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Making Connections: Network Theory, Embodied Mathematics, and Mathematical Understanding

by

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Abstract

In this dissertation, I propose that network theory offers a useful frame for informing mathematics education. Mathematical understanding, like the discipline of formal mathematics within which it is subsumed, possesses attributes characteristic of complex systems. As the techniques of network theorists are often used to explore such forms, a network model provides a novel and productive way to interpret individual comprehension of mathematics.

A network structure for mathematical understanding can be found in cognitive mechanisms presented in the theory of embodied mathematics described by Lakoff and Núñez. Specifically, conceptual domains are taken as the nodes of a network and conceptual metaphors as the connections among them. Examination of this 'metaphoric network of mathematics' reveals the scale-free topology common to complex systems.

Patterns of connectivity in a network determine its dynamic behavior. Scale-free systems like mathematical understanding are inherently vulnerable, for cascading failures, where misunderstanding one concept can lead to the failure of many other ideas, may occur. Adding more connections to the metaphoric network decreases the likelihood of such a collapse in comprehension.

I suggest that an individual's mathematical understanding may be made more robust by ensuring each concept is developed using metaphoric links that supply patterns of thought from a variety of domains. Ways of making this a focus of classroom instruction are put forth, as are implications for curriculum and professional development. A need for more knowledge of metaphoric connections in mathematics is highlighted.

To exemplify how such research might be carried out, and with the intent of substantiating ideas presented in this dissertation, I explore a small part of the proposed metaphoric network around the concept of EXPONENTIATION. Using collaborative discussion, individual interviews and literature, a search for representations that provide varied ways of making sense of EXPONENTIATION is carried out. Examination of the physical and mathematical roots of these conceptualizations leads to the identification of domains that can be linked to EXPONENTIATION.

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Chapter 1

Introduction

> Some of the deepest truths of our world may turn out to be truths about organization, rather than about what kinds of things make up the world and how those things behave as individuals. (Buchanan, 2002, p. 19)

Throughout the history of modern schooling, the pedagogy of mathematics has been organized around prevailing beliefs regarding the nature of mathematical knowledge. Euclid's development of geometry as a rigid and logical structure helped to give shape to linear curricula formulated in terms of assumed-to-be-basic concepts that were elaborated incrementally and hierarchically. The work of the Formalists in the early twentieth century was taken up in North America within the New Math Curriculum by a focus on axioms, laws and proofs. In contrast, the view of mathematics as fallible, tentative, and formed through communication among individuals underlies constructivist approaches where learners actively develop their own understandings of mathematical ideas.

As a student, I experienced mathematics instruction that was influenced by these and other philosophies. In many classes, teachers defined perfect mathematical forms and described their properties. Logical and structured proofs in Euclidean geometry were presented as the ideal way to establish rigor and truth. I studied New Math, where sets and properties of number systems were used to develop concepts. And in my first calculus course, I was forced to acknowledge that to comprehend mathematics I had to make sense of concepts on my own.

As a mathematics major at university, I learned far more than course content; mathematics was more than just a collection of rules and techniques. There was excitement in creating mathematical ideas that were new – at least to me. I saw and appreciated the beauty and elegance of the discipline for the first time. And the unexpected connections I found among diverse branches of mathematics not only fascinated me, but also showed that more than one approach could be used when solving most problems. These revelations were not part of any formal instruction at university or, for that matter, at public school; I became aware of them quite incidentally in my own work.

As a teacher, I tried to share my appreciation of these often-obscured qualities of mathematics with learners. Classroom walls were plastered with quotations from mathematicians about their conceptions of and love for the discipline. Stepping back, I provided chances for young people in my classes to come up with their own solutions, to do mathematics. We had fun together in units on Statistics, where students chose questions they wanted to investigate, and designed and carried out inquiries in groups. And I tried to recreate in them the sense of wonder and power that I had experienced when I recognized that no field of mathematics was isolated, that all were linked together in some way.

This last point seemed particularly important, for all too often learners' understandings of mathematics seemed segmented, as if information about different concepts was stored away in distinct mental 'files'. Students had difficulty relating topics learned one month to those studied in the next. Deliberately pointing out relationships among various ideas was not successful in breaking down the walls of these separated compartments. Students might nod their heads and take down notes when connections were mentioned, but there still was little carry over from one unit to another. Learners perceived mathematics as a collection of discrete topics, each with its own particular set of techniques. I was at a loss; I had no idea how this view came to be established or why it persisted so stubbornly. Nor did I know how to counter its influence in an effective manner.

When I returned to university to undertake a doctoral program, I was introduced to complex systems – self-organizing, self-maintaining wholes whose behaviors emerge from the interactions of their components. Similarities in the dynamics of colonies of ants, social behavior in communities, links in the World Wide Web and many other phenomena became evident. And I learned how the mathematics of network theory provided ways of developing insights into both how complex forms are structured and how they behave.

Some researchers have suggested and presented evidence that a variety of disciplines – language (Barabási, 2003; Cilliers, 1998; Motter, de Moura, Lai & Dasgupta, 2002), music (Zanette, 2006), and science (Cilliers, 1998) – display properties that are consistent with those seen in complex unities. Reflecting on these works, I began to wonder whether mathematics too could be profitably viewed as a self-constituting complex system. Looking at mathematics in this way offered a novel perspective on the discipline; the connections among different branches and concepts played a critical role in determining the dynamic behavior of such a form. Could this new conception enhance knowledge of students' comprehension of mathematics? Would explanations emerge that shed light on why a learner's mathematical knowledge is all too often segmented and why this is problematic? What effects on pedagogy might a complexity and network-based interpretation of mathematics have? I found the possibilities intriguing.

By applying the techniques of network theory to mathematics – viewed as a complex system with an underlying web-like structure – I hoped to understand some of the difficulties learners experience in their studies of mathematics and perhaps find ways to assist them to overcome these problems. Exploring what such a network structure might consist of, how it would act, and what this could mean for teaching and learning mathematics occupied the better part of the next four years of my work at the University of Alberta.

1.1 Commencing with Network Theory

> Networks are in the news and will likely remain there. To understand our world, we need to start thinking in these terms. (Buchanan, 2002, p. 22)

In this dissertation, I develop the suggestion that the field of network theory, a new development in mathematics arising in the last decade or so, may present a novel way of understanding the structure of mathematical understanding and, in consequence, of informing pedagogy. Briefly, network theory examines the various ways in which a group of objects can be connected in some fashion. Originally developed as a branch of applied mathematics (specifically, an extension of graph theory), techniques developed in network theory have been employed to analyze diverse complex systems in nature, society, and business. The significance of the field lies in the finding that the conclusions of network theory arise from the underlying structure and topology of a complex system rather than from the particular objects of which it is comprised.

In the last sixty years or so, scientists in many fields have examined the dynamic, integrated, and unpredictable world of complexity (e.g., Johnson, 2001). Most research done during this period has concentrated on the study of complex systems, where the independent interactions of multiple entities lead to collective behavior. The abilities and potentials exhibited by these systems are different from and potentially far more sophisticated than those possessed by the agents themselves. Until recently, the study of complex systems, from collections of neurons in the brain to species in an ecosystem, has attempted to explain how such coherent and purposive wholes can emerge out of the apparently autonomous actions of individuals.

In the late 1990s, researchers began to develop the field of network theory as a means to explore the structural dynamics of the networks underlying complex systems. Specifically, their foci were the interactions among the system's components or agents, rather than the entities themselves or their particular characteristics. Viewing the elements of a system as *nodes* in a network and their interactions as *links* among nodes, the system of entities and their connections can be portrayed by a graph. Using this technique, Watts and Strogatz (1998) and Barabási and Albert (1999) identified patterns not previously seen in complex phenomena and formulated simple, yet comprehensive, laws that describe network structure and evolution. Scientists in many disciplines, ranging from physics to sociology, have found these principles invaluable in explaining how and why complex systems behave as they do.

1.2 Interrelating Complexity Science, Network Theory, and Mathematical Understanding

Although the use of network theory in analyzing complex systems is rapidly expanding, it has not previously been applied to the field of mathematics education. Thus, it was necessary to look for possible network structures that would reflect the complexity I saw in mathematical understanding. In the literature of mathematics education, I found several graphic structures that have been developed to assist in the learning and teaching of mathematics.

One of these – the ‘conceptual field’ – is defined as “a set of situations, the mastery of which requires mastery of several concepts of different natures” (Vergnaud, 1988, p. 141). For example, the multiplicative conceptual field comprises activities related to proportion, multiplication, and division. These situations have been analyzed according to a variety of task variables and arranged in hierarchical structures (e.g., Behr & Harel, 1990; Nesher, 1988; Vergnaud, 1988; see Figure 1). This tree diagram is seen “to reflect the nature of the [multiplicative conceptual field], both mathematically and cognitively” (Behr & Harel, 1990, para. 2). Conceptual fields have proved useful in understanding the “filiations and jumps” in students’ competencies of multiplication and related topics (Vergnaud, 1988, p. 151) and in designing classroom instruction (Behr & Harel, 1990).

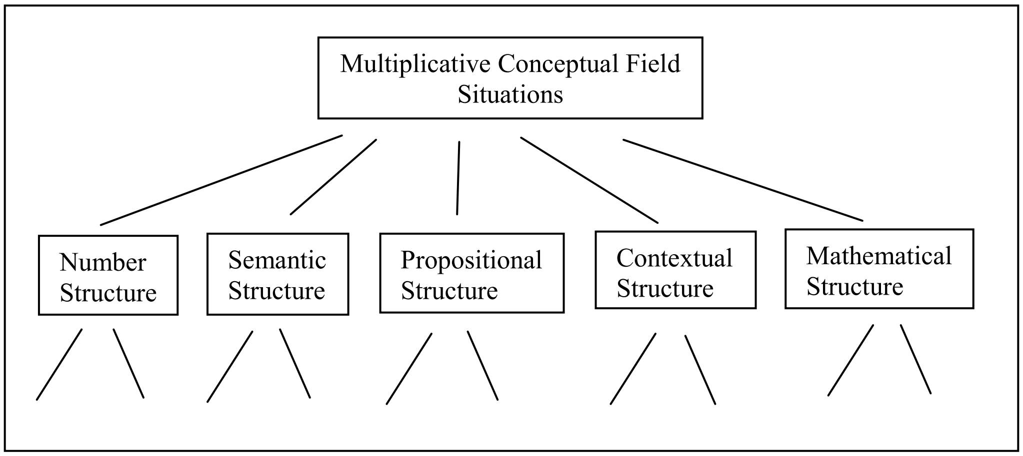

Figure 1: Types of task variables that constitute the nodes of a graph for the Multiplicative Conceptual Field.

Another type of graphic representation is the 'concept map', used to clarify the meaning of a chosen domain by identifying subordinate concepts and their relationships (Novak & Gowin, 1984). Skemp (1989) suggests the need to analyze a mathematical idea before introducing it to students –" [to] 'take it to pieces' … to see what are the contributory concepts" (p. 67). Each of these instances is in turn examined for subordinate topics, and so on, until the many ideas upon which a concept is based are identified. The results of this analysis are represented in a hierarchical graph – similar to a tree diagram, but with the possibility of additional links joining some subsidiary nodes – that displays topics with different levels of specificity that contribute to the meaning of a more general, superordinate concept (Skemp, 1987; see Figure 2). Concept maps can assist in designing instruction (e.g., Schmittau & Vagliardo, 2006; Skemp, 1989) and have been used to trace changes in students' and teachers' understandings of mathematical concepts (Hough, O'Rode, Terman & Weissglass, 2007; McGowen & Tall, 1999).

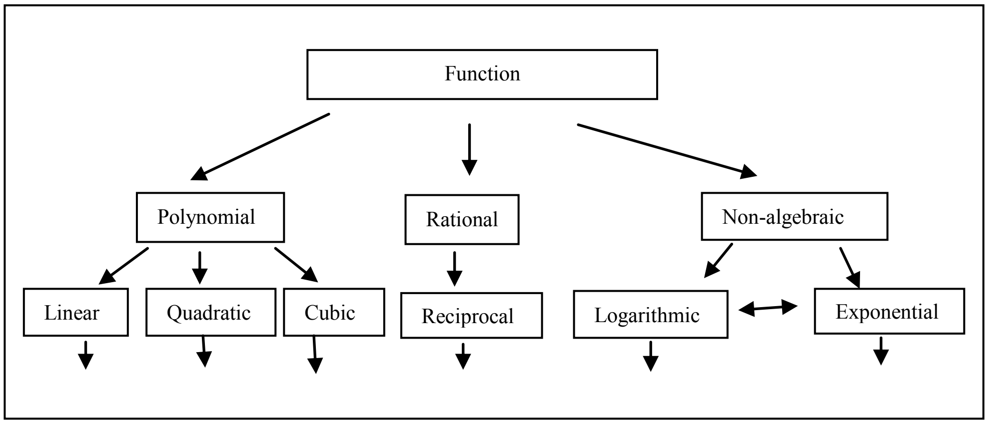

Figure 2: The beginnings of a simple concept map for 'Function'.

While each of these structures is undoubtedly a graph, I found that I could not accept either as a network, as the term is used in complexity science. Conceptual fields and concept maps just did not 'feel right'; somehow they did not reflect the essence of areas that I had come to understand as complex systems. To clarify my thoughts, I found my reflections on the term 'structure' and its different connotations useful.

'Structure' can refer to an architectural edifice that is designed and constructed by a person or group of persons to serve a particular purpose. One can understand the formation and its behavior by examining its elements separately; the system is understood simply as the sum of its parts. A blueprint or hierarchical schematic shows what role these components play in providing a sound foundation for the whole.

When I looked at conceptual fields and concept maps, I saw architectural structures. Both types of representation are designed by an external observer – a researcher, teacher, or student. A mathematical concept is analyzed (taken apart) and portrayed as the sum of its many components, whether these are related situations or subsidiary topics, and hierarchical tree diagrams can serve as useful blueprints.

But complex systems tend to call up images of biological structures (Davis, Sumara & Luce-Kapler, 2000). These are not hierarchical, for "in nature there … are no hierarchies. There are only networks nesting in other networks" (Capra, 1996, p. 35). A complex form does possess components, but these are not simpler than the whole; each part is itself a complex, dynamic entity. An ecosystem is comprised of many organisms, each life form is constituted by many organs and each organ contains many cells. Such systems are not planned or constructed, but rather emerge and evolve from the interactions of their elements.

Thus, the network I envisaged underlying the complex system of mathematical understanding would be quite different from the hierarchical, pyramidal structure of conceptual fields and concept maps. Forms would be nested within forms, and concepts would not be broken up, for "the important characteristics of a complex system are destroyed when it is taken apart" (Cilliers, 2000, p. 41). Instead, connections that affect the evolution of the system as a whole would be emphasized. I found these characteristics in the web of cognitive mechanisms described in the *theory of embodied mathematics*, as set forth by

Lakoff and Núñez (2000); this was the type of network structure that I had been looking for.

1.3 Linking Embodied Mathematics

In this chapter, I have suggested that complexity science and network theory may present a novel and productive way to interpret mathematical understanding. Applying the techniques developed for identifying and investigating network behavior in other disciplines to mathematics offers educators a different view of the nature of the discipline. Insights that emerge may prove useful for developing understandings and informing actions in mathematics education.

In particular, I propose that a possible network model for mathematical understanding may be found in the *conceptual domains* and *conceptual metaphors* presented in the theory of embodied mathematics (Lakoff & Núñez, 2000). In the discussion that follows, I will argue that using such cognitive mechanisms as components of a network for mathematics offers an appropriate and fruitful means for exploration into the understanding of mathematics.

In developing this argument, I acknowledge that network theory and the theory of embodied mathematics offer only one possible construal of what mathematical understanding is and how it is structured. Both offer rich sources of images and insights, with which I attempt the beginnings of a scientific explanation for mathematical comprehension and learning. In other words, I work from the assumption that the discussion should be treated in terms of a useful –

not an objectively real – interpretation of the structure of mathematical understanding.

1.4 Developing New Connections

Three interconnected themes permeate this dissertation: complexity science, the theory of embodied mathematics, and network theory. To frame the work that follows, I begin, in Chapter 2, by developing the suggestion that mathematics and mathematical understanding might be understood as a complex unity. That is, informed by the transdisciplinary realm of complexity science, I argue that these forms manifest properties that are typical of those studied by complexivists. I open with this point because, as Cilliers (1998) argues, “a complex system is constituted by a large number of … units forming nodes in a *network* [italics added] with a high degree of non-linear connection” (p. 91).

I have suggested that a suitable and productive model of mathematical understanding can be found in the cognitive mechanisms described in the *theory of embodied mathematics* (Lakoff and Núñez, 2000). In Chapter 3, I describe particular aspects of this theory that are critical for this argument.

Chapter 4 explores how the conceptual domains and conceptual metaphors described in this theory can be seen to constitute a network, which I refer to throughout the dissertation as the *metaphoric network of mathematics*. The topology and dynamic behavior of this structure are examined in Chapter 5. In particular, the effect these network characteristics have for the learning and teaching of mathematics are explored.

Most of the discussion to this point is quite abstract and, in an effort to illustrate the patterns described and to provide some substantiation for the conclusions made regarding mathematical pedagogy, Chapter 6 discusses my exploration of a very small portion of the proposed metaphoric network of mathematics, one centered on the concept of EXPONENTIATION.

Chapter 7 looks back on different aspects of the dissertation research. I reflect on the model set up for mathematical understanding, the implications it had for the learning of mathematics, and both the method and results of my investigation of the concept of EXPONENTIATION. Possible future directions for research are also discussed.

Chapter 2

Mathematics is a Complex System

In the second half of the 20th century, there was a confluence of interest among a group of scientists drawn from many fields who began to realize that the phenomena they studied, while tremendously varied, had some deep commonalities. It was noted, for example, that anthills, brains, and cities seemed to obey analogous dynamics and to have structures that were oddly reminiscent of one another (cf., Johnson, 2001). This realization of common interest prompted the discourse field of complexity.

Complex systems cannot be investigated in the same ways that trajectories, forces, or orbits are. Nor can they be analyzed using techniques suitable for studies of more difficult problems like the inheritance of genetic traits or the laws of thermodynamics. The well-understood rules and analytic methods that proved so successful in research into such fields are based on the long-held philosophy that the universe and all things in it are mechanical assemblages of components. But complex phenomena, like the flocking of birds or the behaviors of economies, are anything but machine-like.

2.1 The Mechanical Universe

During the Enlightenment, principles were set forth which laid the foundations for science as it has largely been conducted into the twenty-first century. Galileo (1564 – 1642) was perhaps the first to state clearly that the universe operates according to predicable mathematical laws (Hooker, 1999).

Shortly after, Descartes (1596 – 1650) set forth a rigorous approach for establishing scientific truth. This Cartesian method included two tenets that were to have a profound influence not only on how research was conducted, but on views of the world and knowledge: to learn about an area of difficulty, one divides the topic into as many small parts as possible, and then develops understanding by examining the simplest components first, gradually assembling them into more and more complicated groups, until one is able to comprehend the whole (Latham & Smith, 1925).

Descartes's ideas, specifically the complementary processes of analysis (breaking up) and synthesis (combining together), played a major factor in transforming science from speculative descriptions and conjectures to a more certain enterprise, grounded in the physical world and mathematics (Scientific Revolution, 2004). When Isaac Newton proposed the laws of universal gravitation, confidence in this conception of knowledge as machine-like grew. Subsequent work by other scientists confirmed "a mechanical world-view that regarded the Universe as something that unfolded according to mathematical laws with all the precision and inevitability of a well-made clock" (Bolton, Durrant, Lambourne, Manners, & Norton, 2000, para. 2).

The perception that the universe and all things in it were as rational and predictable as the mathematical formulae used to describe them became the predominant model of European scientific thought (Hooker, 1999). Everything was seen as the sum of its parts; comprehension of the whole required little more than a sound understanding of each of its components. Systems could be designed to fulfill specific purposes and each would operate according to its particular design (Davis, Sumara & Luce-Kapler, 2000). Phenomena reflecting the characteristics of this mechanical world-view have come to be known as *complicated* (Waldrop, 1992).

2.2 Complicated vs. Complex

Several distinct types of systems can described as complicated phenomena. 'Simple' problems, like the swing of a pendulum or the motion of a billiard ball, involve a limited number of components that do behave according to mathematical laws (Weaver, 1948). Other fields involve a very large number of variables that behave in an erratic or unknowable manner, but can be analyzed to reveal typical conditions. For instance, the average number of phone calls in an hour can be forecast, as can the number of insurance claims due to car accidents in a month. Weaver describes such problems as 'disorganized complexity'. Using a variety of mathematical techniques, both kinds of situations can be made sense of by closely examining their components and by determining how these parts are related to each other. The aim in studies of these complicated areas is to understand the causes of particular effects and to identify the fundamental principles underlying their behavior (Davis, Sumara & Luce-Kapler, 2000).

Although this mechanistic approach had been applied to problems in politics, history, and economics (Hooker, 1999), by the late 18th and early 19th centuries, it became evident that such a mindset might not be appropriate for studying all phenomena. For example, the works of Adam Smith (1723-1790), Charles Darwin (1809-1882), and Friedrich Engels (1820-1858) examined systems that could not conceived of as machine-like (Johnson, 2001). By 1900, a number of scientists in both the physical and social sciences were using evolutionary rather than mechanical models in their studies (Dewey, 1910). At the time, similarities among these works went unnoticed, but the fields they examined shared characteristics of what came to be known as *complex* systems.

Such studies– whether in biology, psychology or epidemiology – involve a "*sizable number of factors which are interrelated into an organic whole*" (Weaver, 1948, Problems of Organized Complexity section, para. 3; *italics in original*). Organization of this unity is not determined by external laws; it is shaped instead by the independent and interdependent actions of its elements. As these respond in a mutual and recursive manner to each other and to their environment, changes in the operations of the structure occur. Thus, a particular stimulus may not always produce the same result (Davis & Simmt, 2003). Complex fields are not predictable, nor can they be understood merely as the sum of their components: such systems are not machine-like, but more closely resemble living entities.

These kinds of phenomena – from collections of neurons in the brain to relationships among species in an ecosystem – share the property of *emergence*, where coherent and seemingly purposive wholes emerge out of the apparently independent actions of individual elements. These entities follow no external rules, but somehow their dynamic interactions lead to perceptible macrobehaviors for the group in its entirety (Johnson, 2001). Moreover, the abilities and potentials exhibited by such systems are different from and more sophisticated than those possessed by the agents themselves (Davis & Simmt, 2003).

2.3 Formal Mathematics is a Complex System[1]

Because of the very involved and ever-changing nature of interactions among such entities, complex systems cannot be described easily. Despite this, Cilliers (1998) suggests that general characteristics of complex forms can be identified, based on his survey of the complexity literature. In particular, he sees ten qualities as necessary to complex systems:

> (i) *Complex systems consist of a large number of elements. ...*
> (ii) *The elements in a complex system interact dynamically. …*
> (iii) *The level of interaction is fairly rich. ...*
> (iv) *Interactions are non-linear. …*
> (v) *The interactions have a fairly short range. …*
> (vi) *There are loops in the interconnections. …*
> (vii) *Complex systems are open systems. …*
> (viii) *Complex systems operate under conditions far from equilibrium. …*
> (ix) *Complex systems have histories. ...*
> (x) *Individual elements are ignorant of the behavior of the whole system in which they are embedded.* (p. 119-123)

Cilliers (1998) posits that any system possessing these properties can be analyzed in terms of a neural-network model, originally developed from comparisons to the human brain (e.g., Edelman, 1987; Rumelhart & McClelland, 1986). He demonstrates how this model can be used in analysis of postmodern society and suggests that it may also be appropriate for examining language and scientific knowledge. I attempt here to show that the discipline of mathematics

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1 *A version of Section 2.3 is drawn from an article accepted for publication. Mowat & Davis (in press). Complicity: An International Journal of Complexity and Education.*

also possesses the requisite characteristics of complex systems and that, in consequence, a network model of mathematics is meaningful.[2]

(i) *Mathematics consists of a large number of elements.*

At first sight, it might seem that this claim requires no justification or elaboration. However, it is important to be clear as to the nature of the elements, which varies according to how one defines mathematics. For example, if defined as 'what mathematicians do', it would seem that the interacting elements are human individuals. Conversely, if defined in terms of the contents of a standardized examination, the elements might be taken to be discrete technical competencies. In this chapter, I argue that mathematics comprises a large number of ideas or concepts; these are the elements of the complex unity of mathematics.

In imposing this sort of definition, however, I am compelled to acknowledge that it represents an artificial and ultimately untenable delimitation – but, nonetheless, a necessary one. Complex systems do not have fixed or tidy edges. They intersect with, subsume, and are embedded in other complex unities. Hence, any study of a specific complex form inevitably entails an imposition of some sort of artificial boundaries on the part of the observer.

(ii) *The elements in mathematics interact dynamically*.

