^2\beta_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^N \end{vmatrix}}{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \lambda_1^2 & \lambda_1^2\beta_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\beta_1 \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \lambda_2^2 & \lambda_2^2\beta_2 & \cdots & \lambda_2^{N-1} & \lambda_2^{N-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_{2N} & \lambda_{2N} & \lambda_{2N}\beta_{2N} & \lambda_{2N}^2 & \lambda_{2N}^2\beta_{2N} & \cdots & \lambda_{2N}^{N-1} & \lambda_{2N}^{N-1}\beta_{2N} \end{vmatrix}} \tag{4.11}$$

where $\beta_j = c\,\dfrac{\varphi_{01}(x,t,\lambda_j)}{\psi_{02}(x,t,\lambda_j)}$, $\lambda_j = \xi_j + i\eta_j$, $c=$ *constant.*

For the NLS equation, $r = \pm\bar{q}$, $\lambda_{j+N} = \bar{\lambda}_j$, and $\beta_{j+N} = \dfrac{\nu}{\bar{\beta}_j}$ for some constants η_j, ξ_j, with $j \leq N$.

The 1-soliton solution of the NLS is:

$$u(x,t) = \begin{cases} \dfrac{2\eta\ \exp(-2i(\xi x + 2(\xi^2 - \eta^2)t + \gamma_0))}{\cosh(2\eta(x - p + 4\xi t))} & \text{if } \nu = 1 \\ \\ \dfrac{2\eta\ \exp(-2i(\xi x + 2(\xi^2 - \eta^2)t + \gamma_0))}{\sinh(2\eta(x - p + 4\xi t))} & \text{if } \nu = -1 \end{cases} \tag{4.12}$$

and the corresponding Φ functions given by :

Case I: $\nu = -1$

$$\Phi = \frac{1}{2sinh(\theta)} \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \tag{4.13}$$

where:

- $\Phi_{11} = ((\lambda - \xi - i\eta)e^{\theta} + (-i\eta + \xi - \lambda)e^{-\theta})\ e^{-i\lambda x - 2i\lambda^2 t}$
- $\Phi_{12} = 2i\eta\ e^{i(x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi)}$
- $\Phi_{21} = -2i\eta\ e^{-i(x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi)}$
- $\Phi_{22} = ((\lambda - \xi + i\eta)e^{\theta} + (i\eta + \xi - \lambda)\ e^{-\theta})\ e^{i\lambda x + 2i\lambda^2 t}$
- $\theta = 2(x - p + 4t\xi)\eta$

Case II: $\nu = +1$

$$\Phi = \frac{-1}{2cosh(\theta)} \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \tag{4.14}$$

where:

- $\Phi_{11} = ((-\lambda + \xi + i\eta)e^{\theta} + (-i\eta + \xi - \lambda)e^{-\theta})\ e^{-i\lambda x - 2i\lambda^2 t}$
- $\Phi_{12} = 2i\eta\ e^{i(x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi)}$
- $\Phi_{21} = 2i\eta\ e^{-i(x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi)}$

- $\Phi_{22} = ((-\lambda + \xi - i\eta)e^{\theta} + (i\eta + \xi - \lambda)\ e^{-\theta})\ e^{i\lambda x + 2i\lambda^2 t}$

- $\theta = 2(x - p + 4t\xi)\eta$

The constants $\eta = \Im(\lambda_1)$ and $\xi = \Re(\lambda_1)$ determine the amplitude and velocity ($v = -4\xi$) of the soliton respectively, [1]. Unlike the KdV soliton, η and ξ are independent and can be choosen arbitrary. In the general N-case, the N-soliton solution of the NLS equation depends on $4N$ arbitrary constants; η_j, ξ_j, p_j, γ_j.

For distinct ξ_j, as $t \longrightarrow +\infty$ the N-soliton breaks up into individual solitons in such a way that the fastest soliton is always in front of the slowest one in the rear, and vice-versa as $t \longrightarrow -\infty$. In this case, p_j (the center coordinates) and γ_j (phase angle) are no longer fixed. For identical ξ_j, the solitons form a bound state, [9].

4.2 Harmonic Solitons

The NLS with $\nu = 1$ (self-focusing case) has regular solitons that describe modulated pulses. For the NLS with $\nu = -1$ (de-focusing case) we get singular solitons, due to the singularity they can no longer be modulated pulses the way the regular solitons are. We will show that we can still obtain modulation for $\nu = -1$ using what we call **harmonic solitons**. These harmonic solitons appear when $\Im(\lambda_j) = \eta_j = 0$, i.e when both the numerator and denominator of (4.11) are zero. By taking $\lim\limits_{\eta \to 0}$ of (4.11) we get (4.15):

$$q = -2i\ AB^{-1} \tag{4.15}$$

where:

$$A = \begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1\beta_1 & \xi_1^2 & \xi_1^2\beta_1 & \cdots & \xi_1^{n-1} & \xi_1^n \\ 1 & \beta_2 & \xi_2 & \xi_2\beta_2 & \xi_2^2 & \xi_2^2\beta_2 & \cdots & \xi_2^{n-1} & \xi_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_n & \xi_n & \xi_n\beta_n & \xi_n^2 & \xi_n^2\beta_n & \cdots & \xi_n^{n-1} & \xi_n^n \\ 0 & \mu_1\beta_1 & 1 & (1+2\mu_1\xi_1)\beta_1 & 2\xi_1 & (2\xi_1+\mu_1\xi_1^2)\beta_1 & \cdots & (n-1)\xi_1^{n-2} & n\xi_1^{n-1} \\ 0 & \mu_2\beta_2 & 1 & (1+2\mu_2\xi_2)\beta_2 & 2\xi_2 & (2\xi_2+\mu_2\xi_2^2)\beta_2 & \cdots & (n-1)\xi_2^{n-2} & n\xi_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_n\beta_n & 1 & (1+2\mu_n\xi_n)\beta_n & 2\xi_n & (2\xi_n+\mu_n\xi_n^2)\beta_n & \cdots & (n-1)\xi_n^{n-2} & n\xi_n^{n-1} \end{vmatrix}$$

$$B = \begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1\beta_1 & \xi_1^2 & \xi_1^2\beta_1 & \cdots & \xi_1^{n-1} & \xi_1^{n-1}\beta_1 \\ 1 & \beta_2 & \xi_2 & \xi_2\beta_2 & \xi_2^2 & \xi_2^2\beta_2 & \cdots & \xi_2^{n-1} & \xi_2^{n-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_n & \xi_n & \xi_n\beta_n & \xi_n^2 & \xi_n^2\beta_n & \cdots & \xi_n^{n-1} & \xi_n^{n-1}\beta_n \\ 0 & \mu_1\beta_1 & 1 & (1+\mu_1\xi_1)\beta_1 & 2\xi_1 & (2+\mu_1\xi_1)\xi_1\beta_1 & \cdots & (n-1)\xi_1^{n-2} & ((n-1)+\mu_1\xi_1)\xi_1^{n-2}\beta_1 \\ 0 & \mu_2\beta_2 & 1 & (1+\mu_2\xi_2)\beta_2 & 2\xi_2 & (2+\mu_2\xi_2)\xi_2\beta_2 & \cdots & (n-1)\xi_2^{n-2} & ((n-1)+\mu_2\xi_2)\xi_2^{n-2}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_n\beta_n & 1 & (1+\mu_n\xi_n)\beta_n & 2\xi_n & (2+\mu_n\xi_n)\xi_n\beta_n & \cdots & (n-1)\xi_n^{n-2} & ((n-1)+\mu_n\xi_n)\xi_n^{n-2}\beta_n \end{vmatrix}$$

