UNIVERSITY OF ALBERTA

LIVING ONE'S PHILOSOPHY OF MATHEMATICS: GRAPHING CALCULATORS IN HIGH SCHOOL MATHEMATICS

BY

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of MASTER OF EDUCATION.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled LIVING ONE'S PHILOSOPHY OF MATHEMATICS: IMPLEMENTING GRAPHING CALCULATORS IN HIGH SCHOOL MATHEMATICS submitted by ELAINE SIMMT in partial fulfillment of the requirements for the degree of MASTER OF EDUCATION.

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ABSTRACT

In this study I explored teachers' expressed and manifested philosophies of mathematics and mathematics education as these philosophies were articulated and worked out in the context of the teachers making decisions for utilizing graphing calculators in their instruction of high school mathematics. Six teachers were observed while using graphing calculators to teach lessons on the quadratic function. Then they were interviewed regarding their beliefs, attitudes, and conceptions of mathematics and mathematics education. The data, my analysis, and my interpretations were then worked into cases about each of the teachers. Each case study consists of a vignette from one of the teacher's lessons, a discussion of how the teacher used the graphing calculators, an exposition of the teacher's espoused philosophies of mathematics and mathematics education and my interpretative analysis of the teacher's espoused philosophies as they were manifested in the teacher's instruction of mathematics and interactions with students.

Graphing calculators were used primarily as a device to provide graphical images from which the students were expected to observe and make generalizations about transformations of the quadratic function. The calculators were also used as a means of verifying student work. In only a couple of classes, did teachers encourage students to do graphical analyses of application problems. For the most part, teachers did not use the graphing calculators to enable students to investigate more open-ended questions.

The teachers' philosophies of mathematics and mathematics education varied considerably. The beliefs, attitudes, and conceptions of mathematics and mathematics education which contributed to the teachers' philosophies, were manifested in the sum of the teachers' experiences in class with students. Their choices for activities with graphing calculators, their questioning, and their interactions with students all reflected aspects of their philosophies of mathematics and mathematics education.

The teachers in this study all taught the same curriculum and all of them were faced with the same perturbation to their instruction (the availability of graphing calculators); however, each of the teachers brought-forth the curriculum within the context of their personal philosophies of mathematics and mathematics education.

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LIST OF ABBREVIATIONS

Math 10	- Mathematics 10
Math 20	- Mathematics 20
Math 30	- Mathematics 30
Math 33	- Mathematics 33
Math 31	- Mathematics 31
Max/min	- Maximum/minimum
NCTM	- National Council of Teachers of Mathematics

CHAPTER I

Distinguishing a Focus for Study

Introduction

Why is it that, given the same content, teachers teach so differently? While teaching high school mathematics in a school with five other math teachers, I realized that our approaches to instruction often varied dramatically. This realization grew out of sharing strategies, ideas, worksheets, and tests, when more often than not, my colleagues contributions had to be modified or rewritten to be useful to me. We had various approaches to teaching mathematics; some teachers lectured and had students do exercises as independent seat-work; others had students engaged in activities with manipulatives; still others provided students with worksheets that guided the students through the mathematics. We all were responsible for the same curriculum, but we all implemented that curriculum differently.

Research provides few clues as to why we teach so differently. Some research suggests that teachers have varying levels of comfort with the content because of their knowledge of mathematics (Thompson, 1984); have different theories on how children learn (Romberg and Carpenter, 1986); have different views on what students need to know (Anyon, 1981); have different perspectives on the nature of mathematics (Lerman, 1990); and have different ways of motivating students (Blaire, 1981). Much of the research on instruction has focused on the behaviours teachers exhibit in the classroom; little research has been done to try to examine teachers' beliefs or the assumptions underlying mathematics instruction (Romberg and Carpenter, 1986). Obviously we design our lessons based on our comfort level with the content, our beliefs about how children learn and what they should learn, and the demands of the particular class we are teaching. I think there are other reasons. McBride (1989) suggests that:

> Mathematics education, regardless of the approach, is based on commonly held views of mathematics philosophy. For example, if teachers view mathematics as objective and without emotion, their teaching practices will reflect their ideology. Beliefs formed about the academic study of mathematics affect mathematics pedagogy. Therefore to describe a completely different view of mathematics education would involve reconceptualizing new theories to comprise an adequate philosophy of mathematics. (p. 46)

Or as Thom (1973) so succinctly puts it, "all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204).

With the call for reform in mathematics education (NCTM, 1989), the emergence of new learning theories based on constructivism (see von Glasersfeld, 1984, 1987; Pirie and Kieren, 1990) and the availability of new and inexpensive technology there is a need to further investigate the beliefs that underlie the teacher's approach to instruction. In looking back to the "modern mathematics" reform of the 1960s, Steffe (1992) observed:

> The old mathematics curriculum was replaced by the new curriculum in the form of new school mathematics textbooks, and institutes were held for mathematics teachers to prepare them to teach the concepts and principles of the new curriculum. But little attention was given, in either the institutes or the new textbooks, to the nature of mathematical knowledge that would influence the practice of mathematics education. As a consequence, teachers simply taught modern mathematics in the same way that they had taught the old mathematics... most of them knew little about a philosophy of mathematics education, nor had they an adequate understanding of what the modern mathematics movement was all about. (p.446)

Prawat (1992) claims that the present reform movement (led by the constructivists) is very different than the one of the 1960s because of the way those in the present movement view knowledge. He warns that many of the implications of constructivism have not yet been "spelled out" (p. 360). Much of the focus in constructivism has been on developing learning theories and not much has been said about teaching. He explains the discrepancy this way:

> At this point, these views [constructivists views of teaching] are considerably less developed than are constructivist views of learning. This partly reflects the fact that researchers in the two domains are pursuing somewhat different agendas. Learning theories tend to be descriptive, theories of instruction prescriptive; as a result, one cannot directly inform the other.... Most of the problems with implementing a constructivist approach to teaching could be overcome if teachers were willing to rethink not only what it means to know subject matter, but also what it takes foster this sort of understanding in students. This is a tall order. Such change is unlikely to occur without a good deal of discussion and reflection on the part of teachers. Identifying what is problematic about existing beliefs, however, is an important first step in the change process. (p. 360 - 361)

I would suggest locating and exposing those beliefs is the first step in identifying what is problematic with them. The intention of this study was to do just that.

These observations and the needs suggested above, indicate to me the value of studying the nature of teachers' philosophies of mathematics, particularly the nature of these philosophies as they are manifested in teaching practices. **In order to focus this rather broad area of study, I am looking at both expressed and**

manifested philosophies of teachers, as they are articulated and worked out against the background of teachers coming to use new technology (graphing calculators) in their teaching.

Teachers have beliefs about the nature of mathematics and how children come to learn mathematics. Some teachers are very aware of their personal philosophy of mathematics and philosophy of learning mathematics and other teachers are not very aware of their personal philosophies. It is my contention that, regardless of whether or not the teacher's philosophical perspectives are implicit or explicit, these perspectives strongly influence his or her instructional strategies. If perspectives do influence instruction, then, as Thompson (1984) points out, this is an important area to try and understand.

> If teachers' characteristic patterns of behaviour are indeed a function of their views, beliefs, and preferences about the subject matter and its teaching, then any attempt to improve the quality of mathematics teaching must begin with an understanding of the conceptions held by teachers and how these are related to their instructional practice (p. 106).

I believe we must look for the underlying assumptions and beliefs that teachers have if we are to begin to understand why teachers teach the way they do.

An Introduction to the Problem in this Study

Mathematics educators, teachers, and the public once thought that computers would strongly influence mathematics instruction (Howson and Kahane, 1986). This is evident by the vast amount of research that was done with microcomputers[1], the amount of material on computer use in mathematics that was published in teaching magazines[2], and the number of articles in the popular press[3]. But, as Barrett and Goebel (1990) suggest, research has shown that microcomputers have not had the impact in mathematics that was predicted. Two reasons are cited: (1) the computers are not readily available in most classrooms on an everyday or easily accessible basis and (2) the teachers of mathematics have not defined the role of the microcomputer in the mathematics classroom.

[1]The research began to appear in the scholarly journals, such as the *Journal for Research in Mathematics*, in the early 1970s and continues to be reported in these journals.
[2]*The Mathematics Teacher* and *The Arithmetic Teacher* are two examples.
[3]Articles began to appear in the popular press in the early 1980s. Magazines such as *Better Homes and Gardens*, *Parents*, *Time*, *Psychology Today* and daily newspaper have featured such articles

The first problem has been rectified as a result of the introduction of inexpensive graphing calculators. Graphing calculators, which are a form of a hand-held computer, are now widely available and senior high school mathematics departments are buying them by the class set. The mathematics curriculum in this province has been rewritten to include the use of this technology in senior high school mathematics and the use of the graphing calculators is permitted for the provincial examinations in mathematics, chemistry, and physics. Once again mathematics educators are predicting "enormous changes in both fundamental school mathematics and fundamental ways of doing mathematics" (Shumway, 1990, p. 2). But will availability alone result in *enormous* changes? Howson and Kahane (1986) point out to us that:

> Mathematics and mathematics teachers have been provided with a new tool, the computer.... But, like all tools, the computer by itself does not supply the solution to our problems, not least to the problems of mathematics education. There is no automatic beneficial effect linked to a computer: the mere provision of micros in a class or lecture room will not solve teaching problems (p. 24)

We must consider Barrett's second point and address Howson's and Kahane's concern then ask what role will teachers define for this new technology. The curriculum changes mandated by governments must be made by teachers. (I am interested in the curricular decisions that teachers make and their reasons for making those decisions.) Toomey (1977) suggests teachers interpret the curriculum and make decisions for their instruction of mathematics based on a variety of constraints.

> The operational curriculum [is] being shaped by a variety of external constraints. Teachers' perceptions about how learning occurs, exigencies of the classroom situation and influences of other participants in the curriculum planning process are but some of the constraints that are blended together in ways that are not well understood. (p. 122-123)

My intention is to use the opportunity provided by the introduction of graphing calculators, in high school mathematics, to examine some of the underlying assumptions teachers have about mathematics and mathematics education and to explore how teachers define the role of this new technology.

Insights from the Literature

A number of constraints underpin teachers' instructional practices, as Toomey suggested, but my study is focused on teachers' philosophies of mathematics and mathematics education. I have chosen to include, in these areas, the role of technology, the nature of mathematical knowledge, what it means to know mathematics, and what constitutes appropriate instruction. I will begin by reviewing the literature in the field to try to establish the nature of our present understanding on these issues. Then I will present six case studies and try to demonstrate how one's philosophy of mathematics and mathematics education is manifested in one's instruction of mathematics.

On Technology

Computers have been available on a more or less limited basis in schools since the late 1960s. Graphing calculators are much newer. Most schools in the local area purchased the graphing calculators late in 1990 or early in 1991. There was some use of the graphing calculators prior to that date, in isolated cases, in the United States (Demana and Waits, 1990) and Great Britain (Ruthven, 1990). Therefore the research involving graphing calculators is extremely limited. However there is abundant, albeit narrowly focused, research on computers in secondary school mathematics. When I use the word technology then, in this paper, it refers to both computers and graphing calculators.

Studies involving calculators and computers tend to be focused on student learning[4] (Ruthven, 1990; Heid, Sheets, Matras, 1990; Lesh, 1987; Wright, 1989; Stuessy and Rowland, 1989; Hatfield and Kieren, 1972). Discussion about the use of computers and graphing calculators in mathematics is centred around how the tools can best be used to enhance such learning (Leinhardt, Zaslvsky and Stein, 1990; Demana and Waits, 1990; Shumway, 1990). These discussions appear to assume that teachers welcome the technology and are going to use it extensively to provide rich and meaningful learning experiences for mathematics students. Thus the findings from research and the projections of the theorists appear to be contingent on the provision of appropriate implementation of technology in mathematics classrooms.

Howson and Kahane (1986) suggest two distinct ways the teacher can choose to use computers in the mathematics classroom. The first is to use the computer as an "aid for the teacher." The second is to "allow and expect the students to interact with

[4]Many studies could be cited here. Where possible I will only cite research studies that have had an algebraic focus (specifically studies on functions if available).

the computer" (p.27). Some of the ways in which students can interact with computers include using computers to check student answers; to provide drill and practice routines; to provide micro-environments for discovery learning; and to provide students with unlimited computational power (Bialo and Erickson, 1985).

There is much research concentrated at the early and middle school level on using the computer to provide micro-environments (for example with logo). Other studies like Hatfield and Kieren (1972) and Lesh (1987) have developed micro-environments for the study of algebra. Studies involving the use of computers to provide unlimited computational power are not common; however, if we consider the studies that have been done with calculators and the computational power these provide, then here too you can find a number of studies that have examined early school and middle school students' learning.

In spite of the research and its findings, technology has not had the impact on mathematics instruction that was anticipated. Is it possible that the technology is not the proper focus of all the attention by mathematics educators and researchers? Computers have been shown to be effective when used in specific ways (Hatfield and Kieren, 1972; Lesh, 1987). But researchers have failed to take a careful look at at the assumptions that go along with such uses of the technology. Many of the studies have been clinical in nature or have been teaching experiments and they have not examined the affect mathematics teachers have on the effectiveness of computer assisted instruction.

So often in educational research, studies about learning are conducted without fully examining the consequences for teaching. (Even when this is done, we still fail to address, what I believe to be the, issues at the heart of instruction.) What are the teachers' beliefs, attitudes, and conceptions that influence their instruction and how might these beliefs, attitudes, and conceptions change the carefully researched instructional strategies for teaching mathematics? It seems to me that we must probe into the teacher's selection of tasks when we examine the usefulness of technology in mathematics education. We must also consider the question "Why do teachers select the tasks they do?" It is not enough to look at this situation as a dichotomy between using technology and not using technology in the classroom; nor is it enough to simply look at the behaviours and choices displayed by the teachers. We must note the choices the teachers make for their instruction of mathematics and then dig deeper to determine how and why the teachers made the choices they do. Once we have examined these underlying reasons for the teachers' actions only then might we be able to improve mathematics instruction or, at the very least, set the wheels of reform into motion.

On the Nature of Mathematics

The nature of human knowledge has been discussed from the time of the Greeks. "For Plato, the mission of philosophy was to discover true knowledge behind the veil of opinion and appearance, the change and illusion of the temporal world. In this task mathematics had a central place, for mathematical knowledge was the outstanding example of knowledge independent of sense experience, knowledge of eternal and necessary truths" (Davis and Hersh, 1980, p. 325). Since Plato much of the discussion on knowledge has taken place in the context of mathematical concepts (Higginson, 1980). This discussion continues amongst philosophers, mathematicians, and mathematics educators.

Four movements in the foundations of mathematics are generally recognized.[5] Logicism reduces mathematics to logic (Frege and Russell). Formalism treats mathematics as the manipulation of objects in a formal system (Hilbert). Intuitivism[6] sees mathematics as a reflection of man's way of reasoning (Brouwer). Blaire (1981, p.147) called the newest of the four movements the hypothetical movement (Wittgenstein, Lakatos). Others (Ernest, 1991; Lerman, 1983) refer to mathematics presented as models of possibilities, subject to proof and refutation (hypothetical movement) as the quasi-empirical movement. Davis and Hersh (1981) refer to this same movement, headed by Lakatos, as the philosophy of dubitability.

Within each of these movements a particular view of mathematical knowledge is espoused -- either absolutist or fallibilist. The absolutist view suggests that mathematical knowledge "consists of certain and unchallengeable truths" (Ernest, 1991, p.7). The fallibilist view of knowledge asserts that mathematical knowledge is "fallible and corrigible, and can never be regarded as beyond revision and correction (Ernest, 1991, p. 18).

Paul Ernest (1991) suggests that the Platonists, logicists, formalists, and intuitivists all take an absolutist view toward knowledge; however they differ in what constitutes valid mathematics and mathematical knowledge. The Platonists suggest that mathematical objects exist prior to their discovery, and that "any meaningful question about a mathematical object has a definite answer, whether we are able to determine it or not" (Davis and Hersh, 1981, p. 318). According to this view, mathematicians do

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[5]See Davis and Hersh (1980), Blaire (1981), and Ernest (1991) for discussions on the foundational movements in mathematics.

[6]Intuitivism is also known as constructivism. Later in this paper I will use the word constructivism again but in reference to psychological constructivism, therefore in order to prevent confusion I will only use the word intuitivism when referring to the mathematical school of thought called constructivism and the word constructivism to mean psychological constructivism.

not invent or create mathematics -- they discover mathematics. The logicists, who believe that "all the concepts of mathematics can ultimately be reduced to logical concepts" and "all mathematical truths can be proved from the axioms and rules of inference and logic alone," (Ernest, 1991, p. 9) demonstrate an absolutist view of knowledge. The formalists do not believe that mathematics is discovered; they believe mathematics is simply a "game," created by mathematicians, based on strings of symbols which have no meaning (Davis and Hersh, 1981, p. 319). They are absolutists yet they do not believe that mathematical objects are real. The intuitivists differ from the logicists and the formalists in method and in source of knowledge. They suggest that "human mathematical activity is fundamental in the creation of new knowledge" (Ernest, 1991, p. 29) and "that both mathematical truths and the existence of mathematical objects must be established by constructive methods" (Ernest, 1991, p. 11).

Imre Lakatos (1976) in Proofs and Refutations introduces two forms of logic. Infallibilistic logic is based on the premise that the logic will infallibly lead to correct results. This is an absolutist view of knowledge. The converse, fallibilistic logic, is developed through a systematic process of conjecture, proof, and refutation. Only the hypothetical movement in mathematics asserts that mathematical knowledge is fallible and continually subject to refutation.

Paul Ernest (1991) in The Philosophy of Mathematics Education discusses the relationship between mathematics education and philosophies of mathematics. He presents five ideologies of mathematics education structured around the ideology of five social groups and suggests that each of these groups has a different view of mathematics. He claims that "ideologies have a powerful almost determining impact on mathematics pedagogy" (p.137). These social groups and their corresponding views of mathematics are noted in Table 1.

Ernest has tried to establish that there are five distinct ideologies in mathematics education.[7] These groupings are not simply based on the foundational schools in mathematics, since one could be a formalist, logicist or an intuitivist and presumably fit into almost any of the first three or possibly four groups. However there seems to be the assumption (although not explicitly stated) that only a person with a fallibilistic view of mathematical knowledge could fit into the fifth group of public educator. I would argue that a progressive educator could also have a fallibilistic orientation, since many

[7]Ernest (1991) admits that his model is "speculative" although he claims "it is well grounded in different theoretical disciplines.... Consequently no finality is claimed for the list of components in the model which are joined together by associations of plausibility rather than logic" (p. 214).

of the views ascribed to this group could be interpreted within a fallibilistic framework. For example the view of mathematics as a process and the belief that mathematics be personalized, as well as the theory of learning which suggests activity, play, and exploration, all fit into a constructivist theory of learning quite well. Ernest seems to suggest that one must have democratic and social underpinnings to be a fallibilist and that fallibilistic knowledge must be a social construction. I disagree. Fallibilism implies that mathematical knowledge is subject to error and refutation; whether it is socially constructed or personally constructed is debatable.

Table 1
Paul Ernest's Five Educational Ideologies[8]

Social Group	Industrial Trainer	Technological Pragmatist	Old Humanist	Progressive Educator	Public Educator
View of Mathematics	Set of truths and rules	Unquestioned body of useful knowledge	Body of structured pure knowledge	Process view: Personalized mathematics	Social Constructivism
Mathematics Knowledge	Absolute	Absolute	Absolute	Absolute	Fallible
Mathematical Aims	'Back-to-Basics' numeracy and social training in obedience	Useful maths to appropriate level and Certification (Industry Centred)	Transmit body of mathematical knowledge (Mathematics Centred)	Creativity, Self-realization through mathematics (Child-centred)	Critical awareness and democratic citizenship via mathematics
Theory of Learning	Hard work, effort, practice, rote	Skill acquisition, practical experience	Understanding and application	Activity, play, exploration	Questioning, decision making, negotiation
Theory of Teaching Math	Authoritarian, transmission drill, no 'frills'	Skill Instructor Motivate through work-relevance	Explain, motivate, pass on structure	Facilitate personal exploration Prevent failure	Discussion, conflict, questioning of content and pedagogy
Theory of Resources	Chalk and talk only anti-calculator	Hands on and microcomputers	Visual Aids to motivate	Rich environment to explore	Socially relevant, authentic
Theory of Assessment	External testing of simple basics	External tests and certification Skill profiling	External examinations based on hierarchy	Teacher led internal assessment Avoid failure	Various modes. Use of social issues and content
Theory of Child	empty vessel	empty vessel	can build character	child centred	clay to be molded
Theory of Ability	fixed realized by effort	inherited	inherited cast of mind	varies, needs cherished	cultural product

8From The Philosophy of Mathematics Education (p. 138 - 139) by Paul Ernest, (1991), Basingstoke: The Falmer Press. Copyright 1991 by Paul Ernest. Adapted with permission.

Although reform movements in education are complex, one can see how some of the different philosophies of mathematics have underlain reform movements in mathematics education and others paralleled attempted reform movements. For example the attempts to unify mathematics content at the turn of the century and in the early 1900s must have been influenced by people like Bertrand Russell as he attempted to unify mathematics. Another example is the influence of formalism in the "new math" movement of the 1960s or the influence of logicism on mathematics education reform after World War II. The present reform movement, for example, as represented by the NCTM in their Curriculum and Evaluation Standards for School Mathematics, (1989) would appear to be strongly influenced by psychological constructivism but there are also signs of Lakatos' philosophy of dubitability -- that is, mathematics as conjecture, proof, and refutation.

Mathematicians and mathematics educators continue to discuss changes in mathematics. Some (for example, Howson and Kahane, 1986) suggest that computers are resulting in the emergence of a new view of mathematics and of mathematical activity; that is, experimental mathematics. Mathematics, they suggest, is changing from a methodology of deduction to a methodology of induction. The traditional mathematical skills of proving, generalizing, and abstracting are being supplemented by scientific skills such as observing, exploring, forming insights and intuitions, making predictions, testing hypotheses and conducting trials (Howson and Kahane, 1986). But are these really changes? Maybe, as Davis and Hersh (1981) suggest, this is the way mathematics has always been done.

Lakatos' view of mathematics as an ongoing process of conjecture, proof, and refutation, Ernest's view of mathematics as a social construction and Davis and Hersh's views on the "Mathematical Experience," help us to better understand the nature of mathematics. No longer can philosophers turn to mathematical knowledge as the epitome of truth. The examples just discussed demonstrate how mathematics itself is now being tested for its truth value.

On Knowing

Our theories as to the nature of knowledge are in a state of upheaval as we near the twenty-first century. In the past one hundred years Popper tackled the nature of scientific knowledge, Lakatos suggested mathematical knowledge is always subject to proof and refutation, Wittgenstien addressed the nature of language, Bruner discussed the role of narrative in knowing, Merleau-Ponty examined the relationship between our experience and knowing; Dennett argued for a representationalist view of knowing;

and biologists like Varela and Maturana addressed knowing from a nonrepresentationalist, biological point of view. Unique to the current philosophical discussions, in all of these areas, is the growing fuzziness between knowing and knowledge. When knowledge is absolute and knowing is representing knowledge or storing knowledge, it is easy to separate the two in discussions. But when the noun loses its independent existence and the action is the only aspect of cognition that we can observe, we can no longer discuss the two in isolation from one another. The action specifies the object and the object specifies the action. We are now in this grey or fuzzy region and we discuss object and action in the same breath. As Varela, Thompson, and Rosch (1991) say, the object and the action co-emerge; they specify one another (p.172). With the changes we see in the above areas, it is not surprising that the nature of mathematical knowledge/knowing is also being debated.

Knowing is a special concern for mathematics educators since they could take a representationist view or they could take a constructivist point of view. Up until now, no one but mathematicians (and a few others in special circumstances) have been given credit for creating mathematics. Yet for mathematics educators and mathematics education researchers, viewing the learner as a creator of mathematics has a great deal of potential. This is the new position; at most the division between mathematics students and mathematicians must be between creating mathematics and creating original mathematics.

VonGlasersfeld (1984) asserts that all knowledge is created by individual learners through constructing reality. This school of thought is called radical constructivism. Radical constructivism involves two principles. The first is that knowledge is constructed by the cognizing subject not passively transferred from the environment. The second is that coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower (Kilpatrick, 1987).

Although popular in the literature, constructivism is not without its critics. Some of the problems that the constructivists are facing include attacks on constructivism itself as a philosophy, the role constructivism has in mathematics education, and the tying together of constructivist learning theories with constructivist teaching theories. "The expectations of mathematics educators that constructivism can improve practice could create a backlash not unlike the severe backlash to the modern mathematics reform if the way constructivism is used in mathematics education does not take full advantage of the available principles" (Steffe and Wiegel, 1992, p. 445). Kilpatrick (1987) has a concern with radical constructivism. He takes exception to the radical constructivist's view of reality and suggests that mathematics educators "must take ontology into account" (p.19). He further suggests that constructivism must come to terms with mathematical realism and address the issue of mathematics in a socio-historical context (a context which he sees as compatible with both realists and constructivists) if constructivists are to have significant impact on mathematics education. Lerman (1989) tries to find the middle ground in this debate by suggesting that constructivism is a relativistic view of mathematical knowledge and understanding. He proposes that knowledge is subject to change since it is a social construction (a publicly negotiated concept) relative to a particular culture, time, and place.

Although "constructivism has emerged as one of the main philosophies of mathematics education." (Steffe and Wiegel, 1992, p. 445), mathematics education has been under the influence of the other major schools of thought in cognition and learning. The study of learning has included movements such as behaviourism, representationalism, and constructivism. A behaviourist stance in mathematics education would view the learning of mathematics as conditioning appropriate responses to given stimuli. This view of learning would suggest methods such as memorization of facts and the rote learning of algorithms through practice. The cognitivists and the respresentationalists would most likely be quite close in their approaches for the learning of mathematics. Concepts and skills learnt by teaching small units and building on these to present the whole picture. This learning could be either instrumental (memorize how things work and fit together) or relational (learn how and why things work and fit together) (Skemp, 1987). The constructivists, because of their unique view of the world, see the creation or construction of mathematics as the learner's act. The knowledge/knowing is the human interaction with the environment. Knowing mathematics is doing mathematics (Kieren, 1990).

On Teachers' Beliefs

Micheal Crowe (1988) suggests that the following common claims about the nature of mathematics "are both seriously wrong and a hindrance to the historical study of mathematics."

1. The methodology of mathematics is deduction.
2. Mathematics provides certain knowledge.
3. Mathematics is cumulative.
4. Mathematical statements are invariably correct.
5. The structure of mathematics accurately reflects its history.
6. Mathematical proof is unproblematic.
7. Standards of rigor are unchanging.
8. The methodology of mathematics is radically different from the methodology of science.

9. Mathematical claims admit of decisive falsification.
10. In specifying the methodology used in mathematics, the choices are empiricism, formalism, intuitionism, and platonism. (p.260 - 274)

The limited studies done on teachers' beliefs demonstrate that many teachers hold a variety of these misconceptions. Some teachers believe that mathematics is cumulative or hierarchical and it should be taught in a "particular order" as well as in a "particular manner (Bishop & Nickson 1983, p. 64)." Borasi (1990) found that teachers and students view mathematical knowledge as absolute truth. You can also find these and other "misconceptions" of mathematics implicitly stated in a number of journal articles and reports on mathematics instruction and teachers' beliefs (McBride, 1989; Anyon, 1981; Civil, 1990; Thompson, 1984).

Studies that have investigated how mathematics teachers view mathematics (Civil, 1990; Brandau, 1981, cited in Romberg & Carpenter, 1987; Anyon, 1981) help us to better understand some of the beliefs that teachers have of mathematics instruction and of the nature of mathematical knowledge. For example, Civil (1990) found that elementary preservice teachers' beliefs include notions of neatness, speed, adult-like procedures, rule-orientated algorithms, and authoritarian attitudes. Brandau (cited in Romberg and Carpenter, 1981) found that mathematics teachers believe mathematics is that knowledge which you find in textbooks; it is the mathematics of experts, and the job of teachers is to cover the text materials. Anyon (1981) found teachers who talked of knowledge in terms of facts and simple skills taught by having students carry out procedures with few or no explanations. Teachers who held the attitude that knowledge involved individual discovery and creativity provided activities that reflected this attitude.

Some studies have tried to uncover why teachers teach the way they do. Thompson (1984) observed three elementary school teachers and did intensive interviews to investigate the teachers' conceptions of mathematics and mathematics teaching. She found their instructional behaviour was significantly affected by their views of mathematics. Toomey (1977) studied the planning of four teachers and found "The perception of the teacher-student relationship, of the appropriateness of content materials, and of the nature of instructional process may well help to shape his [the teacher's] decision about how to frame his objectives" (p.126). Lerman (1990) used Lakatos' definitions to separate preservice teachers into two categories -- those who have an absolutist epistemology and those who have a fallibilist epistemology. This study found that preservice teachers with an absolutist orientation favored teacher-centred and teacher-directed instructional practices. Whereas preservice teachers with a fallibilist orientation favored instructional practices which are more open to the influence of the students.

Blaire (1981) suggests that a teacher's philosophy of mathematics does not influence the decisions the teacher makes, rather one's philosophy of mathematics teaching influences one's instruction. He uses a number of hypothetical scenarios to persuade us that teachers instruct from one of four teaching perspectives[9] at will. Lerman (1983) disagrees with Blaire's theory and claims that the connection between philosophical perspective and teaching style is stronger than Blaire suggests. "It must be emphasized here that this is not a question of pedagogical method in the first instance, but of the logical consequences of a theory of knowledge" (p. 63). The philosophical perspective (even if implicit) of the teacher results in "choice of syllabus content, teaching style and students' attitudes towards mathematics" (p. 62).

In 1978 Fenstermacher noted that beliefs had not been a major focus in the study of teaching partly because researchers believed that "what the teacher thinks is not the proper object of empirical inquiry" (p.173), instead what the teacher does is the proper focus of study. Today many educational researchers have moved from this behaviourist stance and are researching things that might not be directly observable.

VonGlasersfeld (1990), in arguing the importance of developing new theories of learning, said, "If educational efforts are, indeed, failing, the presuppositions on which, implicitly or explicitly, these efforts have been founded must be questioned" (p. 3). The direction the search for the presuppositions must take is debatable. Like Fenstermacher suggested, some researchers claim that the teacher's philosophical perspective is not the proper focus of research. But there are those who believe that the philosophical perspectives of the teacher is the right direction for the search (Thom, 1973; Lerman, 1983; McBride, 1989). This is the direction this study takes. I am interested in teachers' beliefs about the nature of mathematics and mathematics education. I wish to examine these things which I believe underpin teaching.

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[9]The four teaching perspectives are "(1) the teaching of mathematics as an art form; (2) the teaching of mathematics as a game (or family of games); (3) the teaching of mathematics as a member of the natural sciences; (4) the teaching of mathematics as technology oriented." (Blaire, 1981, p.148)

The Problem

Francisco Varela (1987) offers a metaphor that I would like to borrow to demonstrate why I believe examining teachers' beliefs is an appropriate and important area to study. He asks us to imagine a mobile (wind chime) hanging in a breeze.

> Clearly how a mobile sounds is not determined or instructed by the wind or the gentle push we may give it. The way it sounds has more to do with (is easier to understand in terms of) the kinds of structural configurations it has when it receives a perturbation or imbalance. Every mobile will have a typical melody and tone proper to its constitution. In other words it is obvious in this example that in order to understand the sound patterns we hear, we turn to the nature of the chimes, and not to the wind that hits them. (p. 50)

Now let me make a small change to this example.

> Clearly how a teacher teaches is not determined or instructed by the curriculum change the government may give him. The way he teaches has more to do with (is easier to understand in terms of) the kinds of structural configurations he has when he receives a perturbation or imbalance. Every teacher will have a typical teaching methods and style proper to his constitution. In other words it is obvious in this example that in order to understand the teaching patterns we see, we turn to the nature of the teacher, and not to the curriculum change that is given to him.[10]

Varela said, "the point of this example is to suggest the relative ease with which a degree of self-involvedness immediately gives the system a desire for autonomy vis-à-vis its medium. That is to say, the fact that it handles its medium according to its internal structure becomes the predominant phenomenon" (p. 51). Maturana (1987) said this somewhat differently: "the interactions they undergo will only trigger changes in them; they will not specify what happens to them....nothing can happen which is not determined by it -- how it is made, its structure" (p. 73-74).

Throughout the previous sections in this thesis I have attempted to point in the direction of this study. Although I am interested in teaching in general, this study focuses in on the methods teachers choose for instruction and the beliefs that underpin those choices. Varela suggested, "we turn to the nature of the chimes." I have chosen to turn to the belief systems of the teacher. Bruner (1990) tells us that in folk psychology "people are assumed to have world knowledge that takes the form of beliefs, and are assumed to use that world knowledge in carrying out any program of

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[10]Unlike chimes, teachers are autopoietic systems with plastic structures which are continuously subject to self-modification.

desire or action" (p. 40). I too have made this assumption about beliefs; furthermore, I view beliefs as being a significant way of observing and reviewing the teachers' structure[11]. The introduction of graphing calculators as a tool in high school mathematics classrooms provided an occasion to study teaching by examining the beliefs of teachers as they were manifested in changes teachers made to their instructional strategies. It was this particular "wind" which I used as the perturbation to help me observe the teacher's structure -- their mode of being, or mode of activity in the classroom.

I hoped by observing teachers in their instructional roles I would better understand the relationship between teachers' philosophical perspectives and the choices they make for instruction. I thought that the introduction of graphing calculators would provide me with an opportunity to learn about the pedagogical decisions teachers make. Crocker (1983) suggests that the "implementation of innovations may represent a 'strategic site'" for studying teaching since "essential features ... are much more likely to be laid bare under conditions of change" (p. 354). Since teachers were revising their instructional strategies to include the use of graphing calculators, I hoped they would "lay bare" some of the reasons they had for making the decisions they did. The point of this study is to demonstrate the ways in which the teachers used the graphing calculators in their instruction and to show how in making curricular decisions teachers' philosophies of mathematics and mathematics education are manifested.

The main questions of this study are:

1. How does the teacher implement the use of graphing calculators in instruction?
2. What are the teacher's reasons for using the graphing calculator in particular ways?
3. What are the teacher's philosophies of mathematics and mathematics education?
4. How are a teacher's philosophies of mathematics and mathematics education manifested in instructional practices?
5. How are a teacher's philosophies of mathematics and mathematics education manifested in interactions with students?

[11] I am using the word structure here as I understand Maturana and Varela to use the word structure.

6. In what ways does the availability and use of graphing calculators in mathematics instruction affect the teacher's beliefs about mathematics and mathematics education?

The general purpose of this study, then, is to **better understand how a teacher's philosophies of mathematics and mathematics education are manifested in the teacher's instruction of mathematics.** If we can better understand how one's philosophies are manifested in instructional styles and choices for instruction then possibly we can suggest strategies for teachers' professional development that will help teachers understand and implement curricular changes and in fact better understand and deal with all kinds of pedagogical and epistemological developments.

This study focused on Math 20 and Math 33 teachers who intended to use graphing calculators in their instruction of the quadratic function. This unit (as presented in the textbooks for the two courses) included the topics: transformations of the quadratic function written in the form $y=a(x-p)^2+q$; completing the square of quadratic equations in the form $y=ax^2+bx+c$; and solving maximum and minimum (max/min) word problems.