Pickering (1995) comments that practice in mathematics is "organized around the production of associations, the making of connections and the creation

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[2] When referring to instances from the historical development of mathematics in this section and following chapters, I must acknowledge that what I see has "at least as much to with the interpreter as it does with any set of historical 'facts' " (Dubinsky, 1994, p. 158). My perceptions are shaped not only by the framework underlying this work, but also by current social and cultural conceptions of the past.

of alignments between disparate … elements" (p. 22). Of course, few would argue with the suggestion that mathematical ideas interact; the critical point here is that they interact *dynamically* rather than coming together to form some sort of rigid architecture. Each idea informs and influences others in a constant choreography of emergent meaning. Thus, as mathematics historian Eric Bell (1945) describes change in the system, some ideas die if they prove to be trivial, inadequate, or incorrect, while others survive, often with modification to ensure coherence with other concepts.

(iii) *The level of interaction among concepts in mathematics is fairly rich.*

Relationships among concepts are rich and intricate. New mathematical concepts are formed from interactions among those already existing (Hersh, 1998). These novel ideas may enrich established mathematical knowledge, and may, in turn, interact with other concepts to spur the development of new areas of mathematics (Struik, 1987). Mathematicians have developed a number of means to enable these rich interactions of ideas. As Rotman (2000) notes,

> … it would be perverse not to infer that for large stretches of time [mathematicians] are engaged in a process of communicating with themselves and one another; an inference prompted by the constant presence of standardly presented formal written texts (notes, textbooks, blackboard lectures, articles, digests, reviews, and the like) being read, written, and exchanged, and of all informal signifying activities that occur when they talk, gesticulate, expound, make guesses, draw pictures, and so on. (pp. 7–8)

Such communication ensures that each instance of mathematical knowledge influences and is influenced by many other ideas.

(iv) *Interactions in mathematics are non-linear.*

There are two aspects to the non-linearity of complex unities. The first has to do with the manner in which systems unfold through time. Within complex systems, certain perturbations can prompt unpredictable consequences. Thus, a small idea may spark large-scale changes in the system of mathematical knowledge as a whole. For example, Fermat and Pascal's discussion of a game of chance in 1654 led to the formation of the theories of probability and statistics (Bell, 1945).

The second aspect of non-linearity involves potentially asymmetrical relations among elements as they interact in the moment. For example, a concept may be associated in different ways with different ideas. Analogous to the way in which the relationship of 'red' to 'blue' is not similar to its relationship to 'blood' (Cilliers, 1998), the relationship of $\sin\theta$ to $\cos\theta$ and $\tan\theta$ is not the same as its relation to the integral or the power series.

(v) *Interactions among mathematical concepts have a fairly short range.*

Most mathematical ideas interact primarily with the other elements of their particular branch of mathematics. Thurston (1994) observes that "basic concepts used every day within one subfield are often foreign to another subfield" (p. 6). Such relatively local ideas may play an important role in providing specific instances that lead to the formulation of generalizations in an area (Bell, 1945).

However, as Cilliers (1998) notes, "despite the short range of immediate interactions, nothing precludes wide-ranging influence" (p. 121). Ideas, such as set, permeate many branches of mathematics and, in the process, provide a unifying structure. Other long-range interactions may lead to novel concepts in mathematics. For example, the connections between algebra and geometry, made in efforts to give meaning to complex numbers, led to the development of alternate algebras and foreshadowed modern vector analysis (Pickering, 1995).

(vi) *There are loops in the interconnections among mathematical ideas.*

"Feedback is an essential aspect of complex systems. Not feedback as understood simply in terms of control theory, but as intricately interlinked loops in a large network" (Cilliers, 1998, p. 121). There are many examples of these interconnected loops in the history of mathematics, such as the deep connections involved in the emergence of number systems. For example, the sexagesimal positional notation used by Babylonians may have influenced the development of the decimal system (Cajori, 1896). This more efficient numeration system, in turn, led to a decline in the use of the earlier notation for most purposes. Another instance took place when work with sets led to the development of Russell's Paradox and Gödel's Incompleteness Theorem. These results forced the re-examination of the entire field; set theory was not discarded, but was reshaped in the search for a foundation of mathematics (Hersh, 1998). Thus, loops in the interconnections among mathematical concepts can affect both the survival and the meaning of those ideas.

(vii) *Mathematics is an open system.*

Mathematics is constantly bombarded with input from its physical, cultural and intellectual environment. The conceptual content of mathematics is influenced by other fields, from astronomy and agriculture in ancient times to psychology and physics today (Struik, 1987). Moreover, the “configuration and content of mathematical knowledge is properly and intimately defined by the culture in which it develops and in which it is subsumed” (Radford, 1997, p. 32).

(viii) *Mathematics operates under conditions far from equilibrium*.

One of the challenges to commonsense belief presented by complexity thinking is the assertion that living and learning forms do not seek or operate in equilibrium (Kelly, 1994). Rather, they exist in imbalance and cannot survive in a static condition. Examination of the history of mathematics reveals that the discipline is not fixed (cf., Bell, 1945; Struik, 1987), but “evolves by rather organic … processes” (Thurston, 1994, p. 169). As with any evolving system, changes in mathematics may occur in a sequence of small steps or through major revolutions (Grabiner, 1998). Its openness to external influences and the many loops in interactions among its elements ensure that the system of mathematical concepts is not at equilibrium, but is continually changing.

(ix) *Mathematics has a history.*

Mathematics has a history and mathematical concepts carry with them vestiges of their past. Residues of once commonly used notions can be seen in notation, terminology, and procedures (Bell, 1945). Such traces persist long after the original idea has been changed beyond recognition. Thus, the meaning of a mathematical concept is dependent on both its past and its present interactions with other elements of mathematics.

(x) *Individual elements are ignorant of the behavior of the whole system in which they are embedded*.

Since the 19th century, mathematics has been splintered into many specialized fields (Struik, 1987; Thurston, 1994). Consequently, concepts interact primarily with ideas that lie within the same branch of mathematics, that is, they respond to relatively local information. While each is, in itself, intricate, it is unlikely that a concept could reflect the complexity of the subfield, much less the entire system of mathematics. Therefore, it would seem that the complexity of mathematics, as with any other complex system, is "the result of a rich interaction of … elements that only respond to the limited information each of them are presented with. … The complexity emerges as a result of the patterns of the interaction between the elements" (Cilliers, 1998, p. 5).

Given that the field of mathematics appears to manifest the defining characteristics of complex systems set forth by Cilliers (1998), it would seem reasonable to consider it as a complex unity. While I realize that this point has not been demonstrated conclusively, I proceed here under the assumption that it is appropriate to use a neural network as a model for mathematics in general, and for related systems involved in mathematical pedagogy.

2.4 Complex Systems in Mathematics Education

Like other complex phenomena, educational systems might be argued to be forms nested in and interacting with other forms (Davis & Simmt, 2006). Just as the personal, subjective mathematical understanding of an individual shapes and is shaped by the collective knowledge of the classroom, understandings developed in a class influence and are influenced by mathematics as portrayed in a curriculum, which itself affects and is affected by the system of formal mathematics (see Figure 3). Other levels of organization could be considered as well. On a smaller – but not less complex – scale, one can find the neurological structure in the brain that contributes to innate mathematical abilities; on a larger one, there exists the culture within which formal mathematics is subsumed. In complexivist terms, the entire series of nested systems can be viewed as constituting mathematics.

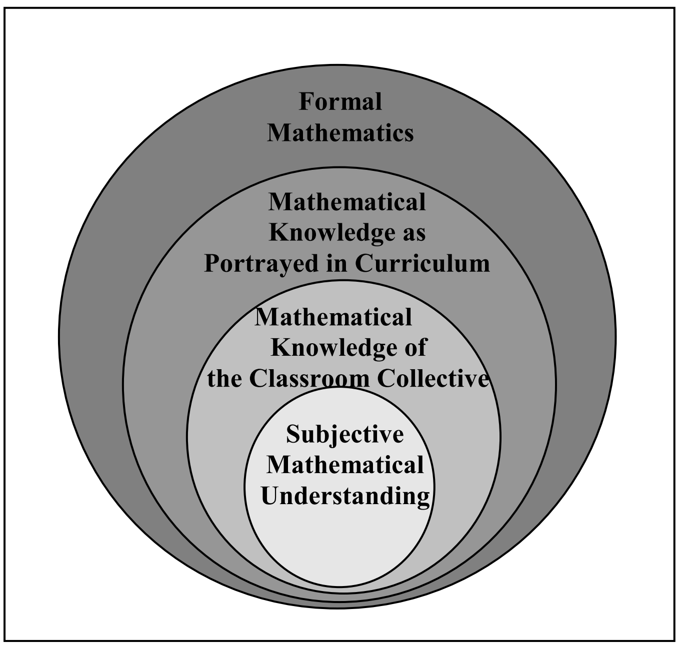

Figure 3: Some dynamic co-evolving complex phenomena of concern to the mathematics teacher (adapted with permission from Davis & Simmt, 2006, p. 296).

Clearly defined distinctions between forms are difficult to establish, as the many layers of this structure simultaneously enable and constrain each other. All stages display similar emergent qualities and dynamic behaviors (Davis & Simmt, 2006); boundaries are fuzzy and layers overlap. Thus, patterns found at one level – for instance, the characteristics of complex phenomena found in formal mathematics – can be seen in the many other forms nested within the system. Thus, any of the layers of organization can be "properly identified as complex" (Davis & Simmt, 2006, p. 4).

To illustrate, curricula can be described as emergent forms, although they change at a different rate from mathematics as a whole. Grumet (1988) uses the image of a stream – a complex system that cannot be understood just by examining the behavior of individual drops of water – to depict the always-moving, ever-changing nature of curriculum. More specifically, in their discussion of programs of study based on fractal geometry, Davis and Sumara (2000) describe characteristics of recursion, embedded forms, similarity over a range of scales, and the inseparability of part from whole – all properties of a complex form. Moreover, likenesses in the underlying dynamics of the evolution of mathematics as a discipline and in the production of curricula have been noted (Davis & Simmt, 2006).

Similarly, the classroom collective displays attributes of a complex system. In the 1980s, several researchers applied the work of Ilya Prigogine (1980) on dissipative structures – where order emerges from internal interactions – to the field of education. Groups of students can become self-organizing systems (Sawada & Caley, 1985), if teachers motivate reorganization of the classroom and allow learners the "opportunity to reflect, to try alternatives and to disagree" (Doll, 1986, p. 15). Davis and Simmt (2003) found not only selforganization, but the emergence of collective knowledge in mathematics communities that display the characteristics of internal diversity, redundancy, decentralized control, organized randomness, and neighbor interactions. In a related study, Proulx (2004) concludes that the absence or neglect of these five conditions prevents collective knowledge from developing and ensures that understanding remains at the level of individual agents.

Subjective understanding of mathematics can also be considered a complex phenomenon, for " … the same evolutionary dynamics and same complex emergent qualities seem to be at play in the cognitive processes of individual and collective" (Davis & Sumara, 2000, p. 832). For this reason and because the boundaries of nested complex phenomena are difficult to determine, Davis and Simmt (2006) "refuse a rigid distinction between collective and individual in [their] research" (p. 4).

2.5 Subjective Mathematical Understanding is a Complex System

I suggest here that the descriptions offered and the implications developed by Cilliers (1998) are relevant not just to the formal knowledge of the discipline of mathematics (the outer layer of Figure 3), but also to the subjective understanding of individuals (the inner layer of Figure 3). As elements in a series of nested systems, the two levels of understanding enable and constrain each other, and "*these forms obey similar dynamics*" (Davis & Simmt, 2006, p. 5; *italics in original*). Developing these insights, it becomes evident that:

(i) subjective understanding of mathematics consists of a large number of concepts;

(ii) relationships among these ideas change continually;

(iii) each concept interacts with a large number of other ideas;

(iv) interactions among concepts vary in response to changing circumstances;

(v) concepts primarily interact with ideas in the same branch of mathematics;

(vi) feedback loops can be found in the interactions among concepts;

(vii) subjective understanding is an open system continually influenced by physical, cognitive, and cultural factors;

(viii) subjective understanding cannot exist in a state of equilibrium, but constantly grows and changes;

(ix) subjective understanding of mathematics has a history – understanding at a particular moment depends on what had been learned before; and

(x) individual concepts are constructed on a relatively limited amount of information; the learner is not aware of all of the relationships among the mathematical ideas he or she comprehends.

Dynamic interactions and adaptations transpire in similar ways in the systems of formal mathematics and subjective understanding. Changes in structure occur at both levels, but at vastly different rates; networks of formal mathematical concepts are transformed more slowly as new conjectures are gradually developed, vetted, and adopted by communities of mathematicians, while modifications may occur quite quickly as students construct their own understanding of topics new to them. In spite of this, Davis and Simmt (2006) state that drawing a clear distinction between what Cooney and Wiegel (2003) term 'fixed' and 'constructed' mathematics is problematic.

Likenesses between formal mathematics and individual understanding have been noticed by many scholars. Haeckel's (1874) biological 'law of recapitulation' had a strong influence on social theories. In particular, Piaget and Garcia (1983) elaborate on this idea, stating that the construction of knowledge in history involves processes related (if not parallel) to those used in an individual's construction of understanding. More recently, Sfard (1995) claims, "There are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities" (p. 15).

Others feel that it would be naive to accept this view, as it ignores the sensitivity of the human mind to its culture (Radford, 1997; Radford & Puig, 2007). Mathematical findings of the past were developed within the contexts of their own times and societies. And for individual learners, it is impossible to separate natural lines of conceptual development from the cultural environment. Thus, "conditions of the actual psychological genesis of a mathematical concept are *ineluctably different* from their historical genesis" (Radford, 1997, p. 28).

Despite these observations, similarities can be identified in the evolution of some topics on the two levels, notably in the development and use of algebraic symbolism (Harper, 1987). Moreover, correspondences have been noted between obstacles from the historical development of mathematics and difficulties understanding particular concepts experienced by students (Artigue, 1992; Brousseau, 1983; Herscovics, 1989; Thomaidis and Tzanakis, 2007). Similar likenesses are found in learners' approaches to problems and solutions that appeared in mathematical history (Moreno & Waldegg, 1991; Thomaidis and Tzanakis, 2007). These findings are perhaps not surprising, given that "the contexts in which we think are anchored on an ubiquitous stratum of historically constituted cognitive activity from which we draw in a fundamental way – even if not consciously" (Radford & Puig, 2007, p. 148).

2.6 Summary

In this chapter, I have demonstrated that subjective understanding of mathematics, like the discipline of formal mathematics within which it is subsumed, shares attributes that comprise Cilliers' (1998) general description of complex systems. Thus, mathematical understanding that develops in the mind of an individual – subjective mathematics – can be explored using a network model.

I posit that a possible network structure for subjective mathematics may be found in the *conceptual domains* and *conceptual metaphors* presented in the theory of embodied mathematics, specifically the version put forth by Lakoff and Núñez (2000). Their work sets forth a proposal describing how cognitive superstructures are constructed, beginning with inborn abilities and physical experiences from everyday life. The cognitive mechanisms through which this process is accomplished this are discussed in the next chapter.

Chapter 3

The Theory of Embodied Mathematics[3]

The theory of embodied mathematics was first expounded by Lakoff and Núñez in *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being* (2000). In this work, convergent evidence from a wide variety of disciplines is used to demonstrate that mathematical understanding is shaped by certain properties that typify human reasoning. Key among these characteristics is the embodiment of the human mind, the unconscious nature of cognition, and the metaphoric nature of thought.

Lakoff and Núñez (2000) assert that mathematics exists by virtue of the embodied mind. Cognitive structures used in all modes of thinking are initially developed from physical sensations and activities. The brain receives input exclusively from other parts of the body. Therefore, distinctive characteristics of our corporeal structure – what can be perceived, how we physically function in the world, and ways in which neurological configurations process information – determine the form and content of the mind. "There is no … fully autonomous faculty of reason separate from and independent of bodily capacities such as perception and movement" (Lakoff & Johnson, 1999, p. 17).

Most reasoning is largely unconscious; cognitive scientists believe that humans are not aware of ninety-five percent of their thought processes (Lakoff & Johnson, 1999). Most every-day thinking takes place at too fast a pace and at too

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[3] *A version of this chapter has been published. Mowat 2005. delta-K. 42(2): 20-29.*

low a level for people to be aware of exactly what occurs. For example, in an informal conversation with friends, one speaks using a language that does have a formal grammar, but one does not consciously check these rules as one talks. Nor in casual speech are words deliberately chosen from one's vocabulary. Phrases flow effortlessly, conveying intended meanings. Inferences are drawn from tones of voices and expressions on faces without having to examine intentionally how these conclusions are arrived at. Choices are made with every sentence, although the speaker is unable to explain how he or she knew what to do and say. Lakoff and Núñez (2000) argue that as with most of human thinking, mathematical thought, "involves automatic, immediate, implicit rather than explicit understanding – making sense of things without having conscious access to the cognitive mechanisms by which you make sense of things" (p. 28).

Metaphor, more than a mere figure of speech, is a central part of everyday thought. Metaphoric mappings are reflected in systems of expressions. For example, many statements reflect ways in which quantity is conceptualized in terms of vertical motion (Lakoff & Johnson, 1980) – 'newspaper sales *went up* this month', 'his batting average is *high*', 'my income *fell* this year', and 'Susan is *under*age'. Such wordings reveal that when individuals think about abstract concepts, "much of the way we conceptualize them, reason about them, and visualize them comes from other domains of experience" (Lakoff & Johnson, 1999, p. 45). Lakoff and Núñez (2000) argue that mathematical thought also draws on metaphoric reasoning, as when we think of numbers as points on a line or equations as balances.

Thus, most mathematical reasoning is not unlike other types of human thinking; it is embodied, largely unconscious, and metaphoric in nature. Lakoff and Núñez (2000) use this premise to address the following questions: “What are the simplest mathematical ideas and how do we build on them and extend them to develop complex mathematical ideas” (p. 15). The answer to the first of these inquiries lies in descriptions of mathematical abilities that humans possess from birth – in what is called *innate arithmetic* (Butterworth, 1999; Lakoff & Núñez, 2000).

3.1 Innate Arithmetic

Humans are born with certain arithmetic capacities; the very notion of ‘number’ is engraved on our brains (Lakoff & Núñez, 2000). Highly specialized sets of neural circuits enable us to subitize, that is, instantly and accurately recognize the number of objects in small collections (Kaufman, Lord, Reese & Volkman, 1949). Moreover, the number of flashes of light or bursts of sound in a sequence can be quickly and accurately identified (Davis & Pérusse, 1988). Even babies can immediately discriminate between groups of one, two, three or four elements (Antell & Keating, 1983; Strauss & Curtis, 1981). While neural processes enabling this ability are not yet known, it is accepted that subitizing is more than just pattern-recognition (Lakoff & Núñez, 2000).

By the age of four or five months, infants possess limited understanding of addition and subtraction with small numbers of objects (Wynn, 1992; Wynn, 1995). Other capacities needed for simple counting and numerosity – the ability to make rough consistent estimates for larger numbers – also develop at an early age. All these facets of innate arithmetic involve a sense of quantity, which is thought to be located in the inferior parietal cortex where vision, hearing and touch are linked (Dehaene, 1997).

However important these inborn capacities are for the beginnings of mathematical thinking, they cannot account for the understandings required to do arithmetic and more advanced mathematics. When investigating how individuals can make such enormous growth in mathematical knowledge, Lakoff and Núñez (2000) point to similarities between ordinary sense making and mathematical thinking. They conclude, "a great many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas" (p. 28).

3.2 Cognitive Mechanisms[4]

Cognitive mechanisms are referential systems that assist people in understanding and employing concepts (Lakoff & Núñez, 2000). The term cognitive mechanism broadly refers to "anything that plays a causal role in guiding behavior on the basis of neurally coded information" (Barrett, 2008, p.

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[4] The word 'mechanism' often indicates a complicated system, as characterized in Chapter 2. However, many cognitive scientists, including Lakoff and Núñez, utilize the term to describe mental processes that are certainly complex in nature. In this dissertation, while I keep with this convention and refer to 'cognitive mechanisms' when discussing mental processes, I feel that it is important always to keep in mind the organic, non-mechanical character of the systems indicated.

174). Such structures provide the means by which simple ideas can be made sense of and extended to more sophisticated conceptions.

In their discussion of the theory of embodied mathematics, Lakoff and Núñez (2000) highlight the role that cognitive mechanisms play in the creation and evolution of mathematical ideas. Specifically, they discuss the importance of three types of cognitive mechanisms: the conceptual domain, the conceptual metaphor, and the conceptual blend.

3.2.1 The Conceptual Domain[5]

Although Lakoff and Núñez (2000) refer to conceptual domains throughout their work, a definition of what characterizes this cognitive mechanism is not provided. However, accounts that clarify its nature can be found in writings by other cognitive scientists: Kövecses (2002) describes a conceptual domain as a "coherent organization of experience" (p. 4); Gentner (1983) defines a domain as a system of "objects, object attributes, and relations between objects" (p. 156); and Clausner and Croft (1999) declare, "A concept is a mental unit, a domain is the background knowledge for representing concepts" (p. 3). While a clear distinction between concept and domain is implied in this last statement, the line between them is blurred. A concept may become a domain (that is, background information) for another concept. Thus, conceptual domains are often embedded in other domains, forming intricate structures.

[5] The words 'concept' and 'domain' are often used in the literature to refer to this mental phenomenon, as are the terms 'idea' and 'topic'.

Many conceptual domains are embodied, arising from basic experiences. For example, BALANCE is part of everyday life for all humans.[6] We first encounter equilibrium, or the lack thereof, as babies wobbling across the floor. Over the years, BALANCE becomes such an intrinsic part of our lives that we are hardly aware of it, but it is extremely important for our coherent perception of the world (Johnson, 1987). Since physical experiences in situations involving BALANCE make use of sight, touch, language and reasoning, corresponding regions of the brain are activated (Lakoff & Núñez, 2000). Consequently, this type of body-based experience forms a general and flexible neural structure that can be and is used repeatedly. Such a recurring configuration establishes patterns of understanding and analysis in cognitive processes. Thus, the conceptual domain of BALANCE can be utilized to make sense of more abstract situations involving chequebooks, relationships, or equations.

Other domains, like FUNCTION, are more abstract and comprise more elaborate propositional configurations. Regardless of their intricacy, conceptual domains are not just mental pictures, but act as general, flexible, and evolving patterns providing the inferential structure that makes our perceptions of the world meaningful. Of particular importance for human reasoning are *image schemas,* domains that derive from spatial relations (Clausner & Croft, 1999; Lakoff & Núñez, 2000).

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[6] Throughout this work, the convention of identifying cognitive mechanisms using small capitals is used.

3.2.1.1 The Image Schema

An image schema like CENTRE, STRAIGHTNESS or VERTICALITY is a conceptual domain that represents the spatial logic inherent in a physical situation (Lakoff & Núñez, 2000). As such, it is a fundamental mode of cognition positioned between abstract domains and concrete experiences (Johnson, 1987). These cognitive mechanisms are recognitions of recurrent patterns across different physical activities and perceptions, and are not specific to a particular sense (Johnson, 1987; Lakoff, 1987). For example, SCALE may refer to differences in position, temperature, pitch or brightness (Clausner & Croft, 1999). Moreover, image schemas appear to be universal – independent of culture and language (Lakoff & Núñez, 2000).

Image schemas "emerge as meaningful structures for us chiefly at the level of our bodily movements through space, our manipulation of objects and our perceptual interactions" (Johnson, 1987, p. 29). To illustrate, the CONTAINER image schema develops from early bodily actions involving material containers. Common statements often refer to components of the container: its boundary (he's on the *brink* of disaster), its exterior (she's *out* of her league), and its interior (he's always getting *into* trouble). Thus, normal language use illustrates how often the CONTAINER image schema is used to reason about nonspatial situations (Johnson, 1987; Lakoff & Núñez, 2000).

The CONTAINER image schema is particularly important for mathematical reasoning. The logic inherent in dealings with physical containers can be projected onto 'cognitive' containers (Lakoff & Núñez, 2000), as represented by sets in Figure 4.

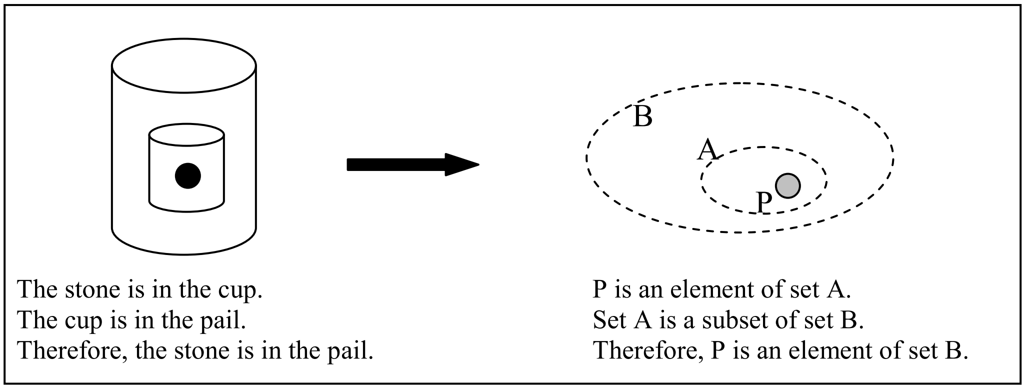

Figure 4: Reasoning from physical experience transferred to abstract mathematics through the CONTAINER image schema.

