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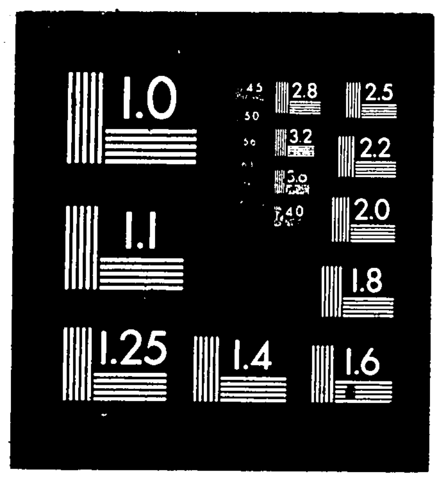

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NAME OF AUTHOR........RAMACHANDRA GOPALAN........

TITLE OF THESIS....SATURATION AND INVERSE PROBLEMS OF APPROXIMATION PROCESSES ON BANACH SPACES OF DISTRIBUTIONS

UNIVERSITY........University of ALBERTA........

DEGREE FOR WHICH THESIS WAS PRESENTED.....Ph.D..........

YEAR THIS DEGREE GRANTED.........1974..................

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THE UNIVERSITY OF ALBERTA

SATURATION AND INVERSE PROBLEMS OF APPROXIMATION PROCESSES ON BANACH SPACES OF DISTRIBUTIONS

by

RAMACHANDRA GOPALAN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1974

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Saturation and Inverse Problems of Approximation Processes on Banach Spaces of Distributions" submitted by Ramachandra Gopalan in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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Supervisor

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Date ..September 30, 1974..........

This Thesis is Respectfully Dedicated to

KAVIYOGI SHUDDHANANDA BHARATIAR

and to

MY PARENTS

ABSTRACT

A sequence of bounded linear operators $\{\Gamma_n\}_{n \in N}$ $(N = \{0,1,2,3 \ldots\})$ on a Banach space X is called an approximation process on X, if $\Gamma_n f \to f$ in X for all $f \in X$. A process $\{\Gamma_n\}$ is said to be of multiplier type if each Γ_n is defined by means of its fourier coefficients in X with respect to a biorthogonal system in X; of convolution type if $\{\Gamma_n\}$ is based on convolution with a sequence of L^1-kernels.

For suitable subspaces Y, A of X, (A being fixed, with $\dim(A) < \infty$) and function $\phi(n) \to 0$ on N, an approximation process $\{\Gamma_n\}_{n \in N}$ on X is said to

(A) Satisfy a Jackson type inequality of order $\phi(n)$ on X w.r.t. Y, if, for all $f \in Y$, $||\Gamma_n f - f||_X \le C\phi(n)||f||_Y$ (where C is a constant independent of $n \in N$);

(B) Satisfy a Bernstein type inequality of order $\phi(n)$ on X w.r.t. Y, if $\bigcup_{n \in N} \Gamma_n(X) \subset Y$ and $||\Gamma_n f||_Y \le D(\phi(n))^{-1}||f||_X$ for every $f \in X$ (where D is a constant independent of $n \in N$);

(C) be saturated with order $\phi(n)$ on X with saturation class Y, if, for $f \in X$

$$||\Gamma_n f - f||_X = \begin{cases} o(\phi(n)) & \text{iff } f \in A \\ O(\phi(n)) & \text{iff } f \in Y, \quad Y - A \neq \emptyset \end{cases}$$

For an approximation process $\{\Gamma_n\}$ as in (C) above, the inverse problem is the characterization of elements of the classes $\{f \in X \mid ||\Gamma_n f - f||_X = O(\eta(n))\}$ for some $\eta(n) \downarrow 0$ such that

$\frac{\phi(n)}{\psi(n)} \to 0$ as $n \to \infty$.

Given an approximation process $\{F_n\}$ on X satisfying a Jackson type inequality on X with respect to its subspace Y with certain order $\phi(n)$, we obtain, in this thesis, a family of Banach spaces related to X , on each of which $\{F_n\}$ is an approximation process and satisfies the Jackson type inequality with the same order. Similar investigations are also carried out for approximation processes satisfying a Bernstein type inequality on X or having saturation and inverse theorems on X .

We considered multiplier type and convolution type approximation processes. In the case of multiplier type processes, we find it most convenient to consider Banach subspaces of the space A' of generalized functions constructed by A.H. Zemanian. Here, each member of A' has Fourier expansion with respect to an orthonormal system. In the case of convolution type processes, we consider Banach subspaces of $S'(R^n)$, the Schwartz space of tempered distributions on R^n . The member spaces of the resulting family, in each case, are either constructed as Sobolev type spaces like $H^{p,m}(R^n)$, $H^{p,-m}(R^n)$ following F. Treves or obtained as intermediate spaces constructed by K-method of J. Peetre.

The above results are applied to various summability approximation methods of expansions with respect to various classical orthonormal systems such as Hermite, Jacobi, or Laguerre functions.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks and deep gratitude to my supervisor, Professor Zeev Ditzian, for suggesting the thesis topic, his continuous encouragement and help, and for all the time he spent in discussions.

I would also like to thank the University of Alberta the Killam Education Foundation and the National Research Council of Canada for the financial support.

Finally I wish to thank Mrs. Mary Easwaran for her patience, efficiency and the great care she took in typing this thesis.

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CHAPTER I

INTRODUCTION

1.1 Introduction

On a Banach space X, a sequence $\{P_n\}_{n \in N}$ $(N = \{0,1,2,3,\ldots\})$ of bounded linear operators is called an approximation process on X, if $P_n f \to f$ in X for all $f \in X$. Given suitable subspaces Y and A of X (A being fixed with $\dim(A) < \infty$) and a nonnegative function $\varphi(n) \to 0$ on N, an approximation process $\{P_n\}$ on X is said to (A) satisfy a Jackson type inequality of order $\varphi(n)$ on X with respect to Y, if for all $f \in Y$,

$$\|P_n f - f\|_X \leq c\,\varphi(n)\,\|f\|_Y \qquad \ldots\ldots(1.1.1)$$

(B) satisfy a Bernstein type inequality of order $\varphi(n)$ on X with respect to Y if $\bigcup_{n \in N} P_n(X) \subset Y$ and

$$\|P_n f\|_Y \leq c\,(\varphi(n))^{-1}\,\|f\|_X \quad (f \in X) \qquad \ldots\ldots(1.1.2)$$

(C) be saturated with order $\varphi(n)$ on X with saturation class Y, if for $f \in X$

$$\|P_n f - f\|_X = \begin{cases} o(\varphi(n)) \quad (n \to \infty) & \text{iff } f \in A, \\ O(\varphi(n)) \quad (n \to \infty) & \text{iff } f \in Y, \quad Y - A \neq \emptyset \end{cases} \qquad \ldots\ldots(1.1.3)$$

For an approximation process $\{P_n\}$ as in (C) above, the inverse problem is the characterization of elements of some classes

$$\{f \in X \mid \|P_n f - f\|_X = O(\psi(n))\} \quad \text{with } \psi(n) \searrow 0, \quad \frac{\varphi(n)}{\psi(n)} \to 0 \text{ as } n \to \infty \qquad \ldots\ldots(1.1.4)$$

The problem we tried to solve in this thesis is: For an approximation process $\{T_n\}$ which, for some particular space, satisfies (A) Jackson type inequality, or (B) Bernstein type inequality, or for that space, has (C) saturation and inverse theorems, for what other spaces would the process $\{T_n\}$ satisfy the above (i.e. (A), (B) or (C)?).

This is partially answered by the results outlined below: Both the multiplier type approximation processes on a general Banach space X with respect to a biorthogonal system on X, and the convolution type approximation processes on a general Banach subspace Y of the space of tempered distributions $S'(R^n)$ were studied. The spaces $L^p(I)$, $1 \leq p < \infty$ (I an open cube rectangle in R^n) is a particular special example of the spaces X or Y above; the resulting family of related Banach spaces, for which the above problem was solved, includes $L_q(I)$, (q lying between p and p'); the Lorentz spaces $L^{q,s}$ $1 \leq s \leq \infty$ [Butzer-Berens [1], p. 181]; Sobolev spaces $H^{q,m}(I)$, $H^{q,-m}(I)$ and the intermediate spaces lying between them [Treves [1], p. 323] and some other spaces of generalized functions.

Application of the above results is made to summability approximation methods of expansions with respect to various classical orthonormal systems such as Hermite, Jacobi or Laguerre functions and to Convolution-type approximation methods defined through various $L_1(R^n)$ - kernels.

Notations and definitions of operators and spaces are given in Section I.2. The main results of this thesis are presented in Section I.3. Some examples on the Convolution-type approximation processes on subspaces of $S'(R^n)$ satisfying the hypothesis of Theorem I.3.5 are presented in Section I.4. For more examples on these processes, refer to Butzer-Nessel [[1], p. 463] and H.S. Shapiro [[1], pp. 259-265]. In Chapter II, we present the proof of the main results of this thesis involving multiplier type approximation operators on Banach subspaces of Λ'. In Section II.3 the application of these results is made to classical summability methods involving the various classical orthonormal systems mentioned above. In Chapter III, the proof of Theorems I.3.4 and I.3.5 involving Convolution type approximation operators on Banach subspaces of $S'(R^n)$ are presented. In Section III.1, under Remarks 1 - 4, some examples of Homogeneous Banach Spaces and Abstract Homogeneous Banach Spaces are given. For more examples on these spaces, refer to H.S. Shapiro [[1], pp. 200-206].

I.2 Notations and Definitions

We shall give, in [illegible] ction, some of the usual notations necessary in the later chapters to describe the main results. The letters X, Y, Z, E ... denote Banach spaces; $P = \{1,2,3,4,\ldots\}$ will be the set of positive integers, and $[X,Y]$ denote the space of all bounded linear operators from X into Y, with $[X,X] = [X]$. We say $X \cong Y$ if X and Y are equal with equivalent norms; $X \subset Y$ if X is continuously contained in Y. For $X \subset Y$, let cl (X,Y) denote the closure of X in the topology of Y. If $X \subseteq Y$ with $||f||_Y \le ||f||_X$ $(f \in X)$ then X is said to be a normalized subspace of Y [Gagliardo [1]]. The letter D denotes the differential operator $\frac{d}{dx}$.

In Section I.2.1, we w[illegible]l present the following:

(i) Definitions and a brief summary of the properties of elements of the spaces A , A' introduced by Zemanian [[1],[2]];

(ii) Definitions of the operators U^{α} ; V^{α} $(\alpha > 0)$ and of the spaces X_{α} , $X_{-\alpha}$ involving U^{α} ;

(iii) some sufficient conditions for a sequence of real or complex numbers to define multiplier type operators as given in Trebels [[1], p. 30].

Definitions and a brief summary of the properties of the Schwartz spaces of distributions [Schwartz [1], Vol. II, p. 75] , of the Sobolev spaces [Treves [4], p. 323] , of the intermediate spaces [Butzer and Berens [1], p. 165], of relative completion [Butzer-Nessel [1], p. 373], and of the Abstract Homogeneous Banach Spaces [H.S. Shapiro, p. 201] are given in Sections I.2.2, I.2.3, I.2.4, I.2.5 and I.2.6 respectively.

I.2.1 Zemanian Spaces A , A'

For approximation processes of multiplier type, it seems natural to investigate the problem dominating this thesis for subspaces of the generalized function space A' introduced by Zemanian [[1],[2]]. Let I be an open interval of the real line R .

Let U be a self adjoint differential operator of the form

$$U = \theta_0 D^{n_1} \theta_1 D^{n_2} \dots D^{n_\nu} \theta_\nu = \bar{\theta}_\nu (-D)^{n_\nu} \dots \bar{\theta}_1 (-D)^{n_1} \bar{\theta}_0 \quad \text{.....(I.2.1)}$$

$\theta_i \in C^{\infty}(I)$, $n_i \in N$, $1 \le i \le \nu$, with discrete spectrum, with $\{\psi_n\}_{n \in N}$ a sequence of orthonormal C^{∞}-functions on I as its eigen-functions, corresponding to the eigenvalues $\{\lambda_n\}_{n \in N}$ with $|\lambda_n| \to \infty$ as $n \to \infty$. There exist only a finite number of $i_k \in N$, $0 \le k \le \ell$, with $\lambda_k = 0$, $0 \le k \le \ell$. Let Λ denote the linear span of $\{\psi_k\}_{k=0}^{\ell}$.

For $n \in N$, $\langle \phi, \psi_n \rangle = \int_I \phi(x) \overline{\psi_n(x)} \, dx$ .....(I.2.2) denotes the Fourier coefficients of [illegible] $(I) \cap L^2(I)$. The space of test functions A as defined by A.H. Zemanian in [[2], p. [illegible]] is $A = \{\phi \in C^\infty(I) \mid \|\mathfrak{U}^k \phi\|_{L^2(I)} < \infty$ for all $k \in N\}$; For all $k, n \in N$,

$$\mathfrak{U}^k \psi_n = \lambda_n^k \psi_n \quad \ldots\ldots(1.2.3)$$

The space A has the following properties:

(A_1) A is a Fréchet space with respect to the countable seminorms $\{\gamma_k\}_{k \in N}$ [Zemanian [2], p. 253], where

$$\gamma_k(\phi) = \|\mathfrak{U}^k \phi\|_{L^2(I)} \quad (\phi \in A,\ k \in N) \quad \ldots\ldots(1.2.4)$$

(A_2) For $\phi \in A$, $\sum_{k=0}^{n} [\text{illegible}]_k$ converges to [illegible] as $n \to$ [illegible] in the topology of A [Zemanian [2], p. 254].

(A_3) For a sequence $\{a_n\}$ of complex numbers, $\sum_{n=0}^{\infty} a_n \psi_n$ converges in A if and only if $\sum_{k=0}^{\infty} |\lambda_k|^{2r} |a_k|^2 < \infty$ for every $r \in N$. [Zemanian [2], p. [illegible]].

(A_4) $\mathfrak{U} : A \to A$ is a continuous linear operator [Zemanian [2], p. 252].

The space A', defined by A.H. Zemanian in [[1],[2]] as the dual space of A, with the ordinary topology on a dual space, has the following Properties (A_1'), (A_2'), (A_3'):

(A_1') If $\mathcal{D}(I) = \{\phi \in C^\infty(I) \mid$ support of ϕ is compact$\}$, then $\mathcal{D}(I) \subset A \subset L^2(I) \subset A'$. [Zemanian [2], p. 258].

(A_2') For $f \in A'$, $\phi \in A$, [illegible] is defined to be equal to $f(\bar{\phi})$ and $\sum_{k=0}^{n} \langle f, \psi_k \rangle \psi_k \to f$ $(n \to \infty)$ in A'. [Zemanian [2], p. 259].

(A_3') For $f, g \in A'$, if $\langle f, \psi_n \rangle = \langle g, \psi_n \rangle$ for all $n \in N$, then $f = g$ in A'. [Zemanian [2], p. 260].

(A_4') For a sequence $\{b_n\}$ of complex numbers, $\sum_{n=0}^{\infty} b_n \psi_n$ converges in A' if and only if there exists a $q \in N$ such that

$$\sum_{\lambda_n \neq 0} |\lambda_n|^{-2q} |b_n|^2 < \infty . \qquad .....(I.2.5)$$

If f denotes the sum in A' of $\sum_{n=0}^{\infty} b_n \phi_n$, then $b_n = \langle f, \phi_n \rangle$ for all $n \in N$. [Zemanian [2], p. 261].

Using the property (A_4'), we can define new elements $U^{\delta}f$, $V^{\delta}f \in A'$ $(f \in A', \delta > 0)$;

Definition I.2.1 For $f \in A'$, $\delta > 0$, $U^{\delta}f \in A'$ is defined uniquely by means of its Fourier coefficients

$$\langle U^{\delta}f, \phi_k \rangle = \lambda_k^{\delta} \langle f, \phi_k \rangle \quad (k \in N) \qquad .....(I.2.6)$$

and $V^{\delta}f \in A'$ is uniquely defined by means of its Fourier coefficients

$$\langle V^{\delta}f, \phi_k \rangle = \nu_{k,\delta} \langle f, \phi_k \rangle \quad (k \in N) \qquad .....(I.2.7)$$

where

$$\nu_{k,\delta} = \begin{cases} \lambda_k^{-\delta} & \text{if } \lambda_k \neq 0 \quad k \in N \\ 0 & \phantom{\text{if }} \lambda_k = 0 \quad k \in N \end{cases} \qquad .....(I.2.8)$$

Remark For $\delta > 0$, both $U^{\delta}f$ and $V^{\delta}f$ are uniquely defined by property (A_3'). For $f \in A$, the functions $U^{\delta}f$ and $V^{\delta}f$ both belong to A by property (A_3) of A.

The case when $\delta = k$, a non-negative integer, is of particular interest. By definition (I.2.6), U^k is a linear operator with discrete spectrum with $\{\phi_n\}$, as its eigenfunctions with $\{\lambda_n^k\}$ as the corresponding eigenvalues. It is easy to observe that the operator U^k coincides with the generalized differential operator obtained by k-times successive application of the self-adjoint differential operator U given by I.2.1. In fact, it is the observation of the nature of the differential operator U^k $(k \in N)$ that has led to the intuitive definition of the operator U^δ $(\delta > 0)$.

The operator V^δ operates like "a kind of inverse" to the operator U^δ. The identity relating the operators U^δ and V^δ,

$$U^\delta V^\delta f = V^\delta U^\delta f = f - \phi_f \,,$$

$$\text{with } \phi_f = \sum_{k=0}^{r} \langle f, \psi_k \rangle \phi_k \qquad (f \in A') \qquad \ldots\ldots(1.2.9)$$

will be derived in Chapter II and will be extensively used in the proof of Lemmas II.1.3 to II.1.7; (For ϕ_k see page 4).

The spaces X_δ, $X_{-\delta}$ related to a Banach subspace X of A' are defined as follows:

Definition 1.2.2 Let X be a Banach subspace of A', and $\delta > 0$. The space X_δ is $\{f \in X \mid U^\delta f \in X\}$ with norm

$$||f||_{X_\delta} = ||f||_X + ||U^\delta f||_X \qquad (f \in X_\delta) \qquad \ldots\ldots(1.2.10)$$

The space $X_{-\delta}$ is $\{f \in A' \mid f = f_o + U^\delta f_1 \text{ with } f_o, f_1 \in X\}$ with norm $||f||_{X_{-\delta}} = \inf \{||f_o||_X + ||f_1||_X \mid f_o, f_1 \in X \text{ and } f = f_o + U^\delta f_1\}$ $\ldots\ldots(1.2.11)$

Since U^δ is a closed operator from X_δ into X and X is complete, the spaces X_δ, $X_{-\delta}$ are Banach spaces. [Butzer-Berens [1], p. 11, 165].

Many operators which we may come across in Chapter II are from one Banach subspace of A' into another and are defined by means of their Fourier coefficients.

For Banach subspaces X, Y of A', we need the notion of the space of such multiplier type operators [Hille & Phillips [1], p. 544] from X into Y, which we define as follows: An operator $\Gamma \in [X,Y]$ is said to be defined by a sequence of complex numbers $\{\gamma_k\}$ if $\langle \Gamma f, \psi_k \rangle = \gamma_k \langle f, \psi_k \rangle$ $(f \in X,\ k \in N)$. $\ldots\ldots(1.2.12)$

Let $M(X,Y)$ be the space of all real sequences $\{\gamma_k\}$ defining some $F \in [X,Y]$ with norm $||\{\gamma_k\}||_{M(X,Y)} = ||F||$. .....(1.2.13)

Let $FM(X,Y)$ be the space of all real double sequences $\{\gamma_{i,k}\}_{i\in\Omega, k\in P}$ (Ω a parameter set) such that, for every $i \in \Omega$, $\{\gamma_{i,k}\} \in M(X,Y)$ defining $F_i \in [X,Y]$ with $\sup_i ||F_i|| < \infty$ and norm

$$||\{\gamma_{i,k}\}||_{FM(X,Y)} = \sup_i ||\{\gamma_{i,k}\}||_{M(X,Y)} \qquad .....(1.2.14)$$

[Ref. Favard [2],[3]].

Some sufficient conditions for a sequence to be a member of $M(X,Y)$ have been obtained both by the author and by Butzer and his colleagues [W. Trebels [1], p. 30; Butzer-Nessel-Trebels [2], p. 132-133]. To describe these results, we need the following definitions:

<u>Definition</u> 1.2.5 The pair $X, \{\varphi_n\}$ is said to satisfy the property (C^α) for some $\alpha > 0$, if X is a Banach subspace of A' and $cl([\{\varphi_n\}],X) = X$, such that, for some constant $C_\alpha > 0$

$$||(C,\alpha)_n f||_X \le C_\alpha ||f||_X \qquad (f \in X,\ n \in N) \qquad .....(1.2.15)$$

where $$(C,\alpha)_n f = \sum_{k=0}^{n} \frac{A^\alpha_{n-k}}{A^\alpha_n} \langle f, \varphi_k \rangle \varphi_k \qquad (f \in X) \qquad .....(1.2.16)$$

denotes the Cesaro means of the Fourier expansions of f, of order α, with

$$A^\alpha_n = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \qquad \alpha \ge 0 \qquad .....(1.2.17)$$

The spaces BV_{j+1} $(j \in N)$ [used by Butzer and his colleagues] [See Trebels [1], p. 24] are defined as follows: For $j \in N$,

$$BV_{j+1} = \{e \in L^\infty_0 \mid e, e^{(1)}, \ldots, e^{(j-1)} \in AC_{loc}(0,\infty),\ e^{(j)} \in BV_{loc}(0,\infty), \text{ and}$$

$$||e||_{BV_{j+1}} = \sup_{x>0} |e(x)| + \frac{1}{\Gamma(j+1)} \int_0^\infty x^j |de^{(j)}(x)| < \infty\}$$

.....(1.2.18)

where $\quad L_0^\infty = \{f \in L^\infty[0,\infty) \mid \lim_{x\to\infty} f(x) = 0\}$, ...(1.2.19)

$AC_{loc}(0,\infty)$ is the [illegible] of all locally absolutely continuous functions on $(0,\infty)$, ...(1.2.20)

and $BV_{loc}(0,\infty)$ is the space of all functions, locally of bounded variation on $(0,\infty)$ (excepting the origin). ......(1.2.21)

The author obtained the following sufficient condition for a sequence to be a member of $M(X)$.

<u>Theorem</u> 1.2.1 Let X, $\{\lambda_n\}$ satisfy the condition (C^1). Then, the space of bounded quasi-convex sequences

(1) $bqc = \{\lambda \in s \mid \|\lambda\|_{bqc} = \sum_{k=0}^{\infty} (k+1) |\Delta^2 \lambda_k| + \lim_{n\to\infty} |\lambda_n| < \infty\} \subset M(X)$

(2) $\{\{f(k/\lambda)\}_{k=0,\lambda>0} \mid f \in BV_2\} \subset M(X)$; ......(1.2.22)

(3) $\{\{f(\mu_k/\lambda)\}_{k\in N,\lambda>0} \mid$ for $a > 0$, $b > 0$, $k \in N$, $k + 1 \leq b\mu_k$,

$0 < \mu_k < \mu_{k+1}$; $\mu_{k+1} - \mu_k = a\} \subset M(X)$;

(4) Let $\{\eta_k\}_{k>0} \in BV_2$, and $\eta \in BV_2$ with $\eta_k(v) = \eta(v^k)$, $k > 0$, $v \in (0,\infty)$. Let $\{\mu_k\}_{k\in N}$ be a real sequence such that, for $a > 0$, $b > 0$, $k \in N$, $0 < \mu_k < \mu_{k+1}$, $\mu_{k+1} - \mu_k = a$, $k + 1 \leq b\mu_k$.

Then $\{\eta_k(\mu_k^s/\lambda)\}_{k\in N,\lambda>0} \subset M(X)$ for $k > 0$, $s > 0$. ......(I.2.23).

Butzer-Nessel-Trebels [[2], I, pp. 132-133] independently obtained the results (1) and (2) of Theorem 1.2.1, with the same proof. W. Trebels [[1], Theorem 3.9, p. 30] obtained the following more general result, which includes the results of Theorem 1.2.1.

<u>Theorem</u> I.2.2 Let X, $\{\psi_n\}$ satisfy the condition (C^α) for some $\alpha = j \in N_p$. Let $\Psi(\delta)$ be a positive monotone increasing function on $(0,\infty)$. Let $\phi(t)$ be a nonnegative function with $\lim_{t\to 0^+} \phi(t) = 0$ and

$\lim_{t\to\infty} \varphi(t) = \infty$, and let φ possess $(j+1)$ continuous derivatives on $(0,\infty)$ (the origin is excluded) with $|t^k \varphi^{(k+1)}(t)| \le C \varphi'(t)$ $(0 \le k \le j)$. Then

$$\{\{e(\varphi(k)/\varphi(\rho))\}_{k\in N,\, \rho>0} \mid e \in BV_{j+1}\} \subset UM(X). \qquad \text{......(I.2.24)}$$

For further results on multipliers, refer to Trebels [1].

