University of Alberta

Imperfect Hedging on Equity-Linked Life Insurance with Market Constraints: Stochastic Interest Rate and Transaction Costs

by

Shuo Tong

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Abstract

Equity-linked life insurance contracts are a type of investment product issued by insurance companies to provide the insured with more appealing benefits, compared with the traditional insurance policy. Such benefits are not only linked to the performance of the underlying investments in the financial market, but also related with some insurance type events, such as death and survival to the contract maturity. Therefore, the equity-linked life insurance contract includes both the financial risk generated from the performance of the risky assets and the insurance risk reflected by the policyholders' survival probability. In this thesis, we consider the problem of utilizing imperfect hedging techniques to value equity-linked life insurance contract with market restrictions: stochastic interest rate and transaction costs. We employ two powerful imperfect hedging techniques to investigate the problem – quantile hedging and efficient hedging. We show that they are effective tools for managing both financial and insurance risk inherent in equity-linked life insurance contracts in a stochastic interest rate economy. Moreover, we incorporate transaction costs in the analysis of quantile hedging on equity-linked life insurance contract.

In chapter 2 and chapter 3, we hedge a single premium equity-linked life insurance contract with a stochastic guarantee from quantile and efficient hedging with a stochastic interest rate respectively. We present the explicit theoretical results for the premium of a contract paying the maximum of two risky asset values at maturity, providing the insured can survive to this date. These results allow the straightforward calculation of survival

probabilities for the contract owner, which can quantify the insurance companies' mortality risk and target their potential clients. Meanwhile, the numerical examples illustrate the corresponding risk management strategies for insurance companies by applying quantile and efficient hedging.

Chapter 4 analyzes the application of quantile hedging on equity-linked life insurance contracts in the presence of transaction costs. We obtain the explicit expressions for the expected present values of hedging errors and transaction costs. Furthermore, the estimated expected present values of hedging errors, transaction costs and total hedging costs are also computed from a simulation approach to compare with the theoretical ones. Finally, the quantile hedging costs of the contract's maturity guarantee inclusive of transaction costs are discussed.

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1. Introduction

Traditional life insurance products are designed to provide financial security for the policy holders and their families. The benefits are always fixed and provided on the death or survival of the insured life. The policy holder can pay single or periodic premiums during the contract term. Insurance companies only need to consider insurance risk, which is reflected by mortality for the purpose of risk management. However, the insurance market has been changing tremendously around the world over the past decades. Policyholders have become more aware of investment opportunities outside the insurance market, particularly in the financial market such as stocks, mutual funds etc. They prefer to enjoy the benefits of equity investment with mortality protection. In order to meet this challenge, insurance companies have issued a new type of investment products - equity-linked life insurance contracts, such as the variable annuities in USA, Segregated funds in Canada.

In contrast to the traditional life insurance policy, the benefit provided by an equity-linked life insurance policy depends on both the performance of some financial assets, such as stocks and foreign currencies, as well as insurance type-events, such as how long the contract owner will survive. Therefore, this benefit is stochastic and appealing to policyholders because of higher yields from the financial market and a variety of guarantees. Equity-linked business has been especially successful and contains variety types of products. For instance, variable annuities and equity-indexed annuities have become popular these years in the United States, which offer different forms of equity-linking guarantees. In Canada, segregated funds have been popular since the late 1990s, which offer complex guaranteed values on death or maturity. In the United Kingdom, unit-linked insurance typically combines a guaranteed minimum payment on death or maturity with a type of mutual fund investment. While in Germany, equity-linked endowment insurance is being introduced.

With the growth in equity-linked business, issuing such products also brings pricing and risk management challenges to insurance companies. They are facing not only the traditional insurance risk but also the financial risk generated from the investment. Therefore, the topic of finding the appropriate hedging approach to value the contracts, risk elements and to provide the efficient risk management strategies is important from both theoretical and practical perspectives. The main goal of this thesis is to investigate the problem of appropriate valuing equity-linked life insurance contracts and hedging the involved risks.

Equity-linked life insurance contracts have been studied since the mid 1970s. As will be discussed in Section 2.1, applying the perfect hedging approach from the Black-Scholes option pricing theory has become a popular method since the initial papers Brennan & Schwartz (1976, 1979) and Boyle & Schwartz (1977). However, the fact that the insurance market is incomplete was not considered. This motivated a number of research works devoted to the application of imperfect hedging techniques to value the contract such as mean-variance (Moeller (1998, 2001)), utility-based approach (Hodges & Neuberger (1989), Young & Zariphopoulou (2002)), etc. In this thesis, we select two well-accepted imperfect hedging techniques: quantile hedging and efficient hedging to value the equity-linked life insurance contract. Quantile hedging was developed by Follmer & Leukert (1999), which can obtain the optimal hedge by maximizing the probability of successful hedging. A closely related idea appears by Browne (2000). Efficient hedging was also developed by the same authors in Follmer & Leukert (2000) which aims to minimize the expected shortfall risk. The idea to use both powerful methods in the area of equity-linked life insurance was proposed in two papers by Melnikov (2004a, 2004 b).

In current literatures, such as Bacinello & Ortu (1993), Ekern & Persson (1996), Milevsky & Posner (2001), Melnikov & Romaniuk (2006, 2008), Melnikov &

Skornyakova (2005, 2011)), many financial market restrictions are placed on the models, one of which is to assume a zero or constant interest rate. As equity-linked life insurance contracts usually have long-term maturities, the valuation results may be sensitive to a change of interest rates. Therefore, our study of quantile and efficient hedging on equity-linked life insurance contracts considers a stochastic interest rate environment, which is more representative of the real world situation. Another important restriction is the frictionless market, which does not consider the transaction costs. In the real financial market, the investors must pay transaction costs during each trading. As the benefit of equity-linked life insurance is based on the investment in financial market, insurance managers are also involved in hedging and trading in the financial market due to the selected portfolio's performance. Therefore, in this thesis, we also incorporate the transaction costs as a factor into the study of quantile hedging costs for the premium of equity-linked life insurance contract and the costs for maturity guarantees.

The outline of the thesis is arranged as followed:

In Chapter 2, we first give a short review of the imperfect hedging techniques on equity-linked life insurance. Then we discuss the problem of applying quantile hedging on equity-linked life insurance contracts in a stochastic interest rate economy. We work with a single premium contract with a stochastic guarantee. The contracts under consideration are based on two risky assets, which satisfy a two-factor jump-diffusion model: one asset is responsible for future gains, and the other one is a stochastic guarantee. In our setting, the stochastic interest rate behavior is described in the Heath-Jarrow-Morton framework. In addition, explicit formulas for both the premium of the contracts and the implied survival probability are obtained by the changing of measures technique under an initial budget constraint. Risk management strategy from quantile hedging for the insurance contract is also discussed.

In Chapter 3, instead of quantile hedging, we study the same problem of hedging equity-linked life insurance contracts in a stochastic interest economy with efficient hedging technique. We work on the same contract, which has a stochastic guarantee as in Chapter 2. In Section 3.3, we present our theoretical results of a contract's premium and the implied survival probability where the two risky assets' financial models are driven by the same Wiener process. It can be considered as a special case where the models are generated from two correlated Wiener processes with correlation $\rho = 1$. In Section 3.4, we conduct the research in the case of $\rho < 1$ and give the theoretical results. Moreover, numerical examples illustrate how the efficient hedging technique can be applied to manage the balance between financial and insurance risks for a risk-taking insurance company.

In Chapter 4, we analyze the application of quantile hedging on equity-linked life insurance contracts in the presence of transaction costs. Following the similar time-based replication strategy discussed by Leland (1985) and Toft (1996), we present the explicit expressions for the expected present values of hedging errors and transaction costs. The results are derived by using Leland's transaction costs adjusted hedging volatility $\overline{\sigma}$. For the purpose of comparison, the estimated expected present values of hedging errors, transaction costs and total quantile hedging costs are obtained from a similar simulation approach utilized by Boyle & Hardy (1997). Finally, the costs of maturity guarantee for equity-linked life insurance contract inclusive of transaction costs are discussed.

In Chapter 5, we simply conclude the thesis and suggest several directions for future studies in the area of imperfect hedging on equity-linked life insurance.

The thesis also concludes with bibliography, Appendix 1 which contain the proof of Theorem 3.1 from the Multi-Asset Theorem in Melnikov & Romaniuk (2008), and Appendix 2 with some formulas in Toft (1996) which can be used to prove the Theorems 4.2 ~ 4.5.

2. Quantile hedging on equity-linked life insurance contracts in a stochastic interest rate environment

2.1. Brief history of imperfect hedging in insurance market

Using different hedging strategies for pricing has been common in financial mathematics since the celebrated results by Black & Scholes (1973) and Merton (1973) on the pricing of call options. In a complete financial market, the so-called fair price is the minimal capital required to replicate the contingent claim. These results from the financial market shed light on the area of valuation for equity-linked life insurance products in insurance market.

Brennan & Schwartz (1976, 1979) and Boyle & Schwartz (1977) were the first papers to investigate the problem of premium calculation for equity-linked life insurance contracts. They decomposed the payoff of a single premium equity-linked life insurance contract into a call/put European option on the reference asset and some guaranteed amount. Then, the Black-Scholes option pricing results was applied to evaluate the equity-linked life insurance contract embedded with some financial guarantee.

Since then, it has becomes a conventional practice to reduce the payoff of the contract into a call (put) European option and apply perfect hedging techniques to calculate the premium. Bacinello & Ortu (1993) extended the results of Brennan & Schwartz (1976, 1979) to the case where the contract's minimum guarantees depend functionally on the premiums and are determined endogenously. Aase & Persson (1994) assumed that the number of shares of the reference portfolio included in the benefit is non-random, and they derived the analytical results in the case of periodic premiums. Later, Ekern & Persson (1996) applied fair pricing valuation to price the contracts with various kinds of guarantees, including fixed deterministic and stochastic guarantees. Boyle & Hardy (1997) examined the pricing and reserving for the contract's maturity guarantee. The authors provided a minimum level of benefit at contract maturity based on stochastic simulation and option pricing approaches. Milevsky & Posner (2001) applied the risk-neutral options pricing to value the various kinds of variable annuities with the minimum death benefit guarantee.

Issuing equity-linked life insurance contracts as the investment products, insurance companies are involved in the risk from both financial and insurance markets. The risk from the insurance market is also called mortality risk. The standard actuarial practice assumes that sufficient contracts can be written to eliminate the mortality risk. However, it will be explained in detail in Section 2.5 that the insurance market is incomplete so that the mortality risk cannot be offset by trading in the insurance market. As a result, perfect hedging from the Black-Scholes and Merton framework applied on the valuation of equity-linked life insurance contract is questionable. Many studies turned to investigate the application of imperfect hedging techniques on the valuation research.

Moeller (1998) exploited mean-variance hedging on equity-linked life insurance contracts. The method is determined by minimizing the squared of the difference between the terminal value of the hedging strategy and the value of contingent claim at contract maturity. Moeller (2001) further examined a portfolio of equity-linked life insurance with risk-minimizing hedging strategy within a discrete-time setup in the Cox-Ross-Rubinsten (CRR) model. Generalized from the utility-based indifference pricing approach in Hodges & Neuberger (1989), Young & Zariphopoulou (2002) introduced an expected utility approach to value the insurance risk in a dynamic financial market setting. Young (2003) determined the risk-adjusted single and continuous premiums and the corresponding reserves for equity-indexed term life insurance by extending the principle of equivalent utility. Moore (2009) studied the optimal surrender strategy on an equity-indexed annuity by maximizing the expected utility of bequest and discussed the optimal time to surrender the contract.

Follmer & Leukert (1999) developed the quantile hedging technique, which can optimally hedge the option with maximal probability in a class of self-financing strategies with restricted initial capital. This technique was applied to value equity-linked life insurance contracts by Melnikov (2004a, 2006) in the Black-Scholes framework. Melnikov & Skornyakova (2005) extended the risky assets' financial model into a two-factor jump-diffusion model, where the second risky asset could be considered as a stochastic guarantee for the contracts. Melnikov & Romaniuk (2006) studied the effect of different mortality models on risk management with unit-linked life insurance contracts where the risks are assessed from quantile hedging. Yumin Wang (2009) presented how to optimally hedge the variable annuity contracts with guaranteed minimum death benefits.

Compared with quantile hedging, efficient hedging introduced in Follmer & Leukert (2000) is a more general imperfect hedging approach, which focuses on minimizing the expected size of shortfall risk. The shortfall risk is defined as the expectation of the positive difference between the terminal value of the hedging strategy and the value of contingent claim at contract maturity weighted by a loss function. Efficient hedging technique has also been applied to hedge equity-linked life insurance contracts. Melnikov (2004b) investigated the problem in a diffusion financial setting. Kirch & Melnikov (2005) utilized the efficient hedging for optimal pricing equity-linked life insurance contracts in a jump-diffusion framework, where the models of two underlying risky assets are driven by the same Wiener process. Melnikov & Romaniuk (2008) made the contribution considering the contracts whose payoff depends on the performance of several risky-assets. Melnikov & Skornyakova (2011) worked on the contracts with stochastic guarantee and the shortfall risk measured by a power loss function.

2.2. Description of the problem

Different imperfect hedging techniques have been intensively studied for pricing equity-linked life insurance contracts. The risk management strategies are also designed based on the corresponding techniques. Most research papers mentioned in Section 2.1

assumed that the interest rate is either constant or deterministic throughout the entire life of the contract. Figure 1 shown below is the historical data on 20 years monthly Treasury Bills rates with 3-month maturity. Data range is from July-1993 to July 2013 (Data is available at Board of Governors of the Federal Reserve System). It is observed from Figure 1 that rate is randomly distributed over 20 years. It sticks near 5% over the first 7 years. However, it is not able to maintain the same trajectory for the long term. The value of the rate after 15 years drops as low as 0.04%. The assumption of a flat interest rate for hedging equity-linked life insurance contract may be possible for short-term ones. However, life insurance products usually have long-term maturities, and it is more practical to incorporate a stochastic interest rate in the analysis.

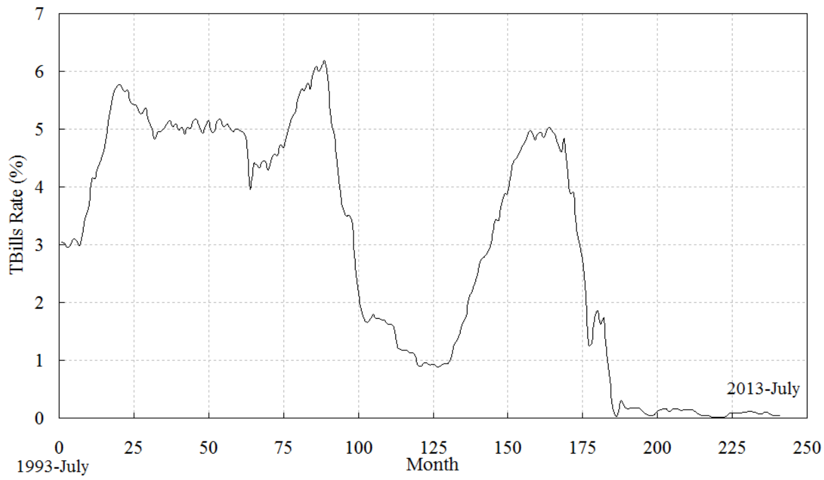

Figure 2.1 Three-month Treasury Bills rate from July 1993 till July 2013.

Many research papers are devoted to value equity-linked life insurance contract with stochastic interest rate. For example, Nielsen & Sandmann (1995) utilized the random interest rate options to value equity-linked life insurance with periodic premium. As no closed form solution was obtained, numerical techniques were applied to approximate the contract value. Nielsen & Sandmann (1996) derived the existence of fair premium principles for the guarantee within equity-linked life insurance contract. Miltersen &

Persson (1999) derived the explicit pricing formulas for maturity and multi-period guarantees on both the stock market return processes and the short-term interest rate return processes. The price calculation based on the perfect hedging in Brennan & Schwarz (1976). Bacinello & Persson (2002) further generalized the pricing formulas for equity-linked life insurance contract from deterministic interest rate to stochastic interest rate by a similar strategy. One important assumption in their study is the independence between financial and mortality factors.

As the insurance market is incomplete, Gao et al. (2010) first studied the effect of stochastic interest rates on quantile hedging of equity-linked life insurance contracts. The authors considered a contract with deterministic guarantee that depends on a constant rate of return. They obtained the results in the context of diffusion models for simplicity. Due to the infrequent and large surprises to the investor's information set, there exist jumps in various financial assets prices. It has been shown that jumps have significant implications for derivative pricing, risk management and portfolio allocation. From a practical point of view, discontinuous models inclusive jump components are more realistic for both the stochastic interest rate and the value of risky assets.

In the following chapter, our aim is to examine the jump-diffusion model for quantile hedging in a stochastic interest rate environment. We focus on a single premium equity-linked life insurance contract with more a appealing maturity guarantee dominated by the performance of a risky asset. The Heath-Jarrow-Morton (HJM) term structure model is widely accepted as the most general framework to value the stochastic behavior of interest rate. It includes the term structure models of Vasicek (1977), Cox, Ingersoll and Ross (1985) as special cases. Here we use a generalized HJM jump-diffusion model of the term structure of interest rate, which is similar to the framework of Shirakawa (1991), and Chiarella & Sklibosios (2003).

In Section 2.3, the quantile hedging technique developed in Follmer & Leukert (1999) is discussed in detail. In Section 2.4, some well-known stochastic interest rate models in literatures are introduced briefly. Section 2.5 presents the theoretical results of applying quantile hedging on equity-linked life insurance contracts. Section 2.6 provides a numerical example to demonstrate the risk management strategy for an insurance company from quantile hedging. Note that all the analysis in this thesis does not consider either model risk or parameter risk. Numerical examples are only for illustration purpose. The employed or estimated parameters might not be accurate for different time periods or other situations. The idea is to provide a convenient way to focus on the main purpose of this thesis.

2.3. Quantile hedging

In a complete financial market, every contingent claim can be hedged perfectly. However, in an incomplete market, it is possible to stay on the safe side by superhedging (See El Karoui & Quenez (1995), Karatzas (1997)). In many situations, superhedging needs a large amount of initial capital to set up the portfolio, which seems costly from the practical point of view. In this case, it naturally arises the following questions: what if the investor is unwilling or unable to put up the large amount of initial capital required for super-hedging? Are we able to construct a hedging strategy so that the investor can achieve the maximal probability of a successful hedge with a smaller amount of initial capital? The answers can be found in Follmer and Leukert (1999) in which the authors developed an imperfect hedging technique - quantile hedging. In this section, we will introduce this approach in detail, where the hedge is implemented with probability less than 1.

There exist two forms of problems involved with quantile hedging: the primary one is to minimize the value of a minimal hedging given the hedging probability; the dual problem is to maximize the hedging probability given a constraint on the initial value of a minimal hedge. In this sense, such hedging problem is methodologically related with the problem of statistical confidence estimation. Quantile is one of the main concepts in the general theory of estimation, which is the boundary of the domain of estimation with a specific probability. So the approach of hedging with probability less than 1 has come to be named as "quantile" hedging.

As an imperfect hedging technique, quantile hedging is based on the important statistical result of the Neyman-Pearson fundamental lemma. Suppose we want to test the null hypothesis H_0 with probability measure P_0, against the alternative hypothesis H_1 with probability measure P_1. Type *I* error α is defined as rejecting H_0 when it is true, and Type *II* error β is defined as accepting H_0 when it is false. Generally, the probability measure corresponding to the alternative hypothesis is considered as the real-world probability measure or the objective measure. The aim of the test is to reject H_0 when it is indeed false. During the test of two hypotheses, we need to control the size of Type *I* error α while minimizing the Type *II* error β. Equivalently, we can fix α and maximize the power of the test $1-\beta$. Referring to Lehmann (1986), the Neyman-Pearson lemma is stated as following:

Lemma 2.1: Neyman-Pearson lemma

A sufficient condition for a most powerful test: If a test ϕ satisfies

$$E_0\left(\phi(X)\right)=\alpha \tag{2.1}$$

$$\phi=\begin{cases}1 & when\dfrac{dP_1}{dP_0}>c\\ 0 & when\dfrac{dP_1}{dP_0}<c\end{cases} \tag{2.2}$$

for some constant c, then it is the most powerful for testing measure P_0 against measure P_1 at level α.

The conclusion of the Neyman-Pearson lemma is the basis for the analysis of both quantile hedging and efficient hedging which will be discussed in Chapter 3. It provides the structure of the set on which the power of the test $1-\beta$ is maximized with a given significance level α.

We follow the main ideas in Follmer & Leukert (1999) and assume the risk-free interest rate is a constant r. Note that Follmer & Leukert (1999) conducted the calculations assuming that the risk-free interest rate r equals zero. Suppose that the discounted price process $X=\left(X_t\right)_{t\in[0,T]}$ of the underlying risky asset is a semimartingale on a probability space $\left(\Omega,F,P\right)$ with the filtration $\left(F_t\right)_{t\in[0,T]}$. Let $\mathbf{P}$ denote the set of all equivalent martingale measures. If the set $\mathbf{P}$ is non-empty, there is no arbitrage opportunity in the market.

A self-financing strategy $\pi=\left(V_0,\xi_t\right)$ is defined by an initial capital $V_0\geq 0$ and a predictable process ξ_t which is the number of units invested on risky asset, if the corresponding value process V_t satisfies:

$$V_t=V_0+\int_0^t \xi_s dX_s\ ,\ \ \forall t\in\left[0,T\right],\ \ P-a.s. \tag{2.3}$$

Such strategy π is admissible if

$$V_t\geq 0\ ,\ \ \forall t\in\left[0,T\right],\ \ P-a.s. \tag{2.4}$$

We consider a contingent claim whose payoff H is F_T -measurable. A successful hedging set A is defined as $A=\left\{\omega: V_T\geq He^{-rT}\right\}$. The completeness of the market implies an unique martingale measure $P^*\approx \mathrm{P}$, such that the payoff H can be hedged perfectly under the martingale measure P^* with the required initial cost $H_0=E^*\left(He^{-rT}\right)$, where E^* denotes the expectation with respect to P^*. With the unique hedging strategy, we also have $P\left(A\right)=1$. However, if the investor is unable to allocate the required initial capital for perfect hedging, what is the best hedge he can achieve with a smaller amount $\tilde{V}_0<H_0$? The problem can be formulated as to construct an admissible

strategy (V_0, ξ_t) such that

$$P\left(V_T = V_0 + \int_0^T \xi_s dX_s \geq H\right) = \max \tag{2.5}$$

Under constraint $$V_0 \leq \tilde{V}_0 < H_0 \tag{2.6}$$

Follmer and Leukert provided the answer by quantile hedging, where the conclusions can be summarized into the following lemma in Melnikov, et. al (2002, pp105-107).

Lemma 2.2: Let $\tilde{A} \in F_T$ be the solution of the following problem:

$$P(A) \longrightarrow \max, \tag{2.7}$$

$$E^*\left(H_T e^{-rT} I_A\right) \leq \tilde{V}_0 < H_0 \tag{2.8}$$

where $I\{\cdot\}$ is the indicator function. Then a perfect hedge $\tilde{\pi}$ with initial value V_0 for the contingent claim $\tilde{H} = H \cdot I_{\tilde{A}}$ is the solution of the problem (2.5)-(2.6), and the successful hedging set A coincides with $\tilde{A}$.

Lemma 2.2 states that the constructed optimal hedge $\tilde{\pi}$, so called quantile hedge, is the perfect hedge for a modified contingent claim $\tilde{H}$. The payoff H can be hedged with maximal probability on the set $\tilde{A}$, which is also called the maximal success set. Based on Neyman-Pearson lemma, the structure of $\tilde{A}$ can be obtained. We can define a probability measure Q^* with density:

$$\frac{dQ^*}{dP^*} = \frac{H}{H_0 e^{rT}} \tag{2.9}$$

It is assumed that Q^* corresponds to measure P_0 and P corresponds to measure P_1 during the hypothesis test in Lemma 2.1. Under the measure Q^*, the budget constraint $E^*\left(He^{-rT} I_A\right) \leq \tilde{V}_0$ becomes

$$Q^*(A) \leq \frac{\tilde{V}_0}{H_0} = \frac{\tilde{V}_0}{E^*\left(He^{-rT}\right)} \tag{2.10}$$

We maximize the power of the hypothesis test given the Type I error α equal to $\frac{\tilde{V}_0}{H_0}$.

From Neyman-Pearson lemma, the maximal success set $\tilde{A}$ has the following structure:

$$\tilde{A} = \left\{ \frac{dP}{dP^*} > \tilde{a} H e^{-rT} \right\} \tag{2.11}$$

where $\frac{dP^*}{dP}$ is the density of the equivalent martingale measure, and $\tilde{a}$ is a constant determined from the condition $\tilde{a} = \inf \left\{ a : Q^* \left(\frac{dP}{dQ^*} > a \right) \leq \alpha \right\}$. With the structure for the maximal success set $\tilde{A}$, it allows us to calculate the explicit expressions for initial cost V_0 and the units of risky asset $\overline{\xi}$.

2.4. Brief Introduction on interest rate models

Interest rate is one of the most important factors for pricing and hedging derivatives, determining the cost of capital, and managing risk from both financial and insurance markets. The topic of term-structure modeling has been covered for many years, which aims to create the plausible projections of future interest rate paths or scenarios. Extensive amount of stochastic term-structure models have been developed for forecasting the short, medium and long term interest rate values and reflecting the shape of the yield curve. Some equilibrium models are derived from proposed relationships between supply and demand for funds. Some no-arbitrage models use the current term structure as a starting point and generate changes from the current values. Furthermore, the term-structure models range from very simple to extremely complex. Some single-factor interest rate models contain only one stochastic variable, while other two-factor and multi-factor models include more stochastic variables which can better capture the term-structure behaviour in the real world as well as reflecting the mean-reversion speed and the volatility factor. In this section, we will briefly introduce some well-known stochastic interest rate models in literatures.

Merton (1973) was the first one to introduce a single-factor model for the term-structure of interest rates. The model can be described as following:

$$dr_t = \mu_r dt + \sigma_r dW_t \tag{2.12}$$

where μ_r and σ_r are constants, and W_t is the standard Wiener process. Even though the Merton's model has bias, such as the negative interest rate values and constant risk-premium, it is important and can be considered as the starting point from which the variety of short term rate models are developed.

Vasicek (1977) modeled the short term interest rate as an Ornstein-Uhlenbeck process, which is also known as a mean reverting process. The model is given by

$$dr_t = a\left(b - r_t\right)dt + \sigma dW_t \tag{2.13}$$

where a , b and σ are positive constants; W_t is a standard Wiener process. The parameter b reflects the long run equilibrium level towards which r_t reverts. The drift factor $a\left(b - r_t\right)$ represents the expected instantaneous change on the interest rate at time t. The advantage of Vasicek model is to reduce the probability of unreasonable large or low interest rates by utilizing the mean reverting process.

One of the main disadvantages of Vasicek model is the negative values of interest rate, which is improved by the model proposed in Cox-Ingersoll-Ross (CIR) (1985). The CIR model is also known as the square root model:

$$dr_t = a\left(b - r_t\right)dt + \sqrt{\sigma} dW_t \tag{2.14}$$

where a determines the speed of adjustment. The CIR model takes only positive values of interest rate due to the presence of the square root in the diffusion coefficient. Even if the interest rates approaches zero, it can still become positive. Besides, the interest rate can follow a steady state distribution.

Ho & Lee (1986) described a single-factor no-arbitrage model, which is the first arbitrage-free model of term-structure of interest rates. The continuous time model is described as following:

$$\begin{cases} dr_t = \theta_t dt + \sigma dW_t \\ \theta_t = f_t(0,t) + \sigma^2 t \end{cases} \tag{2.15}$$

where σ is a constant volatility, $f_t(0,t)$ is a instantaneous forward rate at time $t = 0$, θ_t is a function of time t. With Ho-Lee model, the initial term-structure is exogenously obtained. It follows a lattice model with upward and downward movements, tending towards a binomial tree. The probabilities of upward and downward movements are determined by risk-neutral probability measures. However, Ho-Lee model contains no mean-reverting characteristic and is not arbitrage-free in all cases.

Hull & White (1990) developed a class of models which incorporate mean-reverting features and have more flexibility to fit a given yield curve. The generalized Hull-White model has the expression as:

$$dr_t = (\theta_t - a_t r_t) dt + \sigma_t dW_t \tag{2.16}$$

where θ_t is selected to ensure that the model fits the initial term-structure. Functions a_t and σ_t are parameters which are chosen to fit the market price of a set of actively traded interest-rate options. It is noted that when $a_t = 0$ and σ_t is a constant, Ho-Lee (1986) is a special case of the generalized Hull-White model.

Heath, Jarrow and Morton (HJM) (1992) constructed a family of continuous-time term-structure models in an arbitrage-free framework. The behavior of instantaneous forward rate $f(t,T)$, which is locked at time t for investing at time T, was modelled as:

$$df(t,T) = \alpha(t,T) dt + \sigma(t,T) dW_t, \quad 0 \le t \le T \tag{2.17}$$

where processes $\alpha(t,T)$ and $\sigma(t,T)$ may be random, W_t is the Wiener process under actual measure P. It is also proved that under risk-neutral measure P^*, the non-arbitrage drift is a function of selected forward rate volatility as:

$$\alpha(t,T) = \sigma(t,T)\int_0^t \sigma(t,s)ds \tag{2.18}$$

which brings a big advantage for calibration the parameters of the term-structure model. The spot rate r_t is defined as the instantaneous forward rate at time t for date t, i.e.,

$$r_t = f(t,t) \quad \text{for all} \quad t \in [0,T] \tag{2.19}$$

The HJM model is easy to understand and naturally calibrated to the currently observed yield curve. It is widely accepted as the most general and consistent framework, which includes Vasicek, CIR and Hull-White models as special case. With the rapid advances in computer technology, HJM model is becoming increasingly practical. Various generalized models are being adopted by practitioners for pricing and hedging of interest rate derivatives. Brace, Gatarek & Musiella (1997), Jamshidian (1997), and Miltersen, Sandmann & Sondermann (1997) extended HJM model to capture non-instantaneous forward rates. The modification comes to be known as the Libor Market Model (LMM). Amin & Jarrow (1992) made the contribution on modifying HJM model to include additional risky assets and also considered American-type options. Shirakawa (1991) incorporated jump component into HJM model to solve the call option pricing problem for European pure discount bond. The author assumed only a finite number of possible jump sizes and there exist a sufficient number of traded bonds to hedge away all of the jump risks to guarantee the completeness of the market. Chiarella & Sklibosios (2003) generalized HJM model into a jump-diffusion framework assuming a specific formulation of level and time-dependent volatility. They obtained the corresponding Markovian representation of the spot rate and bond price dynamics in terms of a finite number of state variables.

Besides the above described term-structure models, there exist other well-known interest rate models such as volatility models including Exponentially Weighted Moving

Average models (EWMA), Generalized ARCH models (GARCH). There is no single model which is best for all applications. In this thesis, the HJM model is utilized to capture the stochastic interest rate behaviour and quantify the insurance risk associated with interest rate.

2.5. Application of quantile hedging on equity-linked life insurance

2.5.1 Financial settings

Let $\left(\Omega, \mathbb{F}, \left(F_t\right)_{t\geq 0}, P\right)$ be a standard stochastic basis, where the filtration $\left(F_t\right)_{t\geq 0}$ satisfies the usual conditions and represents a flow of available information. We assume that all processes are adapted to this filtration. We work in a financial market which contains two risky assets S^1 and S^2, where $S^i = \left(S_t^i\right)_{t\in[0,T]}$ satisfy the following two factor jump-diffusion model:

$$dS_t^i = S_{t-}^i\left(\mu_i dt + \sigma_i dW_t - \nu_i d\Pi_t\right), \quad i = 1, 2, \tag{2.20}$$

where μ_i, σ_i and ν_i are all constants, T is a finite time horizon. $\mu_i \in R$ are the instantaneous rate of return; $\sigma_i > 0$ are the instantaneous volatility of the risky assets; $\nu_i < 1$ represent the jump size for the price process of risky asset. W_t is a standard Wiener process under measure P, and Π_t is a Poisson process with a constant intensity $\lambda > 0$. We assume the first asset S^1 is more risky than the second one S^2. Equivalently, we have $\sigma_1 > \sigma_2$. In addition, all trades are assumed to be taken place in a frictionless market, i.e. no transaction costs or taxes.

Information flows affect interest rates continuously in small amounts, best described by diffusion processes. Yet, on the rare occasion, surprise information events have large economic impact, causing interest rates to have jumps. Therefore, the choice of jump-diffusion processes to describe the movements in interest rates trajectories is natural. As in Chiarella & Sklibosios (2003), we consider a default-free bond market where arbitrary maturity bonds are traded continuously within a finite time horizon $[0,T]$. $f(t,T)$ represents the instantaneous forward rate at time t for instantaneous borrowing at time $T(\geq t)$. Let $P(t,T)$ be the price of a default-free discount zero-coupon bond at time t with maturity T, which pays \$1 at maturity time, i.e. $P(T,T)=1$. By the definition of forward rate $f(t,T)$, $P(t,T)$ is given by:

$$P(t.T) = \exp\left(-\int_t^T f(t,s)\,ds\right) \tag{2.21}$$

The spot interest rate $\{r(t)\}_{t\leq T}$ at time t is also defined by the instantaneous forward rate $f(t,T)$ as:

$$r(t) = f(t,t) \tag{2.22}$$

We define an accumulation factor by $B_t = \exp\left(\int_0^t r(s)\,ds\right)$, which is a money market account starting with a dollar investment at time 0. Besides, B_t also satisfies the following dynamics:

$$dB_t = B_{t\_}\, r(t)\,dt \tag{2.23}$$

Following the extended HJM framework, the stochastic differential equation for the instantaneous forward rate $f(t,T)$ is given by:

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t + \beta\left[d\Pi_t - \lambda dt\right] \tag{2.24}$$

where $\alpha:[0,T]\rightarrow R_+$ is the drift function, $\sigma:[0,T]\rightarrow R_+$ is the volatility function, λ is the constant intensity for the Poisson process Π_T, β is the constant jump size for the forward rate process.

Based on (2.24), the forward rate $f(t,T)$ can also be expressed as:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW_s + \int_0^t \beta[d\Pi_s - \lambda ds] \quad (2.25)$$

where $f(0,T)$ is the given initial forward rate curve.

Substituting $T = t$ into (2.25), we arrive at the stochastic integral equation for the instantaneous spot rate $r(t)$ as:

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s + \int_0^t \beta[d\Pi_s - \lambda ds] \quad (2.26)$$

By Ito's lemma, we obtain the dynamics for zero-coupon bond price $P(t,T)$ as following:

$$\begin{aligned} dP(t,T) = P(t_-,T)\Big\{ & \Big[r(t) - \alpha^*(t,T) + \beta(T-t)\lambda + \frac{1}{2}\big[\sigma^*(t,T)\big]^2 \Big] dt \\ & -\sigma^*(t,T)dW_t + \left(e^{-\beta(T-t)} - 1\right)d\Pi_t \Big\} \end{aligned} \quad (2.27)$$

where $\alpha^*(t,T) = \int_t^T \alpha(t,s)ds$, $\sigma^*(t,T) = \int_t^T \sigma(t,s)ds$.

For a security market under consideration, one can determine conditions under which a unique equivalent martingale measure P^* does exist. Following the techniques in According to Melnikov et al. (2002), there exits the unique martingale measure P^* under the conditions:

$$\frac{(\mu_1 - r(t))\sigma_2 - (\mu_2 - r(t))\sigma_1}{\sigma_2 \nu_1 - \sigma_1 \nu_2} > 0, \quad \text{given } \sigma_2 \nu_1 - \sigma_1 \nu_2 \neq 0. \quad (2.28)$$

By Girsanov's theorem, under the measure P^*, $W_t^* = W_t - \int_0^t \phi_s ds$ is a standard Wiener process, Π_t is a Poisson process associated with new intensity λ_t^*, and the processes W_t^* and Π_t are independent. The process ϕ_t can be interpreted as the market price of the diffusion risk, while λ_t^* represents the market price of jump risk generated by the Poisson process.