In this study I assume that teachers' instructional strategies are a result of teachers' decision-making and these decisions are affected by perturbations (such as student behaviour and curriculum requirements and changes) and framed by the teachers' structure. I assume the teachers' beliefs about the nature of mathematics, mathematics education and pedagogy are all part of this structure and that this structure is explicated through the teachers' articulation of their beliefs and actions.

This study is limited by the research methodology I chose to use: specifically the data collection methods I used, the amount of contact I had with each teacher, each teacher's ability tc articulate their beliefs to me, and by my interpretation of our conversations. Maturana (1987) points out, "everything is said by an observer" (p.65). As the researcher who conducted and documented this study I had two explicit roles. The first was that of observer; the second was that of narrator. As an observer I separate that which I believe to be important from all that which confronts me while acting as observer. As a narrator I tell not *a story* but *my story* based on my interpretation of my observations. Bruner (1990) reminds us: "A story is *somebody's* story [itallics original]. Despite past literary efforts to stylize the narrator into an 'omniscient I,' stories have inevitably have a narratorial voice: events are seen through a particular set of personal prisms" (p. 54). So here are two other limitations of this study It was conducted through the eyes of an observer and was told through the voice of a narr or.

You will notice that I am writing this document in the first person. The knowing expressed in this document is knowing in the domain of an observer and is not to be taken as "truths" in the worlds of the teachers studied. My use of the first person highlights my role as an observer in this work.

> [F]or all constructivists, all communication and all understanding are a matter of interpretive construction on the part of the experiencing subject, and therefore, in the last analysis, I alone can take the responsibility for what is being said in these pages. (VonGlasersfeld, 1987, p.19)

Finally I must recognize that I too am a perturbation (a trigger) for each of my teacher informants. The teachers were fully aware that I was interested in their use of graphing calculators in their classes and most of them viewed me as an expert with the graphing calculator. I will discuss this further in the few instances where I think this perturbation resulted in exaggerated behaviours or single time decisions for a teacher. Even though I think there was at least one instance where the perturbation caused by me was quite significant I still assume that the teachers' choices were a result of their structure.

CHAPTER II:
Research Methodology

Introduction

Most research about the instruction of mathematics has been empirical in nature looking for correlations between teacher behaviours and student achievement, measured by standardized test scores (for example, Leinhardt, 1989). A few naturalistic studies have focused on descriptions of the teacher and instructional strategies (for example, Thompson, 1984) and even fewer studies have tried to uncover the underlying assumptions and beliefs that influence the instructional strategies of teachers (for example, McBride, 1989) or preservice teachers (for example, Lerman, 1990; and Ball, 1988).

The intention of this study was to observe how teachers' philosophical perspectives are manifested in their implementation of new curriculum. I chose to examine this problem by doing a qualitative study based on classroom observations, interviews, and display collection. Thus, it was necessary to find teachers who would inform me about their philosophies of mathematics and mathematics education; their use of the graphing calculators and their instructional strategies. I chose to do several cases to exemplify this relationship between philosophical perspectives and instruction. Six teachers who taught high school mathematics and who intended to use graphing calculators in their instruction on the quadratic function were selected as informants for the study.

The Teacher Informants

The teachers, Troy, Mark, Wally, Henry, Pheonix, and Jack, taught either grade 11, Mathematics 20 (Math 20), or grade 12, Mathematics 33 (Math 33)[1]. The first four teachers teach in large urban composite high schools. Pheonix and Jack teach in smaller rural junior-senior high schools. Although they are all experienced mathematics teachers, there is considerable variation in the types of experiences and the amount of experience they bring with them to this particular course. In addition to their teaching responsibilities, they also have other professional responsibilities (See Table

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[1]In the province of Alberta where this study was conducted there are three streams of mathematics. Mathematics 10/20/30 are the courses intended for students who plan to go on to post-secondary education which requires extensive mathematics. Mathematics 13/23/33 are the courses intended for students who plan to go on to post-secondary education which does not require extensive mathematics. Mathematics 14/24 are the courses in the non-academic mathematics program. Mathematics 31 is designed to be taken after Math 30 and is much like an advanced placement course.

2). After having spent considerable time with each teacher, both in their classes and one-on-one, I have no reason to believe that they are atypical examples of high school mathematics teachers.

Table 2
A Description of the Teachers in the Study

Teacher	Education	Years Teaching	Years Teaching Math	Extra Duties
Troy	B.Ed., M.Sc.	21	21	Department head - math
Mark	B.Ed.	16	8	Department head - student activities
Wally	B.Ed.	20	10	Program coordinator - continuing education
Henry	B.Ed.	8	8	
Pheonix	B.Ed.	11	7	Coordinator-"Youth Pride"
Jack	B.Sc., B.Ed.	24	24	Coach - Badminton

Data Collection

In order to learn how teachers implement graphing calculators into their existing instructional strategies and to understand the teachers' philosophical perspectives on the nature of mathematics and mathematics education, a variety of data sources were used. Following Merriam's (1988) suggestions for case study research, data were collected by means of classroom observations, informal discussions, a formal interview, and display collection. Qualitative data, in the form of field notes, class transcripts, interview transcripts, and lesson displays were compiled. It was hoped that these data would provide "a source of well-grounded, rich description" (Miles and Huberman, 1984, p. 21) of the teachers' instructional strategies and of the teachers' beliefs, attitudes, and conceptions of mathematics and mathematics education. Although there was considerable variation in the amount of data collected from each teacher, all forms of data were gathered from each of them.

Classroom Observations

A significant part of the data collection was by means of classroom observations. I observed, took field notes, and audio recorded the lessons on those occasions when the teachers used graphing calculators as part of their instructional strategy. The volume of data collected by this means varied considerably between teachers, since they used the calculator in different ways and over different spans of time. Because Henry, Pheonix, and Jack only relied on graphing calculators once in the course of the unit, they were also observed on a few occasions when they did not use the calculators. This was done in order to witness the calculator-use in context with the rest of the unit. The amount of time I observed each class ranged from 3.25 h to 7.75 h (see Table 3). From these observations, information about how the teachers used the graphing calculators in their classes and examples of the teachers' lived philosophies were identified. The transcripts from the lessons were essential to the writing of the vignettes presented in chapter 3.

Table 3
Number of Hours of Classroom Observations

Teacher	Hours Observed
Troy	3.25
Henry	3.25
Wally	7.75
Mark	6.75
Pheonix	4.00
Jack	4.00

Display Collection[2]

The lesson displays made up an important component of the data for this study. These displays included any displayed objects or verbal interactions to which an action, by either the teacher or the students or both, was directed (Aoki, 1979, p. 5). The verbal interactions which constitute displays were recorded in the transcripts. The teachers' board-work and textbook references (examples and exercises) were recorded in the fieldnotes. Copies of handouts and exams the teachers either created or selected for use in class were collected. The display collection was a valuable source of

[2]The data collected has been saved and can be retrieved upon request.

information on the teachers' methods of instruction. This data helped demonstrate the teachers' various uses of the graphing calculator and also pointed to the kinds of activities and questions the teachers focused on in their instruction of this particular unit.

Informal Discussions

At the conclusion of each class the teacher and I briefly discussed the lesson he or she had just taught. The teachers used this time to reflect on how the calculator had been used in class and to think about how they could best use the calculator in future lessons (both in this unit and for the next year). I used this opportunity to ask the teachers to clarify observations I made during the lesson and to elaborate on the instructional choices they made. Notes from these informal discussions were used both to help formulate interview questions and as general data in the case analyses.

Formal Interview

An important part of the data for this study was gathered in a one to two hour interview, at the conclusion of the observation period, with each teacher. This interview had three intents. The first intention was to uncover the teachers' attitudes, beliefs, and conceptions of mathematics and mathematics education. The second intention was to have the teachers talk about their methods of instruction that I witnessed while observing their classes. The third intention was to find out how useful and the effective the teachers thought the graphing calculators were for teaching the quadratic unit in either Math 20 or Math 33.

Each teacher completed an interview guide (see Appendix A) prior to the interview. The guide was developed, in part, from a questionnaire Lerman used in his doctoral research and which he refers to in his 1990 paper, *Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics*. The interview guide consisted of statements about mathematics and mathematics education to which the respondent was asked to strongly agree, agree, neither agree nor disagree, disagree, or strongly disagree. This guide provided the teachers with an opportunity to think about the things I was interested in before the interview. This pre-organization proved beneficial to both myself and the teachers since there were some statements they previously had not considered. Reading the statements ahead of time not only prepared the teacher to respond in the limited time we had, but also allowed for a more thoughtful response. I noticed that frequently the responses needed the elaboration the teachers provided in the interview; otherwise some of their responses would have been misinterpreted. The need for elaboration was particularly evident with one teacher who demonstrated a multiplistic view on many of the things we discussed.

The interview format was flexible and the guide simply provided some structure for us in the interviews when we were ready to move on in our discussion. The interviews began by reading through the interview guide with the teacher. As we went through the guide, I asked the teacher why she or he agreed or disagreed with the statement. On a number of occasions the teacher asked me what the statement meant or told me how he or she interpreted the statement. A number of times we discussed other issues that came up because of a statement on the guide. We also discussed issues that the teacher brought up (for example the problem with junior high school mathematics). As noted earlier we also discussed their instructional strategies and the pros and cons of graphing calculator technology. The data gathered from the interviews provided me with the basis for my analysis of the teachers' philosophical perspectives on mathematics and mathematics education and were used extensively in the development of the case reports.

Methodology for Data Analysis

Miles and Huberman (1984) suggest a methodology for data analysis, in qualitative research, which consists of four (overlapping) stages beginning with data collection (discussed in the previous section) and including data reduction, data display, and conclusion drawing and verification. I followed this methodology in the study, although I would include a fifth category (report writing) since the data analysis was an ongoing process from data collection through writing the case studies.

Data Reduction

Miles and Huberman (1984) insightfully point out that data reduction occurs before, during, and after the collection of data (p. 24). They suggest the conceptual framework surrounding the research questions, the sampling, and the instrumentation are all forms of anticipatory data reduction (p. 24 - 25). Interim data reduction, Miles and Huberman suggest, is "perhaps the main corrective for the potential blinders of excessive prefocusing and bounding. Essentially the field worker cycles back and forth between thinking about the existing data set and generating strategies for collecting new data" (p. 25). Finally there is the post data collection reduction which I will discuss as part of the data display section since "the methods used after data collection are integrally related to methods of data display, conclusion-drawing, and verification" (p. 25).

The field work was done in three distinct periods. For the first two teachers observed, I attended all but one of their classes during the course of the unit on the quadratic function. I purposely attended both on days when they intended to use the calculators and on days when they had no intention of using the calculators. I ended up with hours of audio tapes from the lessons. Since much of the data was repetitive and my perspectives of the teachers' instruction were developed early in the observational period I decided only to observe the next four teachers on days when they indicated, to me, they would be using the graphing calculators in their classes. Two of these four teachers only used the calculator once during the course of the particular unit I was interested in, so I observed the teachers the day following the use of the graphing calculator as well as the day on which it was used. For another one of the four teachers I missed one class when he used the graphing calculator and sat in on another class when he did not use the graphing calculator.

Data Display

The data from the formal interviews and the classroom observations were transcribed from audio tapes. A file was prepared for each teacher in the study. This file included field notes, transcriptions from both classroom observations and interviews, and displays collected from the teachers. After I reviewed my proposal, I went through the transcriptions and noted interesting things said that related to the teachers' beliefs, attitudes, and conceptions of mathematics, instruction, learning, technology, and mathematics education. Then, for each teacher, I drew a web that reflected my impressions of that teacher (see Figure 1 for an example). The webs included some biographical information, the teacher's view of students, key words and phrases that the teachers used, and some general impressions I had of the teacher. These webs helped me organize my thoughts, identify patterns, construct themes, and articulate my perspectives of the teachers. I went back to the transcripts and looked for connections between my webs and the teachers' comments, and for themes to organize the data around.

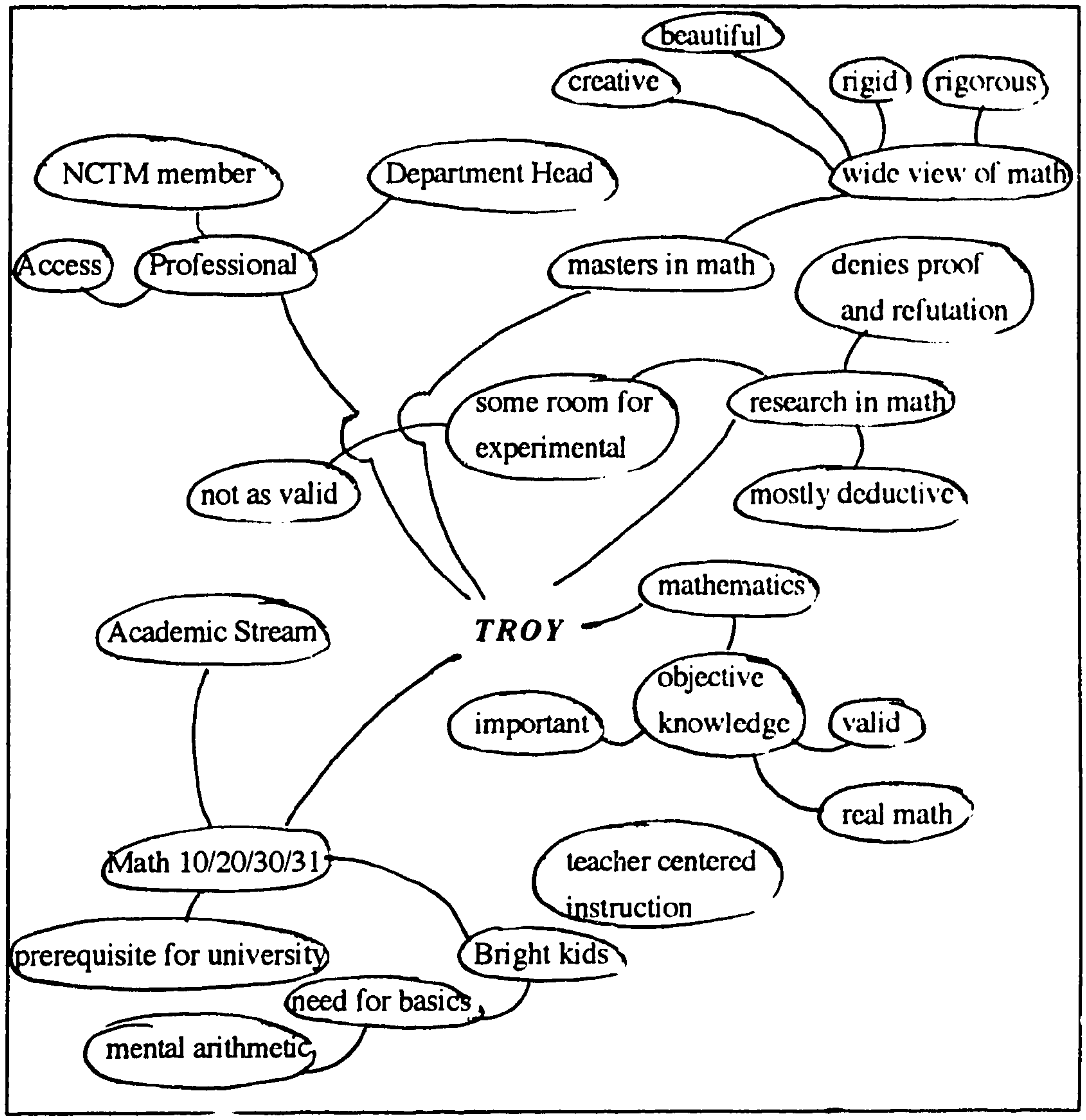

Figure 1
My Web About Troy

Contas (1992) suggests, "Contrary to what some have claimed, categories do not simply 'emerge' from the data. In actuality categories are created, and meanings are attributed by researchers who, wittingly or unwittingly, embrace a particular configuration of analytical preferences" (p. 254). Initially I took data out of the transcripts from the interviews and organized the data according to the themes (technology, nature of mathematics, and knowing) I used for Chapter I. When I approached the data from these perspectives I found data which fit nicely into some of those themes, but the creation of new themes became necessary based on my interpretation of the interviews (see Table 4). The new set of themes provided me with the framework within which I further examined my data. After classifying the data from the interviews (for each of the teachers) I proceeded to examine the data from the classroom observations and the displays I collected. I looked for the ways in which the graphing calculator was used by each of the teachers in their instruction and I also noted where the calculator was not used.

Table 4
Themes for Data Display

Theme
High School Mathematics
Nature of Mathematics
Students
Instruction
Calculators

Conclusion-Drawing and Verification

After organizing and displaying the data, I returned to each teacher for a follow-up interview to verify (from the teachers' perspectives) my data and analyses (Miles and Huberman, 1984, p. 28). This was an important part of the analysis since I checked my perceptions with those of the teachers. I asked the teachers to read my statements about them, their beliefs, and their instruction and then to point out inaccurate or misleading statements. I also took this opportunity to ask questions that needed to be clarified.

The data analysis, verification, and conclusion-drawing was an ongoing process in this research and in the writing of this thesis. Not only did I, as an observer, distinguish in the very act of observing but I also drew tentative conclusions throughout the whole process from data collection to data analysis. As I drew conclusions and wrote the case studies I could not help but continuously analyse the data. The selection of certain quotes and the omission of others was an interpretive act. "We are, in the very act of story making, deciding what to tell and what to leave out and imposing structure and meaning on events" (Carter, 1993, p. 9). The case reports are my interpretation of the data, my meaning, my story. The analysis was a recursive process and was not completed until the writing of the thesis was finished. This was the point at which I said "I have examined the data and here is what I have to say about it." If I were to again analyse the data I would undoubtedly bring to it new meaning and therefore new conclusions (although not necessarily different) would be made.

> A written document appears to stand still; the narrative appears finished. It has been written, characters's [sic] lives constructed, social histories recorded, meaning expressed for all to see. Yet, anyone who has written a narrative knows that it, like life, is a continual unfolding where the narrative insights of today are the chronological events of tomorrow. Such writers know in advance that the task of conveying a sense that the narrative is unfinished and that stories will be retold and lives relived in new ways is likely to be completed in less than satisfactory ways. (Connelly and Clandinin, 1990, p. 9)

Report Writing

Just like I had two explicit roles (observer and narrator) in this study, the study itself has (at least) two distinct and explicit roles. The first was to inform me (the graduate student) about doing research; the second is to inform me and others about my research questions. I will leave the first of these aims to discuss informally with those people around me, but the second aim must be considered here. Jerome Bruner (1986) suggests:

> There are two modes of cognitive functioning, two modes of thought, each providing distinctive ways of ordering experience, of constructing reality. The two (though complementary) are irreducible to one another....A good story and a well-formed argument are different natural kinds. Both can be used as a means for convincing another. Yet what they convince of is fundamentally different: arguments convince one of their truth, stories of their lifelikeness. (p. 11)

I have chosen to inform readers about teachers' philosophies of mathematics and mathematics education as they are manifested in teachers' instruction and interaction with students by means of descriptive case studies[3]. Within the case studies are narratives; stories told by the teachers, stories of events I observed, and stories that reflect my understanding of the teachers' beliefs, conceptions, and concerns about mathematics and mathematics education.

I will introduce each of the teachers by means of a vignette of one of their lessons. This study is about real teachers in real classrooms with real students. I believe the most important aspect of this study is this context, so I feel strongly about providing a sense of what fifty (or eighty) minutes in each of their classes is like. I

[3]The words of the teachers provided in quotes are unaltered. Where necessary I have added words for clarification in brackets. I also removed some of the idiosyncrasies of speech such as the ums and uhs.

think I can most practically do this with the vignettes. Bruner (1990) believes, "we shall be able to interpret meanings and meaning-making in a principled manner only in the degree to which we are able to specify the structure and coherence of the larger contexts in which specific meanings are created and transmitted" (p. 64) A discussion of how the teachers used graphing calculators in their instruction follows the vignettes. This is followed by an exposition of the teachers' stated beliefs, attitudes, and conceptions of mathematics and mathematics education. Finally I try to point out where and how the teachers' philosophical perspectives are manifested in their instruction and interactions with their students as well as where the teachers' actions appear to contradict their espoused philosophies.

Because of the number of cases involved in this study, each case will be presented in a chapter of its own. The data for each teacher will be presented, analysed and interpreted within each chapter. In the last chapter of this thesis I will briefly summarize the findings from across the cases with respect to the research questions and then move on to the implications of this study.

CHAPTER III

Case I: Troy

Biography

Troy Brown is the head of a large mathematics department consisting of 12 teachers in a large urban high school. Of all the informants in this study, he had the strongest background for teaching mathematics with a masters degree in mathematics from the local university. He has taught mathematics for 18 years, and he attends a variety of professional development activities such as mathematics conferences and workshops. (He had just attended a session on the graphing calculator at one of these conferences, prior to participating in this study.) He shares his knowledge in mathematics and mathematics education with his fellow staff members at department meetings and has written curriculum material for various provincial agencies.

Troy's Lesson on Max/Min Problems: A Vignette

Troy, dressed in a suit and tie, stood in front of his classroom waiting for his students to file in. With math texts under their arms, they entered chatting quietly and took their seats. When the bell rang Troy walked to the door and closed it.

"Okay let's begin." Not one minute was wasted.

Immediately the students quieted down and turned their attention to the teacher at the front of the class.

"Take out last night's homework."

The students knew the routine; they opened their books and grabbed their pencils. A couple of students looked around trying to figure out what the previous night's homework was.

Attempting to look very serious, Troy told the students, "I take great pleasure in watching students squirm so you won't get off the hook with the answer, I didn't do my homework last night. I'll say, Well that's fine; try and figure it out now."

The students, some of them squirming, waited for their turn to be called on to provide an answer to one of the homework questions.

"Okay let's look at number one quickly. You had questions one to four on page 219. Let's start with question one. In each case you are to identify what the maximum or the minimum value of the function is and what is the value of x at which the maximum or the minimum occurs. Let's look at question one part a." With a quick glance away from his textbook Troy called on a student, "Dick would you give us the answer to that one please?"

Question one consisted of six parabolas in their graphical forms. The students had spent the previous five classes studying the quadratic function graphically and expressed in the form $y=a(x-p)^2+q$.

"The maximum value is four."

"And that occurs when x is --"

"One."

"Right. So, the statement should be like this. The maximum value of y is four when x is one. Part *b*. How do you answer that one? Let's go down to Jerry please."

The remainder of these questions were handled with ease by the students. It was not clear if there were any students who did not have the work done.

"Do I get the wrong idea that this stuff is easy or is it that I have done such a great job of teaching, you all know it really well?"

"That's it," quickly replied one of the students taking part in the light-hearted conversation.

"That's it. At least we are all agreeing. Okay, I want to look at number two. You see it's easy when you've got the graph in front of you. The question is, can you do the same thing when we've got the equation of the function in front of you. Well, let's do exactly the same thing. But we want to look at the following functions in question two. The first one is y equals in brackets x minus three all squared add five. Joyce."

"I wasn't here."

"You weren't here." Apparently this excuse was acceptable. "Two *a* -- Shane."

"Five."

"Okay and is that a maximum or a minimum?"

"A minimum."

"Yeah, and what's the x value?"

"Three"

"Okay. Uh," Troy looked bothered. "I would suggest -- I know that is the way the question is asked but let's try to give the answer the same way that we did in question one. Okay? The maximum or the minimum value in this case is five when x is three. Part *b* how would you answer that question? Let's see -- who haven't I talked to for a while? Jason."

"Me?"

"Yep."

"The maximum value is three. The maximum value for y is three when x is four.

"Okay. How many people agree with that? Hands up." No one raised their hand. "You don't have too many friends there."

"One?" Jason offered as he attempted to correct his mistake.

"Okay, so let's change Jason's answer. The minimum value of y is three when x is one. How many people agree with that? Hands up. Quick." Again no one raised their hand. "You still don't have many friends. Are you guessing?"

Jason did not respond.

"Jason I asked you a question." Troy paused for a few seconds and then asked, "Are you guessing?"

"Not really."

"So you understand this do you?"

"A little."

"Okay. Well, that's the problem," Troy said with a serious tone. "You only understand it a little bit but you don't understand it enough to in fact give correct responses, because virtually every thing you said is wrong. If I had to draw the graph here Jason, which way would the parabola go? Up or down?"

"Up."

"Okay. Now, do you have a maximum value or a minimum value?"

"Minimum"

"Okay. So, we've got one thing right now. Can you tell me where the vertex of this parabola is? I need the coordinates."

After a short pause that seemed like a long one. Jason came up with the correct coordinates.

"That's the problem when you only *kinda* understand what is going on. Those are the kind of mistakes you make. You have to understand the whole ball of wax. It's not good enough to just get a bit of the surface. You have to understand the whole thing Jason and right now I'm afraid you do not understand very much."

The rest of question two was checked. It was a little more obvious now who was squirming and was not, although, it still was not obvious who had the homework done and who did not.

After going over two of the four homework questions Troy decided to leave the remaining questions (much to the relief of some students) and he moved on to the new lesson.

"Let's have a look at some of the applications of what we've been learning. There is a special type of problem in mathematics referred to as a maximum or a minimum problem. We are going to learn to solve some fairly basic maximum or minimum problems. Those of you who go on next year to take Math 31 in addition to Math 30 are going to learn how to solve some much more complicated maximum and minimum problems because then you will have the tools of calculus to use. Now the tools of calculus open up a whole bunch of doorways for you. There are a lot of things you can't do with simple mathematics. The kind of mathematics we are doing now are pretty basic, but certainly we can use our understanding of the quadratic function to solve some of the maximum/minimum type problems. Let's take a look at this one."

Troy wrote the problem on the board as he read it out loud to his students. "*Mr. Johnson plans to build a rectangular kennel for his dogs. He has 40 m of fencing he may use and he wants the area inside the kennel to be as large as possible.* So the question is: *What should the dimensions be if the kennel is built in the middle of his yard?* While you are finishing writing that down I will once again pass out these calculators. I want to show you how we can use graphing calculators to solve this problem. But at the same time we have to know how to do it algebraically as well; so we will look at both methods of solving."

One student suggested to Troy that the kennel should be circular. But Troy instructed the student that it must be rectangular. The student did not bother to challenge the teacher about the maximizing solution and none of the other students picked up on the first student's suggestion of a circular kennel.

"While I am doing this, I want you to think about a couple of things. What are some of the possibilities here? I guess what I am saying is, we've got forty metres of fencing here. Think of some of the ways in which I could use that forty metres of fencing and build a rectangular kennel. Draw yourself a couple of diagrams. Draw yourself a couple of pictures here."

Troy passed calculators out to all the students.

"Okay, let's think of some of the possibilities."

A number of students offered suggestions for the dimensions of the kennel. As they did Tony kept track of their suggestions on the board by drawing and labeling the dimensions for the rectangles that were suggested.

"Okay. So, what we are finding out here is that Mr. Johnson has a lot of choices. There are many different combinations of a length and a width which will use up his forty metres of fencing material and produce a rectangular dog pen. We have to find the one that generates the largest area and I think, probably, some of you are already thinking it's one of these. It kind of looks like it might be this one -- right?' Troy said as he pointed to a rectangle with dimensions ten by ten.

"Unfortunately, looks like, is not acceptable as proof. We have to come up with something a little more substantial than, looks like. So let's see if we can up with a mathematical solution to this problem."

Troy developed his mathematical solution to the problem for the students.

"So here is what my rectangle looks like. I have a rectangular kennel of width *w* and the length is *l*. See, what we want to be able to do is examine all possibilities at once. Why do we have to start listing them like this?" he said as he pointed to all the diagrams on the board. "One of the things that mathematics allows us to do is to consider all possibilities at the same time and allow us to pick out one and in this case the one that produces the maximum area so that we don't have to start listing the possibilities one by one, because if you ever started doing that you would never finish. There are an infinite number of possibilities. Now what must be true about the length and width here? Give me one condition."

"They have to equal twenty."

"Okay, they have to equal twenty -- or more specifically the total perimeter is forty. So, two lengths plus two widths must equal forty, from which it follows that the length plus the width must equal twenty. In other words the length of this rectangle must be twenty minus the width. What we're going to do now is examine the area function."

The explanation turned out to be a long one but the students listened attentively; they seemed to recognize the potential for running into difficulties with these problems.

Once Troy had the equation for this problem on the board he asked the students to graph the function on their graphing calculators. He provided them with a suitable scale to use on the viewing window of their calculators. He suggested to the students that he knew what the scale should be on the calculator because he had done the problem before. The students were instructed to use the trace and zoom keys on the calculators to examine the ordered pairs on the parabola. Troy talked them through this process giving them instructions on using the calculator and interpreting the values given by the trace function. The students worked independently with a half an ear open to those instructions. Once the students had gotten the cursor on the calculator to the top of the parabola there was some work trying to get the 99.999 and the 9.72 closer to one hundred and ten respectively.

"Boy you can't get much closer than that." Troy said. "Chances are you'll never get to one-hundred and ten. Sometimes it happens. We could use that integer function that we were talking about the other day. The thing we have to recognize about the graphing calculator is that it's good to give you values that are very close, but it doesn't examine every single point on the graph."

"I got ten," interrupted one student.

"Okay something a little different is happening on your calculator. That's fine. How did you do it?"

"I don't know."

When Troy looked at the student's calculator he realized that the student had zoomed in on the graph a few more times than the other students had. He pointed out to the others that the more times you zoom in on the function, at the vertex, the closer you would get to the actual value.

"Everybody listen up. Let's make the same kind of statement we made yesterday in the exercises we were taking up. The maximum value of *a* appears to be one-hundred when *w* is ten. Well I've got one of the dimensions. I've got the width but I also need the length. So when *w* is ten, the length will also be ten, meaning that we get the square that we were talking about earlier. So there is a way of solving the problem. Lets have a look at an algebraic solution of the same problem."

Troy led the students through the algebraic solution. First by having them complete the square and then having the students determine the maximum width by finding the vertex of the quadratic.

"Hey but here is the proof people." Troy exclaimed. "There's the proof of what we read off the graph. See, that is proof-positive people. There's no guess work involved. There's no statements like, it appears to be. I know precisely. Now I know that this function has a maximum value of one-hundred which occurs when the width is ten. I know that for sure. That gives me the algebraic solution to this problem."

He continued, "These problems can be solved algebraically. They can be solved using graphing technology. But again the advantage of doing them algebraically is that you know that when you solve them algebraically you know exactly what the solution is. When you do it using the technology, it will certainly support what you are saying. It will give you some direction. It will show you what you should be looking at, but you have trouble sometimes getting the exact answers off the calculator."

"I got one-hundred." pointed out the persistent student.

"Well you can get close but when you look -- when you got exactly one-hundred for the *a* value what did you set for the *w* value? It was ten point zero, zero, zero. Okay? It wasn't exact."

"I zoomed again and I got ten." This student was not about to give up on his graphical solutions.

"Did you go another one? You got it exact? Okay. So what you are saying is, If you go through enough zooming of the graph you can eventually get the exact value. Again, I think we have to know the algebraic solution. Does anybody have any questions about this?"

Troy went over another example, this time doing only the algebraic solution. A number of questions came up both in setting up the function and in completing the square of the quadratic. With just ten minutes left in the class, Troy assigned four problems out of the textbook for homework.

"What I would suggest you do people; you've got about fifteen minutes here. Why don't you do the first two problems using the graphing calculator and I would suggest that you would confirm your results by also doing the problem algebraically. Let's do the last two strictly the algebraic way. You only have use of the calculator till the end of class."

Most of the students spent the remainder of the time starting the homework questions. A number of the students had questions and Troy responded to them individually. One of the students packed up his books. I guess he decided he would risk having to squirm the next day in class if he got called on to answer one of the homework problems.

Troy's Use of the Graphing Calculator

Troy's first use of graphing calculators with his Math 20 students was in the unit on the quadratic function. Over the course of this eight day unit, the graphing calculator was used in five classes and in three different ways. At the beginning of the unit he used the it as a tool to generate graphs so that his students could investigate transformations of the parabola. Throughout the unit the calculator was used as a device for checking either the accuracy of a sketch or an algebraic solution. The third way the calculator was used was to find graphical solutions to maximum and minimum word problems.

The Graphing Calculator as a Checking Device

Probably the most obvious use of a graphing calculator in the instruction of mathematics is as a tool to verify either the sketch of a graph or an ordered pair. Troy frequently used the calculator in this way. Students were instructed to graph parabolas based on their knowledge of the role of the parameters a, p, and q in the equation

$y=a(x-p)^2+q$ and then check their sketches by graphing the function on the graphing calculator.

> I am going to put some functions on the board and I want you to think about it for a minute and tell me what the graph looks like and then we will check it out by putting it on the graphing calculator.
>
> I want each and every one of you to draw the graph of the function just free-hand. Plot it out. Draw yourself a little grid and locate where you think the vertex is going to be and draw the graph and then we will check your answers using the graphing calculator.

When working with the question, ***find the equation of a function whose vertex is at the origin and passes through the point (-2,-2),*** the students were warned, "This is one that is going to test your understanding....You will not have the graphing calculator to fall back on." Troy instructed the students to plot the given points (by hand) and get an idea of what the graph should look like. Then he took them through his method to solve the problem. Once he came up with an equation for the function he advised the students to use the little sketch they made to confirm the correctness of the equation. Although earlier he suggested the calculator would not be useful for this problem he told them,"While you can't use the calculator directly here, it certainly can be used as a device to see if it's done correctly."

Throughout the unit Troy stressed the importance of the students understanding the relationship between the shape and position of a parabola on a Cartesian Plane with the algebraic expression for the parabola. A student's understanding of this relationship was determined by the student's ability to accurately sketch the graph without the use of a graphing calculator.

> I am going to expect you to know what the graphs look like and know [how] to draw them by hand.
>
> [A]t the end of this unit we don't want to have to depend on a graphing calculator to produce these graphs... You will be able to produce the graphs manually but without having to make a table of values.

Troy moved the students quickly from graphing quadratic functions with the graphing calculator to simply using the calculator as a means for checking the graphs they drew "by hand." Three classes were used to investigate the transformations of the parabola. The calculator was not used even as a checking device after the initial three lessons on transformations. After those lessons Troy used the calculators in class only on one more occasion, when he introduced max/min problems; therefore, only those students who owned their own graphing calculators had access to the calculators all the time.

The Graphing Calculator as a Tool for Producing Graphs

A second way of using the graphing calculator in mathematics instruction is to use the calculator to produce a number of graphs from which the students can look for a pattern or relationship. Troy introduced the transformations of the parabola in this way. For each of the parameters a, p, and q, Troy set up "investigations" that consisted of a couple of examples of the equation $y=a(x-p)^2+q$ with changes to each of the parameters.

The investigation to determine the role q plays on the parabola consisted of three equations (see Figure 2) with the instructions that the students were to graph the functions on their calculators, sketch the graphs and then try to determine what was happening to the graph.

$y = x^2$
$y= x^2 + 2$
$y = x^2 - 3$

Figure 2
Troy's Investigation 2

Although the teacher called this an investigation, it was highly structured and carefully monitored. The students were given even more direct guidance if they requested help. They were told to put all of the graphs on the same coordinate system since it would be easier to see differences and notice they were given the equation $y=x^2$ as a guide to work from.

