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Shallow-Water Models for Gravity Currents

by

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Dedication

For G.A. Barker and C.W. Montgomery.

Abstract

Gravity currents, produced by the instantaneous release of a finite volume of dense fluid beneath a layer of lighter fluid and overlying a spatially-varying rigid bottom boundary, are modelled as discontinuous solutions to the systems of nonlinear hyperbolic conservation laws arising from a shallow-water model.

Equations of motion for two stably-stratified fluids of constant density are derived for the incompressible Navier-Stokes Equations for small aspect ratio flow in an Eulerian fluid, and the equations are nondimensionalized using a gravity current scaling so that they may be stated as a first order system of partial differential equations. The model equations neglect the effects of turbulence, entrainment, density stratification, and viscosity, but include the Coriolis force, variable topography, and bottom friction. Special cases are stated for one-layer three-dimensional axisymmetric flow, and in the two-dimensional case for flow with a free surface, rigid lid, thin upper or lower layer, or small density differences. These equations are then stated as a nonlinear system of conservation laws.

The model equations are classified as hyperbolic, with defined regions of hyperbolicity stated where possible. When in conservation form, discontinuous solutions are considered, and the Rankine-Hugoniot jump conditions derived for solutions which are trivial on one side of the shock. The initial release problem is shown to be well-posed by the method of localization.

By approximating a gravity current front as a vertical discontinuity, the initial

release problem is solved numerically by use of a relaxation method designed for systems of hyperbolic conservation laws and adapted to include boundary conditions and forcing terms. The usefulness of this method is demonstrated by several diagrams which show the effects of bottom slope and friction in the two-dimensional case, and of bottom slope and rotation in the three-dimensional one.

Since the relaxation method is applicable to systems in conservation form, a result is proved showing that an infinite number of polynomial conservation laws do not exist for the two-layer shallow-water equations in one spatial dimension, and it is conjectured that this is the case for one layer in two dimensions. The conservation laws which are known to exist are described, and correspond to the conserved quantities of mass, momentum, and energy.

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Chapter 1

Introduction

The flow of cold air along the floor of a warm room after opening a door on a cool day is an everyday example of the interaction between two fluids of slightly different densities. Although more complex examples can be described, the fundamental ideas are similar to the notion above. For example, the motion of cold fronts in the atmosphere, fresh water rivers meeting a salt water ocean, and even snow powder avalanches may all be considered as the primarily horizontal motion of one fluid below a lighter fluid or above a denser fluid. These phenomena have been described under such general terms as density driven flow, density currents, or the term most commonly used in this thesis: gravity currents. An excellent review of the many examples of gravity currents has recently been updated by Simpson (1997).

When studying gravity currents questions abound as to their rate of motion, heights or thicknesses of layers, effects of topography and turbulence, entrainment between the fluids, importance of boundaries and surface friction, sources and sinks, and other physical processes. These factors and physical mechanisms are all of varying degrees of importance when studying a specific physical example. Before stating specific results of this thesis it is therefore useful to review some of the relevant literature which has motivated this study.

One of the first attempts made to describe gravity currents was by Von Kármán (1940), who considered an idealized steady flow between two immiscible and inviscid fluids in two dimensions (horizontal and vertical) of infinite horizontal domain. His analysis, although flawed, led to the result which was later rederived correctly by Benjamin (1968) that the advancing front of a gravity current in this special case moved at a speed u, which was related to the asymptotic height h behind the front and the gravitational acceleration constant g, by the formula

$$u^2 = 2\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) gh, \tag{1.1}$$

where ρ_2 and rho_1 represent the densities of the lower and upper layers, respectively. The layers are assumed to be statically stable so that $\rho_2 > \rho_1$. Benjamin's result was obtained by balancing the momentum fluxes against forces in the fluid, instead of the previously used method of hydraulic jumps. Variations on the formula (1.1) were also described in Benjamin's (1968) comprehensive review, which considered a variety of special cases of flow, but primarily concerned steady-state, semi-infinite immiscible and inviscid flow over a horizontal bottom boundary. Benjamin (1968) did acknowledge that variations in velocity profile, bottom shape, density stratification, etc., (although not considered in deriving (1.1)) would likely have an effect on the gravity current speed, thus warranting further investigation. Almost for the past 50 years, the study of the 'front condition' of a gravity current has been undertaken, with results arising from experiments, theoretical considerations and numerical calculations.

Experiments concerning fluid entrainment for gravity currents (for example, Ellison and Turner (1959), Hallworth *et al.*(1996), and Huq (1996)) showed that for low velocity flows, entrainment and turbulent effects are initially small and concentrated near the advancing front. Additional experiments studied other aspects of gravity currents such as its internal structure (Britter and Simpson, 1978), the effects of opposing flow (Simpson and Britter, 1979), or flow over small obstacles (Lane-Serff *et al.*, 1995).

Notable experiments which investigate non-horizontal bottom topography are those by Middleton (1966), Britter and Linden (1979), Beghin *et al.*(1980), and Alavian (1986). In general, the resulting description for gravity currents flowing over constantly downward sloping planes with small slope, less than about 1 in 10 or approximately 5 degrees, is that the behaviour of the gravity current is fairly close to the horizontal case. This observation resulted in the notion that the alongslope component of gravitational force which acts to accelerate the flow is balanced by opposing forces due to bottom friction effects. Such a concept was mentioned by Benjamin (1968), and has been used by Hay (1983) to propose a different style of front condition which incorporates both bottom friction and slope.

With the assumption of a front condition in the style of (1.1), shallow-water theory has been extensively applied to predict the behaviour of gravity currents resulting from the instantaneous release of volumes of dense fluid. Shallow-water theory has been quite successful in this regard, and is typified by the assumption that the velocity in the horizontal direction is independent of the vertical spatial variable. That is, at a given horizontal point, the horizontal velocity is independent of height within the gravity current, a phenomenon which is observed for low aspect ratio flow (the aspect ratio is the ratio of typical vertical height to the horizontal length scales of the flow). Similarity solutions such as those found by

Hoult (1972), or Grundy and Rottman (1985), and box-models (see, for example Hallworth *et al.*(1996)) often employ the shallow-water assumption. These results are all dependent on the knowledge of a front condition which must be fixed prior to the determination of a solution. Corresponding results found via the methods of hydraulic theory have been stated for such examples as exchange flow (Barr, 1967), or dam-break problems (Klemp *et al.*, 1994). These examples also included specification of a front condition, which in these cases, was necessary to provide closure in the shallow-water equations.

When modelling gravity current behaviour by the methods of shallow-water theory, the effects of density stratification, friction, viscosity, entrainment, and turbulence, all contain difficulties in that they each introduce a height dependence in the horizontal velocity field, contrary to the shallow-water assumption. It is therefore a practical matter to determine a front condition from experiments, and impose this on the shallow-water equations, subsequently solving them numerically through the method of characteristics (Abbot and Basco, 1989). This approach has been taken by Rottman and Simpson (1983), Bonnecaze (1993), Huppert and Lister (1993), Klemp *et al.*(1994) and others. By using an experimentally derived front condition, which includes all of the physical factors mentioned above, these methods work well for the specific examples for which they are designed. However, they do not generalize easily to many situations for which there is scant knowledge of the front condition, for example with variations in volume or bottom topography. This observation was made by Klemp *et al.*(1994), who mentioned that, "...front propagation remains dependent upon the specific source conditions and cannot be generalized."

In addition to the existing large number of publications which concern two-dimensional gravity currents, similar shallow-water methods have been applied to the three-dimensional case simplified for axisymmetric flow. In this case, the equations are similar to the two-dimensional case, and an early review by Griffiths (1986) identified many of the specific applications. Various results have been published by Grundy and Rottman (1985), Webber and Brighton (1986), Bonnecaze *et al.*(1993, 1995), and Hallworth *et al.*(1996), to cite a few. Non-axisymmetric cases such as the wedge-shaped releases reported by Huq (1996), or the horizontal plumes created by Beghin *et al.*(1980) have also been considered. In most of these cases, the previous front condition methodology is adapted for use in the radial case. Recently, the addition of rotation has been considered by Ungarish and Huppert (1998), who noted in essence that it is still unclear how to include the effects of rotation in a shallow-water based theoretical description of the front of an advancing gravity current. Experimental observation is apparently also lacking since Ungarish and Huppert (1998) stated that, "For a rotating axisymmetric current... no investigation on the (front) condition has been performed." The effects of rotation on gravity currents is still under investigation and a more comprehensive list of references may be found in Hacker (1996).

It is clear from the recent literature, and the past history of publications which has motivated recent work, that the determination of the front position of a gravity current plays a crucial part in the search for both numerical and analytic solutions to the shallow-water equations which have been used, with some success, to model gravity current behaviour. Theoretical results have, in general, been well-supported by specific laboratory experiments. As recently emphasized by Ungarish and Huppert (1998), the additional complication created by rotation due to the Coriolis force, has yet to be explained in a satisfactory way theoretically and investigated experimentally. There is most certainly an identified need for an alternate approach which will permit a generalization from the present theory, and incorporate the additional complications of rotation, variations in bottom topography, and changes in dense current volume. An addition of these factors of rotation, topography and volume changes in gravity currents is required to assist in the connection between the laboratory and the physical world. Such physical examples are widespread in scope, and range from the oceans to the atmosphere.

A large amount of research has been published concerning turbidity currents, which are sometimes called particle-laden gravity currents, for example, Hallworth *et al.*(1998), Moodie *et al.*(1998), Dade and Huppert (1995), Bonnecaze *et al.*(1993, 1995), Garcia (1994), and Sparks *et al.*(1993). The main motivation for such study is the predictability of sediment depositions in the ocean or in rivers and estuaries, Wright *et al.*(1990). The physical factor of nonzero bottom slope is often of some importance in many of these examples, because sedimentation on the sloping abyssal plain is considered in the study of turbidites (see, for example Moodie *et al.*, 1998, Muck and Underwood, 1990, or Dolan *et al.*, 1989).

The theories of turbidity currents have also been applied extensively to pyroclastic flows on slopes by Huppert *et al.*(1986), or Dade and Huppert (1986). Off-world examples exist where variable viscosity gravity currents have been used to explain certain geologic structures on Venus (Sakimoto, 1995). Snow avalanches, as well as some types of rock avalanches are also of some relevance to gravity currents, as reviewed by Simpson (1997).

Some larger scale physical examples exist in the oceans and atmosphere. Dense clouds of gas, which may be fed by an accidental industrial release, in general are bound to spread out over various topography, and have been investigated by Webber *et al.*(1993), Fanneløp (1984), and Hought and Isaacson (1970), among others. In the ocean, examples which include the rotation of the earth are important in describing the movement and circulation of water masses along slopes (Condie, 1995), or deep-water renewal (Karsten *et al.*, 1995, or Quadfasel *et al.*, 1990).

These physical examples listed above certainly differ from the laboratory setting in scale, but also in the inclusion of the somewhat poorly understood aforementioned factors of non-horizontal bottom topography, rotation, and changes in volume due to either a fluid source, entrainment of surrounding fluids, or density variations due to particles and sediment. Considering the progress of shallow-water theory for modelling gravity currents, this approach has been quite well validated, and it is likely that it can still be quite useful when other effects, such as those listed above, are included. The necessity of a front condition, when used in conjunction with the shallow-water equations, is not as widely accepted, as it is often dependent on previously mentioned physical parameters.

An alternate way of viewing a gravity current front is as a vertical discontinuity in the fluid, separating two regions of different density. This is most certainly an approximation, as such factors as entrainment, turbulence, and other non-shallow-water effects are present at this front (Simpson and Britter, 1979). However, there is some usefulness in taking such a mathematical viewpoint, as discontinuous solutions to conservation laws have been well-studied, see for example, John (1982). This idea has a certain parallel to the following example, for the inviscid Burgers equation, stated for a real function $u(x,t)$ in hyperbolic form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

or in the equivalent conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0. \tag{1.3}$$

If a solution, u, to (1.2) is continuous, then it is also a solution to (1.3), and vice-versa (John, 1982). However, a solution to (1.2) which is continuous to the left of a prescribed position $x_f(t)$, and zero to the right, may be thought of in a mathematically acceptable way as a discontinuous solution to (1.3), with $x_f(t)$ defined by the equation (1.3) and the solution itself.

In light of this simple example, it is proposed that shallow-water models for gravity currents may be modelled via the conservation form of the equations. It has been noted, for example by Whitham (1974), that the single layer shallow-water equations admit an infinite number of conservation laws, corresponding to such quantities as mass, momentum, energy, and other terms which do not readily translate into physical quantities. Often, when stating the shallow-water equations, as in Vreugdenhil (1994), they are simply stated in this form as a matter of course. Consequently, a shallow-water model for gravity currents may be stated as a system of equations in conservation form. This approach removes the difficulty of how to define a front condition, replacing it with the problem of how to deal with discontinuities in a solution. Thus, the gravity current front is treated as a vertical discontinuity in the fluid, whose position is unknown and evolves with the flow as part of the solution of the system.

The concept of solving the front position while at the same time finding a solution to the shallow-water equations is not new, and has been attempted numerically over three decades ago, for example by Reid and Bodine (1968) in modelling large water surges due to storms. The difficult problem of the run-up distance on beaches is also connected with this problem, and numerical studies have been conducted for some time (Sielecki and Wurtele, 1970). An early review of the problem of dealing with discontinuous solutions has been completed by Sod (1978), while a more general treatment can be found in Press *et al.*(1986), or the recent work by Godlewski and Raviart (1996). These last two references, as well as the book by Vreugdenhil (1994) describe numerical methods for solving the shallow-water equations.

With the creation of computing methods which permit better resolution, the prediction of discontinuities has received specialized attention by Hyman (1984) and Davis (1992), who, among others, investigated methods for 'tracking' discontinuities, or interfaces. For hyperbolic systems of conservation laws, various means of tracking discontinuities have been proposed, such as the high-resolution techniques by LeVeque and Shyue (1995, 1996). A series of finite difference schemes, known as relaxation methods, have been employed by Jin (1995), and Jin and Xin (1995), which resolve discontinuities well, without resorting to such front tracking methods. The book by LeVeque (1992) describes most of the finite difference methods which are in use today in relation to such hyperbolic systems of equations.

Having briefly described some of the difficulties in modelling gravity currents with shallow-water theory, the stage is now set for stating the main concepts involved in this thesis. There are three main themes which comprise any standard attempt to describe a physical phenomenon. These are: experiments and observation, theoretical description, and numerical simulation. Due to the large amount of experimental observations published, see for example Simpson (1997), the bulk of the research herein is concerns the latter two areas.

The plan of this thesis is then as follows. In Chapter 2, an in-depth derivation of the shallow-water model equations as applied to gravity currents is completed. The shallow-water equations are stated for a two-layer problem in three spatial dimensions, with and without the rigid lid assumption at the upper boundary to close the problem as a system of six partial differential equations. Special cases of these equations are subsequently derived for axisymmetric flow, thin lower and upper layers, layers with small density differences, and flow in two spatial dimensions with and without a free surface. Where it is physically appropriate for three-dimensional flow, Coriolis effects are included, while bottom topography is accommodated in all cases.

The development of the shallow-water equations is fairly standard, and is included in many textbooks such as the one written by Pedlosky (1987). Notwithstanding this, a comprehensive exposition of the assumptions imposed upon the Navier-Stokes Equations, resulting in the shallow-water equations in the several related settings mention above, was felt to be required. The new contribution to the theory is made by omitting the standard front condition through considering the problem as a system of conservation laws. This theoretical framework allows a second novel addition to the theory in the form of a lower layer forcing term which is introduced to capture the effects of entrainment and bottom friction on the lower layer in such a way that it is consistent with the shallow-water assumptions. This new forcing term is based on a Chezy-type bottom friction form, with the addition of a spatially varying truncation which is dependent on the gravity current geometry.

To allow a thorough analysis of the model equations presented in Chapter 2, some theoretical concepts are described in Chapter 3. Definitions of hyperbolic systems, conservation laws, initial value problems (IVPs), and initial boundary value problems (IBVPs) are stated for first order systems of partial differential equations in several space dimensions. For systems of conservation laws, the notion of weak solutions which admit simple discontinuities is discussed in some detail, including a standard derivation of the Rankine-Hugoniot jump conditions at a discontinuity. A generalization of this analysis results in a new type of jump condition applicable to a system of conservation laws with a discontinuous forcing term, where the latter discontinuity corresponds to a simple jump in the solution. This concept is described using a simple example involving Burgers' Equation. Chapter 3 concludes with a short review of the method of localization to determine well-posedness of an IBVP for a nonlinear system, stated as a lemma for later use.

With the model equations outlined, and a rigorous theoretical framework presented, an analysis of the properties can proceed. Chapter 4 is concerned with an investigation of the equations for flow in only the horizontal and vertical directions, and is divided into three sections which discuss the two-layer, thin layer, weakly stratified, and rigid-lid equations. The first of these is the derivation of conditions on the flow for which the various systems are indeed hyperbolic. Although a portion of these results have been published previously (Montgomery and Moodie, 1998 a,b and 1999 a,b), a new result showing hyperbolicity of the two-layer equations when the flow velocities are low is presented. Such a situation arises when the sudden release IBVP is considered. Proceeding under the assumption that the equations are hyperbolic, the second section concerns the derivation of various jump conditions at discontinuities for the various systems as they are expressed in conservation form. These results are entirely new, although some selected equations have been included in a recent paper submitted for publication (Montgomery and Moodie, 1999b). Finally, sudden release IBVPs are shown to be well-posed using the method of localization. Despite a previous preliminary result (Montgomery and Moodie, 1999a), such classification of the various IBVPs represents a new addition to the field.

Chapter 5 mirrors the methods of Chapter 4, but is concerned instead with three-dimensional flow. A new result describing sufficient conditions for hyperbolicity of the two-layer case is shown, and the known results for the single layer and axisymmetric case are included for completeness. The analysis differs from the two-dimensional case where jump conditions for discontinuous solutions are derived, since only the single-layer case is considered. Due to the degeneracy of these equations, an expansion technique is used to yield a new result for predicting discontinuous shock speeds for almost axisymmetric flow. This technique is based on an amplitude dependent plane wave solution to the axisymmetric equations over a horizontal boundary.

In Chapter 6, a numerical method for solving the model equations is presented and used to portray selected results. The finite difference relaxation method developed by Jin and Xin (1995) for the solution of an IVP for a system of nonlinear hyperbolic conservation laws is a conservative, explicit, total variation diminishing iterative technique which removes oscillations near discontinuities through the use of a spatially second-order slope limiter. The modifications to the numerical scheme which have been completed generalize the numerical scheme to include boundary conditions, spatially dependent flux functions, and nonzero forcing terms. Although these new additions to the method have been mentioned previously (Montgomery and Moodie, 1998a,b, 1999a,b), a complete description has been completed in the first section of Chapter 6, along with some preliminary investigation of the properties of the scheme as it concerns the resolution of discontinuities in the solution compared with various parameters. The second section of the chapter contains an application of the numerical scheme to the various models for two-dimensional gravity currents. Height profiles of gravity currents arising from the sudden release IBVP are displayed for various forms of bottom topography, and discontinuous lower layer forcing terms. In addition, curves are plotted which show the lower layer front position as it progresses in time, with and without forcing terms over nonzero bottom slope. The flow values just behind the front are used to calculate values which validate the jump conditions derived in Chapter 4. The final section in Chapter 6 gives some selected results for the initial release problem in three spatial dimensions. The effects of rotation and a constantly sloping bottom are examined separately and together for comparison.

As the numerical technique relies on the ability to state the governing equations as a system of conservation laws, it is desirable to find all possible forms of such equations. Chapter 7 investigates this question in an attempt to classify the standard types of conservation equations which exist for the two-layer two-dimensional situation, and the one-layer three-dimensional case. The method is a generalization of one used by Whitham (1974), who showed that the single layer two-dimensional equations admit an infinite number of conservation laws. The new results for the other physical cases are not directly relevant to the preceeding body of work; however, the results contained in Chapter 7 are important in a more general sense, as the methods used may be generalized in a straightforward way to other first-order systems of partial differential equations. In addition, the equations which are chosen to model gravity currents carry with them a conserved quantity, which may have a bearing on the gravity current front speed.

Chapter 2

Development of the Model Equations

The connection between the physical real-world setting and the mathematical framework for gravity currents is described in this chapter. A time-dependent shallow-water theory for two-layer gravity currents in three spatial dimensions is derived, and the inherent physical assumptions are examined individually to ensure their validity. The development of the two-layer shallow-water equations is not new (Pedlosky 1987). However, the various models used to describe two-layer shallow-water descriptions of gravity currents require that a careful and detailed approach be taken, which is not found elsewhere. A description of these models in a logical format allows the analysis and results of the subsequent chapters to be portrayed in a more understandable and transparent manner than if the equations are simply stated axiomatically.

The first section of this chapter contains a definition of the physical variables and labels which will be used in developing the equations of motion, and a description of the physical geometry which is considered. The general equations for two fluid layers of differing densities moving in three spatial dimensions are developed. When the motion is considered to be of a large enough scale that the rotation of the earth cannot be neglected, the Coriolis parameter is included. Simplifying assumptions are imposed from physical considerations; namely, the fluids are assumed to be inviscid and immiscible. In addition, surface tension and effects due to turbulent entrainment are excluded from the model. These assumptions are the standard ones made when modelling stably-stratifies fluid layers.

Standard scaling arguments (Pedlosky, 1987) are then used in Section 2.2 to simplify the equations, thus obtaining a shallow-water model based on the fact that vertical length scales and velocities are small when compared with the corresponding horizontal ones. This small aspect-ratio flow is subsequently shown to correspond to hydrostatic flow, with horizontal velocities independent of the vertical position, a property typical of shallow-water flow. This results in the equations being stated as a closed system of six partial differential equations in six variables for either a free surface or rigid-lid case.

The derived shallow-water equations for two-layer flow are then specified to model gravity currents resulting from initial release problems. Notably, a new addition to the theory is made in Section 2.3. This is the inclusion of a nonlinear forcing term which is effective at the advancing head of the gravity current, a region of the flow in which the shallow-water assumptions do not apply. Specifically, the front of a current is characterized by large vertical accelerations and non-hydrostatic pressures, and is an area of mixing between the fluids. Notwithstanding these physical processes, some good successes have been achieved by ignoring these effects and making the assumption that shallow-water theory may be applied everywhere within the gravity current (Rottman and Simpson, 1983). The effects of the ignored physical processes at the front are typically modelled by imposing a vertical boundary which moves at a rate determined experimentally via a Froude number front condition. The new nonlinear forcing term introduced maintains the present successful shallow-water assumptions, while removing reliance on experiments to determine specific parameters. It is therefore a natural addition to existing theory which may be compared to standard and established models.

Model equations for two-layer gravity currents are expressed as six equations in six variables, which are reduced in number for several special cases considered in Section 2.4. The system of equations is simplified to three variables and equations by neglecting the upper layer to consider the 'one and a half' layer model in three spatial dimensions. A further reduction to the axisymmetric situation, formulated in polar coordinates, is also stated. In two spatial dimensions (vertical and horizontal) the number of equations is reduced by two to obtain only four variables. Special cases of these equations such as rigid-lid, shallow layer, or weakly stratified (small density differences) are also stated as simplifications, each consisting of only two nonlinear partial differential equations.

2.1 Physical Formulation

The variables for two fluid layers on the earth's surface are defined in this section, with the accompanying geometry also introduced. The equations for an incompressible Newtonian Fluid are stated in a rotating frame, and approximated in the standard way in a locally Cartesian reference frame on the surface of the earth.

The physical situation considered is that of two layers of fluid, blanketed horizontally over each other, with an impermeable bottom boundary below the lower layer and an air/liquid interface above the top layer. The bottom boundary is meant to represent solid topography, such as an ocean floor, whose height profile is assumed to be a known function which then affects any time-evolution of the

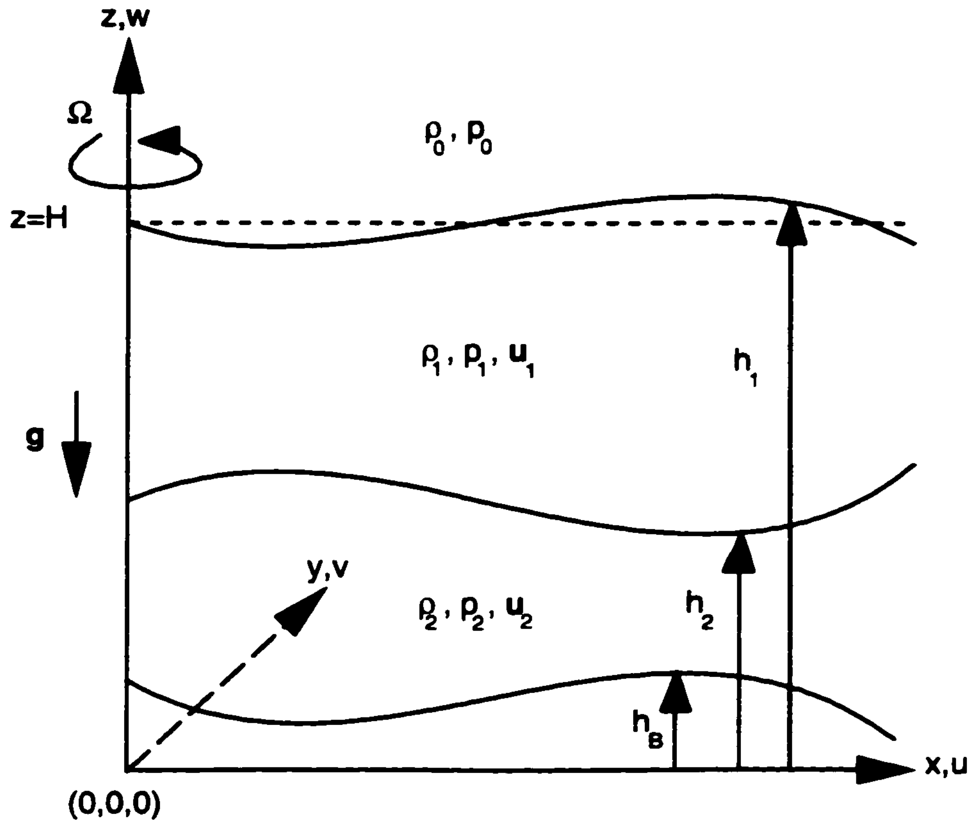

FIGURE 2.1. The two-layer system geometry and variables.

two-layer flow. The upper boundary is such that the pressure above the surface, p_0, is constant and the density difference is large, i.e. $\rho_0 << \rho_1$. The upper and lower layer state variables are denoted with a subscript 1 for the upper layer and 2 for the lower layer, as depicted in Fig. 2.1.

The labels shown in Fig. 2.1 are defined as follows, where the subscripts are irrelevant to the definition and are temporarily neglected. The fluid density is represented by ρ, pressure by p, and velocity by the three component vector $\mathbf{u} = (u, v, w)$. The components u and v are the horizontal velocities in the x and y directions respectively, and the vertical velocity w is in the z direction. The force of gravity acts in the downward vertical direction and is represented by the vector $\mathbf{g} = (0, 0, -g)$ where $g = 9.8\text{ms}^{-2}$. Additionally, the entire coordinate system is assumed to be rotating about the z axis with the frequency of the earth, $\Omega = 2\pi$ radians per day or $7.27 \times 10^{-5}\text{s}^{-1}$.

The three surfaces displayed in Fig. 2.1 are all measured above the plane $z = 0$.

The known solid bottom contour is given by the surface $z = h_B(x, y)$, and the upper boundary of the fluid layers are shown as the surfaces $z = h(x, y, t)$. The plane $z = H$ is noted for later use and should be thought of as the value which the upper layer of an undisturbed system at rest may assume, namely $H = h_1$. Additional notation which will be used where appropriate is $\zeta_2 = h_2 - h_B$ and $\zeta_1 = h_1 - h_2$. These represent the layer thicknesses and are always nonnegative functions of x, y and t.

To state the equations of motion which permit a description of the evolution of the system depicted in Fig. 2.1, some specifications as to the physical nature of the fluids need to be made. Given clearly stated assumptions, researchers will be able to determine the applicability to physical examples of the various models described in this chapter. With these assumptions clearly stated, it will be the case that the application of theory may be facilitated so that anyone considering an application will find their task simplified. Underlying the terminology is the idea that the two fluid layers are watery in nature, and it is sufficient for the reader to form a mental picture of a salt-water bottom layer underlying one of fresh water. Although liquids are almost exclusively considered, many of the following assumptions are also appropriate in some atmospheric situations. In that case, an equation of state may be necessary to account for changes in density, which beyond the scope of the constant-density currents considered in this thesis.

As a starting point, this thesis is concerned with situations in which the continuum hypothesis holds, so that the molecular properties of matter may be ignored and replaced by the large scale behaviour of the fluids. This is generally valid when the scales of the motion (perhaps as small as 10^{-2}m) are much larger than the mean free path of the molecules (perhaps as large as 10^{-9}m) (Kundu, 1990 p.5). With this assumption, the standard methodology of Fluid Mechanics (see, for example, Kundu, 1990) is followed instead of the kinetic theory approach necessary for the consideration of a rarefied gas.

Both layers are assumed to be Newtonian fluids, an assumption which specifies the type of constitutive equation which may be used (Kundu, 1990 p.92). This assumption is commonplace, and is accurate for fluids such as air and water (Kundu, 1990 p.93). The theory is not applicable to fluids that exhibit non-Newtonian behaviour, such as some emulsions and slurries, or fluids with viscoelastic properties.

The next fundamental assumption concerns the question of incompressibility of the fluids. An incompressible fluid is generally defined as one whose density does not change with pressure (Kundu, 1990 p.79, Baines, 1995 p.4). Although the assumption of incompressibility is generally a good one for water there are many circumstances in which air can be considered incompressible (Baines, 1995 p.4,5) or weakly compressible through the anelastic approximation. The subject of gas dynamics considers compressible flows, where the effects of pressure on density are characterised through the Mach number, defined as the ratio of the flow speed to the local speed of sound (Kundu, 1990 p.580). In general, a gas is considered to be incompressible for Mach numbers lower than 0.3 (Kundu, 1990 p.581), for which density variations due to pressure can be neglected. For example, in air at 15°C, the speed of sound is approximately 340ms^{-1} (Kundu, 1990 p.585). Typical atmospheric gravity currents travel at speeds less than about 30ms^{-1} (Simpson, 1997 p.12-19). Ignoring small variations in the sound speed due to temperature variation from 15°C, a rough estimate yields a Mach number below 0.1 for atmospheric gravity currents, which is less than the cut-off of 0.3. Therefore, considering an atmospheric gravity current example, the incompressibility assumption for watery fluids is extended to air, and is assumed to hold.

The assumption of incompressibility allows the continuity equation in differential form,

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \qquad (2.1.1)$$

to be simplified. This simplification has been made under the label of the Boussinesq approximation (Kundu, 1990 p.97), or strictly by stating that $\frac{D\rho}{Dt} = 0$ for an incompressible fluid. Regardless of the style of the argument, an incompressbile fluid permits the term $\rho^{-1}D\rho/Dt$ in equation (2.1.1) to be neglected in favour of the divergence term. Thus the incompressible form of (2.1.1) to be used is

$$\nabla \cdot \mathbf{u} = 0. \qquad (2.1.2)$$

The simpler form of equation (2.1.2) over (2.1.1) makes it a much more convenient equation to use in practice.

In addition to equation (2.1.2) for an incompressible fluid, other equations of motion, for the fluid velocity or momentum, may be stated in a rotating frame (Kundu, 1990 p.97, Baines, 1995 p.10, Pedlosky, 1987 p. 19) as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F} + (\mathbf{g} + \Omega^2\mathbf{R}) - 2\Omega \times \mathbf{u}. \qquad (2.1.3)$$

This equation, in spherical coordinates, contains $\mathbf{R}$ as a radial vector from the axis of rotation which is rotating at angular velocity Ω, and a vector $\mathbf{F}$ to represent

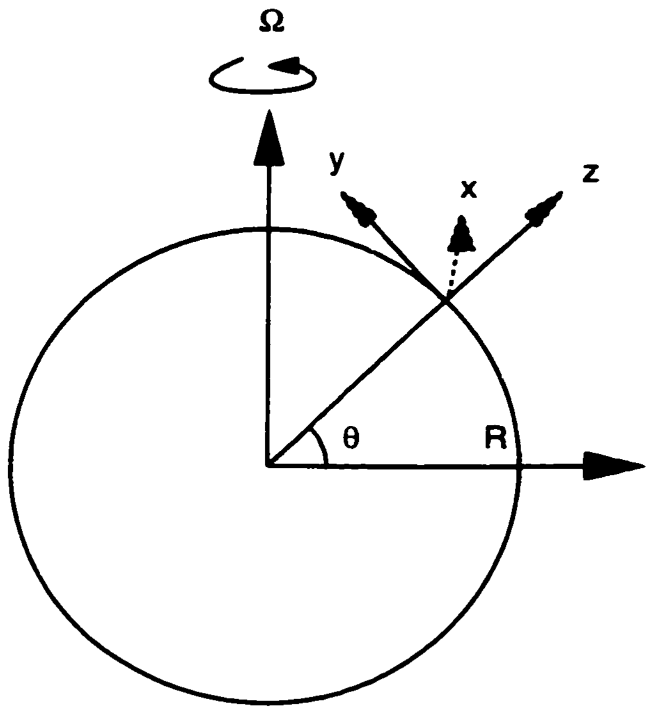

FIGURE 2.2. Cartesian Coordinates on the Earth's Surface.

some general, unspecified forces per unit mass, such as viscosity or diffusion. **F** will be discussed in more detail in Section 2.3.

Although equation (2.1.3) is appropriate for large scale problems, it may be simplified if the horizontal length scales of the motion involved are small when compared to the radius of the Earth, which has an average value of 6371 km. In this case, the motion may be approximately described in a locally Cartesian coordinate system which is rotating at a point on the Earth, having been approximated as a perfect sphere (Kundu, 1990 p. 481). This set of axes may be defined with the x-axis parallel to lines of constant latitude, the y-axis parallel to lines of constant longitude, and the z-axis given in an upward normal direction to the surface. These are sketched in Figure 2.2.

The angular velocity of the Earth resolved in this coordinate system is specified at a given latitude by the angle θ measured from the equator, as

$$\Omega = (0, \Omega \cos\theta, \Omega \sin\theta). \tag{2.1.4}$$

Thus, the Coriolis force term, $-2\Omega \times \mathbf{u}$ from equation (2.1.3), may be expressed in components as

$$-2\Omega \times \mathbf{u} = -2\Omega(w\cos\theta - v\sin\theta, u\sin\theta, -u\cos\theta). \tag{2.1.5}$$

Standard approximations for a thin layer on a rotating sphere simplify this term (2.1.5) even further. The first of these is that the x-component $w\cos\theta$ is assumed to be much smaller than $v\sin\theta$, since in general, $w << v$. Second, the vertical component of the Coriolis force is usually negligible when compared to the vertical gravitational forces; hence $2\Omega u\cos\theta$ is also neglected in the vertical momentum equation. These assumptions to equation (2.1.5) simplify the momentum equations (2.1.3) to the usual ones given for a thin layer, stated from Kundu (1990, p.483) as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \mathbf{g} + (fv, -fu, 0), \tag{2.1.6}$$

where the Coriolis parameter f is inroduced in (2.1.6) and is defined as

$$f = 2\Omega\sin\theta. \tag{2.1.7}$$

In the gravity current setting, a few more physical assumptions are made about the nature of the fluids and their interaction. The fluids are assumed to be inviscid and of constant density within each layer. In addition, entrainment and turbulent effects between fluid layers are not included. These assumptions may be quantified by the parameters of Reynolds and Richardson Numbers (Baines, 1995, Kundu, 1990). These effects are most definitely important near gravity current fronts, however they generally play only a small role elsewhere and may be ignored as is done in standard shallow-water theory (Rottman and Simpson, 1983). As mentioned in the introduction, the effects of entrainment and turbulence will be modelled through the addition to standard theory of a nonlinear forcing term, to be described later in Section 2.3.

Finally, a few comments will be made on the meaning of the word *Boussinesq* as it applies in this context. The Boussinesq approximation pertains to a stratified fluid in which the density changes with position in the fluid (Baines, 1995 p.7), and is usually applicable to liquids in geophysical situations. The Boussinesq approximation may be phrased as the statement that in the momentum equations (2.1.3), density changes are neglected except where they arise in the buoyancy terms. Since only constant density layers are considered herein, this approximation is required within each layer. However, it is noted that in such a case, Boussinesq flow is said to occur when the relative density differences between layers is small (Lawrence, 1990 p.459). It is in this sense that the flow is considered Boussinesq in most situations, and the constant density assumption may often be relaxed with the same resulting equations.

2.2 Two-layer Equations of Motion

Under the assumptions in the previous section, the equations of motion (2.1.1) and (2.1.6) in each layer are written for each layer with subscripts $i = 1$ or $i = 2$ to denote the upper or lower layer, respectively. The equation of mass conservation, or continuity equation, for an incompressible fluid is given in each layer by (2.1.2) stated as

$$\nabla \cdot \mathbf{u}_i = 0. \tag{2.2.1}$$

The corresponding momentum equations (2.1.6) in component form are given by

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + v_i\frac{\partial u_i}{\partial y} + w_i\frac{\partial u_i}{\partial z} = -\frac{1}{\rho_i}\frac{\partial p_i}{\partial x} + fv_i + F_{i_x}, \tag{2.2.2}$$

$$\frac{\partial v_i}{\partial t} + u_i\frac{\partial v_i}{\partial x} + v_i\frac{\partial v_i}{\partial y} + w_i\frac{\partial v_i}{\partial z} = -\frac{1}{\rho_i}\frac{\partial p_i}{\partial y} - fu_i + F_{i_y}, \tag{2.2.3}$$

and

$$\frac{\partial w_i}{\partial t} + u_i\frac{\partial w_i}{\partial x} + v_i\frac{\partial w_i}{\partial y} + w_i\frac{\partial w_i}{\partial z} = -\frac{1}{\rho_i}\frac{\partial p_i}{\partial z} - g + F_{i_z}. \tag{2.2.4}$$

These equations may be simplified by using the hydrostatic approximation, which can be stated simply as

$$\frac{\partial p_i}{\partial z} = -\rho_i g + O(\delta^2), \tag{2.2.5}$$

where δ is the aspect ratio of the flow, defined as the ratio of typical height to length scales. The scaling arguments leading up to this approximation are given succinctly in Pedlosky (1987 p.58) and are not reproduced, but hold for arbitrary frequencies of the angular velocity Ω. For small aspect ratio flows, the small $O(\delta^2)$ term is neglected so that equation (2.2.5) may be integrated vertically to give

$$p_i(x, y, z, t) = -\rho_i gz + \tilde{p}_i(x, y, t). \tag{2.2.6}$$

The hydrostatic approximation is used in the presence of bottom topography, which will excite vertical accelerations. However, small variations in bottom topography with gentle slopes are consistent with shallow-water theory and may be used in conjunction with the hydrostatic approximation. Typical slopes of less than 1/10 are thus considered to be compatible with the hydrostatic approximation.

Equation (2.2.6) may be combined with the boundary conditions of pressure continuity at the fluid interfaces, so that the extra terms $\tilde{p}_i$ may be specified and removed from the governing equations. The two boundary conditions discussed simultaneously are that of the top boundary of the upper layer is either a free surface or a rigid lid. In either case, the pressure in each layer may be simplified somewhat. Letting $p_s(x, y, t)$ denote the pressure at the top boundary of the upper layer, equation (2.2.6) gives the perturbation pressure $\tilde{p}_1$ as

$$\tilde{p}_1 = \rho_1 g h_1 + p_s.$$

This may be then substituted back into equation (2.2.6) for $i = 1$ to give the upper layer pressure expressed as the sum of a hydrostatic part and an external surface pressure as

$$p_1(x, y, z, t) = \rho_1 g(h_1 - z) + p_s. \tag{2.2.7}$$

Similarly, the pressure in the lower layer may be written using the continuity of pressure at the layer interface: $p_1 = p_2$ at $z = h_2$. Equation (2.2.6) may be evaluated at $z = h_2$, and (2.2.7) substituted into the result to find $\tilde{p}_2$ as

$$\begin{aligned} \tilde{p}_2 &= p_2(x, y, h_2, t) + \rho_2 g h_2 \\ &= p_1(x, y, h_2, t) + \rho_2 g h_2 \\ &= \rho_1 g(h_1 - h_2) + p_s + \rho_2 g h_2. \end{aligned}$$

This manipulation allows equation (2.2.6), for $i = 2$, to yield the lower layer pressure expressed as the sum of hydrostatic pressure and surface pressure as:

$$p_2(x, y, z, t) = \rho_2 g(h_2 - z) + \rho_1 g(h_1 - h_2) + p_s. \tag{2.2.8}$$

The only differences between the two boundary conditions considered at the upper interface are that a free upper surface contains the simplification of constant surface pressure p_s, while a rigid lid upper surface has the constraint $h_1 = H$.

At this point, an observation may be made from equations (2.2.7) and (2.2.8). Since $p_s = p_s(x, y, t)$, taking derivatives of (2.2.7), (2.2.8) with respect to either x or y shows that the horizontal pressure gradients are independent of the variable z. Therefore, the horizontal accelerations in equations (2.2.2), (2.2.3) are also independent of z, or at least any changes in z dependence are necessarily independent of pressure. For consistency, it is assumed that if the horizontal velocities are initially independent of z then they remain so. That is, it is assumed that $u_i = u_i(x,y,t)$ and $v_i = v_i(x,y,t)$ for both $i = 1,2$. Along with the hydrostatic assumption, the often termed ***shallow-water*** approximation is the basis of shallow-water theory, and one which has been used successfully (Pedlosky, 1987, Kundu, 1990, Baines, 1995).

An important point to note in making the shallow-water approximation is that to maintain the z-independence of u_i and v_i, the forcing terms F_{ix} and F_{iy} must also be independent of z. This property of the forcing term $\mathbf{F}$ is assumed at this point without justification and will be discussed in Section 2.3.

In addition to simplifying the horizontal momentum equations (2.2.2), (2.2.3) by the removal of the vertical derivatives, $\frac{\partial u_i}{\partial z}$ and $\frac{\partial v_i}{\partial z}$, the shallow-water approximation simplifies the continuity equation (2.2.1) so that it may be integrated over the vertical domain. This is observed clearly when equation (2.2.1) is expanded from its vector form as

$$\frac{\partial w_i}{\partial z} = -\frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y}, \qquad (2.2.9)$$

which integrates vertically, using the fact that the right hand side is independent of z, to allow the vertical velocity w_i to be stated as

$$w_i(x,y,z,t) = -z\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) + \widetilde{w}_i(x,y,t), \qquad (2.2.10)$$

where the terms $\widetilde{w}_i$ are the unknown functions of the integration.

The vertical velocities may be further specified through application of kinematic boundary conditions at the appropriate vertical boundaries of the system. First, at the bottom boundary $z = h_B(x,y)$, the condition of no net flow across the boundary is imposed. This is written as

$$\mathbf{u}_2 \cdot \nabla(z - h_B) = 0 \quad \text{at } z = h_B,$$

which simplifies to

$$w_2(x,y,h_B,t) = u_2\frac{\partial h_B}{\partial x} + v_2\frac{\partial h_B}{\partial y}. \qquad (2.2.11)$$

The condition (2.2.11) now allows the lower layer ($i = 2$) specification of $\widetilde{w}_2$ in equation (2.2.10). The unknown function $\widetilde{w}_2$ is found by evaluating (2.2.10) for

$i = 2$ at $z = h_B$, and using equation (2.2.11) as follows:

$$\begin{aligned}\widetilde{w}_2(x,y,t) &= w_2(x,y,h_B,t) + h_B\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right)\\ &= u_2\frac{\partial h_B}{\partial x} + v_2\frac{\partial h_B}{\partial y} + h_B\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right)\\ &= \frac{\partial}{\partial x}(h_B u_2) + \frac{\partial}{\partial y}(h_B v_2).\end{aligned}$$

This result, substituted into equation (2.2.10) for the lower layer, then yields the lower layer vertical velocity as

$$w_2(x,y,z,t) = -z\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right) + \frac{\partial}{\partial x}(h_B u_2) + \frac{\partial}{\partial y}(h_B v_2). \tag{2.2.12}$$

A similar procedure may be carried out for the lower boundary of the upper layer, although the boundary condition at the top of the upper layer affects the analysis and subsequent equations of motion, and the calculations must be completed separately for the free surface and rigid lid cases. For a free surface, the kinematic condition at the free surface $z = h_1$ is given by

$$w_1(x,y,h_1,t) = \frac{Dh_1}{Dt}, \tag{2.2.13}$$

which becomes upon substitution of equation (2.2.10) for the upper layer,

$$-h_1\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + \widetilde{w}_1(x,y,t) = \frac{\partial h_1}{\partial t} + u_1\frac{\partial h_1}{\partial x} + v_1\frac{\partial h_1}{\partial y}. \tag{2.2.14}$$

This boundary condition (2.2.14) then specifies the function $\widetilde{w}_1$ which may be substituted back into equation (2.2.10) for $i = 1$ to allow the upper layer vertical velocity to be stated similarly to (2.2.12) as

$$\begin{aligned}w_1(x,y,z,t) &= -z\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + \frac{\partial h_1}{\partial t} + u_1\frac{\partial h_1}{\partial x} + v_1\frac{\partial h_1}{\partial y} + h_1\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)\\ &= -z\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + \frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(h_1 u_1) + \frac{\partial}{\partial y}(h_1 v_1).\end{aligned} \tag{2.2.15}$$

For the case of a rigid boundary at $z = h_1$, no net flow across $z = h_1$ may be expressed as

$$w_1(x,y,h_1,t) = 0.$$

This allows the upper layer vertical velocity to be expressed, through the use of equation (2.2.10), as

$$\begin{aligned} w_1(x,y,z,t) &= -z\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+h_1\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right) \\ &= -z\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+\frac{\partial}{\partial x}(h_1u_1)+\frac{\partial}{\partial y}(h_1v_1). \end{aligned}$$

It is somewhat surprising at first to notice that this expression is the same as (2.2.15) once the rigid lid specification $h_1 = H$ is made, since the vertical velocity is therefore independent of the choice of boundary condition. However, equation (2.2.15) will be used exclusively to represent the upper layer vertical velocity for both the free surface and rigid lid equations.

The equations of motion for the two-layer system with either a free surface or a rigid lid at $z = h_1$ may now be specified as a closed system of equations, up to a discussion of the forcing terms $\mathbf{F}_i$. Equations (2.2.2) and (2.2.3) may be simplified through use of the shallow-water approximation, and expressions (2.2.7), (2.2.8). With the definition of the reduced gravity,

$$g' = \frac{(\rho_2-\rho_1)}{\rho_2}g, \tag{2.2.16}$$

the momentum equations, written explicitly for each layer, are:

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+v_2\frac{\partial u_2}{\partial y}-fv_2=-g'\frac{\partial h_2}{\partial x}-(g-g')\frac{\partial h_1}{\partial x}-\frac{1}{\rho_2}\frac{\partial p_s}{\partial x}+F_{2x}, \tag{2.2.17}$$

$$\frac{\partial v_2}{\partial t}+u_2\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}+fu_2=-g'\frac{\partial h_2}{\partial y}-(g-g')\frac{\partial h_1}{\partial y}-\frac{1}{\rho_2}\frac{\partial p_s}{\partial y}+F_{2y}, \tag{2.2.18}$$

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}-fv_1=-g\frac{\partial h_1}{\partial x}-\frac{1}{\rho_1}\frac{\partial p_s}{\partial x}+F_{1x}, \tag{2.2.19}$$

and

$$\frac{\partial v_1}{\partial t}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}+fu_1=-g\frac{\partial h_1}{\partial y}-\frac{1}{\rho_1}\frac{\partial p_s}{\partial y}+F_{1y}. \tag{2.2.20}$$

The system above is almost closed by inclusion of two more equations derived from the kinematic boundary condition similar to (2.2.13) at the interface $z = h_2$. The first of these may be found from consideration of the lower layer velocity, (2.2.12), substituted into the boundary condition at the layers' interface,

$$w_2(x,y,h_2,t) = \frac{Dh_2}{Dt}.$$

This substitution yields

$$-h_2\left(\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}\right)+\frac{\partial}{\partial x}(h_B u_2)+\frac{\partial}{\partial y}(h_B v_2)=\frac{\partial h_2}{\partial t}+u_2\frac{\partial h_2}{\partial x}+v_2\frac{\partial h_2}{\partial y},$$

which simplifies readily through use of the product rule to the standard surface equation from shallow-water theory,

$$\frac{\partial h_2}{\partial t}+\frac{\partial}{\partial x}\left[(h_2-h_B)u_2\right]+\frac{\partial}{\partial y}\left[(h_2-h_B)v_2\right]=0. \qquad (2.2.21)$$

A second equation may be derived similarly via the kinematic boundary condition for the upper layer at the layer interface $z = h_2$,

$$w_1(x,y,h_2,t)=\frac{Dh_2}{Dt}.$$

Substitution of the upper layer vertical velocity w_1 given in the form of equation (2.2.15) yields the evolution equation for the surface $z = h_1$,

$$-h_2\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+\frac{\partial h_1}{\partial t}+\frac{\partial}{\partial x}(h_1 u_1)+\frac{\partial}{\partial y}(h_1 v_1)=\frac{\partial h_2}{\partial t}+u_1\frac{\partial h_2}{\partial x}+v_1\frac{\partial h_2}{\partial y},$$

which subsequently simplifies via the product rule to

$$\frac{\partial}{\partial t}(h_1-h_2)+\frac{\partial}{\partial x}\left[(h_1-h_2)u_1\right]+\frac{\partial}{\partial y}\left[(h_1-h_2)v_1\right]=0. \qquad (2.2.22)$$

Equations (2.2.21) and (2.2.22) are often written, in following chapters, with the variables $\zeta_1 = h_1 - h_2$ and $\zeta_2 = h_2 - h_B$ to simplify the notation.

The six equations (2.2.17)-(2.2.22) are the two-layer equations which describe shallow-water motions for constant density layers. For a free surface, p_s = constant, the system becomes closed since the terms involving p_s are removed from equations (2.2.17)-(2.2.20). For the rigid lid case, h_1 = constant, the addition of the resulting equations (2.2.21) and (2.2.22) yield

$$\frac{\partial}{\partial x}\left[(h_1-h_2)u_1+(h_2-h_B)u_2\right]+\frac{\partial}{\partial y}\left[(h_1-h_2)v_1+(h_2-h_B)v_2\right]=0. \quad (2.2.23)$$

Therefore, for the rigid lid case, the equations also become closed in the six variables u_2, v_2, h_2, u_1, v_1, and p_s, and the six equations (2.2.17)-(2.2.20), (2.2.21), and (2.2.23).

2.3 Two-layer Gravity Currents

Equations (2.2.17)–(2.2.22) developed in the previous section describe two fluid layers of constant density and are applicable in the case of small aspect ratio flow which validates the hydrostatic approximation and shallow-water formulation. Discussion of the forcing terms $\mathbf{F}_i$ previously introduced in equation (2.1.3) is completed in this section, and the equations of motion are revisited with the aim of creating a model for gravity currents.

2.3.1 The Forcing Terms, $\mathbf{F}_i$

The two-layer system depicted in Figure 2.1 differs from the gravity current picture through the absence of a vertical material interface which marks the possible horizontal extent of a layer. The lower layer is now considered not to be semi-infinite in extent, as Fig. 2.1 suggests, but a finite volume of dense fluid which is bounded over a finite domain. The lower layer is assumed to have a relatively simple geometry, and its horizontal profile is taken to be a simply connected (Marsden and Hoffman, 1987 p.146) and (finitely) bounded region in the (x, y)-plane. More generally the boundary of a finite convex region in the plane may be considered, although convexity is not necessary for the analysis, with a smooth (differentiable) boundary. One such possible shape is portrayed in Figure 2.3, which appears above.

It is assumed that the curve Γ represents a well-defined vertical interface. In the case that there is a smooth transition of the lower layer to zero thickness, as would occur if the interface was horizontal or nearly so, then an effective height cut-off at a specific value would be necessary. This difficulty exists when calculating the front position numerically, and will be discussed in Chapter 6. An arbitrary cutoff of 0.1% of the maximum layer thickness is a reasonable value to use in these circumstances.

The lateral extent of the lower layer is denoted by the curve Γ which may evolve in time as the flow progresses. A point lying on Γ at a certain time is usually given a subscript F, which should not be confused with the Coriolis parameter, to denote that the point is on the front, i.e. (x_F, y_F). The dense lower layer is considered to be inside Γ, where the less dense upper layer exists both outside Γ, and above the lower layer.

Since the curve $\Gamma(x, y) = 0$ marks the horizontal extent of the gravity current, it follows that Γ evolves in a direction proportional to the frontal velocity. That

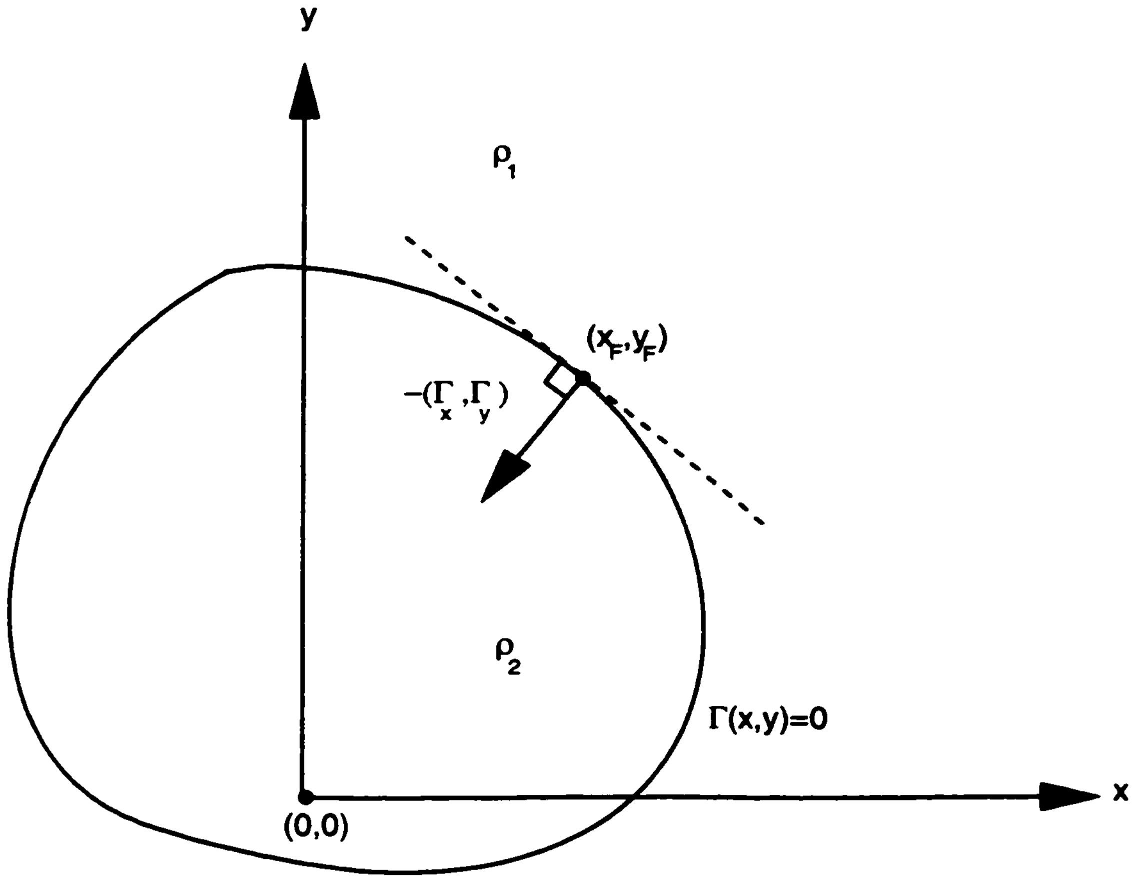

FIGURE 2.3. Front Position Geometry of a Finite Dense Volume (top view).

is,

$$\nabla\Gamma = \left(\frac{\partial\Gamma}{\partial x}, \frac{\partial\Gamma}{\partial y}\right) = \kappa(u_2, v_2)|_\Gamma, \tag{2.3.1}$$

where κ is a constant of proportionality. That this is the case follows from physical reasoning that the front is a material interface and that fluid parcels at the front do not leave the fluid. This includes the assumption that mixing at the interface does not occur.

The forcing term $\mathbf{F}_i$ may now be introduced as a drag force per unit mass which acts only on the lower layer as the gravity current spreads outward from its source. Although the actual physical mechanisms which would be normally included in $\mathbf{F}_i$ are from effects such as viscosity, turbulence, and entrainment, these effects are not modelled explicitly and are gathered as a single term. In words, the forcing term acts in a direction perpendicular to the front curve Γ, and points inwards; it is strongest near to the front, and decays inwardly away from Γ. The form of $\mathbf{F}_i$ mirrors a Chézy-type frictional form, such as that in Whitham (1974, p.83). The forcing terms are thus introduced as

$$\mathbf{F}_2 = -C_f \frac{(\mathbf{u}_2 \cdot \nabla\Gamma)}{h_2 - h_B} T(x, y, \Gamma)\hat{\mathbf{n}}, \tag{2.3.2}$$

and

$$\mathbf{F}_1 = \mathbf{0}. \tag{2.3.3}$$

In equation (2.3.2), C_f is a constant dimensionless coefficient of friction, $\hat{\mathbf{n}}$ is the outward normal unit vector, and T is a truncation function. Note that the sign of expression (2.3.2) ensures that it is in an inward direction.

As discussed in Section 2.2, the forcing terms $\mathbf{F}_i$ must have certain properties if the shallow-water approximation is to remain valid. Specifically, $\mathbf{F}_i$ must be independent of height so that $\mathbf{F}_i = \mathbf{F}_i(x, y, t)$. This restriction is satisfied by equations (2.3.2)-(2.3.3), although it is not very satisfactory in a physical sense since vertical structure does exist at a gravity current front. However, to be consistent with shallow-water theory while maintaining the effective forcing of entrainment and bottom friction in the momentum equations, this type of definition for $\mathbf{F}_i$ suffices.

The truncation function T is somewhat arbitrary, and a numerical investigation of various forms will be completed later in Chapter 6, where several choices of T are considered. Here, for want of a simple example, the truncation T is considered in the form of a quadratic function independent of the t variable, which has the value of unity on Γ, and zero at a specified distance from the curve Γ. To state such a function, it is necessary to define the function $\xi : \text{int}(\Gamma) \to [0, \infty)$ as the Euclidean distance from the front, i.e.

$$\xi(x, y, \Gamma) = \min_{(x_f, y_f) \in \Gamma} \|(x, y) - (x_f, y_f)\|. \tag{2.3.4}$$

Due to the property of the closure of the interior of Γ being a compact set in $\mathbb{R}^2$, such a function is well-defined. The function T in (2.3.2), redefined as $T \circ \xi$, may now be thought of as a real-valued function, $T : [0, \infty) \to [0, 1]$ which is one-to-one, and, for example, could be defined as the lower half of a parabola as shown in Fig. 2.4, with vertex $(0, 1)$ and varying foci dependent on a parameter l. The equation for such a parabola,

$$\xi = l(T - 1)^2$$

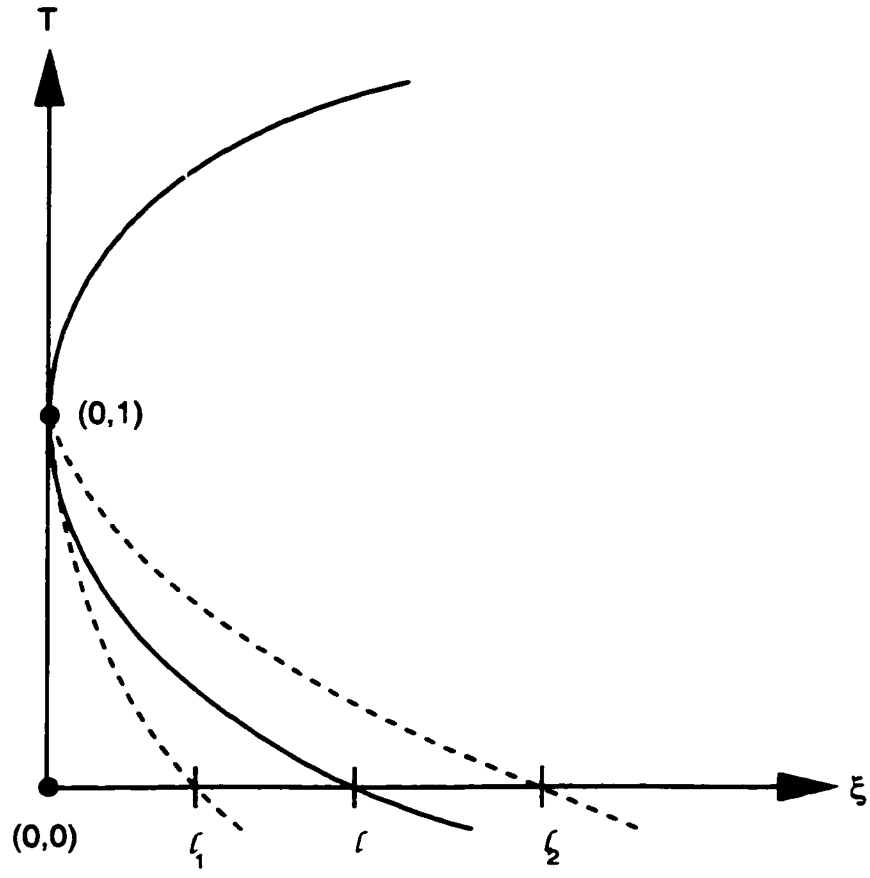

FIGURE 2.4. The families of truncation parabolas, T.

is then quickly inverted, choosing the lower half only to match the stated domain above, as

$$T(\xi) = 1 - \sqrt{\frac{\xi}{l}}. \tag{2.3.5}$$

The parameter l in the definition (2.3.5) is called the *length parameter*, and represents the length of the truncation function T.

To extend T to the entire region inside Γ, it is extended from (2.3.5) to the piecewise continuous function

$$T(\xi) = \begin{cases} 1 - \sqrt{\frac{\xi}{l}}, & \text{if } 0 \leq \xi \leq l, \\ 0, & \text{if } l < \xi. \end{cases} \tag{2.3.6}$$

The definition (2.3.6) is now used to define the forcing term $\mathbf{F}_2$ in the definition (2.3.2).

The equations of motion now consist of a closed system of the six equations (2.2.17)-(2.2.22) and the six variables $u_2, v_2, h_2, u_1, v_1, h_1$. The lower layer bounding curve Γ represents a theoretical difficulty only since it is not assumed to be a known function. Γ is therefore determined as the gravity current develops, and the position taken into account, when the equations are solved numerically, as a known function at each time step as the solution progresses forward in time.

2.3.2 Nondimensionalization of the Equations of Motion

The equations of motion are nondimensionalised using a scaling which focuses on the typical gravity current scalings which are observed experimentally (Rottman and Simpson, 1983). Nondimensional quantities, signified with a tilde, are therefore introduced as:

$$\begin{aligned}
&(x,y) = L(\tilde{x},\tilde{y}), \quad (h_B,h_1,h_2) = H(\tilde{h}_B,\tilde{h}_1,\tilde{h}_2),\\
&(u_1,u_2) = U(\tilde{u}_1,\tilde{u}_2), \quad (v_1,v_2) = U(\tilde{v}_1,\tilde{v}_2),\\
&t = \frac{L}{U}\tilde{t}, \quad C_f = \frac{H}{L}\tilde{C}_f, \quad p_s = \rho_2 g' H \tilde{p}_s.
\end{aligned} \tag{2.3.7}$$

The scalings in the above consist of a characteristic horizontal length scale L, vertical length scale H, and horizontal velocity scale U. Specifying that $U^2 = g'H$, two-layer shallow water internal gravity wave speed, removes the addition of any new parameters. The nondimensionalization scheme (2.3.7), upon substitution into equations (2.2.17)-(2.2.23) gives, upon utilization of (2.3.1) into the definitions (2.3.2)-(2.3.3),

$$\frac{U^2}{L}\left[\frac{\partial \tilde{u}_2}{\partial \tilde{t}} + \tilde{u}_2\frac{\partial \tilde{u}_2}{\partial \tilde{x}} + \tilde{v}_2\frac{\partial \tilde{u}_2}{\partial \tilde{y}} - \frac{fL}{U}\tilde{v}_2\right] = \frac{g'H}{L}\left[-\frac{\partial \tilde{h}_2}{\partial \tilde{x}} - \frac{(g-g')}{g'}\frac{\partial \tilde{h}_1}{\partial \tilde{x}} - \frac{\partial \tilde{p}_s}{\partial \tilde{x}}\right] - \frac{HU^2}{LH}\tilde{C}_f\kappa\tilde{u}_2\frac{\sqrt{\tilde{u}_2^2+\tilde{v}_2^2}}{\tilde{h}_2-\tilde{h}_B}T, \tag{2.3.8}$$

$$\frac{U^2}{L}\left[\frac{\partial \tilde{v}_2}{\partial \tilde{t}} + \tilde{u}_2\frac{\partial \tilde{v}_2}{\partial \tilde{x}} + \tilde{v}_2\frac{\partial \tilde{v}_2}{\partial \tilde{y}} + \frac{fL}{U}\tilde{u}_2\right] = \frac{g'H}{L}\left[-\frac{\partial \tilde{h}_2}{\partial \tilde{y}} - \frac{(g-g')}{g'}\frac{\partial \tilde{h}_1}{\partial \tilde{y}} - \frac{\partial \tilde{p}_s}{\partial \tilde{y}}\right] - \frac{HU^2}{LH}\tilde{C}_f\kappa\tilde{v}_2\frac{\sqrt{\tilde{u}_2^2+\tilde{v}_2^2}}{\tilde{h}_2-\tilde{h}_B}T, \tag{2.3.9}$$

$$\frac{U^2}{L}\left[\frac{\partial \tilde{u}_1}{\partial \tilde{t}} + \tilde{u}_1\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \tilde{v}_1\frac{\partial \tilde{u}_1}{\partial \tilde{y}} - \frac{fL}{U}\tilde{v}_1\right] = \frac{g'H}{L}\left[-\frac{g}{g'}\frac{\partial \tilde{h}_1}{\partial \tilde{x}} - \frac{\rho_2}{\rho_1}\frac{\partial \tilde{p}_s}{\partial \tilde{x}}\right], \tag{2.3.10}$$

$$\frac{U^2}{L}\left[\frac{\partial \tilde{v}_1}{\partial \tilde{t}} + \tilde{u}_1\frac{\partial \tilde{v}_1}{\partial \tilde{x}} + \tilde{v}_1\frac{\partial \tilde{v}_1}{\partial \tilde{y}} + \frac{fL}{U}\tilde{u}_1\right] = \frac{g'H}{L}\left[-\frac{g}{g'}\frac{\partial \tilde{h}_1}{\partial \tilde{y}} - \frac{\rho_2}{\rho_1}\frac{\partial \tilde{p}_s}{\partial \tilde{y}}\right], \qquad (2.3.11)$$

$$\frac{HU}{L}\left[\frac{\partial \tilde{h}_2}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}[(\tilde{h}_2 - \tilde{h}_B)\tilde{u}_2] + \frac{\partial}{\partial \tilde{y}}[(\tilde{h}_2 - \tilde{h}_B)\tilde{v}_2]\right] = 0, \qquad (2.3.12)$$

$$\frac{HU}{L}\left[\frac{\partial}{\partial \tilde{t}}(\tilde{h}_1 - \tilde{h}_2) + \frac{\partial}{\partial \tilde{x}}[(\tilde{h}_1 - \tilde{h}_2)\tilde{u}_1] + \frac{\partial}{\partial \tilde{y}}[(\tilde{h}_1 - \tilde{h}_2)\tilde{v}_1]\right] = 0, \qquad (2.3.13)$$

and

$$\frac{HU}{L}\left[\frac{\partial}{\partial \tilde{x}}[(\tilde{h}_1 - \tilde{h}_2)\tilde{u}_1 + (\tilde{h}_2 - \tilde{h}_B)\tilde{u}_2] + \frac{\partial}{\partial \tilde{y}}[(\tilde{h}_1 - \tilde{h}_2)\tilde{v}_1 + (\tilde{h}_2 - \tilde{h}_B)\tilde{v}_2]\right] = 0. \qquad (2.3.14)$$

To simplify the nondimensional equations (2.3.8)-(2.3.14), some additional terms are defined, and the tilde notation is dropped for convenience. Two new parameters are introduced to this end. These are the ratio of the reduced gravity to gravitational acceleration,

$$\gamma = \frac{g'}{g} = \frac{\rho_2 - \rho_1}{\rho_2}, \qquad (2.3.15)$$

and the inverse of the Rossby number (R_o),

$$\varepsilon = \frac{1}{R_o} = \frac{fL}{U}. \qquad (2.3.16)$$

With this new notation and some rearrangement of terms, equations (2.3.8)-(2.3.14) are written in the final form:

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial y} + \frac{\partial h_2}{\partial x} + (\gamma^{-1} - 1)\frac{\partial h_1}{\partial x} = \varepsilon v_2 - \frac{\partial p_s}{\partial x} - \kappa C_f u_2\frac{\sqrt{u_2^2 + v_2^2}}{h_2 - h_B}T, \qquad (2.3.17)$$

$$\frac{\partial v_2}{\partial t} + u_2\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y} + \frac{\partial h_2}{\partial y} + (\gamma^{-1} - 1)\frac{\partial h_1}{\partial y} = -\varepsilon u_2 - \frac{\partial p_s}{\partial y} - \kappa C_f v_2\frac{\sqrt{u_2^2 + v_2^2}}{h_2 - h_B}T, \qquad (2.3.18)$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + \gamma^{-1}\frac{\partial h_1}{\partial x} = \varepsilon v_1 - (1 - \gamma)^{-1}\frac{\partial p_s}{\partial x}, \qquad (2.3.19)$$

$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + \gamma^{-1} \frac{\partial h_1}{\partial y} = -\varepsilon u_1 - (1-\gamma)^{-1} \frac{\partial p_s}{\partial y}, \tag{2.3.20}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}[(h_2 - h_B)u_2] + \frac{\partial}{\partial y}[(h_2 - h_B)v_2] = 0, \tag{2.3.21}$$

$$\frac{\partial}{\partial t}(h_1 - h_2) + \frac{\partial}{\partial x}[(h_1 - h_2)u_1] + \frac{\partial}{\partial y}[(h_1 - h_2)v_1] = 0, \tag{2.3.22}$$

and

$$\frac{\partial}{\partial x}[(h_1 - h_2)u_1 + (h_2 - h_B)u_2] + \frac{\partial}{\partial y}[(h_1 - h_2)v_1 + (h_2 - h_B)v_2] = 0. \tag{2.3.23}$$

Equations (2.3.17)-(2.3.23) represent the general nondimensional equations for both of the upper boundary conditions considered. The free surface case is obtained from the first six equations (2.3.17)-(2.3.22) with p_s = constant, and the rigid lid case is obtained through equations (2.3.17)-(2.3.21), and (2.3.23) with h_1 a constant, usually equal to 1. These equations can be further considered by imposing various physical restrictions, which will be completed in the following section.

2.4 Physical Limits and Special Cases

The general equations (2.3.17) - (2.3.23) for a two-layer gravity current may be simplified to consider specific cases. These are listed and described in the following subsections. For later use and reference, the equations are rewritten as necessary in the vector system form,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}^{(1)} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{A}^{(2)} \frac{\partial \mathbf{u}}{\partial y} = \mathbf{b}, \tag{2.4.1}$$

where $\mathbf{u}(x, y, t)$ and $\mathbf{b}(\mathbf{u}, x, y, t)$ are n-dimensional vectors, and $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ are $n \times n$ matrices whose components are invariably functions of $\mathbf{u}$, x, y, and t. For example, equations (2.3.17)-(2.3.22) with a free surface (p_s = constant) may be written in system form as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ v_1 \\ h_1 \\ u_2 \\ v_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} u_1 & 0 & \gamma^{-1} & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & 0 & 0 \\ h_1 - h_2 & 0 & u_1 & h_2 - h_B & 0 & u_2 - u_1 \\ 0 & 0 & \gamma^{-1} - 1 & u_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & h_2 - h_B & 0 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ v_1 \\ h_1 \\ u_2 \\ v_2 \\ h_2 \end{bmatrix} +$$

$$
+\begin{bmatrix} v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_1 & \gamma^{-1} & 0 & 0 & 0 \\ 0 & h_1-h_2 & v_1 & 0 & h_2-h_B & v_2-v_1 \\ 0 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & \gamma^{-1}-1 & 0 & v_2 & 1 \\ 0 & 0 & 0 & 0 & h_2-h_B & v_2 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u_1 \\ v_1 \\ h_1 \\ u_2 \\ v_2 \\ h_2 \end{bmatrix} =
$$

$$
= \begin{bmatrix} \varepsilon v_1 \\ -\varepsilon u_1 \\ u_2 \frac{\partial h_B}{\partial x} + v_2 \frac{\partial h_B}{\partial y} \\ \varepsilon v_2 - \kappa C_f u_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B} T \\ -\varepsilon u_2 - \kappa C_f v_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B} T \\ u_2 \frac{\partial h_B}{\partial x} + v_2 \frac{\partial h_B}{\partial y} \end{bmatrix} . \qquad (2.4.2)
$$

In addition to the system form (2.4.1), the equations are often recast in conservative form, as

$$
\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}^{(1)}}{\partial x} + \frac{\partial \mathbf{f}^{(2)}}{\partial y} = \mathbf{b} \qquad (2.4.3)
$$

for vector-valued functions $\mathbf{f}^{(1),(2)} = \mathbf{f}^{(1),(2)}(\mathbf{u}, x, y, t)$. The motivation and definitions relating to equation (2.4.3) will be discussed in chapter 3.

2.4.1 Three spatial dimensions with a thick upper layer and a free surface

The three dimensional equations with a free surface, which consist of equations (2.3.17)-(2.3.22) with p_s = constant, have already been written as a system (2.4.2) above. The first simplification results from the case that the upper layer is very thick compared to the lower layer, so that the horizontal motion in the upper layer (layer 1) has negligible effect on the lower layer (layer 2) and may be neglected in the equations of motion. Neglecting these terms reduces the six equations to a one and a half layer model consisting of the simplified forms of (2.3.17), (2.3.18) and (2.3.21). This consists of the following three equations:

$$
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} + \frac{\partial h_2}{\partial x} = \varepsilon v_2 - \kappa C_f u_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B} T, \qquad (2.4.4)
$$

$$
\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + \frac{\partial h_2}{\partial y} = -\varepsilon u_2 - \kappa C_f v_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B} T, \qquad (2.4.5)
$$

and

$$
\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}\left[(h_2-h_B)u_2\right] + \frac{\partial}{\partial y}\left[(h_2-h_B)v_2\right] = 0. \qquad (2.4.6)
$$

Equations (2.4.4)-(2.4.6) are similar to those found in either Baines (1995, p.18) or Whitham (1970, p.454), with the additional Coriolis and nonlinear forcing terms present here.