The risk-neutral measure P^* has a local density with expression as: (presented in Shirakawa (1991), Melnikov & Skornyakova (2005))

$$Z_t = \left.\frac{dP^*}{dP}\right|_t = \exp\left\{\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \left(\lambda - \lambda_s^*\right) ds + \left(\ln \lambda_t^* - \ln \lambda\right)\Pi_t\right\} \tag{2.29}$$

where the pair $\left(\phi_t, \lambda_t^*\right)$ satisfies the following equations

$$\begin{cases} \mu_1 - r(t) + \phi_t \sigma_1 - \nu_1 \lambda_t^* = 0 \\ \mu_2 - r(t) + \phi_t \sigma_2 - \nu_2 \lambda_t^* = 0 \end{cases} \tag{2.30}$$

Solving the above equations, we get

$$\phi_t = \frac{\left(\mu_1 - r(t)\right)\nu_2 - \left(\mu_2 - r(t)\right)\nu_1}{\sigma_2 \nu_1 - \sigma_1 \nu_2} \tag{2.31}$$

$$\lambda_t^* = \frac{\left(\mu_1 - r(t)\right)\sigma_2 - \left(\mu_2 - r(t)\right)\sigma_1}{\sigma_2 \nu_1 - \sigma_1 \nu_2} \tag{2.32}$$

Under the risk-neutral measure P^*, we can calculate the dynamics of the forward interest rate $f(t,T)$ and the spot interest rate $r(t)$ and obtain:

$$\begin{aligned} f(t,T) = {} & f(0,t) + \int_0^t \sigma(s,T)\sigma^*(s,T) ds + \int_0^t \sigma(s,T) dW_s^* \\ & + \int_0^t \beta \lambda_s^* \left[1 - e^{-\beta(T-s)}\right] ds + \int_0^t \beta \left[d\Pi_s - \lambda_s^* ds\right] \end{aligned} \tag{2.33}$$

$$\begin{aligned} r(t) = {} & f(0,t) + \int_0^t \sigma(s,t)\sigma^*(s,t) ds + \int_0^t \sigma(s,t) dW_s^* \\ & + \int_0^t \beta \lambda_s^* \left[1 - e^{-\beta(t-s)}\right] ds + \int_0^t \beta \left[d\Pi_s - \lambda_s^* ds\right] \end{aligned} \tag{2.34}$$

where the forward rate drift function $\alpha(t,T)$ in (2.24) satisfes the following condition:

$$\alpha(t,T) = \sigma(t,T)\left[-\phi_t + \int_t^T \sigma(t,s) ds\right] - \beta\left[\lambda_t^* e^{-\beta(T-t)} - \lambda\right] \tag{2.35}$$

In this circumstances, the evolutions of risky asset price S_t^i, $i = 1,2$, can be rewritten as:

$$S_t^i = S_0^i \exp\left\{\sigma_i W_t^* + \left[\mu_i - \frac{1}{2}\left(\sigma_i\right)^2\right] t + \int_0^t \sigma_i \phi_s ds + \Pi_t \ln\left(1 - \nu_i\right)\right\} \tag{2.36}$$

which is the solution of the stochastic differential equation (2.20).

2.5.2 Insurance setting

In this section, we work on a single premium equity-linked life insurance contract, which is also called "pure endowment". We assume the insured does not receive any economic compensation for accepting mortality risk, which means the insured receives the payoff of the contract provided that he/she is alive at the contract maturity. The benefit of the contract is linked with both the financial performance of risky assets S^1, S^2 and the insured's life. The risky asset S^1 is responsible for the maximal size of future profits, while the risky asset S^2 provides a stochastic guarantee to the insured.

Let T_x be a nonnegative random variable defined on another probability space $\left(\tilde{\Omega}, \tilde{F}, \tilde{P}\right)$. This random variable T_x represents the remaining life time of a policyholder with current age x-year old. ${}_T p_x = \tilde{P}\left(T_x > T\right)$ is called the survival probability which denotes the probability of a life aged x surviving T more years. We take a natural assumption as in Bacinello and Persson (2002) that the financial market risk and the insurance risk reflected by the insured mortality have no effect on each other. Therefore, T_x is independent of all processes reflecting financial quantities. We use H_T to denote the benefit of the contract paid at time T, which depends on the market value of S_T^1 and on the guaranteed value S_T^2 at the maturity, i.e.

$$H_T = \max\left(S_T^1, S_T^2\right) \tag{2.37}$$

Considering the mortality risk, we are interested in the benefit $H_T \cdot I\left\{T_x > T\right\}$. Under the independence assumption for the financial and mortality risk, the premium X_0 for the contract with payoff (2.37) can be calculated as following;

$$
\begin{aligned}
X_0 &= E^*\left\{\tilde{E}\left[H_T B_T^{-1} I\left\{T_x > T\right\}\right]\right\} \\
&= E^*\left[H_T B_T^{-1}\right]\tilde{E}\left[I\left\{T_x > T\right\}\right] \\
&= E^*\left[H_T B_T^{-1}\right]\cdot {}_T p_x
\end{aligned} \tag{2.38}
$$

where $I\left\{\cdot\right\}$ is the indicator function.

Because of $0 < {}_T p_x < 1$, formula (2.38) implies that

$$
X_0 < H_0 = E^*\left(H_T B_T^{-1}\right) \tag{2.39}
$$

It is noticed that the mortality risk makes it impossible for the insurance company to hedge the payoff of equity-linked life insurance contract with probability 1. Moreover, the mortality risk can not be eliminated by trading directly. Therefore, the insurance market can be considered as an incomplete market. (2.39) can also be treated as a initial budget constraint for insurance company to achieve the perfect hedging. The initial amount X_0 collected by insurance company from selling the contracts is less than the initial amount H_0 needed to hedge the contract perfectly.

2.5.3 Application of quantile hedging on equity-linked life insurance contract

In the situation of an initial budget constraint (2.39) which is short of initial capital for the perfect hedging, quantile hedging technique can be utilized to provide the optimal hedging outcomes. In this section, we can extend the previous study on discussion of quantifying the insurance risk and developing the corresponding risk management strategies by taking quantile hedging strategy for the insurance company.

The premium of the contract X_0 is less than the amount H_0 for the perfect hedging, which is similar as the given constraint for quantile hedging in (2.6). The Insurance company aims to construct a strategy which can maximize the probability of successful hedging with a smaller amount X_0. According to Follmer and Leukert (1999), the initial amount X_0 is also the cost of the constructed optimal strategy π^* (the quantile hedge). Meanwhile, the quantile strategy π^* can perfectly hedge the modified contingent claim $\tilde{H}_T = H_T \cdot I_{A^*}$, where A^* (denoted as $\tilde{A}$ in Section 2.3) is the maximal set of successful hedging which is in the form 0f $A^* = \left\{ \frac{dP}{dP^*} \geq a^* \cdot H_T {B_T}^{-1} \right\}$. a^* is a constant which can be determined from initial budget constraint (2.39). Compared with (2.38), the premium X_0 can also be calculated from using the perfect hedging on modified payoff $\tilde{H}_T$:

$$X_0 = E^* \left(\tilde{H}_T B_T^{-1} \right) \tag{2.40}$$

Taken (2.38) into consideration, we can obtain the following equalities for the premium X_0 :

$$X_0 = E^* \left(\tilde{H}_T B_T^{-1} \right) = E^* \left[H_T {B_T}^{-1} \right] \cdot {}_T p_x \tag{2.41}$$

Therefore, the implied survival probability ${}_T p_x$ is obtained from (2.41) as:

$${}_T p_x = \frac{E^* \left(\tilde{H}_T B_T^{-1} \right)}{E^* \left[H_T {B_T}^{-1} \right]} = \frac{X_0}{H_0}. \tag{2.42}$$

Equation (2.42) is called the key balance equation. It is essential to risk management analysis of quantile hedging on equity-linked life insurance contract, as the quantitative connection between the financial and insurance risk components are given in one formula. This connection can allow the insurance company to evaluate the bearing risks accurately and to implement specific risk management strategies for controlling the corresponding risks. Besides the mortality risk reflected by clients' survival probability ${}_T p_x$, insurance company also faces a default financial risk ε which measures the probability that the quantile hedging fails. The firm issues the equity-linked life insurance contracts to the clients while collecting the premium X_0. Then X_0 is invested as the initial cost into quantile hedging to maintain the maximum probability of the successful hedging $1-\varepsilon$. Alternatively, insurance company can determine some acceptable level of default financial risk ε first, and then they can determine the survival probability ${}_T p_x$ for the potential clients. In the end, based on available mortality life tables, the ages of the corresponding clients can be obtained.

Equation (2.42) provides a guidance for insurance company to quantify the mortality risk. In order to obtain the survival probability ${}_T p_x$ for the contract policy holders, we present explicit formula for the premium X_0 of equity-linked life insurance contract in the following theorem:

Theorem 2.1: We consider a financial market model (2.20) with a HJM framework (2.17) and (2.19). A single premium equity-linked life insurance contract has the payoff $H_T = \max\left(S_T^1, S_T^2\right)$ at maturity. The Brennan-Schwartz price for contract from quantile hedging is

$$\begin{aligned} X_0 = \sum_{n=0}^{\infty} p_{n,T}^* \Big[& e^{v_1 \int_0^T \lambda_t^* dt} \left(1-v_1\right)^n S_0^1 \Psi^2\left(\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2\right) \\ & + e^{v_2 \int_0^T \lambda_t^* dt} \left(1-v_2\right)^n S_0^2 \Psi^2\left(\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2\right) \Big] \end{aligned} \tag{2.43}$$

where S_0^1, S_0^2 are the initial risky assets prices; $p_{n,T}^* = e^{-\int_0^T \lambda_t^* dt} \dfrac{\left(\int_0^T \lambda_t^* dt\right)^n}{n!}$ are the probabilities of a non-homogeneous Poisson distribution with intensity λ_t^*; $\Psi^2(\cdot,\cdot)$ are the two-dimensional cumulative normal distribution function with correlation ρ and $\tilde{\rho}$ separately. The other parameters are shown as following:

$$\Gamma_1 = -\ln \frac{a_n \cdot S_0^1 \left(\lambda_T^1\right)^n}{\lambda^n} - \frac{1}{2}\delta_1^2 - \int_0^T \left(\lambda - \lambda_s^1\right) ds, \qquad \delta_1^2 = \int_0^T \left(\phi_s + \sigma_1\right)^2 ds,$$

$$\Gamma_2 = \ln \frac{S_0^1 \left(1-v_1\right)^n}{S_0^2 \left(1-v_2\right)^n} - \int_0^T \lambda_s^* \left(v_2 - v_1\right) ds + \frac{1}{2}\delta_2^2, \qquad \delta_2^2 = \left(\sigma_2 - \sigma_1\right)^2 T,$$

$$\tilde{\Gamma}_1 = -\ln\frac{a_n \cdot S_0^2\left(\lambda_T^2\right)^n}{\lambda^n} - \frac{1}{2}\tilde{\delta}_1^2 - \int_0^T \left(\lambda - \lambda_s^2\right) ds, \qquad \tilde{\delta}_1^2 = \int_0^T \left(\phi_s + \sigma_2\right)^2 ds,$$

$$\tilde{\Gamma}_2 = \ln\frac{S_0^2\left(1-v_2\right)^n}{S_0^1\left(1-v_1\right)^n} - \int_0^T \lambda_s^*\left(v_1 - v_2\right) ds + \frac{1}{2}\tilde{\delta}_2^2, \qquad \tilde{\delta}_2^2 = \left(\sigma_2 - \sigma_1\right)^2 T,$$

$$\lambda_t^1 = \lambda_t^*\left(1-v_1\right), \;\; \lambda_t^2 = \lambda_t^*\left(1-v_2\right).$$

Proof: Follow the approach of Amin & Jarrow (1992), we can first rewrite the explicit representations of B_t and S_t^i in terms of the parameters of the system by Ito's formula:

$$\begin{aligned} B_t = \frac{1}{P(0,t)} \exp\Big\{ & \frac{1}{2}\int_0^t \left(\sigma^*(s,T)\right)^2 ds + \int_0^t \left[e^{-\beta(T-s)} - 1\right]\lambda_s^* ds \\ & + \int_0^t \sigma^*(s,T)\, dW_s^* - \int_0^t \beta(T-s) d\Pi_s \Big\} \end{aligned} \tag{2.44}$$

$$S_t^i = S_0^i B_t \exp\left\{\sigma_i W_t^* + \Pi_t \ln\left(1-v_i\right) + \int_0^t \left(v_i \lambda_s^* - \frac{1}{2}\sigma_i^2\right) ds\right\}, \;\; i = 1,2 \tag{2.45}$$

Conditioning on each set $\{\Pi_T = n\}$, $n = 1,2,\ldots$, we can decompose the initial price X_0 into two parts X_0^1, X_0^2:

$$\begin{aligned} X_0 &= E^*\left[H_T B_T^{-1} I\{A^*\}\right] \\ &= E^*\left[\frac{\max\left(S_T^1, S_T^2\right)}{B_T} I\left\{1 \geq a_n Z_T \frac{\max\left(S_T^1, S_T^2\right)}{B_T}\right\}\right] \\ &= E^*\left[\frac{S_T^1}{B_T} I_{\left\{1 \geq a_n Z_T \frac{S_T^1}{B_T}\right\}} \cdot I_{\left\{S_T^1 > S_T^2\right\}}\right] + E^*\left[\frac{S_T^2}{B_T} I_{\left\{1 \geq a_n Z_T \frac{S_T^2}{B_T}\right\}} \cdot I_{\left\{S_T^1 \leq S_T^2\right\}}\right] \\ &= X_0^1 + X_0^2 \end{aligned} \tag{2.46}$$

Then, we can use similar approach to calculate X_0^1, X_0^2 separately.

Part 1: Calculation of X_0^1.

$X_0^1 = E^*\left[\frac{S_T^1}{B_T} I_{\left\{1>a_n Z_T \frac{S_T^1}{B_T}\right\}} \cdot I_{\left\{S_T^1 > S_T^2\right\}}\right]$, where Z_T, B_T, S_T^i satisfy the dynamics (2.29), (2.44), (2.45).

By the change of measure approach, we can define another measure Q_1 such that

$$\left.\frac{dQ_1}{dP^*}\right|_{F_T} = \exp\left[\sigma_1 W_T^* + \Pi_T \ln(1-v_1) + \int_0^T \left(v_1\lambda_s^* - \frac{1}{2}\sigma_1^2\right) ds\right] \tag{2.47}$$

Under the new measure Q_1, $\widehat{W}_t^1 = W_t^* - \sigma_1 T$ is another Wiener process, and Π_t is the Poisson process with the corresponding new intensity $\lambda_t^1 = \lambda_t^*(1-v_1)$.

Therefore, on each set $\{\Pi_T = n\}$, $n = 1, 2, \ldots$, we calculate

$$\begin{aligned}
X_0^1 = S_0^1 E^{Q_1} \Bigg\{ I\Bigg[1 > a_n \exp\left[\int_0^T \phi_s d\widehat{W}_s^1 + \int_0^T \phi_s\sigma_1 ds + \frac{1}{2}\int_0^T \phi_s^2 ds + \int_0^T (\lambda - \lambda_s^*) ds + n \ln\frac{\lambda_T^*}{\lambda}\right] \\
S_0^1 \cdot \exp\left[\sigma_1 \widehat{W}_T^1 + \sigma_1^2 T + n\ln(1-v_1) + \int_0^T\left(v_1\lambda_s^* - \frac{1}{2}\sigma_1^2\right)ds\right]\Bigg] \\
\cdot I\Bigg[\frac{S_0^1}{S_0^2} > \exp\Bigg[(\sigma_2 - \sigma_1)\widehat{W}_T^1 + n\ln(1-v_2) + \int_0^T\left(v_2\lambda_s^* - \frac{1}{2}\sigma_2^2\right)ds + \sigma_1\sigma_2 T \\
-\sigma_1^2 T - n\ln(1-v_1) - \int_0^T\left(v_1\lambda_s^* - \frac{1}{2}\sigma_1^2\right)ds\Bigg]\Bigg]\Bigg\} \\
= S_0^1 E^{Q_1} \Bigg\{ I\Bigg[-\ln(a_n \cdot S_0^1) \geq \int_0^T (\phi_s + \sigma_1) d\widehat{W}_s^1 + n\left(\ln\lambda_T^1 - \ln\lambda\right) \\
+ \int_0^T \left[\frac{1}{2}(\phi_s + \sigma_1)^2 + (\lambda - \lambda_s^1)\right] ds\Bigg] \\
\times I\left[\ln\left(\frac{S_0^1}{S_0^2}\right) > (\sigma_2 - \sigma_1)\widehat{W}_T^1 + n\ln\frac{1-v_2}{1-v_1} + \int_0^T \left[\lambda_s^*(v_2 - v_1) - \frac{1}{2}(\sigma_2 - \sigma_1)^2\right] ds\right]\Bigg\}
\end{aligned} \tag{2.48}$$

Let us define two new random variables $y_1 = \int_0^T (\phi_s + \sigma_1) d\widehat{W}_s^1$ and $y_2 = (\sigma_2 - \sigma_1)\widehat{W}_T^1$. It is obvious that y_1, y_2 follow normal distribution under measure Q_1 : $y_1 \sim N\left(0,\delta_1^2\right)$, $y_2 \sim N\left(0,\delta_2^2\right)$, where $\delta_1^2 = \int_0^T \left(\phi_s + \sigma_1\right)^2 ds$, $\delta_2^2 = \left(\sigma_2 - \sigma_1\right)^2 T$. For any constants $k_1, k_2 \neq 0$, the linear combination of y_1, y_2 is

$$\begin{aligned} k_1 y_1 + k_2 y_2 &= \int_0^T k_1\left(\phi_s + \sigma_1\right) d\widehat{W}_s^1 + k_2\left(\sigma_2 - \sigma_1\right)\widehat{W}_T^1 \\ &= \int_0^T \left[k_1\phi_s + \left(k_1 - k_2\right)\sigma_1 + k_2\sigma_2\right] d\widehat{W}_s^1 \end{aligned} \tag{2.49}$$

Clearly, the above linear combination is still a normal random variable. So the random vector $\left(y_1, y_2\right)^T$ is normally distributed with mean equals $\left(0,0\right)^T$, and the correlation between y_1, y_2 is $\rho = \int_0^T \left(\phi_s + \sigma_1\right)\left(\sigma_2 - \sigma_1\right) ds$. Following (2.48), we obtain:

$$\begin{aligned} X_0^1 = S_0^1 E^{Q_1} &\left\{ I\left[y_1 \le -\ln\frac{a_n \cdot S_0^1\left(\lambda_T^1\right)^n}{\lambda^n} - \frac{1}{2}\delta_1^2 - \int_0^T \left(\lambda - \lambda_s^1\right) ds \right] \right. \\ &\left. I\left[y_2 < \ln\frac{S_0^1\left(1-v_1\right)^n}{S_0^2\left(1-v_2\right)^n} - \int_0^T \lambda_s^*\left(v_2 - v_1\right) ds + \frac{1}{2}\delta_2^2 \right] \right\} \\ = S_0^1 Q_1 &\left(y_1 \le -\ln\frac{a_n \cdot S_0^1\left(\lambda_T^1\right)^n}{\lambda^n} - \frac{1}{2}\delta_1^2 - \int_0^T \left(\lambda - \lambda_s^1\right) ds, \right. \\ &\left. y_2 < \ln\frac{S_0^1\left(1-v_1\right)^n}{S_0^2\left(1-v_2\right)^n} - \int_0^T \lambda_s^*\left(v_2 - v_1\right) ds + \frac{1}{2}\delta_2^2 \right) \\ = S_0^1 \Psi^2 &\left(\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2\right) \end{aligned} \tag{2.50}$$

where $\Psi^2\left(\cdot,\cdot\right)$ is the two-dimensional cumulative normal distribution function.

We can also get the maximal set of successful hedging A^* :

$$A^* = \left\{ y_1 \le -\ln\frac{a_n \cdot S_0^1\left(\lambda_T^1\right)^n}{\lambda^n} - \frac{1}{2}\delta_1^2 - \int_0^T \left(\lambda - \lambda_s^1\right) ds \right\} \tag{2.51}$$

Part 2: Calculation of X_0^2.

The calculation of X_0^2 can be treated in the similar way. We can define another new measure Q_2:

$$\left.\frac{dQ_2}{dP^*}\right|_{F_T} = \exp\left[\sigma_2 \tilde{W}_T + \Pi_T \ln\left(1-v_2\right) + \int_0^T \left(v_2 \lambda_s^* - \frac{1}{2}\sigma_2^2\right) ds\right] \tag{2.52}$$

where $\widehat{W}_t^2 = W_t^* - \sigma_2 T$ is a Wiener process under the measure Q_2, and Π_t is a Poisson process with new intensity $\lambda_t^2 = \lambda_t^*\left(1-v_2\right)$.

On each set $\{\Pi_T = n\}$, $n = 0,1,2...$, we can rewrite X_0^2 as

$$\begin{aligned} X_0^2 = S_0^2 E^{Q_2} \Big\{ & I\Big[-\ln\left(a_n \cdot S_0^2\right) \geq \int_0^T \left(\phi_s + \sigma_2\right) d\widehat{W}_s^2 + n\left(\ln \lambda_T^2 - \ln \lambda\right) \\ & + \int_0^T \left[\frac{1}{2}\left(\phi_s + \sigma_2\right)^2 + \left(\lambda - \lambda_s^2\right)\right] ds\Big] \\ & \cdot I\left[\ln\left(\frac{S_0^2}{S_0^1}\right) \geq \left(\sigma_1 - \sigma_2\right)\widehat{W}_T^2 + n \ln\frac{1-v_1}{1-v_2} + \int_0^T \left[\lambda_s^*\left(v_1 - v_2\right) - \frac{1}{2}\left(\sigma_2 - \sigma_1\right)^2\right] ds\right]\Big\} \end{aligned} \tag{2.53}$$

Let us define new normal random variables $\tilde{y}_1 = \int_0^T \left(\phi_s + \sigma_2\right) d\widehat{W}_s^2$, $\tilde{y}_2 = \left(\sigma_1 - \sigma_2\right)\widehat{W}_T^2$, where $\tilde{y}_1 \sim N\left(0, \tilde{\delta}_1^2\right)$, $\tilde{\delta}_1^2 = \int_0^T \left(\phi_s + \sigma_2\right)^2 ds$; $\tilde{y}_2 \sim N\left(0, \tilde{\delta}_2^2\right)$, $\tilde{\delta}_2^2 = \left(\sigma_2 - \sigma_1\right)^2 T$. The random vector $\left(\tilde{y}_1, \tilde{y}_2\right)'$ is still normally distributed with mean equals to $\left(0,0\right)^T$, and correlation $\tilde{\rho} = \int_0^T \left(\phi_s + \sigma_2\right)\left(\sigma_1 - \sigma_2\right) ds$, by checking the linear combination of $\tilde{y}_1, \tilde{y}_2$ similar as the calculation for part I.

Then, we arrive to

$$
\begin{aligned}
H_2(0) = S_0^2 Q_2 \Bigg(& \tilde{y}_1 \le -\ln \frac{a_n \cdot S_0^2 \left(\lambda_T^2\right)^n}{\lambda^n} - \frac{1}{2}\tilde{\delta}_1^2 - \int_0^T \left(\lambda - \lambda_s^2\right) ds \\
& \tilde{y}_2 < \ln \frac{S_0^2 \left(1-v_2\right)^n}{S_0^1 \left(1-v_1\right)^n} - \int_0^T \lambda_s^* \left(v_1 - v_2\right) ds + \frac{1}{2}\tilde{\delta}_2^2 \Bigg) \\
& = S_0^2 \Psi\left(\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2\right)
\end{aligned}
\tag{2.54}
$$

In addition, the maximal set of successful hedging A^* has the expression:

$$
A^* = \left\{ \tilde{y}_1 \le -\ln \frac{a_n \cdot S_0^2 \left(\lambda_T^2\right)^n}{\lambda^n} - \frac{1}{2}\tilde{\delta}_1^2 - \int_0^T \left(\lambda - \lambda_s^2\right) ds \right\} \tag{2.55}
$$

Finally, we combine the results (2.50) and (2.55), and obtain the expression of X_0 as:

$$
\begin{aligned}
X_0 &= X_0^1 + X_0^2 \\
&= \sum_{n=0}^{\infty} e^{-\int_0^T \lambda_t^1 dt} \frac{\left(\int_0^T \lambda_t^1 dt\right)^n}{n!} S_0^1 \Psi\left(\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2\right) \\
&\quad + \sum_{n=0}^{\infty} e^{-\int_0^T \lambda_t^2 dt} \frac{\left(\int_0^T \lambda_t^2 dt\right)^n}{n!} S_0^2 \Psi\left(\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2\right) \\
&= \sum_{n=0}^{\infty} p_{n,T}^* \Big[e^{v_1 \int_0^T \lambda_t^* dt} \left(1-v_1\right)^n S_0^1 \Psi\left(\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2\right) \\
&\quad + e^{v_2 \int_0^T \lambda_t^* dt} \left(1-v_2\right)^n S_0^2 \Psi\left(\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2\right) \Big]
\end{aligned}
\tag{2.56}
$$

where $n = 0,1,2...$, $p_{n,T}^* = e^{-\int_0^T \lambda_t^* dt} \frac{\left(\int_0^T \lambda_t^* dt\right)^n}{n!}$ are the probabilities of a non-homogeneous Poisson distribution with intensity λ_t^*.

Remark 2.1: In order to obtain the premium of the contract X_0, we can also apply the "Multi-Asset Theorem" in Melnikov & Romaniuk (2008) for the calculation. It is found that the calculation by Multi-Asset Theorem leads to the same result as one in Theorem

2.1. The detailed proof using Multi-Asset Theorem is shown in Appendix 1.

Remark 2.2: The payoff of the equity-linked life insurance contract with flexible guarantee H_T can be decomposed into the payoff of an European exchange option plus a pure equity-linked life insurance contract: $H_T = \max\left\{S_t^1, S_t^2\right\} = \left(S_t^1 - S_t^2\right)^+ + S_t^2$. It gives a possibility to reduce the valuation of the initial contract to the embedded exchange option $\left(S_T^1 - S_T^2\right)^+$, and construct the maximal successful hedging set A^* for it (see Melnikov & Skornyakova (2005)).

Following the result in Theorem 2.1, we can derive the expression for the survival probability ${}_T p_x$ formulated in Theorem 2.2:

Theorem 2.2: Suppose that the insurance company sells an equity-linked life insurance contract to the clients and decides to apply quantile hedging to maximize the probability of successful hedging. The survival probability of a potential insured is given by:

$$
{}_T p_x = \frac{\sum_{n=0}^{\infty} p_{n,T}^* \left[p_n \Psi^2\left(\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2\right) + q_n \Psi^2\left(\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2\right) \right]}{\sum_{n=0}^{\infty} p_{n,T}^* \left[p_n \Phi\left(\frac{d_1}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}} \right) + q_n \Phi\left(\frac{d_2}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}} \right) \right]} \tag{2.57}
$$

where the notations $p_{n,T}^*$ and $\Psi^2\left(\cdot,\cdot\right), \Gamma_1, \Gamma_2, \delta_1, \delta_2$, ρ, $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\delta}_1, \tilde{\delta}_2$, $\tilde{\rho}$ are the same as in Theorem2.1, Φ denotes the cumulative distribution function of the standard normal distribution, and other parameters d_1, d_2, p_n, q_n are:

$$
d_1 = \ln \frac{S_0^1\left(1-v_1\right)^n}{S_0^2\left(1-v_2\right)^n} - \int_0^T \lambda_s^*\left(v_2 - v_1\right) ds + \frac{1}{2}\delta_2^2, \qquad p_n = e^{v_1 \int_0^T \lambda_t^* dt}\left(1-v_1\right)^n S_0^1 \ ,
$$

$$
d_2 = \ln \frac{S_0^2\left(1-v_2\right)^n}{S_0^1\left(1-v_1\right)^n} - \int_0^T \lambda_s^*\left(v_1 - v_2\right) ds + \frac{1}{2}\tilde{\delta}_2^2, \qquad q_n = e^{v_2 \int_0^T \lambda_t^* dt}\left(1-v_2\right)^n S_0^2 \ .
$$

Proof: We know from key balance equation (2.42) that the survival probability ${}_T p_x$ admits the following representation:

$$ {}_T p_x = \frac{E^*\left(\tilde{H}_T B_T^{-1}\right)}{E^*\left[H_T B_T^{-1}\right]} = \frac{X_0}{H_0} $$

Since the numerator X_0 is given in Theorem 2.1, we only need to compute the denominator H_0 to get the result.

Let us first decompose $E^*\left[H_T B_T^{-1}\right]$ into two parts: $\Delta_1 = E^*\left[\frac{S_T^1}{B_T} \cdot I_{\{S_T^1 > S_T^2\}}\right]$, and $\Delta_2 = E^*\left[\frac{S_T^2}{B_T} \cdot I_{\{S_T^1 \le S_T^2\}}\right]$. As in the proof of Theorem 2.1, we can calculate $\Delta_1(0)$ as:

$$ \begin{aligned} \Delta_1(0) &= E^*\left[\frac{S_T^1}{B_T} \cdot I_{\{S_T^1 > S_T^2\}}\right] \\ &= S_0^1 E^{Q_1}\left\{ I\left[\ln\left(\frac{S_0^1}{S_0^2}\right) > (\sigma_2 - \sigma_1)\widehat{W}_T^1 + \Pi_T \ln\frac{1-v_2}{1-v_1} \right.\right. \\ &\qquad \left.\left. + \int_0^T \left[\lambda_s^*(v_2 - v_1) - \frac{1}{2}(\sigma_2 - \sigma_1)^2\right] ds \right]\right\} \end{aligned} \tag{2.58} $$

Conditioning on each set $\{\Pi_T = n\}$, $n = 0, 1, 2...$, we can obtain:

$$ \begin{aligned} \Delta_1(0) &= S_0^1 E^{Q_1}\left\{ I\left[\ln\left(\frac{S_0^1}{S_0^2}\right) > (\sigma_2 - \sigma_1)\widehat{W}_T^1 + n\ln\frac{1-v_2}{1-v_1} \right.\right. \\ &\qquad \left.\left. + \int_0^T \left[\lambda_s^*(v_2 - v_1) - \frac{1}{2}(\sigma_2 - \sigma_1)^2\right] ds \right\}\right] \\ &= S_0^1 Q_1\left[(\sigma_2 - \sigma_1)\widehat{W}_T^1 < \ln\frac{S_0^1(1-v_1)^n}{S_0^2(1-v_2)^n} - \int_0^T \lambda_s^*(v_2 - v_1)ds + \frac{1}{2}\delta_2^2 \right] \\ &= S_0^1 \Phi\left[\frac{d_1}{(\sigma_1 - \sigma_2)\sqrt{T}}\right] \end{aligned} \tag{2.59} $$

where $d_1 = \ln\frac{S_0^1(1-v_1)^n}{S_0^2(1-v_2)^n} - \int_0^T \lambda_s^*(v_2 - v_1)ds + \frac{1}{2}\delta_2^2$.

We can treat $\Delta_2(0)$ in the same way:

$$
\begin{aligned}
\Delta_2(0) &= E^*\left[\frac{S_T^2}{B_T}\cdot I_{\{S_T^1 \le S_T^2\}}\right] \\
&= S_0^2 E^{Q_2}\left\{ I\left[\ln\left(\frac{S_0^2}{S_0^1}\right) \ge (\sigma_1-\sigma_2)\widehat{W}_T^2 + n\ln\frac{1-v_1}{1-v_2}\right.\right. \\
&\quad \left.\left. +\int_0^T\left[\lambda_s^*(v_1-v_2)-\frac{1}{2}(\sigma_2-\sigma_1)^2\right]ds\right]\right\} \\
&= S_0^2 Q_2\left[(\sigma_1-\sigma_2)\widehat{W}_T^2 \le \ln\frac{S_0^2(1-v_2)^n}{S_0^1(1-v_1)^n} - \int_0^T \lambda_s^*(v_1-v_2)\,ds + \frac{1}{2}\tilde{\delta}_2^2\right] \\
&= S_0^2\Phi\left[\frac{d}{(\sigma_1-\sigma_2)\sqrt{T}}\right]
\end{aligned}
\tag{2.60}
$$

where $d_2 = \ln\frac{S_0^2(1-v_2)^n}{S_0^1(1-v_1)^n} - \int_0^T \lambda_s^*(v_1-v_2)\,ds + \frac{1}{2}\tilde{\delta}_2^2$.

Therefore, we combine the expression of $\Delta_1(0)\Delta_2(0)$ and obtain

$$
\begin{aligned}
E^*\left[H_T B_T^{-1}\right] &= \sum_{n=0}^{\infty} e^{-\int_0^T \lambda_t^1 dt}\frac{\left(\int_0^T \lambda_t^1 dt\right)^n}{n!} S_0^1\Phi\left[\frac{d_1}{(\sigma_1-\sigma_2)\sqrt{T}}\right] \\
&\quad + \sum_{n=0}^{\infty} e^{-\int_0^T \lambda_t^2 dt}\frac{\left(\int_0^T \lambda_t^2 dt\right)^n}{n!} S_0^2\Phi\left[\frac{d_2}{(\sigma_1-\sigma_2)\sqrt{T}}\right] \\
&= \sum_{n=0}^{\infty} p_{n,T}^*\left[p_n\Phi\left[\frac{d_1}{(\sigma_1-\sigma_2)\sqrt{T}}\right] + +q_n\Phi\left[\frac{d_2}{(\sigma_1-\sigma_2)\sqrt{T}}\right]\right]
\end{aligned}
\tag{2.61}
$$

which leads to the expression (2.57).

2.5.4 Numerical Illustration

In this section, we give a numerical example to illustrate how insurance company can use quantile hedging technique to deal with initial budget constraint situation. Assume that the insurance company will select some acceptable level of financial risk ε first, then figure out the corresponding risk management strategy by quantifying the mortality risk reflected by survival probabilities ${}_T p_x$.

For illustration purpose, we consider a simplified stochastic interest rate model without its jump component: one factor Vasicek-Hull-White model. We set the drift function $\alpha(t,T)$ for forward rate $f(t,T)$ as $\sigma(t,T)\int_t^T \sigma(t,s)ds$. $\alpha(t,T)$ can also be explained as the mean rate of return for the long term interest rate. In addition, we also assume the volatility structure $\sigma(t,T)$ satisfies $\sigma(t,T)=\beta\exp(-\alpha(T-t))$, where $\alpha>0, \beta>0$. This expression leads to one factor Vasicek-Hull-White model so that the dynamics of the instantaneous spot rate r_t is

$$dr_t = \alpha(m(t)-r_t)dt + \beta dW_t \tag{2.62}$$

where $m(t)=f_0+\frac{\eta}{\alpha}+\eta t+\frac{\beta^2}{2\alpha^2}\left(1-e^{-2\alpha t}\right)$ by setting $f(0,t)=f_0+\eta t$. We assume $f_0=0.01$ and $\eta=0$ so that initial term structure is flat. The values of parameters α and β are also assumed to be constant and selected as $\alpha=0.32$, $\beta=0.06$.

For the two-factor jump-diffusion model of risky asset, we apply the approach in Mancini (2004) to estimate the corresponding parameters. There is one Poisson process in our model which determines jumps appeared in two risky assets' price process. However, in Mancini's paper, the Poisson process specifies jumps for only one asset. So we modify the estimator for the number of jumps in Mancini's approach slightly. We consider financial index Russell 2000 (RUT-I) as risky asset S^1, and S&P 500 as risky asset S^2. As Russell 2000 measures the performance of small US companies while S&P 500 is the index of the prices of 500 large-cap common stocks traded in US, it is naturally supposed that RUT-I is more risky than S&P 500. Therefore, it is reasonable to consider S&P 500 as the flexible guarantee S^2 for the contract. Using monthly observations of prices over 23 years from September 1987 to September 2010 (Data source: Yahoo! Finance), we can estimate the parameters of the two-factor jump diffusion model with values:

$$\mu_1 = 0.2763,\ \sigma_1 = 0.19, \nu_1 = -0.27, \tag{2.63}$$

$$\mu_2 = 0.2898, \sigma_2 = 0.15, \nu_2 = -0.2, \lambda = 0.17 \tag{2.64}$$

The initial indices of Russell 2000 and S&P 500 are 167.44 and 329.81. In order to make the initial values of two assets S^1, S^2 the same, we change the value of S^1 as $\frac{329.81}{167.44} S_t^1$. For the following calculations, we select $S_0^1 = S_0^2 = 1000$.

According to proof of Theorem 2.1, we can see that the maximal set of successful hedging $A^* = \left\{ 1 > a_n Z_T \frac{H_T}{B_T} \right\}$ admits two types of expression: If $\Delta_1 > \Delta_2$, $A^* = \left\{ Y^* < \Delta_1 \right\}$, otherwise $A^* = \left\{ Y^* < \Delta_2 \right\}$, where variable Y^* follows the standard normal distribution and

$$\Delta_1 = \frac{-\ln \frac{a_n \cdot S_0^1 \left(\lambda_T^1\right)^n}{\lambda^n} - \frac{1}{2}\delta_1^2 - \int_0^T \left(\lambda - \lambda_s^1\right) ds}{\delta_1} \tag{2.65}$$

$$\Delta_2 = \frac{-\ln \frac{a_n \cdot S_0^2 \left(\lambda_T^2\right)^n}{\lambda^n} - \frac{1}{2}\tilde{\delta}_1^2 - \int_0^T \left(\lambda - \lambda_s^2\right) ds}{\tilde{\delta}_1} \tag{2.66}$$

A sequence of constant a_n can be determined by firstly fixing the probability of the set of successful hedging which is also related with default financial risk ε as $P\left(A^* \middle| \pi_T = n\right) = 1 - \varepsilon = \Phi\left(\Delta_i\right)$, $i = 1,2$, and then using the log-normality of this conditional distribution to estimate the values.

We work on a single premium equity-linked life insurance contracts with maturity $T = 1,3,5,10,15,20$ years separately. Formula (2.57) from Theorem 2.2 can be utilized to calculate the quantile hedging ratio in financial market which also reflects the survival probability ${}_T p_x$ by choosing different acceptable levels of financial risk for insurance company where $\varepsilon = 0.01, 0.025, 0.05$. The results are displayed in Table 2.1.

Table 2.1 Hedging ratios with stochastic guarantee

T	ε=0.01	ε=0.025	ε=0.05
1	0.9885	0.9718	0.9447
3	0.9878	0.9705	0.9426
5	0.9874	0.9697	0.9413
10	0.9867	0.9684	0.9391
15	0.9859	0.9667	0.9364
20	0.9853	0.9656	0.9345

We also calculate the corresponding potential clients' ages based on the well-accepted 2005 United States life Table listed in Arias, Rostron & Tejada-Vera (2010). The ages are presented in Table 2.2.