1.2.2 In Chapter III, we will be considering the problem of extending either a Jackson type inequality, Bernstein type inequality or some saturation and inverse theorems satisfied by convolution type approximation processes on some Banach subspace of $S'(R^n)$ to other related Banach subspaces of $S'(R^n)$. Some of the usual notations and definitions of spaces needed for the above are specified below.

For $x = (x_1, x_2, \dots, x_n) \in R^n$, $j = (j_1, j_2, \dots j_n) \in N^n$, we set $x^j = x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$, $D^j = (\frac{\partial}{\partial x_1})^{j_1} \dots (\frac{\partial}{\partial x_n})^{j_n}$ and $|j| = \sum_{i=1}^{n} j_i$. Let M be the space of all bounded measures μ on R^n with norm

$$||\mu||_M = \int_{R^n} |d\mu| . \qquad \text{......(I.2.25)}$$

The convolution of $\mu \in M$ and $f \in L_p(R^n)$, $1 \le p \le \infty$, is given by

$$(f*d\mu)(x) = \int_{R^n} f(x-y)\, d\mu(y) , \qquad \text{......(I.2.26)}$$

and we have

$$||f*d\mu||_p \le ||f||_p ||\mu||_M . \qquad \text{......(I.2.27)}$$

The Fourier-Stieltjes transform of $\mu \in M$ and the Fourier transform of $f \in L_1$ are defined as

$$[d\mu]^\wedge(v) = (2\pi)^{-n/2} \int_{R^n} e^{-iv\cdot x}\, d\mu(x) \quad \text{and}$$

$$f^\wedge(v) = (2\pi)^{-n/2} \int_{R^n} e^{-iv\cdot x} f(x)\, dx , \qquad \ldots\ldots(1.2.28)$$

respectively.

We also use the following common notation for distribution spaces. [See Schwartz [1]]. The space $\mathcal{D}$ is the space of infinitely differentiable functions on R^n with compact support. The Schwartz space $\mathcal{S}$ of rapidly decreasing functions is

$$\mathcal{S} = \{\phi \in C^\infty(R^n) \mid \sup_{x\in R^n} |x|^k |D^j\phi(x)| < C_{k,j} < \infty \text{ for all } k \in N, j \in N^n\}.$$

The Schwartz space of slowly increasing functions at infinity is

$$\mathcal{O}_M = \{\phi \in C^\infty(R^n) \mid \text{For every } j \in N^n \text{, there exists } k = k(\phi,j) \in R \text{ with } \sup_{x\in R^n} \frac{|D^j\phi(x)|}{|x|^k} < \infty\}.$$

The spaces $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{O}_C'$ are the dual spaces of $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{O}_M$ respectively, with the usual dual topology.

The Fourier transform of $\phi \in \mathcal{S}(R^n)$ is the function $\phi^\wedge(\xi)$ [Treves [1], p. 268] given by 1.2.28. The Fourier transform is an isomorphism of $\mathcal{S}(R^n)$ onto $\mathcal{S}(R^n)$ (for the topological vector spaces), [Treves [1], p. 268].

The distributional Fourier transform $T^\wedge$ of $T \in \mathcal{S}'(R^n)$ [Treves [1], p. 275] is defined by

$$\langle T^\wedge, \phi\rangle = \langle T, \phi^\wedge\rangle \qquad (\phi \in \mathcal{S}). \qquad \ldots\ldots(1.2.29)$$

The Fourier transformation is an isomorphism (for the topological vector spaces) of S' onto S', [Treves [1], p. 269]. The Fourier transformation is one-to-one linear map of $\mathcal{O}_c'$ onto $\mathcal{O}_M$ and of $\mathcal{O}_M$ onto $\mathcal{O}_c'$ [Treves [1], p. 318]. Let, for $u \in R^n$,

$$(\tau_u \phi)(x) = \phi(x - u) \qquad (\phi \in S) . \qquad \ldots\ldots(1.2.30)$$

The group of generalized translation operators $\{\tau_u\}_{u \in R^n}$ on S' is defined by

$$\langle \tau_u f, \phi \rangle = \langle f, \tau_{-u} \phi \rangle \qquad (f \in S', \phi \in S) . \qquad \ldots\ldots(1.2.31)$$

The convolution of $f \in \mathcal{O}_c'$ and $T \in S'$, given by

$$\langle f * T, \phi \rangle = \langle f_y, \langle T_x, \phi(x + y) \rangle \rangle \qquad \ldots\ldots(1.2.32)$$

(T acts on $\phi(x + y)$ as a function of x and f acts on $\langle T_x, \phi(x + y) \rangle$ as a function of y), exists in S' and its Fourier transform is $(f * T)^\wedge = f^\wedge T^\wedge$, the product being defined by $\langle f^\wedge T^\wedge, \phi \rangle = \langle T^\wedge, f^\wedge \phi \rangle$ $(\phi \in S)$. For further details, we refer to Schwartz [1].

In Cha[illegible] will consider a family of Banach subspaces of S' related to [illegible] subspace X of S' obtained by convoluting elements [illegible] $G_\alpha \in \mathcal{O}_c'$ $(\alpha \in R)$, (Calderon, [[1], p. 34] considered [illegible] $X = L_p(R^n)$, $1 \le p < \infty$). For $\alpha \in R$, G_α is [illegible] transform

$$G_\alpha^\wedge([illegible]) = ([illegible]) \qquad (v \in R^n) \qquad \ldots\ldots(1.2.33)$$

$G_\alpha^\wedge(v) \in \mathcal{O}_M$. For $\alpha > 0$, [illegible] called the Bessel kernel, given by

$$G_\alpha(x) = \frac{2^{(2-\alpha)/2}}{\Gamma(\alpha/2)} \left(\frac{|x|}{2}\right)^{-(n-\alpha)/2} [illegible]\, K_{(n-\alpha)/2}(|x|) , \quad [illegible] > 0$$

where $K_\nu(x) = (\pi/2)\,[(I_{-\nu}(x) - I_\nu(x))/\sin \pi\nu]$,

$$I_\nu(x) = \sum_{m=0}^{\infty} (x/2)^{\nu+2m} / (m!\,\Gamma(\nu + m + 1))$$

are the modified Bessel functions of order ν of the third and the first kind respectively. For $\alpha > 0$; $G_\alpha(x) > 0$, and

$$\int_{R^n} G_\alpha(x)\, dx = (2\pi)^{n/2} .$$

For further properties, see Aronszajn-Smith [[1], p. 413-417], Aronszajn-Mulla-Szeptycki [[1], Chapter I, §1] and Calderon[1].

The spaces X^δ , $X^{-\delta}$, $(\delta > 0)$, and $X^\wedge$ related to a Banach subspace X of S' are defined as:

$$X^\delta = \{f \in X \mid G_{-\delta} * f \in X\} \text{ normed by}$$

$$||f||_{X^\delta} = ||f||_X + ||G_{-\delta} * f||_X \qquad (f \in X^\delta) ; \qquad \ldots\ldots(I.2.34)$$

$$X^{-\delta} = \{f \in S' \mid f = f_0 + G_{-\delta} * f_1 \text{ with } f_0, f_1 \in X\}$$

normed by

$$||f||_{X^{-\delta}} = \inf \{||f_0||_X + ||f_1||_X \mid f = f_0 + G_{-\delta} * f_1 \text{ with } f_0, f_1 \in X\} ; \qquad \ldots\ldots(I.2.35)$$

and $X^\wedge = \{u \in S' \mid u^\wedge \in X\}$ normed by

$$||u||_{X^\wedge} = ||u^\wedge||_X . \qquad \ldots\ldots(I.2.36)$$

For Banach subspaces X , Y of S' with $S \subset X \cap Y$, let $I(X,Y)$ be the space of all tempered distributions u , normed by

$$||u||_{I(X,Y)} = \inf \{A \mid ||u * \phi||_Y \le A||\phi||_X ;\ \phi \in S\} . \qquad \ldots\ldots(I.2.37)$$

If $X = Y = L_p(R^n)$, $1 \le p \le \infty$, the space M_p^p , as defined by Hörmander [[1], page 99] denotes the space of Fourier transforms $\hat{u}$, with $u \in I(L_p, L_p)$, and normed by $\|\hat{u}\|_{M_p^p} = \|u\|_{I(L_p, L_p)}$. ......(I.2.38)

We have $M_p^p \subset M_2^2 = L_\infty(R^n)$, $1 \leq p \leq \infty$,

and ......(I.2.39)

$$M_1^1 = M_\infty^\infty = (M(R^n))^\wedge .$$

For further details about M_p^p - spaces, see Hörmander [[1], pp. 100 - 102]. Similarly, we can define M^p as the space of Fourier transforms $\hat{u}$ of distributions $u \in I(L_{p^\wedge}, L_{p^\wedge})$, $1 \leq p \leq \infty$, with norm

$$\|\hat{u}\|_{M^p_\wedge} = \|u\|_{I(L_{p^\wedge}, L_{p^\wedge})} . \qquad ......(I.2.40)$$

Using the relation, it is easy to check that for all p, q, $1 \leq p, q \leq \infty$, we have $M_p^p \quad M^q \subset M_2^2$. ......(I.2.41)

For the origin of definition I.2.36 and more details of spaces $L_{p^\wedge}$, $1 \leq p \leq \infty$, see Katznelson [[1], p. 150].

I.2.3 Sobolev type spaces

For an open subset I of R^n, let $\mathcal{D}(I)$ be the space of all $f \in C^\infty(I)$ with compact support in I. Let $\mathcal{D}'(I)$ be its dual. Analogous to the definitions of spaces $H^{p,m}(I)$, $H^{p,-m}(I)$, $H_0^{p,m}(I)$ by Treves [1], Chapter 31], we define Sobolev-type spaces $W^m(X)$, $W^{m,0}(X)$, $W^{-m}(X)$ $(m \in P)$ related to a Banach subspace X of $\mathcal{D}'(I)$ as follows: $W^m(X) = \{f \in X \mid$ distributional derivatives $D^j f \in X$, $j \in N^n$, $|j| \leq m\}$ with norm

$$\|f\|_{W^m(X)} = \sum_{|j| \leq m, j \in N^n} \|D^j f\|_X, \quad (f \in W^m(X)); \qquad ......(I.2.42)$$

If $\mathcal{D}(I)$ is dense in X, then $W^{m,0}(X)$ is the closure of $\mathcal{D}(I)$ in $W^m(X)$ with the induced norm; and

$$W^{-m}(X) = \{f \in \mathcal{D}'(I) \mid f = \sum_{|j|\le m,\, j\in N^n} D^j f_j \text{ with } f_j \in X,\ |j| \le m\}$$

normed by

$$||f||_{W^{-m}(X)} = \inf \{ \sum_{|j|\le m} ||f_j||_X \mid f = \sum_{|j|\le m} D^j f_j ,\quad \{f_j\}_{|j|\le m} \subset X\} . \quad ......(I.2.43)$$

I.2.4 Intermediate spaces of K-interpolation.

The K-method of interpolation developed by J. Peetre [1] in the theory of intermediate spaces and the concept of relative completion introduced by Gagliardo [See Aronszajn-Gagliardo [1]] play an important role in approximation theory. Definitions and theorems of this section can be found in greater detail in the book of Butzer-Berens [[1], 3.2].

For Banach spaces X, Y continuously contained in a linear Hausdorff space $\mathcal{X}$, let

$X + Y = \{f \in \mathcal{X} \mid f = f_1 + f_2$ with $f_1 \in X,\ f_2 \in Y\}$ with the norm

$$||f||_{X+Y} = \inf \{ ||f_1||_X + ||f_2||_Y \mid f = f_1 + f_2 ;\quad f_1 \in X ,\ f_2 \in Y\} . \quad ......(I.2.44)$$

For $f \in X + Y$, $0 < t < \infty$, the function norm $k(t,f,X,Y)$ is defined as

$$k(t,f,X,Y) = \inf \{ ||f_1||_X + t||f_2||_Y \mid f = f_1 + f_2 ,\quad f_1 \in X ,\ f_2 \in Y\} . \quad ......(I.2.45)$$

$(X,Y)_{\theta,q}$ is defined to be the space of all $f \in X + Y$ for which the norm

$$||f||_{\theta,q} = \begin{cases} \left[\int_0^\infty [t^{-\theta} k(t,f,X,Y)]^q \frac{dt}{t} \right]^{1/q} & 0 < \theta < 1, \\ & 1 \le q < \infty \\ \sup_{0<t<\infty} [t^{-\theta} k(t;f,X,Y)] & 0 \le \theta \le 1, \ q = \infty \end{cases} \quad \ldots\ldots(1.2.46)$$

is finite.

The spaces $(X,Y)_{\theta,q}$ $0 < \theta < 1$, $1 \le q < \infty$ and/or $0 \le \theta \le 1$, $q = \infty$ are Banach spaces under the norm (1.2.46). Furthermore

$$X \cap Y \subset (X,Y)_{\theta,q} \subset X + Y . \quad \ldots\ldots(1.2.47)$$

An interpolation pair (X_1,X_2) in X, is a couple of Banach spaces X_1 and X_2, continuously contained in a linear Hausdorff space X. Let (X_1,X_2) and (Y_1,Y_2) be two interpolation pairs in X and Y respectively. Let $I(X,Y)$ be the space of linear transformations from $X_1 + X_2$ to $Y_1 + Y_2$ such that for $T \in I(X,Y)$,

$$f_i \in X_i, \quad Tf_i \in Y_i \quad \text{and} \quad ||Tf_i||_{Y_i} \le M_i ||f_i||_{X_i} \quad i = 1,2. \quad \ldots\ldots(1.2.48)$$

i.e. $\quad T|_{X_i} \in [X_i,Y_i]$ with a norm $\le M_i$, $i = 1,2$.

Definition 1.2.4 Let X and Y be two intermediate spaces of X_1 and X_2, and Y_1 and Y_2 respectively, [refer to Butzer-Berens [1], p.]. We say X and Y have the interpolation property, if for each $T \in I(X,Y)$, $T|_X \in [X,Y]$. The spaces X and Y are called interpolation spaces of type θ, $0 < \theta < 1$, if for some constant c

$$||T||_{[X,Y]} \le c\,(||T||_{[X_1,Y_1]})^{1-\theta}(||T||_{[X_2,Y_2]})^{\theta}, \quad (T \in I(X,Y)). \quad \ldots\ldots(1.2.49)$$

Theorem 1.2.3 Let (X_1,X_2) and (Y_1,Y_2) be two interpolation pairs of X and Y respectively. Then the intermediate spaces $\Phi(X_1,X_2)_{\theta,q}$

and $(Y_1,Y_2)_{\theta,q}$ $(0<\theta<1,\ 1\le q<\infty$ and/or $0\le\theta\le 1,\ q=\infty)$ are interpolation spaces of (X_1,X_2) and (Y_1,Y_2) of type θ satisfying (1.2.49) with $c=1$.

1.2.5 Relative Completion

For an approximation process having saturation theorems on a Banach space X, the saturation class is usually given by the relative completion of a normalized Banach subspace Y of X. The relative completion of Y with respect to X, denoted by $\tilde{Y}^X$, is the space

$$\tilde{Y}^X = \{f\in X \mid \{f_n\}\subset Y,\ \|f_n - f\|_X \to 0\ \ (n\to\infty),\ \sup_n \|f_n\|_Y < \infty\}.$$

with norm

$$\|f\|_{\tilde{Y}^X} = \inf\{\sup_n \|f_n\|_Y \mid \{f_n\}\subset Y,\ \sup_n \|f_n\|_Y < \infty,\ \|f_n - f\|_X \to 0\ \ (n\to\infty)\}, \quad ......(1.2.50)$$

and satisfying the following properties:

(a) $(\tilde{Y}^X)^{\sim X} = \tilde{Y}^X$

(b) $\tilde{Y}^X = (X,Y)_{1,\infty}$

(c) $Y = \tilde{Y}^X$ if Y is reflexive

(d) Y is a normalized Banach subspace of $\tilde{Y}^X$.

For proof of these statements, see Butzer-Nessel [[1], p. 373].

1.2.5 Abstract Homogeneous Banach Space

The concepts of Abstract Homogeneous Banach Space (abbreviated as AHBS) on R^n and Homogeneous Banach Space (abbreviated as HBS) on R^n have been found useful as "unifying tools" for considering approximation problems on various spaces simultaneously. [For history and origin of definition of these spaces, see H.S. Shapiro [[1], p. 200] and the references cited there]. Many families of spaces which are to be considered in Chapters II and III involve these spaces. Shapiro [[1], p. 201] defines an AHBS on R^n as a Banach Space B together with a group $\{T_u\}_{u \in R^n}$ of operators in $[B]$ satisfying the following conditions (H1), (H2) and (H3) :

(H1) $T_0 = I$ identity in $[B]$. $T_{u+v} = T_u T_v$, $||T_u||_{[B]} = 1$ $(u,v \in R^n)$;

(H2) For each $f \in B$, the function $\varphi_f : R^n \to B$, given by $\varphi_f(u) = T_u f$ is continuous; and

(H3) For all $\sigma, \mu \in M$, $f \in B$ we have $f*(\sigma*\mu) = (f*\sigma)*\mu$ where for $\sigma \in M$, $f \in B$, the element $f*\sigma$ of B is defined to be the Bochner-Stieltjes integral

$$f*\sigma = \int_{R^n} \varphi_f(u)\, d\sigma(u) \qquad \ldots\ldots(I.2.51)$$

of the (continuous bounded) B-valued function $\varphi_f(u)$ satisfying $||f*\sigma||_B \le ||f||_B ||\sigma||_M$ . ...(I.2.52)

Shapiro [[1], p. 206] defines a HBS B on R^n as a Banach subspace of (Lebesgue) measurable and uniformly locally integrable functions on R^n satisfying (H1) and (H2) above with respect to the group of generalized translations $\{T_u\}_{u \in R^n}$ (defined by 1.2.31). Thus, for each compact set K in R^n, there exists a constant $\alpha(K)$ such that for all $f \in B$, $t \in R^n$

$$\int_{K+t} |f(u)|\, du \le \alpha(K)\, \|f\|_B \; . \qquad \ldots\ldots(1.2.53)$$

Shapiro [[1], p. 204 and 207] proves the following:

(a) If B is an AHBS on R^n and $k(t) \in L_1(R^n)$ with $\int_{R^n} k(t)dt = 1$, then for all $f \in B$,

$$\lim_{a \to 0} \|f - f * k_{(a)}\|_B = 0$$

where $k_{(a)}(t) = a^{-n} k(t/a)$, $a > 0$. ......(1.2.54)

(b) Every HBS B on R^n is an AHBS on R^n with $(f*\sigma)(t)$ to be interpreted as $\int_{R^n} f(t-u)\, d\sigma(u)$

for all $f \in B$, $\sigma \in M$. ......(1.2.55)

1.3 Main Results

The main results of this thesis will be stated in this section. We present the results involving multiplier-type approximation processes on Banach subspaces of 'A' in 1.3.1 and the results involving convolution type approximation processes on Banach subspaces of $S'(R^n)$ in 1.3.2. Let $m \in P$ (m fixed throughout this section.) .

1.3.1 Approximation Results on Banach Subspaces of A'

For $\delta > 0$, we will define $F(m,\delta)$, a family of Banach subspaces of A', from each of which we will extend the Jackson or Bernstein type inequalities satisfied by multiplier-type approximation processes to various other related Banach subspaces of A'.

Definition 1.3.1 Let $F(m)$ be the family of all Banach subspaces Z of A' satisfying the following:

(1) Z contains both $\mathcal{D}(I)$ and $[\{\phi_n\}]$ as dense subspaces;

(2) $A \subset W^{m,0}(Z) \cap Z^*$; $W^{-m}(Z + Z^*) \subset A'$; and

(3) For all $\delta > 0$, $\{\nu_{k,\delta}\}_{k \in N}$ (as defined by 1.2.8) belongs to $M(Z)$ defining U^δ.

Definition 1.3.2 For $\delta > 0$, a space $Z \in F(m,\delta)$, if

(1) $Z \in F(m)$;

(2) Z^* contains $\mathcal{D}(I)$ as dense subspace; and

(3) For all $f \in (Z^*)_{-\delta} + Z_{-\delta}$, $D^k f \in A'$ $0 \le k \le m$.

Definition 1.3.3 Let $\delta > 0$ and $X \in F(m,\delta)$ be reflexive. Then $Y(m,\delta,X)$ is the family consisting of the following spaces:

$$Y = (\text{any one of } X,\ X^*,\ (X,X^*)_{\theta_1,q_1}\quad 0 < \theta_1 < 1\ ,$$

$$1 \le q_1 < \infty)\ ,\quad Y_{-\delta}\ ,$$

$$W^{-m}(Y)\ ,\ W^{m,0}(Y)\ ,\ (W^{-m}(Y),\ W^{m,0}(Y))_{\theta,q}$$

$$W^{-m}(Y_{-\delta})\ ,\ W^{m,0}(Y_{-\delta})\ ,\ (W^{-m}(Y_{-\delta})\ ,\ W^{m,0}(Y_{-\delta}))_{\theta,q}$$

where $1 \le q < \infty$, $0 < \theta < 1$.

Definition 1.3.4 A space $X \in Q(m)$, if $X \in I(m)$ and there exists $X' \in I(m)$ with $X = (X')^*$, $X' \subset X^*$, on X, $||\cdot||_X = ||\cdot||_{(X')^*}$; on X', $||\cdot||_{X'} = ||\cdot||_{X^*}$. For $\delta > 0$ for $X \in Q(m)$, let $Q(m,\delta,X)$ be the family consisting of the following spaces:

E_1 (= any one of X, X', $(X,X')_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$),
$W^{-m}(E_1)_q$, $(W^{-m}(E_1),E_1)_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$;
E_2 (= any one of X^*_-, $(X')^*_-$, $(X^*_-,(X')^*_-)_{\theta,q}$, $0 < \theta < 1$,
$1 \leq q \leq \infty$)

E_3 (= any one of X^*, $(X')^*$, $(X^*,(X')^*)_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$)
E_4 (= any one of $(X,X^*)_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$).

Remark: We can cite many examples of the space X' related to $X \in Q(m)$. $L_1(R^n)$, $(L_1(R^n))$, $C_0(R^n)$, $(C_0(R^n)) \in Q(m)$; when $X = L_1(R^n)$, $X' = C_0(R^n)$; when $X = C_0(R^n)$, $X' = L_1(R^n)$; when $X = (L_1(R^n))$, $X' = (C_0(R^n))$; when $X = (C_0(R^n))$, $X' = (L_1(R^n))$.

Having defined these families of subspaces of A' we state the main results of this chapter.

Theorem 1.3.1 (a) Let $\delta > 0$ and $X \in F(m,\delta)$ be reflexive. Then $M(X) \subset M(Z)$ and $UM(X) \subset UM(Z)$ for all $Z \in F(m,\delta,X)$.

(b) For $\delta > 0$ and $X \in Q(m)$.. There holds $M(X) \subset M(Z)$, $UM(X) \subset UM(Z)$ for all $Z \in Q(m,\delta,X)$.

Remarks: Assertion (a) implies that every multiplier type operator on X defines a multiplier type operator on members of $Y(m,\delta,X)$ or $Q(m,\delta,X)$. Assertion (b) implies, with the help of the Banach-Steinhaus Theorem, that every approximation process related to $\{\cdot_n\}_{n \in N}$ on X, defines an approximation process related to $\{\cdot_n\}$ on every

$Z \in Y(m,\delta,X)$ or $Q(m,\delta,X)$ with $cl(A,Z) = Z$.

Let us assume henceforth (throughout this thesis) that $\rho(t)$ is a non-negative function defined on a parameter set Λ with t_0 as its limit point such that $\rho(t) \to 0$ as $t \to t_0$ . ...(I.3.1)

Theorem I.3.2 Let $\delta > 0$ and $\alpha > 0$. Suppose $X \in Y(m)$ is reflexive (respectively $X \in Q(m)$) and for all $t \in \Lambda$, the sequences $\{\gamma_{t,k}\}_{k \in N} \in M(X)$ define $\Gamma_t \in [X]$. Then the following statements hold:

(a) If for all $f \in X_\delta$, $||\Gamma_t f - f||_X \leq c_1 \rho(t)||f||_{X_\delta}$,

then for all $Z \in Y(m,\delta,X)$ (respectively for all $Z \in Q(m,\delta,X)$), there holds $||\Gamma_t f - f||_Z \leq c_{11}\rho(t)||f||_{\tilde{Z}_\delta}$ for all $f \in \tilde{Z}_\delta = \tilde{Z}^Z_\delta$;

(b) If $\bigcup_{t \in \Lambda} \Gamma_t[X] \subset X_\delta$ and $||\Gamma_t f||_{X_\delta} \leq c_2[\rho(t)]^{-1}||f||_X$

$(f \in X)$, then for all $Z \in Y(m,\delta,X)$ (respectively for all $Z \in Q(m,\delta,X)$), $\bigcup_{t \in \Lambda} \Gamma_t[Z] \subset Z_\delta$ and $||\Gamma_t f||_{Z_\delta} \leq c_{22}(\rho(t))^{-1}||f||_Z$ $(f \in Z)$.