The system form, (2.4.1), of equations (2.4.4)-(2.4.6) may be stated as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ v_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} u_2 & 0 & 1 \\ 0 & u_2 & 0 \\ h_2 - h_B & 0 & u_2 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ v_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} v_2 & 0 & 0 \\ 0 & v_2 & 1 \\ 0 & h_2 - h_B & v_2 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u_2 \\ v_2 \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon v_2 - \kappa C_f u_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B}T \\ -\varepsilon u_2 - \kappa C_f v_2 \frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B}T \\ u_2\frac{\partial h_B}{\partial x} + v_2\frac{\partial h_B}{\partial y} \end{bmatrix}. \tag{2.4.7}$$

Derivation of a conservative form, (2.4.3), of equations (2.4.4)-(2.4.6) is slightly more involved, but the results are easily verifiable, and the calculations are completed with a view to the outcome. Since equation (2.4.6) is already in the desired form, two equations representing conservation of linear momentum in the x and y direction need to be derived. First, the vertically integrated linear momentum in the x direction is given, using equation (2.4.4) and (2.4.6) as

$$\begin{aligned}\frac{\partial}{\partial t}[(h_2 - h_B)u_2] &= \frac{\partial h_2}{\partial t}u_2 + (h_2 - h_B)\frac{\partial u_2}{\partial t} \\ &= -u_2\frac{\partial}{\partial x}[(h_2 - h_B)u_2] - u_2\frac{\partial}{\partial y}[(h_2 - h_B)v_2] - (h_2 - h_B)u_2\frac{\partial u_2}{\partial x} \\ &\quad - (h_2 - h_B)v_2\frac{\partial u_2}{\partial y} - (h_2 - h_B)\frac{\partial h_2}{\partial x} + \varepsilon(h_2 - h_B)v_2 \\ &\quad - \kappa C_f(h_2 - h_B)u_2\frac{\sqrt{u_2^2 + v_2^2}}{h_2 - h_B}T \\ &= -\frac{\partial}{\partial x}[(h_2 - h_B)u_2^2] - \frac{\partial}{\partial y}[(h_2 - h_B)u_2v_2] - \frac{\partial}{\partial x}[\frac{1}{2}(h_2 - h_B)^2] \\ &\quad - (h_2 - h_B)\frac{\partial h_B}{\partial x} + \varepsilon(h_2 - h_B)v_2 - \kappa C_f u_2\sqrt{u_2^2 + v_2^2}T.\end{aligned} \tag{2.4.8}$$

Similarly, linear momentum in the y direction can be found using equations (2.4.5) and (2.4.6) as

$$\begin{aligned}\frac{\partial}{\partial t}[(h_2-h_B)v_2] &= \frac{\partial h_2}{\partial t}v_2+(h_2-h_B)\frac{\partial v_2}{\partial t}\\ &= -v_2\frac{\partial}{\partial x}[(h_2-h_B)u_2]-v_2\frac{\partial}{\partial y}[(h_2-h_B)v_2]-(h_2-h_B)u_2\frac{\partial v_2}{\partial x}\\ &\quad -(h_2-h_B)v_2\frac{\partial v_2}{\partial y}-(h_2-h_B)\frac{\partial h_2}{\partial y}-\varepsilon(h_2-h_B)u_2\\ &\quad -\kappa C_f(h_2-h_B)v_2\frac{\sqrt{u_2^2+v_2^2}}{h_2-h_B}T\\ &= -\frac{\partial}{\partial x}[(h_2-h_B)u_2v_2]-\frac{\partial}{\partial y}[(h_2-h_B)v_2^2]-\frac{\partial}{\partial y}[\frac{1}{2}(h_2-h_B)^2]\\ &\quad -(h_2-h_B)\frac{\partial h_B}{\partial y}-\varepsilon(h_2-h_B)u_2-\kappa C_f v_2\sqrt{u_2^2+v_2^2}T.\end{aligned} \tag{2.4.9}$$

Defining the variables $\mu=(h_2-h_B)u_2$, $\nu=(h_2-h_B)v_2$, and $\zeta=h_2-h_B$ allows equations (2.4.8), (2.4.9) and (2.4.6) to be written in the conservative form (2.4.3) as

$$\frac{\partial}{\partial t}\begin{bmatrix}\mu\\ \nu\\ \zeta\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\frac{\mu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \frac{\mu\nu}{\zeta}\\ \mu\end{bmatrix}+\frac{\partial}{\partial y}\begin{bmatrix}\frac{\mu\nu}{\zeta}\\ \frac{\nu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \nu\end{bmatrix}= \\ =\begin{bmatrix}-\zeta\frac{\partial h_B}{\partial x}+\varepsilon\nu-\kappa C_f\mu\frac{\sqrt{\mu^2+\nu^2}}{\zeta^2}T\\ -\zeta\frac{\partial h_B}{\partial y}-\varepsilon\mu-\kappa C_f\nu\frac{\sqrt{\mu^2+\nu^2}}{\zeta^2}T\\ 0\end{bmatrix}. \tag{2.4.10}$$

2.4.2 Axisymmetric flow in three spatial dimensions with a thick upper layer and a free surface

The simplified equations of motion (2.4.4)-(2.4.6) derived in the previous subsection may be written in polar co-ordinate form, with the variables (r,θ) where $x=r\cos\theta$ and $y=r\sin\theta$. The velocity components (u_r,u_θ) in the (r,θ) direction, respectively, are given by the trigonometric transformation $u_r=v_2\sin\theta+u_2\cos\theta$ and $u_\theta=v_2\cos\theta-u_2\sin\theta$ (Kundu, 1990 p.69). This polar co-ordinates change of variables, along with the partial derivatives, is written as

$$\begin{aligned}u_2&=u_r\cos\theta-u_\theta\sin\theta\\ v_2&=u_r\sin\theta+u_\theta\cos\theta\end{aligned}\quad\text{and}\quad\begin{aligned}\frac{\partial}{\partial x}&\to\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial y}&\to\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\end{aligned} \tag{2.4.11}$$

This transformation (2.4.11) is then applied to equation (2.4.4)-(2.4.6) separately as follows. First, note the simplification available via the spatial derivative operator which transforms by (2.4.11) as:

$$
\begin{aligned}
(u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}) \rightarrow (u_r\cos\theta-u_\theta\sin\theta)(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta})+ \\
(u_r\sin\theta+u_\theta\cos\theta)(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}) \\
=u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}.
\end{aligned}
\tag{2.4.12}
$$

Using this, equations (2.4.4) and (2.4.5) become, in polar co-ordinates,

$$
\begin{aligned}
&\cos\theta\frac{\partial u_r}{\partial t}-\sin\theta\frac{\partial u_\theta}{\partial t}+u_r(\cos\theta\frac{\partial u_r}{\partial r}-\sin\theta\frac{\partial u_\theta}{\partial\theta})+\frac{\partial u_\theta}{\partial r}(\cos\theta\frac{\partial u_r}{\partial\theta}-\sin\theta\frac{\partial u_\theta}{\partial\theta} \\
&-\sin\theta u_r-\cos\theta u_\theta)+\cos\theta\frac{\partial h_2}{\partial r}-\frac{\sin\theta}{r}\frac{\partial h_2}{\partial\theta} \\
&=\varepsilon(u_r\sin\theta+u_\theta\cos\theta)-\kappa C_f(u_r\cos\theta-u_\theta\sin\theta)\frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T,
\end{aligned}
\tag{2.4.13}
$$

and

$$
\begin{aligned}
&\sin\theta\frac{\partial u_r}{\partial t}+\cos\theta\frac{\partial u_\theta}{\partial t}+u_r(\sin\theta\frac{\partial u_r}{\partial r}+\cos\theta\frac{\partial u_\theta}{\partial r})+\frac{u_\theta}{r}(\sin\theta\frac{\partial u_r}{\partial\theta}+\cos\theta\frac{\partial u_\theta}{\partial\theta} \\
&+\cos\theta u_r-\sin\theta u_\theta)+\sin\theta\frac{\partial h_2}{\partial r}+\frac{\cos\theta}{r}\frac{\partial h_2}{\partial\theta} \\
&=-\varepsilon(u_r\cos\theta-u_\theta\sin\theta)-\kappa C_f(u_r\sin\theta+u_\theta\cos\theta)\frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T.
\end{aligned}
\tag{2.4.14}
$$

Combinations of equations (2.4.13) and (2.4.14) easily isolate $\frac{\partial u_r}{\partial t}$ and $\frac{\partial u_\theta}{\partial t}$. This can be done by simple manipulations making judicious use of the trigonometric identity $\cos^2\theta+\sin^2\theta=1$. For example, forming the combination $\cos\theta\times$(2.4.13)+ $\sin\theta\times$ (2.4.14) gives

$$
\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}+\frac{\partial h_2}{\partial r}=\varepsilon u_\theta-\kappa C_f u_r\frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T. \tag{2.4.15}
$$

Similarly, $-\sin\theta\times$ (2.4.13) + $\cos\theta\times$ (2.4.14) gives

$$
\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+\frac{1}{r}\frac{\partial h_2}{\partial\theta}=-\varepsilon u_r-\kappa C_f u_\theta\frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T. \tag{2.4.16}
$$

Equation (2.4.6) simplifies directly without resorting to trigonometric combinations as

$$\frac{\partial h_2}{\partial t} + \cos\theta\frac{\partial}{\partial r}[(h_2 - h_B)u_2] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}[(h_2 - h_B)u_2] + \sin\theta\frac{\partial}{\partial r}[(h_2 - h_B)v_2]$$
$$+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}[(h_2 - h_B)v_2] = 0$$

$$\Longrightarrow \frac{\partial h_2}{\partial t} + \cos\theta\frac{\partial}{\partial r}[(h_2 - h_B)(u_r\cos\theta - u_\theta\sin\theta)]$$
$$+\sin\theta\frac{\partial}{\partial r}[(h_2 - h_B)(u_r\sin\theta + u_\theta\cos\theta)]$$
$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}[(h_2 - h_B)(u_r\cos\theta - u_\theta\sin\theta)]$$
$$+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}[(h_2 - h_B)(u_r\sin\theta + u_\theta\cos\theta)] = 0.$$

$$\Longrightarrow \frac{\partial h_2}{\partial t} + u_r\frac{\partial}{\partial r}(h_2 - h_B) + (h_2 - h_B)\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta}(h_2 - h_B)$$
$$+\frac{(h_2 - h_B)}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right) = 0$$

$$\Longrightarrow \frac{\partial h_2}{\partial t} + \frac{\partial}{\partial r}[(h_2 - h_B)u_r] + \frac{\partial}{\partial\theta}[(h_2 - h_B)\frac{u_\theta}{r}] + (h_2 - h_B)\frac{u_r}{r} = 0. \quad (2.4.17)$$

As a system, equations (2.4.15)-(2.4.17) may be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ u_\theta \\ h_2 \end{bmatrix} + \begin{bmatrix} u_r & 0 & 1 \\ 0 & u_r & 0 \\ h_2 - h_B & 0 & u_r \end{bmatrix}\frac{\partial}{\partial r}\begin{bmatrix} u_2 \\ u_\theta \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{u_\theta}{r} & 0 & 0 \\ 0 & \frac{u_\theta}{r} & \frac{1}{r} \\ 0 & \frac{h_2-h_B}{r} & \frac{u_\theta}{r} \end{bmatrix}\frac{\partial}{\partial\theta}\begin{bmatrix} u_2 \\ u_\theta \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{u_\theta^2}{r} + \varepsilon u_\theta - \kappa C_f u_r \frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T \\ -\frac{u_r u_\theta}{r} - \varepsilon u_r - \kappa C_f u_\theta \frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B}T \\ u_r\frac{\partial h_B}{\partial r} + \frac{u_\theta}{r}\frac{\partial h_B}{\partial\theta} - \frac{(h_2-h_B)u_r}{r} \end{bmatrix}. \quad (2.4.18)$$

A simplification of equations (2.4.15)-(2.4.17) is the so-called *axisymmetric case,* found by assuming that the variables are independent of the azimuthal angle θ. Thus, the variables are defined for $u_r = u_r(r,t)$, $u_\theta = u_\theta(r,t)$, and $h_2 = h_2(r,t)$ only. The axisymmetric equations are stated as a reduction of equations (2.4.15)-(2.4.17) as

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + \frac{\partial h_2}{\partial r} = \varepsilon u_\theta - \kappa C_f u_r\frac{\sqrt{u_r^2 + u_\theta^2}}{h_2 - h_B}T, \quad (2.4.19)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} = -\varepsilon u_r - \kappa C_f u_\theta \frac{\sqrt{u_r^2 + u_\theta^2}}{h_2 - h_B} T, \tag{2.4.20}$$

and

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial r}[(h_2 - h_B)u_r] + (h_2 - h_B)\frac{u_r}{r} = 0. \tag{2.4.21}$$

Equations (2.4.19)-(2.4.21) may be expressed in system form (2.4.1) as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ u_\theta \\ h_2 \end{bmatrix} + \begin{bmatrix} u_r & 0 & 1 \\ 0 & u_r & 0 \\ h_2 - h_B & 0 & u_r \end{bmatrix} \frac{\partial}{\partial r}\begin{bmatrix} u_2 \\ u_\theta \\ h_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{u_\theta^2}{r} + \varepsilon u_\theta - \kappa C_f u_r \frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B} T \\ -\frac{u_r u_\theta}{r} - \varepsilon u_r - \kappa C_f u_\theta \frac{\sqrt{u_r^2+u_\theta^2}}{h_2-h_B} T \\ u_r \frac{\partial h_B}{\partial r} - \frac{(h_2-h_B)u_r}{r} \end{bmatrix}, \tag{2.4.22}$$

which may be seen as the system (2.4.18) restricted to dependence only on r and t. This simplification has some obvious importance by the reduction not of the number of variables (this still stands at three) but by the number of spatial derivatives and matrices in the partial differential equation.

The axisymmetric equations (2.4.22) are not valid if the horizontal length scales are such that they are large enough to notice a change in the Coriolis parameter with latitude. Thus, the axisymmetric equations are only considered to be applicable if the horizontal scales are less than the deformation radius (Pedlosky, 1987).

2.4.3 Two spatial dimensions with a free surface

Equations (2.3.17)-(2.3.22) may be simplified in a different manner by considering only motion in one horizontal direction. Without loss of generality, motion in the y-plane is now taken to be zero, so that two variables (v_1, v_2) may be removed from the equations of motion, along with dependence on the transverse (y) variable. The resulting equations, simplified from (2.3.17)-(2.3.23), may be considered to determine the flow in only one spatial variable (x), without any effects of the Coriolis Force. The resultant equations of motion, with v_2 constant, $u_2 = u_2(x,t)$, $h_2 = h_2(x,t)$, $h_1 = h_1(x,t)$ and $p_s =$ constant reduce to:

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{\partial h_2}{\partial x} + (\gamma^{-1} - 1)\frac{\partial h_1}{\partial x} = -\kappa C_f \frac{u_2^2}{h_2 - h_B} T, \tag{2.4.23}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \gamma^{-1} \frac{\partial h_1}{\partial x} = 0, \tag{2.4.24}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}\left[(h_2 - h_B)u_2\right] = 0, \tag{2.4.25}$$

and

$$\frac{\partial}{\partial t}(h_1 - h_2) + \frac{\partial}{\partial x}\left[(h_1 - h_2)u_1\right] = 0. \tag{2.4.26}$$

Equations (2.4.23)-(2.4.26) are similar in form to those developed previously by Montgomery and Moodie (1999a), stated as equations (2.24)-(2.27) therein. This can be seen by some straightforward changes to the notation, and some rearrangement of equations (2.4.25) and (2.4.26). The equations may be stated in conservation form (2.4.3),

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ h_1 \\ u_2 \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u_1^2 + \gamma^{-1}h_1 \\ (h_1 - h_2)u_1 + (h_2 - h_B)u_2 \\ \frac{1}{2}u_2^2 + h_2 + (\gamma^{-1} - 1)h_1 \\ (h_2 - h_B)u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\kappa C_f \frac{u_2^2}{h_2 - h_B} T \\ 0 \end{bmatrix}, \tag{2.4.27}$$

or in system form (2.4.1) as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ h_1 \\ u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} u_1 & \gamma^{-1} & 0 & 0 \\ h_1 - h_2 & u_1 & h_2 - h_B & u_2 - u_1 \\ 0 & \gamma^{-1} - 1 & u_2 & 1 \\ 0 & 0 & h_2 - h_B & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ h_1 \\ u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \frac{\mathrm{d}h_B}{\mathrm{d}x} \\ -\kappa C_f \frac{u_2^2}{h_2 - h_B} T \\ u_2 \frac{\mathrm{d}h_B}{\mathrm{d}x} \end{bmatrix}. \tag{2.4.28}$$

These equations (2.4.27), (2.4.28) may be rewritten in a more recognizable form by making the change of variable $\zeta_1 = h_1 - h_2$ and $\zeta_2 = h_2 - h_B$. With this notation change, equations (2.4.23)-(2.4.26) may be written as a system,

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ \zeta_1 \\ u_2 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} u_1 & \gamma^{-1} & 0 & \gamma^{-1} \\ \zeta_1 & u_1 & 0 & 0 \\ 0 & \gamma^{-1} - 1 & u_2 & \gamma^{-1} \\ 0 & 0 & \zeta_2 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ \zeta_1 \\ u_2 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x} \\ 0 \\ -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T \\ 0 \end{bmatrix}, \tag{2.4.29}$$

or in conservation form,

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ \zeta_1 \\ u_2 \\ \zeta_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u_1^2 + \gamma^{-1}(\zeta_1+\zeta_2) \\ \zeta_1 u_1 \\ \frac{1}{2}u_2^2 + (\gamma^{-1}-1)\zeta_1 + \gamma^{-1}\zeta_2 \\ \zeta_2 u_2 \end{bmatrix} = \begin{bmatrix} -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x} \\ 0 \\ -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T \\ 0 \end{bmatrix}. \tag{2.4.30}$$

In Montgomery and Moodie (1998a), three special cases of equation (2.4.30) were considered for the case of a horizontal bottom boundary ($h_B = 0$) and zero truncation function $T = 0$, i.e. the right hand forcing vector is identically zero. These were labelled therein as a weak-stratification model for small values of γ, and two shallow layer models for a thin lower layer and a thin upper layer. Similar models which include more general bottom topography are derived in the following.

For a *weak stratification* model, the notation of the variables is altered so that the dimensional free surface is given by η, where $h_1 = H + \eta$. Using the nondimensionalization $h_1 = H\tilde{h}_1$, $\eta = \gamma H\tilde{\eta}$ then gives the nondimensional relationship between $\tilde{h}_1$ and $\tilde{\eta}$ as $\tilde{h}_1 = 1+\gamma\tilde{\eta}$. This notation may be substituted into equations (2.4.23)-(2.4.24), and terms which are $O(\gamma)$ may be neglected. The subsequent weak stratification momentum equations, with the tilde notation suppressed, are then

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + \frac{\partial h_2}{\partial x} + \frac{\partial \eta}{\partial x} = -\kappa C_f \frac{u_2^2}{h_2 - h_B} T, \tag{2.4.31}$$

and

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + \frac{\partial \eta}{\partial x} = 0. \tag{2.4.32}$$

The conservation of mass equation (2.4.25) remains unaltered, while equation (2.4.26) becomes

$$\frac{\partial}{\partial t}(\gamma\eta - h_2) + \frac{\partial}{\partial x}\left[(1+\gamma\eta - h_2)u_1\right] = 0, \tag{2.4.33}$$

which simplifies by neglecting $O(\gamma)$ terms, and adding (2.4.25) to give the equation

$$\frac{\partial}{\partial x}\left[(1-h_2)u_1 + (h_2 - h_B)u_2\right] = 0. \tag{2.4.34}$$

Equation (2.4.34) then integrates to

$$(1-h_2)u_1 + (h_2-h_B)u_2 = Q(t), \tag{2.4.35}$$

or

$$u_1 = \frac{Q - (h_2 - h_B)u_2}{1 - h_2}, \tag{2.4.36}$$

where $Q(t)$ depends on the initial conditions, and represents the total volume flux.

The momentum equations may be simplified through the use of (2.4.35) by subtracting (2.4.32) from (2.4.31). The resulting equation is calculated in Appendix 2, and may be expressed as

$$\frac{\partial}{\partial t}(u_2 - u_1) + u_2\frac{\partial u_2}{\partial x} + \frac{\partial h_2}{\partial x} - u_1\frac{\partial u_1}{\partial x} = -\kappa C_f\frac{u_2^2}{h_2 - h_B}T, \tag{2.4.37}$$

where u_1 is given in terms of h_2 and u_2 via (2.4.36). The weak-stratification equations are then given to be the two partial differential equations (2.4.37) and (2.4.25) in the unknowns h_2 and u_2. The upper layer dynamics are then found by solving for u_1 from (2.4.36), then integrating (2.4.32) once spatially to find η.

As a system, the weak stratification equations (2.4.37) and (2.4.25) may be stated as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ h_2 - h_B & u_2 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ u_2\dfrac{\mathrm{d}h_B}{\mathrm{d}x} \end{bmatrix}. \tag{2.4.38}$$

In equation (2.4.38), the terms a_{11}, a_{12} and b_1 are given by

$$a_{11} = \frac{2Q(h_2 - h_B) + [(1-h_2)^2 - (1 + h_2 - 2h_B)(h_2 - h_B)]u_2}{(1 - h_2)(1 - h_B)},$$

$$a_{12} = \Big\{2Q(1-h_2)u_2 - (1 + h_2 - 2h_B)(1-h_2)u_2^2 + (1-h_2)^3 - \\ - [Q - (h_2 - h_B)u_2]^2\Big\}\Big/\Big\{(1-h_2)^2(1-h_B)\Big\},$$

and

$$b_1 = \frac{Q'}{1 - h_B} + a_{12}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \frac{(1-h_2)\kappa C_f u_2^2 T}{(1-h_B)(h_2 - h_B)}.$$

Q' denotes the time derivative of $Q(t)$.

Although the system of equations (2.4.38) cannot be recast directly in conservation form, equations (2.4.37) and (2.4.25) are already almost in the desired form. By writing these equations as a system in conservation form

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 - u_1 \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u_2^2 + h_2 - \frac{1}{2}u_1^2 \\ (h_2 - h_B)u_2 \end{bmatrix} = \begin{bmatrix} -\kappa C_f\frac{u_2^2}{h_2 - h_B}T \\ 0 \end{bmatrix}, \tag{2.4.39}$$

with u_1 given by equation (2.4.36), a more general system of conservation laws is thus stated. By employing a change of notation, the system (2.4.39) may be stated explicitly in the form (2.4.3) for one spatial variable. Letting $\overline{u} = u_2 - u_1$, and employing equation (2.4.35) gives

$$\overline{u} = u_2 - \left(\frac{Q - (h_2 - h_B)u_2}{1 - h_2}\right) = \left(\frac{1 - h_B}{1 - h_2}\right) u_2 - \frac{Q}{1 - h_2},$$

which may be solved for u_2 as

$$u_2 = \left(\frac{1 - h_2}{1 - h_B}\right) \overline{u} + \frac{Q}{1 - h_B}. \tag{2.4.40}$$

Similarly, $\overline{u}$ and u_1 are related through the use of (2.4.35) as

$$\overline{u} = \frac{Q - (1 - h_2)u_1}{h_2 - h_B} - u_1 = \frac{Q}{h_2 - h_B} - \left(\frac{1 - h_B}{h_2 - h_B}\right) u_1,$$

which solves for u_1 as

$$u_1 = \frac{Q}{1 - h_B} - \left(\frac{h_2 - h_B}{1 - h_B}\right) \overline{u}. \tag{2.4.41}$$

Expressions (2.4.40) and (2.4.41) may be combined to yield the result

$$\begin{aligned} u_2^2 - u_1^2 = &\left(\frac{1 - h_2}{1 - h_B}\right)^2 \overline{u}^2 + \frac{Q^2}{(1 - h_B)^2} + 2\left(\frac{1 - h_2}{(1 - h_B)^2}\right) Q\overline{u} \\ &- \frac{Q^2}{(1 - h_B)^2} + 2\left(\frac{h_2 - h_B}{(1 - h_B)^2}\right) Q\overline{u} - \left(\frac{h_2 - h_B}{1 - h_B}\right)^2 \overline{u}^2, \end{aligned}$$

which simplifies to

$$u_2^2 - u_1^2 = \frac{(1 - h_2)^2 - (h_2 - h_B)^2}{(1 - h_B)^2} \overline{u}^2 + 2\left(\frac{1 - h_B}{(1 - h_B)^2}\right) Q\overline{u}. \tag{2.4.42}$$

Removing the variables u_2 and u_1 from equation (2.4.39), in favor of $\overline{u}$ with the aid of the expressions (2.4.40) and (2.4.42), then allows the system to be expressed using the new variable $\overline{u}$ and h_2 as

$$\begin{aligned} \frac{\partial}{\partial t}\begin{bmatrix} \overline{u} \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{(1-h_2)^2 - (h_2 - h_B)^2}{2(1-h_B)^2}\overline{u}^2 + \frac{1-h_B}{(1-h_B)^2} Q\overline{u} + h_2 \\ \frac{(1-h_2)(h_2-h_B)}{1-h_B}\overline{u} + \frac{Q(h_2-h_B)}{1-h_B} \end{bmatrix} &= \\ = \begin{bmatrix} -\kappa C_f \frac{[(1-h_2)\overline{u}+Q]^2}{(1-h_B)^2(h_2-h_B)} T \\ 0 \end{bmatrix}&. \end{aligned} \tag{2.4.43}$$

The system (2.4.43) is now in the desired form (2.4.3).

In the special case with flat bottom topography, $h_B = 0$, zero forcing, $C_f = 0$, and no net mass flux at a point so that $Q(t) = 0$, the model may be simplified further. The results, given previously in Montgomery and Moodie (1998a), are two partial differential equations for the lower layer dynamics in conservation form, with two algebraic equations to solve for the upper layer dynamics. Some additional comments are given in Appendix 2, but for completeness, this simple case may be stated as a system

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} (1-h_2)u_2 & (1-h_2) - \frac{1-2h_2}{(1-h_2)^2}u_2^2 \\ h_2 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.4.44}$$

or in conservation form as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u_2^2 + \eta + h_2 \\ h_2 u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.4.45}$$

In equation (2.4.45), the notation for η has been used as described in Appendix 2, equation A2.8. The conservation form (2.4.45) is not a special case of equation (2.4.39), but rather another, simpler conservation law.

2.4.4 Two spatial dimensions with shallow layers

Thin layer models for shallow upper and lower layer cases have been discussed previously in Montgomery and Moodie (1998a) for the case of $h_B = 0$ and $C_f = 0$. The methodology developed there is now used to generalize the results for shallow layer models in this subsection.

The first case considered is a shallow lower layer model, which may be derived by considering the dimensional equations (2.2.17), (2.2.19), (2.2.21) and (2.2.22) restricted to the case with $v_1 = v_2 = 0$, a free surface $p_s =$ constant and $f = 0$ (i.e. no rotation) with the forcing term developed as in Section 2.3. The resulting equations are

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + g'\frac{\partial h_2}{\partial x} + (g-g')\frac{\partial h_1}{\partial x} = -\kappa C_f\frac{u_2^2}{h_2 - h_B}T, \tag{2.4.46}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + g\frac{\partial h_1}{\partial x} = 0, \tag{2.4.47}$$

$$\frac{\partial h_2}{\partial t} + (h_2 - h_B)\frac{\partial u_2}{\partial x} + u_2\frac{\partial}{\partial x}(h_2 - h_B) = 0, \tag{2.4.48}$$

and

$$\frac{\partial}{\partial t}(h_1 - h_2) + (h_1 - h_2)\frac{\partial u_1}{\partial x} + u_1\frac{\partial}{\partial x}(h_1 - h_2) = 0. \tag{2.4.49}$$

Equations (2.4.46)-(2.4.49) may be rewritten to include the height variations as a free surface $\eta = h_1 - H$ and total lower layer thickness $\zeta_2 = h_2 - h_B$. Using this notation, these equations become

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + g'\frac{\partial \zeta_2}{\partial x} + (g - g')\frac{\partial \eta}{\partial x} = -g'\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f\frac{u_2^2}{\zeta_2}T, \tag{2.4.50}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \tag{2.4.51}$$

$$\frac{\partial \zeta_2}{\partial t} + \zeta_2\frac{\partial u_2}{\partial x} + u_2\frac{\partial \zeta_2}{\partial x} = 0, \tag{2.4.52}$$

and

$$\frac{\partial}{\partial t}(\eta - \zeta_2) + (H + \eta - \zeta_2)\frac{\partial u_1}{\partial x} + u_1\frac{\partial}{\partial x}(\eta - \zeta_2) = h_B\frac{\partial u_1}{\partial x} + u_1\frac{\mathrm{d}h_B}{\mathrm{d}x}. \tag{2.4.53}$$

A nondimensionalization of equations (2.4.50)-(2.4.53) may now be done to emphasize the thin lower layer ζ_2 with a small dimensionless parameter ε, which measures the small percentage of height H given to a layer thickness. The motivation for this choice of nondimensionalization may be found in Montgomery and Moodie (1998a) but rests on the conservation of mass and momentum. The nondimensional variables (denoted with a tilde) are given by

$$x = L\tilde{x}, \quad h_B = \varepsilon H\tilde{h}_B, \quad \zeta_2 = \varepsilon H\tilde{\zeta}_2, \quad \eta = \frac{g'}{g}\varepsilon^2 H\tilde{\eta}$$

$$u_2 = \sqrt{\varepsilon g' H}\tilde{u}_2, \quad u_1 = \varepsilon\sqrt{\varepsilon g' H}\tilde{u}_1, \quad \frac{L}{T} = \sqrt{\varepsilon g' H}, \quad C_f = \frac{\varepsilon H}{L}\tilde{C}_f. \tag{2.4.54}$$

With this nondimensionalization, equations (2.4.50)-(2.4.53) become

$$\frac{\partial \tilde{u}_2}{\partial \tilde{t}} + \tilde{u}_2\frac{\partial \tilde{u}_2}{\partial \tilde{x}} + \frac{\partial \tilde{\zeta}_2}{\partial \tilde{x}} + (1 - \gamma)\varepsilon\frac{\partial \tilde{\eta}}{\partial \tilde{x}} = -\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T, \tag{2.4.55}$$

$$\frac{\partial \tilde{u}_1}{\partial \tilde{t}} + \varepsilon\tilde{u}_1\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\partial \tilde{\eta}}{\partial \tilde{x}} = 0, \tag{2.4.56}$$

$$\frac{\partial \tilde{\zeta}_2}{\partial \tilde{t}} + \tilde{\zeta}_2\frac{\partial \tilde{u}_2}{\partial \tilde{x}} + \tilde{u}_2\frac{\partial \tilde{\zeta}_2}{\partial \tilde{x}} = 0, \tag{2.4.57}$$

and

$$\frac{\partial}{\partial \tilde{t}}(\gamma\varepsilon\tilde{\eta} - \tilde{\zeta}_2) + (1 + \gamma\varepsilon^2\tilde{\eta} - \varepsilon\tilde{\zeta}_2)\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \tilde{u}_1\frac{\partial}{\partial \tilde{x}}(\gamma\varepsilon^2\tilde{\eta} - \varepsilon\tilde{\zeta}_2) = \varepsilon\tilde{h}_B\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \varepsilon\tilde{u}_1\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}}. \tag{2.4.58}$$

The leading order equations from (2.4.55)-(2.4.58) are found by neglecting terms which are $O(\varepsilon)$ for a thin lower layer. The resulting equations are expressed as two nonlinear partial differential equations for the lower layer,

$$\frac{\partial \tilde{u}_2}{\partial \tilde{t}} + \tilde{u}_2\frac{\partial \tilde{u}_2}{\partial \tilde{x}} + \frac{\partial \tilde{\zeta}_2}{\partial \tilde{x}} = -\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T, \tag{2.4.59}$$

and equation (2.4.57) expressed as

$$\frac{\partial \tilde{\zeta}_2}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(\tilde{\zeta}_2\tilde{u}_2) = 0. \tag{2.4.60}$$

These expressions may be written as a system,

$$\frac{\partial}{\partial \tilde{t}}\begin{bmatrix}\tilde{u}_2\\ \tilde{\zeta}_2\end{bmatrix} + \begin{bmatrix}\tilde{u}_2 & 1\\ \tilde{\zeta}_2 & \tilde{u}_2\end{bmatrix}\frac{\partial}{\partial \tilde{x}}\begin{bmatrix}\tilde{u}_2\\ \tilde{\zeta}_2\end{bmatrix} = \begin{bmatrix}-\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T\\ 0\end{bmatrix}. \tag{2.4.61}$$

In conservation form, equation (2.4.61) becomes

$$\frac{\partial}{\partial \tilde{t}}\begin{bmatrix}\tilde{u}_2\\ \tilde{\zeta}_2\end{bmatrix} + \frac{\partial}{\partial \tilde{x}}\begin{bmatrix}\frac{1}{2}\tilde{u}_2^2 + \tilde{\zeta}_2\\ \tilde{\zeta}_2\tilde{u}_2\end{bmatrix} = \begin{bmatrix}-\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T.\\ 0\end{bmatrix} \tag{2.4.62}$$

The other equations (2.4.56) and (2.4.58) are linear, and may be solved directly with some quick manipulations. The reduced equations with the $O(\varepsilon)$ terms removed are

$$\frac{\partial \tilde{u}_1}{\partial \tilde{t}} + \frac{\partial \tilde{\eta}}{\partial \tilde{x}} = 0, \tag{2.4.63}$$

and

$$\frac{\partial \tilde{\zeta}_2}{\partial \tilde{t}} - \frac{\partial \tilde{u}_1}{\partial \tilde{x}} = 0. \tag{2.4.64}$$

Subtracting equation (2.4.64) from (2.4.60) gives

$$\frac{\partial}{\partial \tilde{x}}(\tilde{u}_1 + \tilde{\zeta}_2\tilde{u}_2) = 0$$

which may be integrated (assuming zero initial conditions) as

$$\tilde{u}_1 = -\tilde{\zeta}_2\tilde{u}_2. \tag{2.4.65}$$

Thus, for equation (2.4.63), substitution of the above yields

$$\begin{aligned}\frac{\partial \tilde{\eta}}{\partial \tilde{x}} &= -\frac{\partial \tilde{u}_1}{\partial \tilde{t}} \\ &= \frac{\partial \tilde{\zeta}_2}{\partial \tilde{t}}\tilde{u}_2 + \tilde{\zeta}_2\frac{\partial \tilde{u}_2}{\partial \tilde{t}} \\ &= -\tilde{u}_2\frac{\partial}{\partial \tilde{x}}(\tilde{\zeta}_2\tilde{u}_2) - \tilde{\zeta}_2\tilde{u}_2\frac{\partial \tilde{u}_2}{\partial \tilde{x}} - \frac{\partial \tilde{\zeta}_2}{\partial \tilde{x}} - \frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T,\end{aligned}$$

which may be integrated as

$$\tilde{\eta} = \tilde{\zeta}_2\tilde{u}_2^2 - \frac{1}{2}\tilde{\zeta}_2 - \tilde{h}_B - \kappa\tilde{C}_f\int\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}Td\tilde{x}. \tag{2.4.66}$$

The thin lower layer model then consists of the two nonlinear partial differential equations (2.4.59), (2.4.60) for the lower layer dynamics and the algebraic relations (2.4.65), (2.4.66) for the upper layer. The system form is given by equations (2.4.61), (2.4.62). A simplification of $\tilde{h}_B = 0$, and $\tilde{C}_f = 0$ reduces these equations to those given in Montgomery and Moodie (1998a), which has the benefit of simplifying equations (2.4.61), (2.4.62) to having a zero vector on the right hand side of the equation.

The second case which is considered is that of a thin upper layer. Equations (2.4.46)-(2.4.49) are written with the change of notation $\zeta_1 = h_1 - h_2$ and $h_2 = H - \zeta_1 + \eta$ as

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} - g'\frac{\partial \zeta_1}{\partial x} + g\frac{\partial \eta}{\partial x} = -\kappa C_f\frac{u_2^2}{H - \zeta_1 + \eta - h_B}T, \tag{2.4.67}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \tag{2.4.68}$$

$$\frac{\partial}{\partial t}(-\zeta_1 + \eta) + \frac{\partial}{\partial x}[(H - \zeta_1 + \eta - h_B)u_2] = 0, \tag{2.4.69}$$

and

$$\frac{\partial \zeta_1}{\partial t} + \frac{\partial}{\partial x}(\zeta_1 u_1) = 0. \tag{2.4.70}$$

A scaling which focuses on the thin upper layer is given by

$$x = L\tilde{x}, \quad h_B = \varepsilon H\tilde{h}_B, \quad \zeta_1 = \varepsilon H\tilde{\zeta}_1, \quad \eta = \gamma\varepsilon H\tilde{\eta},$$

$$u_2 = \varepsilon\sqrt{\varepsilon g'H}\tilde{u}_2, \quad u_1 = \sqrt{\varepsilon g'H}\tilde{u}_1, \quad \frac{L}{T} = \sqrt{\varepsilon g'H}, \quad C_f = \frac{H}{L}\tilde{C}_f. \tag{2.4.71}$$

The nondimensional forms of (2.4.67)-(2.4.70) using the scaling (2.4.71) are then

$$\varepsilon\frac{\partial\tilde{u}_2}{\partial\tilde{t}}+\varepsilon^2\tilde{u}_2\frac{\partial\tilde{u}_2}{\partial\tilde{x}}-\frac{\partial\tilde{\zeta}_1}{\partial\tilde{x}}+\frac{\partial\tilde{\eta}}{\partial\tilde{x}}=\frac{\varepsilon^2\kappa\tilde{C}_f\tilde{u}_2^2T}{1-\varepsilon\tilde{\zeta}_1+\gamma\varepsilon\tilde{\eta}-\varepsilon\tilde{h}_B}, \tag{2.4.72}$$

$$\frac{\partial\tilde{u}_1}{\partial\tilde{t}}+\tilde{u}_1\frac{\partial\tilde{u}_1}{\partial\tilde{x}}+\frac{\partial\tilde{\eta}}{\partial\tilde{x}}=0, \tag{2.4.73}$$

$$\frac{\partial}{\partial\tilde{t}}(-\tilde{\zeta}_1+\gamma\tilde{\eta})+\frac{\partial}{\partial\tilde{x}}[(1-\varepsilon\tilde{\zeta}_1+\gamma\varepsilon\tilde{\eta}-\varepsilon\tilde{h}_B)\tilde{u}_2]=0, \tag{2.4.74}$$

and

$$\frac{\partial\tilde{\zeta}_1}{\partial\tilde{t}}+\frac{\partial}{\partial\tilde{x}}(\tilde{\zeta}_1\tilde{u}_1)=0. \tag{2.4.75}$$

The leading order equations may be obtained from neglecting any terms which are $O(\varepsilon)$ from equations (2.4.72)-(2.4.75). These equations are the partial differential equations (2.4.73) and (2.4.75) which remain unchanged, and the simplified linear equations

$$-\frac{\partial\tilde{\zeta}_1}{\partial\tilde{x}}+\frac{\partial\tilde{\eta}}{\partial\tilde{x}}=0, \tag{2.4.76}$$

and

$$\frac{\partial}{\partial\tilde{t}}(-\tilde{\zeta}_1+\gamma\tilde{\eta})+\frac{\partial\tilde{u}_2}{\partial\tilde{x}}=0. \tag{2.4.77}$$

Integrating equation (2.4.77) with respect to $\tilde{x}$ and assuming zero initial values gives the relation

$$\tilde{\eta}=\tilde{\zeta}_1, \tag{2.4.78}$$

which may be substituted into equation (2.4.76) to give

$$-\frac{\partial\tilde{\zeta}_1}{\partial\tilde{t}}+\frac{1}{1-\gamma}\frac{\partial\tilde{u}_2}{\partial\tilde{x}}=0.$$

Addition of equation (2.4.75) to this result gives

$$\frac{\partial}{\partial\tilde{x}}(\tilde{\zeta}_1\tilde{u}_1+\frac{1}{1-\gamma}\tilde{u}_2)=0,$$

a result which integrates to give

$$\tilde{u}_2=-(1-\gamma)\tilde{\zeta}_1\tilde{u}_1,$$

or, using equation (2.4.78),

$$\tilde{u}_2=-(1-\gamma)\tilde{\eta}\tilde{u}_1. \tag{2.4.79}$$

The thin upper layer model then may be written as a set of two partial differential equations in conservation form for the upper layer,

$$\frac{\partial}{\partial \tilde{t}}\begin{bmatrix}\tilde{u}_1\\ \tilde{\eta}\end{bmatrix}+\frac{\partial}{\partial \tilde{x}}\begin{bmatrix}\frac{1}{2}\tilde{u}_1^2+\tilde{\eta}\\ \tilde{\zeta}_1\tilde{u}_1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{2.4.80}$$

As a system, equation (2.4.80) may be written as

$$\frac{\partial}{\partial \tilde{t}}\begin{bmatrix}\tilde{u}_1\\ \tilde{\eta}\end{bmatrix}+\begin{bmatrix}\tilde{u}_1 & 1\\ \tilde{\zeta}_1 & \tilde{u}_1\end{bmatrix}\frac{\partial}{\partial \tilde{x}}\begin{bmatrix}\tilde{u}_1\\ \tilde{\eta}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{2.4.81}$$

It should be noted that the variables in equation (2.4.80) and (2.4.81) for the thin lower layer model arise from a slightly different nondimensionalization than the variables in expressions (2.4.61), (2.4.62).

The lower layer dynamics are then recovered from the algebraic equations (2.4.78) and (2.4.79). Unlike the thin lower layer model, the bottom topography is removed from the thin upper layer equations and the equations obtained are precisely those given in Montgomery and Moodie (1998a) once a change of variables has been made. This result may be interpreted physically through stating that the bottom topography may be thought of as effecting primarily the lower layer, with a smaller order effect on the upper layer or free surface.

2.4.5 Two spatial dimensions with a rigid lid

The rigid lid equations from (2.3.17)-(2.3.23) with the simplification $h_1 =$ constant and simplified for one spatial variable x as in equations (2.4.23)-(2.4.26) are given by

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+\frac{\partial h_2}{\partial x}=-\frac{\partial p_s}{\partial x}-\kappa C_f\frac{u_2^2}{h_2-h_B}T, \tag{2.4.82}$$

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}=-(1-\gamma)^{-1}\frac{\partial p_s}{\partial x}, \tag{2.4.83}$$

$$\frac{\partial h_2}{\partial t}+\frac{\partial}{\partial x}\left[(h_2-h_B)u_2\right]=0, \tag{2.4.84}$$

and

$$\frac{\partial}{\partial x}\left[(h_1-h_2)u_1+(h_2-h_B)u_2\right]=0. \tag{2.4.85}$$

Multiplying equation (2.4.83) by $-(1-\gamma)$ and adding the result to equation (2.4.82) allows the pressure term p_s at the interface to be removed, resulting in the combined equation

$$\frac{\partial}{\partial t}\big(u_2-(1-\gamma)u_1\big)+u_2\frac{\partial u_2}{\partial x}-(1-\gamma)u_1\frac{\partial u_1}{\partial x}+\frac{\partial h_2}{\partial x}=-\kappa C_f\frac{u_2^2}{h_2-h_B}T. \tag{2.4.86}$$

A further reduction may be obtained by integrating equation (2.4.86) with respect to x to give

$$(h_1 - h_2)u_1 + (h_2 - h_B)u_2 = Q(t), \tag{2.4.87}$$

where $Q(t)$ is the total volume flux, independent of position x. Assuming that Q is known, equation (2.4.87) may be used to solve for the variable u_1 as

$$u_1 = \frac{Q - (h_2 - h_B)u_2}{h_1 - h_2}. \tag{2.4.88}$$

The removal of u_1 from the system may be effected by substituting (2.4.88) back in to equation (2.4.86). The result, along with (2.4.84) consists of the much simpler system of two equations in the two unknowns h_2 and u_2. Such an approach has been used previously, for example by Baines (1995, p.97), where the equations are solved using Riemann Invariants. Of direct consequence to gravity currents, similar equations have been developed and used by Rottman and Simpson (1983) or Bonnecaze, Huppert and Lister (1993, p. 346).

After some algebra, contained in Appendix 2, equation (2.4.86) can be written in the form (A2.11) which is used to state the rigid lid problem as a system. This is:

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ h_2 - h_B & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ u_2 \dfrac{\mathrm{d}h_B}{\mathrm{d}x} \end{bmatrix}, \tag{2.4.89}$$

where, in equation (2.4.89), the terms a_{11}, a_{12} and b_1 are given by

$$a_{11} = \frac{(h_1 - h_2)^2 u_2 + (1-\gamma)(h_2 - h_B)[2Q - (h_1 + h_2 - 2h_B)u_2]}{[h_1 - \gamma h_2 - (1-\gamma)h_B](h_1 - h_2)},$$

$$a_{12} = \frac{(h_1 - h_2)^3 + (1-\gamma)(h_1 - h_2)u_2[2Q - (h_1 + h_2 - 2h_B)u_2]}{[h_1 - \gamma h_2 - (1-\gamma)h_B](h_1 - h_2)^2} - \frac{(1-\gamma)[Q - (h_2 - h_B)u_2]^2}{[h_1 - \gamma h_2 - (1-\gamma)h_B](h_1 - h_2)^2},$$

and

$$b_1 = \frac{(1-\gamma)Q'}{[h_1 - \gamma h_2 - (1-\gamma)h_B]} + \left(\frac{(1-\gamma)u_2[2Q - (h_1 + h_2 - 2h_B)u_2]}{[h_1 - \gamma h_2 - (1-\gamma)h_B]}\right)\frac{\mathrm{d}h_B}{\mathrm{d}x} - \frac{(h_1 - h_2)\kappa C_f u_2^2 T}{[h_1 - \gamma h_2 - (1-\gamma)h_B](h_2 - h_B)}.$$

Although the system (2.4.89) cannot be directly written as a system of conservation laws, a straightforward employment of equations (2.4.86) and (2.4.84) yields

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2-(1-\gamma)u_1 \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u_2^2-\frac{1}{2}(1-\gamma)u_1^2+h_2 \\ (h_2-h_B)u_2 \end{bmatrix} = \begin{bmatrix} -\kappa C_f \frac{u_2^2}{h_2-h_B}T \\ 0 \end{bmatrix}. \tag{2.4.90}$$

In equation (2.4.90), u_1 is considered to be given by (2.4.88) so that only h_2 and u_2 are unknown functions.

Equation (2.4.90) may be expressed in the simplified form (2.4.3) in a similar manner as was completed for the weak-stratification system (2.4.39), written as the system (2.4.43). By introducing the new variable $\overline{u} = u_2-(1-\gamma)u_1$, equation (2.4.87) allows u_2 and u_1 to be removed from (2.4.90). The new variable, $\overline{u}$, is therefore related to u_1 via

$$\overline{u} = \frac{Q-(h_1-h_2)u_1}{h_2-h_B}-(1-\gamma)u_1 = \frac{Q}{h_2-h_B}-\frac{h_1-\gamma h_2-(1-\gamma)h_B}{h_2-h_B}u_1, \tag{2.4.91}$$

and to u_2 by

$$\overline{u} = u_2-(1-\gamma)\frac{Q-(h_2-h_B)u_2}{h_1-h_2} = \frac{h_1-\gamma h_2-(1-\gamma)h_B}{h_1-h_2}u_2-\frac{(1-\gamma)Q}{h_1-h_2}. \tag{2.4.92}$$

Using expressions (2.4.91) and (2.4.92), a simplifying calculation may be made for $u_2^2-(1-\gamma)u_1^2$ as

$$u_2^2-(1-\gamma)u_1^2 = \left[\frac{h_1-h_2}{h_1-\gamma h_2-(1-\gamma)h_B}\overline{u}+\frac{(1-\gamma)Q}{h_1-\gamma h_2-(1-\gamma)h_B}\right]^2 - (1-\gamma)\left[\frac{Q}{h_1-\gamma h_2-(1-\gamma)h_B}-\frac{h_2-h_B}{h_1-\gamma h_2-(1-\gamma)h_B}\overline{u}\right]^2,$$

which simplifies to

$$u_2^2-(1-\gamma)u_1^2 = \frac{(h_1-h_2)^2-(1-\gamma)(h_2-h_B)^2}{(h_1-\gamma h_2-(1-\gamma)h_B)^2}\overline{u}^2 - \frac{2(1-\gamma)(h_1-h_B)}{(h_1-\gamma h_2-(1-\gamma)h_B)^2}Q\overline{u}-\frac{\gamma(1-\gamma)Q^2}{(h_1-\gamma h_2-(1-\gamma)h_B)^2}. \tag{2.4.93}$$

Substituting the calculation (2.4.93) into (2.4.90) gives the conservation form

$$\frac{\partial}{\partial t}\begin{bmatrix} \overline{u} \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{[(h_1-h_2)^2-(1-\gamma)(h_2-h_B)^2]\overline{u}^2-2(1-\gamma)(h_1-h_B)Q\overline{u}-\gamma(1-\gamma)Q^2}{2[h_1-\gamma h_2-(1-\gamma)h_B]^2}+h_2 \\ \frac{[(h_1-h_2)\overline{u}+(1-\gamma)Q](h_2-h_B)}{h_1-\gamma h_2-(1-\gamma)h_B} \end{bmatrix} = \begin{bmatrix} -\kappa C_f \frac{[(h_1-h_2)\overline{u}+(1-\gamma)Q]^2}{(h_2-h_B)[h_1-\gamma h_2-(1-\gamma)h_B]^2}T \\ 0 \end{bmatrix}. \tag{2.4.94}$$

A simplification to systems (2.4.89) and (2.4.94) occurs for the case without any forcing function ($C_f = 0$), bottom slope, $h_B = 0$, or volume source $Q = 0$. In this special circumstance, the system (2.4.89) may be stated as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} u_2 \frac{h_1^2 - 3h_1h_2 + \gamma(h_1+h_2)h_2}{(h_1-\gamma h_2)(h_1-h_2)} & \frac{(h_1-h_2)^3 - (1-\gamma)h_1^2 u_2^2}{(h_1-\gamma h_2)(h_1-h_2)^2} \\ h_2 - h_B & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.4.95}$$

where additional calculations are detailed in Appendix 2, equation (A2.12). Similarly, the corresponding system in conservation form, equation (2.4.94) simplifies to

$$\frac{\partial}{\partial t}\begin{bmatrix} \overline{u} \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{(h_1-h_2)^2-(1-\gamma)h_2^2}{(h_1-\gamma h_2)^2}\overline{u}^2 + h_2 \\ \frac{h_1-h_2}{h_1-\gamma h_2}\overline{u}h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.4.96}$$

The systems (2.4.95) and (2.4.96) are similar to (2.4.44) and (2.4.43), respectively given the simplification of notation $h_1 = 1$, which is usually chosen via the nondimensionalization (2.3.7). Making such a simplification reduces the system (2.4.95) to

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1-3h_2+\gamma(1+h_2)h_2}{(1-h_2)(1-\gamma h_2)}u_2 & \frac{(1-h_2)^3-(1-\gamma)u_2^2}{(1-\gamma h_2)(1-h_2)^2} \\ h_2 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.4.97}$$

and the system (2.4.96) to

$$\frac{\partial}{\partial t}\begin{bmatrix} \overline{u} \\ h_2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{(1-h_2)^2-(1-\gamma)h_2^2}{2(h_1-\gamma h_2)^2}\overline{u}^2 + h_2 \\ \frac{(1-h_2)\overline{u}h_2}{1-\gamma h_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.4.98}$$

In the limit as $\gamma \to 0$, the weak-stratification limit, equations (2.4.97) and (2.4.98) become increasingly similar to the simplified weak-stratification equations (2.4.44) and (2.4.43).

Chapter Summary

The model equations which will be used throughout this thesis have been carefully set forth in this chapter, with attention paid to the physical setting in which the equations are relevant. This has been done in order to discuss some details of the gravity current problem which are more relevant to their mathematical treatment. A theoretical development which includes the effect of the Coriolis force on gravity currents is, to the best of the author's knowledge, not covered in any other source. For the more general shallow-water theory, Pedlosky's (1987) derivation of the shallow-water equations was used quite closely in some places. The resulting two-layer equations of motion are hydrostatic, and for small-aspect ratio flows are simplified to the shallow-water equations in Section 2.2, resulting in equations (2.2.16)-(2.2.22).

The new addition to the theory is that for two-layer gravity currents, the specific front geometry related to the initial release problem is defined, and a height independent frictional drag term, is introduced. Nondimensionalization then allows the two-layer gravity current equations to be stated in the form (2.3.17)-(2.3.23).

In Section 2.4, these equations are then examined and restated in various system (2.4.1) or conservation (2.4.3) forms for specific physical limitations. For the case of gravity currents in three dimensions, equations are stated for a thick upper layer with a free surface, and for the special case of axisymmetric flow. In two spatial dimensions, the governing equations are given as systems in the most general case, and in the simplifying cases of small density differences (weakly stratified), thin upper and lower layers, and a fixed upper layer height (rigid lid). The various forms of these equations will be used in subsequent chapters.

Chapter 3

Theoretical Concepts

Partial Differential Equations which exhibit certain types of behaviour have led to a classification scheme which separates the equations based loosely on the behaviour of the solutions, and the type of conditions required on the boundaries of the solution domain. Equations may be classified as elliptic, parabolic, or hyperbolic (John, 1982), although cases exist which are of mixed types and do not readily fall into one of the three main categories. Although such classification has become somewhat standardized over the last several decades, the strength of a thesis rests upon attention to detail. Therefore, in this chapter, a review of some common definitions and notions is conducted first so that the remainder of the equation classification may be completed in later sections without confusion.

3.1 Hyperbolic Systems and Conservation Laws

The notation and definitions for hyperbolic systems is be divided into two sections covering first the standard case with two variables, typically x and t. The more general case with several spatial variables follows in a second subsection.

3.1.1 First order systems in one spatial variable

The notation in this section is generic, and disconnected from the previous usage of variables. This is done to make the development easier to read, and avoid unnecessary subscripts and symbols. Much of the definitions herein have been adapted from Godlewski and Raviart (1996), Whitham (1974), John (1982), or Kreiss and Lorenz (1989), and are restricted to first-order systems of equations in several spatial dimensions.

Consider n differentiable real-valued functions $u_i(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, for $i = 1,...,n$ which are the components of the vector-valued function $\mathbf{u} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^n$. The general *first order quasi-linear system* of n partial differential equations can be written as,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{b}, \quad x \in \mathbb{R}, t > 0. \tag{3.1.1}$$

The $n \times n$ matrix $\mathbf{A}$, and the vector $\mathbf{b}$ may be functions of $u_1,...,u_n$ as well as x and t (John, p.56, Whitham, p.113).

The system (3.1.1) is usually considered with the restriction that

$$\det(\alpha \mathbf{I}_n + \beta \mathbf{A}) \neq 0, \tag{3.1.2}$$

for some α, β real constants, not both zero, and $\mathbf{I}_n$ the identity matrix. this restriction (Whitham, 1974 p.116) is imposed so that systems which are too degenerate are not included for consideration and classification.

Such a system as (3.1.1), satisfying (3.1.2), is said to be *hyperbolic* if n linearly independent non-zero real eigenvectors $\mathbf{r}^{(i)}$ with real eigenvalues $\lambda^{(i)}$ can be found for the matrix $\mathbf{A}$ such that

$$\mathbf{A}\mathbf{r}^{(i)} = \lambda^{(i)}\mathbf{r}^{(i)}, \tag{3.1.3}$$

for $i = 1, ..., n$. The system is said to be *totally hyperbolic* (Whitham, 1974 p.116) or *strictly hyperbolic* (Godlewski and Raviart, 1996 p.2) if there are n distinct real eigenvalues $\lambda^{(i)}$. A standard result from linear algebra is that if $\mathbf{A}$ is a real and symmetric matrix, then n independent real eigenvalues can be found, and the system is hyperbolic.

Equation (3.1.3) is often dependent on $u_1, ..., u_n$, x and t. For that reason, quasi-linear systems may be classified as hyperbolic only in certain domains. In such a domain, a *characteristic curve* or simply *characteristic* is a curve in the (x, t) plane, $X(\eta), T(\eta)$ parametrized by η, which satisfies

$$\det\left(\frac{\mathrm{d}X}{\mathrm{d}\eta}\mathbf{I}_n - \mathbf{A}\frac{\mathrm{d}T}{\mathrm{d}\eta}\right) = 0. \tag{3.1.4}$$

On such a characteristic, the system (3.1.1) may be written as a system of ordinary differential equations, simplifying the overall problem considerably (Whitham, p. 115).

Equation (3.1.1) may be written in *conservation form* if the matrix $\mathbf{A}$ is derived from a *flux function* $\mathbf{f}$ with components $f_i(\mathbf{u}, x, t) : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R}, i = 1, ..., n$. The relationship necessary is that the n^2 components of $\mathbf{A}$ are given by

$$a_{ij} = \frac{\partial f_i}{\partial u_j}, \quad i, j = 1, ..., n. \tag{3.1.5}$$

In this situation, assumption (3.1.5) allows the system (3.1.1) to be written, via the chain rule, as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u}, x, t)}{\partial x} = \overline{\mathbf{b}}, \quad x \in \mathbb{R}, t > 0, \tag{3.1.6}$$

where the forcing term $\overline{\mathbf{b}}(\mathbf{u}, x, t)$ is different from $\mathbf{b}$ unless $\mathbf{f}$ is independent of x. Such a differential equation is usually derived from an integral conservation equation for a physical quantity (mass, momentum, energy) and allows discontinuities in $\mathbf{u}$ to be included, as well as a formal definition of weak solutions (Whitham, 1974 p.139). The function $\overline{\mathbf{b}}$ allows for a source term, such as a body force in the momentum equations.

3.1.2 First order systems in more than one spatial variable

The concepts introduced in the previous subsection generalize to situations for more spatial variables, although the algebra necessarily becomes more complex, and fewer results are known.

A first order quasi-linear system of n partial differential equations in several (p) spatial variables may be denoted as

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{k=1}^{p} \mathbf{A}^{(k)} \frac{\partial \mathbf{u}}{\partial x_k} = \mathbf{b}, \quad \mathbf{x} = (x_1, ..., x_p) \in \mathbb{R}^p, t > 0. \tag{3.1.7}$$

The notation in (3.1.7) is similar to (3.1.1), with the addition of more derivatives in the extra spatial variables for $p > 1$. In (3.1.7), we have the components $u_i : \mathbb{R}^p \times [0, \infty) \to \mathbb{R}$, and the matrices $\mathbf{A}^{(k)}$, $k = 1, ..., p$, as well as $\mathbf{b}$, are functions of $\mathbf{u}$, $\mathbf{x}$, and t.

Assuming that the system (3.1.7) is nondegenerate in a similar way as stated in the constraint (3.1.2), it is said to be *hyperbolic* if for any $\omega = (\omega_1, ...\omega_p) \in \mathbb{R}^p, \omega \neq \mathbf{0}$, the matrix

$$\mathbf{A} = \sum_{k=1}^{p} \omega_k \mathbf{A}^{(k)} \tag{3.1.8}$$

has n linearly independent eigenvectors, and n real eigenvalues (Godlewski and Raviart, 1996 p.2). If, in addition, the eigenvalues are distinct, the system (3.1.8) is said to be *strictly hyperbolic*. Note that for $p = 1$, the condition (3.1.8) reduces to (3.1.2) or (3.1.3) for a single spatial variable.

Much of what is known about hyperbolic systems in more than one spatial variable concerns those that are *symmetrizable*. Symmetrizable systems are ones for which a positive-definite symmetric matrix $\mathbf{A}^{(0)}$ exists such that the matrices $\mathbf{A}^{(0)}\mathbf{A}^{(k)}$ are symmetric for $k = 1, ..., p$. Symmetrizable systems are hyperbolic (Godlewski and Raviart p.2).

In a similar manner to the single spatial variable case (3.1.6) with $p = 1$, the system (3.1.7) may be rewritten in conservation form. Although the notation is necessarily more complex, the procedure is formally the same as that done previously, and is completed in the following several paragraphs.

Let Ω denote an open set which contains the range of the vector-valued function $\mathbf{u}(\mathbf{x},t)$, that is, $\mathbf{u}:\mathbb{R}^p\times[0,\infty)\to\Omega\subset\mathbb{R}^n$. Also, consider the $p\times n$ flux functions $f_i^{(k)}(\mathbf{u},\mathbf{x},t):\Omega\times\mathbb{R}^p\times[0,\infty)\to\mathbb{R}$, for $k=1,...,p$ and $i=1,...,n$ which comprise the p vector-valued flux functions,

$$\mathbf{f}^{(k)}(\mathbf{u},\mathbf{x},t)=\begin{bmatrix} f_1^{(k)}(\mathbf{u},\mathbf{x},t) \\ \vdots \\ f_n^{(k)}(\mathbf{u},\mathbf{x},t)\end{bmatrix},\quad \mathbf{f}^{(k)}:\Omega\times\mathbb{R}^p\times[0,\infty)\to\mathbb{R}^n. \tag{3.1.9}$$

A quasi-linear system of partial differential equations in conservation form may be written as a vector equation (Godlewski and Raviart, 1996 p.11)

$$\frac{\partial\mathbf{u}}{\partial t}+\sum_{k=1}^{p}\frac{\partial}{\partial x_k}\mathbf{f}^{(k)(\mathbf{u},\mathbf{x},t)}=\overline{\mathbf{b}}(\mathbf{u},\mathbf{x},t),\quad \mathbf{x}=(x_1,...,x_p)\in\mathbb{R}^p, t>0. \tag{3.1.10}$$

The n components of the vector equation (3.1.10) are then given individually by the equation

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{p}\frac{\partial}{\partial x_k}f_i^{(k)}(\mathbf{u},\mathbf{x},t)=\overline{b_i}(\mathbf{u},\mathbf{x},t),\quad \text{for } i=1,...,n. \tag{3.1.11}$$

An application of the chain rule allows the derivatives of the flux functions $f_i^{(k)}$ in (3.1.11) to be rewritten as

$$\frac{\partial}{\partial x_k}f_i^{(k)}(\mathbf{u},\mathbf{x},t)=\sum_{j=1}^{n}\frac{\partial}{\partial u_j}f_i^{(k)}(\mathbf{u},\mathbf{x},t)\frac{\partial u_j}{\partial x_k}+\widetilde{b}_i^{(k)}(\mathbf{u},\mathbf{x},t),$$
$$\text{for } i=1,...,n, k=1,...,p. \tag{3.1.12}$$

The terms $\widetilde{b}_i^{(k)}$ in (3.1.12) represent the partial derivatives of $f_i^{(k)}$ with respect to x_k while holding $\mathbf{u}$ constant. Since t and $\mathbf{x}$ are independent, terms due to these derivatives do not appear. Substitution of the expression (3.1.12) into the component equations (3.1.11) yields

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{p}\sum_{j=1}^{n}a_{ij}^{(k)}\frac{\partial u_j}{\partial x_k}=b_i, \tag{3.1.13}$$

where $a_{ij}^{(k)} = \dfrac{\partial f_i^{(k)}}{\partial u_j}$, and the term $b_i = \overline{b_i} - \sum_{k=1}^{p} \widetilde{b}_i^{(k)}$ does not contain any derivatives of $\mathbf{u}$.

The n component equations (3.1.13) may be written in the more recognizable form (3.1.7) by association of the $n \times n$ matrices $\mathbf{A}^{(k)}$ with the components $a_{ij}^{(k)}$ for $i, j = 1, ..., n$. In this way, the system (3.1.7) is associated with the system (3.1.10) in conservation form. Therefore, the system of conservation equations (3.1.10) is said to be *hyperbolic* if the system (3.1.13) is hyperbolic, with a similar identification for *strictly hyperbolic.*

To determine if a general system in conservation form (3.1.10) is hyperbolic, the notion of entropy functions may be used. This method is most often applied to systems in conservation form with spatially and temporally independent flux functions, i.e. $\mathbf{f}^{(k)} = \mathbf{f}^{(k)}(\mathbf{u})$. To keep the generality of spatial dependence, the definitions are modified from Godlewski and Raviart (1996, p.21-25).

Definition: *Let Ω be a convex subset of $\mathbb{R}^n \times \mathbb{R}^p \times [0, \infty)$. A convex function $U : \Omega \to \mathbb{R}$ is called an* entropy *for the system of conservation laws (3.1.10) in Ω if there exist p functions $F^{(k)} : \Omega \to \mathbb{R}$, $k = 1, ..., p$, called* entropy fluxes, *such that*

$$(\nabla U)^T \mathbf{A}^{(k)} = (\nabla F^{(k)})^T, \text{ for } k = 1, ..., p. \tag{3.1.14}$$

This definition allows a corollary to the Godunov-Mock Theorem (Godlewski and Raviart, 1996 p.24) to be stated. The theorem is proved there for the special case of spatially and temporally independent flux functions. The proof generalizes exactly for more general flux functions, and the corollary is stated as the following lemma.

Lemma 3.1 *(Corollary of the Godunov-Mock Theorem):* Let $U : \Omega \to \mathbb{R}$ be a strictly convex function which is an entropy for (3.1.10) in Ω. Then the system (3.1.7) is symmetric, and hence hyperbolic.

This theorem is used in Chapter 6.

3.2 Jump Conditions Across Discontinuities

Solutions of systems of partial differential equations of the form (3.1.6) often may exist which are not continuous. In this section, a discussion of solutions which permit simple jump discontinuities or discontinuous derivatives is conducted. Of particular interest is the modification of the Rankine-Hugoniot jump conditions to include the effect of discontinuous forcing terms in the original equations.

3.2.1 Conditions across discontinuities in $\mathbf{u}$

Partial differential equations in conservative form often can be cast into an integral conservation law. Such integral equations often increase the number of solutions to the original partial differential equation, and these new solutions are called *weak solutions*. Solutions to the original partial differential equation are called *classical solutions*. Ideally, each weak solution would correspond uniquely with a classical solution, but this is often not the case, as usually an entire family of weak solutions appears for each classical solution. A precise definition of weak and classical solutions is given in Godlewski and Raviart (1996, p.15). To remove this difficulty of choosing a single classical solution from a class of weak solutions, only weak solutions which satisfy some form of jump condition are chosen. These Rankine-Hugoniot jump conditions are straightforward to implement for scalar, spatially independent conservation laws, and more general expressions can be derived (Whitham, 1974 p.138, Godlewski and Raviart, 1996 p.18). In this subsection, specialized jump conditions will be derived for later use.

First consider the system of differential equations in conservation form, (3.1.6), with components written as

$$\frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial x} = \overline{b_i}, x \in \mathbb{R}, t > 0, \tag{3.2.1}$$

for $i = 1, ..., n$. Assuming that u_i is a classical solution, this equation may be integrated over an arbitrary constant interval $[x_1, x_2] \subset \mathbb{R}$ to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u_i dx + f_i(\mathbf{u}(x_2,t),x_2,t) - f_i(\mathbf{u}(x_1,t),x_1,t) = \int_{x_1}^{x_2} \overline{b_i} dx, \quad i = 1, ..., n. \tag{3.2.2}$$

Equation (3.2.2), viewed without the preamble leading up to its formulation, does not need the requirements that u_i be continuous, simply integrable. Such weak solutions may satisfy (3.1.6) in a piecewise continuous fashion, although (3.2.2) provides a valuable constraint. To see this, consider a discontinuity in u_i located at the point $s(t) \in [x_1, x_2]$ for some time interval. In this case, the first term in equation (3.2.2) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u_i dx &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{x_1}^{s(t)} u_i dx + \int_{s(t)}^{x_2} u_i dx\right] \\ &= \int_{x_1}^{s(t)} \frac{\partial u_i}{\partial t} dx + \frac{\mathrm{d}s}{\mathrm{d}t} u_i(s^-,t) + \int_{s(t)}^{x_2} \frac{\partial u_i}{\partial t} dx - \frac{\mathrm{d}s}{\mathrm{d}t} u_i(s^+,t)\end{aligned} \tag{3.2.3}$$

In equation (3.2.3), the notation $u_i(s^+, t)$ and $u_i(s^-, t)$ represent the limits as $x \to s(t)$, from the right (+) and left (−) respectively.

Since, in each of the intervals (x_1, s) and (s, x_2) the solution u_i has continuous partial derivatives, by assumption, equation (3.2.1) may be substituted into integrals on the right hand side of expression (3.2.3). The result is simplified by the notation $[u_i]$ representing the jump in u_i across $s(t)$, $[u_i] = u_i(s^+, t) - u_i(s^-, t)$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u_i dx &= -\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] + \int_{x_1}^{s(t)} \overline{b_i} dx - (f_i(\mathbf{u}(s^-, t), s^-, t) - f_i(\mathbf{u}(x_1, t), x_1, t)) \\ &+ \int_{s(t)}^{x_2} \overline{b_i} dx - (f_i(\mathbf{u}(x_2, t), x_2, t) - f_i(\mathbf{u}(s^+, t), s^+, t)) \\ &= -\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] + [f_i] + f_i(\mathbf{u}(x_1, t), x_1, t) - f_i(\mathbf{u}(x_2, t), x_2, t) + \int_{x_1}^{x_2} \overline{b_i} dx. \end{aligned} \quad (3.2.4)$$

This expression can now be substituted into equation (3.2.2), with the result that the remaining equation is

$$-\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] + [f_i] = 0, \quad \text{for } i = 1, ..., n. \quad (3.2.5)$$

This relationship between shock speed and jumps in u_i and f_i across a discontinuity is quoted in Whitham (1974, p.138) without explicit proof.

Of a more specific nature to gravity currents are discontinuous forcing terms $\overline{b_i}$. A general form of $\overline{b_i}$ is considered as

$$\overline{b_i} = H(\sigma(t) - x)\{\frac{\partial}{\partial x} g_i(\mathbf{u}, x, t) + c_i(\mathbf{u}, x, t)\} \quad (3.2.6)$$

where H is the Heaviside function defined as

$$H(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases} \quad (3.2.7)$$

The term $\overline{b_i}$ may be considered as a source term to the left of a discontinuity $x = \sigma(t)$, and zero to the right. The significance of this form (3.2.6)-(3.2.7) in relation to gravity currents will be discussed later in this section, and in Chapter 4, Section 2. Its importance arises when considering the existence of steady-state solutions over non-horizontal bottom slopes.

Such a form of $\overline{b_i}$ permits equation (3.2.3) and (3.2.4) to be generalized as follows. Rewriting $\overline{b_i}$ as $\overline{b_i} = \frac{\partial}{\partial x}(Hg_i) + Hc_i$ (almost everywhere) by the chain rule allows part of $\overline{b_i}$ to be considered as a flux term so that equation (3.2.2) has a right hand side given by

$$\int_{x_1}^{x_2} \overline{b_i} dx = Hg_i|_{x_1}^{x_2} + \int_{x_1}^{x_2} Hc_i dx$$
$$= -g_i(\mathbf{u}(x_1,t),x_1,t) + \int_{x_1}^{\sigma} c_i dx$$

whenever that $x_1 < \sigma < x_2$.

Then the integral split over the discontinuity $s(t)$ may be calculated as follows in the three cases of location of σ and s. It follows that

$$\int_{x_1}^{s(t)} \overline{b_i} dx + \int_{s(t)}^{x_2} \overline{b_i} dx$$

is equal to one of

(i) $(x_1 < \sigma < s < x_2)$

$$\int_{x_1}^{\sigma} \overline{b_i} dx + \int_{\sigma}^{s(t)} \overline{b_i} dx + \int_{s}^{x_2} \overline{b_i} dx = g_i|_{x_1}^{\sigma} + \int_{x_1}^{\sigma} c_i dx + 0,$$

(ii) $(x_1 < \sigma = s < x_2)$

$$\int_{x_1}^{\sigma} \overline{b_i} dx + \int_{\sigma}^{x_2} \overline{b_i} dx = g_i|_{x_1}^{\sigma} + \int_{x_1}^{\sigma} c_i dx,$$

or

(iii) $(x_1 < s < \sigma < x_2)$

$$\int_{x_1}^{\sigma} \overline{b_i} dx + \int_{\sigma}^{s(t)} \overline{b_i} dx + \int_{\sigma}^{x_2} \overline{b_i} dx = g_i|_{x_1}^{s} + g_i|_{s}^{\sigma} + \int_{x_1}^{s(t)} c_i dx + \int_{s(t)}^{\sigma} c_i dx$$
$$= g_i|_{x_1}^{\sigma} + \int_{x_1}^{\sigma} c_i dx. \tag{3.2.8}$$

Therefore, for a discontinuous (weak) solution u_i, the equality of the separate integrations is imposed to yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} u_i dt + f_i|_{x_1}^{x_2} = -g_i(\mathbf{u}(x_1,t),x_1,t) + \int_{x_1}^{\sigma} c_i dx. \tag{3.2.9}$$

Substitution of the split integration (3.2.8) into the integral form of the conservation law (3.2.9) above gives

$$-\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] + [f_i] - f_i|_{x_1}^{x_2} + g_i|_{x_1}^{\sigma} + \int_{x_1}^{\sigma} c_i dx + f_i|_{x_1}^{x_2} = -g_i(\mathbf{u}(x_1,t),x_1,t) + \int_{x_1}^{\sigma} c_i dx,$$

which simplifies to the generalization of the result (3.2.5), namely

$$-\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] + [f_i] + g_i(\sigma^-) = 0 \tag{3.2.10}$$

The further restriction that $s(t) = \sigma(t)$ gives (3.2.10) in the form

$$\frac{\mathrm{d}s}{\mathrm{d}t}[u_i] = [f_i] + g_i(s^-), \tag{3.2.11}$$

an equation which will be utilized later.

As an example of the importance of the source term to conservation laws, a simple one-dimensional equation in one spatial variable is compared in its linear and nonlinear forms with and without discontinuous source terms for the initial value problem.

Example 3.1 (Linear Homogeneous)

Consider the one-dimensional wave equation initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} = 0, \quad t > 0, x \in \mathbb{R}, 0 < c_1 = \text{ constant} \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.2.12}$$

The solution, found by the method of characteristics (John, 1982 p.15) is

$$u(x,t) = u_0(x - c_1 t). \tag{3.2.13}$$

The initial shape is held constant along the characteristic curves with slope $\frac{\mathrm{d}x}{\mathrm{d}t} = c_1$. Any discontinuities which are present in the initial value u_0 therefore must continue to exist, and propagate along the characteristics.

Example 3.2 (Linear Nonhomogeneous)

The problem (3.2.12) is modified through the addition of a source term which has the form (3.2.6),(3.2.7). Consider

$$\begin{aligned} &\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} = xH(c_2 t - x), \quad t > 0, x \in \mathbb{R}, c_1, c_2 \text{ both positive constants} \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.2.14}$$

To solve the first order problem (3.2.14), the method of characteristics may be used (John, 1982) to give the set of ordinary differential equations:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \quad t(s,0) = 0 \tag{3.2.15a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = c_1, \quad x(s,0) = s \tag{3.2.15b}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = xH(c_2 t - x), \quad u(s,0) = u_0(s). \tag{3.2.15c}$$

Solutions of (3.2.15 a-b) are $t = \tau$ and $x = c_1\tau + s$ which allow (3.2.15c) to be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \begin{cases} c_1 t - s, & c_2 t - x \geq 0 \\ 0, & c_2 t - x < 0 \end{cases} \quad \text{and } u(s,0) = u_0(s). \tag{3.2.16}$$

Equation (3.2.16) has solution

$$u(s,t) = \begin{cases} c_1\frac{t^2}{2} - st + u_0(s), & c_2 t - x \geq 0 \\ u_0(s), & c_2 t - x < 0 \end{cases}$$

which becomes, upon removing s,

$$u(x,t) = \begin{cases} \frac{c_1}{2}t^2 - (x - c_1 t) + u_0(x - c_1 t), & c_2 t - x \geq 0 \\ u_0(x - c_1 t), & c_2 t - x > 0 \end{cases}$$

or

$$u(x,t) = u_0(x - c_1 t) + (\frac{3}{2}c_1 t - x)tH(c_2 t - x). \tag{3.2.17}$$

It can be observed that the solution (3.2.17) has a discontinuity along the line $x = c_2 t$ in addition to any possible initial discontinuities which propagate along the lines of slope $\frac{\mathrm{d}x}{\mathrm{d}t} = c_1$. The relation between c_1 and c_2 (i.e. $c_1 < c_2$, $c_1 = c_2$ or $c_1 > c_2$) then allows the front positions, or discontinuities in u, to be determined explicitly. Since this brief discussion is not of direct relevance to the gravity currents considered in this thesis, such an analysis is not included. Rather, the importance of a discontinuous source term is portrayed to show its effect on the solution.

The direct connection here to gravity currents is the case when the discontinuity inherent in the source term travels at the same speed as the discontinuity in the solution arising from the initial value. For this specific example, this would occur when $c_1 = c_2$. A more intriguing situation occurs when $c_2 > c_1$. For instance, suppose that $c_2 = \frac{3}{2}c_1$. The solution (3.2.17) may be portrayed in the (x,t) plane as in Figure 3.1. The discontinuity at $x = \frac{3}{2}c_1 t$ moves faster than the original characteristics for the homogeneous case, modifying (increasing) the initial value $u(x - c_1 t)$ behind the shock.

If equation (3.2.14) is considered in a more general form with discontinuity travelling along the curve $x = s(t)$, the partial differential equation may be replaced by, for example,

$$\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} = xH(s(t) - x). \tag{3.2.18}$$

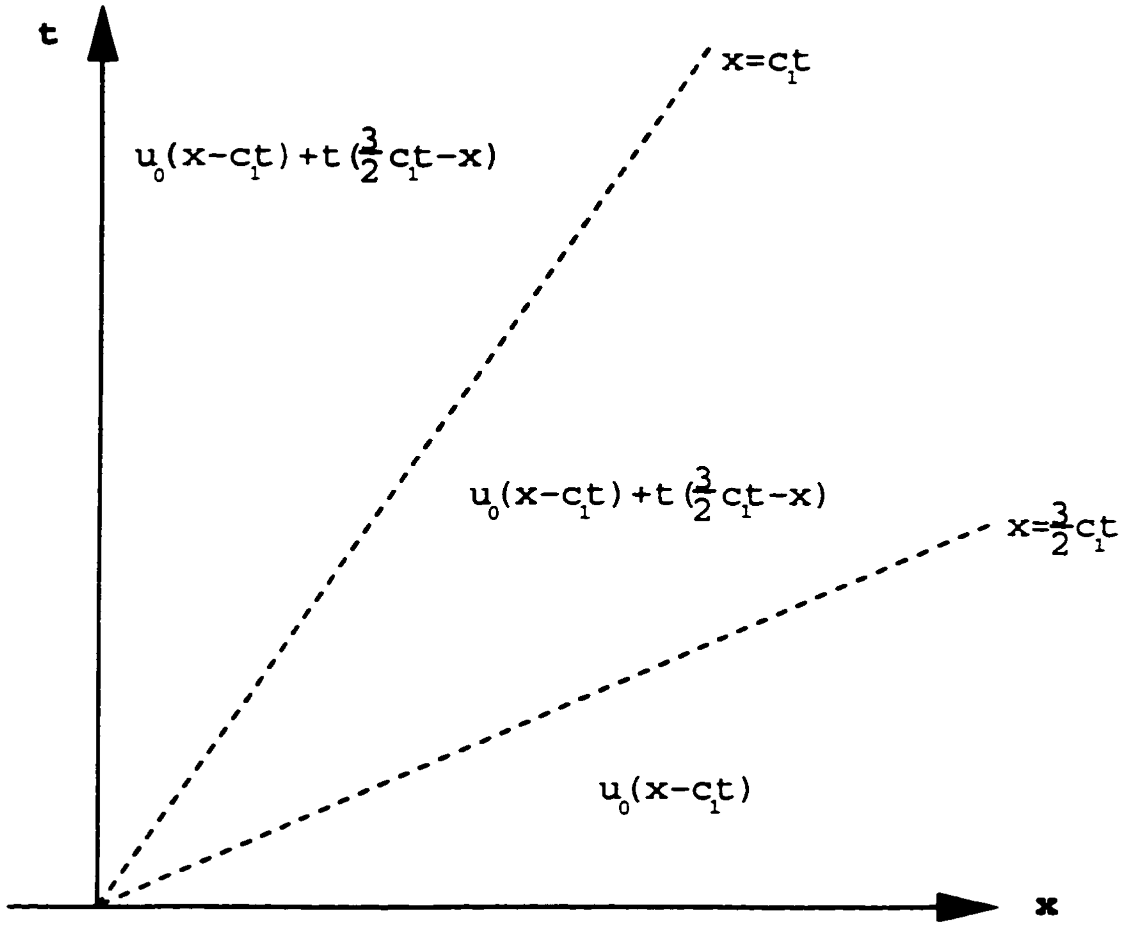

FIGURE 3.1. The Characteristic Plane.

Then the jump condition (3.2.11) may be applied to give

$$\frac{\mathrm{d}s}{\mathrm{d}t}[u] = [c_1 u] + \frac{x^2}{2}|_{x=s}, \tag{3.2.19}$$

which becomes, using $[c_1 u] = c_1[u]$ and assuming $[u] \neq 0$,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = c_1 + \frac{s^2}{2[u]}. \tag{3.2.20}$$

Clearly any weak solution does not satisfy the simple jump condition $\frac{\mathrm{d}s}{\mathrm{d}t} = c_1[u]$ as did (3.2.12). Any weak solutions must satisfy (3.2.20), ensuring that the source term in (3.2.18) affects discontinuities in the solution.

Example 3.3 (Nonlinear Homogeneous)

The nonlinear Burgers equation with a definite initial discontinuous profile is chosen to illustrate the importance of nonlinearities in the partial differential equation. Such a Riemann problem is stated as

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad t > 0, x \in \mathbb{R} \\ &u(x,0) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases} \end{aligned} \tag{3.2.21}$$

The appropriate discontinuous weak solution (LeVeque, 1992 p.29) to (3.2.21) which satisfies the jump condition (3.2.5) is given by a discontinuous function represented in the (x, y) plane as

$$u(x,t) = \begin{cases} 1, & x \le \frac{1}{2}t \\ 0, & x > \frac{1}{2}t \end{cases} \tag{3.2.22}$$

The discontinuity travels the line $x = \frac{1}{2}t$ which is an average of the two speeds impinging on either side of the discontinuity.

Example 3.4 (Nonlinear Nonhomogeneous)

The nonlinear Burgers equation in the form is (3.2.21) with a nonlinear discontinuous forcing term is stated as

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = xH(\sigma - x), \quad t > 0, x \in \mathbb{R} \\ &u(x,0) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases} \end{aligned} \tag{3.2.23}$$

In the problem (3.2.23), $x = \sigma(t)$ describes an arbitrary path in the (x, t) plane with an assumption of differentiability and initial value $\sigma(0) = 0$. For $x < \sigma$, the initial value problem (3.2.23) may be stated as

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x, \quad t > 0, x < \sigma \\ &u(x,0) = 1, x < 0. \end{aligned} \tag{3.2.24}$$

A solution of problem (3.2.24) by the method of characteristics (John p.15) gives the set of ordinary differential equations

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \quad t(s,0) = 0 \tag{3.2.25a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = u, \quad x(s,0) = s \tag{3.2.25b}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = x, \quad u(s,0) = u_0(s). \tag{3.2.25c}$$

Although equation (3.2.25a) integrates directly as $t = \tau$, the other equations may be solved in a fairly straightforward manner. Taking a second derivative of equation (3.2.25b) with respect to τ, and substituting equation (3.2.25c) for $\frac{\mathrm{d}u}{\mathrm{d}\tau}$ gives the second order ordinary differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = x, \quad x(s,0) = s, \quad \frac{\mathrm{d}x}{\mathrm{d}\tau}(s,0) = 1$$

which has the solution

$$x(s,\tau) = \sinh\tau + s\cosh\tau. \tag{3.2.26}$$

The solution u may be found by differentiating (3.2.26) according to equation (3.2.25c). The resulting differentiation gives

$$u(s,\tau) = \cosh\tau + s\sinh\tau. \tag{3.2.27}$$

The variables s and τ may be removed by inverting equation (3.2.26) and using $t = \tau$ to express the solution (3.2.27) as

$$\begin{aligned} u(x,t) &= \cosh t + \left(\frac{x - \sinh t}{\cosh t}\right)\sinh t \\ &= \cosh t + (x - \sinh t)\tanh t. \end{aligned} \tag{3.2.28}$$

Therefore, the solution to the problem (3.2.23) may be given by

$$u(x,t) = \begin{cases} \cosh t + (x - \sinh t)\tanh t, x < \sigma(t) \\ 0, x > \sigma(t). \end{cases} \tag{3.2.29}$$

It is interesting to note the remarkable difference in the solution (3.2.29) and that of (3.2.22) to the homogeneous problem. The effect of the forcing term is not negligible even for small x as the exponential growth in t quickly becomes apparent.

The remaining question from problem (3.2.23) is the determination of the path of the discontinuity $\sigma(t)$. Although it can be imposed arbitrarily, the natural choice is the one which will make the solution (3.2.29) a weak solution. Such a constraint requires that the jump condition across σ must satisfy condition (3.2.11), which may be expressed as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}[u] = [\frac{1}{2}u^2] + \frac{1}{2}x^2|_{\sigma^-}. \tag{3.2.30}$$

This condition is similar to (3.2.19) for the linear case, and may be simplified by dividing by $[u]$ assumed nonzero to obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2}(u|_{\sigma^+} + u|_{\sigma^-}) + \frac{\sigma^2}{2(u|_{\sigma^+} - u|_{\sigma^-})}. \tag{3.2.31}$$

By implementing the solution u in the form (3.2.29), the jump condition (3.2.31) becomes

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{1}{2}\left(\cosh t + (\sigma - \sinh t)\tanh t\right) + \frac{\sigma^2}{2\left(\cosh t + (\sigma - \sinh t)\tanh t\right)} \\ &= \frac{1}{2}\left\{\frac{\cosh^2 t + 2(\sigma - \sinh t)\sinh t + (\sigma - \sinh t)^2\tanh^2 t + \sigma^2}{\cosh t + (\sigma - \sinh t)\tanh t}\right\}\end{aligned} \tag{3.2.32}$$

for $t > 0$, and with the initial condition $\sigma(0) = 0$. Although the constant $\frac{1}{2}$ is recognizable from the jump condition (3.2.22) for the linear case, this ordinary differential equation cannot be solved explicitly. The numerical solution is given in Figure 3.2, where for comparison, the discontinuity position σ for the homogeneous problem (3.2.21) is also shown.

Figure 3.2 also shows the beginning of asymptotic behaviour for later time. As $t \to \infty$, equation (3.2.32) simplifies to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma,$$

which then may be solved with some initial value σ_0 as $\sigma = \sigma_0 \mathrm{e}^t$. Thus, the shock speed grows exponentially, which is quite distinct behaviour from the homogeneous problem.