Table 2.2 Age of insured with stochastic guarantee

T	ε=0.01	ε=0.025	ε=0.05
1	62	73	79
3	49	59	68
5	41	52	61
10	31	41	50
15	22	34	42
20	12	28	36

In order to compare with the results in Melnikov and Skornyakova (2005) which assumed a zero interest rate, we also use the same life table in Boers, et.al (1997) to calculate the ages of the clients shown in Table 2.3.

Table 2.3 Age of insured with stochastic guarantee

T	ε=0.01	ε=0.025	ε=0.05
1	58	69	78
3	45	55	63
5	39	48	56
10	23	39	46

15	12	31	39
20	6	24	33

Compared with Table 2.2 and 2.3, we observe that using the different mortality tables is relevant to the results of potential client ages. The ages obtained from year 2005 Table turns out to be elder than ones calculated from Table in Boers, et.al (1997).

From all the results in Tables 2.1-2.3, it is noted that as the insurance company's financial risk level ε increases (or the probability of successful hedging $1-\varepsilon$ decreases), the survival probability ${}_T p_x$ shows a decreasing trend, while the clients' age increases during the same period. We can conclude that the insurance company should attract elder group of clients in order to compensate for the increasing financial risk. The conclusion is consist with one obtained in Melnikov & Skornyakova (2005).

Furthermore, the results also imply that the survival probability ${}_T p_x$ is decreasing over time as the contract maturity term T is getting longer. Meanwhile, the clients' age also appears a decreasing pattern. Because of mortality risk, the insurance company should attract younger group of clients for the contract with long maturities. However, although the clients' age is still decreasing in the paper by Melnikov and Skornyakova (2005), the survival probability ${}_T p_x$ shows an increasing trend for longer maturity T instead. The difference could be explained by the effect of the stochastic interest rate $r(t)$ in risky assets' model, which leads to a significant change on mortality risk associated with risk management strategies for insurance company.

3. Efficient hedging on equity-linked life insurance contracts in stochastic interest rate environment

3.1. Description of problem

Besides quantile hedging, another well-accepted partial hedging approach is efficient hedging developed by Follmer & Leukert (2000). Efficient hedging is a more general imperfect hedging approach compared with quantile hedging which maximizes the probability of successful hedging under the insufficient initial capital ($X_0 < H_0$). Under the same initial budget constraint, efficient hedging focuses on minimizing the expected size of shortfall risk weighted by some loss functions. It also takes into account the investor's risk preferences towards hedging.

Efficient hedging has been applied on hedging equity-linked life insurance contracts since the first paper Melnikov (2004b). The author obtained the premium of the contract in a diffusion setting. Then, Kirch & Melnikov (2005) extended the valuation work in a jump-diffusion framework. Later on, Melnikov & Romaniuk (2008) made the contribution on this topic by considering the contracts whose payoff depends on the performance of n risky-assets, $n > 2$. Recently, Melnikov & Skornyakova (2011) improved the application of efficient hedging by studying the contracts with stochastic guarantee and they measured the shortfall risk with a special power loss function. In those papers, the interest rate was all assumed either constant or deterministic throughout the entire life of the insurance contract. As far as we know, the issue of valuing equity-linked life insurance contract via efficient hedging in a stochastic interest rate framework has not been studied very much. In this section, we will investigate the effect of stochastic interest rate on efficient hedging and develop the strategies to manage financial and insurance risks inherent in equity-linked life insurance contracts.

In Section 3.2, we introduce the efficient hedging technique. In Section 3.3 and 3.4, we formulate our problem and present the main theoretical results. We consider the diffusion dynamics for both risky assets and interest rates. Section 3.3 focuses on the case that all the financial processes are generated by the same Wiener process. While Section 3.4 considers more advanced models where the financial processes are driven by two correlated Wiener processes.

3.2. Efficient hedging

In this section, we employ the same notations as in Section 2.3 whenever possible. Suppose that the investor is unwilling or unable to put up the initial amount of capital for a perfect hedge, and he is ready to accept some risk. Quantile hedging in Follmer & Leukert (1999) only focuses on the probability of successful hedging. However, the efficient hedging in Follmer & Leukert (2000) takes into account not only the size of the shortfall but also the investor's attitude towards the shortfall risk. The shortfall risk is defined as the expectation of loss from the hedging strategy with expression as $E\left(l\left(\left(H_T - V_T\right)^+\right)\right)$, where $l\left(x\right)$ is a loss function defined on $\left[0,\infty\right)$ and $E\left(l\left(x\right)\right) < \infty$. This loss function $l\left(x\right)$ can represent the investor's risk preference. In Follmer & Leukert (2000), they considered a power loss function as $l\left(x\right) = x^p$. Three types of risk preferences are analyzed based on different values of power p:

(1) when $p > 1$: the investor is risk-aversion, which means the larger the loss is, the less willing the investor wants to take.

(2) when $0 < p < 1$: the investor is risk –taking, such as the addictive gamblers type.

(3) when $p = 1$: the investor is risk-neutral.

Efficient hedging aims to construct an admissible hedging strategy $\pi = \left(V_0, \xi_t\right)$ which can minimize the short fall risk $E\left(l\left(\left(H_T - V_T\right)^+\right)\right)$ under the initial budget constraint $V_0 \le \tilde{V}_0 \le H_0$. Follmer and Leukert obtained the solution of the above optimization problem based on Neyman-Pearson lemma and similar hypothesis testing techniques used to construct quantile hedging. In their paper, first, they prove there exists a unique optimal randomized test $\tilde{\varphi}$ which can solve the equivalent optimization problem $E\left(l\left(\left(1-\varphi\right)H\right)\right)$ under the initial budget constraint $E^*\left(\varphi H\right) \le \tilde{V}_0$. Then, they define the corresponding success ratio as $\varphi\left(V_0, \xi_T\right) = I\left\{V_T \ge H_T\right\} + \frac{V_T}{H_T} I\left\{V_T < H_T\right\}$ for any admissible strategy $\left(V_0, \xi_t\right)$. It is shown in Theorem 3.2 in Follmer & Leukert (2000) that the perfect hedging strategy $\left(\tilde{V}_0, \tilde{\xi}_t\right)$ for the modified contingent claim $\tilde{H}_T = \tilde{\varphi} H_T$ can solve the proposed optimization problem, and the corresponding success ratio $\varphi\left(\tilde{V}_0, \tilde{\xi}_T\right)$ coincides with the optimal randomized test $\tilde{\varphi}$.

Based on Neyman-Pearson lemma, the structure of the success ratio φ is also obtained for the power loss function $l\left(x\right) = x^p$ with $p > 1$, $0 < p < 1$ and $p = 1$ respectively, which will be discussed in more detail in the following Sections. It is noted that the requirement to construct such optimal hedging strategy is on set $\left\{H_T > 0\right\}$. In our case, we work on a single premium equity-linked life insurance contract with a stochastic guarantee, where the benefit at contract maturity equals to the larger value of two risky assets. Therefore, the payoff at maturity H_T is always positive which fulfills the condition $\left\{H_T > 0\right\}$. Next, we will discuss how to apply the efficient hedging technique on equity-linked life insurance.

3.3. Application of efficient hedging on equity-linked insurance: models with one source of randomness

3.3.1 Financial setting

Let $\left(\Omega, \mathbb{F}, \left(F_t\right)_{t\geq 0}, P\right)$ be a standard stochastic basis as described in Section 2.5.1. We consider a continuous time economy with the complete and frictionless financial market. The market is consist of a non-risky asset $\left(B_t\right)_{t\in[0,T]}$ (bank account), and two risky assets $\left(S_t^1\right)_{t\in[0,T]}$ and $\left(S_t^2\right)_{t\in[0,T]}$ (stocks), which satisfy the following dynamics:

$$dB_t = r_t B_t dt \tag{3.1}$$

$$dS_t^i = S_t^i\left(\mu_i dt + \sigma_i dW_t\right), \quad i = 1,2 \tag{3.2}$$

where W_t is a Wiener process under measure P, constants $\mu_i \in R$, $\sigma_i > 0$ are return and volatility of the risky asset S_t^i, respectively. We also assume $\sigma_1 > \sigma_2$, which means S^1 is more risky than S^2. Noted that in Section 3.3, we only restrict our attention to the case where the evolutions of two risky assets' price processes are generated by the same Wiener process W_t. It is a special case ($\rho = 1$) of models driven by two different Wiener processes with some correlation coefficient ρ, which will be discussed in Section 3.4. As the benefit of equity-linked life insurance contracts are mostly related with high positive correlated traditional equities such as traded indices and mutual funds, the restriction of $\rho = 1$ is suitable and convenient. In addition, the case of $\rho = 1$ may not follow the results of the case $0 < \rho < 1$ and should demand a special consideration.

According to Girsanov Theorem, we can define a martingale measure P^* under which the risky-asset $S_t^i, i = 1,2$, satisfies the following risk-neutral dynamics:

$$dS_t^i = S_t^i\left(r_t dt + \sigma_i dW_t^*\right), i = 1, 2, \tag{3.3}$$

where W_t^* is a Wiener process with respect to P^*.

For stochastic interest rate models, we place ourselves in the HJM framework. $f(t,T)$ denotes the forward rate at maturity time $T(\geq t)$ for instantaneous borrowing at time t, which satisfies the following dynamics under the martingale measure P^*:

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)dW_t^* \tag{3.4}$$

and the dynamic of spot interest rate r_t is given by:

$$dr(t) = df(t,t) = \sigma(t,t)\sigma^*(t,t)dt + \sigma(t,t)dW_t^* \tag{3.5}$$

where $\sigma^*(t,T) = \int_t^T \sigma(t,u)du$.

We also consider a zero-coupon bond which pays 1 unit of currency at maturity T, i.e. $P(T,T) = 1$. The bond price $P(t,T)$ at time t is defined as

$$P(t,T) = \exp(-\int_t^T f(t,u)du) \tag{3.6}$$

Since $f(0,t)$ for $0 \leq t \leq T$ is a given constant, the bond price at time zero $P(0,T)$ can also be considered as a constant.

Denote $D_t = \frac{1}{B_t} = \exp(-\int_0^t r(u)du)$, then the discounted bond price $D_t P(t,T)$ admits the following risk-neutral dynamics:

$$d\left(D_t P(t,T)\right) = -D_t P(t,T)\sigma^*(t,T)dW_t^* \tag{3.7}$$

3.3.2 Bond numeraire and forward measure $\tilde{P}$

A numeraire is the unit of account in which other assets are denominated. Sometimes one might change the numeraire to significantly simplify the computation, especially the calculation associated with stochastic interest rate. In principle, we can take any positively priced asset as a numeraire and denominate all other assets in terms of the chosen numeraire. In this section, we will select zero-coupon bond price $P(t,T)$ as the numeraire to calculate the premium of the contract.

Let us define the forward measure $\tilde{P}$ which satisfies:

$$\tilde{P}(A) = \frac{1}{P(0,T)} \int_A D_T dP^*, \text{ for all } A \in \mathbb{F} \tag{3.8}$$

This measure has a local density $\tilde{Z}_t = \left. \frac{d\tilde{P}}{dP^*} \right|_{F_t} = \frac{D_t P(t,T)}{P(0,T)}$, which is a martingale under P^*. We can also construct a new Wiener process $\tilde{W}_t = W_t^* + \int_0^t \sigma^*(u,T) du$ with respect to measure $\tilde{P}$.

A self-financing strategy is given by an initial capital $X_0 > 0$, and a predictable process $\xi_t = (\beta_t, \xi_t^1, \xi_t^2)$ such that the value process X_t is well-defined. Also X_t admits the following expression:

$$X_t = X_0 + \int_0^t \xi_u^1 dS_u^1 + \int_0^t \xi_u^2 dS_u^2 + \int_0^t \beta_u dP(u,T) \tag{3.9}$$

where $\xi_t = (\beta_t, \xi_t^1, \xi_t^2)$ is the number of units invested into bonds and stocks respectively. If the corresponding value process X_t satisfies

$$X_t \geq 0 \qquad \forall t \in (0,T), \quad P - a.s. \tag{3.10}$$

Then the trading strategy (X_0, ξ_t) is called admissible.

By Ito's formula, it is easy to obtain that:

$$d\frac{S_t^i}{P(t,T)} = \frac{S_t^i}{P(t,T)} \left(\sigma^*(t,T) + \sigma_i \right) d\tilde{W}_t, \ i = 1,2 \tag{3.11}$$

Denote $V_X(t) = \frac{X_t}{P(t,T)}$, we can get:

$$V_X(t)=V_X(0)+\int_0^t \xi_u^1 d\frac{S_u^1}{P(u,T)}+\int_0^t \xi_u^2 d\frac{S_u^2}{P(u,T)} \tag{3.12}$$

3.3.3 Applying efficient hedging on equity-linked life insurance

In this section, we follow the same insurance setting as in Chapter 2. Given the constraint on the initial capital available for hedging the payoff $H_T=\max\left(S_T^1,S_T^2\right)$ in $X_0<H_0$, the insurance company needs to seek for some appropriate imperfect hedging techniques which could optimize the hedging outcomes. Efficient hedging can be applied in this situation. Before we introduce our theoretical results of applying efficient hedging on equity-linked life insurance, we modify the original technique discussed in Follmer & Leukert (2000) to make it better suited for the purpose of our analysis with stochastic interest rate. Under our financial setting, we consider the following modified efficient hedging problem:

$$\min E^*\left[l\left(\left(H_T D_T-X_T D_T\right)^+\right)\right] \tag{3.13}$$

Under the constraint: $V_X(0)\le V_0<\tilde{E}(H)$ (3.14)

where $l(x)$ is the loss function defined on $[0,\infty)$ with $l(0)=0$.

Let us introduce the modified contingent claim $\tilde{H}_T=\tilde{\varphi}H_T$, where $\tilde{\varphi}\in\Re$ is the solution of the following problem

$$\min_{\varphi\in\Re} E^*\left[l\left((1-\varphi)HD_T\right)\right] \tag{3.15}$$

Under the constraint: $\tilde{E}(\varphi H)\le\tilde{V}_0$ (3.16)

where $\Re=\left\{\varphi:\Omega\to[0,1]\middle|\varphi\ F_T-measurable\right\}$ is the class of "randomized tests".

For the modified contingent claim $\tilde{H}_T$, we can find an admissible strategy $\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)$, such that

$$\tilde{V}_X(t) = \tilde{E}\left(\tilde{H}_T \middle| F_t\right) = \tilde{V}_X(0) + \int_0^t \tilde{\xi}_u^1 d\frac{S_u^1}{P(u,T)} + \int_0^t \tilde{\xi}_u^2 d\frac{S_u^2}{P(u,T)}, \forall t \in [0,T] \quad (3.17)$$

As a result, we obtain the following proposition:

Proposition 3.1: The admissible strategy $\left(\tilde{V}_X(0), \tilde{\xi}^1, \tilde{\xi}^2\right)$ of the modified contingent claim $\tilde{H}_T = \tilde{\varphi} H_T$ solves the optimization problem (3.13), (3.14), and the success ratio φ is defined as

$$\varphi = 1\left\{\tilde{V}_X(T) \geq H_T\right\} + \frac{\tilde{V}_X(T)}{H_T} 1\left\{\tilde{V}_X(T) < H_T\right\} \quad (3.18)$$

And φ coincides with $\tilde{\varphi} \quad P^* - a.s.$

Proof: According to Theorem 3.2 in Follmer & Leukert (2000), the success ratio φ for any admissible strategy $\left(V_X(0), \xi^1, \xi^2\right)$ with $V_X(0) \leq V_0$ can be defined as

$$\varphi = 1\left\{V_X(T) \geq H_T\right\} + \frac{V_X(T)}{H_T} 1\left\{V_X(T) < H_T\right\} \quad (3.19)$$

Since $\varphi H_T = V_X(T) \wedge H_T$, the shortfall takes the form

$$\left(H_T - V_X(T)\right)^+ = H_T - V_X(T) \wedge H_T = (1 - \varphi) H_T \quad (3.20)$$

Under the forward measure $\tilde{P}$, the value process $V_X(t)$ is a martingale, so we obtain

$$\tilde{E}\left(\frac{\varphi H_T}{P(T,T)}\right) = \tilde{E}(\varphi H_T) \leq \tilde{E}\left(V_X(T)\right) = V_X(0) \leq V_0 \quad (3.21)$$

which shows the success ratio φ satisfies the initial capital constraint (3.14).

As $\tilde{\varphi}$ is the solution for the problem (3.15), (3.16), we can get

$$\begin{aligned} & E^*\left[l\left(\left(D_T H_T - D_T V_X(T)\right)^+\right)\right] \\ & = E^*\left[l\left((1-\varphi) H_T D_T\right)\right] \\ & \geq E^*\left[l\left((1-\tilde{\varphi}) H_T D_T\right)\right] \end{aligned} \quad (3.22)$$

On the other hand, due to (3.22), $\left(\tilde{V}_X(0), \tilde{\xi}^1, \tilde{\xi}^2\right)$ has success ratio $\varphi\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)$ which satisfies

$$\varphi\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)H_T=\tilde{V}_X(T)\wedge H_T\geq\tilde{\varphi}H_T,\quad P^*-a.s.\ \text{ on }\ \{H_T>0\} \tag{3.23}$$

Without loss of generality, we assume that $\tilde{\varphi}=1$ on $\{H_T=0\}$, so we obtain:

$$\varphi\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)H_T=\tilde{\varphi}H_T,\quad \text{on}\{H_T=0\} \tag{3.24}$$

Therefore the success ratio φ coincides $P^*-a.s.$ with $\tilde{\varphi}$. In particular, we get:

$$\left(D_TH_T-D_TV_X(T)\right)^+=(1-\tilde{\varphi})H_TD_T \tag{3.25}$$

So the strategy $\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)$ solves the optimization problem defined by (3.13), (3.14).

To solve the optimization problem, we need to obtain a perfect hedge $\left(\tilde{V}_X(0),\tilde{\xi}^1,\tilde{\xi}^2\right)$ for the modified contingent claim $\tilde{H}_T=\tilde{\varphi}H_T$. We work with a power loss function $l(x)=x^p$. Based on the analysis in Follmer & Leukert (2000), $\tilde{\varphi}$ is unique on set $\{H_T>0\}$, for $p>1$, $0<p<1$, and the special case $p=1$. The value of power p also represents three types of investors: $p=1$ denotes the risk-neutral investor; $0<p<1$ denotes the risk-taker; $p>1$ denotes the risk-averse investor. Neyman-Pearson lemma can be used to find out the structures of $\tilde{\varphi}$ in three different cases. Applying the same technique, if we rewrite the optimal test which is simple hypothesis P against the simple alternative P^* in terms of P^* and $\tilde{P}$, we can obtain the structures of $\tilde{\varphi}$ similar as in Follmer & Leukert (2000) as following:

$$\tilde{\varphi}=1-\left(\frac{I\left(c_p\tilde{Z}_T\right)}{H}\wedge 1\right),\qquad \text{for }\ p>1 \tag{3.26}$$

$$\tilde{\varphi}=1\left\{1>c_pH^{1-p}\tilde{Z}_T\right\},\qquad \text{for }\ 0<p<1 \tag{3.27}$$

$$\tilde{\varphi} = 1\{1 > c_p \tilde{Z}_T\}, \qquad \text{for} \quad p = 1 \tag{3.28}$$

where $I\{\cdot\}$ is the inverse of the derivative of the loss function $l(x)$, $1\{\cdot\}$ is the indicator function, c_p is the constant determined from the condition $\tilde{E} = (\tilde{\varphi} H_T) = \tilde{V}_X(0)$.

In the following, we will apply the efficient hedging approach on equity-linked life insurance contract with a stochastic guarantee. As discussed in Section 2.5.2, the initial cost of hedging equity-linked life insurance contract X_0 is given by:

$$\begin{aligned} X_0 &= E^* \cdot \hat{E}\left(H_T \cdot D_T \cdot 1\{T(x) > T\}\right) \\ &= P(0,T)\tilde{E}(H_T) \cdot {}_T p_x \end{aligned} \tag{3.29}$$

From (3.29), it is obvious that

$$\tilde{V}_X(0) = \frac{X_0}{P(0,T)} = \tilde{E}(H_T) \cdot {}_T p_x \tag{3.30}$$

Applying the efficient hedging, we also obtain that:

$$\tilde{V}_X(0) = \tilde{E}\left(\tilde{H}_T\right) = \tilde{E}(\tilde{\varphi} H_T) \tag{3.31}$$

Hence, we obtain the key balance equation similar as quantile hedging:

$$_T p_x = \frac{\tilde{E}(\tilde{\varphi} H_T)}{\tilde{E}(H_T)} \tag{3.32}$$

The main theoretical results are contained in the following theorems:

Theorem 3.1: Consider an insurance company that sells a single equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$, and the firm's risk preference is risk-aversion with a power loss function $l(x) = x^p$, $p > 1$. Then

(i) The initial price of the contract is

$$\begin{aligned} X_0 &= S_0^1 \Psi^2\left(Y_{1,} Y_2; \rho_{s_1 s_3}\right) + S_0^2 \Psi^2\left(Y_3, Y_4; \rho_{s_2 s_4}\right) \\ &- MP(0,T)\left[\Psi^2\left(Y_5, Y_6; \rho_{s_1 s_3}\right) + \Psi^2\left(Y_7, Y_8; \rho_{s_2 s_4}\right)\right] \end{aligned} \tag{3.33}$$

where S_0^1 and S_0^2 are the initial assets' prices, $\Psi^2(\cdot)$ denotes the two-dimensional cumulative normal distribution function with

$$Y_1 = \frac{\ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_1 - \sigma_2\right)^2 T}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}}, \quad Y_3 = \frac{\ln\frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1 - \sigma_2\right)^2 T}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}},$$

$$Y_2 = \frac{\ln\frac{S_0^1}{\tilde{k}P(0,T)} + \int_0^T \left[\frac{p}{2(p-1)}\left(\sigma^*(u,T)\right)^2 + \frac{\sigma_1^2}{2} + \frac{p}{p-1}\sigma_1\sigma^*(u,T)\right]du}{\delta_1},$$

$$Y_4 \frac{\ln\frac{S_0^2}{\tilde{k}P(0,T)} + \int_0^T \left[\frac{p}{2(p-1)}\left(\sigma^*(u,T)\right)^2 + \frac{\sigma_2^2}{2} + \frac{p}{p-1}\sigma_2\sigma^*(u,T)\right]du}{\delta_2},$$

$$Y_5 = \frac{\ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)T + \frac{p}{1-p}\left(\sigma_1 - \sigma_2\right)\int_0^T \sigma^*(u,T)\,du_u}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}},$$

$$Y_6 = \frac{\ln\frac{S_0^1}{\tilde{k}P(0,T)} + \int_0^T \left[-\frac{p^2+p}{2(p-1)^2}\left(\sigma^*(u,T)\right)^2 - \frac{\sigma_1^2}{2} + \frac{p}{1-p}\sigma_1\sigma^*(u,T)\right]du}{\delta_1},$$

$$Y_7 = \frac{\ln\frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)T + \frac{p}{p-1}\left(\sigma_1 - \sigma_2\right)\int_0^T \sigma^*(u,T)\,du}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}},$$

$$Y_8 = \frac{\ln\frac{S_0^2}{\tilde{k}P(0,T)} + \int_0^T \left[-\frac{p^2+p}{2(p-1)^2}\left(\sigma^*(u,T)\right)^2 - \frac{\sigma_2^2}{2} + \frac{p}{1-p}\sigma_2\sigma^*(u,T)\right]du}{\delta_2},$$

$$\delta_1^2 = \int_0^T \left(\sigma_1 + \frac{p}{p-1}\sigma^*(u,T)\right)^2 du, \quad \delta_2^2 = \int_0^T \left(\sigma_2 + \frac{p}{p-1}\sigma^*(u,T)\right)^2 du$$

$$M = \tilde{k}\exp\left(\frac{p}{2(1-p)^2}\int_0^T \left(\sigma^*(u,T)\right)^2 du\right), \quad \tilde{k} = \left(\frac{c_p}{p}\right)^{\frac{1}{p-1}},$$

and the correlations are

$$\rho_{s_1s_3}=\frac{\int_0^T\left(\sigma_1+\frac{p}{p-1}\sigma^*(u,T)\right)du}{\delta_1\sqrt{T}},\ \rho_{s_2s_4}=\frac{-\int_0^T\left(\sigma_2+\frac{p}{p-1}\sigma^*(u,T)\right)du}{\delta_2\sqrt{T}}.$$

(ii) The survival probability ${}_Tp_x$ is given by

$$\begin{aligned}{}_Tp_x&=\frac{S_0^1\Psi^2\left(Y_1,Y_2;\rho_{s_1s_3}\right)+S_0^2\Psi^2\left(Y_3,Y_4;\rho_{s_2s_4}\right)}{S_0^1\cdot\Psi\left(Y_1\right)+S_0^2\cdot\Psi\left(Y_3\right)}\\&-\frac{M\left[\Psi^2\left(Y_5,Y_6;\rho_{s_1s_3}\right)+\Psi^2\left(Y_7,Y_8;\rho_{s_2s_4}\right)\right]}{S_0^1\cdot\Psi\left(Y_1\right)+S_0^2\cdot\Psi\left(Y_3\right)}\end{aligned}\tag{3.34}$$

where $\Psi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Proof: (i) The success ratio $\tilde{\varphi}$ for risk-aversion case is given by (3.26), so we can simplify the modified contingent claim $\tilde{H}$ as

$$\begin{aligned}\tilde{H}_T=\tilde{\varphi}\cdot H_T&=H_T-\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\wedge H_T\\&=\left(H_T-\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\right)\cdot 1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<H_T\right\}\end{aligned}\tag{3.35}$$

where $\tilde{k}=\left(\frac{c_p}{p}\right)^{\frac{1}{p-1}}$. From (3.31), we can obtain the following expression for $\tilde{V}_X(0)$:

$$\begin{aligned}\tilde{V}_X(0)&=\tilde{E}\left(\tilde{\varphi}H_T\right)\\&=\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^1\right\}\right)\\&+\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)\\&-\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^1\right\}\right)\\&-\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)\end{aligned}\tag{3.36}$$

The indicator functions in (3.36) can be simplified as following:

$$1\left\{S_T^1 \geq S_T^2\right\}$$

$$=1\left\{s_1 \leq \frac{\ln\frac{S_0^1}{S_0^2}+\frac{1}{2}\left(\sigma_2^2-\sigma_1^2\right)T+\left(\sigma_2-\sigma_1\right)\int_0^T\sigma^*\left(u,T\right)du}{\left(\sigma_1-\sigma_2\right)\sqrt{T}}\right\} \quad (3.37$$

$$1\left\{S_T^1 < S_T^2\right\}$$

$$=1\left\{s_2 < \frac{\ln\frac{S_0^2}{S_0^1}+\frac{1}{2}\left(\sigma_1^2-\sigma_2^2\right)T+\left(\sigma_1-\sigma_2\right)\int_0^T\sigma^*\left(u,T\right)du}{\left(\sigma_1-\sigma_2\right)\sqrt{T}}\right\} \quad (3.38)$$

$$1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1\right\}=1\left\{\tilde{k}\left(\frac{D_T B\left(T,T\right)}{B\left(0,T\right)}\right)^{\frac{1}{p-1}} < \frac{S_T^1}{B\left(T,T\right)}\right\}$$

$$=1\left\{s_3 < \frac{\ln\frac{S_0^1}{\tilde{k}P\left(0,T\right)}+\int_0^T\left[-\frac{p}{2\left(p-1\right)}\left(\sigma^*\left(u,T\right)\right)^2-\frac{\sigma_1^2}{2}-\sigma_1\sigma^*\left(u,T\right)\right]du}{\delta_1}\right\} \quad (3.39)$$

$$1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}$$

$$=1\left\{s_4 < \frac{\ln\frac{S_0^2}{\tilde{k}P\left(0,T\right)}+\int_0^T\left[-\frac{p}{2\left(p-1\right)}\left(\sigma^*\left(u,T\right)\right)^2-\frac{\sigma_2^2}{2}-\sigma_2\sigma^*\left(u,T\right)\right]du}{\delta_2}\right\} \quad (3.40)$$

where $s_1=-\frac{\tilde{W}_T}{\sqrt{T}}\sim N\left(0,1\right)$, $s_2=\frac{\tilde{W}_T}{\sqrt{T}}\sim N\left(0,1\right)$,

$$s_3=\frac{-\int_0^T\left(\sigma_1+\frac{p}{p-1}\sigma^*\left(u,T\right)\right)d\tilde{W}_u}{\delta_1}\sim N\left(0,1\right),\ \delta_1^2=\int_0^T\left(\sigma_1+\frac{p}{p-1}\sigma^*\left(u,T\right)\right)^2du,$$

$$s_4 = \frac{-\int_0^T \left(\sigma_2 + \frac{p}{p-1} \sigma^*(u,T) \right) d\tilde{W}_u}{\delta_2} \sim N(0,1), \ \ \delta_2^2 = \int_0^T \left(\sigma_2 + \frac{p}{p-1} \sigma^*(u,T) \right)^2 du .$$

We substitute (3.37) and (3.39) into the expression of $\tilde{V}_X(0)$ in (3.36) and obtain:

$$\begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\{S_T^1 \ge S_T^2\} 1\left\{ \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1 \right\} \right) \\ &= \frac{S_0^1}{P(0,T)} \exp\left(-\int_0^T \frac{\left(\sigma_1 + \sigma^*(u,T)\right)^2}{2} du \right) \tilde{E}\left(e^{-z_1} 1\{s_1 \le \Delta_1\} 1\{s_3 < \Delta_3\} \right) \end{aligned} \tag{3.41}$$

where $z_1 = -\int_0^T \left(\sigma_1 + \sigma^*(u,T)\right) d\tilde{W}_u \sim N\left(0, \zeta_1^2\right)$, $\zeta_1^2 = \int_0^T \left(\sigma_1 + \sigma^*(u,T)\right)^2 du$,

$$\Delta_1 = \frac{\ln \frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right) T + \left(\sigma_2 - \sigma_1\right) \int_0^T \sigma^*(u,T) du}{\left(\sigma_1 - \sigma_2\right)\sqrt{T}},$$

$$\Delta_3 = \frac{\ln \frac{S_0^1}{\tilde{k} P(0,T)} + \int_0^T \left[-\frac{p}{2(p-1)} \left(\sigma^*(u,T)\right)^2 - \frac{\sigma_1^2}{2} - \sigma_1 \sigma^*(u,T) \right] du}{\delta_1} .$$

"Multi-Asset Theorem" (see details in Melnikov & Romaniuk (2008)) can be applied to evaluate the expectation (3.41) for $n = 2$. First, the necessary correlations are calculated as

$$\rho_{s_1 s_3} = \frac{\int_0^T \left(\sigma_1 + \frac{p}{p-1} \sigma^*(u,T) \right) du}{\delta_1 \sqrt{T}}, \ \ \rho_{s_1 z_1} = \frac{\int_0^T \left(\sigma_1 + \sigma^*(u,T) \right) du}{\zeta_1 \sqrt{T}},$$

$$\rho_{s_3 z_1} = \frac{\int_0^T \left(\sigma_1 + \frac{p}{p-1} \sigma^*(u,T) \right) \left(\sigma_1 + \sigma^*(u,T) \right) du}{\delta_1 \zeta_1},$$

Then, we apply the theorem with the above parameters and get:

$$\tilde{E}\left(S_T^1 \cdot 1\{S_T^1 \ge S_T^2\} 1\left\{ \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1 \right\} \right) = \frac{S_0^1}{P(0,T)} \Psi^2\left(Y_1, Y_2; \rho_{s_1 s_3}\right) \tag{3.42}$$

with Y_1 and Y_2 given in Theorem 3.1.

Now let us turn to another term $\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)$ in (3.36).

From (3.38) and (3.40), this term can be simplified as:

$$\begin{aligned}&\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)\\&=\frac{S_0^2}{P(0,T)}\exp\left(-\int_0^T\frac{\left(\sigma_2+\sigma^*(u,T)\right)^2}{2}du\right)\tilde{E}\left(e^{-z_2}1\{s_2<\Delta_2\}1\{s_4<\Delta_4\}\right)\end{aligned}\tag{3.43}$$

where $z_2=-\int_0^T\left(\sigma_2+\sigma^*(u,T)\right)d\tilde{W}_u\sim N\left(0,\zeta_2^2\right)$, $\zeta_2^2=\int_0^T\left(\sigma_2+\sigma^*(u,T)\right)^2du$,

$$\Delta_2=\frac{\ln\frac{S_0^2}{S_0^1}+\frac{1}{2}\left(\sigma_1^2-\sigma_2^2\right)T+\left(\sigma_1-\sigma_2\right)\int_0^T\sigma^*(u,T)du}{\left(\sigma_1-\sigma_2\right)\sqrt{T}},$$

$$\Delta_4=\frac{\ln\frac{S_0^2}{\tilde{k}P(0,T)}+\int_0^T\left[-\frac{p}{2(p-1)}\left(\sigma^*(u,T)\right)^2-\frac{\sigma_2^2}{2}-\sigma_2\sigma^*(u,T)\right]du}{\delta_2}.$$

with the correlations

$$\rho_{s_2s_4=}\frac{-\int_0^T\left(\sigma_2+\frac{p}{p-1}\sigma^*(u,T)\right)du}{\delta_2\sqrt{T}},\quad \rho_{s_2z_2=}-\frac{\int_0^T\left(\sigma_2+\sigma^*(u,T)\right)du}{\zeta_2\sqrt{T}},$$

$$\rho_{s_4z_2=}\frac{\int_0^T\left(\sigma_2+\frac{p}{p-1}\sigma^*(u,T)\right)\left(\sigma_2+\sigma^*(u,T)\right)du}{\delta_2\zeta_2},$$

We apply the Multi-Asset Theorem again, and the expectation in (3.43) becomes:

$$\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)=\frac{S_0^2}{P(0,T)}\Psi^2\left(Y_3,Y_4;\rho_{s_2s_4}\right)\tag{3.44}$$

We can repeat the similar calculations for the other two expectation in (3.36), and obtain the results:

$$\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^1\right\}\right)=M\Psi^2\left(Y_5,Y_6;\rho_{s_1s_3}\right)\tag{3.45}$$

$$\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^2\right\}\right)=M\Psi\left(Y_7,Y_8;\rho_{s_2s_4}\right) \tag{3.46}$$

where $M=\tilde{k}\exp\left(\frac{p}{2(1-p)^2}\int_0^T\left(\sigma^*(u,T)\right)^2du\right)$.

Combining (3.42), (3.44) ~ (3.46), the expression of $\tilde{V}_X(0)$ becomes

$$\begin{aligned}&\tilde{V}_X(0)\\&=\frac{S_0^1}{P(0,T)}\Psi^2\left(Y_{1,}Y_2;\rho_{s_1s_3}\right)+\frac{S_0^2}{P(0,T)}\Psi^2\left(Y_3,Y_4;\rho_{s_2s_4}\right)\\&-M\left[\Psi^2\left(Y_5,Y_6;\rho_{s_1s_3}\right)+\Psi^2\left(Y_7,Y_8;\rho_{s_2s_4}\right)\right]\end{aligned} \tag{3.47}$$

Finally, we arrive at the result (3.33).

(ii) According to the key formula (3.32), we only need to calculate the expectation $\tilde{E}(H)$.