> You want to do the y equals x squared one first then y equals x squared plus two because what you are trying to investigate is what happens when you add two to that function and if you do all three of them at once then you are going to be a little bit confusing [sic].

When the time came to draw conclusions from this investigation the teacher directed the students to focus their attention back on him and then he asked a student to suggest what was happening to the graphs as the parameter q was changed. The student correctly indicated that the graph was moving up and down the y-axis and after prompting by the teacher the student also agreed that the graph was not changing its shape.

The students only used the calculators to investigate the transformations of the parabola in the manner described above. They graphed the examples given and then recorded, in their notebooks, the role of each parameter. The teacher also went on to suggest and make notes on how this form of the quadratic equation is useful for graphing equations by hand (that is, it provides you with information about the shape of the graph, the direction in which the parabola is opening, and its vertex).

The Graphing Calculator as an Aid for Finding Graphical Solutions

The graphing calculator is a new tool for analysing graphical solutions to max/min problems. This is the third way in which Troy used the graphing calculator in his Math 20 unit on the quadratic function. Students were introduced to both graphical and algebraic solutions to max/min problems in the same lesson. After setting up the equation for a max/min word problem, Troy, to graph the function, used the over-head graphing calculator and the students followed on their own calculators (see section titled Troy 's Lesson on Max/Min Problems, pp. 30 - 36). Then everyone used the trace key to find the ordered pair that satisfied the conditions of the problem. As the members of the class worked through the problem Troy explained the graphical analysis.

> I just want to know when the maximum area occurs, so that must be up here at the top of the curve. Okay, so let's trace our graph ... close to one-hundred is the maximum area when x is, well, as close as you want to get to ten.

Troy emphasized the difficulty in obtaining the exact value of the maximum area from a graph on the calculator. The students used the zoom key to try to get the best possible approximation for the maximum. One of the students did get an integer solution but Troy reiterated the problem with the calculator's ability to compute solutions for the function.

> The thing we have to recognize about the graphing calculator is that it's good to give you values that are very close, but it doesn't examine every single point on the graph.

Once the students found the approximation for the maximum area on their graphing calculators, Troy demonstrated a method for finding an algebraic solution to the problem. When he finished he pointed to the completed square form of the quadratic function he had graphed previously and he said, "There's the proof people.

There's the proof of what we read off the graph." Troy did a second example, but he found only an algebraic solution. When he assigned the homework he told the students to do four questions: two problems using the graphing calculator, which he suggested were to be checked using the algebraic method and two problems using the algebraic method only.

Troy's Espoused Philosophies

Troy's strong mathematics background was evident in our discussions of mathematics and mathematics education. He is familiar with the major names and schools of thought in the history of mathematics and, although he claims not to have thought about his beliefs very often, he was able to clearly articulate some of those beliefs.

> When you get into the routine with your classes, you never sit and stop and think in terms of what your own beliefs are. They are in there; they just don't get to the surface very often.

From our discussions of mathematics, I noted a distinct separation between Mathematics (the mathematics of universities and jet propulsion laboratories) and high school mathematics. Mathematics is something you do, whereas high school mathematics is a set of knowledge consisting of facts and skills. High school mathematics is a foundation for doing Mathematics. Troy describes school mathematics as large jigsaw puzzle. The pieces get put together and (hopefully) slowly a coherent whole is formed. This suggests that mathematicians doing Mathematics are involved in creating new jigsaw puzzles which will be boxed and shaken up then presented to mathematics students to build somebody else's creation -- not their own.

In addition to discussing mathematics and high school mathematics we also examined Troy's beliefs concerning the nature of how students come to know mathematics and the role of technology in mathematics instruction and learning. These discussions are the bases for the following expositions on Troy's beliefs, attitudes, and conceptions of mathematics and mathematics education.

On Technology

> I view the calculator as being a tool that can enhance the learning of mathematics, but I think that we have to be very careful how we use them. I really have a concern about Alberta Education [giving] kids blanket approval to use calculators any time, any way they want. I have a problem with that.

Troy's concern, expressed here, was in reference to the graphing calculator but his concern also extends to simple four-function calculators. "It has always been a great concern to me how much pushing of the technology they do at the lower levels. As a consequence, kids have an easy out for [not] doing a lot of arithmetic." He believes over use of technology results in the students failing to develop their understanding of mathematics. "They never do develop the skills of mental calculation. They never really come to an understanding of what numbers are all about." Another time his concern was expressed like this: "We are getting kids at the high school level that are almost mental cripples through their use of the calculator."

Troy is very cautious about the use of graphing calculators in his instruction. He believes the biggest advantage with using the calculators is the time that it saves. "In the past we did the graphs by hand -- plotting ordered-pairs and stuff like that. Looking back that was certainly an inefficient use of time." In fact, if it was not for the time-saving factor, that the graphing calculators produce, Troy does not think there would be any necessity for calculators in high school mathematics.

After all these instruments, as he points out to his students, are not nearly as precise as mathematics is. Like he told them in their lesson on solving max/min problems, "the advantage of solving for the vertex algebraically is that you know precisely what the answer is, if you solve the problem graphically the best you can do is approximate a solution." So instead of risking poor algebraic skills Troy's solution to Alberta Education's requirement of using technology in mathematics is to use the graphing calculator in very limited and structured ways -- that is, to generate examples of graphs and to demonstrate the existence of graphical solutions to word problems.

<u>On High School Mathematics</u>

Two phrases are common when Troy discusses the nature of high school mathematics -- foundational knowledge and basic skills. He believes that school should provide the students with "a foundation of knowledge that will enable them to function on the job and in society." Troy interprets the objective of the high school mathematics program: to provide students with the foundation of mathematical knowledge. This foundation includes the skills and techniques used in algebra, trigonometry, and geometry. He believes the application of these skills in high school mathematics is secondary.

> I prefer to think of the learning of skills in mathematics as, initially providing the kids with a foundation of knowledge in the area of mathematics which will enable them to learn what they need to do in the

real world out there. Now if we can in fact do that by giving them some applications as we're going along -- that's good. But I think that to me that's almost secondary to providing them with a foundation of knowledge, because by the time that a kid gets into a real-life situation a lot of that [specific applications] is going to be lost anyhow. He'll have forgotten.

Troy said he tells his students,"You know, 90% of you will never have to do anything related to having to solve quadratic equations in your life time, but if you do [you've] got the background knowledge."

The curriculum's emphasis on non-routine problem solving in mathematics is seen by Troy as somewhat of an anomaly. He points out that government says we should be doing non-routine problems but he believes there is not enough time in the program to do them. Furthermore (Troy claims) where there has been an emphasis on non-routine problem solving it has been done "with the erosion of their [students] basic mathematical skills." He stresses:

> If a kid does not have the basic foundation of mathematical knowledge that will enable him to practice some of these problems, then teaching them all the non-routine problem-solving strategies is almost wasted on them. So I still believe very strongly in the establishment of [the] basic foundation of mathematical knowledge upon which you can build.

Troy suggested that students coming from junior high school should have the beginning of a foundation of mathematical knowledge. For him this implies students should be "literate in terms of numbers"; "knowledgeable in the use of variables"; "skilled in using numbers and variables"; and have "geometry skills." Once the students have completed high school mathematics these foundations would also include a basic knowledge of algebra and a basic understanding of trigonometry. Troy believes that these (all of the above) are the basis for the student's general understanding of the core of mathematics. As Troy put it, "these are the basics, the things that have been around for centuries."

Proof is another aspect of mathematics that has been around for centuries, yet Troy does not consider proof as a key component of high school mathematics.

> I try to give the kids little glimpses into the development and the proof of certain major results of mathematics ...I just want to give students an appreciation for (you know) what mathematics is about and the kind of thinking that goes into actually developing mathematics. But at this level I think that is entirely an enrichment aspect of this. As a high school teacher, I'm still governed by the fact that my kids at the end of grade 12 -- my kids still have to demonstrate that

> they understand the mathematics and can apply it independently of the development of the proof.

From the discussion of proof in mathematics arises the question of logic and methods of proving something. Troy sees deductive reasoning as a fundamental methodology in Mathematics but does not specifically mention the place for it in high school mathematics; he admits experimental reasoning or inductive proof, although "it does not have the same degree of sophistication [as deductive proof]," has a place in the "development of new mathematics -- to some extent. But certainly, in the learning of mathematics, experimental thinking has its place."

<u>On Mathematics</u>

Troy speaks with a great deal of passion when he discusses mathematics, both in his class with his students and in discussions with me. He talks of the beauty, rigor, and power of mathematics.

> You know exactly what the solution is.
>
> Where something is stated as mathematical knowledge it has been established without a doubt.
>
> I think calculus is one of the most powerful tools ever invented.
>
> Graphical solutions ...[have] a certain beauty to them.
>
> [A]lgebraic solutions could be more perfect [than graphical solutions]. They are generally more precise. ... I find that most appealing.

Troy also speaks of the creative and satisfying nature of mathematics.

> We are going to do algebraic solutions because they seem somewhat more satisfying.
>
> Because of the very nature of mathematics there is often creative ways of dealing with topics that are centuries old.

And he expounds the boundlessness of mathematics.

> [A] mathematician actually presents knowledge which kind of gives us the range of what's possible.

He talks of the contributions mathematics makes.

> [I]t's people like physicists and so on that will find what in fact is real in this mathematics that has been discovered. And it is often the case that it is not until

many years after the fact that in fact uses are found for particular knowledge that was developed in mathematics."

Although Troy discusses the relationship between mathematics and physics, he certainly does not suggest that mathematics can or should be reduced to a supporting role for physics or other applications. Mathematics has a structure of its own independent of science and different than science. Scientific knowledge he claims is hypothetical whereas mathematical knowledge is not.

> Most scientists arrive at their conclusions inductively: they experiment; they observe. And so their view of the world is arrived at inductively. But the mathematicians don't work that way at all. Mathematicians primarily use deductive processes to arrive at their knowledge. Now certainly there are certain things that are hypothesized and then they try to go about providing proof.

He says there is "the slim possibility that something [a piece of mathematical knowledge] may be refuted but where something is stated as mathematical knowledge it has been established *without doubt.*" Until mathematical knowledge has been proved "it is in a different domain." Troy discusses Fermat's Last Theorem[1] to make his point.

> Fermat's Last Theorem has basically been around for several centuries as an hypothesis because while Fermat claimed to have a proof for his theorem nobody in the last 200 years has been able to discover what it is. So it's called Fermat's Last Theorem 'big-question-mark'.... It's conjecture.

Mathematics of the past is established without a doubt and new mathematics is built up from there. Troy sees mathematics as cumulative. Mathematics is built up from its foundations and stretches out to provide us with the "range of what is possible." So it goes for learning mathematics. Knowing your multiplication tables is the basis for your knowledge of arithmetic. Knowing arithmetic forms your basis for understanding algebra and so on.

On Students and Instruction

Troy believes the overall picture of mathematics is elusive to his students. He says they leave high school with such a "narrow view of what the world is about in terms of mathematics" and they do not see the potential mathematics has for describing the world around us. Part of the reason for this lies with the students themselves. "[The students] don't necessarily want an explanation of where this concept fits into the

[1] At the time of the interview when Troy made this statement Fermat's Last Theorem had not yet been proven.

scheme of things. They want to know what kinds of things [they] have to know to pass the next test." Another problem Troy sees is that he believes it is hard to see the big picture before you work with some applications. For example he suggests physics helps students understand the overall picture of mathematics.

> I think it is not until you start using the language of mathematics in conjunction with physics and stuff like that that you really get to the actual description. I think that's one of the things that makes mathematics at this level and at lower levels perhaps difficult. It's tough to start tying it to other external matters that might make it real for the kids."

Unfortunately the applications are not appropriate mathematical experiences until after the basics are learned. So here we have a circle. You cannot understand mathematics unless you can see the whole picture but you cannot see the whole picture until you study some applications of mathematics. You cannot study applications until you have the basics. So, how is it then that students come to understand mathematics?

Troy describes the methods he favors in his mathematics instruction. He suggested that he likes to introduce concepts with illustrations and examples. These, then, are intended to lead to generalizations. Finally the students, by working independently on exercises, learn the concepts and the processes intended. He suggests an important component of this learning process is the independent work. Doing exercises helps students come to understand the mathematics.

> Well I always prefer to have plenty of exercises ... [But] I usually give some indication that it is not necessarily my expectation that they will all do the same amount of homework. Because I recognize that some of the kids at the Math 20 level will in fact learn what I want them to learn very quickly [with few exercises] but others, because of the difficulty that they have had in mathematics, it is going to take them longer. So what I usually tell them is, Here is the minimum amount of exercises that I expect you to do. Now those of you that are experiencing difficulty with this topic, you know the most important question to answer when you get to the end of the exercises is, Do you understand it? Do you know how to do it? Because if you don't, then you better do some more.

In Math 20 Troy believes the students *should* (Troy's emphasis) be able to do the more abstract thinking and therefore he does not extensively use manipulatives. "I would not want to use manipulatives to the exclusion of everything else. I think that at some point kids have to be able to in fact see things visually in their minds and they have to be able to deal with concepts without having things to pick up and turn over."

The goal then of Troy's instruction is the students' understanding. He repeatedly talks about understanding with me and he lectured the students on it in class. I heard him tell his students , "I don't want you to memorize that piece of information; it's better to understand." He told me one of the problems with students is that "some students build their personal knowledge of mathematics by imitating what the teacher is doing." So he warns his students that:

> "You [do not] learn anything worthwhile in my classroom by watching me do mathematics ... the most important thing that [you] do in mathematics is what [you] do after [you] leave the room. And if in fact that part of the mathematical learning process is either ignored or not taken seriously then the kind of mathematics that [you] are going to be learning is very much limited to what you can remember about what I demonstrated. In fact you are not building your own personal knowledge of mathematics you are in fact learning to mimic or parrot what the teacher's doing and that of course will eventually come to a grinding to a full halt."

Troy tells the students that once they understand something its "like riding a bicycle." It's not something you have to go back and learn again. You just need to refresh "your memory of what the skill was and boom you are on your way."

<u>Troy's Philosophies of Mathematics and Mathematics Education Made Manifest</u>[2]

If Margaret Thatcher is Paul Ernest's case study of the Industrial Trainer, then Troy could be his case for the Old Humanist.[3] Troy's beliefs about mathematics and mathematics education are closely aligned with those Ernest attributed to the Old Humanists. In Table 5, I have tried to show the parallels between the two. Even the language that Ernest used to describe the Old Humanists and the language Troy used in our discussions and in his class is remarkably similar. Troy's philosophy of mathematics includes a view of mathematics as an exact and precise body of knowledge consisting of absolute truths. He believes mathematics is (mostly) a deductive system. "Most mathematics," he claimed, "has been developed and proven deductively." Troy was the only teacher whose attitudes about mathematics included its aesthetic qualities. Troy's view of mathematics education is centred around his view of mathematics. He strongly believes that the aim of mathematics education is to provide students with the foundation of mathematics and basic mathematical skills.

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[2]It would be useful for the reader to become reacquainted with the summary chart of Paul Ernest's Ideologies of Mathematics on page 10.
[3]Eisner and Vallance (1973) referred to people with a similar orientation (to curriculum) to the Old Humanists as Academic Rationalists.

Troy's philosophies of mathematics and mathematics education were unaffected by the availibility and use of graphing calculators for his instruction. Instead, the use of graphing calculators seems to have provided an exceptional opportunity to observe Troy's philosophies as he enacted them in his instruction and interaction with his students. Troy's Old Humanist philosophy of mathematics was manifested in his instruction and in his interaction with the students on many occasions and in a number of ways. His investigation of transformations reflects his view that high school mathematics can be done inductively; but, the few examples he used in the investigation reflects the view that the important mathematics is in the algebraic structure of the quadratic function not in its graphical representation. He demonstrated graphical solutions to max/min problems to facilitate understanding but he required the students to do algebraic solutions; that is, the mathematics that requires the mental work which the Old Humanists value. He used, and encouraged his students to use, very precise language in his mathematics class. In class his voice expressed his enthusiasm for the precision of the algebra when he stressed the exactness of the algebra over graphical analysis. He repeatedly emphasized this to his students when demonstrating the superiority of the algebraic solution over the graphical solution to the max/min word problems. When he told his students that they were doing pretty basic mathematics and that in Math 30 and Math 31 they would be doing higher level mathematics, he implied high school mathematics has a hierarchical nature and just like in mathematics, the higher level it is, the more sophisticated and powerful it becomes. He used calculus as his example of powerful of mathematics.

Table 5
Comparing Troy to Ernest's Old Humanists

Views on:	Ernest's Old Humanists	Troy's Espoused Views	Troy's Views Articulated in Class
Mathematics	-a body of pure objective knowledge, based on reason and logic, not authority.	-where something is stated as mathematical knowledge it has been established without a doubt	-you know exactly what the solution is; there is no guess work involved
	-a system of rigour, purity, and beauty (p.169)	-graphical solutions have a certain beauty to them -algebraic solutions are more precise and this is appealing and satisfying	-primarily we are going to use algebraic solutions because they seem somewhat more satisfying [than graphical solutions from the calculators]
School Mathematics	-like the discipline itself, a pure, hierarchically structured self-subsistent body of objective knowledge	-providing kids with a foundation of knowledge in the area of mathematics -Multiplication tables are your basis for your knowledge of arithmetic, which forms your basis for understanding algebra and so on	
	-students are encouraged to climb up this hierarchy as far as possible, according to their 'mathematical ability'.	-numbers, variables, geometry, algebra, and trigonometry are the basis to understanding mathematics (the basics which have been around for centuries)	
	-as they ascend, they get closer to 'real' mathematics, the subject taught and studied at university level" (p. 176)	-I try to give the kids little glimpses into the development and the proof of certain major results in mathematics -- an appreciation for what mathematics is about and the kind of thinking that goes into the actual developing of mathematics.	-the tools of calculus open up a whole bunch of doorways for you

Views on:	Ernest's Old Humanists	Troy's Espoused Views	Troy's Views Articulated in Class
Learning Mathematics	-reception and understanding of a large, logically structured body of knowledge	-the learning of mathematics is like a large jigsaw puzzle: when we are finished we will have a picture that we can look at and appreciate, but if a bunch of those pieces are missing the picture won't make any sense	-This is what we will call our starting point (the vertex) in our study of parabolas. and we should know what that graph looks like because all of our discussions ... centre around that graph....move that graph around...make it wider...narrower...flip it over. That's what the entire discussion ...is going to centre around.
	-internalization of pure conceptual structure of mathematics	-strive for understanding of concepts	-you have to understand the whole ball of wax. It's not good enough just to get a bit of the surface.
	-mathematical knowledge allows learner to solve problems and puzzles (p.177)	-have the knowledge that enables them to solve problems	-90% of you will never have to do anything related to...quadratic equations in your life-time, but if you do you've got the background knowledge.
Teaching Mathematics	-master-possessor of knowledge, passes it on to students	-I try to pass along as many of my own strategies to the kids as possible	
	-electronic calculators only for students who have mastered basics	-it has always been a concern to me how much pushing of the technology they do at the lower levels; they never do develop the skills of mental calculation; they never do have the numerical literacy	
	-hands-on activities inappropriate except for low attainers (p.177-178)	-with Math 13s it is very appropriate but with a Math 10 class I would be reluctant [to use concrete materials]; they should be able to do the more abstract thinking	

Troy's strong, traditional mathematics background contributed to his views and the manifestation of those views in his mathematics instruction. Troy teaches mathematics to his students. His focus is the mathematics. Pheonix, the subject of the next case, on the other hand is student-centred: her focus is on teaching students mathematics.

CHAPTER IV
Case II: Pheonix

Biography

Pheonix has been teaching for 11 years. Of those 11 years she has taught mathematics for seven years and for the past five years has been teaching high school mathematics at a rural school. She has taught all but Math 30 and Math 31. Because Pheonix is in a junior-senior high school she continues to teach both junior high school mathematics and high school mathematics.

Pheonix believes it is her job to excite and motivate the mathematics students she teaches. Every time I was in her classroom I was impressed by her high energy level and the enthusiasm she projected throughout the whole period. For eighty minutes she moved around the classroom helping, praising, and laughing with students. She picks up on any and everything the students do right to try to improve their image and feelings towards mathematics. Her enthusiasm for mathematics seemed to rub off on the kids. She pushed them and encouraged them; she even straightened collars.

Pheonix's Math 33 Class: A Vignette

Sixty minutes into this eighty minute class, Pheonix's Math 33 students were still listening attentively even though they had been engaged in a large class discussion for most of the first sixty minutes of class. During the previous class the students worked on a guided-discovery activity that led them through the transformations of parabolas. This day was spent summarizing the students' findings from the previous day's activity and moving the students on to the next concept in the unit. Pheonix had just explained how to complete the square of a quadratic in the form $y=ax^2+bx+c$ and had written a second example on the board.

"I want you to form an equation so you can graph it," Pheonix instructed her students as she began walking around their desks. "Come on -- do it. Come on -- push yourself."

As the students quietly attempted to work through this on their own, Pheonix added bits of encouragement. "I'm proud."

One student turned to another and jokingly said, "Push yourself, Kal."

Pheonix took this in stride and added, "That's good, Kal."

As the students worked away at the problem, Pheonix made her way around the room looking over their shoulders.

"Mrs. Smith," a male student called out softly.

Pheonix went to his desk, looked over his work and was about to tell him exactly what his problem was but instead offered him some advice.

"The square always -- okay Sandy, one of the things you have to do is pick up your eye for detail. There is a difference between having your square inside the brackets --"

"Oh I get it. This is part of the trinomial," the student interrupted.

"Well put," she said as she mo[illegible] from his desk.

"Mrs. Smith. Do you want us [illegible] that?" One student asked as he pointed to a list on the board that was t[illegible] previous example.

"Yeah."

"The vertex and everything?"

"Get in the habit of vertex, axis of symmetry, domain, range, max, min--" For emphasis, she counted these off on her fingers as she stated each one,"--everything you should know about a graph. This equation tells you all kinds of things."

"Everything?"

"It tells you everything. And then we will look tomorrow for the *x*- and *y*-intercepts."

"We made it all up," another student interjected.

Earlier in class, Pheonix had praised the students for having discovered the process of completing the square on their own. That is, they developed the process as she guided them through a carefully designed example.

"Yes, we invented it." Pheonix replied.

It took the students about five minutes to complete this question. As soon as a few began talking Pheonix signalled the students in the class to move on. She worked through the example on the board by calling on a couple of students to provide answers to various parts of the question. Then she told the students to take out a clean piece of paper. The conversations started up as the students looked for and borrowed paper from each other.

"Is this for marks?" One of the students asked the inevitable question.

When Pheonix replied yes, a number of the boys hooted affirmatively. Pheonix made a point of regularly assigning work for marks and the students responded favorably to this strategy for getting them to do their homework.

"Can you do this one, Jay?" One student teased another.

When Pheonix told them that the assignment was due at the end of class the hooting began again. I assumed they were glad that they would not have to take it home to do.

"Okay. The operative word being -- it's for marks," she reminded them.

"You get marks today," a student near me told a new class member.

"Sketch the graph from the equation. What I would like is -- I will give you a little equation. I would like you to sketch it -- nice and neat. No sloppy [work] or you get zip. Okay?" Pheonix made it clear what they were expected to do.

"You'll ge. zip," one student warned another, in this on-going interaction between classmates.

As Pheonix wrote the assignment questions on the board, a few students started talking and it began to get noisy in the room. One student tried to ask a question but he was not heard. Some of the others who were working started to become irritated with the ones that were noisy. Pheonix put a quick end to the disruptive behaviour by simply saying, "Quiet."

"All you want is the graph?" The student who tried before managed to asked his question.

"All I want is a graph. Please copy the equations and all I want beside it is a little graph."

Now all of the students were back on task. With only ten minutes left of this eighty minute class the students were still very focused on their math.

"Okay you guys. In number five I want you to do some extra. I want you to state the following -- okay. So, I want a sketch but I also want you to *a of s* (axis of symmetry) and the last one -- number six-- I want you to do just what we have learned -- completing the square." Pheonix put her chalk down.

"Okay, Randy just turn around in your desk quietly. You should be concentrating solely on your own work please."

She did not miss a beat.

Pheonix's use of the Graphing Calculator

I was fortunate enough to observe Pheonix use the calculator as she taught the quadratic unit in two different courses: Math 20 (same as the other teachers) and Math 33 (senior level non-academic math course). Pheonix used the same approach for teaching this unit in both classes. I will use data from both groups and indicate which class the data is from by putting M33 after entries from the Math 33 class and M20 following data from the Math 20 class. The units are virtually the same for both groups however the Mathematics 20: Course of Studies (1990, p.11) suggests :

Students will be expected to:

1.1 sketch the graphs of quadratic functions written in standard form, $y=a(x-h)^2+k$

1.1.1 investigate the effects of the parameters a, h, and k in $y=a(x-h)^2+k$ using a calculator or computer

1.2 transform quadratic functions from the general form $y=ax^2+bx+c$ to the standard form $y=a(x-h)^2+k$ by completing the square

1.3 find the vertex, axis of symmetry, domain, range, maximum, minimum values and x- and y-intercepts of a quadratic function **from its equation or from its graph** [emphasis added]

1.3.1 solve problems that involve quadratic functions, by analysing the functions depicted in graphical **and** equation form. [emphasis added]

In the Math 33 program there are slight variations to two of the Math 20 objectives:

1.2.1 (1.3 in the Math 20 program of studies) find the vertex, axis of symmetry, domain, range,maximum or minimum values, and y-intercepts of a quadratic function. [does not specify *x-intercept* nor *in equation or graphical form*]

1.2.2 (1.3.1 in the Math 20 guide) solve problems that involve quadratic functions, by analysing the functions in graphic **or** equation form. [emphasis added] [note the word or was used instead of and] (Mathematics 30/33: Interim Teacher Resource Manual, 1991, p.196)

As you will see in the section that follows, Pheonix used the calculator in only one of the lessons on the quadratic function in both her Math 20 and Math 33 classes.[1] The students used the calculator in one or, as in some cases, two ways: as a graph generating device and as a checking device. In neither course did she use the calculator to find graphical solutions to max/min word problems.

[1] Pheonix had previously used the graphing calculator in her Math 20 and Math 33 classes in an activity that was used to introduce the students to the graphical forms of a number of functions and the concepts of domain and range.

The Graphing Calculator to Generate Graphs

Pheonix used an activity (for both Math 20 and Math 33) that she modified from the professional development workshop she had attended. In this activity the students were instructed to do five investigations. These investigations required the students to graph parabolas for given equations in the forms: $y=ax^2+bx+c$ and $y=a(x-h)^2+k$, where each of the parameters was examined in isolation from the others (see Figure 3). The students worked through the parameters by graphing 3 or 4 graphs on the calculators, sketching the graph beside the equation and then answering specific questions about the graphs of the parabola.

Investigation #1

Explore the role of the coefficient a, where $b=0$ and $c=0$, then $y=ax^2$.

1. Graph $y=x^2$, $y=3x^2$, $y=4x^2$ — What happens to the parabola as value of a increases? $(a>1)$

2. Graph $y=x^2$, $y=\frac{1}{2}x^2$, $y=\frac{1}{3}x^2$ — What happens to the parabola as value of a decreases? $(a<1)$
   What happens to the parabola as a gets closer and closer to zero?
   What similarities do these graphs have?

3. Investigate the effect making a a negative has on the graph of the graph of the parabola. Write you own equations. (Start with $y=x^2$. Why?)

a)What is the direction of the opening of the graph if $a>0$?
b)What is the direction of the opening of the graph if $a<0$?
c)If $a>0$, does $y=ax^2$ have a maximum or minimum point?
d)If $a<0$, does $y=ax^2$ have a maximum or minimum point?

Figure 3
Sample of Pheonix's Investigation of the Quadratic Function

The students were given the opportunity to work on this activity in pairs or individually. Many students moved their desks together; although they still worked mainly on their own, unless they had a question at which point they would consult with each other or the teacher. Occasionally students found discrepancies in their graphs which caused them to collaborate to find the source of the difference. There was some discussion when the students had to answer the questions concerning the role b plays on the graph of ax^2+bx+c. Other than these discussions, most of the students worked through the activity with few questions. As the students worked Pheonix moved around the class answering questions about the investigations, encouraging students to stay on task, and helping with the operation of the graphing calculators.

Investigation 5 consisted of two equations $y=2(x-3)^2-5$ and $y=-3(x+2)^2+4$ and questions that asked the student to find the maximum or minimum point, the equation of the axis of symmetry, and the coordinates of the vertex. The second part of this investigation asked the students to sketch four graphs (see Figure 4) and then check the answers with the calculator.

Using what you have learned sketch the following graphs. a) $y=(x-1)^2-3$ Check with your calculator to see if you are right. b) $y=x^2+3$ c) $y=-2(x+4)^2-1$ d) $y=-\frac{1}{2}(x+3)^2+2$

Figure 4
Pheonix's Investigation #5 - Summary Questions

This activity took up most of an eighty minute class for both the Math 20s and the Math 33s and was summarized the next day in each class. Since the handout for this investigation was very detailed, Pheonix did not have to explain the activity to the class as a group. However she did help out individuals that requested her assistance. The period following the guided discovery activity Pheonix led a large group discussion to summarize the investigation. On the top of the board she wrote "Summary of Quadratic Function Investigation" and then drew the sketch of the graph of $y=x^2$ (M33). She then asked the students, "If we have $y=4x^2$ will the graph be wider or narrower?" As she said this she opened her arms and closed them to demonstrate the parabola. A number of students used their arms to demonstrate as they called out "narrower." She wrote under the sketch of this parabola, "large numerical coefficient: narrow parabola." Next she put the equation $y=\frac{1}{8}x^2$ and used her arms to show that this parabola widens. Then she asked the students what would happen to the parabola if the number got smaller. The following discussion (M33) demonstrates how Pheonix encourages her students to think about mathematics.

> Pheonix: One-sixteenth and then the graph becomes wider. What if I made it one thirty-second x squared?
> A number of students responded: Wider.
> Pheonix: The question was asked, [on her handout] would it ever become a straight line.
> Kal: No.
> Pheonix: Okay. Let's have some comments. We heard Kal and -- okay, Rick what did you think and what were your reasons?
> Rick: Because it's not, uh, a linear equation.
> Pheonix: Okay, but that doesn't make it --
> Joe: Because it keeps going down, down, and getting closer until it goes the other way.
> Rick: Yeah, but it will never be straight --
> Joe: But it has to be.
> Rick: Cause a parabola has a curve.
> Joe: It has to be a curve?
> Matt: It's not a parabola anymore. It's a linear -- linear equation. [Another student tries to say something.]
> Pheonix: Okay, Don has something to add.
> Don: So you get closer to zero but you don't reach it?
> Pheonix: What if you put zero? It is a number.
> Don: Yeah, but zero x squared will still be a --
> Pheonix: Zero x squared equals what?
> Pheonix and Don: Zero
> Pheonix: So where would that line be?
> Don: That would be linear. You're not -- You're not -- You could get one over one billion you know.
> Pheonix: What you are saying then is it is always approaching zero but we never --
> Don: Yeah.

The other parameters (p and q) were discussed in the same way. Pheonix drew two sketches for each and asked the students where the vertex moved to; then she noted the transformations under the appropriate sketches (see Figure 5). She wrote on the board at the end of this summary $y=a(x-p)^2+q$ and used arrows to point to each parameter and noted what the parameter did to the parabola. Before moving on from this discussion of the roles of the parameters, Pheonix worked through an example and asked the students to give her the axis of symmetry and the vertex of the parabola, the domain and range of the function, and whether the parabola had a maximum or minimum point. After giving the students a couple of equations to work through she asked them what would happen to the absolute value function under these same transformations. They very briefly discussed that the same kinds of shifts and distortions would happen.

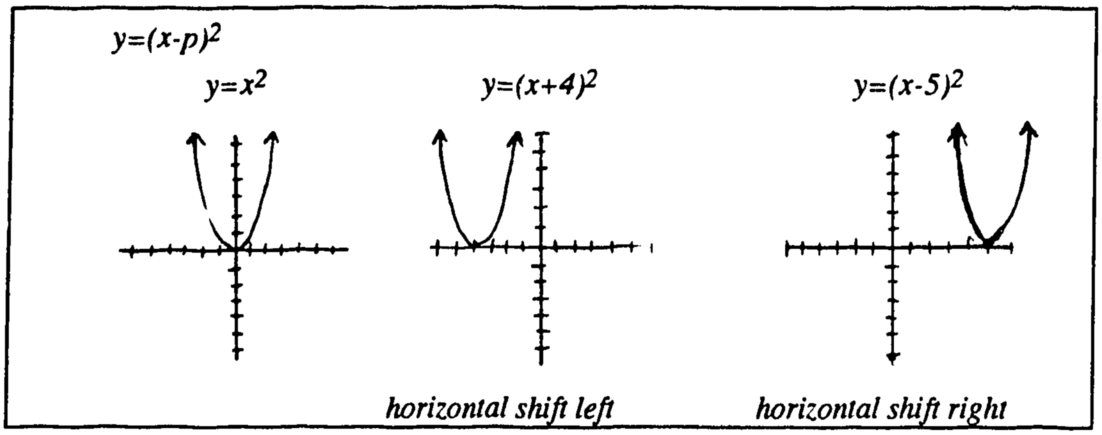

Figure 5
Example of Pheonix's Summary Notes

The Graphing Calculator as a Checking Device

Pheonix gave the students the opportunity to use the graphing calculators as a means of checking their graphs of parabolas from equations in the form $y=a(x-h)^2+k$, but only four of twenty-one students (M33) took advantage of this opportunity. The students seemed to be confident in their own work and did not rely on the calculators to confirm their work. Pheonix is very careful to either mark questions herself or go over responses to assigned questions, so I must assume that the students knew they would check their work in one of those ways.

Pheonix's Espoused Philosophies

Pheonix's beliefs, attitudes, and conceptions of mathematics and mathematics education are framed by her focus in teaching. Let me attempt to demonstrate that focus. When asked to tell me something about herself that she felt was important, she wrote:

> I'm interested and care about the kids -- I want them to do their best. I learn the courses and try to present concepts so the kids will understand them. I offer help classes two nights a week after school.

This attitude of care permeated her instruction and interactions with her students and our discussions about the nature of mathematics and mathematics education.

On Mathematics

Although Pheonix views mathematics as a process that is logical, sequential, and cumulative, she believes that creativity and intuition are important characteristics of good mathematicians and mathematics students. Pheonix said she believes mathematics involves both logical reasoning and creative thought. For example, when I suggested that mathematics could be thought of as a game where you simply learn the rules and play the game, Pheonix responded:

> I think the average player does that. But if you want to become superb, I think you have to learn the different strategies of what goes together. Like you are taking those processes and or the rules and taking it to a higher level. You are creating your own rules.

Later when talking about theorems and proofs she discussed this creative part of mathematics again.

> They are both creative. This [theorem] is more creative because you are coming from nothing. And this one [proof] you are working from something and proving it. I would say the theorem is the more creative of the two.... mathematical theorems are really these kind of great intuitive insights on the part of mathematicians.

I asked Pheonix how she thought new mathematics was created; whether theorems only came from great insights or whether they could be developed from observations. She thought for a moment and then asked, "Why can't they be both?" She then started to say that she was not the right person to be asked this question, but finished her thought by providing an example from her teaching.