Figure 3.2 has a direct physical connection to gravity currents described in the following several paragraphs. Consider the initial release problem of a gravity current obeying (2.4.25) with u_1, ζ_1, ζ_2 all constant and $C_f = 0$. The resulting equation which describes the evolution of u_2 is

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u_2^2) = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x}$$

to the left of the fluid intrusion, and by definition $u_2 = 0$ to the right. That is, u_2 can be thought of as satisfying, at some time t_0, an initial boundary value problem which is similar to the form

$$\begin{aligned}&\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x}H(\sigma(t) - x), \quad t > t_0, x \geq 0, \\ &u_2(x, t_0) = \begin{cases} u_0, & x < \sigma(t_0) \\ 0, & x > \sigma(t_0). \end{cases}\end{aligned} \tag{3.2.33}$$

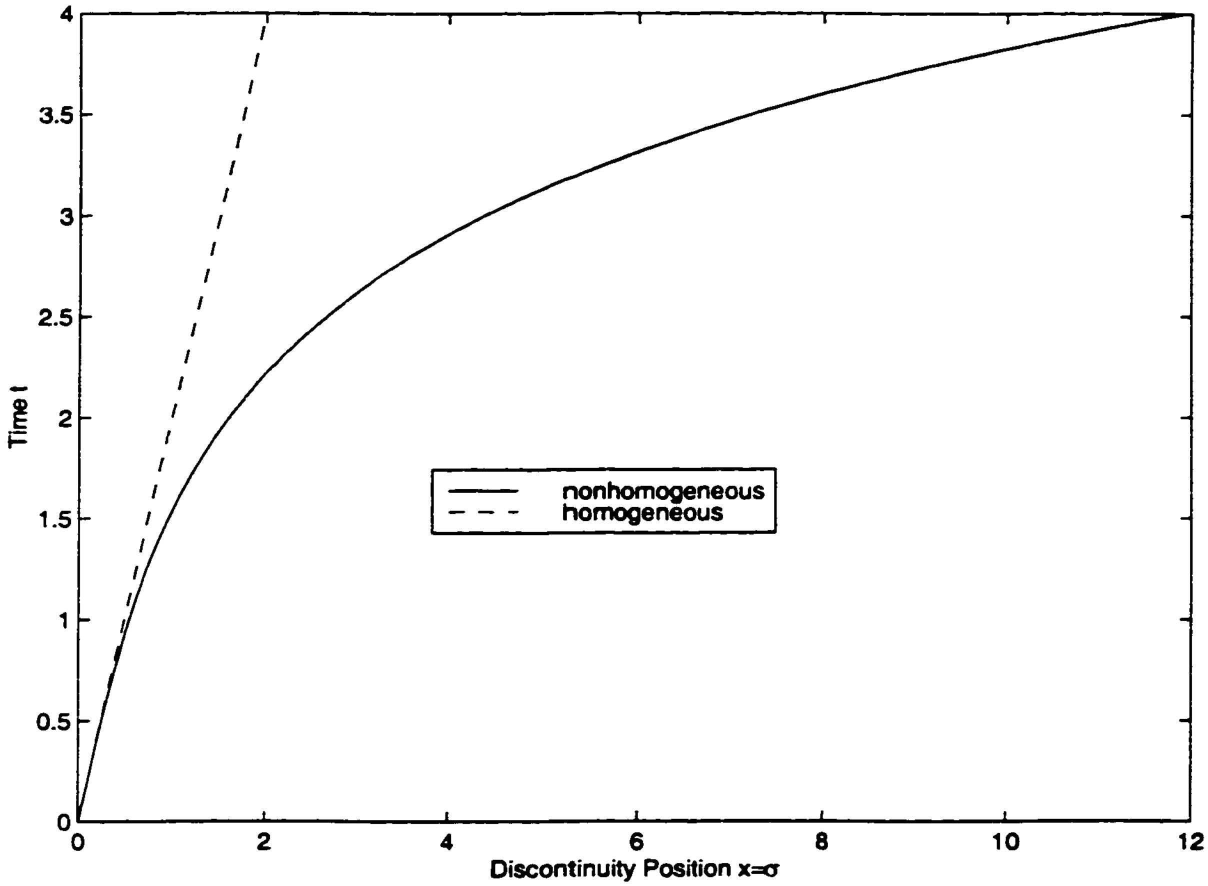

FIGURE 3.2. Numerical Solution to Equation 3.2.32.

Given the similarity of (3.2.33) to (3.2.23), it should be expected that solutions to (3.2.33) may be divided into two types: solutions for which $\sigma(t)$ is known *a priori*, and weak solutions. The latter method of solution is clearly preferable as it is applicable to general problems, and the front position is determined as part of the solution. The former method presupposes knowledge of the front position. Such knowledge may be obtained by physical experiments which include assumptions and factors neglected during the derivation of the equations of motion. The restriction of front position or front speed prior to solving the system does not permit the natural discontinuities which arise from the IVP to propagate at the proper speed. This may be a cause of differences between numerical calculations based on models which ignore physical effects such as entrainment and friction, and experimental results.

This imposition of front position may then lead to differences between numerical calculations meant to model experimental results. Such occurrences will be discussed in later chapters of this thesis.

For systems with more than one spatial variable in the form of equation (3.1.11) a similar result may be derived, although points of discontinuity are replaced by surfaces of discontinuity. Due to the additional complexity of the added spatial variables, extra care to the definition of weak and classical solutions must be taken, as is done in Godlewski and Raviart (1996, p.15) for the special case without forcing $\mathbf{b}$ and with flux vectors $\mathbf{f}^{(k)}$ which are independent of $\mathbf{x}$ and t. However, for the most general case, a generalization of previous results is stated as the following result.

Lemma 3.2 Let $\mathbf{u} : \mathbb{R}^p \times [0, \infty) \to \Omega$ be a continuously differentiable solution of equation (3.1.11) except at a finite number of smooth surfaces Σ in $(\mathbf{x}, t)$ space at which $\mathbf{u}$ may have a jump discontinuity. Denote the normal vector to such a surface Σ as

$$\mathbf{n} = (n_1, ..., n_p, n_t)^T.$$

Then, along surfaces of discontinuity, $\mathbf{u}$ satisfies the jump condition

$$[\mathbf{u}]n_t + \sum_{k=1}^{p}[\mathbf{f}^{(k)}]n_k = \mathbf{0}, \tag{3.2.34}$$

where

$$[\mathbf{u}] = \lim_{\varepsilon\to 0+} \mathbf{u}\left((\mathbf{x}, t) + \varepsilon\mathbf{n}\right) - \lim_{\varepsilon\to 0+} \mathbf{u}\left((\mathbf{x}, t) - \varepsilon\mathbf{n}\right) = \mathbf{u}^+ - \mathbf{u}^-$$

denotes the difference of the limits of $\mathbf{u}$ on each side of Σ.

Proof:

Let $M = (\mathbf{x}_M, t_M)$ be a point on a surface of discontinuity Σ, and B a small open 'cylinder' in $\mathbb{R}^p \times [0, \infty)$ centered at M given by $B = B_{\mathbf{x}_M} \times (t_M - \epsilon, t_M + \epsilon)$ for a small positive real number ϵ and open ball $B_{\mathbf{x}_M} \in \mathbb{R}^p$. Assume that Σ is the only surface of discontinuity which intersects B so that Σ separates B into two disjoint open sets B_+ and B_- as in Figure 3.3.

The normal vector $\mathbf{n} = (n_1, ..., n_p, n_t)^T$ to the surface Σ is assumed to point into the set B_+. Next, let $\mathbf{g}$ denote an appropriate vector 'test function', namely. $\mathbf{g} : \overline{B} \to \mathbb{R}^n$ has continuous partial derivatives in B and is zero on the boundary $\partial B = \overline{B} \backslash B$. Now, form a volume integral over B,

$$\int_B \left\{ \mathbf{u} \cdot \frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)} \cdot \frac{\partial \mathbf{g}}{\partial x_k} \right\} d\mathbf{x}dt. \tag{3.2.35}$$

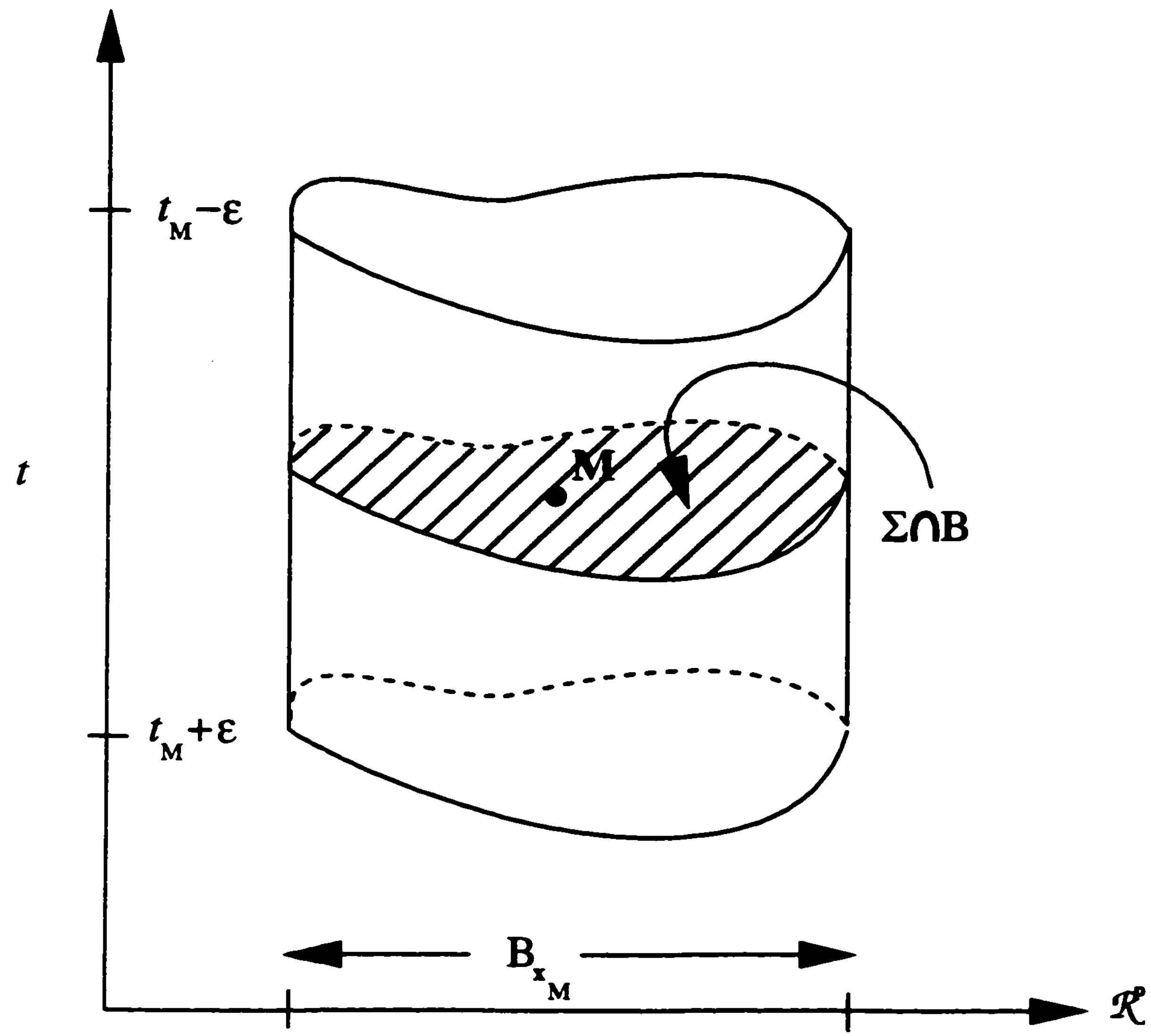

FIGURE 3.3. Geometry of the cylinder B.

Equation (3.2.35) may be split into two integrals, $\int_B = \int_{B_+} + \int_{B_-}$, in each of which the integrand is continuous. In each of these integrals an integration by parts followed by an application of the Gauss Divergence Theorem may be completed. The resulting calculation is

$$\begin{aligned}
\int_{B_+} \{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k}\} d\mathbf{x}dt \\
&= \int_{B_+} \{\frac{\partial}{\partial t}(\mathbf{u}\cdot\mathbf{g}) + \sum_{k=1}^{p} \frac{\partial}{\partial x_k}(\mathbf{f}^{(k)}\cdot\mathbf{g})\} d\mathbf{x}dt - \int_{B_+} \{\frac{\partial \mathbf{u}}{\partial t} + \sum_{k=1}^{p} \frac{\partial \mathbf{f}^{(k)}}{\partial x_k}\}\cdot\mathbf{g}d\mathbf{x}dt \\
&= -\int_{\partial B_+\cap\Sigma} \{(\mathbf{u}^+\cdot\mathbf{g})n_t + \sum_{k=1}^{p}(\mathbf{f}^{(k)+}\cdot\mathbf{g})n_k\} dS + \int_{B_+} \mathbf{b}\cdot\mathbf{g}d\mathbf{x}dt \\
&= -\int_{B\cap\Sigma} \{n_t\mathbf{u}^+ + \sum_{k=1}^{p} n_k\mathbf{f}^{(k)+}\}\cdot\mathbf{g}d\mathbf{x}dt - \int_{B_+} \mathbf{b}\cdot\mathbf{g}d\mathbf{x}dt.
\end{aligned}$$

The compactly supported property of $\mathbf{g}$ is used in the third line above, wherein the surface integral over $\partial B_+ \backslash (\partial B_+ \cap \Sigma)$ does not contribute to the overall boundary integral since $\mathbf{g} = \mathbf{0}$ there. Similarly to the above calculation, the $\int_{B_-}$ may be simplified to

$$\begin{aligned}
\int_{B_-} \{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k}\} d\mathbf{x}dt \\
&= \int_{B\cap\Sigma} \{n_t\mathbf{u}^- + \sum_{k=1}^{p} n_k\mathbf{f}^{(k)-}\}\cdot\mathbf{g}d\mathbf{x}dt + \int_{B_-} \mathbf{b}\cdot\mathbf{g}d\mathbf{x}dt.
\end{aligned}$$

Putting these calculations together, equation (3.2.35) is now written as

$$\begin{aligned}
\int_{B} \left\{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k}\right\} d\mathbf{x}dt = -\int_{B\cap\Sigma} \{n_t[\mathbf{u}] + \sum_{k=1}^{p} n_k[\mathbf{f}^{(k)}]\}\cdot\mathbf{g}dS+ \\
+ \int_{B} \mathbf{b}\cdot\mathbf{g}d\mathbf{x}dt,
\end{aligned}$$

or more simply,

$$\int_{B} \left\{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k} - \mathbf{b}\cdot\mathbf{g}\right\} d\mathbf{x}dt = -\int_{B\cap\Sigma} \{n_t[\mathbf{u}] + \sum_{k=1}^{p} n_k[\mathbf{f}^{(k)}]\}\cdot\mathbf{g}dS. \tag{3.2.36}$$

The volume integral on the left hand side of the above equality (3.2.36) is dependent on the size of ϵ in the interval $(t_M - \epsilon, t_M + \epsilon)$ since it may be expressed as

$$\int_{t_M-\epsilon}^{t_M+\epsilon} \int_{B_{x_M}} \left\{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t} + \sum_{k=1}^{p} \mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k} - \mathbf{b}\cdot\mathbf{g}\right\} d\mathbf{x}dt.$$

Since the surface integral is independent of ϵ (except for perhaps small values for which the cylinder B would not contain Σ), both sides of the equation must be equal to the constant value

$$\lim_{\epsilon\to 0}\int_{t_M-\epsilon}^{t_M+\epsilon}\int_{B_{x_M}}\left\{\mathbf{u}\cdot\frac{\partial \mathbf{g}}{\partial t}+\sum_{k=1}^{p}\mathbf{f}^{(k)}\cdot\frac{\partial \mathbf{g}}{\partial x_k}-\mathbf{b}\cdot\mathbf{g}\right\}d\mathbf{x}dt=0,$$

since the integrand, although discontinuous, is assumed to be bounded in the domain B. Using the arbitrary choice of the function $\mathbf{g}$, it follows that the integrand of the remaining surface integral in (3.2.36) is zero. Therefore, at the original point M, the result (3.2.34),

$$n_t[\mathbf{u}]+\sum_{k=1}^{p}n_k[\mathbf{f}^{(k)}]=\mathbf{0}$$

is shown. □

This proof has been generalized to allow for the spatial dependence and forcing terms. A proof for the simpler case may be found in Godlewski and Raviart p.16. or done as an exercise (John, 1982 exercise 5, p.19).

A special case of the above lemma is given when the surface of discontinuity Σ has a normal vector of the form

$$\mathbf{n}=\begin{bmatrix}-v\\ n_1\\ \vdots\\ n_p\end{bmatrix},n_1^2+\ldots+n_p^2>0.$$

Then, equation (3.2.6) may be written as

$$v[\mathbf{u}]=\sum_{k=1}^{p}n_k[\mathbf{f}^{(k)}]. \tag{3.2.37}$$

Here, if $\mathbf{n}$ is a unit vector, the scalar v and vector $n_1,\ldots,n_p)^T$ may be considered as the speed and direction of propagation of the discontinuity Σ (Godlewski and Raviart, 1996 p.18). This result thus can be seen to recover property (3.2.5) for hyperbolic equations in one dimension.

3.2.2 Conditions across discontinuities in derivatives of $\mathbf{u}$

The general approach in the previous subsection, concerning discontinuities in $\mathbf{u}$, is to use the equations written in conservative form to derive jump conditions across such discontinuities. Of a somewhat different nature are conditions which arise from considering solutions for which the u_i, $i = 1, ..., n$, are continuous, but the partial derivatives $\frac{\partial u_i}{\partial x_k}$ or $\frac{\partial u_i}{\partial t}$ may have simple jump discontinuities. The following results have been collected together by Whitham (1974), and may be summed up by the statement that such discontinuities in the first partial derivatives of $\mathbf{u}$ can occur only on characteristics.

In general, for a hyperbolic system such as (3.1.7), let $\Sigma(\mathbf{x}, t) = 0$ be a surface around which $\frac{\partial u_i}{\partial t}$, $\frac{\partial u_i}{\partial x_k}$, $k = 1, ..., p$ may be piecewise continuous but u_i are continuous. By embedding the surface $\Sigma = 0$ in a family of surfaces Σ =constant, the surface Σ itself can be chosen as a local co-ordinate, and the discontinuities in $\frac{\partial u_i}{\partial x_k}$, or $\frac{\partial u_i}{\partial t}$, which are normal to Σ are expressed as (Whitham, 1974 p.140)

$$\left(\frac{\partial \Sigma}{\partial t}\mathbf{I}_n + \sum_{k=1}^{p} \mathbf{A}^{(k)} \frac{\partial \Sigma}{\partial x_k}\right) \left[\frac{\partial \mathbf{u}}{\partial \Sigma}\right] = 0. \qquad (3.2.38)$$

The discontinuities in the derivatives of u_i occur only on surfaces satisfying

$$\det\left(\frac{\partial \Sigma}{\partial t}\mathbf{I}_n + \sum_{k=1}^{p} \mathbf{A}^{(k)} \frac{\partial \Sigma}{\partial x_k}\right) = 0. \qquad (3.2.39)$$

Equation (3.2.39) is a condition which identifies a surface Σ = constant as a *characteristic surface*, and is a result quoted in Whitham (1974, p. 141).

In one spatial dimension, ($p = 1$), the simplification of (3.2.39) is often expressed simply as

$$\det\left(\frac{\partial \Sigma}{\partial t}\mathbf{I}_n + \mathbf{A}\frac{\partial \Sigma}{\partial x}\right) = 0. \qquad (3.2.40)$$

This condition (3.2.40) is similar to the characteristic condition (3.1.4), so the result that discontinuities in the first derivatives of $\mathbf{u}$ can occur only on characteristics becomes clear (see Whitham, 1974 p.128). This idea is extremely useful when considering the technique of expansion near a wavefront (Whitham, 1974 p.130) for describing discontinuous derivative solutions.

3.3 Initial Boundary Value Problems

When solving problems involving hyperbolic systems such as (3.1.1) in one spatial variable, or (3.1.7) in several (p) variables, knowledge of the hyperbolicity of the coefficient matrices $\mathbf{A}^{(k)}$, $k = 1, \ldots, p$, is not enough to describe the solutions effectively. Various types of initial or boundary information must often be supplied to discern a reasonable solution from a family of prospects. To ensure that a system and associated initial or boundary conditions is sensibly stated, the problem is said to be ***well-posed*** if there exists a unique solution such that small perturbations in the initial data (or forcing term $\mathbf{b}$) do not lead to large variations in the solution, for at least a small time interval. . This concept of well-posedness has been stated with more precision, for example by Kreiss and Lorenz (1989, p.19), but such a development is unnecessary for this thesis and is omitted.

The boundary conditions associated with a hyperbolic system which most often leads to results concerning well-posedness are initial values. This problem is generally referred to as either a *Cauchy Problem* (John, 1982 p.56), or an *Initial Value Problem* (Godlewski and Raviart, 1996 p.2) abbreviated as IVP. A tremendous amount of research has been conducted for initial value problems, with arguably the most famed result being the Cauchy-Kowalevski Theorem (John, 1982 p.74) for the existence of real analytic solutions. Examples of Cauchy problems in one spatial dimension were portrayed in Section 3.2, examples 1-4.

For the physical problem of shallow-water gravity currents which is examined in this thesis, the systems involved often have not only initial values, but also boundary conditions. For example, a physical barrier such as a wall which does not permit flow across it, can be portrayed as a constraint on the solution to a hyperbolic system of partial differential equations. For this reason, an equation with boundary conditions is termed a *Boundary Value Problem*, or BVP. The initial and boundary values may be combined to create additional complexity as an IBVP: *Initial Boundary Value Problem.* It will be shown that the gravity current problem considered herein is an IBVP.

The relevant question to this thesis is then: when is the gravity current problem well-posed? More precisely, this may be phrased as a problem of examining the subsequent IBVP to determine if the initial and boundary conditions arising from the physical problem lead to a well-posed problem. Unfortunately, a complete answer to this question is elusive at this time. However a partial answer is achieved, based on the linear theory provided in Godlewski and Raviart (1996, Chapter V) and Kreiss and Lorenz (1989). For the general IBVP, results may be obtained using the methods of linearization and localization, as discussed in Kreiss and Lorenz (1989, p.20/21). The first notion is stated as follows:

Linearization Principle: A nonlinear problem is well-posed at $\mathbf{u}$ if the linear problems which are obtained by linearizing at all functions near $\mathbf{u}$ are well-posed. This principle is most applicable to coefficient matrices of the form $\mathbf{A}^{(k)}(\mathbf{u})$.

For more general problems with coefficient matrices of the form $\mathbf{A}^{(k)}(x,t)$, a linear problem is related to a constant-coefficient problem by considering the fixed points (x_0, t_0) and freezing the coefficients $\mathbf{A}^{(k)}(x_0, t_0)$. This process is called localization, and the problem is related to the variable-coefficient problem by the following:

Localization Principle: If all frozen (constant) coefficient problems are well-posed, then the corresponding variable coefficient problem is also well-posed.

It should be noted that the linearization and localization principles are not strong results or theorems, but merely methods which turn out to be applicable for hyperbolic equations based on the Navier-Stokes equations (Kreiss and Lorenz, 1989 p.21). Using these principles, it is assumed that for the general problem to be well posed, it is necessary that the constant-coefficient linear problem be well-posed. Results are therefore compiled for these cases.

The applicable result is for the system

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{A}(x,t)\frac{\partial}{\partial x}\mathbf{u} = \mathbf{b}(x,t), \quad 0 < x.0 < t, \tag{3.3.1}$$

with initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad x \geq 0. \tag{3.3.2}$$

At the boundary point $x = 0$, boundary conditions of the form

$$\mathbf{E}(t)\mathbf{u}(0,t) = \mathbf{g}(t), \quad t \geq 0, \tag{3.3.3}$$

are considered where the matrix $\mathbf{E}$ is a $m \times n$ matrix, and $\mathbf{g}$ is a m component vector function.

Consistency of (3.3.2) and (3.3.3) require that

$$\mathbf{E}(0)\mathbf{u}_0(0) = \mathbf{g}(0), \tag{3.3.4}$$

a compatibility condition which is hereafter assumed.

Rather than simply quoting the special case of a more complete result (Kreiss and Lorenz, 1989 p.259 Theorem 7.6.4) concerning the types of matrices $\mathbf{E}$ for which equations (3.3.1)-(3.3.4) are well-posed, it is useful to conduct some calculations. The following analysis is more general than that given in Godlewski and Raviart (1996, p.424-426) for constant-coefficient homogeneous problems, and serves to illuminate the types of boundary conditions (3.3.3) for which the IBVP considered in this thesis makes sense.

The hyperbolicity of system (3.3.1) guarantees that the matrix $\mathbf{A}$ can be diagonalized by the change of basis matrix (see Norman, 1995 for example) $\mathbf{P}$, consisting of the eigenvectors of $\mathbf{A}$, such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}, \quad \mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \lambda_j = \lambda_j(x,t). \tag{3.3.5}$$

The eigenvalues in (3.3.5) are ordered such that $\lambda_1, \ldots, \lambda_p$ are positive, and the remaining eigenvalues, $\lambda_{p+1}, \ldots, \lambda_n$ are nonpositive for some integer $1 \leq p \leq n$ and for all time $t > 0$. The case for which the eigenvalues change sign is not considered, for if this occurs, then the results are only considered valid for small time intervals in which the signs of the eigenvalues do not change.

New (characteristic) variables $\widetilde{\mathbf{u}}$ are defined by

$$\widetilde{\mathbf{u}}(x,t) = \mathbf{P}^{-1}(x,t)\mathbf{u}(x,t). \tag{3.3.6}$$

Such a transformation (3.3.6) transforms equation (3.3.1) to the diagonal form

$$\frac{\partial}{\partial t}\widetilde{\mathbf{u}} + \mathbf{D}\frac{\partial}{\partial x}\widetilde{\mathbf{u}} = \widetilde{\mathbf{b}} + \mathbf{C}(x,t)\widetilde{\mathbf{u}}, \tag{3.3.7}$$

where the vector $\widetilde{\mathbf{b}}$ and matrix $\mathbf{C}(x,t)$ are detailed in Appendix 4 along with the calculation of equation (3.3.7). Similarly, the initial condition (3.3.2) becomes

$$\widetilde{\mathbf{u}}(x,0) = \mathbf{P}^{-1}(x,0)\mathbf{u}_0(x), \quad x > 0, \tag{3.3.8}$$

and the boundary condition (3.3.3) may be written as

$$\mathbf{E}(t)\mathbf{P}(0,t)\widetilde{\mathbf{u}}(0,t) = \mathbf{g}(t), \quad t \geq 0. \tag{3.3.9}$$

Now, split the vector $\widetilde{\mathbf{u}}$ into two parts defined by

$$\widetilde{\mathbf{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_p, \widetilde{u}_{p+1}, \ldots, \widetilde{u}_n)^T = (\widetilde{\mathbf{u}}_+, \widetilde{\mathbf{u}}_-)^T, \tag{3.3.10}$$

where $\tilde{\mathbf{u}}_+$ is a p component vector corresponding to the positive eigenvalues, and $\tilde{\mathbf{u}}_-$ is a $(n-p)$ component vector corresponding to the nonpositive eigenvalues of $\mathbf{A}$. At $x = 0$, $\tilde{\mathbf{u}}_-$ is entirely described by integrating the $n-p$ parts of (3.3.7)-(3.3.8) along those characteristics which initiate from the initial data. Hence, $\tilde{\mathbf{u}}_-(0,t)$ is assumed to be known.

With definition (3.3.10), the boundary condition (3.3.9) becomes

$$\begin{aligned}\mathbf{g}(t) &= \mathbf{E}(t)\mathbf{P}(0,t)\left(\tilde{\mathbf{u}}_+(0,t), \tilde{\mathbf{u}}_-(0,t)\right)^T \\ &= (\mathbf{EP})_+\tilde{\mathbf{u}}_+(0,t) + (\mathbf{EP})_-\tilde{\mathbf{u}}_-(0,t).\end{aligned} \tag{3.3.11}$$

In the last line of (3.3.11), the arguments have been supressed for the two new matrices introduced. $(\mathbf{EP})_+$ is an $m \times p$ matrix which consists of the first p columns of $\mathbf{EP}$, and $(\mathbf{EP})_-$ is an $m \times (n-p)$ matrix consisting of the last $n-p$ columns of $\mathbf{EP}$. The splitting of the matrix $\mathbf{EP}$ is written explicitly in Appendix 4, equations (A4.5). Rewriting equation (3.3.11) as

$$(\mathbf{EP})_+\tilde{\mathbf{u}}_+(0,t) = \mathbf{g} - (\mathbf{EP})_-\tilde{\mathbf{u}}_-(0,t),$$

shows that to compute $\tilde{\mathbf{u}}_+(0,t)$ nontrivially, $(\mathbf{EP})_+$ must be a square matrix which is invertible, that is $m = p$, and $\mathbf{EP}_+$ is an invertible matrix (the result $(\mathbf{EP})_+ = \mathbf{EP}_+$ is given in Appendix 4 as equation (A4.6).) This requirement is now stated as a result which generalizes one from Godlewski and Raviart (1989, Lemma 1.1 p.426).

Lemma 3.3 The hyperbolic IBVP (3.3.3)-(3.3.3) with consistency condition (3.3.4) is well-posed if $\mathbf{E}(t)$ is a $p \times n$ matrix such that $\mathbf{EP}_+$ is invertible, where $\mathbf{P}_+$ is the $n \times p$ matrix whose columns are the eigenvectors of $\mathbf{A}$ corresponding to the positive eigenvalues.

It is useful to apply this result to a simple problem such as the one dimensional wave equation, examined in detail by John (1982, p.40-45) and in many other texts.

Example 3.6 Consider the wave equation in one spatial variable for a C^2 function $u : \mathbb{R}^2 \to \mathbb{R}$ which satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < c = \text{ constant.} \tag{3.3.12}$$

The scalar equation (3.3.12) may be expressed as a system by defining a column vector $\mathbf{u}$ as follows

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial t}. \tag{3.3.13}$$

The components of $\mathbf{u}$ satisfy

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial u_2}{\partial x},$$

and

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial u_1}{\partial x},$$

which may be written in the system form

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.3.14}$$

The eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$ in (3.3.14) can be quickly calculated (Whitham, 1974 p. 118) as

$$\lambda_1 = c, \lambda_2 = -c, \text{ and } \mathbf{v}_1 = \begin{bmatrix} 1 \\ c \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -c \end{bmatrix}, \tag{3.3.15}$$

which are ordered in the desired fashion, so that $n = 2$ and $p = 1$ since $\lambda_1 > 0$ and $\lambda_2 < 0$. The change of basis matrix P is then given along with its inverse as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ c & -c \end{bmatrix} \text{ and } \mathbf{P}^{-1} = \frac{1}{2c}\begin{bmatrix} c & 1 \\ c & -1 \end{bmatrix}. \tag{3.3.16}$$

For initial and boundary data of the form (3.3.2)-(3.3.3), with $n = 2$ and $p = 1$, Lemma 3.3 requires that the matrix $\mathbf{E}(t)$ be a 1×2 matrix such that $\mathbf{EP}_+$ is invertible. A calculation gives

$$\mathbf{EP}_+ = [\, e_{11}(t) \quad e_{12}(t) \,] \begin{bmatrix} 1 \\ c \end{bmatrix} = e_{11} + c e_{12}. \tag{3.3.17}$$

The condition for well-posedness of the hyperbolic system (3.3.14) with initial and boundary conditions (3.3.2), (3.3.3) is then expressed through the invertibility of the scalar in (3.3.17), that is

$$e_{11} + c e_{12} \neq 0. \tag{3.3.18}$$

For the special case of constant matrix $\mathbf{E}$, the compatibility condition (3.3.4) is applied as follows. If $\mathbf{u}_0 = (f_1(x), f_2(x))^T$, then (3.3.4) yields the additional relation between $\mathbf{u}_0$, $\mathbf{E}$ and g as

$$g(0) = [\, e_{11} \quad e_{12} \,] \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = e_{11} f_1(0) + e_{12} f_2(0). \tag{3.3.19}$$

The result (Lemma 3.3) essentially states that the outgoing characteristic variables are described in terms of the incoming ones and a boundary term $\mathbf{g}$. If the function $\mathbf{g}$ is somehow under or over-defining, this contradicts the solution obtained by the method of characteristics. The resulting size restrictions are phrased as a simple condition via linear algebra. A more thorough and comprehensive discussion which includes integral bounds on the solution may be found in Kreiss and Lorenz (1989, Section 7.6) and in the paper by Kreiss (1970). The study of IBVPs is ongoing, and in many circumstances quite difficult. Once a theoretical description of admissible boundary conditions can be determined, in practice, implementation depends on the problem, and '... it mostly remains a matter for the expert, whose know-how is *seldom* (emphasis added) described in detail.' (Godlewski and Raviart, 1996 p.417).

Chapter Summary

Chapter 3 has introduced several classical ideas from the theory of partial differential equations, while rephrasing or generalizing some known results with a view to their application in later chapters.

The concepts of a hyperbolic system of partial differential equations and hyperbolic system of conservation laws discussed in Section 3.1 turns out to be quite applicable to gravity currents. Discontinuous solutions are also investigated in somewhat more detail in Section 3.2, with explicit calculations of Rankine-Hugoniot jump conditions at a shock. The sensitivity of discontinuous solutions to nonhomogeneous forcing terms for several one-dimensional hyperbolic problems is portrayed through some examples, the relevance of which will become clearer throughout the thesis.

As it concerns well-posedness of IBVPs, Section 3.3 describes the principles of localization and linearization which allow statements such as Lemma 3.3 to be applied in more general settings.

Chapter 4

Two-dimensional Gravity Currents

The notation and concepts expounded in chapter **3** will be used to analyze the two-dimensional equations with a free surface from chapter 2. Specifically, the notation leading to equation (2.4.29) in the system form (3.1.1) and equation (2.4.30) in conservation form (3.1.6) are used. Questions concerning hyperbolicity, discontinuities, and well-posedness will be answered for the two-dimensional gravity currents in two layers and for the special cases such as thin layers, small density differences, or rigid lid.

4.1 Hyperbolicity

The first section of theoretical results concerning gravity currents is devoted to investigating the hyperbolicity of the equations when written in the system form (3.1.1). Determining the hyperbolicity aids in classifying the situations for which various numerical schemes are appropriate to use to obtain numerical solutions, as well as determining the type of IBVP which may be stated to obtain a well-posed problem. The most general case, that of two fluid layers, is considered first with special cases and simplifying assumptions examined subsequently.

4.1.1 Two-layer gravity currents with a free surface

Hyperbolicity of the equation (2.4.29), which is stated in the form (3.1.1), is examined where the coefficient matrix is given by

$$\mathbf{A} = \begin{bmatrix} u_1 & \gamma^{-1} & 0 & \gamma^{-1} \\ \zeta_1 & u_1 & 0 & 0 \\ 0 & \gamma^{-1} - 1 & u_2 & \gamma^{-1} \\ 0 & 0 & \zeta_2 & u_2 \end{bmatrix}. \tag{4.1.1}$$

To find the eigenvalues of $\mathbf{A}$, the methods of row reduction (see, for example, Norman, 1995 Section 5.2) are used to calculate the characteristic polynomial,

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}_4) = \det(\mathbf{B}) \tag{4.1.2}$$

with

$$\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}_4 = \begin{bmatrix} u_1 - \lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ \zeta_1 & u_1 - \lambda & 0 & 0 \\ 0 & \gamma^{-1} - 1 & u_2 - \lambda & \gamma^{-1} \\ 0 & 0 & \zeta_2 & u_2 - \lambda \end{bmatrix}. \tag{4.1.3}$$

The matrix **B** can be row reduced through several steps.

$$R_2 - \frac{\zeta_1}{u_1-\lambda}R_1 \to \begin{bmatrix} u_1-\lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ 0 & u_1-\lambda-\frac{\gamma^{-1}\zeta_1}{u_1-\lambda} & 0 & -\frac{\gamma^{-1}\zeta_1}{u_1-\lambda} \\ 0 & \gamma^{-1}-1 & u_2-\lambda & \gamma^{-1} \\ 0 & 0 & \zeta_2 & u_2-\lambda \end{bmatrix}$$
$$= \begin{bmatrix} u_1-\lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ 0 & u_1-\lambda-a & 0 & -a \\ 0 & \gamma^{-1}-1 & u_2-\lambda & \gamma^{-1} \\ 0 & 0 & \zeta_2 & u_2-\lambda \end{bmatrix}$$

where $a = \gamma^{-1}\zeta_1/(u_1-\lambda)$. Further row reduction steps give

$$R_3 - \frac{\gamma^{-1}-1}{u_1-\lambda-a}R_2 \to \begin{bmatrix} u_1-\lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ 0 & u_1-\lambda-a & 0 & -a \\ 0 & 0 & u_2-\lambda & \gamma^{-1}+\frac{(\gamma^{-1}-1)a}{u_1-\lambda-a} \\ 0 & 0 & \zeta_2 & u_2-\lambda \end{bmatrix}$$

and

$$R_4 - \frac{\zeta_2}{u_2-\lambda}R_3 \to \begin{bmatrix} u_1-\lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ 0 & u_1-\lambda-a & 0 & -a \\ 0 & 0 & u_2-\lambda & \frac{\gamma^{-1}(u_1-\lambda)-a}{u_1-\lambda-a} \\ 0 & 0 & 0 & u_2-\lambda-\frac{\gamma^{-1}\zeta_2(u_1-\lambda)-\zeta_2 a}{(u_2-\lambda)(u_1-\lambda-a)} \end{bmatrix}$$

Hence, equation (4.1.1) may be expanded to write the characteristic polynomial as

$$\begin{aligned} 0 &= (u_1-\lambda)(u_1-\lambda-a)(u_2-\lambda)\left(u_2-\lambda-\frac{\gamma^{-1}\zeta_2(u_1-\lambda)-\zeta_2 a}{(u_2-\lambda)(u_1-\lambda-a)}\right) \\ &= (u_1-\lambda)\{(u_2-\lambda)^2(u_1-\lambda-a)-\gamma^{-1}\zeta_2(u_1-\lambda)+\zeta_2 a\}. \end{aligned} \tag{4.1.4}$$

Substituting in $a = \gamma^{-1}\zeta_1/(u_1-\lambda)$ gives

$$\begin{aligned} 0 &= (u_1-\lambda)\{(u_2-\lambda)^2(u_1-\lambda-\frac{\gamma^{-1}\zeta_1}{u_1-\lambda})-\gamma^{-1}\zeta_2(u_1-\lambda)+\frac{\gamma^{-1}\zeta_2\zeta_1}{u_1-\lambda}\} \\ &= (u_2-\lambda)^2[(u_1-\lambda)^2-\gamma^{-1}\zeta_1]-(u_1-\lambda)^2\gamma^{-1}\zeta_2+\gamma^{-1}\zeta_1\zeta_2. \end{aligned}$$

Multiplying by γ^2 and completing the square allows this result to be expressed simply as

$$[\gamma(\lambda-u_2)^2-\zeta_2][\gamma(\lambda-u_1)^2-\zeta_1] = (1-\gamma)\zeta_1\zeta_2. \tag{4.1.5}$$

Equation (4.1.5) is a quartic equation in the variable λ, and as such may be solved exactly. The equation may be rewritten after a few algebraic steps which follow.

$$[\gamma(\lambda^2 - 2u_2\lambda + u_2^2) - \zeta_2][\gamma(\lambda^2 - 2u_1\lambda + u_1^2) - \zeta_1] = (1-\gamma)\zeta_1\zeta_2$$

$$\Rightarrow \gamma^2(\lambda^2 - 2u_2\lambda + u_2^2)(\lambda^2 - 2u_1\lambda + u_1^2) - \gamma\zeta_2(\lambda^2 - 2u_1\lambda + u_1^2)$$
$$- \gamma\zeta_1(\lambda^2 - 2u_2\lambda + u_2^2) + \zeta_1\zeta_2 = (1-\gamma)\zeta_1\zeta_2$$

$$\Rightarrow \gamma^2(\lambda^4 - 2u_2\lambda^3 + u_2^2\lambda^2 - 2u_1\lambda^3 + 4u_1u_2\lambda^2 - 2u_1u_2^2\lambda + u_1^2\lambda^2 - 2u_1^2u_2\lambda + u_1^2u_2^2)$$
$$- \gamma(\zeta_1 + \zeta_2)\lambda^2 + 2\gamma(u_1\zeta_2 + u_2\zeta_1)\lambda - \gamma(u_1^2\zeta_2 + u_2^2\zeta_1) = -\gamma\zeta_1\zeta_2$$

$$\Rightarrow \gamma^2\lambda^4 - 2\gamma^2(u_1 + u_2)\lambda^3 + [\gamma^2(u_2^2 + 4u_1u_2 + u_1^2) - \gamma(\zeta_1 + \zeta_2)]\lambda^2$$
$$+ [\gamma^2(-2u_1u_2^2 - 2u_1^2u_2) + 2\gamma(u_1\zeta_2 + u_2\zeta_1)]\lambda + \gamma^2u_1^2u_2^2 - \gamma(u_2^2\zeta_1 + u_1^2\zeta_2 - \zeta_1\zeta_2) = 0$$

$$\Rightarrow \lambda^4 - 2(u_1 + u_2)\lambda^3 + [(u_1 + u_2)^2 + 2u_1u_2 - \gamma^{-1}(\zeta_1 + \zeta_2)]\lambda^2$$
$$+ 2[\gamma^{-1}(u_1\zeta_2 + u_2\zeta_1) - u_1u_2(u_1 + u_2)]\lambda + u_1^2u_2^2 - \gamma^{-1}(u_1^2\zeta_2 + u_2^2\zeta_1 - \zeta_1\zeta_2) = 0. \tag{4.1.6}$$

After a change of notation, equation (4.1.6) can be observed to be similar to one that was stated previously, (Montgomery & Moodie 1999a), and its roots have been discussed therein with a condition for hyperbolicity described.

Another approach which yields a simpler solution is obtained through the change of variable $\lambda = \eta + (u_1 + u_2)/2$ directly in equation (4.1.5) to get a reduced form. Details are shown in Appendix 3, where the roots of equation (4.1.6) are also described. The result parallels the exposition given in Montgomery & Moodie (1999a) and does not provide any new results. As such, the formulation in Appendix 3 merely provides a convenient outline for use in numerical calculations.

Although it will be used in subsequent chapters, equation (4.1.6) does not always have four distinct roots, and therefore equation (2.4.29) may not always be hyperbolic. For example, if $\zeta_1 = \zeta_2 = \zeta$, and $u_1 = -u_2 = u$, then (4.1.6) reduces to

$$\lambda^4 + 2(u^2 - \gamma^{-1}\zeta)\lambda^2 + u^4 - \gamma^{-1}(u^2 - \zeta)\zeta = 0.$$

This equation has imaginary roots if

$$(u^2 - \gamma^{-1}\zeta)^2 - [u^4 - \gamma^{-1}(u^2 - \zeta)\zeta] < 0,$$

which simplifies for $\zeta > 0$ to the restriction

$$u^2 > (\gamma^{-1} - 1)\zeta.$$

Thus, for large flow velocities, the hyperbolicity assumption is expected to become invalid. However, for gravity currents resulting from initial releases, the speeds are usually slow enough for the equation to be classified as strictly hyperbolic.

The various regions of the variables for which a similar system is hyperbolic has been examined (Lawrence, 1990) in terms of various Stability Froude Numbers for each layer. There, after some simplifying assumptions, a single critical Froude Number for the entire flow was defined for which a simple criterion for hyperbolicity or stability may be quickly observed from a single stability curve (Lawrence, 1990 Fig. 2). Here, the level of detail needed for implementation of boundary conditions to be investigated later requires a more general approach to be taken. Of specific interest is in the region near a vertical end wall, where the horizontal velocities u_1 and u_2 are constrained to be small in magnitude in the vicinity of the wall due to the boundary condition of zero flow at $x = 0$. This physical constraint motivates solutions of the characteristic equation (4.1.6) which may be found from a small parameter (ε) expansion of the form

$$\begin{aligned}
u_1(x,t) &= \varepsilon u_1^{(1)}(x,t) + \varepsilon^2 u_1^{(2)}(x,t) + O(\varepsilon^3),\\
u_2(x,t) &= \varepsilon u_2^{(1)}(x,t) + \varepsilon^2 u_2^{(2)}(x,t) + O(\varepsilon^3),\\
\zeta_1 &= \zeta_1^{(0)}(x,t) + \varepsilon\zeta_1^{(1)}(x,t) + \varepsilon^2\zeta_1^{(2)}(x,t) + O(\varepsilon^3),\\
\zeta_2 &= \zeta_2^{(0)}(x,t) + \varepsilon\zeta_2^{(1)}(x,t) + \varepsilon^2\zeta_2^{(2)}(x,t) + O(\varepsilon^3), \text{ and}\\
\lambda &= \lambda^{(0)}(x,t) + \varepsilon\lambda^{(1)}(x,t) + \varepsilon^2\lambda^{(2)}(x,t) + O(\varepsilon^3).
\end{aligned} \tag{4.1.7}$$

Substitution of the expansion (4.1.7) is completed in Appendix 3, with the second order solution given by the expressions (A3.10), (A3.11), (A3.13) and (A3.15). The result is that to first order, there are four distinct eigenvalues to the matrix (4.1.1) for small velocities u_1 and u_2 and $\gamma < 0.5$. These are given by

$$\lambda_i^{(0)} = \pm\left[\frac{1}{2}\gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)}) \pm \frac{1}{2}\gamma^{-1}\sqrt{(\zeta_1^{(0)} + \zeta_2^{(0)})^2 - 4\gamma\zeta_1^{(0)}\zeta_2^{(0)}}\right]^{\frac{1}{2}} + O(\varepsilon), \tag{4.1.8}$$

where the indices i in (4.1.8) correspond to the signs being chosen as $i = 1$ for $(+,+)$, $i = 2$ for $(+,-)$, $i = 3$ for $(-,-)$, and $i = 4$ for $(-,+)$ so that the eigenvalues are numbered in decreasing order as $\lambda_1 > \lambda_2 > 0 > \lambda_3 > \lambda_4$.

The eigenvectors, $\mathbf{v}$, of the matrix $\mathbf{A}$ given by (4.1.1) satisfy the expression $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for each of the eigenvalues λ_i. Expanding $\mathbf{A}$ as $\mathbf{A} = \mathbf{A}^{(0)} + \varepsilon\mathbf{A}^{(1)} + O(\varepsilon^2)$ and λ according to (4.1.7) gives

$$\left(\mathbf{A}^{(0)} - \lambda^{(0)}\mathbf{I}_4\right)\mathbf{v}^{(0)} = \mathbf{0} + O(\varepsilon). \tag{4.1.9}$$

Neglecting the superscripts (0) for simplicity, the first order equation from (4.1.9) becomes

$$\begin{bmatrix} -\lambda & \gamma^{-1} & 0 & \gamma^{-1} \\ \zeta_1 & -\lambda & 0 & 0 \\ 0 & \gamma^{-1}-1 & -\lambda & \gamma^{-1} \\ 0 & 0 & \zeta_2 & -\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.1.10}$$

For any eigenvalue λ, an eigenvector $\mathbf{v} = (v_1, v_2, v_3, v_4)^T$ must satisfy from (4.1.10) the following three equations, with v_4 considered as a parameter:

$$\begin{aligned} v_3 &= \frac{\lambda}{\zeta_2} v_4, \\ v_2 &= \frac{\lambda v_3 - \gamma^{-1} v_4}{\gamma^{-1} - 1} = \frac{\gamma\lambda^2 - \zeta_2}{(1-\gamma)\zeta_2} v_4, \text{ and} \\ v_1 &= \frac{\lambda}{\zeta_1} v_2 = \frac{(\gamma\lambda^2 - \zeta_2)\lambda}{(1-\gamma)\zeta_1\zeta_2} v_4. \end{aligned} \tag{4.1.11}$$

Choosing v_4 appropriately (i.e. nonzero) in (4.1.11) leads to a general expression for each eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ. One such family of linear independent eigenvectors may therefore be given by

$$\mathbf{v} = \left((\gamma\lambda^2 - \zeta_2)\lambda, (\gamma\lambda^2 - \zeta_2)\zeta_1, (1-\gamma)\lambda\zeta_1, (1-\gamma)\zeta_1\zeta_2\right)^T. \tag{4.1.12}$$

The two-layer equations with a free surface do not admit a simple form for regions of hyperbolicity. However, the low-flow rate expansions are useful in both classifying the equations as hyperbolic, and finding the eigenvalues and eigenvectors as appropriate.

4.1.2 Two-layer gravity currents with simplifying assumptions

The case described in subsection 4.1.1 above is somewhat limited in that the complexity of the characteristic equation (4.1.6) does not permit a simple answer to the question of hyperbolicity for the two-layer equations (2.4.29). Fortunately, the simplified systems described in Section 2.4.3 allow a more comprehensive and complete analysis. Previous results concerning several of some of the simplified two-layer equations can be found elsewhere (Montgomery & Moodie 1998a), however the more general cases are considered herein.

The first simplified equations considered are the weak-stratification equations for situations with small γ. Equation (2.4.38) contains the 2×2 system whose coefficient matrix is written after the substitution $\zeta_2 = h_2 - h_B$ as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ \zeta_2 & u_2 \end{bmatrix}. \tag{4.1.13}$$

The coefficients of $\mathbf{A}$ in (4.1.13) then simplify after the change of notation into

$$\begin{aligned} a_{11} &= \frac{2Q\zeta_2 + [(1-\zeta_2-h_B)^2 - (1+\zeta_2-h_B)\zeta_2]u_2}{(1-h_B-\zeta_2)(1-h_B)} \\ &= \frac{2Q\zeta_2 + [(1-2\zeta_2-2h_B+2\zeta_2 h_B+\zeta_2^2+h_B^2) - (\zeta_2+\zeta_2^2-\zeta_2 h_B)]u_2}{(1-h_B-\zeta_2)(1-h_B)} \\ &= \frac{2Q\zeta_2 + [(1-h_B)^2 - 3\zeta_2 + 3h_B\zeta_2]u_2}{(1-h_B-\zeta_2)(1-h_B)} \\ &= \frac{2Q\zeta_2 + (1-h_B-3\zeta_2)(1-h_B)u_2}{(1-h_B-\zeta_2)(1-h_B)}, \end{aligned} \tag{4.1.14}$$

and

$$\begin{aligned} a_{12} &= \big[2Q(1-h_B-\zeta_2)u_2 - (1-h_B+\zeta_2)(1-h_B-\zeta_2)u_2^2 + (1-h_B-\zeta_2)^3 \\ &\quad - (Q-\zeta_2 u_2)^2\big] / \big[(1-h_B-\zeta_2)^2(1-h_B)\big] \\ &= \big[2Q(1-h_B-\zeta_2)u_2 - (1-2h_B-\zeta_2^2+h_B^2)u_2^2 + (1-h_B-\zeta_2)^3 \\ &\quad - (Q^2 - 2Q\zeta_2 u_2 + \zeta_2^2 u_2^2)\big] / \big[(1-h_B-\zeta_2)^2(1-h_B)\big] \\ &= \frac{-(1-h_B)^2 u_2^2 + 2Q(1-h_B)u_2 + (1-h_B-\zeta_2)^3 - Q^2}{(1-h_B-\zeta_2)^2(1-h_B)}. \end{aligned} \tag{4.1.15}$$

The hyperbolicity of the weak-stratification equations (2.4.38) may now be addressed by solving the characteristic equation for the matrix (4.1.13), $0 = \det(\mathbf{A} - \lambda\mathbf{I}_2)$. Such a calculation yields

$$0 = (a_{11} - \lambda)(u_2 - \lambda) - a_{12}\zeta_2,$$

or

$$\lambda^2 - (u_2 + a_{11})\lambda + a_{11}u_2 - a_{12}\zeta_2 = 0. \tag{4.1.16}$$

Prior to solving the quadratic equation (4.1.16), the two coefficients are simplified. First, the coefficient of λ, upon use of equation (4.1.14) is given by

$$\begin{aligned} u_2 + a_{11} &= \frac{(1-h_B-\zeta_2)(1-h_B)u_2 + 2Q\zeta_2 + (1-h_B-3\zeta_2)(1-h_B)u_2}{(1-h_B-\zeta_2)(1-h_B)} \\ &= \frac{(1-h_B)(1-\zeta_2-h_B+1-3\zeta_2-h_B)u_2 + 2Q\zeta_2}{(1-h_B-\zeta_2)(1-h_B)} \\ &= \frac{2(1-h_B)(1-h_B-2\zeta_2)u_2 + 2Q\zeta_2}{(1-h_B-\zeta_2)(1-h_B)} \end{aligned} \qquad (4.1.17)$$

The other coefficient in (4.1.16) may be calculated similarly as

$$\begin{aligned} a_{11}u_2 - a_{12}\zeta_2 &= \frac{2Q\zeta_2 u_2 + (1-h_B-3\zeta_2)(1-h_B)u_2^2}{(1-h_B-\zeta_2)(1-h_B)} \\ &\quad - \left[\frac{-(1-h_B)^2\zeta_2 u_2^2 + 2Q(1-h_B)\zeta_2 u_2 + (1-h_B-\zeta_2)^3\zeta_2 - Q^2\zeta_2}{(1-h_B-\zeta_2)^2(1-h_B)}\right] \\ &= \big[2Q\zeta_2(1-h_B-\zeta_2)u_2 + (1-h_B-3\zeta_2)(1-h_B-\zeta_2)(1-h_B)u_2^2 \\ &\quad + (1-h_B)^2\zeta_2 u_2^2 - 2Q(1-h_B)\zeta_2 u_2 - (1-h_B-\zeta_2)^3\zeta_2 \\ &\quad + Q^2\zeta_2\big] / \big[(1-h_B-\zeta_2)^2(1-h_B)\big] \\ &= \big[[(1-h_B-3\zeta_2)(1-h_B-\zeta_2) + (1-h_B)\zeta_2](1-h_B)u_2^2 + [1-h_B-\zeta_2 \\ &\quad - (1-h_B)]2Q\zeta_2 u_2 - [(1-h_B-\zeta_2)^3 - Q^2]\zeta_2\big] / \big[(1-h_B-\zeta_2)^2(1-h_B)\big], \end{aligned}$$

which simplifies finally to

$$\begin{aligned} a_{11}u_2 - a_{12}\zeta_2 &= \frac{[(1-h_B)^2 - 3(1-h_B-\zeta_2)\zeta_2](1-h_B)u_2^2 - 2Q\zeta_2^2 u_2}{(1-h_B-\zeta_2)^2(1-h_B)} - \\ &\qquad - \frac{[(1-h_B-\zeta_2)^3 - Q^2]\zeta_2}{(1-h_B-\zeta_2)^2(1-h_B)}. \end{aligned} \qquad (4.1.18)$$

The two eigenvalues for (4.1.13) can now be expressed as the roots of (4.1.16) through expressions (4.1.17) and (4.1.18). The result is

$$\lambda_{\pm} = \frac{1}{2}(u_2 + a_{11}) \pm \sqrt{\frac{1}{4}(u_2 + a_{11})^2 - (a_{11}u_2 - a_{12}\zeta_2)}. \qquad (4.1.19)$$

In keeping with the decreasing ordering, the eigenvalues are given the subscripts $\lambda_1 = \lambda_+$ and $\lambda_2 = \lambda_-$, so that $\lambda_1 > \lambda_2$ when the discriminant in (4.1.19 is positive. The weak-stratification equations (2.4.38) are therefore classified as strictly hyperbolic whenever the term beneath the square root in (4.1.19) is positive. This occurs precisely when

$$(u_2 + a_{11})^2 - 4(a_{11}u_2 - a_{12}\zeta_2) > 0. \tag{4.1.20}$$

Such an inequality as (4.1.20) may be employed with expressions (4.1.17) and (4.1.18) to be restated as a restriction on the numerator since the denominator of (4.1.20), which may be seen to be $(1 - h_B - \zeta_2)^2(1 - h_B)^2$, is assumed to be strictly positive. This results in the inequality,

$$\begin{aligned}&[2(1 - h_B)(1 - h_B - 2\zeta_2)u_2 + 2Q\zeta_2]^2 - 4(1 - h_B)\{[(1 - h_B)^2\\ &\quad - 3(1 - h_B - \zeta_2)\zeta_2](1 - h_B)u_2^2 - 2Q\zeta_2^2 u_2 - [(1 - h_B - \zeta_2)^3 - Q^2]\zeta_2\} > 0,\end{aligned}$$

which simplifies by dividing by 4 and collecting terms in powers of u_2 to give

$$\begin{aligned}&(1 - h_B)^2[(1 - h_B - 2\zeta_2)^2 - (1 - h_B)^2 + 3(1 - h_B - \zeta_2)\zeta_2]u_2^2 + 2Q(1 - h_B)\zeta_2^2]u_2\\ &+ [2Q(1 - h_B)(1 - h_B - 2\zeta_2)\zeta_2 + Q^2\zeta_2^2 + [(1 - h_B - \zeta_2)^3 - Q^2](1 - h_B)\zeta_2 > 0.\end{aligned}$$

After further algebra, this expression may be expressed in the simpler form

$$\begin{aligned}&-(1 - h_B)^2(1 - h_B - \zeta_2)\zeta_2 u_2^2 + 2Q(1 - h_B)(1 - h_B - \zeta_2)\zeta_2 u_2\\ &\qquad + Q^2\zeta_2^2 + [(1 - h_B - \zeta_2)^3 - Q^2](1 - h_B)\zeta_2 > 0. \end{aligned}\tag{4.1.21}$$

This condition of hyperbolicity (4.1.21) depends on u_2, ζ_2, h_B, and Q, and is therefore not simple to interpret algebraically. Thus, (4.1.21) remains a quick check on the hyperbolicity of equation (2.4.38). A much simpler result arises from the case without any endflow, $Q = 0$, for which the condition (4.1.21) simplifies significantly to the equation

$$u_2^2 < \frac{(1 - h_B - \zeta_2)^2}{1 - h_B}. \tag{4.1.22}$$

This condition (4.1.22) simplifies to that obtained by Montgomery and Moodie (1998a) for the case $h_B = 0$.

To aid in a physical interpretation of the condition (4.1.22), it may be expressed dimensionally by replacing the nondimensional variables with the dimensional ones from (2.3.7). After some quick simplification, the result is given by the dimensional inequality

$$u_2^2 < \frac{g'(H - h_2)^2}{H - h_B}. \tag{4.1.23}$$

Condition (4.1.23) may now be interpreted as the physical statement that the system is strictly hyperbolic for flow with velocity below a Froude Number which depends on both the height of the lower layer and bottom depth.

The eigenvectors $\mathbf{v} = (v_1, v_2)^T$ for the matrix (4.1.13) satisfy the second row equation,

$$\zeta_2 v_1 + (u_2 - \lambda)v_2 = 0.$$

An appropriate choice of parameter allows the two eigenvectors for the weak stratification equations (2.4.38) to be expressed as

$$\mathbf{v}_{1,2} = (\lambda_{1,2} - u_2, \zeta_2)\,. \tag{4.1.24}$$

In Section 2.4.4, a thin layer analysis for the equations for two layer flow in two dimensions was completed. For a thin lower layer, equation (2.4.61) has a 2×2 coefficient matrix, given by

$$\mathbf{A} = \begin{bmatrix} \widetilde{u}_2 & 1 \\ \widetilde{\zeta}_2 & \widetilde{u}_2 \end{bmatrix}. \tag{4.1.25}$$

The eigenvalues for this matrix (4.1.25) may be found by solving the characteristic equation

$$(\widetilde{u}_2 - \lambda)^2 - \widetilde{\zeta}_2 = 0,$$

to give two eigenvalues, ordered as

$$\lambda_1 = \widetilde{u}_2 + \sqrt{\widetilde{\zeta}_2}, \text{ and } \lambda_2 = \widetilde{u}_2 - \sqrt{\widetilde{\zeta}_2}. \tag{4.1.26}$$

The eigenvectors to (4.1.25) associated with the eigenvalues (4.1.26) are vectors $\mathbf{v} = (v_1, v_2)^T$ which satisfy the equation

$$(\widetilde{u}_2 - \lambda)v_1 + v_2 = 0.$$

An appropriate choice of eigenvectors may therefore be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{\widetilde{\zeta}_2} \end{bmatrix}, \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{\widetilde{\zeta}_2} \end{bmatrix}. \tag{4.1.27}$$

From the expression (4.1.26), it can be seen that the thin lower layer equations are strictly hyperbolic precisely when $\widetilde{\zeta}_2 > 0$.

For the case with a thin upper layer, equation (2.4.81) has a similar coefficient matrix to (4.1.25) with subscript 2 replaced by the subscript 1. By comparison with the results (4.1.25)-(4.1.27), it may therefore be observed that the thin upper layer equations are strictly hyperbolic precisely when $\widetilde{\zeta}_1 > 0$. The eigenvalues for the thin upper layer equations are thus given by

$$\lambda_1 = \widetilde{u}_1 + \sqrt{\widetilde{\zeta}_1}, \text{ and } \lambda_2 = \widetilde{u}_1 - \sqrt{\widetilde{\zeta}_1}, \tag{4.1.28}$$

with the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{\widetilde{\zeta}_1} \end{bmatrix}, \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{\widetilde{\zeta}_1} \end{bmatrix}. \tag{4.1.29}$$

The last simplifying case to be considered is that of the rigid lid equations (2.4.89). This is also a 2×2 system, with coefficient matrix $\mathbf{A}$ defined by (4.1.13) with the two coefficients stated as

$$a_{11} = \frac{2(1-\gamma)Q\zeta_2 + [\zeta_1^2 - (1-\gamma)(\zeta_1 + 2\zeta_2)\zeta_2]u_2}{\zeta_1[\zeta_1 + (1-\gamma)\zeta_2]}, \tag{4.1.30}$$

and

$$\begin{aligned} a_{12} &= \frac{\zeta_1^3 + (1-\gamma)\zeta_1 u_2[2Q - (\zeta_1 + 2\zeta_2)]u_2 - (1-\gamma)(Q - \zeta_2 u_2)^2}{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]} \\ &= \Big\{\zeta_1^3 + 2Q(1-\gamma)\zeta_1 u_2 - (1-\gamma)(\zeta_1 + 2\zeta_2)\zeta_1 u_2^2 - \\ &\qquad - (1-\gamma)(Q^2 - 2Q\zeta_2 u_2 + \zeta_2^2 u_2^2)\Big\} \Big/ \Big\{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]\Big\} \\ &= \frac{\zeta_1^3 - (1-\gamma)Q^2 + 2Q(1-\gamma)(\zeta_1 + \zeta_2)u_2 - (1-\gamma)(\zeta_1 + \zeta_2)^2 u_2^2}{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]}. \end{aligned} \tag{4.1.31}$$

In equations (4.1.30) and (4.1.31), the notation $\zeta_1 = h_1 - h_2$ and $\zeta_2 = h_2 - h_B$ has been used, leading to a different, yet similar, set of coefficients than (4.1.14) and (4.1.15). The hyperbolicity of the rigid lid equations with coefficient matrix given by (4.1.13), (4.1.30)-(4.1.31) may therefore be found by solving a similar characteristic equation to (4.1.16). The two coefficients in this case are slightly different than (4.1.17) and (4.1.18) and must be recalculated as

$$\begin{aligned} u_2 + a_{11} &= \frac{\zeta_1[\zeta_1 + (1-\gamma)\zeta_2]u_2 + 2(1-\gamma)Q\zeta_2 + [\zeta_1^2 - (1-\gamma)(\zeta_1 + 2\zeta_2)\zeta_2]u_2}{\zeta_1[\zeta_1 + (1-\gamma)\zeta_2)]} \\ &= \frac{2(1-\gamma)Q\zeta_2 + [2\zeta_1^2 + (1-\gamma)(\zeta_1\zeta_2 - \zeta_1\zeta_2 - 2\zeta_2^2)]u_2}{\zeta_1[\zeta_1 + (1-\gamma)\zeta_2]} \\ &= \frac{2(1-\gamma)Q\zeta_2 + 2[\zeta_1^2 - (1-\gamma)\zeta_2^2]u_2}{\zeta_1[\zeta_1 + (1-\gamma)\zeta_2]}, \end{aligned} \tag{4.1.32}$$

and

$$a_{11}u_2 - a_{12}\zeta_2 = \frac{\{2(1-\gamma)Q\zeta_2 + [\zeta_1^2 - (1-\gamma)(\zeta_1 + 2\zeta_2)\zeta_2]u_2\}\,\zeta_1 u_2}{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]}$$
$$- \frac{[\zeta_1^3 - (1-\gamma)Q^2 + 2Q(1-\gamma)(\zeta_1+\zeta_2)u_2 - (1-\gamma)(\zeta_1+\zeta_2)^2u_2^2]\zeta_2}{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]}$$
$$= \{[(1-\gamma)Q^2 - \zeta_1^3]\zeta_2 + [2Q(1-\gamma)\zeta_1\zeta_2 - 2Q(1-\gamma)(\zeta_1+\zeta_2)\zeta_2]u_2 +$$
$$+ [\zeta_1^3 - (1-\gamma)(\zeta_1+2\zeta_2)\zeta_1\zeta_2 + (1-\gamma)(\zeta_1+\zeta_2)^2\zeta_2]u_2^2\}/\{\zeta_1^2[\zeta_1+(1-\gamma)\zeta_2]\}$$
$$= \frac{[(1-\gamma)Q^2 - \zeta_1^3]\zeta_2 - 2Q(1-\gamma)\zeta_2^2u_2 + [\zeta_1^3 + (1-\gamma)\zeta_2^3]u_2^2}{\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]}. \qquad (4.1.33)$$

The eigenvalues for the rigid lid system (2.4.89) are then given by (4.1.19), the same expressions as the weak-stratification results, with the expressions (4.1.32) and (4.1.33) inserted as appropriate. For two distinct and real eigenvalues, the accompanying inequality (4.1.20) must therefore hold. Since the denominator in (4.1.20) becomes $\zeta_1^2[\zeta_1 + (1-\gamma)\zeta_2]^2$, which is assumed to be strictly positive, the resulting inequality which much be satisfied for hyperbolicity is then given by

$$\{(1-\gamma)Q\zeta_2 + [\zeta_1^2 - (1-\gamma)\zeta_2^2]u_2\}^2 - \{(1-\gamma)Q^2\zeta_2 - \zeta_1^3\zeta_2 - 2Q(1-\gamma)\zeta_2^2u_2$$
$$+ [\zeta_1^3 + (1-\gamma)\zeta_2^3]u_2^2\}[\zeta_1 + (1-\gamma)\zeta_2] > 0.$$

Expansion and multiplying of the above allows this inequality to be written as

$$(1-\gamma)^2Q^2\zeta_2^2 + 2(1-\gamma)Q\zeta_2[\zeta_1^2 - (1-\gamma)\zeta_2^2]u_2 + [\zeta_1^2 - (1-\gamma)\zeta_2^2]^2u_2^2 - [(1-\gamma)Q^2 - \zeta_1^3][\zeta_1$$
$$+ (1-\gamma)\zeta_2]\zeta_2 + 2Q(1-\gamma)\zeta_2^2[\zeta_1 + (1-\gamma)\zeta_2]u_2 - [\zeta_1^3 + (1-\gamma)\zeta_2^3][\zeta_1 + (1-\gamma)\zeta_2]u_2^2 > 0.$$

A series of further manipulations and rearrangement yields

$$\{\zeta_1^4 - 2(1-\gamma)\zeta_1^2\zeta_2^2 + (1-\gamma)^2\zeta_2^4 - \zeta_1^4 - (1-\gamma)\zeta_1\zeta_2^3 - (1-\gamma)\zeta_1^3\zeta_2 -$$
$$- (1-\gamma)^2\zeta_2^4\}u_2^2 + 2(1-\gamma)Q\zeta_2\{\zeta_1^2 - (1-\gamma)\zeta_2^2 + \zeta_2\zeta_1 + (1-\gamma)\zeta_2^2]\}u_2$$
$$+ (1-\gamma)^2Q^2\zeta_2^2 - [(1-\gamma)Q^2\zeta_1 + (1-\gamma)^2Q^2\zeta_2 - \zeta_1^4 - (1-\gamma)\zeta_1^3\zeta_2]\zeta_2 > 0,$$

which becomes

$$(1-\gamma)(-2\zeta_1^2\zeta_2^2 - \zeta_1\zeta_2^3 - \zeta_1^3\zeta_2)u_2^2 + 2(1-\gamma)Q\zeta_2\zeta_1(\zeta_1 + \zeta_2)u_2$$
$$+ (1-\gamma)(\zeta_1^3\zeta_2 - Q^2\zeta_1)\zeta_2 + \zeta_1^4\zeta_2 > 0,$$

and finally, since the product of the layer thicknesses, $\zeta_1\zeta_2$, is assumed to be positive,

$$(1-\gamma)(\zeta_1+\zeta_2)^2u_2^2-2(1-\gamma)Q(\zeta_1+\zeta_2)u_2-\zeta_1^3+(1-\gamma)(Q^2-\zeta_1^2\zeta_2)<0. \quad (4.1.34)$$

The rigid lid equations are therefore hyperbolic if the condition (4.1.34) is satisfied, and the eigenvalues and eigenvectors are given by (4.1.19) and (4.1.24) respectively, employing the expressions (4.1.32) and (4.1.33).

To further explore the suitability of the condition (4.1.34), for the special case $Q = 0$, (4.1.34) can be rewritten as

$$u_2^2 < \frac{\zeta_1^3+(1-\gamma)\zeta_1^2\zeta_2}{(1-\gamma)(\zeta_1+\zeta_2)^2}. \quad (4.1.35)$$

This expression simplifies for $\gamma \to 0$ to

$$u_2^2 < \frac{\zeta_1^3+\zeta_1^2\zeta_2}{(\zeta_1+\zeta_2)^2} = \frac{\zeta_1^2}{\zeta_1+\zeta_2}. \quad (4.1.36)$$

Substitution of the variables $\zeta_2 = h_2 - h_B$ and $\zeta_1 = 1 - \zeta_2 - h_B$ (i.e. for rigid lid condition $h_1 = 1$) allows (4.1.36) to be seen as identical to the simplified hyperbolicity condition for the weak stratification equations, (4.1.22).

The inequality (4.1.36) is nondimensional, and can be interpreted dimensionally. Using the variables in the nondimensionalisation (2.3.7), a bit of simplification yields the inequality in dimensional form,

$$u_2^2 < \frac{g'(H-h_2)^2}{H-h_B}. \quad (4.1.37)$$

Expression (4.1.37) is identical to the previous dimensional condition (4.1.23) which characterised hyperbolicity for the weak stratification equations. for the

4.2 Jump Conditions Across Discontinuities

In this section, discontinuous solutions of the two-dimensional gravity current problem are considered. The governing equations written in the various conservation forms from Chapter 2 are examined using the theoretical results developed in Chapter 3. Discontinuities are considered solely as simple jumps in the layer height variables ζ_2 and velocity u_2. The fact that gravity current fronts are not vertical has no bearing on the analysis, only interpretation of validity of the results.

4.2.1 Discontinuities in **u** *for the general two-layer situation*

The specific discontinuous situation which motivates the analysis in this section is that of the gravity current problem for a dense lower layer intruding into a body of lighter fluid, which may not necessarily be at rest. Without loss of generality, the assumption is made that the fluid is moving from left to right, so that for a discontinuity at position $x = s(t)$, s is increasing monotonically, and the lower layer variables for $x > s(t)$ must satisfy $u_2 = 0$ and $\zeta_2 = 0$ in this domain.

The equations for two layers in conservative system form (2.4.30) are considered, with a simple discontinuity in the variables u_1, ζ_1, u_2, and ζ_2. The system is written as the four equations,

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_1^2 + \gamma^{-1}(\zeta_1 + \zeta_2)\right) = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x}, \tag{4.2.1}$$

$$\frac{\partial \zeta_1}{\partial t} + \frac{\partial}{\partial x}(\zeta_1 u_1) = 0, \tag{4.2.2}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)\zeta_1 + \gamma^{-1}\zeta_2\right) = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2}T, \tag{4.2.3}$$

and

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) = 0. \tag{4.2.4}$$

Prior to implementing any jump conditions at $s(t)$, the equations to the right of the discontinuity are considered. Substituting $\zeta = 0$ and $u_2 = 0$ into equation (4.2.3) yields

$$(\gamma^{-1} - 1)\frac{\partial \zeta_1}{\partial x} = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x}, \tag{4.2.5}$$

which may be substituted back into the right hand side of (4.2.1) to yield only two equations which are satisfied in the domain $x > s(t)$. These are the simple one-layer shallow water equations, (4.2.2) and

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_1^2 + \zeta_1\right) = 0. \tag{4.2.6}$$

Considering this, equations (4.2.1)-(4.2.4) may now be assumed to be hyperbolic on both sides of the discontinuity $x = s(t)$. (The fact that the single-layer shallow water equations are hyperbolic follows from comparison with the thin layer models discussed in Section 4.1.)

Equations (4.2.1)-(4.2.4) are of the type (3.2.1), and therefore satisfy the jump conditions (3.2.5). To employ (3.2.5) the notation used in Section 3.2 is used, namely the jump notation $[u] = u^+ - u^-$ for any variable u, where $u^+ = \lim_{x \to s^+} u(x,t)$, and u^- represents the similar left-handed limit. Using this, the jump conditions for (4.2.1)-(4.2.4) become

$$\frac{ds}{dt}[u_1] = \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}(\zeta_1^+ - \zeta_1^- + \zeta_2^+ - \zeta_2^-), \quad (4.2.7)$$

$$\frac{ds}{dt}[\zeta_1] = \zeta_1^+ u_1^+ - \zeta_1^- u_1^-, \quad (4.2.8)$$

$$\frac{ds}{dt}[u_2] = \frac{1}{2}(u_2^+)^2 - \frac{1}{2}(u_2^-)^2 + (\gamma^{-1} - 1)(\zeta_1^+ - \zeta_1^-) + \gamma^{-1}(\zeta_2^+ - \zeta_2^-), \quad (4.2.9)$$

and

$$\frac{ds}{dt}[\zeta_2] = \zeta_2^+ u_2^+ - \zeta_2^- u_2^-. \quad (4.2.10)$$

Since it has been assumed that $u_2^+ = 0$ and $\zeta_2^+ = 0$, a change of notation to $[u_2] = u_2^+ - u_2^- = -u_2$ and $[\zeta_2] = \zeta_2^+ - \zeta_2^- = -\zeta_2$ is employed in (4.2.7)-(4.2.10) so that they simplify somewhat to

$$\frac{ds}{dt}(u_1^+ - u_1^-) = \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}(\zeta_1^+ - \zeta_1^- - \zeta_2), \quad (4.2.11)$$

$$\frac{ds}{dt}(\zeta_1^+ - \zeta_1^-) = \zeta_1^+ u_1^+ - \zeta_1^- u_1^-, \quad (4.2.12)$$

$$u_2 \frac{ds}{dt} = \frac{1}{2}(u_2)^2 + (1 - \gamma^{-1})(\zeta_1^+ - \zeta_1^-) + \gamma^{-1}\zeta_2,, \quad (4.2.13)$$

and

$$\zeta_2 \frac{ds}{dt} = \zeta_2 u_2. \quad (4.2.14)$$

From equation (4.2.14), nontrivial values for ζ_2 immediately reveals the shock speed to be the lower layer velocity behind the shock,

$$\frac{ds}{dt} = u_2. \quad (4.2.15)$$

This result is expected since the shock may be imagined as a vertical material interface which moves at the lower layer horizontal velocity, u_2.

Employing (4.2.15) in the remaining equations (4.2.11)-(4.2.13) to remove the shock speed $\frac{ds}{dt}$ gives the following three equations in six variables:

$$u_2(u_1^+ - u_1^-) = \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}(\zeta_1^+ - \zeta_1^- - \zeta_2), \tag{4.2.16}$$

$$u_2(\zeta_1^+ - \zeta_1^-) = \zeta_1^+ u_1^+ - \zeta_1^- u_1^-, \tag{4.2.17}$$

and

$$\frac{1}{2}u_2^2 = (1 - \gamma^{-1})(\zeta_1^+ - \zeta_1^-) + \gamma^{-1}\zeta_2. \tag{4.2.18}$$

If the terms to the right of the shock, namely u_1^+ and ζ_1^+ are known, then equations (4.2.16)-(4.2.18) reduce to three equations in four unknowns which may, in principle, be solved to provide relationships for u_1^-, ζ_1^-, and u_2 in terms of ζ_2. With such an assumption the following simplification is straightforward, and is similar to a simpler calculation completed by Montgomery and Moodie (1999a).

Equation (4.2.17) may be rewritten as

$$\zeta_1^- = \frac{\zeta_1^+ u_1^+ - \zeta_1^+ u_2}{u_1^- - u_2} = \frac{\zeta_1^+(u_2 - u_1^+)}{u_2 - u_1^-}, \tag{4.2.19}$$

which, when substituted back into equations (4.2.16) and (4.2.18) for ζ_1^- yields the two equations

$$u_2(u_1^+ - u_1^-) = \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}\left(\frac{\zeta_1^+(u_1^+ - u_1^-)}{u_2 - u_1^-} - \zeta_2\right), \tag{4.2.20}$$

and

$$\frac{1}{2}u_2^2 = (1 - \gamma^{-1})\left(\frac{\zeta_1^+(u_1^+ - u_1^-)}{u_2 - u_1^-}\right) + \gamma^{-1}\zeta_2. \tag{4.2.21}$$

Rearranging (4.2.21) as

$$\frac{\zeta_1^+(u_1^+ - u_1^-)}{u_2 - u_1^-} = \frac{\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2}{1 - \gamma^{-1}} = \frac{\gamma u_2^2 - 2\zeta_2}{2(\gamma - 1)}$$

then allows this expression to be substituted into (4.2.20) where the term on the left hand side of the above equation appears. The result is then calculated to be

$$\begin{aligned} u_2(u_1^+ - u_1^-) &= \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}\left(\frac{\gamma u_2^2 - 2\zeta_2}{2(\gamma - 1)} - \zeta_2\right) \\ &= \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}\left(\frac{\gamma u_2^2 - 2\zeta_2 - 2(\gamma - 1)\zeta_2}{2(\gamma - 1)}\right) \\ &= \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \frac{u_2^2 - 2\zeta_2}{2(\gamma - 1)}, \end{aligned} \tag{4.2.22}$$

Expression (4.2.22) is then rewritten after multiplication by $2(\gamma-1)$ as

$$2(\gamma-1)u_2(u_1^+ - u_1^-) = (\gamma-1)[(u_1^+)^2 - (u_1^-)^2] + u_2^2 - 2\zeta_2,$$

which becomes

$$(\gamma-1)[2u_2(u_1^+ - u_1^-) - (u_1^+)^2 + (u_1^-)^2] = u_2^2 - 2\zeta_2,$$

or, after factorization of $(u_1^+)^2 - (u_1^-)^2 = (u_1^+ + u_1^-)(u_1^+ - u_1^-)$,

$$(1-\gamma)(-2u_2 + u_1^+ + u_1^-)(u_1^+ - u_1^-) = u_2^2 - 2\zeta_2. \tag{4.2.23}$$

A different rearrangement of (4.2.21) allows u_1^- to be written in terms of u_2 as follows. Multiplying (4.2.21) by $2\gamma(u_2 - u_1^-)$ gives

$$\gamma u_2^2(u_2 - u_1^-) = -2(1-\gamma)\zeta_1^+(u_1^+ - u_1^-) + 2\zeta_2(u_2 - u_1^-).$$

Rearranging this expression as

$$[-\gamma u_2^2 - 2(1-\gamma)\zeta_1^+ + 2\zeta_2]u_1^- = -\gamma u_2^3 - 2(1-\gamma)\zeta_1^+ u_1^+ + 2\zeta_2 u_2, \tag{4.2.24}$$

allows u_1^- to be expressed as

$$u_1^- = \frac{\gamma u_2^3 + 2(1-\gamma)\zeta_1^+ u_1^+ - 2\zeta_2 u_2}{\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2}. \tag{4.2.25}$$

Now it can be noted from (4.2.25) that

$$\begin{aligned} u_1^+ \pm u_1^- &= \frac{[\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2]u_1^+ \pm [\gamma u_2^3 + 2(1-\gamma)\zeta_1^+ u_1^+ - 2\zeta_2 u_2]}{\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2} \\ &= \frac{\pm\gamma u_2^3 + \gamma u_1^+ u_2^2 \mp 2\zeta_2 u_2 + (2\pm 2)(1-\gamma)\zeta_1^+ u_1^+ - 2\zeta_2 u_1^+}{\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2}, \end{aligned} \tag{4.2.26}$$

which allows equation (4.2.25) to be substituted into equation (4.2.23). After multiplication of the entire equation by $[\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2]^2$, (4.2.23) then becomes

$$\begin{aligned} (1-\gamma)\{ &-2u_2[\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2] + \gamma u_2^3 + \gamma u_1^+ u_2^2 - 2\zeta_2 u_2 + 4(1-\gamma)\zeta_1^+ u_1^+ \\ &- 2\zeta_2 u_1^+]\}[-\gamma u_2^3 + \gamma u_1^+ u_2^2 + 2\zeta_2 u_2 - 2\zeta_2 u_1^+] \\ &= [\gamma u_2^2 + 2(1-\gamma)\zeta_1^+ - 2\zeta_2]^2(u_2^2 - 2\zeta_2). \end{aligned} \tag{4.2.27}$$

This last equation, (4.2.27), contains only the unknowns u_2 and ζ_2 and may be written as

$$(1-\gamma)\{-\gamma u_2^3+\gamma u_1^+ u_2^2-[4(1-\gamma)\zeta_1^+-2\zeta_2]u_2+[4(1-\gamma)\zeta_1^+-2\zeta_2]u_1^+\}[-\gamma u_2^3 + \gamma u_1^+ u_2^2+2\zeta_2 u_2-2\zeta_2 u_1^+]=[\gamma u_2^2+2(1-\gamma)\zeta_1^+-2\zeta_2]^2(u_2^2-2\zeta_2),$$

or

$$(1-\gamma)\{-\gamma u_2^2(u_2-u_1^+)-[4(1-\gamma)\zeta_1^+-2\zeta_2](u_2-u_1^+)\}[-\gamma u_2^2(u_2-u_1^+) + 2\zeta_2(u_2-u_1^+)]=[\gamma u_2^2+2(1-\gamma)\zeta_1^+-2\zeta_2]^2(u_2^2-2\zeta_2),$$

which becomes

$$(1-\gamma)[\gamma u_2^2-2\zeta_2+4(1-\gamma)\zeta_1^+](\gamma u_2^2-2\zeta_2)(u_2-u_1^+)^2 = [\gamma u_2^2-2\zeta_2+2(1-\gamma)\zeta_1^+]^2(u_2^2-2\zeta_2). \quad (4.2.28)$$

Equation (4.2.28) is a sixth order polynomial in u_2 which, for most values of ζ_2, ζ_1^+, etc., will have six distinct solutions. These must be found numerically in order to relate the lower layer velocity (shock speed) to the lower layer height. Fortunately, some analytical progress may be made algebraically by considering an asymptotic expansion for the special case of small density differences.