First, it is easily to observe that

$$\begin{aligned}\tilde{E}\left(H_T\right)&=\tilde{E}\left(\max\left\{S_T^1,S_T^2\right\}\right)\\&=\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}\right)+\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}\right)\end{aligned} \tag{3.48}$$

The first expectation in (3.48) can be calculated as:

$$\begin{aligned}&\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}\right)\\&=\frac{S_0^1}{P(0,T)}e^{-\int_0^T\frac{\left(\sigma_1+\sigma^*(u,T)\right)^2}{2}du}\tilde{E}\left(e^{-z_1}\cdot 1\left\{s_1\leq\Delta_1\right\}\right)\\&=\frac{S_0^1}{P(0,T)}\cdot\Psi\left(Y_1\right)\end{aligned} \tag{3.49}$$

The second expectation in (3.48) can be treated in a similarly way:

$$\begin{aligned}&\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}\right)\\&=\frac{S_0^2}{P(0,T)}e^{-\int_0^T\frac{\left(\sigma_2+\sigma^*(u,T)\right)^2}{2}du}\tilde{E}\left(e^{-z_2}\cdot 1\left\{s_2\leq\Delta_2\right\}\right)\\&=\frac{S_0^2}{P(0,T)}\cdot\Psi\left(Y_3\right)\end{aligned} \tag{3.50}$$

Substitute (3.49), (3.50), we get

$$\tilde{E}(H_T)=\frac{S_0^1}{P(0,T)}\cdot\Psi(Y_1)+\frac{S_0^2}{P(0,T)}\cdot\Psi(Y_3) \tag{3.51}$$

Therefore, the survival probability is given by

$$\begin{aligned}{}_T p_x &= \frac{\tilde{E}(\tilde{\varphi}H_T)}{\tilde{E}(H_T)} \\ &= \frac{S_0^1\Psi^2\left(Y_{1,}Y_2;\rho_{s_1s_3}\right)+S_0^2\Psi^2\left(Y_3,Y_4;\rho_{s_2s_4}\right)}{S_0^1\cdot\Psi(Y_1)+S_0^2\cdot\Psi(Y_3)} \\ &\quad -\frac{M\left[\Psi^2\left(Y_5,Y_6;\rho_{s_1s_3}\right)+\Psi^2\left(Y_7,Y_8;\rho_{s_2s_4}\right)\right]}{S_0^1\cdot\Psi(Y_1)+S_0^2\cdot\Psi(Y_3)}\end{aligned} \tag{3.52}$$

Theorem 3.2: Suppose that an insurance company sells a single equity-linked life insurance contract with payoff $H_T=\max(S_T^1,S_T^2)$, and the firm decides to use efficient hedging to minimize the shortfall risk. It's risk preference is risk-taking with a power loss function $l(x)=x^p$, $0<p<1$. Then we have:

(i) The initial price of the contract is

$$X_0=S_0^1\Psi^2\left(Y_1,\tilde{Y}_2;\rho_{s_1\tilde{s}_1}\right)+S_0^2\Psi^2\left(Y_3,\tilde{Y}_4;\rho_{s_2\tilde{s}_2}\right) \tag{3.53}$$

where S_0^1 and S_0^2 are the initial assets' prices, $\Psi^2(\cdot)$ denotes the two-dimensional cumulative normal distribution function with

$$\tilde{Y}_2=\frac{-\ln\tilde{k}_1+\int_0^T\left[\frac{p}{2}\left(\sigma^*(u,T)\right)^2+\frac{(p-1)}{2}\sigma_1^2+p\sigma_1\sigma^*(u,T)\right]du}{\tilde{\delta}_1},$$

$$\tilde{Y}_4=\frac{-\ln\tilde{k}_2+\int_0^T\left[\frac{p}{2}\left(\sigma^*(u,T)\right)^2+\frac{(p-1)}{2}\sigma_2^2+p\sigma_2\sigma^*(u,T)\right]du}{\tilde{\delta}_2},$$

$$\tilde{\delta}_1^2=\int_0^T\left[(1-p)\sigma_1-p\sigma^*(u,T)\right]^2du,\ \tilde{k}_1=c_p\left(\frac{S_0^1}{P(0,T)}\right)^{1-p}$$

$$\tilde{\delta}_2^2=\int_0^T\left[(1-p)\sigma_2-p\sigma^*(u,T)\right]^2du,\ \tilde{k}_2=c_p\left(\frac{S_0^2}{P(0,T)}\right)^{1-p}$$

The correlations are

$$\rho_{s_1\tilde{s}_1=}\frac{\int_0^T\left(\left(p-1\right)\sigma_1+p\sigma^*\left(u,T\right)\right)du}{\tilde{\delta}_1\sqrt{T}},\ \rho_{s_2\tilde{s}_2=}\frac{\int_0^T\left(\left(1-p\right)\sigma_2-p\sigma^*\left(u,T\right)\right)du}{\tilde{\delta}_2\sqrt{T}}.$$

(ii). The survival probability ${}_T p_x$ is in the form

$$ {}_T p_x=\frac{S_0^1\Psi^2\left(Y_1,\tilde{Y}_2;\rho_{s_1\tilde{s}_1}\right)+S_0^2\Psi^2\left(Y_3,\tilde{Y}_4;\rho_{s_2\tilde{s}_2}\right)}{S_0^1\Psi\left(Y_1\right)+S_0^2\Psi\left(Y_3\right)} \tag{3.54}$$

where $\Psi\left(\cdot\right)$ is the cumulative distribution function of standard normal distribution.

Proof: (i) When $0<p<1$, recall that the success ratio $\tilde{\varphi}$ has the form

$$\tilde{\varphi}=1\left\{1>c_p H_T{}^{1-p}\tilde{Z}_T\right\} \tag{3.55}$$

Then, we can rewrite the expression of $\tilde{V}_X\left(0\right)$:

$$\begin{aligned}&\tilde{V}_X\left(0\right)=\tilde{E}\left(\tilde{\varphi}H_T\right)\\&=\tilde{E}\left(\max\left\{S_T^1,S_T^2\right\}1\left\{1>c_p\max\left\{S_T^1,S_T^2\right\}^{1-p}\tilde{Z}_T\right\}\right)\\&=\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p}<1\right\}\right)\\&+\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^2\right)^{1-p}<1\right\}\right)\end{aligned} \tag{3.56}$$

Here we only show the calculation for $\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p}<1\right\}\right)$, the calculation for $\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1<S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^2\right)^{1-p}<1\right\}\right)$ is symmetric.

The set $\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p}<1\right\}$ is simplified as:

$$\begin{aligned}&\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p}<1\right\}\\&=\left\{\int_0^T\left[\left(1-p\right)\sigma_1-p\sigma^*\left(u,T\right)\right]d\tilde{W}_u<\right.\\&\left.-\ln\tilde{k}_1+\int_0^T\left(-\frac{p}{2}\left(\sigma^*\left(u,T\right)\right)^2-\frac{p-1}{2}\sigma_1^2-\left(p-1\right)\sigma_1\sigma^*\left(u,T\right)\right)du\right\}\\&=\left\{\tilde{s}_1<\tilde{\Delta}_1\right\}\end{aligned} \tag{3.57}$$

where $\tilde{k}_1 = c_p \left(\frac{S_0^1}{P(0,T)} \right)^{1-p}$, $\tilde{\delta}_1^2 = \int_0^T \left[(1-p)\sigma_1 - p\sigma^*(u,T) \right]^2 du$,

$$\tilde{s}_1 = \frac{\int_0^T \left[(1-p)\sigma_1 - p\sigma^*(u,T) \right] d\tilde{W}_u}{\tilde{\delta}_1} \sim N(0,1),$$

$$\tilde{\Delta}_1 = \frac{-\ln \tilde{k}_1 + \frac{1}{2}\int_0^T \left(-p\left(\sigma^*(u,T)\right)^2 + (1-p)\sigma_1^2 + 2(1-p)\sigma_1\sigma^*(u,T) \right) du}{\tilde{\delta}_1}.$$

So the first term in (3.56) becomes

$$\begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\left\{ S_T^1 \geq S_T^2 \right\} 1\left\{ c_p \tilde{Z}_T \left(S_T^1 \right)^{\frac{1}{p-1}} < 1 \right\} \right) \\ &= \frac{S_0^1}{P(0,T)} e^{-\int_0^T \frac{\left(\sigma_1 + \sigma^*(u,T)\right)^2}{2} du} \tilde{E}\left(e^{-z_1} \cdot 1\left\{ \tilde{s}_1 < \tilde{\Delta}_1 \right\} 1\left\{ s_1 \leq \Delta_1 \right\} \right) \end{aligned} \tag{3.58}$$

Based on Multi-Asset Theorem, we take the corresponding correlations as

$$\rho_{s_1\tilde{s}_1} = \frac{\int_0^T \left((p-1)\sigma_1 + p\sigma^*(u,T) \right) du}{\tilde{\delta}_1 \sqrt{T}}, \quad \rho_{s_1 z_1} = \frac{\int_0^T \left(\sigma_1 + \sigma^*(u,T) \right) du}{\zeta_1 \sqrt{T}}$$

$$\rho_{\tilde{s}_1 z_1} = \frac{\int_0^T \left[(p-1)\sigma_1^2 + (2p-1)\sigma_1\sigma^*(u,T) + p\left(\sigma^*(u,T)\right)^2 \right] du}{\tilde{\delta}_1 \zeta_1},$$

and obtain

$$\begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\left\{ S_T^1 \geq S_T^2 \right\} 1\left\{ c_p \tilde{Z}_T \left(S_T^1 \right)^{\frac{1}{p-1}} < 1 \right\} \right) \\ &= \frac{S_0^1}{P(0,T)} \Psi^2\left(Y_1, \tilde{Y}_2; \rho_{s_1\tilde{s}_1} \right) \end{aligned} \tag{3.59}$$

where $\tilde{Y}_2$ is given in Theorem 3.1.

Through similar computations, we have

$$\begin{aligned} &\tilde{E}\left(S_T^2 \cdot 1\left\{ S_T^1 < S_T^2 \right\} 1\left\{ c_p \tilde{Z}_T \left(S_T^2 \right)^{\frac{1}{p-1}} < 1 \right\} \right) \\ &= \frac{S_0^2}{P(0,T)} \Psi^2\left(Y_3, \tilde{Y}_4; \rho_{s_2\tilde{s}_2} \right) \end{aligned} \tag{3.60}$$

With $\tilde{Y}_4$, $\rho_{s_2\tilde{s}_2}$ are shown in Theorem 3.2.

Therefore, (3.59) and (3.60) lead to the expression of

$$\tilde{V}_X(0) = \frac{S_0^1}{P(0,T)} \Psi^2\left(\tilde{Y}_1, \tilde{Y}_2; \rho_{s_1\tilde{s}_1}\right) + \frac{S_0^2}{P(0,T)} \Psi^2\left(\tilde{Y}_3, \tilde{Y}_4; \rho_{s_2\tilde{s}_2}\right) \tag{3.61}$$

So we find $X_0 = S_0^1 \Psi^2\left(Y_1, \tilde{Y}_2; \rho_{s_1\tilde{s}_1}\right) + S_0^2 \Psi^2\left(Y_3, \tilde{Y}_4; \rho_{s_2\tilde{s}_2}\right)$.

(ii) According to the key formula ${}_T p_x = \dfrac{\tilde{E}(\tilde{\varphi} H_T)}{\tilde{E}(H_T)}$ and the calculation of ${}_T p_x$ in Theorem 3.1, we get the formula for ${}_T p_x$ in (3.54).

Theorem 3.3: Consider an insurance company that sells a single equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$, and the firm's risk preference is risk-indifference with a power loss function $l(x) = x^p$, $p = 1$. Then

(i) The initial price of the contract is

$$X_0 = S_0^1 \Psi^2\left(Y_1, \hat{Y}_2; \rho_{s_1\hat{s}_1}\right) + S_0^2 \Psi^2\left(Y_3, \hat{Y}_4; \rho_{s_2\hat{s}_1}\right) \tag{3.62}$$

where S_0^1 and S_0^2 are the initial assets' prices, $\Psi^2(\cdot)$ denotes the two-dimensional cumulative normal distribution function with

$$\hat{Y}_2 = \frac{-\ln c_p + \int_0^T \left[\frac{1}{2}\left(\sigma^*(u,T)\right)^2 + \sigma_1 \sigma^*(u,T)\right] du}{\hat{\delta}_1},$$

$$\hat{Y}_4 = \frac{-\ln c_p + \int_0^T \left[\frac{1}{2}\left(\sigma^*(u,T)\right)^2 + \sigma_2 \sigma^*(u,T)\right] du}{\hat{\delta}_1},$$

$$\hat{\delta}_1^2 = \int_0^T \left(\sigma^*(u,T)\right)^2 du, \quad \rho_{s_1\hat{s}_1} = \frac{\int_0^T \sigma^*(u,T)\, du}{\hat{\delta}_1 \sqrt{T}}, \quad \rho_{s_2\hat{s}_1} = \frac{-\int_0^T \sigma^*(u,T)\, du}{\hat{\delta}_1 \sqrt{T}}.$$

(ii) The survival probability ${}_T p_x$ has the form

$$
{}_T p_x = \frac{S_0^1 \Psi^2\left(Y_1, \hat{Y}_2; \rho_{s_1\hat{s}_1}\right) + S_0^2 \Psi^2\left(Y_3, \hat{Y}_4; \rho_{s_2\hat{s}_1}\right)}{S_0^1 \Psi\left(Y_1\right) + S_0^2 \Psi\left(Y_3\right)} \tag{3.63}
$$

where $\Psi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Proof: (i) If $p = 1$, the structure of success ratio is $\tilde{\varphi} = 1\left\{1 > c_p \tilde{Z}_T\right\}$,

so we rewrite

$$
\begin{aligned}
&\tilde{V}_X(0) = \tilde{E}\left(\tilde{\varphi} H_T\right) \\
&= \tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) + \tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right)
\end{aligned} \tag{3.64}
$$

Simplify the indicator function $1\left\{1 > c_p \tilde{Z}_T\right\}$, we obtain:

$$
\begin{aligned}
&1\left\{1 > c_p \tilde{Z}_T\right\} \\
&= 1\left\{\frac{1}{c_p} > \exp\left(-\int_0^T \sigma^*(u,T)\, d\tilde{W}_u - \frac{1}{2}\int_0^T \left(\sigma^*(u,T)\right)^2 du\right)\right\} \\
&= 1\left\{-\ln c_p - \frac{1}{2}\int_0^T \left(\sigma^*(u,T)\right)^2 du > -\int_0^T \sigma^*(u,T)\, d\tilde{W}_u\right\} \\
&= 1\left\{\hat{s}_1 < \hat{\Delta}_1\right\}
\end{aligned} \tag{3.65}
$$

where $\hat{s}_1 = \dfrac{-\int_0^T \sigma^*(u,T)\, d\tilde{W}_u}{\hat{\delta}_1} \sim N(0,1)$, $\hat{\Delta}_1 = \dfrac{-\ln c_p - \frac{1}{2}\int_0^T \left(\sigma^*(u,T)\right)^2 du}{\hat{\delta}_1}$,

$\hat{\delta}_1^2 = \int_0^T \left(\sigma^*(u,T)\right)^2 du$.

We calculate the first term in (3.64) and get:

$$
\begin{aligned}
&\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) \\
&= \frac{S_0^1}{P(0,T)} e^{-\int_0^T \frac{\left(\sigma_1 + \sigma^*(u,T)\right)^2}{2} du}\, \tilde{E}\left(e^{-z_1} \cdot 1\left\{\hat{s}_1 < \hat{\Delta}_1\right\} 1\left\{s_1 \leq \Delta_1\right\}\right)
\end{aligned} \tag{3.66}
$$

where the correlations are

$$
\rho_{s_1\hat{s}_1 =} \frac{\int_0^T \sigma^*(u,T)\, du}{\hat{\delta}_1 \sqrt{T}}, \quad \rho_{s_1 z_1 =} \frac{\int_0^T \left(\sigma_1 + \sigma^*(u,T)\right) du}{\zeta_1 \sqrt{T}},
$$

$$\rho_{\hat{s}_1 z_1 =} \frac{\int_0^T \left(\sigma_1 + \sigma^*(u,T)\right)\sigma^*(u,T)\,du}{\hat{\delta}_1 \zeta_1}.$$

We can construct $Y_1 = \Delta_1 + \zeta_1 \cdot \rho_{s_1 z_1} = \dfrac{\ln \dfrac{S_0^1}{S_0^2} + \dfrac{1}{2}(\sigma_1 - \sigma_2)^2 T}{(\sigma_1 - \sigma_2)\sqrt{T}}$

$$\hat{Y}_2 = \hat{\Delta}_1 + \zeta_1 \cdot \rho_{\hat{s}_1 z_1} = \frac{-\ln c_p + \int_0^T \left[\frac{1}{2}\left(\sigma^*(u,T)\right)^2 + \sigma_1 \sigma^*(u,T)\right] du}{\hat{\delta}_1}$$

and by Multi-Asset Theorem, (3.66) becomes

$$\begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) \\ &= \frac{S_0^1}{P(0,T)} \Psi^2\left(Y_1, \hat{Y}_2; \rho_{s_1 \hat{s}_1}\right) \end{aligned} \tag{3.67}$$

For the second term in (3.64), we can follow the similar steps and get the expression as:

$$\begin{aligned} &\tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) \\ &= \frac{S_0^2}{P(0,T)} \Psi^2\left(Y_3, \hat{Y}_4; \rho_{s_2 \hat{s}_1}\right) \end{aligned} \tag{3.68}$$

where

$$\hat{Y}_4 = \frac{-\ln c_p + \int_0^T \left[\frac{1}{2}\left(\sigma^*(u,T)\right)^2 + \sigma_2 \sigma^*(u,T)\right] du}{\hat{\delta}_1}, \quad \hat{\delta}_1^2 = \int_0^T \left(\sigma^*(u,T)\right)^2 du,$$

$$\rho_{s_2 \hat{s}_1 =} \frac{-\int_0^T \sigma^*(u,T)\,du}{\hat{\delta}_1 \sqrt{T}}.$$

Therefore, we obtain:

$$\tilde{V}_X(0) = \frac{S_0^1}{P(0,T)} \Psi^2\left(Y_1, \hat{Y}_2; \rho_{s_1 \hat{s}_1}\right) + \frac{S_0^2}{P(0,T)} \Psi^2\left(Y_3, \hat{Y}_4; \rho_{s_2 \hat{s}_1}\right) \tag{3.69}$$

which leads to the expression (3.62) for X_0 in risk-neutral case.

(ii) Using (3.69), (3.51) and performing calculations in the key formula (3.32), we can derive the expression for survival probability in (3.63).

3.3.4 Numerical results

In this section, we will investigate the characteristics of efficient hedging on risk management of equity-linked life insurance contracts with stochastic interest rate from some numerical examples. We consider the insurance company's attitude is risk-taking, i.e., loss function with $0 < p < 1$. For illustrative purpose, we focus on the extreme case as in Melnikov & Skornyakova (2011), i.e. $p \to 0$.

Based on the discussion in Follmer & Luekert (2000), we have the following expression of shortfall risk:

$$\begin{aligned} & E\left[\left(H_T - V_X(T)\right)^p\right] \\ & = E\left[H_T{}^p - \tilde{\varphi}(H)^p\right] \\ & = E\left[H_T{}^p \cdot 1_{\left\{1<cH^{1-p}\tilde{Z}_T\right\}}\right] \end{aligned} \tag{3.70}$$

As the value of contingent claim H should be bounded, we can take the limit of (3.70) when $p \to 0$, then apply the dominated convergence theorem. Thus we obtain:

$$E\left[H_T{}^p \cdot 1_{\left\{1<cH_T{}^{1-p}\tilde{Z}_T\right\}}\right] \underset{p\to 0}{\to} E1_{\left\{1<cH_T\tilde{Z}_T\right\}} = P\left(1 < cH_T\tilde{Z}_T\right) \tag{3.71}$$

We can fix the probability of failing to hedge payoff H_T at maturity time T as $P\left(1 < cH_T\tilde{Z}_T\right) = \varepsilon$, where ε also quantifies the insurance company's financial risk level. With a fixed financial risk level ε, we can estimate the constant c.

To estimate the parameters in risky assets' model (3.2), we use daily stock prices of Russell-2000 (RUT-I) and S&P 500 from August 1, 2006 till July 31, 2011. Based on the discussion in Section 2.5.4, it is reasonable to choose RUT-I as the first risky asset S_t^1 and S&P 500 as the second risky asset S_t^2. We also assume to have 252 days during each business year.

The estimated parameters are:

$$\mu_1 = 0.067\,,\ \ \sigma_1 = 0.2909\,,\ \mu_2 = 0.0312\,,\ \ \sigma_2 = 0.2362 \qquad (3.72)$$

In order to correct the large difference between two risky assets, the initial prices of two assets can be chosen as $S_0^1 = \left(1278.53/900.02\right)\cdot 900.02\,, S_0^2 = 1278.53$.

For simplicity, all the results obtained in this section are under the assumption of a constant volatility structure in HJM model (3.4) (3.5), i.e. $\sigma\left(t,T\right) = \sigma$ for all t and T. Without loss of generality, we select $\sigma = 0.06$.

With very small p $\left(< 0.01\right)$, the initial value of a single equity-linked life insurance contract is chosen as $S_0 = 1000$. To study the effect of the stochastic interest rate on applying efficient hedging to equity-linked life insurance contract, we calculate the survival probabilities ${}_T p_x$ with maturity terms $T = 1 \sim 20$ years in three different specifications for the initial term structure $f(0,t)$, i.e.

Scenario I: flat initial term structure $f\left(0,t\right) = r_0$.

Scenario II: linearly increasing initial term structure $f\left(0,t\right) = r_0 + 0.002\cdot t$.

Scenario III: linearly decreasing initial term structure $f\left(0,t\right) = r_0 - 0.002\cdot t$.

In case of $r_0 = 0.033$, the survival probabilities ${}_T p_x$ which is reflected by the efficient hedging ratios in the financial market at different financial risk level ε are shown in Figure 3.1, 3.2, 3.3 respectively.

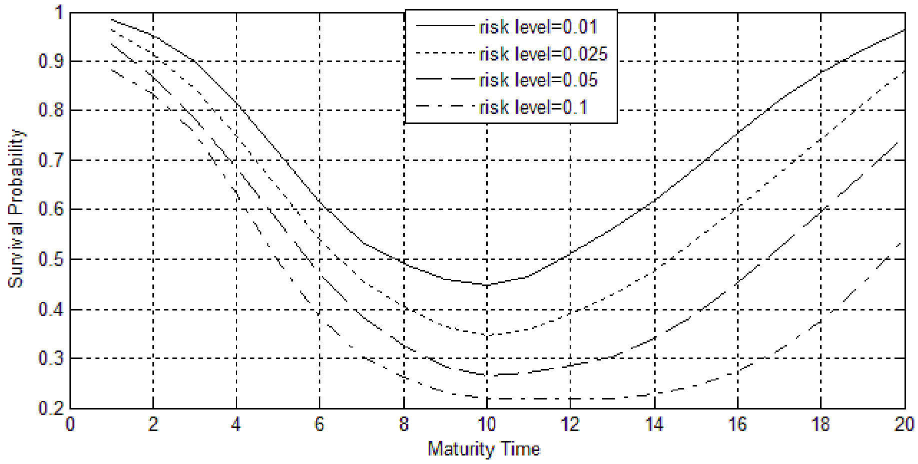

Figure 3.1 Hedging ratios (survival probabilities ${}_T p_x$) at different financial risk ε with $f(0,t) = r_0 = 0.033$.

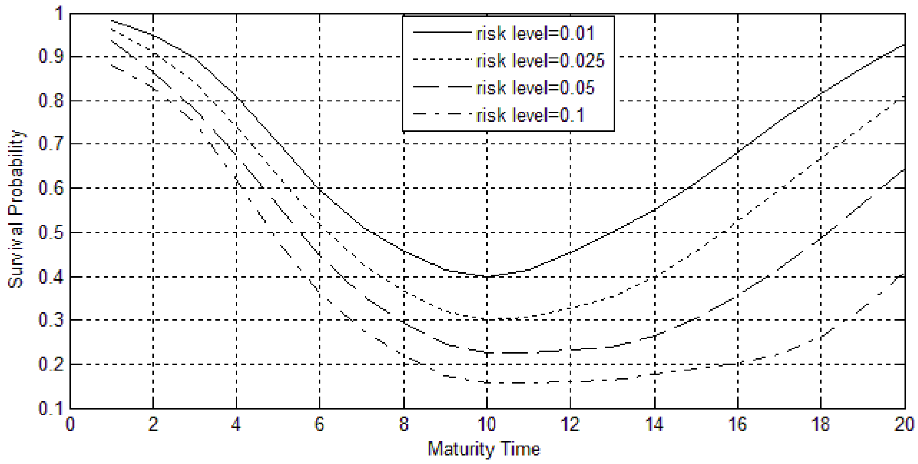

Figure 3.2 Hedging ratios (survival probabilities ${}_T p_x$) at different financial risk ε with $f(0,t) = r_0 + 0.002 \cdot t$.

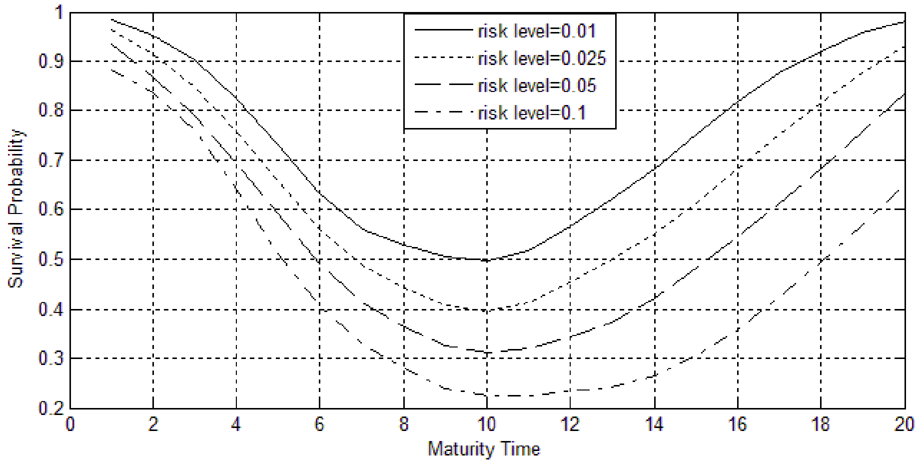

Figure 3.3 Hedging ratios (survival probabilities ${}_T p_x$) at different financial

risk ε with $f(0,t) = r_0 - 0.002 \cdot t$.

We also obtain the ages of clients corresponding to those survival probabilities based on 2005 United States life table listed in Arias, Rostron & Tejada-Vera (2010). The results are shown in Figure 3.3, 3.4, 3.5.

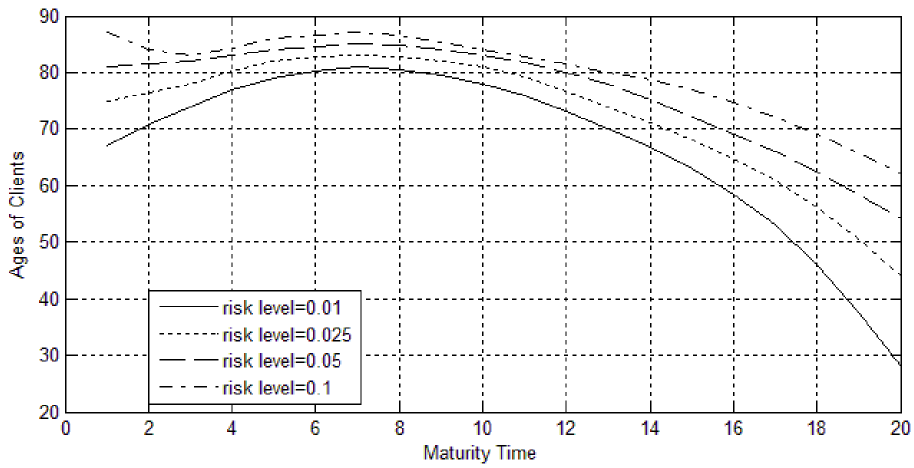

Figure 3.4 Age of clients with flat initial term structure $f(0,t) = r_0 = 0.033$.

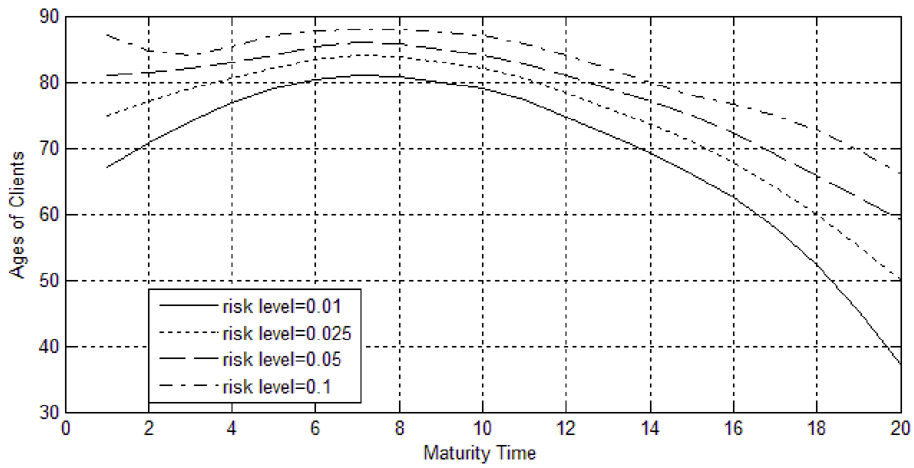

Figure 3.5 Age of clients with linearly increasing initial term structure $f\left(0,t\right)=r_0+0.002\cdot t$.

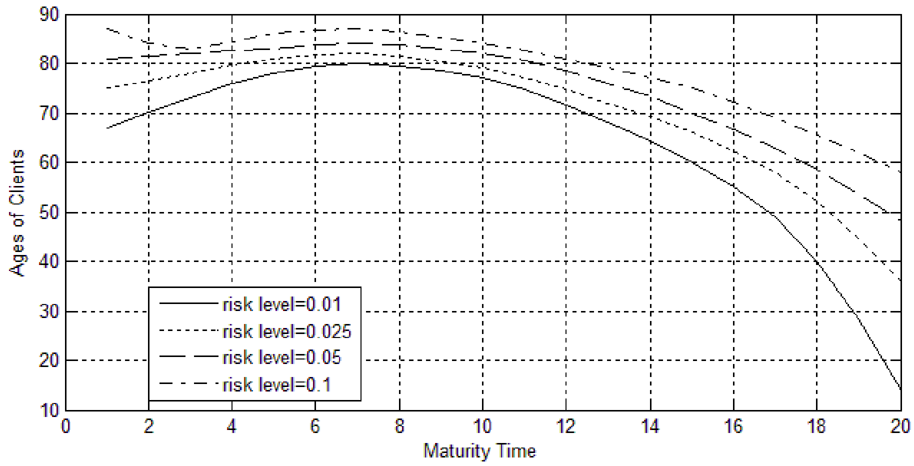

Figure 3.6 Age of clients with linearly decreasing initial term structure $f\left(0,t\right)=r_0-0.002\cdot t$.

According to Figures 3.1~3.3, regardless of the patterns of the initial term structure $f(0,t)$, the survival probabilities ${}_T p_x$ show the same trend of changes. The corresponding ages of the clients also appear the same trend as the survival probabilities, as shown in Figure 3.4~3.6.

In Figures 3.1~3.3, the survival probability ${}_T p_x$ decreases as the insurance company's financial risk ε increases at any fixed point of maturity term T. In addition, Figures 3.4~3.6 shows that the recommended clients' age increases at any fixed

T when ε increases. Therefore, it indicates an offset between financial and mortality risks. As a result, the insurance company may attract elder clients to compensate for the increasing financial risk.

However, at any specific financial risk level ε, survival probability ${}_T p_x$ is concaved up with increased maturity time T. This pattern in survival probability shows dissimilarity with the constant interest rate case discussed in Melnikov & Skornyakova (2011). When a constant interest rate is considered, the survival probability ${}_T p_x$ keeps decreasing all the time. It is noted that in Melnikov & Skornyakova (2011), the successful hedging set only admits one expression $\left\{S_T^1 / S_T^2 < c\right\}$. However, in this section, the success set is in the form of either $\left\{c \geq S_T^1 \tilde{Z}_T\right\}$ or $\left\{c \geq S_T^2 \tilde{Z}_T\right\}$. If $S_T^i \tilde{Z}_T$ is considered as a function of maturity time T, then there exists a critical value $\tilde{T}$ which may lead to different expressions for the successful hedging set. Therefore, the survival probability ${}_T p_x$ shows opposite trends before and after critical year $\tilde{T}$. In our case, this critical time $\tilde{T}$ is calculated to be 10 years.

From the above observations, the stochastic interest rate r_t shows a strong influence on the risk-management for risk-taking companies. At the beginning of an insurance period, the insurance company may attract old group of clients. In our case, it is recommended that the ages between 67 and 87 for different financial risks ε may be selected. However, the company may attract younger group of clients for a longer maturity term T because of the increasing mortality risk.

In addition, we also investigate the sensitivity of survival probability ${}_T p_x$ with respect to three different initial term structure at certain financial risk level $\varepsilon = 0.01$ as shown in Figure.3.7. The differences of ${}_T p_x$ in three patterns are relatively small within first 3 years. After that period, the patterns of initial term structure have larger impact on survival probability ${}_T p_x$ over time T. Survival probability obtained from linearly increasing pattern is lower than the one from flat pattern. The Survival probability generated from linearly decreasing pattern is the highest among them.

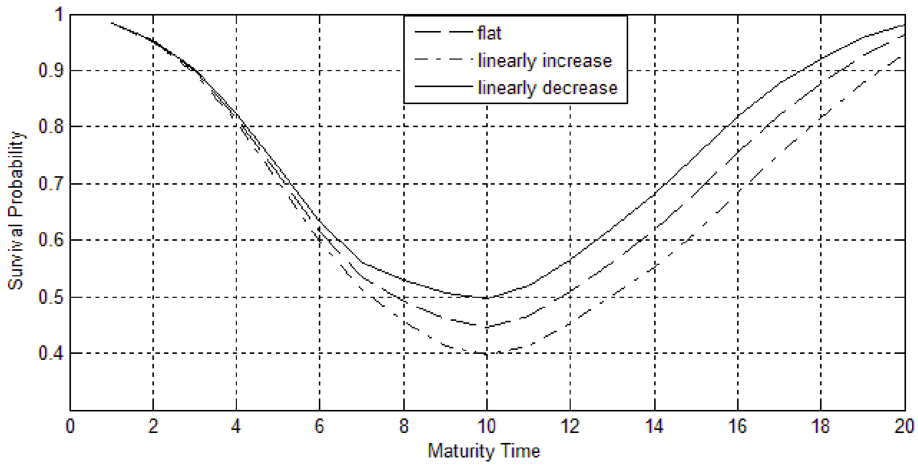

Figure 3.7 Sensitivity of survival probability w.r.t. different $f(0,t)$ at $\varepsilon = 0.01$.

Figure.3.8 shows the change of survival probability ${}_T p_x$ with different values of the initial spot rate r_0 in flat term structure over 20 years. The rate r_0 varies between 0.01 and 0.09 with incremental step 0.02 in this example. It can be seen that the initial spot rate r_0 has a greater effect on survival probability ${}_T p_x$, especially during the years from 5~20. Survival probability ${}_T p_x$ becomes smaller at any fixed time T with an increased initial spot rate r_0. This implies that the insurance company should span its clients to an elder group of clients with a larger initial spot rate.

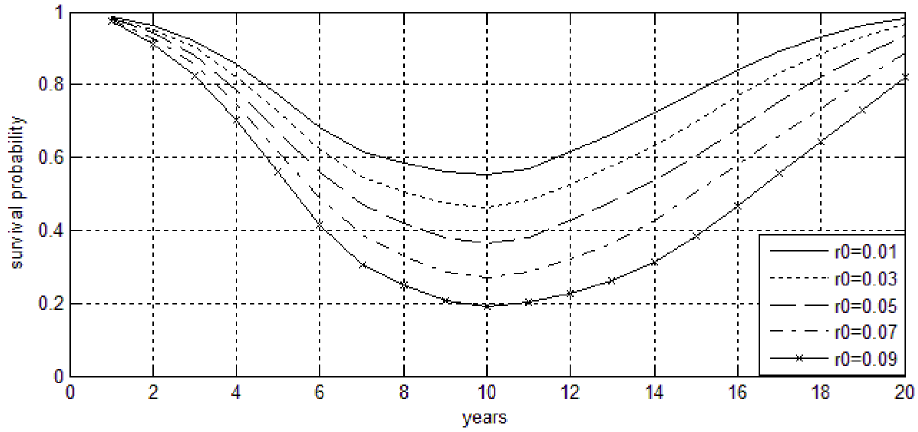

Figure 3.8 Sensitivity of survival probability w.r.t. different r_0 at $\varepsilon = 0.01$, initial term structure is flat.

3.4. Application of efficient hedging on insurance: models with different sources of randomness

In this section, we reconsider the same issue of efficient hedging on equity-linked life insurance with stochastic guarantee in the stochastic interest rate environment. Section 3.3 restricts the financial models restricted to the special case that all the financial processes are generated by the same Wiener process ($\rho = 1$). In Section 3.4, a more general model will be used in the financial settings. We assume that the return processes of two underlying risky assets are driven by different correlated Wiener processes separately with $0 < \rho < 1$. Therefore, the HJM framework utilized in this section contains two sources of randomness: The first Wiener process can be interpreted as a "long-run" factor, while the second one can be interpreted as the spread between a "short" and "long-term" by choosing appropriate volatility function.

3.4.1 Financial settings

As in previous Section, we consider a continuous time economy with the frictionless and complete financial market. For simplicity, we work in a risk-neutral world directly. Let $\left(\Omega, \mathbb{F}, (F_t)_{t\in[0,T]}, P^*\right)$ be a standard stochastic basis. Without loss of generality, we assume P^* is the risk-neutral probability measure. $\left(S_t^1\right)_{t\in[0,T]}$, $\left(S_t^2\right)_{t\in[0,T]}$ are two risky assets, with prices' evolutions as:

$$dS_t^i = S_t^i\left(r_t dt + \sigma_i dW_{t,i}^*\right),\ \ i = 1,2 \tag{3.73}$$

where r_t is the randomly evolving spot interest rate with constant volatility $\sigma_i > 0$ of the risky asset S_t^i, $\left(W_{t,1}^*\right)_{t\in[0,T]}$, $\left(W_{t,2}^*\right)_{t\in[0,T]}$ are two different Wiener processes under P^* with $dW_{1,t}{}^* dW_{2,t}{}^* = \rho dt$. ρ is the correlation between the two Wiener processes and assumed to be $0 < \rho < 1$, which means the risks can not be perfectly correlated. We assume $S_t{}^1$ is more risky than $S_t{}^2$, i.e. $\sigma_1 > \sigma_2$.

For HJM framework, the forward rate $f(t,T)$ satisfies the following dynamic:

$$df(t,T) = A(t,T)dt + \sigma^1(t,T)dW_{t,1}^* + \sigma^2(t,T)dW_{t,2}^* \tag{3.74}$$

and the spot interest rate $r(t)$ is defined by the following expression:

$$dr(t) = A(t,t)dt + \sigma^1(t,t)dW_{t,1}^* + \sigma^2(t,t)dW_{t,2}^* \tag{3.75}$$

where $A(t,T) = \left(\sigma^1(t,T) + \rho\sigma^2(t,T)\right)\sigma^*{}_1(t,T) + \left(\sigma^2(t,T) + \rho\sigma^1(t,T)\right)\sigma^*{}_2(t,T)$, $\sigma_i^*(t,T) = \int_t^T \sigma^i(t,u)du$, $i = 1,2$, and $\sigma^i(\cdot,\cdot)$ are deterministic functions.

We also consider a zero-coupon bond followed the same definition of price $P(t,T)$ as in (3.6), and the bond price at time zero $P(0,T)$ can be considered as constant.