> I'll teach a concept and then I'll say "What if this ...? Let's ponder this. If you did this to this equation what would happen to it?" But by the same token, with the graphing calculator, you are going from the observable and making a generalization.

She could see both experimental and analytical approaches in her own teaching, yet she was reluctant to generalize from her teaching to mathematics in general. After making this comment about her own teaching, she returned to the original question about the development of mathematical theorems and said, "I really don't -- I can't answer that."

Even though Pheonix believes mathematics can be taught and possibly developed through inductive reasoning and experimental approaches she denied that mathematical knowledge is like scientific knowledge. "I don't see math as being -- like science is constantly changing all the time. You can't keep up with it. Like -- I mean we still solve the quadratic equation the same way." Immediately she questioned her own belief. "Or maybe I am being too narrow in my scope. I have taught it this way for so long." Her questions must have been uncomfortable for her because she concluded by saying: "I don't like that question."

On Mathematics Education

When I asked Pheonix about her views on the nature of mathematical knowledge she spoke mostly of the how students view and relate to mathematics. She spoke of the impact mathematics has on kids. Some of our most interesting conversation evolved out of the statement that *mathematical truths have a certain inevitability about them -- a world with different mathematical truths is inconceivable.* She responded lightheartedly that who knows what happens on other planets. So I asked her to think about different cultures and whether she believed the mathematics of different cultures is the same or different.

This is an excerpt from that part of our discussion.

Pheonix: I just couldn't get a handle on what I wanted to say or what I really feel. When I think [of] cultures I look and I see a sort of socialization and I guess I don't look at math as being a very socializing process.

Elaine: Okay, and that is the very question in a sense. Is mathematics influenced by society? Is it just like all other bodies of knowledge? Is it subject to the pressures of society in the thought--

Pheonix: I think -- I don't know. There are so many people that hate mathematics; that are so fearful of it. That the only people who like mathematics have any influence on it are the people who are good at it. You know so many people [say]-- "Oh my God I finished that math. I don't want to think math. I don't want to look at a math book. I don't want anything to do with it.
So I guess from a societal point of view mathematics has -- I don't think they have much input other than hatred.

Elaine: This is good, Pheonix. You are the only one who has gotten political on mathematics.

Pheonix: I went for an eye appointment. There were four girls behind the desk and one said "What do you do?" Teach. "What do you teach?" Mathematics. "Oh my God, I hated math." Everyone of those girls said they hated mathematics. They could hardly wait until they got out They never understood one thing. It never made any sense. Four -- there were only five people. Four of them -- four out of four -- four out of five if I counted. But I didn't count because I like -- I --

Elaine: Yeah you're a math teacher so you don't count just because --

Pheonix: Exactly, because --

Elaine: You're different.

Pheonix: So I don't know what are you asking me with that. Are some day society going to say, to hell with this math. It's not important?

Elaine: Well it might be that, too; but, I guess what I thought of the question it was more like "Do you think the mathematics of the Eskimos is different than the mathematics of the Plains Indians or of the West Coast Indians?"

Pheonix: No, I don't think so. They might explain it differently or they might put it in different words, but I think whether you are talking trig ratios or I think the Plains Indians saw the same sort of relationships that the Eskimos as the white man. But on the alien planet?

Although Pheonix suggested that mathematics is not part of the socialization process the above is a wonderful example of mathematics as a socialization process.

She further discussed the socializing nature of mathematics when I suggested that doing mathematics as a profession is exclusive. She spoke not of the exclusiveness of mathematics as a profession but of the exclusiveness of the learning of mathematics.

> So is the learning of mathematics [exclusive] because people hate it or they like it. It's very exclusive. There are very few people that are okay about it. They are either good or bad at it.... The most often used phrase in mathematics -- I can't do it. It is too hard. I can't do it. I don't know where to begin.

This negative attitude is one Pheonix works hard to overcome by sharing her enthusiasm for mathematics and by showing her concern for the students as learners of mathematics. The best examples of this come directly from her instruction and interactions with her students. She constantly tells them that they "can do it" and encourages them to "push themselves." She spends the whole eighty minutes of her class with her students helping them to see that they can do mathematics.

Pheonix seems to believe there is a great separation between what high school mathematics is about and what it should be about.

> I think with teaching math is more for a process and way of thinking that is different than in any other discipline.
>
> I think some kids can go through all of school and learn absolutely nothing about mathematics. They wrote it. They memorized it. But they haven't got a clue what it's all about.

The way mathematics is taught and the way it could be taught.

> They should [do creative work in mathematics]. I'll put it that way. But I don't think they are.... To get the kids to do any creative work there has to be that self-discipline mode. And usually the kids who are the brightest usually do the best in that situation and the ones who are low they don't have that abstract capability. The majority of them don't and so it's so much of a hassle that I

really don't -- I stick to the things that I think the majority of them can do and that's only because that makes it easier for me to control.[2]

What we should be preparing students for and what we do prepare them for.

> The problem with Math 30 [is] we can't decide whether we are preparing them [students] for university or general [sic]. And like everyone is taking it because they think they need it for university but the university profs are telling us we are not preparing them [students] at all -- that the level is so low. There seems to be a contradiction there.

Pheonix believes mathematics is large body of knowledge that can be approached in a unique way -- a body of knowledge that is based on a systematic method. If you understand mathematics you have a means for approaching problems in a certain way. She said she teaches math as a process and a way of thinking. The way of thinking in math, she claimed, is different than in any other subject. When asked how it was different she responded:

> Well it's just the logistics or the order, the sequence, the process. You start here; one thing leads to another and as you grow your building blocks become more expanded. Then you can see connections between the different concepts and that's -- I think -- the interrelationship between the concepts.

She spoke of this process more than once in our conversation.

> I think this is really the process: Step one, step two, step three, step four, and I believe that math is like that where you have to do step one to get to -- It comes before step two. ... But it could be a step-by-step but your mind shouldn't just be trained in that one way; [be]cause if you are trained in only one way like in the army then you don't see the other sides. If you want creativity to be introduced then you have to have that broader attitude.

Here, too, Pheonix pointed to some of the contradictions in high school mathematics; this desire to teach for creativity but believing that the nature of mathematics is hierarchical and cumulative and logical -- where everything progresses step-by-step. Pheonix partly solves this dilemma by using concrete representations for concepts and asking students to ponder "what if" questions. By using the concrete representations, students have the opportunity to discover for themselves how a solution can be obtained by developing a process to come up with the solution.

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[2] Although Pheonix said here she teaches to the majority she was the only teacher that indicated to me that she, in the past, had developed independent programs for some students to enrich their mathematics.

Even though Pheonix encourages her students to discover and develop mathematics (the work of mathematicians) she does not consider them (or herself for that matter) to be mathematicians.

> I think a mathematician is someone who has little squiggles in his head all the time. You know -- Einstein.
>
> Basically I am not a mathematician. I'm just an average person who likes math, who thinks math is fun. I learn the courses and I try to teach them so the kids will understand them.

On Students and Instruction

Pheonix views herself as a learner of mathematics. She believes she learns new things all the time even after teaching a course many times and believes that she has something worthwhile to share with her students -- the enjoyment and enthusiasm of learning and doing mathematics.

> It's been since I've been teaching it [math] that I really see. It's kind of a discovery for me because I am seeing how all this seems related and I guess I am sharing that enthusiasm with my kids. Look what I discovered! Finally-- after teaching it three times, look what I found out! This is really neat!

This focus on the learners of mathematics strongly influences her approach to the instruction of mathematics. The negative attitudes students have about mathematics is a great concern to her and she works hard to change those attitudes.

She builds a relationship with the students in her class based on trust She needs the students to trust that her primary concern is for them and their learning of mathematics.

> I think there is a certain trust built up between teacher and student and if the kids trust you or think you are trying to do the best by them then they are willing to go along [with new forms of instruction]. I know the first year I taught [Math] 20 ... when we did quadratic equations -- you know the swimming pool is surrounded by a deck. I cut out a piece of blue construction paper. [She said to her students] Here's your pool and the artificial turf in green. And the -- you know -- [she encouraged her students] come on stick with me. [The students said to her] 'Mrs. Smith you don't have to do that.' But really for a lot of them that have trouble visualizing from the problem that was useful.

Pheonix believes it is important for the students to be approached at a level they can handle.

> I try to learn the course and present it to them in a way that they will understand it -- as bizarre as that might be. To put it another way, like some of the math teachers I know are very cerebral -- like they are really up here [pointing to her head]. They see everything on such an abstract level. They function way up

> here and the rest of the populous is way down here. Like I'm down here with the rest of the population. And I try to present it in not such an airy-fairy way, but as simplistic as I can.

To that end she presents material as simplistically as she can and with concrete materials (where appropriate) or other aids like the graphing calculator. Furthermore, she suggested that she does not just explain *how* to do something but *why* it is done in such a way. Her goal is the students' relational understanding of a concept not their memorization of the concept. "I try to get the kids to understand. I'm not always successful but I do it anyway."

One feature of Pheonix's instruction that was prominent on the days I witnessed her teach was her emphasis on cumulative review. In many of her class periods she sets aside time for cumulative review. She works on strengthening student weaknesses by giving questions that her students had trouble with but she also includes questions that the students are good at to build their confidence. One day when going over the previous day's review questions (M20) she said, "I will ask another question like this because everybody is still having trouble." Then a minute later she said with respect to another question, "You did it rapidly, accurately, and well. That means your cumulative knowledge is growing. That's great." She said other things like, "I haven't been looking at exponents for a while. I want you to keep up your skill level so I'll give you some of these. I noticed in the help sessions where a few problems are so I will use these." Where the students were getting real good at something she told them she would ask a higher level question or in another case she said she would not ask a question at all since they "knew it so well." Or with her Math 33 students while outlining their midterm she told them, "I think this is our strongest area [factoring polynomials]."

On Technology

Pheonix does not have many or really strong views or attitudes towards technology in mathematics education. She only brings the graphing calculators into her classes for a few guided-discovery activities. This means that she does not use much technology in class but when she does, it is used to enrich the students' understanding through carefully selected activities. She believes part of the problem with students' poor basic skills is a result of overuse of four function calculators to do homework. This problem is partly the result of easy access to calculators at home. At school she controls the use of the calculators (although a few students have their own graphing calculators). Pheonix makes time available in class for students to do cumulative review without the aid of a calculator; thus providing an opportunity for the students to better develop their mathematical skills.

Pheonix's Philosophies of Mathematics and Mathematics Education made Manifest

Pheonix's beliefs about mathematics education probably best fit into Ernest's category of Progressive Educator because of the focus on the student, however, her philosophy of mathematics fits better in the Old Humanist category; therefore, her overall philosophy does not fit well into any one of Ernest's ideologies thus I have not set up a comparative table here. (In a way this demonstrates at least a categorical weakness in Ernest's system.) Pheonix's philosophy of mathematics includes the beliefs that: mathematical knowledge is absolute; mathematics as a process is logical, cumulative, and sequential: mathematics can be done both inductively and deductively; and the process of doing mathematics although logical is also creative.

Pheonix's beliefs about the nature of mathematics could be considered relativistic (Perry, 1970); that is, not only does she acknowledge different perspectives but she can justify the existence of two seemingly opposed positions. For example, this was evident when she explained that mathematics consists of a logical process where one systematically moves from step *a* to step *b*. Then she said,"But your mind shouldn't be trained in just one way -- like in the army -- then you don't see the other sides." Or, she spoke of the superb player of mathematics who learns rules and strategies and then does not just play by the existing rules but creates new rules and strategies at a higher level. Pheonix's philosophy of mathematics education includes her views about mathematics but it also includes her beliefs of what it means to teach -- particularly, her commitment to students. She views teaching as building a relationship, based on trust, between herself and her students; as was demonstrated, when she asked her Math 20s to stick with her as she used concrete manipulatives in class.

Her aims in teaching mathematics go beyond teaching mathematical concepts; she is determined to help her students believe that they can do and understand mathematics. She expressed concern for some of her students because of their negative attitudes towards mathematics. She told me, "The most often used phrase in mathematics is, 'It's too hard. I can't do it.'" She said of their understanding, "They wrote it [mathematics tests]. They memorized it [the mathematics]. But they haven't got a clue of what it's about at all." She told me about one of her best students who asked her why they were doing a practical example from nature instead of just learning the rule. She realizes these attitudes towards mathematics interfere with students' learning and understanding of mathematics. She seems to believe that one of her responsibilities is to help students overcome negative attitudes towards mathematics. She tries to do this by sharing her enthusiasm for learning mathematics and teaching it, not in an "airy-fairy way," but, in a way her kids will understand.

Pheonix seems to view her role as somewhat like that of a motivator[3]. I view her role as that of a facilitator. In class she encourages and praises her students. Her approach, to teaching mathematics, is to build on the students' strengths and work on their weaknesses; hence her emphasis on cumulative review. She also encourages them to be independent learners by suggesting ways in which they might work out their problems. "Look it up. Be an investigator," she told a student when he asked her a what-would-happen-if question. I believe her methods of instruction and attitude of caring foster a suitable environment to help students move from a position of silence where they experience themselves as "mindless and voiceless and subject to the whims of external authority", or from the position of received knowledge, where they view themselves as receivers or reproducers of knowledge from the "all-knowing external authorities," to subjective or procedural knowers, where the students view mathematics as personal and subjectively known or where they can apply procedures to obtain and communicate mathematical knowledge[4] (Belenky,Clinchy, Goldberger, Tarule, 1986, p 15). By encouraging students to master basic skills in mathematics, to discover mathematical concepts on their own, and, most importantly, to answer their own questions in mathematics, Pheonix is helping students not only to grow in their mathematical understanding but also helping them to recognize themselves as persons with the ability and power to use mathematics and to think mathematically.

The availability and use of graphing calculators did not affect Pheonix's philosophies of mathematics and mathematics education; instead they provided an opportunity for her to live these philosophies as she made choices for utilizing graphing calculators. The guided-discovery activity she modified and used with her Math 20s and Math 33s made space for the students to engage in both experimental and analytical processes. The search for regularities, as parameters were changed, and the discussion that followed from the question, "What happens as parameter *a* gets closer and closer to zero," are examples of places where Pheonix's conception of mathematics as a process was manifested. Pheonix's philosophy of mathematics education was manifested in

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[3]It is interesting to note that Pheonix coordinates the school's student motivational program.
[4]Belenky et al. (1986) did not refer specifically to mathematical knowledge. Their study focus on women's ways of knowing in general. I could find only one reference to mathematics and it was in reference to teaching, not to knowledge per se.

other aspects of her instruction and in interactions with her students as well. Her utilization of concrete materials, diagrams and pictorial representations, class discussions, and what-would-happen-if questions all reflect her espoused belief that mathematics needs to be taught in a way which promotes understanding of mathematics as a process and a connected body of knowledge, not as something that is to be memorized. Finally I would like to suggest that she shows her commitment to the students and care for them by offering tutorials, collecting and marking student work, and developing independent programs for individuals that need to be challenged.

Pheonix's instruction of mathematics is highlighted by her commitment to the students and her efforts to help them grow in mathematical understanding. Mark, the subject of the next case, also is committed to his students as learners of mathematics, but his views of mathematics and mathematics education are somewhat different than Pheonix's views.

CHAPTER V
Case III: Mark

Biography

Mark, a physical education teacher turned mathematics teacher, was active with his teaching duties and responsibilities as teacher representative for the student council the term he agreed to be involved in this study. Mark has been teaching for 16 years and for the last eight of those he has been teaching mathematics in a large urban high school. Like other teachers that cross-over to subject areas they are not trained in, Mark slowly picked up math courses, to teach, one at a time. He has now taught all but the senior level academic courses (Math 30 and Math 31).

Mark's Math Class: A Vignette

On the first day of my observational period I met Mark in the office and walked to his classroom with him. It was the first thing in the morning and the halls were very busy with students arriving and preparing themselves for their period one class. As we made our way to the classroom Mark was approached by a couple of students looking for a key to get into a room. As he dug it out, he said to me, "Student council kids."

When we entered Mark's classroom I was surprised to find myself in a lecture room -- the kind of room with the teacher's desk (more properly, a demonstration bench) at the front of the room and the student desks on three elevated levels rising toward the back of the room. The walls were scantly decorated with posters with inspirational messages. There was a blackboard and an overhead screen beyond the teacher's bench and in the corner of the room were some lockable glass cabinets. (This room must have been designed to be a science classroom.)

The students entered the room at the same time we did and Mark pointed out a desk in the front row that was unoccupied during this class. Mark began the lesson by establishing what they had done the previous day because a substitute teacher had been with them. After talking with the class and going over a question that might have been covered, he established that the students went over some of the terminology that is used in the unit on the quadratic function. For this lesson Mark had prepared a worksheet with a guided discovery of activity that required the students to investigate transformations of the parabola using graphing calculators.

"What we are going to do today [is] we are not going to utilize the books. We are going to utilize the calculators and the exercise we are going to do involves getting some of this information down so that for later reference we are able to look back and see what's what. So to start with I want everybody take two of the graphing sheets and then one of the actual instruction sheet."

Mark passed the handouts to a student at the front of the row.

"Okay. Let's send that sheet around."

As the sheets were being passed around the room Mark handed out the graphing calculators. The students immediately started playing with them. They had used the calculators once the previous week and it was clear, to me, that most of them remembered how to operate them and were eager to do their math with the aid of a calculator. As the students played with the calculators, Mark gave them instructions.

"There are two copies of the graphing sheets. All your books and stuff you won't need so you can put them away."

The students were obviously listening, because most of them placed their books under their desks. Mark turned on the overhead calculator and looked up to the students. Most of the students turned their attention to him and he began.

"Even though there may be one section in here that you have already covered we are going to record our results. Okay. Now, here is what we are going to do with this."

Not everyone was paying attention.

"You've got to pay attention, in case I've left out any instructions," Mark warned the students.

"The problem's a little vague. What we are going to do is we are going to look at what happens when we change the value of different variables associated with the quadratic function in completed square form, $y=a(x-p)^2 + q$." As he wrote the equation on the overhead he continued explaining. "All those variables a, p, and q, when you change them, have an effect on where the parabola is and what happens to it and we compare it to $y=x^2$. We compare everything to $y=x^2$."

Mark used the overhead calculator to help guide the students through the first graph. I noticed how careful his instructions were.

"Here's what we are going to do."

The students followed along on their worksheets and had their graphing calculators ready.

"You've got some information on the left and where it says sketch of function you're simply going to go $y= 2x^2$. You just write that in so you know what the sketch to the right actually represents. The next thing you are going to do is on the graph. If you have a different coloured pen, I would like you to sketch in $y=x^2$. So in one colour you are going to sketch in $y=x^2$ that is x^2 not $2x^2$. Then you can go ahead and punch it in. You can actually do that on your calculator overlap some of the graphs so you can actually see what is happening. And after doing that punch in $y=2x^2$. Everybody do that -- very simple."

"Now is there anybody here -- I see that there are three unfamiliar faces -- that do not know how to use the calculators? I would like you to sit with somebody to get a little more familiar with the calculators." Mark asked three other students to move their desks with the three new students. After class Mark explained to me that these three students just had come to his class from an English-as-a-second-language math class. For those students today was the first day in his class.

As the students moved their desks together, Mark entered the equation for the first graph into his calculator and watched it draw the graph up on the screen.

"Okay. Magic -- you graph the functions. What is happening?" He looked up and noticed that a number of students were still entering the equations into their calculators. "I'll give you half a second here. So what you do is graph $y=x^2$ and then over top of it you graph $y=2x^2$. So overlap it. Just graph it."

A voice from above and behind me said, "Skinnier."

"Someone said skinnier. Great result. That is what you are supposed to get. It is suppose to be tighter to the y-axis. Now in the information to the left of the graph, you have to tell me if the graph is opening up or down -- you just have to circle that or underline it, you have to tell me what the equation of the axis of symmetry is."

"Zero," a few students mumbled.

"*X* equals zero. Okay. Good. What is the vertex?"

"Zero, zero."

"Zero, zero. Okay. Good. Is it a minimum or a maximum?"

"Minimum," a number of students responded together.

"It's a minimum point. What is the minimum point or the minimum value?"

"Zero."

"Zero. Good. Give me the range."

"Greater than or equal to zero," somebody replied.

"I've set it up so these are really easy to fill in." He had set it up so it was very easy to fill in. Mark had written out everything except the number. He left a blank for those.

"The range is greater than or equal to zero. And any x-intercepts other than it makes contact with the origin?"

A few people shook their head no.

"Any y-intercept?"

"Zero."

"Then sketch that graph and go to the next one. Do the same process for $y=3x^2$ You don't have to take the previous graph off there but you have to record it on the next section. You will do that process for each one of those. It won't take you very long. You'll just have to sketch them. Remember, what you record on this sheet is the most valuable bit of information you are going to have because you are going to see what happens to all of these. In some cases it won't have much of an effect, but when you go to the next section it will. Once you finish a section, turn to the second sheet. Okay? And where it says 'What effect does *a* have on the graph,' I want you to think mathematically. When you give me a description of what's happening you'll do it in writing. Then I want to see if you can do it using variables and mathematical symbols to give me what the rule would be. Then next class we will take all of the information that you have collected from that and we will try to come up with a standard statement that is easy to remember and that sums up everything as concisely as possible. Any questions? It looks like a lot of work, but you will get through it," he assured the students.

"Now just one thing before I let you go. On question four it says graph each quadratic function without the graphing calculator. So what you have to do is -- I want you to tuck away the calculator. You can check your graph with the calculator after. You should have enough information at that point to know what each one of the little variables has on the transformation of the function. Don't be afraid to ask one another if you have any problems or us too."

Many of the students had stopped listening and were already working on the activity.

<u>Mark's Use of the Graphing Calculator</u>

Mark used the graphing calculators quite extensively in teaching this unit on the quadratic function, even though this was the first time he used them in his instruction. The calculators just had arrived in the school and the teachers had just been in-serviced on their uses. It was at the in-service that Mark indicated to me he would be willing to participate in this research and I believe his extensive use of the calculators was influenced, at least partly, by the fact that I was interested in how he would use them.

Mark spent considerable time developing handouts and thinking through ways in which he would use the calculators to teach this unit. I witnessed Mark use the calculators in five of the ten periods used to teach this unit and in three different ways.

The calculator was used as a tool for drawing graphs in a guided discovery activity, as a checking device, and to demonstrate a graphical representation to max/min problems.

The Graphing Calculator as a Checking Device

Mark's students used the calculator not only to check their understanding of the parameters a, p, and q and their hand sketched graphs; but, they also used the calculator to verify intercepts and range determined by algebraic means, and to verify algebraic manipulations -- a use only Mark suggested.

Like we saw in the case about Troy, Mark had the students use the graphing calculator as a checking device. He had the students sketch graphs of the parabola from the parameters a, p, and q, in the expression $y=a(x-p)^2+q$. He then told students to check their sketches by graphing the function on the graphing calculator.

One of Mark's summary activities consisted of the students completing a chart (Figure 6) using their understanding of the algebraic form of the quadratic function. The students were instructed to check the chart by graphing the functions on their calculators.

Function	Direction of Opening	Width Compare to $y=x^2$	Coordinate of the Vertex	Equation of the Axis of Symmetry	Range	Max/Min Value	y-intercept
$y = 4x^2$							
$y=5x^2+1$							

Figure 6
Sample of Mark's Summary Activity

Mark suggested a use of the calculator as a checking device to his students that I found particularly interesting. The suggestion came after Mark showed the students how to complete the square. He asked the students how the calculators might be helpful with this work. One student responded, "We wouldn't have to put them [the equations] in that [completed square] form." Mark agreed but suggested, "I think you have to put it in that form because some equations, if they are fractions you are not going to be able to read that off the graph. Alright. But the graph gives you enough information to be able to check if your result of completing the square is correct." The calculator was thus used to check that they completed the square correctly by first graphing the function in the form $y = ax^2 + bx + c$, and then checking to see that the vertex of the parabola was in the position the completed square form of the function indicated it should be in. This was a unique way of utilizing the graphing calculator as a checking device.

The Graphing Calculator as a Tool for Drawing Graphs

Mark used the graphing capabilities of the calculator to provide the opportunity for the students to determine the role of the parameters: a, p, and q on the graph of the quadratic function. This activity (Figure 7) required the students to graph different functions for each parameter and to compare those graphs with the graph of $y=x^2$. The students were to keep track of their work on a separate observation sheet (Figure 8). After working through all of the parameters the students were to summarize their findings (Figure 9).

Math 20 Transformations of Quadratic Functions

The purpose of this activity is to explore the effect of changing the parameters of the different variables associated with quadratic functions. The explorations below will provide you with the necessary information to help you sketch functions of the form $y=a(x-p)^2+q$.

Using the graphing calculator, sketch the graph of each function and fill in the information on the graphing sheets provided. After completing each section, summarize your findings on the summary sheet provided.

1. Explore the effect of a on the graph $y=ax^2$. Compare the transformations to $y=x^2$.

a) $y=2x^2$
$y=3x^2$
$y=5x^2$
$y=1/2x^2$
$y=1/3x^2$
$y=1/5x^2$

b) What happens when the values for a are negative?

2. Explore the effect of q on the graph of $y=ax^2+q$. Compare the transformations to $y=ax^2$

$y=x^2+1$
$y=x^2+3$
$y=x^2-5$
$y=x^2-2$

3. Explore the effect of p on the graph of $y=a(x-p)^2$. Compare the transformations to $y=x^2$

$y=(x-2)^2$
$y=(x+2)^2$
$y=(x+3)^2$
$y=(x-4)^2$

Graph each quadratic without he graphing calculator. Use the information you have learned form the above explorations. Describe the shape of the parabola when compared to $y=x^2$.

$y=-3x^2+1$
$y=-1/4(x+2)^2-5$
$y=2(x-3)^2+2$
$y=(x-1/4)^2-1$
$y=-(x+1)^2-3$

Figure 7
Mark's Guided Discovery Worksheet

Sketch of function_______________
Indicate the direction the parabola opens (upwards/downwards)
Equation of the axis of symmetry $x=$_____
Vertex (____,____)
Maximum or Minimum Value________
Range $\{y|y$________$\}$
x-intercept(s) ___________, y-intercept _____

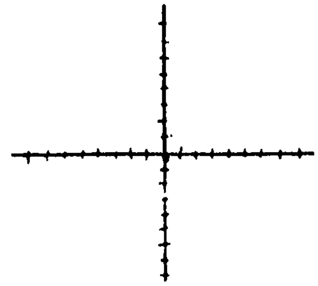

Figure 8
Sample of Mark's Observation Sheet

TRANSFORMATIONS OF THE PARABOLA
SUMMARY SHEET

For each of the following forms of a QUADRATIC FUNCTION, describe the effect of changing the indicated variable and its effect on the graph of the parabola.

1. $y=x^2+q$ What effect does "q" have on the graph?

2. $y=ax^2$ What effect does "a" have on the graph?

3. $y=(x-p)^2$ What effect does "p" have on the graph?

Figure 9
Mark's Summary Questions

Mark introduced this activity by telling the students what they were going to be doing and then working through the first equation with the students on the overhead calculator. He instructed the students to "record the results of what you see on the calculator." He suggested to them, "What you record on this sheet is the most valuable bit of information you are going to have because you are going to see what happens to all of these." The students worked alone but often consulted each other and Mark. The activity took approximately 50 minutes. Towards the end of the class Mark noticed a number of students were having difficulty with the last part of the activity -- sketching parabolas (without the calculators) using the parameters, so he went through one of the equations with the students. After class he commented to me that he felt the students had not done the summary before attempting to sketch these graphs. The next day he went over the summary with the students and had them record the roles of the various parameters in their notebooks. After summarizing the transformations, Mark carefully demonstrated how to sketch the parabolas under the various transformations. He told the students "it's one thing to be able to describe them it's another to be able to go and move them [get an accurate sketch on a coordinate system]."

Mark encouraged the students to use the graphing calculators when they were unsure or hesitant. At one point when going over this activity Mark suggested the students use their graphing calculators to see what happened to a graph under a certain transformation. "If the q is positive what is the transformation? [no response from any of the students] If you are unsure what you do, try one." At another point he said, "If you didn't off the top of your head know what happens, what do you do? Punch it in and you will see exactly what happens."

[illegible] resting note regarding Mark's use of the calculator for this guided - d [illegible] ty; in the summary discussion regarding the effect of the parameter a on the [illegible] function Mark asked the students to check what happens to the cubic function $y=ax^3$ if a is equal to two and if a is equal to a negative one-half. Here he used the calculator to foreshadow the things to come in a later unit -- transformations of various functions. He told his students, "What you should understand is it doesn't matter what the function is."

The activities that Mark designed for his students to use with the calculator could accurately be thought of as guided-discovery activities. Mark designed a highly structured activity that he encouraged the students to work through on their own. Only the next day, in the form of a summary, did he tell them what they should have discovered. The second feature of Mark's use of the calculators was the space that the calculators provided for which he could encourage the students to just "try it." This is different from checking an answer because it was used for learning and discovering something the students did not know, not for checking what they already knew.

Graphing Calculator as Aid for Understanding Max/Min Problems

Although Mark did not demonstrate the possibility of graphical solutions to max/min problems directly, he did use the calculator to help the students understand what he meant by a max/min problem. He asked the students to graph an equation that expressed the relationship between a diver's height and the length of time from when the diver left a three metre diving board. The equation, $h = -5t^2 + 6t + 3$, was given in the textbook (Kelly, Alexander, Atkinson, 1990, p. 220). After the students graphed the equation, Mark had the students use the trace function on the calculator to examine ordered pairs on the graph.

> Mark: I want you to stop when it hits the y-axis. What is that point?
> Student: Three.
> Mark: It represents?
> Student: Height.
> Mark: Of?
> [no response]
> Mark: That's the starting point of the dive. Isn't that the height of a three metre board. Isn't that what he dives from -- A three metre board? Okay so there he is on the three metre board, at that particular position. What happens? [Mark continues without pausing] He jumps up to a maximum height. Which in this particular case I want you to continue tracing until you get to that point up there -- 4.78.

To finish this activity Mark had the students complete the square for this equation. Then he asked them what the value $\frac{3}{5}$ represented. One of the students suggested it represented time. Then Mark repeated and elaborated on the students response. "Three-fifths of a second represents the time that the person is at a maximum height of $\frac{24}{5}$. Which is the same as $4\frac{4}{5}$. So that is the maximum height. Okay, so does everybody understand you will have to be able to interpret graphs like that?" Mark did this example with the calculator to help the students understand what a max/min problem was, not to show them that one could solve these problems graphically or with graphing technology. The students did two more of these type of interpretive problems on the calculator before they moved on to the standard max/min problems which involve writing equations from word problems, and then completing the square to solve for a min/max point.

Mark's Espoused Philosophy

Mark's sensitivity for his students permeated our discussions and his interactions with the kids at school. Our discussions were about mathematics, and what mathematics means for his students and the people he knows. The discussions rarely turned to the mathematicians and the mathematics you find in textbooks, universities, or highly specialized industries. When we discussed mathematics we were talking about Mark and his world. Early in our interview Mark paraphrased a poster he has in his room. "The problem must involve the student and he must search for the answer. He might not reach the answer but sometimes the search is more important than the [goal]."

On Technology

Mark claimed to welcome the new technology (graphing calculators and computers) to his classroom He said that the availability of the graphing calculators allowed him and his students to do more. He could extend topics and provide many more instances or experiences with a concept. Mark was taken up by the challenge of modifying his instruction to include the use of graphing calculators. He had a computer at home and expressed a great deal of interest in a graphing package for his personal use. As soon as the graphing calculators became available at school Mark made a point of trying to determine where they would best fit into his mathematics program.

He is critical, however, of the new mathematics curriculum. He mentioned that the curriculum requires the use of the new technology in mathematics but at the same time includes topics that are fast becoming obsolete as a result of that very technology. "From my point of view -- for example, radicals -- they are having less and less of a function in math because of the onset of calculators." He wonders, "why are we teaching them?" Another example: "For someone who is going further in math, I think it is interesting the relationship between exponents and logarithms.... But the thing is, for a Math 33 student, is that important?" Mark is more than willing to use the new technology but would like to see further curriculum changes that reflect new ways of doing old mathematics.

On Mathematics and High School Mathematics

When talking about mathematics, Mark almost always directed the discussion towards everyday or real-world mathematics. His view of mathematics, or the mathematics he seemed most interested in, is the mathematics you find in newspapers, or when watching people working or when working yourself. Mark takes mathematics personally. He sees it as something that every person does at one level or another. At one point in our discussion of what mathematics is he said to me, "Do you know what a dutch gable is? ... Well my dad who had never built one of those before, built one using a regular square using rise-over-run." Or he spoke about himself using mathematics, 'When I put my tile floor in my kitchen -- Pythagoras! That's the first thing I did when the guy gave me instructions to grid off the floor."

High school mathematics for some students is a building block for further mathematics (Mark implied in our discussions) however, he does not think this is its ultimate goal. For all students, high school mathematics is "a building block for life." Mark left me with the impression that he believes that mathematical knowledge keeps growing after a student leaves school -- as it did for his dad, or as it does for the person figuring out where the best place to invest retirement money, or as it does for a couple figuring out how much interest they will have to pay on a mortgage.

Even when teaching Mark brought in this real world aspect. "Today in class we started functions... and I said [to the students], 'So, what's in the news today? What do you see plastered up on every bank window?'" Or at other times he showed the human side of mathematics. Mark told me that when he taught logarithms, "I'd bring in a stack of old tables, all dusty and dirty. And I'd say 'John Napier. You know to me it is fascinating how he came up with the tables. You know coming up with a theory we just take for granted. We look in the table and we take it down. But we really don't understand why it's there and who put it there and why they were able to come up with that.'"

I tried to discuss the nature of mathematical knowledge with Mark and how mathematical knowledge is created, but I only got limited responses. Mark believes that the mathematics of textbooks is cumulative, constructed by the students learning the rules at each level. "You have to know the rules in order to make it to the next level. It [math] builds on itself." He agreed that mathematical knowledge is absolute and not subject to revolutionary change but these were my words and I sensed he felt uneasy about using them. I do not think he had a strong opinion about these statements. He did indicate he strongly agreed with the statements like "the applications of mathematics follow once mathematical knowledge is acquired" and that "mathematics helps you develop abstract thinking" and "mathematics is exercise for the brain."

Not having a clear sense of Mark's view of the nature of mathematics I asked him again in the follow-up interview, "What is mathematics?" Mark replied:

> "What's math? To me, I think math is a process.... [W]e do math to challenge the mind, to be able to -- I guess [get] a better understanding of where it [mathematics] fits out there." [He paused and then continued] "For the majority of people, I don't think we are taking math for the knowledge base.... I think we are taking math so we are able to understand there is a process that we go through and -- that we are able to take our minds through that process to come to a final outcome.

On Students and Instruction

Mark was very sensitive when talking about and interacting with his students. Although he admitted some students do not work hard enough and others are not very interested in math class, he viewed them all as learners with varying abilities, differing needs, and genuine difficulties. Mark's concern for his students as learners was evident in our discussions and in his actions. He carefully thought through the lessons he planned for his students. Part of this sensitivity may be a result of viewing himself as a learner. "I just can't go in and wing it. I have to give thought to Where are they having problems? What were some of the problems I had when I first [did] this particular question? I know the kinds of problems I had with those. Once I understand where the meat of the problem -- where the focus -- is, then I'm able to focus my students in on that." He wondered if some teachers take the content for granted. "I'm wondering if at some times there are people out there that have a really good knowledge base and lose interest in the delivery [teaching] because it's not exciting anymore."