(c) If X and $\{\varphi_n\}$ satisfy property (C_1) (given by I.2.15) and $||\Gamma_t f - f||_X \leq c_1\rho(t)||f||_{X_\delta}$ $(f \in X_\delta)$ for some $\delta > 0$ and for some $c \neq 0$, $(1 - \gamma_{t,k})/\rho(t) \to c\lambda_k^\delta$ as $t \to t_0$ for every fixed $k \in N$, then for all $Z \in Y(m,\delta,X)$ (respectively for all $Z \in Q(m,\delta,X)$) there holds for $f \in Z$,

$$||\Gamma_t f - f||_Z = \begin{cases} o(\rho(t)) & (t \to t_0) \text{ iff } f \in A \\ O(\rho(t)) & (t \to t_0) \text{ iff } f \in \tilde{Z}_\delta . \end{cases}$$

In the following theorem, some sufficient conditions for members of $Y(m,\delta,X)$ and $Q(m,\delta,X)$ to be subspaces of A' , are given.

Theorem I.3.3 Let X and Y be Banach subspaces of Lebesgue measurable, real or complex valued functions on I such that $X \subset Y^*$, $Y \subset X^*$,

$cl(P(I),X) = X$ and $cl(P(I),Y) = Y$. Let $D = \frac{d}{dx}$.

(a) Suppose $||U^k D\psi_n||_{L^2(I)} = O(|\lambda_n|^{s+k})$ $(n,k \in P$, $s \in P$ independent of $n,k)$. Then $D: \Lambda' \to \Lambda'$ is a continuous linear operator of Λ' into Λ' and hence the spaces under consideration are subspaces of Λ'.

(b) Suppose for all $k \in P$, $0 \le k \le m$, $||D^k\psi_n||_{X \cap X^*} = O(|\lambda_n|^{s_k})$ $(s_k \in P$, depending only on $k)$. Then for all $k \in P$, $0 \le k \le m$, the mapping $D^k : X + X^* \to \Lambda'$ is continuous, when $\langle D^k f, \varphi\rangle = (-1)^k \langle f, D^k \varphi\rangle$, $(f \in X + X^*$, $\varphi \in \Lambda)$.

(c) Suppose $||\psi_n||_{X \cap Y} = O(|\lambda_n|^s)$ $(s \in P$, independent of $n \in P)$ and for all $n \in P$, there exists $\ell_1 \in P$, $\{n_q\}_{q=0}^{\ell_1}$ in P, a finite sequence $\{C_q^n\}_{q=0}^{\ell_1}$ of constants with

$$D\psi_n = \sum_{q=0}^{\ell_1} C_q^n \psi_{n_q}, \quad \text{and} \quad \sum_{q=0}^{\ell_1} |C_q^n| \le C_1|\lambda_n|^{q_1}, \quad \sup_{0 \le q \le \ell_1} |\lambda_{n_q}| \le C_2|\lambda_n|^{q_2}$$

$(q_1, q_2 \in P$, $C_1 > 0$, $C_2 > 0$; q_1, q_2, C_1, C_2 all independent of $n \in P)$. Then we have (i) $\Lambda \subset X \cap X^*$; X, X^*, $W^{-m}(X+X^*)$, $W^{-m}(X_{-r}+X^*_{-r})$, $r \ge 0$, are all subspaces of Λ'.
(ii) $cl([\psi_n], W^{m,0}(X)) = W^{m,0}(X)$ and hence $cl(\Lambda, W^{m,0}(X)) = W^{m,0}(X)$.

(d) Let $k_0 \in P$ (k_0 fixed). Suppose for all $k \in P$, $0 \le k \le m$, $||U^{k_0}D^k\psi_n||_{X \cap X^*} = O(|\lambda_n|^{s_{k,k_0}})$ $(s_{k,k_0} \in P$, depending only on k, $k_0)$. Then for all $k \in P$, $0 \le k \le m$, $D^k U^{k_0}: X + X^* \to \Lambda'$ is continuous. Hence $W^{-m}(X_{-k_0}+X^*_{-k_0}) \subset \Lambda'$.

I.3.2 Approximation Results on Banach Subspaces of $S'(R^n)$

First of all, we will define a family of Banach subspaces $\mathcal{F}$ of $S'(R^n)$, from each of which we would like to extend Jackson or Bernstein type inequalities satisfied by convolution type approximation processes, to various other related Banach subspaces of $S'(R^n)$.

Definition I.3.5 An MHBS B on R^n belongs to $\mathcal{F}$ if

(1) $S \subset B \subset S'$ with $T_u = \mathcal{T}_u$ ($u \in R^n$) the generalized translation on R^n (see I.2.31);

(2) S is dense in B;

(3) $\|e^{ixt}\varphi\|_B \le C\|\varphi\|_B$ ($\varphi \in S$, C a constant).

Remark: In Lemma III.1.. Chapter III we will show that, for $B \in \mathcal{F}$, $B^\wedge$ is also a member of $\mathcal{F}$. (see I.2.36 for definition). Thus, $L_p(R^n)$, $(L_p)^\wedge(R^n)$, $1 \le p \le \infty$, $C_0(R^n)$, $(C_0(R^n))^\wedge$ are all members of $\mathcal{F}$. Please note that, for $1 \le p \le 2$, $(L_p)^\wedge$ is a HBS on R^n. But, for $2 < p < \infty$, $(L_p)^\wedge$ is an AHBS, not a HBS on R^n. If $B \in \mathcal{F}$, without loss of generality, we can assume that for all $f \in B$, $\tilde{f} \in B$ with $\|\tilde{f}\|_B = \|f\|_B$. ......(I.3.2)
Since $\tilde{B} = \{f \in S' \mid \tilde{f} \in B$ with $\|f\|_{\tilde{B}} = \|\tilde{f}\|_B\} \in \mathcal{F}$ ......(I.3.3)
and $B + \tilde{B} \in \mathcal{F}$, if (I.3.2) is not satisfied, we can consider $B + \tilde{B}$ instead of B.

For $B \in \mathcal{F}$, the family of related Banach subspaces of S' can be defined as follows:

Definition I.3.6 Let $B \in \mathcal{F}$ and $\delta > 0$. Let $\mathcal{B}(m,\delta,B)$ denote the family consisting of the following spaces:

E_i (= any one of X_i , $W^{-m}(X_i)$, $W^m(X_i)$, $(W^{-m}(X_i),W^m(X_i))_{\theta,q}$ $0<\theta<1$, $1\le q\le\infty$) , $(E_i,E_j)_{\theta,q}$ $1\le i\ne j\le 6$, $0<\theta<1$, $1\le q\le\infty$ where $X_1=B$, $X_2=B^*$, $X_3=B^\wedge$, $X_4=(B^*)^\wedge$, $X_5=B^{-\wedge}$, $X_6=(B^*)^{-\wedge}$.

When B is any one of $L^p(R^n)$, $(L_p)^\wedge$, $1\le p\le\infty$, $C_0(R^n)$ or $(C_0(R^n))^\wedge$, we define a family $R(m,B)$; so that an approximation process saturated on B with a certain order will be saturated on each member of $R(m,B)$ with the same order.

Definition I.3.7 Let $R_1(m,L_1(R^n))$ be the family consisting of the following spaces:

Y [= any one of $L_1(R^n)$, $C_0(R^n)$, $(L_1(R^n))^\wedge$, $(C_0(R^n))^\wedge$] , $W^{-m}(Y)$, $W^m(Y)$.

Let $R_2(m,L_1(R^n))$ be the family consisting of the following spaces:

Y_1 [= any one of $L_\infty(R^n)$, $(L_\infty(R^n))^\wedge$, $(L_p,L_\infty)_{\theta,q}$, $((L_p)^\wedge,(L_\infty)^\wedge)_{\theta,q}$, $(L_p,L_{p_1})_{\theta,\infty}$, $((L_p)^\wedge,(L_{p_1})^\wedge)_{\theta,\infty}$, $1<p\ne p_1<\infty$, $0<\theta<1$, $1\le q\le\infty$] , $W^{-m}(Y_1)$, $W^m(Y_1)$, $(W^{-m}(Y_1),Y_1)_{\theta,q}$ $0<\theta<1$, $1\le q\le\infty$;

Y_2 (= any one of $L_p,(L_p)^\wedge$, $(L_p,L_{p_1})_{\theta,q}$, $((L_p)^\wedge,(L_{p_1})^\wedge)_{\theta,q}$ $1<p\ne p_1<\infty$, $0<\theta<1$, $1\le q<\infty$) , $W^{-m}(Y_2)$, $W^m(Y_2)$, $(W^{-m}(Y_2),W^m(Y_2))_{\theta,q}$, $0<\theta<1$, $1\le q\le\infty$.

Let $R(m,L^1(R^n)) = R(m,C_0(R^n)) = R_1(m,L^1(R^n)) \cup R_2(m,L_1(R^n))$. ...(I.3.4)

Definition I.3.8 For, $1<p<\infty$, let $R(m,L_p)$ denote the family consisting of the following spaces:

Y_1 [= any one of $L_q(R^n)$, with q between p and p' , $(L_{p_1})^\wedge$, $(L_p,L_{p'})_{\theta,q}$, $((L_{p_1})^\wedge,(L_{p_2})^\wedge)_{\theta,q}$, $1<p_1\ne p_2<\infty$, $0<\theta<1$,

$1 \le q \le \infty$] , $W^{-m}(Y_1)$, $W^{m}(Y_1)$, $(W^{-m}(Y_1),W^{m}(Y_1))_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$;

Y_2 (= any one of $(L_1)^\wedge$, $Co^\wedge$, $(L_p,L_{p'})_{\theta,\infty}$, $((L_{p_1})^\wedge,(L_{p_2})^\wedge)_{\theta,\infty}$, $0 < \theta < 1$, $1 \le p_1 \ne p_2 \le \infty$) , $W^{-m}(Y_2)$, $W^{m}(Y_2)$, $(W^{-m}(Y_2),Y_2)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

Definition I.3.9 For $m \in P$, let $R(m,(L_1)^\wedge)$ denote the family consisting of the following spaces:

E [= any one of $(L_p)^\wedge$, $((L_p)^\wedge,(L_{p_1})^\wedge)_{\theta,q}$, $1 \le p \ne p_1 < \infty$, $0 < \theta < 1$, $1 \le q \le \infty$] , $W^{-m}(E)$, $W^{m}(E)$, $(W^{-m}(E),W^{m}(E))_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

E_1 [= any one of $(L_1)^\wedge$, $(C_0(R^n))^\wedge$, $((L_p)^\wedge,(L_{p_1})^\wedge)_{\theta,\infty}$, $1 \le p \ne p_2 \le \infty$, $0 < \theta < 1$], $W^{-m}(E_1)$, $W^{m}(E_1)$, $(W^{-m}(E_1),E_1)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

Let $R(m,(C_0(R^n))^\wedge)=R(m,(L_1)^\wedge)$.

Definition I.3.10 For $m \in P$, $1 < p < \infty$, let $R(m,(L_p)^\wedge)$ denote the family consisting of the following spaces:

Y_1 (= any one of $(L_q)^\wedge$, with q between p and p' , $(L_p^\wedge,(L_{p'})^\wedge)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$) $W^{-m}(Y_1)$, $W^{m}(Y_1)$, $(W^{-m}(Y_1),W^{m}(Y_1))_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

Y_2 [= any one of $((L_p)^\wedge , (L_{p'})^\wedge)_{\theta,\infty}$, $0 < \theta < 1$] $W^{-m}(Y_2)$, $W^{m}(Y_2)$, $(W^{-m}(Y_2),Y_2)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$.

The structural relationship between the above defined families of spaces is given by the following result: Let $X(R^n)$ = any one of $L_p(R^n)$, $(L_p)^{\sim}$, $1 \le p < \infty$, $C_0(R^n)$, $(C_0(R^n))^{\sim}$.

Theorem 1.3.4 Let $\delta > 0$. Let $B \in \Gamma$ (respectively $B = X(R^n)$) Then for all $Z \in B(m,\delta,B)$ (respectively $Z \in R(m,X(R^n))$. There holds $I(B,B) \subset I(Z,Z)$. (Refer to 1.2.37).

The main result of this section is as follows:

Theorem 1.3.5 Suppose $B \in \Gamma$ (respectively $B = X(R^n)$). Let $\gamma > 0$, $\delta > 0$. Let $Z \in B(m,\gamma,B)$ (respectively $Z \in R(m,X(R^n))$. Let $\rho(t)$ be as given by (1.3.1) and $\{\mu_t\}_{t\in\Omega}$ be a family of elements of $I(B,B)$.

(1) If $||\mu_t * f - f||_B \le C_1\rho(t)||f||_{B^\delta}$ $(f \in B^\delta)$ then for all $f \in \tilde{Z}^\delta = \tilde{Z}^{\delta Z}$, $||\mu_t * f - f||_Z \le C_1\rho(t)||f||_{\tilde{Z}^\delta}$.

(2) If for all $f \in B$, $\mu_t * f \in B^\delta$ for all $t \in \Omega$ and $||\mu_t * f||_{B^\delta} \le C_2(\rho(t))^{-1}||f||_B$ $(f \in B)$, then for all $f \in Z$, $\mu_t * f \in Z^\delta$ for all $t \in \Omega$ and $||\mu_t * f||_{Z^\delta} \le \tilde{C}_2||f||_Z$ $(f \in Z)$.

(3) Suppose $\psi(v)$ is a Lebesgue measurable function on R^n satisfying

$$\psi(v) = [d\mu_1]^{\wedge}(v)\ (1 + |v|^2)^{\delta/2}$$

$$(1 + |v|^2)^{\delta/2} = (d\mu_2)^{\wedge}(v) + \psi(v)\ [d\mu_3]^{\wedge}(v) \qquad \ldots\ldots(1.3.5)$$

for some $\delta > 0$ and bounded measures μ_i , i= 1,2,3 in $M(R^n)$, and suppose for all fixed $v \in R^n$

$$([\mu_t]^{\wedge} - 1)/\rho(t) \to \psi(v) \quad \text{as} \quad t \to t_0 \qquad \ldots\ldots(1.3.6)$$

and for B equal any one of $L_p(R^n)$, $(L_p)^{\sim}$, $1 \le p < \infty$, $C_0(R^n)$ and

$(C_0(R^n))$, then $\|\mu_t * f - f\|_B = O(\varphi(t))$ $(t \to t_0)$ $(f \in B^{\delta})$.
Then for all $Z \in \mathcal{R}(m,B)$ (See Definitions I.3.8 to I.3.11) and for $f \in Z$, there holds:

$$\|\mu_t * f - f\|_Z = \begin{cases} o(\varphi(t)) & \text{iff } \hat{f}\psi(v) = 0 \\ O(\varphi(t)) & \text{iff } f \in \tilde{Z}^{\delta} . \end{cases}$$

I.4 Some Examples on Convolution-type Approximation Processes

We will cite some examples of functions $\psi(v)$ and measures $\{\mu_t\}$ in $M(R^n)$ satisfying conditions I.3.5 and I.3.6 .

(1) Fejer-type kernels

The function $\psi(v)$ = constant $|v|^{\kappa}$, $\kappa > 0$ satisfies I.3.5. [E.M. Stein [1] , p. 102 and Butzer-Nessel [1], P. 253] . There are many examples of measures $\{\mu_t\}_{t>0}$ in $M(R^n)$ with $(\mu_t)(x) = t^{-n}k(x/t)$ $t > 0$ with $k(x) \in L^1(R^n)$, $\int_{R^n} k(t)dt = 1$, satisfying $(\hat{\mu}_t(v) - 1)/\varphi(t) \to c|v|^{\delta}$ for some $\delta > 0$ and some $\varphi(t) \to 0$ as $t \to 0$. See Butzer-Nessel[[1] , Chapter 12 and H.S. Shapiro [1], pp. 259 to 265].

(2) Examples of kernels $\{\mu_t\}$ which are not of Fejer type

For $\eta > 0$ and $c = \pi/\eta$, $0 < y < \eta/2$, let

$$\mu_{s,y}(x) = \sqrt{(2/\pi)}\ [c(\sin cy)(\cosh cx)/(\sinh^2 cx + \sin^2 cy)]$$

$$\mu_{a,y}(x) = \sqrt{(1/2\pi)}\ [c(\sin 2cy)/(\sinh^2 cx + \sin^2 cy)] .$$ The kernels $\mu_{s\{a\},y}$ are not of FEJER'S TYPE, i.e. $\neq t^{-n}\chi(x/t)$ with $\chi(x) \in L^1(R^n)$.
Butzer-Kolbe-Nessel [[1], p.. 333, Lemma 2] have shown, using a lemma due to Stein [1], that

$$\frac{\mu_{s,y}(v) - 1}{y} \to \psi_s(v) = -v \tan\left(\pi \frac{v}{2}\right) \quad \text{as} \quad y \to 0+$$

$$\frac{\mu_{a,y}(v) - 1}{y} \to \psi_a(v) = -v \cosh\left(\pi \frac{v}{2}\right) \quad \text{as} \quad y \to 0+$$

and the functions $\psi_{s(a)}(v)$ satisfy 1.3.5 with appropriate measures μ_i, $i = 1,2,3$.

CHAPTER II

APPROXIMATION PROCESSES ON Λ'

II.1.1 The proof of the main results of this thesis, related to the multiplier-type approximation processes on Banach subspaces of Λ', (theorems I.3.1, I.3.2, and I.3.3) are presented in this chapter. Various structural relationships among members of the families $Y(m,\alpha,X)$, $Q(m,\alpha,X)$ (see Definitions I.3.3, I.3.4) with $\alpha > 0$ and $X \in \Gamma(m,\alpha)$ (see Definition I.3.2, $m \in P$ being fixed throughout the thesis), are investigated in Section II.1.2. We show that every multiplier type operator on X can be extended to a similar operator on every member of the families $Y(m,\alpha,X)$ and $Q(m,\alpha,X)$. A simple characterization of the elements of $M(X_\alpha, X)$ is given in Lemma II.1.7.

The proof of the theorems I.3.2 and I.3.3 is given in Section II.2, utilizing a result due to H. Berens [1] [see Butzer-Nessel [1], p. 502]. Application is made in Section II.3 of these results to classical summability methods involving various classical orthonormal systems like Hermite, Laguerre, Jacobi, Trignometric and Bessel functions.

II.1.2 We state and prove certain lemmas here which are needed in the proof of Theorem I.3.2.

<u>Lemma</u> II.1.1 Let X be a Banach subspace of $\mathcal{D}'(I)$ with $\mathcal{D}(I)$ as a dense subspace. Then

(a) $(W^{m,0}(X))^* \cong W^{-m}(X^*)$

(b) If X is reflexive, so is $W^{m,0}(X)$. ......(II.1.1)

We give the proof of the above Lemma for the general case when $I \subset R^n$. This case is needed in Chapter III.

Proof (i) This proof is analogous to that of Prop. 31.3, page 325, Trèves [1]. Let ℓ_m be the number of elements in the set $\{\alpha \in P^n \mid |\alpha| \le m\}$. Let $E = \underbrace{X \times X \times \ldots \times X}_{\ell_m \text{ times}}$ with the norm

$$\|\{f_\alpha\}_{\alpha \in P^n, |\alpha| \le m}\|_E = \sum_{|\alpha| \le m, \alpha \in P^n} \|D^\alpha f\|_X \qquad \ldots\ldots(II.1.1)$$

Then dual of $E = E^* \cong \underbrace{X^* \times X^* \times \ldots \times X^*}_{\ell_m \text{ times}}$ with a norm similar to (II.1.1). The map $T : W^{m,0}(X) \to E$ given by

$$Tf = \{D^\alpha f\}_{|\alpha| \le m, \alpha \in P^n} \qquad (f \in W^{m,0}(X)) \qquad \ldots\ldots(II.1.2)$$

is an isometry of $W^{m,0}(X)$ into E. The transpose of T, $T^* : E^* \to (W^{m,0}(X))^*$, is a continuous linear map of E^* onto $(W^{m,0}(X))^*$. That T^* is onto follows from the Hahn-Banach Theorem. Indeed, for $f' \in (W^{m,0}(X))^*$, f' can be transferred as a continuous linear form on $T(W^{m,0}(X))$, and then extended to the whole of E as an element $f_0' \in E^*$. Then $T^* f_0' = f'$. For $v \in W^{-m}(X)$ with $v = \sum_{|j| \le m} D^j f_j$, $f_j \in X^*$, $|j| \le m$, $j \in P^n$, and for $\phi \in \mathcal{D}(I)$, define

$$\langle v, \phi \rangle = \sum_{|j| \le m} (-1)^{|j|} \langle f_j, D^j \phi \rangle \qquad \ldots\ldots(II.1.3)$$

Then $|\langle v, \phi \rangle| \le \|v\|_{W^{-m}(X^*)} \|\phi\|_{W^m(X)}$. Thus, $v : \mathcal{D}(I) \to C$, defined in (II.1.3), can be uniquely extended to $\bar{v} \in (W^{m,0}(X))^*$. Hence, the map $I : W^{-m}(X^*) \to (W^{m,0}(X))^*$ given by $Iv = \bar{v}$ $(v \in W^{-m}(X^*))$ is one to one as $\mathcal{D}(I)$ is dense in $W^{m,0}(X)$. Clearly, $\|Iv\|_{(W^{m,0}(X))^*} \le \|v\|_{W^{-m}(X^*)}$. We will show that I is onto.

Let $f' \in (W^{m,0}(X))^*$. Since T^* is onto, $f' = T^*\{f_\alpha^*\}_{\alpha\in P^n, |\alpha|\le m}$ for some $\{f_\alpha^*\}_{|\alpha|\le m, \alpha\in P^n}$ in E^*. Let $v = \sum_{|j|\le m, j\in P^n} (-1)^{|j|} D^j f_j^*$.

Then $v \in W^{-m}(X^*)$ and $Iv = \bar{v}$. Thus, I is onto.

(ii) Let X be reflexive. Then $W^{m,0}(X)$ is reflexive, since $W^{m,0}(X)$ can be embedded as a closed linear subspace of the reflexive space E under the map (II.1.2).

Lemma II.1.2 Let X and Y be Banach subspaces of $\mathcal{D}'(I)$. Then there exists an extension of $T \in [X,Y]$, $\bar{T}$, $\bar{T} \in [W^{-m}(X), W^{-m}(Y)]$ such that

$$||\bar{T}|| \le ||T||, \qquad \text{......(II.1.4)}$$

and when $\mathcal{D}(I)$ is dense in X, $\bar{T}$ is uniquely determined.

Proof: On $f \in W^{-m}(X)$, define $\bar{T}$ by $\bar{T}f = T(\sum_{|j|\le m} D^j f_j) = \sum_{|j|\le m} D^j Tf_j$. This definition is independent of the representation of f, since $f = \sum_{|j|\le m} D^j f_j = \sum_{|j|\le m} D^j g_j$ implies $0 = \bar{T}0 = \bar{T}(\sum_{|j|\le m} D^j(f_j - g_j)) = \sum_{|j|\le m} D^j Tf_j - \sum_{|j|\le m} D^j Tg_j$. Also, for $f = \sum_{|j|\le m} D^j f_j$, $f_j \in X$, $|j| \le m$, $||\bar{T}f||_{W^{-m}(Y)} \le \sum_{|j|\le m} ||Tf_j||_Y \le ||T|| \sum_{|j|\le m} ||f_j||_X$. Hence $||\bar{T}|| \le ||T||$.

In Lemmas II.1.3, II.1.4 and II.1.6, we investigate various structural relationships among members of each of the families $Y(m,\delta,X)$ and $Q(m,\delta,X)$, ($X \in F(m)$ and $\delta > 0$). In Lemma II.1.5, we extend multiplier type operators on X to its related Sobolev type spaces.

The following relations II.1.5, II.1.6, and II.1.7 on the operational calculus of the operators V^δ, U^δ ($\delta > 0$) on elements of Λ', play an important role in the proof of Lemma II.1.3 (a), (b), (e), (g) and Lemmas II.1.4 and II.1.6. They help in obtaining f from $U^\delta f$ ($f \in \Lambda'$). They are derived from the knowledge of the "trivial space" Λ.

The space Λ is the linear span of $\{\phi_k\}_{k=0}^{\ell}$ ($\lambda_s = 0$ if $s = i_k$, $k = 0, \ldots \ell$). For $h \in \Lambda'$, let ϕ_h be defined as

$$\phi_h = \sum_{k=0}^{\ell} \langle h, \phi_k \rangle \phi_k \quad , \qquad \ldots\ldots(II.1.5)$$

$\phi_h \in \Lambda$. For $\delta > 0$

$$U^\delta h = \sum_{\substack{s=0 \\ s \neq i_k, k=0,\ldots\ell}}^{\infty} \lambda_s^\delta \langle h, \phi_s \rangle \phi_s \; , \qquad \text{(see I.2.6)}$$

$$V^\delta h = \sum_{\substack{s=0 \\ s \neq i_k, k=0,\ldots\ell}}^{\infty} \lambda_k^{-\delta} \langle h, \phi_s \rangle \phi_s \; , \qquad \text{(see I.2.7)}$$

$$U^\delta (V^\delta h) = \sum_{\substack{s=0 \\ s \neq i_k, k=0,\ldots\ell}}^{\infty} \lambda_s^\delta \langle V^\delta h, \phi_s \rangle \phi_s$$

$$= \sum_{\substack{s=0 \\ s \neq i_k, k=0,\ldots\ell}}^{\infty} \lambda_s^\delta \lambda_s^{-\delta} \langle h, \phi_s \rangle \phi_s = h - \phi_h \; .$$

Similarly

$$V^\delta (U^\delta h) = \sum_{\substack{s=0 \\ s \neq i_k, k=0\ldots\ell}}^{\infty} \lambda_s^{-\delta} \langle U^\delta h, \phi_s \rangle \phi_s \; ,$$

$$= \sum_{\substack{s=0 \\ s \neq k,\ k=0,\dots,\ell}} \langle U_s^{\delta} h, \varphi_s \rangle \psi_s = h - \phi_h .$$

Thus, $V^{\delta} U^{\delta} h = U^{\delta} V^{\delta} h = h - \phi_h$ . ...(II.1.6)

If $Z \in F(m)$ (see Definition I.3.1) and $h \in Z$, then

$$||\phi_h||_Z \leq c||h||_Z \quad \text{with} \quad c = \sum_{k=0}^{\ell} ||\psi_k||_Z ||\varphi_k||_{Z^*}$$

......(II.1.7)

since, by definition of Z , $A \subset Z \cap Z^*$.