Moreover, an image schema can go beyond the original context and introduce new ideas or extensions that do not arise from physical experiences (Lakoff & Núñez, 2000). Figure 5 illustrates how one can imagine two sets overlapping even though two separate material containers cannot be made to intersect in this way.

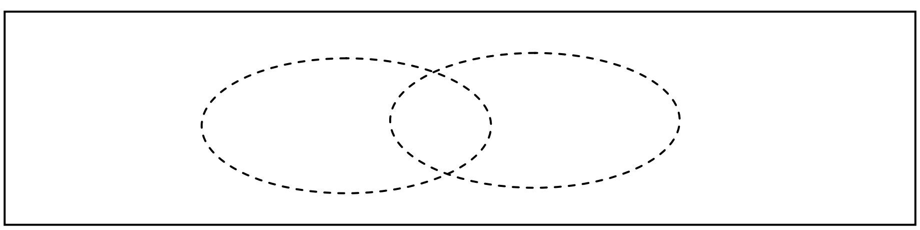

Figure 5: The concept of intersecting sets as introduced by the CONTAINER image schema.

3.2.2 The Conceptual Metaphor

Metaphors, which are often perceived as literary devices, have been shown to be important cognitive mechanisms. Conceptual metaphors project inferential structure; particular features from one domain – the *source* – are mapped onto corresponding aspects of another domain – the *target* (Lakoff & Johnson, 1999; Lakoff & Núñez, 2000). Ways of perceiving and thinking abstracted from a relatively concrete domain are used, automatically and unconsciously, in reasoning about the more abstract concept (Lakoff & Núñez, 2000). [7] An essential part of all types of human understanding, conceptual metaphors provide structure for our thought, experiences and language (Lakoff & Johnson, 1980).

Many conceptual metaphors arise initially from the everyday experiences of children. A baby, held in his mother's arms, feels both love and warmth. The two sensations are so often conflated that, for some time, the infant cannot distinguish between them. Later, differences are noticed, but the associations that have been created persist (Johnson, 1997); they provide the basis for the AFFECTION IS WARMTH[8] metaphor. Perceptions from incidents involving the source domain of warmth are mapped onto relationships in the target domain of affection. Because the inferential structure inherent in these experiences is preserved, the more abstract concept of affection can be understood in terms of the more concrete area of warmth.

While evidence of the existence of metaphors is seen in everyday language – 'they *warmed up* to each other' or 'she gave him an *icy* stare' – they are not just linguistic devices. Gesture analysis shows that conceptual metaphors

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[7] Other mental processes, such as metonymy, also allow us to go from concrete to abstract thinking. In this chapter, following the example of Lakoff and Núñez (2000), I focus on the role conceptual metaphors play in this transition.

[8] Metaphors will be identified using the convention TARGET IS SOURCE. In the example discussed here, AFFECTION IS WARMTH, the source of inferential structure is WARMTH, while the target of the metaphoric mapping is AFFECTION.

have a very real psychological existence (Edwards, 2005; Núñez, 2004). Moreover, conceptual metaphors are empirically observable processes in the mind; their use results in the simultaneous activation of two different areas of the brain (Lakoff & Núñez, 2000; Narayanan, 1997). This establishes new neural connections between the regions and generates a single multifaceted experience.

As well as transferring inferential structure from one domain to another, conceptual metaphors can also introduce new elements or *extensions* into the target domain (Lakoff & Núñez, 2000). The statement 'I had to work hard on that question' provides evidence of several metaphors, among them LEARNING IS A JOB. Subtle aspects of this metaphor, which are not initially evident (see Figure 6), are absorbed and unconsciously influence thinking.

Learning is work.
Learning is routine.
Learning is difficult.
I deserve some compensation for learning.

Learning is not play.
Learning is not fun.

Figure 6: Entailments of the LEARNING IS A JOB conceptual metaphor.

3.2.2.1 Grounding metaphors

Conceptual metaphors that establish correlations between physical experiences and abstract concepts – like AFFECTION IS WARMTH – are called *grounding metaphors*. In mathematics, the grounding metaphor is the primary tool that enables the extension of inborn numerical abilities to arithmetic within the set of natural numbers, and ultimately to more sophisticated concepts. Such metaphors ground arithmetic in everyday activities like “forming collections, putting objects together, using measuring sticks, and moving through space” (Lakoff & Núñez, 2000, p. 102).

For example, a child who puts blocks into piles is establishing neural connections between areas of the brain responsible for the physical action and innate arithmetic. This initiates the ARITHMETIC IS OBJECT COLLECTION metaphor, where numbers are identified with groups of objects (Lakoff & Núñez, 2000). Adding is derived from putting two collections together, while subtracting involves taking a small collection from a larger one. The Natural Number system, which includes quantities too large to be subitized, is formed. Properties of number-collection entities are consistent with those of inborn mathematical abilities, but are extended to include new properties. Since the sum of any two collections is another collection, the sum of any two numbers must be another number. Thus, the natural numbers possess the property of Closure, which is not a characteristic of innate arithmetic.

Similar grounding metaphors are ARITHMETIC IS OBJECT CONSTRUCTION, where items are put together to form new objects, the MEASURING STICK metaphor, in which objects are measured using physical segments, and ARITHMETIC IS MOTION ALONG A PATH, where numbers are point locations on a line (Lakoff & Núñez, 2000). Each is based on the bodily experiences of children and possesses its own set of entailments and extensions (see Table 1).

Table 1
Characteristics of the Four Grounding Metaphors

Grounding metaphor	ARITHMETIC IS OBJECT COLLECTION	ARITHMETIC IS OBJECT CONSTRUCTION	THE MEASURING STICK METAPHOR	ARITHMETIC IS MOTION ALONG A PATH
Physical basis	Manipulation	Manipulation	Manipulation	Ambulation
Numbers are …	Collections of objects	Wholes with parts	Physical segments	Points on a line
Addition is …	Adding objects	Adding parts	Putting segments end-to-end	Moving away from the origin
Subtraction is …	Taking objects away	Removing parts	Taking segments away	Moving towards the origin
Entailments and Extensions	Natural numbers	Natural numbers Fractions	Natural numbers Fractions Irrational numbers	Natural numbers Fractions Irrational numbers Integers Real numbers
Properties	Discrete	Discrete	Discrete	Continuous Zero is the origin

These four grounding metaphors are not imaginary; evidence of their existence can be found in language and in mathematical constructs of the past. ARITHMETIC IS OBJECT COLLECTION appears in such expressions as '*add* some lettuce *to* the salad' and '*take* a log *from* the woodpile'. ARITHMETIC IS OBJECT CONSTRUCTION is seen in Roman numerals like IX and VII, where parts are being added to or subtracted from a whole. The MEASURING STICK metaphor is revealed by the use of units of measurement like cubits, feet and paces. And ARITHMETIC IS MOTION ALONG A PATH is reflected in expressions like '6 is *close* to 8' and '*starting* at 20 count to 50'.

Nor are these four grounding metaphors arbitrarily chosen. Of the many grounding metaphors that have been identified, Lakoff and Núñez (2000) find that only these four have physical sources with properties and logic sufficient to form a connection with inborn numerical capacities. "Each of them forms just the right kind of [correlation] with innate arithmetic to give rise to just the right kind of metaphorical mappings so that the inferences of the source domains will map correctly onto arithmetic…" (Lakoff & Núñez, 2000, p. 102).

3.2.2.2 Linking metaphors

While grounding metaphors tie conceptual domains directly to physical experiences, abstract domains can also be connected to each other by conceptual metaphors. These *linking metaphors* may project inferential structure from domains that are directly grounded to more sophisticated concepts. To illustrate, the metaphor FUNCTIONS ARE NUMBERS transfers ways of reasoning from its source, NUMBERS, which can be made sense of using a variety of physical representations.[9] Thus, components of the inferential structure of the domain of NUMBERS, including the operations of addition, subtraction, multiplication and division, can be meaningfully applied to FUNCTIONS.

Other linking metaphors carry modes of reasoning even farther – from one abstract mathematical domain to another. Lakoff and Núñez (2000) refer to the PROPERTIES ARE FUNCTIONS metaphor. For example, a dimension-function assigns the number zero to a point, one to a line, and two to a square. Ultimately, through elaborate chains of metaphors, all conceptual domains possess grounding, however, distant, in bodily perceptions and actions (Lakoff & Johnson, 1999; Lakoff & Núñez, 2000).

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[9] The term 'representation' refers to external manifestations of a concept – examples, images, gestures, phrases – that assist learners to construct understanding of the domain.

3.2.3 The Conceptual Blend

Distinct concepts can be combined into a new domain; the *conceptual blend* projects some of the corresponding features of two or more different cognitive sources onto a new conceptual domain (Fauconnier & Turner, 1998; Lakoff & Núñez, 2000). For example, the UNIT CIRCLE is a conceptual blend of the circle in the Euclidean plane and the Cartesian plane with coordinate axes (Lakoff & Núñez, 2000; see Figure 7)[10].

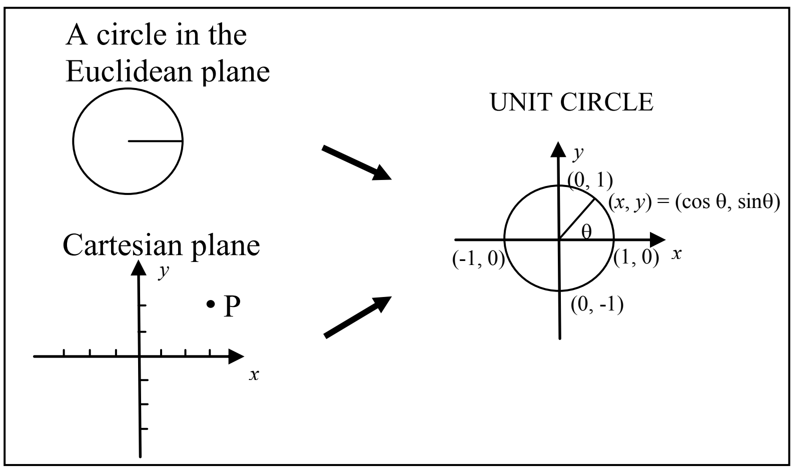

Figure 7: Features of Euclidean and Cartesian geometry combined into the UNIT CIRCLE.

[10] Figure 7 was my own attempt to illustrate the UNIT CIRCLE conceptual blend. Later, I discovered that it has a remarkable similarity to figures on pages 390-392 in *Where Mathematics Comes From* (Lakoff & Núñez, 2000). Independent development of the diagram illustrates how particular metaphors have entailments that compel certain interpretations. It is likely that any graphic representation of the UNIT CIRCLE conceptual blend would closely resemble Lakoff and Núñez's images.

In the Euclidean plane, a circle consists of all points in the plane a fixed distance, called the radius, from a fixed point, called the center. The two-dimensional Cartesian plane is defined by two axes set at right angles to each other. The horizontal or *x*-axis and the vertical or *y*-axis intersect at a point called the origin, O. By using a unit length on each axis and forming a grid, the position of any point on the Cartesian plane can be described using (*x, y*) coordinates.

The UNIT CIRCLE conceptual blend inherits characteristics from both input domains. A circle is still composed of points a set distance from its center. But now this center is at the origin, the radius of the circle has a length of one unit, and coordinates can be used to describe points on the circle. Moreover, new characteristics may arise in the conceptual blend (Fauconnier, 1997; Lakoff & Núñez, 2000); the UNIT CIRCLE has emergent properties related to trigonometry that are not part of either of the original source domains.

3.3 Summary

The theory of embodied mathematics, as set forth by Lakoff and Núñez (2000), portrays mathematics as being extended from a rather limited set of inborn skills through bodily experiences to an ever-growing web of conceptual domains. These are connected by conceptual metaphors – cognitive mechanisms used automatically and unconsciously – which project patterns of inference. Grounding metaphors make basic arithmetic possible by forming correlations between innate abilities and physical actions. Linking metaphors connect arithmetic to more abstract mathematical concepts, each metaphor carrying inferential structure systematically from one domain to another. New concepts are formed as domains fuse and create conceptual blends; new metaphors involving these blends are subsequently formed.

This ‘network of ideas’ is the basis of mathematical knowledge and knowing (Lakoff & Núñez, 2000, p. 376). Although it is not clear that Lakoff and Núñez mean to evoke the mathematical discipline of network theory with this phrase, I would contend that closer examination of the structure of cognitive mechanisms in terms of networks sheds light on the subjective understanding of mathematics. In cognition, as with other complex phenomena, “what happens and how it happens depends on the network” (Watts, 2002, p. 28).

Chapter 4

The Metaphoric Network of Mathematics[11]

In preceding chapters, I have argued that mathematical understanding is a complex system and, therefore, that representing this form and investigating its properties using a network model is appropriate. I have also described the conceptual domains and conceptual metaphors, as put forth in the theory of embodied mathematics (cf., Lakoff & Núñez, 2000), that can be used to account for the creation and development of mathematical ideas. Connecting these frames, I propose that such cognitive mechanisms comprise the basis for a network structure for subjective comprehension of mathematics.

Viewing mathematical understanding, or any complex system, as a network necessitates stripping away specific details and focusing on characteristics of the formation that lies beneath it. As with any mathematical model, the network structure of a complex form is a drastic simplification of the reality it represents; important properties of the system are inevitably missed (Cilliers, 1998). But one is enabled to draw on the techniques and understandings established by network theorists and, in doing so, it is possible to develop new insights that are ordinarily obscured by the very richness of features that make complex systems fascinating objects of study.

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11 *Parts of this chapter have been drawn from an article that is published. Mowat 2008. For the Learning of Mathematics. 28(3): 20-27. Used with permission. Other elements come from an article accepted for publication. Mowat & Davis (in press). Complicity: An International Journal of Complexity and Education.*

In any attempt to understand a network as a whole, it is necessary to carefully examine the components of the structure (Barabási, 2003). The nature of the vertices or nodes of the network and the edges or links that connect them are critical in determining characteristics of the system. In particular, the ways in which these constituent parts interact govern the structure's overall configuration or *topology*.

In this chapter, I discuss attributes of the conceptual domains and conceptual metaphors that I have suggested constitute a network for individual mathematical understanding – henceforth referred to as the *metaphoric network of mathematics*. Characteristics of the network structure that emerges from these properties are also explored. In particular, I develop the argument that subjective mathematics exhibits the *scale-free* topology that distinguishes all complex systems.

4.1. Conceptual Domains as Nodes

I propose to take, as nodes in the metaphoric network of mathematics, conceptual domains like the CONTAINER image schema, ARITHMETIC, or FUNCTION. Even the simplest of these domains possesses considerable internal structure (Johnson, 1987); each contains interconnected elements related to a variety of sensory experiences, language, and related concepts. Thus, every conceptual domain is a subnetwork of the larger network that forms the cognitive system (Kimmel, 2002; Kövacses, 2002; Lamb, 1999).

While most work in this area has been done in other fields, I am applying the principles to mathematical domains. For example, the concept CIRCLE contains many nodes representing a person's knowledge of and experiences with circles, all held together by a central coordinating conceptual node. This network is dynamic, changing with new experiences and interpretations, and differs from person to person.

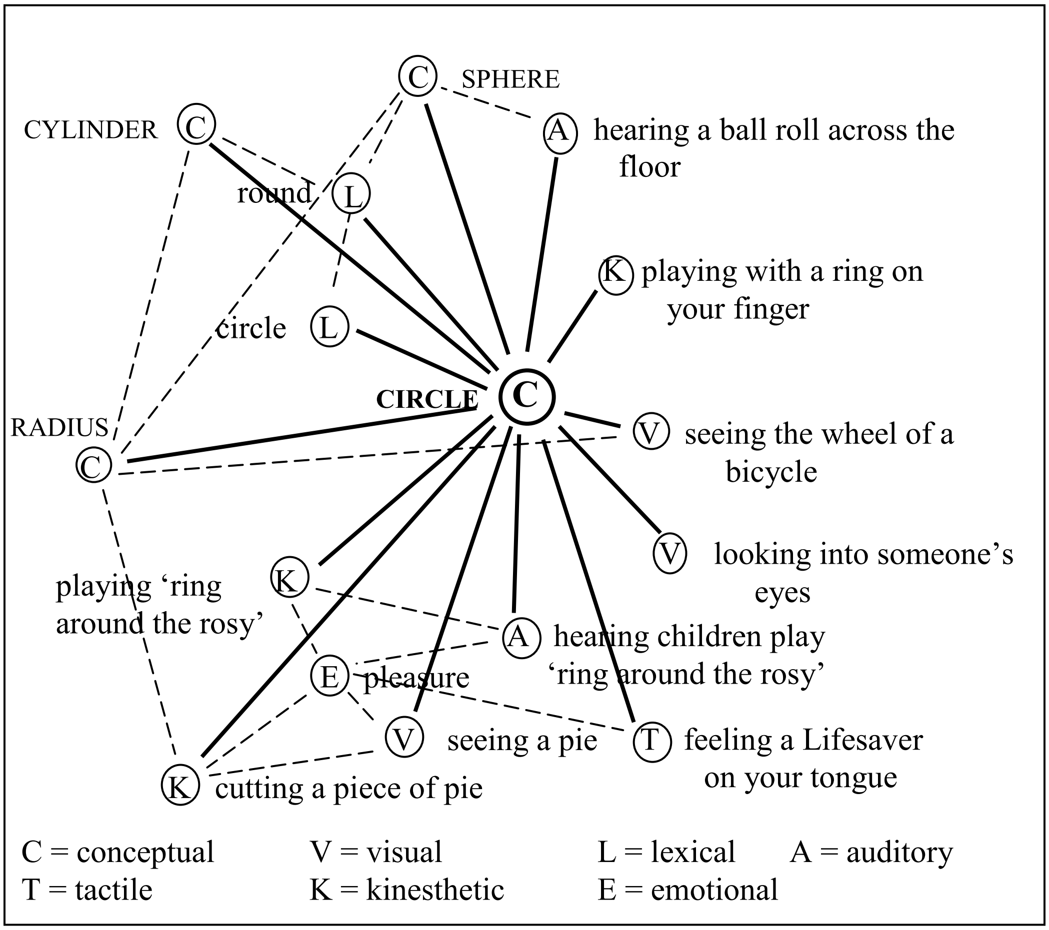

Figure 8: Part of a conceptual domain of CIRCLE following Lamb's (1999) example of CAT.

Figure 8 illustrates only part of a possible network for the concept of CIRCLE: it is not possible to provide a complete map, for any concept may contain thousands of nodes (Lamb, 1999). Moreover, no component of the subnetwork can be considered an indivisible unit. For example, the visual node labeled 'seeing a pie' in Figure 8 includes many different optical features (color, size, topping) that might be involved in a person 'seeing a pie'. Each node is itself a subnetwork of the conceptual domain (Lamb, 1999).

Patterns of spatial relations also have their own network structure. Figure 9 presents some of the nodes contained in the subnetwork that an individual might construct for the CONTAINER image schema. As mentioned earlier, this conceptual domain is particularly important for reasoning in mathematics (Lakoff & Núñez, 2000).

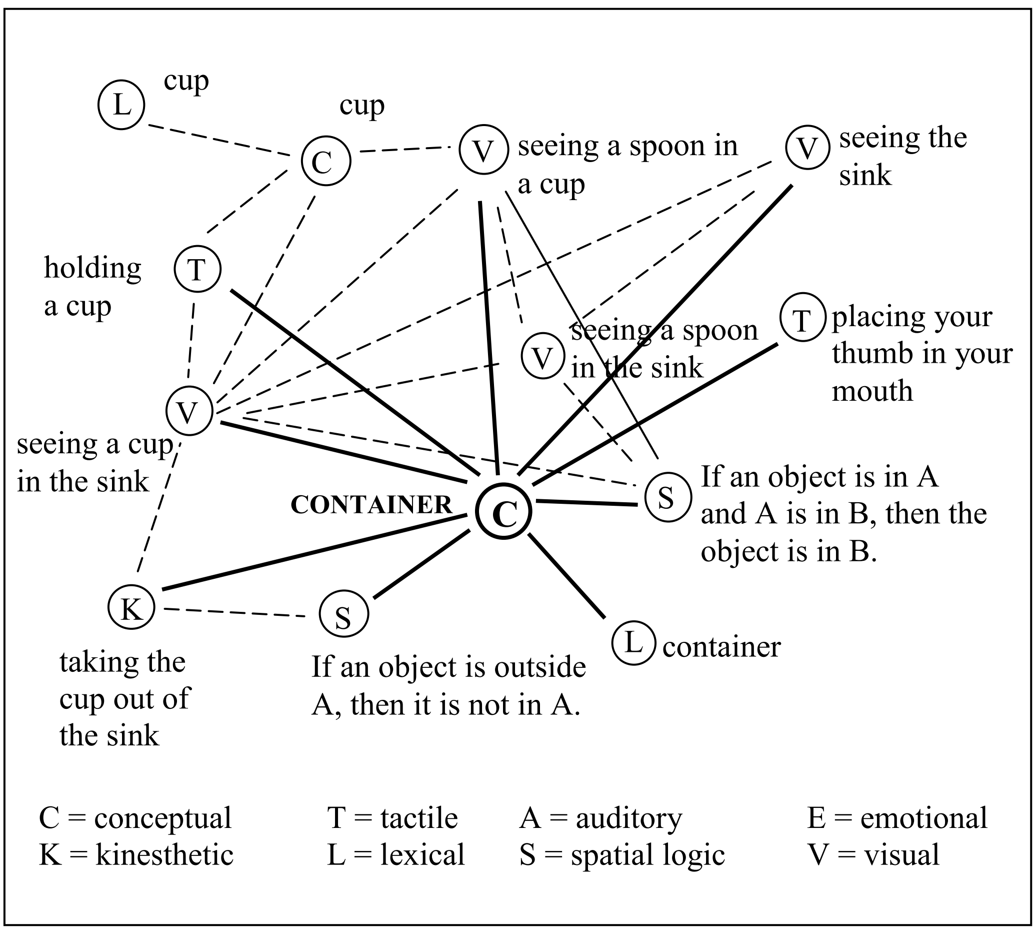

Figure 9: Part of the CONTAINER image schema.

When sensori-motor, lexical, and conceptual nodes in a subnetwork are activated, they operate together and form an individual's perception of the domain (Lakoff, 1987; Lakoff & Núñez, 2000; Lamb, 1999). Not all of the nodes in a conceptual domain must be stimulated in order to awaken this gestalt. "For most … concepts there are many properties, and if *enough* of them are present in a given situation, the [concept] is activated" (Lamb, 1999, p. 154). Thus, each concept has a threshold of activation – the number of nodes, not all of equal weight, that must function in order to make the domain active. For some concepts, the threshold may be just one. The sight of a bicycle wheel, for instance, might be enough to stimulate the central conceptual node for CIRCLE and, through it, the rest of the nodes in the subnetwork. Other domains may have higher activation thresholds. There is no simple way of determining how many properties need to be satisfied for a conceptual domain to become active (Lamb, 1999).

Moreover, traits of individual elements of the subnetwork have an effect on the activation of the domain. Each node requires a particular number of interactions with other vertices to initiate a change in state. Some "early adaptors" are easily altered; if even one nearby node is activated, its status will be modified (Watts, 2002, p. 233). Others are more stable, but if the number of active neighbors is large enough, they too will be energized. Thus, the likelihood of a conceptual domain becoming operational is affected by both the propensity of different nodes to be activated and the number of connections they have to other parts of the subnetwork.

4.2 Conceptual Metaphors as Links

Following the work of Lakoff and Núñez (2000), I propose that links in the metaphoric network of mathematics be conceptual metaphors. These cross-domain mappings establish connections between concepts by projecting inferential structure from one domain to another. Each metaphor transfers an entire cluster of nodes and their relations from the subnetwork of the source domain to that of the target (Gentner, 1983; Gentner & Toupin, 1986; Gholson, Smither, Buhrman, Duncan & Pierce, 1997).

However, not all characteristics of the source are mapped onto the target domain. It is typical of a metaphor that it has “unused parts” (Lakoff & Johnson, 1980, p. 54). If aspects of a source domain do not correspond to some portion of the target domain, they will simply not be projected onto it (Kövecses, 2002), for “metaphorical mappings preserve the cognitive topology … of the source domain *in a way consistent with the inherent structure of the target domain* [italics added]” (Lakoff, 1993, p. 11).