where $\beta_j = e^{2i\xi_j(\gamma_j+x+2\xi_j t)}$, $\mu_j = -2i(p_j - x - 4\xi_j t)$, γ_j's, ξ_j's, p_j's are some constants ,for $j \le n$. The $(n+j)$-th row of A or B is obtained by subtracting j-th row of (4.11) from the $(n+j)$-th row of (4.11), dividing the result by η_j and taking limit as $\eta_j \to 0$, [11]. The one harmonic soliton solution, $q(x,t)$, (Fig 3), of the NLS can be obtained by taking $n = 1$ in (4.15) :

$$q(x,t) = \lim_{\eta \to 0} \frac{2\eta \;\exp(-2i\xi x - 4i(\xi^2 - \eta^2)t - i\gamma)}{\sinh(2\eta(x - p + 4\xi t))} = \frac{\exp(-2i\xi x - 4i\xi^2 t - i\gamma)}{x - p + 4\xi t} \tag{4.16}$$

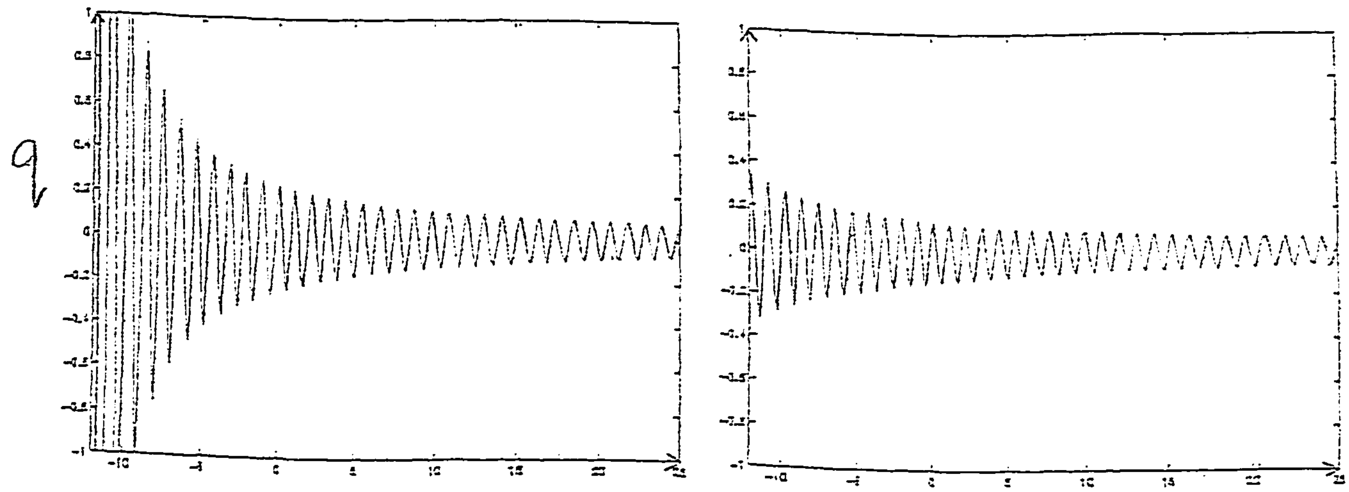

Fig.3(a) Fig.3(b)

Figure 3 : *Shows snapshots of the single harmonic soliton solution: (a) at time* $t = 1.00$ *and (b) at time* $t = 1.50$ *with* $p = -10$, $\xi = 2.5$, $\gamma = 0$.

The eigenfunction corresponding to the harmonic soliton that satisfies the compatability conditions (4.2), (4.3) is obtained by taking $\lim_{\eta \to 0}$ in equation (4.13), yielding:

$$\Phi = \begin{pmatrix} \dfrac{[i+2(\lambda-\xi)(p-x-4t\xi)]\exp(-i\lambda(x+2\lambda t))}{p-x-4t\xi} & \dfrac{-i\exp(i[-2\gamma\xi-4t\xi^2-2x\xi+\lambda x+2\lambda^2 t])}{p-x-4t\xi} \\ \dfrac{i\exp(i[2\gamma\xi+4t\xi^2+2x\xi-\lambda x-2\lambda^2 t])}{p-x-4t\xi} & \dfrac{[-i+2(\lambda-\xi)(p-x-4t\xi)]\exp(i\lambda(x+2\lambda t))}{p-x-4t\xi} \end{pmatrix} \tag{4.17}$$

Each harmonic soliton is determined by the constants λ, p and γ which corresponds to frequency, displacement and phase shift respectively. By analogy with solitons, we define the nonlinear superposition of N harmonic solitons given by the the sets of parameters $\Lambda = (\lambda_1, \cdots, \lambda_N)$, $P = (p_1, \cdots, p_n)$ and $\Gamma = (\gamma_1, \cdots, \gamma_N)$ to be the potential obtained by taking $r(k,t) = r(k,0)\exp(4i\xi^2 t)$ in the form:

$$r(k) = \lim_{\epsilon \to 0} \sum_{j=1}^{N} \left(e^{i\gamma_j + p_j\epsilon}\chi_{[\lambda_j-\epsilon,\lambda_j+\epsilon]}(k) + e^{-i\gamma_j+p_j\epsilon}\chi_{[-\lambda_j-\epsilon,-\lambda_j+\epsilon]}(k)\right) \tag{4.18}$$

$$\chi_{[k_1,k_2]}(k) = \begin{cases} 1 & \text{if } k_1 < k < k_2 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

The corresponding potential is given by (4.15).

4.3 Nonlinear Interference

Formulas (4.18)-(4.19) describe the interaction of N-harmonic solitons. Although we cannot prove it, it certainly seems to be true that each N-harmonic soliton (4.18) is a meromorphic function with at most N real poles. The 2-harmonic soliton solution of the NLS equation is:

$$q = 4(\lambda_1 - \lambda_2)\left[\frac{\exp[-2i(x\lambda_1 + \lambda_1^2 t + \gamma_1\lambda_1)]\cdot(-i + (\lambda_1-\lambda_2)\tau_2)}{\sin^2[\gamma_1\lambda_1 - \gamma_2\lambda_2 + t(\lambda_1^2 - \lambda_2^2) + x(\lambda_1-\lambda_2)] - 4(\lambda_1-\lambda_2)^2\tau_1\tau_2} + \right.$$
$$\left. + \frac{\exp[-2i(x\lambda_2 + \lambda_2^2 t + \gamma_2\lambda_2)]\cdot(i + (\lambda_1-\lambda_2)\tau_1)}{\sin^2[\gamma_1\lambda_1 - \gamma_2\lambda_2 + t(\lambda_1^2 - \lambda_2^2) + x(\lambda_1-\lambda_2)] - 4(\lambda_1-\lambda_2)^2\tau_1\tau_2}\right] \tag{4.20}$$

where

$$\tau_k = x - p_k + 2t\lambda_k, \qquad k = 1, 2 \tag{4.21}$$

Formula (4.20) makes sense only when $\lambda_2 \neq \pm\lambda_1$. Yet for $\lambda_2 = \lambda_1 + n\pi$, $n \in \mathbb{Z}$, the concept of superposition of two harmonic solitons can be naturally extended to the case $\lambda_2 = \pm\lambda_1$ by taking the limit of (4.20) as $\lambda_2 \longrightarrow \lambda_1$ and $\gamma_2\lambda_2 \longrightarrow \gamma_1\lambda_1$. This yields a harmonic soliton with:

- $\lambda = \lambda_1 = \lambda_2$
- $\gamma = \gamma_1 = \gamma_2$
- and p satisfying

$$\frac{1}{p + \frac{\partial}{\partial\lambda_2}(\gamma_2\lambda_2)} = \frac{1}{p_1 + \frac{\partial}{\partial\lambda_2}(\gamma_2\lambda_2)} + (-1)^n \frac{1}{p_2 + \frac{\partial}{\partial\lambda_2}(\gamma_2\lambda_2)}$$

For p_i's large we can neglect the $\frac{\partial}{\partial\lambda_2}(\gamma_2\lambda_2)$ term to get,

$$\frac{1}{p} \approx \frac{1}{p_1} + (-1)^n \frac{1}{p_2} \tag{4.22}$$

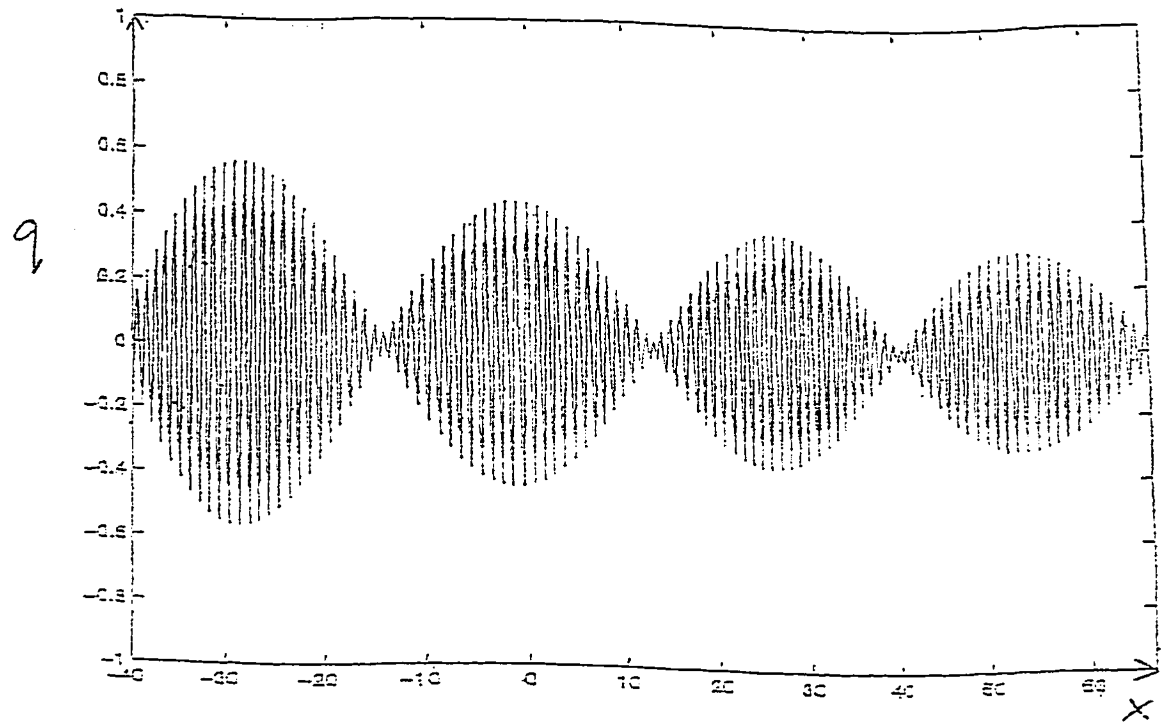

Figure 4 : *Shows snapshots of the two harmonic soliton solution of the NLS for t=0, $\Gamma = (0, 0)$, $\Lambda = (3, 3.1)$ and $p_n = -120\ exp(3(\lambda_n - \lambda_{avg})^2)$.*

Let us take a finite region D= $\{x, t|\ |x| \le X,\ 0 \le t \le T\}$ with $|p_1|, |p_2|, |p| \gg X, T$ and D sufficiently far away from the poles of both harmonic solitons. The first and the second harmonic solitons in D are now of the oscillatory form (4.23) with amplitude $\frac{1}{p_k}$:

$$\frac{\exp\left(-2i\lambda_k(\lambda_k t + \gamma_k + x)\right)}{p_k} + O\left(\frac{1}{p_k}\right), \qquad k = 1,\ 2 \tag{4.23}$$

The harmonic soliton obtained as the result of their nonlinear superpositioning in D is of the form:

$$\frac{\exp\left(-2i\lambda_1(\lambda_1 t + \gamma_1 + x)\right)}{p} + O\left(\frac{1}{p}\right) \tag{4.24}$$

Away from the poles the superpositioning of the harmonic solitons is practically reduced to adding their absolute values, just like in the linear case. In linear theory this phenomenon is called linear interference so we call its nonlinear analogue nonlinear interference. It is exactly this phenomenon that is responsible in linear theory for the formation of wavepackets. The nonlinear analogue also leads to the formation of wavepackets, in this case nonlinear. Since there is no theory for the nonlinear case, we simply construct some of the wavepackets.

Figure 5 shows the time evolution of the 8 harmonic soliton solution. The N-harmonic soliton solution does not necessarily vanish outside **D** as $x \longrightarrow \pm\infty$. Comparison with the linear Schrödinger equation suggest that by choosing λ_n's sufficiently small such that $\lambda_n \leq \lambda_{avg} + |\Delta\lambda|$ then we can choose X and T arbitrarily large. This allows us to create a wavepacket of desired magnitude and halflife. Equations (4.11), (4.15) have been used to verify this numerically. Figures 6-9 shows some other interesting properties exhibited by the harmonic soliton solutions of the NLS. The values of p_n are chosen to be of the forms $p_0 n^3$, $p_0 n^2$, $p_0 n$ and p_0 so that we can mimic the δ function with its antiderivatives. Figures 8 and 9 shows the time evolution of some $N - wave$ type and δ-type solutions respectively.