Mark's views of his students' needs as learners underpin the methods he chooses for his instruction. He told me:

> [F]or some of these units I could send most of my students off and say, You've got to get from here to here [in the textbook]. Go ahead and discover it. Utilize me for whenever you need help. I'll give you direction. -- And that would work. I think there's a group of students, that don't have those skills, that need a type of introduction to just about every section.

Mark used what he calls a "stand-and-deliver" approach to introduce his students to new concepts. He explained the stand-and-deliver approach this way:

> The stand-and-deliver approach is one where you're trying to give them [students] an idea of what we're doing ... or a base for them to get started. You're introducing [the new concept to] them and they're going to learn [it] by doing the exercises; because the exercises are all in progression.

He said one of the problems students have is "they want you to do bits and pieces in order to get to a common goal at the end." The point then of doing the introduction is "to bring a whole concept in right off the bat. So they will see where you are going.... not necessarily giving them all the information."

Mark said he believes students learn when the work is structured in such a way that they get a general overview first, then the teacher shows the concept worked three or four different ways based on the students' questions. Finally the students need to work through questions and exercises for themselves. The instruction I witnessed had this overview component and was followed by guided discovery activities and a number of exercises that reinforced the same concepts in a variety of ways. Although Mark did not elaborate, on the interview guide he indicated that he strongly agrees that repetition and plenty of textbook exercises are necessary for learning most mathematical concepts. He also strongly agreed that concrete materials, pictures, and diagrams are useful for teaching mathematical concepts at the high school level.

Mark's Philosophies of Mathematics and Mathematics Education Made Manifest

Mark focused on mathematics as a human activity. Mark was the only person to acknowledge the mathematics done by people in their everyday lives[1]. He talked about his father framing a house, himself tiling his kitchen floor, and he spoke of the mathematics his students would be doing when they left school and were faced with economic decisions. He was also the only teacher who told me that he likes to tell his students about the mathematicians who developed the mathematics they study in school. He does not give history lessons, but he does try to place the development of mathematics into a human context by asking his students to consider the things that might have motivated the mathematician in developing his mathematics.

Mark's philosophies of mathematics and mathematics education fit into Ernest's category of a Progressive Educator. Ernest (1991) describes the Progressive Educators as viewing the processes in mathematics as more important than the specific mathematical content. Mark believes the purpose of teaching high school mathematics is not to teach "basic skills" but to teach a process. He said,"I think we are taking math so we are able to understand there is a process that we go through and we are able to take our minds through that process to come to a final outcome." The Progressive Educators' aim of mathematics education is to help the child develop into an "autonomous inquirer and knower" (p. 191). Mark's ethnomathematics view and his view of mathematics as a human activity underlie his teaching as he helps students see mathematics as something that they can subjectively or procedurally know (autonomous and inquiring would certainly be part of this), therefore something that they can use to act on their world not only inside the classroom but more importantly outside of the classroom. There is a discrepancy between Mark's ethnomathematics view and that of Ernest's. Ernest suggests ethnomathematics belongs under the ideology of the Public Educators; however this implies "democratic socialist" underpinnings which I do not believe fit with Mark's view. Another view of the Progressive Educators, that Mark

[1]This is often referred to this as ethnomathematics.

seems to share, is that "mathematics is perceived in humanistic and personal terms, and mathematics as a language, its creative and human side, and subjective knowledge are valued and emphasized" (p.182).

Mark's philosophy of mathematics as a human activity (and therefore experience) was manifested in his instruction by using examples from the real-world and encouraging students to consider what motivated mathematicians to do the work they did. Mark's philosophies of mathematics and mathematics education were not affected by the availability of the graphing calculators; rather these philosophies were manifested in his choice of activities for use with the graphing calculators. He did not have the students use the graphing calculators to look for right answers rather he used activities that encouraged students to interact with the graphing calculators to learn about the parabola and to investigate transformations. Mark indicated to me that the next time he taught the course he was going to let them work things out on their own more, rather than him "steering them" through it. This, too, demonstrates his view of mathematics as a human activity, one that he wants his students to engage in. I believe Mark's philosophy, of mathematics as a process, was reflected in the structure of the activities that Mark gave the students. These activities required the students to investigate, to observe, to look for patterns, and to make generalizations.

Mark's instruction and interaction with his students demonstrated the view that, he, his students, his father, John Napier, and Pythagoras all share in the mathematical experience.

Mark's perspective of mathematics as a human activity in which people engage olve specific problems is in contrast with Wally's (the subject of case four) dents doing mathematics by working in the abstract and with general cases.

CHAPTER VI
Case IV: Wally

Biography

Prior to teaching mathematics, Wally taught physical education and spent many years coaching athletics. Wally has a weak background in mathematics so most of what he teaches he has had to learn on the fly. But this doesn't seem to intimidate him. He told me that he was quite good at mathematics when he was in high school; however, he did not take any university mathematics. He has taught for 20 years and for the last ten years he has taught some mathematics courses in a large urban high school. He began by teaching the lower level mathematics courses and now teaches Math 20 -- the second level of the academic mathematics stream. Since he found himself teaching quite a lot of mathematics he decided to return to university, a couple of years ago, and he picked up a couple of methods courses. For the most part, however, Wally has taught himself the high school mathematics as the need arose.

A Day in Wally's Math 20 Class: A Vignette

If it were not for his age Wally might be mistaken for a student. Wally was casually dressed (like the other members in this math department) in a sport shirt and dress pants and he interacted with the students in a very friendly manner. He conversed with them about their lives outside the classroom and I heard him share some jokes and gossip with them. In general, I found the atmosphere of Wally's class to be very casual and relaxed. Students paid attention to Wally once he had their attention, but freely conversed with each other when they were doing seat work.

When I first entered Wally's classroom one of the things I noticed was a bulletin board on the back wall full of cartoons clipped from the newspaper. There must have been hundreds of all types posted up for the students to view. That bulletin board reflected the image I formed of Wally.

This particular day was the fourth in which the students had been working with the graphing calculators and investigating transformations of the quadratic function. As I walked into the classroom I found Wally putting the outline of a chart up on the blackboard. It had a number of general forms of equations. Listed in the chart were the equations $y=mx+b$, $y=x^2$, $y=x^2+q$, $y=ax^2$, $y=ax^2+q$, $y=(x-p)^2$, *and* $y=a(x-p)^2+q$. There were seven columns that were titled shape, vertex, axis of symmetry, direction of opening, y-intercept, x-intercept, and stretch factor. The students were familiar with some of the equations and most of the column headings, since some of the terminology had already been used and some of the equations had been investigated previously in class.

After I greeted Wally, I made my way to the back of the room to find a place from where I could watch the class. He didn't have time to talk right then because he was busy with the chart. As Wally completed his chart, the student's wandered into the classroom. There was a constant buzz of conversation as they took their seats. Friends in the class all sat near each other; this made for easy conversation. Wally acknowledged a number of them as they entered the room. He asked them how it was going and about a basketball game that was played the previous night. One student tried to convince Wally that she would make up for her absence the previous day by catching up on the work that was covered at home that night. Wally put on his serious face and emphasized that she had better because otherwise she would get behind.

Just then the bell rang and Wally turned back to the board to finish his chart. The students continued their conversations until Wally took up his position in front of the class. Finally the students' conversations slowly died down.

"You'll notice up on the board; there's the form of a chart up there. I would like you to fill it in. Let me give you some help here now. Here are the headings," he said as he pointed to the chart on the board. "On the left hand column," Wally stopped in mid-sentence when he realized that a number of students were writing the headings on their paper across the top of the page and were running out of room. He held up the page sideways and told them, "You'll probably have to get a sheet and go like this to get them all in."

All of a sudden the noise from the crumpling of paper filled the room as students decided to start over. This caused a little disturbance but a minute later Wally started again.

"So far, some of you have some knowledge of $y=mx+b$."

There were still a couple of boys talking.

"The people that do not want to pay attention can go to the library, okay. That means let's stop talking."

He waited for their silence.

"Okay, we've got $y=mx+b$. You have investigated these four," he pointed to the second through fifth equations on the chart, "however, I want you to make room for these two new ones that you are going to look at a little later today."

Wally told the students they were going to be given twenty minutes to complete the chart, but before he let them begin he explained what they were to write in each column.

"So you are going to make a series of comparisons. I also want you to have a little picture here -- a little picture. You make a little diagram of what it could look like."

In an attempt to tie this work with a previously studied unit, Wally asked the students, "Alright now -- is it going to be possible, if we have function written up there, to have a sideways parabola?"

"No," responded one student.

"No the only parabolas that we can have are the ones that open up and open down. *Vertex* -- what are we talking about when we talk about the vertex of our parabola?"

There was no immediate response so Wally continued by answering his own question."I would like you to put there, the coordinates. Now in some cases they are going to be abstract. Right? You might have to involve *q* or something. We'll leave that up to you and your partner to develop and work out. *The axis of symmetry*. Assume this -- up to now with all parabolas on these worksheets the axis of symmetry has been what?

A faint response came from one student.

"The *y* axis -- very good," Wally responded.

"So we will assume that until we might find something different -- hint, hint. Okay?"

Wally continued in the same fashion to explain to the students what they should write down in the other columns. Then he gave the students twenty-five minutes to work on this chart.

The students sat in pairs and triplets to work on this assignment. The noise level from the conversations rose immediately after Wally finished talking. As I walked around the room, I noticed that some of the students were trying to work on the assignment but a number of them seemed to have picked up on their social conversations that they were engaged in at the beginning of the class.

About thirty minutes later Wally took up his position in front of the class and said to the students, "I know you say, 'You are giving me all this abstract stuff,' and you are right. It is kind of abstract, but that promotes discussion."

At this moment it was not the mathematical discussion Wally had in mind.

"I'm going to wait until it's quiet." He paused until the talking had all stopped.

"Now we just made up a straight line equation here," he said pointing to the $y=mx+b$ entry. "What are some of the things about it that we know?"

No one responded. He continued, "Well we know that *m* stands for slope -- this thing has a slope of what? Negative or positive?"

"Negative," a few voices called out.

"Negative. It's going down from left to right. Is it going to have an *x* and *y* intercept?"

"Yep," a male voice from near the back of the room called out.

"Yep," Wally affirmed the student's response and moved on to the next column.

"What about *stretch factor*? It doesn't really apply. That seems to be a term we use for parabolas. What about *direction of opening*?" Wally answered his own questions. "Well there's no real opening, so I guess we'll put *n/a*. *Axis of symmetry*? There's not going to be an axis of symmetry. Axis of symmetry is supposed to divide something in half. Is there a vertex?"

A student was fooling around so Wally interrupted his instruction to deal with the student. As soon as he finished he said, "Okay. So let's give me the abstract coordinate -- if you will -- of this line."

No one responded.

"What is the *x*-coordinate's or *x*-intercept's coordinates going to be?" he prompted.

He waited for a response but no one volunteered one.

"Nina do you want to try it? What will the coordinates be, of this point?"

She did not say anything.

"Well it has an ordered pair," Wally prompted.

"*b*," Nina responded with a shrug.

"Okay. Not a bad guess, but that's going to be up here." Wally pointed to the *y*-intercept.

"Would you agree that it is *x* and zero?" He asked her.

"Yeah," she responded timidly.

"In the abstract form -- I mean obviously, if we have it written on here," he pointed to the axis, "we can figure it out exactly but we don't."

"So what about this one," he said as he pointed to the line crossing the *y*-axis.

Another student quickly responded, "Zero, *y*."

"Okay. Zero, *y* or I think Lee-Anne said -- what about zero, *b*? Certainly that would be acceptable."

The discussion and filling in of the chart continued in this same way for the next twenty minutes. The students made weak attempts to give the answers to fill in the chart but most of the time they needed very specific prompts. A few times the students were tempted to read the sketches as if they were for specific equations and Wally reminded them that the sketches were the general cases. Although he referred to them as *in the abstract.*

"What is the significance of having q in here?"

A student suggested that it is the y-intercept.

Wally said, "Okay, but on my little diagram I could just make up an abstract value here. Let's say that is q."

The student interrupted and said, "One."

Wally, trying to lead the student in the right direction, responded, "Okay, so what is the vertex in this particular abstract example?"

"Zero and one." The student stated.

"Zero and one? Zero and q --q in the abstract. If this was," Wally hesitated and then said, "Okay I can see why you said one. Because you thought I meant this particular one but I should have put the coordinate here."

"Yeah," said the student appearing somewhat confused.

"This is in the abstract." Wally repeated.

"Okay," but he student did not sound convinced.

Wally led them through the rest of the chart in much the same way. For most of the columns he ended up telling the students what the "abstract" response should have been. Once he had to go back and rewrite the response he put on the chart because he wrote it for a specific case instead of the general case.

"No I goofed there," he said. "So we better do it like this and we'll call that body *in the abstract.* What were the coordinates for there?"

Now the students started to use his language.

"In the abstract?" One of them asked.

"Yeah, for the vertex." Wally said.

"Zero, q," the student quickly responded. He must have worked out the meaning of *in the abstract.*

With ten minutes left in the class, Wally explained to the students that he had one more thing to show them that would help them with their homework. It was the process for completing the square. He explained to them how to complete the square by asking one-word-answer questions. There was this interchange of question and response between teacher and the students throughout the explanation for completing-the-square. Not only was this a means for teaching the new content but it helped keep everybody on task.

Wally's Use of the Graphing Calculator

Wally had never previously taught this course nor had he ever used graphing calculators. Prior to teaching this unit on the quadratic function he attended a graphing calculator workshop. He used materials from this workshop and materials developed by another teacher to incorporate graphing calculators into the instruction of this unit. The graphing calculators were used extensively throughout this unit. Although an overhead model of the graphing calculator was available, Wally did not use it; only the hand-held calculators were used.

The students had use of the calculators in class for most of the unit. The exception was for the max/min problems. Wally encouraged the students to work in pairs over the duration of this unit because there were not enough calculators for each student. This, he claimed, changed the nature of the classroom environment slightly from the norm. Wally suggested the calculator be used in two ways, although the students used them in a third way as well. The graphing calculators were used to provide graphical images so students could discover the role of the parameters a, p, and q of the quadratic function, in the form $y=a(x-p)^2+q$. Wally also suggested the students used the calculators to identify the max/min point of the parabolas with the trace function. Without being directed to do so, a number of students used the calculators as checking devices throughout the course of the unit.

The Graphing Calculator as a Tool for Providing Students with a Picture

Wally used an activity with his students, received during the graphing calculator workshop he attended, to investigate transformations of the parabola. This activity was much like the activities the other teachers used for teaching this same concept. The students were expected to graph a number of different parabolas and discover the role of each of the parameters, a, p, and q. Wally told the students to use the graphing calculator to draw "pictures" of the graphs from which they were to compare and infer the roles of each parameter.

> Each time you get a diagram or a picture or a graph, I want you to sketch that graph as well on a piece of paper.... And then we will be able to make comparisons.

This activity had questions that asked the students to state in their own words what happened to the parabola under the various transformations.

Wally divided the activity into three parts, separating each of the parameters, and spread the activity out over three periods with partial summaries at the beginning of the second, third, and fourth class periods. The summaries were most often completed by Wally asking very specific questions about the role of the parameters. Once Wally heard an appropriate response, he moved on to the next task. Wally also used a chart (see Figure 10) on the fourth day of this unit to summarize all of the parameters and properties of the parabolas that he had the students consider in their investigations. He asked students to provide him with information to put in the charts; however, here, too, the students often needed very specific prompts and in many instances Wally just answered his own questions. The calculators were not used when Wally and the students were doing these oral summaries.

	Shape	Vertex	Axis of Symmetry	Direction of Opening	*y*-intercept	*x*-intercept	Stretch Factor
$y=mx+b$							
$y=x^2$							
$y=x^2+q$							
$y=ax^2$							
$y=ax^2+q$							
$y=(x-p)^2$							
$y=a(x-p)^2+q$							

Figure 10
Wally's Summary Chart

Using the Trace Function on the Graphing Calculator

The second way Wally had the students use the graphing calculators was by having them use the trace function to identify the parabolas' vertex. He suggested to the students that they trace to find the coordinates of the vertex when they were investigating the role of the parameters p and q. "For each graph, use the trace function and note the vertex. The vertex is the point where the graph of the parabola changes directions." Although Wally did not specifically suggest this, the students continued to use the trace function when they were doing other assignments in this unit -- finding the range and intercepts of functions, for example.

The Graphing Calculator as a Checking Device

Although Wally never specifically suggested to the students that they use the calculator as a checking device, they were given the calculators to use even after the investigations of the transformations were completed, and some of them did seem to use the calculator for checking their work. On one day the students were given an exercise (see Figure 6, from Case III: Mark on page 71) and the graphing calculators, and were told to "discuss it [they were working in pairs] and make a little picture." However Wally added, "You won't even need the calculator." I observed a number of students use the calculator to provide the image of the graph to complete the table and I witnessed other students sketch the graph, complete the table, and then check it on the calculator.

Wally's Espoused Philosophies

Wally eagerly discussed his beliefs, attitudes, and conceptions of mathematics. He prefaced our discussions of the nature of mathematics by telling me that his experience with high school mathematics and for teaching high school mathematics was quite limited. More than anything else, he impressed on me that the students were his reason for being there. It did not seem to matter much that he was teaching math -- only that he was teaching kids.

It was very difficult to point to Wally's philosophies of mathematics and mathematics education and to clearly articulate those philosophies. Before I discuss Wally's beliefs in mathematics, I feel it is necessary to demonstrate one of Wally's attitudes that was salient in our discussions, but not articulated until late in the interview. Very close to the end of our discussion on the nature of mathematics, Wally told me.

> I'm a firm believer that anything is possible. Okay? That's kind of my credo of life. And that's why I answered it[1] the way I did. Certainly out there in the scheme of things it could be a lot different.

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[1]This was Wally's response to, *Mathematical truths have a certain inevitability about them. A world with different mathematical truths is inevitable.*

Frequently, as we worked our way through the interview guide Wally hesitated with his responses because (I later learned) of the way the statements were phrased. He had strong objections to statements that made use of adjectives such as the word "all" or statements that were exclusive. These statements conflicted with his "credo of life." He made a point of elaborating on such statements in order to clarify his point of view for me. Table 6 lists some of the statements on the interview guide that Wally insisted upon clarifying because of his objection to the way they were worded.

Table 6
Examples of Qualifications Wally Made

Statement	**Wally's Qualifier**
Diploma Exams should be developed by mathematics teachers.	I said agree but you see I'd qualify that by saying other people could certainly be part of the process, too.
The process of doing mathematics in school can be seen to be a model of all mathematical experiences.	[He responded strongly agree on the guide.] Now that I read this again, I'm not -- I see this word *all* and all of a sudden I would change my mind only because of the word *all.*
Math is the best subject to train you to have a logical mind.	In my mind it is the best subject, but I would never say that others couldn't train you to be logical as well.

In the next sections I will attempt to demonstrate some of Wally's beliefs, attitudes, and conceptions of mathematics and mathematics education.

On Mathematics and High School Mathematics

Repeatedly Wally told me he (or sometimes they) "believes in mathematics." I had a difficult time trying to determine what he meant, because he used this expression in more than one way. For example he told me, "I believe in math, not just for the sake of crunching numbers." Here, I believe, he was suggesting mathematics is more than just arithmetic. It is also a way of thinking or reasoning. "Whether it's just the thinking component of mathematical experiences that allows you to think something else out -- abstract or whatever," he explained.

Another sense of this expression, as he used it, was related to the logical nature of mathematics. I think he pointed to this when he told me the following little story he sometimes tells his students.

> It's the old story about the track. Okay? Where you've got a bunch of runners around the track and if we go and witness [the race] you would all agree that so-and-so finished first and so-and-so finished second and -- [he faded out]
> If you get a bunch of math -- bunch of guys that believe totally in math -- I'm not that devout of a believer. I rely on my senses or my experiences. If they get you to agree with this statement -- mathematicians -- they can prove or suggest that what you saw at the race was an illusion [referring to Zeno's paradox].
> And it's all based on the premise that if you run a full race you must agree that before you run the full race you are going to run half the race. So there is a half to run and before you complete that second half you must run half of that. So you see what's coming. Here is an infinity concept. Isn't it? You're going to say to the guy "No you never finished the race. That's and illusion because you have to agree that even in this microism [sic] they still have to run half of that real little bit.
> Now it's absurd.
> However can you dispute it if you believe totally in mathematics?

As he was telling me this story, Wally made a point of emphasizing that he is not that "devout of a believer" because he "relies on his senses" and "experiences."

When I specifically asked Wally what he meant by "I believe in mathematics," he stumbled for a response and then replied, "I believe in math well in the sense that I think it should be one of the core. I think it is one of the most important subjects." Then a few sentences later, in response to a question about whether he was going to use the calculators to find graphical solutions to max/min problems or whether he was going to do algebraic solutions, once again he used this phrase. He replied, "Algebraic solutions -- I believe strongly in them." This statement seems to go back to the first meaning attributed to his belief in mathematics. Hence, Wally's use of the phrase " I believe in mathematics" seems to have three meanings. He believes: 1) There is truth value attributed to mathematical knowledge derived from logical reasoning; 2) Mathematical thinking is an important aspect of teaching mathematics; 3) Mathematics is an important component of the curriculum.

Throughout our discussion on the nature of mathematics Wally slipped between a general notion of mathematics and specifically high school mathematics; although he most often referred to high school mathematics. He told me he sees high school mathematics as one of the better subjects for training one to have a logical mind.

> In my mind it [mathematics] is the best subject; but you see I would never say that the others couldn't train you to be logical as well. Particularly physics and stuff which falls right in. So much of it is similar to what we [math teachers] are teaching.

> Well, I think it teaches people organization skills -- concrete blocks if I can use that kind of terminology. That they can get from point *a* to point *b* in a logical fashion and it makes some sense. And it can be used in life; it can be used in their career later on.

Wally indicated that he believes it is important for high school students to be able to work "in the abstract." He used this phrase often in class when going over concepts with his students. The following exchange demonstrates what Wally meant when he said "in the abstract."

Elaine: [Reading a statement from the interview guide.]
Mathematics helps you develop abstract thinking.
Wally: Sure.
Elaine: Okay. That's one of the things that ...
Wally: That's close to a strongly agree.
Elaine: [I pointed out to Wally that he often uses the phrase "in the abstract" with his students and I tried to flush out what he meant when he used that phrase with his students.]
Let me tell you. One of the things that is not uncommon for you to say is "in the abstract."
Wally: Did I?
Elaine: And so when you used it I think you wanted them to use variables. You were trying to bring them to the general case.
Wally: Exactly!
Elaine: So that's kind of what you mean by the abstract?
Wally: So I should use -- Yeah.
Elaine: Oh. I'm just commenting that those are words you commonly use.
Wally: Well I think too that they are forced all the time to be kind of thinking how am I going to solve this. So that's to me a form of the abstract. Particularly on a final exam, where if it is a chapter six quiz they've got a good idea what's being tested on. Right? But if it's a full year test how do they differentiate between when to use one variable and when to use two variables for a word problem. That's to me where the classic abstract thing comes in.
So maybe like you say I use the word abstract wording in a different sense and I am applying it here. That could well be to a certain degree but did I explain how? Does that make sense; how I interpret abstract thinking?
Elaine: I think so. So you are saying that abstract thinking is talking more about general cases.
Wally: Yeah. Exactly. Not which came first the chicken or the egg. I mean that's philosophy.
Elaine: So instead of saying I've got two *x* squared plus three. In the abstract means that ...
Wally: You have *a x* squared plus *b* or something. That's exactly what I mean by abstract there.

Elaine: So abstract for you means in the general case?
Wally: Okay. Yeah.

Other than his strong beliefs in the value of mathematics to teach logical and abstract thinking, Wally did not have many explicit beliefs about the nature of mathematics. I prompted some views out of him with the interview guide. He agreed with the statements *mathematics helps you find patterns and relationships in the world* and *mathematics provides a means for describing the world.* "Oh, I believe that strongly. As a matter of fact, one of the definitions in a dictionary is the study of patterns. That's what math is. So that word pattern really sold me there." He agreed with the statement that *mathematics was cumulative* and after discussing the meaning of the word hierarchical he agreed that *mathematics was hierarchical* as well. He agreed with the statement that *mathematical knowledge is subject to proof and refutation* and he disagreed with the statement that said *mathematical knowledge is absolute.* He had trouble with the word absolute, since he felt absolute was like the words all and inclusive. He disagreed that *mathematical truths have an inevitability about them. A world with different mathematical truths is inconceivable.* I said to him, "You are saying that it is possible to come up with different mathematical truths?" He replied:

> Oh for sure. Mainly because -- What is that show I saw? One of those wild wooly sci-fi things where some guy ... Anyway the jist of it was there was another earth on the other side of the sun. Left and right were opposite and stuff. So to me, I guess I thought of it in a totally abstract. I'm a firm believer that anything is possible.

Wally discussed the nature of proof only with respect to the role of proofs and their value in high school mathematics. He suggested that if one can see how a proof is developed then maybe a student who forgot a formula could work their way "backwards through a common-sense proof." He agreed that proofs are technical in the sense that they are already discovered. "But," he said," I could plan a class for them to discover it. Which would be the creative [way]. They could find out the proof if I gave them enough clues. Right?" His only reference to mathematicians and their work with proofs came in response to a statement about mathematical truths. Here he agreed with the statement; *Mathematical knowledge is close to scientific knowledge in the sense that its conclusions are tested for their truth.* He elaborated, "I mean, to me, here I'm saying to myself -- Yes somebody could find some new proofs. They are probably finding new stuff all the time."

On Students and Instruction

When Wally talked about himself as a teacher (not necessarily a mathematics teacher) and his beliefs about student learning and instruction, he spoke with excitement and confidence. Near the end of the interview I asked Wally about a comment he made to me during the observational period. At some point he had told me that he was glad to be teaching just math. Wally elaborated for me.

> Well let's just say I taught phys. ed. for about eighteen years. [I] loved it during those eighteen years. I made sure I tried other subject areas thinking -- knowing full well that at some point in time I'm going to have to gravitate to a different -- You know I didn't want to get a mixed bag all the time. You know some guys are going to be here and there. Forget that. I decided I was going to have to pick a certain subject area and I love CALM. I don't know if you are familiar with it? I can go in there and do my thing. I'm doing all kinds of things. Math doesn't necessarily lend itself to that. However I got in [teaching math] two years ago thanks to [our department head]. He opened the door and said "Yeah try some courses." I took a couple of university courses [methods courses] and that confirmed in my own mind.

Wally went on to talk about his preferred methods of instruction.

> My style is very much *let them interact.* If they go too far, then I pull in the reins. I'm not a straight lecture style. I've used a lot of cooperative learning techniques. Not so much at the 10-20-30 level. I find it is not as effective for those types of kids because they are more thinkers. But the 13s and the 14 - 24s they like that hands on. They are more the tactile -- you know what I mean -- type learner.

More than once in the interview and in discussions with Wally he mentioned that he prefers to use alternative (to lecture) instructional techniques in his classes. He welcomed the opportunity to use the graphing calculators in his instruction because they gave the students an opportunity for discovery learning. He said, "I like this [the approach he used for teaching the transformations]. I think it gives them a lot more opportunity for them to have hands on. Developing how this thing comes about or seeing it for themselves. I think it is an excellent way of learning."

For the most part Wally instructs the Math 10 and Math 20 students using what he referred to as a "guided practice approach." This approach includes beginning the class with a bit of review ("not only pick up where you left off but review"), developing the new concept with the students ("I think they should help develop them first"), and having them practice. Wally believes that repetition is the way most of us learn and therefore his job is to "make sure that [he] goes over the material a number of different times but not in the same way every time to bore them to tears." He said, "the more times they see it, whether it is combination of [him] putting it on the board, them discovering it, the homework," the better opportunity there is for the students to learn the math. When asked about teaching applications, he suggested that you "give them [the students] the basics and then go on to the applications." He continued, "Some people would argue you show them [the students] first how it fits in then work [back to the basics]. I mean both ways can be effective."

Wally had some interesting ideas as to why some students do not do well in mathematics. He believes the Math 20 students must be able to see things in the abstract if they are going to succeed in the long run. That is, they must know what the general cases are and what they mean. Second, students must learn the rules. "You need to know some formulas or you are going to get clobbered -- right? Whether it's common denominators for fractions or -- You don't have a hope right?" Third, some students "have the blinders on when it comes to this math thing." Wally made this reference based on his experience with students and the methods that they are taught to subtract integers with. They have been taught to do it one way but they do not really remember it completely, therefore they get the work wrong. Wally claimed when he tries to teach them a different method, they do not want to change their old ways. He said "they keep failing and -- guess what -- I keep putting on the next test the review questions. I carry [them] over and over again until they get it right."

When discussing the students difficulties with the quadratic function unit (the one I was interested in), Wally said:

> They are always used to "here's the formula.-- plug them in" kind of thing. To me this whole stuff now is kind of backwards relative to that. And that's probably another reason why I used the word abstract a lot. I'm surprised this group has not clued in quicker on this and maybe that's because I haven't explained it very well. But I think with the sheets you have given me, the repetition -- you know. I think it's fair to say a lot of them are not getting it because they are not working at it. I think it fits with the homework. If they went home at the end of the day and did those there should be a lot of "ah-hahs!" taking place at home. I still see a high number of blank looks; although the last couple days --the day that we did the word problems and the day before that -- now this is only my intuitiveness that they were starting to get it. It usually it comes quicker -- you know.

<u>On Technology</u>

Wally readily accepted the graphing calculators into his math classes. I believe they very much fit with his teaching style. When asked if he would use the calculator the next time he taught the unit he replied, "With the 20s I'm going to do it again just so I can improve myself and improve what I did the last time, because I think it is an effective way to do it."

However, like some of the other teachers, he has concerns about overuse of technology. "I'm getting worried that the calculator is getting to the point that you turn it on and you talk into it -- the problem-- and then the problem comes up on the screen.... I'm saying from now on ten percent of the unit on the calculator -- or something like that -- and the rest without it. Hopefully we are going to get some people thinking that way."

Wally claimed that some of the kids are becoming so dependent on calculators (four function calculators) that "we are missing some of the basics." By basics he seemed to be referring to arithmetic skills. When asked if Math 20 is basic he said "Well when you compare it with some of the kids that come from Hong Kong. They saw it in about grade eight. It's diluted."

In spite of his concerns, Wally repeated his enthusiasm for using this new instructional tool in his mathematics classes. "I'm learning myself. I think it is going to work really well with my 20s [Math 20 students] because I'll be better at letting them have the right amount of calculator versus practicing examples out of the book without the calculator on."

<u>Wally's Philosophies of Mathematics and Mathematics Education Made Manifest</u>

It is difficult to place Wally's views and beliefs about mathematics and mathematics education completely into one of Ernest's (1991) categories; Wally expressed a range of sometimes conflicting opinions when we talked about mathematics and mathematics education. While he believes mathematics is logical and it teaches students organizational skills, he stresses the importance of students being able to use formulas to solve problems. He believes mathematics is useful for teaching logical reasoning (which mathematicians say leads to truthful conclusions); however, personally, he relies on his senses and experiences to come to truthful conclusions. Wally qualified the statement that mathematics is good for teaching logic and organizational skills by indicating that other subjects can also teach these. He suggested that when he teaches mathematics, he teaches the basics first then applications; but, he acknowledged that teaching applications first and then basics could be effective too. He suggested there is more than one way to solve most problems, but, in class students had to use his methods for doing their mathematics exercises. Wally's diversity of opinions about mathematics and mathematics education and his "credo of life that anything is possible" leads me to suggest that his views of mathematics are multiplistic (Perry, 1970). Perry describes multiplicity:

> A plurality of "answers", points of view, or evaluations, with reference to similar topics or problems. This plurality is perceived as an aggregate of discretes without internal structure or external relation, in the sense, "Anyone has the right to his own opinion," with the implication that no judgements among opinions can be made. (fold out)

Wally's weak formal background in mathematics and his unfamiliarity with the course may be part of this perceived multiplicity. If Wally had taught the course prior to the study, he would have been familiar with the Math 20 content and it may have resulted in more specific and fewer conflicting responses to questions in the interview. Having said that, I believe Wally's espoused and manifested views of mathematics are only partly tied to his experience teaching this particular course; after all, he has been teaching mathematics for the past eight years and the questions in the interview guide were not specific to this particular course.

Ernest suggests that the Technological Pragmatists have a multiplistic view of mathematical knowledge. This view, he claims, leads people with this ideology to rely on experts for the best methods to apply mathematics and solve mathematical problems. The experts may have different methods to solve a single problem; however, behind these different methods lies an absolute mathematical truth. Wally is much like the Technological Pragmatists in that he seems to place authority for the mathematics he teaches with the textbook, his colleagues, and other experts. In class he told his students that they would call "*a*" the stretch factor since another teacher indicated this was an appropriate term for this parameter. Also his response to the statement, *Mathematics helps you find patterns and relationships*, helps demonstrate his reliance on others as authorities in mathematics. He responded by telling me how strongly he agreed with the statement and elaborated by pointing out that even the dictionary has this as a definition of mathematics. Further, the activities he used with the graphing calculators to teach the unit on the quadratic function seemed to reflect this reliance on others as experts and authorities in mathematics. Recall that Wally did not design the activities he used during the course of the unit. All of these activities, exercises, and worksheets came from other sources -- the textbook, a fellow teacher, and the graphing calculator workshop he attended. Wally did not modify the textbook exercises or the activities and the worksheets he received from others; although he did determine the nature of the discussion surrounding these materials and suggestions and he indicated that he was going to limit the amount of time he gave the students for working with the graphing calculators the next time he taught the course. Like the Technological Pragmatists he relies on the experts, but, unlike the Technological Pragmatists he does not believe mathematical knowledge is absolute nor does he emphasize applications of mathematics.

Since Wally specifically stated that he does not believe mathematical truths are absolute, he probably would not agree with the statement that he views mathematics as a set of truths and rules or as an unquestioned body of useful knowledge; but, this is the way he taught the quadratics unit. He gave the students the opportunity to discover the rules for transformations on their own, but, when he summarized the activity he asked simple one-word-response questions and in most cases he had to answer the questions himself. He suggested that the students were not as responsive (in this summary activity) as he expected them to be. He thought this was possibly because they were used to being told the formula so they could work on questions -- not developing the formulas on their own. This view of mathematics as an unquestioned body of knowledge also was evidenced when one student suggested he had a different way to solve a max/min problem. Wally told the student to use the method on the board to ensure a correct answer. These two examples imply a view of teaching mathematics which is more closely aligned with Ernest's Industrial Trainers than it is with the Technological Pragmatists -- that is, learning rules and formulas as opposed to doing applications.

Even though Wally's espoused philosophy of mathematics seems to be partly in contradiction with his instruction of mathematics, we do see some of his beliefs, attitudes, and conceptions of mathematics manifested in his instruction. The way in which Wally summarized the transformation activity with a strong focus on the general ("abstract") cases reflected his view that mathematics is abstract. It might also be thought to reflect his belief that academic students should be able to work "in the abstract." Some of Wally's other beliefs about mathematics education were also manifested in his instruction of mathematics. Consistent with his favorable attitude towards manipulatives for mathematics instruction, Wally accepted the graphing calculators as a welcome addition to his instruction of mathematics. As he indicated in the interview, he likes to show students many ways to approach mathematics; the graphing calculator provided him with the opportunity to teach transformations of the quadratic function differently than he teaches other topics in the Math 20 program.