For example, the metaphor AN ACTIVITY IS A SUBSTANCE (Lakoff & Johnson, 1980) is revealed in sentences like ‘How *much* homework did you do last weekend?’ and ‘I did *a lot* of marking’. While the quantifiability of a material SUBSTANCE is projected onto an ACTIVITY like schoolwork, other characteristics of the source domain are not mapped onto the target; the actions of marking papers and doing assignments cannot be touched or put in a container. Sections of the SUBSTANCE subnetwork that correspond to these qualities do not correspond to innate properties of ACTIVITY and, thus, are not transferred onto the domain.

Some source domains provide a framework for a variety of targets (Kövecses, 2002). For example, consider the many metaphors (some of which are listed in Figure 10) that can be based on the conceptual domain of SET. These metaphors all project significant characteristics from the source of SET onto various targets, thus developing common inferential structures in disparate domains. These shared modes of reasoning can provide powerful tools for mathematical understanding. For instance, the metaphors, AN ORDERED PAIR IS A SET and A NUMBER IS A SET, provide a means for understanding all of mathematics using set theory (Lakoff and Núñez, 2000), an insight that, from the perspective of cognitive science, underlies the work done by the foundationalist philosophers of mathematics in the early 1900s.

AN ORDERED PAIR IS A SET.	A NUMBER IS A SET.
A FUNCTION IS A SET.	A LINE IS A SET.
A LOGICAL PROPOSITION IS A SET.	A GRAPH IS A SET.

Figure 10: Metaphors with a source domain of SET.

Just as source domains may be metaphorically linked to more than one target, some targets are connected to a variety of sources. For many conceptual domains, a single source does not possess enough structure to support all features of the concept (Bills, 2004; Lakoff & Johnson, 1999). For example, Lakoff and Núñez (2000) state that four grounding metaphors are needed to fully capture the many characteristics of ARITHMETIC (see Figure 11). These metaphors not only provide structures for distinct aspects of the target (Chiu, 2001; Kimmel, 2002; Kövecses, 2002; Sfard, 1997; Sinclair & Schiralli, 2003), but also enable alternative interpretations of the target domain in different contexts (Kimmel, 2002). Together, a collection of conceptual metaphors from distinct sources can construct a coherent understanding of their common target domain (Lakoff & Johnson, 1980).

ARITHMETIC IS OBJECT COLLECTION

ARITHMETIC IS OBJECT CONSTRUCTION

ARITHMETIC IS MEASURING WITH PHYSICAL SEGMENTS
(the MEASURING STICK metaphor)

ARITHMETIC IS MOTION ALONG A PATH

Figure 11: Grounding metaphors for the target domain ARITHMETIC.

The likelihood of a conceptual domain being activated is increased when it is structured by a collection of metaphors. As each conceptual metaphor transfers a group of related nodes, the target contains tightly knit clusters that are loosely connected to each other. Stimulating any one of the source domains from which one of these bits of inferential structure is inherited may provide the trigger that initiates activation of the entire target domain.

The network topology as portrayed to this point is very intricate. Some sources are connected to and provide modes of reasoning for many target domains; some targets are metaphorically linked to and receive structure from a variety of different sources. This depiction implies that metaphors are directional mappings from source domain to target domain – a view accepted by many researchers (e.g., Falkenheiner, Forbus & Gentner, 1989; Lakoff & Núñez, 2000).

However, there is evidence that metaphors may not always transfer inferential structure in just one direction (Barnden, Glasbey, Lee & Wallington, 2004; Black, 1993; Danziger, 1990). Metaphoric projections involve the simultaneous activation of two distinct neural structures, resulting in the establishment and reinforcement of neural connections between them. Neuroscientists have found that such links are generally reciprocal, with activation flowing both ways between the source and target regions. (Lamb, 2005; Narayanan, 1997).

Other researchers confirm the bidirectional nature of metaphor. Meisner (1995) observes that, just as an idiom like 'he is a pig' creates particular images of the person referred to, repeated use of the expression increases negative perceptions about the appetite and cleanliness of swine. The metaphor not only defines features of the target domain, but also modifies the source.

On occasion, the source and target of a metaphor exchange places when the perceived importance of the domains is altered. Kimmel (2002) relates how Newton initially explained gravitation in terms of sociability in groups of people. But as scientific reasoning became the accepted standard of truth in Europe, people started describing social interactions using the language of physical gravity. Over time, the target of a metaphor may gradually come to be viewed as a source, changing understanding of the original source domain or domains that shaped it (English, 1997; Sfard, 1997).

This transformation is most likely to occur when the target is structured by more than one source (Meisner, 1995). To illustrate, the metaphor ARITHMETIC IS OBJECT CONSTRUCTION is revealed in expressions like '2 plus 3 *makes* 5' and '6 can be *broken up* into 3 groups of 2'. Experiences constructing objects affect our thinking about ARITHMETIC and NUMBER. In turn, these imported modes of reasoning influence perceptions in other source domains for ARITHMETIC, such as MOTION ALONG A PATH; we speak of '*breaking* a journey up into three easy stages'. Something of the structure of numbers that originates in experiences constructing objects – not part of the MOTION ALONG A PATH domain – has been added. A conceptual metaphor is not static, but involves "ceaseless two-way interaction between the old and the new … [in a] process of coemergence" (Sfard, 1997, p. 355).

4.3 The Topology Of The Network

Cognitive mechanisms offer a network structure for mathematics, with conceptual domains (nodes) connected to each other in intricate ways by conceptual metaphors (links). Within this formation, each concept is a subnetwork encompassing a multitude of sensory perceptions, linguistic forms, and related domains. A closer look at such a formation reveals yet more deeply embedded structures, like the optical web that might represent the many different aspects of a particular visual perception. Figure 12 offers an image of this nested organization, which is typical of complex systems (Davis & Simmt, 2006).

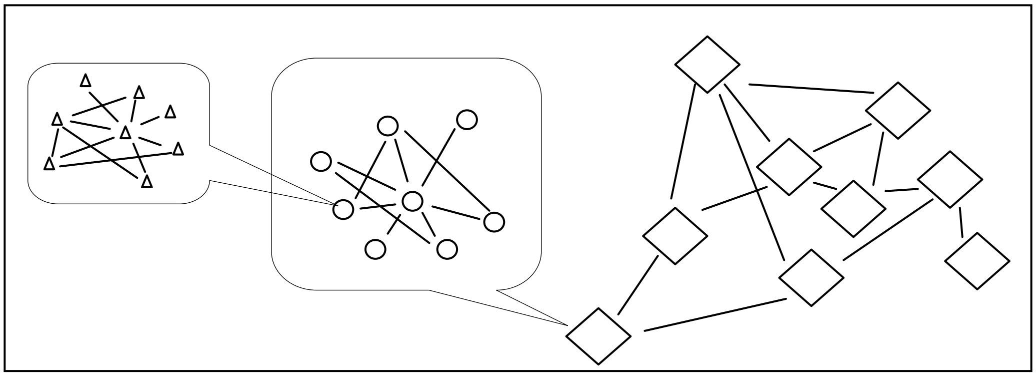

Figure 12: Nested networks: a subnetwork (perhaps of the many visual features involved in 'seeing a pie') nested within the subnetwork of the CIRCLE domain (see Figure 8), which is itself nested within the metaphoric network of mathematics.

Networks underlying complex forms share a number of attributes, including the arrangement of forms fitted inside each other discussed above. Many other characteristic patterns are found in the structures and evolution of these configurations (Barabási, 2003; Watts, 2002). In particular, network models of complex systems tend to display a *scale-free* topology (Barabási & Albert, 1999).

4.3.1 The Scale-free Topology of Networks

Connectivity in a scale-free network displays several distinctive properties. A graph representing the number of links per node in such a structure does not form the normal curve that characterizes data for many phenomena, but is more properly determined by a power law (Barabási & Albert, 1999).[12] Such a distribution lacks the characteristic peak of the bell shape that indicates an

[12] A power law distribution is formed when N(k), the number of nodes with k links, satisfies the relation, $N(k) \sim k^{-\gamma}$, where γ lies between 2 and 3 for most complex systems (Barabási, 2003).

‘average’. Instead, it is depicted by a continuously decreasing function (see Figure 13), revealing that a few nodes possessing very many connections coexist with numerous vertices that have only a small number of links. The degree of connectivity varies so much that no node can be considered representative and, therefore, there is no intrinsic *scale*, or typical number of links, in the network (Barabási, 2003).

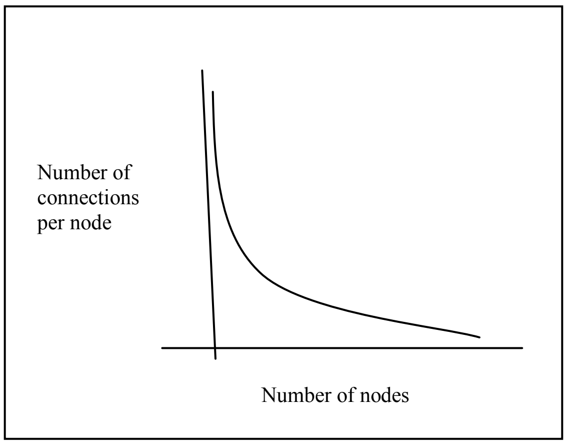

Figure 13: The power law distribution of connectivity in a scale-free network.

This pattern of connectivity arises from the organization of complex networks (Barabási, 2003; Watts, 2002). Highly connected nodes or *hubs* play a key role in the organization of the structure. In a scale-free network, clusters are formed within which every vertex is connected to a hub; these are in turn linked to more central nodes, and so on (see Figure 14).

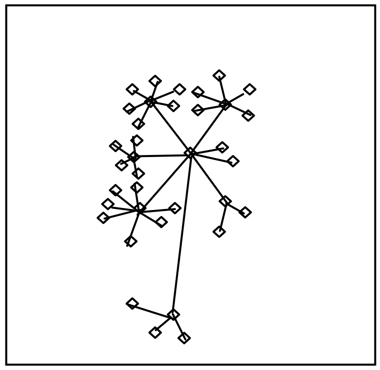

Figure 14: A simple network displaying a scale-free topology.

Much research has been done investigating complex systems and the networks that underlie them. Barabási and Albert (1999) state that two conditions – *growth* and *preferential attachment* – are necessary and sufficient conditions for a network to display the scale-free pattern of organization. I argue here that the metaphoric network of mathematics possesses these attributes and is, thus, a scale-free structure.

4.3.2 Growth: How Nodes are Formed, Modified and Reorganized.

The metaphoric network of mathematics comprises an intricate web of conceptual domains linked by conceptual metaphors. As a model for subjective mathematics, the network can only be truly understood by considering the nature of the processes through which this complex system evolves (Watts, 2002). An individual's mathematical understanding is not static, but develops as he or she continues to learn; existing domains change and novel concepts come into being. Metaphors provide the "scaffolding for [this] … growing conceptual system" (Sfard, 1997, p. 350).

The conceptual metaphor is such an intrinsic part of human reasoning and human language that one is constantly exposed to this cognitive mechanism. Many metaphors are so deeply embedded in everyday life and mathematical culture that individuals are hardly conscious of their use (Lakoff & Johnson, 1980; Lakoff & Johnson, 1999). For example, people are unaware that they are using a metaphor when they say, '97 is *close to* 100', '12 can be *broken up* into its prime factors', or 'the curve *reaches* its maximum at (1, 5)'. But these *invisible* metaphors are so much a part of thought patterns that most individuals would be hard pressed to think of other ways of expressing the ideas (Sfard, 1997).

Learners are introduced to conceptual metaphors from many sources – peers, parents, and teachers. *Exegetical* metaphors, that is, metaphors used for schooling, like AN EQUATION IS A BALANCE, play an important role in assisting students to learn mathematical concepts needed for every-day routines and to acquire the mathematical competencies necessary for becoming part of the larger mathematics community (Travers, 1996). Some people also construct *idiosyncratic* metaphors, relating newly encountered concepts to personal experiences or previously understood ideas (Pimm, 1987; Presmeg, 1992; Sinclair & Schiralli, 2003).[13] Metaphors, regardless of the source, both bring new concepts into being and modify existing domains.

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[13] There is some controversy about the difference between the conceptual metaphor and the idiosyncratic metaphor. Many researchers characterize the conceptual metaphor as a "publicly accessible tool" (Sinclair & Schiralli, 2003, p. 5) and contrast it to the "private, personal" idiosyncratic metaphor (Presmeg, 1997a, p. 277) that spontaneously evolves when an individual tries to make sense

A conceptual metaphor may possess entailments that can lead to new understandings of previously encountered conceptual domains and to the development of novel concepts. For example, the MEASURING STICK metaphor portrays numbers as physical segments (Lakoff & Núñez, 2000). Using this metaphor, anything that can be measured – not just using a ruler or other rigid item, but with any device, perhaps a measuring tape – can be considered a number; this provides some inferential structure for a previously unknown domain, the irrational numbers (see Figure 15).

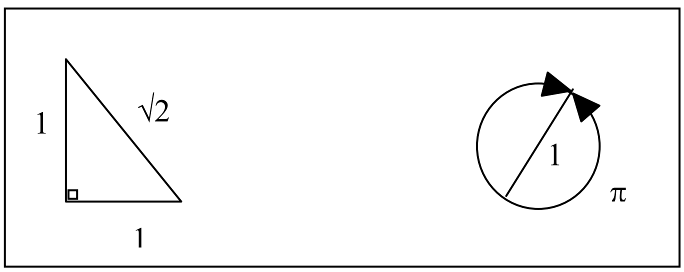

Figure 15: Grounding √2 and π using the MEASURING STICK metaphor.

Nodes are also added to the metaphoric network of mathematics through conceptual blends. These cognitive mechanisms construct a partial correspondence between two unrelated sources and project this onto the novel blended domain (Fauconnier & Turner, 1998; Lakoff & Núñez, 2000). For example, ARITHMETIC IS MOTION ALONG A PATH can be blended with ROTATION

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of mathematics on their own (Sinclair & Schiralli, 2003; Sfard, 1997). But an idiosyncratic metaphor may in fact be a conceptual metaphor, that is, "a grounded, inference-preserving cross-domain mapping" (Lakoff & Núñez, 2000, p. 6). For example, Machtinger (1965) creates a metaphor, GROUPS OF CHILDREN ARE NUMBERS, to assist kindergarten children in conjecturing and justifying theorems about number theory. The metaphor, while certainly idiosyncratic, is just as surely conceptual. Viewed from this perspective, the distinctions drawn between idiosyncratic and conceptual metaphors seem perhaps artificial.

BY 180° to form a conceptual blend that accounts for MULTIPLICATION BY A NEGATIVE NUMBER (Lakoff & Núñez, 2000; see figure 16).[14]

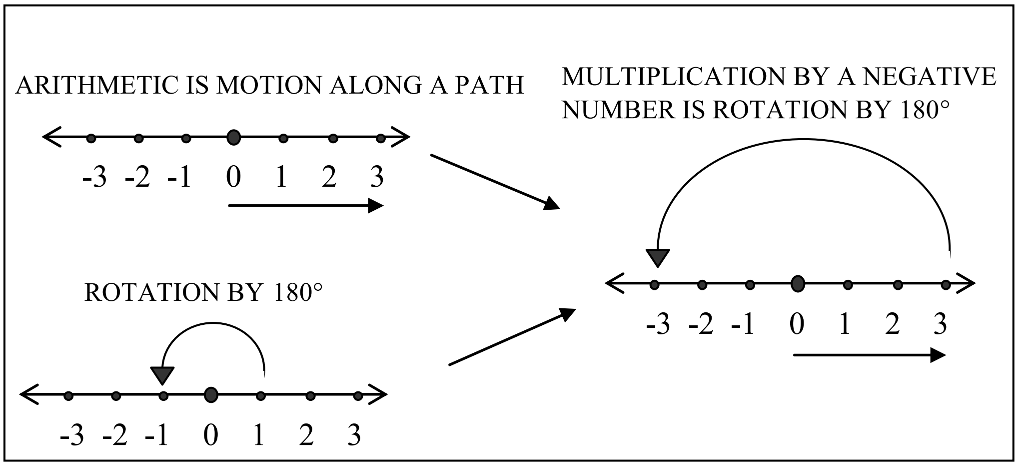

Figure 16: The conceptual blend for MULTIPLICATION BY A NEGATIVE NUMBER IS ROTATION BY 180°.

Most significantly, metaphors have the power to create previously unknown concepts; metaphoric mappings do not merely highlight a domain that already exists, but bring new concepts into existence by transplanting inferential structure from one context into another (Boyd, 1993; Chiu, 2000; Lakoff & Johnson, 1980; Presmeg, 1997a; Sfard, 1997). For example, the metaphor linking rotation by 90° to *i,* which was used by Argand to give geometric meaning to complex numbers (O'Connor & Robertson, 2000), is a classic use of this type of *constitutive* metaphor in mathematics.

While often unrecognized or ignored, the role that metaphor plays in the creation of new conceptual domains in the subjective understandings of

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14 While much of the discussion in this dissertation is organized around examples from Lakoff and Núñez (2000), I acknowledge that mathematicians throughout history originally developed many of these conceptions.

mathematicians is sometimes acknowledged. Sfard (1994) quotes a mathematician who describes the important function metaphor serves in his work:

> To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then can I answer questions, solve problems. I may even be able then to perform some manipulation on the concept. Only when I have the metaphor. Without the metaphor I just can't do it. (p. 48)

As learners assimilate and construct metaphors, the domain structure that comprises mathematical understanding grows. Entailments and extensions of existing metaphors, conceptual blends, and constitutive metaphors all lead to the creation of new concepts. Thus, the metaphoric network of mathematics satisfies the criterion of growth that is necessary for any scale-free structure.

4.3.3 Preferential Attachment: Why some Nodes Attract more Connections.

Although it is clear that the network model for mathematical understanding expands, perhaps unexpectedly, it does not grow in an evenly distributed manner. Not all nodes are equally likely to make new connections; "if a source domain is used to shed light on one or more salient target domains … this increases its likelihood to be chosen as a source domain in the future" (Kimmel, 2002, p. 108). When a concept is used as the source for a number of targets, it not only creates coherence among them by providing a common basis for their grounding, but it is strengthened in itself, simply because it is used repeatedly (Boyer, 1994). Recurrent use contributes to the source domain becoming an "attractor of meaning" (Kimmel, 2002, p. 113) – a *hub* in the metaphoric network of mathematics.

The term 'preferential attachment' is used to describe the pattern of development in networks where a disproportionately large number of new links involve nodes that are already highly connected (Barabási, 2003). Researchers have identified a number of factors that contribute to a node's attractiveness to connections.

Domains added to the network early in its development have more time to acquire links (Adamic & Huberman, 2000; Barabási, 2003; Barabási, Albert, Jeong & Bianconi, 2000; Krapivsky & Redner, 2001; Wagner & Fell, 2001). Thus, sensori-motor image schemas learned in early childhood, like the CONTAINER image schema or ROTATION, act as the source domain for many grounding metaphors and become hubs which are connected to a large number of concepts.

Concepts that have a greater degree of 'fitness' also tend to have more connections than other nodes (Bianconi & Barabási, 2001). Certain domains are repeatedly employed as sources, because of the power and utility of their particular inferential structures. Projections from such conceptual domains become not just acceptable, but expected. History provides examples; soon after Cantor's development of set theory in the late 1800s, the domain of SET was taken as the foundation for newly developed concepts in many other branches of mathematics (Eves, 1997; see Figure 10).

Other conditions may be involved. People are likely to use sources that they are already familiar with; “in the end, we all follow an unconscious bias, linking with higher probability to the nodes we know which are inevitably the more connected nodes of the [network]” (Barabási, 2003, p. 85). As well, formal education might be understood in terms of reinforcing the use of commonly accepted metaphors.

For many reasons, metaphors tend to link new concepts to domains that are already used as sources for a number of other concepts. Age, fitness, familiarity, and possibly other factors determine whether a node is likely to attract new links. Source domains with numerous links are likely to become even more well connected, causing the evolving network of metaphors in mathematics to exhibit the property of preferential attachment.

The two features of growth and preferential attachment are the requisite characteristics of a scale-free network topology (Barabási & Albert, 1999). As the metaphoric network of mathematics exhibits these characteristics, it possesses a scale-free structure. Thus, it shares the “common blueprint … [that governs] the structure and evolution of all the complex networks that surround us” (Barabási, 2003, p. 6).

4.4 Summary

In mathematics, as in all areas of human understanding, “from the day we are born, we use metaphoric projections to construct intricate conceptual systems” (Sfard, 1997, p. 343). New vertices are continually being added to the metaphoric network of mathematics through entailments of metaphors, conceptual blends and constitutive metaphors. Each node might be the target for projections from several sources, just as each concept might serve as the source for metaphors to a variety of targets. Domains that supply inferential structure to a number of concepts are likely to attract even more connections. Thus, the metaphoric network of mathematics – exhibiting both growth and preferential attachment – possesses a scale-free topology.

But there is more to understanding a network than determining its topological structure. One also needs to investigate its behaviors and discover how these actions are influenced by structural properties. For, in complex phenomena, "which outcomes occur, how frequently they occur, and with what consequences, are all questions that can only be resolved by thinking jointly about structure and dynamics, and the relationship between the two " (Newman, Barabási & Watts, 2006, p. 7). In next chapter, I will examine dynamic behaviors of the metaphoric network of mathematics and explore how these affect mathematical understanding. Implications for the learning and teaching of mathematics are also discussed.

Chapter 5

Dynamic Behaviors of the Metaphoric Network: How these Impact the Learning and Teaching of Mathematics[15]

Cognitive mechanisms outlined in the theory of embodied mathematics are seen to offer a useful model for representing mathematical understanding, with conceptual domains taken as the nodes of a network and conceptual metaphors viewed as the connections among them. When examined, this structure is seen to display properties of growth and preferential attachment. Thus, the metaphoric network of mathematics possesses the scale-free topology that is characteristic of all complex forms.

To understand a complex network, however, it is necessary to consider more than its structure; one must also examine the dynamics of the system (Barabási, 2003). In particular, one must not "overlook or oversimplify the relationship between structural properties of a networked system and its behavior" (Newman, Barabási & Watts, 2006, p. 7). In this chapter, I explore how the topology of the metaphoric network of mathematics influences the interactions that take place among its many nodes. These dynamical traits are of particular interest for they have a direct impact on subjective understanding

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[15] *Parts of this chapter have been drawn from an article that is published. Mowat 2008. For the Learning of Mathematics, 28(3): 20-27. Used with permission. Other elements come from an article accepted for publication. Mowat & Davis (in press). Complicity: An International Journal of Complexity and Education.*

of mathematics. This finding is discussed below, as are implications for the learning and teaching of mathematics.

5.1 The Dynamics of the Network

Much research has been done investigating how the structure of a complex network affects its overall behavior. The most conspicuous feature of a scale-free network is its distribution of connectivity; most nodes do not have many connections to other domains, but a few vertices have large numbers of links (Barabási, 2003; Watts, 2002; see Figures 13 and 14). Such highly connected nodes have an influence that is out of proportion to their number (Watts, 2002), for they not only ensure the ease and efficacy of interactions across the web, but "determine the structural stability, dynamic behavior, robustness, and error and attack tolerance of real networks" (Barabási, 2003, p. 72).

5.1.1 Interactions within the Network

Vertices with a great many connections have a major effect on relationships within the network, for "hubs create short paths between any two nodes in the system" (Barabási, 2003, p. 64). A scale-free structure is often considered to be a *small world* (Watts, 2002), as the 'distance' or number of links between different vertices is not large. Thus, chains of interactions can spread quickly throughout the web.

Despite this characteristic, if a network is made up of links that project inferential structure in just one direction, communication inside the formation is limited (Barabási, 2003). While a one-way path from node A to node B may consist of 2 or 3 short steps, the reverse path from B to A may not even exist. The network is segmented into several distinct regions and interactions between these areas are restricted. In such networks, paths only exist among approximately one quarter of the vertex pairs in the network (Broder et al., 2000).

In contrast, where connections are not directional, the network acts as a single homogeneous structure, ensuring that short paths can be found between any two nodes (Barabási, 2003). As discussed in Chapter 4, while conceptual metaphors in mathematics may appear to project modes of perception and reasoning in just one direction, there is evidence that such mappings come to exhibit reciprocal behavior. Sources for conceptual metaphors do develop co-emergent meanings with their target domains. Therefore, it is likely that the metaphoric network of mathematics is a unified structure. One might thus expect that any conceptual domain in mathematics can be linked to any other by a sequence of relatively few conceptual metaphors, a point that might be supported through reference to recent examinations of the figurative underpinnings of some mathematical concepts (cf., Lakoff & Núñez, 2000; Mazur, 2003).

5.1.2 The Stability of the Metaphoric Network

As well as having this efficient pattern of connectivity, scale-free networks are generally very robust. Since the majority of nodes have only a few links, a significant number of vertices can fail with little or no effect on the system. Yet the network can be subject to severe collapses.