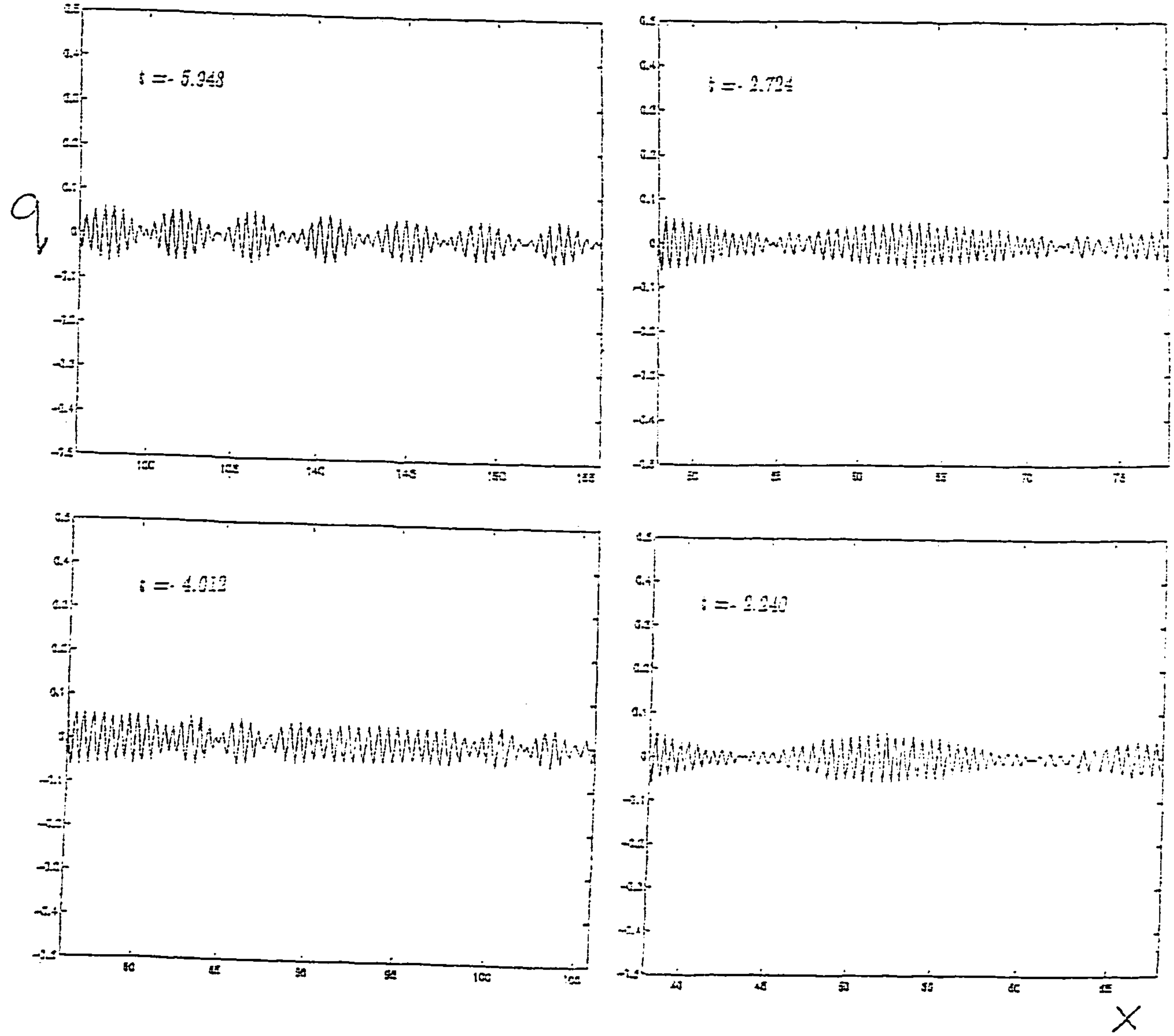

Figure 5 : *Time evolution of the wavepacket generated by 8 harmonic solitons with* $P = (p_1, ..., p_n)$, $p_n = -68000\ exp(3(\lambda_n - \lambda_{avg})^2)$: $\Lambda = (\lambda_1, ..., \lambda_8)$, $\lambda_n = 3 + 0.056(n-1)$ *and* $\Gamma = (0, ..., 0)$.

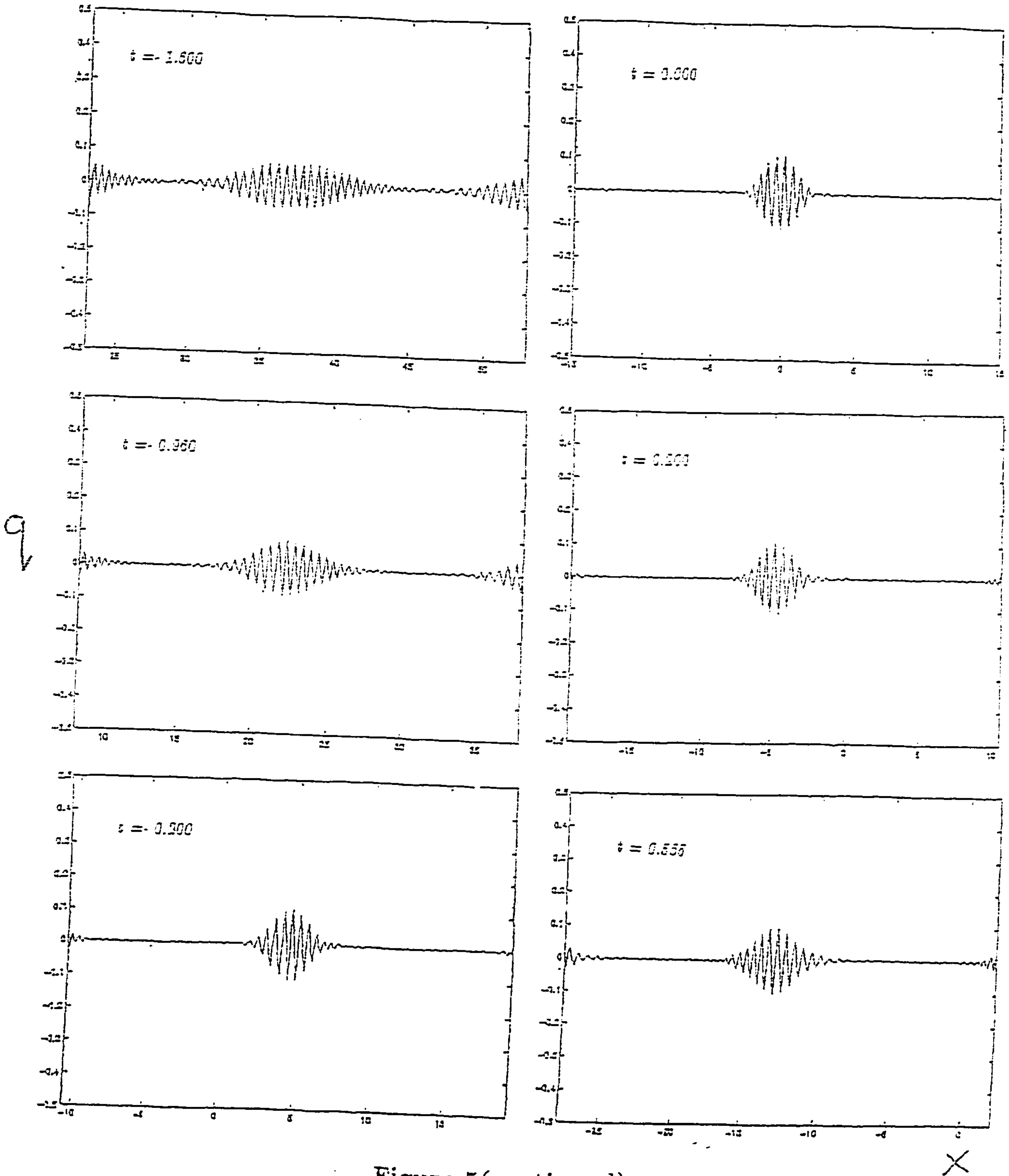

Figure 5(continued)