<u>Lemma</u> II.1.3 Let $Z \in F(m)$. [see Definition I.3.1] and $\delta > 0$. Th

(a) A is dense in Z_{δ} and Z_{δ} is a Banach space with the norm (I.2.10)

(b) $(Z_{\delta})^* = (Z^*)_{-\delta}$, $Z^* \subset (Z^*)_{-\delta}$.

(c) If A is dense in Z^* , then A is dense in $(Z^*)_{-\delta}$.

(d) For $0 < \alpha < \delta$, there holds $Z_{\delta} \subset Z_{\alpha}$ and $(Z^*)_{-\alpha} \subset (Z^*)_{-\delta}$

(e) If Z is reflexive, so is Z_{δ}

(f) $M(Z) \subset M(W^{-m}(Z))$

(g) $W^{-m}(Z_{\delta}) = (W^{-m}(Z))_{\delta}$

(h) If $Z^* \in F(m)$ and Z is reflexive, then $(W^{-m}_{*}(Z_{\delta}))^* = (W^{m,0}(Z^*))_{-\delta}$

(i) $((Z^*)_{-\delta})_{\delta} = Z^*$, $(Z_{\delta})_{-\delta} = Z$.

<u>Proof of (a)</u> Z_{δ} is the space of all $f \in Z$ with $U^{\delta} f \in Z$. (See Definition I.2.10). Since $U^{\delta} A \subset A \subset Z$, we obtain $A \subset Z_{\delta} \subset Z$. We want to prove that A is dense Z_{δ}. Let $f \in Z_{\delta}$ and $\varepsilon > 0$. Since A is dense in Z and $U^{\delta} f \in Z$, there exists $\phi \in A$ such that $||U^{\delta} f - \phi||_Z < \varepsilon$. We have $\phi + V^{\delta} \phi \in A$. By definition of Z (see definition I.3.1 and I.2.8) $V^{\delta} \in [Z]$. If we let $g = U^{\delta} f - \phi$, then $g \in Z$, $V^{\delta} g \in Z$, $||g||_Z < \varepsilon$. By II.1.6, we have $V^{\delta} g =$

$V^{\delta}U^{\delta}f - V^{\delta}\varphi = f - \varphi_f - V^{\delta}\varphi$ and $U^{\delta}V^{\delta}g = g - \varphi_g$. By II.1.7, we have $||V^{\delta}g||_Z \leq ||V^{\delta}||_{[Z]}||g||_Z \leq ||V^{\delta}||_{[Z]}\varepsilon$ and

$$||U^{\delta}V^{\delta}g||_Z = ||g - \varphi_g||_Z \leq (1 + \varepsilon)||g||_Z \leq (1 + \varepsilon)\varepsilon .$$

Thus, $||f - (\varphi + V^{\delta}\psi)||_{Z_{\delta}} = ||V^{\delta}g||_Z + ||U^{\delta}V^{\delta}g||_Z$

$$\leq (||V^{\delta}||_{[Z]} + 1 + \varepsilon)\varepsilon .$$

This implies that Λ is dense in Z_{δ} .

The operator U^{δ} , being of multiplier type on Λ' , is closed on Λ' [Hille-Phillips [1], p. 545]. Since $\Lambda \subset Z$, for any sequence $\{x_n\}$ in Z , $x_n \to 0$ in Z implies $x_n \to 0$ in Λ' ..(*) This property (*) implies that the operator $U^{\delta} : Z_{\delta} \to Z$ is closed with Z_{δ} as its domain.

The graph of U^{δ} is a closed subspace of the Banach space $Z \times Z$ with the usual norm of $Z \times Z$. This implies that Z_{δ} is Banach with the norm $||f||_{Z_{\delta}} = ||f||_Z + ||U^{\delta}f||_Z \qquad (f \in Z_{\delta})$

<u>Proof of (b)</u> The space $(Z^*)_{-\delta}$ is, by definition I.2.11, the space of all $f \in \Lambda'$ with a respresentation $f = f_0 + U^{\delta}f_1$ $(f_0, f_1 \in Z^*)$ and with the norm $||f||_{(Z^*)_{-\delta}} = \text{Inf}\ \{||f_0||_{Z^*} + ||f_1||_{Z^*}\ |\ f_0 + U^{\delta}f_1 = f\ ,$ $f_0, f_1 \in Z^*\}$. The essential steps in this proof are the same as those of proposition 31.3, page 325, Treves [1].

The map $T : Z_{\delta} \to Z \times Z$ given by $Tf = (f, U^{\delta}f) \quad (f \in Z_{\delta})$

......(II.1.8)

is an isometry. The transpose map $T^* : Z^* \times Z^* \to (Z_{\delta})^*$ is <u>onto</u> by the Hahn-Banach Theorem, as every continuous linear functional f on Z_{δ} identified under (II.1.8) as a subspace of $Z \times Z$ can be extended as a continuous linear functional F on $Z \times Z$ such that $T^*F = f$. For $f \in (Z^*)_{-\delta}$ with $f = f_0 + U^\delta f_1$, $(f_0, f_1 \in Z^*)$, define $\bar{f}$ on Z_δ into C as

$$\bar{f}(\phi) = f_0(\phi) + f_1(U^\delta \phi) \qquad (\phi \in Z_\delta) . \qquad \text{......(II.1.9)}$$

The function $\bar{f}$ is well defined for the following reasons:

(i) From the definition 1.3.1 of Z, $[\{\phi_n\}]$ is dense in Z. By similar arguments employed in the proof of the fact that Λ is dense in Z_δ, we can prove that $[\{\phi_n\}]$ is dense in Z_δ.

(ii) $\langle U^\delta h, \phi_k\rangle = \langle h, U^\delta \phi_k\rangle$ $(h \in \Lambda', k \in N)$. This implies that $\langle U^\delta h, \phi\rangle = \langle h, U^\delta \phi\rangle$ $(h \in \Lambda', \phi \in [\{\phi_n\}])$.

(iii) Let $\phi \in Z_\delta$. Since $[\{\phi_n\}]$ is dense in Z_δ, there exists a sequence $\{\phi_\ell\}$ in $[\{\phi_k\}]$ such that $\phi_\ell \to \phi$ in Z and $U^\delta \phi_\ell \to U^\delta \phi$ in Z as $\ell \to \infty$. (**)

$$\bar{f}(\phi) = \langle f_0, \phi\rangle + \langle f_1, U^\delta \phi\rangle$$

$$= \lim_{\ell\to\infty} [\langle f_0, \phi_\ell\rangle + \langle f_1, U^\delta \phi_\ell\rangle] = \lim_{\ell\to\infty} [\langle f_0, \phi_\ell\rangle + \langle U^\delta f_1, \phi_\ell\rangle]$$

$$= \lim_{\ell\to\infty} [\langle f_0 + U^\delta f_1, \phi_\ell\rangle] . \qquad (***)$$

Suppose, f has two representations in $(Z^*)_{-\delta}$ as $f = f_0 + U^\delta f_1 = \lambda_0 + U^\delta \lambda_1$; $f_0, f_1, \lambda_0, \lambda_1 \in Z^*$. Then, by (***) above,

$$\langle f_0, \phi\rangle + \langle f_1, U^\delta \phi\rangle = \lim_{\ell\to\infty} [\langle f_0 + U^\delta f_1, \phi_\ell\rangle]$$

$$= \lim_{\ell\to\infty} [\langle \lambda_0 + U^\delta \lambda_1, \phi_\ell\rangle]$$

$$= \lim_{\ell\to\infty} [\langle \lambda_0, \phi_\ell\rangle + \langle U^\delta \lambda_1, \phi_\ell\rangle]$$

$$= \lim_{\ell\to\infty} [\langle \lambda_0, \phi_\ell\rangle + \langle \lambda_1, U^\delta \phi_\ell\rangle]$$

$$= \langle g_0, \phi\rangle + \langle g_1, U^\delta \phi\rangle .$$

Thus, $\langle f_0,\phi\rangle + \langle f_1, U^\delta\phi\rangle = \langle g_0,\phi\rangle + \langle g_1, U^\delta\phi\rangle$ $(\phi \in Z_\delta)$; i.e., $\tilde{f}(\phi)$ is uniquely defined for all $\phi \in Z_\delta$, i.e., $\tilde{f} : Z_\delta \to C$ is well defined (independent of the representations of f in $(Z^*)_{-\delta}$) . Since f_0 , $f_1 \in Z^*$ with $f = f_0 + U^\delta f_1$, $\tilde{f} \in (Z_\delta)^*$. The map $I : (Z^*)_{-\delta} \to (Z_\delta)^*$ given by $If = \tilde{f}$ $(f \in (Z^*)_{-\delta})$ is well defined. I is one to one, since $\{\xi_n\}$ is dense in Z_δ . For, if $f,g \in (Z^*)_{-\delta}$ with $f = f_0 + U^\delta f_1$, $g = g_0 + U^\delta g_1$, f_0 , f_1 , g_0 , $g_1 \in Z^*$ such that $If = \tilde{f} = \tilde{g} = Ig$. Then $\tilde{f}(\xi) = \tilde{g}(\xi)$ for all $\xi \in Z_\delta$. Thus, $\tilde{f}(\xi_k) = \tilde{g}(\xi_k)$ $(k \in N)$; i.e. $\langle f_0,\xi_k\rangle + \langle f_1, U^\delta \xi_k\rangle = \langle g_0,\xi_k\rangle + \langle g_1, U^\delta \xi_k\rangle$, $(k \in N)$, i.e., $\langle f_0 + U^\delta f_1, \xi_k\rangle = \langle g_0 + U^\delta g_1, \xi_k\rangle$ $(k \in N)$, i.e. $\langle f - g, \xi_k\rangle = 0$, $(k \in N)$. Since $f - g \in A'$, the above implies $f = g$ by the property (A_3') of A' . [Section 1.2]. Further I is continuous, since $\|If\|_{(Z_\delta)^*} \le \|f\|_{(Z^*)_{-\delta}}$. We want to prove that I is onto. Let $\tilde{f} \in (Z_\delta)^*$. Since T^* is onto, there exist h_0 , $h_1 \in Z^*$ such that $T^*(h_0,h_1) = \tilde{f}$. Define $v \in (Z^*)_{-\delta}$ as $v = h_0 + U^\delta h_1$. Then $Iv = \tilde{f}$. Hence, $(Z^*)_{-\delta} \cong (Z_\delta)^*$. Since, for $f \in Z^*$, $f = f + U^\delta 0 \in (Z^*)_{-\delta}$, we have $Z^* \subset (Z^*)_{-\delta}$.

Proof of (c) Let A be dense in Z^* . We want to prove that A is dense in $(Z^*)_{-\delta}$. Let $f \in (Z^*)_{-\delta}$ and $\varepsilon > 0$. Let $f = f_0 + U^\delta f_1$ with f_0 , $f_1 \in Z^*$. Since A is dense in Z^* , there exist ϕ_0 , $\phi_1 \in A$ with $\|f_i - \phi_i\|_{Z^*} < \varepsilon/2$, $i = 0,1$. Thus,

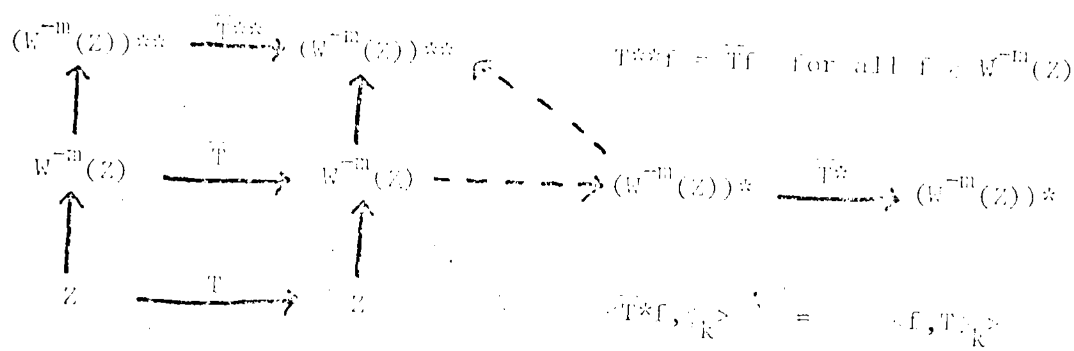
$$\phi = \phi_0 + U^\delta \phi_1 \in A \text{ and } \|f - \phi\|_{(Z^*)_{-\delta}} \le \sum_{i=0}^{1} \|f_i - \phi_i\|_{Z^*} < \varepsilon .$$

Hence A is dense in $(Z^*)_{-\delta}$.

Proof of (d) Let $0 < \alpha < \delta$. For $f \in Z_\delta$, $U^\delta f \in Z$ and $U^\alpha f = U^{\alpha-\delta} U^\delta f \in Z$, since $U^{\alpha-\delta} \in [Z]$ by definition of Z (see Definition I.3.1). Hence $Z_\delta \subset Z_\alpha$. Since A is dense in Z_α and $A \subset Z_\delta \subset Z_\alpha$, we have Z_δ is dense in Z_α and hence $(Z_\alpha)^* = (Z^*)_{-\alpha} \subset (Z^*)_{-\delta} \subset A'$.

Proof of (e) If Z is reflexive, then Z_α, identified with a closed subspace of the reflexive space $Z \times Z$ under (II.1.8), is reflexive.

Proof of (f)



In this diagram, - - - -> (resp. ——>) denotes the direction to which proof proceeds, taking transpose (resp. extension) of the operator under consideration. Since A is dense in A' , A is dense in $W^{-m}(Z)$ and $W^{-m}(Z) \subset A'$, we have $A \subset (W^{-m}(Z))^* \subset A'$. Let $\{\gamma_k\} \in M(Z)$ defining $T \in [Z]$. By Lemma II.1.2, there exists $\bar{T} \in [W^{-m}(Z)]$. $(\bar{T})^* \in [(W^{-m}(Z))^*]$ satisfying

$$\langle \bar{T}^* f, \gamma_k \rangle = \langle f, \bar{T}\gamma_k \rangle = \lambda_k \langle f, \gamma_k \rangle \qquad (k \in N\ ,\ f \in (W^{-m}(Z))^*) \quad .....(II.1.10)$$

Hence, $\{\gamma_k\} \in M((W^{-m}(Z))^{**})$ defining $\bar{T}^{**} \in [(W^{-m}(Z))^{**}]$. Since $\bar{T}^{**} f = \bar{T} f$ for all $f \in W^{-m}(Z)$, $\{\gamma_k\} \in M(W^{-m}(Z))$ defining

$T \in [W^{-m}(Z)]$.

Proof of (g) $U \in [Z]$. Both Lemma II.1.2 and Lemma II.1.3(f) imply that for $f = \sum_{j=0}^{m} D^j f_j \in W^{-m}(Z)$ with $f_j \in Z$, $j = 1,2,\ldots m$, we have

$$U^{\varepsilon}\left(\sum_{j=0}^{m} D^j f_j\right) = \sum_{j=0}^{m} D^j U^{\varepsilon} f_j \qquad \ldots\ldots(II.1.11)$$

Let $f \in W^{-m}(Z_\varepsilon)$. Then f can be written as $f = \xi_f + \sum_{j=0}^{m} D^j U^{\varepsilon} g_j$, with $g_j \in Z$, $j = 1,2,\ldots m$. By II.1.11, if $\xi = \sum_{j=0}^{m} D^j g_j$, then, $f = \xi_f + U^{\varepsilon}[\sum_{j=0}^{m} D^j g_j]$ and hence $U^{\varepsilon} f = \xi - \xi_f \in W^{-m}(Z)$. Thus, $W^{-m}(Z_\varepsilon) \subset (W^{-m}(Z))_\varepsilon$. Conversely, let $f \in (W^{-m}(Z))_\varepsilon$. Then, $U^{\varepsilon} f = \sum_{j=0}^{m} D^j g_j \in W^{-m}(Z)$ with $g_j \in Z$, $0 \le j \le m$. $f = \xi_f + U^{\varepsilon} U^{\varepsilon} f = \xi_f + \sum_{j=0}^{m} D^j U^{\varepsilon} g_j \in W^{-m}(Z_\varepsilon)$ by II.1.6 and II.1.11. This implies $(W^{-m}(Z))_\varepsilon \subset W^{-m}(Z_\varepsilon)$.

Proof of (h) If $Z^* \in \Gamma(m)$ and Z is reflexive, then $W^{m,0}(Z^*)$ is reflexive. The rest follows by steps similar to those of (b) of this lemma.

The result (i) is easy to prove.

Lemma II.1.4 Suppose X and Y are Banach subspaces of A' , each containing A as a dense subspace. Let, for $\varepsilon > 0$, $\{v_{k,\varepsilon}\} \in M(X) \cap M(Y)$ (refer to Definition I.2.8) . Then, for $0 < \theta < 1$, $1 \le q < \infty$ and/or $0 \le \theta \le 1$, $q = \infty$.

$$((X,Y)_{\theta,q})_\varepsilon = (X_\varepsilon, Y_\varepsilon)_{\theta,q} \quad ; \quad ((X^*,Y^*)_{\theta,q})_{-\varepsilon} = (X^*_{-\varepsilon}, Y^*_{-\varepsilon})_{\theta,q} \qquad (II.1.12)$$

Proof: First, we want to prove

$$((X,Y)_{\theta,q})_{\delta} \subset (X_{\delta},Y_{\delta})_{\theta,q} .$$

(See 1.2.46 for definition of above spaces). Let $f \in ((X,Y)_{\theta,q})_{\delta}$. We have to prove that $||t^{-\theta}K(t,f,X_{\delta},Y_{\delta})||_{L^q_*} < \infty$. (See 1.2.45).

Let $f_1 , g_1 \in X$, $f_2 , g_2 \in Y$ be such that $f = f_1 + f_2$, $U^{\delta} f = g_1 + g_2$.

$$\varphi_f = \varphi_{f_1} + \varphi_{f_2}$$

$$f = \varphi_f + V^{\delta} U^{\delta} f = (\varphi_{f_1} + V^{\delta} g_1) + (\varphi_{f_2} + V^{\delta} g_2) .$$

$$\varphi_{f_1} + V^{\delta} g_1 \in X_{\delta} \quad \text{and} \quad \varphi_{f_2} + V^{\delta} g_2 \in Y_{\delta} .$$

Let $C_1 = \sum_{k=0}^{\ell} ||\varphi_{i_k}||_X ||\psi_{i_k}||_{X^*}$, $C_2 = \sum_{k=0}^{\ell} ||\varphi_{i_k}||_Y ||\psi_{i_k}||_{Y^*}$.

Let $C_{11} = 1 + C_1 + C_2 + ||V^{\delta}||_{[X]} + ||V^{\delta}||_{[Y]}$.

Then $K(t,f,X_{\delta},Y_{\delta}) \le ||\varphi_{f_1} + V^{\delta} g_1||_{X_{\delta}} + t||\varphi_{f_2} + V^{\delta} g_2||_{Y_{\delta}}$

$$\le ||\varphi_{f_1} + V^{\delta} g_1||_X + ||\varphi_{U^{\delta} V^{\delta} g_1}||_X$$

$$+ t[||\varphi_{f_2} + V^{\delta} g_2||_Y + ||\varphi_{U^{\delta} V^{\delta} g_2}||_Y]$$

$$\le ||\varphi_{f_1}||_X + t||\varphi_{f_2}||_Y + ||V^{\delta}||_{[X]}||g_1||_X$$

$$+ t||V^{\delta}||_{[Y]}||g_2||_Y + ||g_1 - \varphi_{g_1}||_X + t||g_2 - \varphi_{g_2}||_Y$$

$$\le C_{11}[||f_1||_X + t||f_2||_Y + ||g_1||_X + t||g_2||_Y]$$

The above inequality holds true for every representation of f and $U^{\delta} f$ in $X + Y$ as $f = f_1 + f_2$, $U^{\delta} f = g_1 + g_2$: $f_1 , g_1 \in X$,

$f_2, g_2 \in Y$. Hence $K(t,f,X_\theta,Y_\theta) \leq C_{11}(K(t,f,X,Y) + K(t,\Omega f,X,Y))$.

This implies that $||f||_{(X_\theta,Y_\theta)_{\theta,q}} = ||t^{-\theta}K(t,f,X_\theta,Y_\theta)||_{L^q_*}$,

and $||f||_{(X_\theta,Y_\theta)_{\theta,q}} \leq C_{11}(||f||_{(X,Y)_{\theta,q}} + ||\Omega f||_{(X,Y)_{\theta,q}})$

$$= C_{11}||f||_{((X,Y)_{\theta,q})_\Omega} . \qquad ......(II.1.13)$$

Thus, $((X,Y)_{\theta,q})_\Omega \subset (X_\theta,Y_\theta)_{\theta,q}$.

Conversely, we want to prove that $(X_\theta,Y_\theta)_{\theta,q} \subset ((X,Y)_{\theta,q})_\Omega$.

Let $f \in (X_\theta,Y_\theta)_{\theta,q}$. For a fixed t with $0 < t < \infty$, there exists $f_1 \in X_\theta$, $f_2 \in Y_\theta$ with $f = f_1 + f_2$ and

$$||f_1||_{X_\theta} + t||f_2||_{Y_\theta} \leq 2K(t,f,X_\theta,Y_\theta) ,$$

$$K(t,f,X,Y) \leq ||f_1||_X + t||f_2||_Y \leq 2K(t,f,X_\theta,Y_\theta)$$

$$K(t,\Omega f,X,Y) \leq ||\Omega f_1||_X + t||\Omega f_2||_Y \leq 2K(t,f,X_\theta,Y_\theta) .$$

These inequalities imply that $f, \Omega f \in (X,Y)_{\theta,q}$ and

$||f||_{(X,Y)_{\theta,q}} \leq 2||f||_{(X_\theta,Y_\theta)_{\theta,q}}$ and $||\Omega f||_{(X,Y)_{\theta,q}} \leq 2||f||_{(X_\theta,Y_\theta)_{\theta,q}}$.

i.e. $||f||_{((X,Y)_{\theta,q})_\Omega} \leq 4||f||_{(X_\theta,Y_\theta)_{\theta,q}}$ $\qquad ......(II.1.14)$

II.1.13 and II.1.14 imply that

$$((X,Y)_{\theta,q})_\Omega = (X_\theta,Y_\theta)_{\theta,q} .$$

Using the above identity and the result of Lemma II.1.3(i) , we will prove the second identity of II.1.12

$$[(X^*,Y^*)_{\theta,q}]_- = ((\ (X^*)_-)\ ,((Y^*)_-)\ )_{\theta,q})_-$$

$$= ((((X^*)_{-\alpha}, (Y^*)_{-\beta})_{\theta,q})_{\alpha-\beta}$$

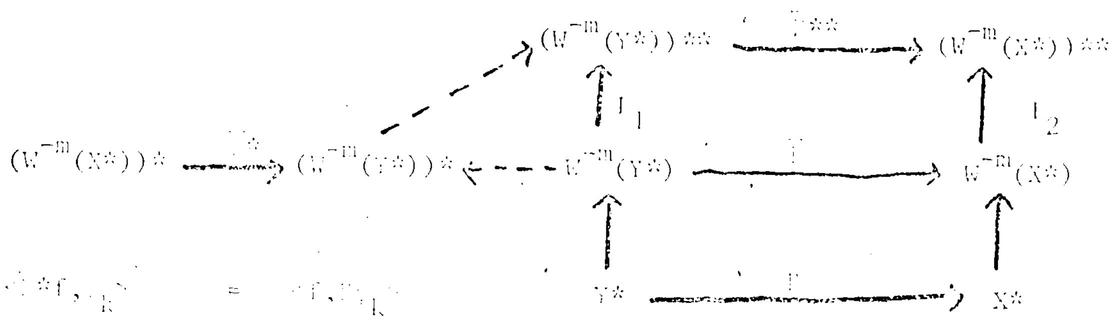
$$= ((X^*)_{-\alpha}, (Y^*)_{-\beta})_{\theta,q} \qquad \ldots\ldots(II.1.15)$$

Lemma II.1.5 Let X , Y be Banach subspaces of A' such that A is dense in both X and Y^* , $A \subset Y$, $W^{-m}(Y^*) \subset A'$. Then

$$M(X,Y) \subset M(Y^*,X^*) \subset M(W^{-m}(Y^*),W^{-m}(X^*))$$

$$\ldots\ldots(II.1.16)$$

$$M(X,Y) \subset M((W^{-m}(X^*))^*,(W^{-m}(Y^*))^*)$$



$(k \in N)$ $\qquad T^{**}I_1 = I_2 T$, I_1, I_2 are identity maps.