This characteristic is highlighted when the scale-free topology is compared to two other structures – the centralized and distributed systems (see Figure 17). As nodes in a centralized system are separated by at most one vertex, such webs are vulnerable to massive failures. If the core dies, the network disintegrates into isolated nodes. Interactions become impossible. In contrast, because of its many connections, a distributed system is extremely robust. Many vertices can be removed from the mesh-like structure before relationships among nodes are hampered and the system begins to fail (Baran, 1964).

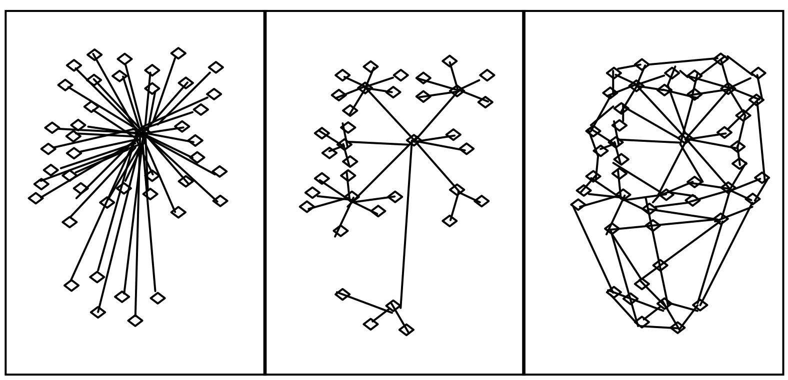

Figure 17: Centralized, scale-free, and distributed networks.

A scale-free network lies somewhere between these types of formations in terms of its robustness (Watts, 2002). The breakdown of a single central node does not cause the structure to become completely fragmented, but, if a highly connected hub is damaged, the network's stability may be seriously compromised (Barabási, 2003). The collapse may result in what is perhaps the most significant threat to the strength of a scale-free network – the *cascading failure*.

5.1.3 Cascading Failures in the Metaphoric Network of Mathematics

In a cascading failure, the weakening of a key concept reverberates throughout the network; nodes directly connected to the hub fail first, nodes linked to these fall next and so on (see Figure 18). While this damage can go unnoticed for a long time, the collapse of one highly connected vertex may eventually cause a large part of the network to break down and become fragmented (Albert, Jeong & Barabási, 2000; Barabási, 2003; Watts, 2002).

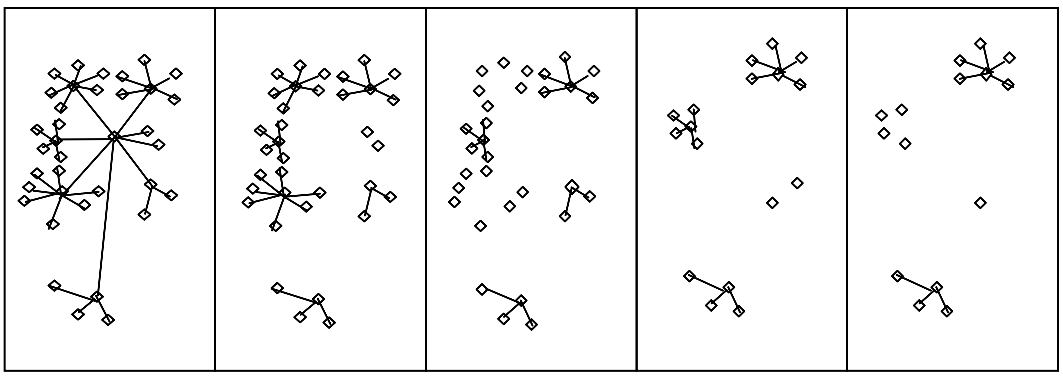

Figure 18: The effect of a cascading failure on a scale-free network.

To illustrate this dynamic process in the metaphoric network of mathematics, consider the many topics that are based on the image schema of ROTATION. Figure 19 displays just some of the domains – from ANGLES to ROOTS OF REAL NUMBERS[16]– that may be jeopardized if the concept of ROTATION breaks down. Mathematical ideas linked to these conceptual domains may in turn collapse. Consequently, the failure of this important

[16] Any real number like 1 has 3 complex cube roots. The principal cube root of 1 is $1 = 1 + 0i = (1, 0)$. The two non-real cube roots of 1 can be found by repeatedly rotating the line segment from the origin to (1, 0) through an angle of $120^\circ = 360^\circ/3$.

source domain ripples throughout the metaphoric structure and an individual's understanding of mathematics may be severely compromised.

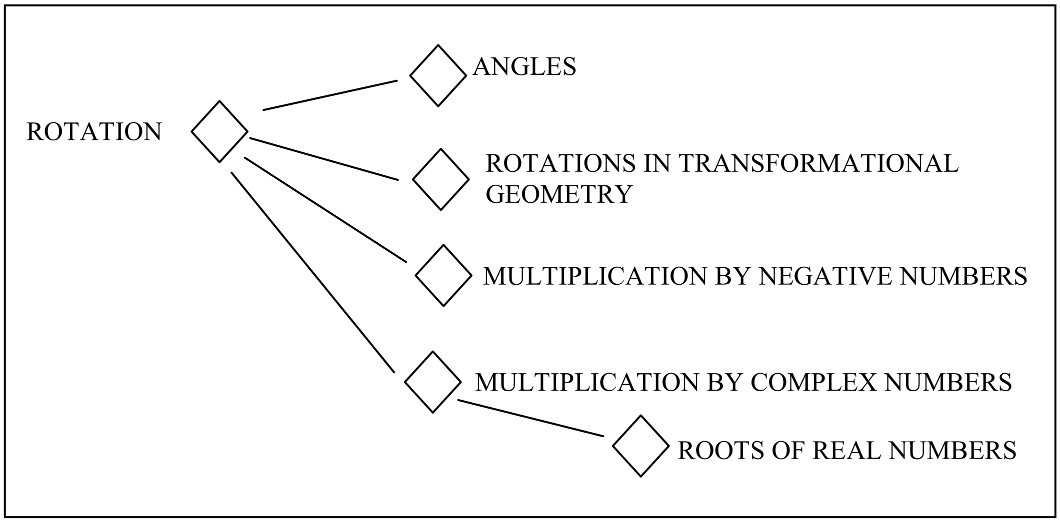

Figure 19: Some concepts metaphorically linked to ROTATION.

Thus, the scale-free structure formed by conceptual domains and conceptual metaphors influences the dynamic behavior of the network as a whole. Lines of communication go through key mediating nodes, which are powerful, commonly used sources. These hubs serve to connect clusters of related concepts and, ultimately, all nodes in the network. At the same time, dependence on such vertices puts the network at risk; if a hub fails, sizable portions of the web may break down. Those parts of the formation that remain are isolated, fragmented, and consequently more vulnerable. The scale-free topology of the metaphoric network of mathematics proves to be both a strength and a weakness of its structure.

There is some intrinsic credibility in the idea of the vulnerability of concepts and cascading failures in an individual's knowledge of mathematics.

Experience in the classroom leads one to recognize situations where the catastrophic collapse of a student's understanding does occur. A learner may seem to comprehend a mathematical topic well, and then something happens. Perhaps one too many idea is introduced, or some critical piece of background material is shaken, but suddenly the student's understanding of the concept falls apart.

As an educator, I believe I have witnessed this in the middle school classroom. For example, the concept of MULTIPLICATION tends to be constrained by 'definitions' of repeated addition and grouping. These interpretations work well for elementary arithmetic, but, when learners encounter multiplication of negative numbers, they lack a key metaphor – MULTIPLICATION BY -1 IS ROTATION BY 180º – needed to make sense of the new situation. A once well-understood idea is no longer clear.

As discouraging as this is for a teacher to witness, it is even harder to see comprehension of a whole group of related concepts shatter. The student whose understanding of a key idea is limited will inevitably have difficulty with associated topics. In high school classes, I have seen the trouble Grade 10 students have comprehending EXPONENTIATION and their consequent problems working with connected domains such as POLYNOMIALS, QUADRATIC EQUATIONS, and LOGARITHMS. A cascading failure, described previously in theoretical terms, becomes visible in the classroom when it is set in motion by the weakness of a single mathematical concept.

The dynamics that occur in the metaphoric network of mathematics offer some explanation for why students experience these types of difficulties. As a scale-free structure, mathematical understanding is inherently vulnerable because of the crucial role that certain conceptual domains play in ensuring the connectivity of the system. As in other complex networks, some nodes are simply more important than others. An inadequate understanding of one of these hubs has the potential to handicap severely a student's comprehension of concepts that are linked to it and, perhaps, of mathematics as a whole.

5.1.4 Increasing the Network's Stability

To support the learning of mathematics, there is a need to counteract the damaging effect that the weakness of a highly connected node can have on the stability of the metaphoric network – to limit cascading failures. Various options present themselves. For instance, attention might be focused on strengthening a learner's grasp of major source domains used in mathematical metaphors. There are two difficulties with this approach. First, little research has been done examining which source domains might be hubs; such research is confounded by the fact that hubs are not necessarily the same for everyone. Second, teachers can assist learners to construct more stable conceptions of source domains that are seen as key, but this will not eliminate the vulnerability that is characteristic of a scale-free network. Hubs are still hubs; they retain their central position as attractors of meaning in the network of mathematics.

In order to improve the stability of the network of metaphors, one must change its structure. There are several ways in which this might be accomplished. Watts (2002) suggests that reducing the number of connections to a hub would lessen the likelihood of network failure. He states, "even in the event a hub did fail, fewer [nodes] would be affected, causing the system as a whole to suffer less" (p. 193).

This course is not one that a mathematics teacher can readily choose; particular concepts are repeatedly selected as source domains because of their usefulness and because the mathematics community has traditionally employed them to develop new concepts. It is not likely that a teacher would deliberately refuse to use domains that do provide a coherent structure for developing mathematical knowledge, nor would this be responsible.

It would seem that another approach is required. Increasing the number of links among conceptual domains would have the desired effect of reducing the network's dependence on its hubs. Adding even a few connections – called *weak links* – between clusters of nodes decreases the network's vulnerability (Barabási, 2003; Buchanan, 2002; see Figure 20). The more distributed structure that results has sufficient redundancy to ensure that "even if some nodes [go] down, alternative paths [maintain] the connections between the rest of the nodes" (Barabási, 2003, p. 144).

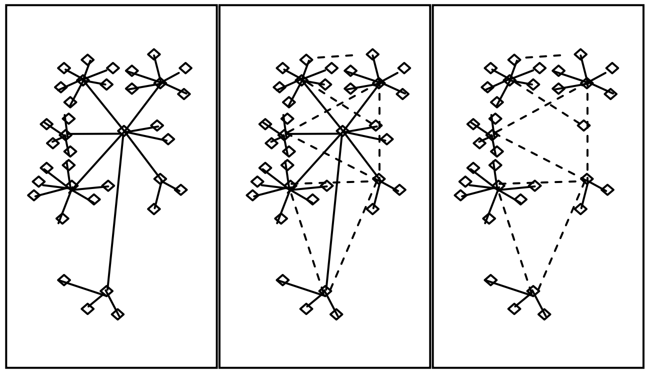

Figure 20: Increasing stability by adding a few weak links to a scale-free network.

Thus, connections added to a scale-free network play a key role in tying the structure together (Barabási, 2003; Buchanan, 2002). By serving as 'bridges' between different segments of the network, these new links decrease the chance of a cascading failure occurring. The collapse of one node is therefore less likely to cause the catastrophic fall of many other vertices; the network remains a functional whole. Following a tenet of complex dynamics, even a small change, like a slight increase in connectivity, can make a tremendous difference to the dynamics and robustness of the network.

This insight offers a more effective, and more acceptable, pedagogical approach. By promoting the establishment of new connections among mathematical concepts, teachers should be able to assist students to construct more robust understandings of mathematics. Suggestions for implementing and supporting such a methodology are discussed in the following section.

5.2 Implications for Mathematics Education

Awareness of the dynamic behaviors described above offers a way for educators to enhance their understandings of mathematics. In particular, recognizing events – more specifically, cascading failures – that are characteristic of a scale-free structure can shed light on students' difficulties in learning and utilizing mathematics. Such insights have significant implications for the learning and teaching of mathematics.

5.2.1 Adding Connections to Subjective Mathematics

In mathematics education, some metaphors are traditionally employed to make sense of certain concepts (e.g., EQUATIONS ARE BALANCES); such connections are strong because they are widely used and constantly reinforced. However, relying on a single link to provide structure for an idea can be dangerous. Failure to comprehend the source a concept is connected to not only jeopardizes understanding of the topic, but threatens comprehension of many related mathematical domains.

It is desirable, therefore, that students draw on a variety of conceptual metaphors when making sense of a mathematical idea. Then, if a learner's comprehension of one source breaks down, he or she can rely instead on metaphors linking the topic to other domains. When concepts are not dependent on the strength of a single key source domain, then the network of metaphors that constitutes a student's understanding of mathematics should become more robust and not subject to the cascading failures and fragmentation that are characteristic of scale-free networks. "If our mathematical conceptions are to be sound and stable, they must stand on more than one metaphorical leg" (Sfard, 1997, p. 367).

Support for this approach can be found in recent studies in mathematics education (although the work does not refer explicitly to network theory). Tall (2003) states that the automatic use of previously learned metaphors for arithmetic can cause confusion for students. He describes how young children feel that adding two numbers should always yield a larger sum. This perception is justified if one is using the ARITHMETIC IS OBJECT COLLECTION metaphor, but the OBJECT COLLECTION source domain cannot deal with situations where the addition of integers leads to smaller sums (adding 2 to -7 gives you -5). Moses and Cobb (2001) find the same reliance on ARITHMETIC IS OBJECT COLLECTION. They suggest that ARITHMETIC IS MOTION ALONG A PATH is more useful in this context and develop a series of activities using experiences familiar to students, like riding on the subway, to strengthen this metaphor. With such techniques, they are successful in improving children's understanding of integer arithmetic.

Davis and Simmt (2006) report on a study of teachers' understanding of the concept of MULTIPLICATION. While the most common definition for MULTIPLICATION given in primary textbooks is that of repeated addition ($3 \times 4 = 4 + 4 + 4$), participants identify a large cluster of different representations. In a more recent work, still more interpretations of MULTIPLICATION are found (Davis, 2008); conceptualizations from the two articles are presented in Figure 21. Teachers in both studies come to realize that MULTIPLICATION does not have a single meaning, but many, as revealed by the images, actions, and analogies they list. Moreover, they concur that if learners are made aware of such multiple interpretations, they will be able to appreciate how MULTIPLICATION is used in so many diverse contexts.

Multiplication is …	
- repeated addition	- equal grouping
- number-line hopping	- sequential folding
- many-layered	- ratios and rates
- array-generating	- area-producing
- dimension changing	- number-line stretching or compressing
- steady rise/slope	- branching
- number-line rotation (for multiplication of integers)	

Figure 21: Representations for MULTIPLICATION.

While metaphoric reasoning is not explored in either work, both concept studies can be viewed as suggesting metaphoric connections to MULTIPLICATION. Indeed it would be strange if metaphors were not revealed, for in all discourse language and gestures are replete with surface manifestations of underlying conceptual metaphors (Lakoff, 1993; Núñez, 2004). Thus, varied representations for MULTIPLICATION expose bits of the complex inferential structures they embody. Many of the images brought to light in these studies, while they cannot simply be reduced to the four grounding metaphors of ARITHMETIC, point to links with the image schemas upon which these are based (see Figure 22). Connections to other domains may also be indicated.

OBJECT COLLECTION	OBJECT CONSTRUCTION	MEASURING STICK	MOTION ALONG A LINE
repeated addition equal grouping many-layered	repeated addition sequential folding array generating area producing branching	repeated addition	repeated addition number-line hopping number-line stretching

Figure 22: Representations of MULTIPLICATION linked to the four grounding metaphors.

These four studies confirm that it is advantageous to utilize more than one metaphor when making sense of a concept. This approach is consistent with what is known about the internal structure of conceptual domains. As discussed in Chapter 4, a domain actively contributes to an individual's understanding only when a certain number of nodes are activated (Lamb, 1999). If a concept is connected to only one source and it fails, then the cluster of vertices that would have been transplanted from the domain's subnetwork to that of the target is no longer available. Consequently, the threshold of activation for the target domain may not be met. If several links from different source domains to the concept exist, there is less chance that this will occur. Although nodes corresponding to properties of one source may remain dormant, clusters of vertices associated with different sources are still available for activation. Augmenting the number of domains that map onto a target increases the likelihood that the concept and other topics, which in turn depend on its inferential structure, are activated.

Thus, in the metaphoric network of mathematics, it is important to develop additional connections among domains. A collection of different metaphors can provide varied ways of comprehending the many distinct aspects of a concept;

these mappings ensure a rich and complex subnetwork for the mathematical idea. If topics are not each dependent on a single source – if weak links are made to several domains – then subjective understanding of mathematics should become more stable and less prone to the cascading failures and fragmentation that can hamper scale-free networks.

5.2.2 Influences on Pedagogy

A central role in building these connections is played by teachers, for they largely determine students' exposure to mathematical ideas. To facilitate the construction of 'metaphorical legs', instruction could be oriented around the utilization of varied metaphors when a new concept is being introduced or a previously encountered one is being extended. It would not be advisable to present multitudes of metaphors all at once, for students would almost surely find this confusing. But during a unit or course – or even throughout classes over a several years of schooling – a number of different metaphoric connections to each mathematical topic could be established. Regular use of a variety of metaphors from different domains to make sense of concepts ought to benefit students in their learning of mathematics.

From the point of view of a classroom teacher, this makes sense. In discussions with colleagues, a number commented that using several metaphoric approaches to a topic increases the likelihood of students understanding the concept. That some of these peers were speaking of their experiences teaching subjects other than mathematics is particularly interesting. It appears that teachers in many disciplines can appreciate the desirability of establishing connections to a target from a number of source domains.

Teachers play a critical part in this process, for they introduce the images, models, and other representations that trigger metaphoric links in students' subjective understandings of mathematics. Moreover, instructors design the activities and explanations that assist learners to understand and internalize these connections. Neither of these undertakings is simple for there are many different factors that must be considered.

Metaphors need to be selected carefully; it is important that the source domain accurately reflects some aspect of the structure of the target domain (Chiu, 2001; English 1997). For example, the activity of walking along a street might appropriately be used as a source for the concept of a number line (Chiu, 2000). The ideas of starting point, location, distance, and directions correspond to features of the target domain like zero, number, absolute value, and positive or negative quantities. When a natural correlation between source and target domains exists, learners find it easier to transfer appropriate inferential structures from one to the other (English, 1997).

In contrast, consider the metaphor Sandra chooses to develop the concept of inverse functions – as gears in a car that are shifted into reverse (Heaton, 1992). There is no real sense of 'undoing' – the essence of the inverse of a function – in Sandra's image; thus understanding of situations involving inverse functions is limited. Students' familiarity with cars may be motivational, but the comparison provides "an ambiguous (and potentially deceitful) impetus with which to make sense of the problem" (Sinclair & Schiralli, 2003, p. 85).

It is also important for students to be familiar with the source of a metaphoric mapping (Chiu, 2000; Davis & Maher, 1997; Schifter, 2001), for "inadequate understanding of the source domain of a metaphor limits a person's reasoning through that metaphor" (Chiu, 2000, p. 7). For example, Edwards (2003) finds that, while rotations about a point that lies within an object are well understood, learners have difficulties when the centre of rotation is outside the object. Although not stated explicitly in Edwards' work, it appears that limited comprehension of the source of the ROTATION IS TURNING metaphor handicaps people studying transformation geometry. Individuals interpret TURNING using physical experiences like rolling over – with themselves as the centre of rotation – and do not consider other 'turning' situations like children playing on a swing or satellites orbiting a planet. As the degree of understanding of a source both facilitates and restricts reasoning using a conceptual metaphor, it is desirable to ensure that students comprehend the many different features of source domains.

In addition, each conceptual metaphor has limitations that could be brought to students' attention. Even if learners comfortably draw on a metaphor to make sense of a concept, they are not likely to be aware of everything that is implicit in the mapping – what it introduces and what it hides – and unquestioned use may lead to "invalid inferences, unreliable justifications, and inefficient procedures" (Chiu, 2001, p. 94). Nolder's (1991) discussion of the PRIME NUMBERS ARE PRIMARY COLORS metaphor exemplifies some of these liabilities.

From the source domain of PRIMARY COLORS, basic relations are projected onto the target of PRIME NUMBERS; ‘secondary colors are composed of primary colors’ becomes ‘natural numbers are composed of prime numbers’. But use of this metaphor may cause misconceptions. Individuals may incorrectly assume that, because there are three primary colors, only a finite number of prime numbers exist. Clarifying the boundaries of metaphoric projections is an important part of the teacher’s task when conceptual metaphors are introduced and utilized in classroom activities.

Moreover, when several appropriate metaphoric links have been established from well-understood sources to a target, the mathematics educator can integrate these connections. Distinct metaphors each have their own modes of reasoning and “lead to different conscious and unconscious beliefs that can cause obstacles to drawing various aspects [of target domains] into a central core mathematical concept” (Watson, Spyrou & Tall, 2003, p. 74). Núñez, Edwards and Matos (1999) illustrate this in their study of conflicting metaphors used in calculus. In high school, the metaphor A LINE IS THE MOTION OF A TRAVELER TRACING THAT LINE shapes introductory descriptions of the limit. Later, at university, a different analogy – A LINE IS A SET OF POINTS – underlies the Cauchy-Weierstrass ε-δ definition of the concept. As students are never told that these conceptualizations have dissimilar embodied foundations, difficulties naturally arise (Lakoff & Núñez, 2001; Núñez, 1997; Núñez, Edwards, & Matos, 1999). To avoid such problems, special efforts are needed to combine inferential structures from different source domains into a coherent whole.

Classroom communication plays a significant role in addressing these tasks. When interacting with students, teachers cannot help but use metaphoric utterances for “metaphor is as central to the expression of mathematical meaning, as it is to the expression of meaning in everyday language” (Pimm, 1987, p. 11-12). Metaphors may be explicitly stated or implicit in representations hidden in diagrams drawn on the board, in displays on the wall and in a teacher’s casual expressions and gestures (Bolite Frant, Avecedo & Font, 2005). It is particularly important that educators be aware of the effect their words and actions have on students. For since a learners’ mathematical development is shaped significantly by the instructor’s preferred modes of reasoning, the ways metaphors are used by teachers have a real effect on their students’ understandings (Bolite Frant, Acevedo & Font, 2005; Presmeg, 1997b).

References to metaphors could also be an integral part of classroom discourse. Attention can be drawn to the use of metaphoric terms, notations and images, and, as individuals are likely to understand a metaphor in terms of their own personal experience (Pimm, 1987), learners can be encouraged to discuss the associations these bring to mind. Students can also be invited to articulate ways in which various metaphors linked to a concept are different and similar, and to identify features in various sources that correspond to certain aspects of the target domain. Such discussion might not only help learners realize which metaphors are appropriate for use in particular situations, but may also emphasize the need for having more than one metaphoric perspective from which to view a concept.

Social interaction among students could also be encouraged, as this has been shown to be important in determining the efficacy and usefulness of metaphoric thought in the classroom (Madden, 2001). As learners work together on mathematical activities, they can collectively explore and reinforce their understandings of the mappings conveyed by different conceptual metaphors. Moreover, problems are likely to be interpreted with idiosyncratic metaphors. Since life experiences vary from child to child, the comparison one student uses to understand a mathematical concept may not be the same as that constructed by another (Presmeg, 1997a). Discussions provide opportunities for learners to explore and evaluate these personal analogies. Thus, private metaphoric links may develop taken-as-shared meanings and become part of the classroom culture (Presmeg, 1997a; Sfard, 1997). Moreover, opportunities emerge for teachers and students to integrate idiosyncratic metaphors, when appropriate, with the conceptual metaphors of formal mathematics (Bolite Frant, Acevedo & Font, 2005; Presmeg, 1997a; Sinclair & Schiralli, 2003; Wood, Cobb & Yackel, 1991).

As metaphors are an intrinsic part of mathematical thinking, paying attention to the reasoning revealed in classroom interactions can provide educators with insights into learners' understandings or misunderstandings (Bolite Frant, Acevedo & Font, 2005; McClain & Cobb, 2001; Presmeg, 1997a; Schifter, 2001). For student errors are not random; they contain an element of logic, even though it is incorrect (Schifter & Fosnot, 1993; Wood, Cobb & Yackel, 1991). Thus, it is important for teachers to "dig under [the] children's words to find the sense in their perplexities" (Schifter, 1998, p. 78). One learner may base his or her reasoning on a domain that does not provide appropriate inferential structure for a particular topic; a second may misconstrue correspondences between source and target. Still another may hang on to “a concrete metaphor that refuses to die” (Sfard, 1997, p. 368), not making the necessary transition to using more abstract concepts as source domains. Awareness of these problems is important if mathematics educators are to assist students in forming sound understandings of metaphoric projections and in leaving inappropriate links behind and moving on to more suitable connections.