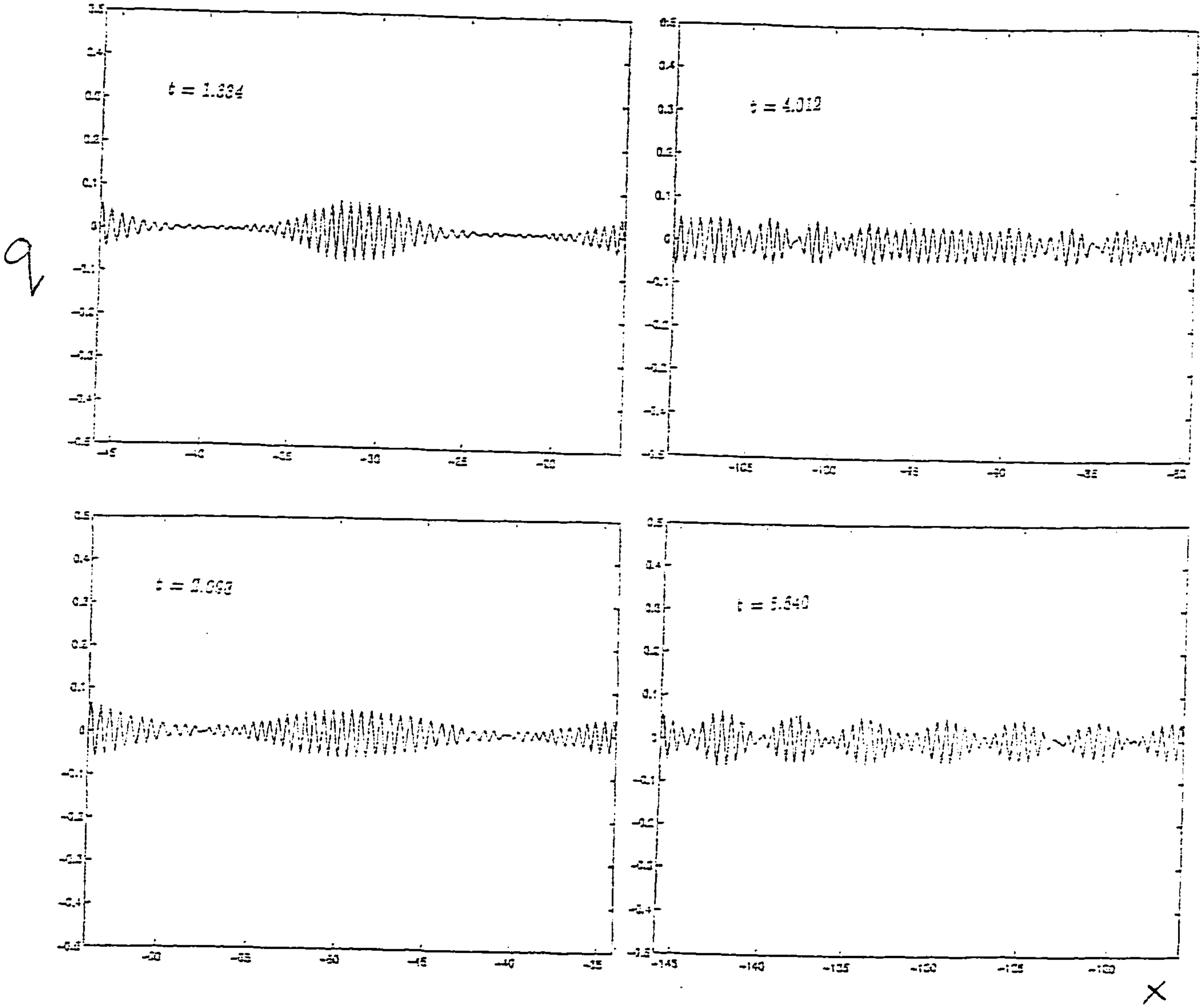

Figure 5 : *Time evolution of the wavepacket generated by 8 harmonic solitons with* $P = (p_1, \ldots, p_n)$, $p_n = -68000\ exp(3(\lambda_n - \lambda_{avg})^2)$; $\Lambda = (\lambda_1, \ldots, \lambda_8)$, $\lambda_n = 3 + 0.056(n-1)$ *and* $\Gamma = (0, \ldots, 0)$.

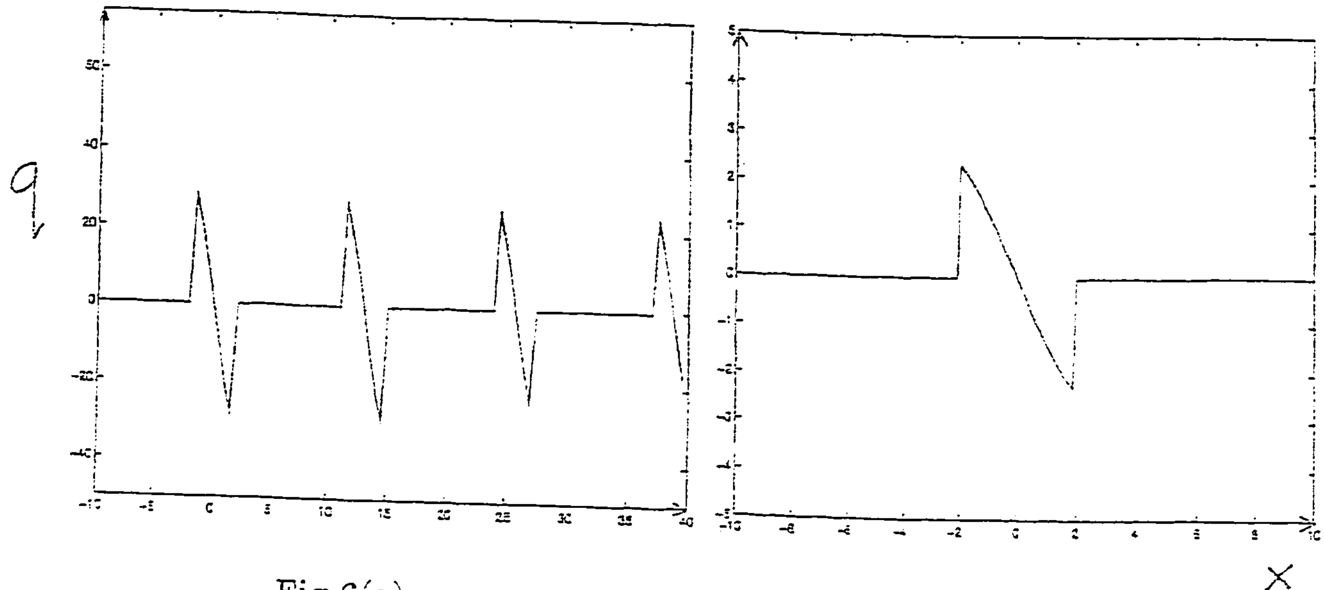

Fig.6(a) Fig.6(b)

Figure 6 : *The 13-harmonic soliton solution at $t = 0$ with* $P = (p_1, \dots, p_{13})$, $p_n = -2.8n^2 \times 10^7$, $\Gamma = (0, \dots, 0)$ *and* (a) $\lambda = .21n$ (b) $\lambda = .17n$.