Under the assumption that $\rho_2-\rho_1 << \rho_2$, the parameter γ defined by (2.3.15) is then much less than 1, i.e. $0<\gamma<<1$. With γ a small parameter, equation (4.2.28) may be rewritten as a polynomial in powers of γ. The result follows after a few rearrangements of (4.2.28), to expand as such a polynomial.

$$(1-\gamma)(u_2-u_1^+)^2[\gamma^2u_2^4-2\zeta_2u_2^2\gamma+4\zeta_1^+u_2^2(1-\gamma)\gamma-8\zeta_1^+\zeta_2(1-\gamma)-2u_2^2\zeta_2\gamma+4\zeta_2^2] = [u_2^4\gamma^2+2u_2^2[2(1-\gamma)\zeta_1^+-2\zeta_2]\gamma+[2(1-\gamma)\zeta_1^+-2\zeta_2]^2](u_2^2-2\zeta_2)$$

$$\Longrightarrow (1-\gamma)(u_2-u_1^+)^2[(u_2^4-4\zeta_1^+u_2^2)\gamma^2+(-2u_2^2\zeta_2+4\zeta_1^+u_2^2+8\zeta_1^+\zeta_2-2u_2^2\zeta_2)\gamma - 8\zeta_1^+\zeta_2+4\zeta_2^2]=[(u_2^4-4\zeta_1^+u_2^2)\gamma^2+4u_2^2(\zeta_1^+-\zeta_2)\gamma+4(1-\gamma)^2(\zeta_1^+)^2 - 8\zeta_1^+\zeta_2(1-\gamma)+4\zeta_2^2](u_2^2-2\zeta_2)$$

$$\Longrightarrow (1-\gamma)(u_2-u_1^+)^2[(u_2^4-4\zeta_1^+u_2^2)\gamma^2+(4\zeta_1^+u_2^2+8\zeta_1^+\zeta_2-4u_2^2\zeta_2)\gamma-8\zeta_1^+\zeta_2 + 4\zeta_2^2]=[(u_2^4-4\zeta_1^+u_2^2+4(\zeta_1^+)^2)\gamma^2+(4\zeta_1^+u_2^2-8(\zeta_1^+)^2+8\zeta_1^+\zeta_2-4u_2^2\zeta_2)\gamma + 4(\zeta_1^+)^2-8\zeta_1^+\zeta_2+4\zeta_2^2](u_2^2-2\zeta_2)$$

$$\Longrightarrow -(u_2-u_1^+)^2(u_2^4-4\zeta_1^+u_2^2)\gamma^3+\{(u_2-u_1^+)^2(u_2^4-4\zeta_1^+u_2^2)-(u_2-u_1^+)^2\times$$
$$\times(4\zeta_1^+u_2^2+8\zeta_1^+\zeta_2-4u_2^2\zeta_2)-(u_2^2-2\zeta_2)(u_2^4-4\zeta_1^+u_2^2+4(\zeta_1^+)^2)\}\gamma^2$$
$$+\{(u_2-u_1^+)^2(4\zeta_1^+u_2^2+8\zeta_1^+\zeta_2-4u_2^2\zeta_2)-(u_2-u_1^+)^2(-8\zeta_1^+\zeta_2+4\zeta_2^2)$$
$$-(u_2^2-2\zeta_2)(4\zeta_1^+u_2^2-8(\zeta_1^+)^2+8\zeta_1^+\zeta_2-4u_2^2\zeta_2)\}\gamma$$
$$+\{(u_2-u_1^+)^2(-8\zeta_1^+\zeta_2+4\zeta_2^2)-(u_2^2-2\zeta_2)(4(\zeta_1^+)^2-8\zeta_1^+\zeta_2+4\zeta_2^2)\}=0$$

$$\Longrightarrow -(u_2-u_1^+)^2(u_2^2-4\zeta_1^+)u_2^2\gamma^3+\{(u_2-u_1^+)^2(u_2^4-8\zeta_1^+u_2^2+4u_2^2\zeta_2-8\zeta_1^+\zeta_2)$$
$$-(u_2^2-2\zeta_2)(u_2^2-2\zeta_1^+)^2\}\gamma^2+\{(u_2-u_1^+)^2(4\zeta_1^+u_2^2-4u_2^2\zeta_2+16\zeta_1^+\zeta_2-4\zeta_2^2)$$
$$-(u_2^2-2\zeta_2)(4\zeta_1^+u_2^2-4u_2^2\zeta_2+8\zeta_1^+\zeta_2-8(\zeta_1^+)^2)\}\gamma$$
$$+\{(u_2-u_1^+)^2(\zeta_2-2\zeta_1^+)4\zeta_2-(u_2^2-2\zeta_2)(2\zeta_1^+-2\zeta_2)^2\}=0. \qquad (4.2.29)$$

Neglecting terms which are $O(\gamma^2)$ in equation (4.2.29) simplifies this equation to

$$0=(u_2-u_1^+)^2(\zeta_2-2\zeta_1^+)\zeta_2-(u_2^2-2\zeta_2)(\zeta_1^+-\zeta_2)^2+\big[(u_2-u_1^+)^2(\zeta_1^+u_2^2-\zeta_2u_2^2+4\zeta_1^+\zeta_2$$
$$-\zeta_2^2)-(u_2^2-2\zeta_2)(\zeta_1^+u_2^2-\zeta_2u_2^2+2\zeta_1^+\zeta_2-2(\zeta_1^+)^2)\big]\gamma+O(\gamma^2). \quad (4.2.30)$$

Now, an expansion solution about small γ may be utilized, and is considered in the form

$$u_2=u_2^{(0)}+\gamma u_2^{(1)}+O(\gamma^2). \qquad (4.2.31)$$

Substitution of the expansion (4.2.31) into equation (4.2.30) yields

$$0=[{u_2^{(0)}}^2+2\gamma u_2^{(0)}u_2^{(1)}-2(u_2^{(0)}+\gamma u_2^{(1)})u_1^++(u_1^+)^2](\zeta_2-2\zeta_1^+)\zeta_2-({u_2^{(0)}}^2$$
$$+2\gamma u_2^{(0)}u_2^{(1)}-2\zeta_2)(\zeta_1^+-\zeta_2)^2+\big[({u_2^{(0)}}^2-2u_2^{(0)}u_1^+-(u_1^+)^2)(\zeta_1^+u_2^{(0)}$$
$$+\zeta_2{u_2^{(0)}}^2\quad+4\zeta_1^+\zeta_2-\zeta_2^2)-({u_2^{(0)}}^2-2\zeta_2)(\zeta_1^+{u_2^{(0)}}^2-\zeta_2{u_2^{(0)}}^2$$
$$+2\zeta_1^+\zeta_2-2(\zeta_1^+)^2\big]\gamma+O(\gamma^2),$$

which simplifies to

$$0=({u_2^{(0)}}^2-2u_2^{(0)}u_1^++(u_1^+)^2)(\zeta_2-2\zeta_1^+)\zeta_2-({u_2^{(0)}}^2-2\zeta_2)(\zeta_1^+-\zeta_2)^2$$
$$+\gamma\{2(u_2^{(0)}u_2^{(1)}-u_2^{(1)}u_1^+)(\zeta_2-2\zeta_1^+)\zeta_2-2u_2^{(0)}u_2^{(1)}(\zeta_1^+-\zeta_2)^2$$
$$+({u_2^{(0)}}^2-2u_2^{(0)}u_1^+-(u_1^+)^2)(\zeta_1^+u_2^{(0)}+\zeta_2{u_2^{(0)}}^2+4\zeta_1^+\zeta_2-\zeta_2^2)$$
$$-({u_2^{(0)}}^2-2\zeta_2)(\zeta_1^+{u_2^{(0)}}^2-\zeta_2{u_2^{(0)}}^2+2\zeta_1^+\zeta_2-2(\zeta_1^+)^2\}\gamma+O(\gamma^2). \qquad (4.2.32)$$

The $O(1)$ problem from (4.2.32) is then given by the first line of the above, which is

$$(u_2^{(0)^2} - 2u_2^{(0)}u_1^+ + (u_1^+)^2)(\zeta_2^2 - 2\zeta_1^+\zeta_2) - (u_2^{(0)^2} - 2\zeta_2)[(\zeta_1^+)^2 - 2\zeta_1^+\zeta_2 + \zeta_2^2] = 0. \tag{4.2.33}$$

This equation (4.2.33) may be reorganized so that it may be written as a quadratic equation for $u_2^{(0)}$. This is completed through the following steps:

$$(\zeta_2^2 - 2\zeta_1^+\zeta_2 - (\zeta_1^+)^2 + 2\zeta_1^+\zeta_2 - \zeta_2^2)u_2^{(0)^2} + (-2u_1^+\zeta_2^2 + 4u_1^+\zeta_1^+\zeta_2)u_2^{(0)}$$
$$+ (u_1^+)^2\zeta_2^2 - 2(u_1^+)^2\zeta_1^+\zeta_2 + 2(\zeta_1^+)^2\zeta_2 - 4\zeta_1^+\zeta_2^2 + 2\zeta_2^3 = 0,$$

which becomes

$$-(\zeta_1^+)^2u_2^{(0)^2} - 2u_1^+\zeta_2(\zeta_2 - 2\zeta_1^+)u_2^{(0)} + [(u_1^+)^2\zeta_2 - 2(u_1^+)^2\zeta_1^+ + 2(\zeta_1^+)^2$$
$$- 4\zeta_1^+\zeta_2 + 2\zeta_2^2]\zeta_2 = 0,$$

and finally

$$(\zeta_1^+)^2u_2^{(0)^2} + 2(\zeta_2 - 2\zeta_1^+)u_1^+\zeta_2u_2^{(0)} - [(u_1^+)^2(\zeta_2 - 2\zeta_1^+) + 2(\zeta_1^+ - \zeta_2)^2]\zeta_2 = 0. \tag{4.2.34}$$

Equation (4.2.34) now solves for $u_2^{(0)}$ via the quadratic formula as

$$u_2^{(0)} = \frac{u_1^+\zeta_2(2\zeta_1^+ - \zeta_2)}{(\zeta_1^+)^2}$$
$$\pm\sqrt{\frac{(u_1^+)^2\zeta_2^2(\zeta_2 - 2\zeta_1^+)^2}{(\zeta_1^+)^4} + \frac{\zeta_2[(u_1^+)^2(\zeta_2 - 2\zeta_1^+) + 2(\zeta_1^+ - \zeta_2)^2]}{(\zeta_1^+)^2}},$$

which simplifies to

$$u_2^{(0)} = \frac{1}{(\zeta_1^+)^2}\Big\{u_1^+\zeta_2(2\zeta_1^+ - \zeta_2)$$
$$\pm\sqrt{(u_1^+)^2\zeta_2^2(\zeta_2 - 2\zeta_1^+)^2 + (\zeta_1^+)^2\zeta_2[(u_1^+)^2(\zeta_2 - 2\zeta_1^+) + 2(\zeta_1^+ - \zeta_2)^2]}\Big\}. \tag{4.2.35}$$

To choose the correct sign in (4.2.35), the physical limit of an almost undisturbed upper layer to the right of the discontinuity is considered, that is, $\zeta_1^+ \approx 1 - h_B$ and $u_1^+ \approx 0$. For such a special limit, the solution (4.2.35) becomes

$$u_2^{(0)} = \frac{1}{(1-h_B)^2}\left\{0 \pm \sqrt{0 + (1-h_B)^2\zeta_2[0 + 2(1 - h_B - \zeta_2)^2]}\right\}$$
$$= \pm\frac{1 - h_B - \zeta_2}{1 - h_B}\sqrt{2\zeta_2}. \tag{4.2.36}$$

Since the physically acceptable root in (4.2.36) corresponds to the choice of the + sign, this choice is imposed on the solution (4.2.35) to yield a single asymptotic solution $u_2^{(0)}$.

The first order correction solution to $u_2^{(0)}$ is given by solving the $O(\gamma)$ problem from (4.2.32) which is given by

$$\begin{aligned} 2(u_2^{(0)} - u_1^+)(\zeta_2 - 2\zeta_1^+)\zeta_2 u_2^{(1)} - 2u_2^{(0)}(\zeta_1^+ - \zeta_2)^2 u_2^{(1)} &= ({u_2^{(0)}}^2 - 2\zeta_2)(\zeta_1^+ {u_2^{(0)}}^2 \\ - \zeta_2 {u_2^{(0)}}^2 + 2\zeta_1^+\zeta_2 - 2(\zeta_1^+)^2) - ({u_2^{(0)}}^2 - 2u_2^{(0)}u_1^+ - (u_1^+)^2)(\zeta_1^+ u_2^{(0)} \\ + \zeta_2 {u_2^{(0)}}^2 + 4\zeta_1^+\zeta_2 - \zeta_2^2). \end{aligned} \tag{4.2.37}$$

This equation (4.2.37) provides a single solution $u_2^{(1)}$ once $u_2^{(0)}$ is specified by the positive root in (4.2.35). The first correction term to u_2 in (4.2.31) is thus given by

$$\begin{aligned} u_2^{(1)} = \{ &({u_2^{(0)}}^2 - 2\zeta_2)(\zeta_1^+ {u_2^{(0)}}^2 - \zeta_2 {u_2^{(0)}}^2 + 2\zeta_1^+\zeta_2 - 2(\zeta_1^+)^2) - ({u_2^{(0)}}^2 - 2u_2^{(0)}u_1^+ \\ &- (u_1^+)^2)(\zeta_1^+ u_2^{(0)} + \zeta_2 {u_2^{(0)}}^2 + 4\zeta_1^+\zeta_2 - \zeta_2^2)\}/\{2(u_2^{(0)} - u_1^+)(\zeta_2 - 2\zeta_1^+)\zeta_2 \\ &- 2u_2^{(0)}(\zeta_1^+ - \zeta_2)^2\}. \end{aligned} \tag{4.2.38}$$

With the asymptotic solution (4.2.31) found, equation (4.2.25) may be used to determine an asymptotic solution to u_1^-. It is easier to employ expression (4.2.24) so that u_1^- satisfies

$$[-\gamma {u_2^{(0)}}^2 - 2(1-\gamma)\zeta_1^+ + 2\zeta_2]u_1^- = -\gamma {u_2^{(0)}}^3 - 2(1-\gamma)\zeta_1^+ u_1^+ + 2\zeta_2(u_2^{(0)} + \gamma u_2^{(1)}) + O(\gamma^2). \tag{4.2.39}$$

This expression simplifies by considering an asymptotic solution u_1^- of the equation (4.2.39) which is assumed to be of the form

$$u_1^- = u_1^{(0)} + \gamma u_1^{(1)} + O(\gamma^2). \tag{4.2.40}$$

Such a substitution into (4.2.39) yields the equation

$$\begin{aligned} (-2\zeta_1^+ + 2\zeta_2)u_1^{(0)} &+ \gamma[(-{u_2^{(0)}}^2 + 2\zeta_1^+)u_1^{(0)} + (-2\zeta_1^+ + 2\zeta_2)u_1^{(1)}] \\ &= -2\zeta_1^+ u_1^+ + 2\zeta_2 u_2^{(0)} + \gamma[-{u_2^{(0)}}^3 + 2\zeta_1^+ u_1^+ + 2\zeta_2 u_2^{(1)}] + O(\gamma^2). \end{aligned} \tag{4.2.41}$$

This equation (4.2.41) yields the $O(1)$ solution

$$u_1^{(0)} = \frac{\zeta_2 u_2^{(0)} - \zeta_1^+ u_1^+}{\zeta_2 - \zeta_1^+}, \tag{4.2.42}$$

and the $O(\gamma)$ correction

$$u_1^{(1)} = \frac{-u_2^{(0)^3} + 2\zeta_1^+ u_1^+ + 2\zeta_2 u_2^{(1)} - (-u_2^{(0)^2} + 2\zeta_1^+)u_1^{(0)}}{-2\zeta_1^+ + 2\zeta_2},$$

which yields, upon substitution of $u_1^{(0)}$ from (4.2.42),

$$\begin{aligned}
u_1^{(1)} &= \frac{[-u_2^{(0)^3} + 2\zeta_1^+ u_1^+ + 2\zeta_2 u_2^{(1)}](\zeta_2 - \zeta_1^+) + (u_2^{(0)^2} - 2\zeta_1^+)(\zeta_2 u_2^{(0)} - \zeta_1^+ u_1^+)}{2(\zeta_2 - \zeta_1^+)^2} \\
&= \{ - u_2^{(0)^3}\zeta_2 + 2\zeta_1^+ u_1^+ \zeta_2 + 2\zeta_2 u_2^{(1)}\zeta_2 + u_2^{(0)^3}\zeta_1^+ - 2(\zeta_1^+)^2 u_1^+ - 2\zeta_1^+ \zeta_2 u_2^{(1)} \\
&\quad u_2^{(0)^3}\zeta_2 - 2\zeta_1^+\zeta_2 u_2^{(0)} - u_2^{(0)^2}\zeta_1^+ u_1^+ + 2(\zeta_1^+)^2 u_1^+\}/\{2(\zeta_2 - \zeta_1^+)^2\}. \\
&= \frac{2\zeta_1^+ u_1^+ \zeta_2 - 2\zeta_1^+\zeta_2 u_2^{(0)} + u_2^{(0)^3}\zeta_1^+ - u_2^{(0)^2}\zeta_1^+ u_1^+ + 2u_2^{(1)}\zeta_2^2 - 2\zeta_1^+ u_2^{(1)}\zeta_2}{2(\zeta_2 - \zeta_1^+)^2} \\
&= \frac{2\zeta_1^+\zeta_2(u_1^+ - u_2^{(0)}) - u_2^{(0)^2}\zeta_1^+(u_1^+ - u_2^{(0)}) + 2u_2^{(1)}\zeta_2(\zeta_2 - \zeta_1^+)}{2(\zeta_2 - \zeta_1^+)^2} \\
&= \frac{(2\zeta_2 - u_2^{(0)^2})(u_1^+ - u_2^{(0)})\zeta_1^+ + 2u_2^{(1)}\zeta_2(\zeta_2 - \zeta_1^+)}{2(\zeta_2 - \zeta_1^+)^2}.
\end{aligned} \tag{4.2.43}$$

Similarly, to find ζ_1^-, equation (4.2.17) may be employed to give an expansion

$$(u_2^{(0)} + \gamma u_2^{(1)})(\zeta_1^+ - \zeta_1^-) = \zeta_1^+ u_1^+ - \zeta_1^-(u_1^{(0)} + \gamma u_1^{(1)}) + O(\gamma^2),$$

which simplifies to

$$u_2^{(0)}(\zeta_1^+ - \zeta_1^-) + \zeta_1^- u_1^{(0)} - \zeta_1^+ u_1^+ + \gamma[u_2^{(1)}(\zeta_1^+ - \zeta_1^-) - \zeta_1^- u_1^{(1)}] = O(\gamma^2). \tag{4.2.44}$$

An expansion solution may be considered of the form

$$\zeta_1^- = \zeta_1^{(0)} + \gamma\zeta_1^{(1)} + O(\gamma^2). \tag{4.2.45}$$

Substitution of (4.2.45) into equation (4.2.44) then gives

$$u_2^{(0)}(\zeta_1^+ - \zeta_1^{(0)}) - \gamma u_2^{(0)}\zeta_1^{(1)} + \zeta_1^{(0)}u_1^{(0)} + \gamma\zeta_1^{(1)}u_1^{(0)} + \gamma[u_2^{(1)}(\zeta_1^+ - \zeta_1^{(0)}) - \zeta_1^{(0)}u_1^{(1)}] = O(\gamma^2),$$

which can be expressed as

$$\zeta_1^{(0)}(u_1^{(0)} - u_2^{(0)}) + u_2^{(0)}\zeta_1^+ + \gamma[\zeta_1^{(1)}(u_1^{(0)} - u_2^{(0)}) + u_2^{(1)}(\zeta_1^+ - \zeta_1^{(0)}) - \zeta_1^{(0)}u_1^{(1)}] = O(\gamma^2). \tag{4.2.46}$$

The order solutions to (4.2.46) are then given by the $O(1)$ part,

$$\zeta_1^{(0)} = \frac{u_2^{(0)}\zeta_1^+}{u_2^{(0)} - u_1^{(0)}}, \tag{4.2.47}$$

and the first order correction,

$$\zeta_1^{(1)} = \frac{\zeta_1^{(0)}u_1^{(1)} - u_2^{(1)}(\zeta_1^+ - \zeta_1^{(0)})}{u_1^{(0)} - u_2^{(0)}}. \tag{4.2.48}$$

Substituting the solution (4.2.47) for $\zeta_1^{(0)}$ into the above $O(\gamma)$ solution (4.2.48) yields a simplified version of this solution as:

$$\begin{aligned}
\zeta_1^{(1)} &= \frac{u_2^{(0)}\zeta_1^+u_1^{(1)} - u_2^{(1)}\zeta_1^+(u_2^{(0)} - u_1^{(0)}) + u_2^{(1)}u_2^{(0)}\zeta_1^+}{-(u_1^{(0)} - u_2^{(0)})^2} \\
&= \frac{[u_2^{(0)}u_1^{(1)} - u_2^{(1)}u_2^{(0)} + u_2^{(1)}u_1^{(0)} + u_2^{(1)}u_2^{(0)}]\zeta_1^+}{-(u_1^{(0)} - u_2^{(0)})^2} \\
&= \frac{(u_2^{(0)}u_1^{(1)} + u_2^{(1)}u_1^{(0)})\zeta_1^+}{-(u_1^{(0)} - u_2^{(0)})^2}.
\end{aligned} \tag{4.2.49}$$

Equations (4.2.31) for u_2^-, (4.2.40) for u_1^-, and (4.2.45) for ζ_1^-, now provide a solution expanded about the small parameter γ for the conditions (4.2.16)-(4.2.18) about a discontinuity. Each of these allows the limit of the solution on the left-hand side of the discontinuity to be expressed in terms of the solution to the right and the height of the lower layer on the left, $\zeta_2^- = \zeta_2$.

4.2.2 Discontinuities in **u** *for a lower layer intrusion into a quiescent upper layer*

For a lower layer moving into a quiescent fluid, the type of discontinuous solution to be considered is more specific than that used in Section 4.2.1. Here, to the right of a discontinuity again denoted by $x = s(t)$, the desired solution is

$$u_2 = 0, \quad \zeta_2 = 0, \quad u_1 = 0, \text{ and } \zeta_1 = 1 - h_B. \tag{4.2.50}$$

A difficulty quickly arises when equations (4.2.1)-(4.2.4) are used, since the desired solution (4.2.50) does not satisfy equation (4.2.3), which results in

$$-(\gamma^{-1} - 1)\frac{\mathrm{d}h_B}{\mathrm{d}x} = -\gamma^{-1}\frac{\mathrm{d}h_B}{\mathrm{d}x},$$

or, more simply,

$$\frac{\mathrm{d}h_B}{\mathrm{d}x} = 0. \tag{4.2.51}$$

For general topography, $h_B(x)$, the condition (4.2.51) is too restrictive to consider quiescent solutions of the form (4.2.50). For physical reasons, it is expected that (4.2.50) should be a solution of the equations of motion. Therefore, the incompatibility must lie in the governing partial differential equations themselves. It is simpler to consider in this case the equations (2.4.27) in conservation form, with the notational simplification $\zeta_2 = h_2 - h_B$ for the lower layer, but without any such notation for the upper layer. The resulting system is expressed as the four equations

$$\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_1^2 + \gamma^{-1}h_1\right) = 0, \tag{4.2.52}$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}[(h_1 - \zeta_2 - h_B)u_1 + \zeta_2 u_2] = 0, \tag{4.2.53}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1 + \zeta_2\right] = \frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T, \tag{4.2.54}$$

and

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) = 0. \tag{4.2.55}$$

Equations (4.2.52)-(4.2.55) are four partial differential equations in the four unknown variables u_1, h_1, u_2, and ζ_2.

A steady-state quiescent solution of (4.2.52)-(4.2.55) of the form (4.2.50) is written as

$$u_2 = 0, \quad \zeta_2 = 0, \quad u_1 = 0, \text{ and } h_1 = \text{ constant}, \tag{4.2.56}$$

where h_1 is usually, but not necessarily, chosen to be 1. Substitution of the desired solution (4.2.56) into equations (4.2.52)-(4.2.55) shows that the solution satisfies three of these equations (4.2.52), (4.2.53) and (4.2.55) trivially, but leaves equation (4.2.54) as a restatement of the condition (4.2.51) which restricts the topography. To modify the equations of motion so that a solution such as (4.2.56) is satisfied, the lower layer momentum equation (4.2.54) is written as

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1 + \zeta_2\right] = H(s(t) - x)\left[\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T\right]. \tag{4.2.57}$$

In equation (4.2.57) the Heaviside function H is defined by (3.2.7). Now, any solution of the form (4.2.56) satisfies the four equations of motion (4.2.52), (4.2.53), (4.2.55) and (4.2.57). Therefore, the shock conditions (3.2.11) for this system of equations are employed to determine the form of discontinuous solutions which are possible.

Using the same notation as introduced previously for equations (4.2.7)-(4.2.10), the jump conditions (3.2.11) are employed to yield the four equations

$$\frac{ds}{dt}[u_1] = \frac{1}{2}(u_1^+)^2 - \frac{1}{2}(u_1^-)^2 + \gamma^{-1}(h_1^+ - h_1^-), \tag{4.2.58}$$

$$\frac{ds}{dt}[h_1] = (h_1^+ - \zeta_2^+ - h_B^+)u_1^+ - (h_1^- - \zeta_2^- - h_B^-)u_1^- + \zeta_2^+ u_2^+ - \zeta_2^- u_2^-, \tag{4.2.59}$$

$$\frac{ds}{dt}[u_2] = \frac{1}{2}(u_2^+)^2 - \frac{1}{2}(u_2^-)^2 + (\gamma^{-1} - 1)(h_1^+ - h_1^-) + \zeta_2^+ - \zeta_2^- - h_B^-, \tag{4.2.60}$$

and

$$\frac{ds}{dt}[\zeta_2] = \zeta_2^+ u_2^+ - \zeta_2^- u_2^-. \tag{4.2.61}$$

For the solution (4.2.56) employed as the right-hand limit, the conditions (4.2.58)-(4.2.61) simplify to

$$u_1^- \frac{ds}{dt} = \frac{1}{2}(u_1^-)^2 + \gamma^{-1}(h_1^+ - h_1^-), \tag{4.2.62}$$

$$\frac{ds}{dt}(h_1^+ - h_1^-) = -(h_1^- - \zeta_2^- - h_B^-)u_1^- - \zeta_2^- u_2^-, \tag{4.2.63}$$

$$u_2^- \frac{ds}{dt} = \frac{1}{2}(u_2^-)^2 + (1 - \gamma^{-1})(h_1^+ - h_1^-) + \zeta_2^- + h_B^-, \tag{4.2.64}$$

and

$$\zeta_2^- \frac{ds}{dt} = \zeta_2^- u_2^-. \tag{4.2.65}$$

Equations (4.2.62)-(4.2.65) now represent five equations in four unknown variables, with the assumption that h_1^+ and h_B^- are known.

To employ equations (4.2.62)-(4.2.65), some simplification is necessary, and the superscript ($^-$) is omitted where obvious, although the positive sign is still used for h_1^+. The last equation, (4.2.65), yields the same result as (4.2.15) which fixes the shock speed as the lower layer horizontal velocity u_2^-. Substituting this simplification back into (4.2.62)-(4.2.64) to remove $\frac{ds}{dt}$ gives

$$u_1 u_2 = \frac{1}{2}u_1^2 + \gamma^{-1}(h_1^+ - h_1), \tag{4.2.66}$$

$$u_2(h_1 - h_1^+) = (h_1 - \zeta_2 - h_B)u_1 + \zeta_2 u_2, \tag{4.2.67}$$

and

$$u_2^2 = \frac{1}{2}u_2^2 + (1-\gamma^{-1})(h_1^+ - h_1) + \zeta_2 + h_B. \tag{4.2.68}$$

Expanding equation (4.2.67) and reorganizing the terms gives

$$u_2 h_1 - u_1 h_1 = u_2 h_1^+ - \zeta_2 u_1 - h_B u_1 + \zeta_2 u_2,$$

from which h_1 may be isolated as

$$h_1 = \frac{\zeta_2(u_2 - u_1) + u_2 h_1^+ - h_B u_1}{u_2 - u_1} = \zeta_2 + \frac{u_2 h_1^+ - h_B u_1}{u_2 - u_1}. \tag{4.2.69}$$

This result may be used to obtain the expression

$$\begin{aligned} h_1^+ - h_1 &= h_1^+ - \zeta_2 - \left(\frac{u_2 h_1^+ - h_B u_1}{u_2 - u_1}\right) \\ &= -\zeta_2 + \frac{h_1^+(u_2 - u_1) - u_2 h_1^+ + h_B u_1}{u_2 - u_1} \\ &= -\zeta_2 - \left(\frac{h_1^+ - h_B}{u_2 - u_1}\right) u_1. \end{aligned} \tag{4.2.70}$$

Substituting the result (4.2.70) back into equations (4.2.66) and (4.2.68) reduces the number of unknowns by one to yield

$$u_1 u_2 = \frac{1}{2}u_1^2 - \gamma^{-1}\left(\zeta_2 + \frac{h_1^+ - h_B}{u_2 - u_1}u_1\right), \tag{4.2.71}$$

and

$$\begin{aligned} \frac{1}{2}u_2^2 &= -(1-\gamma^{-1})\left[\zeta_2 + \frac{h_1^+ - h_B}{u_2 - u_1}u_1\right] + \zeta_2 + h_B \\ &= \frac{(\gamma^{-1} - 1)(h_1^+ - h_B)u_1}{u_2 - u_1} + \gamma^{-1}\zeta_2 + h_B. \end{aligned} \tag{4.2.72}$$

To remove u_1 from (4.2.71) and (4.2.72), the first line of (4.2.72) is rearranged as

$$\zeta_2 + \left(\frac{h_1^+ - h_B}{u_2 - u_1}\right) u_1 = \frac{\frac{1}{2}u_2^2 - \zeta_2 - h_B}{\gamma^{-1} - 1}. \tag{4.2.73}$$

This is substituted into equation (4.2.71) to yield

$$u_1 u_2 = \frac{1}{2}u_1^2 - \gamma^{-1}\left(\frac{\frac{1}{2}u_2^2 - \zeta_2 - h_B}{\gamma^{-1} - 1}\right)$$
$$= \frac{1}{2}u_1^2 - \frac{1}{1-\gamma}\left(\frac{1}{2}u_2^2 - \zeta_2 - h_B\right). \tag{4.2.74}$$

Next, the expression (4.2.72) is rewritten to solve for u_1 by multiplying by $(u_2 - u_1)$ to give

$$\frac{1}{2}u_2^2(u_2 - u_1) = (\gamma^{-1} - 1)(h_1^+ - h_B)u_1 + (\gamma^{-1}\zeta_2 + h_B)(u_2 - u_1),$$

which allows the terms to be rearranged as

$$\left[-\frac{1}{2}u_2^2 - (\gamma^{-1} - 1)(h_1^+ - h_B) + \gamma^{-1}\zeta_2 + h_B\right] u_1 = (\gamma^{-1}\zeta_2 + h_B)u_2 - \frac{1}{2}u_2^3,$$

so that the variable u_1 may be expressed as

$$u_1 = \frac{(\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2 - h_B)u_2}{\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B}. \tag{4.2.75}$$

Substituting expression (4.2.75) into (4.2.74) yields a relationship between the two unknowns, ζ_2 and u_2. Straightforward substitution gives

$$\frac{(\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2 - h_B)u_2^2}{\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B} = -\frac{1}{(1-\gamma)}\left(\frac{1}{2}u_2^2 - \zeta_2 - h_B\right)$$
$$+ \frac{1}{2}\left[\frac{(\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2 - h_B)u_2}{\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B}\right]^2,$$

which, after multiplication by a common denominator becomes

$$\left(\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2 - h_B\right)u_2^2\left(\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B\right)$$
$$= -\frac{1}{1-\gamma}\left(\frac{1}{2}u_2^2 - \zeta_2 - h_B\right)\left(\frac{1}{2}u_2^2 + (\gamma^{-1} - 1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B\right)^2$$
$$+ \frac{1}{2}\left(\frac{1}{2}u_2^2 - \gamma^{-1}\zeta_2 - h_B\right)^2 u_2^2. \tag{4.2.76}$$

The expression (4.2.76) is a cubic polynomial in u_2^2. To write such an equation in standard form, several algebraic steps are completed, which follow. First, isolating the powers of u_2^2 may be achieved by expanding (4.2.76) as

$$\frac{1}{4}u_2^6 + \left\{\frac{1}{2}(-\gamma^{-1}\zeta_2 - h_B) + \frac{1}{2}[(\gamma^{-1}-1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B]\right\} u_2^4$$
$$+(-\gamma^{-1}\zeta_2 - h_B)[(\gamma^{-1}-1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B]u_2^2 = \frac{-1}{1-\gamma}\left(\frac{1}{2}u_2^2\right.$$
$$\left. - \zeta_2 - h_B\right)\left\{\frac{1}{4}u_2^4 + [(\gamma^{-1}-1)h_1^+ - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B]u_2^2 + [(\gamma^{-1}-1)h_1^+\right.$$
$$\left. - \gamma^{-1}\zeta_2 - \gamma^{-1}h_B]^2\right\} + \frac{1}{2}\left[\frac{1}{4}u_2^4 + (-\gamma^{-1}\zeta_2 - h_B)u_2^2 + (-\gamma^{-1}\zeta_2 - h_B)^2\right]u_2^2.$$

This reorganizes further as

$$\frac{1}{4}u_2^6 + \frac{\gamma^{-1}}{2}[-\zeta_2 - \gamma h_B + (1-\gamma)h_1^+ - \zeta_2 - h_B]u_2^4 - \gamma^{-1}(\zeta_2 + \gamma h_B)\gamma^{-1}[(1-\gamma)h_1^+$$
$$-(\zeta_2 + h_B)]u_2^2 = \frac{-1}{1-\gamma}\left\{\frac{1}{8}u_2^6 - \frac{1}{4}(\zeta_2 + h_B)u_2^4 + \frac{1}{2}[(\gamma^{-1}-1)h_1^+ - \gamma^{-1}(\zeta_2\right.$$
$$+h_B)]u_2^4 - (\zeta_2 + h_B)[(\gamma^{-1}-1)h_1^+ - \gamma^{-1}(\zeta_2 + h_B)]u_2^2 + \frac{1}{2}[(\gamma^{-1}-1)h_1^+$$
$$\left. -\gamma^{-1}(\zeta_2 + h_B)]^2u_2^2 - (\zeta_2 + h_B)[(\gamma^{-1}-1)h_1^+ - \gamma^{-1}(\zeta_2 + h_B)]^2\right\} + \frac{1}{8}u_2^6$$
$$-\frac{\gamma^{-1}}{2}(\zeta_2 + \gamma h_B)u_2^4 + \frac{\gamma^{-2}}{2}(\zeta_2 + \gamma h_B)^2u_2^2,$$

or

$$\frac{1}{4}u_2^6 + \frac{1}{2}\gamma^{-1}[(1-\gamma)h_1^+ - 2\zeta_2 - (1+\gamma)h_B]u_2^4 - \gamma^{-2}(\zeta_2 + \gamma h_B)[(1-\gamma)h_1^+$$
$$-(\zeta_2 + h_B)]u_2^2 = \frac{-1}{1-\gamma}\left\{\frac{1}{8}u_2^6 + \frac{1}{4}\gamma^{-1}[2(1-\gamma)h_1^+ - (2+\gamma)(\zeta_2 + h_B)]u_2^4\right.$$
$$+\frac{1}{2}\gamma^{-2}[(1-\gamma)h_1^+ - (\zeta_2 + h_B)][(1-\gamma)h_1^+ - (\zeta_2 + h_B) - 2\gamma(\zeta_2 + h_B)]u_2^2$$
$$\left. -\gamma^{-2}(\zeta_2 + h_B)[(1-\gamma)h_1^+ - (\zeta_2 + h_B)]^2\right\} + \frac{1}{8}u_2^6 - \frac{1}{2}\gamma^{-1}(\zeta_2 + \gamma h_B)u_2^4$$
$$+\frac{1}{2}\gamma^{-2}(\zeta_2 + \gamma h_B)^2u_2^2.$$

Collecting terms in powers of u_2^2 gives the previous equation as

$$\left(\frac{1}{4}+\frac{1}{8(1-\gamma)}-\frac{1}{8}\right)u_2^6+\left\{\frac{1}{2}\gamma^{-1}[(1-\gamma)h_1^+-2\zeta_2-(1+\gamma)h_B+(\zeta_2+\gamma h_B)]\right.$$
$$\left.+\frac{\gamma^{-1}}{4(1-\gamma)}[2(1-\gamma)h_1^+-(2+\gamma)(\zeta_2+h_B)]\right\}u_2^4+\left\{-\gamma^{-2}(\zeta_2+\gamma h_B)[(1-\right.$$
$$-\gamma)h_1^+-(\zeta_2+h_B)]+\frac{\gamma^{-2}}{2(1-\gamma)}[(1-\gamma)h_1^+-(\zeta_2+h_B)][1-\gamma)h_1^+-$$
$$\left.-(1+2\gamma)(\zeta_2+h_B)]-\frac{1}{2}\gamma^{-2}(\zeta_2+\gamma h_B)^2\right\}u_2^2=$$
$$=\frac{-\gamma^{-2}}{1-\gamma}(\zeta_2+h_B)[(1-\gamma)h_1^+-(\zeta_2+h_B)]^2,$$

which simplifies to

$$\frac{2-\gamma}{8(1-\gamma)}u_2^6+\frac{\gamma^{-1}}{4(1-\gamma)}\left\{2(1-\gamma)^2h_1^+-2(1-\gamma)\zeta_2-2(1-\gamma)h_B\right.$$
$$\left.+2(1-\gamma)h_1^+-(2+\gamma)(\zeta_2+h_B)\right\}u_2^4+\frac{\gamma^{-2}}{1-\gamma}\left\{[(1-\gamma)h_1^+\right.$$
$$-(\zeta_2+h_B)][-(1-\gamma)(\zeta_2+\gamma h_B)+\frac{1}{2}(1-\gamma)h_1^+-\frac{1}{2}(1+2\gamma)(\zeta_2+h_B)]$$
$$\left.-\frac{1}{2}(1-\gamma)(\zeta_2+\gamma h_B)^2\right\}u_2^2=\frac{-\gamma^{-2}}{1-\gamma}(\zeta_2+h_B)[(1-\gamma)h_1^+-(\zeta_2+h_B)]^2.$$

Multiplying by $\gamma^2(1-\gamma)$ allows this to be then expressed as

$$\frac{1}{8}\gamma^2(2-\gamma)u_2^6+\frac{1}{4}\gamma\left\{2(1-\gamma)(2-\gamma)h_1^+-(4-\gamma)(\zeta_2+h_B)\right\}u_2^4+\left\{[(1-\gamma)h_1^+\right.$$
$$-(\zeta_2+h_B)][\frac{1}{2}(1-\gamma)h_1^+-\frac{3}{2}\zeta_2+(\gamma^2-2\gamma-\frac{1}{2})h_B]$$
$$\left.-\frac{1}{2}(1-\gamma)(\zeta_2+\gamma h_B)^2\right\}u_2^2=-(\zeta_2+h_B)[(1-\gamma)h_1^+-(\zeta_2+h_B)]^2. \tag{4.2.77}$$

Finally, equation (4.2.77) may be multiplied by 8 to allow it to be written as

$$\gamma^2(2-\gamma)u_2^6+2\gamma\left\{2(1-\gamma)(2-\gamma)h_1^+-(4-\gamma)(\zeta_2+h_B)\right\}u_2^4$$
$$+4\left\{[(1-\gamma)h_1^+-(\zeta_2+h_B)][(1-\gamma)h_1^+-3\zeta_2+(2\gamma^2-4\gamma-1)h_B]\right.$$
$$\left.-(1-\gamma)(\zeta_2+\gamma h_B)^2\right\}u_2^2=-8(\zeta_2+h_B)[(1-\gamma)h_1^+-(\zeta_2+h_B)]^2. \tag{4.2.78}$$

Equation (4.2.78) may be solved explicitly using the methods outlined in Appendix 3; however, it is useful to examine the solutions for small γ. Taking the limit of equation (4.2.78) as $\gamma \to 0$ reveals

$$4\left[(h_1^+ - \zeta_2 - h_B)(h_1^+ - 3\zeta_2 - h_B) - (\zeta_2 + h_B)^2\right] u_2^2 = -8(\zeta_2 + h_B)(h_1^+ - \zeta_2 - h_B)^2,$$

which may be solved for u_2^2 as

$$u_2^2 = \frac{2(\zeta_2 + h_B)(h_1^+ - \zeta_2 - h_B)^2}{(\zeta_2 + h_B)^2 - (h_1^+ - \zeta_2 - h_B)(h_1^+ - 3\zeta_2 - h_B)}. \tag{4.2.79}$$

The solution (4.2.79) with $h_1^+ = 1$ is similar in form to the special limit of (4.2.36) for the quiescent upper layer case. Differences between the two expressions reflect the choice of equations (4.2.7)-(4.2.10) or the modified ones (4.2.52)-(4.2.55) with the lower layer momentum equation (4.2.54) replaced by (4.2.57). Although (4.2.79) is an exact calculation of the jump condition into a quiescent layer, it is easier to compare experimental evidence to the approximate expression (4.2.36) which is simpler and is of a similar form to (4.2.79). These expressions will be compared in Chapter 6.

4.2.3 Discontinuities in **u** *for the simplified systems*

The first simplified case considered is the weak-stratification equations developed in Chapter 2. These were expressed in the conservative form (2.4.39) and are written here with $Q = 0$ and $\zeta_2 = h_2 - h_B$, employing (2.4.36) to remove u_1. The result is

$$\frac{\partial}{\partial t}\left[u_2 - \frac{-\zeta_2 u_2}{1 - \zeta_2 - h_B}\right] + \frac{\partial}{\partial x}\left[\frac{1}{2}u_2^2 + \zeta_2 + h_B - \frac{1}{2}\left(\frac{-\zeta_2 u_2}{1 - \zeta_2 - h_B}\right)^2\right] = -\kappa C_f \frac{u_2^2}{\zeta_2} T, \tag{4.2.80}$$

and

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) = 0. \tag{4.2.81}$$

Equation (4.2.80) is simplified to

$$\frac{\partial}{\partial t}\left[\frac{(1 - h_B)u_2}{1 - \zeta_2 - h_B}\right] + \frac{\partial}{\partial x}\left[\frac{[(1 - h_B)^2 - 2\zeta_2(1 - h_B)]u_2^2}{2(1 - \zeta_2 - h_B)^2} + \zeta_2 + h_B\right] = -\kappa C_f \frac{u_2^2}{\zeta_2} T, \tag{4.2.82}$$

so that the two variables u_2 and ζ_2 are described by the system (4.2.81) and (4.2.82).

To consider a discontinuous solution of equations (4.2.81) and (4.2.82), it is assumed that for $x > s(t)$, the lower layer is nonexistent so that the equations of motion admit a solution of the form

$$u_2 = 0 \text{ and } \zeta_2 = 0. \tag{4.2.83}$$

This solution clearly satisfies (4.2.81), however when substituted into the second equation (4.2.82), the result is $\dfrac{\mathrm{d}h_B}{\mathrm{d}x} = 0$, as found previously with (4.2.15). Therefore, equation (4.2.82) is restated in the form (3.2.6) by replacing the right hand side with a discontinuous forcing term to yield

$$\frac{\partial}{\partial t}\left[\frac{(1-h_B)u_2}{1-\zeta_2-h_B}\right] + \frac{\partial}{\partial x}\left[\frac{[(1-h_B)^2 - 2\zeta_2(1-h_B)]u_2^2}{2(1-\zeta_2-h_B)^2} + \zeta_2\right]$$
$$= H(s(t)-x)\left[-\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T\right]. \tag{4.2.84}$$

The shock conditions (3.2.11) may not be employed directly to the system (4.2.81) and (4.2.84) since these equations are not precisely in the form (3.2.1). However, the argument leading to the conditions (3.2.11) generalizes simply to include such cases as (4.2.84). The results are then

$$\frac{\mathrm{d}s}{\mathrm{d}t}\left[\frac{(1-h_B)u_2}{1-\zeta_2-h_B}\right] = \left[\frac{[(1-h_B)^2 - 2\zeta_2(1-h_B)]u_2^2}{2(1-\zeta_2-h_B)^2} + \zeta_2\right] + \lim_{x\to s^-}\left(-h_B(x)\right), \tag{4.2.85}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t}[\zeta_2] = [\zeta_2 u_2], \tag{4.2.86}$$

where the square brackets denote the change in the function across the discontinuity.

Imposing the solution (2.4.87) on the jump conditions (4.2.85) and (4.2.86) results in

$$-\frac{\mathrm{d}s}{\mathrm{d}t}\frac{(1-h_B^-)u_2^-}{1-\zeta_2^- - h_B^-} = -\frac{[(1-h_B^-)^2 - 2\zeta_2^-(1-h_B^-)]u_2^{-2}}{2(1-\zeta_2^- - h_B^-)^2} - \zeta_2^- - h_B^-, \tag{4.2.87}$$

and

$$-\frac{\mathrm{d}s}{\mathrm{d}t}\zeta_2^- = -\zeta_2^- u_2^-. \tag{4.2.88}$$

Simplifying the notation in (4.2.87) and (4.2.88) so that the superscripts are neglected yields the conditions

$$\frac{\mathrm{d}s}{\mathrm{d}t}\frac{(1-h_B)u_2}{1-\zeta_2-h_B}=\frac{[(1-h_B)^2-2\zeta_2(1-h_B)]u_2^2}{2(1-\zeta_2-h_B)^2}+\zeta_2+h_B, \tag{4.2.89}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t}=u_2. \tag{4.2.90}$$

Equation (4.2.90) is familiar, and allows $\frac{\mathrm{d}s}{\mathrm{d}t}$ to be removed from (4.2.89) to give a single equation in u_2 and ζ_2 as

$$\frac{(1-h_B)u_2^2}{1-\zeta_2-h_B}=\frac{[(1-h_B)^2-2\zeta_2(1-h_B)]u_2^2}{2(1-\zeta_2-h_B)^2}+\zeta_2+h_B. \tag{4.2.91}$$

Equation (4.2.91) may be solved for u_2^2 by multiplying by the denominator $2(1-\zeta_2-h_B)^2$ to yield

$$2(1-\zeta_2-h_B)(1-h_B)u_2^2=[(1-h_B)^2-2\zeta_2(1-h_B)]u_2^2+2(\zeta_2+h_B)(1-\zeta_2-h_B)^2.$$

This equation may be rewritten as

$$[2(1-\zeta_2-h_B)(1-h_B)-(1-h_B)^2+2\zeta_2(1-h_B)]u_2^2=2(\zeta_2+h_B)(1-\zeta_2-h_B)^2,$$

which simplifies to

$$(1-h_B)^2u_2^2=2(\zeta_2+h_B)(1-\zeta_2-h_B)^2,$$

then solves for u_2^2 as

$$u_2^2=\frac{2(\zeta_2+h_B)(1-\zeta_2-h_B)^2}{(1-h_B)^2}. \tag{4.2.92}$$

This expression (4.2.92) is identical to (4.2.36) except for the inclusion of h_B in (4.2.92). It should be noted that if $h_B=0$, then the square root of (4.2.92) becomes identical to (4.2.36).

The next simplified equations to be considered are those for a thin lower layer, (2.4.60). These equations were stated in different nondimensional variables given by (2.4.54) and the two partial differential equations are given by

$$\frac{\partial}{\partial\tilde{t}}\tilde{u}_2+\frac{\partial}{\partial\tilde{x}}\left(\frac{1}{2}\tilde{u}_2^2+\tilde{\zeta}_2\right)=-\frac{\mathrm{d}\tilde{h}_B}{\mathrm{d}\tilde{x}}-\kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T, \tag{4.2.93}$$

and

$$\frac{\partial\tilde{\zeta}_2}{\partial\tilde{t}}\frac{\partial}{\partial\tilde{x}}(\tilde{\zeta}_2\tilde{u}_2) = 0. \tag{4.2.94}$$

Again, a discontinuous solution is desired such that for $x > s(t)$, the lower layer has zero thickness. Such a solution is expressed as (4.2.83), which transforms through the scaling (2.4.52) as

$$\tilde{\zeta}_2 = 0, \text{ and } \tilde{u}_2 = 0. \tag{4.2.95}$$

Since this is not a solution of equation (4.2.93), as it again produces the topographical restriction (4.2.15), this equation must be restated as

$$\frac{\partial}{\partial\tilde{t}}\tilde{u}_2 + \frac{\partial}{\partial\tilde{x}}\left(\frac{1}{2}\tilde{u}_2^2 + \tilde{\zeta}_2\right) = H(\tilde{s}(\tilde{t}) - \tilde{x})\left[-\frac{d\tilde{h}_B}{d\tilde{x}} - \kappa\tilde{C}_f\frac{\tilde{u}_2^2}{\tilde{\zeta}_2}T\right]. \tag{4.2.96}$$

In equation (4.2.96), the new term $\tilde{s}$ is nondimensionalized as $\tilde{x}$ in (2.4.52).

This equation (4.2.96), along with (4.2.94) admit the solution (4.2.95), and the jump conditions (3.2.11) may be employed across the discontinuity. The result is

$$\frac{d\tilde{s}}{d\tilde{t}}[\tilde{u}_2] = \left[\frac{1}{2}\tilde{u}_2^2 + \tilde{\zeta}_2\right] - \lim_{\tilde{x}\to\tilde{s}^-}\tilde{h}_B(\tilde{x}), \tag{4.2.97}$$

and

$$\frac{d\tilde{s}}{d\tilde{t}}[\tilde{\zeta}_2] = [\tilde{\zeta}_2\tilde{u}_2]. \tag{4.2.98}$$

Substitution of the solution (4.2.95) for $\tilde{u}_2^+$ and $\tilde{\zeta}_2^+$ simplifies these two equations. Neglecting the superscript $(^-)$ then permits the jump conditions to be expressed as

$$\frac{d\tilde{s}}{d\tilde{t}}\tilde{u}_2 = \frac{1}{2}\tilde{u}_2^2 + \tilde{\zeta}_2 + \tilde{h}_B, \tag{4.2.99}$$

and

$$\frac{d\tilde{s}}{d\tilde{t}}\tilde{\zeta}_2 = \tilde{\zeta}_2\tilde{u}_2. \tag{4.2.100}$$

The shock speed (4.2.100) is the same result as found previously, and may be substituted into (4.2.99) to yield

$$\tilde{u}_2^2 = \frac{1}{2}\tilde{u}_2^2 + \tilde{\zeta}_2 + \tilde{h}_B,$$

which solves for $\tilde{u}_2^2$ as

$$\tilde{u}_2^2 = 2(\tilde{\zeta}_2 + \tilde{h}_B). \tag{4.2.101}$$

This result may be obtained from (4.2.92) by taking the limit for thin lower layer variables, $\zeta_2 << 1$ and $h_B << 1$.

In the case of a thin upper layer, the system of conservation laws derived in Chapter 2 may be read directly from equation (2.4.80) without any change of notation as

$$\frac{\partial \tilde{u}_1}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left(\frac{1}{2}\tilde{u}_1^2 + \tilde{\eta}\right) = 0, \tag{4.2.102}$$

and

$$\frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(\tilde{\eta}\tilde{u}_1) = 0. \tag{4.2.103}$$

The nondimensional variables in (2.4.102)-(4.2.103) are defined through the scaling (2.4.71).

The solution discontinuity desired for the upper layer is that for $x > s(t)$, $u_1 = 0$, and $h_1 =$ constant. From the paragraph prior to equation (2.4.67), such a condition may be rewritten in nondimensional variables as $\tilde{u}_1 = 0$, and

$$\eta = h_2 - H + (h_1 - h_2) = h_1 - H,$$

which nondimensionalizes via (2.4.71) to

$$\tilde{\eta} = \frac{\eta}{\gamma\varepsilon H} = \frac{h_1 - H}{\gamma\varepsilon H}.$$

Thus, when h_1 is constant, so is $\tilde{\eta}$, and notably $\tilde{\eta} = 0$ when $h_1 = H$.

Unlike previously for the thin lower layer, this solution satisfies equations (4.2.102) and (4.2.103) precisely, allowing the jump conditions (3.2.5) to be applied directly. This results in

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}[\tilde{u}_1] = \left[\frac{1}{2}\tilde{u}_1^2 + \tilde{\eta}\right], \tag{4.2.104}$$

and

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}[\tilde{\eta}] = [\tilde{\eta}\tilde{u}_1]. \tag{4.2.105}$$

Denoting the jump in $\tilde{\eta}$ by $[\tilde{\eta} = \tilde{\eta}^+ - \tilde{\eta}_-$ and using $\tilde{u}_1^+ = 0$ allows the conditions (4.2.104) and (4.2.105) to be expanded to

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{t}}\tilde{u}_1^- = \frac{1}{2}(\tilde{u}_1^-)^2 + \tilde{\eta}^- - \tilde{\eta}^+, \tag{4.2.106}$$

and

$$\frac{d\tilde{s}}{d\tilde{t}}(\tilde{\eta}^{+} - \tilde{\eta}^{-}) = -\tilde{\eta}^{-}\tilde{u}_1^{-}. \tag{4.2.107}$$

The shock speed may then be calculated from (4.2.107) as

$$\frac{d\tilde{s}}{d\tilde{t}} = \frac{\tilde{\eta}^{-}\tilde{u}_1^{-}}{\tilde{\eta}^{-} - \tilde{\eta}^{+}}. \tag{4.2.108}$$

Substituting (4.2.108) back into equation (4.2.106) yields

$$\left(\frac{\tilde{\eta}^{-}\tilde{u}_1^{-}}{\tilde{\eta}^{-} - \tilde{\eta}^{+}}\right)\tilde{u}_1^{-} = \frac{1}{2}(\tilde{u}_1^{-})^2 + \tilde{\eta}^{-} - \tilde{\eta}^{+},$$

which then simplifies to

$$\tilde{\eta}^{-}(\tilde{u}_1^{-})^2 - \frac{1}{2}(\tilde{\eta}^{-} - \tilde{\eta}^{+})(\tilde{u}_1^{-})^2 = (\tilde{\eta}^{-} - \tilde{\eta}^{+})(\tilde{\eta}^{-} - \tilde{\eta}^{+}).$$

Collecting terms in the above reveals a simpler equation

$$\frac{1}{2}(\tilde{\eta}^{-} + \tilde{\eta}^{+})(\tilde{u}_1^{-})^2 = (\tilde{\eta}^{-} - \tilde{\eta}^{+})^2,$$

which solves for $\tilde{u}_1^{-}$ as

$$(\tilde{u}_1^{-})^2 = \frac{2(\tilde{\eta}^{-} - \tilde{\eta}^{+})^2}{\tilde{\eta}^{-} + \tilde{\eta}^{+}}. \tag{4.2.109}$$

The special case previously mentioned of $\tilde{\eta}^{+} = 0$ reduces the relation (4.2.109) to the much simpler expression,

$$(\tilde{u}_1^{-})^2 = 2\tilde{\eta}^{-}, \tag{4.2.110}$$

which is similar in character to the previous shock speed (4.2.101) determined for the thin lower layer.

The last special case to be considered are the rigid lid equations (2.4.90) with a constant value for h_1. With the change of notation $\zeta_2 = h_2 - h_B$, the system (2.4.90) may be expressed as the following two equations:

$$\frac{\partial}{\partial t}\left[u_2 + (1-\gamma)\frac{\zeta_2 u_2}{h_1 - \zeta_2 - h_B}\right] + \frac{\partial}{\partial x}\left[\frac{1}{2}u_2^2 + \zeta_2 + h_B - \right.$$
$$\left. - \frac{(1-\gamma)}{2}\left(\frac{\zeta_2 u_2}{h_1 - \zeta_2 - h_B}\right)^2\right] = -\kappa C_f \frac{u_2^2}{\zeta_2} T. \tag{4.2.111}$$

and

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) = 0. \tag{4.2.112}$$

Some simplification of (4.2.111) allows it to be restated in the more useful form

$$\frac{\partial}{\partial t}\left[\frac{(h_1 - \gamma\zeta_2 - h_B)u_2}{h_1 - \zeta_2 - h_B}\right] + \frac{\partial}{\partial x}\left[\frac{[(h_1 - h_B)^2 - 2\zeta_2(h_1 - h_B) + \gamma\zeta_2^2]u_2^2}{2(h_1 - \zeta_2 - h_B)^2} + \zeta_2\right]$$
$$= -\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T. \tag{4.2.113}$$

It is desired to find a discontinuous solution such that the solution to the right of the discontinuity $x = s(t)$ is given by (4.2.83). As with the weak-stratification case for equation (4.2.82), this is not a solution of equation (4.2.113). Consequently, equation (4.2.113) must be replaced by the associated equation with discontinuous forcing term,

$$\frac{\partial}{\partial t}\left[\frac{(h_1 - \gamma\zeta_2 - h_B)u_2}{h_1 - \zeta_2 - h_B}\right] + \frac{\partial}{\partial x}\left[\frac{[(h_1 - h_B)^2 - 2\zeta_2(h_1 - h_B) + \gamma\zeta_2^2]u_2^2}{2(h_1 - \zeta_2 - h_B)^2} + \zeta_2\right]$$
$$= H(s(t) - x)\left[-\frac{\mathrm{d}h_B}{\mathrm{d}x} - \kappa C_f \frac{u_2^2}{\zeta_2} T\right]. \tag{4.2.114}$$

A discontinuous solution to equations (4.2.112) and (4.2.114) must satisfy the jump conditions (3.2.11). Employing this condition on (4.2.112) and (4.2.114) yields

$$\frac{\mathrm{d}s}{\mathrm{d}t}\left[\frac{(h_1 - \gamma\zeta_2 - h_B)u_2}{h_1 - \zeta_2 - h_B}\right] = \left[\frac{[(h_1 - h_B)^2 - 2\zeta_2(h_1 - h_B) + \gamma\zeta_2^2]u_2^2}{2(h_1 - \zeta_2 - h_B)^2} + \zeta_2\right]$$
$$- \lim_{x \to s^-} h_B(x), \tag{4.2.115}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t}[\zeta_2] = [\zeta_2 u_2]. \tag{4.2.116}$$

Substituting the zero-valued solution to the right of the $s(t)$ discontinuity allows (4.2.116) to simplify as the shock speed,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = u_2^-. \tag{4.2.117}$$

Using this result in the second jump condition (4.2.115) yields

$$\frac{(h_1 - \gamma\zeta_2^- - h_B^-)(u_2^-)^2}{h_1 - \zeta_2^- - h_B} = \frac{[(h_1 - h_B^-)^2 - 2\zeta_2^-(h_1 - h_B^-) + \gamma(\zeta_2^-)^2](u_2^-)^2}{2(h_1 - \zeta_2^- - h_B^-)^2} + \zeta_2^- + h_B^-. \tag{4.2.118}$$

The notation h_1^- is not used since h_1 is a constant value.

Equation (4.2.118) is expressed in the usual fashion with the suppressed superscripts, and simplified as

$$2(h_1 - \zeta_2 - h_B)(h_1 - \gamma\zeta_2 - h_B)u_2^2 = [(h_1 - h_B)^2 - 2\zeta_2(h_1 - h_B) + \gamma\zeta_2^2]u_2^2 \\ + 2(h_1 - \zeta_2 - h_B)^2(\zeta_2 + h_B).$$

Collecting the coefficient of u_2^2 in the above equation then yields

$$[2(h_1 - h_B)^2 - 2(1-\gamma)\zeta_2(h_1 - h_B) + 2\gamma\zeta_2^2 - (h_1 - h_B)^2 + 2\zeta_2(h_1 - h_B)^2 - \gamma\zeta_2^2]u_2^2 \\ = 2(\zeta_2 + h_B)(h_1 - \zeta_2 - h_B)^2,$$

which then simplifies to

$$[(h_1 - h_B)^2 + 2\gamma\zeta_2(h_1 - h_B) + \gamma\zeta_2^2]u_2^2 = 2(\zeta_2 - h_B)(h_1 - \zeta_2 - h_B)^2.$$

Finally, this expression solves for u_2^2 as

$$u_2^2 = \frac{2(\zeta_2 + h_B)(h_1 - \zeta_2 - h_B)^2}{(h_1 - h_B)^2 + 2\gamma\zeta_2(h_1 - h_B) + \gamma\zeta_2^2}. \tag{4.2.119}$$

The expression (4.2.119) reduces to (4.2.92) in the limit as $\gamma \to 0$, but is more general due to the presence of γ in the denominator.

4.3 Initial Boundary Value Problems

Assuming that the various two-dimensional gravity current problems are indeed hyperbolic, the various initial and boundary conditions are examined to determine whether or not they are appropriate. The question of well-posedness for the resulting initial boundary value problems (IBVPs) was introduced theoretically in Chapter 3.3, and the framework extended therein is employed for the specific examples of concern to this thesis.

The physically motivated problem is that of the initial release of a finite volume of dense fluid in a semi-infinite region of less dense fluid at rest. It is seen that this initial value is also accompanied by possible boundary values which restrict the horizontal flow along a vertical boundary. The general case for two-layer gravity currents with a free surface is considered first in section 4.3.1, with the remaining special cases and simplified systems discussed in the following section 4.3.2.

4.3.1 IVPs and IBVPs for the two-layer equations with a free surface

The two-layer equations with a free surface were written as a 4×4 system in Chapter 2, equation (2.4.29). For the variables, u_1, ζ_1, u_2 and ζ_2, a typical initial value for a finite volume at rest takes the form

$$\mathbf{u}(x,0) = \begin{bmatrix} u_1(x,0) \\ \zeta_1(x,0) \\ u_2(x,0) \\ \zeta_2(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \zeta_{1_0}(x) \\ 0 \\ \zeta_{2_0}(x) \end{bmatrix}. \tag{4.3.1}$$

Typically, a block profile is considered to be existent for $0 \leq x \leq 1$, since the horizontal extent may be altered via nondimensionalization. This leads to a form of the initial data in (4.3.1) as

$$\zeta_{2_0}(x) = \begin{cases} \text{constant}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.3.2}$$

where the constant in (4.3.2) is chosen so that it is between 0 and 1. This avoids the non-physical problems where the lower layer is initially thicker than the typical vertical length scale H. The initial value for the upper layer profile is chosen to match the physical character of the problem as portrayed in Figure 2.1 This is then

$$\zeta_{1_0}(x) = 1 - \zeta_{2_0}(x) - h_B(x). \tag{4.3.3}$$

The initial data (4.3.1) is noncharacteristic for the first order hyperbolic system (2.4.29), which permits the existence of a solution for at least a short time (John, p.46-51). The IVP (2.4.29) with initial values (4.3.1)-(4.3.3) is therefore properly stated. For initial release of a parcel of fluid, this initial value is then sufficient.

For the case where a vertical boundary is important, without loss of generality an impermeable barrier at the point $x = 0$ is considered across which there is no flow. This may be expressed mathematically by the boundary values

$$u_1(0,t) = 0 \text{ and } u_2(0,t) = 0 \text{ for } t > 0. \tag{4.3.4}$$

The IBVP is a half-space problem which consists of the system (2.4.29) for $x \geq 0$, the initial value (4.3.1), and the boundary value (4.3.4). To determine if this IBVP is well-posed, the method of localization described in Chapter 3.3 is used.

The boundary value (4.3.4) may be expressed in the form (3.3.3) as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(0,t) \\ \zeta_1(0,t) \\ u_2(0,t) \\ \zeta_2(0,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, t \geq 0. \tag{4.3.5}$$

The notation used in (3.3.3) is satisfied with the identification of the matrix $\mathbf{E}$ as

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

That this boundary condition satisfies the consistency condition (3.3.4) for the initial value (4.3.1) may be seen from a quick substitution.

To make use of Lemma 3.3, the eigenvalues of the matrix (4.1.1) must be considered at the point $x = 0$ where the boundary value is in effect. This calculation was completed in Section 4.1.1 where the four eigenvalues were given by formula (4.1.8). Substituting in the initial values at $x = 0$ allows these eigenvalues to be simplified as

$$\lambda_i = \pm\left[\frac{1}{2}\gamma^{-1}\left(\zeta_1(0,t) + \zeta_2(0,t) \pm \sqrt{(\zeta_1(0,t) + \zeta_2(0,t))^2 - 4\gamma\zeta_1(0,t)\zeta_2(0,t)}\right)\right]^{\frac{1}{2}},$$

which becomes

$$\lambda_i = \pm\left[\frac{1}{2}\gamma^{-1}\left(1 - h_B(0) \right.\right.$$
$$\left.\left. \pm \sqrt{(1 - h_B(0))^2 - 4\gamma(1 - \zeta_2(0,t) - h_B(0))\zeta_2(0,t)}\right)\right]^{\frac{1}{2}}.$$

Since $h_B(0) = 0$ by the choice of the system variables depicted in figure 2.1, this simplifies further to

$$\lambda_i = \pm\left[\frac{1}{2}\gamma^{-1}\left(1 \pm \sqrt{1 - 4\gamma(1 - \zeta_2(0,t))\zeta_2(0,t)}\right)\right]^{\frac{1}{2}},$$

for $i = 1, \ldots, 4$. Since $0 < \zeta_2(0,t) < 1$ and $\gamma \leq 0.25$, there are two positive eigenvalues corresponding to the choice of signs as

$$\lambda_{1,2} = \left[\frac{1}{2}\gamma^{-1}\left(1 \pm \sqrt{1 - 4\gamma(1 - \zeta_2(0,t))\zeta_2(0,t)}\right)\right]^{\frac{1}{2}}, \tag{4.3.6}$$

where the subscript 1 corresponds to the positive square root, and the subscript 2 corresponds to the negative one.

The eigenvectors corresponding to the positive eigenvalues (4.3.6) are given by equation (4.1.12). Substituting (4.3.6) into this expression yields the two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ stated together as

$$\mathbf{v}_{1,2} = \begin{bmatrix} \left(\gamma\frac{1}{2}\gamma^{-1}(1 \pm \sqrt{1 - 4\gamma(1 - \zeta_2(0,t))\zeta_2(0,t)} - \zeta_2(0,t)\right)\lambda_{1,2} \\ \left(\gamma\frac{1}{2}\gamma^{-1}(1 \pm \sqrt{1 - 4\gamma(1 - \zeta_2(0,t))\zeta_2(0,t)} - \zeta_2(0,t)\right)\zeta_1(0,t) \\ (1-\gamma)\lambda_{1,2}\zeta_1(0,t) \\ (1-\gamma)\zeta_{1_0}(0)\zeta_2(0,t) \end{bmatrix}. \tag{4.3.7}$$

Using $\zeta_1 = 1 - \zeta_2 - h_B$ and the fact that $h_B(0) = 0$ simplifies the eigenvectors (4.3.7) to

$$\mathbf{v}_{1,2} = \begin{bmatrix} \left(\frac{1}{2} - \zeta_2(0,t) \pm \frac{1}{2}\sqrt{1 - 4\gamma(1-\zeta_2(0,t))\zeta_2(0,t)}\right)\lambda_{1,2} \\ \left(\frac{1}{2} - \zeta_2(0,t) \pm \frac{1}{2}\sqrt{1 - 4\gamma(1-\zeta_2(0,t))\zeta_2(0,t)}\right)(1-\zeta_2(0,t)) \\ (1-\gamma)\lambda_{1,2}(1-\zeta_2(0,t)) \\ (1-\gamma)(1-\zeta_2(0,t))\zeta_2(0,t) \end{bmatrix}. \quad (4.3.8)$$

The 4×2 matrix $\mathbf{P}_+$ introduced in Section 3.3 is then given by the block notation

$$\mathbf{P}_+ = \begin{bmatrix} (\mathbf{v}_1)_1 & (\mathbf{v}_2)_1 \\ (\mathbf{v}_1)_2 & (\mathbf{v}_2)_2 \\ (\mathbf{v}_1)_3 & (\mathbf{v}_2)_3 \\ (\mathbf{v}_1)_4 & (\mathbf{v}_2)_4 \end{bmatrix} = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,], \quad (4.3.9)$$

where $\mathbf{v}_{1,2}$ are the column vectors from (4.3.8), whose components are denoted, for example, by $(\mathbf{v}_1)_1$.

For Lemma 3.3 to be applied, the matrix product $\mathbf{EP}_+$ must be shown to be invertible. A calculation of this matrix, with the simplification $\zeta_2 = \zeta_2(0,t)$, follows to give

$$\begin{aligned} \mathbf{EP}_+ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \\ &= \begin{bmatrix} (\mathbf{v}_1)_1 & (\mathbf{v}_2)_1 \\ (\mathbf{v}_1)_3 & (\mathbf{v}_2)_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2} - \zeta + \frac{1}{2}\sqrt{1-4\gamma(1-\zeta)\zeta}\right)\lambda_1 & \left(\frac{1}{2} - \zeta - \frac{1}{2}\sqrt{1-4\gamma(1-\zeta)\zeta}\right)\lambda_2 \\ (1-\gamma)\lambda_1(1-\zeta) & (1-\gamma)\lambda_2(1-\zeta) \end{bmatrix}. \end{aligned} \quad (4.3.10)$$

The matrix (4.3.10) is invertible if and only if its determinant is nonzero. Such a calculation follows as

$$\begin{aligned} \det(\mathbf{EP}_+) &= \left(\frac{1}{2} - \zeta_2 + \frac{1}{2}\sqrt{1-4\gamma(1-\zeta_2)\zeta_2}\right)\lambda_1(1-\gamma)\lambda_2(1-\zeta_2) \\ &\quad - \left(\frac{1}{2} - \zeta_2 - \frac{1}{2}\sqrt{1-4\gamma(1-\zeta_2)\zeta_2}\right)\lambda_2(1-\gamma)\lambda_1(1-\zeta_2) \\ &= \lambda_1\lambda_2(1-\gamma)(1-\zeta_2)\left[\frac{1}{2} - \frac{1}{2} - \zeta_2 + \zeta_2 + \left(\frac{1}{2} + \frac{1}{2}\right)\sqrt{1-4\gamma(1-\zeta_2)\zeta_2}\right] \\ &= (1-\gamma)(1-\zeta_2)\sqrt{1-4\gamma(1-\zeta_2)\zeta_2}\,\lambda_1\lambda_2. \end{aligned} \quad (4.3.11)$$

The expression (4.3.11) is nonzero precisely when the term within the square root is nonzero, which is the condition for hyperbolicity. Therefore, if the system is assumed to be hyperbolic, then $\det(\mathbf{EP}_+) \neq 0$, and Lemma 3.3 may be applied. This may be summed up as the following statement: the IBVP consisting of the system (2.4.29) with the initial value (4.3.1) and boundary value (4.3.4) is well-posed whenever it is strictly hyperbolic.

4.3.2 IVPs and IBVPs for the simplified systems

In this section, the preceding methodology is applied to the special cases of the two-layer two-dimensional gravity current problem described in Chapter 2. These are the weak-stratification equations, the thin lower and thin upper layer equations, and the rigid lid equations, which are all 2×2 systems of first-order equations. The results are all quite simple to show when compared to the previous section 4.3.1; however, for completeness, the results are included.

First, for the weak-stratification equations (2.4.38), an initial value for the instantaneous release of a block of fluid may be expressed as

$$\mathbf{u}(x,0) = \begin{bmatrix} u_2(x,0) \\ \zeta_2(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \zeta_{2_0}(x) \end{bmatrix}. \tag{4.3.12}$$

The relation (4.3.12) can be seen as a specialization of the more general initial condition (4.3.1), so that (4.3.2) still may be assumed for a description of $\zeta_{2_0}(x)$. When the weak-stratification system (2.4.38) is hyperbolic, the initial value (4.3.12) for $x \in \mathbb{R}$ is noncharacteristic, so that the IVP (2.4.38) and (4.3.12) is well-posed.

To include a boundary value in the description of the problem, no flow across a vertical barrier is considered as in section 4.3.1 to give a physical boundary condition of

$$u_2(0,t) = 0, \text{ for } t \geq 0. \tag{4.3.13}$$

This boundary condition may be expressed in the matrix form (3.3.3) as

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u_2(0,t) \\ \zeta_2(0,t) \end{bmatrix} = 0, \tag{4.3.14}$$

where the notation $\mathbf{E} = [1 \quad 0]$ is used. This condition (4.3.14) also satisfies the consistency condition with the initial value (4.3.12).

To apply the results from Section 3.3, the eigenvectors and eigenvalues for the system (2.4.38) must be examined. From equation (4.1.19) it can be recalled that this system has one positive and one negative eigenvalue. Using the expressions (4.1.17) and (4.1.18) with $h_B(0) = 0$ and $u_2(0,t) = 0$, the positive eigenvalue at the boundary $x = 0$ from (4.1.19) is written as

$$\lambda_1 = \frac{1}{2}\left(\frac{2Q\zeta_2(0,t)}{1-\zeta_2(0,t)}\right) + \sqrt{\frac{1}{4}\left(\frac{2Q\zeta_2(0,t)}{1-\zeta_2(0,t)}\right)^2 - \frac{-[(1-\zeta_2(0,t))^3 - Q^2]\zeta_2(0,t)}{(1-\zeta_2(0,t))^2}},$$

which simplifies to

$$\lambda_1 = \frac{Q\zeta_2(0,t)}{1-\zeta_2(0,t)} + \sqrt{(1-\zeta_2(0,t))\zeta_2(0,t)}. \tag{4.3.15}$$

The eigenvector corresponding to the eigenvalue (4.3.15) is then given by equation (4.1.24) as

$$\mathbf{v}_1 = \begin{bmatrix} \lambda_1 - u_2(0,t) \\ \zeta_2(0,t) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \zeta_2(0,t) \end{bmatrix}. \tag{4.3.16}$$

Lemma 3.3 examins questions the invertibility of the matrix $\mathbf{EP}_+$. For the weak stratification equations, this matrix is given by

$$\mathbf{EP}_+ = [1 \quad 0] \begin{bmatrix} \lambda_1 \\ \zeta_2(0,t) \end{bmatrix} = \lambda_1. \tag{4.3.17}$$

By the condition (4.3.17), $\mathbf{EP}_+$ is invertible precisely when $\lambda_1 \neq 0$, which is exactly the condition of hyperbolicity. Therefore, the IBVP for the weak-stratification equations (2.4.38) with initial value (4.3.12) and boundary value (4.3.13) is well-posed if the equations (2.4.38) are hyperbolic.

The next special case to be considered is the thin lower layer first order system (2.4.61). This situation is similar to the previous one in that the initial release problem has the initial condition formulated as

$$\mathbf{u} = \begin{bmatrix} \tilde{u}_2(x,0) \\ \tilde{\zeta}_2(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{\zeta}_{2_0}(x) \end{bmatrix}. \tag{4.3.18}$$

The shape of $\tilde{\zeta}_2(x,0)$ is similar to that given by the finite volume profile (4.3.2) with a tilde. As mentioned previously, the initial data (4.3.18) is noncharacteristic, leading to a well posed IVP for the equations (2.4.61) with (4.3.18) over $x \in \mathbb{R}$.

To include the standard boundary condition at $x = 0$, it may be written as (4.3.13) or (4.3.14) with superscript tildes, and is not restated. There is one positive eigenvalue given by (4.1.26) at $x = 0$ as

$$\lambda_1 = \sqrt{\tilde{\zeta}_2(0,t)}, \tag{4.3.19}$$

with associated eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}. \tag{4.3.20}$$

Since $\mathbf{P}_+ = \mathbf{v}_1$ in this case, the matrix $\mathbf{EP}_+$ is given by

$$\mathbf{EP}_+ = [1 \quad 0] \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = 1. \tag{4.3.21}$$

By Lemma 3.3 and equation (4.3.21), since $\mathbf{EP}_+$ is invertible, it follows that the IBVP for the thin lower layer equations consisting of (2.4.61), (4.3.18) and (4.3.13) is well-posed.

Similarly to the thin lower layer, the thin upper layer problem consists of equations (2.4.81) with the initial value

$$\mathbf{u} = \begin{bmatrix} \tilde{u}_1(x,0) \\ \tilde{\eta}(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{\eta}_0(x) \end{bmatrix}. \tag{4.3.22}$$

The function $\tilde{\eta}_0(x)$ in (4.3.22) may be expressed similarly to (4.3.2) as

$$\tilde{\eta}_0(x) = \begin{cases} \text{constant}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.23}$$

As in the previous discussion, the IVP (2.4.81) and (4.3.23) is well-posed for $x \in \mathbb{R}$ since the data is noncharacteristic.

The boundary value at $x = 0$ may be expressed as

$$\tilde{u}_1(0,t) = 0, \text{ for } t > 0, \tag{4.3.24}$$

or in the matrix form

$$\mathbf{E}(t)\mathbf{u}(0,t) = [1 \quad 0][\tilde{u}_1(0,t)\tilde{\eta}(0,t)] = 0. \tag{4.3.25}$$

To apply Lemma 3.3, the eigenvalues and eigenvectors at $x = 0$ must be found. The positive eigenvalue and eigenvector from equations (4.1.28) and (4.1.29) turn out to be

$$\lambda_1 = \sqrt{\tilde{\eta}(0,t)}, \tag{4.3.26}$$

and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}. \tag{4.3.27}$$

Comparison with equations (4.19)-(4.3.21) then gives the matrix product $\mathbf{EP}_+ = 1$.

Again through use of Lemma 3.3, since $\mathbf{EP}_+$ is invertible it follows that the IBVP for the thin upper layer equations (2.4.81) with initial value (4.3.22) and boundary value (4.3.24) is well-posed.

The last case considered is the rigid lid equations (2.4.89). The similarities between the rigid lid system and the weak-stratification system are such that the result (4.3.19) holds here, with a slightly different value of λ_1 than (4.3.15). The initial value (4.3.12) and boundary value (4.3.13) are identical to the previous weak-stratification case. The same result, that the IVP is well-posed for $x \in \mathbb{R}$, and the IBVP (2.4.89), (4.3.12) and (4.3.13) is well-posed thus follows from Lemma 3.3, as long as the system (2.4.89) remains hyperbolic.

Chapter Summary

Two-dimensional gravity currents have been considered in this chapter, expanding on the equation development from Chapter 2, and the theoretical results discussed in Chapter 3.

For the two-layer equations with a free surface, it was shown that the equations are in general strictly hyperbolic, as long as the flow is of a sufficiently slow nature, such as that resulting from an initial release problem. Discontinuous solutions may be considered to satisfy one of two conditions, depending on the assumption of movement of the lower layer into a fluid at rest or not. The natural physical boundary conditions then lead to a well-posed problem.

Similar results are obtained for the special cases from Chapter 2, namely the weak-stratification equations, the thin lower and upper layer equations, and the rigid lid equations. It can be observed that the results for the rigid lid situation reduces to the weak-stratification equations, suggesting that the rigid lid equations may be more usefully employable.

Chapter 5

Three-dimensional Gravity Currents

With the previous chapter's analysis concerning two-layer and two-dimensional gravity currents complete, a similar approach is now taken towards generalizing these results for the three-dimensional situation. The results mostly pertain to the simpler one and a half layer model rather than the full two-layer case, since the consideration of two spatial variables adds a complexity which is not present in the two-dimensional equations.

In the first section, the systems of partial differential equations are examined to determine whether or not they are indeed hyperbolic. Both the two-layer and the single layer equations are shown to be hyperbolic for most domains of the flow variables. This allows associated full-space initial value problems, such as the instantaneous release of a volume of dense fluid, to be stated as well-posed. Boundary conditions such as impermeable vertical barriers considered in the two-dimensional case are not considered here.

In the second section, the idea of discontinuous solutions is considered, with the aim of determining expressions for front speeds which generalize those discovered in Chapter 4. Since the two-layer equations were not easily expressed as a closed system in conservation form (this topic will be addressed in Chapter 7), the results are obtained only for the single layer case. The method used to obtain expressions for front speeds is a novel idea based existing perturbation methods such as acceleration fronts (Seymour, 1975) and nonlinear optics (Whitham, 1974 p.533). A general development of this method is not conducted as it is an intuitive technique developed for this case alone in order to investigate the effects of rotation on a spreading gravity current.

5.1 Hyperbolicity

In this section, the hyperbolicity of the two-layer model, equations (2.3.8)-(2.3.13), is examined first, followed by conditions for hyperbolicity for the simpler single layer model, equation (2.4.6) and the axisymmetric case, equation (2.4.22). Since both of these systems have two spatial variables, the definition of hyperbolicity from equation (3.1.8) is used. In calculating the determinants of the matrices, the method of row reduction is used. This is a change from the more common techniques of expansion by cofactors, and they can be found in most introductory texts of linear algebra (for example Norman, 1995 Ch. 5).

5.1.1 The Two-Layer Equations

For the full two-layer system, the system of equations (2.4.2) may be written as the system

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ v_1 \\ \zeta_1 \\ u_2 \\ v_2 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} u_1 & 0 & \gamma^{-1} & 0 & 0 & \gamma^{-1} \\ 0 & u_1 & 0 & 0 & 0 & 0 \\ \zeta_1 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}-1 & u_2 & 0 & \gamma^{-1} \\ 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & \zeta_2 & 0 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ v_1 \\ \zeta_1 \\ u_2 \\ v_2 \\ \zeta_2 \end{bmatrix}$$
$$+ \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_1 & \gamma^{-1} & 0 & 0 & \gamma^{-1} \\ 0 & \zeta_1 & v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & \gamma^{-1}-1 & 0 & v_2 & \gamma^{-1} \\ 0 & 0 & 0 & 0 & \zeta_2 & v_2 \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} u_1 \\ v_1 \\ \zeta_1 \\ u_2 \\ v_2 \\ \zeta_2 \end{bmatrix} = \mathbf{b}, \qquad (5.1.1)$$

where the change of notation $\zeta_1 = h_1 - h_2$ and $\zeta_2 = h_2 - h_B$ has been employed, and $\mathbf{b}$ is given by the column vector

$$\mathbf{b} = \left(\varepsilon v_1 - \gamma^{-1}\frac{\partial h_B}{\partial x}, -\varepsilon u_1 - \gamma^{-1}\frac{\partial h_B}{\partial y}, 0, -\gamma^{-1}\frac{\partial h_B}{\partial x} + \varepsilon v_2 - \kappa C_f u_2 \frac{\sqrt{u_2^2+v_2^2}}{\zeta_2},\right.$$
$$\left. -\gamma^{-1}\frac{\partial h_B}{\partial y} - \varepsilon u_2 - \kappa C_f v_2 \frac{\sqrt{u_2^2+v_2^2}}{\zeta_2}, 0\right)^T. \qquad (5.1.2)$$

Hyperbolicity depends upon a new matrix, $\mathbf{A}$, which is defined in (3.1.8), for equation (5.1.1). It can be recalled that a single matrix is formed as a scalar linear combination of the coefficient matrices in (5.1.1) as

$$\mathbf{A} = \omega_1 \mathbf{A}_x + \omega_2 \mathbf{A}_y$$
$$= \begin{bmatrix} \overline{\omega}_1 & 0 & \omega_1\gamma^{-1} & 0 & 0 & \omega_1\gamma^{-1} \\ 0 & \overline{\omega}_1 & \omega_2\gamma^{-1} & 0 & 0 & \omega_2\gamma^{-1} \\ \omega_1\zeta_1 & \omega_2\zeta_1 & \overline{\omega}_1 & 0 & 0 & 0 \\ 0 & 0 & \omega_1(\gamma^{-1}-1) & \overline{\omega}_2 & 0 & \omega_1\gamma^{-1} \\ 0 & 0 & \omega_2(\gamma^{-1}-1) & 0 & \overline{\omega}_2 & \omega_2\gamma^{-1} \\ 0 & 0 & 0 & \omega_1\zeta_2 & \omega_2\zeta_2 & \overline{\omega}_2 \end{bmatrix}, \qquad (5.1.3)$$

where the notational simplification $\overline{\omega}_1 = \omega_1 u_1 + \omega_2 v_1$ and $\overline{\omega}_2 = \omega_1 u_2 + \omega_2 v_2$ has been used. The scalars ω_1 and ω_2 are arbitrary, but not both zero as discussed Subsection 3.1.2.