In this diagram, - - -> (respectively ——>) denotes the direction to which proof proceeds, taking transpose (respectively extension) of the operator under consideration.

Proof: Since A is dense in $W^{-m}(Y^*)$ and $W^{-m}(Y^*)$ is dense in A', we get $A \subset (W^{-m}(Y^*))^* \subset A'$. We have $M(X,Y) \subset M(Y^*,X^*)$. Let $\{\phi_k\} \in M(Y^*,X^*)$, defining $T \in [Y^*,X^*]$. By Lemma II.1.2, there exists a unique extension $\tilde{T} \in [W^{-m}(Y^*),W^{-m}(X^*)]$ of T . The transpose of $\tilde{T}$, $\tilde{T}^* \in [(W^{-m}(X^*))^*,(W^{-m}(Y^*))^*]$ such that

$$\langle \tilde{T}^*f, \phi_k \rangle = \langle f, \tilde{T}\phi_k \rangle$$

$$= \langle f, T\phi_k \rangle$$

$$= \langle \Phi_k f, \phi_k \rangle \qquad (k \in N,\ f \in (W^{-m}(X^*))^*)$$

(Since $cl(A,X) = X$, we get $A \subset W^{-m}(X^*) \subset A'$, $(W^{-m}(X^*))^* \subset A'$)

Thus, $\{\Phi_k\} \in M((W^{-m}(X^*))^*, (W^{-m}(Y^*))^*)$ defining $\Phi^* \in [(W^{-m}(X^*))^*, (W^{-m}(Y^*))^*]$.

$\Phi^{**} \in [(W^{-m}(Y^*))^{**}, (W^{-m}(X^*))^{**}]$ and $\Phi^{**}f = \Phi f$ for all $f \in W^{-m}(Y^*)$. Hence $\{\Phi_k\} \in M(W^{-m}(Y^*), W^{-m}(X^*)]$ defining $\Phi \in [W^{-m}(Y^*), W^{-m}(X^*)]$ and $||\Phi|| \le ||\Phi||$. Thus

$$M(X,Y) \subset M(Y^*,X^*) \subset M(W^{-m}(Y^*), W^{-m}(X^*)) .$$

The other identity follows easily.

Corollary II.1.1 Let $X \in F(m)$. [Refer to Definition I.3.1] and A is a dense subspace of X^*. Then, for $0 < \theta < 1$, $1 \le q \le \infty$

$$M(X) \subset M(W^{m,0}(X)) \subset M(W^{-m}(X)); \qquad \ldots\ldots.(II.1.17)$$

$$M(X) \subset M((W^{-m}(X), W^{m,0}(X))_{\theta,q}) .$$

Proof: Since $X \in F(m)$, $M(X) \subset M(W^{-m}(X))$ by Lemma II.1.3(f) and $(W^{m,0}(X))^* = W^{-m}(X^*)$ by Lemma II.1.1(a). Since $A \subset X \subset A'$, A is dense in both X and X^*, $W^{-m}(X + X^*) \subset A'$, by Lemma II.1.5, we have $M(X) \subset M((W^{-m}(X^*))^*) = M((W^{m,0}(X))^{**})$. The space $W^{m,0}(X)$ can be embedded as a closed subspace of $(W^{m,0}(X))^{**}$. We will prove that

$$M(X) \subset M(W^{m,0}(X)) .$$

Let $\{\Phi_k\} \in M(X)$ defining $\Phi \in [X]$. Then by the above observation $\{\Phi_k\} \in M((W^{m,0}(X))^{**})$ defining $\Phi \in [(W^{m,0}(X))^{**}]$. Let

$\Phi_0 = \Phi|_{W^{m,0}(X)}$, the restriction of Φ to $W^{m,0}(X)$. Let $f \in W^{m,0}(X)$.

Since $\{\phi_n\}$ is dense in $W^{m,0}(X)$, there exists a sequence $\{f_\ell\}$ in $\{\phi_n\}$ such that $f_\ell \to f$ in $W^{m,0}(X)$. Then $Jf_\ell \to Jf$ in $(W^{m,0}(X))^{**}$. Since $Jf_\ell \in \{J\phi_n\}$ for all ℓ , $Jf \in$ cl $(\{J\phi_n\}, (W^{m,0}(X))^{**}) = W^{m,0}(X)$. i.e. $Jf \in W^{m,0}(X)$. Thus, $\{\lambda_k\} \in M(W^{m,0}(X))$ defining $T_\lambda \in [W^{m,0}(X)]$. Since $M(X) \subset M(W^{m,0}(X)) \cap M(W^{-m}(X))$, by Theorem I.2.3, it follows that

$$M(X) \subset M((W^{m,0}(X),W^{-m}(X))_{\theta,q}) .$$

<u>Lemma</u> II.1.6 (a) Suppose $Z \in F(m,\delta)$, for some $\delta > 0$. Then

(1) $\{v_{k,\delta}\} \in M(Z,Z_\delta) \subset M(Z^*_{-\delta},Z^*) \subset M(W^{-m}(Z^*_{-\delta}),W^{-m}(Z^*))$ ......(II.1.18)

(2) $(W^{-m}(Z^*_{-\delta})) = W^{-m}(Z^*)$

(3) $W^{-m}(Z^*_{-\delta}) = (W^{-m}(Z^*))_{-\delta}$

(b) If, <u>in addition</u>, Z is reflexive, then

(1) $(W^{m,0}(Z))_\delta = (W^{-m}(Z^*_{-\delta}))^*$

(2) $W^{m,0}(Z^*_{-\delta}) = (W^{m,0}(Z^*))_{-\delta}$. Here, $Z^*_{-\delta}$ denotes $(Z^*)_{-\delta}$.

<u>Proof of (a)</u>

(Since $Z \in F(m,\delta)$, $Z \in F(m)$ (See Definitions I.3.1, I.3.2) . Hence $\{v_{k,\delta}\} \in M(Z)$ defining $U^\delta \in [Z]$. (Refer to I.2.7) . For each $f \in Z$, $U^\delta f \in Z_\delta$ and $\|U^\delta f\|_{Z_\delta} = \|U^\delta f\|_Z + \|H^\delta(U^\delta f)\|_Z$

$\le (1 + C + \|U^\delta\|_{[Z]})\|f\|_Z$. (See II.1.7) Thus, $\{v_{k,\delta}\} \in M(Z,Z_\delta)$ defining $U^\delta \in [Z,Z_\delta]$. By Lemma II.1.3(b), we have $(Z_\delta)^* = (Z^*)_{-\delta}$ Since Z , and Z_δ are Banach subspaces of A' with $A \subset Z_\delta \subset Z$, cl $(A,Z) = Z$, cl $(A,(Z_\delta)^*) = (Z_\delta)^*$, and $W^{-m}((Z^*)_{-\delta}) \subset A'$, by

Lemma II.1.5 we have

$$M(Z,Z_\delta) \subset M((Z^*)_{-\delta},Z^*) \subset M(W^{-m}((Z^*)_{-\delta}),W^{-m}(Z^*)) .$$

(2) $W^{-m}(Z^*) \subset W^{-m}((Z^*)_{-\delta})$. We will prove that for $f \in W^{-m}(Z^*)$, $U^\delta f \in W^{-m}((Z^*)_{-\delta})$. Let $f \in W^{-m}(Z^*)$ with $f = \sum_{j=0}^{m} D^j f_j$; $f_j \in Z^*$ $0 \le j \le m$. Let $g_\delta(f) = \sum_{j=0}^{m} D^j U^\delta f_j$, $g_\delta(f) \in W^{-m}((Z^*)_{-\delta})$.

$V^\delta U^\delta$ is a linear operator from Z^* into Z^* with $||V^\delta U^\delta f||_{Z^*} = ||f - \phi_f||_{Z^*} \le (1 + C)||f||_{Z^*}$ $(f \in Z^*)$ (Refer to II.1.7). Thus, $V^\delta U^\delta \in [Z^*]$. By Lemma II.1.6.a(1), $V^\delta \in [(Z^*)_{-\delta},Z^*]$. Using the results of Lemmas II.1.2 and II.1.3(f), we have

$$f = \phi_f + V^\delta U^\delta f = \sum_{j=0}^{m} D^j V^\delta U^\delta f_j + \phi_f = \phi_f + V^\delta g_\delta(f)$$

(By Lemma II.1.2)

$$U^\delta f = V^\delta U^\delta g_\delta(f) = g_\delta(f) - \phi_{g_\delta(f)} \in W^{-m}((Z^*)_{-\delta})$$

and

$$||f||_{(W^{-m}(Z^*_{-\delta}))_\delta} = ||f||_{W^{-m}(Z^*_{-\delta})} + ||U^\delta f||_{W^{-m}(Z^*_{-\delta})}$$

$$\le (2 + C)||f||_{W^{-m}(Z^*)} .$$

Thus, $W^{-m}(Z^*) \subset (W^{-m}((Z^*)_{-\delta}))_\delta$.

Conversely, we will prove that, for $f \in (W^{-m}((Z^*)_{-\delta}))_\delta$, $f \in W^{-m}(Z^*)$. Let $f \in (W^{-m}((Z^*)_{-\delta}))_\delta$. Then $f, U^\delta f \in W^{-m}((Z^*)_{-\delta})$. By II.1.18, $V^\delta(U^\delta f) \in W^{-m}(Z^*)$;i.e. $f = \phi_f + V^\delta(U^\delta f) \in W^{-m}(Z^*)$ with $\phi_f \in A$. (See II.1.5 for definition of ϕ_f). This proves that

$$W^{-m}(Z^*) = (W^{-m}((Z^*)_{-\delta}))_\delta .$$

(3) $(W^{-m}(Z^*))_{-\delta} = ((W^{-m}(Z^*_{-\delta}))_\delta)_{-\delta} = W^{-m}(Z^*_{-\delta})$

Proof of (b)

(1) If Z is reflexive, so are $W^{m,0}(Z)$ and $(W^{m,0}(Z))_\delta$. Hence $(W^{m,0}(Z))_\delta = ((W^{m,0}(Z))_\delta)^{**} = (((W^{m,0}(Z))_\delta)^*)^* = ((W^{-m}(Z^*))_{-\delta})^*$

$= (W^{-m}(Z^*_{-\delta}))^*$.

(2) $W^{m,0}(Z^*_{-\delta}) = (W^{-m}(Z_\delta))^* = ((W^{-m}(Z))_\delta)^* = (W^{-m}(Z))^*_{-\delta} = (W^{m,0}(Z^*))_{-\delta}$

Using the results of Lemmas II.1.2 to II.1.6, we will prove Theorem I.3.1.

Proof of Theorem I.3.1

(a) Let $\delta > 0$, and a reflexive space $X \in F(m,\delta)$. Using Theorem I.2.3, we have $M(X) \subset M(Y)$, where Y = any one of X , X^* , $(X,X^*)_{\theta,q}$ $0 < \theta < 1$, $1 < q < \infty$. $\{\lambda_k\} \in M(Y)$ defining $\Gamma \in [Y]$, define $\overline{\Gamma} \in [Y_{-\delta}]$ as follows.

For $f \in Y_{-\delta}$ with $f = f_1 + U^\delta f_2$; f_1 , $f_2 \in Y$ define

$$\overline{\Gamma} f = \Gamma f_1 + U^\delta \Gamma f_2 \qquad \text{......(II.1.19)}$$

It is easy to check that $\{\lambda_k\} \in M(Y_{-\delta})$ defining $\overline{\Gamma} \in [Y_{-\delta}]$. Corollary II.1.1 implies that $M(X) \subset M(E)$ for all $E \in Y(m,\delta,X)$.

(b) Let $\delta > 0$ and $X \in Q(m)$. Let X' , E_i , $1 \le i \le 4$ be as in Definition I.3.4. $M(X) \subset M(X')$ since $[\{\phi_n\}]$ is dense in X' . Hence $M(X) \subset M(E_i)$ $I = 1,3,4$ by Theoren I.2.3, and Lemma II.1.3(f). When $Z = X^*$, or Y^* , we can prove $M(Z) = M(Z_{-\delta})$ as in (II.1.19) . Hence (b) follows by theorem I.2.3 and Lemma II.1.4. Q.E.D.

Using the definition of $M(X,Y)$, we would like to give a simple characterization of the elements of $M(X_\delta,X)$ for a Banach subspace X of A' and $\delta > 0$.

Indeed, for $\{\gamma_k\} \in M(X_\delta,X)$ defining $\Gamma \in [X_\delta,X]$, we have for every $f \in X$, $V^\delta f \in X_\delta$ and hence $\Gamma(V^\delta f) \in X$. Thus, $\{\gamma_k \nu_{k,\delta}\} \in M(X,X)$ defining $\Gamma V_\delta \in [X]$ with

$$||\Gamma(V^\delta f)||_X \leq ||\Gamma||_{[X_\delta,X]}(C + ||V^\delta||_{[X]})||f||_X$$

giving the inequality $||\{\gamma_k \nu_{k,\delta}\}||_{M(X)} \leq c_1||\{\gamma_k\}||_{M(X_\delta,X)}$

......(II.1..0)

where $c_1 = (C + ||V^\delta||)$ where C is given by II.1.7

and $\gamma_k = \delta_k \lambda_k^\delta \quad k \in N$, $k \neq i_0 \ldots i_\ell$

......(II.1.21)

with $\{\delta_k\} \in M(X)$, $||\{\delta_k\}||_{M(X)} \leq c_1||\{\gamma_k\}||_{M(X_\delta,X)}$

Conversely, for $\{\eta_k\} \in M(X)$, $\{\eta_k \lambda_k^\delta\} \in M(X_\delta,X)$ with

$$||\{\eta_k \lambda_k^\delta\}||_{M(X_\delta,X)} \leq ||\{\eta_k\}||_{M(X)} \qquad ......(II.1.22)$$

Thus we have proved the following.

<u>Lemma</u> II.1.7 Let X be a Banach subspace of A' and $\delta > 0$. The $\{\gamma_k\} \in M(X_\delta,X)$ if and only if there exists $\{\eta_k\} \in M(X)$ satisfying

$$\gamma_k = \eta_k \lambda_k^\delta \quad k \in N , k \neq i_0,\ldots i_\ell . \qquad ......(II.1.23)$$

In this case $||\{\gamma_k\}||_{M(X_\delta,X)} \leq ||\{\eta_k\}||_{M(X)} \leq c_1||\{\gamma_k\}||_{M(X_\delta,X)}$

II.2 <u>Results related to multiplier type processes</u>

In this section, we will prove the main results of this thesis, related to multiplier type approximation processes on Banach subspaces of $\mathcal{X}'$ (i.e. Theorems I.3.2 and I.3.4). We will show that an approximation process on some $X \in \Gamma(m,\varphi)$ for some $\varphi > 0$ (respectively $X \in \Omega(m)$) [Refer to Definitions I.3.2 and I.3.4] satisfying either (A) Jackson type inequality or (B) Bernstein type inequality or (C) having saturation theorems on X with certain order, satisfies the same (i.e. (A), (B) or (C)) on each $Z \in Y(m,\varphi,X)$ (respectively $\Omega(m,\varphi,X)$) (Refer to Definitions I.3.3, I.3.5) with the same order. In this context, we state below a result due to H. Berens [1] [For proof, see Butzer-Nessel [1], p. 502] on the saturation of approximation processes, which we use in the proof of Theorem I.3.2.

Theorem II.2.1. Let $\varphi(t)$ be a nonnegative function on $\Re$ such that $\varphi(t) \to 0$ as $t \to t_0$. Let $\{T_t\}$ be commutative strong approximation process on X and let B be a closed linear operator with domain $D(B)$ dense in X and range in X such that for every $f \in D(B)$

$$\lim_{t \to t_0} \|(T_t f - f)/\varphi(t) - Bf\|_X = 0. \qquad \ldots\ldots(II.2.1)$$

Suppose there exists a regularization process $\{J_n\}$ i.e. $\{J_n\}_{n \in N} \subset [X]$, $\bigcup_{n \in N} J_n[X] \subset D(B)$, $\lim_{n \to \infty} \|J_n f - f\| = 0$ (for all $f \in X$) and J_n and T_t commute for all $n \in N$, $t \in \Re$. Then

(a) If $f \in X$ is such that $\|T_t f - f\|_X = o(\varphi(t))$, then $f \in D(B)$ and $Bf = 0$.

(b) The following statements are equivalent: For $f \in X$

(i) $\|T_t f - f\|_X = O(\varphi(t))$ $(t \to t_0)$, (ii) $f \in D(B)^X$ where the norm on $D(B)$ is given by $||f||_{D(B)} = ||f||_X + ||Bf||_X$ $(f \in D(B))$.

Proof of Theorem I.3.2 Let $\rho(t) > 0$, $\alpha > 0$, X , $\{\gamma_{t,k}\}$, Γ_t and Z be as given in Theorem I.3.2.

(a) Suppose for some $\delta > 0$ we have for all $f \in X_\delta$,

$$||\Gamma_t f - f||_X \leq C_1 \rho(t) ||f||_{X_\delta}$$

Then, for all t , $\{(\gamma_{t,k} - 1)/\rho(t)\}_{k \in N} \in M(X_\delta, X)$ with

$$\sup_t |||\{(\gamma_{t,k} - 1)/\rho(t)\}|||_{M(X_\delta, X)} = \sup_t ||(\Gamma_t - I)/\rho(t)||_{[X_\delta, X]}$$

$$\leq C_1 < \infty \qquad \text{......(II.2.2)}$$

By Lemma II.1.7, for each t, there exists $\{\eta_{t,k}\}_{k \in N}$ belonging to $M(X)$ satisfying

$$(\gamma_{t,k} - 1)/\rho(t) = \eta_{t,k} \lambda_k \qquad k \in N , \quad k \neq i_0 \ldots i_\ell$$

$$\text{......(II.2.3)}$$

and $\sup_t |||\{\eta_{t,k}\}|||_{M(X)} \leq e_1 C_1 < \infty$, with e_1 given by II.1.20. By Theorem I.3.1, we have for each t ,

$$\{\eta_{t,k}\} \in M(Z) \text{ with } \sup_t |||\{\eta_{t,k}\}|||_{M(Z)} \leq e_1 C_1 < \infty .$$

Hen[illegible] n by Lemma II.1.7, we obtain for each t ,

$$\{(\gamma_{t,k} - 1)/\rho(t)\}_{k} \in M(Z_\delta, Z) \text{ defining } (\Gamma_t - I)/\rho(t)$$

and satisfying for all $f \in Z_\delta$, $||(\Gamma_t f - f)/\rho(t)||_Z \leq e_1 C_1 ||f||_{Z_\delta}$.

$$\text{......(II.2.4)}$$

Let $f \in Z_\delta$. Let $\{f_k\}$ be a sequence in Z_δ satisfying

$\sup_k ||f_k||_{Z_\delta} \le 2||f||_{Z_\delta}$, $||f_k - f||_Z \to 0$ as $k \to \infty$.

Then for each t, $(\Gamma_t f_k - f_k)/\lambda(t) \to (\Gamma_t f - f)/\lambda(t) \qquad (k \to \infty)$.

Thus

$$||(\Gamma_t f - f)/\lambda(t)||_Z \le \limsup_k ||(\Gamma_t f_k - f_k)/\lambda(t)||_Z \le e_1 C_1 \sup_k ||f_k||_{Z_\delta}$$

$$\le 2e_1 C_1 ||f||_{Z_\delta} \qquad \ldots\ldots (II.2.5)$$

(b) Suppose for some $\delta > 0$ we have for all $f \in X$, and for all t, $\Gamma_t f \in X_\delta$ and $||\Gamma_t f||_{X_\delta} \le C_2(\lambda(t))^{-1}||f||_X$. Then we have for every t, $\{\lambda(t)\gamma_{t,k}\lambda_k^\delta\} \in M(X)$ with $||\{\lambda(t)\gamma_{t,k}\lambda_k^\delta\}||_{M(X)} \le C_2$.

By theorem I.3.1, we have $\{\lambda(t)\gamma_{t,k}\lambda_k^\delta\}_{k \in P} \in M(Z)$ for all t.

i.e. $\lambda(t)||U^\delta \Gamma_t f||_Z \le C_2 ||f||_Z \qquad (f \in Z) \qquad \ldots\ldots (II.2.6)$

By Theorem I.3.1, $\{\lambda_k^{-\delta}\} \in M(Z)$ defining $U^{-\delta} \in [Z]$. For $f \in Z$,

$\lambda(t)\Gamma_t f = U^{-\delta}[\lambda(t)U^\delta \Gamma_t f] + \lambda(t)\Gamma_t \phi_f \qquad \ldots\ldots (II.2.7)$

(ϕ_f defined by II.1.3) and hence for a constant $C = C(\lambda_{i_0}, \ldots, \lambda_{i_\ell}) > 0$

we have $\lambda(t)||\Gamma_t f||_Z \le C_2(C + ||U^{-\delta}||_{[Z]})||f||_Z \qquad \ldots\ldots (II.2.8)$

Hence, for all $f \in Z$ and for all t, $\Gamma_t f \in Z_\delta$, and

$$||\Gamma_t f||_{Z_\delta} \le C_{22}(\lambda(t))^{-1}||f||_Z \qquad (f \in Z)$$

($C_{22} = C_2 e_1$ where e_1 is given by II.1.20).

(c) Let $Z = Y(m, \beta, X)$ (respectively, $Q(m, \beta, X)$). Suppose the conditions of the hypothesis of Theorem I.3.2(c) are satisfied by X, $\{\phi_n\}$, Γ_t, $\{\gamma_{t,k}\}$. Then, by (a) of this theorem, we have

$$\|G_t f - f\|_Z \le C_{11}(t)\|f\|_{Z_\phi} \qquad (f \in Z_\phi) \qquad \ldots\ldots(II.2.9)$$

Then (c) follows by Theorem II.2.1.

For $f \in Z_\phi$, let $T_t f = (G_t f - f)/\Lambda(t) + \psi(\mathcal{U}) f$ ......(II.2.10)

By Uniform boundedness principle, we have

$$\sup_{t} \|T_t\|_{[Z_\phi, Z]} < \infty \qquad \ldots\ldots(II.2.11)$$

For every $k \in N$, $T_t \phi_k = [(G_{t,k} - 1)/\Lambda(t) + \psi_k]\phi_k \to 0$ as $t \to t_0$.

Since $\{\phi_k\}$ is dense in Z_ϕ, the Banach Steinhaus Theorem implies that for all $f \in Z_\phi$, $(G_t f - f)/\Lambda(t) \to -\psi(\mathcal{U}) f$ in Z as $t \to t_0$

......(II.2.12)

Case 1 Suppose $\{\phi_n\}$ is dense in Z. $-\psi(\mathcal{U})$ is a closed operator with a dense domain Z_ϕ and range in Z. We will show that

(i) For every $f \in Z_\phi$, $(G_t f - f)/\Lambda(t) \to -\psi(\mathcal{U}) f$ in Z

......(II.2.13)

(ii) There exists $\{J_n\}_{n \in N} \subset [Z]$ such that $\bigcup_{n \in N} J_n(Z) \subset Z_\phi$;

$J_n f \to f$ in Z for every $f \in Z$ and J_n and G_t commute for all n and t.

Let $R_n f = (1 - k/(n+1))\langle f, \phi_k^* \rangle \phi_k$ ......(II.2.14)

By Theorem I.3.1, the pair Z, $\{\phi_n\}$ satisfy the property (C) (refer to definition I.2.3). Since the function

$$r(x) = \begin{cases} 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

is quasiconvex on $[0, \infty)$, by Theorem I.2.1, we obtain that

$\{R_n\} \subset [Z]$, $R_n(Z) \subset A \subset Z_\phi$ for all n, $\sup_n \|R_n\|_{[Z]} \le M_1$,

......(II.2.15)

$R_n f \to f$ in Z for all $f \in Z$; R_n and G_t commute for all n and t.

Thus J_n R_n satisfies (II.2.13) .

Case 2 Z = dual of a Banach subspace $F \subset A'$ with F containing A as a dense subspace. We only have to prove the following:

For $f \in Z$

$$||T_t f - f||_Z = \begin{cases} o(\phi(t)) & \text{implies } f \in A \\ O(\phi(t)) & \text{implies } f \in \widetilde{Z} \end{cases}$$

Let $f \in Z$ be such that $||T_t f - f||_Z = O(\phi(t))$. Since bounded sets are weak* compact, there exists $f^0 \in Z$ and $\{t_q\}_{q \in P}$ such that $t_q \to t_0$ as $q \to \infty$, $(T_{t_q} f - f)/\phi(t_q) \to f^0$ as $q \to \infty$, in the weak* topology on Z . For each $k \in N$,

$$<(T_{t_q} f - f)/\phi(t_q), \psi_k> = ((\tau_{t_q}^k - 1)/\phi(t_q)) <f, \psi_k> \quad \ldots\ldots(II.2.16)$$

$\to -c(\mathcal{U}^\lambda f, \psi_k> = <f^0, \quad$ as $t \to t_0$.