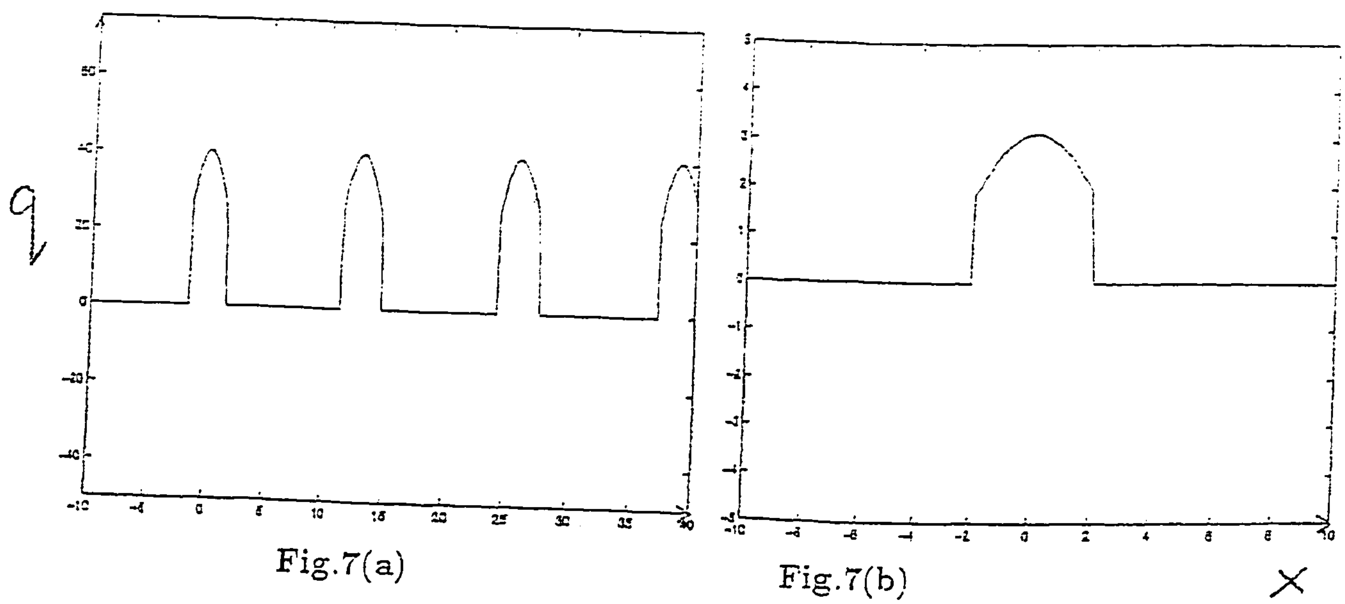

Fig.7(a) Fig.7(b)

Figure 7 : *The 13-harmonic soliton solution at $t = 0$ with* $P = (p_1, \dots, p_{13})$, $p_n = -2.8n^2 \times 10^7$, $\Gamma = (\frac{\pi}{2}, \dots, \frac{\pi}{2})$ *and* (a) $\lambda = .21n$ (b) $\lambda = .17n$.

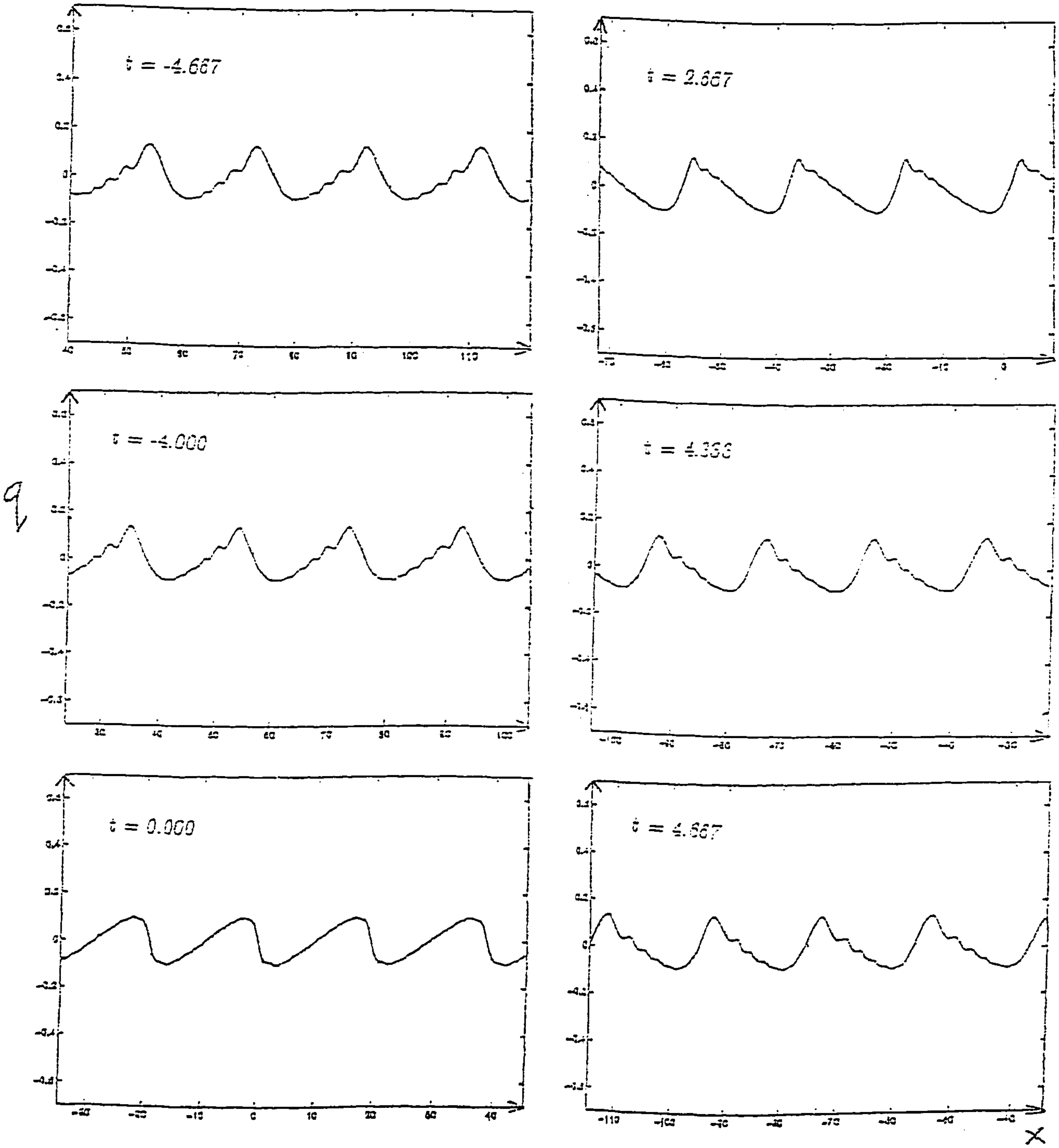

Figure 8 : *Time evolution of the wavepacket generated by 10 harmonic solitons with* $P = (p_1, \ldots, p_{10})$, $p_n = -1.5n^3 \times 10^4$; $\Lambda = (\lambda_1, \ldots, \lambda_{10})$, $\lambda_n = 0.14(n-1)$ *and* $\Gamma = (0, \ldots, 0)$.