To determine the hyperbolicity of equation (5.1.1), the eigenvalues of the matrix (5.1.3) must be found by solving the equation

$$\det(\mathbf{A} - \lambda \mathbf{I}_6) = 0. \tag{5.1.4}$$

To solve equation (5.1.4) for the scalar λ, row reduction methods are employed using the following two properties (Wilde, 1988 p.137) for a square n by n matrix $\mathbf{B}$:

(1) If $\mathbf{C}$ is obtained from $\mathbf{B}$ by the addition of $k \in \mathbb{R}$ times row i to row j, then $\det(\mathbf{C}) = \det(\mathbf{B})$, and
(2) If $\mathbf{C}$ is upper triangular, then $\det(\mathbf{C}) = \prod_{i=1}^{i=n} c_{ii}$.

The matrix (5.1.3) can now be employed in the condition (5.1.4) by several row reduction steps. First, the matrix in (5.1.4) is written as

$$\mathbf{A} - \lambda \mathbf{I}_6 = \begin{bmatrix} \varpi_1 - \lambda & 0 & \omega_1 \gamma^{-1} & 0 & 0 & \omega_1 \gamma^{-1} \\ 0 & \varpi_1 - \lambda & \omega_2 \gamma^{-1} & 0 & 0 & \omega_2 \gamma^{-1} \\ \omega_1 \zeta_1 & \omega_2 \zeta_1 & \varpi_1 - \lambda & 0 & 0 & 0 \\ 0 & 0 & \omega_1(\gamma^{-1} - 1) & \varpi_2 - \lambda & 0 & \omega_1 \gamma^{-1} \\ 0 & 0 & \omega_2(\gamma^{-1} - 1) & 0 & \varpi_2 - \lambda & \omega_2 \gamma^{-1} \\ 0 & 0 & 0 & \omega_1 \zeta_2 & \omega_2 \zeta_2 & \varpi_2 - \lambda \end{bmatrix}. \tag{5.1.5}$$

Addition of a scalar multiple of the first row to the third row, symbolized by the row reduction notation $R_3 - (\frac{\omega_1 \zeta_1}{\varpi_1 - \lambda}) R_1 \to R_3$, gives the new matrix

$$\begin{bmatrix} \varpi_1 - \lambda & 0 & \omega_1 \gamma^{-1} & 0 & 0 & \omega_1 \gamma^{-1} \\ 0 & \varpi_1 - \lambda & \omega_2 \gamma^{-1} & 0 & 0 & \omega_2 \gamma^{-1} \\ 0 & \omega_2 \zeta_1 & \varpi_1 - \lambda - \frac{\omega_1^2 \zeta_1 \gamma^{-1}}{\varpi_1 - \lambda} & 0 & 0 & -\frac{\omega_1^2 \zeta_1 \gamma^{-1}}{\varpi_1 - \lambda} \\ 0 & 0 & \omega_1(\gamma^{-1} - 1) & \varpi_2 - \lambda & 0 & \omega_1 \gamma^{-1} \\ 0 & 0 & \omega_2(\gamma^{-1} - 1) & 0 & \varpi_2 - \lambda & \omega_2 \gamma^{-1} \\ 0 & 0 & 0 & \omega_1 \zeta_2 & \omega_2 \zeta_2 & \varpi_2 - \lambda \end{bmatrix}.$$

Similarly, the addition of another scalar row multiple, denoted $R_3 - (\frac{\omega_2 \zeta_1}{\varpi_1 - \lambda}) R_2 \to R_3$ yields the matrix

$$\begin{bmatrix} \varpi_1 - \lambda & 0 & \omega_1 \gamma^{-1} & 0 & 0 & \omega_1 \gamma^{-1} \\ 0 & \varpi_1 - \lambda & \omega_2 \gamma^{-1} & 0 & 0 & \omega_2 \gamma^{-1} \\ 0 & 0 & \varpi_1 - \lambda - \frac{\omega^2 \zeta_1 \gamma^{-1}}{\varpi_1 - \lambda} & 0 & 0 & -\frac{\omega^2 \zeta_1 \gamma^{-1}}{\varpi_1 - \lambda} \\ 0 & 0 & \omega_1(\gamma^{-1} - 1) & \varpi_2 - \lambda & 0 & \omega_1 \gamma^{-1} \\ 0 & 0 & \omega_2(\gamma^{-1} - 1) & 0 & \varpi_2 - \lambda & \omega_2 \gamma^{-1} \\ 0 & 0 & 0 & \omega_1 \zeta_2 & \omega_2 \zeta_2 & \varpi_2 - \lambda \end{bmatrix}, \tag{5.1.6}$$

where the notation $\omega^2 = \omega_1^2 + \omega_2^2$ in (5.1.6) has been used to simplify the two nontrivial expressions in the third row. The bottom right hand four rows and columns of the matrix (5.1.6) may be written, for convenience of notation, by the 4×4 matrix

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ \omega_1(\gamma^{-1} - 1) & \overline{\omega}_2 - \lambda & 0 & \omega_1\gamma^{-1} \\ \omega_2(\gamma^{-1} - 1) & 0 & \overline{\omega}_2 - \lambda & \omega_2\gamma^{-1} \\ 0 & \omega_1\zeta_2 & \omega_2\zeta_2 & \overline{\omega}_2 - \lambda \end{bmatrix}, \tag{5.1.7}$$

where the letter a in (5.1.7) represents $\omega^2\zeta_1\gamma^{-1}/(\overline{\omega}_1 - \lambda)$. The row reduction procedure on the matrix (5.1.6) can now be completed by considering the smaller matrix (5.1.7), keeping the same row numbers as for the 6×6 matrix. The step $R_4 - (\frac{\omega_1(\gamma^{-1}-1)}{\overline{\omega}_1-\lambda-a})R_3 \to R_4$ alters the matrix (5.1.7) to

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & \overline{\omega}_2 - \lambda & 0 & \omega_1\gamma^{-1} + a\frac{\omega_1(\gamma^{-1}-1)}{\overline{\omega}_1-\lambda-a} \\ \omega_2(\gamma^{-1} - 1) & 0 & \overline{\omega}_2 - \lambda & \omega_2\gamma^{-1} \\ 0 & \omega_1\zeta_2 & \omega_2\zeta_2 & \overline{\omega}_2 - \lambda \end{bmatrix},$$

which may be further reduced via $R_5 - (\frac{\omega_2(\gamma^{-1}-1)}{\overline{\omega}_1-\lambda-a})R_3 \to R_5$ as

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & \overline{\omega}_2 - \lambda & 0 & \frac{\omega_1\gamma^{-1}(\overline{\omega}_1-\lambda-a)+a\omega_1(\gamma^{-1}-1)}{\overline{\omega}_1-\lambda-a} \\ 0 & 0 & \overline{\omega}_2 - \lambda & \omega_2\gamma^{-1} + a\frac{\omega_2(\gamma^{-1}-1)}{\overline{\omega}_1-\lambda-a} \\ 0 & \omega_1\zeta_2 & \omega_2\zeta_2 & \overline{\omega}_2 - \lambda \end{bmatrix}. \tag{5.1.8}$$

The matrix (5.1.8) becomes, using the notation $b = [\gamma^{-1}(\overline{\omega}_1 - \lambda) - a]/(\overline{\omega}_1 - \lambda - a)$,

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & \overline{\omega}_2 - \lambda & 0 & \omega_1 b \\ 0 & 0 & \overline{\omega}_2 - \lambda & \omega_2 b \\ 0 & \omega_1\zeta_2 & \omega_2\zeta_2 & \overline{\omega}_2 - \lambda \end{bmatrix}. \tag{5.1.9}$$

Further row reduction steps for the matrix (5.1.9) complete the procedure, and are listed as
$R_6 - (\frac{\omega_1\zeta_2}{\overline{\omega}_2-\lambda})R_4 \to R_6$

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & \overline{\omega}_2 - \lambda & 0 & \omega_1 b \\ 0 & 0 & \overline{\omega}_2 - \lambda & \omega_2 b \\ 0 & 0 & \omega_2\zeta_2 & \overline{\omega}_2 - \lambda - \frac{\omega_1^2\zeta_2 b}{\overline{\omega}_2-\lambda} \end{bmatrix},$$

and then $R_6 - (\frac{\omega_2\zeta_2}{\overline{\omega}_2-\lambda})R_5 \to R_6$ resulting in

$$\begin{bmatrix} \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & \overline{\omega}_2 - \lambda & 0 & \omega_1 b \\ 0 & 0 & \overline{\omega}_2 - \lambda & \omega_2 b \\ 0 & 0 & 0 & \overline{\omega}_2 - \lambda - \frac{\overline{\omega}_1^2\zeta_2 b}{\overline{\omega}_2-\lambda} - \frac{\omega_2^2\zeta_2 b}{\overline{\omega}_2-\lambda} \end{bmatrix}. \tag{5.1.10}$$

Through the previous steps of simple replacement of a row with a scalar mulitple of another row, the matrix (5.1.5) has been reduced through matrices (5.1.6)-(5.1.10), giving the upper triangular form

$$\mathbf{C} = \begin{bmatrix} \overline{\omega}_1 - \lambda & 0 & \omega_1\gamma^{-1} & 0 & 0 & \omega_1\gamma^{-1} \\ 0 & \overline{\omega}_1 - \lambda & \omega_2\gamma^{-1} & 0 & 0 & \omega_2\gamma^{-1} \\ 0 & 0 & \overline{\omega}_1 - \lambda - a & 0 & 0 & -a \\ 0 & 0 & 0 & \overline{\omega}_2 - \lambda & 0 & \omega_1 b \\ 0 & 0 & 0 & 0 & \overline{\omega}_2 - \lambda & \omega_2 b \\ 0 & 0 & 0 & 0 & 0 & \overline{\omega}_2 - \lambda - \frac{\omega^2\zeta_2 b}{\overline{\omega}_2-\lambda} \end{bmatrix}. \tag{5.1.11}$$

Using the previously stated property (2) of the determinant, det($\mathbf{C}$) may be calculated easily as

$$\begin{aligned} \det(\mathbf{C}) &= (\overline{\omega}_1 - \lambda)^2(\overline{\omega}_1 - \lambda - a)(\overline{\omega}_2 - \lambda)^2 \left[\overline{\omega}_2 - \lambda - \frac{\omega^2\zeta_2 b}{\overline{\omega}_2 - \lambda}\right] \\ &= (\overline{\omega}_1 - \lambda)^2(\overline{\omega}_2 - \lambda)(\overline{\omega}_1 - \lambda - a)\left[(\overline{\omega}_2 - \lambda)^2 - b\omega^2\zeta_2\right]. \end{aligned} \tag{5.1.12}$$

Now, since the matrix (5.1.11) was obtained from (5.1.5) by row reduction steps consisting of the addition of scalar multiples of one row to another, property (1) for determinants gives $\det(\mathbf{C}) = \det(\mathbf{A} - \lambda\mathbf{I}_6)$. Using this in (5.1.12), and reintroducing the full expression $b = [\gamma^{-1}(\overline{\omega}_1 - \lambda) - a]/(\overline{\omega}_1 - \lambda - a)$ yields

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}_6) &= (\overline{\omega}_1 - \lambda)^2(\overline{\omega}_2 - \lambda)(\overline{\omega}_1 - \lambda - a)\Bigg[(\overline{\omega}_2 - \lambda)^2 - \\ &\quad - \frac{\omega^2\zeta_2[\gamma^{-1}(\overline{\omega}_1 - \lambda) - a]}{\overline{\omega}_1 - \lambda - a}\Bigg] \\ &= (\overline{\omega}_1 - \lambda)^2(\overline{\omega}_2 - \lambda)\left\{[(\overline{\omega}_2 - \lambda)^2(\overline{\omega}_1 - \lambda - a) - \omega^2\zeta_2\gamma^{-1}(\overline{\omega}_1 - \lambda) + \omega^2\zeta_2 a\right\}. \end{aligned} \tag{5.1.13}$$

Now, substituting the value $a = \omega^2\zeta_1\gamma^{-1}/(\overline{\omega}_1 - \lambda)$ into (5.1.13), the resulting expression may be reformulated as

$$\det(\mathbf{A}-\lambda\mathbf{I}_6) = (\overline{\omega}_1-\lambda)^2(\overline{\omega}_2-\lambda)\bigg[(\overline{\omega}_2-\lambda)^2(\overline{\omega}_1-\lambda)-(\overline{\omega}_2-\lambda)^2\frac{\omega^2\zeta_1\gamma^{-1}}{\overline{\omega}_1-\lambda}$$
$$-\omega^2\zeta_2\gamma^{-1}(\overline{\omega}_1-\lambda)+\frac{\omega^4\zeta_1\zeta_2\gamma^{-1}}{\overline{\omega}_1-\lambda}\bigg]$$
$$=(\overline{\omega}_1-\lambda)(\overline{\omega}_2-\lambda)\big[(\overline{\omega}_2-\lambda)^2(\overline{\omega}_1-\lambda)^2-\omega^2\zeta_1\gamma^{-1}(\overline{\omega}_2-\lambda)^2$$
$$-\omega^2\zeta_2\gamma^{-1}(\overline{\omega}_1-\lambda)^2+\omega^4\zeta_1\zeta_2\gamma^{-1}\big]$$
$$=(\overline{\omega}_1-\lambda)(\overline{\omega}_2-\lambda)\big\{(\overline{\omega}_1-\lambda)^2(\overline{\omega}_2-\lambda)^2-\omega^2\gamma^{-1}[\zeta_1(\overline{\omega}_2-\lambda)^2$$
$$+\zeta_2(\overline{\omega}_1-\lambda)^2]+\omega^4\zeta_1\zeta_2\gamma^{-1}\big\}. \quad (5.1.14)$$

This result (5.1.14) may be stated finally as

$$\det(\mathbf{A}-\lambda\mathbf{I}_6)=(\overline{\omega}_1-\lambda)(\overline{\omega}_2-\lambda)\big\{[(\overline{\omega}_1-\lambda)^2-\omega^2\zeta_1\gamma^{-1}][(\overline{\omega}_2-\lambda)^2-\omega^2\zeta_2\gamma^{-1}]$$
$$+\omega^4\zeta_1\zeta_2\gamma^{-1}(1-\gamma^{-1})\big\}. \quad (5.1.15)$$

Using equation (5.1.15), equation (5.1.4) may now be solved. Clearly, two eigenvalues are given by

$$\lambda_1=\overline{\omega}_1=\omega_1u_1+\omega_2v_1 \text{ and } \lambda_2=\overline{\omega}_2=\omega_1u_2+\omega_2v_2. \quad (5.1.16)$$

The other roots may be found by solving the quartic equation

$$[(\overline{\omega}_1-\lambda)^2-\omega^2\zeta_1\gamma^{-1}][(\overline{\omega}_2-\lambda)^2-\omega^2\zeta_2\gamma^{-1}]=\omega^4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1}-1). \quad (5.1.17)$$

Equation (5.1.17) may be solved directly via the methods for solving quartic equations. However, it is more useful, when investigating hyperbolicity of the two-layer system (5.1.1), to obtain more specific information from the quartic equation (5.1.17). This is phrased as the following result.

Lemma 5.1 Equation (5.1.17) has four real and distinct solutions if and only if the following two conditions hold:

(i)

$$\begin{cases} \zeta_1\neq\zeta_2 & \text{if } \overline{\omega}_1=\overline{\omega}_2,\\ |\overline{\omega}_1-\overline{\omega}_2|\neq\omega(\sqrt{\gamma^{-1}\zeta_1}+\sqrt{\gamma^{-1}\zeta_2}) & \text{if } \overline{\omega}_1\neq\overline{\omega}_2. \end{cases} \quad (5.1.18)$$

and (ii)

$$\left[\left(\lambda_c+\frac{\overline{\omega}_2}{2}-\frac{\overline{\omega}_1}{2}\right)^2-\omega^2\zeta_1\gamma^{-1}\right]\left[\left(\lambda_c+\frac{\overline{\omega}_1}{2}-\frac{\overline{\omega}_2}{2}\right)^2-\omega^2\zeta_2\gamma^{-1}\right]$$
$$>\omega^4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1}-1), \quad (5.1.19)$$

where $\lambda_c \in \mathbb{R}$ is given by

$$\lambda_c = \frac{1}{2}(\varpi_1 + \varpi_2) - \sqrt{\frac{1}{3}[2\omega^2\gamma^{-1}(\zeta_1 + \zeta_2) + (\varpi_1 - \varpi_2)^2]} \times$$
$$\times \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{\sqrt{27}(\varpi_1 - \varpi_2)\omega^2\gamma^{-1}(\zeta_1 - \zeta_2)}{[2\omega^2\gamma^{-1}(\zeta_1 + \zeta) + (\varpi_1 - \varpi_2)^2]^3}\right] - \frac{\pi}{3}\right\}.$$

Proof: Consider a change of variable to (5.1.17) given by

$$x = \lambda - \left(\frac{\varpi_1 + \varpi_2}{2}\right). \tag{5.1.20}$$

Removing λ from (5.1.17) in preference of x via (5.1.20) gives

$$\left[\left(x + \frac{1}{2}\varpi_2 - \frac{1}{2}\varpi_1\right)^2 - \omega^2\zeta_1\gamma^{-1}\right]\left[\left(x + \frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2\right)^2 - \omega^2\zeta_2\gamma^{-1}\right]$$
$$= \omega^4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1} - 1). \tag{5.1.21}$$

Introducing the notation for ϖ as

$$\varpi = \frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2 = \frac{1}{2}\omega_1(u_1 - u_2) + \frac{1}{2}\omega_2(v_1 - v_2), \tag{5.1.22}$$

simplifies equation (5.1.21) to

$$f(x) = \omega^4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1} - 1), \tag{5.1.23}$$

where $f(x)$ is defined by the fourth degree polynomial,

$$f(x) = [(x - \varpi)^2 - \omega^2\zeta_1\gamma^{-1}][(x + \varpi)^2 - \omega^2\zeta_2\gamma^{-1}]. \tag{5.1.24}$$

It should be observed that the right hand side of equation (5.1.23) is strictly positive since $0 < \gamma < 1$.

Equation (5.1.24) is already factored, so that the roots of the equation $f(x) = 0$ may be directly found to be the set

$$x \in \left\{-\varpi - \omega\sqrt{\zeta_2\gamma^{-1}}, -\varpi + \omega\sqrt{\zeta_2\gamma^{-1}}, \varpi - \omega\sqrt{\zeta_1\gamma^{-1}}, \varpi + \omega\sqrt{\zeta_1\gamma^{-1}}\right\}. \tag{5.1.25}$$

The signs of the square roots are chosen so that $\omega = \sqrt{\omega_1^2 + \omega_2^2} > 0$ and ζ_1, ζ_2, γ^{-1} are all strictly positive by assumption: thus, the roots (5.1.25) are all real. If

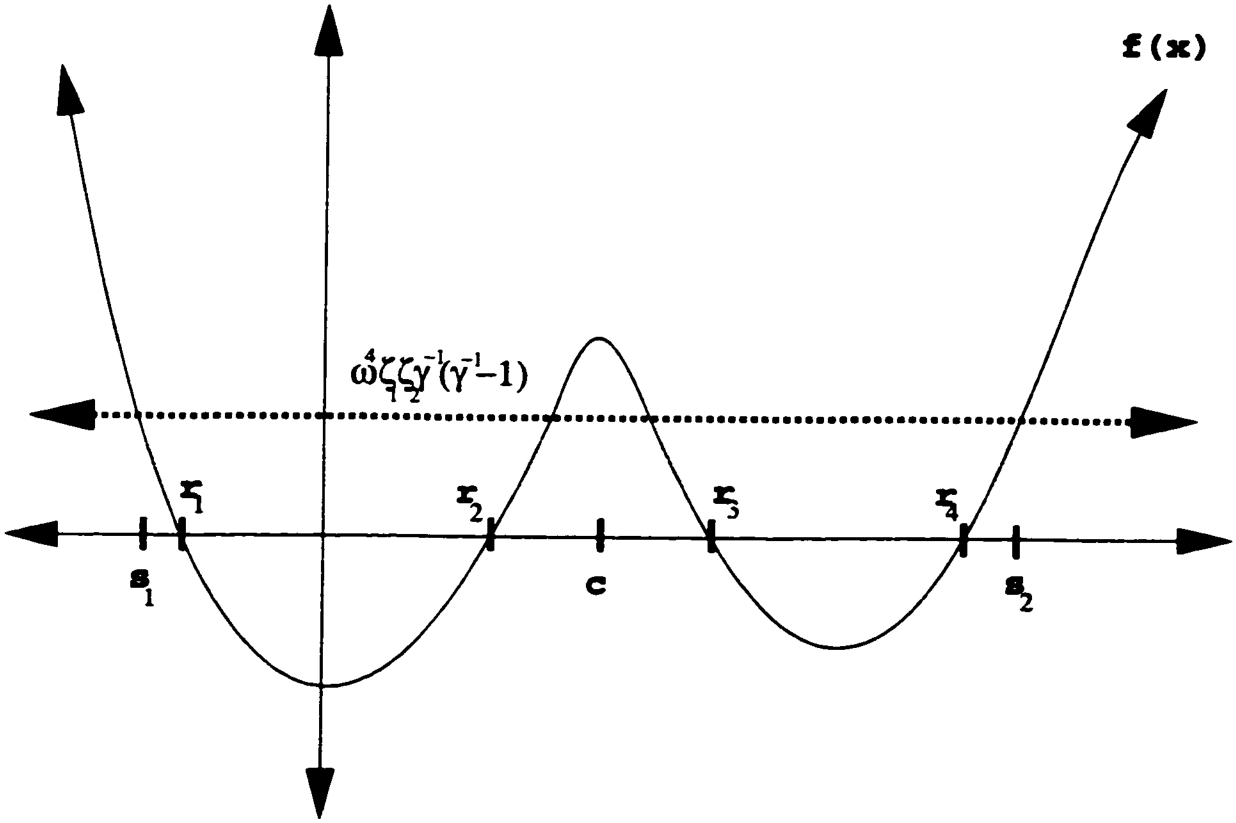

FIGURE 5.1. Sketch of $f(x)$ for distinct roots, and $\omega^4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1} - 1)$.

the roots are all distinct, then the graph of $f(x)$ must be of the same nature as portrayed in Figure 5.1, which is shown on the following page.

From Fig. 5.1, equation (5.1.23) may be seen to have two solutions at $x = s_1$ and $x = s_2$, for all possible cases of the roots of $f(x)$ in (5.1.25). Assuming distinct roots r_1, r_2, r_3, r_4, it can be seen from Fig. 5.1, that equation (5.1.23) has four distinct real solutions if

$$f(c) > \omega_4\zeta_1\zeta_2\gamma^{-1}(\gamma^{-1} - 1), \tag{5.1.26}$$

where $c \in \mathbb{R}$ is the critical number corresponding to the local maximum in Fig 5.1.

The value of c may be found by solving for the derivative, $f'(x)$, from (5.1.24) as

$$\begin{aligned} f'(x) &= 2(x-\varpi)[(x+\varpi)^2 - \omega^2\zeta_2\gamma^{-1}] + [(x-\varpi)^2 - \omega^2\zeta_1\gamma^{-1}]2(x+\varpi) \\ &= 2\left\{(x-\varpi)(x+\varpi)^2 + (x-\varpi)^2(x+\varpi) - \omega^2\gamma^{-1}[(x-\varpi)\zeta_2 + (x+\varpi)\zeta_1]\right\} \\ &= 2\left\{(x-\varpi)(x+\varpi)(x+\varpi+x-\varpi) - \omega^2\gamma^{-1}[(\zeta_1+\zeta_2)x + (\zeta_1-\zeta_2)\varpi]\right\} \\ &= 2\left[(x^2-\varpi^2)2x - \omega^2\gamma^{-1}(\zeta_1+\zeta_2)x - \omega^2\gamma^{-1}(\zeta_1-\zeta_2)\varpi\right], \end{aligned}$$

which gives the cubic equation, $f'(x) = 0$, after dividing by 4 to be

$$x^3 - \frac{1}{2}\left[2\varpi^2 + \omega^2\gamma^{-1}(\zeta_1+\zeta_2)\right]x - \frac{1}{2}\omega^2\gamma^{-1}(\zeta_1-\zeta_2)\varpi = 0. \qquad (5.1.27)$$

From the ordering of the roots to (5.1.27) calculated in Appendix 1, c is given from equations (A1.17)-(A1.19) as

$$c = -2\sqrt{\frac{1}{6}[2\varpi^2 + \omega^2\gamma^{-1}(\zeta_1+\zeta_2)]}\cos[(\theta-\pi)/3], \text{ where}$$

$$\theta = \cos^{-1}\left[\sqrt{\frac{27}{2}}\frac{\varpi\omega^2\gamma^{-1}(\zeta_1-\zeta_2)}{[2\varpi^2+\omega^2\gamma^{-1}(\zeta_1+\zeta_2)]^3}\right]. \qquad (5.1.28)$$

To guarantee four distinct roots (5.1.25), three cases are considered for the signs of ϖ.

Case (i): $\varpi = 0$. The roots (5.1.25) in this case simplify greatly by dividing by ω to give

$$x \in \left\{-\omega\sqrt{\zeta_2\gamma^{-1}}, \omega\sqrt{\zeta_2\gamma^{-1}}, -\omega\sqrt{\zeta_1\gamma^{-1}}, \omega\sqrt{\zeta_1\gamma^{-1}}\right\}.$$

Since $\omega > 0$, it follows that there are four distinct roots if and only if $\zeta_2 \neq \zeta_1$.

Case (ii): $\varpi > 0$. Clearly, the roots from (5.1.25) may be ordered almost completely since

$$-\varpi - \omega\sqrt{\zeta_2\gamma^{-1}} < -\varpi + \omega\sqrt{\zeta_2\gamma^{-1}}, \quad \varpi - \omega\sqrt{\zeta_1\gamma^{-1}} < \varpi + \omega\sqrt{\zeta_1\gamma^{-1}},$$

and

$$-\varpi - \omega\sqrt{\zeta_1\gamma^{-1}} < \varpi + \omega\sqrt{\zeta_1\gamma^{-1}}.$$

Therefore, there are at least three distinct roots (5.1.25). For there to be only one root, the only way for this to occur is for the remaining two roots not covered by the inequality above to be equal, i.e.

$$-\varpi + \omega\sqrt{\zeta_2\gamma^{-1}} = \varpi - \omega\sqrt{\zeta_1\gamma^{-1}},$$

which may be rewritten as

$$2\varpi = \omega\sqrt{\gamma^{-1}}(\sqrt{\zeta_2} + \sqrt{\zeta_1}).$$

Thus, four distinct roots exist for

$$\varpi \in (0, \infty) \setminus \frac{\omega}{2}\sqrt{\gamma^{-1}}(\sqrt{\zeta_2} + \sqrt{\zeta_1}). \tag{5.1.29}$$

Case (iii): $\varpi < 0$. This is similar to case (ii), in that there are precisely three roots (5.1.25) if

$$-\varpi - \omega\sqrt{\zeta_2\gamma^{-1}} = \varpi + \omega\sqrt{\zeta_1\gamma^{-1}},$$

and four distinct roots otherwise. Since this condition is rewritten as

$$-2\varpi = \omega\sqrt{\gamma^{-1}}(\sqrt{\zeta_2} + \sqrt{\zeta_1}),$$

a similar conclusion to (5.1.29) can be stated, namely that four distinct roots exist for

$$\varpi \in (-\infty, 0) \setminus -\frac{\omega}{2}\sqrt{\gamma^{-1}}(\sqrt{\zeta_2} + \sqrt{\zeta_1}). \tag{5.1.30}$$

In summary, the polynomial equation (5.1.23) has four real and distinct roots precisely when either (i) $\zeta_1 \neq \zeta_2$ if $\varpi = 0$, or (ii) $|\varpi| \neq \frac{\omega}{2}\sqrt{\gamma^{-1}}(\sqrt{\zeta_1} + \sqrt{\zeta_2})$ if $\varpi \neq 0$. This statement may be reworked via the notation ϖ from (5.1.22) to obtain the statement (5.1.18). Once four real roots are assumed, the condition (5.1.26) with c given by (5.1.28) is sufficient and necessary to guarantee four distinct real solutions to (5.1.22), thereby showing through the change of variable (5.1.20) that there exist four distinct real solutions to (5.1.17). The condition (5.1.26) is rewritten in terms of $\lambda_c = c + \frac{1}{2}(\varpi_1 + \varpi_2)$ as the statement (5.1.19) with the aid of the solution (5.1.28) □

Returning to the question of hyperbolicity of the system (5.1.1), the result may now be encapsulated as a theorem to complete this subsection.

Theorem 5.1 If Lemma 5.1 is applicable, that is equations (5.1.18) and (5.1.19) both hold, then the system (5.1.1) is hyperbolic for $\zeta_2 > 0$, $\zeta_1 > 0$, and $0 < \gamma < 1$.

Proof: To show that the system (5.1.1) is hyperbolic, it is sufficient to show that the matrix $\mathbf{A}$ in (5.1.3) has 6 real and independent eigenvectors for every scalar pair $(\omega_1, \omega_2) \neq (0, 0)$. From linear algebra, it is known that this is the case if $\mathbf{A}$ has 6 real and distinct eigenvalues, λ_i, for every scalar pair ω_1, ω_2 not both zero. Two of these eigenvalues are given by equation (5.1.16), and the remaining four must solve equation (5.1.17). If the conditions of Lemma (5.1) hold, then (5.1.17) has four distinct real solutions, and it suffices to show that the eigenvalues (5.1.16) do not solve (5.1.17), so that there are indeed six distinct eigenvalues of **A**. First, if $\lambda = \overline{\omega}_1$, then this corresponds by equation (5.1.20) to

$$x = \overline{\omega}_1 - \left(\frac{\overline{\omega}_1 + \overline{\omega}_2}{2}\right) = \frac{\overline{\omega}_1}{2} - \frac{\overline{\omega}_2}{2} = \overline{\omega}.$$

The value $x = \overline{\omega}$ is not one of the roots (5.1.25) unless $\zeta_1 = 0$, a condition which would be contrary to assumption. Hence, $\lambda = \overline{\omega}_1$ is not a repeated root. Next, if $\lambda = \overline{\omega}_2$, then a similar calculation to the above yields $x = -\overline{\omega}$, which is not one of the roots (5.1.25) unless $\zeta_2 = 0$, which also contradicts assumptions. Therefore, $\lambda = \overline{\omega}_2$ is not a repeated root either. Therefore, as long as Lemma 5.1 is satisfied, there are six real and distinct solutions of equation (5.1.4), requiring that **A** has six distinct real eigenvalues for every pair of scalars $(\omega_1, \omega_2) \in \mathbb{R}^2$ such that $\omega_1^2 + \omega_2^2 > 0$. □

5.1.2 The Single Layer Equations

In this section, the single layer model (2.4.7) from Section 2.4.1 is examined, when written in the system form with a change of variable $\zeta_2 = h_2 - h_B$. The system (2.4.7) is then rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_2 \\ v_2 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} u_2 & 0 & 1 \\ 0 & u_2 & 0 \\ \zeta_2 & 0 & u_2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u_2 \\ v_2 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} v_2 & 0 & 0 \\ 0 & v_2 & 1 \\ 0 & \zeta_2 & v_2 \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} u_2 \\ v_2 \\ \zeta_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \varepsilon v_2 - \dfrac{\partial h_B}{\partial x} - \kappa C_f u_2 \dfrac{\sqrt{u_2^2+v_2^2}}{\zeta_2} T \\ -\varepsilon u_2 - \dfrac{\partial h_B}{\partial y} - \kappa C_f v_2 \dfrac{\sqrt{u_2^2+v_2^2}}{\zeta_2} T \\ 0 \end{bmatrix}. \quad (5.1.31)$$

As with the matrix (5.1.3), hyperbolicity of the system (5.1.31) depends on a matrix **A** defined via (3.1.8) for $(\omega_1, \omega_2) \in \mathbb{R}^2$ as

$$\mathbf{A} = \omega_1 \mathbf{A}_x + \omega_2 \mathbf{A}_y = \begin{bmatrix} \omega_1 u_2 + \omega_2 v_2 & 0 & \omega_1 \\ 0 & \omega_1 u_2 + \omega_2 v_2 & \omega_2 \\ \omega_1 \zeta_2 & \omega_2 \zeta_2 & \omega_1 u_2 + \omega_2 v_2 \end{bmatrix}. \quad (5.1.32)$$

Since the system (5.1.31) is hyperbolic if **A** has three real and linearly independent eigenvectors, it is sufficient to show that there are three real and distinct eigenvalues. These are obtained from solving

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}_3), \quad (5.1.33)$$

for a scalar eigenvalue λ. Substituting (5.1.32) into (5.1.33) gives the equation to be solved as

$$0 = \det \begin{bmatrix} \omega_1 u_2 + \omega_2 v_2 - \lambda & 0 & \omega_1 \\ 0 & \omega_1 u_2 + \omega_2 v_2 - \lambda & \omega_2 \\ \omega_1 \zeta_2 & \omega_2 \zeta_2 & \omega_1 u_2 + \omega_2 v_2 - \lambda \end{bmatrix}. \qquad (5.1.34)$$

Expanding the determinant (5.1.34) via cofactors in the first column allows this to be expressed as

$$0 = (\omega_1 u_2 + \omega_2 v_2 - \lambda)[(\omega_1 u_2 + \omega_2 v_2 - \lambda)^2 - \omega_2^2 \zeta_2] \\ + \omega_1 \zeta_2 [0 - \omega_1(\omega_1 u_2 + \omega_2 v_2 - \lambda)]$$

which simplifies to

$$0 = (\omega_1 u_2 + \omega_2 v_2 - \lambda)[(\omega_1 u_2 + \omega_2 v_2 - \lambda)^2 - (\omega_1^2 + \omega_2^2)\zeta_2]. \qquad (5.1.35)$$

Equation (5.1.35) therefore has three real roots, given by

$$\lambda_1 = \omega_1 u_2 + \omega_2 v_2, \text{ and } \lambda_{2,3} = \omega_1 u_2 + \omega_2 v_2 \pm \sqrt{(\omega_1^2 + \omega_2^2)\zeta_2}. \qquad (5.1.36)$$

It follows that for $\zeta_2 > 0$, equation (5.1.36) yields three real and distinct eigenvalues for all $(\omega_1, \omega_2) \in \mathbb{R}^2$ such that $\omega_1^2 + \omega_2^2 > 0$, and the system (5.1.31) is therefore hyperbolic whenever $\zeta_2 > 0$. It should be noted that this result may be obtained from a simplification of the determinant for the two-layer case (5.1.15) with $\overline{\omega}_1 = 0$, $\overline{\omega}_2 = \omega_1 u_2 + \omega_2 v_2$, and $\zeta_1 = 0$.

Another form of the simplified equations for the single lower layer is the conservative form (2.4.10) with the notation $\mu = \zeta_2 u_2$, $\nu = \zeta_2 v_2$ representing vertically integrated momentum in the two horizontal directions. The conservative form (2.4.10) may be easily expanded to be written as a system, with $\zeta_2 \to \zeta$, as

$$\frac{\partial}{\partial t}\begin{bmatrix} \mu \\ \nu \\ \zeta \end{bmatrix} + \begin{bmatrix} \frac{2\mu}{\zeta} & 0 & \zeta - \frac{\mu^2}{\zeta^2} \\ \frac{\nu}{\zeta} & \frac{\mu}{\zeta} & -\frac{\mu\nu}{\zeta^2} \\ 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \mu \\ \nu \\ \zeta \end{bmatrix} \begin{bmatrix} \frac{\nu}{\zeta} & \frac{\mu}{\zeta} & -\frac{\mu\nu}{\zeta^2} \\ 0 & \frac{2\nu}{\zeta} & \zeta - \frac{\nu^2}{\zeta^2} \\ 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} \mu \\ \nu \\ \zeta \end{bmatrix} = \\ = \begin{bmatrix} -\zeta \dfrac{\partial h_B}{\partial x} + \varepsilon\nu - \kappa C_f \mu \sqrt{u_2^2 + v_2^2}\, T \\ -\zeta \dfrac{\partial h_B}{\partial y} - \varepsilon\mu - \kappa C_f \nu \sqrt{u_2^2 + v_2^2}\, T \\ 0 \end{bmatrix}. \qquad (5.1.37)$$

To discern hyperbolicity of the system (5.1.37), the matrix $\mathbf{A}$ from (3.1.8) is formed as

$$\mathbf{A} = \omega_1 \mathbf{A}_x + \omega_2 \mathbf{A}_y = \frac{\omega_1}{\zeta^2}\begin{bmatrix} 2\mu\zeta & 0 & \zeta^3 - \mu^2 \\ \nu\zeta & \mu\zeta & -\mu\nu \\ \zeta^2 & 0 & 0 \end{bmatrix} + \frac{\omega_2}{\zeta^2}\begin{bmatrix} \nu\zeta & \mu\zeta & -\mu\nu \\ 0 & 2\nu\zeta & \zeta^3 - \nu^2 \\ 0 & \zeta^2 & 0 \end{bmatrix}$$
$$= \frac{1}{\zeta^2}\begin{bmatrix} (2\omega_1\mu + \omega_2\nu)\zeta & \omega_2\mu\zeta & \omega_1(\zeta^3 - \mu^2) - \omega_2\mu\nu \\ \omega_1\nu\zeta & (\omega_1\mu + 2\omega_2\nu)\zeta & -\omega_1\mu\nu + \omega_2(\zeta^3 - \nu^2) \\ \omega_1\zeta^2 & \omega_2\zeta^2 & 0 \end{bmatrix}. \qquad (5.1.38)$$

To find the eigenvalues of the matrix (5.1.38), the characteristic equation (5.1.33) is solved with $\mathbf{A}$ from (5.1.38). To simplify the algebra, equation (5.1.33) is multiplied by ζ^6 (assumed to be nonzero) to give

$$0 = \zeta^6 \det(\mathbf{A} - \lambda \mathbf{I}_3) = \det(\zeta^2 \mathbf{A} - \zeta^2 \lambda \mathbf{I}_3)$$
$$= \det \begin{bmatrix} (2\omega_1\mu + \omega_2\nu)\zeta - \zeta^2\lambda & \omega_2\mu\zeta & \omega_1(\zeta^3 - \mu^2) - \omega_2\mu\nu \\ \omega_1\nu\zeta & (\omega_1\mu + 2\omega_2\nu)\zeta - \zeta^2\lambda & -\omega_1\mu\nu + \omega_2(\zeta^3 - \nu^2) \\ \omega_1\zeta^2 & \omega_2\zeta^2 & -\zeta^2\lambda \end{bmatrix}.$$

This equation is a cubic equation in λ, which is obtained by evaluating the determinant via the method of cofactors on the first column. The resulting polynomial is then

$$\begin{aligned} 0 = {} & \big[(2\omega_1\mu + \omega_2\nu)\zeta - \zeta^2\lambda\big]\big\{\big[(\omega_1\mu + 2\omega_2\nu)\zeta - \zeta^2\lambda\big](-\zeta^2\lambda) \\ & \quad - [-\omega_1\mu\nu + \omega_2(\zeta^3 - \nu^2)]\omega_2\zeta^2\big\} \\ & - \omega_1\nu\zeta\big\{\omega_2\mu\zeta(-\zeta^2\lambda) - [\omega_1(\zeta^3 - \mu^2) - \omega_2\mu\nu]\omega_2\zeta^2\big\} + \omega_1\zeta^2\big\{\omega_2\mu\zeta[-\omega_1\mu\nu \\ & + \omega_2(\zeta^3 - \nu^2)] - [\omega_1(\zeta^3 - \mu^2) - \omega_2\mu\nu][(\omega_1\mu + 2\omega_2\nu)\zeta - \zeta^2\lambda]\big\}. \end{aligned} \qquad (5.1.39)$$

Equation (5.1.39) may be simplified greatly, although the algebra is a bit tedious. The steps involved in expanding and simplifying (5.1.39) follow first by expanding as

$$\begin{aligned} 0 = {} & [-\zeta^2\lambda + (2\omega_1\mu + \omega_2\nu)\zeta][\zeta^4\lambda^2 - (\omega_1\mu + 2\omega_2\nu)\zeta^3\lambda + \omega_1\omega_2\mu\nu\zeta^2 - \\ & - \omega_2^2(\zeta^3 - \nu^2)\zeta^2] - \omega_1\nu\zeta[-\omega_2\mu\zeta^3\lambda - \omega_1\omega_2(\zeta^3 - \mu^2)\zeta^2 + \omega_2\mu\nu\zeta^2] \\ & + \omega_1\zeta^2\{[\omega_1(\zeta^3 - \mu^2) - \omega_2\mu\nu]\zeta^2\lambda - \omega_1\omega_2\mu^2\nu\zeta + \omega_2^2\mu\zeta(\zeta^3 - \nu^2) - \\ & - \omega_1(\zeta^3 - \mu^2)(\omega_1\mu + 2\omega_2\nu)\zeta + \omega_2\mu\nu(\omega_1\mu + 2\omega_2\nu)\zeta\}. \end{aligned}$$

The previous equation may then be further expanded to obtain

$$\begin{aligned}0 = &-\zeta^6\lambda^3 + 3\zeta^5(\omega_1\mu + \omega_2\nu)\lambda^2 + [-(2\omega_1\mu + \omega_2\nu)(\omega_1\mu + 2\omega_2\nu)\zeta^4 - \omega_1\omega_2\mu\nu\zeta^4 \\ &+ \omega_2^2(\zeta^3 - \nu^2)\zeta^4 + \omega_1\omega_2\mu\nu\zeta^4 + \omega_1^2(\zeta^3 - \mu^2)\zeta^4 - \omega_1\omega_2\mu\nu\zeta^4]\lambda + (2\omega_1\mu \\ &+ \omega_2\nu)[\omega_1\omega_2\mu\nu - \omega_2^2(\zeta^3 - \nu^2)]\zeta^3 + \omega_1^2\omega_2\nu(\zeta^3 - \mu^2)\zeta^3 - \omega_1\omega_2^2\mu\nu^2\zeta^3 \\ &- \omega_1^2\omega_2\mu^2\nu\zeta^3 + \omega_1\omega_2^2\mu(\zeta^3 - \nu^2)\zeta^3 - \omega_1^2(\zeta^3 - \mu^2)(\omega_1\mu + 2\omega_2\nu)\zeta^3 \\ &+ \omega_1\omega_2\mu\nu(\omega_1\mu + 2\omega_2\nu)\zeta^3.\end{aligned}$$

Gathering terms in like powers of λ allows this expression to be compressed in the following three steps:

$$\begin{aligned}0 = &-\zeta^6\lambda^3 + 3\zeta^5(\omega_1\mu + \omega_2\nu)\lambda^2 + \zeta^4[(\omega_1^2 + \omega_2^2)\zeta^3 - 2\omega_1^2\mu^2 - 5\omega_1\omega_2\mu\nu - 2\omega_2^2\nu^2 \\ &- \omega_2^2\nu^2 - \omega_1^2\mu^2 - \omega_1\omega_2\mu\nu]\lambda + \zeta^3[2\omega_1^2\omega_2\mu^2\nu + \omega_1\omega_2^2\mu\nu^2 + 2\omega_1\omega_2^2\mu\nu^2 + \omega_2^3\nu^3 \\ &- \omega_1^2\omega_2\mu^2\nu - \omega_1\omega_2^2\mu\nu^2 - \omega_1^2\omega_2\mu^2\nu - \omega_1\omega_2^2\mu\nu^2 + \omega_1^3\mu^3 + 2\omega_1^2\omega_2\mu^2\nu + \omega_1^2\omega_2\mu^2\nu \\ &+ 2\omega_1\omega_2^2\mu\nu^2 + \zeta^3(-2\omega_1\omega_2^2\mu - \omega_2^3\nu + \omega_1^2\omega_2\nu + \omega_1\omega_2^2\mu - \omega_1^3\mu - 2\omega_1^2\omega_2\nu)],\end{aligned}$$

$$\begin{aligned}0 = &-\zeta^6\lambda^3 + 3\zeta^5(\omega_1\mu + \omega_2\nu)\lambda^2 + \zeta^4[(\omega_1^2 + \omega_2^2)\zeta^3 - 3(\omega_1\mu + \omega_2\nu)^2]\lambda \\ &+ \zeta^3[\omega_1\mu(\omega_1^2\mu^2 + 2\omega_1\omega_2\mu\nu + \omega_2^2\nu^2) + \omega_2\nu(\omega_1^2\mu^2 + 2\omega_1\omega_2\mu\nu + \omega_2^2\nu^2) \\ &- \zeta^3(\omega_1^3\mu + \omega_1\omega_2^2\mu + \omega_1^2\omega_2\nu + \omega_2^3\nu)],\end{aligned}$$

and

$$\begin{aligned}0 = &-\zeta^6\lambda^3 + 3\zeta^5(\omega_1\mu + \omega_2\nu)\lambda^2 + \zeta^4[(\omega_1^2 + \omega_2^2)\zeta^3 - 3(\omega_1\mu + \omega_2\nu)^2]\lambda \\ &+ \zeta^3(\omega_1\mu + \omega_2\nu)[(\omega_1\mu + \omega_2\nu)^2 - \zeta^3(\omega_1^2 + \omega_2^2)].\end{aligned}$$

This last expression simplifies finally, after multiplication by $-\zeta^3$, to

$$0 = \zeta^3\lambda^3 - 3\zeta^2\overline{\omega}\lambda^2 - \zeta(\omega^2\zeta^3 - 3\overline{\omega}^2)\lambda - \overline{\omega}(\overline{\omega}^2 - \omega^2\zeta^3) \tag{5.1.40}$$

where equation (5.1.40) is simplified through introduction of the notation

$$\overline{\omega} = \omega_1\mu + \omega_2\nu, \quad \text{and} \quad \omega^2 = \omega_1^2 + \omega_2^2.$$

Equation (5.1.40) shown in Appendix 1, equation (A1.8) to have three real and unequal roots

$$\{\lambda_0, \lambda_1, \lambda_2\} = \{\frac{\overline{\omega}}{\zeta} - \sqrt{\omega^2\zeta}, \frac{\overline{\omega}}{\zeta}, \frac{\overline{\omega}}{\zeta} + \sqrt{\omega^2\zeta}\}, \tag{5.1.41}$$

which are unequal whenever $\zeta > 0$. Therefore, the system (5.1.37) is hyperbolic if $\zeta = \zeta_2 > 0$, as seen for the system (5.1.31).

The final system which is examined is the axisymetric case (2.4.22), in which the lower layer variables are expressed in polar coordinates (r, θ, t) with the notation $u_r = u_r(r,t)$ representing the radial velocity, $u_\theta = u_\theta(r,t)$ the tangential velocity. and $h_2 = h_2(r,t)$ the thickness of the lower layer as before. Restating the system (2.4.22), with the familiar notation $\zeta_2 = h_2 - h_B(r)$, gives

$$\frac{\partial}{\partial t}\begin{bmatrix} u_r \\ u_\theta \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} u_r & 0 & 1 \\ 0 & u_r & 0 \\ \zeta_2 & 0 & u_r \end{bmatrix} \frac{\partial}{\partial r}\begin{bmatrix} u_r \\ u_\theta \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \frac{u_\theta^2}{r} + \varepsilon u_\theta - \dfrac{\partial h_B}{\partial r} - \kappa C_f u_r \frac{\sqrt{u_r^2+u_\theta^2}}{\zeta_2} T \\ -\frac{u_r u_\theta}{r} - \varepsilon u_r - \kappa C_f u_\theta \frac{\sqrt{u_r^2+u_\theta^2}}{\zeta_2} T \\ -\frac{\zeta_2 u_r}{r} \end{bmatrix}. \tag{5.1.42}$$

The axisymetric case is thus seen as a single spatial variable case (3.1.1), and as such is hyperbolic if the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} u_r & 0 & 1 \\ 0 & u_r & 0 \\ \zeta_2 & 0 & u_r \end{bmatrix}, \tag{5.1.43}$$

has three real and linearly independent eigenvectors.

A quick analysis of the matrix (5.1.43) in a similar manner to that completed after equation (5.1.33) gives the eigenvalues of (5.1.43) from

$$\begin{aligned} 0 &= \det \begin{bmatrix} u_r - \lambda & 0 & 1 \\ 0 & u_r - \lambda & 0 \\ \zeta_2 & 0 & u_r - \lambda \end{bmatrix} \\ &= (u_r - \lambda)[(u_r - \lambda)^2 - 0] + \zeta_2[0 - (u_r - \lambda)] \\ &= (u_r - \lambda)[(u_r - \lambda)^2 - \zeta_2]. \end{aligned} \tag{5.1.44}$$

The characteristic equation (5.1.44) has the solution

$$\lambda_1 = u_r, \text{ and } \lambda_{2,3} = u_r \pm \sqrt{\zeta_2}. \tag{5.1.45}$$

Since the real roots (5.1.45) are distinct precisely when $\zeta_2 > 0$, the system (5.1.42) is strictly hyperbolic whenever $\zeta_2 > 0$.

5.2 Discontinuous Solutions

For the two-layer shallow-water equations in two dimensions, Section 4.2 contains various expressions for determining the front speed of a finite volume of dense fluid spreading laterally into a quiescent volume of lighter fluid. When the two-layer equations in three spatial dimensions are examined, the corresponding system of six equations contains a complication additional to the increase in the number of variables. This is that there are fewer than six known corresponding conservation laws, the number that is required to close the system of algebraic conditions at a discontinuity. Since the Rankine-Hugoniot jump conditions hold at a discontinuity for a system of conservation laws, there is a requirement for the two-layer, three-dimensional system to be expressed as a closed system of conservation laws, which so far has not been possible. An investigation into this problem is postponed until Chapter 7.

Due to this difficulty, only the simplified case, where the motion of the upper layer is neglected, is considered. Such equations are discussed in Sections 2.4.1 and 2.4.2. In this section, the single layer equations (2.4.4)-(2.4.6) are considered, in the conservative form (2.4.10) or the axisymmetric form (2.4.22). The general type of discontinuous solutions considered are those of a finite volume of dense fluid spreading outwards from an initial resting state.

5.2.1 Discontinuous Solutions for the Single Layer Equations

The conservative form of the single layer equations (2.4.10) is given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\mu\\ \nu\\ \zeta\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\frac{\mu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \frac{\mu\nu}{\zeta}\\ \mu\end{bmatrix}+\frac{\partial}{\partial y}\begin{bmatrix}\frac{\mu\nu}{\zeta}\\ \frac{\nu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \nu\end{bmatrix}=\mathbf{b}(\mu,\nu,\zeta,x), \tag{5.2.1}$$

where $\mathbf{b}(0,0,0,x)=\mathbf{0}$. Equation (5.2.1) is of the form (3.1.11) for $p=2$ where

$$\mathbf{f}^{(1)}=\begin{bmatrix}\frac{\mu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \frac{\mu\nu}{\zeta}\\ \mu\end{bmatrix} \text{ and } \mathbf{f}^{(2)}=\begin{bmatrix}\frac{\mu\nu}{\zeta}\\ \frac{\nu^2}{\zeta}+\frac{1}{2}\zeta^2\\ \nu\end{bmatrix}. \tag{5.2.2}$$

A discontinuous solution to equation (5.2.1) is considered by restricting the solution to having only one discontinuity along a simple curve $\Sigma(x,y,t)=0$, such that in the interior of Σ, the solution is continuous, and outside of Σ, the solution is trivial, i.e. $(\mu,\nu,\zeta)=0$. This solution satisfies (5.2.1) outside Σ since $\frac{\mu^2}{\zeta}=u_2$,

$\frac{\nu^2}{\zeta} = v_2$, and $\frac{\mu\nu}{\zeta} = u_2 v_2 \zeta_2$, which all have limits of zero as u_2, v_2, and ζ_2 vanish. For such a situation, Lemma 3.2 may be applied to give the jump conditions (3.2.34) along the curve Σ.

A specialization of the jump condition is noted in the form (3.2.37), which may be applied for curves Σ of the form $\Sigma = \sigma(x, y) - s(t)$, which has an outward normal vector $\mathbf{n}$ given by

$$\mathbf{n} = \left(-\frac{\mathrm{d}s}{\mathrm{d}t}, \frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial y}\right).$$

The condition (3.2.37) for such a curve may be expressed as

$$\frac{\mathrm{d}s}{\mathrm{d}t}[\mathbf{u}] = \frac{\partial \sigma}{\partial x}[\mathbf{f}^{(1)}] + \frac{\partial \sigma}{\partial y}[\mathbf{f}^{(2)}]. \tag{5.2.3}$$

In (5.2.3), the notation used is

$$\begin{aligned}[\mathbf{u}] &= \lim_{\varepsilon \to 0+} \mathbf{u}((\mathbf{x}, t) + \varepsilon \mathbf{n}) - \lim_{\varepsilon \to 0+} \mathbf{u}((\mathbf{x}, t) - \varepsilon \mathbf{n}) \\ &= 0 - \mathbf{u}^- = -\begin{bmatrix} \mu^- \\ \nu^- \\ \zeta^- \end{bmatrix},\end{aligned} \tag{5.2.4}$$

where μ^-, ν^-, and ζ^- are the limits of the inner solution on the interior of Σ. Similarly, the jumps in the flux vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are given by

$$[\mathbf{f}^{(1)}] = -(\mathbf{f}^{(1)})^- = -\begin{bmatrix} \frac{(\mu^-)^2}{\zeta^-} + \frac{1}{2}(\zeta^-)^2 \\ \frac{\mu^- \nu^-}{\zeta^-} \\ \mu^- \end{bmatrix}, \tag{5.2.5}$$

and

$$[\mathbf{f}^{(2)}] = -(\mathbf{f}^{(2)})^- = -\begin{bmatrix} \frac{\mu^- \nu^-}{\zeta^-} \\ \frac{(\nu^-)^2}{\zeta^-} + \frac{1}{2}(\zeta^-)^2 \\ \nu^- \end{bmatrix}. \tag{5.2.6}$$

Substituting the notation (5.2.4)-(5.2.6) into the condition (5.2.3), and suppressing the superscript ($^-$) allows the jump conditions to be written as three equations:

$$\frac{\mathrm{d}s}{\mathrm{d}t}\mu = \frac{\partial \sigma}{\partial x}\left(\frac{\mu^2}{\zeta} + \frac{1}{2}\zeta^2\right) + \frac{\partial \sigma}{\partial y}\left(\frac{\mu\nu}{\zeta}\right), \tag{5.2.7}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\nu = \frac{\partial \sigma}{\partial x}\left(\frac{\mu\nu}{\zeta}\right) + \frac{\partial \sigma}{\partial y}\left(\frac{\nu^2}{\zeta} + \frac{1}{2}\zeta^2\right), \tag{5.2.8}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t}\zeta = \frac{\partial\sigma}{\partial x}\mu + \frac{\partial\sigma}{\partial y}\nu. \tag{5.2.9}$$

The three equations (5.2.7)-(5.2.9) contain six unknowns, and should be viewed as a restriction on the shock speed $\frac{\mathrm{d}s}{\mathrm{d}t}$ in terms of the system variables μ, ν and ζ given that the geometry, or direction, of the discontinuity $\sigma(x, y)$ is known. With $\frac{\partial\sigma}{\partial x}$ and $\frac{\partial\sigma}{\partial y}$ assumed determined, equations (5.2.7)-(5.2.9) may now be solved to find $\frac{\mathrm{d}s}{\mathrm{d}t}$ in terms of ζ. To achieve such a simplification, it is assumed that $\zeta \neq 0$, so that multiplication of equation (5.2.7) by ζ yields the equation

$$\mu\left(\zeta\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{\partial\sigma}{\partial x}\left(\mu^2 + \frac{1}{2}\zeta^3\right) + \frac{\partial\sigma}{\partial y}\mu\nu. \tag{5.2.10}$$

Substitution of (5.2.9) into (5.2.10) then results in

$$\mu\left(\frac{\partial\sigma}{\partial x}\mu + \frac{\partial\sigma}{\partial y}\nu\right) = \frac{\partial\sigma}{\partial x}\left(\mu^2 + \frac{1}{2}\zeta^3\right) + \frac{\partial\sigma}{\partial y}\mu\nu,$$

which reduces to simply $0 = \frac{\partial\sigma}{\partial x}\zeta^3$. Since $\zeta \neq 0$ is assumed, the resulting restriction becomes $\frac{\partial\sigma}{\partial x} = 0$. Similarly, multiplying (5.2.8) by ζ and substituting in equation (5.2.9) to remove $\frac{\mathrm{d}s}{\mathrm{d}t}$ produces the analogous result $\frac{\partial\sigma}{\partial y} = 0$. Using this knowledge in (5.2.9), it follows that $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$, and hence that Σ is a constant curve.

This rather surprising result leads to the conclusion that the conditions (5.2.7)-(5.2.9) are inappropriately imposed for such a discontinuous solution. This degeneracy has been investigated by Renardy (1998), who showed that near the front a degenerate hyperbolic system exists which may change type and lose its hyperbolic character. The jump conditions are an over-specification of the solution, and lead to an alternate method to consider the problem, which follows in the next section. The difficult occurs in reconciling integrated jump conditions over a radial domain with those in the azimuthal direction.

5.2.2 The Axisymmetric, Radial-motion Equations

A special case of the system (5.2.1) is considered for which the jump conditions make sense. This is the axisymmetric equations (2.4.19)-(2.4.21) with the restriction that the azimuthal velocity, u_θ, is zero. This assumption ensures that the velocity is entirely in the radial direction, which restricts the Coriolis parameter to also vanish, i.e. $\varepsilon = 0$. The simplified equations (2.4.19) and (2.4.21), with the notation $\zeta_2 = h_2 - h_B$, may be stated as the system

$$\frac{\partial}{\partial t}\begin{bmatrix} u_r \\ \zeta_2 \end{bmatrix} + \frac{\partial}{\partial r}\begin{bmatrix} \frac{1}{2}u_r^2 + \zeta_2 \\ \zeta_2 u_r \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{d}h_B}{\mathrm{d}r} - \kappa C_f \frac{u_r^2}{\zeta_2} T \\ -\frac{\zeta_2 u_r}{r} \end{bmatrix}. \tag{5.2.11}$$

The system (5.2.11) is remarkably similar to the equations for a thin lower layer (4.2.93) and (4.2.94). From this comparison, for a solution to (5.2.11) which has a single jump discontinuity at a radius given by $r = s(t)$, where the solution is zero for $r > s(t)$, the results (4.2.100) and (4.2.101) may be applied directly after changing the notation. This gives the jump condition at $r = s(t)$ as

$$u_r^2 = 2(\zeta_2 - h_B), \tag{5.2.12}$$

and the shock speed

$$\frac{\mathrm{d}s}{\mathrm{d}t} = u_r. \tag{5.2.13}$$

Contrasting the results (5.2.12)-(5.2.13) with the lack of information obtained for the system (5.2.1) via the same methods motivates an expansion technique which attempts to expand upon the information for the axisymmetric, radial-motion equations (5.2.11). In this way, an inclusion of azimuthal velocity and the Coriolis parameter may be achieved. The use of this method recognizes the conservation of linear radial momentum, raher than the x and y components separately with a small variation due to angle.

5.2.3 An Expansion Technique for Discontinuous Solutions to the Axisymmetric Equations

The full axisymmetric equations (2.4.19)-(2.4.21) are considered, with the only simplifications assumed as $C_f = 0$, and $h_B = 0$. Also, for ease of the following analysis, a change of variable notation is made to:

$$u = u_r, \quad v = u_\theta, \text{ and } h = h_2. \tag{5.2.14}$$

The new notation (5.2.14), along with the assumptions above give the equations (2.4.19)-(2.4.21) as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{\partial h}{\partial r} = \frac{v^2}{r} + \varepsilon v, \tag{5.2.15}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} = -\frac{uv}{r} - \varepsilon u, \tag{5.2.16}$$

and

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial r}(hu) = -\frac{hu}{r}. \tag{5.2.17}$$

For equations (5.2.14)-(5.2.17), discontinuous solutions are desired of the form

$$(u, v, h) = \begin{cases} (0,0,0) \text{ for } r > s(t), \\ (u, v, h) \in C^1 \text{ for } r < s(t). \end{cases} \tag{5.2.18}$$

At the single discontinuity $r = s(t)$, it is desired to find some form of jump conditions which relate the solution and the speed of propagation of this curve.

Equation (5.2.17) is in the form (3.2.1), and as such the jump condition (3.2.5) may be applied directly, to yield

$$-\frac{\mathrm{d}s}{\mathrm{d}t}[h] + [hu] = 0,$$

which simplifies, from the solution (5.2.18), to

$$\frac{\mathrm{d}s}{\mathrm{d}t}h^- - h^- u^- = 0, \tag{5.2.19}$$

where $h^- = \lim_{r\to s(t)^-} h(r,t)$ and $u^- = \lim r \to s(t)^- u(r,t)$. Assuming that $h^- > 0$, the condition (5.2.19) then becomes

$$\frac{\mathrm{d}s}{\mathrm{d}t} = u^-, \tag{5.2.20}$$

which is the familiar physical result that the discontinuity travels at the speed of the horizontal radial velocity, and in this way the discontinuity may be thought of as a material interface.

To determine the relationship between u^-, v^-, and h^-, normally, two more conservation equations may be used to produce an algebraic relationship between these variables, as done in Chapter 4. However, equation (5.2.16) is not in conservation form, and an alternate method of obtaining the information from this equation is required. The idea behind this method is that of expansion about a plane wave solution, which is sometimes called linear geometrical optics (see Whitham, 1974 or Seymourm, 1975), to obtain the important effects of v^- and ε on the front speed. This method cannot be applied directly, however, since plane wave solutions to equations (5.2.15)-(5.2.17) do not exist.

A solution to equation (5.2.16), motivated by geometrical optics, is considered which consists of a plane wave, slowly varying along the characteristic variable $\lambda = r - ut$, multiplied by an attenuation factor which is dependent on the radial variable. Such a solution would be of the form

$$v(r,t) = f(r-ut)g(r). \tag{5.2.21}$$

If it is assumed, momentarily, that u in (5.2.21) is constant, then substitution of this desired solution (5.2.21) into equation (5.2.16), using the notation $f' = \dfrac{\mathrm{d}f}{\mathrm{d}(r-ut)}$, reveals

$$-uf'g + u\left(f'g + f\frac{\mathrm{d}g}{\mathrm{d}r}\right) = -\frac{ufg}{r} - \varepsilon u,$$

which simplifies to

$$u\left(f\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{fg}{r} + \varepsilon\right) = 0. \tag{5.2.22}$$

Since $u \neq 0$ identically, equation (5.2.22) may be rewritten as an ordinary differential equation for g whenever $f \neq 0$ as

$$\frac{\mathrm{d}g}{\mathrm{d}r} = -\frac{g}{r} - \frac{\varepsilon}{f}. \tag{5.2.23}$$

The choice of the solution (5.2.21) may now be seen to have the advantage that, once f is known, equation (5.2.23) may be integrated to solve for g, hence determining v by (5.2.21). The resulting solution v can then be substituted into equation (5.2.15) which, in turn, when written in conservation form, can be used to yield a relationship between u^- and h^-. Such a calculation follows, with the assumption that locally, near the discontinuity $r = s(t)$, the value of f is given by a constant, which is arbitrary, and the assumption of constant u leading in (5.2.21) is thus not necessary.

The local analysis is completed by assuming that at some time $t = t_0$, the variables are denoted by

$$r_0 = s(t_0), \quad f_0 = f(r_0 - u^- t_0), \text{ and } g_0 = g(r_0). \tag{5.2.24}$$

For $r - r_0$ small, the function g is expanded in a Taylor series as

$$g(r) = g_0 + (r-r_0)\frac{\mathrm{d}g}{\mathrm{d}r}(r_0) + O((r-r_0)^2). \tag{5.2.25}$$

This series (5.2.25) may be written using the relation (5.2.23) to give

$$g(r) = g_0 + (r - r_0)\left(-\frac{g(r_0)}{r_0} - \frac{\varepsilon}{f_0}\right) + O((r-r_0)^2)$$
$$= \left(2 - \frac{r}{r_0}\right) g_0 - (r - r_0)\frac{\varepsilon}{f_0} + O((r-r_0)^2). \tag{5.2.26}$$

Now, equation (5.2.15) may be rewritten using the approximations above to rewrite the right hand side of (5.2.15) by substituting (5.2.21) for v, obtaining

$$\frac{v^2}{r} + \varepsilon v = \frac{f^2 g^2}{r} + \varepsilon f g. \tag{5.2.27}$$

Near $t = t_0$, equation (5.2.27) is replaced with the aid of the approximations (5.2.24) and (5.2.26) as

$$\begin{aligned}
\frac{v^2}{r} + \varepsilon v &= \frac{f_0^2}{r}\left[\left(2 - \frac{r}{r_0}\right) g_0 - (r-r_0)\frac{\varepsilon}{f_0} + O((r-r_0)^2)\right]^2 \\
&\quad + \varepsilon f_0 \left[\left(2 - \frac{r}{r_0}\right) g_0 - (r-r_0)\frac{\varepsilon}{f_0} + O((r-r_0)^2)\right] \\
&= \frac{f_0^2}{r}\left[\left(2 - \frac{r}{r_0}\right)^2 g_0^2 - 2(r-r_0)\left(2 - \frac{r}{r_0}\right)\frac{\varepsilon g_0}{f_0} + O((r-r_0)^2)\right] \\
&\quad + \varepsilon\left(2 - \frac{r}{r_0}\right) f_0 g_0 + O(\varepsilon^2, (r-r_0)^2) \\
&= \left(2 - \frac{r}{r_0}\right)^2 \frac{f_0^2 g_0^2}{r} - 2\varepsilon(r-r_0)\left(2 - \frac{r}{r_0}\right)\frac{f_0 g_0}{r} + \varepsilon\left(2 - \frac{r}{r_0}\right) f_0 g_0 \\
&\quad + O((\varepsilon^2, (r-r_0)^2).
\end{aligned} \tag{5.2.28}$$

Expanding the expressions, and rewriting (5.2.28) then gives

$$\frac{v^2}{r} + \varepsilon v = \left(\frac{4}{r} - \frac{4}{r_0} + \frac{r}{r_0^2}\right) f_0^2 g_0^2 + \varepsilon f_0 g_0 \left[\left(2 - \frac{r}{r_0}\right)\left(1 - 2\frac{r - r_0}{r}\right)\right] + O((\varepsilon^2, (r-r_0)^2),$$

which may be simplified further to

$$\frac{v^2}{r} + \varepsilon v = f_0^2 g_0^2 \left(\frac{4}{r} - \frac{4}{r_0} + \frac{r}{r_0^2}\right) + \varepsilon f_0 g_0 \left(\frac{4r_0}{r} - 4 + \frac{r}{r_0}\right) + O((\varepsilon^2, (r-r_0)^2). \tag{5.2.29}$$

Now, the expression (5.2.29) is rewritten in gradient form as

$$\begin{aligned}\frac{v^2}{r}+\varepsilon v &= f_0^2 g_0^2\frac{\partial}{\partial r}\left(4\ln r-4\frac{r}{r_0}+\frac{r^2}{2r_0^2}\right)+\varepsilon f_0 g_0\frac{\partial}{\partial r}\left(4r_0\ln r-4r+\frac{r^2}{2r_0}\right)\\ &\quad+O((\varepsilon^2,(r-r_0)^2)\\ &=\frac{\partial}{\partial r}\left[f_0^2g_0^2\left(4\ln r-4\frac{r}{r_0}+\frac{r^2}{2r_0^2}\right)+\varepsilon f_0g_0\left(4r_0\ln r-4r+\frac{r^2}{2r_0}\right)\right]\\ &\quad+O((\varepsilon^2,(r-r_0)^2). \qquad (5.2.30)\end{aligned}$$

This result, (5.2.30) may be substituted into the partial differential equation (5.2.15) to rewrite it in conservation form as

$$\begin{aligned}\frac{\partial u}{\partial t}+\frac{\partial}{\partial r}\Bigg[\frac{1}{2}u^2+h-f_0^2g_0^2\left(4\ln r-4\frac{r}{r_0}+\frac{r^2}{2r_0^2}\right)\\ -\varepsilon f_0g_0\left(4r_0\ln r-4r+\frac{r^2}{2r_0}\right)\Bigg]=O((\varepsilon^2,(r-r_0)^2). \quad (5.2.31)\end{aligned}$$

Equation (5.2.31) is now in the necessary form (3.2.1), and an application of the jump condition (3.2.5) at $r=s(t)$ is possible. So that the condition (3.2.5) does not reduce to the result (5.2.12), the Taylor series for $g(r)$ given in (5.2.5) is restricted to $r<r_0$. In this way, the term g_0 in (5.2.31) may be substituted with a discontinuous function,

$$g_0(r)=\begin{cases}0, & \text{for } r>s(t),\\ g_0 & \text{for } r<s(t).\end{cases}$$

This allows the jump in the term f_0g_0 to be given as $[f_0g_0]=0-f_0g_0$.

Applying condition (3.2.5) to the partial differential equation (5.2.31) at the discontinuity $r=s(t)$ then yields

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}u^- = \frac{1}{2}(u^-)^2+h^- -(f_0g_0)^2\left(4\ln r_0-4\frac{r_0}{r_0}+\frac{r_0^2}{2r_0^2}\right)\\ -\varepsilon f_0g_0\left(4r_0\ln r_0-4r_0+\frac{r_0^2}{2r_0}\right), \quad (5.2.32)\end{aligned}$$

which simplifies to

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}u^- &= \frac{1}{2}(u^-)^2+h^- -(f_0g_0)^2\left(4\ln r_0-4+\frac{1}{2}\right)-\varepsilon f_0g_0r_0\left(4\ln r_0-4+\frac{1}{2}\right)\\ &=\frac{1}{2}(u^-)^2+h^- -\left[(f_0g_0)^2+\varepsilon f_0g_0r_0\right]\left(4\ln r_0-\frac{7}{2}\right). \qquad (5.2.33)\end{aligned}$$

Using the notation $v^- = g_0 g_0$ to denote the azimuthal velocity limit in the interior solution, and using (5.2.24), equation (5.2.33) becomes

$$2\frac{\mathrm{d}s}{\mathrm{d}t}u^- = (u^-)^2 + 2h^- - v^-[v^- + \varepsilon s(t_0)](8\ln s(t_0) - 7). \tag{5.2.34}$$

Substituting equation (5.2.20) for $\frac{\mathrm{d}s}{\mathrm{d}t}$ into the result (5.2.34) then yields a final relationship between the variables at the discontinuity $r = s(t)$, for any t, which is restated in the original variables from (5.2.14) as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = u_r^- = \sqrt{2h_2^- - u_\theta^-(u_\theta^- + \varepsilon s)(8\ln s - 7)}. \tag{5.2.35}$$

Some physical interpretation of the condition (5.2.35) assists in its comprehension. For example, if $u_\theta^- = 0$ and there is no rotation, then the previous result (5.2.12)-(5.2.13) is recovered, as expected. Also, for small values of ε, and when $\ln r_0 > \frac{7}{8}$ (which occurs for $r_0 \gtrsim 2.4$), the leading term in (5.2.35) is

$$u_r^- = \sqrt{2h_2^- - pu_\theta^-}, \tag{5.2.36}$$

where $0 < p < \mathbb{R}$ is constant. It can now be seen from (5.2.36) that the radial velocity is reduced from its nonrotating value of $\sqrt{2h_2^-}$ independently of the direction of spin. This is also intuitive physically, since if a particle is travelling radially, and is deflected azimuthally, the net radial distance it will have travelled will be reduced. It should be emphasized that this is a local expansion, and does not assume any balance (e.g. geostrophic) which may exist for long time periods.

Chapter Summary

In Chapter 5, the two-layer and single-layer equations for gravity currents in three spatial dimensions have been examined. The two-layer equations were shown to be hyperbolic if two conditions, (5.1.18) and (5.1.19) were satisfied. The single-layer equations simplified this problem greatly, and in all cases of the equations, the systems are strictly hyperbolic precisely when the thickness of the lower layer, ζ_2, is positive.

When considering discontinuous solutions, only the single-layer cases were examined since the two-layer system cannot yet be expressed in conservative form. For the general single layer equations in conservative form (5.2.1), the jump conditions were found to be degenerate. This difficulty was circumvented through a novel method of analysis. A slowly varying plane wave solution for the azimuthal velocity u_θ was assumed which allowed a jump condition to be expressed as (5.2.35) for the flat bottom case without any forcing terms, C_f. This front speed condition was shown to be physically reasonable for the limits when $u_\theta = 0$, and when the radius of the discontinuity is greater than about 2.4 for small Coriolis parameters ε.

Chapter 6

A Numerical Model For Gravity Currents

The study of gravity currents by numerical methods is not new, and many high-resolution computing techniques have been developed to simulate the various physical problems (Simpson, 1997). In fact, when considering the present-day calibre of computing abilities, it is quite appealing to simply purchase one of the available packages of commercial Computational Fluid Dynamics software, adapt it to the desired physical IBP or IBVP, compute, and present the results in a visually attractive graphical format. While this approach is certainly useful, it does have some severe limitations, in addition to the academic leap of faith required when using commercial software.

Arguably, high-resolution computing methods can be expensive, both monetarily and in required computing time, and may not permit the desired amounts of detail for the specific problem in question. Notwithstanding some of the more controversial issues of numerical computing, it is often the case that when using the unsimplified Navier-Stokes Equations, any analytical results or theoretical concepts can be obscured. This may cause the researcher not to pursue various methods of analysis which may eventually prove to be productive. It is the author's belief that relatively simple numerical methods are an important tool for any researcher in the natural, and especially the mathematical, sciences. Such methods are easy to comprehend and adapt, are well-studied and widespread, allow results to be produced quickly and cheaply, and often permit the researcher to gain insight and new ideas pertaining to the problem under study. It is with this philosophy in mind that this chapter is written. The numerical concepts discussed serve to help explain some of the theoretical results, as this is the main intent of the completed calculations.

In the first section, a numerical method is described in detail prior to its subsequent implementation for gravity currents. This method is a generalization of a finite-difference relaxation scheme adeveloped recently to solve hyperbolic systems of nonlinear conservation laws (Jin and Xin, 1995). This previously developed numerical method is generalized herein so that it is applicable to systems of nonlinear hyperbolic conservation laws which may have: spatial dependence in the flux vector, both boundary and initial values, and nonzero forcing terms. The method is described sufficiently so that it may be applied to systems in either one spatial variable, (3.1.6), or several spatial variables (3.1.10). This numerical approach to modelling gravity currents is new (Montgomery and Moodie 1998a,b and 1999a,b), and contrasts sharply with other methods in use such as the method of characteristics (Bonnecaze *et. al* 1993). It is shown to be more adaptable to many applications of gravity current models.

Subsequent Sections 6.2 and 6.3 contain an application of the relaxation method to gravity currents in two and three spatial dimensions, respectively. The two-dimensional case considered in Section 6.2 concerns some of the various cases from Chapter 2, such as the two-layer, weak-stratification, rigid lid, and thin layer equations, while the three-dimensional case examined in Section 6.3 is restricted to the single-layer problem. The three-dimensional two-layer problem is not solved numerically since this case cannot be stated as a closed system of conservation laws. Questions concerning the effects of front speed for the instantaneous release problem are discussed in both sections.

6.1 The Relaxation Method

The numerical method presented by Jin and Xin (1995) is a system of relaxation schemes for systems of conservation laws in several space dimensions. These schemes are finite-difference, iterative, and have been shown to be total variation diminishing (TVD) for scalar hyperbolic conservation laws. For nonlinear hyperbolic systems, the second-order relaxation schemes are stable, conservative, and capture discontinuous solutions in a nonoscillatory manner corresponding to the correct shock speed for initial value problems.

In this section, the scheme proposed by Jin and Xin (1995) is generalized to include problems for which the system has spatial dependence in the flux vector, boundary values and nonzero forcing terms. This generalization of the method allows it to be employed later to solve the gravity current equations developed in Chapter 2. It is described first for systems of conservation laws in one spatial dimension so that the ideas may be expressed in as simple a notation as possible. The subsequent generalization to more spatial dimensions is then somewhat condensed since the main ideas are all contained in the previous case.

6.1.1 The Relaxation Method in One Spatial Dimension

The relaxation scheme is designed for application to an initial value problem for a system of the form

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f} = \mathbf{b}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^{+}, \tag{6.1.1}$$

where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^n$, $\mathbf{f} = \mathbf{f}(\mathbf{u},x) \in \mathbb{R}^n$, and $\mathbf{b} = \mathbf{b}(\mathbf{u},x) \in \mathbb{R}^n$ are all vector-valued functions. The variables are all assumed to be nondimensional so that units may be neglected throughout the expressions in this Chapter.

Associated with the system (6.1.1) is a larger system, called the *modified relaxation system*. This system consists of $2n$ equations derived from the n equations in (6.1.1), and it can be expressed as

$$\frac{\partial}{\partial t}\mathbf{w} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{w}) = \mathbf{B}(\mathbf{w},\mathbf{b},\mathbf{f}), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{6.1.2}$$

where $\mathbf{w}$, $\mathbf{F}$, and $\mathbf{B}$ are all vector-valued functions in $\mathbb{R}^{2n}$. $\mathbf{f}$ in (6.1.2) is the same vector-valued function that appears in equation (6.1.1). These new vectors are defined as

$$\mathbf{w} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ \alpha u_1 \\ \vdots \\ \alpha u_n \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \\ \frac{1}{\varepsilon}(f_1 - v_1) \\ \vdots \\ \frac{1}{\varepsilon}(f_n - v_n) \end{bmatrix}, \tag{6.1.3}$$

for real scalars α and ε. The symbol ε introduced in (6.1.3) is a positive parameter, and is not related to the Coriolis parameter discussed previously. Although this notation is somewhat ambiguous, the Coriolis parameter will not be used until Section 3 of this chapter, and specific reference will be made there to avoid confusion of the terms.

Using the definition (6.1.3), the system (6.1.2) may be restated as two separate systems of n equations each by

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{v} = \mathbf{b}, \tag{6.1.4}$$

and

$$\frac{\partial}{\partial t}\mathbf{v} + \alpha \mathbf{I}_n \frac{\partial}{\partial x}\mathbf{u} = \frac{1}{\varepsilon}(\mathbf{f} - \mathbf{v}). \tag{6.1.5}$$

The modified relaxation system interpreted in the form (6.1.4), (6.1.5) may now be described as linear, and is such that in the limit as $\varepsilon \to 0$, equation (6.1.5) has the solution $\mathbf{f} = \mathbf{v}$, which subtitutes back into (6.1.4) to yield the original system (6.1.1). In this way, it is theorized that for small enough values of ε, solutions of

(6.1.1) can be obtained as limits of solutions to the linear, and much simpler to deal with, system (6.1.2).

The system (6.1.2) is stated in more general terms than that considered previously (Jin and Xin, 1995) where the simpler systems $\mathbf{f} = \mathbf{f}(\mathbf{u})$ and $\mathbf{b} = \mathbf{0}$ were considered exclusively. Before the numerical method to solve the linear system (6.1.2) is outlined, a few comments about the parameters α and ε are necessary.

The easiest explanation concerns ε, in light of the desired relaxation limit as $\varepsilon \to 0$. It was shown by Jin and Xin (1995) that when solving the system (6.1.2) numerically, the solutions tended to solutions of (6.1.1) for

$$\varepsilon = O\big((\Delta t)^3\big), \tag{6.1.6}$$

where Δt is the width of the discretization for the time variable t. In general, for the present calculations, a value of $\varepsilon = 10^{-10}$ or 10^{-11} was found to be sufficient to satisfy (6.1.6) while being small enough so that the approximate solutions to (6.1.2) were indistinguishable from those of (6.1.1).