It is quite difficult to determine what, if any, impact the graphing calculators had on Wally's beliefs about mathematics. He expressed the concern that he allowed his students to do to much work on the graphing calculators and he plans on changing the amount of time he gives his students with the calculators the next time he teaches the unit. This indicates that he is using his experience to determine the direction his instruction will take in the future, rather than relying so heavily on the experts (like he did the first time he taught the unit) to teach this unit. One of his strongest views, which came out in our discussions, was that Wally believes he is a good teacher. After teaching the unit once, he can now rely on his teaching expertise, as opposed to the mathematics expertise of others, to determine the methods he will use to teach the quadratic function.

Although many of Wally's multiplistic views of mathematics and mathematics education appear to fit into the category of the Technological Pragmatist, some of his beliefs, attitudes, and conceptions about mathematics and mathematics education do not fit as well. His emphasis on the basics and formulas, his view that mathematical knowledge is not absolute, and his view of himself as a teacher are views that do not fit with those of the Technological Pragmatists.

From Wally who articulated a multiplistic view of mathematics and mathematics education, we move on to Jack, the subject of the next case, who is very pragmatic and has specific beliefs, attitudes, and conceptions of mathematics and mathematics education.

CHAPTER VII
Case V: Jack

Biography

Jack has a B.Sc. in chemistry and a B.Ed. with some university level mathematics and at least one methods course in mathematics. However Jack has not taught much chemistry; he is one of the senior mathematics teachers in this rural junior-senior high school. Jack has taught mathematics for 24 years and high school mathematics for 17 of those years. Numerous times in the past he has held the mathematics coordinator position at his school. Jack has taught all of the senior high school mathematics courses except Math 31. This year in addition to his teaching duties, which included both junior and senior high school mathematics, he also coached badminton.

A Day in Jack's Class: A Vignette

Jack's classroom had an unkempt ambience about it. Although the school had just undergone a major renovation, the mathematics department (Jack included) was assigned to the portables which were the only classrooms that had not been renovated. The walls of Jack's classroom were decorated with some photographs (that he had taken), some golfing posters (Jack is an avid golfer), and a number of faded posters (with math formulas and descriptions of number systems). There were also a few posters that advertised textbook companies (the kind you pick up free at conferences).

For a person who had taught for 24 years Jack had a very youthful appearance. He often wore plaid shirts, work pants, and cowboy boots. His hair was kind of long and it kept falling in his eyes. When he addressed the students he was quite soft-spoken but firm. They listened respectfully and seemed to have no fear of asking questions or making conjectures. However, by his own admission, Jack did most of the talking in his math classes.

On this particular day, one of the things that immediately struck me about Jack's Math 20 class was the few students that he had. I had been in his class earlier in the term and knew he did not have many students but on this day there were only eleven kids. Jack later told me some of his students were away for a concert and a few others had been selected to complete a questionnaire. Jack spent the first few minutes of the period taking attendance while the students listened to the announcements. Once the announcements were finished Jack got right down to business and introduced the students to the activity he planned for the day.

Jack wrote, $y=ax^2+bx+c$, on the board and he began, "What we want to do is try to figure out the effect of a, the effect of b, the effect of c, and these are what we call transformations. Alright? And so I want you to go through the sheets. I don't think this will take a lot of time. After you are through this, concentrate on the parabola. What I want you to do is, I want you to see if you can think of any other curves -- and there is one that should definitely come to mind -- that would behave in a very similar manner. And make sure you can see what the effect is of putting some numbers in there and see if it does the same sort of thing. So, what I would like you to do is come up with your own curves -- something different -- and see if some of the effects, at least, are the same."

Jack continued, "Now there are some things that are kind of hard to look at. For instance, we have already talked about $y=x^2$ and you already know what happens when we do this sort of thing -- at least you should. But sometimes there are numbers that it's just harder to see what's happening. So if we put a number in here it makes describing what's going on a little bit harder. But you are going to have a little bit of that. We will talk -- we will look at forms like this in detail a little later in the unit so that you can get some more specific information about it. All I want at the moment is to see what's happening. Remember the one term I gave you yesterday? The bottom most -- for lack of better definition at the moment -- the bottom most part of a parabola is where the what occurs?"

Jack paused for a response but there wasn't one.

"Okay, the vertex. Alright? And if the graph is opening up we have what all the time? Okay Sam.

"Maximum."

"Careful."

"Minimum." Sam offered the only other possibility.

"Okay the minimum. And if the graph opens down we will have a maximum. Call that point where the minimum or the maximum occurs -- just call that point the vertex and I'll give you a more formal definition later."

"Okay, so pick up the calculator. Again, if some of them aren't functioning right there are so few of you here today that you can probably just grab another one. Also at the same time pick up these sheets and start going through them. If you have questions again Mrs. Simmt would be -- I'm sure -- very interested in hearing what you are asking because that is part of what she is doing for her masters. So don't hesitate to ask her questions or what would happen if or what ever you are thinking of or what ever you are considering. So feel free to just say anything you want."

"So pick up a sheet, pick up a calculator, and go to it. And I don't think this will take much more than a period so try to get through them pretty rapidly but make sure you understand what is going on."

For the next sixty minutes the students worked on their investigation. Jack and I walked around helping students operate their calculators; however, there were very few questions. Occasionally a couple of students consulted with each other over a graph or a conjecture but for most of the sixty minutes it was very quiet as the students did their work. Only once in this sixty minutes did Jack ask the students for their attention. It was when one student asked him about writing an equation for the axis of symmetry. Since he had not discussed this with the students prior to the investigation he made time for a mini-lesson on writing equations of vertical lines.

In the last twenty minutes of the period Jack worked through the results of the investigation with his students. For most of this summary the students provided Jack with the observations they made and Jack provided the students with explanations for the observations. They reviewed the role of the parameter *a* and then discussed the role of the parameter *c*.

"We were supposed to look at the *c* value. What I have to do is hold some of the other values. We told you, let *b* be zero, because I didn't want that in there. And we also told you to make *a* one, so if *b* is gone and if *a* is one, now if you just change *c* the effect must be straight due to the *c*. If we allowed you to move or take two numbers, let's say *a* as seven *x* in here," he pointed to the equation on the board, "and a four [for *c*] you wouldn't be sure if the seven is affecting what is happening to the graph or if the four is affecting the graph. So make sure if you come up with something yourself that you only change one value at a time otherwise you are not sure which value is causing the effect on the graph. Alright so we just want to look at *c* and so, if $y=x^2+0, 2, 3, 8$, Robert what is happening with that?"

"It sort of shifts."

"Okay it shifts the graph up and down the --"

"*y* --"

"Axis. So as *c* gets larger the graph would just move higher up on the *y*-axis And if we had negative one, negative two, negative three, etc., it would just shift down. Okay so a nice clear effect. *c* will simply move the graph up and down the *y*-axis. Is there anything we could say about domain?"

"It changes the range," offered one student

"Okay. Does it change the domain?"

"No."

"Everybody okay on that?" he paused. "Are you?"

A few students responded together, "Yeah."

"Okay -- like does changing c affect the domain of that? Okay if I just change c, if I hold -- let's say I keep $y=1x^2$, I still got b equal to zero and all I'm doing is putting plus or minus or anything I feel like here. Does it change the domain?"

A few more students this time said, "No."

"Okay domain is going to stay the Reals [the Real numbers]. And Greg said -- it certainly does affect the range. Is everybody clear?"

No one responded.

"Alright. So write the coordinates of the vertex of each of these. So, Jerry give me the vertex of all four of these: for the $y=x^2+0$, $y=x^2+2$, etc."

"The first would be zero, zero. The second is zero, two -- zero, three and zero, eight."

"And make sure when you are writing these you do what -- or don't forget to do what ?"

"Brackets," another student responded.

"Brackets. Okay, zero comma four sitting in space -- I don't know what that is. And, for heavens sake, don't use these kind of brackets." Jack drew them on the board. "These are solution set brackets and they don't mean the same thing as this," he said as he drew a second type of bracket on the board. "So just make sure you are using your ordered-pair brackets. What will happen if c is negative? We already said that. So if c is negative it shifts the graph downwards."

Jack worked through the remaining questions in the same manner. Just before the bell rang to signal the end of the class, Jack asked the students if they had thought of another function that would behave in much the same way under the transformations they had just studied.

One student called out, "x cubed."

This was not the function Jack was looking for. "Alright, but most of you didn't get to that. Before something like the x cubed" he prompted the students.

"Absolute value," another student offered.

"So, if tomorrow if you have a couple minutes maybe you can use the calculator again with the absolute value function." The class was almost over. "Anybody questions? I know we did this fairly rapidly but I'm really trying to pick up time in this unit because the next one we've got to do probability and if we do that fast you guys will get into big trouble. Any questions? None. Okay tomorrow we will just summarize this again."

The students packed up their books, returned the graphing calculators to Jack and stood at the door waiting for the bell to ring.

Jack's Use of the Graphing Calculator

Jack used the calculator in two ways: the first was to produce graphical images and the second was to encourage enrichment work. He used the graphing calculator for only one activity in the quadratic function unit but I also observed him use the calculator on another occasion in the unit on functions and relations. In both activities the calculator was used to produce graphical images, but, what was done with the graphical images was different in the two situations. Jack did not use the graphing calculators for introducing or solving max/min word problems.

The Graphing Calculator to Produce Graphs

Jack described what he does with the graphing calculator as "using it as a learning tool." One with which he can "expose the kids to many many more examples" than if they did the graphs by hand.

The first activity Jack used the calculator for was a classification investigation[1] designed to introduce the students to relations and functions and the concepts of domain and range. The students were given 41 equations and instructed to graph each equation on their calculators, record the equation, sketch the graph, and write out observations about the shape of the graph. The students were asked on the handout to cut out the graphs and group them together based on their similarities. Jack told the students it was not necessary to cut them out[2], instead the students could simply observe the graph on there screen and then draw it with other graphs that looked similar. Once the students grouped the graphs based on their similar shapes the students were then to try to find the patterns in the equations of similar graphs. The equations were of many types; there were linear equations, quadratic equations, hyperbolic equations, the absolute value function and some trigonometric functions, and there were also some relations (circles and ellipses) in the list.

Jack said he likes this activity because it gives the students the opportunity to look at a whole bunch of functions just to see "what the things look like." The problem he said, with the students attempting to graph them by hand is they "don't have a clue

[1]Pheonix used this same activity with her Math 20 and Math 33 classes.
[2]This activity was originally Jack's idea but I included the instructions to cut and group the graphs into families. Then I used this handout in a workshop Jack attended. This is where he picked up this handout for the activity.

what they [the graphs] look like," therefore they don't know what their sketch should look like. He said, "hopeful!y [by doing this activity] they can tell me what the thing is in the equation that makes the different thing -- function -- to come up."

The second activity that Jack used the graphing calculator for was the same one that Pheonix's students used to investigate transformations of the quadratic function. The students investigated transformations of the quadratic function in the forms $y=ax^2+bx+c$ and $y=a(x-h)^2+k$. Jack's class did this activity and the summary in one eighty minute period. Although he said he had used this activity in the past, I do not think he would have taught the transformations with the calculators this term if it had not been for the fact that I wanted to observe him using the calculator. I approached Jack and asked him when he would be doing that lesson since I wanted to observe that class. He told me that he would be ready in a day or two but he was not sure he was going to use the calculators because his students had a good grasp of graphing. A little while later I met him in the hallway again and he informed me that he would do the transformation investigation (with the calculators) the next day. This is one instance where I believe I may have strongly influenced a teacher's decision to use the calculators.

This activity was different than the first in that it explicitly stated the differences in the algebraic form of the functions and the students were to observe the changes to the graphical forms of the functions. The students had little difficulty with this activity. Jack finds this activity worthwhile because it gives the students an opportunity to "discover the transformations" in half the time than it took before the calculators were available. He said he always taught the transformations with graphical examples but it used to take so long just to sketch a few graphs. Now with the graphing calculator he has found he "makes up time" in this section. In the past, transformations used to take him three classes now they only take one class (for both the investigation and the summary).

<u>The Graphing Calculator as a Tool to Facilitate Enrichment Work</u>

In addition to investigating transformations of the parabola, Jack suggested to his students that that they might use the calculator to help them investigate transformations of other functions. He told them to try to come up with other examples of functions that would behave the same way as the quadratic function did under the various transformations. Since Jack treated this as an enrichment activity, only a few of the students who finished the transformation investigation early played with the calculators and did some transformations on other functions. I noted one student explore with the cubic function and a few others with the absolute value function.

Jack's Espoused Philosophies

In our discussions about mathematics and mathematics education I noticed Jack's comments focused on the mathematics content prescribed for his students. We discussed the nature of mathematics and mathematics education, but for the most part our conversations were centred on the content of the programs and how he approaches that content.

On High School Mathematics

Jack's emphasis appears to be on the *ends* of his mathematics instruction as opposed to the *means* of his instruction. He often spoke of covering the content, preparing students for the next level of mathematics, preparing his students for exams (particularly Math 30 students for the provincial diploma examination), and preparing his academic students for the possibility of further schooling. One of the aims of the academic mathematics programs is to prepare students for post-secondary education. Jack suggested that many of his students do not know what they are going to do in the future so he suggests to them that it is best for them to take as much math as they can get "just to leave all the options open."

Covering the content came up in both our conversations and in Jack's class. By covering the content Jack seemed to mean transmitting a prescribed body of mathematical knowledge consisting of facts, skills, and problem types to his students. Jack said, "I'm always so worried about trying to complete the so called course and trying to expose them to as many twists and tough stuff." Jack told me he simply does not have the time to use concrete materials even if students might understand a little better with manipulatives.

> Concrete objects? As much as we probably should [use them] because -- I think when we use those we've got to have a little bit more time. So, um, we are kind of in a situation where if we do that they may understand a little better but we may not get a whole bunch of sections. So, I'm not sure what the best way to handle that is.

Here we see Jack sacrifice depth for breath because of time constraints. He referred to this time constraint on a number of occasions.

> My kids sometimes, especially my [Math] 20s and [Math] 30s, are used to me talking for about one hour.... I find by going over the problems from the

> previous day and giving them the new theory -- in those courses the new theory is so much that it takes me a long time and in fact with some of the numbers [of students] we are getting now it's just so [difficult].

In class he told his students:

> I'm really trying to pick up time in this unit because the next one is probability and if we do that too fast you guys will get into big trouble.

I interpreted Jack's tight control of his class discussions as another example of his concern with the content. When covering material with his students, he limited the scope of the discussion or lecture to the particular topic the students were studying on that day. For example, when the students were classifying the functions Jack responded to questions about the trigonometric functions by telling the students they did not need to know about those until next year in Math 30. Or, later that same day, he told the students, "I'm leading into this. It's called domain and range and the third thing we are going to ask you a number of times is whether this thing is called a function or not." Since this particular class was to introduce graphs and graphing they did not need to know that yet so Jack mentioned these things in passing but did not elaborate on them.

<u>On Mathematics</u>

Jack has a Platonist view of mathematics. He clearly stated, "mathematical laws are constant. We may not understand them but they're there. They don't change." I asked him if that meant that there exists mathematical truths." He replied, "right and it's constant and we may change our method of looking at it, or our degree of understanding, or what ever, but there's a basic truth that doesn't get changed." That is not to say however that mathematical theorems and conclusions are not tested for truth. Jack explained, "If I come up with a little formula or truth or whatever, it is the first thing I'm going to do: test it to see if it is working."

Jack described mathematics as a "bunch of rules, there are algorithms -- what ever you want to call them -- there is a bunch of things you have to follow and if you follow them, step *a* often leads to *b* which leads to *c*. There is a very logical, sequential approach to it. And if you don't know the rules you are going to have trouble playing it. And if you don't understand how to go from step *a*, to step *b*, etcetera you are going to have trouble with it."

When I asked him about the nature of theorems and proofs, he indicated that proofs are not being done anymore (or not many) in school. He explained that he did not like Euclidean geometry. "That is another bias of mine. I didn't like Euclidean geometry. I didn't like those proofs and especially when they had to be written out as rigorously as they were. I thought, this is nuts."

I asked him about the apparent deductive nature of mathematics and the mathematics done at universities. Jack indicated that he believed the mathematics of universities was much like the mathematics of high schools. He related his experience with university mathematics. "I just thought it was the same basic approach. I just thought some of the teaching wasn't very good. Their examples weren't very clear."

On Instruction and Students

Jack described his method of instruction as "lecture, lecture, lecture." He explained:

> I still think it [lecturing] is the most efficient method of getting it across to a majority of your students. It doesn't work for some and especially when it's often pretty verbal and pretty quick. So I try to break it down as much as I can and give them as many concrete examples as I can. But it is still lecture, listen; they ask questions, I answer; [and if they still don't get it] they ask for further illustrations... I depend on them to let me know where they are.

He described the students' role this way:

> Their responsibility is to do the work... the textbook, handouts -- well it is almost always textbook and handouts. But I think I break it down enough that if they do what I ask, and if the kid is in the appropriate course they will do fine. And I do require they do a self-check. I don't go around and check to see of they have done page fifty-seven -- that sort of stuff. It is totally up to them to check the exercises, to recognize where they know or don't know the material. If they don't know the material they are expected to ask for more illustrations.

As I already mentioned, Jack wished he had more time to use concrete materials and manipulatives. The only manipulative he had used was the graphing calculator. He thought the students really liked to use the graphing calculators. He said that this was one opportunity where they got to do the work and not just listen to him talk. "Maybe that's why they like it so much, [be]cause they usually just get the calculators and go to it. That's one of the few instances where they really have a lot of time to work in class."

The introduction of graphing calculators to Jack's classes resulted in a couple of opportunities where students got to "find out for themselves" instead of being lectured to, but in Jack's class (as previously noted) the usual approach was lecture.

> In high school classes again we tend to -- especially the academic -- say, this is it. I don't even show how we develop the cosine law and sine law and all that

stuff anymore because 98 percent of the kids are just -- give me the formula and I'm quite willing to accept that you know more math than I do and that formula is probably correct. We don't ask the kids to determine whether something is true, hardly ever. We say this is the way it is done. Here are some examples. Do it.

On Technology

Jack was quick to introduce the graphing calculators to his high school mathematics classes. Prior to the purchase of the graphing calculators he and the students did all of the graphing by hand (they did not have easy access to computers). Jack expressed a number of reasons why he had chosen to use graphing calculators with his students. He said the graphing calculator gives the students "something concrete they [can] trust in. Nine times out of ten they weren't sure if they did it right. They'd always look at me like, Is this even remotely close to what I'm suppose to get? But they believe the calculator." Another advantage of using the graphing calculator is the time it saves graphing functions. The third advantage is Jack can give the students many more examples than he had before.

Jack views the calculator as a learning tool that was useful for introducing graph related concepts. However, he does not believe it is good to over use the graphing calculator or for the students to become dependent on the calculators. Jack indicated to me that once the students know what the graphs should look like then he said they should graph them by hand. He told the students that it would be much quicker to get an image of a graph in your head from an algebraic expression than it would be to use the calculator.

Jack pointed out, to me, a few of the limitations and difficulties the graphing calculator has resulted in for him and his students. The major difficulties are associated with the viewing window. The first problem is that circles appear to look like ellipses. He told his students not to worry about it since it did not really matter for the unit (domain and range) and he then demonstrated the algebraic difference between a circle and an ellipse (which was his purpose for giving the activity). The second problem is that he has to make sure that all of the examples he uses on his handouts fit in the viewing window. He thinks that it is "just too messy" when they do not. The third problem is with his tests. He has had to design his tests so that students with their own graphing calculators would not be at an advantage. He indicated he solved this problem by using graphs that would not fit the standard viewing window.

Jack's Philosophies of Mathematics and Mathematics Education Made Manifest

Jack's views on mathematics and mathematics education are consistent with one another. He believes that mathematical knowledge is "out there," fixed and permanent. Although we may look at these truths in different ways, their essence is absolute; they are fixed mathematical laws. Mathematics education, he believes, consists of covering a prescribed curriculum. The content is the given and, although you may use different methods of instructing or approaching concepts, the goal is to cover that prescribed curriculum. In both cases, Jack acknowledges that there may be different methods but that is not important; the end product is.

With this focus on ends instead of means and his views of mathematics, Jack's philosophies of mathematics and mathematics education fit nicely with those in Ernest's (1991) category of Technological Pragmatist. His aim is to teach useful mathematics, which he defines to be those concepts, skills, and techniques which are mandated in the curriculum, required for the students to pass the necessary tests, and necessary to prepare the students for further education. His views on mathematics as a body of absolute knowledge are also compatible with those of the Technological Pragmatists. Jack's method of teaching (lecture format to provide detailed explanations), at first glance, seems to fit better with the Old Humanists. However his choice of lecture (to transmit the content) as an instructional style seems to be the result of practical concerns. He told me that he covers more content by lecturing. Therefore I would place his theory of teaching not with the Old Humanists, but with the Technological Pragmatists. His comments implied that in his instruction (as it is for the Technological Pragmatist) "choices between approaches are not on the basis of principles, but on pragmatic utilitarian grounds" (Ernest, 1991, p. 153). When involved in class discussions, the students contribute observations and answers and Jack provides explanations. This format could also be one of expediency. Jack also seems to have similar beliefs to the Technological Pragmatists' with respect to their theory of learning. That is, the students need to do as many exercises as necessary to develop their skills, and be exposed (in Jack's words) to as many "twists" and "tough stuff" as possible if they are to be successful in the course. Jack indicated that concepts often do not need to be justified to the students since they trust that he knows more math than they do. He said the students are usually satisfied with his showing them how to do the concept or procedure and then giving them some examples to work on.

Jack's view of mathematics as a body of absolute truths which are independent of the method at which you arrive at them was manifested in his instruction. For example, he told his students not to worry about the circles that looked like ellipses (on the graphing calculators) since the students could use the algebraic form to recognize the circles. I realize he was trying to make a point about the algebraic form of a circle but this example points to the view that the method is not as important as the end. He could have told the students to use the square window[3] on the graphing calculator so they could get an accurate image of the circle but he did not. His emphasis on the ends of high school mathematics was manifested in both his choice to use the graphing calculator and in his choices on how it was used.

His reasons for using the calculators were for the most part pragmatic; the fact that the students enjoyed working on the guided-discovery activity was a bonus. He uses the graphing calculator because it is efficient. Not only does it save time but the students do not depend on him to confirm their work. He indicated he used to teach the transformations in much the same way but he had to graph the examples on the board, by hand, now the students can do this lesson on their own. He did not indicate to me that he thought they were learning the content any better or developing a deeper understanding for the transformations only that they enjoyed it. Another example of how Jack's focus on the ends as opposed to the means can be found in his choice of examples and questions. Recall that Jack fixed his examples and questions so they would always fit into the standard viewing window. Jack indicated this helps prevent confusion but it also prevents the students from going off in new directions and exploring mathematics beyond the prescribed curriculum, by restricting the mathematical images the students see and base their observations on.

Jack's philosophies of mathematics and mathematics education were, for the most part, unaffected by the availability of graphing calculators for his mathematics classes. However, the calculators did take him out of his usual mode of instruction (lecture) for at least a couple of lessons. His belief that lecturing is the most efficient means for instructing mathematics has been swayed, if only to a limited degree.

Henry, the subject of the next case and a relatively new teacher compared to Jack, has a very different view towards the introduction of the graphing calculator to high school mathematics.

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[3] A pre-set function to change the scale of the axes to accommodate for the rectangular pixels on the screen.

CHAPTER VIII
Case VI: Henry

Biography

Henry has a B.Ed. with a major in physical education and a minor in mathematics. He has been teaching for eight years. After his initial experience teaching junior high school for three years, he moved to a large urban high school to teach mathematics full-time. Henry has taught all the senior high school mathematics courses except Math 31.

Inside Henry's Classroom: A Vignette

Henry grabbed a cup of coffee on his way out the staff room door. As he headed towards class, I followed him, dodging students in the crowded hallway, and hoping I would not get lost. When we got to his room, he indicated that I could share the table at the back of his room, with some plants. I sat down and watched the students as they made their way into their desks. The classroom seemed quite new to me, maybe because the other schools I had been in were considerably older than this school or maybe it was because Henry kept the room neat and tidy. Besides the plants that decorated the room, there were a few posters, some mark sheets, and a long banner around all four walls which had pi expanded out to a room's worth of decimal places.

Henry had put his coffee down and was engaged in, what must have been, some left-over business with a couple of students. From what I could gather, one student had to write a test and another student was looking for a test he had already written. Even though the bell had rung, the students talked until Henry indicated he was ready to begin. Today, for the first time, the students were to use the graphing calculators in their math class.

"Now listen please."

Most of the students turned their attention to Henry.

"These calculators are worth about a hundred and twenty bucks. If you break it, it will cost you about a hundred and twenty bucks." Henry looked up from the case containing the calculators and gave the students an annoyed look.

"Shhh. The only way this is going to work is if I have everyone's attention so I don't have to repeat myself and you guys actually get something out of this as quickly as possible. These are very easy machines to use, assuming you learn some basic things."

Henry passed calculators out to the students. "Thing number one. After you take it out of the case -- I'm just going to hold this up because I don't have -- if you turn the bottom button down here, it will turn it on. And if you just do exactly what I tell you to do, I will get you graphing things really quickly."

It took a minute for everyone to turn their calculators on. Henry waited for the students to do this and then step-by-step he showed them how to set the scale for the axes. There was a constant hum of voices as the students consulted with each other about using the calculators. Henry walked around helping out students but most of them managed things on their own and with the help of their classmates. Once it appeared as though they were ready, Henry explained to the students how to graph a line. The students were pretty excited about all of this and were showing each other their graphs and asking each other for help. It began to get quite loud in the classroom.

"Hello, hello." Then a little louder he said, "Hello."

The students stopped talking and listened.

"I'm getting a little annoyed. That is one of the problems with these toys. Now if you have other things and/or you don't want to graph $y=x+2$ anymore, you need to press clear. Which is right here," he said as he pointed to button on a calculator. "Now if you had a couple of equations that you didn't want then you simply have to move down by pressing your down arrow and then *clear* and this will enable you -- if you want -- to graph up to four different things. Would you please clear all of your equations in *y-is-equal-to* so we can graph the things and talk about them."

Henry sipped his coffee. "So clear everything on that. Any problems with that?"

The students were still consulting with each other but many of them seemed to be catching on.

"Listen. Hello." Henry got their attention back again. "From your test yesterday," he wrote the equation, $y=(x+3)^2+5$, on the board. "What would the vertex -- this was going to be what shape?"

"Parabola," responded a couple of students.

"This is going to be a parabola. Some of you who graphed this did not show me a parabola on the test, with this particular equation. I was a little disappointed when we spent a week and a half on parabolas and I get lines on the test. That's disappointing. Anyway, this turns out to be a parabola with a vertex at negative three, five. If I drew a rough sketch of this negative three, five -- Is it going up or down?"

Most of the students responded to this question.

"Thanks. Now I am going to get you people to graph it on your calculator just to see that what I have been telling you for the last while is right. So we need an *x* plus three in brackets. There is a brackets -- there on your calculator -- and then you also have to press *x squared* after the brackets. Which is above your *log* button here. Now are there any problem getting a parabola opening up with a vertex of negative three, five?"

The students were all busy working through this on their calculators. The noise level rose again as the students solicited help from classmates and compared their graphs with each other.

"Anyone got a problem? Does everyone have a parabola? Okay, what happens if we put a one-half here?" Henry replaced the negative sign with the fraction one-half. "What do you think is going to happen?"

There were a few suggestions like, "It is going to get wider", and, "It is going to open up."

"Try it," Henry responded.

Try it, was a key phrase for Henry. This whole class was spent having the students graph a number of functions, first to become familiar with the graphing calculator but second to re-introduce the students to intercepts and to introduce students to roots of quadratic equations. The students worked through a number of linear equations and identified the *y*-intercepts of the equations graphically. Henry reminded them they could simply look at the equation and identify the *y*-intercept algebraically -- a skill he told them they had learned in grade ten. From the linear equations they moved on to finding the *y*-intercepts of quadratic equations.

"Now remember we talked about standard form and graphing form for quadratic functions right? Standard form, graphing form. Okay, let me give you this, *x* plus three all squared minus two. I want to know what the *y*-intercept is for that particular function. Write down the equation in your book and then write down the *y*-intercept."

The students were very quiet now. They seemed to have the calculators under control and were busy keeping up with Henry's try-it exercises.

"That's the y-intercept?" He said to a student as he looked over her shoulder.

She shook her head yes.

"That is it where does it cross the *y*-axis?"

She looked up at him.

"Where does it cross the *y*-axis?" Henry asked her.

"Where does it cross the *y*-axis people?" He asked the whole class this time.

"Eight," a few people responded.

"Okay, let's go with eight. We can go with eight. It is about eight. Yeah. Let's try another one. Let's go $5(x-2)^2-6$. And I would like the y-intercepts. "

The students went back to work. As they worked Henry walked up and down the aisles asking various students where the y-intercept was as he passed their desks. When he saw that most of the students were finished he asked the whole class, "What's the y-intercept people?"

There were a few responses that indicated some confusion.

"There isn't one?" Henry was puzzled. "Oh I find that hard to believe. Doesn't it ever touch --"

"You have to change the range," interrupted one student.

"You have to change the range." Henry shook his head in agreement. "Are you guys capable of changing the range? Try changing the range to find out where it crosses. Now for some of these it might be a little tough to tell. It is just the limits of the calculator. There is a way of going around it but we are not going to worry about that today."

A number of the students had now found the intercept.

"So, what do you think is the number?"

"Thirteen," said one student

"Twelve," another student replied. There was some disagreement about this intercept.

"Thirteen about? Thirteen, about thirteen? Twelve, thirteen? Okay, well I'm going to give you another one. Are you guys ready? Now I'm going to give you this in standard form and we are trying to tell y-intercepts in that crazy graphing form. Try x squared plus six x plus eight, " he called out. "Where does that cross the y-axis."

A minute or two later he said, "Where does it cross about?"

Again there were a few different responses.

"Eight, nine, or ten. Okay, I'll just put down eight, nine, or ten. It crosses somewhere along there. We are going to discover some amazing thing here pretty quick, but I don't want to give it away." He wrote x^2+4x-5 on the board. "Try that one for me. We will discover some amazing thing. It's amazing. You'll say, that's amazing."

As Henry walked by one student, she asked him if they would be using the calculators all the time now.

"No way," he said. "You'll have to get back to graphing them by hand."

"The y-intercept now," he turned his attention back to the class. "This is in standard form -- a little different now. A different form. I'm curious. Where does this cross the y-axis?"

"Five."

"Five. The guess is -- that's interesting " He circled the five on the example on the board. "Hum, I wonder. This one," he pointed to the previous example. "Could that have been eight maybe?"

Many of the students either shook their heads in agreement or said, "yeah."

"Let's try another one here. $3x^2+7x+1$. Now I'm thinking it is going to cross somewhere around one. That is what I'm thinking -- guided discovery. I'm guiding you. Where does it cross?"

"One," a number of students respond.

"Around one. Yeah, yeah. Now let's just do an amazing thing here. Okay? Let's change this to thirteen and change this to seventeen." He changed the coefficients in front of the x squared term and the x term. "Which that would -- I mean, that would be a crazy one for me to give you in class to do by hand. Try this one. The y-intercept is all we are looking for." As he went through this he scratched his head and tried to act very contemplative.

"I'm thinking it's going one. Am I right?"

"It's really steep."

"Right, it'll be really steep, but is it going to cross at one?"

Now most of the students were nodding in agreement.

"This is in standard form. Now we are sort of running out of time in terms of what else I wanted to get done, but let's look at x-intercepts for a little bit. We will come back to this y-intercept. I want to know how many places this parabola crosses the x-axis." He wrote $y=x^2+6x+9$ on the board. "And I want to know what they are -- the x-intercepts. Then you can try this one, x^2+4x+4. Then we can try this one, $y=x^2+5x+6$. And by all means you could --"

"Can it be just once?" A student asked.

"Well it could be once or twice, but if there is only one place -- so how many places does this cross?"

One and none were two of the responses from the students.

"One? None? Two?" Henry pointed to the students giving the responses. "Where does this hit the x-axis?"

The students stopped volunteering responses.

"I think it only hits in one place and I think actually you people are telling me negative three. I think the vertex is right on the x [axis]. Do you guys recognize this trinomial as anything. This is a perfect --"

"Square trinomial."

"This is a perfect square trinomial." Henry wrote the equation of another one on the board. "You guys graph this and see where it hits the x-axis."

"Zero," somebody suggested.

"Zero, zero? It's undefined? Infinity?" Henry repeated some of the student responses. "I got a negative two on this side."

The students were talking amongst themselves.

"Can I get confirmation on negative two?" Henry said and regained their attention.

"So it is at negative two and it only hits at one place. Hey and this is a perfect square trinomial," he tried to act surprised. "Try this one. Is this a perfect square trinomial? See if it only hits once."

Now the students responded even before they had time to enter the function into their calculators.

Henry asked, "Is it only crossing in one place? Is the vertex on the x-axis again? What's the value?"

Most of the students appeared to understand now. Henry led them through some more examples so they could discover that it was possible to have quadratic equations with no x-intercepts, one x-intercept, or two x-intercepts.

Henry wrapped the lesson up by having the students factor the examples of quadratics that they had just graphed. Then he told them, "That's what we are going to find out about in this chapter."

It seemed to me that they had a pretty good grasp of the things that were to come.

<u>Henry's Use of the Graphing Calculator</u>

Henry used the calculator only to provide examples of graphs when teaching two different concepts: transformations and intercepts[1]. In one case he used only the overhead graphing calculator and in the second case each student had a calculator to use.

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[1]Most of the teachers in this study taught the unit on the quadratic equation (which examines roots) prior to teaching the unit on the quadratic function (transformations and max/min problems) and did not use the graphing calculator. Since Henry used the graphing calculator I decided to observed this lesson and include it in my observations.

One interesting note about Henry's use of the graphing calculator in his instruction is that he used it only in the term that I was doing my research with him. In my follow-up interview with Henry, he indicated that although he had taught the course again he had not used the calculators. It is important for me to indicate that I do not think that his choice to use the calculators the term I observed his instruction was in response to this research, because Henry had used the graphing calculator before this project was initiated. I believe the perturbation for his use of the calculators that term was probably their recent arrival in the school.

Henry's Approach with the Overhead Graphing Calculator

Henry used the overhead graphing calculator to demonstrate transformations of the parabola. For each parameter, Henry produced on the overhead calculator a number of graphs (as examples) for the students to observe and examine. With Henry's guidance, the students discovered the role of each parameter for the quadratic function in the form $y=a(x-p)^2+q$. Henry told me that using the calculator made no real difference to the way he always taught this unit: "It was just a different way of doing the same thing." He thought the calculator saved him a bit of time, suggesting, "We probably got through more or if we didn't get through more we certainly did the same number quicker." This must not have been a great enough benefit since Henry did not use the calculator the next time he taught this concept.