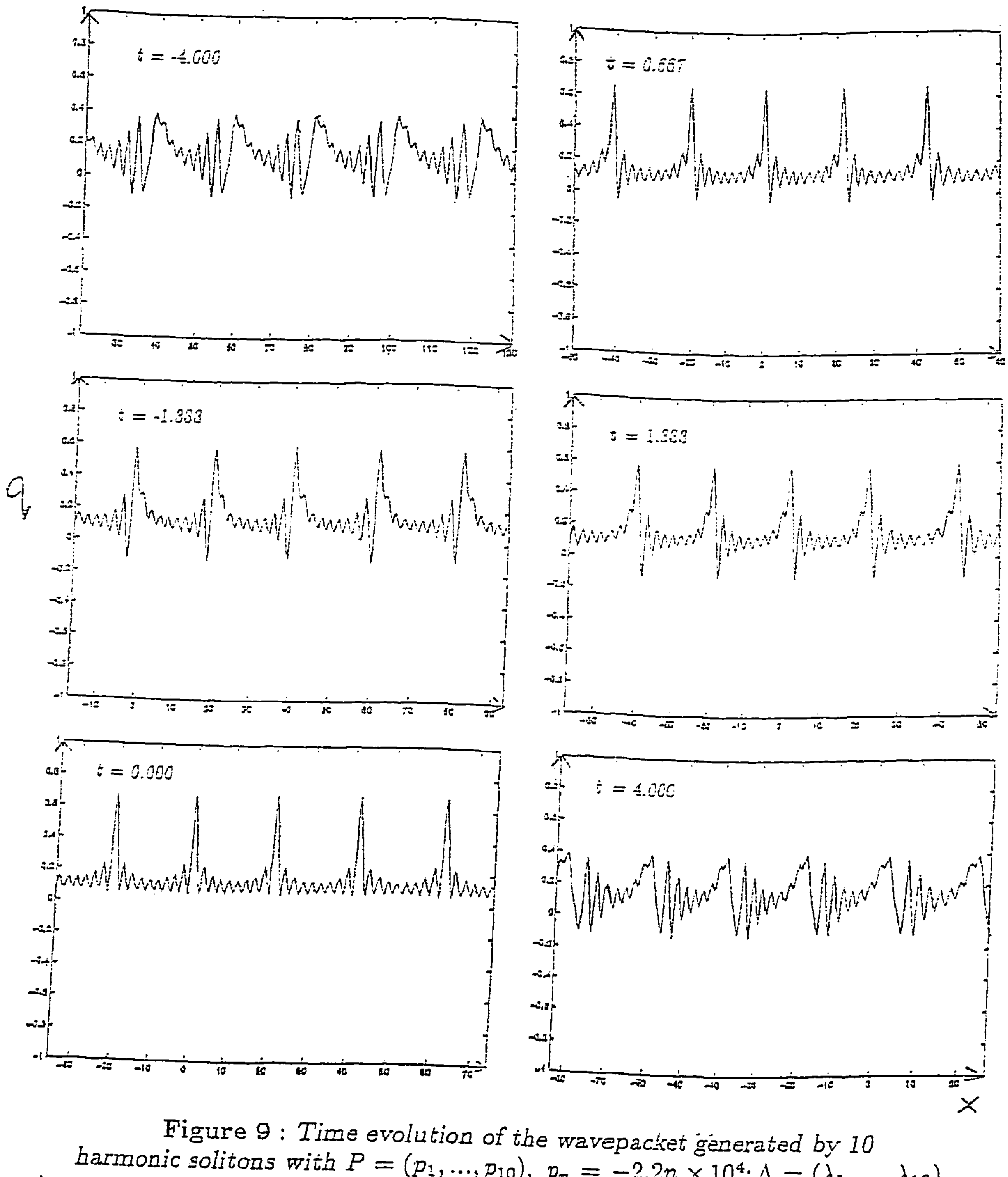

Figure 9 : *Time evolution of the wavepacket generated by 10 harmonic solitons with* $P = (p_1, \ldots, p_{10}),\ p_n = -2.2n \times 10^4; \Lambda = (\lambda_1, \ldots, \lambda_{10}),$ $\lambda_n = 0.13(n-1)$ *and* $\Gamma = (\frac{\pi}{2}, \ldots, \frac{\pi}{2})$.

Remarks

A standard MATLAB package was used to obtain these graphics. However when ξ was chosen such that $\xi \notin Q^c$ we obtained fake wavepackets Fig.10(a),this can either be corrected by choosing $\xi \in Q^c$, Fig.10(b), or by increasing the resolution.

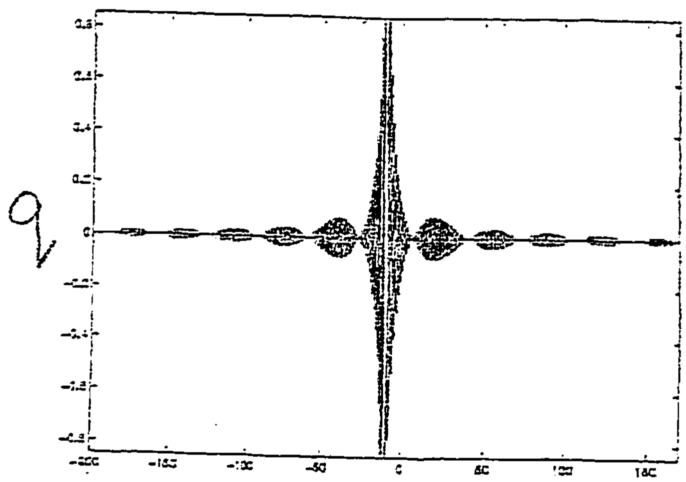

Fig.10(a)

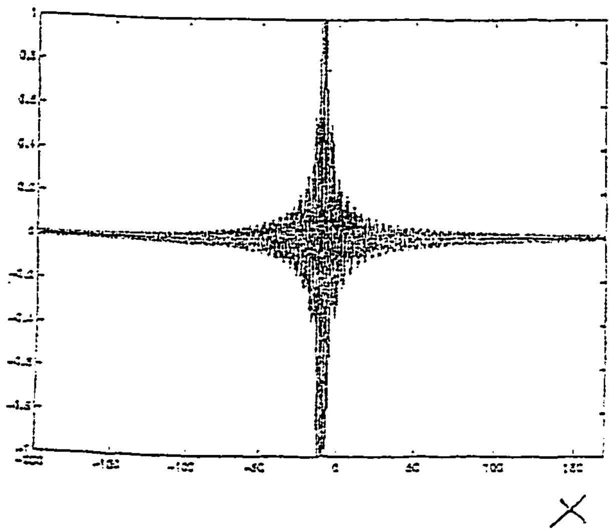

Fig.10(b)

The Matlab Programme (I) was constructed using equation (4.11),see figures 3,4,6-10 and Matlab Programme (II) was constructed from equation (4.15) yielding figures 5.