The parameter α requires more description than ε. α is a constant parameter whose choice depends on the magnitude of the eigenvalues of the Jacobian matrix $\mathbf{f}'(\mathbf{u}, x)$, where the derivative is taken with respect to $\mathbf{u}$ alone. This matrix is precisely that examined in detail in previous chapters when discussing hyperbolicity. In general, for the system to be dissipative, the condition

$$\alpha > \lambda^2, \tag{6.1.7}$$

is necessary, where $\lambda = \max_{i=1,\dots,n} |\lambda_i(\mathbf{u}, x)|$ is the supremum of the eigenvalues λ_i of $\mathbf{f}'(\mathbf{u}, x)$. Thus, from the knowledge of the eigenvalues for the hyperbolic system considered in Chapters 4 and 5, a lower bound for α is easily determined. An upper bound for α arises quite naturally from the Cauchy-Friedrichs-Lewy (CFL) condition for the numerical stability of a linear system. The CFL number for the system (6.1.2) is defined following Jin and Xin (1995) by the expression

$$\text{CFL}\ \# = \frac{\sqrt{\alpha}\Delta t}{\Delta x}. \tag{6.1.8}$$

The CFL number, in general, must be less than 1 for numerical stability. This stability condition, is then expressed as either an upper bound for the grid width Δt, or a lower bound for the parameter α as

$$\alpha \leq \left(\frac{\Delta x}{\Delta t}\right)^2, \tag{6.1.9}$$

where Δx is the grid width of the spatial discretization for x.

Equations (6.1.8)-(6.1.9) may at first glance seem somewhat unusual due to the lack of an apparent velocity term. However, closer inspection reveals that the velocity is present through the parameter α. From the form of the system (6.1.2) expressed as (6.1.4)-(6.1.5), the left hand side is seen to be a linear system, with a natural velocity of α.

The effects of choosing α are such that it is desirable to choose α as small as practicable while satisfying (6.1.7), and then to fix the grid widths Δt and Δx so that (6.1.9) holds. Conceptually, α must be large enough such that the characteristic curve $x - \alpha t$ creates a wide enough cone in (x, t) space to encompass the characteristic curves $x - \lambda_i t$ from (6.1.1), while being small enough so that shocks are permitted to remain for enough time steps to be observable. If α is too large, then the resolution of discontinuities is poor, and the system becomes first order (Jin and Xin, 1995).

With α and ε fixed, a finite-difference numerical method is now described to calculate solutions of (6.1.2). This method is generalized from that proposed by Jin and Xin (1995), and is a second-order TVD Runge-Kutta splitting scheme, which employs Van Leer's slope limiter (LeVeque, 1992) to remove oscillations near any shocks. It is an iterative scheme on a numerical approximation of $\mathbf{w}$ which is discretized on a uniform grid. To maintain the conservative nature of this scheme, cell averages are used (LeVeque, 1992) as described in the following paragraph.

The spatial domain $x \in [0, \infty)$ and $t \in [0, \infty)$ is discretized by the points $x_{j+\frac{1}{2}}$ for $j = 0, 1, 2, \ldots$ with $x_{\frac{1}{2}} = 0$, and t_k for $k = 0, 1, 2, \ldots$ with $t_0 = 0$. The uniform grid width is denoted by $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j > 0$, and $\Delta t = t_{k+1} - t_k$ for $k \geq 0$. At an arbitrary grid point $(x_{j+\frac{1}{2}}, t_k)$, the approximate point value of $\mathbf{w}$ is denoted by $\mathbf{w}^k_{j+\frac{1}{2}} \approx \mathbf{w}(x_{j+\frac{1}{2}}, t_k)$. Additionally, for the j^{th} cell, defined as $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j > 0$, $\mathbf{w}$ is approximated by the cell average $\mathbf{w}^k_j$ given as

$$\mathbf{w}^k_j = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{w}(x, t_k) dx. \tag{6.1.10}$$

The splitting scheme is defined as a five step iterative process for each vector $\mathbf{w}^k_j$, and is written in the following general notation without the subscript j:

$$\mathbf{w}^* = \mathbf{w}^k - \Delta t \mathbf{B}^*, \tag{6.1.11}$$

$$\mathbf{w}^{(1)} = \mathbf{w}^* - \Delta t D(\mathbf{F}^*), \tag{6.1.12}$$

$$\mathbf{w}^{**} = \mathbf{w}^{(1)} + \Delta t \mathbf{B}^{**} + 2\Delta t \mathbf{B}^{*}, \tag{6.1.13}$$

$$\mathbf{w}^{(2)} = \mathbf{w}^{**} - \Delta t D(\mathbf{F}^{**}), \tag{6.1.14}$$

and

$$\mathbf{w}^{k+1} = \frac{1}{2}\left(\mathbf{w}^{k} + \mathbf{w}^{(2)}\right). \tag{6.1.15}$$

The new notation in (6.1.12) and (6.1.14) is given by the spatial discretization operator $D : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined for the j^{th} cell as

$$D(\mathbf{F}^{*}) = \frac{1}{\Delta x}\left(\mathbf{F}^{*}_{j+\frac{1}{2}} - \mathbf{F}^{*}_{j-\frac{1}{2}}\right), \tag{6.1.16}$$

with a similar definition for $\mathbf{F}^{**}$. The abridged notation in expressions (6.1.11)-(6.1.16) is defined as $\mathbf{F}^{*} = \mathbf{F}^{*}(\mathbf{w}^{*})$, $\mathbf{F}^{*}_{j+\frac{1}{2}} = \mathbf{F}^{*}(\mathbf{w}^{*}_{j+\frac{1}{2}})$, and similarly for $\mathbf{B}^{*}$, $\mathbf{F}^{**}$, etc.

The spatial discretization to find $\mathbf{F}_{j+\frac{1}{2}}$ for either $\mathbf{F}^{*}$ or $\mathbf{F}^{**}$ is given through equation (6.1.3) by Van Leer's second-order MUSCL scheme (Jin and Xin, 1995) via

$$\mathbf{v}_{j+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{v}_{j+1} + \mathbf{v}_{j}\right) - \frac{\sqrt{\alpha}}{2}\left(\mathbf{u}_{j+1} - \mathbf{u}_{j}\right) + \frac{\Delta x}{4}\left(\sigma^{+}_{j} - \sigma^{-}_{j+1}\right), \tag{6.1.17}$$

and

$$\mathbf{u}_{j+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{u}_{j+1} + \mathbf{u}_{j}\right) - \frac{1}{2\sqrt{\alpha}}\left(\mathbf{v}_{j+1} - \mathbf{v}_{j}\right) + \frac{\Delta x}{4\sqrt{\alpha}}\left(\sigma^{+}_{j} + \sigma^{-}_{j}\right). \tag{6.1.18}$$

The slope vectors in (6.1.18) are given by

$$\sigma^{\pm}_{j} = \frac{1}{\Delta x}\left[\left(\mathbf{v}_{j+1} - \mathbf{v}_{j}\right) \pm \sqrt{\alpha}\left(\mathbf{u}_{j+1} - \mathbf{u}_{j}\right)\right]\phi(\theta^{\pm}_{j}), \tag{6.1.19}$$

where each component of the n-vector $\theta^{\pm}_{j}$ is given by

$$\theta^{\pm}_{j} = \frac{(v_j - v_{j-1}) \pm \sqrt{\alpha}(u_j - u_{j-1})}{(v_{j+1} - v_j) \pm \sqrt{\alpha}(u_{j+1} - u_j)}, \tag{6.1.20}$$

and the slope limiter function $\phi : \mathbb{R}^{n} \to \mathbb{R}^{n}$ is defined by

$$\phi(\theta) = \begin{bmatrix} \phi_1(\theta_1) \\ \vdots \\ \phi_n(\theta_n) \end{bmatrix}, \quad \phi_i(\theta_i) = \frac{|\theta_i| + \theta_i}{1 + |\theta_i|}, \quad i = 1, \ldots, n. \tag{6.1.21}$$

It can be observed from (6.1.11)-(6.1.15) that there are only two implicit steps, namely (6.1.11) and (6.1.13), with the other three steps being explicit. This does not present as many difficulties as it may seem to at first glance, since, for example, (6.1.11) can be written in components from (6.1.3) as

$$u_i^* = u_i^k - \Delta t b_i(\mathbf{u}^*, x), \quad i = 1, \dots, n, \tag{6.1.22}$$

and

$$v_i^* = v_i^k - \frac{\Delta t}{\varepsilon}(f_i(\mathbf{u}^*, x) - v_i^*), \quad i = 1, \dots, n. \tag{6.1.23}$$

Therefore, only $\mathbf{b}$ needs to be inverted from (6.1.22), a difficulty which is not great since $\mathbf{b}$ is usually straightforward in most of the gravity current equations to be solved numerically. Consequently, $\mathbf{v}^*$ can be solved explicitly by rearranging (6.1.23) as the vector equation

$$\mathbf{v}^* = \left(1 + \frac{\Delta t}{\varepsilon}\right)^{-1} \left[\mathbf{v}^* - \frac{\Delta t}{\varepsilon}\mathbf{f}(\mathbf{u}^*, x)\right], \tag{6.1.24}$$

once $\mathbf{u}^*$ is determined from (6.1.22). A similar analysis for $\mathbf{B}^{**}$ may be completed for equation (6.1.13) to get resulting equations which are quite similar to (6.1.22)-(6.1.24)

6.1.2 The Relaxation Method in Several Space Dimensions

The numerical scheme in Section 6.1.1 for systems of hyperbolic conservation laws may be expanded to encompass systems in several space variables by generalization of the results stated by Jin and Xin (1995). Instead of (6.1.1), the problem to be considered in p spatial dimensions is

$$\frac{\partial}{\partial t}\mathbf{u} + \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\mathbf{f}^{(i)} = \mathbf{b}, \quad (\mathbf{x}, t) \in \mathbb{R}^p \times \mathbb{R}^+, \tag{6.1.25}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$, $\mathbf{f} = \mathbf{f}(\mathbf{u}, \mathbf{x}) \in \mathbb{R}^n$, and $\mathbf{b} = \mathbf{b}(\mathbf{u}, \mathbf{x}) \in \mathbb{R}^n$. Associated with (6.1.25) is the relaxation system, which is stated in a style similar to equations (6.1.4)-(6.1.5) as

$$\frac{\partial}{\partial t}\mathbf{u} + \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\mathbf{v}^{(i)} = \mathbf{b}, \tag{6.1.26}$$

and

$$\frac{\partial}{\partial t}\mathbf{v}^{(i)} + \alpha_i \mathbf{I}_n \frac{\partial}{\partial x_i}\mathbf{u} = \frac{1}{\varepsilon}(\mathbf{f}^{(i)} - \mathbf{v}^{(i)}), \quad i = 1, \dots, p. \tag{6.1.27}$$

In the relaxation system (6.1.26)-(6.1.27), ε is the relaxation constant, which is considered to be a small parameter. The choice of the constants α_i requires more consideration than was done for the $p = 1$ case, and uses the notion of an entropy function for the system (6.1.25). The entropy is not the same as the physical concept, and a definition is stated from Godlewski and Raviart (1996, p.21). A convex function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called an *entropy* for (6.1.25) if there exist p functions $\mathbf{q}^{(i)} : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, p$, called *entropy fluxes* such that

$$\eta'(\mathbf{u})\mathbf{f}^{(i)\prime}(\mathbf{u},\mathbf{x}) = \mathbf{q}^{(i)\prime}(\mathbf{u},\mathbf{x}), \quad i = 1, \ldots, p, \tag{6.1.28}$$

holds, where the prime notation denotes the Jacobian matrix with respect to $\mathbf{u}$.

The condition that the constants α_i must satisfy can now be stated from Jin and Xin (1995), which is that the system (6.1.26)-(6.1.27) is dissipative if

$$\sum_{i=1}^{p} \frac{(\lambda^{(i)})^2}{\alpha_i} \le 1, \tag{6.1.29}$$

where

$$\lambda^{(i)} = \max_{i \le j,k \le n} \left\{ |\lambda_j^{(i)}|, |\mu_k^{(i)}| \right\}, \quad i = 1, \ldots, p. \tag{6.1.30}$$

In equation (6.1.30), the scalars $\lambda_j^{(i)}$ and $\mu_k^{(i)}$ are the eigenvalues of the matrices $\mathbf{f}^{(i)\prime}(\mathbf{u},\mathbf{x})\eta''(\mathbf{u})^{-1}$ and $\eta''(\mathbf{u})\mathbf{f}^{(i)\prime}(\mathbf{u},\mathbf{x})$, respectively. The matrix η'' is the Hessian matrix, and $\mathbf{f}^{(i)\prime}$ is the Jacobian matrix, which are both symmetric matrices for which real eigenvalues always exist.

Once the α_i are chosen appropriately, the second-order Runge-Kutta splitting scheme (6.1.11)-(6.1.15) applies, with $\mathbf{w}$, $\mathbf{F}$, and $\mathbf{B}$ generalized to be vectors of length $(p+1)n$ in the style of definition (6.1.3). The only difference is that the spatial discretization operator D becomes multi-dimensional. In addition, given a regular partition of width Δx for each of the variables x_i, $i = 1, \ldots, p$, the CFL condition for stability, stated as (6.1.8) in one spatial variable, becomes

$$\text{CFL } \# = \max_{1 \le i \le p} \left\{ \frac{\sqrt{\alpha_i}\Delta t}{\Delta x} \right\}. \tag{6.1.31}$$

For numerical stability, the condition that the CFL number is ≤ 1, may be rewritten via (6.1.31) as the more useful equation

$$\max_{1 \le i \le p} \sqrt{\alpha_i} < \frac{\Delta x}{\Delta t}. \tag{6.1.32}$$

This constraint restricts the relative size of the grid widths which may be used so that the numerical scheme remains stable.

An example of the spatial discretization is stated for $p = 2$, since this is the situation for the gravity current equations stated in Chapters 2 and 5, wherein $x_1 = x$ and $x_2 = y$. A grid is created for the spatial grid points $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ for i and j integers. The uniform grid is denoted with a grid width of Δx for both the horizontal directions, $\Delta x = x_{i+\frac{1}{2},j+\frac{1}{2}} - x_{i-\frac{1}{2},j+\frac{1}{2}}$ and $\Delta x = y_{i+\frac{1}{2},j+\frac{1}{2}} - y_{i+\frac{1}{2},j-\frac{1}{2}}$ (Δy is not used in favour of the simpler notation $\Delta x = \Delta y$). With the point values for a vector $\mathbf{u}(x, y)$ given by $\mathbf{u}_{i+\frac{1}{2},j+\frac{1}{2}} = \mathbf{u}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$, the spatial cell average, introduced in one variable as (6.1.10), is generalized to

$$\mathbf{u}_{i,j} = \frac{1}{(\Delta x)^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{u}(x, y)\,dy\,dx. \tag{6.1.33}$$

With the time variable discretized as before, the discretization operator D in (6.1.12) and (6.1.14) becomes

$$D(\mathbf{F}^*) = \begin{cases} D_x \mathbf{v}^{(1)^*} + D_y \mathbf{v}^{(2)^*} & \text{for the first } n \text{ components,} \\ \alpha_1 D_x \mathbf{u}^* & \text{for the second } n \text{ components, and} \\ \alpha_2 D_y \mathbf{u}^* & \text{for the last } n \text{ components.} \end{cases} \tag{6.1.34}$$

The notation D_x and D_y denotes the spatial discretization with respect to the x and y variables, and is given for each component by

$$D_x u^*_{i,j} = \frac{1}{\Delta x}\left(u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j}\right), \tag{6.1.35}$$

and for the y variable as

$$D_y u^*_{i,j} = \frac{1}{\Delta x}\left(u^*_{i,j+\frac{1}{2}} - u^*_{i,j-\frac{1}{2}}\right). \tag{6.1.36}$$

Similar definitions may be stated for $D_x \mathbf{v}^{(1)^*}$ and $D_y \mathbf{v}^{(2)^*}$, with u replaced by $v^{(1)}$ or $v^{(2)}$, and an expression using $\mathbf{F}^{**}$ can be stated identically to (6.1.34)-(6.1.36) with the superscript * replaced by **.

The second-order MUSCL discretization with Van Leer's slope limiter is given componentwise, without the * superscript, by

$$u_{i+\frac{1}{2},j} = \frac{1}{2\sqrt{\alpha_1}}\left(v^{(1)}_{i,j} - v^{(1)}_{i+1,j}\right) + \frac{1}{2}\left(u_{i,j} + u_{i+1,j}\right) + \frac{\Delta x}{4\sqrt{\alpha_1}}\left(\sigma^{x,+}_{i,j} + \sigma^{x,-}_{i+1,j}\right). \tag{6.1.37}$$

and

$$v^{(1)}_{i+\frac{1}{2},j} = \frac{1}{2}\left(v^{(1)}_{i+1,j} + v^{(1)}_{i,j}\right) + \frac{\sqrt{\alpha}}{2}\left(u_{i,j} - u_{i+1,j}\right) + \frac{\Delta x}{4}\left(\sigma^{x,+}_{i,j} - \sigma_{i+1,j}.^{x,-}\right), \tag{6.1.38}$$

The slopes $\sigma^{x,\pm}_{i,j}$ are defined as

$$\sigma^{x,\pm}_{i,j} = \frac{1}{\Delta x}\left[(v^{(1)}_{i+1,j} - v^{(1)}_{i,j}) \pm \sqrt{\alpha_1}(u_{i+1,j} - u_{i,j})\right]\phi(\theta^{x,\pm}_{i,j}), \tag{6.1.39}$$

where

$$\theta^{x,\pm}_{i,j} = \frac{(v^{(1)}_{i,j} - v^{(1)}_{i-1,j}) \pm \sqrt{\alpha_1}(u_{i,j} - u_{i-1,j})}{(v^{(1)}_{i+1,j} - v^{(1)}_{i,j}) \pm \sqrt{\alpha_1}(u_{i+1,j} - u_{i,j})}, \tag{6.1.40}$$

and ϕ is defined as before in (6.1.20). For $\mathbf{v}^{(2)}$, the formulas (6.1.37)-(6.1.40) remain the same, with the change of notation $\alpha_1 \to \alpha_2$ and the components $v^{(1)}_{i,j} \to v^{(2)}_{i,j}$. The discretization in the y direction, for $\mathbf{u}_{i,j+\frac{1}{2}}$ and $\mathbf{v}^{(2)}_{i,j+\frac{1}{2}}$, may be obtained by interchanging x and y where they appear in the expressions (6.1.37)-(6.1.40), except for the Δx term, and by switching i and j, for example $u_{i+\frac{1}{2},j} \to u_{i,j+\frac{1}{2}}$.

6.1.3 Numerical Implementation of Initial and Boundary Values

For the system (6.1.1) or (6.1.24), an initial value problem may be stated with a known vector function $\mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^n$ such that $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^p$, with $p = 1$ for the system (6.1.1). Using this, an initial value for the relaxation system (6.1.4)-(6.1.5) or (6.1.26)-(6.1.27) is chosen to be

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \text{ and } \mathbf{v}^{(i)}(\mathbf{x},0) = \mathbf{f}^{(i)}(\mathbf{u}_0(\mathbf{x}),\mathbf{x}), \text{ for } i = 1,\ldots,p. \tag{6.1.41}$$

Such a choice of initial value for the vectors $\mathbf{v}^{(i)}$ was suggested by Jin and Xin (1995) for spatially independent flux functions, and the generalization is suggested as (6.1.41). This choice avoids the introduction of an initial boundary layer (Jin and Xin, 1995), and is consistent with the equilibrium solution $\mathbf{v}^{(i)} = \mathbf{f}^{(i)}(\mathbf{u},\mathbf{x})$ which is desired in the limit as $\varepsilon \to 0$.

In the case with $p = 1$, for gravity currents in two spatial dimensions, the system of conservation laws which will be solved numerically is of the form (6.1.1), and there is a boundary value $\mathbf{u}(0,t)$ of the form (4.3.4). Although it would seem straightforward to follow the pattern suggested in (6.2.39) by stating the boundary values for $\mathbf{v}$ as

$$\mathbf{v}(0,t) = \mathbf{f}(\mathbf{u}(0,t),0), \tag{6.1.42}$$

this is not possible since only two of the four components of $\mathbf{u}(0,t)$ are specified at the boundary $x = 0$ through the physical boundary condition (4.3.4). For this reason, a new approach to implementation of this boundary condition must be devised.

The innovation chosen here is to state the unknown values of $\mathbf{u}(0,t)$, discretized at each time step as $\mathbf{u}^k_{\frac{1}{2}}$, in terms of the known values $\mathbf{u}^k_{i+\frac{1}{2}}$ for $i > 0$. These are computed for each k from the previous time step's vectors $\mathbf{u}^{k-1}_{i+\frac{1}{2}}$ for $i \geq 0$. This may be done, for example, as described by Leveque (1992), through a Taylor Series approximation of the cell averages. Such a calculation for a single unknown component follows which is then applied in the subsequent section.

For an arbitrary component of $\mathbf{u}^k_i$, denoted simply by u_i, it is assumed that u_i is known for $i > 1$, and u_1 must be calculated through an interpolation using these values. A Taylor series, valid near $x = 0$, is assumed for the solution u of the form

$$u(x) = u(0) + u'(0)x + \frac{1}{2}u''(0)x^2 + O(x^3), \tag{6.1.43}$$

which is approximated by the cell averages u_i at the midpoints of each cell. That is, from (6.1.43), and the definition of the spatial cells in the paragraph prior to equation (6.1.10), it follows that

$$u_1 = u(0) + u'(0)\left(\frac{\Delta x}{2}\right) + \frac{1}{2}u''(0)\left(\frac{\Delta x}{2}\right)^2 + O((\Delta x)^3), \tag{6.1.44}$$

$$u_2 = u(0) + u'(0)\left(\frac{3\Delta x}{2}\right) + \frac{1}{2}u''(0)\left(\frac{3\Delta x}{2}\right)^2 + O((\Delta x)^3), \tag{6.1.45}$$

and

$$u_3 = u(0) + u'(0)\left(\frac{5\Delta x}{2}\right) + \frac{1}{2}u''(0)\left(\frac{5\Delta x}{2}\right)^2 + O((\Delta x)^3). \tag{6.1.46}$$

To remove numerical oscillations introduced at the boundary, the smoothness assumption $u'(0) = 0$ is made which allows (6.1.44) to be restated as

$$u_1 = u(0) + \frac{1}{8}u''(0)(\Delta x)^2 + O((\Delta x)^3). \tag{6.1.47}$$

Solving (6.1.45) and (6.1.46) for $u(0)$ and $u''(0)$ in terms of u_2 and u_3 gives

$$u''(0) = \frac{1}{2(\Delta x)^2}(u_3 - u_2) + O(\Delta x), \tag{6.1.48}$$

and

$$u(0) = \frac{25}{16}u_2 - \frac{9}{16}u_3 + O((\Delta x)^3). \tag{6.1.49}$$

Expressions (6.1.48) and (6.1.49) can now be substituted back into equation (6.1.47) to give the result

$$\begin{aligned} u_1 &= \frac{25}{16}u_2 - \frac{9}{16}u_3 + \frac{1}{16}(u_3 - u_2) + O((\Delta x)^3) \\ &= 1.5u_2 - 0.5u_3 + O((\Delta x)^3). \end{aligned} \tag{6.1.50}$$

It is this expression, (6.1.50), which is subsequently implemented, neglecting the smaller terms, in the boundary conditions at $x = 0$ for determining the first cell's unknown portions.

With the description of a general relaxation scheme for conservation laws in one or more spatial variables complete, the second-order discretization can be applied to specific systems. The equations for gravity currents in two dimensions are mostly systems of the form (6.1.1) in one spatial variable; consequently, the numerical methods are applied first to these systems in the following Section.

6.2 Numerical Solutions for Gravity Currents in Two Dimensions

To portray the usefulness of the relaxation method to hyperbolic systems of conservation laws of the form (6.1.1), the discretization described in Section 6.1 can be applied to compute solutions to some of the IVPs for gravity currents in two spatial dimensions. Application of the relaxation method to the two-layer gravity current equations in two spatial dimensions has been published previously (Montgomery and Moodie, 1998a, 1999a). The first application (Montgomery and Moodie, 1998a) showed the usefulness of the method for shock determination within the initial release problem, and compared numerical solutions to the two-layer equations with calculations for the weak-stratification and thin-lower/upper layer equations. Later, constant slope bottom topography was included, with an introduction of the forcing term $\kappa C_f u_2^2/(h_2 - h_B)T$ discussed in Chapter 2, in Montgomery and Moodie (1999a). The addition of a following flow was completed in Montgomery and Moodie (1998b).

The aim of this section is to establish the appropriateness of the relaxation method to the model equations developed in Chapter 2 for gravity currents in two spatial dimensions, without simply restating the previous results (Montgomery and Moodie, 1998a, 1999a). To do this, the relaxation method will be examined using various values of the parameters α and ε, and the grid widths Δx and Δt, to display stability and resolution of the method. In addition, a discussion of the numerical determination of front position, and the front speed graphs to be portrayed, will be included. The numerical scheme will then be applied to the two-layer equations with varying topography, and the rigid lid equations will be solved for the first time by the modified relaxation method outlined in Section 6.1.

6.2.1 Resolution of the Relaxation Method

The two-layer equations to be solved are those given in the nondimensional conservation form (2.4.30), which are in the desired form (6.1.1). The problem of instantaneous release of a fixed volume of dense fluid consists of these four equations, with the initial value and a boundary value given by (4.3.1)-(4.3.4) where the initial value chosen for the next few calculations is the constant value $\zeta_{2_0} = 0.75$. Throughout this ubsection the equations (2.4.30) will be considered without the effects of topography or lower layer forcing, and it is assumed that $h_B = 0$ and $C_f = 0$, so that these factors do not affect any changes in the parameters α, ε, CFL number, and Δx which are varied.

To decrease the amount of computing time required in solving the system of four equations (2.4.30), a reduced form is considered when the lower layer height is negligible. Physically, when the lower layer height is positive, equations (2.4.30) hold in their entirety; however, if the lower layer thickness, ζ_2, is zero, then only the two equations from (2.4.30) relating to the upper layer variables u_1 and ζ_1 need to be solved. This approach is implemented numerically by keeping track of the advancing lower layer gravity current front position which moves from the left to the right in the following diagrams. This front tracking is achieved at an arbitrary position, and for the calculations presented herein, a point at which ζ_2 is less than 10^{-10} was considered sufficiently negligible such that only the single upper layer exists to the right of this point. Thus, for such grid points x_j, the reduced system consisting of the two upper layer equations from (2.4.30) was solved, rather than solving the larger system containing a redundant number of zeros. This reduction in the size of the system decreased the required computing time, which, although not always important as most calculations are completed in a matter of several minutes, sets a desirable precedent for subsequent calculations which are more involved.

The first parameter to be investigated is the dissipation parameter α which, from (6.1.7), is bounded below by the eigenvalues from (2.4.30). Although these are not known prior to solving the equations, the experience gained from previous calculations (Montgomery and Moodie, 1998a) places the supremum of the eigenvalues to be somewhere around 3 for most calculations. Using this value, a minimum choice for α which would then satisfy all of the subsequent calculations, would be $\alpha = 9$.

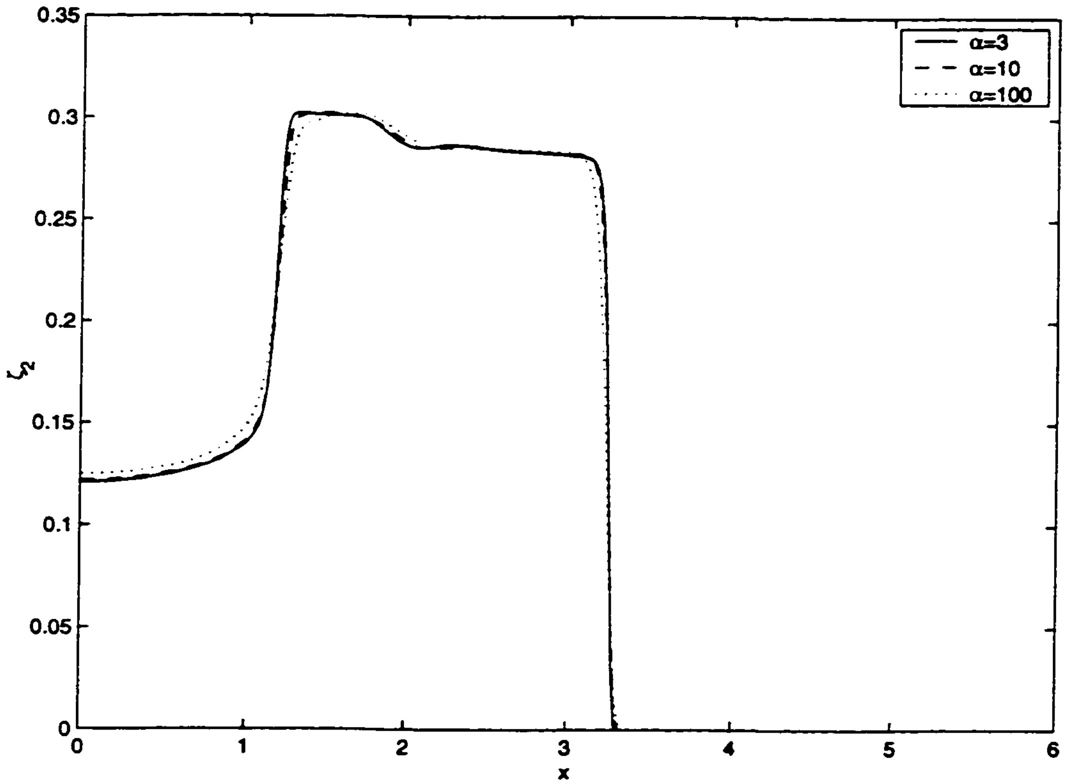

FIGURE 6.1. Graph of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of α. Relevant parameter values: $\varepsilon = 10^{-10}$, $\Delta x = 0.02$, and CFL $\# = 0.75$.

As can be seen from Figure 6.1, the resolution of the shocks as strictly vertical discontinuities decreases with increasing values of α. This is not as large a drawback as is imposed by the upper bound (6.1.9) on Δt, which requires that if α is increased by a factor of 10, then Δt is decreased by a factor of 10^2, thus increasing the computing time required. From results such as those portrayed in Figure 6.1, a standard value of $\alpha = 10$ was chosen for use in subsequent calculations for Section 6.2. This value is large enough to satisfy (6.1.7), without substantially sacrificing the resolution of discontinuities, and is small enough so that the computation time is reasonable. Figure 6.1 also displays a rear bore which travels catches up to the initial front prior to the gravity current similarity solution. This type of behaviour has been observed experimentally (Rottman and Simpson, 1983) for large enough initial release height values.

The second parameter to consider is the relaxation parameter, ε. Variation of this parameter does not have any effect on the calculation time, or the choice of grid width, but its choice is very important to shock resolution. To display the effects of variation in ε, calculations are portrayed in Figure 6.2 for three differing values.

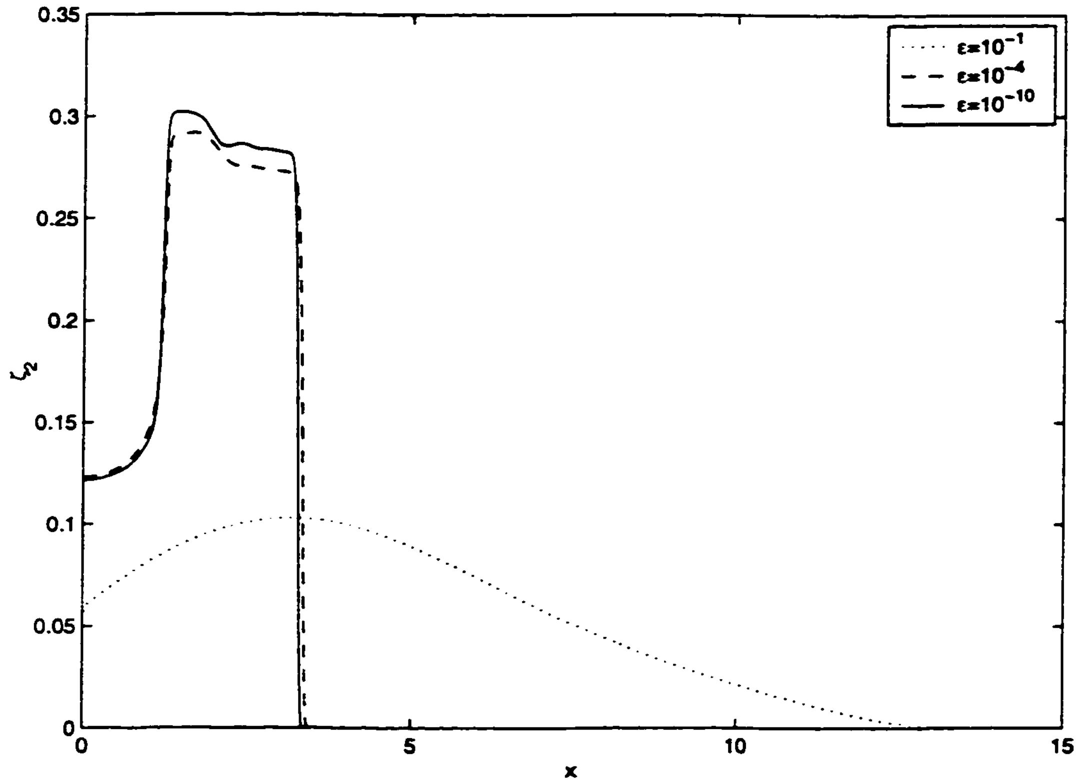

FIGURE 6.2. Graph of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of ε. Relevant parameter values: $\alpha = 10$, $\Delta x = 0.02$, and CFL $\# = 0.75$.

It can clearly be seen in Figure 6.2 that the larger ε values tend to cause the shock resolution to be quite limited. In fact, the numerical scheme does not approximate a solution to any useful degree of accuracy, and choosing sufficiently small values of

ε is quite important. The curve corresponding to a value of $\varepsilon = 10^{-1}$, for example, exhibits a blurring of the fine features such that they become unrecognizable. For these reasons, as well as ensuring that ε satisfies (6.1.6), a value for the relaxation parameter of $\varepsilon = 10^{-10}$ was chosen for subsequent calculations.

The third parameter which effects the numerical stability of the calculations is the CFL number (6.1.8). Although linear stability theory gives a requirement that this number must be less than 1 (LeVeque, 1992), choosing this number to be small again forces Δt to be small via the inequality (6.1.9). A few initial calculations showed that the relaxation method proved fairly robust to changes in the CFL number, in the range of $0.2 \leq$ CFL number ≤ 2.5. Since numerical instability has been observed in past calculations (Montgomery and Moodie 1999a), a CFL number of 0.75 was chosen for use in subsequent calculations.

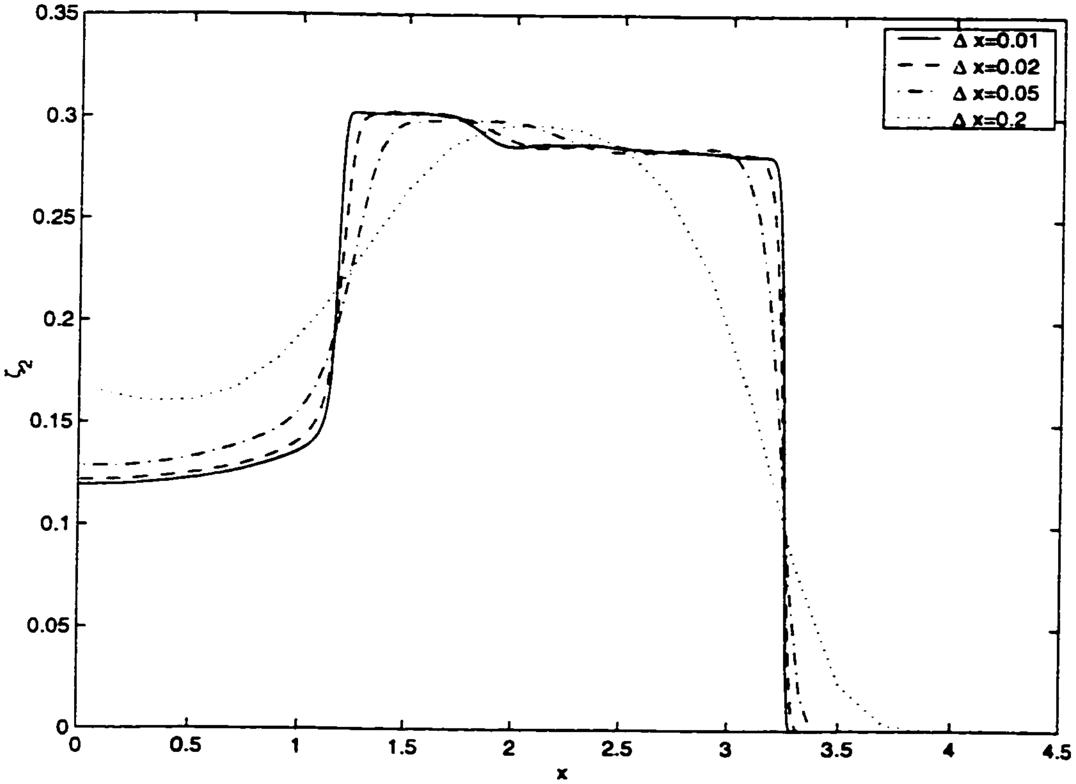

FIGURE 6.3. Graph of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of Δx. Relevant parameter values: $\alpha = 10$, CFL number $= 0.75$, and $\varepsilon = 10^{-10}$.

With α and the CFL number fixed, equation (6.1.8) gives a relationship between Δt and Δx, $\Delta t = \text{CFL } \#\Delta x/\sqrt{\alpha}$. It is a standard practice to first choose Δx, and then fix Δt in this manner. The question of numerical stability is therefore not important when varying Δx; rather, the resolution and refinement of any discontinuities is limited by the spatial grid width, Δx.

A similar portrayal in the style of the previous results is given in Figure 6.3, which shows the lower layer height for varying grid widths. Although smaller values for Δx will lead to better spatial resolution, there is a limitation imposed by the computational power available. For the purposes of this thesis, a grid width of $\Delta x = 0.02$ was found to be sufficient to determine the front position adequately without necessitating lengthy computations, and was used exclusively for the remaining diagrams in Section 6.2.

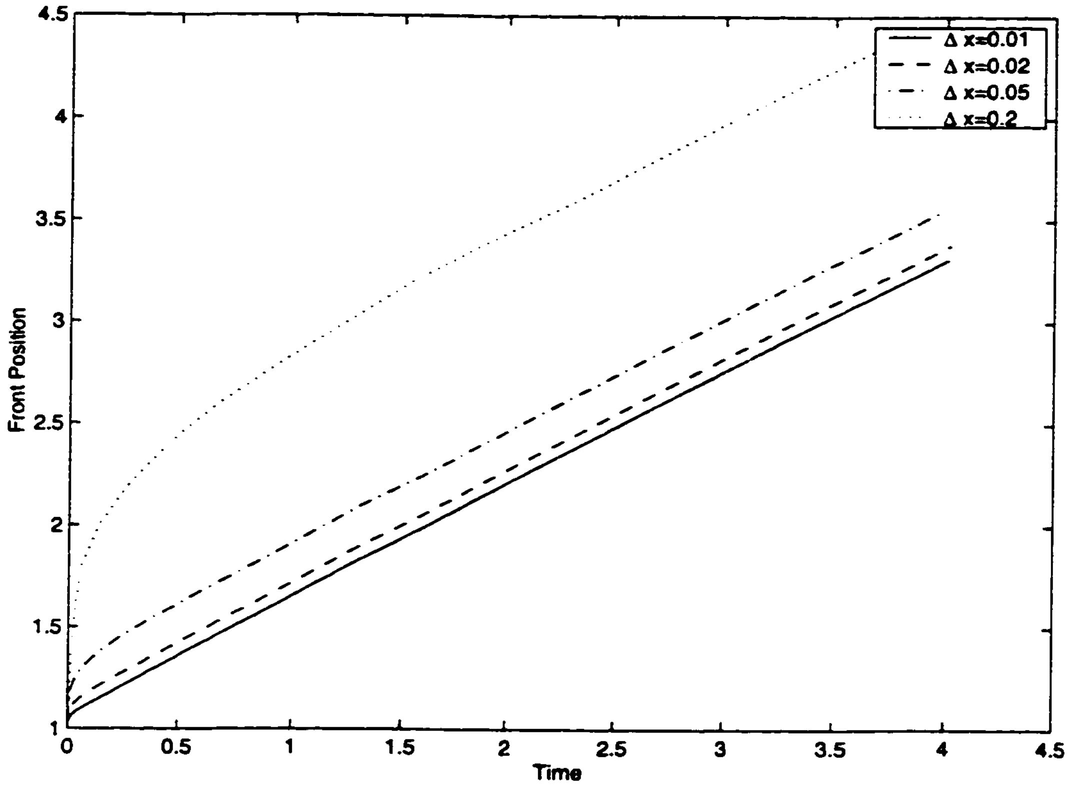

FIGURE 6.4. Graph of Front Position versus time for varying values of grid width Δx. Relevant parameter values: $\alpha = 10$, CFL number=0.75, and $\varepsilon = 10^{-10}$.

The last diagram of this subsection is a portrayal, not of the lower layer height as shown in Figures 6.1-6.3, but of the tracked front position as a function of time. This type of curve is produced because it mimics the most common style of portraying experimental results of gravity front position versus time. The frontal speed (i.e. the slopes of the lines in Figure 6.4) is independent of the gridth width only after the initial starting time due to the varying widths of the initial value disconinuity at $x = 1$. Although Figure 6.4 shows a similar qualitative nature for the slopes of the front position, i.e. front velocity, the actual position is quite dependent on the choice of grid width. For this reason, as well as the shock resolution in figure 6.3, $\Delta x = 0.02$ is felt to be a good compromise between capturing the shocks and front position accurately and keeping the computation time and memory requirements down. This is not a serious problem, since computation times of only several hours are not considered too lengthy for practical purposes.

6.2.2 Comparison of the Weak-Stratification and Rigid Lid models to the Two-Layer Equations

With the properties of the relaxation method established, the shock resolution and simplicity of the scheme's implementation suggest that it is a useful method for solving systems of hyperbolic conservation laws. As a first application, this method is used to examine the differences between three models: the weak-stratification equations (2.4.43), the rigid lid equations (2.4.98), and the two-layer equations (2.4.30). Although the weak-stratification equations have been considered previously (Montgomery and Moodie, 1998a), the form of the equations solved therein differs from the form (2.4.43), and it is useful to contrast the simplified equations with the general case.

The equations are solved in the case without the forcing term C_f, bottom slope, h_B, or source terms Q. This simplification removes any additional parameters which may tend to obscure the relevant parameter here, which is the density difference parameter, γ. In addition, other parameters are fixed at $\alpha = 10$, $\varepsilon = 10^{-10}$, and the CFL number at 0.75 as determined in section 6.2.1. The single parameter change is to the initial value problem, which is implemented with an initial height of $\zeta_{2_0} = 0.9$, a value greater than the value of 0.75 considered in the previous subsection.

The first diagram, Figure 6.5, shows a comparison between numerical solutions to the weak-stratification equations (2.4.43), and the two-layer equations (2.4.30) for decreasing values of γ at a fixed time. The agreement between the solutions is quite obvious as γ decreases, and it is observed that this solution to the simpler weak-stratification system (2.4.43) exhibits similar qualitative properties to the

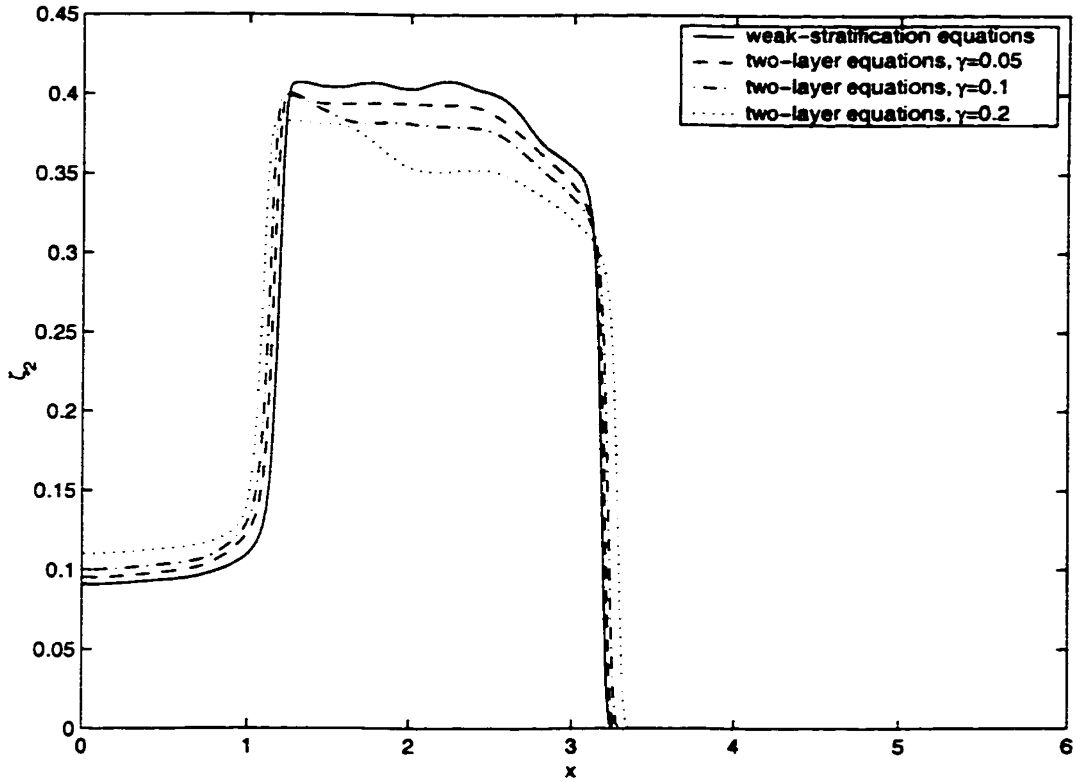

FIGURE 6.5. Graphs of $\zeta_2(x)$ versus x at time $t = 4$ for the two-layer equations with varying values of γ, and the weak- stratification equations. Relevant parameter: $\zeta_{2_0} = 0.9$.

solution to the larger two-layer system (2.4.30). Figure 6.5 is similar to a previously published diagram (Fig 2D of Montgomery and Moodie, 1998a) with the differences of a larger grid width ($\Delta x = 0.02$ instead of $\Delta x = 0.01$), and a later time ($t = 4$ instead of $t = 3$). The front position is very close for all four cases portrayed in Fig. 6.5. The curves in Fig. 6.5 all show the emergence of the rear bore which has been observed experimentally (Rottman and Simpson, 1983) to occur initially, and overtake the front after a short time. In between the rear bore and the front are several smaller waves which exist due to the high shear between the layers at the interface. These do not have time to grow into Kelvin-Helmholtz type billows since the rear bore catches up to the front in a fairly short time and removes the smaller waves.

In company with the lower layer height resolution portrayed in Figure 6.5, is the lower layer velocity, u_2, which is graphed in Figure 6.6 to portray any differences in velocity calculation between the two-layer and weak-stratification equations.

The difference between the graphs is even less pronounced in Figure 6.6 than in Figure 6.5, which suggests that a comparison of lower layer height values is an effective method for measurement of the main differences between the model predictions. Similar results hold for the upper layer variables, ζ_1 and u_1, and are not included since the resulting plots are similar in nature to Figures 6.5 and 6.6.

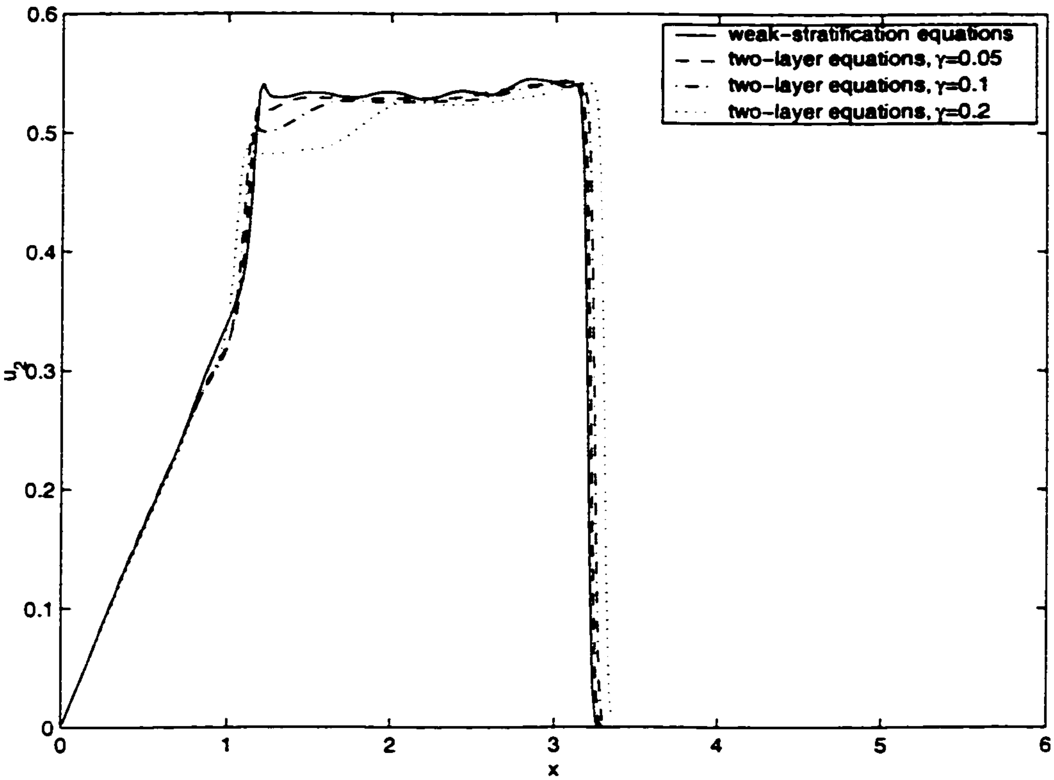

FIGURE 6.6. Graphs of $u_2(x)$ versus x at time $t = 4$ for the two-layer equations with varying values of γ, and the weak- stratification equations. Relevant parameter: $\zeta_{2_0} = 0.9$.

The second comparison is between the rigid-lid equations (2.4.98) and the weak-stratification equations (2.4.43). A comparison of lower layer thickness ζ_2 is portrayed in Figure 6.7, again at the same time and initial height, for decreasing values of γ. The graphs in Figure 6.7 are similar to those in Figure 6.5, suggesting that the solutions to the rigid-lid equations approximate the solutions to the simpler weak-stratification equations as $\gamma \to 0$.

The results of Figures 6.5-6.7 allow the following suggestion to be made when approximating two-layer gravity currents by simpler models. For $\gamma \lesssim 0.05$, the

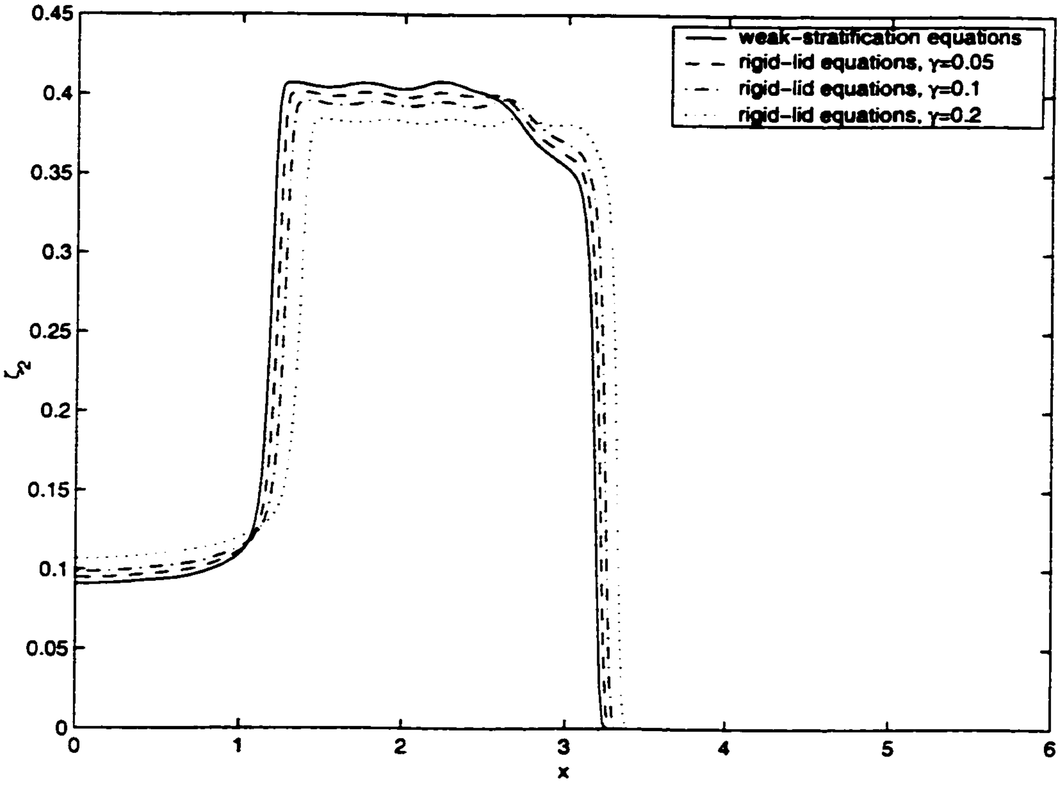

FIGURE 6.7. Graphs of $\zeta_2(x)$ versus x at time $t = 4$ for the rigid-lid equations with varying values of γ, and the weak- stratification equations. Relevant parameter: $\zeta_{2_0} = 0.9$.

weak-stratification equations are a better approximation than the rigid-lid equations, while the situation is reversed for $0.05 \lesssim \gamma \lesssim 0.2$. To make this notion more quantitative, a measure of the difference between two solutions, denoted $\zeta_{2_a}(x,t)$ and $\zeta_{2_b}(x,t)$ can be given by the time-dependent error function, denoted by $E(t)$, and defined via the L^2-norm as

$$E(t)^2 = \int_0^{\infty} \left(\zeta_{2_a}(x,t) - \zeta_{2_b}(x,t)\right)^2 dx. \tag{6.2.1}$$

This integral can be approximated by a finite spatial discretization, evaluated at discrete time steps, t_k, resulting in the notation

$$E_k^2 \approx \sum_{i=1}^{\infty} \left[(\zeta_{2_a})_i^k - (\zeta_{2_b})_i^k\right]^2 \Delta x. \tag{6.2.2}$$

This discretized error function, $E_k = E(t_k)$ from (6.2.2), may be used to compare the differences between the numerical solutions to the two-layer equations, the rigid-lid equations, and the weak-stratification equations. The results, plotted in Figure 6.8 for $t \leq 30$ and $\gamma = 0.05$, show that the error between the weak-stratification equations and the two-layer equations is less than the other two error estimates.

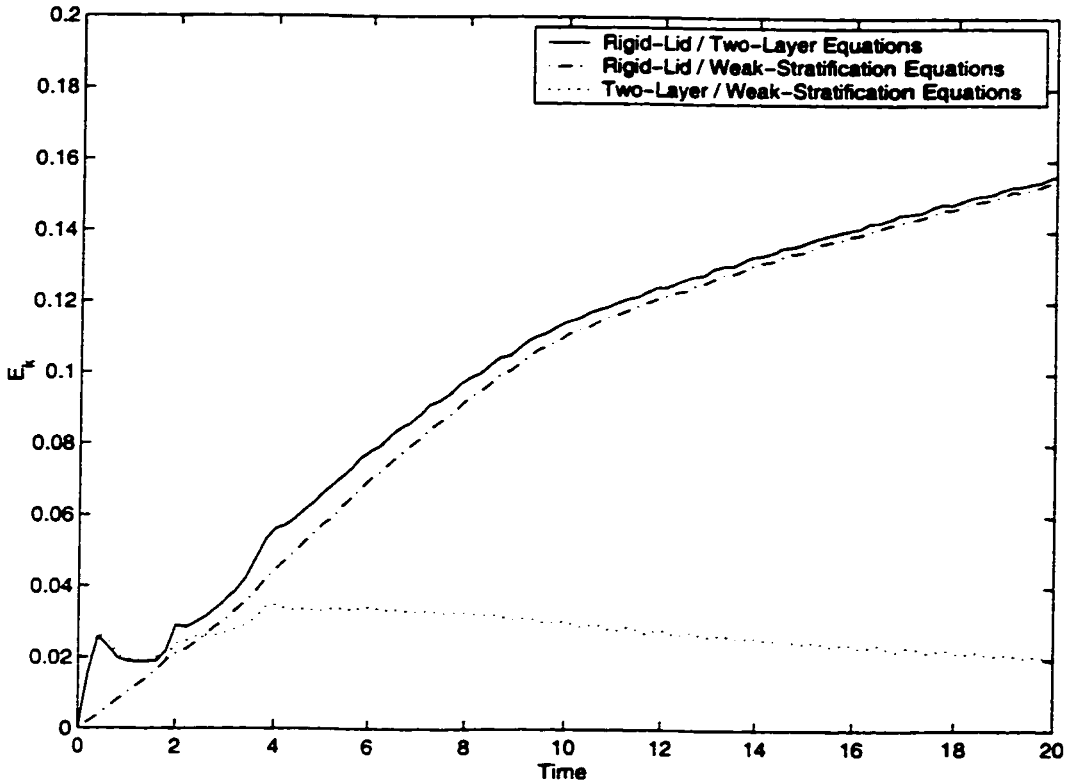

FIGURE 6.8. Graphs of E_k versus t for differences between the weak-stratification, two-layer ($\gamma = 0.05$), and rigid-lid equations ($\gamma = 0.05$). Relevant parameter: $\zeta_{2_0} = 0.9$.

Although these errors are dependent on γ, and only measure the differences in the calculated lower layer heights ζ_2, such calculations as portrayed in Figure 6.8 give a quantifiable support for choosing one approximation over another. Additional results may be calculated and obtained for various values of γ as required by the researcher.

6.2.3 Front Speeds for Discontinuous Solutions

Since the question, "How fast does a gravity current travel?" is of some importance, a feature of the numerical simulations is portrayed to help investigate gravity current speeds when they are thought of as discontinuous solutions to systems of hyperbolic conservation laws. Again, the situation with $h_B = 0$ and $C_f = 0$ is considered exclusively to avoid any effects which may obscure the front speed calculations.

For the system with a discontinuity moving to the right into a quiescent fluid, several expressions relating the lower layer velocity, u_2 and the lower layer height, ζ_2, were derived in Section 4.2. These discontinuities may be perceived as a vertical interface which separates the two fluids of densities ρ_1 and ρ_2, and is well-defined through the relaxation method as shown in Figures 6.5-6.7, for example. The jump condition (4.2.79) across this discontinuity for the two-layer equations with $h_1^+ = 1$ and $h_B = 0$ simplies to the expression

$$u_2^2 = \frac{2\zeta_2(1-\zeta_2)^2}{-2\zeta_2^2 - 4\zeta_2 + 1}. \tag{6.2.3}$$

Similarly, the weak-stratification relation (4.2.92) with $h_B = 0$ yields

$$u_2^2 = 2\zeta_2(1-\zeta_2)^2. \tag{6.2.4}$$

Finally, the rigid-lid equations were used to produce the jump condition (4.2.119), which simplifies using $h_B = 0$ and $h_1 = 1$ to

$$u_2^2 = \frac{2\zeta_2(1-\zeta_2)^2}{1+\gamma\zeta_2(2+\zeta_2)}. \tag{6.2.5}$$

To employ expressions (6.2.3)-(6.2.5), the appropriate equations are solved via the relaxation method, and the variables used to calculate the right and left hand sides of (6.2.3)-(6.2.5). The usefulness of (6.2.3)-(6.2.5) may then be found through plotting a general function for all of the expressions which is of the form

$$\frac{u_2^2}{f(\zeta_2)}. \tag{6.2.6}$$

In (6.2.6), the various forms of $f(\zeta_2)$ are chosen as the right-hand sides of (6.2.3)-(6.2.5) as appropriate. In this way, the expressions (6.2.6) are in agreement with numerical calculations whenever the graph of (6.2.6) plotted against time is equal to 1.

A comparison to standard experimental values, for example Bonnecaze *et al.*(1993), can be achieved through use of the experimental expression,

$$u_2^2 = \mathrm{Fr}\zeta_2, \tag{6.2.7}$$

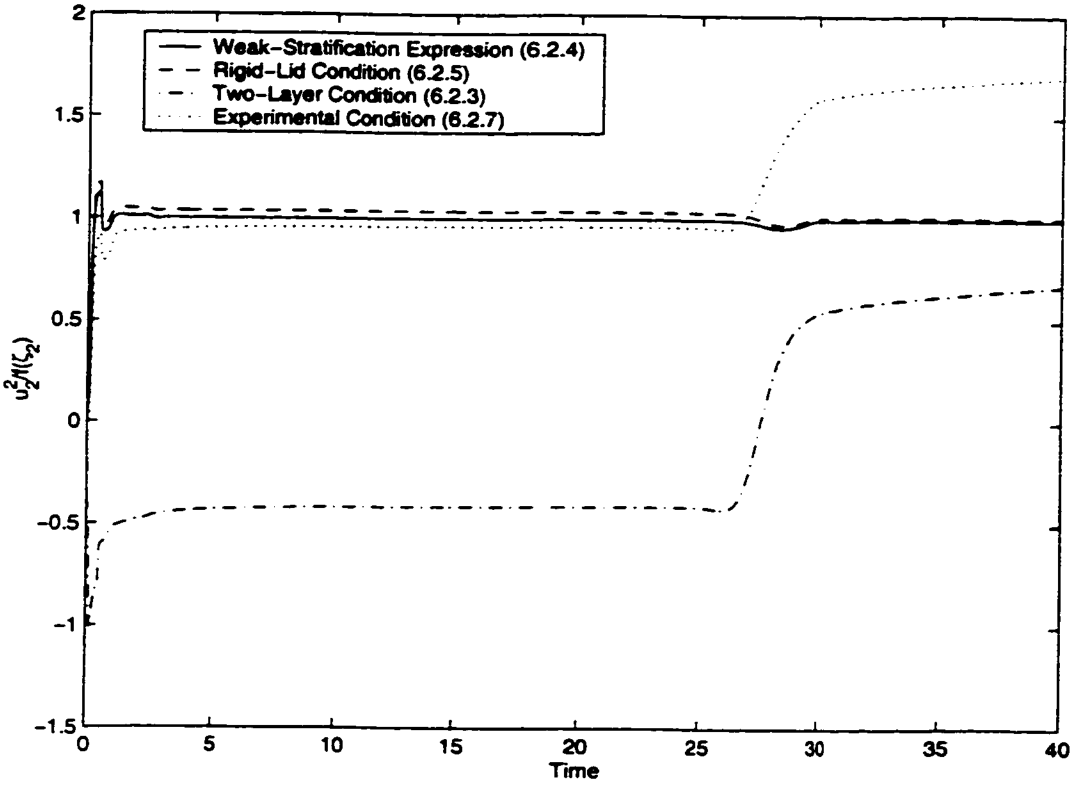

FIGURE 6.9. Graphs of $u_2^2/f(\zeta_2)$ versus t for expressions (6.2.3)-(6.2.5) and experimental expression (6.2.7). Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.05$.

for which the Froude Number is usually fixed in the range of $1.2 - \sqrt{2}$. By using the calculated values for u_2 and ζ_2 near the front, portrayal of the ratio u_2^2 to ζ_2 will yield a value which may be compared directly with the constant Fr.

These ratios are portrayed together in Figure 6.9, where the changes in time show distinct domains of agreement for each of the expressions (6.2.3)-(6.2.5) and (6.2.7). For very early times, there is a short adjustment phase as the gravity current is established from the initial release. After this initial adjustment, the ratios are all constant until approximately $t = 26$, at which point the rear bore overtakes the front. During this phase, the experimental Froude Number can be seen as approximately 0.95, and there is a good correlation for the expressions (6.2.4) and (6.2.5). The two-layer condition (6.2.3) displays a rather poor correlation. This is expected since this expression is achieved for the special limit of $\gamma = 0$, and intrusion into a quiescent upper layer.

For $t \gtrsim 26$, the approximate expression for the two-layer equations becomes more valid, and the correlation of expression (6.2.3) improves. In this region, it is the experimental Froude Number which jumps drastically, showing that there are different Froude Numbers for different flow regions. However, the correlation for expressions (6.2.4) and (6.2.5) remains quite close to 1 throughout the evolution of the gravity current. This verifies the calculations in Chapter 4 leading to the various theoretical jump conditions.

6.2.4 Two-layer Gravity Currents With Varying h_B and C_f.

In the previous calculations for Figures 6.1-6.9, neither bottom topography, h_B, nor nonlinear forcing terms C_f are present to affect the results. The addition of a linearly sloping bottom has been considered previously in Montgomery and Moodie (1999a); however, the introduction of the truncation function C_f therein was somewhat limited. The relaxation method is therefore applied in this section to the two-layer equations with varying forms of bottom topography, and various forms of forcing term C_f, so that the relative importance may be described. Three different types of C_f will be examined in conjunction with four different bottom topographies: horizontal, constant positive slope, constant negative slope, and sinusoidally varying.

In Montgomery and Moodie (1999a), it was observed that for a bottom boundary with a constant negative slope, there is a linear acceleration due to the component of gravity acting along the slope. This will, after a certain time which is dependent on both the slope and gravity current parameters, cause the two-layer equations to become non-hyperbolic, which may lead to instabilities at the interface which are characterised in the laboratory by strong regions of entrainment between the flow. Since, as mentioned in Chapter 1, this phenomenon is not observed in the laboratory for small slopes near the horizontal (Middleton, 1966), the standard method of avoiding this runaway acceleration problem is by including a basal frictional drag term of the form $C_f u_2^2/\zeta_2$ (see Whitham, 1974 or Baines, 1995 p.48). Such a forcing term on the lower layer equations only permits steady-state solutions which, through bifurcation theory, lead to the phenomenon of roll waves (Dressler, 1949).

In Figure 6.10, solutions to the two-layer equations (2.4.30) are solved using three non-horizontal bottom topographies, with the results compared for the lower layer height. The comparison is between the actual layer thickness relative to the bottom height, and the cross-sections are not measured in relation to an absolute value of $z = 0$. The effects of the bottom shape may be observed, and compared to

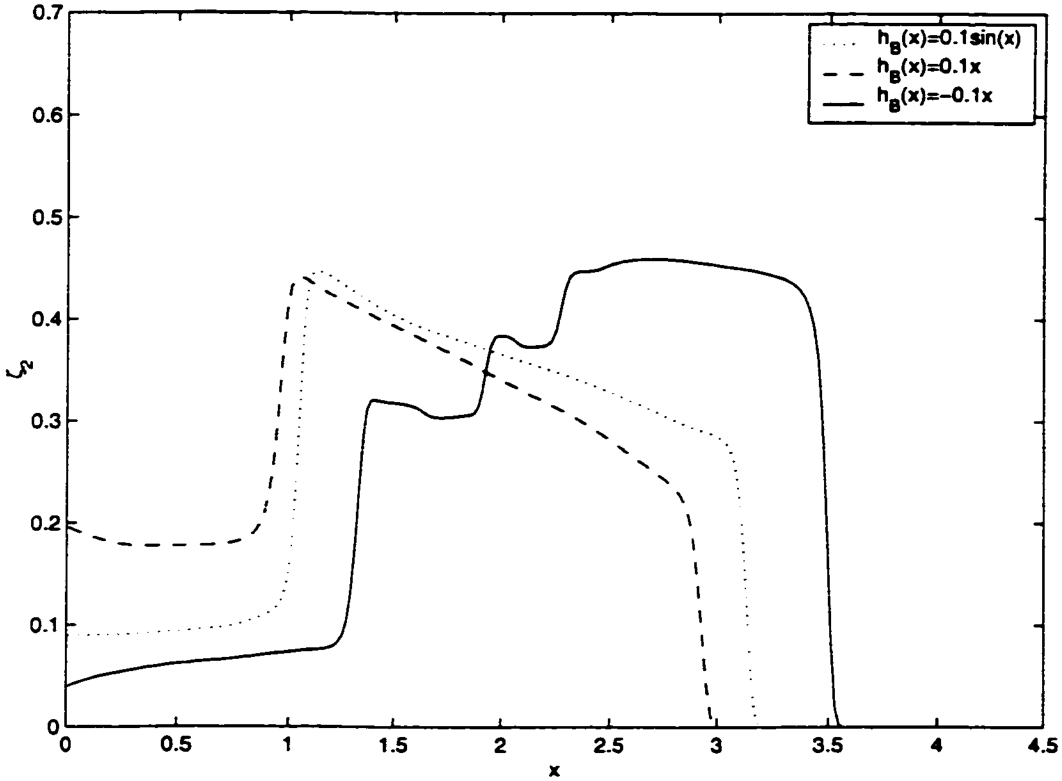

FIGURE 6.10. Graphs of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of the bottom topography function, $h_B(x)$. Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0$.

the horizontal case in Figure 6.5. The important features of the gravity current are changed by the topography. With positive and sinusoidally varying slope, the gravity current front position is behind that corresponding to the negatively sloping bottom. Also, some additional shocks and discontinuities develop behind the front for the downslope case, which are dependent on the initial height and disappear after a short time.

The corresponding front positions are displayed in Figure 6.11 for $t \leq 20$, where the effects of the three different functions $h_B(x)$ may be observed in a different manner. In contrast with Figure 6.4, which only portrays the front position for a short time for a gravity current travelling over a horizontal surface, the curves in Figure 6.11 vary substantially. For example, the curve corresponding to $h_B = 0.1x$, i.e. a positive bottom slope, only exists for a short time. This is due to the breakdown of solutions when the lower layer height overlaps the upper layer height; that is, the lower layer 'breaks through' the surface numerically. In

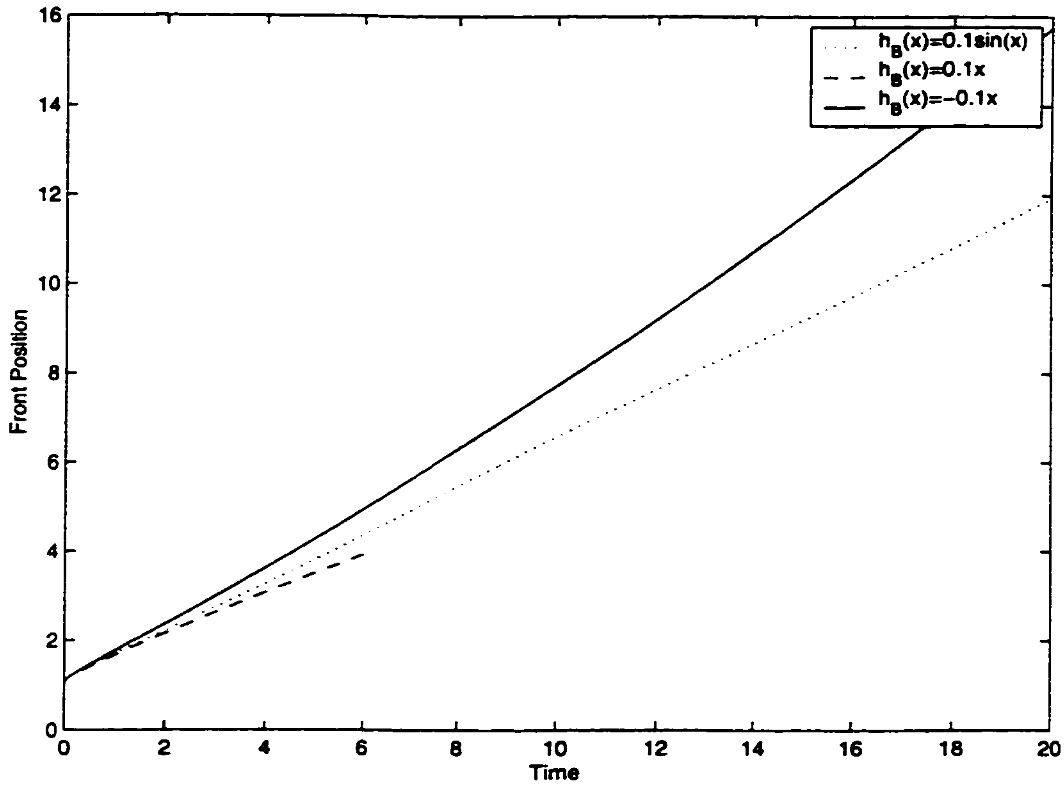

FIGURE 6.11. Graphs of Front Position versus time for varying values of the bottom topography function, $h_B(x)$. Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0$.

this case, for example, at about the time $t = 6$, the lower layer has advanced to approximately $x_F = 5$, corresponding to $h_B = 0.5$. The lower layer thickness here, of approximately $\zeta_2 \approx 0.3$ then overlaps with the upper layer thickness, $\zeta_1 \approx 0.3$. Thus a total heightof value $h_B + \zeta_1 + \zeta_2 \approx 1.1$ is greater than 1, leading to the breakdown of the hyperbolicity of the equations, and hence the numerical solution technique.

To gain a better insight into the nature of solutions having positive slope, a smaller slope is used, $h_B = 0.05x$, to calculate the time evolution of the lower layer, which is portrayed in Figure 6.12. In Fig. 6.12, the lower layer is graphed as its absolute height, with the bottom topography, h_B also portrayed. It can be observed that the lower layer velocity begins to reverse direction so that a reverse flow occurs down the slope, tending towards, for long time, the stable constant-height solution consisting of a triangle of dense fluid with a horizontal upper interface.

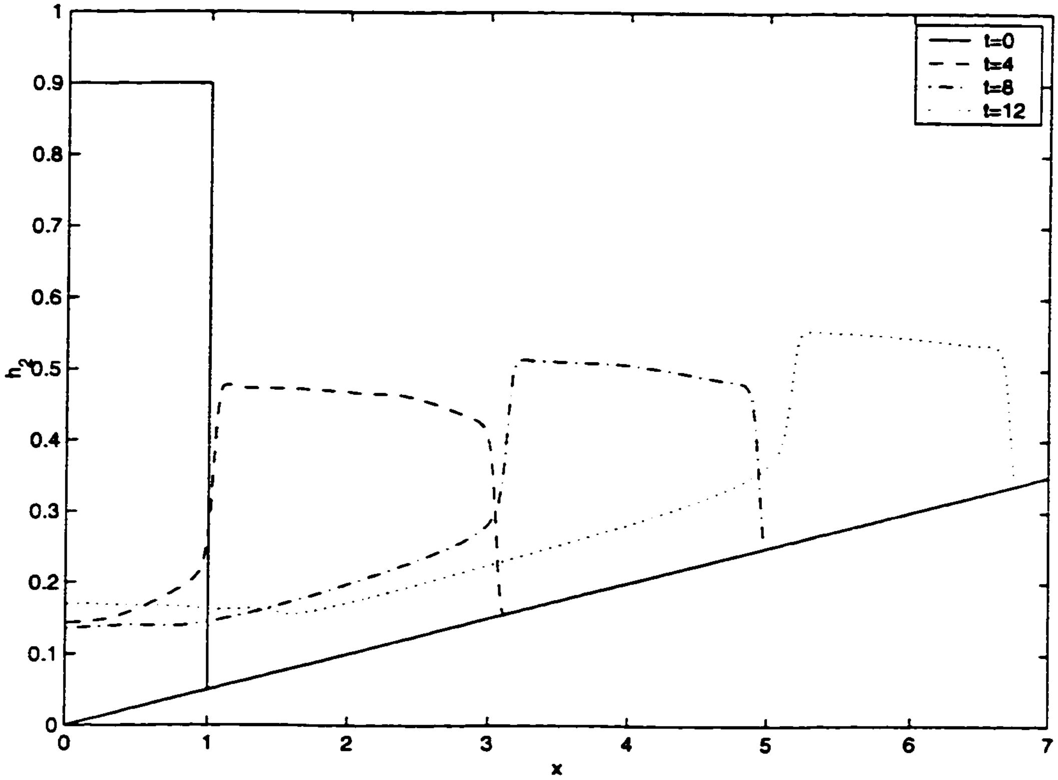

FIGURE 6.12. Graphs of lower layer height, h_2, relative to the bottom topography $h_B(x) = 0.05x$, for several times. Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0$.

Gravity currents which are created in the laboratory from instantaneous release of a dense fluid are observed not to have a steady-state velocity profile within the current. The front speed usually decays with time, once the gravity current has been observed in for several stages of the flow (Huppert and Simpson, 1980). Indeed, the fluid behind the head generally travels faster than the front, while the front speed controls the evolution of the current (Simpson, 1997). To capture this observed behaviour numerically, and avoid the kinds of difficulties in equation classification apparent from Figure 6.11, Montgomery and Moodie (1999a) introduced a weighted forcing term which multiplied the standard force $C_f u_2^2/\zeta_2$ by a spatially dependent truncation function, $T(x)$, which was developed in Chapter 2.3. The introduction of $T(x)$ specifically neglects the existence of any vertical structure so that it may be used consistently with the shallow-water approximations (see Chapter 2, Section 3). The effect of the truncation is solely to act as a control on the retarding force which is in effect at the front of the gravity current while not present behind the head so that the following flow is unchanged.

The three truncation functions $T(x)$ are defined as follows. The first, $T_1(x) = 1$, leads to the standard constant forcing term already mentioned, and is employed to serve as a benchmark in comparing the effects of the other two types. By denoting the front position as x_F, this truncation may be extended to the entire half line as

$$T_1(x) = \begin{cases} 0 & \text{if } x > x_F, \\ 1 & \text{if } 0 \le x \le x_F. \end{cases} \tag{6.2.7}$$

The second truncation function is that developed in Chapter 2.3, a parabolic truncation. For a specific length parameter, $l > 0$, this may be expressed directly from equation (2.3.6), limited to one spatial dimension as

$$T_2(x) = \begin{cases} 0 & \text{if } x > x_F, \\ 1 - \sqrt{\frac{x_F - x}{l}} & \text{if } x_F - l \le x \le x_F, \\ 0 & \text{if } 0 \le x \le x_F - l. \end{cases} \tag{6.2.8}$$

Introduction of another parameter, the length l, is problematic since it requires some sort of knowledge of the effective head width parameterized as a ratio of the total length scale.

The third, and last, truncation function to be used is the one proposed previously by Montgomery and Moodie (1999a), which is an exponential truncation designed to maintain an infinitely smooth transition. This contains an additional parameter similarly to (6.2.8), which is denoted as well by $l > 0$, and is defined as

$$T_3(x) = \begin{cases} 0 & \text{if } x > x_F, \\ \exp[-\left(\frac{x_F - x}{l}\right)^2] & \text{if } 0 \le x \le x_F. \end{cases} \tag{6.2.9}$$

This truncation function (6.2.9) has a value of 1 at the front, decaying to zero behind it. Thus although it is in effect throughout the entire length of the lower layer, unlike the parabolic truncation (6.2.8), the exponential decay distinguishes T_3 from the constant forcing term T_1.

The effect of the three truncation functions on gravity current evolution is portrayed for a horizontal bottom slope in Figure 6.13 so that a reference may be made with similar calculations for $h_B \neq 0$. Two different length parameters are chosen, so that the effective cutoffs behind the front for the truncations (6.2.8) and (6.2.9) are similar, although not identical. Two observations which may be made from Figure 6.13 are: there is little difference between the parabolic truncation T_3 and the exponential truncation T_2, and the constant forcing term arising from

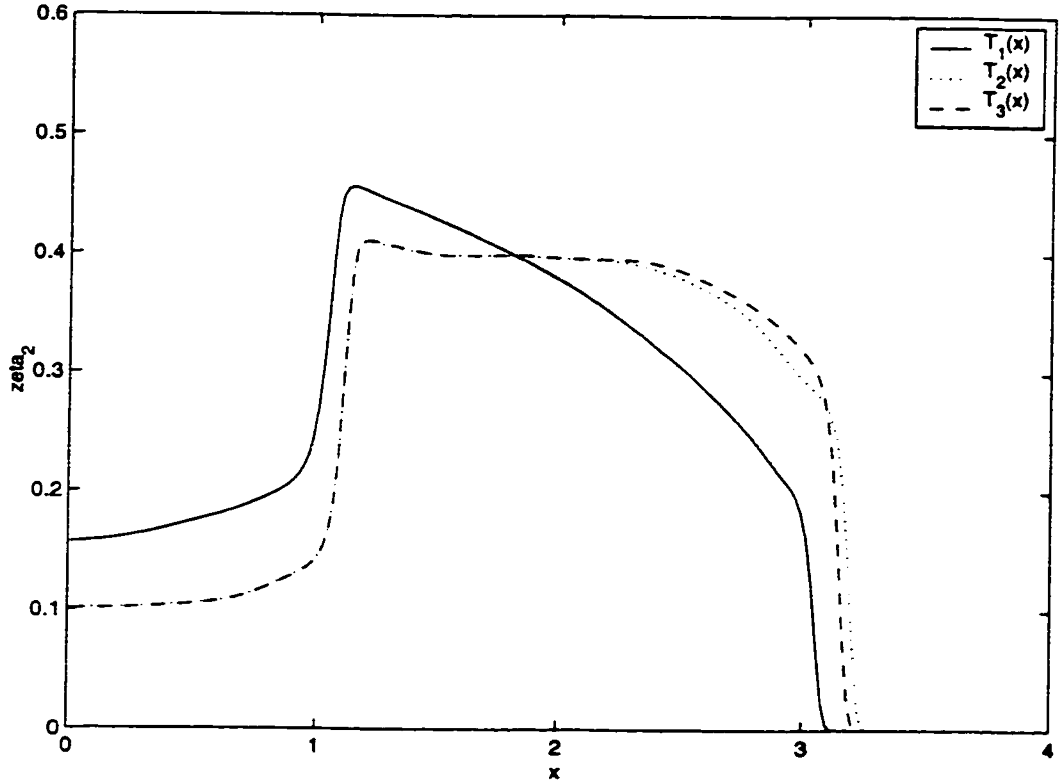

FIGURE 6.13. Graphs of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of the truncation function, $T_i(x)$. Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0.25$, $l = 0.5$ for T_2, $l = 0.2$ for T_3, and $h_B = 0$.

the truncation function T_1 affects the shape of the gravity current substantially. This diagram reinforces the result from Montgomery and Moodie (1999a) that the truncated forcing terms retain the essential shapes of the gravity current without any forcing. On the basis of the curves in Figure 6.13, it seems reasonable to conclude that the exponential truncation T_3 gives the sharpest discontinuity and is likely the best choice.