3.4.2 Bond numeraire and forward measure $\tilde{P}$

Similarly as in Section 3.3, we select the price of zero-coupon bond $P(t,T)$ as the numeraire to simplify the computation in the following Sections. Let us define the forward measure $\tilde{P}$ which satisfies:

$$\tilde{P}(A)=\frac{1}{P(0,T)}\int_A D_T dP^*\text{ , for all } A\in\mathbb{F} \tag{3.76}$$

where $D_t=\exp(-\int_0^t r(u)du)$.

This measure has the local density $\tilde{Z}_t=\left.\frac{d\tilde{P}}{dP^*}\right|_{F_t}=\frac{D_tP(t,T)}{P(0,T)}$, which is a martingale under P^*.

By Ito's formula, we obtain the discounted price of S_t^1 and S_t^2 as follows:

$$\begin{aligned}\frac{S_t^1}{P(t,T)}=\frac{S_0^1}{P(0,T)}\exp\Big\{&\int_0^t\left(\sigma_1^*(u,T)+\sigma_1\right)d\tilde{W}_{u,1}+\int_0^t\sigma^*_2(u,T)d\tilde{W}_{u,2}\\&-\frac{1}{2}\int_0^t\Big[\left(\sigma_1^*(u,T)+\sigma_1\right)^2+2\rho\sigma^*_2(u,T)\left(\sigma_1^*(u,T)+\sigma_1\right)+\left(\sigma^*_2(u,T)\right)^2\Big]du\Big\}\end{aligned} \tag{3.77}$$

$$\begin{aligned}\frac{S_t^2}{P(t,T)}=\frac{S_0^2}{P(0,T)}\exp\Big\{&\int_0^t\left(\sigma_2^*(u,T)+\sigma_2\right)d\tilde{W}_{u,2}+\int_0^t\sigma^*_1(u,T)d\tilde{W}_{u,1}\\&-\frac{1}{2}\int_0^t\Big[\left(\sigma_2^*(u,T)+\sigma_2\right)^2+2\rho\sigma^*_1(u,T)\left(\sigma_2^*(u,T)+\sigma_2\right)+\left(\sigma^*_1(u,T)\right)^2\Big]du\Big\}\end{aligned} \tag{3.78}$$

where $\tilde{W}_{t,1}$ and $\tilde{W}_{t,2}$ are new Wiener processes under forward measure $\tilde{P}$, which satisfy:

$$\tilde{W}_{t,1}=W_{t,1}^*+\int_0^t\left(\sigma^*_1(u,T)+\rho\sigma^*_2(u,T)\right)du,$$

$$\tilde{W}_{t,2}=W_{t,2}^*+\int_0^t\left(\sigma^*_2(u,T)+\rho\sigma^*_1(u,T)\right)du.$$

A self-financing strategy is given by an initial capital $X_0>0$, and a predictable process $\xi_t=\left(\beta_t,\xi_t^1,\xi_t^2\right)$ such that the value process X_t is well defined. Also X_t admits the following expression:

$$X_t=X_0+\int_0^t\beta_u dB(u,T)+\int_0^t\xi_u^1 dS_u^1+\int_0^t\xi_u^2 dS_u^2 \tag{3.79}$$

where $\xi_t = \left(\beta_t, \xi_t^1, \xi_t^2\right)$ is the number of units invested to bonds and stocks respectively.

Following Geman, Karoui & Rochet (1995), we know that the self-financing portfolios will remain be the self-financing after a numeraire change. By the Ito formula, we obtain that

$$V_X(t) = \frac{X_t}{P(t,T)} = V_X(0) + \int_0^t \xi_u^1 d\frac{S_u^1}{P(u,T)} + \int_0^t \xi_u^2 d\frac{S_u^2}{P(u,T)} \tag{3.80}$$

As a result, $\left(V_X(0), \xi^1, \xi^2\right)$ is a self-financing strategy under the forward measure $\tilde{P}$.

3.4.3 Applying efficient hedging to life insurance contracts

We keep working on the same single premium equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$. The modified efficient hedging technique discussed in Section 3.3.3 will be utilized here to value the contract first and then to imply the survival probabilities which reflect the mortality risk. The main theoretical results are formulated in the following three Theorems devoted to the case $p > 1$, $0 < p < 1$ and $p = 1$ correspondently.

Theorem 3.4 Consider an insurance company that sells a single equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$, and the firm's risk preference is risk-aversion with a power loss function $l(x) = x^p$, $p > 1$. Then

(i) The initial price of the contract is

$$\begin{aligned} X_0 &= S_0^1 \Psi^2\left(\Delta_1^{'}, \Delta_3^{'}, \rho_{s_1 s_3}\right) + S_0^2 \Psi^2\left(\Delta_2^{'}, \Delta_4^{'}, \rho_{s_2 s_4}\right) \\ &\quad - P(0,T) M\left[\Psi^2\left(\tilde{\Delta}_1, \tilde{\Delta}_3, \rho_{s_1 s_3}\right) + \Psi^2\left(\tilde{\Delta}_2, \tilde{\Delta}_4, \rho_{s_2 s_4}\right)\right] \end{aligned} \tag{3.81}$$

where S_0^1 and S_0^2 are the initial prices for risky assets, $P(0,T)$ is the initial price of a zero-coupon bond, $\Psi^2(\cdot)$ denotes the two-dimensional cumulative normal distribution function with

$$\Delta_1^{'} = \frac{\ln \frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\right)T}{\tilde{\sigma}\sqrt{T}}, \Delta_2^{'} = \frac{\ln \frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)T}{\tilde{\sigma}\sqrt{T}},$$

$$\Delta_3^{'} = \frac{\ln \frac{S_0^1}{\tilde{k}P(0,T)} + \frac{1}{2}\sigma_1^2 T + \int_0^T \frac{p}{(p-1)}\sigma_1^*(u,T)\left(\sigma_1 + \frac{1}{2}\sigma_1^*(u,T)\right)du}{\delta_1} + \frac{\int_0^T \frac{p}{(p-1)}\sigma_2^*(u,T)\left(\rho\left(\sigma_1 + \sigma_1^*(u,T)\right) + \frac{1}{2}\sigma_2^*(u,T)\right)du}{\delta_1},$$

$$\Delta_4^{'} = \frac{\ln \frac{S_0^2}{\tilde{k}P(0,T)} + \frac{1}{2}\sigma_2^2 T + \int_0^T \frac{p}{(p-1)}\sigma_2^*(u,T)\left(\sigma_2 + \frac{1}{2}\sigma_2^*(u,T)\right)du}{\delta_2} + \frac{\int_0^T \frac{p}{(p-1)}\sigma_1^*(u,T)\left(\rho\left(\sigma_2 + \sigma_2^*(u,T)\right) + \frac{1}{2}\sigma_1^*(u,T)\right)du}{\delta_2}$$

$$\tilde{\Delta}_1 = \frac{\ln \frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)T + \int_0^T \left[\frac{p}{p-1}\sigma_1^*(u,T)\left(\rho\sigma_2 - \sigma_1\right) + 2\sigma_2^*(u,T)\left(\sigma_2 - \sigma_1\rho\right)\right]du}{\tilde{\sigma}\sqrt{T}},$$

$$\tilde{\Delta}_2 = \frac{\ln \frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)T + \frac{p}{p-1}\int_0^T \left[\left(\sigma_1\rho - \sigma_2\right)\sigma_2^*(u,T) + \sigma_1^*(u,T)\left(\sigma_1 - \rho\sigma_2\right)\right]du}{\tilde{\sigma}\sqrt{T}},$$

$$\tilde{\Delta}_3 = \frac{\ln \frac{S_0^1}{\tilde{k}P(0,T)} - \frac{1}{2}\sigma_2^2 T + \int_0^T \frac{2-p}{(p-1)}\sigma_1\left(\rho\sigma_2^*(u,T) + \sigma_1^*(u,T)\right)du}{\delta_1} + \frac{\int_0^T \frac{p}{(p-1)^2}\left(\rho\sigma_1^*(u,T)\sigma_2^*(u,T) + \frac{(3-p)}{2}\left(\left(\sigma_1^*(u,T)\right)^2 + \left(\sigma_2^*(u,T)\right)^2\right)\right)du}{\delta_1},$$

$$\tilde{\Delta}_4 = \frac{\ln \frac{S_0^2}{\tilde{k}P(0,T)} - \frac{1}{2}\sigma_2^2 T - \int_0^T \frac{p}{(p-1)}\sigma_2\left(\sigma_2^*(u,T) + \rho\sigma_1^*(u,T)\right)du}{\delta_2}$$

$$- \frac{\int_0^T \frac{p(p+1)}{2(p-1)^2}\left[\left(\sigma_1^*(u,T)\right)^2 + \left(\sigma_2^*(u,T)\right)^2 + 2\rho\sigma_1^*(u,T)\sigma_2^*(u,T)\right]du}{\delta_2}$$

$$\delta_1^2 = \int_0^T \left[\left(\sigma_1 + \frac{p}{p-1}\sigma_1^*(u,T)\right)^2 + \frac{p}{p-1}\sigma_2^*(u,T)\left[\frac{p}{p-1}\sigma_2^*(u,T) + 2\rho\left(\sigma_1 + \frac{p}{p-1}\sigma_1^*(u,T)\right)\right]\right]du$$

$$\delta_2^2 = \int_0^T \left[\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T)\right)^2 + \frac{p}{p-1}\sigma_1^*(u,T)\left[\frac{p}{p-1}\sigma_1^*(u,T) + 2\rho\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T)\right)\right]\right]du$$

$$\rho_{s_1 s_3} = \frac{\int_0^T \left[(\sigma_1 - \sigma_2\rho)\left(\frac{p}{p-1}\sigma_1^*(u,T) + \sigma_1\right) + \frac{p}{p-1}\sigma_2^*(u,T)(\sigma_1\rho - \sigma_2)\right]du}{\tilde{\sigma}\delta_1\sqrt{T}},$$

$$\rho_{s_2 s_4} = \frac{\int_0^T \left[(\sigma_2 - \sigma_1\rho)\left(\frac{p}{p-1}\sigma_2^*(u,T) + \sigma_2\right) + \frac{p}{p-1}\sigma_1^*(u,T)(\sigma_2\rho - \sigma_1)\right]du}{\tilde{\sigma}\delta_2\sqrt{T}},$$

$$\tilde{\sigma}^2 = \sigma_2^2 + \sigma_1^2 - 2\sigma_1\sigma_2\rho, \quad \tilde{k} = \left(\frac{c_p}{p}\right)^{\frac{1}{p-1}}, \quad M = \tilde{k}\exp\left(\frac{p}{2(1-p)^2}\int_0^T \Pi(u,T)du\right).$$

$$\Pi(t,T) = \left(\sigma^*_1(t,T)\right)^2 + 2\rho\sigma^*_2(t,T)\sigma_1^*(t,T) + \left(\sigma^*_2(t,T)\right)^2.$$

(ii) The survival probability ${}_T p_x$ is given by

$$ {}_T P_x = \frac{S_0^1\Psi^2\left(\Delta_1', \Delta_3', \rho_{s_1 s_3}\right) + S_0^2\Psi^2\left(\Delta_2', \Delta_4', \rho_{s_2 s_4}\right)}{S_0^1\Psi(d_1) + S_0^2\Psi(d_2)} - \frac{MP(0,T)\left(\Psi^2\left(\tilde{\Delta}_1, \tilde{\Delta}_3, \rho_{s_1 s_3}\right) + \Psi^2\left(\tilde{\Delta}_2, \tilde{\Delta}_4, \rho_{s_2 s_4}\right)\right)}{S_0^1\Psi(d_1) + S_0^2\Psi(d_2)} \quad (3.82)$$

where $\Psi(\cdot)$ is the cumulative distribution function of standard normal distribution, and

$$d_2 = \frac{\ln \frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\right)T}{\tilde{\sigma}\sqrt{T}}.$$

Proof: (i) The success ratio $\tilde{\varphi}$ for risk-aversion case is given by (3.26), so we can simplify the modified contingent claim $\tilde{H}_T$ as

$$\begin{aligned}\tilde{H}_T &= \tilde{\varphi}\cdot H_T = H_T - \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} \wedge H_T \\ &= \left(H_T - \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\right)\cdot 1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < H_T\right\}\end{aligned} \tag{3.83}$$

where $\tilde{k} = \left(\dfrac{c_p}{p}\right)^{\frac{1}{p-1}}$. The expression for $\tilde{V}_X(0)$ can be decomposed into:

$$\begin{aligned}\tilde{V}_X(0) = \tilde{E}(\tilde{\varphi}H_T) &= \tilde{E}\left(S_T^1\cdot 1\left\{S_T^1 \ge S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1\right\}\right) \\ &+\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1 < S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}\right) \\ &-\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1 \ge S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1\right\}\right) \\ &-\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}\cdot 1\left\{S_T^1 < S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}\right)\end{aligned} \tag{3.84}$$

The indicator functions in (3.84) can be simplified as follows:

$$\begin{aligned}&1\left\{S_T^1 \ge S_T^2\right\} \\ &= 1\left\{s_1 \le \frac{\ln\frac{S_0^1}{S_0^2} + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)T + \int_0^T\left[\sigma_1^*(u,T)\left(\rho\sigma_2 - \sigma_1\right) + \sigma_2^*(u,T)\left(\sigma_2 - \rho\sigma_1\right)\right]du}{\tilde{\sigma}\sqrt{T}}\right\}\end{aligned} \tag{3.85}$$

$$\begin{aligned}&1\left\{S_T^1 < S_T^2\right\} \\ &= 1\left\{s_2 < \frac{\ln\frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)T + \int_0^T\left[\sigma_1^*(u,T)\left(\sigma_1 - \rho\sigma_2\right) + \sigma_2^*(u,T)\left(\rho\sigma_1 - \sigma_2\right)\right]du}{\tilde{\sigma}\sqrt{T}}\right\}\end{aligned} \tag{3.86}$$

$$1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1\right\} = 1\left\{\tilde{k}\left(\frac{D_T P(T,T)}{P(0,T)}\right)^{\frac{1}{p-1}} < \frac{S_T^1}{B(T,T)}\right\}$$

$$= 1\left\{ s_3 < \frac{\ln\frac{S_0^1}{\tilde{k}P(0,T)} - \frac{1}{2}\int_0^T\left[\frac{p}{(p-1)}\Pi(u,T) + \sigma_1\left(\sigma_1 + 2\sigma_1^*(u,T) + 2\rho\sigma_2^*(u,T)\right)\right]du}{\delta_1}\right\}$$

(3.87)

$$1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}$$

$$= 1\left\{ s_4 < \frac{\ln\frac{S_0^2}{\tilde{k}P(0,T)} - \frac{1}{2}\int_0^T\left[\frac{p}{(p-1)}\Pi(u,T) + \sigma_2\left(\sigma_2 + 2\sigma_2^*(u,T) + 2\rho\sigma_1^*(u,T)\right)\right]}{\delta_2}\right\}$$

(3.88)

where $s_1 = \frac{\hat{W}_T^{\ 1}}{\sqrt{T}} \sim N(0,1), \hat{W}_T^{\ 1} = \frac{\sigma_2\tilde{W}_{T,2} - \sigma_1\tilde{W}_{T,1}}{\tilde{\sigma}}$, $\tilde{\sigma}^2 = \sigma_2^{\ 2} + \sigma_1^{\ 2} - 2\sigma_1\sigma_2\rho$

$$s_2 = \frac{\hat{W}_T^{\ 2}}{\sqrt{T}} \sim N(0,1),\ \hat{W}_T^{\ 2} = \frac{\sigma_1\tilde{W}_{T,1} - \sigma_2\tilde{W}_{T,2}}{\tilde{\sigma}},$$

$$s_3 = \frac{-\int_0^T\left(\sigma_1 + \frac{p}{p-1}\sigma_1^*(u,T)\right)d\tilde{W}_{u,1} - \int_0^T\left(\frac{p}{p-1}\sigma_2^*(u,T)\right)d\tilde{W}_{u,2}}{\delta_1} \sim N(0,1),$$

$$\delta_1^2 = \int_0^T\left[\left(\sigma_1 + \frac{p}{p-1}\sigma_1^*(u,T)\right)^2 + \frac{p}{p-1}\sigma_2^*(u,T)\left[\frac{p}{p-1}\sigma_2^*(u,T) + 2\rho\left(\sigma_1 + \frac{p}{p-1}\sigma_1^*(u,T)\right)\right]\right]du$$

$$s_4 = \frac{-\int_0^T\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T)\right)d\tilde{W}_{u,2} - \int_0^T\left(\frac{p}{p-1}\sigma_1^*(u,T)\right)d\tilde{W}_{u,1}}{\delta_2} \sim N(0,1),$$

$$\delta_2^2 = \int_0^T\left[\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T)\right)^2 + \frac{p}{p-1}\sigma_1^*(u,T)\left[\frac{p}{p-1}\sigma_1^*(u,T) + 2\rho\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T)\right)\right]\right]du$$

Consider the first expectation in the expression of $\tilde{V}_X(0)$(3.84), (3.85) and (3.87) can lead to obtain:

$$\begin{aligned}&\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1\geq S_T^2\right\}1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}}<S_T^1\right\}\right)\\&=\frac{S_0^1}{P(0,T)}\exp\left(-\frac{1}{2}\zeta_1^2\right)\tilde{E}\left(e^{-z_1}1\{s_1\leq\Delta_1\}1\{s_3<\Delta_3\}\right)\end{aligned}\tag{3.89}$$

where $z_1=-\int_0^T\left(\sigma_1+\sigma_1^*(u,T)\right)d\tilde{W}_{u,1}-\int_0^T\sigma_2^*(u,T)\,d\tilde{W}_{u,2}\sim N\left(0,\zeta_1^2\right)$,

$$\zeta_1^2=\int_0^T\left[\left(\sigma_1^*(u,T)+\sigma_1\right)^2+2\rho\sigma^*{}_2(u,T)\left(\sigma_1^*(u,T)+\sigma_1\right)+\left(\sigma^*{}_2(u,T)\right)^2\right]du\,,$$

$$\Delta_1=\frac{\ln\dfrac{S_0^1}{S_0^2}+\dfrac{1}{2}\left(\sigma_2^2-\sigma_1^2\right)T+\int_0^T\left[\sigma_1^*(u,T)\left(\rho\sigma_2-\sigma_1\right)+\sigma_2^*(u,T)\left(\sigma_2-\rho\sigma_1\right)\right]du}{\tilde{\sigma}\sqrt{T}}\,,$$

$$\Delta_3=\frac{\ln\dfrac{S_0^1}{\tilde{k}P(0,T)}-\dfrac{1}{2}\int_0^T\left[\dfrac{p}{(p-1)}\Pi(u,T)+\sigma_1\left(\sigma_1+2\sigma_1^*(u,T)+2\rho\sigma_2^*(u,T)\right)\right]du}{\delta_1}\,.$$

Similarly as the proof of the Theorems in Section 3.3, "Multi-Asset Theorem" is applied again to evaluate the expectation in (3.89). First, the necessary correlations are given by:

$$\rho_{s_1s_3}=\frac{\int_0^T\left[\left(\sigma_1-\sigma_2\rho\right)\left(\dfrac{p}{p-1}\sigma_1^*(u,T)+\sigma_1\right)+\dfrac{p}{p-1}\sigma_2^*(u,T)\left(\sigma_1\rho-\sigma_2\right)\right]du}{\tilde{\sigma}\delta_1\sqrt{T}}\,,$$

$$\rho_{s_1z_1}=\frac{\int_0^T\left[\left(\sigma_1+\sigma_1^*(u,T)\right)\left(\sigma_1-\sigma_2\rho\right)+\sigma_2^*(u,T)\left(\sigma_1\rho-\sigma_2\right)\right]du}{\tilde{\sigma}\zeta_1\sqrt{T}}\,,$$

$$\begin{aligned}\rho_{s_3z_1}=&\frac{\int_0^T\left(\sigma_1+\dfrac{p}{p-1}\sigma_1^*(u,T)\right)\left(\sigma_1+\sigma_1^*(u,T)+\rho\sigma_2^*(u,T)\right)du}{\delta_1\zeta_1}\\&+\frac{\int_0^T\dfrac{p}{p-1}\sigma_2^*(u,T)\left(\rho\sigma_1+\sigma_2^*(u,T)+\rho\sigma_1^*(u,T)\right)du}{\delta_1\zeta_1}\,.\end{aligned}$$

Then, applying the theorem with the above parameters, we get

$$\tilde{E}\left(S_T^1 \cdot 1\left\{ S_T^1 \geq S_T^2 \right\} 1\left\{ \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1 \right\} \right) = \frac{S_0^1}{P(0,T)} \Psi^2\left(\Delta_1^{'}, \Delta_3^{'}, \rho_{s_1 s_3}\right) \quad (3.90)$$

with $\Delta_1^{'}$ and $\Delta_3^{'}$ are given in Theorem 3.4.

Now let us turn to the second expectation $\tilde{E}\left(S_T^2 \cdot 1\left\{ S_T^1 < S_T^2 \right\} 1\left\{ \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2 \right\} \right)$ in (3.84). From (3.86) and (3.88), this expectation can be simplified to:

$$\begin{aligned} &\tilde{E}\left(S_T^2 \cdot 1\left\{ S_T^1 < S_T^2 \right\} 1\left\{ \tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2 \right\} \right) \\ &= \frac{S_0^2}{P(0,T)} \exp\left(-\frac{1}{2}\zeta_2^2 \right) \tilde{E}\left(e^{-z_2} 1\left\{ s_2 \leq \Delta_2 \right\} 1\left\{ s_4 < \Delta_4 \right\} \right) \end{aligned} \quad (3.91)$$

where $z_2 = -\int_0^T \left(\sigma_2^*(u,T) + \sigma_2\right) d\tilde{W}_{u,2} - \int_0^T \sigma_1^*(u,T)\, d\tilde{W}_{u,1} \sim N\left(0, \zeta_2^2\right)$,

$$\zeta_2^2 = \int_0^T \left[\left(\sigma_2^*(u,T) + \sigma_2\right)^2 + 2\rho\sigma^*_1(u,T)\left(\sigma_2^*(u,T) + \sigma_2\right) + \left(\sigma^*_1(u,T)\right)^2 \right] du,$$

$$\Delta_2 = \frac{\ln\frac{S_0^2}{S_0^1} + \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)T + \int_0^T \left[\sigma_1^*(u,T)\left(\sigma_1 - \rho\sigma_2\right) + \sigma_2^*(u,T)\left(\rho\sigma_1 - \sigma_2\right) \right] du}{\tilde{\sigma}\sqrt{T}},$$

$$\Delta_4 = \frac{\ln\frac{S_0^2}{\tilde{k}P(0,T)} - \frac{1}{2}\int_0^T \left[\frac{p}{(p-1)}\Pi(u,T) + \sigma_2\left(\sigma_2 + 2\sigma_2^*(u,T) + 2\rho\sigma_1^*(u,T)\right) \right]}{\delta_2}.$$

with the correlations

$$\rho_{s_2 s_4 =} \frac{\int_0^T \left[\left(\sigma_2 - \sigma_1\rho\right)\left(\frac{p}{p-1}\sigma_2^*(u,T) + \sigma_2 \right) + \frac{p}{p-1}\sigma_1^*(u,T)\left(\sigma_2\rho - \sigma_1\right) \right] du}{\tilde{\sigma}\delta_2\sqrt{T}},$$

$$\rho_{s_2 z_2 =} \frac{\int_0^T \left[\left(\sigma_2 + \sigma_2^*(u,T)\right)\left(\sigma_2 - \sigma_1\rho\right) + \sigma_1^*(u,T)\left(\sigma_2\rho - \sigma_1\right) \right] du}{\tilde{\sigma}\zeta_2\sqrt{T}},$$

$$\rho_{s_4 z_2} = \frac{\int_0^T \left(\sigma_2 + \sigma_2^*(u,T)\right)\left(\sigma_2 + \frac{p}{p-1}\sigma_2^*(u,T) + \frac{p}{p-1}\rho\sigma_1^*(u,T)\right)du}{\delta_2\zeta_2} + \frac{\int_0^T \sigma_1^*(u,T)\left(\rho\sigma_2 + \frac{p}{p-1}\rho\sigma_2^*(u,T) + \frac{p}{p-1}\sigma_1^*(u,T)\right)du}{\delta_2\zeta_2}.$$

From Multi-Asset Theorem, the expectation in (3.91) becomes

$$\tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}\right) = \frac{S_0^2}{P(0,T)}\Psi^2\left(\Delta_2^{'}, \Delta_4^{'}, \rho_{s_2 s_4}\right) \tag{3.92}$$

We can follow the similar steps for the other two expectations in (3.84), and get the results:

$$\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^1\right\}\right) = M\Psi^2\left[\tilde{\Delta}_1, \tilde{\Delta}_3, \rho_{s_1 s_3}\right] \tag{3.93}$$

$$\tilde{E}\left(\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{\tilde{k}\left(\tilde{Z}_T\right)^{\frac{1}{p-1}} < S_T^2\right\}\right) = M\Psi^2\left[\tilde{\Delta}_2, \tilde{\Delta}_4, \rho_{s_2 s_4}\right] \tag{3.94}$$

where $M = \tilde{k}\exp\left(\frac{p}{2(1-p)^2}\int_0^T \Pi(u,T)du\right)$.

Using (3.90), (3.92) ~ (3.94), the expression of $\tilde{V}_X(0)$ becomes

$$\begin{aligned}\tilde{V}_X(0) = &\frac{S_0^1}{P(0,T)}\Psi^2\left(\Delta_1^{'}, \Delta_3^{'}, \rho_{s_1 s_3}\right) + \frac{S_0^2}{P(0,T)}\Psi^2\left(\Delta_2^{'}, \Delta_4^{'}, \rho_{s_2 s_4}\right) \\ &- M\left[\Psi^2\left(\tilde{\Delta}_1, \tilde{\Delta}_3, \rho_{s_1 s_3}\right) + \Psi^2\left(\tilde{\Delta}_2, \tilde{\Delta}_4, \rho_{s_2 s_4}\right)\right]\end{aligned} \tag{3.95}$$

which leads to the result of (3.81).

(ii) According to the key balance equation, we only need focus on expectation $\tilde{E}(H)$. Based on (3.48), we can compute the first expectation as:

$$\begin{aligned}\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\}\right) &= \frac{S_0^1}{P(0,T)} e^{-\frac{1}{2}\zeta_1^2}\tilde{E}\left(e^{-z_1} \cdot 1\left\{s_1 \leq \Delta_1\right\}\right) \\ &= \frac{S_0^1}{P(0,T)} \cdot \Psi(d_1)\end{aligned} \tag{3.96}$$

where $d_1 = \dfrac{\ln \dfrac{S_0^1}{S_0^2} + \dfrac{1}{2}\left(\sigma_2{}^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2\right)T}{\tilde{\sigma}\sqrt{T}}$.

The second expectation in (3.48) can be treated in a similarly way so that we get

$$\tilde{E}\left(H_T\right) = \frac{S_0^1}{B\left(0,T\right)} \cdot \Psi\left(d_1\right) + \frac{S_0^2}{B\left(0,T\right)} \Psi\left(d_2\right) \tag{3.97}$$

where $d_2 = \dfrac{\ln \dfrac{S_0^2}{S_0^1} + \dfrac{1}{2}\left(\sigma_1{}^2 + \sigma_2{}^2 - 2\sigma_1\sigma_2\rho\right)T}{\tilde{\sigma}\sqrt{T}}$.

Therefore, the survival probability is given by

$$\begin{aligned} {}_T P_x &= \frac{\tilde{E}\left(\tilde{\varphi} H_T\right)}{\tilde{E}\left(H_T\right)} \\ &= \frac{S_0^1 \Psi^2\left(\Delta_1^{'}, \Delta_3^{'}, \rho_{s_1 s_3}\right) + S_0^2 \Psi^2\left(\Delta_2^{'}, \Delta_4^{'}, \rho_{s_2 s_4}\right)}{S_0^1 \Psi\left(d_1\right) + S_0^2 \Psi\left(d_2\right)} - \frac{M\left(\Psi^2\left(\tilde{\Delta}_1, \tilde{\Delta}_3, \rho_{s_1 s_3}\right) + \Psi^2\left(\tilde{\Delta}_2, \tilde{\Delta}_4, \rho_{s_2 s_4}\right)\right)}{S_0^1 \Psi\left(d_1\right) + S_0^2 \Psi\left(d_2\right)} \end{aligned} \tag{3.98}$$

Theorem 3.5 Suppose that an insurance company sells a single equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$, and the firm decides to use efficient hedging to minimize the shortfall risk. It's risk preference is risk-taking with a power loss function $l\left(x\right) = x^p$, $0 < p < 1$. Then we have:

(i).The initial price of the contract is

$$X_0 = S_0^1 \Psi^2\left(\Delta_1^{'}, \Lambda_1^{'}, \rho_{s_1 \tilde{s}_1}\right) + S_0^2 \Psi^2\left(\Delta_2^{'}, \Lambda_2^{'}, \rho_{s_2 \tilde{s}_2}\right) \tag{3.99}$$

where S_0^1 and S_0^2 are the initial assets' prices, $\Psi^2\left(\cdot\right)$ denotes the two-dimensional cumulative normal distribution function with

$$\Lambda_1^{'} = \frac{-\ln \tilde{k}_1 + \int_0^T \left(p\sigma_1\left(\sigma^*{}_1\left(u,T\right) + \rho\sigma^*{}_2\left(u,T\right)\right) - \frac{1-p}{2}\sigma_1^2 + \frac{p}{2}\Pi\left(u,T\right)\right) du}{\tilde{\delta}_1},$$

$$\Lambda_2^{'} = \frac{-\ln\tilde{k}_2 + \int_0^T \left(p\sigma_2\left(\sigma^*{}_2(u,T)+\rho\sigma^*{}_1(u,T)\right) - \frac{1-p}{2}\sigma_2^2 + \frac{p}{2}\Pi(u,T)\right)du}{\tilde{\delta}_2},$$

$$\tilde{\delta}_1^2 = \int_0^T \left(p\Pi(u,T) + (1-p)\sigma_1\left((1-p)\sigma_1 - 2p\sigma^*{}_1(u,T) - 2p\rho\sigma^*{}_2(u,T)\right)\right)du,$$

$$\tilde{\delta}_2^2 = \int_0^T \left(p\Pi(u,T) + (1-p)\sigma_2\left((1-p)\sigma_2 - 2p\sigma^*{}_2(u,T) - 2p\rho\sigma^*{}_1(u,T)\right)\right)du,$$

$$\tilde{k}_1 = c_p\left(\frac{S_0^1}{P(0,T)}\right)^{1-p}, \quad \tilde{k}_2 = c_p\left(\frac{S_0^2}{P(0,T)}\right)^{1-p},$$

$$\rho_{s_1\tilde{s}_1} = \frac{\left[\int_0^T (\sigma_1 - \rho\sigma_2)\left(p\sigma_1^*(u,T) - (1-p)\sigma_1\right) + p\sigma_2^*(u,T)(\sigma_1\rho - \sigma_2)\right]du}{\tilde{\sigma}\tilde{\delta}_1\sqrt{T}},$$

$$\rho_{s_2\tilde{s}_2} = \frac{\left[\int_0^T (\sigma_2 - \rho\sigma_1)\left(p\sigma_2^*(u,T) - (1-p)\sigma_2\right) + p\sigma_1^*(u,T)(\sigma_2\rho - \sigma_1)\right]du}{\tilde{\sigma}\tilde{\delta}_2\sqrt{T}}.$$

(ii). The survival probability ${}_T p_x$ is in the form

$$ {}_T P_x = \frac{S_0^1\Psi^2\left(\Delta_1^{'}, \Lambda_1^{'}, \rho_{s_1\tilde{s}_1}\right) + S_0^2\Psi^2\left(\Delta_2^{'}, \Lambda_2^{'}, \rho_{s_2\tilde{s}_2}\right)}{S_0^1\cdot\Psi(d_1) + S_0^2\Psi(d_2)} \tag{3.100}$$

where $\Psi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Proof: (i). When $0 < p < 1$, recall that the success ratio $\tilde{\varphi}$ has the form (3.27) from which we can rewrite $\tilde{V}_X(0)$ as:

$$\begin{aligned}\tilde{V}_X(0) = \tilde{E}(\tilde{\varphi}H_T) = \tilde{E}&\left(S_T^1\cdot 1\left\{S_T^1 \geq S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p} < 1\right\}\right) \\ +\tilde{E}&\left(S_T^2\cdot 1\left\{S_T^1 < S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^2\right)^{1-p} < 1\right\}\right)\end{aligned} \tag{3.101}$$

The calculation for $\tilde{E}\left(S_T^1\cdot 1\left\{S_T^1 \geq S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p} < 1\right\}\right)$ is shown in this proof, and the expression for $\tilde{E}\left(S_T^2\cdot 1\left\{S_T^1 < S_T^2\right\}1\left\{c_p\tilde{Z}_T\left(S_T^2\right)^{1-p} < 1\right\}\right)$ can be obtained in a similarly way.

The set $\left\{c_p\tilde{Z}_T\left(S_T^1\right)^{1-p} < 1\right\}$ is simplifies as:

$$\left\{c_p \tilde{Z}_T \left(S_T^1\right)^{1-p} < 1\right\} = \left\{\Lambda_1 > \tilde{s}_1\right\} \tag{3.102}$$

where $\tilde{s}_1 = \dfrac{\int_0^T \left((1-p)\sigma_1 - p\sigma^*{}_1(u,T)\right) d\tilde{W}_{u,1} - \int_0^T p\sigma^*{}_2(u,T) d\tilde{W}_{u,2}}{\tilde{\delta}_1} \sim N(0,1)$

$$\Lambda_1 = \frac{-\ln \tilde{k}_1 - \int_0^T \left[\frac{p}{2}\Pi(u,T) - (1-p)\sigma_1\left(\frac{\sigma_1}{2} + \sigma^*{}_1(u,T) + \rho\sigma^*{}_2(u,T)\right)\right] du}{\tilde{\delta}_1},$$

$$\tilde{\delta}_1^2 = \int_0^T \left[p\Pi(u,T) + (1-p)\sigma_1\left((1-p)\sigma_1 - 2p\sigma^*{}_1(u,T) - 2p\rho\sigma^*{}_2(u,T)\right)\right] du,$$

$$\tilde{k}_1 = c_p \left(\frac{S_0^1}{P(0,T)}\right)^{1-p}.$$

So the first term in (3.101) becomes

$$\begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{c_p \tilde{Z}_T \left(S_T^1\right)^{\frac{1}{p-1}} < 1\right\}\right) \\ &= \frac{S_0^1}{P(0,T)} \exp\left(-\frac{1}{2}\zeta_1^2\right) \tilde{E}\left(e^{-z_1} 1\left\{s_1 \leq \Delta_1\right\} 1\left\{\tilde{s}_1 < \Lambda_1\right\}\right) \end{aligned} \tag{3.103}$$

Now we apply Multi-Asset Theorem and take the corresponding correlations as

$$\rho_{s_1\tilde{s}_1} = \frac{\int_0^T \left((\sigma_1 - \rho\sigma_2)\left(p\sigma_1^*(u,T) - (1-p)\sigma_1\right) + p\sigma_2^*(u,T)(\sigma_1\rho - \sigma_2)\right) du}{\tilde{\sigma}\tilde{\delta}_1\sqrt{T}},$$

$$\rho_{s_1 z_1} = \frac{\int_0^T \left[\left(\sigma_1 + \sigma_1^*(u,T)\right)(\sigma_1 - \sigma_2\rho) + \sigma_2^*(u,T)(\sigma_1\rho - \sigma_2)\right] du}{\tilde{\sigma}\zeta_1\sqrt{T}}$$

$$\rho_{\tilde{s}_1 z_1} = \frac{\int_0^T \left[p\Pi(u,T) + (2p-1)\sigma_1\left(\sigma_1^*(u,T) + \rho\sigma^*{}_2(u,T)\right) - (1-p)\sigma_1^2\right]}{\tilde{\delta}_1\zeta_1}.$$

We obtain

$$\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{c_p \tilde{Z}_T \left(S_T^1\right)^{1-p} < 1\right\}\right) = \frac{S_0^1}{P(0,T)} \Psi^2\left(\Delta_1', \Lambda_1', \rho_{s_1\tilde{s}_1}\right) \tag{3.104}$$

After the similar calculations, we also have

$$\tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{c_p \tilde{Z}_T \left(S_T^2\right)^{\frac{1}{p-1}} < 1\right\}\right) = \frac{S_0^2}{P(0,T)} \Psi^2\left(\Delta_2', \Lambda_2', \rho_{s_2\tilde{s}_2}\right) \tag{3.105}$$

Therefore, (3.104) and (3.105) lead to the expression of

$$\tilde{V}_X(0) = \frac{S_0^1}{P(0,T)} \Psi^2\left(\Delta_1^{'}, \Lambda_1^{'}, \rho_{s_1 \tilde{s}_1}\right) + \frac{S_0^2}{P(0,T)} \Psi^2\left(\Delta_2^{'}, \Lambda_2^{'}, \rho_{s_2 \tilde{s}_2}\right) \tag{3.106}$$

So that we find $X_0 = S_0^1 \Psi^2\left(\Delta_1^{'}, \Lambda_1^{'}, \rho_{s_1 \tilde{s}_1}\right) + S_0^2 M_1 \Psi^2\left(\Delta_2^{'}, \Lambda_2^{'}, \rho_{s_2 \tilde{s}_2}\right)$.