The Students' Use of the Graphing Calculators

In the unit on the quadratic equation, Henry had the students use the calculators to graph various lines and quadratics in order to re-introduce the notion of intercepts and to introduce the concept of roots. (This lesson is described in detail in the vignette.) Each student was given a calculator and used it to graph examples. Henry spent the whole period calling out examples for the students to graph and examine. He kept reminding them, as they did this, to compare the graphs with the equations (the algebraic forms of the functions). Even though (from my point of view) the students enjoyed this activity and seemed to learn from it, Henry did not use the activity again the next term. He said, "I didn't have the time and I just never -- I don't think it hurt them any."

Henry's Espoused Philosophies

Henry was one of the teachers who really limited the discussions of our topics by making brief responses and making most of his references to high school mathematics. We followed the interview guide but he did not expand on many of his responses; however, there were a few items that he did elaborate on. Henry clearly articulated his view that doing algebra is the focus of high school mathematics.

On High School Mathematics

Henry has very specific and strong beliefs of what is appropriate mathematics for high school students, but his views on why we teach mathematics are less clear. He does not agree that we teach math because it is useful (for work or university), but he does admit that one of the reasons we go to school is to "develop some logic" and math is good for this since it is a little bit more focused" than some [of the] other subjects. Henry said when students ask him why they have to do math he tells them "society deems it necessary" by requiring it for a high school diploma.

Henry has a very specific conception of what high school mathematics is -- doing algebra. I think I can best demonstrate Henry's views of mathematics by relating one of the most focused pieces of our discussion. This part of the discussion came after we started talking about the use of graphing calculators to teach transformations of the quadratic function.

Elaine: If you start with your transformations of functions let's say, and you show them the vertex moving or --
Henry: Right but you still -- if your going to teach them the math, then you have to teach them the math. Now sure it moves.
Elaine: The math being the algebra part?
Henry: Yeah.
Elaine: Right. So, in our high school mathematics program we really equate math and algebra?
Henry: Well I think that's where -- that's the level that it's at.
Elaine I mean, because it could be -- it could be coordinate geometry or it could be, um, it could just be trig I guess.
Henry: Well there's still algebra behind it.
Elaine: Right, so algebra --
Henry: I mean I don't see high school math [as saying to students], okay, I want you to memorize that. When this happens, this happens. [Rather] I want you to be able to show me how it works and why it works and things like that.

Technology might be available to do applications of mathematics but in Henry's view that is not mathematics.

Henry: Like we're not -- it's sort of like the computers when computers first came in. They used to teach kids how to program computers. Well they don't do that anymore.
Elaine: No?

Henry: No. They teach kids how to use the programs.
Elaine: Right.
Henry: Okay, well now the issue I have is, if we have these graphing calculators are we supposed to be teaching the kid to carry a calculator around with them and be able to graph things? Or -- and therefore use a graphing calculator. Or are you trying to teach them math? And those are two completely different things. Programming a computer these days is unrealistic; but, assuming you are going to use a computer then you have it there and you know -- and so you can use programs that are out.
Elaine: Right and a select group will go out and program computers and the rest of the world will --
Henry: Right, but I don't know that graphing calculators are [in] so broad of use that you'll run into a lab and everyone will have a graphing calculator and then you'll graph things all day.
Elaine: I would say that they're not going to be.
Henry: No.
Elaine: But, on the other hand, what will be available is every engineer will have a computer.
Henry: That's right, but, it's not math though. I see them as two completely different things.
It's nice to be able to graph a bunch in a row and see some general things but that's not the math behind it.

Again, Henry tried to explain what he thinks the mathematics is.

Elaine: Okay and this is key. I mean this is exactly what my study is. So what then is math? Like what --
Henry: Well, math -- are you talking at the high school level?
Elaine: Sure. What is math -- like for you?
Henry: Well math is basically the logical -- showing logical relationships between things.
Elaine: -- and the work we can do with algebra?
Henry: But that just follows from -- I mean you start at a point, say this is a number and this is what it means and then it's just developed up to here. Now this is where we are at in the high school and that obviously can go a lot further.
Elaine: And we're still showing this logical progression of things? If you have an equation or a system of equations there's this logical process that will get you to a solution?
Henry: If you want to relate two things then you can show that relationship graphically with a line. Now if you want to relate those two things as they relate to these other two things, then you might want to solve a system. So the algebra is the tool that you use to figure out things.

Not quite understanding why he thought kids can not do mathematics with a computer, I pushed Henry a little bit more.

Elaine: The premise is, by the people who have brought in the graphing calculators -- the people who brought it in, they say that graphical solutions are as valid as algebraic solutions. So if your students have a system of four equations they can, in fact, graph them and they can use the trace key and they can come up with a solution
Henry: Sure.

Elaine: Graphically.
Henry: That's right, and you can punch in, uh, a system (in a calculator that doesn't graph things) of three equations. You punch in all nine numbers and you'll get an answer. Does that mean anything to the student? No.
Elaine: No, I probably guess not --
Henry: You can program a calculator to solve any triangle. Does that mean anything to the student? No.
Elaine: Okay, so in solving things --
Henry: So you get five -- the number is five. So --
Elaine: So in solving things then that must -- not only does solving things have something to do with mathematics but it has something to do with kids learning mathematics? Like it's in solving --
Henry: The mathematics is the solving.
Elaine: Okay, it is the doing -- like the process.
Henry: Yeah.
Elaine: Okay, that is meaningful to me.
Henry: Now one part of mathematics might be using a graphing calculator and graphically showing that these are going to cross in two places, but the actual math as far as I see it is, okay, well let's actually use some algebra to find these things --
Elaine: So mathematics is in doing?
Henry: Yeah - the sign I have in my room -- *mathematics is not a spectator sport* -- you don't sit and watch math.

The bottom line for Henry is you "do algebra" when you do high school mathematics. You manipulate variables and equations and do things with a paper and pencil. Henry defines mathematics as an action, not as an object.

On Mathematics

This view of mathematics, as an action, came up time and time again in our conversation. I suggested to Henry that *mathematics helps you find patterns and relationships in the world;* and, Henry said that *helps* was not a good word. He suggested it showed patterns and relationships. I asked him if *a world with different mathematical truths was inconceivable* and he said yes because, if you go back to the way the number system is, and work your way up to where we are now, then it would make sense that things would make the same kind of sense now. Again he described mathematics with an action. Another case of this action sense of mathematics was demonstrated by his reaction to the idea that *mathematical objects are constructed by the mind.* He said, "well, I wasn't really sure what a mathematical object was." I suggested the number three, pi, or triangles might be thought of as objects constructed by the mind. His response was, "About a circle -- I don't know that a circle is constructed by the mind. I think it's been around and therefore we have analysed it." Once again the mathematics is in the action not in the object.

I asked if he thought mathematical truths were subject to revolutionary change in the way scientific truths might be. He suggested that the way in which mathematics is developed would prevent that, since it is "based on something previous, which is based on something previous which is based on -- If you are talking about university [mathematics] then there might be a mess up. But in high school... It [mathematical truths (content)] is time tested and pretty basic as far as math goes." So at the "university level" there is the possibility that somebody may mess up with a proof or a theorem or a technique, but, the implication is when doing mathematics you will eventually come up with an absolute truth. Henry suggested the bottom line is, "Things have to jive."

<u>On Students and Instruction</u>

Henry's views of students and instruction fit with his views of mathematics. Henry suggested that students do not learn math by sitting and watching him do math. He said:

> They need to participate. If I write down an example they need to write down the example. They need to try it. I tend to let kids try things before I actually show them, a lot more than just showing them, so that they appreciate later the fact that there is something involved in doing it. They just can't do it by thinking, they've got to know it.... You can't memorize a solution, ...by doing it you supposedly understand.

His instructional style reflects this belief. Henry said he does not usually lecture or write notes. His lessons consist of an ongoing interaction with the students. He suggests examples and problems and the students are continuously engaged in solving these. He called his style, "guided discovery," with an emphasis on the guided. He explained:

> I think you have to [guide] because kids are not generally going to sit and discover things unless you prod them along -- and find things that are interesting to you as a teacher, that they'll only find interesting because of the fact that I do. So you have to know where you're headed and have to know where some highlights are and bring those to their attention.

He structures his lessons so the students discover the concepts on their own "You set it up so they are actually doing something in the process rather than me telling them this is how it is." He wants his students to be able to show him "how it works and why it works."

When asked if mathematics helps one develop abstract thinking, he responded, "Well it depends what level you are at. I don't think we [Math 20] do a lot of abstract thinking." I asked him about all the algebra the students do, if he thought this was abstract. Henry explained that he did not think the students find algebra abstract.

> Well I don't know that it's abstract to them though. Abstract to me means, not concrete. Now algebra to them, I think, [as they see it] there's only one way to do it. I don't view that as abstract. Now maybe that's the problem with the way I look at the word.

On Technology

As I have already pointed out, Henry does not believe graphing calculators and computers have a place in the high school mathematics classroom. He has some concerns with the students using technology. Use of the calculators, he believes, might result in the students neither learning nor understanding the mathematics "behind the graphs." He said, the graphing calculator is a "tool to show them, fairly quickly, what's happened. But realistically they are not learning any math from that." His concern seems to be that the students will look at the images but not try to make any sense of them in the context they are being used. Furthermore Henry seems to be assuming that by plotting a table of values, the students will engage in what ever action it takes to understand this graph they draw. Even though Henry has chosen not to use graphing calculators in his instruction, their availability has influenced his instruction; he said has had to rewrite his tests so that students with graphing calculators are not at an advantage.

I do not think that Henry is opposed to the use of technology in other situations. Henry prepared his unit test on the word processor, and I noticed a box of discs on his desk, so obviously Henry is familiar with computers and uses them. He just does not see any point in using them to teach mathematics.

Henry's Philosophy of Mathematics and Mathematics Education Made Manifest

Henry's philosophy of mathematics education can be deduced from his beliefs about the importance of algebra and his beliefs about what it means to do mathematics. Henry's views and beliefs, like Pheonix's, do not correspond with any one particular ideology in Ernest's classification system. Table 7 indicates the different views in Ernest's classification system that are similar to Henry's. This spread of Henry's views on the chart seem to once again point to a potential flaw in Ernest's system. (I will discuss the problems with Ernest's classification system in Chapter IX)

Henry's view of high school mathematics, as we saw, is focused on algebra. This view seems be very similar to the Old Humanist perspective of mathematics as a body of pure knowledge. However, he did not express many of the views of mathematics, that are important to this group. That is, he never spoke of the aesthetic qualities of mathematics, he did not emphasize its structure or its precision, nor did he express an image of high school mathematics as the foundational knowledge for the study of higher level and more powerful mathematics. Henry did indicate that he expects his students to be able to know how to do the math and explain why it is done in such a way. This is very much like the Old Humanists' theory of learning: understanding and application. Henry's views on resources are also compatible with this group; he sees the value in well chosen examples and try-it exercises, but he does not see the value in using technological devices such as the graphing calculators. He also seems to project the Old Humanist attitude (in his classes) that he is the possessor of the knowledge which he is responsible for passing on to his students.

The way Henry teaches mathematics, by guided discovery, and his emphasis on student activity in his lessons, suggest at least implicitly that mathematics is a process. This process view of mathematics fits with the Progressive Educators' view of mathematics and partly with their theory of learning which says learning is "first and foremost active." Recall that this action notion of mathematics was one of Henry's most clearly articulated conceptions; however I do not want to lose sight of the fact that Henry's guided discovery activities were *very closely guided.* There is a difference between Henry 's process view of mathematics and that of the Progressive Educators, since the latter stress personal exploration and I am not suggesting that Henry facilitated any exploration. So, here we may see an instructional style that Ernest did not account for. The other possibility is that Henry's theory of learning is a variation of the Technological Pragmatist's, for whom the acquisition of knowledge and skills is an objective. If we take away the emphasis on applications from the Technological Pragmatists, or view algebra as an application of mathematics, then Henry's instructional style would fit with their theory of learning. This may be a problem with Ernest's classification system or it may be that Henry has an unique view of algebra. Although Henry probably best fits into the category of the Old Humanist. Ernest's classification system does not help me better understand Henry's philosophy of mathematics education.

Table 7
Placing Henry's Views into Ernest's Classification System

Social Group	Technological Pragmatist	Old Humanist	Progressive Educator
View of Mathematics		-structured body of pure knowledge	-problem solving, investigating, generalizing
Theory of Learning	-acquisition of knowledge and skills, practical experience	-teacher transmits body of knowledge	-learning first and foremost active
Theory of Teaching	-skill instructor, development and awareness of effective strategies	-various approaches to motivate, -transmit knowledge from teacher, the master, to the students	

Henry's philosophy of mathematics education "as doing" was manifested in his instruction in a few different ways. For example, it was manifested in his instructional style, with the try-it examples he used to teach the concepts of roots and intercepts. Henry spent very little (to the point of almost no) time explaining to the students; instead, students were always engaged in doing: entering functions, examining functions, and comparing equations and graphs. He led the students to the concepts through the examples and questions he asked them. He would tease out the ideas, concepts, or relationships he wanted the students to find.

His way of interacting with students in class reflected his belief that the teacher knows not only the mathematics, but also knows what is important and interesting. When introducing the concept of roots, the students engaged in a search for the important and interesting by entering suggested functions on their calculators and then examining the graphs and comparing them with their equations. Henry would hint at what they should notice and then give then another function. This was almost like a cat and mouse game, where the teacher hides the mouse in his pocket and the cats chase him around trying to catch it.

Henry's focus on doing algebra was manifested in his decision not to use the graphing calculators the next time he taught the course. Not only did he make it very clear that he believes that doing math is pretty much about doing algebra but he made it very clear that you don't do mathematics by having a calculator provide a solution (whether it is a graphical solution or a numerical solution). Henry's philosophy of mathematics seems to be unaltered by the availability of graphing calculators in his school. After all, one does mathematics by working out solutions algebraically. Having said this, I must point out that Henry's decision not to use graphing calculators the next time he taught the course could have been as a result of very little in-servicing. This is possible since Henry had attended a only one session on graphing calculators and that session was focused on their operation.

This is the last one of the case studies. In the next chapter I will briefly summarize the study, the research questions, and then discussion some implications.

CHAPTER IX

The Wind and the Chimes: A Discussion and Some Implications

Introduction

Varela (1987) suggests that if we want to better understand the melody of a wind chime we turn, not to the nature of the wind, but, to the structure of the wind chime itself. In this study I turned to the nature of the teachers' philosophical perspectives of mathematics and mathematics education so I could better understand why teachers implemented the use of graphing calculators in the ways that they did.

In this study I explored six teachers' expressed and manifested philosophies as they were articulated and worked out in the context of the teachers making choices and decisions for implementing graphing calculators in their instruction of high school mathematics. The introduction of graphing calculators into high school mathematics programs provided an ideal opportunity to carry out this study, since teachers were being faced with a perturbation to their instruction. I hoped their philosophies of mathematics and mathematics education underlying the decisions they made would be more visible to both themselves and an observer, thus enabling me to examine their philosophies and decisions. This opportunity also seemed valuable since there was very little research on graphing calculators available, yet, the number of recommendations on how the graphing calculators could best be used were growing rapidly.

In this chapter I will summarize the study and the responses to the research questions. Since, Ernest's (1991) classification system of "Ideologies of Mathematics Education" was used extensively in my interpretation of the teachers' philosophies of mathematics and mathematics education, I have included a critique of this system based on my examination of teachers' philosophies in an instructional context. Finally, I will discuss some implications of my study for graphing calculator use, teacher education, and further research.

Summary of the Study

Method

In an effort to learn about teachers' philosophies of mathematics and mathematics education and how these philosophies are manifested in the teachers' instruction of mathematics, I studied teachers faced with making curricular decisions for implementation of graphing calculators in high school mathematics. Six teachers were observed in their classroom settings (two to eight times) as they used calculators in their instruction of lessons on the quadratic function in either Math 20 or Math 33. Field notes and audio tapes were taken as part of these observations. Following the classroom observations each teacher was interviewed twice. The first interview immediately followed the classroom observations and focused on the teachers' beliefs, attitudes, and conceptions of mathematics and mathematics education. The second interview was conducted after I made a preliminary analysis of the data and was used to verify my interpretations of the data. Complete transcriptions of the teachers' lessons and interviews, my field notes, and documents such as handouts and tests the teachers gave to students comprised the data display from which I analysed the teachers use of the graphing calculators and their philosophies of mathematics and mathematics education.

The data, my analyses, and my interpretations were then worked into cases about each of the six teachers. Within each case is a brief biography, a vignette from one of the teacher's lessons, expositions of the teacher's utilization of the graphing calculator and the teacher's espoused philosophies of mathematics and mathematics education, and a discussion of the teacher's philosophies and the manifestation of those philosophies in their instruction of mathematics.

The number of teachers studied in this research necessitated an interpretive analysis within the body of each case report. Since I did not devote a chapter to the analysis, I believe it is useful to address the research questions again in this chapter in order to bring the highlights of the study together into one place. I am intentionally not doing a cross-case analysis since I do not believe one would be in the spirit of the research and I do not think it would help the reader to better understand the teachers in the context of the research questions. I will now turn to the research questions.

The Research Questions

In this section I will summarize the findings from the research with respect to the research questions. Assuming that teachers have philosophies of mathematics and mathematics education and that these philosophies even if "scarcely coherent" underlie mathematics pedagogy, I posed six questions at the beginning of the study to help me better understand teachers' philosophies and the manifestation of those philosophies as they implemented graphing calculators in their instruction of mathematics.

1) How does the teacher implement the use of graphing calculators in instruction?

2) What are the teacher's reasons for using the graphing calculators in particular ways?

3) What are the teacher's philosophies of mathematics and mathematics education?

4) How are the teacher's philosophies of mathematics and mathematics education manifested in instructional practice?

5) How are a teacher's philosophies of mathematics and mathematics education manifested in interactions with students?

6) In what ways does the availability and use of graphing calculators in mathematics instruction affect the teacher's beliefs about mathematics and mathematics education?

Following the interpretive form I established within the case chapters, I will discuss the research questions under two themes: graphing calculators and teachers' philosophies. First, I will respond to questions one and two, those that are focused on the graphing calculators. Then, I will respond to the last four questions on teachers' philosophies. Finally, I will attempt to integrate the responses to all the questions.

Teachers' Uses of Graphing Calculators

How does the teacher implement the use of graphing calculators in instruction?

All of the teachers used the graphing calculator to teach transformations of the quadratic function expressed in the form, $y=a(x-p)^2+q$. Although the availability of the tool and the curriculum requirements were the same for all the teachers, their uses for the graphing calculator varied considerably. One teacher wrote a few examples (for each parameter) on the board, which each student graphed on a graphing calculator. In another teacher's class, the students worked through the transformations, from worksheets with both examples and questions, over a period of a few days. Pheonix, Mark, Wally, and Jack used the graphing calculators so students could draw accurate graphs of many functions with which they, independently of the teacher, then could use to discover the role of each parameter on transformations of the parabola. Their worksheets required the students make and record observations (observing, pattern making) as they carried out the investigation and then summarize their findings (generalizing, abstracting) at the end of the investigation. Students were given sufficient time to work through the activities before the teacher confirmed or summarized student findings. Troy did this same type of activity but with many fewer examples and much more teacher intervention. Henry used the calculator to generate examples of graphs from which he could demonstrate to the students the effects the parameters had on the shape and position of the parabola.

In the course of this study, I witnessed two other activities that were not part of the quadratic function unit I was observing. Jack and Pheonix gave their students a classification activity which entailed the students graph many functions and relations and then group these graphs based on similarities. Henry used the graphing calculator to introduce his students to the concept of roots. Students examined intercepts on the graphs of quadratic equations and compared these with the corresponding algebraic expression.

The availability of graphing calculators provided teachers with the opportunity to not only teach the concepts in Math 20 differently than they had in the past, but also to extend the content. Some extensions led to new concepts; some extensions simply meant a change in the particular types of examples studied. Pheonix asked the students to explore the possibility of the parameter "*a*" getting closer and closer to zero (conjecturing and exploring the concept of limits). Jack suggested to his students that they look for another function that behaved in the same way as the quadratic function (generalizing and foreshadowing future content). Henry asked students to graph functions that he would have never asked the students to do by hand (using the graphical power of the calculator) and Mark had the students think about what would happen to the cubic function under a transformation (generalizing) and then he had them test their conjecture with the graphing calculator.

The graphing calculators were not only used to provide the graphical images for the students to study, but they were also used as a checking device to confirm students hand sketches of graphs and as a checking device to confirm the manipulation of the algebraic expressions of the quadratic function from standard form to completed-square form. All of the teachers taught transformations with the explicit intent that the students would learn to graph parabolas by hand from the quadratic equation, $y=a(x-p)^2+q$. However all of them (except Henry) gave the students some opportunity to check their hand drawn graphs with the calculators.

Although using graphical solutions for max/min problems is stated as an objective in the provincial curriculum, only two of the teachers (Troy and Mark) used the graphing calculator to investigate graphical solutions to max/min word problems. Both of them went through an example with the students and suggested that the students try to solve at least a couple of problems graphically. The other teachers used algebraic means to solve these problems and not one of the teachers requested or encouraged a graphical solution to a max/min word problem in the unit test they gave the students, even though they all asked the students to solve at least one max/min word problem. These instructions and this question from Troy's test are typical of the word problems the teachers gave on their tests

> Instructions: Write complete solutions for the following questions. All pertinent work must be shown to receive full credit.
>
> Question: A rectangular lot is to be enclosed on 3 sides by 160 metres of fencing. What are the dimensions of the lot of maximum area that can be enclosed?

In summary, the graphing calculators were used primarily to provide graphical images so the students could investigate transformations of the quadratic function. These investigations intended the students observe, generalize, and abstract. The calculators were also used as a means of verifying student work. In only a couple of instances teachers encouraged students to use the graphing calculator to investigate what-happens-if questions. Although most of the teachers used the calculators to facilitate one or two guided-discovery activities, they (for the most part) did not use the calculators to facilitate and/or encourage the students to conjecture and prove or refute ideas.

What are the teacher's reasons for using the graphing calculator in particular ways?

Teachers gave a number of pragmatic reasons for using graphing calculators: to offer instructional variety, to save time, to generate many more examples, and to motivate students. All of the teachers suggested that using the graphing calculator saved time. Troy and Jack both indicated that the time they saved in this unit could be better spent in other places in the Math 20 curriculum. Pheonix, Mark, Jack, and Henry said they could provide or request many more examples when using the graphing calculators. Mark, Pheonix, Wally, and Jack saw the implementation of graphing calculators into their lessons as providing an opportunity for varying their instructional strategy; they thought this was important for their students. Wally and Jack noted the change in routine it resulted in for their students, whereas, Pheonix and Mark noted the different type of learning experience it resulted in for their students. Jack expressed his observation that the students had more confidence in the accuracy of the graphs they produced with the graphing calculators (Ruthven , 1991 also found this) and it enabled the students to work on their own with less need to ask him if what they were doing was right. All of the teachers but Henry expressed the view that the graphing calculators were motivating for their students. Pheonix and Mark both articulated the view that their students have a better understanding of the concepts in this unit because of the activities that they did with the calculators. This is not to be mistaken for attributing their better understanding to the calculators alone but to the activity with the calculators. The guided-discovery activity gave the students the opportunity to investigate and "find out", for themselves, the effects of the parameters on the shape and position of the parabolas. Finally, although Mark did not suggest that this was his reason for using the graphing calculators he did say that the calculators in high school represent a modernization of the mathematics curriculum.

To summarize, the teachers' most common reasons for using the graphing calculators in this unit on the quadratic function were: they saved time and they were highly motivating for the students. Some of the teachers also indicated they were able to use a guided-discovery approach to teaching transformations of the quadratic function because the graphing calculators were available for their students to use.

Teachers' Philosophies of Mathematics and Mathematics Education

The response to the research questions about teachers' philosophies will be in the form of a brief summary of the data presented in the cases As a way of reminding the reader of the six teachers in this study, I have noted some of their key beliefs and conceptions about mathematics and mathematics education in Table 7. As I indicated earlier in this document, I believe teachers need to be observed in the context of their classes and clearly a summary such as this strips away any context that I have tried to preserve for the reader in each of the case chapters. Having said that, I will proceed with the summary assuming readers are familiar with the case reports.

Table 8
A Brief Look, in Summary, at the Six Teachers in this Study[1]

Teacher	Math	High School Math	Students	Instruction	Calculator	Ernest's Ideology
Troy	large jigsaw puzzle	laying a foundation	strive for image of large picture	transmit and explain	requirement, saves time	Old Humanist
Pheonix	process	teach a way of thinking	turn on to math	make understandable	tool for understanding	Old Humanist/ Progressive Educator
Mark	ethno-mathematics view	teach a process	engage in mathematical activity	demonstrate connections and incorporate human experience	modernization of the curriculum	Progressive Educator/ Public Educator
Wally	multiplicity of views	content of textbooks	have to be able to work in the abstract	guided practice	a need to balance calculator use and practice	Technological Pragmatist/ Industrial Trainer
Jack	mathematical laws that are independent of method	teach content of program, focus on ends	trust he knows more math than they do, satisfied with being told	lecture	save time and provide an alternative instructional mode	Technological Pragmatist
Henry	mathematics as doing	doing algebra	must be active	possessor of knowledge/ leads students to interesting mathematics	mathematics is not in the solution from the calculator, -did not use calculators the following term	Old Humanist/ Technological Pragmatist/ Progressive Educator (but not a good fit in any of these)

What are the teachers philosophies of mathematics and mathematics education?

Troy expressed an Old Humanist view of mathematics and mathematics education. His philosophy of mathematics includes these beliefs and conceptions: mathematics is a structured body of pure and absolute knowledge; mathematical

[1]This table is an oversimplification of the teachers' beliefs and positions on some topics. It is intended as device to help the reader remember the teachers without referring back to the cases. It is not meant to be used to describe the teachers without the support of the case reports.

processes are rigorous, precise, and logical; mathematics is, for the most part, derived through deductive reasoning; some mathematics is done inductively; and mathematics provides us with a "range of what is possible." His philosophy of mathematics education includes the following beliefs and conceptions: the aim of school mathematics is to provide a foundation in the area of mathematics and give the students an appreciation for mathematics; providing applications is secondary to providing the foundation of knowledge; inductive reasoning is appropriate for high school mathematics even though it is not as sophisticated as deductive reasoning; use of four function calculators has contributed to the erosion of students' basic skills; and graphing calculators can be used to demonstrate concepts in high school mathematics, but, doing mathematics with a calculator is neither as precise as algebraic means nor as satisfying.

Pheonix, like Troy, has views about mathematics which are similar to the Old Humanists' but her beliefs about the learning of mathematics are much closer aligned with the Progressive Educators. Her philosophy of mathematics includes the following beliefs and conceptions: mathematics involves a logical, cumulative, and sequential process; mathematics consists of rules and processes that can be used to develop new rules and processes; mathematical knowledge is derived through logical reasoning; mathematics can be done both inductively and deductively; and mathematical truths are absolute. Pheonix's philosophy of mathematics education includes her philosophy of mathematics but it is strongly flavored by her beliefs and conceptions of the students as learners. She believes all students can do mathematics, although, some find it more difficult than others; many people "hate" mathematics; and excessive use of calculators can result in the erosion of students basic arithmetic skills. Her beliefs and conceptions about teaching include: teaching mathematics requires a relationship between the students and teacher; teachers need to motivate and show students they can do mathematics and use mathematics; teachers need to help students master basics and build on what they already know; teachers need to teach in a way that "kids can understand"; mathematics can be taught both inductively and deductively; graphing calculators provide an opportunity for the students to discover mathematical concepts.

Mark's views of the students as learners of mathematics are closely aligned with the Progressive Educators, as were Pheonix's. The most dominant aspect of Mark's philosophy of mathematics, however, is the view of mathematics as a human activity. He also express other views of mathematics which contribute to his philosophy of mathematics. These include the beliefs that: mathematics is found in our everyday world; mathematics is something we use in our lives; mathematics is a logical process that we can think through; and mathematics builds on itself. His philosophy of mathematics education is based on the premise that we do not teach mathematics for the knowledge base, but, rather we teach mathematics to show there is a process that we can go through to solve problems. Mark's philosophy of mathematics education also includes his views on students as learners of mathematics and on teaching mathematics. He believes all students can learn even though they have varying abilities; graphing calculators provide an opportunity for students to engage in mathematical activity; in his lessons he must provide a base or foundation from which his students can work; and in learning mathematics, he also learns about teaching mathematics.

Wally, in comparison to Mark's focused perspective on mathematics, has a multiplicity of views and opinions about mathematics and mathematics education without a clear focus. His espoused views on mathematics include: mathematics is logical and abstract; mathematics helps you find patterns and relationships; mathematical knowledge is not absolute, since "anything is possible"; mathematics can be used to help train a person to be logical; mathematics teaches people organization skills; and mathematics can be used in life and in careers. His philosophy of mathematics education includes a variety of views and beliefs on students, learning, and teaching. He believes students need to see things in many different ways; graphing calculators provide another way of seeing the mathematics; and academic kids (the thinkers) are not as well suited to cooperative learning as the non-academic students. His beliefs about teaching and learning mathematics include: students need repetition; students have to learn formulas to be successful in mathematics; and limited use of the graphing calculator will get people thinking but over use promotes dependency. Wally's views of learning mathematics by learning rules and formulas seem to fit with the views of the Industrial Trainers; however his multiplicity of opinions and views about mathematics is more reflective of the Technological Pragmatists.

Jack's philosophy of mathematics and mathematics education seem to focus on ends as opposed to means which led me to classify Jack as a Technological Pragmatist. Jack believes there is a logical and sequential approach to mathematics; mathematics is discovered; mathematical laws are independent of the means in which they are derived; there may be different ways to approach concepts, but there is only one answer; mathematical knowledge is absolute; you need to know the rules to do mathematics; and proving theorems is a rigorous task (which he does not care for). His philosophy of mathematics education is centred around his aim of getting through the content. Other beliefs and attitudes which contribute to his philosophy of mathematics education include: lecturing is the most efficient way of getting through the content; students accept the teacher as the authority for knowing what is true in mathematics; students are usually content with having the formula provided to them without a proof to demonstrate that it works; students are responsible for doing their work and identifying where they have problems; and graphing calculators provide an efficient means for teaching some concepts

Henry's philosophy of mathematics is similar to Jack's but his views about appropriate instruction are quite different. His beliefs, attitudes, and conceptions of mathematics include: mathematics is developed by building on to previously established mathematical truths; mathematics is consistent; mathematical truths are well founded, therefore, they are not prone to error; and mathematics is used to analyse objects of the world. Henry's philosophy of mathematics education is specifically focused on doing algebra. His philosophy includes the following beliefs and attitudes: students do mathematics when they do algebra; students do not find algebra abstract; students do not learn math by sitting and watching him do math; teachers bring the highlights of mathematics to the attention of the students; students need to be guided and prodded along; and students are not doing mathematics when they are using the graphing calculators to do their graphing for them.

Although these six teachers share some common beliefs and conceptions about mathematics (for example, all of them expressed the view that doing mathematics involves using a logical process, one that is unique to mathematics and all but Wally said mathematical knowledge is absolute), they all have quite different philosophies of mathematics education. These differences were articulated as different focuses in our discussions. Troy focused on teaching mathematics to provide a foundation. Pheonix talked about helping students to see that they can do mathematics. Mark spoke of mathematics as a human experience. Wally expressed a mutiplicity of views. Jack focused on the pragmatic aspects of teaching mathematics. Henry focused on mathematics as doing algebra. Both the similarities and the differences, in the teachers' perspectives, were manifested in the teachers' instruction and interactions with their students.

How are the teachers' philosophies of mathematics and mathematics education manifested in their instruction and interaction with their students?

My observations of teachers in their classroom environments, as they taught and interacted with students, leads me to suggest that teachers' philosophies of mathematics and mathematics education are manifested in their choices of activities for use with the graphing calculator, the kinds of questions they ask students, and in other interactions with their students. Ernest (1991) suggests teachers with differing ideologies of mathematics education have different theories about the use of resources. I found this to be the case but the differences were not clearly manifested in the actual use of graphing calculators; rather they were most clearly articulated when, in the interviews, the teachers discussed their beliefs, attitudes, and conceptions about the appropriateness and the usefulness of such resources.

Teachers' complete philosophies of mathematics or mathematics education are not manifested in a particular decision, a single lesson, or, even over the course of a few lessons; rather, it is their beliefs, attitudes, and conceptions (which define their philosophies) which are made manifest. Since this manifestation is conscious in the domain of the observer, the more one observes a particular teacher's lessons the more opportunity there is for the observer to make distinctions thus the clearer the manifestations becomes to the observer. Here are some examples of teachers philosophies made manifest in their instruction.

All of the teachers in my study used the graphing calculator to teach transformations of the quadratic function. Students used the graphing calculator as a tool to graph functions from which they were to make observations and from these determine the role of the various parameters. This activity reflected Troy's and Pheonix's explicit beliefs that mathematics can be done inductively. It also reflected beliefs such as Jack's and Henry's that when using mathematical processes (if used correctly) a person will be led to the mathematical truth behind the task. In other words, they believe a given mathematical task leads to one correct answer. Mark's views of mathematics as a human activity (experience) might also thought to be reflected in his choice of a guided-discovery activity. This was made clearer when Mark indicated to me that the next time he taught the unit he was not going to "steer" the students as much. All of the teachers spoke of mathematics as logical and sequential. The way in which the handouts and the examples were structured, to move from one parameter of quadratic functions to the next, seemed to reflect the belief that mathematics is logical and sequential in nature[2]. None of the teachers used an open approach to teach the concepts related to the quadratic function; that is, they did not have students define their own questions or generate questions the class could explore together to investigate the mathematics of the quadratic function.

[2]Pheonix even attributed her organizational skills and her preference for order to her mathematical nature. She was quite surprised when I indicated I was not well organized; she thought all math teachers should be.

A well-structured activity cannot be solely attributed to a teacher's belief about mathematics since teachers' choices for activities are also a consequence of their beliefs about learning. For example, Wally suggested students need repetition and practice to learn mathematics and Pheonix suggested students need things broken down in ways they can understand. These guided-discovery activities provided the opportunities for repetition and practice and they broke things down into pieces.

Some explicit instructional decisions can be interpreted in light of the teacher's philosophies. For example, Troy's use of the graphing calculator to discuss graphical solutions to max/min problems at first glance seems to contradict his beliefs about the usefulness of technology in mathematics, but when Troy's discussion of graphical solutions is taken into consideration the contradiction is dismissed. He made it clear to his students that he found graphical solutions less acceptable than algebraic solutions and that they would be required solve max/min problems without the aid of the graphing calculator after the introductory lessons. For Troy, a graphical analysis does not constitute "real" mathematics or at least not sophisticated mathematics.

Henry's decision not to use graphing calculators in his future lessons seems to me one of the clearest manifestations of one's beliefs about mathematics. Henry does not think students are doing mathematics when they "look at graphs on a calculator." In accordance with his beliefs, he found no good reasons to use the graphing calculators to teach lessons on the quadratic function, consequently, he did not plan to use them in the future.