The truncation functions also have a pronounced effect on the position of the advancing front of the lower layer, as shown in Figure 6.14. There, although the front positions are almost identical for either T_2 or T_3, the constant forcing term associated with the truncation function T_1 has a slowing effect on the front of the gravity current. This shows that the following flow in the lower layer, which is only affected by T_1, is important in determining the front speed of the current. That is, the gravity current travels at a roughly constant speed for both T_2 and

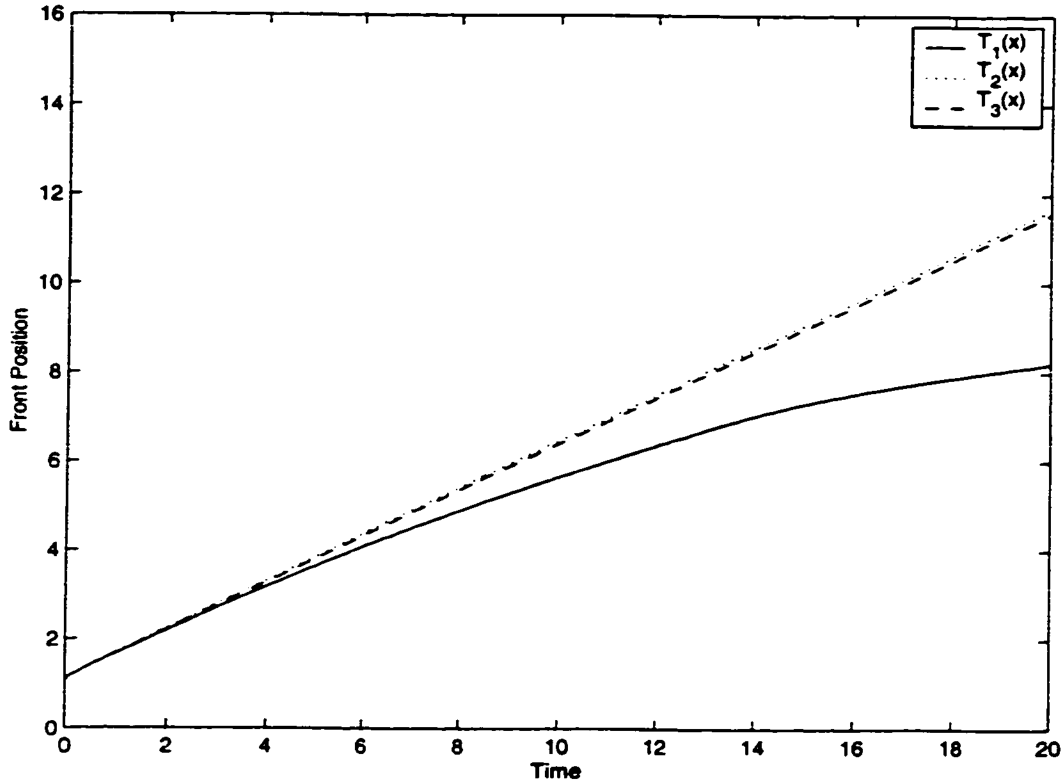

FIGURE 6.14. Graphs of Front Position versus time for varying values of the truncation functions, T_i in equations (6.2.7)-(6.2.9). Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0.25$, $l = 0.5$ for T_2, $l = 0.2$ for T_3, and $h_B = 0$.

T_3, as is the case for the case without bottom friction, $C_f = 0$ (Montgomery and Moodie, 1998a).

The previous calculations for Figures 6.13 and 6.14 can be examined with a change in bottom topography to a gradually sloping downward plane, $h_B = -0.1x$. From Figure 6.15, the truncation functions may be observed to have similar effects as seen in Figure 6.13. The change in bottom topography has the effect of changing the profile within the lower layer, when compared with the horizontal case in Figure 6.13. Again, there is a marked difference between the effects of T_1 and T_2, T_3 in both the heights, and position of the front.

The corresponding diagram for the position of the front is plotted in Figure 6.16. There, the effect of bottom slope is observed through the increase of the front positions when compared with the results from Figure 6.14. Qualitatively, no observable differences are created due to the bottom slope in conjunction with

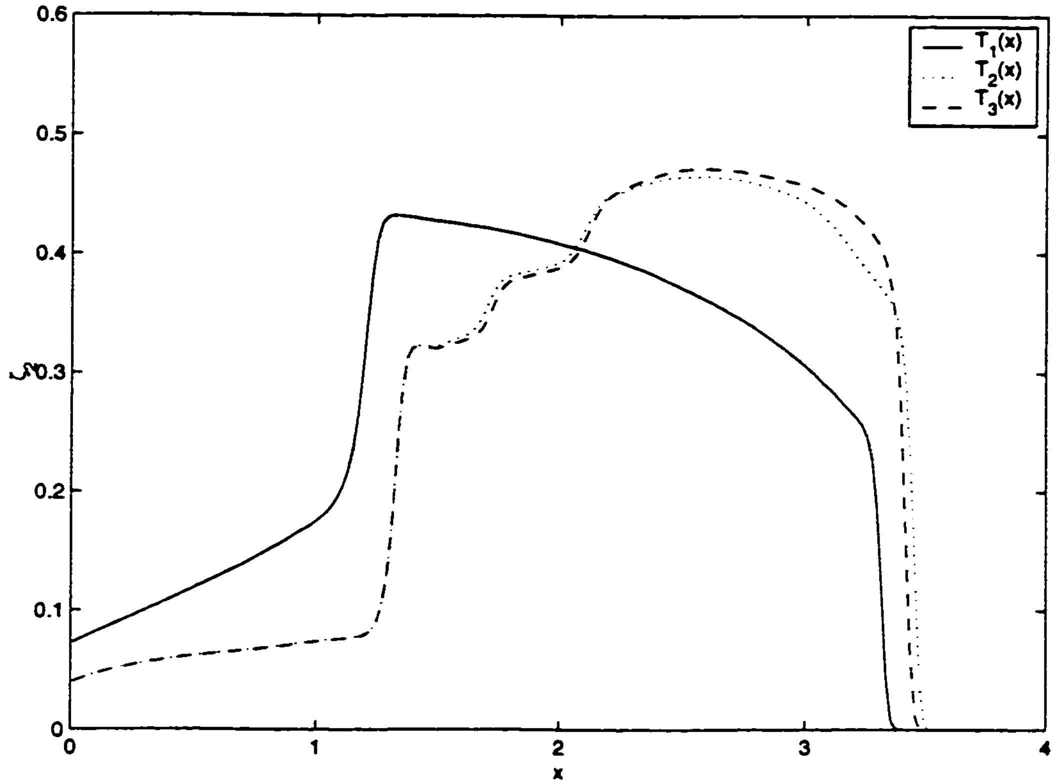

FIGURE 6.15. Graphs of $\zeta_2(x)$ versus x at time $t = 4$ for varying values of the truncation function, $T_i(x)$. Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0.25$, $l = 0.5$ for T_2, $l = 0.2$ for T_3, and $h_B = -0.1x$.

the truncation functions. However, there is a noticeable acceleration of the front for the two curves corresponding to the truncations T_2 and T_3. Although not displayed, this acceleration occurs until the rear bore catches the front, and is present for the $C_f = 0$ case. Additional results are shown in Montgomery and Moodie (1999a).

The results portrayed in Figures 6.10-6.16 show that the relaxation method generalizes easily to encompass non-horizontal bottom topography while retaining the essential shallow-water nature of the flow. The addition of a nonlinear lower-layer forcing term with an assumed truncation function is an interesting new addition to the theory which has the benefit of retaining the qualitative nature of the solutions used to predict gravity current behaviour. The obvious advantage to the practical researcher is that a front condition such as (6.2.4) needs not be assumed prior to any experiments, and may be calculated as the flow evolves. This

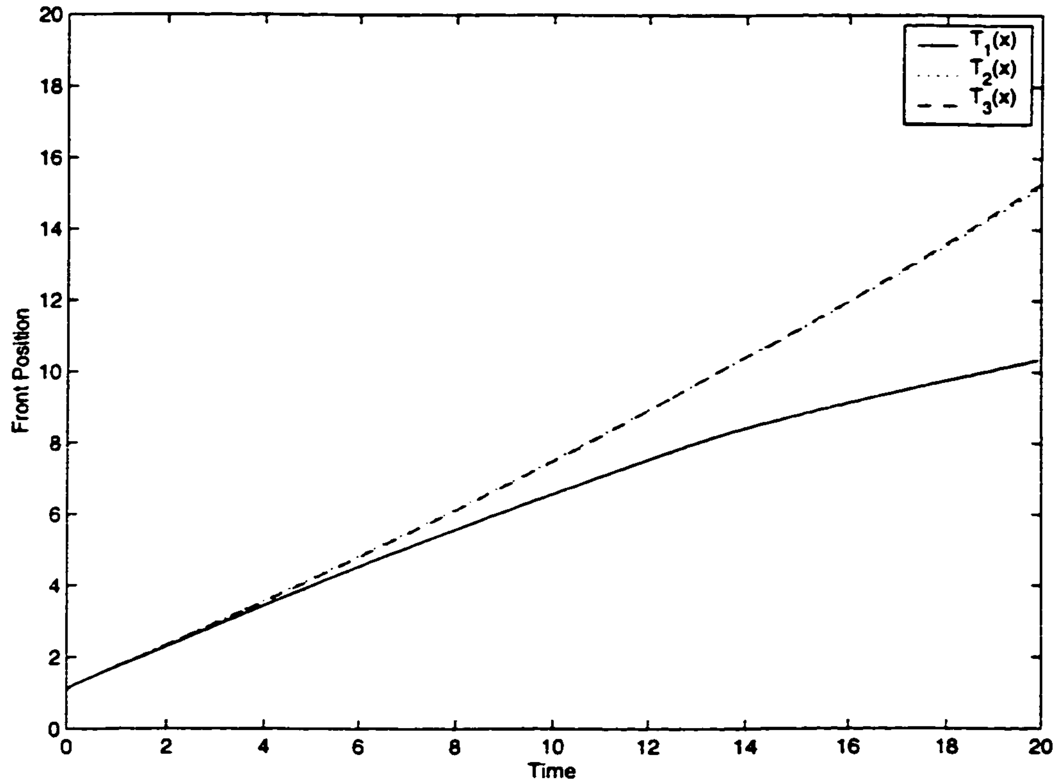

FIGURE 6.16. Graphs of Front Position versus time for varying values of the truncation functions, T_i in equations (6.2.7)-(6.2.9). Relevant parameter values: $\zeta_{2_0} = 0.9$, $\gamma = 0.1$, $C_f = 0.25$, $l = 0.5$ for T_2, $l = 0.2$ for T_3, and $h_B = -0.1x$.

allows the flexibility accorded to variations from horizontal bottom topography, and changes in volume of the current.

These results are at this point only described in a numerical fashion. Comparison with experimental results was beyond the scope of this thesis, and was not conducted. Such a comparison was identified as a topic for future research.

6.3 Numerical Solutions for Gravity Currents in Three Dimensions

For systems of hyperbolic conservation laws in more than one spatial dimension, the relaxation method may be employed, as described in Section 6.1.2. For a single layer gravity current, the shallow-water equations (2.4.7) were shown to be hyperbolic in Section (5.1.2). The conservation form of these equations, (2.4.10), is considered in this section with $C_f = 0$ exclusively, i.e. no nonlinear forcing terms, to yield a well-posed initial value problem to model finite volume gravity currents in three spatial dimensions. Results are obtained for various initial volume geometry, bottom topography h_B, and Coriolis parameter ε.

The initial value problem to be solved via the relaxation method is stated as an equation in the conservative form (6.1.25) with $p = 1$ and $n = 3$, where, from (2.4.10), the functions $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are given by

$$\mathbf{f}^{(1)} = \begin{bmatrix} \frac{\mu^2}{\zeta} + \frac{1}{2}\zeta^2 \\ \frac{\mu\nu}{\zeta} \\ \mu \end{bmatrix}, \text{ and } \mathbf{f}^{(2)} = \begin{bmatrix} \frac{\mu\nu}{\zeta} \\ \frac{\nu^2}{\zeta} + \frac{1}{2}\zeta^2 \\ \nu \end{bmatrix}. \tag{6.3.1}$$

The forcing term, $\mathbf{b}$, and generic vector $\mathbf{u}$ in (6.1.25) are also given by

$$\mathbf{b} = \begin{bmatrix} -\zeta\dfrac{\partial h_B}{\partial x} + \varepsilon\nu \\ -\zeta\dfrac{\partial h_B}{\partial y} - \varepsilon\mu \\ 0 \end{bmatrix}, \text{ and } \mathbf{u} = \begin{bmatrix} \mu \\ \nu \\ \zeta \end{bmatrix}. \tag{6.3.2}$$

In (6.3.1), (6.3.2), the terms ζ, μ and ν represent the layer thickness, x momentum, and y momentum, respectively. The initial value, stated with the aid of Figure 2.3, may be expressed as

$$\mathbf{u}(x, y, 0) = \begin{bmatrix} 0 \\ 0 \\ \zeta(x, y, 0) \end{bmatrix}, (x, y) \in \mathbb{R}^2, \tag{6.3.3}$$

where $\zeta(x, y, 0)$ is a discontinuous initial function

$$\zeta(x, y, 0) = \begin{cases} \zeta_0(x, y), & \text{for } (x, y) \in \text{int}(\Gamma_0) \\ 0, & \text{otherwise.} \end{cases} \tag{6.3.4}$$

The notation Γ_0 in (6.3.4) is assumed to be a simple, closed, and convex initial curve in $\mathbb{R}^2$, and $\zeta_0(x, y)$ is assumed to be a smooth (at least C^2) function, $\zeta_0 : \text{int}(\Gamma_0) \to \mathbb{R}$.

The initial value problem consists of equation (2.4.10), with the definitions (6.3.1)-(6.3.4). As discussed in Section 6.1.2, several scalars α_i must be chosen, in this case with $i = 2$, before any numerical calculations may be completed. This choice of scalars is the topic of the next subsection.

6.3.1 The Entropy and Entropy Fluxes

To implement the relaxation method, two scalars α_1 and α_2 must be chosen to satisfy the dissipative condition (6.1.29). Thus, an entropy η and entropy fluxes $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$ satisfying (6.1.28) need to be defined for the system (2.4.10).

A convex function, which will subsequently be shown to be an entropy for (2.4.10), is the function $\eta : \mathbb{R}^3 \to \mathbb{R}$, defined as

$$\eta(\mathbf{u}) = \frac{1}{2}\left(\frac{\mu^2+\nu^2}{\zeta}\right) + \frac{1}{2}\zeta^2. \tag{6.3.5}$$

This function, (6.3.5), may be considered as the sum of the kinetic and potential energies of the lower layer. Since $0 < \zeta$, this function is well-defined and convex over the region $\mathbb{R}^2 \times (0,\infty)$. The gradient vector, η', is given by a single row matrix

$$\eta' = \nabla_{\mathbf{u}}\eta = \begin{bmatrix} \frac{\mu}{\zeta} & \frac{\nu}{\zeta} & -\frac{1}{2}\frac{(\mu^2+\nu^2)}{\zeta^2} + \zeta \end{bmatrix}. \tag{6.3.6}$$

For the vectors $\mathbf{f}^{(i)}$ given by (6.3.1), the Jacobian matrices are

$$\mathbf{f}^{(1)\prime} = \begin{bmatrix} 2\frac{\mu}{\zeta} & 0 & -\frac{\mu^2}{\zeta^2} + \zeta \\ \frac{\nu}{\zeta} & \frac{\mu}{\zeta} & -\frac{\mu\nu}{\zeta^2} \\ 1 & 0 & 0 \end{bmatrix}, \tag{6.3.7}$$

and

$$\mathbf{f}^{(2)\prime} = \begin{bmatrix} \frac{\nu}{\zeta} & \frac{\mu}{\zeta} & -\frac{\mu\nu}{\zeta^2} \\ 0 & 2\frac{\nu}{\zeta} & -\frac{\nu^2}{\zeta^2} + \zeta \\ 0 & 1 & 0 \end{bmatrix}. \tag{6.3.8}$$

Equations (6.3.6)-(6.3.8) then may be used to calculate the desired matrix products $\eta'\mathbf{f}^{(i)\prime}$ in (6.1.28) for $i = 1, 2$ as

$$\eta'\mathbf{f}^{(1)\prime} = \begin{bmatrix} 2\frac{\mu^2}{\zeta^2} + \frac{\nu^2}{\zeta_2} - \frac{1}{2}\left(\frac{\mu^2+\nu^2}{\zeta^2}\right) + \zeta \\ 0 + \frac{\mu\nu}{\zeta^2} + 0 \\ -\frac{\mu^3}{\zeta^3} + \mu - \frac{\mu\nu^2}{\zeta^3} + 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\frac{\mu^2}{\zeta^2} + \frac{1}{2}\frac{\nu^2}{\zeta^2} + \zeta \\ \frac{\mu\nu}{\zeta^2} \\ -\frac{\mu(\mu^2+\nu^2)}{\zeta^3} + \mu \end{bmatrix}, \tag{6.3.9}$$

and

$$\eta'\mathbf{f}^{(2)\prime} = \begin{bmatrix} \frac{\mu\nu}{\zeta^2} + 0 + 0 \\ \frac{\mu^2}{\zeta^2} + 2\frac{\nu^2}{\zeta^2} - \frac{1}{2}\frac{(\mu^2+\nu^2)}{\zeta^2} + \zeta \\ -\frac{\mu^2\nu}{\zeta^3} - \frac{\nu^3}{\zeta^3} + \nu \end{bmatrix} = \begin{bmatrix} \frac{\mu\nu}{\zeta^2} \\ \frac{1}{2}\frac{\mu^2}{\zeta^2} + \frac{3}{2}\frac{\nu^2}{\zeta^2} + \zeta \\ -\frac{\nu(\mu^2+\nu^2)}{\zeta^3} + \nu \end{bmatrix}. \tag{6.3.10}$$

Defining two functions

$$\mathbf{q}^{(1)} = \frac{1}{2}\frac{\mu(\mu^2+\nu^2)}{\zeta^2} + \mu\zeta, \tag{6.3.11}$$

and

$$\mathbf{q}^{(2)} = \frac{1}{2}\frac{(\mu^2+\nu^2)\nu}{\zeta^2} + \nu\zeta, \tag{6.3.12}$$

gives the gradient vectors

$$\nabla_{\mathbf{u}}\mathbf{q}^{(1)} = \left(\frac{3}{2}\frac{\mu^2}{\zeta^2} + \frac{1}{2}\frac{\nu^2}{\zeta^2} + \zeta, \frac{\mu\nu}{\zeta^2}, -\frac{\mu(\mu^2+\nu^2)}{\zeta^3} + \mu\right), \tag{6.3.13}$$

and

$$\nabla_{\mathbf{u}}\mathbf{q}^{(1)} = \left(\frac{\mu\nu}{\zeta^2}, \frac{1}{2}\frac{\mu^2}{\zeta^2} + \frac{3}{2}\frac{\nu^2}{\zeta^2} + \zeta, -\frac{(\mu^2+\nu^2)\nu}{\zeta^3} + \nu\right). \tag{6.3.14}$$

By writing (6.3.13) and (6.3.14) as column matrices, they are seen to be equal to the products (6.3.9) and (6.3.10), respectively. Therefore, by a direct calculation, (6.1.28) is shown to be satisfied so that the vectors $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$, given by (6.3.11) and (6.3.12), are entropy fluxes for the system (2.4.10) with entropy (6.3.5).

With the entropy pairs η and $\mathbf{q}^{(i)}$, an estimate of the scalars α_1 and α_2 through (6.1.29) and (6.1.30), requires an approximate knowledge of the eigenvalues for the following four matrices: $\mathbf{f}^{(1)\prime}\eta''^{-1}$, $\mathbf{f}^{(2)\prime}\eta''^{-1}$, $\eta''\mathbf{f}^{(1)\prime}$, and $\eta''\mathbf{f}^{(2)\prime}$. The Hessian matrix η'' may be calculated as

$$\eta''(\mathbf{u}) = \begin{bmatrix} \frac{\partial^2\eta}{\partial\mu^2} & \frac{\partial^2\eta}{\partial\mu\partial\nu} & \frac{\partial^2\eta}{\partial\mu\partial\zeta} \\ \frac{\partial^2\eta}{\partial\nu\partial\mu} & \frac{\partial^2\eta}{\partial\nu^2} & \frac{\partial^2\eta}{\partial\nu\partial\zeta} \\ \frac{\partial^2\eta}{\partial\zeta\partial\mu} & \frac{\partial^2\eta}{\partial\zeta\partial\nu} & \frac{\partial^2\eta}{\partial\zeta^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\zeta} & 0 & -\frac{\mu}{\zeta^2} \\ 0 & \frac{1}{\zeta} & -\frac{\nu}{\zeta^2} \\ -\frac{\mu}{\zeta^2} & -\frac{\nu}{\zeta^2} & \frac{\mu^2+\nu^2}{\zeta^3}+1 \end{bmatrix}. \tag{6.3.15}$$

This matrix has an inverse, η''^{-1}, which may be calculated easily through the cofactor method, and expressed as

$$\eta''^{-1} = \begin{bmatrix} \zeta + \frac{\mu^2}{\zeta^2} & \frac{\mu\nu}{\zeta^2} & \frac{\mu}{\zeta} \\ \frac{\mu\nu}{\zeta^2} & \zeta + \frac{\nu^2}{\zeta^2} & \frac{\nu}{\zeta} \\ \frac{\mu}{\zeta} & \frac{\nu}{\zeta} & 1 \end{bmatrix}. \tag{6.3.16}$$

By use of the matrices (6.3.7), (6.3.8), (6.3.15) and (6.3.16), the necessary matrix products may be calculated, and labelled, as

$$\mathbf{E}_1 = \mathbf{f}^{(1)\prime}\eta''^{-1} = \begin{bmatrix} \frac{\mu^3}{\zeta^3} + 3\mu & \frac{\mu^2\nu}{\zeta^3} + \nu & \frac{\mu^2}{\zeta^2} + \zeta \\ \frac{\mu^2\nu}{\zeta^3} + \nu & \frac{\mu\nu^2}{\zeta^3} + \mu & \frac{\mu\nu}{\zeta^2} \\ \frac{\mu^2}{\zeta^2} + \zeta & \frac{\mu\nu}{\zeta^2} & \frac{\mu}{\zeta} \end{bmatrix}, \tag{6.3.17}$$

$$\mathbf{E}_2 = \mathbf{f}^{(2)'}\eta''^{-1} = \begin{bmatrix} \frac{\mu^2\nu}{\zeta^3} + \nu & \frac{\mu\nu^2}{\zeta^3} + \mu & \frac{\mu\nu}{\zeta^2} \\ \frac{\mu\nu^2}{\zeta^3} + \mu & \frac{\nu^2}{\zeta^3} + 3\nu & \frac{\nu^2}{\zeta^2} + \zeta \\ \frac{\mu\nu}{\zeta_2} & \frac{\nu^2}{\zeta^2} + \zeta & \frac{\nu}{\zeta} \end{bmatrix}, \tag{6.3.18}$$

$$\mathbf{E}_3 = \eta''\mathbf{f}^{(1)'} = \begin{bmatrix} \frac{\mu}{\zeta^2} & 0 & -\frac{\mu^2}{\zeta^3} + 1 \\ 0 & \frac{\mu}{\zeta^2} & -\frac{\mu\nu}{\zeta^3} \\ -\frac{\mu^2}{\zeta^3} + 1 & -\frac{\mu\nu}{\zeta^3} & \frac{\mu(\mu^2+\nu^2)}{\zeta^4} - \frac{\mu}{\zeta} \end{bmatrix}, \tag{6.3.19}$$

and

$$\mathbf{E}_4 = \eta''\mathbf{f}^{(2)'} = \begin{bmatrix} \frac{\nu}{\zeta^2} & 0 & -\frac{\mu\nu}{\zeta^3} \\ 0 & \frac{\nu}{\zeta^2} & -\frac{\nu^2}{\zeta^3} + 1 \\ -\frac{\mu\nu}{\zeta^3} & -\frac{\nu^2}{\zeta^3} + 1 & \frac{(\mu^2+\nu^2)\nu}{\zeta^4} - \frac{\nu}{\zeta} \end{bmatrix}. \tag{6.3.20}$$

The matrices (6.3.17)-(6.3.20) are all real and symmetric, and therefore each have real eigenvalues. These may be expressed analytically, however for the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$, the expressions for the eigenvalues are lengthy. Since all that is needed are upper bounds on the eigenvalues in (6.1.30), some straightforward approximation simplifies the analysis to obtain a sufficient maximum value. The first property involves expressing the maximum of the eigenvalues through the 2-norm (Golub, 1996 p.394) as

$$||\mathbf{E}_i||_2 = \max\left\{|\lambda_1^{(i)}|, |\lambda_2^{(i)}|, |\lambda_3^{(i)}|\right\}, \tag{6.3.21}$$

where $\lambda_k^{(i)}$ represents an eigenvalue of $\mathbf{E}_i$. The 2-norm of a 3×3 matrix $\mathbf{A}$ is defined as

$$||\mathbf{A}||_2 = \sup_{0 \neq \mathbf{x} \in \mathbb{R}^3} \frac{||\mathbf{A}\mathbf{x}||_2}{||\mathbf{x}||_2}, \tag{6.3.22}$$

with the usual Euclidean vector norm, $\mathbf{x} = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

A useful approximation for an upper bound on $||\mathbf{A}||_2$ is given in terms of the elements of the matrix (Golub, 1996 p.56) as

$$||\mathbf{A}||_2 \leq \sqrt{3}||\mathbf{A}||_1, \tag{6.3.23}$$

where

$$||\mathbf{A}||_1 = \max_{j=1,2,3}\left\{\sum_{i=1}^{i=3} |a_{ij}|\right\}. \tag{6.3.24}$$

Thus, for the matrix $\mathbf{E}_1$ from (6.3.17), for example, an upper bound is given by

$$\|\mathbf{E}_1\|_1 = \max\left\{\left|\frac{\mu^3}{\zeta^3} + 3\mu\right| + \left|\frac{\mu^2\nu}{\zeta^3} + \nu\right| + \left|\frac{\mu^2}{\zeta^2} + \zeta\right|, \left|\frac{\mu^2\nu}{\zeta^3} + \nu\right| + \left|\frac{\mu\nu^2}{\zeta^3} + \mu\right| \right.$$
$$\left. + \left|\frac{\mu\nu}{\zeta^2}\right|, \left|\frac{\mu^2}{\zeta^2} + \zeta\right| + \left|\frac{\mu\nu}{\zeta^2}\right| + \left|\frac{\mu}{\zeta}\right|\right\}. \quad (6.3.25)$$

By the triangle inequality, and using the fact that $\zeta > 0$, the expression (6.3.25) may be further bounded above as

$$\|\mathbf{E}_1\|_1 = \max\left\{\frac{|\mu|^3}{\zeta^3} + 3|\mu| + \frac{\mu^2|\nu|}{\zeta^3} + |\nu| + \frac{\mu^2}{\zeta^2} + \zeta, \frac{\mu^2|\nu|}{\zeta^3} + |\nu| + \frac{\nu^2|\mu|}{\zeta^3} + |\mu| \right.$$
$$\left. + \frac{|\mu\nu|}{\zeta^2}, \frac{\mu^2}{\zeta^2} + \zeta + \frac{|\mu\nu|}{\zeta^2} + \frac{|\mu|}{\zeta}\right\}. \quad (6.3.26)$$

Assuming that $\zeta \leq 1$, $|\mu| \leq 1$, and $|\nu| \leq 1$, the upper bound (6.3.26) can be replaced by a numeric expression, and evaluated as

$$\|\mathbf{E}_1\|_1 = \max\{8, 5, 4\} = 8. \quad (6.3.27)$$

For these ranges of ζ, μ, and ν, similar expressions to (6.3.25) and (6.3.26) can be derived for the other matrices in (6.3.18)-(6.3.20), which then give upper bounds on the 1-norms as

$$\|\mathbf{E}_2\|_1 \leq 8, \quad \|\mathbf{E}_3\|_1 \leq 6, \text{ and } \|\mathbf{E}_4\|_1 \leq 6. \quad (6.3.28)$$

Now, using the relation (6.3.23), the 2-norms, and hence the maximum eigenvalues of the matrices $\mathbf{E}_i$ via the equality (6.3.21), may be bounded by (6.3.27) and (6.3.28) after multiplication by $\sqrt{3}$. It then follows from (6.1.30) that

$$\begin{aligned}\lambda^{(1)} &= \max\{\|\mathbf{E}_1\|_2, \|\mathbf{E}_3\|_2\} \\ &\leq \sqrt{3}\max\{\|\mathbf{E}_1\|_1, \|\mathbf{E}_3\|_1\} \\ &\leq \sqrt{3}\max\{8, 6\} = 8\sqrt{3}, \end{aligned} \quad (6.3.29)$$

and similarly

$$\lambda^{(2)} = \max\{\|\mathbf{E}_2\|_2, \|\mathbf{E}_4\|_2\} \leq 8\sqrt{3}, \quad (6.3.30)$$

whenever $\zeta \leq 1$, $|\mu| \leq 1$, and $|\nu| \leq 1$.

Estimates (6.3.29) and (6.3.30) then allow the restriction (6.1.29) to hold by the ordering

$$\frac{(\lambda^{(1)})^2}{\alpha_1} + \frac{(\lambda^{(2)})^2}{\alpha_2} \leq \frac{192}{\alpha_1} + \frac{192}{\alpha_2}, \tag{6.3.31}$$

so that if the right hand side of (6.3.31) is less than or equal to 1, then (6.1.29) holds. With the restriction that $\alpha_2 = \alpha_1$, it can be seen that (6.1.29) holds, provided that

$$\frac{192}{\alpha_1} + \frac{192}{\alpha_1} \leq 1,$$

which occurs precisely when

$$\alpha_1 \geq 384. \tag{6.3.32}$$

In practice the constraint (6.3.32) produces a value of α_1 which causes the grid width for the time variable to be quite small, through the CFL condition (6.1.32). To obtain a quicker computation time, smaller values of $\alpha_1 = \alpha_2$, denoted simply by α, were used to compute solutions to the initial value problem (6.3.1)-(6.3.4) with Γ_0 a circle of diameter 1, and $\zeta_0(x, y) = 1$. A cross section of the solution after a short time, with varying values of α, is plotted in Figure 6.17.

From Figure 6.17, the parameter $\alpha = 10$ was chosen for use in the subsequent calculations for Section 6.3, since, although this is certainly less than the estimate (6.3.32), the savings in computation time were quite appreciable, and worth the occasional appearance of small oscillations such as those observed at the peak in Figure 6.17 for the $\alpha = 10$ plot. The other parameters were fixed as: the relaxation parameter as 10^{-10} and the CFL number as 0.5. The notation ε, which previously was used to denote the relaxation parameter, is now used solely for the Coriolis Parameter from the notation (6.3.2), and future reference to the relaxation parameter will be stated explicitly without use of potentially confusing symbols.

6.3.2 Numerical Solutions With Horizontal Bottom Topography ($h_B = 0$)

The relaxation method, with the aforementioned parameter restrictions, is now employed to solve the initial value problem (6.3.1)-(6.3.4) for varying values of the Coriolis parameter, ε, and various initial geometries, Γ_0. Boundary values are not necessary to use in the calculations as the number of grid points required to solve the IVP simply increases in time as the support for the non-zero solution covers more area. The initial height profile, ζ_0, is not varied, and a constant value of $\zeta_0 = 1$ is to remove any effects due to the surface height, and allow easier

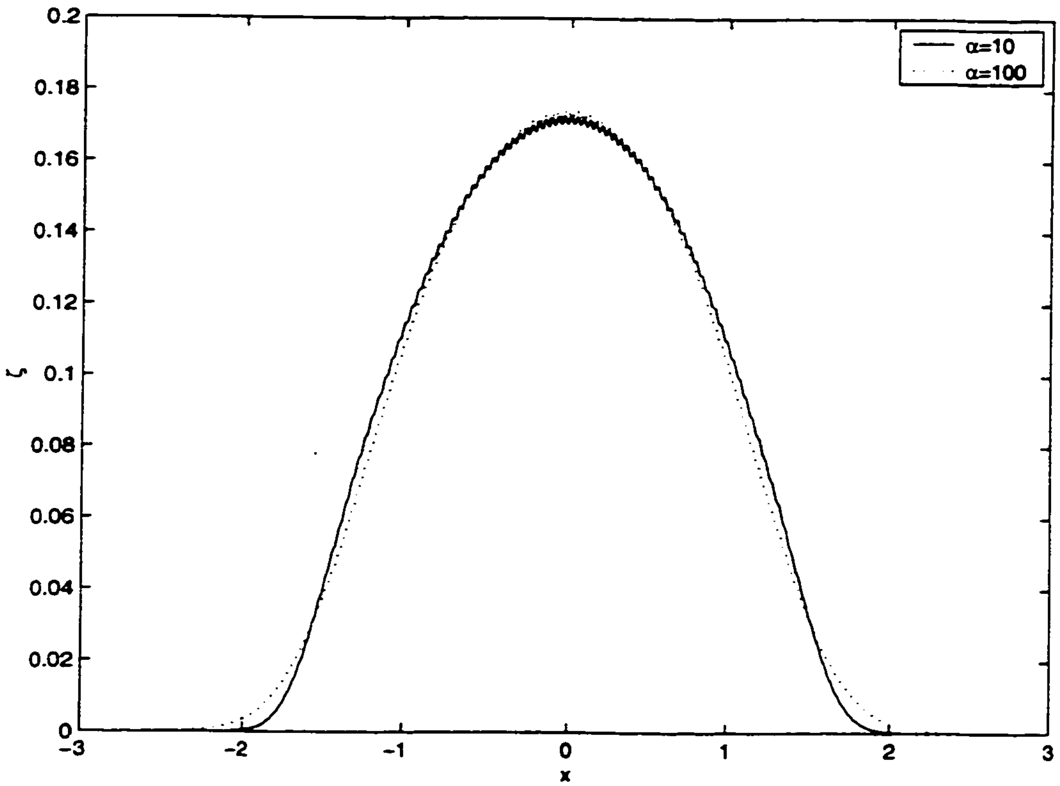

FIGURE 6.17. Cross-Section profile of the lower layer height, ζ, at time $t = 1$ for two values of the parameter α. Relevant parameter values: $\Delta x = 0.02$, $\varepsilon = 0$, $h_B = 0$, relaxation parameter 10^{-10}, and CFL number 0.5.

interpretation of the results. Before varying any parameters, a straightforward diagram is shown to act as a reference of some of the later figures.

The first diagram, Figure 6.18, shows a three-dimensional plot of the lower layer height, computed at the shown time step, for the initial value problem starting with a cylinder of height and diameter both equal to 1. It is easy to see that the cross-section profile from Figure 6.17 matches with one from Figure 6.18, with the only substantial difference being the relative proportions. The dome-shaped surface of Figure 6.18 was found to be the most prevalent configuration for the time-dependent initial release problem. In fact, variation of the initial geometry from a cylinder to a cube, or a rectangle, only caused observable effects for a short time after release.

The time evolution of the initial release problem, as shown in Figure 6.19, suggests that cylindrical geometry is the most general one, and that the instan-

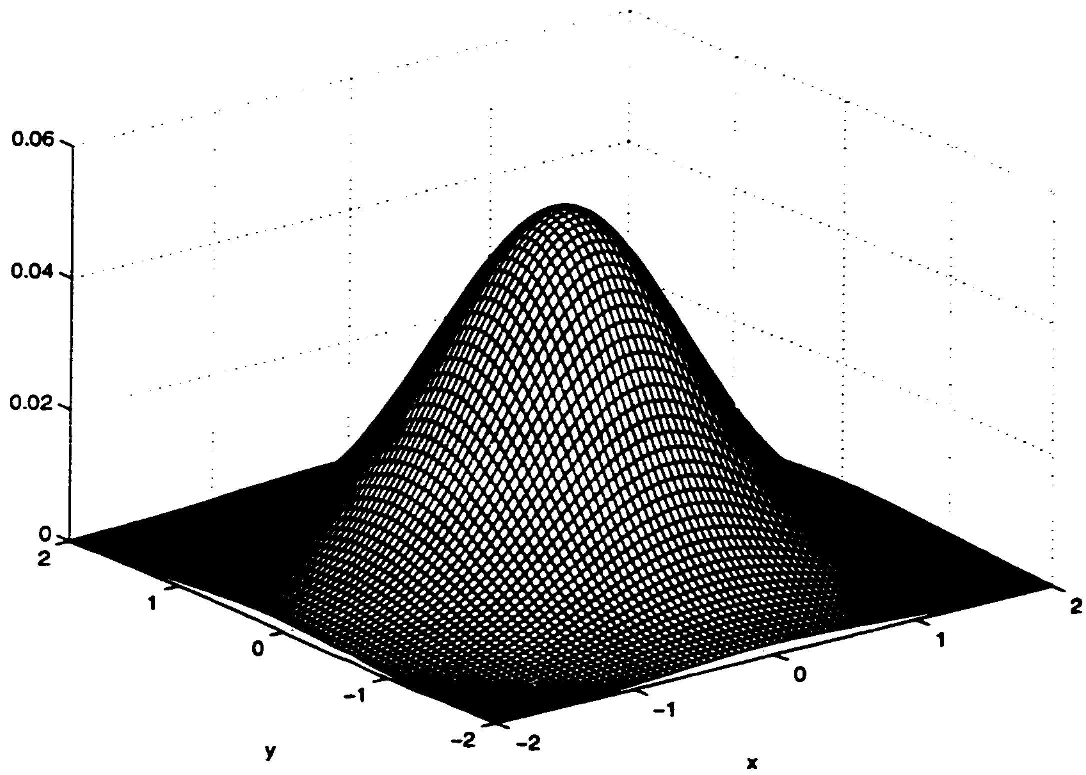

FIGURE 6.18. Lower layer height ζ at $t = 10$. Relevant parameters: $\varepsilon = 0$, $h_B = 0$, Γ_0 a circle of diameter 1, and $\zeta_0 = 1$.

taneous release problem will tend to this geometry after a short period of time. This occurs since a collapsing volume will tend to smooth out any irregularities in curvature by balancing the flow speeds near a bend or corner. Although a deeper analysis could be completed by examining the eccentricity of the geometry to quantify the observation that the front positions are nearly circular, the object of this diagram is to show that study of circular initial releases is a basic geometry which should be considered. Figure 6.19 also shows the slowing down of the radially expanding gravity current, observed by the shrinking distances between time steps.

When the Coriolis parameter is non-zero, the effects of rotation on the calculations may be portrayed in two ways, either as a vector plot, in which the velocities are superimposed over a contour diagram, or as a line diagram which displays the front position versus time, as done in Section 6.2. This latter type of diagram is portrayed below in Figure 6.20.

The three curves in Figure 6.20 show that the front position slows with time,

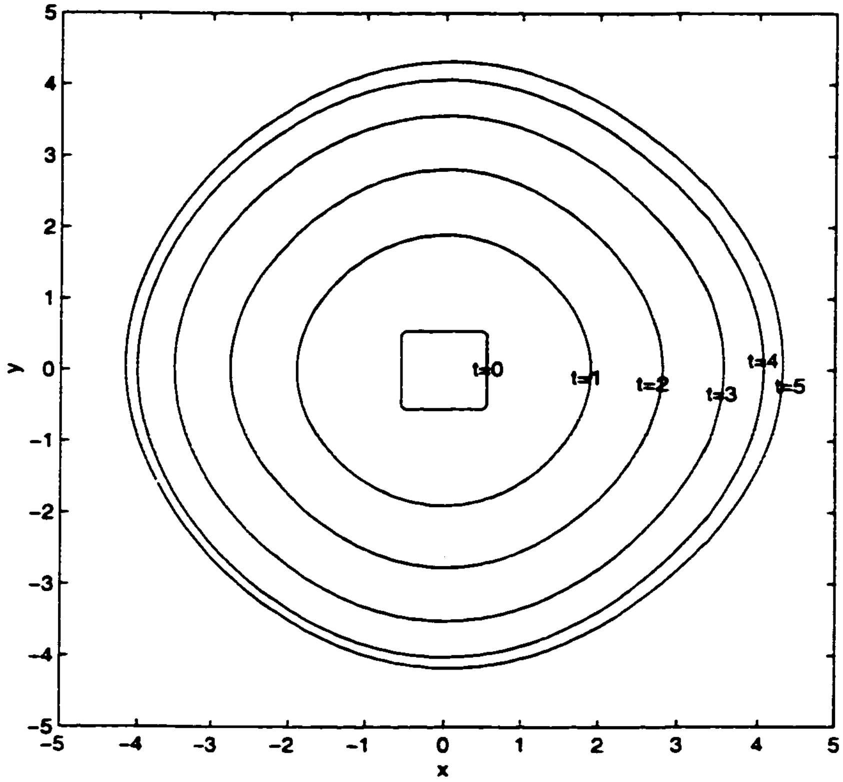

FIGURE 6.19. Top-view contour plot for cubic initial geometry, with Γ_0 a square of length 1, at increasing times with the contour drawn at $\zeta = 0.01$. Relevant parameters: $\varepsilon = 0$, $h_B = 0$, $\zeta_0 = 1$.

as expected as the gravity current expands radially. As in the two-dimensional case, the front position was determined as the radius at which the layer height, averaged over the azimuthal angle, was below a specific value, here chosen as 10^{-10}. Once the Coriolis parameter is increased, it acts to slow the spread of the gravity current. The two bottom curves, with $\varepsilon = 1/10$ and $\varepsilon = 2$, both are truncated since at that time the front position ceases to advance. The small jumps in the curves, at around $t = 10$ for example, are due to a rescaling of the grid width once the arrays reach the limit of memory constraints, and are a numerical effect only.

A second method of portraying the effect of the Coriolis parameter, is to view the results from above, with vector representation of the lower layer velocity. This is done in Figure 6.21, which shows that the gravity current, after this amount of time, is flowing towards a steady rotating state in the clockwise direction, and the solution begins to approach a geostrophic balance. This state occurs, according

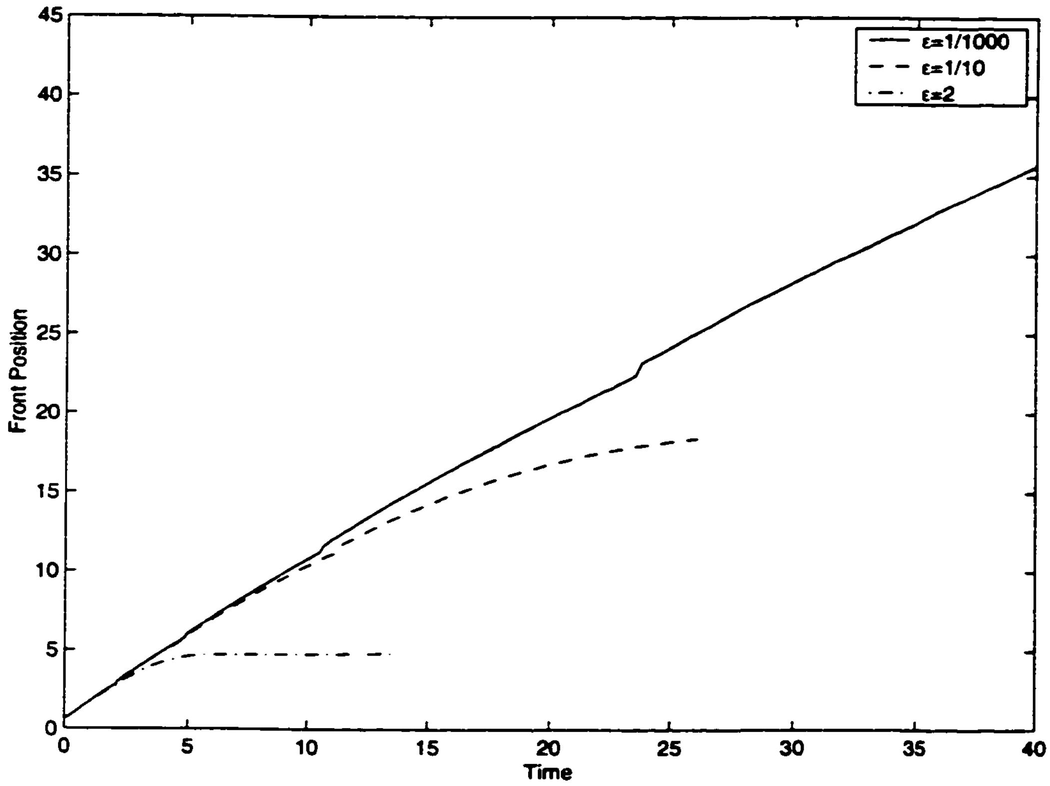

FIGURE 6.20. Front position versus time for increasing levels of the Coriolis parameter, ε. Relevant parameters: $h_B = 0$, $\zeta_0 = 1$.

to Figure 6.20, at approximately time $t = 26$. It should be noted that although the velocities are non-zero in a region outside of the contours shown in Figure 6.21, the actual calculated values for the x and y momenta are small, and do decrease with radius. The calculated values of velocity, are portrayed by dividing the momentum by the lower layer thickness, ζ, which also decreases with radius.

The effect of rotation portrayed in Figure 6.21 is intuitive in the following sense. With the Coriolis parameter set at $\varepsilon = 1/10$, this may be expressed physically by stating that the ratio of the effects of inertial spreading to rotation is 10 to 1. Thus, after 10 time steps, the effects of rotation should be observable. Hence, at $t = 20$ as shown in Figure 6.21, the effects of rotation are such that they have had an equal effect as the spreading. For short times, $t \leq 10$, the inertial spreading would be the primary source of fluid motion, a fact which is observed numerically, while for later times rotation becomes increasingly important.

6.3.3 Numerical Solutions With Gradually Sloping Bottom Topography

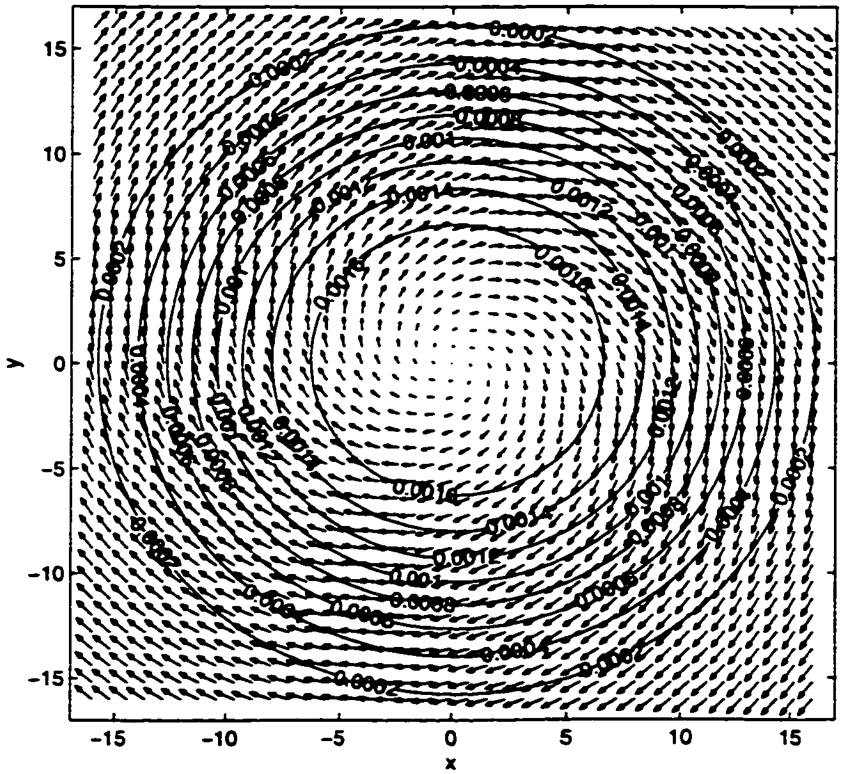

FIGURE 6.21. Height contour and velocity vector plot at time $t = 20$. Relevant parameters: $h_B = 0$, $\varepsilon = 1/10$, $\zeta_0 = 1$.

The effects of non-horizontal bottom topography on three-dimensional gravity currents are quite unlike that of two-dimensional currents, once rotation is included. As a simple example, a constantly sloping bottom, with height given by the function $h_B = 0.1x$, is considered in this subsection.

The initial release problem previously investigated is restated as a cylinder of diameter 1, as before, but with a horizontal lower layer height of 1, as opposed to a constant lower layer thickness. The addition of nonzero bottom topography then gives the function ζ_0 chosen in (6.3.4) as

$$\begin{aligned} \zeta_0(x, y) &= 1 - h_B(x, y) \\ &= 1 - 0.1x. \end{aligned} \tag{6.3.33}$$

Such an initial function as (6.3.33) gives a horizontal height profile of the lower layer, $h = \zeta + h_B$, while keeping the initial volume the same as that considered previously.

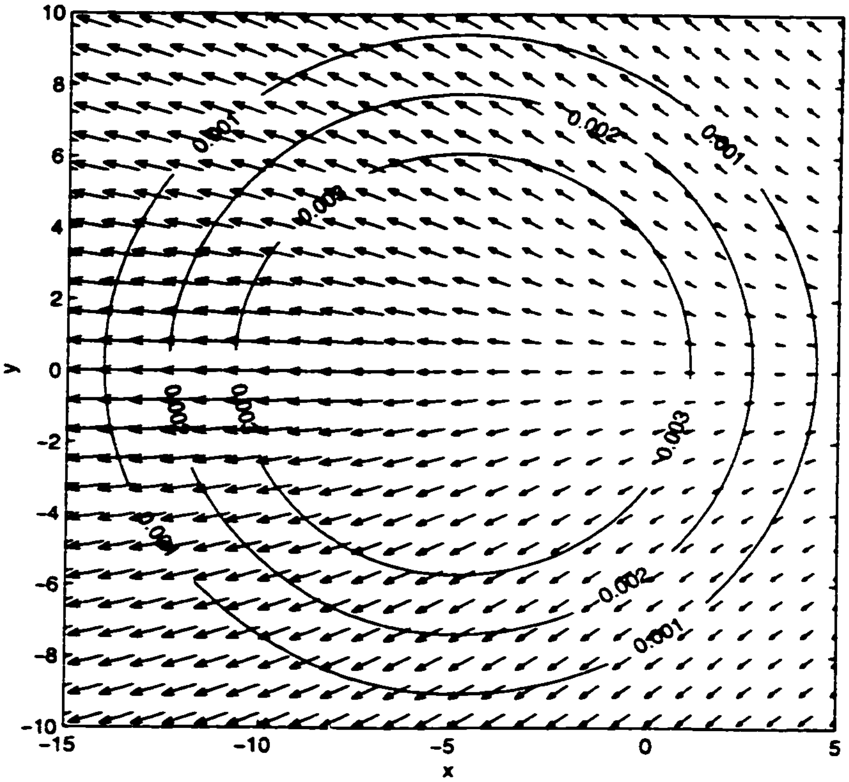

FIGURE 6.22. Height contour and velocity vector plot at time $t = 10$ without rotation. Relevant parameters: $h_B = 0.1x$, $\varepsilon = 1/1000$, $\zeta_0 = 1 + 0.1x$.

To show the effects of nonzero bottom topography, a similar diagram to Figure 6.21 is plotted as Figure 6.22. This diagram, plotted at $t = 10$, should be viewed while keeping in mind that the bottom topography is such that the height of the bottom on the left side is lower than that on the right side. It can be observed from Figure 6.22, that the sloping bottom tends to push the centre of the gravity current down the slope, as the original position was at $x = 0$. In addition, the velocity, at this time is such that the upslope spreading has been checked, and the entire current begins to move down the slope.

When the additional factor of rotation is included in the calculation leading up to Figure 6.22, it has quite a dramatic effect. Initially, as the gravity current begins to spread, the Coriolis parameter is not very important. However, as seen in Figure 6.23, at a later time $t = 10$, the effects of rotation are such that the centre of the gravity current has been deflected in the y-direction, and now has an approximate centre $(-5, 2)$, as opposed to the value of approximately $(-5, 0)$

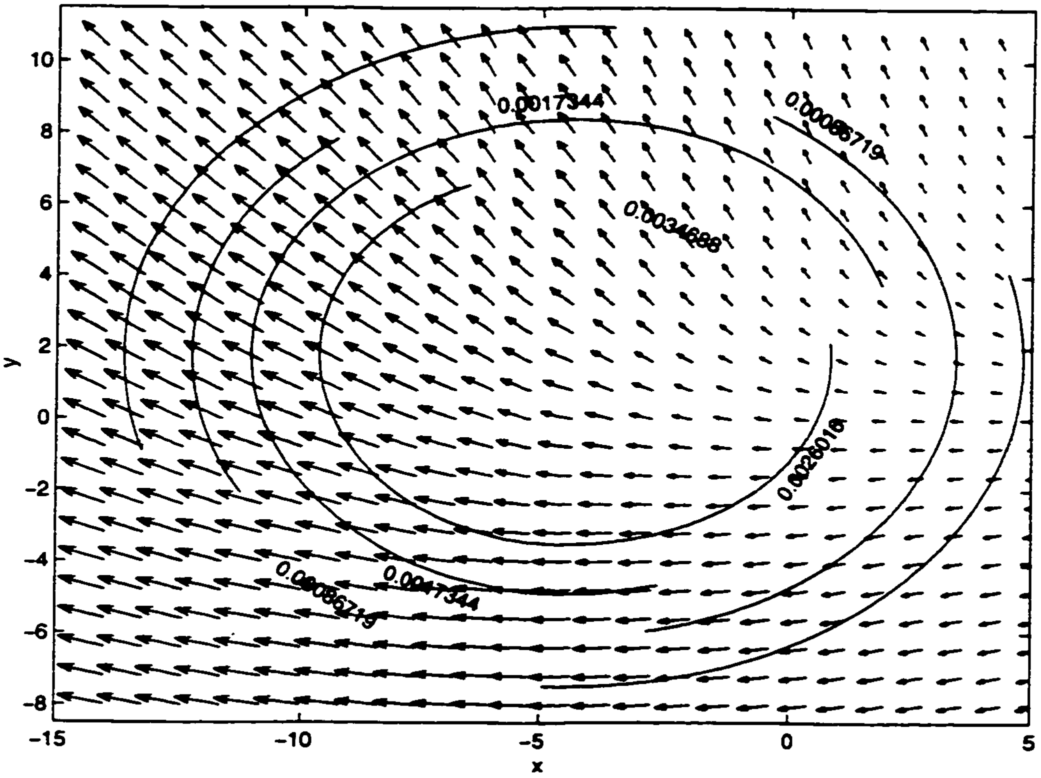

FIGURE 6.23. Height contour and velocity vector plot at time $t = 10$ with rotation. Relevant parameters: $h_B = 0.1x$, $\varepsilon = 1/10$, $\zeta_0 = 1 + 0.1x$.

without rotation. (Unfortunately, the aspect ratio shown in Figure 6.23 is not 1:1, so the diagram looks skewed. This is a limitation of the plotting routine, which omits portions of the height contours when labelling the diagram.)

Figure 6.23 clearly shows that the effects of rotation and bottom slope cause the gravity current to travel in a direction which is neither down the slope, nowralong the slope, but along a path on which the effects of inertial spreading, gravitational acceleration, and rotation are balanced.

Chapter Summary

In Chapter 6, a finite-difference numerical scheme, the so-called relaxation method, has been described and generalized to include non-zero forcing terms and boundary conditions. The method is applicable to nonlinear systems of hyperbolic conservation laws in both single and several spatial variables, and has the advantage of not requiring calculation of the eigenvalues of the system during calculation of numerical solution. Since the gravity current equations developed previously are, in general, hyperbolic, the method is applicable in this case.

In Section 6.2, the relaxation method was investigated as to the parameters involved, and then applied to solve the two-dimensional gravity current initial value problem. The results and calculations portrayed are well supplemented by additional calculations published and submitted for publication by Montgomery and Moodie (1998a, 1998b, 1999a, and 1999b). The results suggest that the method is quite useful for calculating solutions to the nonlinear problem without relying on an experimentally or theoretically determined front condition. As such, the method is generalizable to include non-horizontal bottom topography, spatially dependent forcing terms such as friction, and variable-volume gravity currents (Montgomery and Moodie 1998b).

In Section 6.3, the relaxation method was applied to calculate numerical solutions to the initial value problem in three dimensions. The initial release of a cylinder is such that the external radius slows with time, and is affected by the Coriolis parameter such that the rate of spreading slows with an increase in the effects of rotation. The addition of a non-zero bottom slope acts to drag the ensuing gravity current in the down-slope direction, which is balanced by inertial spreading and the effects of rotation.

Chapter 7

Multinomial Conservation Equations

In chapter 2, various cases of the gravity current equations were introduced, and where possible, stated as systems of conservation equations. Physically, a conservation equation often represents a quantity which is conserved, such as mass, momentum, or energy. In this chapter, a classification of the conservation equations arising from multinomial flux functions are characterized for three dimensional (one and two-layer) and two dimensional (two-layer) gravity currents with a free surface.

It is known (Whitham, 1974 p.459) that the shallow-water equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

admit an infinite number of conservation equations of the form

$$\frac{\partial}{\partial t}P(u,h) + \frac{\partial}{\partial x}Q(u,h) = 0,$$

where P, Q are polynomials (more correctly, multinomials) in u and h. This result is useful since either inverse scattering methods or the hodograph transformation may be used to solve the system (Whitham, 1974). In addition, the situation in two spatial dimensions, specified as the fully nonlinear long wave equations which are a simplification of the equations considered in this chapter, has been shown by Miura (1974) to admit an infinite number of conservation equations. This chapter will examine the more general shallow-water equations for gravity currents which have been previously investigated, to discover whether or not these situations admit an infinite number of multinomial conservation equations. The various conservation forms and equations will be examined in subsequent sections, which are arranged in increasing order of complexity: the two-dimensional two-layer case is discussed in section 7.1, three-dimensional one-layer system in section 7.2, and the three-dimensional two-layer equations in section 7.3.

The equations considered in this chapter are the dimensional ones from chapter 2 with a free surface. A change of notation $\zeta_1 = h_1 - h_2$, and $\zeta_2 = h_2 - h_B$, is affected so that the variables represent the actual thickness or height of the respective layers. Equations (2.2.17)-(2.2.22) with the simplifications $F_2 = 0$,

$F_1 = 0$, p_s =constant and the above substitution are stated for use in this chapter as

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial y} + (1-\gamma)g\frac{\partial \zeta_1}{\partial x} + g\frac{\partial \zeta_2}{\partial x} = -g\frac{\partial h_B}{\partial x} + fv_2, \tag{7.1}$$

$$\frac{\partial v_2}{\partial t} + u_2\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y} + (1-\gamma)g\frac{\partial \zeta_1}{\partial y} + g\frac{\partial \zeta_2}{\partial y} = -g\frac{\partial h_B}{\partial y} - fu_2 \tag{7.2}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + g\frac{\partial \zeta_1}{\partial x} + g\frac{\partial \zeta_2}{\partial x} = g\frac{\partial h_B}{\partial x} + fv_1, \tag{7.3}$$

$$\frac{\partial v_1}{\partial t} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} + g\frac{\partial \zeta_1}{\partial y} + g\frac{\partial \zeta_2}{\partial y} = g\frac{\partial h_B}{\partial y} - fu_1, \tag{7.4}$$

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) + \frac{\partial}{\partial y}(\zeta_2 v_2) = 0, \tag{7.5}$$

and

$$\frac{\partial \zeta_1}{\partial t} + \frac{\partial}{\partial x}(\zeta_1 u_1) + \frac{\partial}{\partial y}(\zeta_1 v_1) = 0. \tag{7.6}$$

7.1 Two-Dimensional, Two-Layer Conservation Equations

The two-dimensional, two-layer simplification consists of equations (7.1), (7.3). (7.5) and (7.6) with the restriction that the functions are independent of the variable y, and the transverse velocities v_1, v_2 both vanish. The simplified equations are

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + (1-\gamma)g\frac{\partial \zeta_1}{\partial x} + g\frac{\partial \zeta_2}{\partial x} = -g\frac{\mathrm{d}h_B}{\mathrm{d}x}, \tag{7.1.1}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + g\frac{\partial \zeta_1}{\partial x} + g\frac{\partial \zeta_2}{\partial x} = g\frac{\mathrm{d}h_B}{\mathrm{d}x}, \tag{7.1.2}$$

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial}{\partial x}(\zeta_2 u_2) = 0, \tag{7.1.3}$$

and

$$\frac{\partial \zeta_1}{\partial t} + \frac{\partial}{\partial x}(\zeta_1 u_1) = 0. \tag{7.1.4}$$

Combinations of equations (7.1.1)-(7.1.4) for the homogeneous case with constant bottom height h_B are desired to be in the form

$$\frac{\partial}{\partial t}P(u_1, u_2, \zeta_1, \zeta_2) + \frac{\partial}{\partial x}Q(u_1, u_2, \zeta_1, \zeta_2) = 0, \tag{7.1.5}$$

where P and Q are constant coefficient multinomials in u_1, u_2, ζ_1, and ζ_2.

The functions P and Q in (7.1.5) are by assumption infinitely differentiable in their arguments. Differentiation of (7.1.5) with several applications of the chain rule yields the expression

$$\frac{\partial P}{\partial u_1}\frac{\partial u_1}{\partial t} + \frac{\partial P}{\partial u_2}\frac{\partial u_2}{\partial t} + \frac{\partial P}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial t} + \frac{\partial P}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial t} + \frac{\partial Q}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial Q}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial Q}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial Q}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} = 0. \quad (7.1.6)$$

Substitution of the temporal derivatives from equations (7.1.1)-(7.1.4) into equation (7.1.6) then gives

$$\begin{aligned} &\frac{\partial P}{\partial u_1}\left(-u_1\frac{\partial u_1}{\partial x} - g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right) + \frac{\partial P}{\partial u_2}\left(-u_2\frac{\partial u_2}{\partial x} - (1-\gamma)g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right) \\ &\quad + \frac{\partial P}{\partial \zeta_1}\left(-\zeta_1\frac{\partial u_1}{\partial x} - u_1\frac{\partial \zeta_1}{\partial x}\right) + \frac{\partial P}{\partial \zeta_2}\left(-\zeta_2\frac{\partial u_2}{\partial x} - u_2\frac{\partial \zeta_2}{\partial x}\right) \\ &\quad\quad + \frac{\partial Q}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial Q}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial Q}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial Q}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} = 0. \end{aligned}$$

This expression simplifies, upon collection of the common $\frac{\partial}{\partial x}$ factors to

$$\begin{aligned} &\left(-u_1\frac{\partial P}{\partial u_1} - \zeta_1\frac{\partial P}{\partial \zeta_1} + \frac{\partial Q}{\partial u_1}\right)\frac{\partial u_1}{\partial x} + \left(-u_2\frac{\partial P}{\partial u_2} - \zeta_2\frac{\partial P}{\partial \zeta_2} + \frac{\partial Q}{\partial u_2}\right)\frac{\partial u_2}{\partial x} \\ &\quad + \left(-g\frac{\partial P}{\partial u_1} - (1-\gamma)g\frac{\partial P}{\partial u_2} - u_1\frac{\partial P}{\partial \zeta_1} + \frac{\partial Q}{\partial \zeta_1}\right)\frac{\partial \zeta_1}{\partial x} \\ &\quad + \left(-g\frac{\partial P}{\partial u_1} - g\frac{\partial P}{\partial u_2} - u_2\frac{\partial P}{\partial \zeta_2} + \frac{\partial Q}{\partial \zeta_2}\right)\frac{\partial \zeta_2}{\partial x} = 0. \end{aligned} \quad (7.1.7)$$

Since u_2, u_2, ζ_1, and ζ_2 are in general independent for all x and t in the domain of interest, it is necessary that this property holds for each of their spatial derivatives. Therefore, using this property of independence, for equation (7.1.7) to be satisfied the following four equations must hold:

$$\frac{\partial Q}{\partial u_1} = u_1\frac{\partial P}{\partial u_1} + \zeta_1\frac{\partial P}{\partial \zeta_1}, \quad (7.1.8)$$

$$\frac{\partial Q}{\partial u_2} = u_2\frac{\partial P}{\partial u_2} + \zeta_2\frac{\partial P}{\partial \zeta_2}, \quad (7.1.9)$$

$$\frac{\partial Q}{\partial \zeta_1} = g\frac{\partial P}{\partial u_1} + (1-\gamma)g\frac{\partial P}{\partial u_2} + u_1\frac{\partial P}{\partial \zeta_1}, \quad (7.1.10)$$

and

$$\frac{\partial Q}{\partial \zeta_2} = g\frac{\partial P}{\partial u_1} + g\frac{\partial P}{\partial u_2} + u_2\frac{\partial P}{\partial \zeta_2}. \tag{7.1.11}$$

From these expressions, the various second order derivatives of Q can be evaluated separately and used to derive compatibility conditions for the function P. The relevant second derivatives are calculated from (7.1.8)-(7.1.11) to derive six compatibility conditions.

The first compatibility condition arises from equating $\frac{\partial}{\partial u_2}$ of equation (7.1.8),

$$\frac{\partial^2 Q}{\partial u_1 \partial u_2} = u_1\frac{\partial^2 P}{\partial u_1 \partial u_2} + \zeta_1\frac{\partial^2 P}{\partial \zeta_1 \partial u_2},$$

with $\frac{\partial}{\partial u_1}$ of equation (7.1.9),

$$\frac{\partial^2 Q}{\partial u_2 \partial u_1} = u_2\frac{\partial^2 P}{\partial u_2 \partial u_1} + \zeta_2\frac{\partial^2 P}{\partial \zeta_2 \partial u_1},$$

to obtain

$$(u_1 - u_2)\frac{\partial^2 P}{\partial u_1 \partial u_2} + \zeta_1\frac{\partial^2 P}{\partial \zeta_1 \partial u_2} = \zeta_2\frac{\partial^2 P}{\partial \zeta_2 \partial u_1}. \tag{7.1.12}$$

The second compatibility condition is obtained by calculating $\frac{\partial}{\partial \zeta_1}$ of equation (7.1.8),

$$\frac{\partial^2 Q}{\partial u_1 \partial \zeta_1} = u_1\frac{\partial^2 P}{\partial u_1 \partial \zeta_1} + \frac{\partial P}{\partial \zeta_1} + \zeta_1\frac{\partial^2 P}{\partial \zeta_1^2},$$

and equating this with $\frac{\partial}{\partial u_1}$ of equation (7.1.10),

$$\frac{\partial^2 Q}{\partial \zeta_1 \partial u_1} = g\frac{\partial^2 P}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_2 \partial u_1} + \frac{\partial P}{\partial \zeta_1} + u_1\frac{\partial^2 P}{\partial \zeta_1 \partial u_1}.$$

The resulting equation is

$$\zeta_1\frac{\partial^2 P}{\partial \zeta_1^2} = g\frac{\partial^2 P}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_1 \partial u_2}. \tag{7.1.13}$$

A third equation may be found by equating $\frac{\partial}{\partial \zeta_2}$ of equation (7.1.8),

$$\frac{\partial^2 Q}{\partial u_1 \partial \zeta_2} = u_1\frac{\partial^2 P}{\partial u_1 \partial \zeta_2} + \zeta_1\frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2},$$

and $\dfrac{\partial}{\partial u_1}$ of equation (7.1.11),

$$\frac{\partial^2 Q}{\partial \zeta_2 \partial u_1} = g\frac{\partial^2 P}{\partial u_1^2} + g\frac{\partial^2 P}{\partial u_2 \partial u_1} + u_2\frac{\partial^2 P}{\partial \zeta_2 \partial u_1},$$

yielding

$$(u_1 - u_2)\frac{\partial^2 P}{\partial \zeta_2 \partial u_1} + \zeta_1\frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2} = g\frac{\partial^2 P}{\partial u_1^2} + g\frac{\partial^2 P}{\partial u_1 \partial u_2}. \tag{7.1.14}$$

For the fourth compatibility condition, $\dfrac{\partial}{\partial \zeta_1}$ of equation (7.1.9),

$$\frac{\partial^2 Q}{\partial u_2 \partial \zeta_1} = u_2\frac{\partial^2 P}{\partial u_2 \partial \zeta_1} + \zeta_2\frac{\partial^2 P}{\partial \zeta_2 \partial \zeta_1},$$

and $\dfrac{\partial}{\partial u_2}$ of equation (7.1.10),

$$\frac{\partial^2 Q}{\partial \zeta_1 \partial u_2} = g\frac{\partial^2 P}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_2^2} + u_1\frac{\partial^2 P}{\partial \zeta_1 \partial u_2},$$

equate to

$$(u_2 - u_1)\frac{\partial^2 P}{\partial \zeta_1 \partial u_2} + \zeta_2\frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2} = g\frac{\partial^2 P}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_2^2}. \tag{7.1.15}$$

Another manipulation, equating $\dfrac{\partial}{\partial \zeta_2}$ of equation (7.1.9),

$$\frac{\partial^2 Q}{\partial u_2 \partial \zeta_2} = u_2\frac{\partial^2 P}{\partial u_2 \partial \zeta_2} + \frac{\partial P}{\partial \zeta_2} + \zeta_2\frac{\partial^2 P}{\partial \zeta_2^2},$$

with $\dfrac{\partial}{\partial u_2}$ of equation (7.1.11),

$$\frac{\partial^2 Q}{\partial \zeta_2 \partial u_2} = g\frac{\partial^2 P}{\partial u_1 \partial u_2} + g\frac{\partial^2 P}{\partial u_2^2} + \frac{\partial P}{\partial \zeta_2} + u_2\frac{\partial^2 P}{\partial \zeta_2 \partial u_2},$$

results in the fifth compatibility condition

$$\zeta_2\frac{\partial^2 P}{\partial \zeta_2^2} = g\frac{\partial^2 P}{\partial u_1 \partial u_2} + g\frac{\partial^2 P}{\partial u_2^2}. \tag{7.1.16}$$

The sixth (and final) compatibility condition arises when $\dfrac{\partial}{\partial \zeta_2}$ of equation (7.1.10),

$$\frac{\partial^2 Q}{\partial \zeta_1 \partial \zeta_2} = g\frac{\partial^2 P}{\partial u_1 \partial \zeta_2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_2 \partial \zeta_2} + u_1\frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2},$$

and $\dfrac{\partial}{\partial \zeta_1}$ of equation (7.1.11),

$$\frac{\partial^2 Q}{\partial \zeta_2 \partial \zeta_1} = g\frac{\partial^2 P}{\partial u_1 \partial \zeta_1} + g\frac{\partial^2 P}{\partial u_2 \partial \zeta_1} + u_2\frac{\partial^2 P}{\partial \zeta_2 \partial \zeta_1},$$

are equated to obtain

$$(u_1 - u_2)\frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2} + g\frac{\partial^2 P}{\partial u_1 \partial \zeta_2} + (1-\gamma)g\frac{\partial^2 P}{\partial u_2 \partial \zeta_2} = g\frac{\partial^2 P}{\partial u_1 \partial \zeta_1} + g\frac{\partial^2 P}{\partial u_2 \partial \zeta_1}. \quad (7.1.17)$$

With compatibility conditions (7.1.12)-(7.1.17) imposed on the multinomial P, the constraints on the form of P may then be used to describe the form of Q via equations (7.1.8)-(7.1.11). General binomial functions satisfying equation (7.1.5) are then assumed to be of the form

$$P(u_1, u_2, \zeta_1, \zeta_2) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} p_{ij}\zeta_1^i\zeta_2^j, \quad p_{ij} = p_{ij}(u_1, u_2). \quad (7.1.18)$$

Binomials of the form (7.1.18), although not the most general, are sufficiently general to achieve the desired result.

To apply the conditions (7.1.12)-(7.1.17), it is desirable to write down the partial derivatives involved by differentiating equation (7.1.18) as required. The first derivatives are

$$\frac{\partial P}{\partial u_1} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial p_{ij}}{\partial u_1}\zeta_1^i\zeta_2^j, \quad \frac{\partial P}{\partial u_2} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial p_{ij}}{\partial u_2}\zeta_1^i\zeta_2^j,$$
$$\frac{\partial P}{\partial \zeta_1} = \sum_{i=1}^{n}\sum_{j=0}^{n-i} i p_{ij}\zeta_1^{i-1}\zeta_2^j, \text{ and } \frac{\partial P}{\partial \zeta_2} = \sum_{i=0}^{n}\sum_{j=1}^{n-i} j p_{ij}\zeta_1^i\zeta_1^{j-1}. \quad (7.1.19)$$

Equation (7.1.19) allows the second partial derivatives to be written as

$$\frac{\partial^2 P}{\partial u_1^2} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_1^2}\zeta_1^i\zeta_2^j, \quad \frac{\partial^2 P}{\partial u_2^2} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_2^2}\zeta_1^i\zeta_2^j,$$
$$\frac{\partial^2 P}{\partial \zeta_1^2} = \sum_{i=2}^{n}\sum_{j=0}^{n-i} i(i-1)p_{ij}\zeta_1^{i-2}\zeta_2^j, \text{ and } \frac{\partial^2 P}{\partial \zeta_2^2} = \sum_{i=0}^{n}\sum_{j=2}^{n-i} j(j-1)p_{ij}\zeta_1^i\zeta_2^{j-2}. \quad (7.1.20)$$

The six mixed partial derivatives may be stated similarly as

$$\frac{\partial^2 P}{\partial u_1 \partial u_2} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\zeta_1^i\zeta_2^j, \quad \frac{\partial^2 P}{\partial u_1 \partial \zeta_1} = \sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_1}\zeta_1^{i-1}\zeta_2^j,$$
$$\frac{\partial^2 P}{\partial u_1 \partial \zeta_2} = \sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_1}\zeta_1^i\zeta_2^{j-1}, \quad \frac{\partial^2 P}{\partial u_2 \partial \zeta_1} = \sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_2}\zeta_1^{i-1}\zeta_2^j, \quad (7.1.21)$$
$$\frac{\partial^2 P}{\partial u_2 \partial \zeta_2} = \sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_2}\zeta_1^i\zeta_2^{j-1}, \text{ and } \frac{\partial^2 P}{\partial \zeta_1 \partial \zeta_2} = \sum_{i=1}^{n}\sum_{j=1}^{n-i} ijp_{ij}\zeta_1^{i-1}\zeta_2^{j-1}.$$

The partial derivatives (7.1.19)-(7.1.21) may now be substituted into the compatibility conditions (7.1.12)-(7.1.17). These results are calculated for each compatibility condition to determine restrictions on the functions p_{ij}.

The first equation (7.1.12) is employed by substituting the appropriate derivatives from (7.1.21) to give

$$(u_1 - u_2)\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\zeta_1^i\zeta_2^j + \zeta_1\sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_2}\zeta_1^{i-1}\zeta_2^j = \zeta_2\sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_1}\zeta_1^i\zeta_2^{j-1}.$$

After simplifying, the above equation becomes

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\left((u_1 - u_2)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + i\frac{\partial p_{ij}}{\partial u_2} - j\frac{\partial p_{ij}}{\partial u_1}\right)\zeta_1^i\zeta_2^j = 0,$$

which gives, upon using the independence of ζ_1 and ζ_2, the result

$$(u_1 - u_2)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} = j\frac{\partial p_{ij}}{\partial u_1} - i\frac{\partial p_{ij}}{\partial u_2} \text{ for } i = 0, \dots, n, \text{ and } j = 0, \dots, n - i. \quad (7.1.22)$$

The next condition, (7.1.13) is slightly more involved, but substitution of the derivatives from (7.1.20) and (7.1.21) into equation (7.1.13) gives

$$\zeta_1\sum_{i=2}^{n}\sum_{j=0}^{n-i} i(i-1)p_{ij}\zeta_1^{i-2}\zeta_2^j = g\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_1^2}\zeta_1^i\zeta_2^j + (1-\gamma)g\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\zeta_1^i\zeta_2^j.$$

Gathering terms and shifting indices allows this expression to be rewritten as follows:

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} i(i-1)p_{ij}\zeta_1^{i-1}\zeta_2^j - \sum_{i=0}^{n}\sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\right)\zeta_1^i\zeta_2^j = 0$$

$$\Longrightarrow \sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}(i+1)ip_{i+1,j}\zeta_1^i\zeta_2^j - \sum_{i=0}^{n}\sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\right)\zeta_1^i\zeta_2^j = 0$$

$$\Longrightarrow \sum_{i=1}^{n-1}\left[\sum_{j=0}^{n-i-1}(i+1)ip_{i+1,j}\zeta_2^j - \sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\right)\zeta_2^j\right]\zeta_1^i$$
$$-\sum_{j=0}^{n} g\left(\frac{\partial^2 p_{0j}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2}\right)\zeta_2^j - g\left(\frac{\partial^2 p_{n0}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{n0}}{\partial u_1 \partial u_2}\right)\zeta_1^n = 0.$$

Independence of the powers of ζ_1 allow this result to be stated as a condition for each i. Therefore, it follows that

$$\frac{\partial^2 p_{n0}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{n0}}{\partial u_1 \partial u_2} = 0, \tag{7.1.23}$$

and

$$\sum_{j=0}^{n} g\left(\frac{\partial^2 p_{0j}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2}\right)\zeta_2^j = 0,$$

from which it follows that

$$\frac{\partial^2 p_{0j}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2} = 0 \text{ for } j = 0, \ldots, n. \tag{7.1.24}$$

For the remaining terms, with $i = 1, \ldots, n-1$, equating the coefficients of ζ_1^i to zero yields

$$\sum_{j=0}^{n-i-1}(i+1)ip_{i+1,j}\zeta_2^j - \sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\right)\zeta_2^j = 0,$$

which may be rewritten as

$$\sum_{j=0}^{n-i-1}\left((i+1)ip_{i+1,j} - g\frac{\partial^2 p_{ij}}{\partial u_1^2} - (1-\gamma)g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\right)\zeta_2^j$$
$$- g\left(\frac{\partial^2 p_{i,n-i}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{i,n-1}}{\partial u_1 \partial u_2}\right)\zeta_2^{n-i} = 0.$$

From this sum, it follows that for $i = 1, \ldots, n-1$,

$$(i+1)ip_{i+1,j} = g\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} \text{ for } j = 0, \ldots, n-i-1. \tag{7.1.25}$$

An additional equation follows from the coefficient of ζ_2^{n-i} which can be made to include equation (7.1.23) and (7.1.24) with $j = n$ by writing it as

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1^2} + (1-\gamma)\frac{\partial^2 p_{i,n-1}}{\partial u_1 \partial u_2} = 0, \qquad (7.1.26)$$

for $i = 0, \ldots, n$.