5.2.3 Other Consequences for Mathematics Education

The teacher cannot alone implement recommended changes to instructional practices. In particular, support from curriculum structures is desirable. Current programs of study focus on distinct mathematical topics, arranging them in an essentially linear fashion. Mathematics is presented as a hierarchical structure, with concepts at each level being built on those taught in previous grades. Students accumulate a collection of techniques from isolated units to use in prescribed ways.

This presentation of mathematics inevitably conflicts with what complexity science and network theory reveal about mathematical understanding. The structure of mathematics is more akin to an ecosystem of vibrant notions, rather than a tower of static ideas. Concepts are important, but the connections between them are even more vital; it is the metaphoric links in mathematics that determine a concept’s inferential structure, connect it to clusters of associated topics, and ensure its stability. Curricula could highlight multiple interpretations of mathematical concepts at each level – indeed across grades – and mandate their inclusion in classroom instruction. A syllabus might also actively encourage teachers to seek out and make use of activities that present and reinforce metaphoric representations differing from those explicitly included in programs of study or authorized texts. Without changes to curriculum, attempts to make systematic use of metaphors in mathematics education are not likely to be successful.

Programs of professional development could also assist teachers in developing a repertoire of teaching strategies that facilitates the effective introduction and use of metaphors in classrooms. One approach that has been shown to be effective in assisting students reason metaphorically involves setting forth a familiar situation or object, discussing it using ordinary, everyday language, and finally introducing mathematical terminology (Nesher, 1989). Knowledge of this technique and other methods for enabling students to successfully acquire and utilize metaphoric connections would be invaluable for educators.

As well, awareness of the network structure of mathematics and the important role conceptual metaphors play in cognition could be increased. Classroom teachers, authors of textbooks, and designers of programs of study can learn more about the many metaphors that connect mathematical concepts together. Facilitating this is not easy, for identification of metaphors requires sensitive attention to language, gestures, and images. The multifaceted meanings of mathematical concepts are “constructed on the basis of scattered cues and sustained innuendo” (Kimmel, 2002, p. 518). Providing opportunities for a more stable understanding of mathematics requires educators to pick up on these subtle hints, to identify underlying metaphors, and to share them with students. This is a daunting task for there are so many conceptual metaphors linking multitudes of domains, and very little is known about which cognitive mechanisms are related to particular mathematical ideas.

While metaphoric structures have not been studied extensively, some progress has been made. In the field of cognitive science, some researchers try to clarify the precise nature of particular mathematical concepts (cf., Lakoff & Núñez, 2000). But a thorough analysis of mathematical ideas – explicating what the structure of each conceptual domain is, showing how it is ultimately grounded, and elucidating how it is metaphorically connected to other mathematical concepts – is lacking. Without this information, it is difficult to imagine teachers presenting mathematics as a system of complex domains knitted together by metaphorical reasoning. Designing programs of study and classroom activities to encourage students to construct their understandings of mathematical concepts using inferential structures conveyed by clusters of metaphors would also be problematic. Only when more is known about the many connections among conceptual domains can educators assist students to make the metaphoric links needed for more robust understanding of mathematics.

5.3 Summary

As with other complex systems, network theory provides valuable tools for examining subjective mathematics. By looking past the specific details of the system to the decontextualized structure of nodes and links, it is possible to see patterns of dynamic behavior not otherwise evident (Barabási, 2003). Specifically, the innate vulnerability of the scale-free topology of mathematical understanding to cascading failures becomes apparent, as does the possibility of decreasing the chance of such a collapse by adding more connections to the metaphoric network of mathematics.

In particular, I have proposed that mathematical understanding may be made more robust by ensuring concepts are linked to multiple source domains. Ways of making this a focus of classroom instruction have been put forth and different aspects involved in realizing this goal have been described. Modifications to curriculum design and professional development to support this strategy have also been suggested, as has the need for systematic research into the structure of the metaphoric network of mathematics. Attending to what is known of the dynamics of complex systems, it is hoped that these ideas are useful for ongoing cyclical elaborations of school mathematics and pedagogical practice.

In Chapter 6, I seek to contribute to the needed understanding of metaphoric webs of mathematical concepts with the intent of substantiating ideas presented to this point. However, carrying out a comprehensive "mathematical idea analysis" (Lakoff & Núñez, 2000, p. 29) would require the life work of many researchers and is clearly far beyond the scope of what I might attempt in my investigation. Therefore, I will explore a very small part of the proposed metaphoric network of mathematics, focusing on the concept of EXPONENTIATION. More specifically, I will search for representations identifying conceptual domains that provide varied ways of making sense of EXPONENTIATION; these can function as sources for the multiple metaphoric connections to the topic that should increase the stability of students' understandings of mathematics.

Chapter 6

Exploring the Metaphoric Network of Mathematics

In this dissertation, I have proposed that complexity science and, hence, network theory offer a new way of looking at subjective mathematics. I have further suggested that the theory of embodied mathematics (Lakoff & Núñez, 2000) provides a possible network model for mathematical understanding, where conceptual domains represent nodes in the network and conceptual metaphors provide the links among them. By applying techniques developed by network theorists to this structure, I have demonstrated that it exhibits the scale-free topology and consequent dynamic behaviors commonly found in complex systems. Specifically, the metaphoric network of mathematics is prone to cascading failures, which threaten the robustness of individual mathematical understanding. Adding to the connectivity of the network structure lessens the chance of such systemic collapses and increases the stability of subjective mathematics.

In particular, I have presented evidence that this reinforcement of the metaphoric network can be accomplished by ensuring that the meaning of a mathematical idea does not depend solely on patterns and modes of reasoning mapped from a single conceptual domain. Thus, for each concept there is a need to develop metaphoric connections to a variety of sources. This conclusion has implications for mathematics education; classroom instruction, curriculum, and professional development are affected.

Before changes in these areas can be implemented, however, more needs to be known about metaphors that can provide important inferential structure for specific concepts. This is a daunting task, for "it is a non-trivial process to tease out the connotations and meanings … which underlie fundamental relationships among fields of mathematical ideas" (Presmeg, 2002, p. 59). While some progress has been made, only a few topics have been explored to date, and these areas have been interpreted using small numbers of metaphoric connections. Thus, a more systematic investigation of the cognitive structure of mathematics, identifying conceptual metaphors that support understandings of different concepts, is badly needed.

In this chapter, I report on my efforts to carry out such an inquiry. This work was done with the intent of substantializing the claims described above. My study followed recent trends in complexity science that develop understanding of a network structure based on dynamics that occur at a local level (Newman, Barabási & Watts, 2006). Thus, I focused on the nodes, connections, and dynamics of one small neighborhood in the metaphoric network of mathematics. In doing so, I hoped to illustrate that 'metaphorical legs' for a mathematical concept can be identified and to exemplify ways in which research in this area can be conducted.

6.1 EXPONENTIATION: An Illustrative Example

With these goals in mind, I chose to explore the cognitive structure of EXPONENTIATION. More specifically, I looked for representations – examples, images, models, gestures, turns of phrase – that can be used to make sense of the concept. Such depictions and interpretations are more than just illustrations; drawn from a variety of physical and social experiences, "these elements are … embodiments of the concept" (Davis & Simmt, 2006, p. 314). Such conceptualizations, therefore, should indicate sources from which important inferential structures can be projected onto EXPONENTIATION. These metaphoric mappings would provide the additional connections needed to enhance the robustness of mathematical understanding.

My decision to focus on this concept was influenced by several factors. EXPONENTIATION plays an important role in many branches of mathematics. A wide range of topics – basic arithmetic, elementary algebra, trigonometry, logarithms, calculus, abstract algebra, and so on – incorporate exponential ideas. By virtue of its numerous links to substantial numbers of concepts, it is possible that EXPONENTIATION may serve as a hub in the metaphoric network of mathematics.

Reflecting this high degree of connectivity, EXPONENTIATION is related to a substantial part of school mathematics. In one form or another, it permeates classroom activities – from the introduction of units of measurement and place value in primary grades to the use of polynomials and equations at secondary levels. To successfully construct mathematical understanding of such topics, students need a sound comprehension of the concept.

Such knowledge is also important for individuals' participation in public life. EXPONENTIATION plays a significant role in modeling many phenomena, both scientific and social. For appreciating and dealing with such issues as uncontrolled population growth, global warming, or the disposal of radioactive elements, the importance of understanding EXPONENTIATION cannot be doubted (Confrey, 1994).

Thus, my intention to explore the structure and dynamics manifest in this conceptual domain seemed worthwhile. The discovery of 'metaphorical legs' upon which the concept of EXPONENTIATION stands would clarify the nature of one small section of the metaphoric network of mathematics. And increased understanding of this cognitive structure would assist educators in their efforts to support students' constructions of sound and stable mathematical understanding.

I must admit that, when I first outlined my proposal for this task, I expected the work to be accomplished relatively quickly. However, this part of my doctoral project soon grew beyond all expectation; it became much more involved and took a good deal longer to complete than I had planned. In retrospect, I should have anticipated this, as research into complex systems and their models is not likely to be straightforward.

What was meant to be a brief example, describing some metaphoric connections to EXPONENTIATION and illustrating how such links could be found, became a major piece of work. As my understanding of EXPONENTIATION grew, my interpretations of results from previous stages were affected and ways in which I carried out research changed. The following sections of the chapter relate how this study evolved and present the many unforeseen findings that emerged during the project.

6.2 The Methodology of the Study

Over a period of twenty-two months, I looked for representations that pointed to conceptual metaphors for EXPONENTIATION; explorations consisted of three, often concurrent, components. The first involved reading about the many conceptualizations identified by others. Throughout the study, I examined textbooks and teaching materials, read articles reporting on classroom activities and research studies, and investigated the historical development of concepts related to EXPONENTIATION.

The second and central stage involved a 'concept study'; that is, a collaborative examination of representations – gestures, images, analogies, metaphors, models, activities, and applications – used to develop understanding of a topic (Davis, 2008). Addressing questions like, "When you hear the term EXPONENTIATION, what do you think of?", a group of six mathematics teachers who were enrolled in graduate programs developed a variety of conceptualizations. The concept study discussion was videotaped and field notes were taken.

Audiotaped interviews with different groups were also conducted. I met with each participant in the concept study, posing a series of questions directed at pedagogy. I hoped that this shift in focus would elicit additional images and analogies for EXPONENTIATION. As well, I interviewed five mathematicians and seven other mathematics educators, with the expectation that these individuals would mention novel representations for the concept. Notes were taken of gestures and other details not recorded on the tapes.

These parts of the study are described below in more depth. For each phase, representations for EXPONENTIATION that emerged from the work are outlined and conceptual domains associated with them identified. My reactions to and reflections on each stage of the work are also included, for I found that I was a participant in all levels of the study. Changes that occurred in my own understandings of representations and conceptual metaphors are also acknowledged.

6.3 Preliminary Readings

My first explorations in the literature led to two conceptualizations for EXPONENTIATION: a definition – "the mathematical operation of raising one quantity (the base) to the power of another (the exponent)" (Exponentiation, 1989); and numerous references to repeated multiplication. Many dictionaries, encyclopedias, articles, textbooks, and teaching materials alluded to one or both of these interpretations.

Examination of meanings of related terms yielded the same ideas. An *exponent* sets forth the number of identical factors in a product. The word *power* indicates a value formed through the multiplication of a number by itself one or more times. An *index* points to a smaller symbol written to the right and above a numerical quantity. These themes – a particular notation and repeated multiplication – occurred again and again. Some sources even stated that only these representations are meaningful; "a whole-number exponent is simply shorthand for repeated multiplication of a number times itself, … that is the only conceptual knowledge required" (Van de Walle & Folk, 2005, p. 424).

In spite of this, a few writings encouraged my search for different ways of thinking about EXPONENTIATION. One medieval source suggested a connection to geometry. The term 'zenzizenzizenzic' was used to represent a number raised to an exponent of eight; it was defined as "the square of squares squared" in *The Whetstone of Witte* (Recorde, 1557, p. 150).[17] At that time, there was no easy way of denoting a larger power except by breaking it down into a combination of squares and cubes (Cajori, 1928). And in modern work, Lakoff and Núñez (2000) refer to an exponential as a function or mapping "from the domain of real numbers under addition to the range of positive real numbers under multiplication" (p. 405). These two representations – pointing to geometric shapes and to functions – stimulated my search for metaphoric connections to EXPONENTIATION.

6.4 The Concept Study

The core of my investigation was a concept study involving six mathematics teachers. Participants had widely differing backgrounds; they had taught at various levels, had been educated in different countries, and had

[17] The root word 'zenzic' was derived from the Italian 'censo' meaning squared (Zenzic, 1989).

experienced teaching many subject areas. All were members of a graduate mathematics education course that met once a week.

Before the session, information on the theory of networks and embodied mathematics was shared with participants, in the form of several related articles. Subsequently, they talked about their understandings of EXPONENTIATION, engaging with and reacting to each other's ideas. While the concept study was originally scheduled for the second half of one class, the group revisited the topic a week later. Both of these sessions are analyzed below.

6.4.1 The Initial Discussion

The concept study began with questions: "When I mention the word exponentiation, what does that mean to you? What do you start thinking of? What kind of images leap into your mind?". After a brief pause, participants began to comment. The first responses echoed dictionary definitions:

> Yvette: I always think of 2 to the power *x*, you know just the basic simple base 2.
>
> Ellen: I think of *x* squared.

But different representations soon emerged. The following exchange took place in the fifteen-minute period immediately following the above comments.[18]

> Ardis: I always think of the **growth curve[19]** and its contrast to linearity [sketching the shapes of an exponential curve and a linear curve with her hands].

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[18] Parts of the discussion – expressions of agreement, long explanations of ideas, etc – are omitted to reduce the length of this extract from the transcript.

Gary: I usually think of growth as well. ... I always think about it as a growth, I never think about it as **decay**.

Ardis: And that's really interesting because the specific context I was thinking about is decay and yet I called it growth.

Yvette: It's like rabbits **doubling** [using her hand to shape an exponential curve]. … But then that's so funny to think about rabbits multiplying right. … When they're breeding, rabbits aren't there with a little pencil in their paw, … but doubling, doubling, **spreading out**.

Gary: To me it's like addition and multiplication. Addition is the basic compilation and multiplication is **a more expedient way of writing it.**

Odette: I couldn't help but think outside the mathematical context and I think it's because I am a biology teacher, so I thought more about **population growth rate.**

Ivan: Mine's more simple that that. …What I see is a picture; it's not necessarily a symbol. **It's just a little picture with one [number] as a superscript in the corner,** so that's the symbol there [drawing the image $\bigcirc^{\circ}$ on the whiteboard].

Ellen: One [idea] that immediately came to mind was **decimals and place value in a variety of bases.** … When you're talking about **rotations**, that is represented as the equivalent of exponentiation.

Ardis: I think of activities I used to do with **chain letters, pyramid marketing … email spam –** 'Bill Gates is giving away his fortune'**.** … But all these things go back to that **growth pattern**. To me, this image says exponentiation and then I think of **bronchioles** and **blood vessels** and **tree roots** and **tree branches** and **drainage systems** and **road maps**. … The whole structure, somehow, even if it's not represented in symbols, could mean this is exponentiation. … This **branching structure,** each time it branches, you're growing exponentially.

Odette: **Family trees!** … Family trees make sense. … They are very basic.

Ellen: Well along the same lines, by the time I was in fifth grade, we had been through three **generations** of cats, and we started with two and each of them had a few babies.

---

[19] I have emphasized, using bold type, utterances in this discussion that I see as referring to representations of EXPONENTIATION.

Yvette: You know a lot of classrooms have some kinds of pets or **something that would grow**.

Ardis: Or flies.

Yvette: I like the **blocks for base arithmetic**, like the single unit, which is any number, … and then you have the first power [moving her hand along an imaginary **rod**] and then **flat** [shaping a square] which is the second power and then the **cube** which is to the third power [using her hands to form a cube].

Gary: **Cuisenaire rods**

Yvette: They can even build up to the next exponent. They take 10 of the **rods** and that becomes the **flat** … and then ten of the flats and [that becomes] the **cube** and then where do you go from there? I have ten of the cubes [that represent the third power] in a **rod** [using her hands to shape a rod].

Ellen: I wonder if it's too early to introduce **Zeno's paradox** at that point - where **you drink half your glass of milk, and then drink half of what's left, and half of what's left.**

Yvette: That's fantastic!

Although the discussion occasionally drifted back to standard definitions, a wide variety of conceptualizations for EXPONENTIATION emerged. While the range of representations was substantial, many shared certain characteristics. Each cluster of ideas hinted at a source domain for connections to EXPONENTIATION.

I must acknowledge that, when I interpreted participants' contributions to the talk about EXPONENTIATION, I could not help looking for links to the image schemas upon which Lakoff and Núñez's (2000) grounding metaphors for ARITHMETIC are based. Because their inferential structures give meaning to MULTIPLICATION, it seemed likely that these domains would also be connected to EXPONENTIATION. It is not surprising, therefore, that I saw collections of objects, object construction, measuring with segments, and motion along a line in the conceptualizations mentioned by members of the concept study. But representations also arose that suggested links to other domains. Some echoed ideas found in my early reading about the concept; others were quite unanticipated. Participants' understandings revealed that EXPONENTIATION is a far more complex concept than I had imagined.

Findings from this first meeting are presented below. These are organized according to the various sources that I perceived as having been used to make sense of EXPONENTIATION. Illustrative representations and brief explanations are provided for each conceptual domain to show how connections to EXPONENTIATION were made in the concept study. For reasons of length, some variations of conceptualizations mentioned by participants are not included here.

OBJECT CONSTRUCTION:

- Repeated multiplication[20] – The face of a combination lock shows thirty-nine numbers; the dial must point to three values, not necessarily distinct, in the right order. The number of possible combinations for the lock is 39 5 39 5 39 = 39^3.
- Geometric shapes – Students can get a physical sense of what powers of one, two and three mean by looking at geometric shapes

[20] Lakoff and Núñez (2000) state that repeated multiplication implicitly reflects use of the OBJECT CONSTRUCTION domain. See Section 6.6.2 for further discussion of this point.

such as lines, squares, and cubes. Base 10 blocks make use of this connection.

- Changing dimensions – As the power increases, so does the associated dimension. The unit cm measures a one-dimensional length, cm^2 refers to a two-dimensional area and cm^3 is used to quantify a three-dimensional volume.
- Constructing sequences of shapes – Illustrative patterns can be formed by doubling a square sideways to form a rectangle, doubling this rectangle vertically to form a larger square, doubling this square sideways to form a new rectangle, etc. In this progression, powers that are 'perfect squares' always have a square shape.
- Fractals – The Koch snowflake and Sierpinski triangle exhibit many properties involving exponents. With each iteration, the number of units, the perimeter and the area change exponentially.
- Creating new units – The act of renaming a collection of items effectively creates a new entity, thus involving the construction of a novel object. A loaf can be cut into ten slices, each slice cut into ten toast fingers, and each toast finger cut into ten croutons. Articles at each stage are different from those at other levels: a crouton does not have the same properties or purposes as a loaf of bread.

- Positional number systems – A positional number is an ordered set of symbols in which the value of each character is determined by its place. The decimal number 327 can be *broken up* into three hundreds, two tens and seven ones.
- An object changing at a constant rate – Radioactive decay can be simulated using a cube made up of sixty-four blocks. After the first half-life, thirty-two blocks decay and are broken off. Next, sixteen blocks are taken away, then eight, etc.

OBJECT COLLECTION:

- Comparing sets that have a common ratio – The traditional story of the King of Persia rewarding the inventor of chess tells of gold coins piled on the squares of a chessboard. Each square has twice as many coins as the one before.
- A set changing at a constant rate – Radioactive decay can be simulated by repeatedly rolling a collection of dice. When the dice are thrown, those showing an even number are considered to have decayed and are removed. Dice with an odd number are rolled again. Compound interest and population growth involve sets of units – dollars and living entities respectively – which grow at a constant rate.
- Branching structures – This pattern is exemplified by a family tree, where one person has two parents, four grandparents, eight great-grandparents, and so on. Each generation contains twice as

many people as the previous one. Tree diagrams are often used to display such structures (see Section 6.6.3 for discussion of natural branching structures like bronchioles and see Section 6.6.4 regarding the related notion of 'splitting').

- Sets that form a geometric series – In a chain letter, where each person writes three letters to friends, the total number of people sending letters can be represented by the series 1 + 3 + 9 + 27 + ….

MOTION ALONG A PATH:

- Motions along path that form a geometric series – Zeno's paradox of the Dichotomy describes a very fast runner who needs to reach a particular location. He or she must run halfway to the goal first, then go half of the remaining distance, and so on. This could be represented by the series:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

FUNCTION:

- Mapping an arithmetic progression onto a geometric progression – For the function, $y = 2^n$, $n \varepsilon N_0$, 0 is mapped onto 1, 1 is mapped onto 2, 2 is mapped onto 4, 3 is mapped onto 8, etc.
- An exponential curve – The curve of an exponential function was indicated in many gestures (see Figure 23).

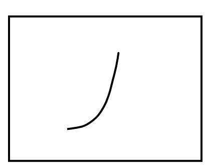

Figure 23: The gestural path signifying an exponential curve.

- Rate of change of slope – The rate at which the slope of an exponential curve increases surpasses that of linear and quadratic functions.

ROTATION:

- Rotating in the complex plane – Powers of complex numbers involve rotation around the origin in the complex plane.[21]

The concept study proved to be the richest source of representations in all of the stages of research into EXPONENTIATION. Several factors may have contributed to this. Before the session began, participants had read and discussed articles about the theory of embodied mathematics. As well, the idea of mathematical concepts comprising a network was familiar, for I had explained my hypothesis when obtaining each individual's consent to take part in the project. Knowledge of the theories upon which this work is based may have facilitated the fruitful discussion of EXPONENTIATION that occurred.

As often happens in discourse, understandings that emerged from the concept study surpassed those held by individuals before the session. One participant expressed her appreciation of this situation.

> Ellen: What really surprised me … was … how absolutely stuck I was on exponentiation as repeated multiplication and could not get that out of my head until you kind of seeded me [gesturing toward the group] with a few other ways to look at it and it was only then that I found more experiences

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[21] Lakoff and Nüñez (2000) construct a complex blend of domains to account for MULTIPLICATION in the complex plane. These include ROTATION, the NUMBER LINE, the CARTESIAN PLANE and the ROTATION PLANE. EXPONENTIATION in the complex plane must necessarily involve these, and possibly other, sources.

> that were related to it. … You know those things were in there, but … I literally had to get out of the textbooks and start putting the links together.

The added stimulation of exchanging ideas with peers may have contributed to the generation of new insights. For as Surowiecki (2004) comments, “A face-to-face group … makes everyone work harder, think smarter, and reach better conclusions than they would have on their own” (p. 176).

But more is involved here. Davis and Simmt (2003) discuss the delicate balances – diversity with redundancy and order with randomness – necessary for the generation of collective knowledge. Shared understandings that go beyond that of individuals in a group often emerge under such conditions. As the exchange of ideas among participants in the concept study displayed these properties, it is not surprising that particularly astute insights were developed.

6.4.2 One Week Later

The group of mathematics teachers met for the next class in seven days and spontaneously started talking about the concept study and the representations of EXPONENTIATION that had emerged from it. I was surprised, but delighted. The discussion was not videotaped or audiotaped, but field notes were taken.

Participants had not stopped thinking about EXPONENTIATION when our session ended the week before. Many ideas from the concept study were revisited and extended. For example, the root *quad* and its connections to both geometric shapes and algebraic forms were explored. One teacher brought a tape entitled *Powers of 10* to show the group; the creation of new units – angstroms, microns and light years – was exemplified in the brief video clip. And new representations for EXPONENTIATION emerged; these conceptualizations are described below.

OBJECT CONSTRUCTION:

- Sequential folding – Repeated folding of a piece of paper involves the exponential growth in the number of smaller shapes and the exponential decay of their areas.

MEASURING WITH SEGMENTS:[22]

- Comparing segmented distances that have a common ratio – A group of students lines up against a wall. The first takes one step, the second takes two, the third takes four, the next eight, etc. Distances traveled have a common ratio. Moreover, final positions trace an exponential curve.

LOGARITHMS:

- Mapping an arithmetic progression onto a geometric progression – The source domain LOGARITHM subsumes this representation, which was previously related to FUNCTION in Section 6.4.1.
- Orders of magnitude – Logarithmic scales for sound (dB), earthquakes (the Richter scale) and acidity (pH) represent sensations occurring over extremely wide ranges – from barely detectable to overwhelming.

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[22] Lakoff and Núñez (2000) use the phrase "THE USE OF A MEASURING STICK" (p. 68) to describe the source for the MEASURING STICK metaphor. However, I prefer the expression MEASURING WITH SEGMENTS, which seems to reflect the essence of the domain more clearly.

Participants spoke favorably about the experience of being involved in the concept study. Several stated they had enjoyed the discussion; others felt that similar experiences exploring other concepts would be beneficial. Comments included, “These activities are often seen as enrichment – why are they not core activities?” and “If only we could talk about every topic in math in this way!”.