The teachers' philosophies of mathematics education become more evident when you examine the kinds of questions and discussions they engaged their students in as they worked through their lessons. The way in which Troy used questions with his investigations seems to be a manifestation of his conception of mathematics as well-structured. The students would do a couple of examples and then he would ask them for the generalization. Students were not expected to explain their generalizations, but, instead just relate them so the class members could note them and move on. Troy used the students to "transmit information" in much the same way that a teacher might "pass on the information." Troy's questioning in this lesson was directed at getting the structure of the equation $y=a(x-p)^2+q$ on the board, with the consequence of each parameter noted, so the students could move on to interpreting graphs algebraically. Pheonix's views of mathematics as a process and her belief that we teach mathematics for this process was manifested in her questioning. She asked students for both generalizations and explanations; not only did she want them to be able to graph by hand, but she wanted them to engage in the mathematical thinking required to justify their generalizations. Jack's pragmatic views were evidenced by his questioning. Students provided the observations and he provided any necessary explanations. He also used questions as a way of foreshadowing upcoming concepts -- things they would have to know in later chapters. Wally's belief that the students must know the formulas or they will have problems with the mathematics was reflected by the way in which he summarized the transformation activity. The end result of this graphing calculator assignment (for some students) was a large summary chart on the board with the general form of the equation and the role of each of the parameters noted for the students to "learn" so they could do the questions from the text and on upcoming tests. Henry's view of mathematics as "doing" and his belief that mathematical processes lead to absolute truths were manifested in his style of questioning. He used questions to carefully guide the students to the mathematics they were to discover. Recall how he would hint at the answers or towards the places the students should be looking for the answers. For all of the teachers, their style of questioning was perhaps the most revealing of their philosophies of mathematics and mathematics education.

Finally, the teachers philosophies of mathematics and mathematics education were also reflected in other types of interactions with the students. Troy's views of mathematics as precise and well-structured were clearly expressed in his requests of the students to use precise language and a specific form for their answers. He explicitly stated his view of mathematics as satisfying when he explained why he preferred algebraic solutions to graphical solutions. Pheonix's beliefs about her role as teacher (to motivate and encourage students) were clear in her actions. She praised students, she demonstrated enthusiasm for their work and efforts, and she encouraged students to try things on their own. Mark's views of mathematics as a human experience were manifested in his stories about mathematicians, in asking students to consider what might have motivated mathematicians in developing certain concepts, and in using examples from the world outside of their classroom.

The beliefs, attitudes, and conceptions of mathematics and mathematics education which contributed to the teachers philosophies, were manifested in the sum of their experiences in class with students. Their choices for lessons, their questioning, and their interactions with students all reflected aspects of their philosophies of mathematics and mathematics education. These six teachers all taught the same curriculum but all interpreted that curriculum in light of their philosophy of mathematics and mathematics education. Although they may not have done so with out my questioning, they all could react to philosophical question and, at least informally, enunciate their philosophies of mathematics and mathematics education, or at least specific tenets of such philosophies.

In what ways does the availability and use of graphing calculators in mathematics instruction affect the teacher's beliefs about mathematics and mathematics education?

I believe, none of the teachers' philosophies were radically changed (or turned in a different direction) as a result of the introduction of graphing calculators. Instead, their existing beliefs, attitudes, and conceptions were reinforced and in some cases strengthened. Troy continued to find satisfaction in algebraic solutions, Henry continued to believe you do not do mathematics by looking at graphs on a calculator, and Jack continued to focus on the ends of the program rather on the means, although, his view of lecturing as the most efficient means for teaching now admits to at least one exception. Pheonix, Wally, and Mark all seem to value the calculator for the different type of experience they can now give to their students; but, these teachers valued the opportunity to vary their instructional strategies prior to the introduction of the graphing calculator. The availability of the graphing calculator has simply provided these teachers with the opportunity to further live their philosophies of mathematics and mathematics education.

The opportunity to observe teachers teach and discuss their philosophies with them has enabled me to develop a better understanding of how teacher's philosophies of mathematics are manifested in their instruction of mathematics. Paul Ernest (1991) provided me with a framework from which I could examine the teachers' philosophies, and, the teachers provided me with the experiences to examine Ernest's "Ideologies of Mathematics Education." In the next section I will critique Ernest's classification system.

A Critique of Paul Ernest's Five Ideologies of Mathematics Education

> [T]he experiential world, be it that of everyday life or of the laboratory, constitutes the testing ground for our ideas. (VonGlasersfeld, 1984, p.22)

Since, I made extensive use of Paul Ernest's (1991) "Ideologies of Mathematics Education" in this study of teacher's philosophies of mathematics and mathematics education, I believe it is appropriate to critique this classification system. The critique is based on the findings from my study of teachers in an instructional context and Maturana's and Varela's (1987) notions of organization and structure. In this critique I will argue that the information from this study shows the classification system Ernest developed is incomplete and therefore does not help us better understand *all* teachers' philosophies of mathematics and mathematics education. I will try to show how this classification system can be improved for the practical purposes to which I put it.

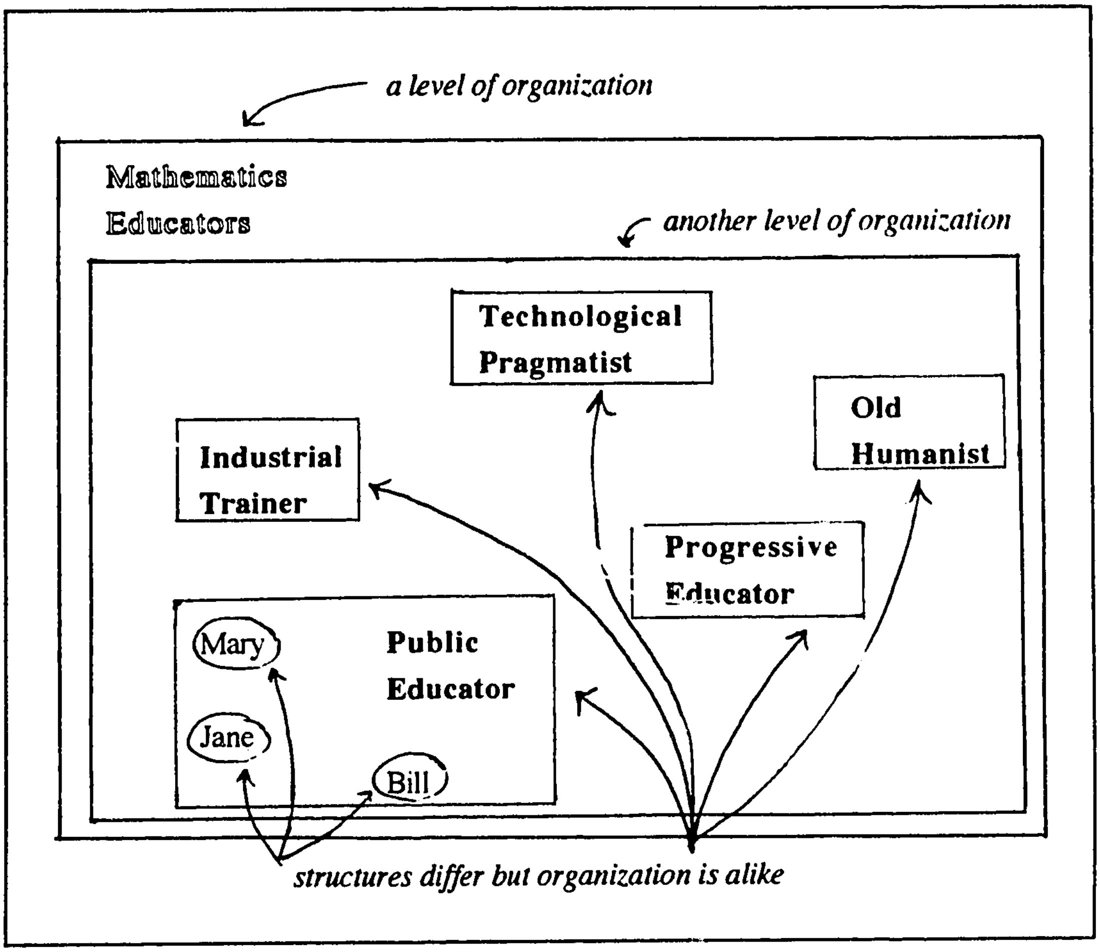

Figure 11

Viewing Ernest's Classification System as an Unity with Levels of Organization

Maturana and Varela (1987) explain organization as "those relations that must exist among the components of a system for it to be a member of a specific class" (p.47) and structure as "the components and relations that actually constitute a particular unity and make its organization real" (p. 47). Ernest has distinguished classes of mathematics educators (unities[3]) based on their organization consisting of political ideologies, views of mathematics, morals, society, and ability, and their theories of

[3]Maturana and Varela (1987) call social groups third-order unities. There is a discussion of these in The Tree of Knowledge.

learning, teaching, resources, assessment, and social diversity. These classes differ from each other in their structure but they are alike in their organization. (Refer back to Table 1, p. 10 for an overview.) Figure 11 demonstrates the levels of organization in this classification system.

In this study I re-presented the structures of six mathematics teachers. Part of the analysis was to compare their observed structures with the classes Ernest developed. I did not examine all of the organizational features outlined in Ernest's overview. However, I did consider those that I was able to distinguish while observing the teachers in their classes and in talking with them about mathematics. Recall that Troy's philosophies of mathematics and mathematics education were very similar to the Old Humanists and Jack's to the Technological Pragmatists. However the philosophies of the other four teachers could not be said to fit with any one of the classes Ernest suggested. For example, Ernest defines ethnomathematics as permeated with social and political views which he associates with his class of Public Educators. Yet Mark's ethnomathematical view is different; mathematics is a personally constructed human activity without political and social overtones. Ernest might argue that Mark did not express a mature philosophy, but Mark's actions and expressions call into question Ernest's reserving non-absolutist views of mathematics for only the Public Educator class. Pheonix's beliefs, attitudes, and conceptions partly fit into two categories: that of the Old Humanists and that of the Progressive Educators. Ernest's system does not take into account teachers with seemingly conflicting positions on the nature of athematical knowledge and the nature of the learner, such as Pheonix expressed. While for Pheonix the nature of the learner is paramount, manifestations of this aspect of her philosophy do not fit well with Ernest's concerns. Henry's structure was re-presented and then re-presented again in trying to determine which of Ernest's ideologies related to his views. Henry's views of mathematics could be interpreted to fit with those of the Technological Pragmatists, the Old Humanists, or the Progressive Educators. Ernest has not taken into account those with a unique view of mathematics, like Henry's view of mathematics as algebra. It is possible that Ernest might question Henry's view of mathematics as incomplete and philosophically inadequate; however, even scarcely coherent philosophies of mathematics underlie mathematics pedagogy. If Ernest has managed to clearly distinguish the classes (unities) these discrepancies should not be found in the experiential world.

Into which classes would Ernest fit Pheonix , Mark, and Henry? Or, What structures would classes that these teachers fit into have? Ernest's "Five Ideologies of Mathematics Education" is his bringing-forth of his world. These people's structures were brought forth in my study by the distinctions I made. What about the distinctions another person would have made?[4] Ernest has tried to examine social groups without looking at the individuals that comprise those social groups. He made an initial attempt by using Margaret Thatcher as a case study but he did not carry this any further. My study tests his theory with "real" people and it identified some inconsistencies. As VonGlasersfeld (1984) suggested it is the "experiential world" where we test our theories.

Ernest discussed some of the limitations of his overview of educational ideologies. They included: simplification, more groups exist than are listed, groups are in a state of flux, within a group there are overlapping ideologies, and that his account is tentative. The former two are true limitations; the latter two are not limitations. Rather they specify part of the power in the type of mechanism (social constructivism) he has suggested for examining philosophies of mathematics and mathematics education. Ernest points out later in his book that the power of his social constructivist theory is that it is fallible and subject to modification. I must take issue however with the second limitation he suggests. His classification system brought-forth the Industrial Trainers, the Technological Pragmatists, and the others. He acknowledges that he may not have described all of the groups. But this disclaimer is not sufficient since in distinguishing and naming he has brought those five groups forward into existence while the other groups do not exist until he brings them into existence. There is a problem with this other [sic] unnamed category. When you observe and talk to mathematics teachers you find the "others" (and they do exist) and they too should be named. They need to be brought-forth into our existence so they too will be subjects of the research and theory-making community. This is a problem with naming; when you name you both include and exclude. It is useful to classify since it provides a way of knowing and being; but, it usually leaves out as much as it encompasses. Maturana and Varela make (as I interpret it) a similar point:

> We make descriptions of the descriptions that we make. Indeed we are observers and exist in a semantic domain created by our operating in language where ontogenic adaptation is conserved. (p. 211)

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[4]I suppose just realizing that many take issue with the classifications systems of others, and feel a need to critique them, helps us understand Maturana and Varela's explanation of autopoietic unities bringing-forth their world through reciprocal perturbations. Individuals make different distinctions when they classify objects. The creator of a classification system makes certain distinctions and the researcher using the system makes others, resulting in a critique of that system (reciprocal perturbation), which, in turn can be used by the creator to make new distinctions, and so on.

In conclusion I wish to make the following points about my use and analyses of Ernest's classification system of "Ideologies of Mathematics Education." As is more evident with some teachers than others, the organizational features of Ernest's classification system from which he has distinguished classes -- that is, the elements he looks at within the classes -- have been useful. Furthermore, in this thesis I have attempted to illustrate his more general point that studying teachers' espoused and manifested philosophies is important in understanding the implementation of curricular changes and instructional strategies. Yet my observations and interpretations call into question the adequacy of Ernest's (and perhaps any) classification system for observing teachers' espoused and manifested philosophies of mathematics and mathematics education as they are worked out in teaching practice.

Implications for Teachers and Using Graphing Calculators

Views of Mathematics

The case studies in this thesis demonstrated that the way in which six teachers used graphing calculators was consistent with and reflected their philosophies of mathematics and mathematics education. For example, Troy views mathematics as a well-structured body of pure knowledge from which the fundamental concepts, skills, and techniques are passed on to students. Accordingly, he used the graphing calculator to teach his students an algebraic form of the quadratic function which is useful for discussing graphical properties of quadratic functions. Pheonix, in contrast, views mathematics as a logical and sequential process; therefore, she used the graphing calculator to provide the students with opportunities to classify, observe, and generalize. Fundamentally, Troy, Wally, and Jack all view mathematics as as a large body of knowledge, hence, they shared a static view of mathematics. Pheonix and Mark view mathematics more as a process, hence, they share a dynamic view of mathematics. Henry's view of mathematics was quite unique in that he always spoke of mathematics in terms of the actions that comprise it, yet, the actions were not necessarily generalizing , conjecturing, proving, etcetera; rather, they were actions mostly based on doing algebraic manipulations. This view of mathematics seems to be more resonant with Troy's, Wally's and Jack's static view of mathematics than with Pheonix's and Mark's process view.

All of the teachers, but especially those with the static view of mathematics, seemed to believe high school mathematics is fundamentally about manipulative algebra and providing algebraic solutions for questions. This included not only questions about algebraic expressions but also questions about graphs: finding max/min points, identifying vertices, and finding intercepts. Numerical and graphical analyses were not viewed as appropriate ways of doing high school mathematics. This view of high school mathematics likely co-emerged from the teachers histories and the demands of past curriculums, which filtered the study of functions through algebra. Thus, the curriculums of the past occasioned the teachers to develop and apply these very limited philosophies of mathematics. Although the curriculum and the instructional opportunities have changed it is the teachers' structures which determine their actions in this new space. Perhaps more experience in this space will bring teachers to change their structure: their philosophies and their actions based on their philosophies.

Capturing and Creating Teachable Moments[5]

Even though the teachers share this "algebraic filter," the two views of mathematics, as static and dynamic, seem to provide a natural separation between the ways in which the graphing calculators were used. The teachers with a static view of mathematics used the calculators as a means to an end. Graphing calculators were used to find answers (in this case to find the roles of the parameters). Whereas, the teachers with a more dynamic or process view of mathematics used the calculators to provide an opportunity to reason mathematically (observe patterns and regularities, and generalize). Their views of mathematics were consistent with the way in which they used the calculator. If you view mathematics as static and doing mathematics as finding "answers," then you would not want to encourage too much use of the calculator because the calculator can provide many of the kinds of answers (vertex, roots, and max/min points) you expect students to compute through algebraic means. If, on the other hand, you want your students to note patterns and regularities, make conjectures, and generalize, then you might encourage your students to use the graphing calculators because they provide an efficient means of producing images from which your students could engage in this type of mathematical thinking.

While observing the teachers using graphing calculators in their instruction, I witnessed "teachable moments"; that is, opportunities in lessons that had the potential for promoting a deeper understanding of the mathematics. Teachers try to plan lessons to create teachable moments; however, teachable moments are often occasioned by the students, with a comment, suggestion, or question they interject into the teacher's

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[5] I first heard the term *teachable moment* from Brent Davis while teaching a teacher education course with him.

lesson. I found that the teachers with a static view of mathematics handled these teachable moments quite differently than the teachers with a dynamic view. For example, in Troy's class, one student suggested the way to maximize an area of given perimeter is to make it circular; in Wally's class a student told him that he had a different way of solving a max/min problem; and in Jack's class, a student found that the cubic function behaved the same way the quadratic function did under the transformations. In each of these cases, the students' conjectures were noted by the teachers but the teachable moment slipped away. Troy indicated to the student that he was looking for a region with four sides. Wally suggested to the student that he better use the way just demonstrated to be sure to get the right answer. Jack acknowledged the student was correct, but indicated he had another function in mind. The students' conjectures were noted, but, then dismissed without any further discussion. Pheonix handled such a teachable moment quite differently. When her students provided a response to the question about the shape of the parabola as "*a*" got closer and closer to zero, she let the students freely discuss their answers and justifications for a few minutes. When she moved on she left the discussion open by not providing a definitive answer. One can only speculate why Troy, Wally, and Jack did not follow up on the students' ideas. In the first and third case the conjectures went beyond the topic being taught; perhaps the teachers did not want to go beyond the prescribed content. In the second case, it is possible that the teacher was uneasy with his own understanding of the problem and did not want to get caught up in something he could not handle, or, possibly he felt he had taught the one correct way of solving the problem and the student was expected to use that method. Behind the specific reasons lie the teachers' views of mathematics. The three teachers with a static view of mathematics shut out these diversions. Whereas Pheonix, who has a more dynamic view of mathematics, seemed to welcome and make space for such diversions.

In high school mathematics graphing calculators can be used to facilitate teachable moments. That is, teachers can plan lessons which provide students with opportunities to reason mathematically and explore some "big" ideas in mathematics in an effort to help students develop deeper understandings of mathematical concepts and processes. These are opportunities that encourage students to observe, note regularities, make generalizations, and conjecture, and then try to prove or refute those conjectures. The guided-discovery activities used by the teachers in this study are suitable for doing these things; however the nature of the questions that teachers ask in conjunction with this type of activity must facilitate and promote this kind of thinking. Most of the teachers had their students generalize (which is one aspect of this kind of thinking) but most did not ask students the kinds of questions that can promote further mathematical thinking. Examples of such questions are: "Explain why "q" makes the parabola move up and down along its axis of symmetry. What might happen if we apply this parameter to another function? Under what conditions will a parabola cross through only three quadrants?"

Graphing calculators are appropriate tools for analysing functions graphically; but, again, we must ask suitable questions. Mark used, as a example with his students, a quadratic function which expresses the relationship between the height of a dive and the length of time from when the diver left the diving board. Students can use the graphing calculator to graph the function and then use the trace function to help them consider questions like: "At what time is the diver at his maximum height? How do you know he is at his maximum height? When does he hit the water? How do you know? What is the divers height above the water before he leaves the diving board? What do you know about the divers speed throughout the dive? Explain." Whenever students do graphical analyses one can ask questions about the vertex and the intercepts; not necessarily what their coordinates are but what the coordinates represent: "What does the vertex tell us in this case? What does the root tell us in this case?"

Another way one can use the graphing calculator to create opportunities for teachable moments is to use the "problem with the viewing window" that Jack described. Recall that he "fixed" his examples so they would fit on the standard axes of the graphing calculators that the students were using. Students could graph parabolas which do not fit in the viewing window and then discuss why they cannot see the graph. Or, one could ask students questions like: "Do all parabolas have the domain of the real numbers? Can you find a parabola which would 'use up a lot of y compared to x" values?' Can you tell me why I have put that phrase in the question in quotation marks? Can you find a parabola that has as its range the set of all real numbers? Why, or Why not? Can you find functions which have the set of real numbers for their range but not for their domain?"

These are a few different ways in which teachers can implement graphing calculators in their instruction of mathematics. However, as my study demonstrated, changes in instruction are filtered through the teacher's structure which includes their philosophies of mathematics and mathematics education. Thus, such changes in action must co-emerge with changes in beliefs, attitudes, and conceptions about mathematics and mathematics education.

Teachable Moments in Teacher Education

Now that teachers have access to graphing calculators, how are we going to get them to ask the types of questions which encourage mathematical thinking or to have students examine functions graphically instead of algebraically? Since teachers are self-referencing and self-modifying beings we can provide opportunities which will act as perturbations to promote self-modification. This study indicates we need to find appropriate perturbations for the teachers to modify their philosophies of mathematics and mathematics education.

Teachers' views of mathemat[illegible]matics education have tremendous implications for curriculum developm[illegible] the evidence gathered in this study, I would suggest providing the opportuni[illegible]ular change is not sufficient for changing the curriculum students are tau[illegible]As we saw, teachers (autopoietic unities) with differing philosophical positions (structures) were faced with (perturbed by) the introduction of graphing calculators, and, each of the teachers implemented graphing calculators into their instruction differently and in accordance with their philosophies of mathematics and mathematics education. (This is not to say that there were no other factors in the teachers' structures that also determined the way in which they handled the perturbation; however this study was focused on those aspects just mentioned). Troy, Pheonix, Mark, Wally, Jack, and Henry, all have unique structures that were given a perturbation in their environment and they acted accordingly. Hence six groups of students were presented with six different learning opportunities.

The NCTM (1991) in their Professional Standards for Teaching Mathematics, suggests the following standards for the professional development of teachers. Developmental opportunities should include:

- Experiencing good mathematics teaching
- Knowing mathematics and school mathematics
- Knowing students as learners of mathematics
- Knowing mathematics pedagogy
- Developing as a teacher of mathematics

I will use these to discuss how we might provide opportunities for teachable moments in mathematics teacher education.

Experiencing good mathematics teaching[6]

> We human beings in particular are modified by every experience, even though at times the changes are not wholly visible. (Maturana and Varela, 1987, p. 168).

Research has shown that teachers are influenced by the teaching they experienced (Ball, 1988). Further, based on the view that I have presented of a teacher's plastic structure, it is evident that "experiencing" is necessary for the education of teachers. However, Wilcox, Schram, Lappan, and Lanier (1991) found new teachers, who in their pre-service education experienced non-traditional mathematics instruction and subsequently tried to use these non-traditional methods for teaching mathematics, were sometimes overwhelmed by restraints such as time and peer pressure and retreated to the status quo. So, although, experiencing good mathematics teaching is necessary for teachers, it is not always sufficient.

Once teachers are in the classroom they are often not exposed to alternative strategies for teaching mathematics. As much as possible, in-services should be designed as workshops and seminars -- not lectures. We will not convince many teachers to examine their belief that "lecturing is the most efficient way of getting the content across" if we lecture to them. Teachers need to experience the type of teaching we want them to do.

Knowing mathematics and school mathematics

In light of the discussion on knowing (in the review of the literature), where I have indicated that the nature of knowing is in a state of upheaval and there is a redefining of what it means to know, as well as a growing fuzziness between knowledge and knowing, this statement could be problematic. However, I assume there is a constructivist position on knowing underlying the NCTM's standards.

Taking Maturana and Varela's claim that "all doing is knowing and all knowing is doing," (1987, p. 27) for my position, I believe teachers must continue to *do* mathematics. Do not get this confused with teaching mathematics. This is most often not the same thing as doing mathematics.[7] Teachers usually teach concepts that they either have quite a deep understanding of, or, a shallow understanding which they believe (or act as believing) to be complete and therefore do not view themselves as

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[6]These objectives are elaborated on in the Professional Standards for Teaching Mathematics.

[7]This is determined by the teacher. In this study the teachers, all except Wally, were not learning the mathematics of the quadratic function. They were already quite familiar with the concepts they were teaching.

learners of this mathematics. I would suggest that teachers need to spend time as students of mathematics. They need to engage in the kinds of experiences they set up for their students, but with mathematics that is new to them, so they find themselves in the position of a learner.

With the "computer age" and the development of experimental mathematics, special consideration should be given to teaching teachers "new" mathematics. This would provide teachers with an opportunity to learn about mathematics that is changing in both method and substance. For example, teachers could be introduced to fractal geometry and investigate the mathematics of fractals. This could include ideas about area, perimeter, infinity, sequences, series, logarithms, and limits and could be approached both experimentally and analytically. Another possibility is: teachers could investigate the mathematics of dynamical systems and chaos. The study of quadratic functions as dynamical systems is a new way to study an old concept. These types of experiences have the potential to leave observable changes in teacher's beliefs and attitudes about mathematics -- beliefs such as, there is only one way to do mathematics, and there is not any new mathematics appropriate for high school students. The study of intercepts and roots might provide interesting challenges for teachers if asked to study them graphically instead of algebraically. Another example of an activity for teachers is to have them find graphs of parabolas that will "fit" into the standard viewing window and then ask them to discuss the ways in which they accomplished the task. There are a couple of efficient strategies for doing this and presumably the participants would come up with an assortment of strategies An important part of the activity is the discussion of the ways in which people solved this trivial problem. There is a change in focus that accompanies these kinds of activities; the emphasis becomes finding ways to solve problems as opposed to finding answers to problems. Strategies for solving problems are often far more diverse than the actual solution(s) to the problem.

These type of activities for teachers can provide opportunities for them to not only learn mathematics but at the same time examine the beliefs about the nature of mathematics and mathematics education.

Knowing students as learners of mathematics

I believe an effective way of teaching about learners is to put teachers into the position of being mathematics learners, as I indicated in response to the previous standard. In my study, I found the teachers who viewed themselves a learners of mathematics were very thoughtful about how they could help students understand mathematics. For example, Mark commented, that because he has to think about problems when he does them, he is more aware of where the students might encounter difficulties. Pheonix, too, expressed sensitivity about the students as learners. She collected homework to find where they were having trouble, and she commented that she tries to keep her instruction at a level the kids can understand. Some of the other teachers did not discuss how they could make mathematics more understandable for the students, rather, they discussed what the students had to do to understand. Jack indicated the students have to let him know where they are having trouble and Troy indicated that the students are to decide how much work they have to do in order to understand. This is a subtle difference but one that seems to be drawn along the lines between those who still view themselves as learners of mathematics and those who do not. This is an area that needs to be researched and considered in more detail.

Knowing mathematical pedagogy.

Providing teachers with instructional materials, or new technology, may be a necessary condition to improve teaching but it is not sufficient. These tools, and instructional materials are perturbations and, as I have demonstrated, the perturbations are enacted according to the teacher's structure. As I have said many times now it is not sufficient to look to the perturbations; we must also look to the teacher's structure as we attempt to improve mathematics instruction. Well designed teacher in-services and workshops can provide opportunities for reflecting on and modifying our beliefs, attitudes, and conceptions about mathematics and mathematics education.

The classification activity Jack used for introducing functions could be used in an in-service for teachers with the graphing calculator. This activity requires the participants graph and sketch forty (lots) functions and relations. Then they are to cut out these sketches and group them according to their similarities and discuss not only the similarities in the graphs and their corresponding algebraic expressions but also why they grouped them in the way that they did. In a workshop setting, I would also have the teachers discuss why the students should go to the trouble of cutting and pasting their sketches and justifying their groupings. I would also ask the teachers to discuss where the mathematics in that activity is found. This activity together with the reflection component provides an opportunity for the teachers to reflect on their beliefs about what it means to do mathematics and their ideas of "where the mathematics is found."

Teacher education should be designed to challenge our beliefs, attitudes, and conceptions about issues of pedagogy. When we introduce a new tool, like the graphing calculator, we must provide teachers with an opportunity to not only learn ways in which the new tool can best be utilized, but, also provide opportunities for them to examine their beliefs about its utilization in mathematics education.

Developing as a teacher

Under this standard it is suggested that teachers must examine and revise their assumptions about the nature of mathematics, how it should be taught, and how students should learn mathematics. The point of my study is: the assumptions about mathematics, how it should be taught, and how it should be learned, are part of the teacher's self-referencing and self-modifying structure. The potential exists for examination and reflection about one's beliefs, but, the introduction of a curriculum change (for example) will not necessarily act as a strong perturbation on the teacher's beliefs, attitudes, and conceptions, as was evident in this study.

On the other hand, interview questions did generate some very thoughtful responses, and, for some of the teachers the questions seemed to challenge at least a few of their beliefs. It did this for Troy, when, after some thought he admitted some mathematics is done inductively, and for Pheonix, when, she started to question her conception that there is only one way to solve a quadratic equation. Mathematics teachers do not get many opportunities to discuss their beliefs about the nature of the subject they teach. I found the teachers eager to discuss such beliefs.[8] Troy said it best. "When you get into the routine with your classes you never sit and stop and think in terms of what you own beliefs are. They are in there; they just don't get to the surface very often." This leads me to suggest that one of the ways teachers can be encouraged to examine their beliefs about mathematics and mathematics education is to involve teachers in critical discussions of such things. Periodicals such as, The Arithmetic Teacher and The Mathematics Teacher and books such as, The Mathematical Experience, by Davis and Hersh (1980) and Pi in the Sky: Counting, Thinking, and Being, by John Barrow (1992) are sources of interesting views which could be used as starters for critical discussions in professional development activities.

These approaches to teacher education are based on the assumption that our structures change with every perturbation even if that change is not noticeable to an observer and with appropriate and frequent perturbations a teacher's beliefs about the nature of mathematics and the nature of mathematics education could be altered significantly.

[8]A number of times other teachers interrupted our interviews so they could expressed their views.

Conclusion

In this study I have demonstrated that teachers live their philosophies of mathematics and mathematics education. These philosophies are manifested in their instruction and interactions with their students and these philosophies are a part of the teacher's structure which determines each and every act of the teacher. I have recommended ways in which the graphing calculator can be used in high school mathematics and I have suggested that teacher education include activities that provide opportunities which not only suggest new ways of teaching but also encourage teachers to reflect on their beliefs about mathematics and mathematics education. We may specify changes in curriculum for teachers, but only the teachers can bring-forth changes in their instruction of mathematics.

When the breeze blows through the wind chime the world is filled with the chime's music.

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Appendix A

NATURE OF MATHEMATICS AND NATURE OF LEARNING MATHEMATICS[1]

After reading each question please circle one of the responses provided. The responses are strongly agree (SA), agree (A), neither agree nor disagree (N), disagree (D), and strongly disagree (SD).

Part I

1.	Diploma exams should be developed by mathematicians.	SA	A	N	D	SD
2.	Diploma exams should be developed by mathematics teachers	SA	A	N	D	SD
3.	The major value of teaching problem solving skills is to enable students to tackle unusual examination questions.	SA	A	N	D	SD
4.	It is a consequence of the nature of mathematics itself, that pupils will more often wonder about the purpose of a topic in mathematics than in other subjects.	SA	A	N	D	SD
5.	School mathematics can be seen to provide the basic skills and techniques of mathematics, to be extended into applicable mathematics in work or university situations.	SA	A	N	D	SD
6.	Creative work in mathematics only takes place at the frontiers of mathematical knowledge.	SA	A	N	D	SD
7.	The process of doing mathematics in school can be seen to be a model of all mathematical experiences: industry, research, daily life.	SA	A	N	D	SD
8.	Discovery methods are useful for older pupils.	SA	A	N	D	SD
9.	Children learn math by absorbing mathematical concepts and skills.	SA	A	N	D	SD
10.	Discovery learning of mathematics is relevant for all stages of school mathematics.	SA	A	N	D	SD
11.	Children learn mathematics by the personal building of mathematical knowledge.	SA	A	N	D	SD
12.	Discovery methods of learning mathematics are relevant for the earliest concepts in mathematics only, eg. addition, area.	SA	A	N	D	SD
13.	Expecting pupils to be creative in mathematics is unreasonable.	SA	A	N	D	SD
14.	Time should be set aside for the students to work creatively in mathematics.	SA	A	N	D	SD
15.	If a student understands a mathematical concept then that student will be able to accurately perform algorithms associated with the concept.	SA	A	N	D	SD
16.	Being able to accurately perform algorithms is necessary for understanding mathematical concepts.	SA	A	N	D	SD

[1]This interview quide (in part) was adapted from a questionnaire developed and provided to me by S. Lerman.

17. Students should use algorithms taught by the teacher and not ones they develop on their own. **SA A N D SD**

18. Repetition is necessary for learning most mathematical concepts. **SA A N D SD**

19. Plenty of textbook exercises are necessary for students to learn mathematics **SA A N D SD**

20. Students should be taught to think deductively (from the general to the particular) in mathematics. **SA A N D SD**

21. Students should be taught to think inductively (from the particular to the general) in mathematics. **SA A N D SD**

22. Concrete manipulatives are useful for teaching mathematical concepts at the high school level. **SA A N D SD**

23. Diagrams and pictorial representations are useful for teaching mathematical concepts at the high school level. **SA A N D SD**

Part II

If an intelligent pupil were to ask the purpose of a mathematics or a topic in mathematics, I would answer:

1. Mathematics can be seen to be, like a chess game, a game with rules that have to be learnt. **SA A N D SD**

2. The applications of mathematics follow once mathematical knowledge is acquired. **SA A N D SD**

3. Mathematics is the best subject to train you to have a logical mind. **SA A N D SD**

4. This topic is in the curriculum so you can be sure it is important. **SA A N D SD**

5. School mathematics is a building block for further mathematics. **SA A N D SD**

6. Mathematics helps you find patterns and relationships in the world. **SA A N D SD**

7. Mathematics provides a means for describing the world. **SA A N D SD**

8. Mathematics helps you develop abstract thinking. **SA A N D SD**

9. Jogging is exercise for the body; mathematics is exercise for the brain. **SA A N D SD**

Part III

This next section deal with the nature of mathematics.

1. Once a mathematical structure has been developed, and a theorem formulated, its proof is a technical detail. **SA A N D SD**

2. Mathematical truths are not susceptible to revolutionary change in the way that scientific truths are, e.g. relativity. **SA A N D SD**

3.	Mathematical knowledge is close to scientific knowledge in the sense that conclusions are tested for their truth.	SA	A	N	D	SD
4.	Mathematical truths have an inevitability about them. A world with different mathematical truths is inconceivable.	SA	A	N	D	SD
5.	The general proof is a highly creative part of mathematics, since it can lead to new structures, reformulated hypotheses, etc.	SA	A	N	D	SD
6.	Mathematical knowledge is hypothetical.	SA	A	N	D	SD
7.	Mathematical knowledge is potentially subject to refutation and adaptation.	SA	A	N	D	SD
8.	Experimental mathematics is as valid as analytic mathematics.	SA	A	N	D	SD
9.	Mathematics is essentially hierarchical and cumulative.	SA	A	N	D	SD
10.	Mathematics is strongly influenced by culture.	SA	A	N	D	SD
11.	Mathematics is very intuitive.	SA	A	N	D	SD
12.	Mathematical knowledge is absolute.	SA	A	N	D	SD
13.	Mathematical objects are constructed by the mind.	SA	A	N	D	SD