Hence $(-)\mathcal{U}^\lambda f = -(1/c) f^0 \in Z$; i.e. $f \in Z_\lambda = \widetilde{Z}_\lambda$. If big O is replaced by small o , then $\mathcal{U}^\lambda f = 0$, i.e. $f \in A_\lambda$.

Proof of Theorem I.3.3

$$\text{Let } p \in P \text{ such that } \sum_{\substack{k=0 \\ k \neq i_1 \ldots i_\ell}}^{\infty} |\lambda_k|^{-2p} < M_0 < \infty \quad \ldots\ldots(II.2.17)$$

(a) Suppose $||\mathcal{U}^k D_n||_{L^2(I)} \le M_1(|\lambda_n|^{s+k}) \quad (k \in N \ , \ n \in N)$ ......(II.2.18)

($s \in P$, independent of $n \in N$, $k \in N$)

Let $\xi \in A$. $D_\xi := \sum_{n=0}^{\infty} <\xi, \psi_n> D_n \in C^\infty(I)$

$$||\mathcal{U}^k D_\xi||_{L^2(I)} \le \sum_{n=0}^{\infty} |<\xi, \psi_n>| \, ||\mathcal{U}^k D_n||_{L^2(I)}$$

$$\leq \sum_{n=0}^{\infty} |\langle \phi, \psi_n \rangle| \, |\lambda_n|^{s+k}$$

$$\leq M_0 M_1 \left\{ \sum_{n=0}^{\infty} |\langle \phi, \psi_n \rangle|^2 |\lambda_n|^{2(s+k+m)} \right\}^{1/2} < \infty$$

......(II.2.19)

Hence $\mathcal{U}^k D\phi \in L^2(I)$ for all $k \in N$. Since $D\phi$, ψ_n belong to the domain of $\mathcal{U}^k$ in $L^2(I)$ we have

$$\langle \mathcal{U}^k D\phi, \psi_n \rangle = \langle D\phi, \mathcal{U}^k \psi_n \rangle \qquad (k \in N, \; n \in N) \qquad ......(II.2.20)$$

Hence $D\phi \in A$ (refer to I.2.3) ......(II.2.21)

Let $\{\phi_n\}$ be a sequence in A such that $\phi_n \to \phi$ in A $(n \to \infty)$ i.e. for each $p \in N$, $\sum_{k=0}^{\infty} |\langle \phi_n, \psi_k \rangle - \langle \phi, \psi_k \rangle|^2 |\lambda_k|^{2p} \to 0$ as $n \to \infty$.

For all $p \in N$, $\|\mathcal{U}^p (D\phi_n - D\phi)\|_{L^2(I)} \leq \sum_{k=0}^{\infty} |\langle \phi_n - \phi, \psi_k \rangle| \, \|\mathcal{U}^p D\psi_k\|_{L^2(I)}$

$$\leq M_0 M_1 \left\{ \sum_{k=0}^{\infty} |\langle \phi_n - \phi, \psi_k \rangle|^2 |\lambda_k|^{2(s+p+m)} \right\}^{1/2} \to 0 \text{ as}$$

$n \to \infty$. Hence $D\phi_n \to D\phi$ in A $(n \to \infty)$. This implies that the mappings $D : A \to A$, $D : A' \to A'$ are continuous with $D = \frac{d}{dx}$.

(b) Let $m \in P$. Let, for all $k \in N$ with $0 \leq k \leq m$

$$\|D^k \psi_n\|_{X \cap X^*} \leq M_2 |\lambda_n|^{s_k} \qquad ......(II.2.22)$$

($s_k \in P$, depending only on k; M_2 a constant > 0).

For $\phi \in A$, $\|D^k \phi\|_{X \cap X^*} \leq \sum_{n=0}^{\infty} |\langle \phi, \psi_n \rangle| \, \|D^k \psi_n\|_{X \cap X^*}$

$$\leq M_2 M_0 \|\mathcal{U}^{s_k+m} \phi\|_{L^2(I)} < \infty, \quad 0 \leq k \leq m.$$

Thus, $(-1)^k D^k : A \to X$, $(-1)^k D^k A \to X^*$ are continuous. Hence (b) follows.

(d) By steps similar to those in the proof of (b), we can show,

$$||U^{k}D^{k}\phi||_{X\cap X^*} \leq \text{const.}\ ||U^{s_{k,1}+n}\phi||_{L^2(I)} \qquad \ldots\ldots(II.2.23)$$

$(0 \leq k \leq m)$. Thus $(-1)^k U^{k}D^k : A \to X$, $(-1)^k U^{k}D^k : A \to X^*$ are continuous. Hence, $D^k U^{k}\theta : X + X^* \to A'$, $0 \leq k \leq m$ is continuous.

(e) (i) For all $\phi \in A$, $||\phi||_{X\cap X^*} \leq \text{const.}\ ||\phi||_{X\cap Y}$

$$\leq \text{const.}\ ||U^{c+n}\phi||_{L^2(I)} \qquad \ldots\ldots(II.2.24)$$

This gives $A \subset X \cap Y$. Since $D(I)$ is dense in both X and Y , A is dense in both X and Y . Further $X + X^* \subset Y^* + X^* \subset A'$.

Let $||\phi_n||_{X\cap X^*} \leq a_1|\lambda_n|^{s}$ $(a_1 > 0$, $s \in P$, independent of $n \in N)$

$$||D\phi_n||_{X\cap X^*} \leq \sum_{q=0}^{c_1} |C_q^n|\ ||\phi_{n_q}||_{X\cap X^*} \leq a_1 c_1 c_2^{s}|\lambda_n|^{q_1+c_2 s} \qquad \ldots\ldots(II.2.25)$$

$$D^2\phi_n = \sum_{n=0}^{c_1} C_q^n\ D\phi_{n_q} \qquad \ldots\ldots(II.2.26)$$

This gives $||D^2\phi_n||_{X\cap X^*} = O(|\lambda_n|^{q_1+q_1q_2+q_2^2 s})$

By similar arguments,

$$||D^k\phi_n||_{X\cap X^*} = O(|\lambda_n|^{s_k}) \qquad \ldots\ldots(II.2.27)$$

$(s_k \in P$, depending only on $k \in P)$. Hence

$$W^{-p}(X + X^*) \subset A' \quad \text{for all} \quad p \in P \text{ , by (b) .}$$

For all $n \in N$, $k \in N$, we can write

$$D^k\phi_n = \sum_{q=0}^{N_k} C_{k,q}^n\ \phi_{n,k,q} \qquad \ldots\ldots(II.2.28)$$

where $N_k \in P$, depends only on k , $C_{k,q}^n$ constants, with

$$\sum_{k=0}^{N_k} |C^n_{k,q}| = O(|\lambda_n|^{d_k}) \quad , \quad \sup_{0 \le q \le N_k} \|\psi_{n,k,q}\| = O(|\lambda_n|^{d'_k}) \quad \ldots (II.2.29)$$

$d_k, d'_k \in P$ depending on k . This implies for $r > 0$, $k \in N$,

$$U^r D^k \phi_n = \sum_{q=0}^{N_k} C^n_{k,q} \lambda_{n,k,q} \psi_{n,k,q} \; , \; \|U^r D^k \phi_n\|_{X_r X^*} = O(|\lambda_n|^{s_{k,r}}) \ldots (II.2.30)$$

with $s_{k,r} = d_k + d'_k (r + s)$. Hence by (d)

$$W^{-m}(X^*_{-r} + X_{-r}) \subset A'$$

(ii) The map $T : W^m(X) \to \underbrace{X \times X \times \ldots \times X}_{(m+1) \text{ times}} = E$

given by $Tf = (f, Df, D^2 f, \ldots, D^m f) \qquad (f \in W^m(X)) \qquad \ldots\ldots (II.2.31)$

is an isometry. $T^* : \underbrace{X^* \times X^* \times \ldots \times X^*}_{(m+1) \text{ times}} = E^* \to (W^m(X))^*$ is onto

by the Hahn-Banach Theorem. Suppose, for some $n_0 \in P$, $\phi_{n_0} \notin W^{m,0}(X)$. Since $A \subset W^m(X)$, there exists $f' \in (W^m(X))^*$ with $\langle f', \phi_{n_0} \rangle \neq 0$ such that $\langle f', \phi \rangle = 0$ for all $\phi \in W^{m,0}(X)$. Since T^* is onto, $f' = T^*(f_0, f_1, \ldots, f_m)$ with $f_i \in X^*$, $0 \le i \le m$. Define

$$v = \sum_{j=0}^{m} (-1)^j D^j f_j \qquad \ldots\ldots (II.2.32)$$

Now, $v \in W^{-m}(X^*)$.

$$\langle v, \phi \rangle = \langle \sum_{j=0}^{m} (-1)^j D^j f_j , \phi \rangle = \sum_{j=0}^{m} \langle f_j , D^j \phi \rangle$$

$$= \langle (f_0, f_1, \ldots, f_m), T\phi \rangle = \langle T^*(f_0, f_1, \ldots, f_m), \phi \rangle$$

$$= \langle f', \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{D}(I) .$$

Thus, $v = 0$ in $W^{-m}(X^*) \subset A'$. Hence $\langle v, \phi_k \rangle = 0$ for all $k \in N$. But $\langle v, \phi_{n_0} \rangle \neq 0$. This leads to a contradiction. Hence $\phi_n \in W^{m,0}(X)$ for all $n \in N$. For $\phi \in A$, let $\phi_n = \sum_{k=0}^{n} \langle \phi, \psi_k \rangle \phi_k \quad (n \in P)$.

$\{\phi_n\} \subset W^{m,0}(X)$. Using the estimates II.2.28 and II.2.29, we can prove $\phi_n \to \phi$ in $W^m(X)$-norm. Then $\phi \in W^{m,0}(X)$, since $W^{m,0}(X)$ is a norm-closed subset of $W^m(X)$. Thus $A \subset W^{m,0}(X)$. Since $D(I) \subset A \subset W^{m,0}(X)$, A is dense in $W^{m,0}(X)$. This implies $[\{\phi_n\}]$ is dense in $W^{m,0}(X)$.

II.3 Applications

In this section, we illustrate our main results of this chapter by citing examples of spaces belonging to $F(m,\delta)$ $\delta > 0$ [Refer to definition I.3.2]; of multiplier type approximation processes satisfying Jackson and/or Bernstein type inequalities on Banach subspaces of A'; and of orthonormal systems $\{\phi_n\}$ satisfying the conditions of Theorems I.3.2 and I.3.3.

II.3.1 Examples of Spaces Belonging to $F(m,\delta)$ ($\delta > 0$) or $Q(m)$

Suppose for all $f \in L_1(I) + L_\infty(I)$, $D^k f \in A'$

$$\qquad \text{......(II.3.1)}$$

$0 \le k \le m$, and $A \subset L_1(I) + L_\infty(I)$.

Then $[\{\phi_n(x)\}]$ is dense in $L_p(I)$ $1 \le p < \infty$ $Co(I)$. For $\delta > 0$, let

$$P_{m,\delta} = \{p \mid 1 \le p \le \infty \; ; \; \{\psi_{k,\delta}\} \in M(L_p) \text{ for all } \varepsilon > 0 , \text{ for all } f \in (L_p)_{-\delta} + (L_{p'})_{-\delta} , \; D^k f \in A' , \; 0 \le k \le m\}$$

$$\text{......(II.3.2)}$$

Then, for all $p \in P_{m,\delta}$, $L_p \in F(m,\delta)$ and L_p is reflexive.

$L_1(I)$ and $Co(I) \in Q(m)$. $Q(m,\delta,L_1(I)) \supset \bigcup_{p \in P_{m,\delta}} Y(m,\delta,L_p)$

$$\text{......(II.3.3)}$$

$Y(m,\delta,L_p) = \{L_q(I) \mid q$ lies between p and $p'\}$ $(p \in P_{m,\delta})$

(Here $Co(I) = C(I)$ if I is a finite interval) . For $I = R$, $1 < p < \infty$, $1 < q < \infty$, the reflexive spaces $(L_p)^{\sim}$, $L^{p,q}(R)$ belong to $F(m,\delta)$; $(L_1)^{\sim}$ and $(C_0(R))^{\sim} \in Q(m)$. Refer to Katznelson [1] for $(L_p)^{\sim}$ spaces and to Butzer-Berens [[1], p. 181] for $L^{p,q}$ spaces. In Section I.3.3, we cite examples of orthonormal systems $\{\phi_n\}$ satisfying II.3.1 with $P_{m,\phi} \neq \phi$.

II.3.2 Examples of multiplier type approximation processes

Here, we cite examples of multiplier type approximation processes satisfying (i) Jackson and Bernstein type inequalities on Banach subspaces of A' , and (ii) conditions of the hypothesis of Theorem I.3.2(c).

Let, for $\delta > 0$, $g_\delta(v)$ = any one of the functions $r_{\delta,k}(v)$, $W_\delta(v)$, $C(v)$ ($v > 0$, $k > 1$) , where

$$r_{\delta,k}(v) = \begin{cases} (1 - v^\delta)^k & \text{if } 0 < v < 1 \\ 0 & \text{if } v > 1 \end{cases} \qquad \text{......(II.3.4)}$$

is the fourier transform of Riesz kernel [Butzer-Nessel [1], p. 469]; $W_\delta(v) = e^{-v^\delta}$, $v > 0$ is the fourier transform ......(II.3.5) of the generalized Weierstrass kernel [Butzer-Nessel [1], p. 465]. $C_\delta(v) = 1/(1 + v^\delta)$ $v > 0$ is the fourier transform of ......(II.3.6) the picard kernel [Butzer-Nessel [1] p. 464]. Let $Z \in F(m,\delta)$ be reflexive (respectively $Z \in Q(m)$) such that the pair Z , $\{\phi_n\}$ satisfy (C^{δ}) [Refer I.2.15]. Let

$\lambda_k = (k + b)^s$, $s > 0$, $b > 0$; $\psi_\delta(n) = \lambda_{n+1}^{-\delta} = (n + 1 + b)^{-\delta s}$

and $\gamma_{n,\delta}^{k} = g_\delta(\lambda_k/\lambda_{n+1})$ ......(II.3.7)

Since $g_\delta(v)$, $v^{\delta} g_\delta(v)$, $(1 - g_\delta(v))/v^{\delta}$ are quasi-convex $C_0(0, \infty)$ functions, by Theorem I.2.1, we obtain that

$$\{\gamma^k_{n,\delta}\}_{k \in N, n \in P} , \{(1 - \gamma^k_{n,\delta})/\lambda^{\delta}_{k}\}_{k \in N, n \in P} \subset M(Z) ,$$

$$\{\rho_\delta(n)\lambda^{-\delta}_{k} \gamma^k_{n,\delta}\}_{k \in N, n \in P} \subset M(Z) \qquad \ldots\ldots(II.3.8)$$

This implies that the approximation process $\{T_{n,\delta}\}_{n \in P} \subset [Z]$ defined by

$$T_{n,\delta} f \sim \sum_{k=0}^{\infty} \gamma^k_{n,\delta} \langle f, \phi_k \rangle \phi_k \quad (f \in Z) \qquad \ldots\ldots(II.3.9)$$

satisfies both Jackson and Bernstein type inequalities on Z with respect to Z_δ of order $\rho_\delta(n)$. Further, $(1 - \gamma^k_{n,\delta})/\rho_\delta(n) \to c\lambda^{\delta}_{k}$ $(n \to \infty)$ for each fixed $k \in N$ and for some $c \neq 0$ .....(II.3.10)

Hence $\{T_{n,\delta}\}$ satisfies the hypothesis of Theorem I.3.2(c).

II.3.3 Examples of orthonormal systems $\{\phi_n\}$

Here, we cite many classical orthonormal systems as examples of orthonormal systems $\{\phi_n\}$ satisfying II.3.1 with $P_{m,\delta} \neq \phi$, the various conditions of the hypothesis of Theorem I.3.3, and on a suitable Banach subspace X of the corresponding space A' the pair X , $\{\phi_n\}$ satisfies (CI) [See I.2.15]

(1) Hermite functions: $I = (-\infty, \infty)$, X = any one of $L^p(-\infty, \infty)$, $1 \leq p < \infty$ or $C_0(-\infty, \infty)$. $U = -e^{x^2/2} \frac{d}{dx} e^{-x^2} \frac{d}{dx} e^{x^2/2} = -D^2 + x^2 - 1$.

$\phi_n(x) = (e^{-x^2/2} H_n(x))/[2^n n! \sqrt{\pi}]^{1/2}$, $n \in N$, with $H_n(x)$ = Hermite polynomial of order n . $\lambda_n = 2n$, $(n \in N)$, $\lambda_0 = 0$. Hence, $\Lambda = \{ce^{-x^2/2} \mid c \in R\}$, $A = S$, $A' = S'$. A.H. Zemanian [[1], [2],

p. 265].

(i) X, $\{\varphi_n\}$ satisfy (C^1). [Poiani [1]].

(ii) $\frac{d}{dx}\varphi_n(x) = \sqrt{(n/2)}\,\varphi_{n-1} - \sqrt{((n+1)/2)}\,\varphi_{n+1}$.

(iii) $||\varphi_n||_{X\cap X^*} = O(n^{1/4})$

(iv) $||U^k D\varphi_n||_{L^2} = O(\lambda_n^{k+1})$, $k \in N$

(v) For all $\delta > 0$, $\{\nu_{k,\delta}\}_{k\in N} \in M(X)$.

(2) _Laguerre functions, $\alpha = 0$ case:_ $I = [0,\infty)$, X = any one of $L^p[0,\infty)$, $1 \le p < \infty$, or $C_0[0,\infty)$. $U = -e^{x/2}\frac{d}{dx}\,x e^{-x}\frac{d}{dx}\,e^{x/2} = -xD^2 + D + x/4 - 1/2$, $\varphi_n(x) = e^{-x/2}\sum_{m=0}^{n}\binom{n}{m}(-x)^m/m!$, $(n \in N)$, $\lambda_n = n$, $(n \in N)$.

(i) $\lambda_0 = 0$, $Z = \{ce^{-x/2} \mid c \in R\}$

(ii) X, $\{\varphi_n\}$ satisfy (C^1) [See Poiani [1]].

(iii) $\frac{d}{dx}\varphi_n(x) = -(1/2)\varphi_n - \sum_{k=0}^{n-1}\varphi_k(x)$, $||\varphi_n||_{X\cap X^*} = O(n)$,

$||U^k D\varphi_n||_{L^2(0,\infty)} = O(n^{k+1})$, for all $\delta > 0$, $\{\nu_{k,\delta}\}_{k\in P} \in M(X)$.

(3) _Laguerre functions, $\alpha \neq 0$ case:_ $I = [0,\infty)$, X = any one of $L^p[0,\infty)$, $C_0[0,\infty)$, $1 \le p < \infty$. Let $m \in P$. Let $\alpha > 2m - 1$, _α, m_ fixed. $U_\alpha = -x^{-\alpha/2} e^{x/2}\frac{d}{dx}\,e^{-x}x^{\alpha+1}\frac{d}{dx}\,e^{x/2}x^{-\alpha/2}$

$$= -[xD^2 + D - x/4 + \alpha^2/4x + (\alpha+1)/2] ;$$

$\varphi_n^{(\alpha)} = [\Gamma(n+1)/\Gamma(n+\alpha+1)]^{1/2} x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x)$ with $\{L_n^{(\alpha)}(x)\}_{n\in N}$ [illegible] generalized Laguerre polynomials, $\lambda_n = n$.

(i) $\lambda_0 = 0$, $Z = \{cx^{\alpha/2}e^{-x/2} \mid c \in R\}$,

(ii) X , $\{\phi_n\}$ satisfy (C^1) [See Poiani [1]] ;

(iii) $\|\phi_n\|_{X \cap X^*} = O(n)$

(iv) $$\frac{d}{dx}\phi_n^{(\alpha)} = (\alpha/2) \sum_{k=0}^{n} \sum_{\ell=0}^{k} [n!\,\Gamma(\ell+1)/\Gamma(n+\alpha+1)\,\ell!]^{1/2}\,\phi_k^{(\alpha-2)}$$

$$-(1/2)\phi_n^{(\alpha)}(x) - \sum_{k=0}^{n-1} [(n!/k!)\,\Gamma(k+\alpha+1)/\Gamma(n+\alpha+1)]^{1/2}\phi_k^{(\alpha)}$$

(v) $\|\mathcal{U}^k D\phi_n^{(\alpha)}\|_{L^2[0,\infty)} = O(n^{k+2})$, $0 \le k \le m$: for all $\delta > 0$,

$\{\nu_{k,\delta}\}_{k \in P} \in M(X)$.

(4) Legendre functions: $I = (-1,1)$, X = any one of $L^p(-1,1)$, $1 \le p < \infty$, $C(-1,1)$. $\mathcal{U} = \frac{d}{dx}(x^2-1)\frac{d}{dx} - (1/4)$, $\phi_n(x) = \sqrt{n+(1/2)}\,P_n(x)$

$P_n(x)$ = Legendre polynomial of degree n . $\lambda_n = (n+1/2)^2$, $\Lambda = \{0\}$.

(i) X , $\{\phi_n\}$ satisfy (C^1) , [See Askey-Hirschman Jr. [1]].

(ii) $$\phi_n'(x) = \sum_{k=1}^{[(n+1)/2]} [(n+1-2k)/\sqrt{2n+(7/2)-4k}]\,\phi_{2n-4k+3}(x)$$

(iii) $\|\mathcal{U}^k D\phi_n\|_{L^2(-1,1)} = O(\lambda_n^{k+1})$, $k \in N$.

(iv) For all $\beta > 0$, $\{(k+(1/2))^{-2\beta}\} \in M(X)$, $k \in N$.

(5) Jacobi Functions: $I = (-1,1)$, $m \in P$. Let $\kappa > 0$. Let $\kappa_0 = \kappa$ if $\kappa \in P$, $\kappa_0 = [\kappa] + 1$ otherwise. Let $\alpha > 2(m+\kappa_0) + 1$; $\beta > 2(m+\kappa_0) + 1$, m , κ , α , β all fixed. $W_{\alpha,\beta} = (1-x)^{\alpha}(1+x)^{\beta}$;

$$\mathcal{U}^{\alpha,\beta} = (1/\sqrt{W_{\alpha,\beta}})\frac{d}{dx}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}(1/\sqrt{W_{\alpha,\beta}}) + (\alpha+\beta+1)^2/4 ;$$

$$P_n^{(\alpha,\beta)} = (1/2^n)\sum_{k=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^{n+m}$$ are Jacobi polynomials

[See Pollard [1]]. $\phi_n^{(\alpha,\beta)} = (\sqrt{W_{\alpha,\beta}(x)})\,P_n^{(\alpha,\beta)}/(\sqrt{h_n^{(\alpha,\beta)}})$ where

$h_n^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(\beta+n+1)/n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$;

$\lambda_{n,\alpha,\beta} = [n + ((\alpha+\beta+1)/2)]^2$. Let X = any one of $L^p(-1,1)$, $1 \le p < \infty$ or $C(-1,1)$. Then, by direct computation, it can be shown that

(i) $||D\phi_n^{k(\alpha,\beta)}||_{X \to X^*} = O(|\lambda_{n,\alpha,\beta}|^{s_k})$, $||U^k D^k \phi_n^{(\alpha,\beta)}||_{X \to X^*} = O(\lambda_{n,\alpha,\beta}^{\ell_k})$, $0 \le k \le m$, s_k , $\ell_k \in P$, depending only on k .

(ii) $\Lambda = \{0\}$.

(iii) If $P_{1,\alpha,\beta} = \{p \mid 1 < p < \infty$, L^p and $\{\phi_n^{(\alpha,\beta)}\}$ satisfy $(C^1)\}$ Then $(4/3,4) \subset P_{1,\alpha,\beta}$ (See C.M. Wing [1]) for all $\delta > 0$ $\{(k + (\alpha+\beta+1)/2)^{-2\delta}\}_{k \in N} \in M(L^p(-1,1))$ for all $p \in P_{1,\alpha,\beta}$.

(6) Trignometric functions (First Form): X = any one of $L^p(-\pi,\pi)$, $1 \le p < \infty$ or $C(-\pi,\pi)$. $U = i^{-1/2}\frac{d}{dx}i^{1/2} = -iD$. $\phi_n(x) = e^{inx}/\sqrt{2\pi}$, $n \in Z$, $\lambda_n = n$, $\lambda_0 = 0$, $\Lambda = \{0\}$, $||U^k D\phi_n||_{L^2(I)} = O(|n|^{k+1})$, $(k \in P$, $n \in Z)$, for all $\delta > 0$, $v_{k,\delta} = 1/|k|^\delta$ $k \neq 0$, $k \in Z$, then $\{v_{k,\delta}\} \in M(X)$.

(7) (Second Form): $I = (0,\pi)$, $U = -D^2$, $\phi_n(X) = \sqrt{2/\pi}\cos nx$, $n \in N$, $\lambda_n = n^2$, $\lambda_0 = 0$, Λ = constants , $||U^k D\phi_n(x)||_{L^2(0,\pi)} = O(n^{2k+1})$, $(k \in P$, $n \in N)$. For $\delta > 0$, $\{v_{k,\delta}\} \in M(X)$.