Significantly, there appeared to have been a shift in participants’ conceptualizations of EXPONENTIATION. No one mentioned repeated multiplication or notation; instead, the image of the rapidly growing rate of increase associated with EXPONENTIATION predominated. When the question, “So, what’s it like to be an exponent?”, was posed, Yvette responded, “Powerful!”, tracing an exponential curve with her hand. Around the room, heads nodded.[23]

6.5 Interviews

This phase of the study involved a series of interviews from which I hoped to elicit more and different representations for EXPONENTIATION. A separate meeting was held with each participant of the concept study; these discussions focused not on general thoughts about the concept, but on teaching and learning in the classroom. I also talked one-on-one to five mathematicians and seven

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23 This play on words – using ‘power’ to refer simultaneously to exponents and to the ability to produce a great effect – occurred repeatedly throughout the study.

mathematics educators of varied backgrounds. It seemed likely that these individuals might reveal novel understandings of EXPONENTIATION.

I was not surprised when many conceptualizations that emerged during the concept study were mentioned in these interviews, although the specific images called on were often quite different. But new and insightful perceptions of EXPONENTIATION were found. Following the form of earlier parts of the chapter, discussion will focus on previously unseen representations and conceptual domains. As well, my own realizations about conceptions and metaphoric connections are described.

6.5.1 New Conceptualizations from Members of the Concept Study

Individual interviews were held with teachers from the concept study; these sessions focused on the teaching and learning of EXPONENTIATION. Discussions were centered around a series of questions, which included:

> How do you introduce students to exponents in your classes?
> What are typical problems experienced by students?
> Are there explanations you use to successfully overcome these?
> When talking to students, how do you make sense of 2^{-3}, $2^{1/2}$, or $2^{\sqrt{2}}$?
> What are the prerequisite skills needed to understand exponents?
> Which mathematical concepts require an understanding of exponents?
> Which types of applied problems call for knowledge of exponents?

Although I found that participants often referred to types of representations discussed in the concept study, it was clear that they had continued to reflect on these conceptualizations in the intervening period of time. Individuals elaborated on images and applications; some shared classroom activities that illustrated particular representations. And several new ways to construct understanding of EXPONENTIATION arose in conversations.

OBJECT CONSTRUCTION:

- Comparing objects that have a common ratio – Take a piece of paper and rip it in half; retain one part and tear the remaining piece in two; keep one of these quarters and divide the other in two, and so on. When the bits of paper are placed side by side, their lengths display a common ratio.

LOGARITHMS:

- Scientific notation – …, 2.3×10^{-1}, 2.3×10^{0}, 2.3×10^{1}, …

The only metaphor explicitly stated in the entire study arose in one of these interviews; I mentioned a comparison that I had used in teaching exponents to a grade 10 class. Many years ago it had seemed useful and appropriate, but with all that I had learned about EXPONENTIATION, its weaknesses became obvious.

This idiosyncratic metaphor likens the index of an exponential expression to the turnstile at the door of a library, where a counter records the number of people entering the room. I compared this to an exponential expression like 2^3, where the index 3 indicates the number of 2's in its expansion as 2 5 2 5 2. If four people enter the library (2^4) and a group of seven follow (2^7), then eleven individuals pass the turnstile (2^4 5 $2^7 = 2^{11}$). The similarity can even be stretched to division of exponential terms if one is subtracted from the counter for each person leaving the library. When five people enter and two leave, then three remain ($2^5 \div 2^2 = 2^3$).

In retrospect, it is clear that EXPONENTIATION IS A TURNSTILE is not a satisfactory comparison. While grounded in a situation that is familiar to students, it fails in another, more significant, way. As mentioned in Chapter 5, in a 'good' metaphor, the mathematics of the target domain is embedded in the source. But many properties of EXPONENTIATION can hardly be said to be intrinsic elements of the domain of turnstiles and counters. When thinking of EXPONENTIATION, the image of a turnstile does not leap to mind in the same way geometric shapes or family trees do. No natural correspondence between the domains exists here.

Comparisons of conceptual domains can be assessed, based on different types of structure mapped from source to target (Bowdle & Gentner, 2005; Gentner, 1983). The 'turnstile metaphor' focuses on a surface element of the domains involved – a component is discretely countable. Some inferential structure is also projected onto EXPONENTIATION, as interpretations of multiplication and division involving only whole number exponents are possible. But the comparison cannot provide sufficient modes of reasoning for the target domain as a whole; thinking of turnstiles does not provide any way to understand negative exponents, fractional exponents or exponential growth.

Using Gentner's (1983) classification of mappings, the 'turnstile' comparison can be described as a "literal similarity" (p. 161). As such, it does provide some degree of assistance in understanding the concept of EXPONENTIATION. However, as learners tend to rely on superficial similarities between domains and do not examine relational structures carefully, misconceptions may arise (Gholson et al., 1997; Ratterman, 1997). Hence, the representation has limited value in illuminating EXPONENTIATION.

The 'turnstile' image is not the only conceptualization arising in the study that is inadequate in some way. In Chapter 5, I pointed out that because a representation is often based on a source domain that differs inherently from the target, it inevitably has limitations and may lead to invalid inferences. Thinking of EXPONENTIATION as collections of objects restricts one to natural number exponents. Conceptualizations using geometric shapes cause learners to have difficulty imagining powers larger than three. Tracing the outline of an exponential curve in piles of coins uses discrete units; the continuous nature of an exponential function is obscured. Making sense of EXPONENTIATION in a variety of ways enables individuals to overcome the limitations of individual conceptualizations.

6.5.2 Mathematicians' Conceptualizations

The five mathematicians interviewed in this study differed in many ways. Two had extensive experience teaching EXPONENTIATION to secondary students, undergraduates, or teachers who had returned to university for upgrading, while others only instructed more advanced courses. One participant had a keen interest in the history of mathematics; another had a background in computing science. This diversity contributed to the range of representations voiced in interviews with these individuals. As expected, mathematicians had unique perspectives and novel representations of EXPONENTIATION were revealed.

OBJECT CONSTRUCTION:

- A power of $\frac{1}{n}$ means *breaking up* a number into *n* equal pieces.

MOTION ALONG A LINE:

- Lengths on a line – Reference was made to Descartes, who stated, "Here it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean only simple lines" (Latham & Smith, 1925, p. 5).[24]
- A length changing at a constant rate – A Chinese 'dragging noodle' can be repeatedly stretched so that it doubles in length each time.
- Comparing motions along a path that have a common ratio – When a ball is dropped, the heights it reaches on successive bounces form a geometric sequence.

FUNCTION:

- Derivative – The derivative of e^x is equal to the function itself.
- Limit – $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$
- Power series – The value of e can be approximated using the power series $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

[24] Descartes was responsible for making the final break between geometric shapes and algebraic exponents. With this new perspective, mathematicians could use powers larger than three without being concerned about their lack of geometric meaning (Katz, 1993).

- Recursion – Exponentiation can be viewed as a type of primitive recursion.

  $b^0 = 1$

  $b^{n+1} = b^n \times b, n \in N_0$

I observed several differences between perceptions of mathematicians and views expressed in other phases of the study. The connection between MOTION ALONG A LINE and EXPONENTIATION was pointed out more clearly by a variety of representations. And, most notably, mathematicians found the relationship between EXPONENTIATION and FUNCTION to be paramount. Concept study participants suggested links between the concepts, just as they associated many other domains. In contrast, all five mathematicians emphasized the primary importance of the connection; EXPONENTIATION was viewed as a crucial member of a "menagerie of functions". Indeed, some felt that this was the only conceptualization of any significance. As one mathematician stated, "When you talk about exponentiation, I don't think about the process of exponentiation, I think about function." Moreover, reflections shifted to consideration of a singular function – the exponential, $y = e^x$.

6.5.3 Mathematics Educators' Conceptualizations

The third set of interviews involved seven mathematics educators with different backgrounds. Some had been mathematics teachers, but were now pursuing other avenues in mathematics education. Two were graduate students in Education and others were experienced researchers. While many conceptualizations similar to those found in previous phases of the study were seen, novel representations for EXPONENTIATION surfaced in discussions.

MOTION ALONG A PATH:

- Number-line 'dancing' – Six students stand on a number line at points corresponding to 0, 1, 2, 3, 4, and 5. They do a 'plus 2' dance and return to their starting positions, then they perform a 'times 2' dance, a 'square' dance, and a '2 to the power *x*' dance.

OBJECT CONSTRUCTION:

- Asymptotic behavior – "I take a square, take half the square, take half of the half and this goes to zero".
- Embedded figures – Many embedded geometric figures illustrate exponential decay (see Figure 24).

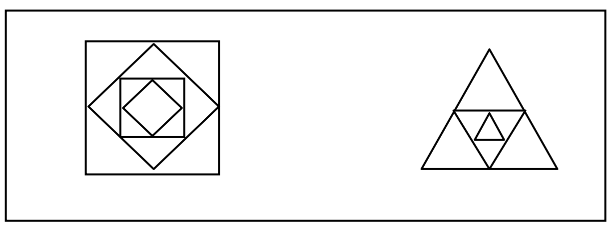

Figure 24: Embedded figures that represent EXPONENTIATION.

FUNCTION:

- Additive rate of change – Consider a table of values for an exponential function with columns for x and $y = 2^x$. A third column for differences in y, Δy, will contain numbers that are identical to those in the column for y, as will columns for Δ(Δy) and Δ(Δ(Δy)). In contrast, values change with every difference column for algebraic expression like $y = x$, $y = x^2$, and $y = x^3$, until eventually numbers in a column all equal zero.

6.6 Further Readings

Throughout all phases of research, I looked for conceptualizations of EXPONENTIATION in a variety of writings. Textbooks, other teaching materials, research studies, reports of classroom activities, and historical accounts were all examined. Not surprisingly, many representations mentioned in earlier sections of this chapter were also found in these sources. But I found much more – novel representations for source domains identified in other parts of the study, fresh insights into previously encountered conceptualizations, and two image schemas that provide patterns of reasoning for EXPONENTIATION.

6.6.1 New Representations for Already-Identified Domains

OBJECT COLLECTION:

- ‘Square’ numbers – Mesopotamian and Greek mathematicians considered a number to be “a multitude composed of units” (Heath, 1956, p. 277). Of special significance, figurative numbers were represented by items arranged in geometric shapes. The numbers 4 and 9 were considered ‘square numbers’ because they correspond respectively to a 2 5 2 and 3 5 3 array of pebbles, cuneiforms, or vertical lines (Cordrey, 1991). This representation is widespread; images of square numbers are found on Neolithic pottery (Struik, 1987) and in present day textbooks (e.g., Small et al., 2008).

OBJECT CONSTRUCTION:

- Objects that form a geometric series – Mason (1992) describes a problem: "The carrot patch is square. One spring day, you dig half of the bed. On each succeeding day, you dig half of what remains. How long until you finish?" (p. 67). The geometric series $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-...$ is indicated (see Figure 25).

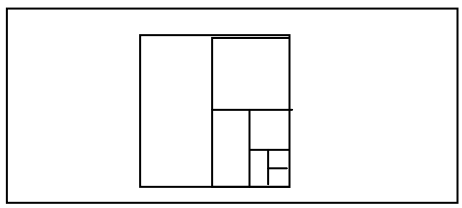

Figure 25: Objects that form a geometric series.
(Printed with permission from Mason, 1992, p. 67).

MEASURING WITH SEGMENTS:

- Segments that form a geometric series – Rinvold (2007) uses segments to justify the statement $1 + 2^1 + 2^2 + 2^3 = 2^4 - 1$. (See Figure 26).

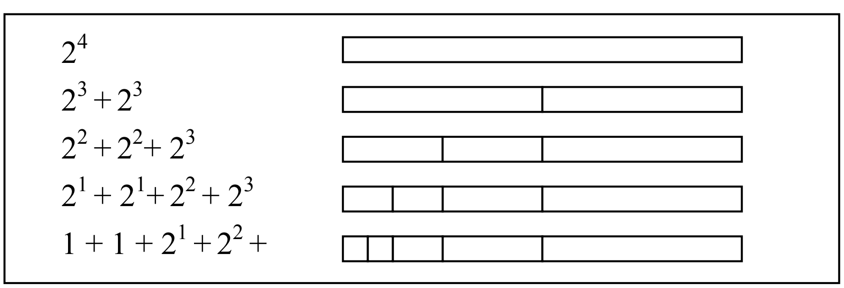

Figure 26: Segments that form a geometric series.

- A segmented length changing at a constant rate – Thompson (1992) illustrates powers of two by placing segments end to end. The length of the resulting shapes grows at a constant rate (see Figure 27).

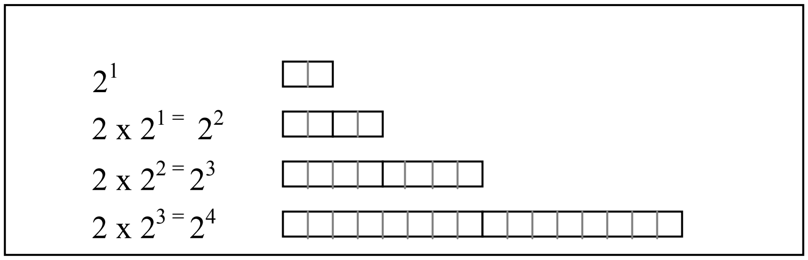

Figure 27: Powers of two conceived by putting segments together.

FUNCTION:

- New-to-old ratio – Confrey and Smith (1994) discuss the "new-to-old" ratio of an exponential function (p. 55). An examination of the bar graph representing natural number powers of a given base (see Figure 28) shows that, for each bar, the original function and the added increment are proportional.

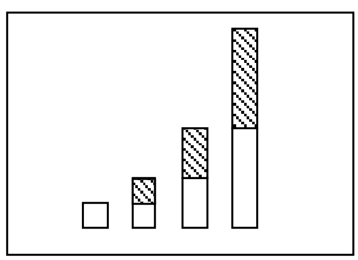

Figure 28: Comparing the new-to-old ratio of $y = 2^x$, for $x = 1, 2, 3$.

6.6.2 Returning to Repeated Multiplication

In early readings and during the first meeting of concept study teachers, repeated multiplication was introduced as a representation of EXPONENTIATION.

Following the example of Lakoff and Núñez (2000), I interpreted this conceptualization as indicating a link to OBJECT CONSTRUCTION. “When we write 32 [or 2^5] as 2 5 2 5 2 5 2 5 2, we are implicitly making use of the grounding metaphor ARITHMETIC IS OBJECT CONSTRUCTION; that is, a whole that is made up of five instances of the same part, 2, put together by multiplication” (Lakoff & Núñez, 2000, p. 404). But this statement does not appear to be entirely consistent with other sections of their work where the foundations of ARITHMETIC are developed.

Just as Lakoff and Núñez describe repeated multiplication as a manifestation of OBJECT CONSTRUCTION, one would expect to find repeated addition regarded in the same way. And this stance is found as part of the ARITHMETIC IS OBJECT CONSTRUCTION metaphor; multiplication is defined as “the repeated addition (A times) of A parts of size B to yield a whole object of size C” (Lakoff & Núñez, 2000, p. 66).

But similar definitions are found in isomorphisms for ARITHMETIC IS OBJECT COLLECTION, the MEASURING STICK metaphor and ARITHMETIC IS MOTION ALONG A PATH (Lakoff & Núñez, 2000). Statements refer to the repeated addition of collections, segments, and movements away from the origin. If repeated addition is connected to all four image schemas upon which the grounding metaphors for ARITHMETIC are based, then it would seem plausible that repeated multiplication also draws inferential structure from each of these domains. And looking back, repeated multiplication can be seen to lie beneath representations associated with these schemas.

OBJECT COLLECTION:

- Repeated multiplication[25] – The face of a combination lock shows thirty-nine numbers; the dial must point to three of these in the right order. The number of possible combination for the lock is 39 5 39 5 39 = 39^3.

OBJECT CONSTRUCTION:

- Repeated multiplication – To calculate the volume of a cube with sides 4 cm in length, one multiplies 4 cm 5 4 cm 5 4 cm = 64 cm^3. A new unit is created.

MEASURING WITH SEGMENTS:

- Repeated multiplication – On a trail, a woman finds that she is three paces from a skunk. She cautiously moves so that the animal is six paces away, but she can still catch a whiff of the skunk's odor. So she steps back until she is twelve paces away and sniffs the air, etc. Figure 29 uses segments to illustrate the pattern of movement.

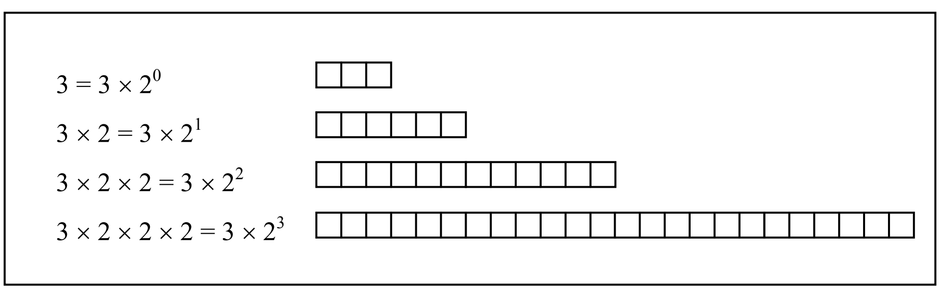

Figure 29: Repeated doubling of segmented lengths.

[25] This example was originally presented as a representation of OBJECT CONSTRUCTION (see Section 6.4.1). But in light of the above discussion, it appears to fit better with OBJECT COLLECTION, as arrangements of three numbers chosen from a set of thirty-nine numbers (with repetition allowed) are being considered.

MOTION ALONG A PATH:

- Repeated multiplication – A ‘Twizzler’ gets shorter when one boy devours half, passing what is left to a friend who consumes half, giving what remains to another boy who eats half and so on. The original length, L, shrinks and becomes $L \times \frac{1}{2} = L \times \left(\frac{1}{2}\right)^1$, $L \times \frac{1}{2} \times \frac{1}{2} = L \times \left(\frac{1}{2}\right)^2$, $L \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = L \times \left(\frac{1}{2}\right)^3$, … .

6.6.3 Reexamining Branching Structures

In the concept study, Ardis mentioned natural branching structures like bronchioles, tree roots and drainage systems (see Section 6.4.1). Such forms closely resemble fractal trees (see Figure 30), in which both the number of branches and the lengths of the branches vary exponentially. These structures and many other fractals, like the Koch snowflake or the Sierpinski triangle, offer conceptualizations for EXPONENTIATION on several levels.

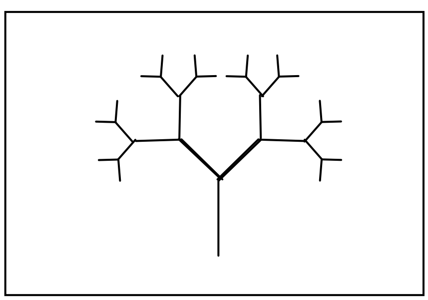

Figure 30: A fractal tree.

However, natural structures like bronchioles do not possess certain key properties that define fractals (Peitgen, Jürgens & Saupe, 1991). They lack the attribute of self-similarity – the exact repetition of every detail at all degrees of magnification. Yet, such forms do exhibit 'statistical' self-similarity over a limited range of levels (Jelinek, Jones & Warfel, 1998). Thus, their fractal or Hausdorff dimension can be estimated and the calculation of this descriptor does involve exponents (Mandelbrot, 1967). While natural forms are certainly connected to EXPONENTIATION, they are not, strictly speaking, true representations of the concept.

6.6.4 Image Schemas Linked to EXPONENTIATION

It was not until I related conceptualizations located in the many texts and accounts I read to representations found in the concept study and interviews that I became aware of two image schemas from which inferential structures could be mapped onto EXPONENTIATION. These organizing patterns – ITERATION and SPLITTING – seemed to permeate representations associated with other conceptual domains. This should not have surprised me, for image schemas are generalized patterns of thought distilled from many bodily and perceptual experiences.

Conceptualizations for EXPONENTIATION often involve recurrent actions such as continually doubling the number of coins in a pile or repeatedly folding a piece of paper in half. Everyday experiences like these form the basis for the image schema of ITERATION (Johnson, 1987; Lakoff, 1987; Lakoff & Núñez, 2000). Inferential structure from this cognitive domain clearly plays an essential role in the process of EXPONENTIATION.

I was directed to a second image schema during an interview, when one mathematics educator suggested that an interesting perspective on

EXPONENTIATION might be found in Confrey's (1994) notion of 'splitting', which is defined as "… an action of creating equal parts or copies of an original" (p. 300). This conception is extended to refer to a multitude of representations, such as doubling, halving, folding, similarity, magnifying, sharing, dividing symmetrically, embedded figures, and growth spirals (Confrey, 1994; Confrey & Smith, 1994; Confrey & Smith, 1995). Such diverse situations are seen as providing important ways of developing division, multiplication and EXPONENTIATION (Confrey & Smith, 1995).

As a class of actions, splitting uses inferential structure from not one, but many conceptual domains. Halving a pile of candies involves OBJECT COLLECTION, while sharing a cookie entails OBJECT CONSTRUCTION. Similarly, magnification might refer to MOTION ALONG A PATH (see Figure 31), whereas growth spirals (see Figure 32) call on the MEASURING STICK metaphor. The use of one expression, 'splitting', to describe all of these conceptualizations tends to obscure the varied nature of these one-to-many correspondences.

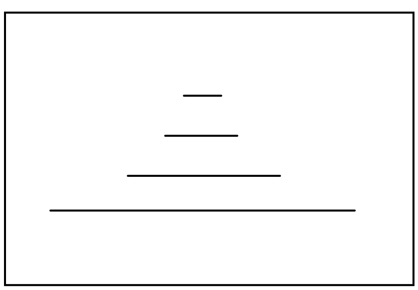

Figure 31: Splitting shown in magnification.

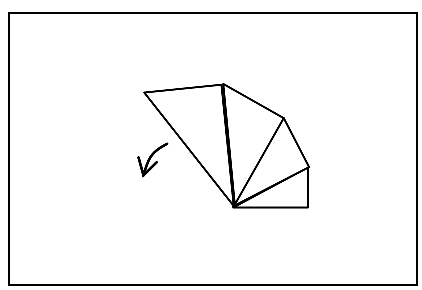

Figure 32: Splitting shown in a growth spiral.

Moreover, use of the generic term 'splitting' is problematic because it refers to two quite different types of actions. The most primitive conception of splitting is copying or "creating simultaneously multiple versions of an original" (Confrey & Smith, 1994, p. 146). A cell repeatedly multiplies to become two, four, eight, sixteen cells and so on. A chain letter starts with one person, then grows to five friends, twenty-five, one hundred twenty-five, etc. This type of multiplicative formation, previously referred to as a 'branching structure' (see Section 6.4.1), is perhaps more appropriately described by the word "replication" (Confrey & Smith, 1994, p. 148), which captures the image of populations growing in size, and highlights its relationship to OBJECT COLLECTION.

But 'splitting' also refers to situations where objects are not copied but divided up; these make use of modes of perception and reasoning from the image schema of SPLITTING. Cognitive scientists envisage SPLITTING as the inferential structure involved in activities such as breaking up a set, object or segment into a number of equal parts (Johnson, 1987; Lakoff & Núñez, 2000). More recently, Peña (2008) finds this the same meaning in language use: "The SPLITTING schema … is described as a notion underlying expressions which convey the idea of a whole separated into several parts" (p. 1062).

Using this conception, new representations for EXPONENTIATION that derive from SPLITTING become apparent. These are characterized by a 'sequence of splits', where a unit is divided into *n* smaller parts, each of which is cut into *n* pieces, and so on (Confrey, 1994). The repetition of action indicates that a conceptual blend with ITERATION is formed; the resulting domain is itself often combined with other image schemas, as is illustrated in the following examples.

OBJECT COLLECTION:

- A sequence of splits – A pile of dollars is divided in two, the resulting stacks of bills are halved, and so on.

OBJECT CONSTRUCTION:

- A sequence of splits – A child wants to share with friends, so he or she cuts a pie in half, splits the halves in two, and then divides the quarters into equal pieces (Confrey & Smith, 1994).

MEASURING WITH SEGMENTS:

- A sequence of splits – One mathematics educator described activities using old-fashioned continuous computer paper for dot matrix printers. Suppose that one pile of paper contains sixty-four sheets, and that this long strip is carefully separated into two equal sections. Each of these parts is split in half; the resulting pieces are themselves divided in two, and so on.

MOTION ALONG A LINE:

- A sequence of splits – To build a picket fence, you might carefully measure where planks and the spaces between them are to go. Or you could nail a board up at the midpoint of the span and place planks in the centers of the two spaces on either side of it. Pickets could then be attached in the middle of each of the four sections, etc. (Confrey, 1994).