(ii). From the key formula ${}_T p_x = \dfrac{\tilde{E}(\tilde{\varphi} H_T)}{\tilde{E}(H_T)}$ and the calculation of ${}_T p_x$ in Theorem 3.4, we get the formula for ${}_T p_x$ in (3.100).

Theorem 3.6 Consider an insurance company that sells a single equity-linked life insurance contract with payoff $H_T = \max(S_T^1, S_T^2)$, and the firm's risk preference is risk-indifference with a power loss function $l(x) = x^p$, $p = 1$. Then

(i). The initial price of the contract is

$$X_0 = S_0^1 \Psi^2\left(\Delta_1^{'}, \bar{\Lambda}_1^{'}; \rho_{s_1 \hat{s}_1}\right) + S_0^2 \Psi^2\left(\Delta_2^{'}, \bar{\Lambda}_2^{'}; \rho_{s_2 \hat{s}_1}\right) \tag{3.107}$$

where S_0^1 and S_0^2 are the initial assets' prices, $\Psi^2(\cdot)$ denotes the two-dimensional cumulative normal distribution function with

$$\bar{\Lambda}_1^{'} = \frac{-\ln c_p + \frac{1}{2}\int_0^T \left[\Pi(u,T) + \sigma_1\left(\sigma_1^*(u,T) + \rho \sigma^*_2(u,T)\right)\right] du}{\hat{\delta}_1},$$

$$\bar{\Lambda}_2^{'} = \frac{-\ln c_p + \frac{1}{2}\int_0^T \left[\Pi(u,T) + \sigma_2\left(\rho \sigma_1^*(u,T) + \sigma^*_2(u,T)\right)\right] du}{\hat{\delta}_1},$$

$$\rho_{s_1 \hat{s}_1} = \frac{\int_0^T \left[\sigma_1^*(u,T)(\sigma_1 - \sigma_2 \rho) + \sigma_2^*(u,T)(\sigma_1 \rho - \sigma_2)\right] du}{\tilde{\sigma} \hat{\delta}_1 \sqrt{T}}, \quad \hat{\delta}_1^2 = \int_0^T \Pi(u,T) du,$$

$$\rho_{\hat{s}_1 s_2} = \frac{\int_0^T \left[\sigma_1^*(u,T)(\sigma_2 \rho - \sigma_1) + \sigma_2^*(u,T)(\sigma_2 - \sigma_1 \rho)\right] du}{\tilde{\sigma} \hat{\delta}_1 \sqrt{T}}.$$

(ii). The survival probability ${}_T p_x$ has the form

$$ {}_{T}P_{x} = \frac{S_0^1 \Psi^2\left(\Delta_1^{'}, \overline{\Lambda}_1^{'}; \rho_{s_1\hat{s}_1}\right) + S_0^2 \Psi^2\left(\Delta_2^{'}, \overline{\Lambda}_2^{'}; \rho_{s_2\hat{s}_1}\right)}{S_0^1 \cdot \Psi\left(d_1\right) + S_0^2 \Psi\left(d_2\right)} \tag{3.108} $$

Proof: (i). If $p = 1$, the structure of success ratio $\tilde{\varphi}$ is given in (3.28),

so we rewrite $\tilde{V}_X\left(0\right)$ as:

$$ \begin{aligned} &\tilde{V}_X\left(0\right) = \tilde{E}\left(\tilde{\varphi} H_T\right) \\ &= \tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) + \tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) \end{aligned} \tag{3.109} $$

Simplify the indicator function $1\left\{1 > c_p \tilde{Z}_T\right\}$, we obtain:

$$ 1\left\{1 > c_p \tilde{Z}_T\right\} = 1\left\{\hat{s}_1 < \overline{\Lambda}\right\} \tag{3.110} $$

where $\hat{s}_1 = \dfrac{-\int_0^T \sigma^*{}_1\left(u,T\right) d\tilde{W}_{u,1} - \int_0^T \sigma^*{}_2\left(u,T\right) d\tilde{W}_{u,2}}{\hat{\delta}_1} \sim N(0,1)$,

$$ \overline{\Lambda}_1 = \frac{-\ln c_p - \frac{1}{2}\int_0^T \Pi\left(u,T\right) du}{\hat{\delta}_1}, \quad \hat{\delta}_1^2 = \int_0^T \Pi\left(u,T\right) du . $$

We calculate the first term in (3.109)

$$ \begin{aligned} &\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\} 1\left\{1 > c_p \tilde{Z}_T\right\}\right) \\ &= \frac{S_0^1}{P\left(0,T\right)} e^{-\frac{1}{2}\zeta_1^2} \tilde{E}\left(e^{-z_1} \cdot 1\left\{\hat{s}_1 < \overline{\Lambda}_1\right\} 1\left\{s_1 \leq \Delta_1\right\}\right) \end{aligned} \tag{3.111} $$

where the correlations are

$$ \rho_{s_1\hat{s}_1} = \frac{\int_0^T \left[\sigma_1^*\left(u,T\right)\left(\sigma_1 - \sigma_2\rho\right) + \sigma_2^*\left(u,T\right)\left(\sigma_1\rho - \sigma_2\right)\right] du}{\tilde{\sigma}\hat{\delta}_1\sqrt{T}}, $$

$$ \rho_{s_1 z_1} = \frac{\int_0^T \left[\left(\sigma_1 + \sigma_1^*\left(u,T\right)\right)\left(\sigma_1 - \sigma_2\rho\right) + \sigma_2^*\left(u,T\right)\left(\sigma_1\rho - \sigma_2\right)\right] du}{\tilde{\sigma}\zeta_1\sqrt{T}}, $$

$$ \rho_{\hat{s}_1 z_1} = \frac{\int_0^T \left[\Pi\left(u,T\right) + \sigma_1\left(\sigma_1^*\left(u,T\right) + \rho\sigma^*{}_2\left(u,T\right)\right)\right] du}{\hat{\delta}_1\zeta_1}. $$

We can construct $\Delta_1^{'} = \dfrac{\ln\dfrac{S_0^1}{S_0^2} + \dfrac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\right)T}{\tilde{\sigma}\sqrt{T}}$

$$\overline{\Lambda}_1^{'} = \frac{-\ln c_p + \frac{1}{2}\int_0^T \left[\Pi(u,T) + \sigma_1\left(\sigma_1^*(u,T) + \rho\sigma^*_2(u,T)\right)\right]du}{\hat{\delta}_1}$$

and by two-asset theorem, (3.111) become

$$\tilde{E}\left(S_T^1 \cdot 1\left\{S_T^1 \geq S_T^2\right\}1\left\{1 > c_p\tilde{Z}_T\right\}\right) = \frac{S_0^1}{P(0,T)}\Psi^2\left(\Delta_1^{'}, \overline{\Lambda}_1^{'}; \rho_{s_1\hat{s}_1}\right) \tag{3.112}$$

For the second term in (3.109), we can follow the similar steps and get the expression as:

$$\tilde{E}\left(S_T^2 \cdot 1\left\{S_T^1 < S_T^2\right\}1\left\{1 > c_p\tilde{Z}_T\right\}\right) = \frac{S_0^2}{P(0,T)}\Psi^2\left(\Delta_2^{'}, \overline{\Lambda}_2^{'}; \rho_{s_2\hat{s}_1}\right) \tag{3.113}$$

Therefore, we obtain:

$$\tilde{V}_X(0) = \frac{S_0^1}{P(0,T)}\Psi^2\left(\Delta_1^{'}, \overline{\Lambda}_1^{'}; \rho_{s_1\hat{s}_1}\right) + \frac{S_0^2}{P(0,T)}\Psi^2\left(\Delta_2^{'}, \overline{\Lambda}_2^{'}; \rho_{s_2\hat{s}_1}\right) \tag{3.114}$$

which leads to final formula (3.107) for X_0 in risk-neutral case.

(ii). From (3.97) and key balance equation, we can derive the expression for survival probability in (3.108).

3.4.4 Numerical Results

In this section, we provide a numerical example to illustrate the effect of both stochastic interest rate and correlated Wiener processes on hedging equity-linked life insurance contracts by efficient hedging technique. For the purpose of comparison, we consider the same extreme case as in Section 3.3.4 that insurance company's attitude is risk-taking and loss function with power $p \to 0$.

Based on the analysis in previous Sections, we can fix the probability of failing to hedge claim H_T at maturity as $P\left(1 < cH\tilde{Z}_T\right) = \varepsilon$ .With the fixed financial risk level ε, constant c can be estimated from $P\left(1 < cH\tilde{Z}_T\right) = \varepsilon$. We use the same estimated volatilities for risky assets as in Section 3.3.4 and estimate the correlation ρ as:

$$\sigma_1 = 0.2909, \quad \sigma_2 = 0.2362, \quad \rho = 0.637 \tag{3.115}$$

For simplicity, all the results obtained in this section are under the assumption of constant volatility structures in HJM model (3.74) (3.75), i.e. $\sigma^i(t,T) = \sigma^i$, $i = 1,2,$ for all t and T. Without loss of generality, we select $\sigma^1 = 0.03$, $\sigma^2 = 0.02$.

With small $p < 0.01$, we assume the initial value of a single equity-linked life insurance contract is $S_0 = 1000$. We calculate the survival probabilities ${}_T p_x$ with maturity terms $T = 1 \sim 20$ years in the same specifications for the initial term structure $f(0,t)$ as in Section 3, i.e.

Scenario I: flat initial term structure $f(0,t) = r_0$.

Scenario II: linearly increasing initial term structure $f(0,t) = r_0 + 0.002 \cdot t$.

Scenario III: linearly decreasing initial term structure $f(0,t) = r_0 - 0.002 \cdot t$.

In case of $r_0 = 0.033$, the survival probabilities ${}_T p_x$ at different financial risk level ε are presented in Figure 3.9, 3.10, 3.11. The ages of clients corresponding to those survival probabilities are also obtained and graphically shown in Figure 3.12, 3.13, 3.14.

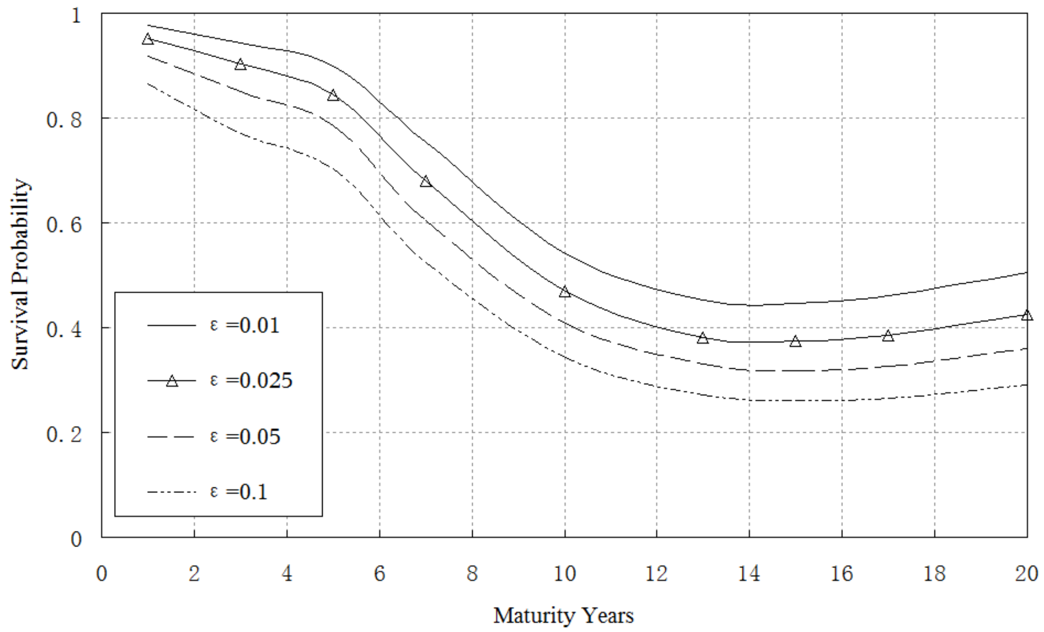

Figure 3.9 Hedging ratios (survival probabilities ${}_T p_x$) at different financial

risk ε with $f(0,t) = r_0 = 0.033$

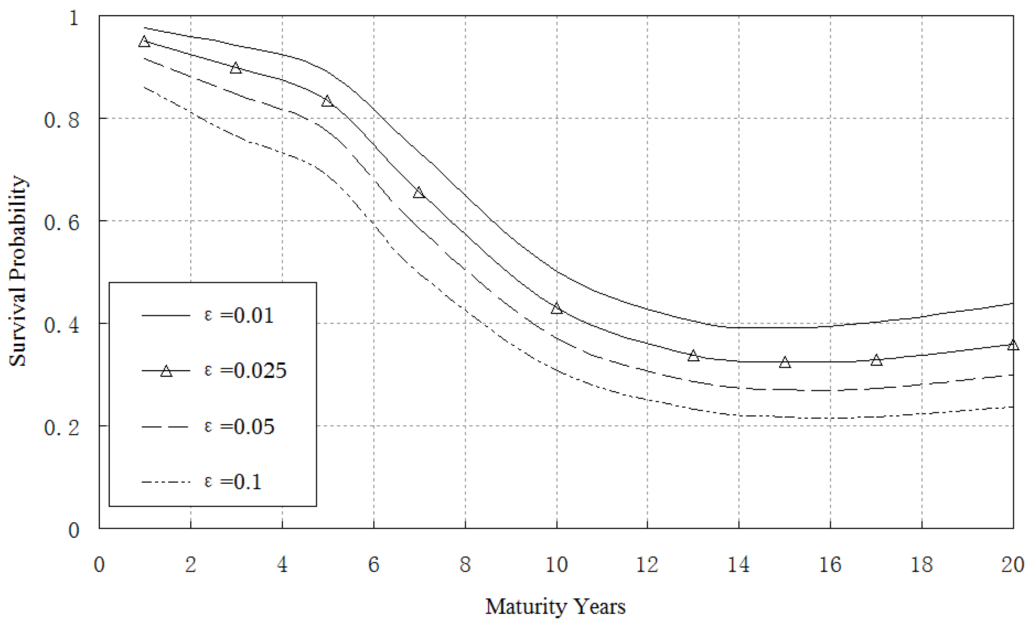

Figure 3.10 Hedging ratios (survival probabilities ${}_T p_x$) at different financial

risk ε with $f(0,t)=r_0+0.002t$

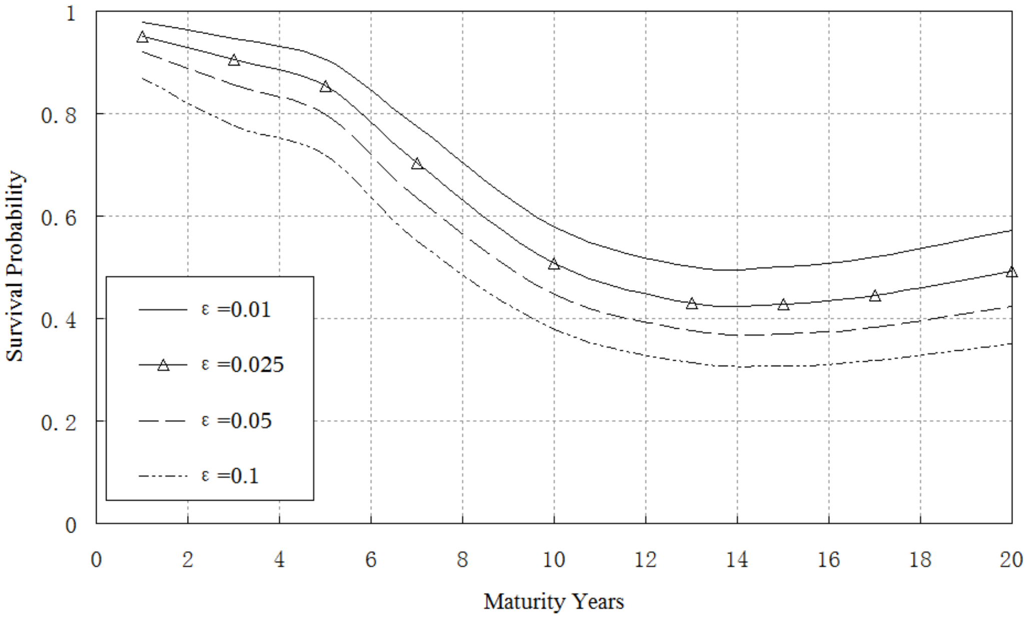

Figure 3.11 Hedging ratios (survival probabilities ${}_T p_x$) at different financial

risk ε with $f(0,t)=r_0-0.002t$

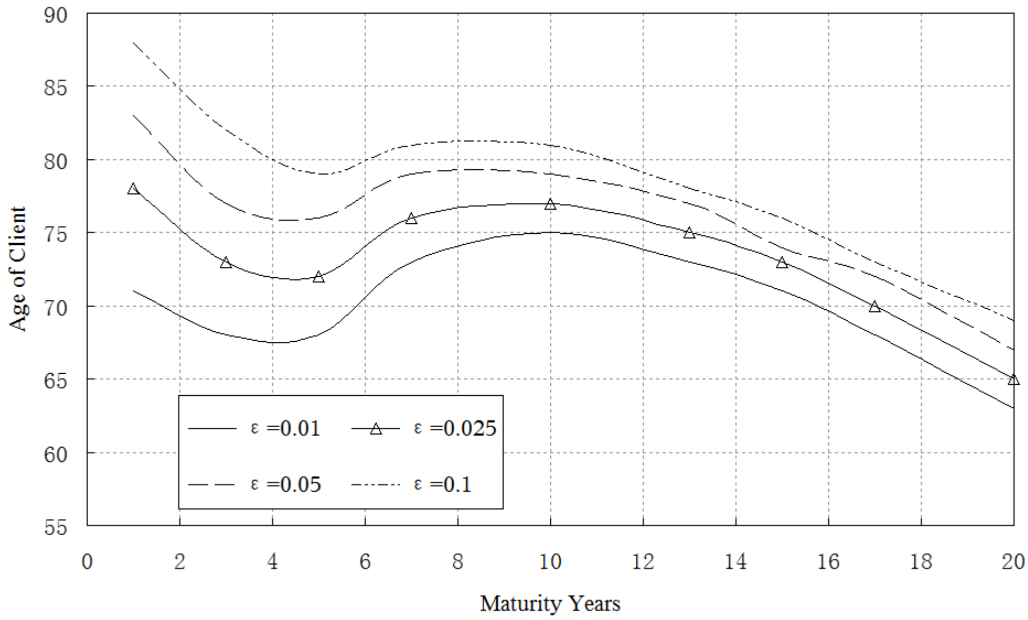

Figure 3.12 Age of clients with flat initial term structure $f(0,t) = r_0 = 0.033$.

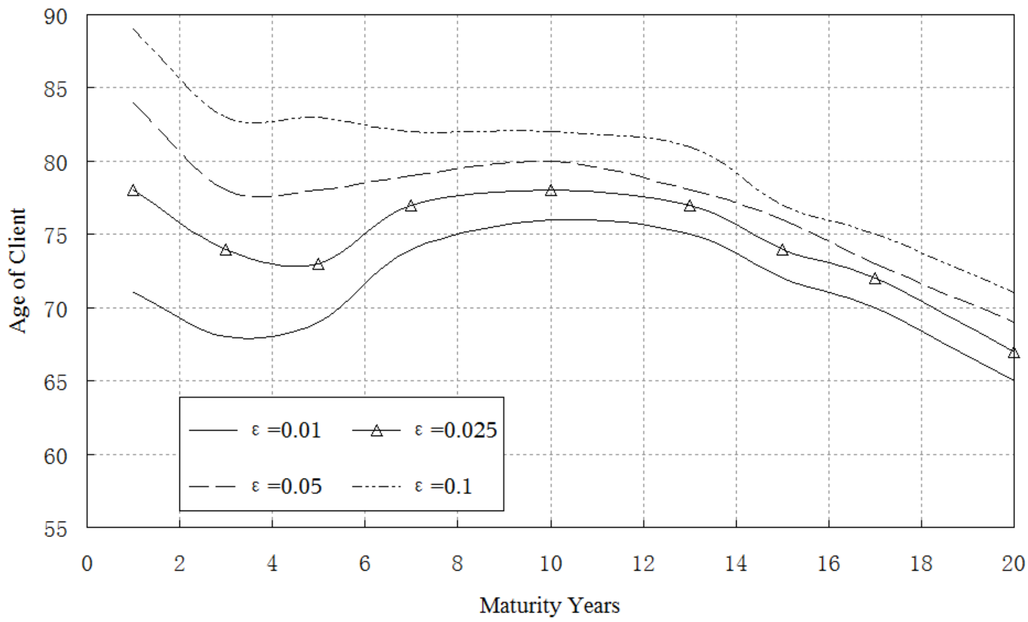

Figure 3.13 Age of clients with linearly increasing initial term structure $f(0,t) = r_0 + 0.002t$.

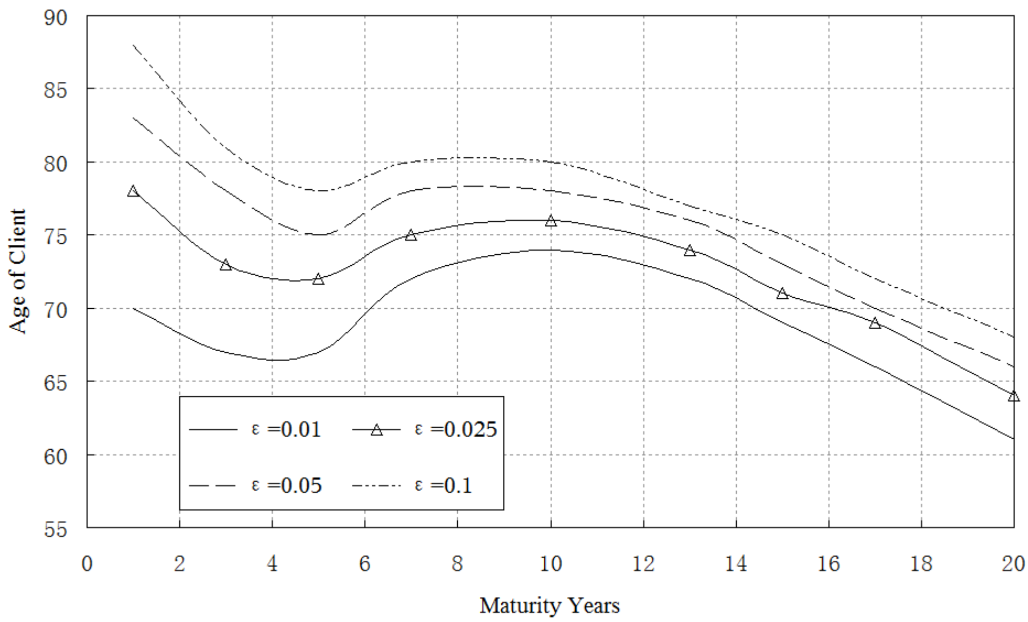

Figure 3.14 Age of clients with linearly decreasing initial term structure $f(0,t) = r_0 - 0.002t$.

For three different term structures $f(0,t)$, we still can observe an offset between the financial risk and the mortality risk. When insurance company faces an increased financial risk ε at fixed time T, the corresponding survival probability ${}_T p_x$ appears a decreasing trend. Besides, the trend of the client age reveals that the more financial risk the insurance company is willing to carry the elder group of clients should be attracted to compensate for the increased financial risk. With longer contract maturities, the company is better to attract younger group of clients while maintaining the same financial risk exposure.

From Figures 3.9 to 3.11, the survival probability ${}_T p_x$ shows a slightly increasing trend after the year of 15, but the trend has an overall decreasing pattern over the time. Additionally, the decreasing rate of the survival probability becomes faster after year

$T = 5$. This phenomenon can be well explained by the result in Miltersen & Persson (1999) which the stochastic interest rate can significantly affect the value of the contract only after a specific insurance period. However, from the results in Section 3.3.4 which the price processes of the risky assets are generated by the same Wiener process, the survival probability ${}_T p_x$ is concaved up obviously as maturity time evolves. The difference in trends of ${}_T p_x$ is caused by more source of randomness appearing in stochastic interest rate model which can make the insurance company facing increased financial risk. Therefore, the decreasing trend of survival probability ${}_T p_x$ is observed in this section because of the offset between financial risk and mortality risk.

As a result of different survival probability patterns, the trend of client age in Figures 3.12 to 3.14 is concaved up during first 10 years, and then it begins to decrease gradually. This observation is noticeably different compared with the overall concave-down pattern derived from special case $\rho = 1$ in Section 3.3.4. This comparison implies that when different Wiener processes are used for financial modeling, the insurance company does not have to attract wider age group of clients. Due to the uncontrollable factors of the volatility of the underlying risky assets and the fluctuations of interest, the insurance company may conservatively consider the elder group of clients (Age>60) to compensate the high potential risks.

Figure 3.15 shows the sensitivity of survival probability ${}_T p_x$ with respect to three different initial term structures at financial risk level $\varepsilon = 0.05$. The survival probabilities for the three $f(0,t)$ are overlapped for the initial time periods (such as $T = 3$). However, with the evolution of time, the difference in three survival probabilities becomes larger, and the survival probability with linearly decreasing $f(0,t)$ is significantly higher than the flat one. In addition, the survival probability for linearly increasing $f(0,t)$ is always lower than the one for flat throughout the rest of the period.

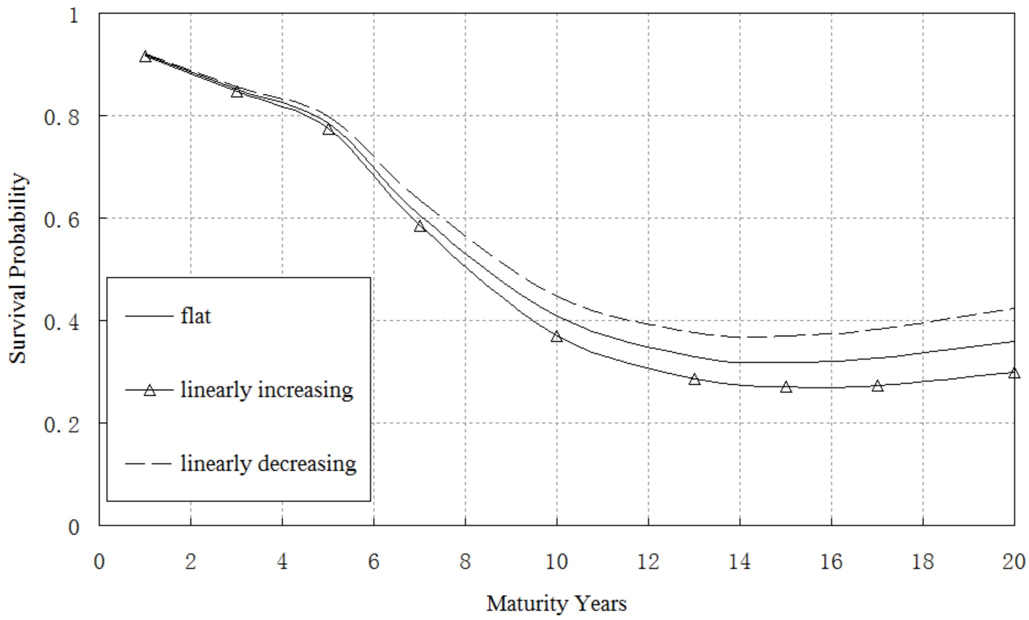

Figure 3.15 Sensitivity of survival probability ${}_T p_x$ w.r.t. different $f(0,t)$ at $\varepsilon = 0.05$.

Figure 3.16 shows the change of survival probability with respect to ρ at $\varepsilon = 0.05$ with flat $f(0,t)$ over 20 years. The correlation ρ varies between 0.15 and 0.95 with increment value 0.2. One can notice that different parameter ρ can lead to significant change in survival probabilities. The difference in survival probabilities becomes longer for maturity time T ($T > 4$). At any fixed time T , the survival probability ${}_T p_x$ keeps increasing with an increased value of ρ . This observation suggests that the insurance company could consider a younger group of clients to compensate the increased risk by investing on more correlated risky assets in financial market.

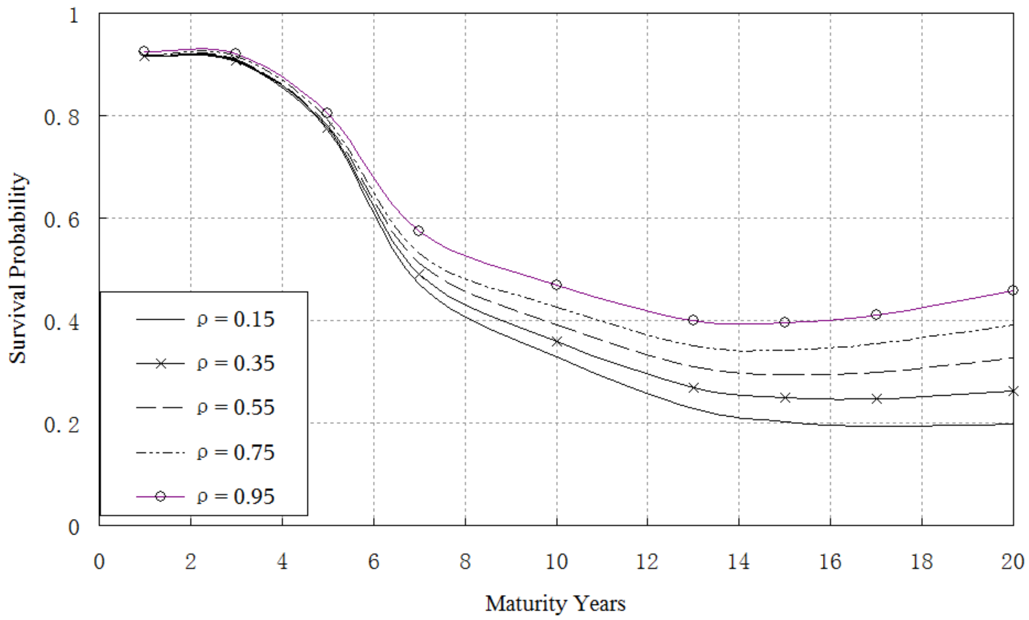

Figure 3.16 Sensitivity of survival probability w.r.t different ρ at $\varepsilon = 0.05$, initial term structure is flat.

4. Application of quantile hedging on equity-linked life insurance with market restriction: transaction costs

4.1. Literatures review of hedging inclusive transaction costs

In Chapter 2 and 3, we have already reviewed many research papers which made significant contributions to the imperfect hedging techniques. In general, one important assumption in these literatures as well as in our financial settings in Chapter 2 and 3 is the frictionless market. This market does not consider transaction costs in the financial trading. However, transaction costs can not be negligible in the real trading world. Transaction costs with frequent trading and large size can considerably affect the financial and insurance companies to properly value their products. They also possibly lead to huge financial losses. As a result, there are substantial amount of theoretical work devoted to option pricing with transaction costs.

Leland (1985) developed a hedging strategy to approximately replicate the European call option's payoff inclusive of transaction costs. The idea is to offset the transaction costs by implementing a modified volatility during hedging. The modified volatility depends on both the rate of transaction costs and the length of the rebalance interval which is also called the revision period. Inspired by Leland's contribution, Toft (1996) obtained the closed-form expressions for the expected transaction costs, hedging errors and variance of the cash flow from a time-based hedging strategy. Hodges & Neuberger (1989) initially developed a utility-based approach on option pricing with transaction costs. They also took into account the investor's behaviour towards the risk during the valuation. Their work was extended by a number of researchers, for instance, Clewlow & Hodges (1997), Barles & Soner (1998), and Zakamouline (2006).

Merton (1990) firstly examined the effects of transaction costs on derivative security pricing by two-period version of the Cox-Ross-Rubinstein (CRR) binomial model. Later on, Boyle & Vorst (1992) introduced an exact replication procedure in the C-R-R binomial model with transaction costs. They also gave a discrete-time variant of Leland's Theorem for the problem in the Black-Scholes market. Afterwards, Palmer (2001) removed Boyle and Vorst's conditions for the replication of short position on options. More recently, Melnikov & Petrachenko (2005) extended the C-R-R binomial model to cover the case of proportional transaction costs for one risky asset which has different interest rates on bank credit and deposit. They also considered the contingent claims which are two-dimensional random variables.

There also exist other approaches devoted to hedging and pricing options considering transaction costs, such as mean-variance hedging techniques, the abstract theory of cones. These techniques are introduced by papers Lamberton et. al. (1998), Reiss (1999), Stettner (1997), and Stettner (2000).

Equity-linked life insurance contracts usually have long term maturities in insurance market. As a result, the insurance companies are necessary to rebalance the hedging portfolio several times within the contract maturity. There are a few theoretical works related with hedging insurance contracts with transaction costs. Boyle & Hardy (1997) worked on the segregated fund which is a popular type of equity-linked product in Canada. They estimated the total hedging costs associated with the corresponding transaction costs for hedging maturity guarantee from a simulation method. Hardy (2000) compared three methods of determining suitable provision for maturity guarantees for single premium segregated fund contracts with transaction costs. Nteukam T., et.al. (2011) analyzed the optimality of several available hedging strategies. This result allows the insurer to reduce the risk related to a portfolio of unit-linked life insurance contracts including transaction costs.

4.2. Description of problem

To the best of our knowledge, although there are some researches considered hedging equity-linked type of insurance contract with transaction costs, few has been done on quantile hedging on insurance contract with transaction costs. The main focus of this Chapter is to discuss the valuation of equity-linked life insurance contracts by quantile hedging strategy in presence of transaction costs. For the sake of simplicity, we work on a single premium equity-linked life insurance contract and assume the guarantee at maturity is deterministic. However, we only consider the investments of the contract on risky asset which is attractive and has good financial performance. In Section 4.3, we calculate the quantile price of the contract without considering the transaction costs. In addition, the quantile hedging portfolio at the beginning of the contract term includes risk-free bonds and risky assets. The explicit expressions for the expected present value of total transaction costs and total hedging errors from a time-based replication strategy are obtained in Section 4.4. The expressions are based on the obtained quantile price formula. To compare with the analytical result obtained previously, a simulation method is introduced which contains the estimated expected present values of transaction costs, hedging errors and total quantile hedging costs, in Section 4.5. We also investigated the quantile hedging costs of maturity guarantee for equity-linked life insurance contract in this section.

Leland's transaction costs adjusted hedging volatility $\bar{\sigma}$ is utilized in this approach, which is different from the underlying risky asset's volatility σ. We also investigate the performance of Leland's adjusted volatility $\bar{\sigma}$ in presence of transaction costs from some numerical examples. As a matter of fact, there are some studies questioned the performance of Leland's adjusted transaction costs volatility $\bar{\sigma}$ in asymptotic case. For example, Kabanov & Safarian (1997) pointed out a flaw in the proof of convergence in Leland's main Theorem. Zhao & Ziemba (2007) numerically confirmed the findings from simulation results. It is noted that the constraint of Leland's adjusted volatility exists only when revision period is extremely small ($\Delta t \to 0$). However, this is the case beyond the consideration by insurance companies. From the practical point of view, insurance companies can not adjust the hedging positions frequently. Otherwise, the premium of the contract will be prohibitively expensive with the appearance of transaction costs. Therefore, Leland's adjusted hedging volatility $\bar{\sigma}$ can still be effectively considered in this study for practical purposes.

4.3. Premium of equity-linked life insurance contract without transaction costs

In this section, we will discuss the application of quantile hedging to calculate the premium for the equity-linked life insurance contract without consideration of transaction costs. We assume that we have a typical Black-Scholes-Merton setting: a financial market with the bond price B_t and the risky asset price S_t, which satisfy the following dynamics:

$$dB_t = rB_t dt \;\; \rightarrow \;\; B_t = B_0 e^{rt} \tag{4.1}$$

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right) \;\; \rightarrow \;\; S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t\right) \tag{4.2}$$

where W_t is a Wiener process defined on a complete probability space $\left(\Omega, F, \left(F_t\right)_{t\in[0,T]}, P\right)$, r is the constant risk-free interest rate, μ is the constant mean-rate of return on the risky asset, and σ is the constant risky asset's volatility. We also assume the financial market is frictionless.

By Girsanov Theorem, the equivalent martingale measure P^* is unique with the density given by

$$Z_t = \left.\frac{dP^*}{dP}\right|_{F_t} = \exp\left(-\theta W_t - \frac{\theta^2}{2}t\right), \;\; \theta = \frac{\mu - r}{\sigma} \tag{4.3}$$

We work on a single premium equity-linked life insurance contract with a maturity guarantee K. Ekern & Persson (1996) introduced varieties types of maturity guarantees for the contract. Here we only consider K is either fixed or deterministic which can both be treated as non-stochastic terms in the later calculation. H_T is defined to represent the payoff at maturity T as following:

$$H_T = \max\left(S_T, K\right) = S_T I\left\{S_T \geq K\right\} + K I\left\{S_T < K\right\} \tag{4.4}$$

where $I\left\{\cdot\right\}$ is the indicator function.

Following the detailed analysis of quantile hedging in Chapter 2, we can apply quantile hedging to value the initial hedging costs for equity-linked life insurance contract given the initial budget constraint $X_0 < H_0 = E^*\left(H_T e^{-rT}\right)$. The initial amount X_0 collected by the insurance company from selling the contract is strictly less than the amount H_0 required for a perfect hedging. Remind that the quantile hedge π^* can maximize the probability of successful hedging and it coincides with the perfect hedge for a modified contingent claim $H_T^* = H_T I\left\{A^*\right\}$. A^* is the maximal set of successful hedging which satisfies $A^* = I\left\{1/Z_T > a^* e^{-rT} H_T\right\}$, where a^* is a constant to be determined from the initial budget constraint with a nonzero interest rate r.