(8) (Third Form): $I = (0,\pi)$, $U = -D^2$, $\phi_n(x) = \sqrt{2/\pi}\sin nx$, $\lambda_n = n^2$.

(9) Multiple Fourier Series: Let R^n be the n-fold Cartesian product of R and denote u , v elements of R^n . Then $v = (v_1, \ldots, v_n)$. Let Z^n be the n-fold Cartesian product of Z with elements

$m = (m_1, \ldots, m_n)$. T^n , the n-dimensional torus, is given by $T^n = \{v \in R^n$, $|v_i| \le \pi$, $1 \le i \le n\}$. Let $X = L^p(T^n)$, $1 \le p < \infty$ or $C(T^n)$, the set of all functions f , 2π-periodic in each coordinate, with the standard norm $\|f\|_X$ equal to

$$\{\int_{T^n} |f(v)|^p dv\}^{1/p} \quad (1 \le p < \infty) \quad \text{or} \quad \max_{v \in R^n} |f(v)|$$

respectively.

$\phi_m(v) = e^{im\cdot v}/(2\pi)^{n/2}$ $(m \in Z^n)$, $\Delta = -\sum_{k=1}^{n} (\frac{\partial}{\partial v_k})^2$, $\lambda_m = |m|^2$

$m \in Z^n$, $\frac{\partial}{\partial x_i} : A_{T^n} \to A_{T^n}$ is a continuous [illegible] ator.

We have

$$\|(C,\alpha)_n f\| \le C \|f\| \quad (f \in X)$$

for [illegible] $|1/p - 1/2|$, $1 \le p$ [illegible] . [See Trebels [1] , p. 78]

$\{\lambda_m^{-\delta}\}_{m \in Z^n}$ is a multiplier of X for $\delta > 0$.

(10) Hermite functions in R^ℓ: Let $P = \{0,1,2,3,\ldots\}$ Let P^ℓ = ℓ-fold Cartesian product of P the letters m , n , [illegible] , [illegible] denote elements of P^ℓ , $m = (m_1, \ldots, m_\ell)$, $m_i \in P$.

$m! = m_1! m_2! \ldots m_\ell!$

$|m| = m_1 + m_2 + \ldots + m_\ell$

$(m)^n = m_1^{n_1} m_2^{n_2} \ldots m_\ell^{n_\ell}$

$m + 1 = (m_1 + 1, m_2 + 1, \ldots, m_\ell + 1)$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_\ell^{\alpha_\ell}} \qquad x^\alpha = x_1^{\alpha_1} \ldots x_\ell^{\alpha_\ell}$$

The letters r , s , i , and j denote elements of P .

The letters a, b, and c denote elements of N^{ℓ}, $N = \{0,1,2,\ldots\}$;

For $I = R^{n}$

$$H = (-(1/2)(\partial/\partial x_1)^2 + x_1^2 + 1)(-(1/2)(\partial/\partial x_2)^2 + x_2^2 + 1) \ldots$$
$$(-(1/2)(\partial/\partial x_{\ell})^2 + x_{\ell}^2 + 1)$$
$$\neq -(1/2)(\partial/\partial x)^2 + x^2 + 1$$

$$\phi_n(x) = \phi_{n_1}(x_1)\,\phi_{n_2}(x_2)\ldots\phi_{n_{\ell}}(x_{\ell}) \qquad (n \in N^{n})$$

where $\phi_{n_i}(x_i)$ denote the Hermite functions in the variable x_i of order n_i, i.e. $(1/(2^{n_i} n_i!\,\pi^{1/2})^{1/2})\, e^{x_i^2/2} (\partial/\partial x_i)^2\, e^{-x_i^2}$

$$\lambda_m = \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_{\ell}} = (n_1+1)(n_2+1)\ldots(n_{\ell}+1) \neq n+1 \qquad (n \in N^{n})$$

$A_{R^{\ell}} = S(R^{\ell})$, $A' = S'(R^{\ell})$

$\{1/(n+1)\}_{n \in N^{\ell}}$ is a multiplier of all $L^p(R^{\ell})$ $1 \le p < \infty$ or $C_0(R^{n})$

$$1/(n+1) = \int_0^1 t_1^{n_1} dt_1 \int_0^1 t_2^{n_2}\, dt_2 \ldots \int_0^1 t_{\ell}^{n_{\ell}}\, dt_{\ell} .$$

Sufficient conditions for a sequence $\{\mu_n\}_{n \in N^{\ell}}$ to be multiplier for $L^p(R^{n})$ $1 \le p < \infty$ can be given by ℓ-dimensional analogues of Theorems 21.3.2, 21.5.1 of Hille and Phillips [[1], pp. 577, 580].

For further details about $S'(R^{\ell})$ see Barry Simon [1].

The results of this chapter hold true if, instead of taking $\lambda > 0$ in the definitions 1.3.1 to 1.3.4 and $\lambda > 0$ in the Theorems 1.3.1 and 1.3.2, we take $\lambda > \lambda_0$; $\lambda > \lambda_0 > 0$ there, for some fixed constant $\lambda_0 > 0$ depending only on $\{\lambda_n\}$. In this case, we can cite orthonormal functions constructed from Bessel functions as examples.

(11) Bessel functions (J-form): I (0,1), $U = -S_\mu$

$= -x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}$, $\mu > -1$

$\phi_n(x) = \sqrt{2x} J_\mu(y_{\mu,n} x)/J_{\mu+1}$ $n = 1,2,3,\ldots$

where $J_\mu(x)$ is the μ^{th} order Bessel function of the first kind and the $y_{\mu,n}$ denote all the positive roots of $J_\mu(y) = 0$ with $0 < y_{\mu,1} < y_{\mu,2} < y_{\mu,3} \ldots\ldots$, $\lambda_n = y^2_{\mu,n}$ $n = 1,2,3,\ldots$

(i) Using the integral representation, for Re(z) > 0

$$J_\mu(z) = (1/\pi) \int_0^\pi \cos(\mu\theta - z \sin\theta)\, d\theta$$

$$-((\sin \mu\pi)/\pi) \int_0^\infty \exp(-\mu\theta - z \sinh\theta)\, d\theta ,$$

[See Rainville [1], p. 114].

We have $J_\mu(x) \leq 1 + e^{-\mu}/\mu$ for all real x .

$$\text{(ii)} \quad J^2_{\mu+1}(y_{\mu,n}) = 2 \int_0^1 t[J_\mu(y_{\mu,n} t)]^2\, dt = (1/y^2_{\mu,n}) \int_0^{y_{\mu,n}} z(J_\mu(z))^2\, dz$$

$$\geq \gamma_2 (y_{\mu,n})^{-1}$$

[See Wing [1], Relation 6.2] γ_2 a constant.

Hence $1/J_{\mu+1}(y_{\mu,n}) \leq \sqrt{(y_{\mu,n}/\gamma_2)}$

(iii) We have the following recurrence relations

$$J_\mu(z) = (2(\mu-1)/z)\, J_{\mu-1}(z) - J_{\mu-2}(z)$$

$$2\, J'_\mu(z) = J_{\mu-1}(z) - J_{\mu+1}(z)$$

[See Rainville [1] p. 111].

Using (i), (ii), and (iii), we can show that

$\|(\frac{d}{dx})^k \psi_n\|_{L^1(I)} = O(n^{s_k})$ $(k \in N,\ s_k \in P$, independent of $n \in N)$.

Hence the hypothesis of Theorem 1.3.3, (a), (b), (d) are all satisfied.

Further $\sum_{n=1}^{\infty} [1/y^{2\lambda}_{\mu,n} - 1/y^{2\lambda}_{\mu,n+1}] = \sum_{n=1}^{\infty} 1/y^{2\lambda}_{\mu,n} < \infty$

for $\lambda = 1,3,5,\ldots,2^k$ $(k \in P)$ [Watson [1], p. 502]. Wing [1] has shown that $L^p(0,1)$ and $\{\psi_n\}_{n=0}^{\infty}$ satisfy (C^1), $1 < p < \infty$ for $\mu \geq -1/2$ and Benedek and Panzone [1] have extended this result to $-1 < \mu < -1/2$, provided $1/(\mu+3/2) < p < 1/(-\mu-1/2)$. Hence $\{1/y^{2\lambda}_{\mu,n}\}_{n=1}^{\infty}$ forms a multiplier of $L^p(0,1)$, $1 < p < \infty$, if $\mu \geq -1/2$; $1/(\mu+3/2) < p < 1/(-\mu-1/2)$ if $-1 < \mu < -1/2$. For $\lambda = 1,3,5,\ldots,2^k$ $(k \in P)$.

(12) Bessel functions (Second Form): $I = (0,1)$. Let $\mu > -1/2$. Let a be a real number with $a \geq |\mu|$. $\Re = S_\mu =$ $-x^{-\mu-1/2} Dx^{2\mu+1} Dx^{-\mu-1/2} + a^2 - \mu^2$. $\psi_n(x) = \sqrt{2x}/h_n \, J_\mu(z_{\mu,n} x)$ $n = 1,2,3,\ldots$ where the $z_{\mu,n}$ denote all the positive roots of

$$zJ^{(1)}_\mu(z) + aJ_\mu(z) = 0$$

with $0 < z_{\mu,1} < z_{\mu,2} < z_{\mu,3} \ldots$ Here $J^{(1)}_\mu(z) = D_z J_\mu(z)$. $\lambda_n = z^2_{\mu,n}$ $(n \in P)$. Also,

$$h_n = [J^{(1)}_\mu(z_{\mu,n})]^2 + [1 - \mu^2/z^2_{\mu,n}][J_\mu(z_{\mu,n})]^2$$

$h_n^2 = 2\int_0^1 x\,[J_\mu(z_{\mu,n} x)]^2\,dx = 1/z^2_{\mu,n} \int_0^{z_{\mu,n}} x[J_\mu(x)]^2 dx \sim Cz^{-1}_{\mu,n}$

hence $1/|h_n| \leq$ constant $\sqrt{z_{\mu,n}}$. Using the results (i), (ii), (iii) quoted above, we can show that

$$\left\|\left(\frac{d}{dx}\right)^k \phi_n\right\|_{L^1 \to L^1} = O(\lambda_n^{s_k}) \quad (C_k' > 0,\ s_k > 0 \text{ independent of } n)$$

Further, $\sum_{n=1}^{\infty} 1/(x_{k,n}^2 + a^2 - \alpha^2) = 1/2(a+\alpha) \cdot$ [See H. Lamb, [1], p. 273] ,

Further, [illegible] $\subset L^p(0,1)$ and $\{\phi_n\}$ satisfy (C^1) for $1 < p <$ [illegible] [see Wing [1]]. Hence

$$\sum_{n=1}^{\infty} [1/(x_{k,n}^2 + a^2 - \alpha^2) - 1/(x_{k,n+1}^2 + a^2 - \alpha^2)]$$

$$\sum_{n=1}^{\infty} 1/(x_{k,n}^2 + a^2 - \alpha^2) = 1/2(a + \alpha)$$

This implies that $\{\lambda_n^{-\alpha}\}_{n=1}^{\infty}$ is a multiplier of L^p, $1 < p < \infty$, for [illegible] < 1.

CHAPTER III

APPROXIMATION PROCESSES ON $S'(R^n)$

In this chapter, we will show that the families $\mathcal{F}$ [Definition 1.3.5] ; $\mathcal{E}(m,\beta,X)$ with $\beta > 0$, $X \in \mathcal{F}$ [Definition 1.3.6] ; $\mathcal{R}(m,B)$ [Definitions 1.3.7 to 1.3.10] with B = any one of $L_p(R^n)$, $(L_p)^\wedge$, $1 \le p < \infty$, $C_0(R^n)$, $(C_0(R^n))^\wedge$ of Banach subspaces of $S'(R^n)$ defined in Section 1.3.2, have the following properties:

(1) A convolution type approximation process $\{I_n\}$ on some $X \in \mathcal{F}$, satisfying either Jackson or Bernstein type inequality on X with a certain order, satisfies the same inequality with the same order on every member of $\mathcal{E}(m,\beta,X)$.

(2) If the above $\{I_n\}$ is saturated on B with a certain order, then $\{I_n\}$ is saturated on every member of $\mathcal{R}(m,B)$ with the same order.

In Section III.1, we will investigate some properties of ABBS on R^n [Refer to 1.2.6]. For a ABBS B on R^n , we will show that $B \subset S'(R^n)$. We will present sufficient conditions on an ABBS B on R^n with $B \subset S'(R^n)$ for the space $\hat{B}$ (See 1.2.35) to be an ABBS on R^n with respect to the generalized translation operators $\{T_u\}_{u \in R^n}$ (given by 1.2.31) and to be a member of $\mathcal{F}$ (See Definition 1.3.5). We investigate some relevant properties of members of $\mathcal{F}$. In Section III.2 proof of Theorems 1.3.4 and 1.3.5 are presented.

III.1 Some Properties of ABBS on R^n

Lemma III.1.1 If B is HBS on R^n, then $B \subset S'$.

Proof: Let $W = \{t \in R^n \mid t = (t_1, t_2, t_3, \ldots t_n),\ 0 \le t_i \le 1,$ $i = 1,2,\ldots n\}$. Since elements of B are uniformly locally integrable, (see I.2.52), there exists $\varepsilon > 0$, depending only on W, such that

$$\int_{W+u} |f(x)|\,dx \le \varepsilon \|f\|_B \qquad (f \in B,\ u \in R^n) \quad \ldots\ldots \text{III.1.1}$$

For $m \in P$, let $P_m = \{0, 1, -1, 2, -2, \ldots +m, -m\}$ containing $2m+1$ elements. Let $C_m = \underbrace{P_m \times P_m \times P_m \times \cdots \times P_m}_{n\text{ - times}} = P_m^n$.

Let $J_m = C_m - C_{m-1} = C_m \cap C'_{m-1} \qquad m = 1,2,3,\ldots$

$$J_0 = \underbrace{(0,0,0\ldots)}_{n\text{ - times}} = C_0$$

$$R^n = \bigcup_{i=0}^{\infty} \bigcup_{\alpha \in J_i} W + \alpha$$

The number of elements in $J_m = (2m+1)^n - (2m-1)^n$. Let $\alpha \in J_m$. Then $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_{i_0} = \pm m$ for some i_0, $1 \le i_0 \le n$. Let $x \in W + \alpha$. Then

$$x = (t_1 + \alpha_1,\ t_2 + \alpha_2, \ldots, t_n + \alpha_n) \text{ for some } (t_1, t_2, \ldots t_n) \in W.$$

$$|x|^2 = \sum_{i=1}^{n} |\alpha_i + t_i|^2 \ge \sum_{i=1}^{n} \big||\alpha_i| - |t_i|\big|^2 \ge (|\alpha_{i_0}| - |t_{i_0}|)^2$$

$$\ge (m-1)^2.$$

$$(3 + 2|x|) \ge 3 + 2(m-1) = 2m+1 \qquad \text{i.e. } \frac{1}{(3+2|x|)} \le \frac{1}{2m+1}$$

(i) Let $f \in B$ and $\varphi \in S$. There exists a constant $c > 0$ such that $|\varphi(x)| \le c/(3+2|x|)^{n+2}$ $(x \in R^n)$. III.1.2

Define $F(\varphi) = \int_{R^n} f(x)\,\overline{\varphi(x)}\,dx$ III.1.3

$$|F(\varphi)| \le \int_{R^n} |f(x)|\,|\varphi(x)| \le \int_W |f(x)|\,|\varphi(x)|$$

$$+ \sum_{m=1}^{\infty} \sum_{\alpha \in J_m} \int_{W+\alpha} |f(x)|\,|\varphi(x)|\,dx$$

$$\le c\Big[\int_W |f(x)|dx + \sum_{m=1}^{\infty} \sum_{\alpha \in J_m} \int_{W+\alpha} \frac{|f(x)|}{(3+2|x|)^{n+2}}\,dx\Big]$$

$$\le c\Big[\int_W |f(x)|dx + \sum_{m=1}^{\infty} \sum_{\alpha \in J_m} (1/(2m+1))^{2+n} \int_{W+\alpha} |f(x)|dx\Big]$$

$$\le c\,||f||_B \Big[1 + \sum_{m=1}^{\infty} ((2m+1)^n - (2m-1)^n)/(2m+1)^{n+2}\Big]$$

$$\le c\,||f||_B \Big[1 + \sum_{k=1}^{\infty} 1/k^2\Big] < \infty. \qquad ......(III.1.4)$$

Thus, $F(\cdot)$ is well defined. F is linear on S. Let $\{\varphi_k\} \subset S$ s.t. $\varphi_k \to 0$ in S. Let $\varepsilon > 0$. Let $\varepsilon' > 0$ be such that $0 < \varepsilon' < \varepsilon/||f||_B(1 + \sum_{k=1}^{\infty} 1/k^2)$.

There exists some $k_0 \in N$ such that for all $k \ge k_0$,

$$|\varphi_k(x)| \le \varepsilon'/(3+2|x|)^{n+2} \qquad (x \in R^n).$$

By computation similar to that in III.1.4, we have

$|F(\varphi_k)| = |\int_{R^n} f(x)\,\overline{\varphi_k(x)}\,dx| \le \varepsilon'||f||_B\,(1 + \sum_{k=1}^{\infty} 1/k^2) < \varepsilon$ for all $k \ge k_0$. Thus, $F(\varphi_k) \to 0$. Thus, F defined by (III.1.3) is a tempered distribution. If we identify $f \in B$ with F defined by (III.1.3), we obtain $f \in S'$.

(ii) Let $\{f_k\} \subset B$ with $f_k \to 0$ in B. We want to show that F_k defined by

$$F_k(\phi) = \int_{R^n} f_k(x)\,\phi(x)\,dx \qquad (\phi \in S)$$

converges to 0 in S' as $k \to \infty$. Let $\phi \in S$. Let C_ϕ be a constant > 0 with $|\phi(x)| \le C_\phi/(1+|x|)^{n+2}$ for all $x \in R^n$. As in (III.1.4), we get $|F_k(\phi)| = |\int_{R^n} f_k(x)\,\phi(x)| \le C_\phi \cdot \|f_k\|_B (1 + \sum_{k=1} 1/k^2)$ $\to 0$ as $k \to \infty$. Thus, for all $\phi \in S$, $F_k(\phi) \to 0$. Hence, "convergence in B" implies "convergence in S'" i.e. $B \subset S'$. Q.E.D.

The following lemma implies that $(L_p(R^n))' = (L_p)'$, $1 \le p < \infty$, and $C_0'(R^n) = (C_0(R^n))'$ are ABBS on R^n with respect to the group of translations $\{\tau_u\}_{u \in R^n}$ as defined in (I.2.30) and are members of F. For $1 \le p \le 2$, $(L_p)'$ are ABBS on R^n, since $(L_p)' \subset L_p(R^n)$. For $2 < p < \infty$, $(L_p)'$ is not a BBS on R^n, since $(L_p)'$ contains some element f_p which is not a measure; i.e. f_p is not a uniformly locally integrable function [Refer to Hörmander [1], page 104-105].

<u>Lemma</u> III.1.2 Let $B \in F$ (see definition I.3.5). Then

(a) B' (see Definition I.1.35) is an ABBS on R^n with respect to the group of generalized translations $\{\mathcal{T}_u\}_{u \in R^n}$ (see I.2.31).

(b) Let $g(x) \in \mathcal{D}(R^n)$ be such that $g(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 2 \end{cases}$

Let $g_k(x) = g(x/k) \qquad (k \in P)$ ......(III.1.5)

Then (i) for all $k \in P$, $\gamma \in P_c^n$, $f \in B$, we have $(g_k f)$, $f(D^\gamma g_k) \in B$.

(ii) For some constants $C_1 > 0$, $C_2 > 0$ (C_1, C_2 independent of $h \in B$) and for all $f \in B$, we have

$$\sup_k \|g_k f\| \le C_1 \|f\|_B , \qquad \sup_{\substack{0 \neq \alpha \in P^n \\ k \in P}} \|f(D^\alpha g_k)\|_B \le C_2 \|f\|_B$$

......(III.1.6)

$$\|g_k f - f\|_B \to 0 \text{ (as } k \to \infty) , \quad \|f(D^\alpha g_k)\|_B \to 0 \text{ as } k \to \infty .$$

Remark 1: There exists a HBS on R^n satisfying conditions (i), (ii), but not (iii) of Lemma III.1.2. Indeed, $W^1(L_1(R))$ is one such space.

$$\phi_0(x) = e^{-x^2} \in S, \quad \left|\frac{d}{dx} e^{-x^2} e^{ixt}\right| \le \phi_0(x) \, |t|$$

and hence $\|\phi_0 e^{ixt}\|_{W^1(L_1(R))} \le \|\phi_0\|_{L^1(R)} (1 + |t|) \quad (t \in R)$.

Remark 2: There exists a HBS on R^n satisfying conditions (i), (iii), but not (ii) of Lemma III.1.2. Indeed, $UCB(R)$ the space of uniformly continuous bounded functions with sup norm, is one such space. The closure of S in sup norm is $C_0(R)$, which is a proper subspace of $UCB(R)$.

Remark 3: There exists a HBS B on R satisfying $S \not\subset B$ and none of the conditions (ii), (iii) of Lemma III.1.2. Indeed $C^1[0,2\pi]$ the space of 2π-periodic continuously differentiable functions on R with a norm $\|f\|_{C^1[0,2\pi]} = \|f\|_{C[0,2\pi]} + \|f^{(1)}\|_{C[0,2\pi]}$ is one such space as [illegible] $\|e^{ixt}\|_{C^1(0,2\pi)} = 1 + |t| \quad (t \in R)$.

Remark 4: Let B be HBS on R^n satisfying the following condition:

"$f, g \in B$, $|f(x)| \le |g(x)|$ a.e. implies $\|f\|_B \le \|g\|_B$"

(III.1.7)

then $\|f(x)\, e^{ixt}\|_B = \|f\|_B$ $(f \in B)$.

The HBS $C^1(T)$ on R^n described in Remark 3 does not satisfy II.3.15. Indeed, $f_1(x) = 1$, $f_2(x) = \sin x \in C^1(T)$, $|f_2(x)| \le |f_1(x)|$ for all $x \in R$. But $\|f_1\|_{C^1(T)} = 1 < \|f_2\|_{C^1(T)} = 2$.

Proof: $\hat{B}$ is a Banach space isometric to B. Since the Fourier transform is an isomorphism of S onto S, B onto $\hat{B}$, S' onto S', we get $S \subset \hat{B} \subset S'$, S is dense in $\hat{B}$. The fact that $S \subset \hat{B}$ implies that, for a sequence $\{\phi_n\}$ in S,

$\phi_n \to 0$ in S implies $\phi_n \to 0$ in $\hat{B}$. ......(III.1.8)

Let $\{T_u\}_{u \in R^n}$ be the group of translation operators defined on S' as given by (I.2.30). Since for all $\phi \in S$, $T_u \phi \to \phi$ as $u \to 0$ in S, (III.1.8) implies that

$T_u \phi \to \phi$ as $u \to 0$ in $\hat{B}$; $(\phi \in S)$. ......(III.1.9)

Condition (iii) implies that $\|T_u \phi\|_{\hat{B}} = \|\phi\|_{\hat{B}}$ $(\phi \in S,\ u \in R)$.
......(III.1.10)

Let, for all $u \in R^n$, S_u denote the unique extension S_u of T_u from S to $\hat{B}$, given by $S_u f = \lim_{n \to \infty} T_u \phi_n$ (in $\hat{B}$-norm) $(\phi_n \in S,\ \phi_n \to f$ in $\hat{B})$

We have $\|S_u f\|_{\hat{B}} = \|f\|_{\hat{B}}$ $(f \in \hat{B},\ u \in R^n)$. ......(III.1.11)

(III.1.9) and (III.1.11), together with the Banach Steinhaus Theorem, imply that $\|S_u f - f\|_{\hat{B}} \to 0$ as $u \to 0$ $(f \in \hat{B})$. ......(III.1.12)

It is easy to check that $S_{u+v} f = S_u S_v f$ $(f \in \hat{B};\ u, v \in R^n)$.

Let $f \in \hat{B}$, $u \in R^n$. Let $g_f, g_{f,u} \in B$, with $\hat{g}_f = f$, $\hat{g}_{f,u} = S_u f$.

Let $\{\phi_n\} \subset \mathcal{S}$ with $\phi_n \to f$ in B. Then $\tau_u \phi_n \to S_u f$ in B'. Hence, $\phi_n \to g_f$ in B and hence in $\mathcal{S}'$, $e^{ixu} \phi_n \to g_{f,u}$ in B and hence in $\mathcal{S}'$. Since e^{ixu} as a function of x, belongs to $\mathcal{O}_M$, we get

$$e^{ix\cdot} g_f = g_{f,u} \quad \text{i.e.} \quad (e^{ix\cdot} g_f)^{\vee} = S_u f \qquad \text{......(III.1.13)}$$

$$\langle S_u f, \phi \rangle = \langle g_{f,u}, \hat{\phi} \rangle = \langle e^{ix\cdot} g_f, \hat{\phi} \rangle = \langle g_f, e^{ix\cdot} \hat{\phi} \rangle = \langle f, \tau_{-u} \phi \rangle$$

for every $f \in B'$; $\phi \in \mathcal{S}$; $u \in R^n$. ......(III.1.14)

Hence the group $\{S_u\}_{u \in R^n}$ coincides with the group of generalized translations on B'.