Conclusion

Although we could not apply the Fourier transform method to solve the NLS, we were still able to obtain exact solutions using the concept of nonlinear modulation. Using the analogy between Fourier analysis and the Inverse Scattering Theory used for solving the Cauchy problem for integrable equations we were able to show that the nonlinear defocusing Schrödinger equation permits nonlinear modulation. Also using the N-harmonic soliton solutions we were also able to construct localized in space solutions for the nonlinear defocusing Schrödinger equation, the so called wave packets.

Some other possible extensions that can be made in this thesis is the further simplification of the the determinants found in (4.15), so we could possibly write the general N-harmonic soliton solutions in an applicable form for large N. One can also try producing an analytic proof for the existence of N-real poles for the N-harmonic soliton solution or question the existence of a generalized formula similar to (4.22) for $N \geqq 3$ that would ultimately link the KdV and the NLS.

Appendix

Matlab Program (I)

```
clear xp up M
disp('started')

% Input Area

remfake=(\pi/3)*2^(1/8)*1.012345678901*exp(1)/2.71;

delxi=0.19*remfake;
halfdim=8;
p0=-11000
xio=1:halfdim;
for n = 1:halfdim,
xio(n)=delxi*(n-1) ;
end

halfgam=1:halfdim;
halfgam=(pi/2)*ones(1,halfdim);

halfp=1:halfdim;
for n = 1:halfdim,
halfp(n)=-p0*n^(0.5);
end

t0=0.0;
tf=100.0;
numt=100;

x0=-40.0;
```

```
xf=60.0;
y0=-4;
yf=4.0;
numx=300;

framespeed=0;

SgnSch=1;

% End of Input Area

epss=0.000000001;

deltax=(xf-x0)/numx;
nx=1:numx;
xp=x0-deltax + nx*deltax;

deltat=(tf-t0)/numt;
nt=1:numt;
tp=t0-deltat+nt*deltat;

frameshift=framespeed*deltat;

dim=2*halfdim;

lam=1:dim;
for n=1:halfdim,
lam(2*n-1)=i*epss+xio(n);
lam(2*n)  =-i*epss+xio(n);
end

gam=1:dim;
for n=1:halfdim,
gam(2*n-1)=halfgam(n);
gam(2*n)  =halfgam(n);
end

p=1:dim;
for n=1:halfdim,
```

```
p(2*n-1)=halfp(n);
p(2*n)  =-halfp(n);
end

disp('plotting loop started')

AN=eye(dim);
AD=eye(dim);
b =1:dim;
for it=1:numt,
t=tp(it);
x0=x0+frameshift;
xf=xf+frameshift;
xp=x0-deltax + nx*deltax;
end

up=1:numx;
up1=1:numx;
for ix = 1:numx,

for n= 1:halfdim,

b(2*n-1)=exp(2*i*gam(n)+2*epss*p(2*n-1))*(exp(2*i*lam(2*n-1)*xp(ix)+
         2*i*t*(lam(2*n-1))^2));

b(2*n)=exp(2*i*gam(n)+2*epss*p(2*n))*(exp(2*i*lam(2*n)*xp(ix)+
       2*i*t*(lam(2*n))^2));
      lamfac=1;
      for k=1:halfdim,
         if  k~=n
         lamfac=lamfac*(lam(2*n-1)-lam(2*k-1))(lam(2*n)-lam(2*k))^(-1);
         else  qqqq=1 ;
         end
      end
b(2*n-1)=b(2*n-1)*lamfac;
b(2*n)=b(2*n)*conj(lamfac);
end

    for n = 1:dim,
```

```
      for m = 1:halfdim,
        AD(n,2*m-1)=lam(n)^(m-1);
        AD(n,2*m)  =(lam(n)^(m-1))*b(n);
     end
   end
   AN=AD;
   for n=1:dim,
   AN(n,dim)=lam(n)^(dim);
   end

  up(ix) = real(det(AN)/det(AD));
  up1(ix) = real(i*det(AN)/det(AD));
  end

figure(it)
plot(xp,up)
axis([x0,xf,y0,yf])
valueoft=t
print  -dps -append pusa.ps

figure(it+numt)
plot(xp,up1)
axis([x0,xf,y0,yf])
valueoft=t
print  -dps -append pusa.ps

disp('finished')
```

Matlab Program (II)

```
clear xp up
disp('started')

% Input Area

remfake=(pi/3)*2^(1/8)*1.012345678901*exp(1)/2.75;
delxi=.1*remfake;

xi0=3;
halfdim=10;

xiave=xi0+delxi*(halfdim-1)/2;
p0=-70;

gam=1:halfdim;

gam=(0)*ones(1,halfdim);

xi=1:halfdim;
for n = 1:halfdim,
xi(n)=xi0+delxi*(n-1);
end

p=1:halfdim;
for n = 1:halfdim,
p(n)=p0*exp(2*(xi(n)-xiave)^2);

end

t0=0.0;
tf=0.0;
numt=1;

x0=-40;
xf=50;
y0=-.04;
yf=.03;
numx=600
```

```
framespeed=4*3.25;

SgnSch=1;

% End of Input Area

dim=2*halfdim;

deltax=(xf-x0)/numx;
deltat=(tf-t0)/numt;

nx=1:numx;
nt=1:numt;

AN=eye(dim);
AD=eye(dim);

b =1:halfdim;
mu =1:halfdim;

up=1:numx;
up1=1:numx;
tp=t0-deltat+nt*deltat;

frameshift=framespeed*deltat;
disp('plotting loop started')

for it=1:numt,
t=tp(it);
xp=x0-deltax+nx*deltax;

for ix = 1:numx,

for n= 1:halfdim,
   b(n)=exp(i*gam(n)+2*i*xi(n)*xp(ix)+4*i*t*(xi(n))^2);

   mu(n)=-2*i*(p(n)-xp(ix)-4*t*xi(n)  );
```

```
end

    for m = 1:halfdim,
      for n = 1:halfdim,
        AD(m,2*n-1)=xi(m)^(n-1);
        AD(m,2*n)  =(xi(m)^(n-1))*b(m);

        AD(halfdim+m,2*n-1)=(n-1)*xi(m)^(n-2);

        AD(halfdim+m,2*n)  =(n-1+mu(m)*xi(m))*...

                                   (xi(m)^(n-2))*b(m);
        end

        AD(halfdim+m,2 ) = mu(m)*b(m);
    end
   AN=AD;
   for m=1:halfdim,
      AN(m,dim)=xi(m)^(halfdim);

      AN(halfdim+m,dim)=halfdim*xi(m)^(halfdim-1);
  end

  up(ix) = imag(-2i*det(AN)/det(AD));

  end

figure(it)
plot(xp,up)
axis([x0,xf,y0,yf])
valueoft=t
print  -dps -append  pusa.ps

x0=x0+frameshift;
xf=xf+frameshift;

end

disp('finished')
```

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