We derive the explicit formula for the quantile premium of equity-linked life insurance contract.

Theorem 4.1: Consider an insurance company that sells a single premium equity-linked life insurance contract with payoff at maturity as $H_T = \max\left(S_T, K\right)$, the initial premium of the contract determined from quantile hedging is:

$$X_0 = S_0 \Psi\left(\sigma\sqrt{T} - \Lambda_2\right) + \frac{K}{e^{rT}}\left(\Psi\left(\Lambda_2\right) - \Psi\left(\Lambda_1\right)\right), \quad \mu - r - \sigma^2 > 0. \tag{4.5}$$

where $\Psi\left(\cdot\right)$ is the cumulative distribution function for standard normal distribution, and

$$\Lambda_1 = \frac{\left(\frac{1}{2}\theta^2 - r\right)T + \ln\left(Ka^*\right)}{\theta\sqrt{T}}, \Lambda_2 = \frac{\ln\frac{K}{S_0} + \left(\frac{\sigma^2}{2} - r\right)T}{\sigma\sqrt{T}}.$$

Proof: Using the evolution of S_t under risk-neutral measure P^* and the structure of the modified contingent claim H_T^*, we can calculate the initial premium X_0 from quantile hedging as

$$\begin{aligned} X_0 &= E^*\left[\frac{H_T}{e^{rT}} I\left\{A^*\right\}\right] \\ &= E^*\left[\frac{\max\left(S_T, K\right)}{e^{rT}} I\left\{1 > a^* Z_T \frac{\max\left(S_T, K\right)}{e^{rT}}\right\}\right] \\ &= E^*\left[\frac{S_T}{e^{rT}} I\left\{1 > a^* Z_T \frac{S_T}{e^{rT}}\right\} \cdot I\left\{S_T \geq K\right\}\right] + E^*\left[\frac{K}{e^{rT}} I\left\{1 > a^* Z_T \frac{K}{e^{rT}}\right\} \cdot I\left\{S_T < K\right\}\right] \end{aligned} \tag{4.6}$$

Then, we can simplify the indicator functions $I\left\{1 > a^* Z_T \frac{S_T}{e^{rT}}\right\}$, $I\left\{S_T \geq K\right\}$ and $I\left\{S_T < K\right\}$.

For $\mu - r - \sigma^2 > 0$, we obtain the result:

$$X_0 = E^*\left[\frac{K}{e^{rT}} I\left\{y > \Lambda_1\right\} \cdot I\left\{y < \Lambda_2\right\}\right] + E^*\left[\frac{S_T}{e^{rT}} I\left\{y > \Lambda_3\right\} \cdot I\left\{y \geq \Lambda_2\right\}\right] \tag{4.7}$$

where $y = \frac{W_T^*}{\sqrt{T}}$ is a standard normal random variable and $\Lambda_1, \Lambda_2, \Lambda_3$ are given by

$$\Lambda_1 = \frac{\left(\frac{1}{2}\theta^2 - r\right)T + \ln\left(Ka^*\right)}{\theta\sqrt{T}}, \Lambda_2 = \frac{\ln\frac{K}{S_0} + \left(\frac{\sigma^2}{2} - r\right)T}{\sigma\sqrt{T}},$$

$$\Lambda_3 = \frac{\frac{1}{2}\left(\theta^2 - \sigma^2\right)T + \ln\left(S_0 a^*\right)}{\left(\theta - \sigma\right)\sqrt{T}}.$$

Next, we will compare $\Lambda_1, \Lambda_2, \Lambda_3$

$$\Lambda_2 - \Lambda_1 = \frac{\theta \ln \frac{K}{S_0} - \frac{\mu + r}{2}(\theta - \sigma)T - \sigma \ln\left(a^* S_0\right)}{\sigma\theta\sqrt{T}} \tag{4.8}$$
$$= \frac{M}{\sigma\theta\sqrt{T}}$$

$$\Lambda_2 - \Lambda_3 = \frac{(\theta - \sigma)\left(\ln \frac{K}{S_0} - \frac{\mu + r}{2}T\right) - \sigma \ln\left(a^* S_0\right)}{\sigma(\theta - \sigma)\sqrt{T}}$$

$$= \frac{M}{\sigma(\theta - \sigma)\sqrt{T}}, \tag{4.9}$$

$$\Lambda_1 - \Lambda_3 = \frac{(\theta - \sigma)\left(\ln\left(a^* S_0\right) - \frac{\mu + r}{2}T\right) - \theta \ln\left(a^* S_0\right)}{\theta(\theta - \sigma)\sqrt{T}}. \tag{4.10}$$
$$= \frac{M}{\theta(\theta - \sigma)\sqrt{T}}$$

where $M = (\theta - \sigma)\left(\ln\left(a^* S_0\right) - \frac{\mu + r}{2}T\right) - \theta \ln\left(a^* S_0\right)$. If $M > 0$, we have

$$\left(\mu - r - \sigma^2\right)\left(\ln \frac{K}{S_0} - \frac{\mu + r}{2}T\right) > \sigma^2 \ln\left(a^* S_0\right) \tag{4.11}$$

Given $\mu - r - \sigma^2 > 0$, (4.11) can lead us to select the value of guarantee K as

$$K > S_0 \exp\left(\frac{\mu + r}{2}T + \frac{\sigma^2 \ln\left(a^* S_0\right)}{\mu - r - \sigma^2}\right) \tag{4.12}$$

Based on (4.8) ~ (4.10), we can get $\Lambda_2 > \Lambda_1 > \Lambda_3$. Thus, the premium X_0 is reduced to

$$X_0 = E^*\left[\frac{S_T}{e^{rT}} I\left\{y \geq \Lambda_2\right\}\right] + E^*\left[\frac{K}{e^{rT}} I\left\{\Lambda_2 > y > \Lambda_1\right\}\right] \tag{4.13}$$

It is clearly that

$$E^*\left[\frac{K}{e^{rT}} I\{\Lambda_2 > y > \Lambda_1\}\right] = \frac{K}{e^{rT}}\left(\Psi(\Lambda_2) - \Psi(\Lambda_1)\right) \tag{4.14}$$

The first term in (4.13) is calculated directly:

$$\begin{aligned} E^*\left[\frac{S_T}{e^{rT}} I\{y \geq \Lambda_2\}\right] &= \frac{S_0}{e^{rT}} E^*\left[\exp\left(\left(r - \frac{\sigma^2}{2}T\right) + \sigma W_T^*\right) I\{y \geq \Lambda_2\}\right] \\ &= S_0 \int_{\Lambda_2}^{+\infty} e^{-\frac{1}{2}\sigma^2 T} \cdot e^{\sigma\sqrt{T}\cdot x} \cdot e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} dx \\ &= S_0 \int_{\Lambda_2 - \sigma\sqrt{T}}^{+\infty} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} du \\ &= S_0 \Psi\left(\sigma\sqrt{T} - \Lambda_2\right) \end{aligned} \tag{4.15}$$

From (4.14) and (4.15), we obtain the formula (4.5) for the contract premium X_0.

Remark 4.1: In this chapter, we only focus on the condition $\mu - r - \sigma^2 > 0$, which implies that the expected return of the risky asset S_t over the risk-free rate r is higher than the risk from the asset (volatility σ^2). This condition also reveals that the insurance companies usually try to select the risky asset which is attractive and have a good investment performance.

4.4. Quantile hedging on equity-linked life insurance contract with transaction costs

4.4.1 Transaction costs adjusted hedging volatility

In this section, we will investigate the impact of transaction costs on the premium of equity-linked life insurance contracts determined from quantile hedging. From intuition, the price inclusive of transaction costs could be higher than the price without transaction costs. The goal of insurance company is to maintain a portfolio that can replicates the payoff H_T. We assume that the replicating portfolio is rebalanced at discrete intervals and trades can only be executed at a set of points in time $\{t_0,...,t_m,...,t_M\}$, where $t_M = T$ is the maturity time for the contract. Hedging period Δt is a constant from t_m to t_{m+1}. At each revision point m, $m = 0,1,...,M-1$, the stock price is given by

$$S_{t_m} = S_{t_0} \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t_m - t_0) + \sigma W_{t_m - t_0}\right) \tag{4.16}$$

According to Toft (1996), while the risky asset's volatility equals to σ, we assume the quantile price at each revision point for equity-linked life insurance contract use another hedging volatility $\bar{\sigma}$ in the presence of transaction costs and discrete trading. The selected hedging volatility $\bar{\sigma}$ results in a hedging error, which can offset the transaction costs paid when the hedging strategy is revised. In this paper, we choose hedging volatility as $\bar{\sigma} = \sigma\sqrt{1 + 2k\sqrt{2/\pi} / \sigma\sqrt{\Delta t}}$, which is Leland's transaction costs adjusted hedging volatility (1985). The parameter k is the one-way transaction cost rate, which is proportional to the stock price.

As we have noted in (4.5), for $\mu - r - \sigma^2 > 0$, the initial quantile price for equity-linked life insurance contract with hedging volatility $\bar{\sigma}$ at t_0 is given as:

$$X_0 = S_{t_0}\Psi\left(\bar{\sigma}\sqrt{T_0} - \frac{\ln\frac{K}{S_{t_0}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_0}{\bar{\sigma}\sqrt{T_0}}\right)$$

$$+Ke^{-rT_0}\left[\Psi\left(\frac{\ln\frac{K}{S_{t_0}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_0}{\bar{\sigma}\sqrt{T_0}}\right) - \Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2 - r\right)T_0 + \ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_0}}\right)\right] \tag{4.17}$$

where $T_0 = t_M - t_0$, $\bar{\theta} = \dfrac{\mu - r}{\bar{\sigma}}$. The price formula (4.17) implies the initial hedging portfolio consist of $\bar{\Delta}_{t_0}$ units in the risky asset and $\bar{B}_{t_0}$ units of the bond which pays value K at maturity T,

$$\text{where }\ \overline{\Delta}_{t_0} = \Psi\left(\bar{\sigma}\sqrt{T_0} - \frac{\ln\frac{K}{S_{t_0}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_0}{\bar{\sigma}\sqrt{T_0}}\right),$$

$$\overline{B}_{t_0} = \left[\Psi\left(\frac{\ln\frac{K}{S_{t_0}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_0}{\bar{\sigma}\sqrt{T_0}}\right) - \Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2 - r\right)T_0 + \ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_0}}\right)\right].$$

At each revision point $t_m, m = 1,...,M-1$, the weights $\overline{\Delta}_{t_0}$ and $\overline{B}_{t_0}$ are adjusted and the quantile price is defined as:

$$X_{t_m} = S_{t_m}\Psi\left(\bar{\sigma}\sqrt{T_m} - \frac{\ln\frac{K}{S_{t_m}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_m}{\bar{\sigma}\sqrt{T_m}}\right)$$
$$+Ke^{-rT_m}\left[\Psi\left(\frac{\ln\frac{K}{S_{t_m}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_m}{\bar{\sigma}\sqrt{T_m}}\right) - \Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2 - r\right)T_m + \ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_m}}\right)\right] \quad (4.18)$$

where $T_m = t_M - t_m$. Over the hedging period Δt (from t_m to t_{m+1}, $m = 0,1,...,M-1$), the pricing formula (4.18) indicates that the shares of risky asset in the hedge portfolio is defined as:

$$\overline{\Delta}_{t_m} = \Psi\left(\bar{\sigma}\sqrt{T_m} - \frac{\ln\frac{K}{S_{t_m}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_m}{\bar{\sigma}\sqrt{T_m}}\right) \quad (4.19)$$

and the units of risk-free bond in the hedging portfolio is given by

$$\overline{B}_{t_m} = Ke^{-rT_m}\left[\Psi\left(\frac{\ln\frac{K}{S_{t_m}}+\left(\frac{\overline{\sigma}^2}{2}-r\right)T_m}{\overline{\sigma}\sqrt{T_m}}\right)-\Psi\left(\frac{\left(\frac{1}{2}\overline{\theta}^2-r\right)T_m+\ln\left(Ka^*\right)}{\overline{\theta}\sqrt{T_m}}\right)\right] \tag{4.20}$$

Both the quantile price (4.18) and the weights of the hedging portfolio (4.19), (4.20) depend on the Leland's adjusted hedging volatility $\overline{\sigma}$.

4.4.2 Expected hedging error

Suppose at time t_0, the initial hedging portfolio includes $\overline{\Delta}_{t_0}$ shares of risky assets and $\overline{B}_{t_0}$ units of risk-free bond. Given by (4.19) and (4.20), the hedging portfolio should consist of $\overline{\Delta}_{t_1}$ shares of risky assets and $\overline{B}_{t_1}$ units of bond at the next rebalanced time t_1. Since the hedging position can not be self-adjusted, the hedging error H_{t_1} at time t_1 is defined as:

$$\begin{aligned} H_{t_1} &= e^{r\cdot\Delta t}\overline{B}_{t_0} + \overline{\Delta}_{t_0} S_{t_1} - S_{t_1}\overline{\Delta}_{t_1} - Ke^{-rT_1}\overline{B}_{t_1} \\ &= e^{r\cdot\Delta t}\overline{B}_{t_0} + \overline{\Delta}_{t_0} S_{t_1} - X_{t_1} \end{aligned} \tag{4.21}$$

Similarly, the portfolio is rebalanced repeatedly at all t_m, $m = 0,1,...,M-1$, the hedging error is then shown as:

$$H_{t_{m+1}} = e^{r\cdot\Delta t}\overline{B}_{t_m} + \overline{\Delta}_{t_m} S_{t_{m+1}} - X_{t_{m+1}} \tag{4.22}$$

Following Toft (1996)'s derivation, we can get the one period expected hedging error and the expected present value of hedging error separately.

Theorem 4.2: We assume the hedging errors are discounted at the risk-free interest rate r. If the quantile hedging strategy is rebalanced at all t_m, $m = 0,1,...,M-1$, with a hedging volatility $\overline{\sigma}$, the one period expected hedging error at time t_{m+1} given S_{t_m}, $m = 0,1,...,M-1$, is:

$$E\left(H_{t_{m+1}}\middle|F_{t_m}\right)$$

$$= K\cdot e^{-rT_{m+1}}\left\{\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_{m+1}+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_{m+1}}}\right)-\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_m+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_m}}\right)\right.$$

$$\left.+\Psi\left(\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)-\Psi\left(\frac{\lambda^*_{m,m+1}+\frac{1}{2}\sigma^{*2}_{m,m+1}}{\sigma^*_{m,m+1}}\right)\right\} \tag{4.23}$$

$$+S_{t_m}e^{\mu\Delta t}\left\{\Psi\left(\frac{\frac{1}{2}\sigma^{*2}_{m,m}-\lambda^*_{m,m}}{\sigma^*_{m,m}}\right)-\Psi\left(\frac{\frac{1}{2}\sigma^{*2}_{m,m+1}-\lambda^*_{m,m+1}}{\sigma^*_{m,m+1}}\right)\right\}$$

where

$$\sigma^*_{m,k}=\sqrt{\bar{\sigma}^2T_k+\sigma^2\left(T_m-T_k\right)},\ \lambda^*_{m,k}=\ln(\frac{K}{S_{t_m}})-rT_k-\mu\left(T_m-T_k\right),\ k=m,\ldots,M-1,$$

$$\Delta t=t_{m+1}-t_m\,,\quad m=0,1,\ldots,M-1\,.$$

Proof: First, we can take conditional expectation on (4.22) and obtain

$$E\left(H_{t_{m+1}}\middle|F_{t_m}\right)$$

$$=Ke^{-rT_{m+1}}\left[\Psi\left(\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)-\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_m+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_m}}\right)\right]$$

$$-Ke^{-rT_{m+1}}E\left(\Psi\left(\frac{\ln\frac{K}{S_{t_{m+1}}}+\left(\frac{\bar{\sigma}^2}{2}-r\right)T_{m+1}}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\middle|F_{t_m}\right)+Ke^{-rT_{m+1}}\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_{m+1}+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_{m+1}}}\right)$$

$$+E\left(S_{t_{m+1}}\Psi\left(\bar{\sigma}\sqrt{T_m}-\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)\middle|F_{t_m}\right)$$

$$-E\left(S_{t_{m+1}}\Psi\left(\bar{\sigma}\sqrt{T_{m+1}}-\frac{\lambda^*_{m,m+1}+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}\bar{\sigma}^2T_{m+1}-\sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\middle|F_{t_m}\right) \tag{4.24}$$

where y is a random variable satisfied the standard normal distribution.

Let us calculate the conditional expectation $E\left(S_{t_{m+1}}\Psi\left(\bar{\sigma}\sqrt{T_m}-\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)\middle|F_{t_m}\right)$

using Formula (49) in Toft (1996), and we have

$$E\left(S_{t_{m+1}}\Psi\left(\bar{\sigma}\sqrt{T_m}-\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)\middle|F_{t_m}\right)$$

$$=S_{t_m}\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\Psi\left(\bar{\sigma}\sqrt{T_m}-\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right)E\left(e^{\sigma W_{t_{m+1}-t_m}}\middle|F_{t_m}\right)$$

$$=S_{t_m}e^{\mu\Delta t}\Psi\left(\bar{\sigma}\sqrt{T_m}-\frac{\lambda^*_{m,m}+\frac{1}{2}\sigma^{*2}_{m,m}}{\sigma^*_{m,m}}\right) \tag{4.25}$$

With the help of formula (49), we can get

$$E\left(S_{t_{m+1}}\Psi\left(\bar{\sigma}\sqrt{T_{m+1}}-\frac{\lambda^*_{m,m+1}+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}\bar{\sigma}^2T_{m+1}-\sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\middle|F_{t_m}\right)$$

$$= S_{t_m} \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t\right)\times$$

$$E\left(e^{\sigma W_{\Delta t}}\Psi\left(\bar{\sigma}\sqrt{T_{m+1}} - \frac{\lambda^*_{m,m+1} + \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\bar{\sigma}^2 T_{m+1} - \sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\right)$$

$$= S_{t_m} \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t\right)\times$$

$$\times\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\sigma\sqrt{\Delta t}\cdot y}e^{-\frac{1}{2}y^2}\Psi\left(\bar{\sigma}\sqrt{T_{m+1}} - \frac{\lambda^*_{m,m+1} + \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\bar{\sigma}^2 T_{m+1} - \sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)dy$$

$$= S_{t_m} e^{\mu\Delta t}\Psi\left(\frac{-\lambda^*_{m,m+1} + \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\bar{\sigma}^2 T_{m+1}}{\sqrt{\bar{\sigma}^2 T_{m+1} + \sigma^2\Delta t}}\right)$$

$$= S_{t_m} e^{\mu\Delta t}\Psi\left(\frac{\frac{1}{2}\sigma^{*2}_{m,m+1} - \lambda^*_{m,m+1}}{\sigma^*_{m,m+1}}\right) \qquad (4.26)$$

Similarly, we can calculate

$$E\left(\Psi\left(\frac{\ln\frac{K}{S_{t_{m+1}}} + \left(\frac{\bar{\sigma}^2}{2} - r\right)T_{m+1}}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\middle| F_{t_m}\right)$$

$$= E\left(\Psi\left(\frac{\ln\frac{K}{S_{t_m}} - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\Delta t\cdot y + \frac{1}{2}\bar{\sigma}^2 T_{m+1} - rT_{m+1}}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\middle| F_{t_m}\right)$$

$$= E\left(\Psi\left(\frac{\lambda^*_{m,m+1} + \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\bar{\sigma}^2 T_{m+1} - \sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)\right)$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\Psi\left(\frac{\lambda^*_{m,m+1}+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}\bar{\sigma}^2T_{m+1}-\sigma\sqrt{\Delta t}\cdot y}{\bar{\sigma}\sqrt{T_{m+1}}}\right)dy$$

$$=\Psi\left(\frac{\lambda^*_{m,m+1}+\frac{1}{2}\sigma^{*2}_{m,m+1}}{\sigma^*_{m,m+1}}\right). \tag{4.27}$$

Substitute (4.25)-(4.27) into (4.24), we obtain the expression of (4.23).

Theorem 4.3: We assume the hedging errors are discounted at the risk-free interest rate r. If the quantile hedging strategy is rebalanced at all t_m, $m=0,1,...,M-1$, with a hedging volatility $\bar{\sigma}$, the expected present value of hedging error at time t_{m+1} given the information at time t_0 is:

$$\begin{aligned}
&E\left(H_{t_{m+1}}e^{-rt_{m+1}}\middle|F_{t_0}\right)\\
&=S_{t_0}e^{(\mu-r)(t_{m+1}-t_0)}\left[\Psi\left(\frac{\frac{1}{2}\sigma^{*2}_{0,m}-\lambda^*_{0,m}}{\sigma^*_{0,m}}\right)-\Psi\left(\frac{\frac{1}{2}\sigma^{*2}_{0,m+1}-\lambda^*_{0,m+1}}{\sigma^*_{0,m+1}}\right)\right]\\
&+Ke^{-rT_0}\left\{\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_{m+1}+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_{m+1}}}\right)-\Psi\left(\frac{\left(\frac{1}{2}\bar{\theta}^2-r\right)T_m+\ln\left(Ka^*\right)}{\bar{\theta}\sqrt{T_m}}\right)\right.\\
&\left.+\Psi\left(\frac{\lambda^*_{0,m}+\frac{1}{2}\sigma^{*2}_{0,m}}{\sigma^*_{0,m}}\right)-\Psi\left(\frac{\lambda^*_{0,m+1}+\frac{1}{2}\sigma^{*2}_{0,m+1}}{\sigma^*_{0,m+1}}\right)\right\}
\end{aligned} \tag{4.28}$$

To prove Theorem 4.3, we have $E\left(H_{t_{m+1}}\middle|F_{t_0}\right)=E\left(E\left(H_{t_{m+1}}\middle|F_{t_m}\right)\middle|F_{t_0}\right)$, and then we can substitute (4.29) into the above equation and apply formula (49) in Toft (1996) again to get (4.28).

Theorem 4.2 can be used to analyze the expected hedging errors at the end of the first hedging period and also provides the size of the expected hedging error in each of the subsequent hedging periods. Theorem 4.3 gives an expression for the expected present value of total hedging error, conditional on the information at time t_0 . Subsequently, let us define the present value of the total expected hedging errors during the contract maturity as:

$$E\left(H \middle| F_{t_0}\right) = \sum_{m=0}^{M-1} E\left(H_{t_{m+1}} e^{-rt_{m+1}} \middle| F_{t_0}\right) \tag{4.29}$$

Providing the numerical values for different model parameters, we assume there are 12 months, 24 biweekly or 48 business weeks in one year. We select the following parameters to calculate the expected present value of total hedging error for contracts with different maturities T , and same values are used for calculations through the rest part of Section 4:

$$\mu = 0.13, \ \sigma = 0.2, \ r = 0.06, \ S_0 = 100.$$

We fix the financial risk for quantile hedging as $\varepsilon = 0.025$, and set the deterministic maturity guarantee for the contract as $K = S_0 e^{gT}$, where the guarantee rate $g = 0.1$.

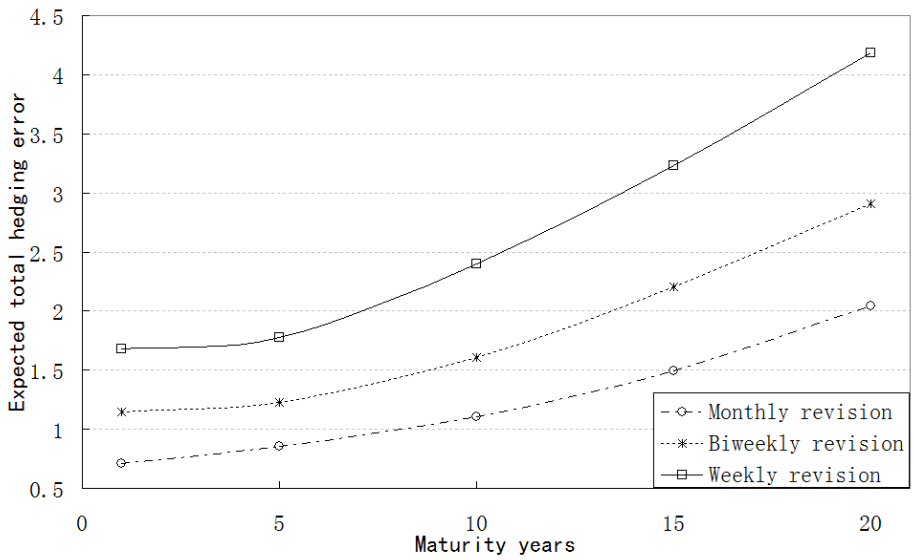

Figure 4.1 The expected present value of total hedging errors (HE) from hedging volatility $\bar{\sigma}$ with the one way transaction costs rate $k = 0.25\%$.

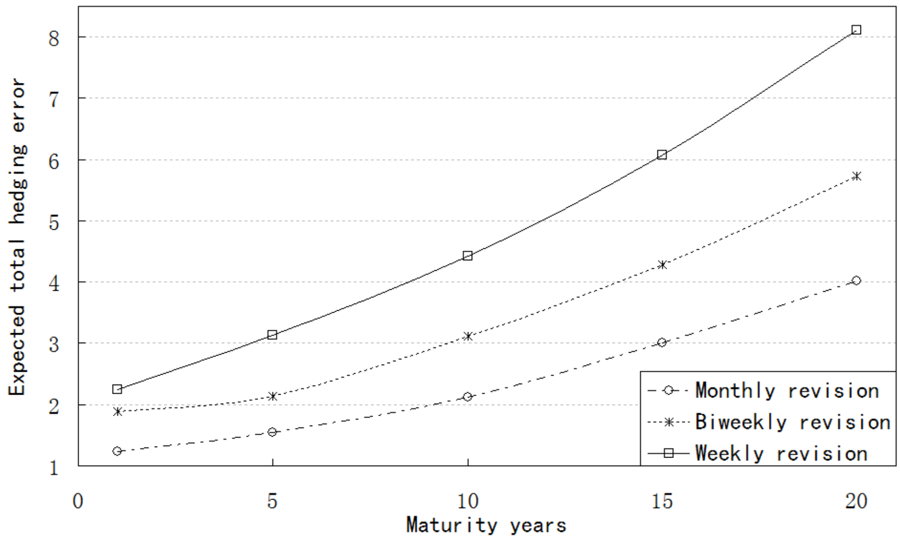

Figure 4.2 The expected present value of total hedging errors (HE) from hedging volatility $\overline{\sigma}$ with the one way transaction costs rate $k = 0.5\%$.

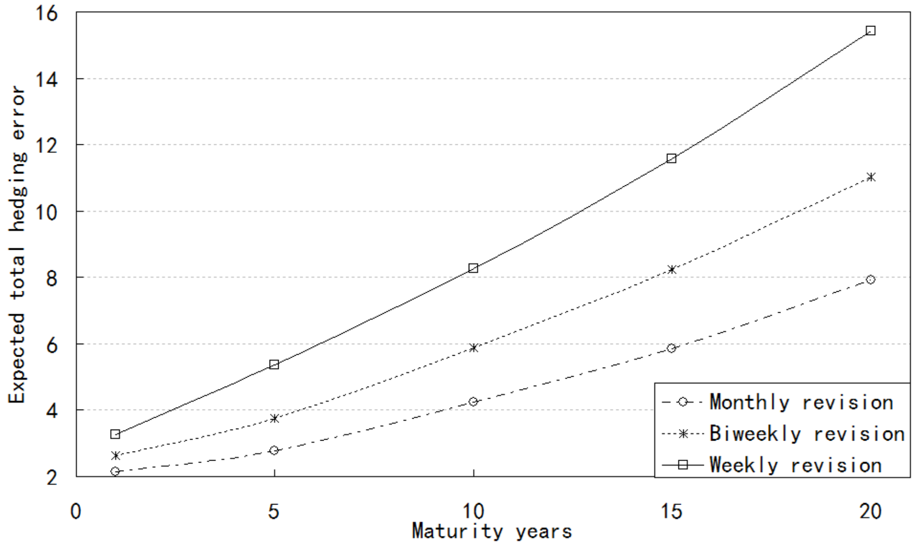

Figure 4.3 The expected present value of total hedging errors (HE) from hedging volatility $\overline{\sigma}$ with transaction costs rate $k = 1\%$.

Figures 4.1~ 4.3 show that the expected present values of total hedging error (HE) obtained from Leland's adjusted hedging volatility $\overline{\sigma}$ with one way transaction cost rate $k = 0.25\%, 0.5\%, 1\%$ respectively. It is observed that the total expected hedging errors with Leland's volatility are not negligible, and the errors turn out to be positive with different rebalancing frequency across different maturity time T. As rebalancing frequency and maturity time T increase, the discounted values of the expected hedging errors also increase significantly. The magnitude of the expected total hedging errors is evidently sensitive to the revision period Δt, contract's maturity T, and the transaction costs rate k.

Remark 4.2: As discussed in Melnikov & Romaniuk (2006), two factors determine the selection of the contract's maturity guarantee K: (i) the return on the stock investment; (ii) the short-term interest rates. The manager should set the guarantee K higher than the amount generated from risk-free interest rate r while less than the return from stock S_T. Otherwise, clients would invest in a more appealing money market and seek for products with less financial risks. For a contract with long term maturity, the manager can set a deterministic guarantee K with a rate g satisfying $r < g < \mu$.

4.4.3 Expected transaction costs

Parameter k represents the one-way transaction cost rate, which is measured as the proportion of risky asset's price. The dollar value of the transaction costs for trading at time t_{m+1} is defined as:

$$TC_{t_{m+1}} = kS_{t_{m+1}} \left| \overline{\Delta}_{t_{m+1}} - \overline{\Delta}_{t_m} \right| \tag{4.31}$$

We can obtain the expected present value of total transaction costs through the following two Theorems:

Theorem 4.4: We assume the transaction costs are discounted at the risk-free interest rate r. If the quantile hedging strategy is rebalanced at all t_m, $m = 0,1,...,M-1$, with a hedging volatility $\overline{\sigma}$, the expected one period transaction costs at time t_{m+1} given the information at time t_m is:

$$E\left(TC_{t_{m+1}}\middle|F_{t_m}\right) = kS_{t_m}e^{\mu\Delta t}\left[\Psi\left(b_m\right)-\Psi\left(c_m\right)+2\left(\Psi\left(a_m,c_m,0\right)-\Psi\left(a_m,b_m,\rho_m\right)\right)\right] \tag{4.32}$$

where

$$a_m = \frac{z_m^* - \left(\mu+\frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}},\ z_m^* = (1-\frac{\sqrt{T_{m+1}}}{\sqrt{T_m}})\ln\frac{K}{S_{t_m}} + \left(\sqrt{T_{m+1}T_m}-T_{m+1}\right)\left(r-\frac{1}{2}\bar{\sigma}^2\right),$$

$$b_m = \frac{\frac{1}{2}\sigma_{m,m+1}^{*2}-\lambda_{m,m+1}^*}{\sigma_{m,m+1}^*},\ c_m = \frac{\frac{1}{2}\sigma_{m,m}^{*2}-\lambda_{m,m}^*}{\sigma_{m,m}^*},\ \rho_m = -\frac{\sigma\sqrt{\Delta t}}{\sigma_{m,m+1}^*}.$$

Proof: Let us take the conditional expectation on definition (4.31). We can obtain

$$\begin{aligned} &E\left(TC_{t_{m+1}}\middle|F_{t_m}\right) \\ &= kE\left(S_{t_{m+1}}\left|\bar{\Delta}_{t_{m+1}}-\bar{\Delta}_{t_m}\right|\middle\|F_{t_m}\right) \\ &= kS_{t_m}e^{\left(\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)}E\left(e^{\sigma\sqrt{\Delta t}\cdot y}\left|Y_1-Y_2\right\|F_{t_m}\right) \end{aligned} \tag{4.33}$$

where y is a standard normal distributed random variable, and

$$Y_1 = \bar{\sigma}\sqrt{T_{m+1}} - \frac{\ln\frac{K}{S_{t_{m+1}}}+\left(\frac{\bar{\sigma}^2}{2}-r\right)T_{m+1}}{\bar{\sigma}\sqrt{T_{m+1}}},\ Y_2 = \bar{\sigma}\sqrt{T_m} - \frac{\ln\frac{K}{S_{t_m}}+\left(\frac{\bar{\sigma}^2}{2}-r\right)T_m}{\bar{\sigma}\sqrt{T_m}}.$$

Given $\bar{x} = \dfrac{\left(1-\frac{\sqrt{T_{m+1}}}{\sqrt{T_m}}\right)\ln\frac{K}{S_{t_m}}+\left(\sqrt{T_{m+1}T_m}-T_{m+1}\right)\left(r-\frac{1}{2}\bar{\sigma}^2\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}}$,

the conditional expectation (4.33) becomes

$$\begin{aligned} &E\left(TC_{t_{m+1}}\middle|F_{t_m}\right) \\ &= \int_{\bar{x}}^{+\infty} e^{\sigma\sqrt{\Delta t}\cdot y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\left[\Psi\left(Y_1\right)-\Psi\left(Y_2\right)\right]dy + \int_{-\infty}^{\bar{x}} e^{\sigma\sqrt{\Delta t}\cdot y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\left[\Psi\left(Y_2\right)-\Psi\left(Y_1\right)\right]dy \\ &= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\Delta t}\cdot y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\left[\Psi\left(Y_1\right)-\Psi\left(Y_2\right)\right]dy + 2\int_{-\infty}^{\bar{x}} e^{\sigma\sqrt{\Delta t}\cdot y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\left[\Psi\left(Y_2\right)-\Psi\left(Y_1\right)\right]dy \end{aligned} \tag{4.34}$$

Using formula (50) in Toft (1996), we get expression (4.32).

Theorem 4.5: We assume the transaction costs are discounted at the risk-free interest rate r. If the quantile hedging strategy is rebalanced at all $t_m, m = 0,1,...,M-1$, with a hedging volatility $\bar{\sigma}$, the expected one period transaction costs at time t_{m+1} given the information at time t_0 is:

$$\begin{aligned} & E\left(TC_{t_{m+1}} e^{-rt_{m+1}} \middle| F_{t_0}\right) \\ & = S_{t_0} k e^{(\mu-r)(t_{m+1}-t_0)} \left\{\Psi\left(e_m\right) - \Psi\left(f_m\right) + 2\left[\Psi\left(d_m, f_m, \rho_1\right) - \Psi\left(d_m, e_m, \rho_2\right)\right]\right\} \end{aligned} \tag{4.35}$$

where

$$e_m = \frac{\frac{1}{2}\sigma^{*2}_{0,m+1} - \lambda^*_{0,m+1}}{\sigma^*_{0,m+1}}, f_m = \frac{\frac{1}{2}\sigma^{*2}_{0,m} - \lambda^*_{0,m}}{\sigma^*_{0,m}}, \hat{\sigma}_m = \sqrt{\tau_m^2\left(t_m - t_0\right) + \Delta t}, \tau_m = 1 - \frac{\sqrt{T_{m+1}}}{\sqrt{T_m}},$$

$$d_m = \frac{\tau_m\left(\ln(\frac{K}{S_0}) - \left(\mu + \frac{\sigma^2}{2}\right)t_m + \sqrt{T_{m+1}T_m}\left(r - \frac{\bar{\sigma}^2}{2}\right)\Delta t\right) - \left(\mu + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\hat{\sigma}_m},$$

$$\rho_1 = \frac{-\tau_m \sigma t_m}{\sqrt{\left(\tau_m^2 t_m + \Delta t\right)\left(\sigma^2 t_m + \bar{\sigma}^2 T_m\right)}}, \quad \rho_2 = \frac{-\sigma\left(\tau_m t_m + \Delta t\right)}{\sqrt{\left(\Delta t + \tau_m^2 t_m\right)\left(\bar{\sigma}^2 T_{m+1} + \sigma^2 t_{m+1}\right)}}.$$

Proof: It is clearly that $E\left(TC_{t_{m+1}} \middle| F_{t_0}\right) = E\left(E\left(TC_{t_{m+1}} \middle| F_{t_m}\right) \middle| F_{t_0}\right)$. Substitute (4.32) into the previous equation and with the help of formula (49), (51) and (52) in Toft (1996), we can get the closed form expression for expected present value of transaction costs in (4.35).

Theorem 4.5 allows the calculation of the expected present value of the transaction costs at each of the future rebalancing points conditional on the information at time t_0. These expected transaction costs subsequently are aggregated into a forward looking measure of the replication strategy's total expected transaction costs. We define the following definition for the expected present value of total transaction costs as:

$$E\left(TC\middle|F_0\right)=\sum_{m=0}^{M-1}E\left(TC_{t_{m+1}}e^{-rt_{m+1}}\middle|F_0\right) \tag{4.36}$$

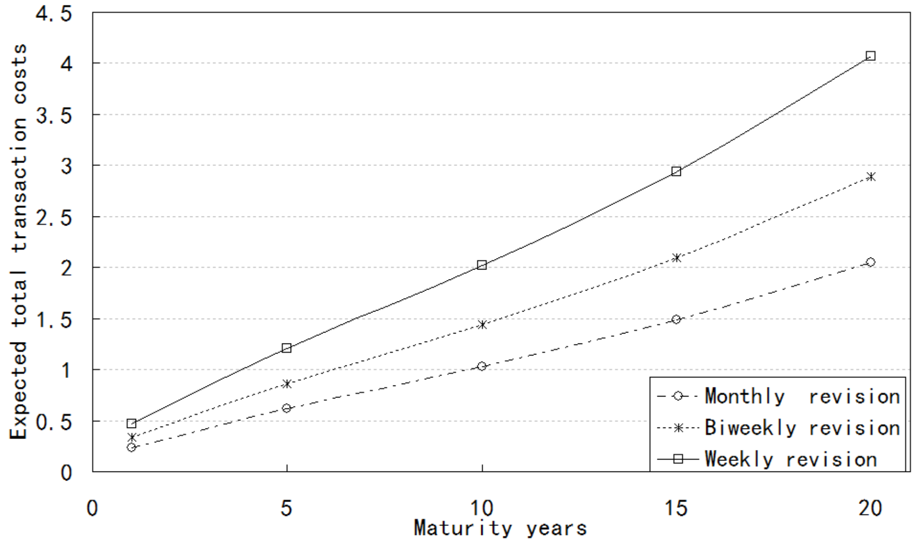

Figure 4.4 The expected present value of total transaction costs (TC) from hedging volatility $\bar{\sigma}$ with the one way transaction costs rate $k=0.25\%$.