Let $\sigma \in M$. For $f \in B'$, we define $f * \sigma$ as

$$f * \sigma = F_\sigma(f) = \int_{R^n} (\tau_u f)\, d\sigma(u) \qquad \text{......(III.1.15)}$$

the integral in (III.1.15) being understood as a Bochner-Stieltjes integral of the continuous, bounded B'-valued function $\tau_u f$ [Refer to Hille-Phillips [1] for definition of Bochner-Stieltjes integral]. F_σ is a linear bounded operator on B', since $\|f * \sigma\|_{B'} = \|F_\sigma(f)\|_{B'} \le \|f\|_{B'} \|\sigma\|_M$. ......(III.1.16)

Let $f \in B'$, $\phi \in \mathcal{S}$. Then

$$\langle f * \sigma, \phi \rangle = \langle \int_{R^n} (\tau_u f)\, d\sigma(u), \phi \rangle = \int_{R^n} \langle \tau_u f, \phi \rangle\, d\sigma(u) = \int_{R^n} \langle f, \tau_{-u} \phi \rangle\, d\sigma(u)$$

$$= \langle f, \int_{R^n} (\tau_{-u} \phi)\, d\sigma(u) \rangle = \langle f, \phi * \check{\sigma} \rangle$$

where $d\check{\sigma}(u) = d\sigma(-u)$, $\forall u \in R^n$ ......(III.1.17)

For $f \in B'$; $\sigma, \beta \in M$, $\phi \in \mathcal{S}$, we have $\check{\sigma} * \check{\beta} = (\sigma * \beta)^{\vee}$,

$\langle (f * \sigma) * \beta, \phi \rangle = \langle f * \sigma, \phi * \check{\beta} \rangle = \langle f, (\phi * \check{\beta}) * \check{\sigma} \rangle = \langle f, \phi * (\check{\beta} * \check{\sigma}) \rangle = \langle f, \phi * (\sigma * \beta)^{\vee} \rangle$

$= f*(\tau * \mu)$, ... (III.1.18)

Hence, $(f*\sigma)*\mu = f*(\sigma*\mu)$ $(f \in B' ; \sigma, \mu \in M)$. Hence, B' an AHBS on R^n with respect to the group of translations $\{\tau\}$.

(b) Let $g(x) \in \mathcal{D}(R^n)$ with $g(x) = 1$ if $|x| \leq 1$, $g(x) = 0$ if $|x| \geq 2$. Then, $h(x) = \hat{g}(x) \in S$. Let $h_k(x) = h_{(1/k)}(x)$

$= k^n h(kx)$ $(k \in P)$ . ... (III.1.19)

Then $g_k(x) = g(x/k) = \hat{h}_k(x)$; $(k \in P)$ . ... (III.1.20)

Since $\{f*h_k \mid f \in B'\} \subset B'$, we have $\{g_k f \mid f \in B\} \subset B$. Since

$\int_{R^n} h(x)dx = g(0) = 1$, by (I.2.54), we get $||f*h_{(1/k)} - f||_{B'} \to 0$

for all $f \in B'$. i.e. for all $f \in B$, $||g_k f - f||_B \to 0$.

Further, for $f \in B$, $\sup_k ||g_k f||_B = \sup_k ||f*h_{(1/k)}||_{B'} \leq$

$||f||_B ||h||_{L_1}$. Let $\beta \in N^n$ with $|\beta| = \sum_{i=1}^{n} \beta_i \neq 0$. $(D^\beta g)(x) \in \mathcal{D}(I)$.

Let $\psi(x) \in S$ with $\hat{\psi}(x) = (D^\beta g)(x)$. Let $\psi_k(x) = \psi_{(1/k)}(x) =$

$k^n \psi(kx)$ $(k \in P)$ . ... (III.1.21)

$$(D^\beta g_k)(x) = (1/k^{|\beta|}) (D^\beta g)(x) = (1/k^{|\beta|}) \hat{\psi}_k(x) .$$

For all $f \in B$, $f(D^\beta g_k(x)) = (f * \psi_k)(x) \in B'$ and

$$||f(D^\beta g_k)||_B \leq (1/k^{|\beta|}) ||f * \psi_k||_{B'} \to 0 \text{ as } k \to \infty ,$$

and $\sup_{k \in P, \beta \in N^n, \beta \neq 0} ||f(D^\beta g_k)||_B \leq ||f||_B ||\psi||_{L_1(R)}$.

Q.E.D.

Lemma III.1.3 Let $B \in \mathcal{F}$. Then $\mathcal{D}$ is dense in both B and $W^m(B)$.

Remark: The proof of the fact that $\mathcal{D}$ is dense in $W^m(B)$ is analogous to that of Proposition 31.5, Treves [1, p. 327]. Here, B need not be a function space.

Proof: Let $f \in B$, and $\varepsilon > 0$. Then $||f - g_{k_0} f||_B < \varepsilon/2$ for some $k_0 \in P$. If $\rho(x) \in L_1(R^n)$ with compact support and $\int_{R^n} \rho(x)\, dx = 1$, then $||g_{k_0} f - (g_{k_0} f) * \rho_{(1/n_0)}||_B < \varepsilon/2$ for some $n_0 \in P$ (by 1.2.54). Hence $\mathcal{D}$ is dense in B. For γ and $\beta \in N^n$, by $\gamma \leq \beta$ we mean $\gamma_i \leq \beta_i$, $1 \leq i \leq n$,

$\binom{\beta}{\gamma} = \prod_{n=1}^{n} \binom{\beta_i}{\gamma_i}$. For $f \in W^m(B)$, let $\alpha \in N^n$, $|\alpha| \leq m$.

$$D^\alpha(g_k f) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} [D^\gamma g_k][D^{\alpha-\gamma} f] \in B . \quad D^\alpha(g_k f) - g_k(D^\alpha f) =$$

$$\sum_{\substack{0 \neq \gamma \leq \alpha \\ \gamma \in N^n}} \binom{\alpha}{\gamma} [D^\gamma g_k]\, [D^{\alpha-\gamma} f] .$$

Hence, $||D^\alpha(g_k f) - g_k(D^\alpha f)||_B \leq \sum_{0 \neq \gamma \leq \alpha} \binom{\alpha}{\gamma} ||[D^\gamma g_k]\, [D^{\alpha-\gamma} f]||_B \to 0$, as $k \to \infty$, and $||(D^\alpha f) g_k - D^\alpha f||_B \to 0$ as $k \to \infty$.

$$||g_k f - f||_{W^m(B)} = \sum_{\substack{|\alpha| \leq m \\ \alpha \in N^n}} ||D^\alpha(g_k f) - D^\alpha f||_B \to 0 \text{ as } k \to \infty .$$

If $\rho(x) \in L_1(R^n)$ with compact support and $\int_{R^n} \rho(x)\, dx = 1$, then for every $k \in P$,

$$||g_k f - (g_k f) * \rho_{(1/n)}||_{W^m(B)} = \sum_{|\alpha| \leq m} ||D^\alpha(g_k f) - D^\alpha(g_k f) * \rho_{(1/n)}||_B \to 0$$

as $n \to \infty$. Hence, for some $k_0 \in P$, $n_0 \in P$,

$\|f - G_{k_0} f\|$ [illegible] $(1/n_0)$ [illegible] and [illegible] $(1/n_0)$ [illegible] . Q.E.D.

III.2 Results Related to Convolution Type Processes

In this section, we give the proof of Theorems I.3.4 [illegible] and I.3.5. Analogous to the Lemma II.1.7, we will derive a simple characterization of elements of $I(X^{\epsilon},X)$, with $\epsilon > 0$, X a Banach subspace of S' containing S, in terms of the elements of $I(X,X)$. (Refer to I.2.37). Indeed, for $u \in I(X,X)$, we have

$$\|u * G_{\epsilon} * \phi\|_X \leq \|u\|_{I(X^{\epsilon},X)} (1 + \|G_{\epsilon}\|_{L_1}) \|\phi\|$$

$(\phi \in S)$. Hence, $u * G_{\epsilon} = v_u$ or $u = v_u * G_{-\epsilon}$, for $v_u \in I(X,X)$ and

$$\|v_u\|_{I(X,X)} \leq C_1 \|u\|_{I(X^{\epsilon},X)} \qquad \text{......(III.2.1)}$$

with $C_1 = 1 + \|G_{\epsilon}\|_{L_1}$.

Conversely, for $v \in I(X,X)$ we have

$$\|v * G_{-\epsilon} * \phi\|_X \leq \|v\|_{I(X,X)} \|\phi\|_{X^{\epsilon}}$$

This implies $v * G_{-\epsilon} \in I(X^{\epsilon},X)$ and

$$\|v * G_{-\epsilon}\|_{I(X^{\epsilon},X)} \leq \|v\|_{I(X,X)} \qquad \text{......(III.2.2)}$$

Hence we have the following lemma.

Lemma III.2.1 Let $\epsilon > 0$ and X satisfy $S \subset X \subset S'$. Then [illegible] if and only if there exists $v_u \in I(X,X)$ such that [illegible]

[illegible] with C_1 given by III.2.1

[illegible] ...... (III.2.3)

Proof of Theorem I.3.4: Let B ∈ F and U ∈ $\mathcal{I}$(B,B) .

Claim 1: $D^{\alpha}(U*\varphi) = U*D^{\alpha}\varphi$; ($\varphi \in S$, $\alpha \in N^n$) ......(III.2.4)

For, let $\varphi \in S$. Since $\mathcal{D}$ is dense in S , there exists a sequence $\{\varphi_{\ell}\}$ in $\mathcal{D}$ such that $\varphi_{\ell} \to \varphi$ in S , $D^{\alpha}\varphi_{\ell} \to D^{\alpha}\varphi$ in S . Since $S \subset B$, $\varphi_{\ell} \to \varphi$ in B , $D^{\alpha}\varphi_{\ell} \to D^{\alpha}\varphi$ in B . This implies $U*\varphi_{\ell} \to U*\varphi$ in B , $U*D^{\alpha}\varphi_{\ell} \to U*D^{\alpha}\varphi$ in B . Since B ⊂ S' , we have $U*\varphi_{\ell} \to U*\varphi$ in S' , $U*D^{\alpha}\varphi_{\ell} \to U*D^{\alpha}\varphi$ in S' . But, $U*D^{\alpha}\varphi_{\ell} = D^{\alpha}(U*\varphi_{\ell}) \to D^{\alpha}(U*\varphi)$ in S' . Hence $D^{\alpha}(U*\varphi) = U*D^{\alpha}\varphi$.

Claim 2: $\check{U} \in \mathcal{I}$(B,B) . ...(III.2.5)

This follows from the identity $\check{U}*f = (U*\check{f})^{\vee}$ (f ∈ B) ......(III.2.6)

Thus, (III.2.5) implies, by arguments similar to those of Theorem 3.20, Stein-Weiss [[1], p. 30] that $U \in \mathcal{I}(Y,Y^{*})$ and hence $\check{U} \in \mathcal{I}(Y^{*},Y^{*})$ (· · (– , ·)) with Y = B or B* . Let X = any one of Y (= any one of B , B*) , Y with (· , ·). (III.2.4) implies that $U \in \mathcal{I}(W^{\ell}(X),W^{\ell}(X))$ ($\ell \in N$) . For m ∈ P , let $f \in W^{-m}(X)$ with $f = \sum_{|\alpha| \le m} D^{\alpha} f_{\alpha}$. Define $\overline{U}(f) = \sum_{|\alpha| \le m} D^{\alpha} U*f_{\alpha}$. By Lemma II.1.1, $\overline{U} \in [W^{-m}(X)]$. For $\varphi \in S$, we have

$$\langle \overline{U}(f), \varphi \rangle = \sum_{|\alpha| \le m} \langle D^{\alpha} U*f_{\alpha}, \varphi \rangle = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle U*f_{\alpha}, D^{\alpha}\varphi \rangle$$

$$= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_{\alpha}, \check{U}*D^{\alpha}\varphi \rangle = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_{\alpha}, D^{\alpha}(\check{U}*\varphi) \rangle$$

$$= \langle f, \check{U}*\varphi \rangle = \langle \overline{U}(f), \varphi \rangle \ ; \ (\varphi \in S) \ ......(III.2.7)$$

Hence, we can define $\overline{U}(f) = U*f$ given by III.2.7 . (III.2.7) also implies that $U \in \mathcal{I}(W^{-m}(X),W^{-m}(X))$. The rest follows from Theorem I.2.3.

Q.E.D.

Proof of Theorem I.3.5

(i) Let m, $\gamma > 0$, $\{G_t\}_{t>0}$, B, Z, be as given in the hypothesis. Then, for some $d_1 > 0$, we have

$$\|G_t * f - f\|_B \leq d_1\, \varphi(t)\, \|f'\|_{B'} \quad (f \in B') \qquad \ldots\ldots(III.2.8)$$

This implies that $\{(G_t - \delta)/\varphi(t)\}_t$ is a family of elements of $I(B',B)$ such that $\sup_t \|(G_t - \delta)/\varphi(t)\|$ [illegible] d_1 [illegible]. Here, δ stands for dirac measure with weight 1 at the origin. By Lemma II.2.2, there exists a family $\{V_t\}_t$ of elements of $I(B,B)$ with $(G_t - \delta)/\varphi(t) = V_t * G$ [illegible], t [illegible], and

$$\sup_t \|V_t\|_{I(B,B)} \leq (1 + \|G\|_{L_1}) \sup_t \|(G_t - \delta)/\varphi(t)\|_{I(B',B)}$$

$$\leq (1 + \|G\|_{L_1})\, d_1 \ldots$$

Theorem I.3.4 implies that $\{(G_t - \delta)/\varphi(t)\}_t$ [illegible] (Z',Z) with

$$\sup_t \|(G_t - \delta)/\varphi(t)\|_{I(Z',Z)} \leq \sup_t \|V_t\|_{I(B,B)} \leq C_{11} \qquad \ldots\ldots(III.2.9)$$

with $C_{11} = d_1(1 + \|G\|_{L_1})$.

This implies that for all $f \in Z'$, $\|G_t * f - f\|_Z \leq C_{11}\, \varphi(t)\, \|f\|_{Z'}$.

For $f \in Z'$, let $\{f_\ell\}_{\ell \in P} \subset Z'$ with $\sup_\ell \|f_\ell\|_Z \leq 2\|f\|_Z$ and $f_\ell \to f$ in Z. Since $(G_t - \delta)/\varphi(t) \in I(Z,Z)$ we have $(G_t * f_\ell - f_\ell)/\varphi(t) \to (G_t * f - f)/\varphi(t)$ in Z as $\ell \to \infty$ for every fixed $t > 0$.

$||(G_t * f - f)Z(t)|| \le \lim\sup_{\ell} ||(G_t * f_\ell - f_\ell)Z(t)||_Z$

$\le C_{11}(\sup_\ell ||f_\ell||_{Z}) \ldots ||_Z$. Q.E.D.

(2) Let, for $f \in B$, $G_t * f \in B$ and $||G_t * f||_B \le C_{22}(Z(t))^{-1}||f||_B$.

This implies that $\{(G_t * G_-)Z(t)\}_{t>0} \subset L(B,B)$ and

$\sup_{t} ||Z(t)(G_t * G_-)||_{L(B,B)} \le d$ . ...(III.2.10)

By Theorem I.3.4, we have $\{(G_t * G_-)Z(t)\}_{t>0} \subset L(Z,Z)$ and

$\sup_{t>0} ||Z(t)(G_t * G_-)||_{L(Z,Z)} \le d$.

This implies that for all $f \in Z$, $G_t * f \in Z$ and

$||G_t * f||_Z \le ||G_t * f||_Z + ||G_t * G_- * f||_Z$

$\le (1 + ||G_-||_{L^1})d_2(Z(t))^{-1}||f||_Z$. Q.E.D.

(3) Let $Z(v)$ and Z satisfy the hypothesis (2) of the Theorem I.3.3. Let F be any one of $L_p(R^n)$, $(L_p)'$, $1 \le p \le \infty$, $C_0(R^n)$, $(C_0(R^n))'$.

By result (1) of this theorem, we have

for all $f \in Z$, $||G_t * f - f||_Z \le C_{11}Z(t)||f||_Z$ . ...(III.2.11)

Further, by I.2.4, we obtain $\{((G_t(v) - 1)/Z(t))(1 + |v|^2)^{-s/2}\}_{t>0}$

$\subset L_\infty(R^n)$,

and $\sup_{t>0} ||((G_t(v) - 1)/Z(t))(1 + |v|^2)^{-s/2}||_{L_\infty} \le d_2$.

......(III.2.

since $(\ldots_t(v) - 1)/\ldots(t) \to \ldots(v)$ $(t \to t_0)$ for each fixed $v \in R^n$.

Hence, by the dominated convergence theorem, we have for all $\ldots \in S$ and for all p, $1 \ldots$, we have

$\ldots(v)\ldots(v) \in L_p(R^n)$, and $((\ldots_t(v) - 1)/\ldots(t))\ldots(v) \to \ldots(v)\ldots(v)$

as $t \to t_0$ in L_p-norm. ......(III.2.13)

We want to prove that for all $\ldots \in S$, and for all $f \in W^{m}(B + B^* + B \ldots (B^*))$, we have

$\ldots(\ldots_t * f - f)/\ldots(t), \ldots \to \ldots_1 * G_\ldots * f, \ldots$ as $t \to t_0$ ...(III.2.14)

Case 1: $B = L_p(R^n)$, $1 \ldots p \ldots$, $f \in B$ and $\ldots \in S$.

Since $f \in L_p$ and $((\ldots_t(v) - 1)/\ldots(t))\ldots(v) \to \ldots(v)\ldots(v)$ $(t \to t_0)$

in L_p-norm,

$\ldots(\ldots_t * f - f)/\ldots(t), \ldots = \ldots f, ((\ldots_t(-v) - 1)/\ldots(t))\ldots(v)\ldots$

$\ldots f, \ldots(-v)\ldots(v)\ldots$

$= \ldots_1 * G_\ldots * f, \ldots$ $(t \to t_0)$.

......(III.2.15)

Case 2: Let $B = L_p(R^n)$, $2 \ldots p \ldots$, $\ldots \in S$. Since $S \ldots (L_p)$, we have $\sup_{t \in I} \|(\ldots_t * f - f)/\ldots(t)\|_{L_p} \ldots$ and as $t \to t_0$, for all $\ldots \in S$,

$\ldots(\ldots_t * f - f)/\ldots(t), \ldots \to \ldots_1 * G_\ldots * f, \ldots$ by III.2.15.

Hence, by the Banach-Steinhaus Theorem, we have, as $t \to t_0$, for all $\ldots \in L_p$,

$\ldots(\ldots_t * f - f)/\ldots(t), \ldots = \ldots f, (\ldots_t * \ldots - \ldots)/\ldots(t) \ldots \to \ldots_1 * G_\ldots * f, \ldots$ ...(III.2.16)

Case 3: Let $B = (L_p)$, $2 < p < \infty$. Let $f \in B'$, $\ldots S$.
Since $f \in L_p'$, and $((G_t(v) - 1)/\lambda(t)) \ldots (v) \ldots (v)$, $(t \to t_0)$
in L_p-norm, we get

$$(G_t * f - f)/\lambda(t), \ldots ((G_t(-v) - 1)/\lambda(t)) \ldots (v)$$

$$\ldots f, a_1(-v)(1 + |v|^2)^{\ldots/2} \ldots$$

$$= \ldots_1 * G_{-} * f, \ldots \text{ as } t \to t_0 \quad \ldots(III.2.17)$$

Case 4: Let $B = (L_p)$, $1 < p < 2$. Let $\ldots S$. Since
$S \subset ((L_p)')'$ we have $\sup_t \|(G_t * \ldots)/\lambda(t)\|_{(L_p)'} < \ldots$, and

$(G_t * \ldots)/\lambda(t), \ldots \to \ldots_1 * G_{-} * \ldots$, as $t \to t_0$, for all $\ldots S$.

Since S is dense in $(L_p)'$, by Banach-Steinhaus Theorem, we
get for all $f \in (L_p)$, for all $\ldots S$

$$\ldots (G_t * f - f)/\lambda(t), \ldots \to \ldots_1 * G_{-} * f, \ldots \to 0 .$$

Case 5: Let $f \in W^{-m}(B + B^* + B + (B^*))$. Let $f = \sum_{|\alpha| \le m} D^\alpha f_\alpha$

with $f_\alpha \in B + B^* + B + (B^*)$ for all $\alpha \in \ldots^n$ with $|\alpha| \le m$.

For $\ldots S$, $\langle (G_t * f - f)/\lambda(t), \ldots \rangle = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_\alpha, D^\alpha((G_t * \ldots)/\lambda(t)) \rangle$

$$= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_\alpha, (G_t * D^\alpha \ldots - D^\alpha \ldots)/\lambda(t) \rangle$$

$$\to \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_\alpha, \ldots_1 * G_{-} * D^\alpha \ldots \rangle$$

$$= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle f_\alpha, D^\alpha(\ldots_1 * G_{-} * \ldots) \rangle$$

$$= \langle f, \ldots_1 * G_{-} * \ldots \rangle = \langle \ldots_1 * G_{-} * f, \ldots \rangle \quad \ldots\ldots(III.2.18)$$

The result (3) of Theorem I.3.5 is proved, if we show that for all $Z \in E(m, L_1(R^n))$ and for $f \in Z$

$$\|G_t * f - f\|_Z = \begin{cases} o(\omega(t)) & \text{implies } \omega(v)f = 0 \\ O(\omega(t)) & \text{implies } f \in \tilde{Z} \end{cases} \qquad \ldots\ldots(III.2.19)$$

This is so since $E(m, L^1(R^n)) \subset E(m, E)$.

Case 1: Let $Z \in E_2(m, L_1(R^n))$. There exists a Banach subspace $F_Z \subset S'$ such that $Z = (F_Z)^*$, S is dense in F_Z. Let $f \in Z$ [illegible] $\|G_t * f - f\|_Z = O(\omega(t))$, $(t \to 0)$. Then there exists [illegible] $\{t_k\}$ with $t_k \to 0$, as $k \to \infty$ and $g_f \in Z$ such that

$$((G_{t_k} * f - f)/\omega(t_k), \varphi) \to (g_f, \varphi) \text{ as } k \to \infty, \text{ for all } \varphi \in S$$

By (III.2.14), we have $g_f =$ [illegible] $\in Z$

[illegible] $G_{-\alpha} * f = $ [illegible] $* f +$ [illegible] $* g_f \in Z$, i.e. $f \in \tilde{Z}$ [illegible].

Case 2: Let $Z \in E_1(m, E_1(R^n))$. Then there exists a Banach subspace F_Z of S', containing S as a dense subspace, and Z is a norm closed subspace of $(F_Z)^*$. Consider the Gauss-Weierstrass kernel $H(x)$ given by $\hat{H}(v) = e^{-|v|^2}$, $H(x) > 0$. Let $W_{\alpha,t}(x) = (c/t^{n/\alpha}) H(x/t^{1/\alpha})$ where c is a constant chosen so that $\int_{R^n} W_{\alpha,1}(x)\,dx = 1$. [See Butzer-Nessel [1], p. 465]. We have, for all $f \in Z$, $\|W_{\alpha,t} * f - f\|_Z \to 0$ as $t \to 0+$, $W_{\alpha,t} * f \in Z$ for $t > 0$.

$\ldots\ldots(III.2.20)$

and, where $Y = Z$ or F_Z^*, $\sup_{t>0} \|W_{\alpha,t} * f\|_Y \le C \|f\|_Y$, $(f \in Y)$.

Further, we have $\|f\|_Z \le d_1\|f\|_{G_Z} \le d_2\|f\|_Z$, $(f \in B)$

for some constants $d_1 > 0$, $d_2 > 0$. Let $f \in Z$ such that

$$\|(G_t * f - f)/Z(t)\|_Z = O(1), \quad (t \to 0) \qquad \ldots\ldots(\text{III.2.21})$$

Then there exists a sequence $\{t_k\}$, $t_k \to t_0$ as $k \to \infty$, and $g_f \in (F_Z)^*$ such that

$$(G_{t_k} * f - f)/Z(t_k) \to g_f, \quad (k \to \infty) \quad (\ldots S).$$

By (III.2.14), we get $g_f = \ldots * G_{-\Delta} * f \in (F_Z)^*$. Hence, $G_{-\Delta} * f = \ldots_2 * f + \ldots_3 * g_f \in (F_Z)^*$.

$$\|W_{\delta,t} * f\|_Z = \|W_{\delta,t} * \ldots\| \le \|W_{\delta,t} * G_{-\Delta} * f\|_Z$$

$$\le C(\|W_{\delta,t} * f\|_Z + \|W_{\delta,t} * (\ldots_2 * f + \ldots_3 * g_f)\|_{(F_Z)^*})$$

$$\le CC_2(\|f\|_Z + \|\ldots_2 * f + \ldots_3 * g_f\|_{F_Z^*}) \ldots$$

where C is a constant independent of $t > 0$. $\qquad \ldots\ldots(\text{III.2.22})$

Hence, III.2.20 and III.2.22 imply that $f \in \tilde{Z}$. If big O is replaced by small o in III.2.20, then we get $\ldots(v) \ldots = 0$.

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