6.7 Discussion of Results

My search for links to EXPONENTIATION was far more productive than anticipated; numerous representations arose from the concept study, individual interviews, and readings. From these clues, a number of conceptual domains to which EXPONENTIATION can be connected were identified. While previous sections of this chapter describe findings for each phase of the study, the sheer volume and variety of data make it difficult to get an overall picture of the complex organization implied. When everything is drawn together (see Table 2), different levels of understanding are possible. For the display not only *represents* findings from all phases of the study, it *re-presents* them so that new interpretive possibilities are enabled.

As expected, source domains for the four grounding metaphors of ARITHMETIC – OBJECT COLLECTION, OBJECT CONSTRUCTION, MEASURING WITH SEGMENTS, and MOTION ALONG A PATH – can be connected to EXPONENTIATION. Furthermore, striking similarities in conceptualizations from these domains exist. Five representations – 'repeated multiplication', ' a sequence of splits', 'a unit changing at a constant rate', 'comparing units that have a common ratio', and 'units that form a geometric series' – draw on inferential structure from each of these sources in only slightly varied forms. As definitions for MULTIPLICATION based on the grounding metaphors of Lakoff and Núñez (2000) show parallel structure and content, it should not be surprising to see comparable likenesses in representations for EXPONENTIATION.

Table 2

Representations linking EXPONENTIATION to different source domains

OBJECT COLLECTION	OBJECT CONSTRUCTION	MEASURING WITH SEGMENTS	MOTION ALONG A PATH	ROTATION	FUNCTION	LOGARITHM
Repeated multiplication	Repeated multiplication	Repeated multiplication	Repeated multiplication			
A sequence of splits$_S$	A sequence of splits$_S$	A sequence of splits$_S$	A sequence of splits$_S$			
A set changing at a constant rate	An object changing at a constant rate	A segmented length changing at a constant rate	A length changing at a constant rate			
Comparing sets that have a common ratio	Comparing objects that have a common ratio	Comparing segmented distances that have a common ratio	Comparing motions along a path that have a common ratio			
Sets that form a geometric series	Objects that form a geometric series	Segments that form a geometric series	Motions along a path that form a geometric series			
Branching structures (replication)	Constructing sequences of shapes		Lengths on a line	Rotating in the complex plane	Recursion	Mapping an arithmetic progression onto a geometric progression
			Number-line dancing		Power series	
Square numbers	Fractals				Limit	
	Embedded figures$_S$				Additive rate of change	Orders of magnitude
	Sequential folding$_S$					
	Creating new units$_S$				Rate of change of slope	Scientific notation
	Positional number systems				Derivative	
	A power of 1/n means breaking up a number into n pieces$_S$				New-to-old ratio	
	Geometric shapes				An exponential curve	
	Changing dimensions					
	Asymptotic behavior					

Representations involving ITERATION are located above the dashed line.

SPLITTING is not just found in "sequences of splits", but is also part of other representations in the table. Some of these are signified by the subscript, S.

A number of other patterns became apparent as I examined the chart. For example, the key role that ITERATION plays in many representations of EXPONENTIATION is highlighted as the image schema spreads in a wide swath across Table 2. While the importance of recursion is not immediately obvious when thinking of ‘geometric shapes’ or ‘square numbers’, in other images like ‘branching structures’ or ‘fractals’ ITERATION is clearly observable. The shift in mode of reasoning required, when moving from conceptualizations that do not require iterative thinking to those that do, may explain some of the difficulties a student experiences in trying to construct an understanding of EXPONENTIATION.

Other schemas can be seen throughout the table. Consider SPLITTING, which is found, not just in ‘sequences of splits’, but also in blends revealed by other representations. The definition ‘a power of $\frac{1}{n}$ means breaking up a number into n pieces’ refers to the combination of SPLITTING and OBJECT CONSTRUCTION, as do ‘sequential folding’ of pieces of paper and ‘creating new units’ of smaller size in the decimal system. It is likely that other themes recur in many scattered conceptualizations.

It is also evident that representations connected to a particular source domain are not completely distinct. Various conceptualizations may be related, with only subtle differences distinguishing them. For example, several representations linked to FUNCTION – ‘rate of change of slope’, ‘derivative’, ‘additive rate of change’ and ‘new-to-old ratio’ – address the same aspect of EXPONENTIATION using different tools and vocabulary.

Table 2 proved to be of great assistance in organizing and summarizing data from the study. Moreover, assembling the chart caused me to reexamine and reflect on relationships among the representations and conceptual domains identified there. But as I explored the information contained there, I began to question the appropriateness of this particular format for displaying these findings. The work in this chapter and throughout the dissertation is based on the supposition that conceptual domains and the connections among them form a complex network. Thus, a network structure would be a more suitable mode for presenting understandings arrived at in this research.

Generating such a portrayal is not without difficulties. Each conceptual domain discussed in this chapter has its own internal structure representing associated language, experiences, and modes of reasoning. Producing a figure that depicts these complex subnetworks and shows the inferential structure from each that is projected onto EXPONENTIATION is far beyond the scope of this study. But part of the network structure can be illustrated; a number of domains that can act as sources for EXPONENTIATION have been identified, as have some of the connections among them. Figure 33, while it cannot be complete, is an attempt to portray a small part of the metaphoric network of mathematics centered on the concept of EXPONENTIATION.

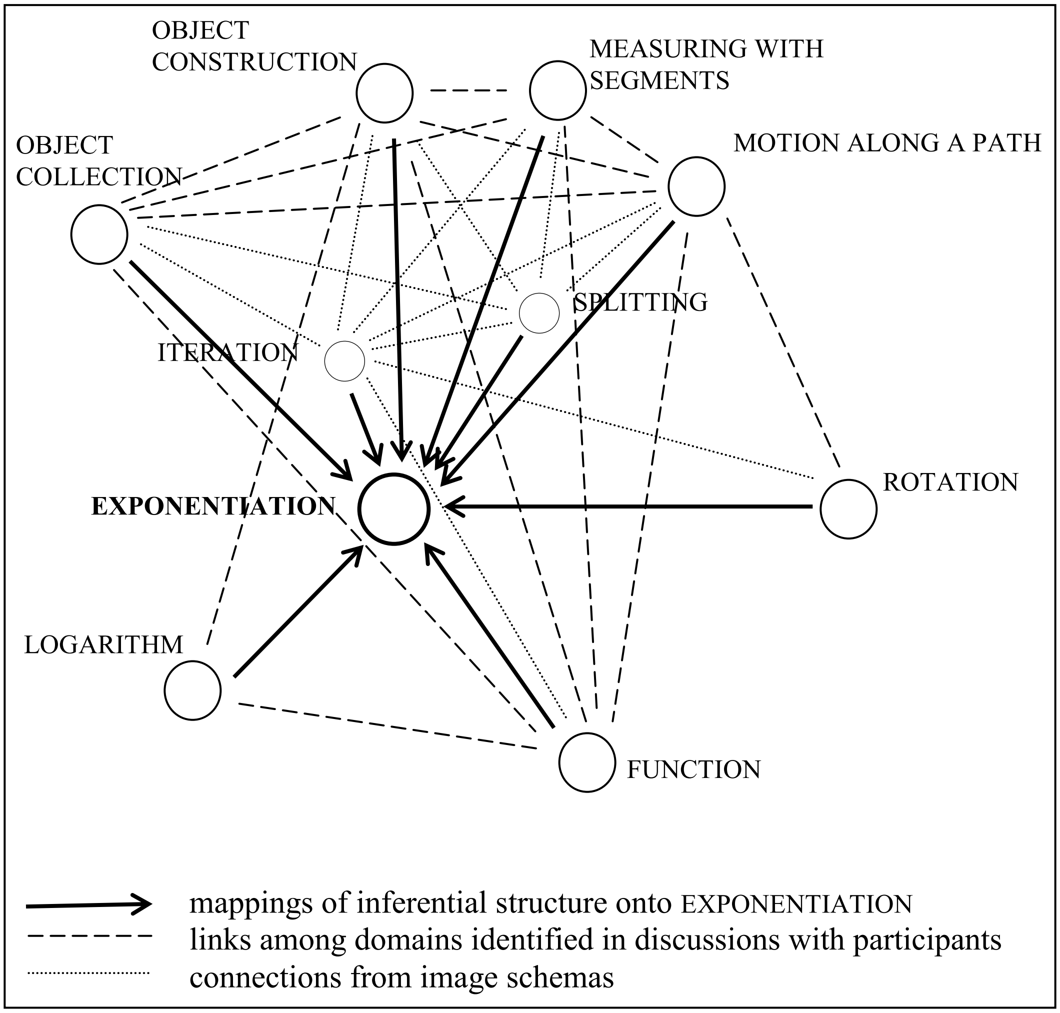

Figure 33: Part of the metaphoric network of mathematics around EXPONENTIATION.

6.8 Summary

In this chapter, I reported on my search for different representations for EXPONENTIATION in a collaborative discussion, individual interviews and writings. The views of mathematics teachers, mathematics educators, and mathematicians were sought with the aim of demonstrating that 'metaphorical legs' for EXPONENTIATION can be identified. The process was not easy; conceptualizations offered only "scattered cues and sustained innuendo"

(Kimmel, 2002, p. 518) and I found it difficult sometimes to look past them and see underlying source domains.

Moreover, inferences were mutable, meanings slippery. Phases of the study formed a series of recursive elaborations. For each comment I heard, each image I saw, each realization I came to at one point in time not only influenced what I perceived in the next, but forced me to reexamine and reanalyze all that I had previously become aware of. It is likely that such ever-evolving understandings are inevitable when exploring a complex system.

Despite these difficulties, connections between EXPONENTIATION and a number of other mathematical concepts emerged from the work. Participants, both individually and collectively, displayed an extraordinary understanding of EXPONENTIATION. Their perceptions, supplemented by what I learned from academic and pedagogical texts, yielded many representations that shed light on different aspects of the concept. Examination of the physical and mathematical roots of these conceptualizations led to the identification of sources from which inferential structure can be projected onto the target of EXPONENTIATION.

While this research yielded much useful information, it should not be viewed as providing a definitive collection of representations and source domains. Further reflection may lead to new interpretations of ideas presented here. And future studies will undoubtedly find conceptualizations that reveal novel ways to link EXPONENTIATION to domains identified in this chapter, or that connect it to distinct and unexpected concepts.

Although this work was successful in locating sources that contribute to an understanding of EXPONENTIATION, I find that I am unable to set down conceptual metaphors stating, "EXPONENTIATION IS …". When developing grounding metaphors for ARITHMETIC, Lakoff and Núñez (2000) found four image schemas that had exactly the right inferential structure to not only "fit and extend what is known about innate arithmetic" (p. 101), but to account for "*all* [italics added] the properties and computational inferences about the mathematical target" (p. 101). Even OBJECT COLLECTION, the most basic and most intuitive of these sources, provides a means for conceptualizing the immense and intricate domain of ARITHMETIC with natural numbers, although simple blends are required to account for groups of collections and zero.

I cannot find this degree of fusion between EXPONENTIATION and any of the conceptual domains discussed in this research. To say, for example, that EXPONENTIATION IS OBJECT CONSTRUCTION is to deny the complexity of the concept. One domain cannot provide all of the modes of reasoning necessary for understanding the many varied aspects of EXPONENTIATION; an elaborate mix of ideas from both concrete and abstract sources is required. Thus, I find it necessary to think of EXPONENTIATION as a conceptual blend, one of such involvedness that I cannot trace the ways in which notions combine and evolve, as Lakoff and Núñez (2000) have done for ARITHMETIC.

But while a list of metaphors for EXPONENTIATION is not compiled here, the existence and characteristics of its 'metaphorical legs' are indicated. Examination of conceptualizations explicit in the words and implicit in the images of mathematics teachers, educators, mathematicians, and researchers who have written about the concept was very productive. It is hoped that the findings of this study, this illustrative example of network analysis, will provide valuable assistance for mathematics educators in designing classroom instruction, in developing curricula, in planning programs of professional development, and in crafting further inquiries into the metaphoric structure of mathematical understanding.

Chapter 7

Re-viewing Ideas

I began this dissertation by noting that, throughout the history of schooling, beliefs regarding the nature of mathematical knowledge have shaped pedagogy. As a student, I experienced many practices that reflect different views of mathematics. These varied approaches helped me to extend my understanding and to develop a love for the discipline, but none emphasized the many relationships among mathematical concepts that I found both fascinating and empowering. Nor, as a mathematics teacher, was I satisfied with my efforts to highlight these connections in the classroom. I was not aware of any effective ways of helping students to overcome what seemed to be a natural tendency to divide mathematical knowledge into discrete sections.

It was not until I encountered the study of complex systems that a new conception of knowledge domains came to my attention – a perspective from which interactions among elements of a field are paramount. As I learned more, I began to wonder whether mathematics is a complex form and, if so, what this might mean for teaching and learning. Was it possible that such a view might shape techniques to help me encourage students' appreciation of the interconnectedness of mathematical ideas? Would I find it easier to understand learners' difficulties in making sense of mathematics? Could methods be developed to enhance students' understandings of mathematics? What changes in mathematics education might be called for? These intriguing questions motivated my explorations of ideas as described in this work.

7.1 A Brief Summary

In earlier chapters, I posited that, since mathematics exhibits properties typical of complex forms, it is appropriate to explore mathematical understanding using a network model. I further suggested that the theory of embodied mathematics provides a possible frame for such a structure, where conceptual domains represent nodes in the web and conceptual metaphors provide the links among them. Examination of this metaphoric network of mathematics provided evidence that it displays the scale-free topology typical of complex systems.

I proposed that closer examination of this network of concepts offers fresh insights into the dynamics of subjective mathematics. In particular, if comprehension of a central concept deteriorates, a debilitating cascading failure may lead to the collapse of many other topics. As constructing additional links among domains increases the robustness of mathematical understanding, I suggested that it is desirable to utilize a variety of metaphors in making sense of each concept. Implications for classroom teaching, curriculum, professional development, and research activities were discussed.

Following the proposal that further investigation of the metaphoric network is warranted, and with the aim of substantiating ideas presented in this work, I explored collective and individual understandings of EXPONENTIATION. A rich assortment of representations emerged, indicating various conceptual domains that can be linked to this concept; patterns of connections in a small region of the metaphoric network of mathematics became clearer.

7.2 Reflections on Complexity

Looking back, I cannot help but ponder the nature of complexity-based research. It is not possible to come to an accurate and complete understanding of subjective mathematics or, indeed, of any complex system. Models used to explore these forms must, by their very nature, be less complex than the systems themselves, for in order to understand the truly complex it is necessary to leave things out, to make patterns easier to understand (Cilliers, 1998). In addition, while a complex system does not have well-defined boundaries and often overlaps, is nested within and subsumes other forms, the choice of a model unavoidably sets boundaries that separate the system from the many other areas with which it interacts (Osberg & Biesta, 2003). Thus, any description will inevitably fail to depict truly the dynamic properties and behaviors of a complex system.

Moreover, a complex form continuously evolves due to the nature of interactions among its components. Characteristics of the system emerge from myriads of interactions; these do not stop when a researcher completes her work, but continue on in an increasingly involved manner. The result of any analysis must, therefore, be thought of as merely a ‘snapshot’, a moment’s insight into an ever-changing, ever-growing form.

The very act of trying to capture an elusive truth changes that reality, for the investigations used to explore a phenomenon influence its structure and behavior (Proulx, 2008). Interpretations cannot be described as objective, for "when we deal with complexity, we cannot avoid framing our descriptions thereof in some way or another" (Cilliers, 2000, p. 43). Thus, the knower shapes the known, regardless of any efforts made to avoid personal predispositions and biases. And, in turn, the knower is shaped by the known. Observations and descriptions of a phenomenon alter how and what the researcher observes and describes (Proulx, 2008). Thus, through continuous cycles of examination, interpretation, and creation of new possibilities for the complex system, the researcher and the researched co-determine one another.

What then can be said of the work presented here? Both my exploration of the larger structure of subjective mathematics and my search for representations that contribute to understandings of EXPONENTIATION are inevitably constrained. Neither can be acclaimed as truly correct or complete, and neither can be viewed as rising from assumption-free and objective observations. But, considering Maturana's (1988) description of the many "different, equally legitimate … explanatory realities" (p. 31) that can be observed, the metaphoric network of mathematics may be taken as a valid construal of the structure of mathematical understanding and connections to EXPONENTIATION revealed in Chapter 6 may be acknowledged as part of the cognitive structure of that concept. As such, I hope that my attempts to develop the beginnings of a scientific explanation – indeed, since I am using insights from network theory, almost a mathematical explanation – for subjective understanding of mathematics are found to be both informative and useful.

7.3 New Perspectives

Developing a new finding or conceptualization through research is a complex process. Novel concepts and theories emerge from the interactions of disciplines and discourses. Just as these frames shape the constructs of research, they are in turn themselves affected. Consequent reinterpretations of these areas lead to still deeper understandings of a study's insights, and so on. Another cycle – where contributing thoughts and emerging realizations recursively influence each other – directs research activities. Thus, just as an inquiry should acknowledge what it has heard from various fields, "it has a concomitant responsibility to in some manner 'reply' to the domains to which it listens" (Davis & Sumara, 2006, p. 165).

My interpretation of mathematical understanding as a network of metaphors is influenced by interactions among ideas from complexity science, network theory, and embodied mathematics. And, as developed above, these fields of thought should, in some respect, be reciprocally acted upon by the ideas presented in earlier chapters. Ways in which both the development of the metaphoric network of mathematics and the subsequent exploration of EXPONENTIATION contribute to these ideational systems are discussed below.

Of the three elements, complexity science plays perhaps the largest role in this dissertation for it not only constitutes one of the main arguments in support of my thesis, but it forms the perspective and shapes the approaches I use to investigate both mathematics as a network structure and EXPONENTIATION as the target of metaphoric projections. It is too much to expect that one study can greatly affect such a major domain of ideas, but those parts of this work providing evidence that mathematics and mathematical understanding are complex systems do make a contribution to the field. For while researchers have shown that many physical and social systems – even language – display characteristics of complex forms, similar work for mathematical cognition has not previously been carried out.

I also draw heavily on the insights and techniques of network theory in exploring the structure of mathematics. Because of the particular nature of mathematical understanding, or indeed cognition in any discipline, I believe that my work offers a new way to explore and analyze network structures.

While the first studies using network theory tended to focus on simple (often random) mathematical models, network theorists soon began to focus on real-world networks and consider empirical details as well (Newman, Barabási & Watts, 2006). In particular, researchers began to question which kinds of nodes possess the largest number of links and how much influence these nodes have on the network as a whole. Statistical studies of data sets and the development of computer-based models representing network dynamics became commonly used modes of inquiry.

These types of studies are not, at present, possible for investigating metaphoric connections in mathematics. While quantitative analysis of mathematical links could potentially be made on the basis of whether two concepts are referred to in the same sentence or text, this type of research would not reveal underlying metaphoric relationships. Detecting these connections requires looking beyond surface details like proximity and being sensitive to underlying conceptual meanings. Qualitative methodologies, like the concept study and personal interviews employed in this dissertation to explore EXPONENTIATION, are more appropriate for obtaining these types of insights.

Research described in this dissertation also adds to understandings of embodied mathematics. A new way of exploring the field is developed; by looking at the network beneath the theory – with specific details removed from the system – patterns in the interactions of conceptual domains and conceptual metaphors become more apparent. As well, the search for metaphors that can be used to make sense of EXPONENTIATION reveals at least part of the underlying cognitive structure of that topic. While “no amount of information at the level of the individual … agent can hope to reveal the patterns of organization that make the collective function as it does” (Buchanan, 2002, p. 15), understanding some of the metaphoric connections to even one concept does shed light on the nature of links among domains in general.

But the concept study exploring EXPONENTIATION contributes more; it demonstrates that the search for connections among concepts is not exclusively the work of the cognitive scientist, but a task for educators as well. Teachers possess powerful understandings of mathematical ideas, as revealed in the images, activities and analogies they use – consciously or unconsciously – with learners in the classroom. These representations reflect, not only teachers’ knowledge of mathematics, but also their knowledge of how learners’ understandings develop.

With this special perspective, educators have an important role to play in the search for conceptual structures.

7.4 What Emerges Next?

Complex systems are never at rest, but continually change and grow. And thus it is with knowledge; when it appears that puzzles are solved, queries are answered and conjectures are confirmed, new questions inevitably arise. From the work in this dissertation, insights and instruments from complexity science, embodied mathematics and network theory widen the universe of possibilities for mathematics education. A multitude of possible avenues for future research suddenly appear.

It is impossible to write down all the ideas that leap to mind; some investigations are not just desirable, but necessary for the implementation of changes to mathematics education proposed in this work. The search for metaphoric connections among mathematical concepts could fully occupy researchers' time and energy. Efforts to focus schooling on building connections among concepts would be facilitated by studies that attempt to identify and evaluate different methods of introducing, discussing and utilizing metaphors in instruction. Along with this, explorations of ways to help individuals become aware of and articulate their tacit metaphoric understandings of mathematical concepts would of great value to teachers in the classroom and in pre-service programs. Students' responses to the development of concepts using multiple metaphors could be explored to determine if attitudes toward mathematics become more positive. Researchers could investigate whether understandings of mathematics become more robust. The opportunities seem endless.

An area that particularly interests me is exploration of teachers' reactions to views of mathematics and pedagogy suggested in this dissertation. While I did not systematically study participants' feelings on these matters during my research into EXPONENTIATION, a variety of opinions were expressed about the desirability of developing concepts from a variety of metaphoric connections. One teacher stated that using rules from the textbook was a much more efficient mode of instruction – that focusing on metaphoric links would be time-consuming, require a great deal of work, and be less effective in preparing students for examinations. Others were more positive. One participant declared:

> You would do well to get students comfortable with switching back and forth among different metaphors, and feeling that sense of power that I can choose the one that works best for this context. … So I think what we're talking about is more than just introducing two ways to look at things. It's changing the way you think about things. And part of that is realizing there are many ways of looking at things, and exploring the connections, and getting that sense of ownership that I can use the one that works for me. … I'm not sure why I'm convinced of this, but I'm convinced kids start that way. And then they're put into rows and lined up and taught a recipe and they lose that.

Such remarks were encouraging and gave me the sense that educators might appreciate the value of using metaphors and multiple connections in the classroom. But, although the temptation is great, I cannot generalize from a limited number of teacher comments to educators in general. Further research in this area is needed if proposed changes for mathematics education are to be implemented.

Developing this suggestion, researchers might also investigate ways of encouraging teachers to accept the use of metaphoric connections in developing mathematical understandings. I have seen how colleagues can quietly and effectively resist when they feel that a particular change mandated for instructional strategies or a program of studies is not desirable. Preventing unfavorable outlooks from forming and overcoming negative attitudes that arise is necessary, for teachers control classroom activities and, ultimately, classroom curriculum.

Concept studies provide one means of modifying educators' opinions, for such group discussions can influence participants' ways of thinking about mathematics (Davis, 2008) and their readiness to adopt the results of research projects (Davis, 2009). By recasting teachers as active partners in the production of knowledge rather than passive recipients of academic findings, a concept study openly values educators' particular knowledge of mathematics and the learning of mathematics. The development of other methods of inquiry that involve educators in collaborative research should increase the likelihood that the recommendations of researchers actually impact pedagogy.

7.5 Making Connections

Throughout my experiences as a student and a teacher, I have been fascinated by the relationships among ideas in mathematics. My work developing and investigating the metaphoric network of mathematics, and subsequently exploring EXPONENTIATION using this frame, was both challenging and exciting.

Long-held intuitions regarding the importance of making connections among concepts were confirmed.

As I developed new perspectives on mathematics and on the learning of mathematics, questions that had intrigued me were answered. Viewing mathematics as a complex network did explain some of students' difficulties learning mathematics. A teaching strategy – making sense of concepts by linking them to a variety of physical and social experiences, and other mathematical ideas – was developed that could help students construct more complete and robust understandings. Changes in curriculum and professional development would support this approach.

Moreover, my studies have opened new areas to investigate. What are the source domains that contribute to understandings of other mathematical concepts? How can the construction of metaphoric links be facilitated? Which methods of instruction are most successful? Will teacher and student reactions be positive? Can opinions be changed? It seems that more puzzles are posed than solved by my proposals.

In the end, I find that my efforts to apply the insights of network theory to mathematical understanding gives me more than solutions and new problems to explore. I have a much keener awareness of complex patterns of interactions in mathematics and in the world that surrounds us. So now, when I hear reports describing massive blackouts caused by the destruction of a power plant in a hurricane, I mutter, "It's a hub". When viewing programs that show images of the effects of global warming, I exclaim, "It's a cascading failure". And when I reflect on teaching mathematics, imagining how I might present material to students, I think, "It's a network".

Just as the network analysis of mathematics put forth in this dissertation has changed my perceptions, it is my hope that the teaching of mathematics and, more generally, mathematics education may also be affected. And perhaps ripples may spread even farther, for I suspect that the significance of network theory extends into the pedagogy of other disciplinary fields. It may also assist in understanding the dynamics of other sorts of systems involved in schooling, such as classroom collectives and programs of study. In other words, my speculations may be only a beginning for an important complex conversation in education.

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