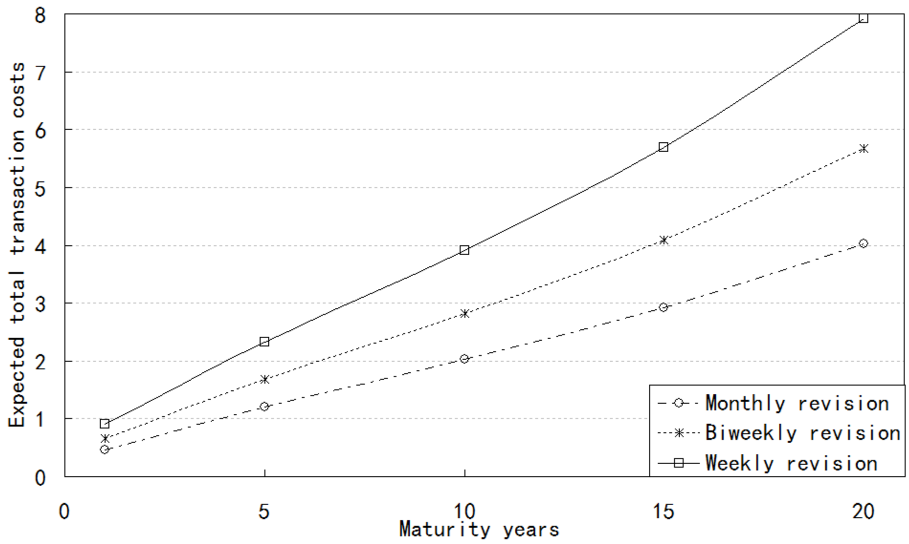

Figure 4.5 The expected present value of total transaction costs (TC) from hedging volatility $\bar{\sigma}$ with the one way transaction costs rate $k=0.5\%$.

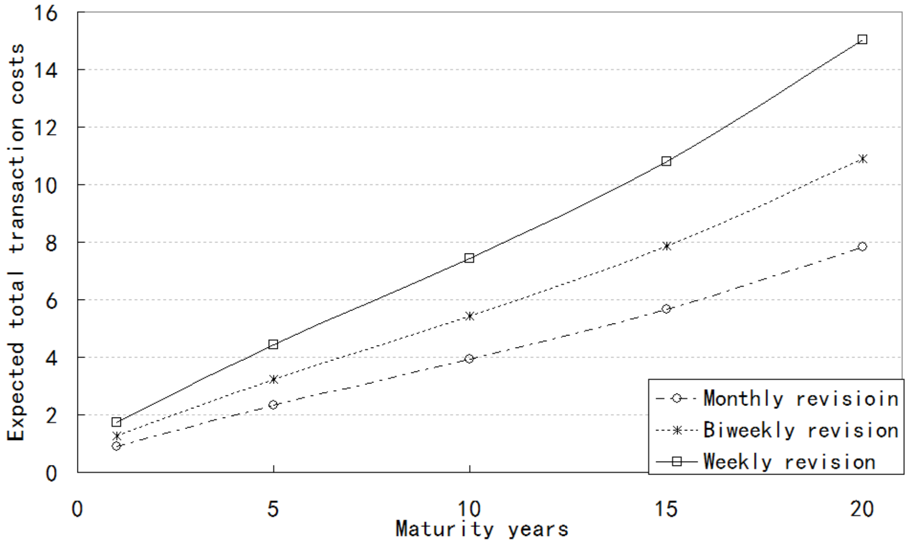

Figure 4.6 The expected present value of total transaction costs (TC) from hedging volatility $\bar{\sigma}$ with the one way transaction costs rate $k = 1\%$.

Figures 4.4~4.6 show the expected present value of total transaction costs (TC) calculated from Leland's adjusted hedging volatility $\bar{\sigma}$ with one way transaction cost rate $k = 0.25\%, 0.5\%, 1\%$ at different maturity T respectively. The base parameters for the calculations are the same as in Section 4.4.2. It is observed that transaction costs are significantly increased in values for more frequent rebalancing and longer maturity. Furthermore, the values almost increase proportionally with the increase of transaction rate k.

4.4.4 Expected total hedging cost

Based on the analysis in Section 4.4.1~4.4.3, the expected total quantile hedging costs for equity-linked life insurance contract is supposed to include two parts: one is the initial cost to set up the hedging portfolio, which is determined by quantile price for the contract; the other one is the difference between total expected hedging error and total expected transaction costs. We could obtain the expected total quantile hedging costs in presence of transaction costs for different maturities and revision periods at transaction costs rate $k = 0.5\%$. For comparison, the results are calculated based on hedging with the underlying asset's volatility σ and with Leland's adjusted hedging volatility $\bar{\sigma}$ separately. Table 4.1 shows the expected total quantile hedging costs with hedging volatility σ, and Table 4.2 displays the expected total quantile hedging costs with the adjusted hedging volatility $\bar{\sigma}$.

Table 4.1 The expected total quantile hedging costs with hedging volatility σ.

Maturity T	Revision	Quantile	HE	TC	HE-TC	Total
	Monthly	10.8478	0.2676	1.2518	-0.9842	11.8320
$T = 5$	Biweekly	10.8478	0.3200	1.7670	-1.4470	12.2948
	Weekly	10.8478	0.3461	2.4984	-2.1523	13.0001
	Monthly	12.1843	-0.0092	2.1068	-2.1160	14.3003
$T = 10$	Biweekly	12.1843	0.0761	2.9679	-2.8918	15.0761
	Weekly	12.1843	0.1188	4.2954	-4.1766	16.3609
	Monthly	12.6638	-0.1584	3.0300	-3.1884	15.8522
$T = 15$	Biweekly	12.6638	-0.0374	4.2954	-4.3328	16.9966
	Weekly	12.6638	0.0230	6.0853	-6.0623	18.7261
	Monthly	12.7852	-0.2770	4.1786	-4.4556	17.2468
$T = 20$	Biweekly	12.7852	-0.1149	5.9411	-6.0570	18.8422
	Weekly	12.7852	-0.0339	8.4336	-8.4675	21.2527

Table 4.2 The expected total quantile hedging costs with Leland's adjusted hedging volatility $\bar{\sigma}$.

Maturity T	Revision	Quantile	HE	TC	HE-TC	Total
$T=5$	Monthly	12.0760	1.5403	1.2073	0.3330	11.7430
	Biweekly	12.5655	2.1312	1.6811	0.4501	12.1097
	Weekly	13.2397	3.1271	2.3339	0.7932	12.4465
$T=10$	Monthly	14.0748	2.1244	2.0329	0.0915	13.9833
	Biweekly	14.9821	3.1202	2.8254	0.2948	14.6855
	Weekly	16.1473	4.4202	3.9166	0.5036	15.6437
$T=15$	Monthly	15.3602	3.0038	2.9247	0.0791	15.2811
	Biweekly	16.9264	4.2907	4.0922	0.1985	16.7279
	Weekly	18.0399	6.0760	5.6975	0.3785	17.6554
$T=20$	Monthly	16.6769	4.0163	4.0268	-0.0105	16.6874
	Biweekly	18.2837	5.7285	5.6682	0.0603	18.2234
	Weekly	20.2723	8.1092	7.9132	0.196	20.0763

As expected, we observe that the initial costs (quantile prices) with the adjusted volatility $\bar{\sigma}$ are higher than those obtained from volatility σ . In fact, inclusive of positive transaction costs, the initial costs to set up the hedging portfolio can be higher than the ones without transaction costs. The total hedging errors from volatility σ are relatively small, which shows an increasing trend with respect to rebalancing frequency for each contract's maturity. However, it is encouraging to see that the expected total hedging errors implied by the adjusted volatility $\bar{\sigma}$ have greater values, which can approximately offset the corresponding total transaction costs. Furthermore, when the adjusted volatility $\bar{\sigma}$ is greater than the underlying asset's volatility σ , the hedging errors in Table 4.2 turn out to be positive. This observation is agreed with the conclusion in Toft (1996).

The differences between the hedging errors and the transaction costs can be considered as the cash flow from the quantile hedging strategy. Positive values imply to sell the asset, while negative values imply that extra asset is required to maintain the quantile hedge. In addition, the differences obtained from volatility σ reach a significantly higher than those calculated from the adjusted volatility $\overline{\sigma}$, especially under monthly rebalancing (see Table 4.1 and 4.2). One can see that the hedging errors generated from adjusted volatility $\overline{\sigma}$ can almost offset the transaction costs during the trading. This observation results from the positive hedging errors along with the similar values of transaction costs.

From Tables 4.1 and 4.2, the expected total costs obtained from adjusted volatility $\overline{\sigma}$ are slightly less than the costs from volatility σ. However, the results also reveal that in the presence of transaction costs, the expected total costs for both cases are prohibitively expensive compared with ones without transaction costs. For longer maturity contract with more frequent revision periods, the expected total costs of quantile hedging can increase as high as 57% by considering transaction costs.

4.5. Quantile hedging inclusive transaction costs based on simulation approach

4.5.1 Quantile hedging costs for equity-linked life insurance contract

In Boyle & Hardy (1997), a time-based simulation method is introduced to estimate the expected present value of total transaction costs and the total costs of hedging. The advantage of this method is that the entire distribution of the transaction costs can be estimated, and it is more flexible to handle the contracts with annual premium. In presence of transaction costs, they assume the replicating portfolio is adjusted at regular time intervals Δt, the revision period. The sequence of transactions that take place will depend on the evolution of the stock price over the contract maturity $\left[0,T\right]$. So a series of stock price values at each revision point $\{t_0,...,t_m,...,t_M\}$ are simulated first, where $t_M = T$, $m = 0,1,...,M-1$. Then, given the definition of transaction costs at time t_{m+1} as $TC_{t_{m+1}} = kS_{t_{m+1}} \left|\overline{\Delta}_{t_{m+1}} - \overline{\Delta}_{t_m}\right|$, the expected discount values of transaction costs at time t_0 are computed. The estimates of the expected total costs are the basic Black-Scholes price plus the expected transaction costs.

Although Boyle & Hardy (1997) discussed the accuracy of the prescribed hedging strategies by tracking errors, they did not take them into account the expected total costs for hedging. In this paper, we use the same simulation method to estimate the expected present value of transaction costs. We also adopt Leland's transaction cost adjusted hedging volatility $\overline{\sigma}$ in calculations. For comparing with the results in Section 4.4, we calculated the hedging error, which is the difference between the prices of the hedge portfolio before and after rebalancing. The cash flows are considered into the expected total hedging costs.

Table 4.3 and 4.4 illustrate that the estimated expected present values of total transaction costs and the percentiles of distribution at $k = 0.5\%$ based on hedging volatility $\sigma = 0.2$ and adjusted volatility $\overline{\sigma}$ separately respectively. It is shown that the estimated expected total transaction costs from σ are higher than the costs based on $\overline{\sigma}$. However, the increment of transaction costs are only within \$0.50. As rebalancing becomes more frequent, the estimated transaction costs of weekly rebalancing are almost doubled compared to the monthly case across different maturities. This observation is agreed with those comparisons between Table 4.1 and 4.2. Similarly, the percentiles estimated from volatility σ are slightly increased compared with the percentiles obtained from adjusted volatility $\overline{\sigma}$. The expected transaction calculated from the simulation approach seems higher than the results obtained from the explicit formulas.

Table 4.3 Estimated expected present value of total transaction costs and percentiles of distribution at $k = 0.5\%$, based on hedging volatility σ.

Maturity T	Revision	TC	95th percentile	99th percentile
$T = 5$	Monthly	1.4466	2.4343	2.7715
	Biweekly	2.0151	3.3984	3.7722
	Weekly	2.8574	4.7432	5.2546
$T = 10$	Monthly	2.3344	4.0007	4.5184
	Biweekly	3.2702	5.5777	6.2695
	Weekly	4.6253	7.8955	8.7705
$T = 15$	Monthly	3.2962	5.7103	6.4329
	Biweekly	4.6607	7.9965	9.0076
	Weekly	6.5826	11.2347	12.6094
$T = 20$	Monthly	4.4629	7.7854	8.8135
	Biweekly	6.2711	10.9339	12.4337
	Weekly	8.8181	15.3950	17.3999

Table 4.4 Estimated expected present value of total transaction costs and percentiles of distribution at $k = 0.5\%$, based on Leland's hedging volatility $\bar{\sigma}$.

Maturity T	Revision	TC	95th percentile	99th percentile
$T = 5$	Monthly	1.3989	2.3188	2.6552
	Biweekly	1.9526	3.1920	3.5495
	Weekly	2.6670	4.3163	4.7979
$T = 10$	Monthly	2.2903	3.8294	4.2708
	Biweekly	3.1525	5.2365	5.8330
	Weekly	4.3689	7.1449	7.9827
$T = 15$	Monthly	3.1916	5.4440	6.1847
	Biweekly	4.4439	7.5071	8.4215
	Weekly	6.1558	10.2783	11.5360
$T = 20$	Monthly	4.3009	7.4111	8.4479
	Biweekly	6.0045	10.2435	11.4828
	Weekly	8.3152	14.0624	15.7021

By using the same parameters as in Table 4.1 and 4.2, we calculated the expected total hedging costs of quantile hedging from simulation approach with hedging volatility σ and adjusted volatility $\overline{\sigma}$ separately. The results are shown in Table 4.5 and 4.6. For hedging with the adjusted volatility $\overline{\sigma}$, the difference between hedging errors and transaction costs from simulation method is lower than the values obtained from explicit formulas overall. Although the values of the total costs from simulation method are slightly higher than the ones obtained from formulas, these values such as hedging errors, the difference of hedging errors and the transaction costs, and the total costs all have the similar trends across different maturities and revision periods for both approaches.

Table 4.5 Expected total hedging costs at $k = 0.5\%$, based on hedging with volatility $\sigma = 0.2$.

Maturity T	Revision	HE	HE-TC	Total Costs
	Monthly	0.2214	-1.2252	12.0730
$T = 5$	Biweekly	0.3925	-1.6226	12.4704
	Weekly	0.3208	-2.5366	13.3844
	Monthly	-0.0624	-2.3968	14.5811
$T = 10$	Biweekly	0.0327	-3.2375	15.4218
	Weekly	0.1449	-4.4804	16.6647
	Monthly	-0.3394	-3.6356	16.2994
$T = 15$	Biweekly	-0.0453	-4.7060	17.3698
	Weekly	0.0250	-6.5576	19.2214
	Monthly	-0.3741	-4.8370	17.6222
$T = 20$	Biweekly	-0.1918	-6.4629	19.2481
	Weekly	0.0086	-8.8095	21.5947

Table 4.6 Expected total hedging costs at $k = 0.5\%$, based on Leland's adjusted volatility $\bar{\sigma}$.

Maturity T	Revision	HE	HE-TC	Total Costs
$T = 5$	Monthly	1.5884	0.1895	11.8865
	Biweekly	2.3383	0.3857	12.1798
	Weekly	3.1288	0.4618	12.7779
$T = 10$	Monthly	2.3246	0.0343	14.0405
	Biweekly	3.2582	0.1057	14.8764
	Weekly	4.6603	0.2914	15.8559
$T = 15$	Monthly	3.0132	-0.1784	15.5386
	Biweekly	4.4329	-0.0110	16.9374
	Weekly	6.2072	0.0514	17.9885
$T = 20$	Monthly	4.0375	-0.2634	16.9403
	Biweekly	5.9391	-0.0654	18.9377
	Weekly	8.2594	-0.0558	20.3281

4.5.2 Quantile hedging costs of maturity guarantees

To have a possibility to compare our findings with the results in Boyle & Hardy (1997), and Hardy (2000), we also discuss the costs of quantile hedging on European put option inclusive transaction costs.

According to Brennan and Schwartz (1976), the benefit of a single premium equity-linked life insurance contract can alternatively be written as:

$$H_T = S_T + \max\left(K - S_T, 0\right) \tag{4.37}$$

(4.37) shows the benefit of the contract is the sum of the value of reference portfolio at maturity and the value of an European put option on the reference portfolio with the strike price as same as the contract's guarantee K. Therefore, the maturity guarantee premium charged by the insurance company can be viewed in terms of the hedging costs on put options.

As mentioned previously, we know that a financial insurance market is incomplete.

As a result, quantile hedging can be considered as one of the appropriate imperfect hedging techniques to price the guarantee. Next, we will introduce the quantile price for European put option without transaction costs.

Theorem 4.6: Consider a European put option with strike price K and maturity T on risky asset S_t from the model (4.2). If the underlying risky asset S_t satisfies the condition $\mu - r - \sigma^2 > 0$, the quantile price for the put option is

$$\begin{aligned} X_0 &= Ke^{-rT}\left[\Psi\left(d_0\right)-\Psi\left(b/\sqrt{T}\right)\right] \\ &-S_0\left[\Psi\left(\sigma\sqrt{T}-b/\sqrt{T}\right)-\Psi\left(\sigma\sqrt{T}-d_0\right)\right] \end{aligned} \tag{4.38}$$

where $d_0 = \dfrac{\ln\dfrac{K}{S_0} - \left(r - \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, b is a constant.

Proof: The quantile price of the put option X_0 can be calculated as

$$X_0 = E^*\left(H_T \cdot e^{-rT} I_{A^*}\right) \tag{4.39}$$

We can take some transformation on the structure of maximal set of success hedging A^*

$$\begin{aligned} A^* &= \left\{Z_T^{-1} > a^* H_T e^{-rT}\right\} \\ &= \left\{\exp\left(\theta W_T^* - \frac{\theta^2}{2}T\right) > a^*\left(K - S_T\right)^+ e^{-rT}\right\} \\ &= \left\{\exp\left\{\frac{\mu - r}{\sigma^2}\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*\right)\right\}\right. \\ &\left. * \exp\left\{-\frac{\mu - r}{\sigma^2}\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T\right) - \frac{1}{2}\left(\frac{\mu - r}{\sigma^2}\right)^2 T + rT\right\} > a^*\left(K - S_T\right)^+\right\} \\ &= \left\{S_T^{\frac{\mu - r}{\sigma^2}} \exp\left\{-\frac{\mu - r}{\sigma^2}\left(\ln S_0 + \frac{\mu + r - \sigma^2}{2}T\right) + rT\right\} > a^*\left(K - S_T\right)^+\right\} \end{aligned}$$

(4.40)

Considering the case $\mu - r - \sigma^2 > 0$, A^* can be written in the following form for some constants b and d :

$$\begin{aligned} A^* &= \{\omega : S_T > d\} \\ &= \left\{\omega : S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*\right\} > d\right\} \\ &= \{\omega : W_T^* > b\} \end{aligned} \tag{4.41}$$

Therefore, the quantile price for the European put option can be calculated as

$$\begin{aligned} X_0 &= E^*\left(H_T \cdot e^{-rT} I_{A^*}\right) \\ &= e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{b/\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}} \left(K - S_T\right)^+ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{b/\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}} \left(Ke^{-rT} - S_0 \exp\left\{\sigma\sqrt{T} \cdot y - \frac{\sigma^2}{2}T\right\}\right)^+ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{b/\sqrt{T}}^{d_0} e^{-\frac{y^2}{2}} \left(Ke^{-rT} - S_0 \exp\left\{\sigma\sqrt{T} \cdot y - \frac{\sigma^2}{2}T\right\}\right) dy \\ &= Ke^{-rT} \int_{b/\sqrt{T}}^{d_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - S_0 \int_{b/\sqrt{T}}^{d_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(y - \sigma\sqrt{T}\right)^2}{2}} dy \\ &= Ke^{-rT}\left[\Psi(d_0) - \Psi\left(b/\sqrt{T}\right)\right] - S_0\left[\Psi\left(\sigma\sqrt{T} - b/\sqrt{T}\right) - \Psi\left(\sigma\sqrt{T} - d_0\right)\right] \end{aligned} \tag{4.42}$$

where $d_0 = \dfrac{\ln\dfrac{K}{S_0} - \left(r - \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$.

Remark 4.3: Solving the problem of quantile hedging, we can choose some acceptable financial risk level ε first. Then, we obtain constant b from the maximal probability of successful hedging:

$$P(A) = 1 - \varepsilon = 1 - \Psi\left(\frac{b - \frac{\mu - r}{\sigma}T}{\sqrt{T}}\right) \tag{4.43}$$

Following the simulation method introduced in Section 4.5.1, we estimate the expected transaction costs and the expected total costs for contracts maturity guarantee from quantile hedging. Table 4.7 includes the estimated expected present value of total transaction costs and the percentiles of distribution at one way transaction rate $k = 0.5\%$. They are calculated based on hedging volatility $\sigma = 0.2$ and Leland's adjusted hedging volatility $\bar{\sigma}$ separately. The results in brackets are calculated from volatility σ. We fix the financial risk $\varepsilon = 0.025$ and select a constant maturity guarantee $K = 100$ as an example. The estimated transaction costs and the corresponding percentiles shows the same changing patterns as the results in Boyle and Hardy (1997), especially on the same rebalancing basis. The expected transaction costs of hedging put options for 5 years are greater than the costs for 10 years. However, for both hedging volatility σ and $\bar{\sigma}$, the expected transaction costs generated from quantile hedging appears less than the values in Boyle and Hardy (1997) which calculated from perfect hedging based on the Black-Scholes formula.

Table 4.7 Estimated expected present value of total transaction costs and percentiles of distribution at $k = 0.5\%$.

Maturity T (Years)	Revision Period	Expected Total TC	95th percentile	99th percentile
$T = 5$	Monthly	0.6407 (0.6631)	1.3573 (1.4066)	1.4828 (1.5335)
	Biweekly	0.8896 (0.9145)	1.8279 (1.9247)	2.0322 (2.2370)
	Weekly	1.2446 (1.2710)	2.4086 (2.6806)	2.8184 (3.0733)
$T = 10$	Monthly	0.5450 (0.5929)	1.3370 (1.3552)	1.4793 (1.5332)
	Biweekly	0.7246 (0.7409)	1.7519 (1.8382)	1.9824 (2.2027)
	Weekly	0.9997 (1.0162)	2.2276 (2.3736)	2.6160 (2.7720)

The Black-Scholes Put option price are \$5.6968 for $T = 5$ and \$4.1685 for $T = 10$. Table 4.8 shows the expected total hedging costs for maturity guarantee at $k = 0.5\%$, based on hedging with volatility $\sigma = 0.2$. Table 4.9 shows expected total hedging costs of the maturity guarantee at k=0.5%, based on Leland's hedging with volatility $\bar{\sigma}$. We observe that with more frequent hedging, the estimated total hedging errors show a decreasing trend in absolute values while the estimated transaction costs increase. This observation consists with the conclusions in Hardy (2000). As expected, the estimated total costs of quantile hedging calculated from both σ and $\bar{\sigma}$ on contract's maturity guarantee are less than the total costs in Boyle & Hardy (1997), because of existence of 2.5% financial risk during quantile hedging. Although the cash flows from quantile hedging using adjusted volatility $\bar{\sigma}$ is slightly less than those from hedging with volatility σ, the total costs based on $\bar{\sigma}$ is higher than the values based on σ. This is mainly caused by the relatively large difference between quantile prices shown in Table 4.8 and 4.9. It is noted that quantile price calculated from volatility σ in Table 4.8 is the price without transaction costs. However, the values obtained from adjusted volatility $\bar{\sigma}$ in Table 4.9 are inclusive of the transaction costs.

Table 4.8 Expected total hedging costs of the maturity guarantee at $k = 0.5\%$, based on hedging with volatility σ.

Maturity	Revision	Quantile	HE	HE-TC	Total Costs
$T = 5$	Monthly	2.0547	-2.8614	-3.5245	5.5792
	Biweekly	2.0547	-2.8003	-3.7148	5.7695
	Weekly	2.0547	-2.7407	-4.0117	6.0664
$T = 10$	Monthly	0.2378	-2.6927	-3.2856	3.5234
	Biweekly	0.2378	-1.8501	-2.5910	2.8288
	Weekly	0.2378	-1.6186	-2.6348	2.8726

Table 4.9 Expected total hedging costs of the maturity guarantee at $k = 0.5\%$, based on Leland's adjusted hedging volatility $\bar{\sigma}$.

Maturity	Revision	Quantile	HE	HE-TC	Total Costs
	Monthly	2.7792	-2.4843	-3.1250	5.9042
$T = 5$	Biweekly	3.0799	-2.2156	-3.1052	6.1851
	Weekly	3.5038	-1.9547	-3.1993	6.7031
	Monthly	0.6801	-1.2257	-1.7707	2.4508
$T = 10$	Biweekly	0.9019	-1.2070	-1.9316	2.8335
	Weekly	1.2418	-1.1118	-2.1115	3.3533

5. Conclusions and future directions

In this thesis, we analyzed the problem of valuation of imperfect hedging techniques on equity-linked life insurance contracts with market restrictions on interest rate and transaction costs. We mainly focused on quantile hedging and efficient hedging approaches.

First, quantile hedging on equity-linked life insurance was discussed. We generalized the results in Melnikov & Skornyakova (2005) by assuming the interest rate is stochastic and follows the HJM framework. We found that the implied survival probability ${}_T p_x$ obtained in a stochastic interest rate environment shows a decreasing trend with longer contract maturities, while the one with constant interest rate is increasing instead. Along with this result, the insurance company may have potential clients with different age levels. Furthermore, we examined the effect of a stochastic interest rate on efficient hedging of equity-linked life insurance. The price of a zero-coupon bond was selected as the numeraire to reduce the complicated calculation when using term-structure models. The risky assets models with the correlation of two Wiener processes $\rho = 1$ and $\rho < 1$ were considered separately. The numerical examples illustrate that the stochastic interest rate has great impact on overall patterns of survival probabilities across different financial risk levels. It is also implied that increasing sources of randomness in the valuation model can affect the trends of survival probability and clients' age dramatically.

Finally, it comes to the transaction costs factor. We consider the case of rebalancing the portfolio with less extreme trading frequencies, such as monthly, biweekly and weekly, which is practical for insurance companies. Our results of expected present value of total transaction costs, hedging errors are derived with Leland's transaction costs adjusted hedging volatility $\bar{\sigma}$, which is used to construct the replicating portfolio. The numerical examples illustrate that the hedging errors generated from Leland's volatility $\overline{\sigma}$ almost offset the transaction costs. Besides, the cash flow from hedging with adjusted volatility $\overline{\sigma}$ at future rebalancing points is less than the one from hedging with volatility σ . These findings are also confirmed by the estimated results obtained from a simulation approach.

For future studies, there are several interesting directions worth exploring. First, more general jump-diffusion models can be considered to capture the trajectories of risky assets. In this thesis, we use the two-factor jump-diffusion models, which assume the jump sizes in stock price are constants. There are some more general jump-diffusion models, which can better describe the characteristics of risky assets; for instance, a model with compound Poisson processes can be utilized to reflect the random jump sizes. Some theoretical results are obtained for quantile hedging based on models with compound Poisson processes in Tong & Melnikov (2013). However, more research work related with efficient hedging and mean-variance hedging could be considered as a future direction.

Another natural extension is to investigate other types of equity-linked life insurance products. In this thesis, we work with a single premium equity-linked life insurance contract, where the benefit is paid only as the insured can survive until maturity. Other more complicated and flexible equity-linked insurance products can be considered, for instance, segregated funds and variable annuities, which can provide not only minimum maturity benefit guarantee but also minimum death benefit guarantee. Sometimes, such contracts are usually combined with periodic premiums, which allow the policyholders to regularly pay a predetermined premium to the insurance company compared with single premium case. Such extensions would illustrate the use of quantile and efficient hedging in a more realistic setting appealing to the practitioners in insurance fields

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Appendix 1: Proof of Theorem 2.1.

We will now calculate X_0 from Multi-Asset Theorem in Melnikov & Romaniuk (2008) to get the expression of X_0. Because we only have two risky assets in our setting, we will use the multi-asset theorem for $n=2$ indicators. Let us review the multi-asset theorem first with the same notations as in Melnikov & Romaniuk (2008).

Multi-Asset Theorem: Let $x_i \sim N\left(\mu_i, \sigma_i^2\right)$, $i=1,...,n$ and $z \sim N\left(\mu_z, \sigma_z^2\right)$ be $n+1$ normally distributed random variables with a variance-covariance matrix R_{n+1} given by

$$R_{n+1} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1\sigma_z\rho_{1z} \\ \vdots & \ddots & \vdots \\ \sigma_1\sigma_z\rho_{1z} & \cdots & \sigma_z^2 \end{bmatrix}$$

Then, for some given constraints X_i,

$$E\left(e^{-z} I\left\{x_1 < X_1\right\} \cdots I\left\{x_n < X_n\right\}\right) = e^{-\left(\mu_z - \frac{\sigma_z^2}{2}\right)} \cdot \Psi^n\left(\hat{X}_1, \ldots, \hat{X}_n\right)$$

$$\hat{X}_i = \frac{X_i - \mu_i}{\sigma_i} + \sigma_z\rho_{iz}$$

Similarly as the proof of Theorem 2.1, we can decompose X_0 into X_0^1 plus X_0^2. Let us calculate X_0^1 first. On each set $\left\{\Pi_T = n\right\}, n = 1, 2...,$ X_0^1 has the expression:

$$X_0^1 = E^*\left[\frac{S_T^1}{B_T} I\left\{1 \geq a_n Z_T \frac{S_T^1}{B_T}\right\} \cdot I\left\{S_T^1 > S_T^2\right\}\right] \tag{A1.1}$$

Let us define the random variables $x_1 = \int_0^T \left(\phi_s + \sigma_1\right) d\tilde{W}_s$, $x_2 = \left(\sigma_2 - \sigma_1\right)\tilde{W}_T$, and

$\alpha = -\sigma_1 \tilde{W}_T - \int_0^T \left(v_1 \lambda_s^* - \frac{1}{2}\sigma_1^2 \right) ds$, where α , x_1 , x_2 satisfy the following normal distribution:

$$\alpha \sim N\left(-\int_0^T \left(v_1 \lambda_s^* - \frac{1}{2}\sigma_1^2 \right) ds, \sigma_1^2 T \right) \tag{A1.2}$$

$$x_1 \sim N\left(0, \int_0^T \left(\phi_s + \sigma_1\right)^2 ds\right), \quad x_2 \sim N\left(0, \left(\sigma_2 - \sigma_1\right) T\right). \tag{A1.3}$$

We can also define random variables X_1, X_2 as

$$X_1 = -\ln a_n S_0^1 - n \ln \frac{\lambda_T^* \left(1 - v_1\right)}{\lambda} + \int_0^T \left[\frac{1}{2}\left(\sigma_1^2 - \phi_s^2\right) + \left(1 - v_2\right) \lambda_s^* - \lambda \right] ds \tag{A1.4}$$

$$X_2 = \ln \frac{S_0^1}{S_0^2} + n \ln \frac{\left(1 - v_1\right)}{\left(1 - v_2\right)} + \int_0^T \left[\frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right) + \left(v_1 - v_2\right) \lambda_s^* \right] ds \tag{A1.5}$$

Applying Multi-Asset Theorem for $n = 2$, we can obtain

$$\begin{aligned} X_0^1 &= S_0^1 \left(1 - v_1\right)^n E^* \left[e^{-\alpha} I_{\{x_1 < X_1\}} \cdot I_{\{x_2 < X_2\}} \right] \\ &= S_0^1 \left(1 - v_1\right)^n e^{\int_0^T v_1 \lambda_s^* ds} \Psi\left(\widehat{X}_1, \widehat{X}_2\right) \end{aligned} \tag{A1.6}$$

where $\widehat{X}_1 = \dfrac{X_1 - \int_0^T \sigma_1 \left(\phi_s + \sigma_1\right) ds}{\bar{\sigma}_1}$, $\widehat{X}_2 = \dfrac{X_2}{\bar{\sigma}_2} + \dfrac{\sigma_1}{\sqrt{T}}$.

We treat X_0^2 in a similar way and define the random variables $x_1^{'} = \int_0^T \left(\phi_s + \sigma_2\right) d\tilde{W}_s$, $\alpha^{'} = -\sigma_2 \tilde{W}_T - \int_0^T \left(v_2 \lambda_s^* - \frac{1}{2}\sigma_2^2 \right) ds$ and $x_2^{'} = \left(\sigma_1 - \sigma_2\right) \tilde{W}_T$,

where $\alpha^{'} \sim N\left(-\int_0^T \left(v_2 \lambda_s^* - \frac{1}{2}\sigma_2^2 \right) ds, \sigma_2^2 T \right)$, $x_1 \sim N\left(0, \int_0^T \left(\phi_s + \sigma_2\right)^2 ds\right)$, and $x_2 \sim N\left(0, \left(\sigma_2 - \sigma_1\right)^2 T\right)$.

Let us define other random variables $X_1^{'}, X_2^{'}$ as

$$X_1^{'} = -\ln a_n S_0^2 - n\ln\frac{\lambda_T^*\left(1-v_2\right)}{\lambda} + \int_0^T \left[\frac{1}{2}\left(\sigma_2^2 - \phi_s^2\right) + \left(1-v_2\right)\lambda_s^* - \lambda\right] ds \quad \text{(A1.7)}$$

$$X_2^{'} = \ln\frac{S_0^2}{S_0^1} + n\ln\frac{\left(1-v_2\right)}{\left(1-v_1\right)} + \int_0^T \left[\frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right) + \left(v_2 - v_1\right)\lambda_s^*\right] ds \quad \text{(A1.8)}$$

We use the Multi-Asset Theorem again and obtain

$$\begin{aligned} X_0^2 &= S_0^2\left(1-v_2\right)^n E^*\left[e^{-\alpha^{'}} I_{\left\{x_1^{'} < X_1^{'}\right\}} \cdot I_{\left\{x_2^{'} < X_2^{'}\right\}}\right] \\ &= S_0^2\left(1-v_2\right)^n e^{\int_0^T v_2\lambda_s^* ds}\,\Psi\left(\widehat{X}_1^{'}, \widehat{X}_2^{'}\right) \end{aligned} \quad \text{(A1.9)}$$

where $\widehat{X}_1^{'} = \dfrac{X_1^{'} - \int_0^T \sigma_2\left(\phi_s + \sigma_2\right) ds}{\bar{\sigma}_1^{'}}$, $\widehat{X}_2^{'} = \dfrac{X_2^{'}}{\bar{\sigma}_2^{'}} - \dfrac{\sigma_2}{\sqrt{T}}$.

(A1.6) and (A1.9) lead to the expression of X_0 as

$$X_0 = \sum_{n=0}^{\infty} p_{n,T}^*\left[S_0^1\left(1-v_1\right)^n e^{\int_0^T v_1\lambda_t^* dt}\,\Psi\left(\widehat{X}_1, \widehat{X}_2\right) + S_0^2\left(1-v_2\right)^n e^{\int_0^T v_2\lambda_t^* dt}\,\Psi\left(\widehat{X}_1^{'}, \widehat{X}_2^{'}\right)\right]$$

(A1.10)

Appendix 2: Useful formulas

Here we list some formulas in Toft (1996) which are used to prove the Theorem 4.2~4.5. In the following list of formulas, $\Psi(\cdot)$ and $\Psi(\cdot,\cdot,\cdot)$ denote the univariate and bivariate standard normal distribution function respectively.

Formula (49): For some constants A, B, C, D, E, we have

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Psi\left(Dx+E\right)dx$$
$$=\frac{\sqrt{2\pi}}{\sqrt{A}}\exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right)\Psi\left(\sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right)\right)$$

(A2.1)

Formula (50): For some constants A, B, C, D, E, we have

$$\int_{-\infty}^{\bar{x}} \exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Psi\left(Dx+E\right)dx$$
$$=\frac{\sqrt{2\pi}}{\sqrt{A}}\exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right)\Psi\left(\bar{x}\sqrt{A}+\frac{B}{2\sqrt{A}},\sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right),-\frac{D}{\sqrt{A+D^2}}\right)$$

(A2.2)

Formula (51): For some constants $A, B, C, D, E, F, G, \rho$, we have

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Psi\left(Dx+E, Fx+G, \rho\right)dx$$
$$=\frac{\sqrt{2\pi}}{\sqrt{A}}\exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right)\Psi\left(\sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right),\right. \qquad \text{(A2.3)}$$
$$\left.\sqrt{\frac{A}{A+F^2}}\left(G-\frac{BF}{2A}\right),-\frac{DF+A\rho}{\sqrt{\left(A+D^2\right)\left(A+F^2\right)}}\right)$$

In case of zero correlation, formula (51) turns to:

Formula (52):

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Psi\left(Dx+E\right)\Psi\left(Fx+G\right)dx$$
$$=\frac{\sqrt{2\pi}}{\sqrt{A}}\exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right)\Psi\left(\sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right),\right. \tag{A2.4}$$
$$\left.\sqrt{\frac{A}{A+F^2}}\left(G-\frac{BF}{2A}\right),-\frac{DF}{\sqrt{\left(A+D^2\right)\left(A+F^2\right)}}\right)$$