The third compatibility condition (7.1.14) is now used to place restrictions on p_{ij}. Substituting the expressions for the derivatives produces the relation

$$(u_1 - u_2)\sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_1}\zeta_1^i\zeta_2^{j-1} + \zeta_1\sum_{i=1}^{n}\sum_{j=1}^{n-i} ijp_{ij}\zeta_1^{i-1}\zeta_2^{j-1}$$
$$= g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_1^2}\zeta_1^i\zeta_2^j + g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2}\zeta_1^i\zeta_2^j.$$

By collecting terms and simplifying, the above expression can be written as

$$\sum_{i=0}^{n}\left[\sum_{j=1}^{n-i}\left((u_1-u_2)j\frac{\partial p_{ij}}{\partial u_1} + ijp_{ij}\right)\zeta_2^{j-1} - \sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1^2} + \frac{\partial^2 p_{ij}}{\partial u_1\partial u_2}\right)\zeta_2^j\right]\zeta_1^i = 0,$$

from which the $n+1$ conditions arise for $i = 0, \ldots, n$ as

$$\sum_{j=0}^{n-i-1}\left((u_1-u_2)(j+1)\frac{\partial p_{i,j+1}}{\partial u_1} + i(j+1)p_{i,j+1} - g\frac{\partial^2 p_{ij}}{\partial u_1^2} - g\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2}\right)\zeta_2^j$$
$$- g\left(\frac{\partial^2 p_{i,n-i}}{\partial u_1^2} + \frac{\partial^2 p_{i,n-i}}{\partial u_1\partial u_2}\right)\zeta_2^{n-i} = 0.$$

By isolating the powers of ζ_2, the above equation gives rise for $i = 0, \ldots, n-1$ to

$$(u_1-u_2)(j+1)\frac{\partial p_{i,j+1}}{\partial u_1} + i(j+1)p_{i,j+1} = g\frac{\partial^2 p_{ij}}{\partial u_1^2} + g\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2} \qquad (7.1.27)$$
$$\text{for } j = 0, \ldots, n-i-1,$$

and for $i = 0, \ldots, n$ to

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1^2} + \frac{\partial^2 p_{i,n-i}}{\partial u_1\partial u_2} = 0. \qquad (7.1.28)$$

Equation (7.1.26) and (7.1.28) can now be subtracted to give the result that, for $\gamma \neq 0$ (an assumption), for $i = 0, \ldots, n$

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1\partial u_2} = 0. \qquad (7.1.29)$$

This result can be substituted back into equation (7.1.28) to give the corresponding condition for $i = 0\ldots,n$

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1^2} = 0. \tag{7.1.30}$$

Equation (7.1.15) gives a similar result to the above as, after substitution of the derivatives (7.1.20), (7.1.21) it becomes

$$(u_2-u_1)\sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_2}\zeta_1^{i-1}\zeta_2^j + \zeta_2\sum_{i=1}^{n}\sum_{j=1}^{n-i} ijp_{ij}\zeta_1^{i-1}\zeta_2^{j-1}$$
$$= g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2}\zeta_1^i\zeta_2^j + (1-\gamma)g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_2^2}\zeta_1^i\zeta_2^j.$$

This expression may be employed after a few manipulations which follow:

$$\sum_{i=1}^{n}\sum_{j=0}^{n-i}\left((u_2-u_1)i\frac{\partial p_{ij}}{\partial u_2} + ijp_{ij}\right)\zeta_1^{i-1}\zeta_2^j$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_2^2}\right)\zeta_1^i\zeta_2^j$$

$$\Longrightarrow \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}\left((u_2-u_1)(i+1)\frac{\partial p_{i+1,j}}{\partial u_2} + (i+1)jp_{i+1,j}\right)\zeta_1^i\zeta_2^j$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_2^2}\right)\zeta_1^i\zeta_2^j$$

$$\Longrightarrow \sum_{i=0}^{n-1}\left[\sum_{j=0}^{n-i-1}(i+1)\left((u_2-u_1)\frac{\partial p_{i+1,j}}{\partial u_2} + jp_{i+1,j}\right)\zeta_2^j\right.$$
$$\left.-\sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_2^2}\right)\zeta_2^j\right]\zeta_1^i = g\left(\frac{\partial^2 p_{n0}}{\partial u_1\partial u_2} + (1-\gamma)\frac{\partial^2 p_{n0}}{\partial u_2^2}\right)\zeta_1^n.$$

The coefficients of the powers of ζ_1 may now be isolated for $i = 0,\ldots,n-1$ as,

$$(i+1)\left((u_2-u_1)\frac{\partial p_{i+1,j}}{\partial u_2} + jp_{i+1,j}\right) = g\left(\frac{\partial^2 p_{ij}}{\partial u_1\partial u_2} + (1-\gamma)\frac{\partial^2 p_{ij}}{\partial u_2^2}\right) \tag{7.1.31}$$
$$\text{for } j = 0,\ldots,n-i-1,$$

and the remaining term, for $i = 0, \ldots, n$

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1 \partial u_2} + (1-\gamma)\frac{\partial^2 p_{i,n-i}}{\partial u_2^2} = 0.$$

The coefficient of ζ_1^n is included in the above equation. Using equation (7.1.29) allows this last result to be simplified to

$$\frac{\partial^2 p_{i,n-i}}{\partial u_2^2} = 0, \text{ for } i = 0, \ldots, n. \tag{7.1.32}$$

The simplification for the compatibility condition (7.1.16) is somewhat shorter than that completed previously. Substituting the expressions for the derivatives gives

$$\zeta_2 \sum_{i=0}^{n}\sum_{j=2}^{n-i} j(j-1)p_{ij}\zeta_1^i\zeta_2^{j-2} = g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}\zeta_1^i\zeta_2^j + g\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^2 p_{ij}}{\partial u_2^2}\zeta_1^i\zeta_2^j,$$

which simplifies as

$$\sum_{i=0}^{n}\left[\sum_{j=2}^{n-i} j(j-1)p_{ij}\zeta_2^{j-1} - \sum_{j=0}^{n-i} g\left(\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{ij}}{\partial u_2^2}\right)\zeta_2^j\right]\zeta_1^i = 0.$$

Isolating the coefficients in the powers of ζ_1 yields, after some manipulation of the indices,

$$\sum_{j=0}^{n-i-1}\left[(j+1)jp_{i,j+1} - g\left(\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{ij}}{\partial u_2^2}\right)\right]\zeta_2^j$$
$$= g\left(\frac{\partial^2 p_{i,n-i}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{i,n-i}}{\partial u_2^2}\right)\zeta_2^{n-i}.$$

This gives the final result, for $i = 0 \ldots, n-1$, as

$$(j+1)jp_{i,j+1} = g\left(\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{ij}}{\partial u_2^2}\right), \text{ for } j = 0, \ldots, n-i-1. \tag{7.1.33}$$

The remaining coefficient yields

$$\frac{\partial^2 p_{i,n-i}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{i,n-i}}{\partial u_2^2} = 0,$$

for $i = 0, \ldots, n$, which after substituting in the result (7.1.29) becomes nothing more than the result (7.1.32).

The last compatibility condition, (7.1.17), becomes

$$(u_1 - u_2)\sum_{i=1}^{n}\sum_{j=1}^{n-i} ijp_{ij}\zeta_1^{i-1}\zeta_2^{j-1} + g\sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_1}\zeta_1^i\zeta_2^{j-1}$$
$$+(1-\gamma)g\sum_{i=0}^{n}\sum_{j=1}^{n-i} j\frac{\partial p_{ij}}{\partial u_2}\zeta_1^i\zeta_2^{j-1} = g\sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_1}\zeta_1^{i-1}\zeta_2^j + g\sum_{i=1}^{n}\sum_{j=0}^{n-i} i\frac{\partial p_{ij}}{\partial u_2}\zeta_1^{i-1}\zeta_2^j,$$

which may be rearranged as

$$\sum_{i=0}^{n-1}\left[\sum_{j=1}^{n-i-1}(i+1)j(u_1-u_2)p_{i+1,j}\zeta_2^{j-1} + \sum_{j=1}^{n-i} jg\left(\frac{\partial p_{ij}}{\partial u_1} + (1-\gamma)\frac{\partial p_{ij}}{\partial u_2}\right)\zeta_2^{j-1}\right.$$
$$\left. - \sum_{j=0}^{n-i-1}(i+1)g\left(\frac{\partial p_{i+1,j}}{\partial u_1} + \frac{\partial p_{i+1,j}}{\partial u_2}\right)\zeta_2^j\right]\zeta_1^i = 0.$$

From this sum, each coefficient of ζ_1^i vanishes for $i = 0, \ldots, n-1$. These are written as

$$\sum_{j=0}^{n-i-2}(i+1)(j+1)(u_1-u_2)p_{i+1,j+1}\zeta_2^j + \sum_{j=0}^{n-i-1}(j+1)g\left(\frac{\partial p_{i,j+1}}{\partial u_1}\right.$$
$$\left. + (1-\gamma)\frac{\partial p_{i,j+1}}{\partial u_2}\right)\zeta_2^j - \sum_{j=0}^{n-i-1}(i+1)g\left(\frac{\partial p_{i+1,j}}{\partial u_1} + \frac{\partial p_{i+1,j}}{\partial u_2}\right)\zeta_2^j = 0,$$

which allows the following restrictions by isolating powers of ζ_2. The result is, for $i = 0, \ldots, n-2$,

$$(i+1)(j+1)(u_1-u_2)p_{i+1,j+1} + (j+1)g\left(\frac{\partial p_{i,j+1}}{\partial u_1} + (1-\gamma)\frac{\partial p_{i,j+1}}{\partial u_2}\right)$$
$$= (i+1)g\left(\frac{\partial p_{i+1,j}}{\partial u_1} + \frac{\partial p_{i+1,j}}{\partial u_2}\right) \quad \text{for } j = 0, \ldots, n-i-2, \qquad (7.1.34)$$

and, for $i = 0, \ldots, n-1$,

$$(n-i)\left(\frac{\partial p_{i,n-i}}{\partial u_1} + (1-\gamma)\frac{\partial p_{i,n-i}}{\partial u_2}\right) = (i+1)\left(\frac{\partial p_{i+1,n-i-1}}{\partial u_1} + \frac{\partial p_{i+1,n-i-1}}{\partial u_2}\right). \qquad (7.1.35)$$

The previous few pages of calculations may now be summarized. The six equations (7.1.12)-(7.1.17) for a general continuous function P, become, upon assumption of the form of P as (7.1.18), eleven equations: (7.1.22), (7.1.24), (7.1.25), (7.1.27), (7.1.29)-(7.1.35). Before making a general statement concerning the possible solutions p_{ij} to these constraints, it is useful to examine the problem in several steps, stated as lemmas.

Lemma 7.1 The coefficients p_{0j}, $j = 1, \ldots, n$ for the multinomial $P(u_1, u_2, \zeta_1, \zeta_2)$ in equation (7.1.18) are functions of u_2 only.

Proof: Equation (7.1.27) with $i = 0$ becomes

$$(u_1 - u_2)(j+1)\frac{\partial p_{0,j+1}}{\partial u_1} = g\frac{\partial^2 p_{0j}}{\partial u_1^2} + g\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2}, \text{ for } j = 0, \ldots, n-1. \quad (7.1.36)$$

From equation (7.1.24), this result may be simplified to

$$(u_1 - u_2)(j+1)\frac{\partial p_{0,j+1}}{\partial u_1} = \gamma g\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2}, \text{ for } j = 0, \ldots, n-1. \quad (7.1.37)$$

Now, equation (7.1.22) with $i = 0$,

$$(u_1 - u_2)\frac{\partial^2 p_{0j}}{\partial u_1 \partial u_2} = j\frac{\partial p_{0j}}{\partial u_1}, \text{ for } j = 0, \ldots, n,$$

may be substituted into equation (7.1.37) above to give

$$(u_1 - u_2)^2(j+1)\frac{\partial p_{0,j+1}}{\partial u_1} = \gamma g j\frac{\partial p_{0j}}{\partial u_1} \text{ for } j = 0, \ldots, n-1. \quad (7.1.38)$$

This recursive relation, by starting with $j = 0$, gives

$$\frac{\partial p_{01}}{\partial u_1} = 0, \frac{\partial p_{02}}{\partial u_1} = 0, \ldots, \frac{\partial p_{0n}}{\partial u_1} = 0. \quad (7.1.39)$$

Since $p_{ij} = p_{ij}(u_1, u_2)$, this last equation (7.1.39), gives the desired result, $p_{0j} = p_{0j}(u_2)$ for $j = 1, \ldots, n$. □

The next result is a similar one to Lemma 1 for the opposite coefficients.

Lemma 7.2 The coefficients p_{i0}, $i = 1, \ldots, n$ for the multinomial $P(u_1, u_2, \zeta_1, \zeta_2)$ in equation (7.1.18) are functions of u_1 only.

Proof: Equation (7.1.31) with $j = 0$ simplifies to

$$(i+1)(u_2 - u_1)\frac{\partial p_{i+1,0}}{\partial u_2} = g\frac{\partial^2 p_{i0}}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 p_{i0}}{\partial u_2^2} \text{ for } i = 0, \ldots, n-1. \quad (7.1.40)$$

By considering equation (7.1.33) for $j = 0$,

$$0 = g\left(\frac{\partial^2 p_{i0}}{\partial u_1 \partial u_2} + \frac{\partial^2 p_{i0}}{\partial u_2^2}\right) \text{ for } i = 0, \ldots, n-1,$$

equation (7.1.40) may be expressed as

$$(i+1)(u_2 - u_1)\frac{\partial p_{i+1,0}}{\partial u_2} = -\gamma g \frac{\partial^2 p_{i0}}{\partial u_1 \partial u_2} \text{ for } i = 0, \ldots, n-1. \tag{7.1.41}$$

Use of equation (7.1.22) can now be made with $j = 0$ as

$$(u_1 - u_2)\frac{\partial^2 p_{i0}}{\partial u_1 \partial u_2} = -i\frac{\partial p_{i0}}{\partial u_2} \text{ for } i = 0, \ldots, n, \tag{7.1.42}$$

which substitutes into (7.1.41) to give

$$(i+1)(u_2 - u_1)^2\frac{\partial p_{i+1,0}}{\partial u_2} = -\gamma g i\frac{\partial p_{i0}}{\partial u_2}, \text{ for } i = 0, \ldots, n-i. \tag{7.1.43}$$

This recursive relation, starting with $i = 0$, yields

$$\frac{\partial p_{10}}{\partial u_2} = 0, \frac{\partial p_{20}}{\partial u_2} = 0, \ldots, \frac{\partial p_{n0}}{\partial u_2} = 0. \tag{7.1.44}$$

Since $p_{ij} = p_{ij}(u_1, u_2)$, this last equation (7.1.44), achieves the desired result, $p_{i0} = p_{i0}(u_2)$ for $i = 1, \ldots, n$. □

Lemma 7.3 The coefficients $p_{i,n-i}$, $i = 0, \ldots, n$ for the multinomial function $P(u_1, u_2, \zeta_1, \zeta_2)$ in equation (7.1.18) are constant for $n \geq 2$ a positive integer. If $n = 1$, then p_{10} and p_{01} are linear in u_1 and u_2, respectively. If $n = 0$ then p_{00} is linear in both u_1 and u_2 separately.

Proof: For $p_{i,n-i}$, equations (7.1.29), (7.1.30) and (7.1.32) state that the second partial derivatives vanish everywhere. The only continuously differentiable coefficients which satisfy this constraint must take a linear form

$$p_{i,n-i} = a_i u_1 + b_i u_2 + c_i \text{ for } i = 0, \ldots, n, \tag{7.1.45}$$

where the terms a_i, b_i, and c_i are all real constants. If $n = 0$, then this is precisely the required result for this special case.

Coefficients of the form (7.1.45), substituted into equation (7.1.22) for $i = 0, \ldots, n$ and $j = n - i$, give a further constraint on the constants,

$$ib_i = (n-i)a_i \text{ for } i = 0, \ldots, n. \tag{7.1.46}$$

From (7.1.46) it is quickly observed that for $n \geq 1$, $a_0 = 0$ and $b_n = 0$. Additionally, for $i = 0, \ldots, n-1$, equation (7.1.35) simplifies to

$$(n-i)[a_i + (1-\gamma)b_i] = (i+1)[a_{i+1} + b_{i+1}]. \tag{7.1.47}$$

Substitution of (7.1.46) into this result (7.1.47) for to remove b_i yields, for $i = 1, \ldots, n-1$,

$$(n-i)[a_i + (1-\gamma)\frac{(n-i)}{i}a_i] = (i+1)[a_{i+1} + \frac{n-(i+1)}{i+1}a_{i+1}],$$

which simplifies as

$$(n-i)[\gamma i + (1-\gamma)n]a_i = ina_{i+1}. \tag{7.1.48}$$

Similarly, for $i = 0$, equation (7.1.46) and (7.1.47) give

$$n(1-\gamma)b_0 = a_1 + (n-1)a_1,$$

which simplifies to

$$(1-\gamma)b_0 = a_1. \tag{7.1.49}$$

Using equations (7.1.46), (7.1.48) and (7.1.49) determines the constants a_i and b_i recursively to depend on the single constant, a_n. If $n = 1$, then only (7.1.49) applies, and the coefficients from (7.1.45) take the form

$$p_{01} = b_0 u_2 + c_0, \quad p_{10} = (1-\gamma)b_0 u_1 + c_1,$$

which is the result for $n = 1$.

To determine a_n, two more constraints are employed. First, consider equation (7.1.25) for $i = n-1$ and $j = 0$, which is

$$n(n-1)p_{n0} = g\frac{\partial^2 p_{n-1,0}}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 p_{n-1,0}}{\partial u_1 \partial u_2}. \tag{7.1.50}$$

Using Lemma 7.2 and substituting in the form (7.1.45) for p_{n0} simplifies (7.1.50) to

$$n(n-1)(a_n u_1 + c_n) = g\frac{\mathrm{d}^2 p_{n-1,0}}{\mathrm{d}u_1^2}. \tag{7.1.51}$$

Similarly, equation (7.1.27) for $i = n - 1$ and $j = 0$ is

$$(u_1 - u_2)\frac{\partial p_{n-1,1}}{\partial u_1} + (n-1)p_{n-1,1} = g\frac{\partial^2 p_{n-1,0}}{\partial u_1^2} + g\frac{\partial^2 p_{n-1,0}}{\partial u_1 \partial u_2}. \qquad (7.1.52)$$

Again, by use of Lemma 7.2 and the form (7.1.45) for $p_{n-1,1}$ this result becomes

$$(u_1 - u_2)a_{n-1} + (n-1)(a_{n-1}u_1 + b_{n-1}u_2 + c_{n-1}) = g\frac{\mathrm{d}^2 p_{n-1,0}}{\mathrm{d}u_1^2},$$

which simplifies further by removing b_{n-1} through (7.1.46) to

$$(u_1 - u_2)a_{n-1} + (n-1)(a_{n-1}u_1 + \frac{1}{n-1}a_{n-1}u_2 + c_{n-1}) = g\frac{\mathrm{d}^2 p_{n-1,0}}{\mathrm{d}u_1^2},$$

or

$$nu_1 a_{n-1} + (n-1)c_{n-1} = g\frac{\mathrm{d}^2 p_{n-1,0}}{\mathrm{d}u_1^2}. \qquad (7.1.53)$$

Comparing equations (7.1.51) and (7.1.53) gives the equality of the respective left hand sides as

$$n(n-1)(a_n u_1 + c_n) = nu_1 a_{n-1} + (n-1)c_{n-1},$$

or

$$n[(n-1)a_n - na_{n-1}]u_1 + (n-1)[nc_n - c_{n-1}] = 0. \qquad (7.1.54)$$

Since u_1 is not, in general, a constant, equation (7.1.54) gives rise to the equality

$$(n-1)a_n = na_{n-1}. \qquad (7.1.55)$$

However, equation (7.1.48) with $i = n - 1$ is

$$(n-\gamma)a_{n-1} = n(n-1)a_n, \qquad (7.1.56)$$

which, after a_{n-1} is removed by use of (7.1.55), yields

$$(n-1)(n-\gamma)a_n = n^2(n-1)a_n.$$

For $n \geq 2$, this result then simplifies to

$$(n-\gamma)a_n = n^2 a_n. \qquad (7.1.57)$$

Since $\gamma \neq 0$, equation (7.1.57) can be used to conclude that $a_n = 0$. By the recursion formula (7.1.48), it follows that $a_i = 0$ for $i = 1, \ldots, n-1$. Finally, equation (7.1.49) gives $b_0 = 0$ as well. Therefore, from the form (7.1.45) it may be seen that $p_{i,n-i} = c_i$, a constant, for $i = 0, \ldots, n$ whenever $n \geq 2$. □

Complementing Lemma 7.3 is an additional result which relates the constant terms $p_{i,n-i}$ by a recursive relation. This is stated as:

Lemma 7.4 The coefficients $p_{i,n-i}$, $i = 0, \ldots, n-1$ $(n \geq 2)$ for the multinomial $P(u_1, u_2, \zeta_1, \zeta_2)$ in equation (7.1.18) satisfy

$$p_{i+1,n-i-1} = \frac{(n-i)[\gamma i + (1-\gamma)(n-1)]}{(i+1)(n-1)} p_{i,n-i} \text{ for } i = 0, \ldots, n-1. \quad (7.1.58)$$

Proof: By Lemma 7.3, the terms $p_{i,n-i}$ are all constant, and are substituted into the appropriate previously derived constraints to determine the recursion relation (7.1.58). First, consider equation (7.1.25) for $j = n-i-1$, which is

$$(i+1)ip_{i+1,n-i-1} = g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2}, \text{ for } i = 1, \ldots, n-1, \quad (7.1.59)$$

and equation (7.1.27) for $j = n-i-1$,

$$i(n-i)p_{i,n-i} = g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1^2} + g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2}, \text{ for } i = 1, \ldots, n-1. \quad (7.1.60)$$

Subtracting equation (7.1.59) from (7.1.60) gives the equation

$$i(n-i)p_{i,n-i} - i(i+1)p_{i+1,n-i-1} = \gamma g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2}, \text{ for } i = 1, \ldots, n-1. \quad (7.1.61)$$

Next, equation (7.1.31) for $j = n-i-1$ may be written as

$$(i+1)(n-i-1)p_{i+1,n-i-1} = g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 p_{i,n-i-1}}{\partial u_2^2} \text{ for } i = 0, \ldots, n-1, \quad (7.1.62)$$

and equation (7.1.33) for $j = n-i-1$ simplifies to

$$(n-i)(n-i-1)p_{i,n-i} = g\frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2} + g\frac{\partial^2 p_{i,n-i-1}}{\partial u_2^2} \text{ for } i = 0, \ldots, n-1. \quad (7.1.63)$$

Subtracting (7.1.62) from $(1-\gamma)$ times (7.1.63) then yields

$$(1-\gamma)(n-i)(n-i-1)p_{i,n-i} - (i+1)(n-i-1)p_{i+1,n-i-1} = -\gamma g \frac{\partial^2 p_{i,n-i-1}}{\partial u_1 \partial u_2} \quad \text{for } i = 0, \ldots, n-1. \tag{7.1.64}$$

For $i = 0$, equation (7.1.64) becomes

$$(1-\gamma)n(n-1)p_{0n} - (n-1)p_{1,n-1} = -\gamma g \frac{\partial^2 p_{0,n-1}}{\partial u_1 \partial u_2}. \tag{7.1.65}$$

By Lemma 7.1, $\dfrac{\partial p_{0,n-1}}{\partial u_1} = 0$, so that the right hand side of (7.1.65) is zero, which then gives (since $n \geq 2$ is assumed)

$$p_{1,n-1} = (1-\gamma)np_{0n}. \tag{7.1.66}$$

For $i = 1, \ldots, n-1$, substituting the result (7.1.61) into (7.1.64) allows the partial derivative to be removed to yield

$$(1-\gamma)(n-i)(n-i-1)p_{i,n-i} - (i+1)(n-i-1)p_{i+1,n-i-1}$$
$$= i(i+1)p_{i+1,n-i-1} - i(n-i)p_{i,n-i}, \text{ for } i = 1, \ldots, n-1,$$

which simplifies to

$$(n-i)[\gamma i + (1-\gamma)(n-1)]p_{i,n-i} = (i+1)(n-1)p_{i+1,n-i-1} \text{ for } i = 1, \ldots, n-1. \tag{7.1.67}$$

Since equation (7.1.67) reduces to the result (7.1.66) for the case $i = 0$, division by $(i+1)(n-1)$ and inclusion of $i = 0$ then permits the result to be expressed in the desired form (7.1.58). □

A final Lemma can now be proved to investigate the form of the coefficients p_{i0} and p_{0i} previously discussed in Lemmas 7.1 and 7.2.

Lemma 7.5 The coefficients p_{i0} and p_{0i}, for $i = 1, \ldots, n$, $n \geq 2$, are polynomials in u_1 and u_2 respectively, with maximum degree of $2(n-i)$. If $n \geq 1$ then p_{00} is linear in u_1 and u_2.

Proof: For $n \geq 2$, first p_{i0} is considered, since the proof for p_{0i} is similar. When the result of Lemma 7.2 is used, equation (7.1.25) with $j = 0$ becomes an ordinary differential equation,

$$(i+1)ip_{i+1,0} = g\frac{d^2 p_{i0}}{du_1^2}, \text{ for } i = 1, \ldots, n-1. \tag{7.1.68}$$

Since p_{n0} is a constant by Lemma 7.3, integration and induction are used to show the desired result.

Inductive hypothesis: (deflation on i) p_{i0} is a polynomial of degree $2(n-i)$ for $n \geq 2$ a fixed positive integer, and $i = 1, \ldots, n-1$.

For $i = n-1$, p_{n0} is a constant, which is a polynomial of degree zero. Then $p_{n-1,0}$ satisfies equation (7.1.68) which can be integrated twice with respect to u_1 to give, for example a quadratic,

$$p_{n-1,0} = \frac{n(n-1)}{2g} u_1^2 + a_1 u_1 + a_2,$$

where a_1, a_2 are arbitrary constants of the integration. $p_{n-1,0}$ is therefore a polynomial of degree 2, and the inductive hypothesis is shown for $i = n-1$.

Assuming that the hypothesis is true for a fixed value of i, $1 < i \leq n-1$, it is necessary to show that $p_{i-1,0}$ is a polynomial of degree $2(n-i+1)$. Again, $p_{i-1,0}$ satisfies equation (7.1.68), so that $\dfrac{\mathrm{d}^2 p_{i-1,0}}{\mathrm{d}u_1}$ is a polynomial of degree $2(n-i)$. Integrating such a polynomial twice with respect to its argument yields a polynomial of degree $2(n-i)+2 = 2(n-i+1)$, which is the desired result.

The inductive hypothesis is now shown for an arbitrary finite value of n, and accounting for zero coefficients in the leading terms for the polynomials, the statement of the lemma has been shown for p_{i0}.

For the terms p_{0i}, equation (7.1.33) with $i = 0$ may be written with the use of Lemma 7.1 as

$$(j+1)j p_{0,j+1} = g \frac{\mathrm{d}^2 p_{0j}}{\mathrm{d}u_2^2}, \text{ for } j = 1, \ldots, n-1. \tag{7.1.69}$$

Since p_{0n} is a constant by Lemma 7.3, an analysis similar to the above argument may be completed, to give the stated result after switching the indices i and j.

For the case $n \geq 1$, equations (7.1.22), (7.1.24) and (7.1.33) respectively become, for $i = 0$ and $j = 0$,

$$\frac{\partial^2 p_{00}}{\partial u_1 \partial u_2} = 0, \quad \frac{\partial^2 p_{00}}{\partial u_1^2} = 0, \text{ and } \frac{\partial^2 p_{00}}{\partial u_2^2} = 0. \tag{7.1.70}$$

This equation implies that the coefficient p_{00} must be of the form

$$p_{00} = a u_1 + b u_2 + c, \tag{7.1.71}$$

with a, b, and c constant, which is the second statement of the lemma. □

With the results of Lemmas 7.1 to 7.5, the possible forms of conservation laws (7.1.5) may now be stated for the possible values of n in the multinomial form (7.1.18) of P. The simplest case is for $n = 0$, which is considered in the following theorem.

Theorem 7.1 (Solution for $n = 0$.) Conservation laws of the form (7.1.5) with P given by (7.1.18) with $n = 0$, must be of the form

$$P = au_1 + bu_2, \quad Q = \frac{a}{2}u_1^2 + \frac{b}{2}u_2^2 + g[a + (1-\gamma)b]\zeta_1 + g(a+b)\zeta_2, \qquad (7.1.72)$$

for a and b arbitrary constants.

Proof: For $n = 0$, Lemma 7.3 requires that the function P must be of the form $P = p_{00}(u_1, u_2) = au_1 + bu_2 + c$ for a, b and c all constants. Since constant terms do not change the differential form of a conservation law, c is assumed to be zero without loss of generality. Equations (7.1.8)-(7.1.11) then describe the first partial derivatives of Q via the following steps with subsequent integration.

$$\frac{\partial Q}{\partial u_1} = au_1 \quad \Rightarrow \quad Q = \frac{a}{2}u_1^2 + Q^{(1)}(u_2, \zeta_1, \zeta_2),$$

$$\frac{\partial Q}{\partial u_2} = bu_2 \quad \Rightarrow \quad Q = \frac{a}{2}u_1^2 + \frac{b}{2}u_2^2 + Q^{(2)}(\zeta_1, \zeta_2),$$

$$\frac{\partial Q}{\partial \zeta_1} = ga + (1-\gamma)gb \quad \Rightarrow \quad Q = \frac{a}{2}u_1^2 + \frac{b}{2}u_2^2 + g[a + (1-\gamma)b]\zeta_1 + Q^{(3)}(\zeta_2),$$

and finally,

$$\frac{\partial Q}{\partial \zeta_2} = ga + gb \quad \Rightarrow \quad Q = \frac{a}{2}u_1^2 + \frac{b}{2}u_2^2 + g[a + (1-\gamma)b]\zeta_1 + g(a+b)\zeta_2. \quad (7.1.73)$$

This last equation allows the result of the theorem to be stated as equation (7.1.72). In addition, it should be noted that the conservation laws (7.1.1) and (7.1.2) are recovered from (7.1.72) by the choice of constants $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$. □

Theorem 7.2 (solution for $n = 1$) Conservation laws of the form (7.1.5) with P given by (7.1.18) with $n = 1$ must be of the form

$$P = (au_2 + b)\zeta_2 + [(1-\gamma)au_1 + c)]\zeta_1,$$

and

$$Q = (au_2^2 + bu_2)\zeta_2 + [(1-\gamma)au_1^2 + cu_1]\zeta_1 + \frac{(1-\gamma)}{2}ga\zeta_1^2 + (1-\gamma)ga\zeta_1\zeta_2 + \frac{1}{2}ga\zeta_2^2. \quad (7.1.74)$$

for a, b, and c all constants.

Proof: For $n \geq 1$, Lemma 7.5 gives $p_{00} = au_1 + bu_2 + c$ for a, b, and c constants. It can be assumed that $a = b = c = 0$ without loss of generality, since if we write $P = P|_{n=0} + P|_{n\geq 1}$ then (7.1.8)-(7.1.11) allow a similar decomposition, $Q = Q_{n=0} + Q|_{n\geq 1}$, where $P|_{n=0}$ and $Q|_{n=0}$ satisfy Theorem 7.1. The linearity of the conservation law in differential form allows P and Q to be written as the sum of such terms, and since the form of p_{00} is the same as for the $n = 0$ case, no new terms appear in $Q|_{n=0}$ and therefore may be ignored, and $p_{00} = 0$ assumed.

By Lemma 7.3, p_{01} and p_{10} are linear in u_2 and u_1, respectively, and are therefore assumed to be of the form

$$p_{01} = au_2 + b, \text{ and } p_{10} = cu_1 + d, \quad (7.1.75)$$

for a, b, c and d all constants. A quick verification of conditions (7.1.22)-(7.1.35) shows that the coefficients in (7.1.75) satisfy all of these conditions trivially so that no constraints on the four constants arise, and P takes on its most general form. To find Q, equations (7.1.8)-(7.1.11) are used with $P = (au_2 + b)\zeta_2 + (cu_1 + d)\zeta_2$. The first derivative, (7.1.8) gives

$$\frac{\partial Q}{\partial u_1} = u_1 c\zeta_1 + \zeta_1(cu_1 + d) = (2cu_1 + d)\zeta_1,$$

which integrates to

$$Q = (cu_1^2 + du_1)\zeta_1 + Q^{(1)}(u_2, \zeta_1, \zeta_2),$$

for the function $Q^{(1)}$ to be determined. Equation (7.1.9) is similar, and becomes

$$\frac{\partial Q}{\partial u_2} = (2au_2 + b)\zeta_2,$$

from which the form of Q written above can be further determined up to another unknown function $Q^{(2)}$ as

$$Q = (cu_1^2 + du_1)\zeta_1 + (au_2^2 + bu_2)\zeta_2 + Q^{(2)}(\zeta_1, \zeta_2).$$

Equation (7.1.10) becomes

$$\frac{\partial Q}{\partial \zeta_1} = g(c\zeta_1) + (1-\gamma)g(a\zeta_2) + u_1(cu_1 + d),$$

from which Q may be further specified to be

$$Q = (cu_1^2 + du_1)\zeta_1 + (au_2^2 + bu_2)\zeta_2 + \frac{c}{2}g\zeta_1^2 + (1-\gamma)ga\zeta_1\zeta_2 + Q^{(3)}(\zeta_2). \quad (7.1.76)$$

In (7.1.76), $Q^{(3)}$ is still undetermined. The last partial differential equation for Q is (7.1.11), which simplifies to

$$\frac{\partial Q}{\partial \zeta_2} = g(c\zeta_1) + ga\zeta_2 + u_2(au_2 + b).$$

Substitution of (7.1.76) into this result yields

$$au_2^2 + bu_2 + (1-\gamma)ga\zeta_1 + \frac{dQ^{(3)}}{d\zeta_2} = g(c\zeta_1) + ga\zeta_2 + u_2(au_2 + b),$$

from which it follows that $c = (1-\gamma)a$ and $\frac{dQ^{(3)}}{d\zeta_2} = ga\zeta_2$. Therefore, Q must be of the form (the constant of integration is zero without loss of generality):

$$Q = [(1-\gamma)au_1^2 + du_1]\zeta_1 + (au_2^2 + bu_2)\zeta_2 + \frac{(1-\gamma)}{2}ga\zeta_1^2 + (1-\gamma)ga\zeta_1\zeta_2 + \frac{1}{2}ga\zeta_2^2. \quad (7.1.77)$$

After relabeling the appropriate constants, equations (7.1.75) and (7.1.77) can be easily restated as the result (7.1.74). □

Some special cases of the result (7.1.74) are observed to occur. First, when $a = 0$, choosing $b = 1$, $c = 0$ gives the conservation law (7.1.3), and choosing $b = 0$, $c = 1$ recovers equation (7.1.4). The interesting new equation is for $a = 1$, $b = 0$, $c = 0$ which gives the conservation law

$$\frac{\partial}{\partial t}[(1-\gamma)u_1\zeta_1 + u_2\zeta_2] + \frac{\partial}{\partial x}[(1-\gamma)u_1^2\zeta_1 + u_2^2\zeta_2 + \frac{(1-\gamma)}{2}g\zeta_1^2 + (1-\gamma)g\zeta_1\zeta_2 + \frac{1}{2}g\zeta_2^2] = 0. \quad (7.1.78)$$

Equation (7.1.78) corresponds the physical to the physical principal of conservation of total horizontal momentum, as the density difference between the layers is accounted for in the terms with $1-\gamma$.

Before examining expansions such as (7.1.18) for $n \geq 2$, some additional development and simplification of the eleven conditions (7.1.22), (7.1.24), (7.1.25), (7.1.27), and (7.1.29)-(7.1.35) is completed in light of the results in Lemmas 7.1-7.5. For example, in equation (7.1.22), the $i = 0$ and $j = 0$ cases are satisfied trivially. As well, the terms with $i + j = n$ are satisfied due to Lemma 7.3. The remaining condition is restated with appropriate range of indices as

$$(u_1 - u_2)\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} = j\frac{\partial p_{ij}}{\partial u_1} - i\frac{\partial p_{ij}}{\partial u_2} \text{ for } \begin{cases} i = 1, \ldots, n-2, \\ j = 1, \ldots, n-i-1. \end{cases} \tag{7.1.79}$$

Although equation (7.1.24) is always trivially satisfied due to Lemma 7.1, equation (7.1.25) is only somewhat simplified by removing a few indices. The resulting condition is then

$$(i+1)ip_{i+1,j} = g\frac{\partial^2 p_{ij}}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} \text{ for } \begin{cases} i = 1, \ldots, n-2, \\ j = 1, \ldots, n-i-1. \end{cases} \tag{7.1.80}$$

The case with $j = 0$ in equation (7.1.25) results in Lemma 7.5's (7.1.68) which is excluded from (7.1.80).

Equation (7.1.27) is reduced somewhat as well since the $i = 0$ case is trivial due to Lemma 7.1. The resulting condition becomes

$$(u_1 - u_2)(j+1)\frac{\partial p_{i,j+1}}{\partial u_1} + i(j+1)p_{i,j+1} = g\frac{\partial^2 p_{ij}}{\partial u_1^2} + g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2}$$
$$\text{for } i = 1, \ldots, n-1 \text{ and } j = 0, \ldots, n-i-1. \tag{7.1.81}$$

The two conditions (7.1.29) and (7.1.30) are satisfied by Lemma 7.3, and equation (7.1.31) without the trivial case for $j = 0$ may be stated as

$$(i+1)(u_2 - u_1)\frac{\partial p_{i+1,j}}{\partial u_2} + (i+1)jp_{i+1,j} = g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 p_{ij}}{\partial u_2^2}$$
$$\text{for } i = 0, \ldots, n-2 \text{ and } j = 1, \ldots, n-i-1. \tag{7.1.82}$$

Of the four remaining equations, equations (7.1.32) and (7.1.35) are satisfied trivially by Lemma 7.3. Condition (7.1.33) simplifies to

$$(j+1)jp_{i,j+1} = g\frac{\partial^2 p_{ij}}{\partial u_1 \partial u_2} + g\frac{\partial^2 p_{ij}}{\partial u_2^2} \text{ for } \begin{cases} i = 1, \ldots, n-2, \\ j = 1, \ldots, n-i-1, \end{cases} \tag{7.1.83}$$

where the $i = 0$ case is removed since it simplifies to (7.1.69) in Lemma 7.5 The remaining equation, (7.1.34) is restated for completeness as

$$(i+1)(j+1)(u_1 - u_2)p_{i+1,j+1} = (i+1)g\left(\frac{\partial p_{i+1,j}}{\partial u_1} + \frac{\partial p_{i+1,j}}{\partial u_2}\right)$$
$$-(j+1)g\left(\frac{\partial p_{i,j+1}}{\partial u_1} + (1-\gamma)\frac{\partial p_{i,j+1}}{\partial u_2}\right) \text{ for } \begin{cases} i = 0, \dots, n-2, \\ j = 0, \dots, n-i-2. \end{cases} \quad (7.1.84)$$

Therefore, for $n \geq 2$, the coefficients of the multinomial solution P of the form (7.1.18) must satisfy the six constraints (7.1.79)-(7.1.84), as well as the two equations (7.1.68) and (7.1.69) found in the proof of Lemma 7.5. It is now easier to prove the following results.

Theorem 7.3 (Solution for $n = 2$.) Conservation laws of the form (7.1.5) with P given by (7.1.18) with $n = 2$, must be of the form

$$P = \frac{1}{2}(1-\gamma)u_1^2\zeta_1 + \frac{1}{2}(1-\gamma)g\zeta_1^2 + \frac{1}{2}u_2^2\zeta_2 + \frac{1}{2}g\zeta_2^2 + (1-\gamma)g\zeta_1\zeta_2, \quad (7.1.85)$$

and

$$Q = \frac{1}{2}(1-\gamma)u_1^3\zeta_1 + (1-\gamma)gu_1\zeta_1^2 + \frac{1}{2}u_2^3\zeta_2 + gu_2\zeta_2^2 + (1-\gamma)g(u_1+u_2)\zeta_1\zeta_2, \quad (7.1.86)$$

or scalar multiples of (7.1.85) and (7.1.86) with scalar additions of the solutions for $n = 1$ and $n = 0$.

Proof: As done in the proof of theorem 7.2, the coefficient $p_{00} = 0$ without loss of generality. Of the remaining 5 coefficients, Lemmas 7.3 and 7.4 describe p_{02}, p_{11} and p_{20} up to an arbitrary constant. Employing the recursive algorithm in Lemma 7.4 gives, for an arbitrary constant a,

$$p_{02} = a, \quad (7.1.87)$$
$$p_{11} = \frac{(2-0)[\gamma 0 + (1-\gamma)(2-1)]}{(0+1)(2-1)}p_{02} = 2(1-\gamma)a, \text{ and} \quad (7.1.88)$$
$$p_{20} = \frac{(2-1)[\gamma + (1-\gamma)(2-1)]}{(1+1)(2-1)}p_{11} = \frac{1}{2}p_{11} = (1-\gamma)a. \quad (7.1.89)$$

By Lemma 7.5, the remaining coefficients p_{10} and p_{01} are polynomials of degree 2, stated with arbitrary constants a_2, a_1, a_0, b_2, b_1, and b_0 as

$$p_{10} = a_2u_1^2 + a_1u_1 + a_0, \text{ and} \quad (7.1.90)$$
$$p_{01} = b_2u_2^2 + b_1u_2 + b_0. \quad (7.1.91)$$

Substituting equations (7.1.87)-(7.1.91) into the constraints (7.1.79)-(7.1.84) reduces the number of constants further. Although the conditions (7.1.79) and (7.1.80) are not applicable for $n = 2$, equation (7.1.81) becomes

$$(u_1 - u_2)\frac{\partial p_{11}}{\partial u_1} + p_{11} = g\frac{\mathrm{d}^2 p_{10}}{\mathrm{d}u_1^2},$$

which simplifies to the result

$$(1-\gamma)a = ga_2. \qquad (7.1.92)$$

Similarly, the condition (7.1.82) is

$$(u_2 - u_1)\frac{\partial p_{11}}{\partial u_2} + p_{11} = (1-\gamma)g\frac{\mathrm{d}^2 p_{01}}{\mathrm{d}u_2^2},$$

which becomes simply

$$a = gb_2. \qquad (7.1.93)$$

The constraint (7.1.83) is not applicable, and the remaining condition (7.1.84) may be applied as follows:

$$(u_1 - u_2)p_{11} = g\frac{\mathrm{d}p_{10}}{\mathrm{d}u_1} - g(1-\gamma)\frac{\mathrm{d}p_{01}}{\mathrm{d}u_2},$$

which becomes

$$2(u_1 - u_2)(1-\gamma)a = g(2a_2u_1 + a_1) - g(1-\gamma)(2b_2u_2 + b_1).$$

Substituting the results (7.1.92) and (7.1.93) into the above for a_2 and b_2 simplify this greatly, revealing the equation

$$0 = a_1 - (1-\gamma)b_1. \qquad (7.1.94)$$

The results (7.1.92)-(7.1.94) then allow the multinomial P to be given as

$$P = \left(\frac{(1-\gamma)}{g}au_1^2 + a_1u_1 + a_0\right)\zeta_1 + (1-\gamma)a\zeta_1^2 + \left(\frac{a}{g}u_2^2 + \frac{a_1}{(1-\gamma)}u_2 + b_0\right)\zeta_2 + a\zeta_2^2 + 2(1-\gamma)a\zeta_1\zeta_2. \qquad (7.1.95)$$

Equation (7.1.95) can be rearranged as

$$P = \frac{a}{g}\left[(1-\gamma)u_1^2\zeta_1 + (1-\gamma)g\zeta_1^2 + u_2^2\zeta_2 + g\zeta_2^2 + 2(1-\gamma)g\zeta_1\zeta_2\right] + \frac{a_1}{1-\gamma}\left[(1-\gamma)u_1\zeta_1 + u_2\zeta_2\right] + a_0\zeta_1 + b_0\zeta_2. \qquad (7.1.96)$$

By the linearity of equations (7.1.8)-(7.1.11) for Q, and the linearity of the conservation form (7.1.5), the previous $n = 1$ solution in (7.1.96) may be neglected without loss of generality. By setting $a_1 = 0$, $a_0 = 0$ and $b_0 = 0$, the desired result (7.1.85) can be derived directly from (7.1.96) with $a = \frac{g}{2}$.

To find the form for the multinomial Q, equations (7.1.8)-(7.1.11) may be integrated individually. For example, for P given by (7.1.85), equation (7.1.8) becomes

$$\frac{\partial Q}{\partial u_1} = u_1[(1-\gamma)u_1\zeta_1] + \zeta_1\left[\frac{1}{2}(1-\gamma)u_1^2 + (1-\gamma)g\zeta_1 + (1-\gamma)g\zeta_2\right],$$

which integrates once as

$$Q = \frac{1}{2}(1-\gamma)u_1^3\zeta_1 + (1-\gamma)gu_1\zeta_1^2 + (1-\gamma)gu_1\zeta_1\zeta_2 + Q^{(1)}(u_2,\zeta_1,\zeta_2). \quad (7.1.97)$$

Similarly, equation (7.1.9) yields

$$\frac{\partial Q}{\partial u_2} = u_2(u_2\zeta_2) + \zeta_2\left[\frac{1}{2}u_2^2 + g\zeta_2 + (1-\gamma)g\zeta_1\right].$$

Substitution of (7.1.97) into the left hand side of the above expression, and integrating with respect to u_2 produces

$$Q = \frac{1}{2}(1-\gamma)u_1^3\zeta_1 + (1-\gamma)gu_1\zeta_1^2 + (1-\gamma)gu_1\zeta_1\zeta_2 + \frac{1}{2}u_2^3\zeta_2 + gu_2\zeta_2^2 + (1-\gamma)gu_2\zeta_1\zeta_2 + Q^{(2)}(\zeta_1,\zeta_2). \quad (7.1.98)$$

Equation (7.1.10) becomes, after substituting equation (7.1.85) for P,

$$\frac{\partial Q}{\partial \zeta_1} = g(1-\gamma)u_1\zeta_1 + (1-\gamma)gu_2\zeta_2 + u_1[\frac{1}{2}(1-\gamma)u_1^2 + (1-\gamma)g\zeta_1 + (1-\gamma)g\zeta_2].$$

Substituting the derivative of (7.1.98) into the above shows that $\dfrac{\partial Q^{(2)}}{\partial \zeta_1} = 0$, i.e. $Q^{(2)} = Q^{(2)}(\zeta_2)$. Similarly, equation (7.1.11) may be written as

$$\frac{\partial Q}{\partial \zeta_2} = g(1-\gamma)u_1\zeta_1 + gu_2\zeta_2 + u_2[\frac{1}{2}u_2^2 + g\zeta_2 + (1-\gamma)g\zeta_1],$$

after which calculating $\dfrac{\partial}{\partial \zeta_2}$ of (7.1.98) to compare terms gives simply $\dfrac{dQ^{(2)}}{d\zeta_2} = 0$. Therefore, without loss of generality, it can be assumed that $Q^{(2)} = 0$. With this assumption, equation (7.1.98) may be rewritten to give the desired result (7.1.86). □

In the $n = 0$ and $n = 1$ cases for Theorems 7.1 and 7.2, the form of P contains easily identifiable terms, such as mass, velocity and momentum. For $n = 2$, the conserved quantity is interpreted physically as the sum of vertically integrated kinetic and potential energy per unit mass (of the lower layer). This can be seen from the following short calculation. The vertically integrated kinetic energy per unit mass is given by

$$\int_0^{\zeta_1} \frac{1}{2}u_2^2 dz + \int_{\zeta_2}^{\zeta_1+\zeta_2} \frac{1}{2}(1-\gamma)u_2^2 dz = \frac{1}{2}u_2^2\zeta_2 + \frac{1}{2}(1-\gamma)u_2^2\zeta_1. \tag{7.1.99}$$

The vertically integrated potential energy per unit mass is similarly obtained via

$$\begin{aligned}\int_0^{\zeta_2} gz dz + \int_{\zeta_2}^{\zeta_1+\zeta_2} g(1-\gamma)z dz &= \frac{1}{2}g\zeta_2^2 + \frac{1}{2}g(1-\gamma)[(\zeta_1+\zeta_2)^2 - \zeta_2^2] \\ &= \frac{1}{2}g\zeta_2^2 + \frac{1}{2}g(1-\gamma)\zeta_1^2 + g(1-\gamma)\zeta_1\zeta_2.\end{aligned} \tag{7.1.100}$$

The sum of the right hand sides of the kinetic energy (7.1.99) and the potential energy (7.1.100) gives the conserved quantity (7.1.85).

An observation may now be made concerning the results of Theorems 7.1-7.3. This is that the number of arbitrary constants involved in the conserved quantities P seems to decrease with increasing n. In equation (7.1.72), for the $n = 0$ case, P contained two arbitrary constants. For the $n = 1$ case, equation (7.1.74) contains three arbitrary constants, two of which are associated with the 'first-order' conservation laws (7.1.3) and (7.1.4). For $n = 2$, equation (7.1.85) does not have any arbitrary constants, although adding scalar multiples of the previous earlier cases would necessarily yield some scalars. This pattern motivates the conjecture that for increasing values of n, ($n \geq 3$), the constraints (7.1.79)-(7.1.84) may result in some inconsistencies which do not permit higher-order conservation laws associated with the two-layer shallow water equations. This question is answered through the final result of this section.

Theorem 7.4 (Solution for $n \geq 3$) Conservation laws of the form (7.1.5) for the two-layer shallow water equations (7.1.1)-(7.1.4) with h_B =constant do not exist for multinomials P of the form (7.1.18) having $n \geq 3$.

Proof: It is assumed that P exists which is non-trivial of order n with $n \geq 3$. That is, the coefficients $p_{i,n-i} \neq 0$ for $i = 0 \ldots, n$, for otherwise it would suffice to consider multinomials P of order $n-1$. From Lemma 7.3, $p_{i,n-i}$ is a constant for $i = 0, \ldots, n$. The four equations (7.1.80)-(7.1.83) exploit this property with the indices chosen as $i = n-2$ and $j = 1$. The simplified results are

$$(n-2+1)(n-2)p_{n-1,1} = g\frac{\partial^2 p_{n-2,1}}{\partial u_1^2} + (1-\gamma)g\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2}, \tag{7.1.101}$$

$$(n-2)2p_{n-2,2} = g\frac{\partial^2 p_{n-2,1}}{\partial u_1^2} + g\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2}, \tag{7.1.102}$$

$$(n-2+1)(1)p_{n-1,1} = g\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2} + (1-\gamma)g\frac{\partial^2 p_{n-2,1}}{\partial u_2^2}, \tag{7.1.103}$$

and

$$2p_{n-2,2} = g\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2} + g\frac{\partial^2 p_{n-2,1}}{\partial u_2^2}. \tag{7.1.104}$$

From Lemma 7.4, the constants $p_{n-2,2}$ and $p_{n-1,1}$ are related by the recursion relation (7,1,58). Stated for $i = n-2$ this is

$$\begin{aligned} p_{n-1,1} &= \frac{[n-(n-2)][\gamma(n-2)+(1-\gamma)(n-1)]}{(n-2+1)(n-1)}p_{n-2,2} \\ &= \frac{2[n\gamma - 2\gamma + n - 1 - \gamma n + \gamma]}{(n-1)^2}p_{n-2,2} \\ &= \frac{2(n-1-\gamma)}{(n-1)^2}p_{n-2,2}. \end{aligned} \tag{7.1.105}$$

Equation (7.1.105) may be substituted into equations (7.1.101)-(7.1.104), while making the change of variables

$$x_1 = g\frac{\partial^2 p_{n-2,1}}{\partial u_1^2}, \quad x_2 = g\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2}, \quad x_3 = g\frac{\partial^2 p_{n-2,1}}{\partial u_2^2}, \text{ and } (n-1)x_4 = p_{n-2,2}. \tag{7.1.106}$$

to give four linear equations. The first two,

$$x_1 + (1-\gamma)x_2 = 2(n-2)(n-1-\gamma)x_4,$$

and

$$x_1 + x_2 = 2(n-1)(n-2)x_4,$$

can be manipulated to give the solution parameterised by x_4 as

$$\begin{aligned} x_2 &= \gamma^{-1}2(n-2)[n-1-(n-1-\gamma)]x_4 = 2(n-2)x_4, \text{ and} \\ x_1 &= 2(n-1)(n-2)x_4 - x_2 = 2(n-2)^2 x_4. \end{aligned} \tag{7.1.107}$$

The last two equations,

$$x_2 + (1-\gamma)x_3 = 2(n-1-\gamma)x_4,$$

and

$$x_2 + x_3 = 2(n-1)x_4,$$

are similarly solved as

$$\begin{aligned} x_3 &= 2\gamma^{-1}[n-1-(n-1-\gamma)]x_4 = 2x_4, \text{ and} \\ x_2 &= 2(n-1)x_4 - x_3 = 2(n-2)x_4. \end{aligned} \tag{7.1.108}$$

Substituting the change of variable (7.1.106) back into the results in (7.1.107) and (7.1.108) gives

$$\begin{aligned} \frac{\partial^2 p_{n-2,1}}{\partial u_1^2} = \frac{2(n-2)^2}{g(n-1)} p_{n-2,2}, \quad \frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2} = \frac{2(n-2)}{g(n-1)} p_{n-2,2}, \\ \text{and } \frac{\partial^2 p_{n-2,1}}{\partial u_2^2} = \frac{2}{g(n-1)} p_{n-2,2}. \end{aligned} \tag{7.1.109}$$

Continuously differentiable solutions to the partial differential equations (7.1.109) are given by

$$\begin{aligned} p_{n-2,1} = \frac{(n-2)^2}{g(n-1)} p_{n-2,2} u_1^2 + \frac{2(n-2)}{g(n-1)} p_{n-2,2} u_1 u_2 \\ + \frac{1}{g(n-1)} p_{n-2,2} u_2^2 + a u_1 + b u_2 + c, \end{aligned} \tag{7.1.110}$$

for a, b, and c arbitrary constants.

Now, since $n \geq 3$, constraint (7.1.79) may be applied to $p_{n-2,1}$ with $i = n-2$ and $j = 1$,

$$(u_1 - u_2)\frac{\partial^2 p_{n-2,1}}{\partial u_1 \partial u_2} = \frac{\partial p_{n-2,1}}{\partial u_1} - (n-2)\frac{\partial p_{n-2,1}}{\partial u_2}. \tag{7.1.111}$$

Calculating the appropriate derivatives from (7.1.110) and substituting the results into equation (7.1.111) gives the result

$$(u_1 - u_2)\frac{2(n-2)}{g(n-1)}p_{n-2,2} = \frac{2(n-2)^2}{g(n-1)}p_{n-2,2}u_1 + \frac{2(n-2)}{g(n-1)}p_{n-2,2}u_2 + a$$
$$- (n-2)\left(\frac{2(n-2)}{g(n-1)}p_{n-2,2}u_1 + \frac{2}{g(n-1)}p_{n-2,2}u_2 + b\right),$$

which simplifies to

$$(u_1 - u_2)\frac{2(n-2)}{g(n-1)}p_{n-2,2} = a - (n-2)b. \qquad (7.1.112)$$

Since u_1 and u_2 are independent, it follows that $u_1 - u_2$ is not a constant. Therefore, equation (7.1.112) yields the result that $p_{n-2,2} = 0$ which contradicts the original assumption, completing the proof. □

Summarizing the results from Theorems 7.1-7.4, it is seen that the four conservation laws (7.1.1)-(7.1.4) give rise to only two additional conserved quantities of the form (7.1.18). These conserved quantities have been interpreted as: total horizontal momentum, given by equation (7.1.78), and energy, given by the conserved quantity. This result stands in contrast to the result of Whitham (1974, p.460) for the one-layer shallow-water equations which have an infinite number of conserved quantities of the form $P = \sum_{i=0}^{n} p_i(u)h^i$. It may be observed that the special cases of Theorem 7.2 and 7.3 for one layer may be obtained by taking $u_1 = 0$ and $\zeta_1 = 0$ to recover the corresponding single layer results derived elsewhere (Whitham, 1974 p.460).

In general, if the shallow-water equations (7.1.1)-(7.1.4) admit any more conservation laws than those found in this section, then the conserved quantities must necessarily be of a more general form than the multinomial (7.1.18).

7.2 Three-Dimensional, One-Layer Conservation Equations

The one-layer simplification of equations (7.1), (7.2) and (7.5) may be obtained by letting the upper layer variables vanish, i.e. $\zeta_1 = 0$, $u_1 = 0$, and $v_1 = 0$. By implementing the change of variable $u_2 \to u$, $v_2 \to v$, and $\zeta_2 \to \zeta$, the three equations simplify with constant bottom topography h_B to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \zeta}{\partial x} = fv, \qquad (7.2.1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \zeta}{\partial y}=-fu, \tag{7.2.2}$$

and

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}(\zeta u)+\frac{\partial}{\partial y}(\zeta v)=0. \tag{7.2.3}$$

It is desired to obtain combinations of equations (7.2.1)-(7.2.3) which may be expressed in the conservative form

$$\frac{\partial}{\partial t}P(u,v,\zeta)+\frac{\partial}{\partial x}Q(u,v,\zeta)+\frac{\partial}{\partial y}R(u,v,\zeta)=0, \tag{7.2.4}$$

for sufficiently continuous functions P, Q, and R. By necessity, only the irrotational case is considered ($f=0$) so that no forcing terms will appear on the right hand side of equation (7.2.4). In a similar (but shorter) manner to section 7.1, the partial differential equations (7.2.1)-(7.2.3) are used to place constraints on the specific forms of P, Q, and R which may exist and still satisfy (7.2.4).

An application of the chain rule allows equation (7.2.4) to be written as

$$\begin{aligned}\frac{\partial P}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial P}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial P}{\partial \zeta}\frac{\partial \zeta}{\partial t}+\frac{\partial Q}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial Q}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial Q}{\partial \zeta}\frac{\partial \zeta}{\partial x}&\\ +\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial \zeta}\frac{\partial \zeta}{\partial y}&=0.\end{aligned} \tag{7.2.5}$$

Substituting the temporal derivatives from equations (7.2.1)-(7.2.3) into the result (7.2.5) gives

$$\begin{aligned}\frac{\partial P}{\partial u}\left(-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-g\frac{\partial \zeta}{\partial x}\right)+\frac{\partial P}{\partial v}\left(-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}-g\frac{\partial \zeta}{\partial y}\right)+\frac{\partial P}{\partial \zeta}\left(-\frac{\partial}{\partial x}(\zeta u)\right.\\ \left.-\frac{\partial}{\partial y}(\zeta v)\right)+\frac{\partial Q}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial Q}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial Q}{\partial \zeta}\frac{\partial \zeta}{\partial x}+\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial \zeta}\frac{\partial \zeta}{\partial y}=0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\left(-u\frac{\partial P}{\partial u}-\zeta\frac{\partial P}{\partial \zeta}+\frac{\partial Q}{\partial u}\right)\frac{\partial u}{\partial x}+\left(-g\frac{\partial P}{\partial u}-u\frac{\partial P}{\partial \zeta}+\frac{\partial Q}{\partial \zeta}\right)\frac{\partial \zeta}{\partial x}\\ &\quad+\left(-u\frac{\partial P}{\partial v}+\frac{\partial Q}{\partial v}\right)\frac{\partial v}{\partial x}++\left(-v\frac{\partial P}{\partial u}+\frac{\partial R}{\partial u}\right)\frac{\partial u}{\partial y}\\ &\quad+\left(-v\frac{\partial P}{\partial v}-\zeta\frac{\partial P}{\partial \zeta}+\frac{\partial R}{\partial v}\right)\frac{\partial v}{\partial y}+\left(-g\frac{\partial P}{\partial v}-v\frac{\partial P}{\partial \zeta}+\frac{\partial R}{\partial \zeta}\right)\frac{\partial \zeta}{\partial y}=0.\end{aligned} \tag{7.2.6}$$

By the linear independence of the first partial derivatives, it follows that for nontrivial or nonconstant solutions to the original equations (7.2.1)-(7.2.3), the coefficients in (7.2.6) must vanish. This may be written as six equations

$$\frac{\partial Q}{\partial u} = u\frac{\partial P}{\partial u} + \zeta\frac{\partial P}{\partial \zeta}, \quad \frac{\partial Q}{\partial v} = u\frac{\partial P}{\partial v}, \quad \frac{\partial Q}{\partial \zeta} = g\frac{\partial P}{\partial u} + u\frac{\partial P}{\partial \zeta}, \quad \frac{\partial R}{\partial u} = v\frac{\partial P}{\partial u},$$
$$\frac{\partial R}{\partial v} = v\frac{\partial P}{\partial v} + \zeta\frac{\partial P}{\partial \zeta}, \text{ and } \frac{\partial R}{\partial \zeta} = g\frac{\partial P}{\partial v} + v\frac{\partial P}{\partial \zeta}. \tag{7.2.7}$$

The relations in (7.2.7) may be differentiated and restated as constraints on the function P alone. First, equality of the second derivatives $\frac{\partial^2 Q}{\partial v \partial u}$ and $\frac{\partial^2 Q}{\partial v \partial u}$ gives

$$u\frac{\partial^2 P}{\partial v \partial u} + \zeta\frac{\partial^2 P}{\partial v \partial \zeta} = \frac{\partial P}{\partial v} + u\frac{\partial^2 P}{\partial v \partial u},$$

which simplifies to the result

$$\frac{\partial P}{\partial v} = \zeta\frac{\partial^2 P}{\partial v \partial \zeta}. \tag{7.2.8}$$

Next, equating $\frac{\partial^2 Q}{\partial \zeta \partial u}$ and $\frac{\partial^2 Q}{\partial u \partial \zeta}$ yields the equation

$$u\frac{\partial^2 P}{\partial \zeta \partial u} + \frac{\partial P}{\partial \zeta} + \zeta\frac{\partial^2 P}{\partial \zeta^2} = g\frac{\partial^2 P}{\partial u^2} + \frac{\partial P}{\partial \zeta} + u\frac{\partial^2 P}{\partial u \partial \zeta},$$

which becomes simply

$$\zeta\frac{\partial^2 P}{\partial \zeta^2} = g\frac{\partial^2 P}{\partial u^2}. \tag{7.2.9}$$

The equality of $\frac{\partial^2 Q}{\partial \zeta \partial v}$ and $\frac{\partial^2 Q}{\partial v \partial \zeta}$ produces

$$u\frac{\partial^2 P}{\partial \zeta \partial v} = g\frac{\partial^2 P}{\partial v \partial u} + u\frac{\partial^2 P}{\partial v \partial \zeta},$$

giving the simplest result

$$\frac{\partial^2 P}{\partial v \partial u} = 0, \tag{7.2.10}$$

which may also be obtained by equating $\frac{\partial^2 R}{\partial u \partial \zeta}$ and $\frac{\partial^2 R}{\partial \zeta \partial u}$. A fourth equation may be derived from the equality of $\frac{\partial^2 R}{\partial v \partial u}$ and $\frac{\partial^2 R}{\partial u \partial v}$ which becomes

$$\frac{\partial P}{\partial u} + v\frac{\partial^2 P}{\partial v \partial u} = v\frac{\partial^2 P}{\partial u \partial v} + \zeta\frac{\partial^2 P}{\partial u \partial \zeta},$$

resulting in

$$\frac{\partial P}{\partial u} = \zeta \frac{\partial^2 P}{\partial u \partial \zeta}. \tag{7.2.11}$$

The last constraint on P arises from equating $\frac{\partial^2 R}{\partial \zeta \partial v}$ and $\frac{\partial^2 R}{\partial v \partial \zeta}$ to obtain

$$v\frac{\partial^2 P}{\partial \zeta \partial v} + \frac{\partial P}{\partial \zeta} + \zeta \frac{\partial^2 P}{\partial \zeta^2} = g\frac{\partial^2 P}{\partial v^2} + \frac{\partial P}{\partial \zeta} + v\frac{\partial^2 P}{\partial v \partial \zeta},$$

which then becomes

$$\zeta \frac{\partial^2 P}{\partial \zeta^2} = g\frac{\partial^2 P}{\partial v^2}. \tag{7.2.12}$$

Equations (7.2.8)-(7.2.12) are now used to determine the form of P. First, equation (7.2.10) implies that P must be of the general form

$$P(u, v, \zeta) = \alpha(u, \zeta) + \beta(v, \zeta), \tag{7.2.13}$$

for arbitrary C^2 functions α and β. Substituting this form of P found in (7.2.13) into the remaining conditions (7.2.8), (7.2.9), (7.2.11) and (7.2.12) give

$$\frac{\partial \beta}{\partial v} = \zeta \frac{\partial^2 \beta}{\partial v \partial \zeta}, \tag{7.2.14}$$

$$\zeta\left(\frac{\partial^2 \alpha}{\partial \zeta^2} + \frac{\partial^2 \beta}{\partial \zeta^2}\right) = g\frac{\partial^2 \alpha}{\partial u^2}, \tag{7.2.15}$$

$$\frac{\partial \alpha}{\partial u} = \zeta \frac{\partial^2 \alpha}{\partial u \partial \zeta}, \tag{7.2.16}$$

and

$$\zeta\left(\frac{\partial^2 \alpha}{\partial \zeta^2} + \frac{\partial^2 \beta}{\partial \zeta^2}\right) = g\frac{\partial^2 \beta}{\partial v^2}. \tag{7.2.17}$$

Comparison of equations (7.2.15) and (7.2.17) reveals that $\frac{\partial^2 \alpha}{\partial u^2} = \frac{\partial^2 \beta}{\partial v^2}$, from which it follows that $\frac{\partial^2 \alpha}{\partial \zeta^2} + \frac{\partial^2 \beta}{\partial \zeta^2} = f(\zeta)$. Substituting this deduction into equation (7.2.15) gives $g\frac{\partial^2 \alpha}{\partial u^2} = \zeta f(\zeta)$, which integrates twice to produce a general expression for α with two arbitrary functions of ζ as

$$g\alpha = \frac{1}{2}\zeta f(\zeta)u^2 + \alpha_0(\zeta)u + \alpha_1(\zeta). \tag{7.2.18}$$

Substituting this form form α into equation (7.2.16) yields, after multiplying the entire equation by g to simplify it,

$$\zeta f u + \alpha_0 = \zeta \left(f u + \zeta \frac{\mathrm{d}f}{\mathrm{d}\zeta} u + \frac{\mathrm{d}\alpha_0}{\mathrm{d}\zeta} \right),$$

from which the following conclusions may be drawn:

$$0 = \zeta^2 \frac{\mathrm{d}f}{\mathrm{d}\zeta} \Rightarrow f = c,$$

and

$$\alpha_0 = \zeta \frac{\mathrm{d}\alpha_0}{\mathrm{d}\zeta} \Rightarrow \alpha_0 = a\zeta, \tag{7.2.19}$$

for a and c both constants.

Similarly, equation (7.2.17) may be integrated to yield an expression for β involving arbitrary functions of ζ,

$$g\beta = \frac{1}{2}\zeta f v^2 + \beta_0(\zeta) v + \beta_1(\zeta). \tag{7.2.20}$$

Using the fact that $f = c$, this expression may be substituted into the equation (7.2.14) resulting in

$$\zeta c v + \beta_0 = \zeta \left(c v + \frac{\mathrm{d}\beta_0}{\mathrm{d}\zeta} \right),$$

which simplifies to a single ordinary differential equation for β_0 as in equation (7.2.19), which then integrates as

$$\beta_0 = b\zeta, \tag{7.2.21}$$

for b a constant.

Using equations (7.2.18)-(7.2.21), the function P may be further described from the form stated in (7.2.13) as

$$P(u, v, \zeta) = \frac{c}{2g}\zeta(u^2 + v^2) + \frac{1}{g}\zeta(au + bv) + \widetilde{P}(\zeta), \tag{7.2.22}$$

where the unknown functions α_1 and β_1 have been combined into $\widetilde{P}$. This last term may be further investigated by substituting the expression (7.2.22) for P into either equation (7.2.9) or (7.2.12), resulting in $\zeta \frac{\mathrm{d}^2 \widetilde{P}}{\mathrm{d}\zeta^2} = g\left(\frac{c}{g}\zeta\right)$, which may be integrated to contain two additional constants, d and e as

$$\widetilde{P}(\zeta) = \frac{1}{2}c\zeta^2 + d\zeta + e. \tag{7.2.23}$$

Since without loss of generality, $e = 0$, equation (7.2.23) may be inserted into (7.2.22), and the constants reorganized to state the most general form of P (without any constant terms) which satisfies the conservative form (7.2.4) as

$$P(u, v, \zeta) = ga\zeta^2 + [a(u^2 + v^2) + bu + cv + d]\zeta. \tag{7.2.24}$$

To determine the flux functions Q and R, the partial derivatives from the statement (7.2.7) can be used knowing P above. For Q, the analysis is completed in three steps. First, the derivative with respect to u is expressed as

$$\begin{aligned}\frac{\partial Q}{\partial u} &= u(2au + b)\zeta + \zeta[2ga\zeta + a(u^2 + v^2) + bu + cv + d] \\ &= 2ga\zeta^2 + (3au^2 + av^2 + 2bu + cv + d)\zeta,\end{aligned}$$

which integrates to an expression for Q containing an unknown function,

$$Q = 2gau\zeta^2 + (au^3 + auv^2 + bu^2 + cuv + du)\zeta + Q_1(v, \zeta). \tag{7.2.25}$$

The next derivative of $\dfrac{\partial Q}{\partial v}$, obtained from (7.2.7) and (7.2.24) is

$$\frac{\partial Q}{\partial v} = u(2av + c)\zeta,$$

from which substitution of (7.2.25) for Q reveals that $\dfrac{\mathrm{d}Q_1}{\mathrm{d}v} = 0$, and hence $Q_1 = Q_1(\zeta)$. The final derivative, $\dfrac{\partial Q}{\partial \zeta}$, yields the equation

$$\begin{aligned}\frac{\partial Q}{\partial \zeta} &= g(2au + b)\zeta + u[2ga\zeta + a(u^2 + v^2) + bu + cv + d] \\ &= (4au + b)g\zeta + au^3 + auv^2 + bu^2 + cuv + du.\end{aligned}$$

Differentiating equation (7.2.25) with respect to ζ and comparing the result with the above equation yields

$$\frac{\mathrm{d}Q_1}{\mathrm{d}\zeta} = bg\zeta,$$

which integrates to (with the constant of integration set equal to zero without loss of generality) give $Q_1 = \frac{1}{2}bg\zeta^2$. This allows the final result of Q, up to addition of a constant, which is

$$Q(u, v, \zeta) = \left(2au + \frac{1}{2}b\right) g\zeta^2 + [au(u^2 + v^2) + u(bu + cv + d)]\zeta. \tag{7.2.26}$$

In a similar manner, the form of the function R can be described. The u partial derivative of R from (7.2.7) and (7.2.24) is given by

$$\frac{\partial R}{\partial u} = v(2au + b)\zeta,$$

which integrates to

$$R(u, v, \zeta) = (au^2v + buv)\zeta + R_1(v, \zeta). \tag{7.2.27}$$

The next derivative, $\dfrac{\partial R}{\partial v}$ from (7.2.7) yields the expression

$$\begin{aligned}\frac{\partial R}{\partial v} &= v(2av + c)\zeta + \zeta[2ga\zeta + a(u^2 + v^2) + bu + cv + d] \\ &= 2ga\zeta^2 + (au^2 + 3av^2 + bu + 2cv + d)\zeta.\end{aligned} \tag{7.2.28}$$

Substituting equation (7.2.27) into the expression (7.2.28) simplifies somewhat to yield

$$\frac{\partial R_1}{\partial v} = 2ga\zeta^2 + (3av^2 + 2cv + d)\zeta,$$

which may be integrated to include an unknown function in ζ as

$$R_1(v, \zeta) = 2gav\zeta^2 + (av^3 + cv^2 + dv)\zeta + R_2(\zeta). \tag{7.2.29}$$

The final equation to be used to help determine R is obtained from $\dfrac{\partial R}{\partial \zeta}$ in (7.2.7), and becomes

$$\begin{aligned}\frac{\partial R}{\partial \zeta} &= g(2av + c)\zeta + v[2ga\zeta + a(u^2 + v^2) + bu + cv + d] \\ &= (4av + c)g\zeta + [a(u^2 + v^2)v + buv + cv^2 + dv].\end{aligned} \tag{7.2.30}$$

Substituting the form of R given by equations (7.2.27) and (7.2.29) into the expression (7.2.30) gives

$$\frac{\mathrm{d}R_2}{\mathrm{d}\zeta} = cg\zeta,$$

which integrates to

$$R_2 = \frac{1}{2}cg\zeta^2, \tag{7.2.31}$$

where the constant of integration in (7.2.31) has been chosen as zero without loss of generality. Equations (7.2.27), (7.2.29) and (7.2.31) may now be used to express the most general form of R, up to addition of a constant, as

$$R(u, v, \zeta) = (2av + \frac{1}{2}c)g\zeta^2 + [a(u^2 + v^2)v + (bu + cv + d)v]\zeta. \tag{7.2.32}$$

Equations (7.2.24), (7.2.26), and (7.2.32) can now be combined as a concise result, stated in the following theorem, which has been proved by the preceding discussion.

Theorem 7.5 The three-dimensional, one-layer irrotational shallow-water equations (7.2.1)-(7.2.3) with $f = 0$ admit conservation laws of the form (7.2.4) with P, Q, and R given by equations (7.2.24), (7.2.26) and (7.2.32), respectively.

In essence, theorem 7.5 states that there are four conserved quantities associated with the three-dimensional one-layer shallow water equations. These are mass, momentum in the x and y directions, and the sum of kinetic and potential energy, all vertically integrated through the height of the layer. This can be observed readily by choosing the constants in the conserved quantity P given by (7.2.24) as required. For example, the energy equation may be obtained by fixing the constants as $a = \frac{1}{2}$, and $b = c = d = 0$.

7.3 Three-dimensional, Two-layer Conservation Equations

To consider the types of conservation equations which may arise from the full shallow-water equations (7.1)-(7.6) without any simplifications other than h_B constant and $f = 0$, an analysis similar to the methodology in sections 7.1 or 7.2 may be employed. However, initial calculations did not prove productive, prompting a different style of result which generalizes the stronger statements of theorems 7.4 and 7.5 to determine the associated mass, momentum and energy conservation equations. Thus, these conserved quantities examined in sections 7.1 and 7.2 are generalized for the three-dimensional two-layer shallow water equations, and the related conservation equations are verified in the following statement.

Theorem 7.6 The three-dimensional two-layer shallow-water equations (7.1)-(7.6), neglecting rotation ($f = 0$) over constant bottom height h_B, admit conservation equations of the form

$$\frac{\partial}{\partial t}P(u_1, u_2, v_1, v_2, \zeta_1, \zeta_2) + \frac{\partial}{\partial x}Q(u_1, u_2, v_1, v_2, \zeta_1, \zeta_2) + \frac{\partial}{\partial y}R(u_1, u_2, v_1, v_2, \zeta_1, \zeta_2) = 0, \tag{7.3.1}$$

with P, Q, and R of the form

$$P = a\left\{\frac{1}{2}(u_2^2 + v_2^2)\zeta_2 + \frac{1}{2}(1-\gamma)(u_1^2 + v_1^2)\zeta_1 + \frac{1}{2}g\zeta_2^2 + \frac{1}{2}(1-\gamma)g\zeta_1^2 \right.$$
$$\left. + (1-\gamma)g\zeta_1\zeta_2\right\} + b\{(1-\gamma)u_1\zeta_1 + u_2\zeta_2\} + c\{(1-\gamma)v_1\zeta_1 + v_2\zeta_2\} + d\zeta_1 + e\zeta_2, \tag{7.3.2}$$

$$Q = a\left\{\frac{1}{2}u_2(u_2^2 + v_2^2)\zeta_2 + \frac{1}{2}(1-\gamma)u_1(u_1^2 + v_1^2)\zeta_1 + g[(1-\gamma)u_1\zeta_1^2 + u_2\zeta_2^2] \right.$$
$$\left. + (1-\gamma)(u_1 + u_2)\zeta_1\zeta_2\right\} + b\left\{(1-\gamma)u_1^2\zeta_1 + u_2^2\zeta_2 + \frac{1}{2}g[\zeta_2^2 + (1-\gamma)\zeta_1^2] \right.$$
$$\left. + (1-\gamma)g\zeta_1\zeta_1\right\} + c\{(1-\gamma)u_1v_1\zeta_1 + u_2v_2\zeta_2\} + du_1\zeta_1 + eu_2\zeta_2, \tag{7.3.3}$$

and

$$R = a\left\{\frac{1}{2}(u_2^2 + v_2^2)v_2\zeta_2 + \frac{1}{2}(1-\gamma)(u_1^2 + v_1^2)v_1\zeta_1 + g[(1-\gamma)v_1\zeta_1^2 + v_2\zeta_2^2] \right.$$
$$\left. + (1-\gamma)g(v_1 + v_2)\zeta_1\zeta_2\right\} + b\{(1-\gamma)u_1v_1\zeta_1 + u_2v_2\zeta_2\} + c\left\{(1-\gamma)v_1^2\zeta_1 + v_2^2\zeta_2 \right.$$
$$\left. + \frac{1}{2}g[(1-\gamma)\zeta_1^2 + \zeta_2^2] + (1-\gamma)g\zeta_1\zeta_2\right\} + dv_1\zeta_1 + ev_2\zeta_2, \tag{7.3.4}$$

where a, b, c, d, and e are arbitrary constants.

Proof: Equations (7.1)-(7.6) are rewritten for subsequent use with constant h_B and $f = 0$. They are also reordered and stated as

$$\frac{\partial u_1}{\partial t} = -u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial u_1}{\partial y} - g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}, \tag{7.3.5}$$

$$\frac{\partial v_1}{\partial t} = -u_1\frac{\partial v_1}{\partial x} - v_1\frac{\partial v_1}{\partial y} - g\frac{\partial \zeta_1}{\partial y} - g\frac{\partial \zeta_2}{\partial y}, \tag{7.3.6}$$

$$\frac{\partial u_2}{\partial t} = -u_2\frac{\partial u_2}{\partial x} - v_2\frac{\partial u_2}{\partial y} - (1-\gamma)g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}, \tag{7.3.7}$$

$$\frac{\partial v_2}{\partial t} = -u_2\frac{\partial v_2}{\partial x} - v_2\frac{\partial v_2}{\partial y} - (1-\gamma)g\frac{\partial \zeta_1}{\partial y} - g\frac{\partial \zeta_2}{\partial y}, \tag{7.3.8}$$

$$\frac{\partial \zeta_1}{\partial t} = -\frac{\partial}{\partial x}(u_1\zeta_1) - \frac{\partial}{\partial y}(v_1\zeta_1), \tag{7.3.9}$$

and

$$\frac{\partial \zeta_2}{\partial t} = -\frac{\partial}{\partial x}(u_2\zeta_2) - \frac{\partial}{\partial y}(v_2\zeta_2). \tag{7.3.10}$$

The linearity of P, Q, and R in the constants $a - e$, and the linearity of the partial derivatives in equation (7.3.1), allows the verification of equations (7.3.1)-(7.3.4) to be completed in five parts, one corresponding to each constant. That is, substitution of equations (7.3.2)-(7.3.4) into (7.3.1) may be written as

$$a\left[\frac{\partial P_a}{\partial t} + \frac{\partial Q_a}{\partial x} + \frac{\partial R_a}{\partial y}\right] + b\left[\frac{\partial P_b}{\partial t} + \frac{\partial Q_b}{\partial x} + \frac{\partial R_b}{\partial y}\right] + c\left[\frac{\partial P_c}{\partial t} + \frac{\partial Q_c}{\partial x} + \frac{\partial R_c}{\partial y}\right]$$
$$+ d\left[\frac{\partial P_d}{\partial t} + \frac{\partial Q_d}{\partial x} + \frac{\partial R_d}{\partial y}\right] + e\left[\frac{\partial P_e}{\partial t} + \frac{\partial Q_e}{\partial x} + \frac{\partial R_e}{\partial y}\right] = 0, \tag{7.3.11}$$

where P_a, Q_a, R_a, etc. are the coefficients of P, Q, R with respect to the constants a through e. Equations (7.3.1)-(7.3.4) may be verified by checking that separate terms in the square brackets in equation (7.3.11) vanish separately. For the constants d and e, this exercise is straightforward from the observation that if $a = b = c = 0$, then equations (7.3.1)-(7.3.4) reduce simply to $d\times$(7.3.9) + $e\times$(7.3.10).

For the term in P which is the coefficient of the constant c, P_c, consider $a = b = d = e = 0$, and $c = 1$ without loss of generality. In this case, $\dfrac{\partial P}{\partial t}$ may be calculated from equation (7.3.2) as

$$\frac{\partial P}{\partial t} = (1-\gamma)\left(\frac{\partial v_1}{\partial t}\zeta_1 + v_1\frac{\partial \zeta_1}{\partial t}\right) + \frac{\partial v_2}{\partial t}\zeta_2 + v_2\frac{\partial \zeta_2}{\partial t}. \tag{7.3.12}$$

Substitution of the time derivatives from equations (7.3.6), and (7.3.8)-(7.3.10) into (7.3.12) yields

$$\frac{\partial P}{\partial t} = (1-\gamma)\left[-u_1\frac{\partial v_1}{\partial x} - v_1\frac{\partial v_1}{\partial y} - g\frac{\partial \zeta_1}{\partial y} - g\frac{\partial \zeta_2}{\partial y}\right]\zeta_1$$
$$+ (1-\gamma)v_1\left[-\frac{\partial}{\partial x}(u_1\zeta_1) - \frac{\partial}{\partial y}(v_1\zeta_1)\right] + \left[-u_2\frac{\partial v_2}{\partial x} - v_2\frac{\partial v_2}{\partial y} - (1-\gamma)g\frac{\partial \zeta_1}{\partial y}\right.$$
$$\left. - g\frac{\partial \zeta_2}{\partial y}\right]\zeta_2 + v_2\left[-\frac{\partial}{\partial x}(u_2\zeta_2) - \frac{\partial}{\partial y}(v_2\zeta_2)\right]. \tag{7.3.13}$$

Reordering the result (7.3.13) and applying the product rule allows the following calculation:

$$\frac{\partial P}{\partial t} = -(1-\gamma)\left(\frac{\partial v_1}{\partial x}u_1\zeta_1 + v_1\frac{\partial}{\partial x}(u_1\zeta_1)\right) - \left(\frac{\partial v_2}{\partial x}u_2\zeta_2 + v_2\frac{\partial}{\partial x}(u_2\zeta_2)\right)$$
$$- (1-\gamma)\left(\frac{\partial v_1}{\partial y}v_1\zeta_1 + v_1\frac{\partial}{\partial y}(v_1\zeta_1)\right) - \left(\frac{\partial v_2}{\partial y}v_2\zeta_2 + v_2\frac{\partial}{\partial y}(v_2\zeta_2)\right)$$
$$- (1-\gamma)g\zeta_1\frac{\partial \zeta_1}{\partial y} - (1-\gamma)g\left(\zeta_1\frac{\partial \zeta_2}{\partial y} + \frac{\partial \zeta_1}{\partial y}\zeta_2\right) - g\frac{\partial \zeta_2}{\partial y}\zeta_2,$$

which then becomes

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\{(1-\gamma)u_1v_1\zeta_1 + u_2v_2\zeta_2\}$$
$$-\frac{\partial}{\partial y}\left\{(1-\gamma)v_1^2\zeta_1 + v_2^2\zeta_2 + \frac{1}{2}(1-\gamma)g\zeta_1^2 + (1-\gamma)g\zeta_1\zeta_2 + \frac{1}{2}g\zeta_2^2\right\}. \quad (7.3.14)$$

This result (7.3.14) may now be seen as equations (7.3.1)-(7.3.4) with $a = b = d = e = 0$ and $c \neq 0$.

The coefficient for the constant b may be handled similarly. With $a = c = d = e = 0$, and $b = 1$ without loss of generality, equation (7.3.2) may be differentiated to give

$$\frac{\partial P}{\partial t} = (1-\gamma)\left(\frac{\partial u_1}{\partial t}\zeta_1 + u_1\frac{\partial \zeta_1}{\partial t}\right) + \frac{\partial u_2}{\partial t}\zeta_2 + u_2\frac{\partial \zeta_2}{\partial t}. \quad (7.3.15)$$

Substitution of equations (7.3.5), (7.3.7), (7.3.9) and (7.3.10) into (7.3.15) gives, along with some reorganization,

$$\frac{\partial P}{\partial t} = (1-\gamma)\left(-u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial u_1}{\partial y} - g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right)\zeta_1 + (1-\gamma)u_1\left(-\frac{\partial}{\partial x}(u_1\zeta_1)\right.$$
$$\left.-\frac{\partial}{\partial y}(v_1\zeta_1)\right) + \left(-u_2\frac{\partial u_2}{\partial x} - v_2\frac{\partial u_2}{\partial y} - (1-\gamma)g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right)\zeta_2$$
$$+ u_2\left(-\frac{\partial}{\partial x}(u_2\zeta_2) - \frac{\partial}{\partial y}(v_2\zeta_2)\right)$$

$$\Longrightarrow \frac{\partial P}{\partial t} = -(1-\gamma)\left(\frac{\partial u_1}{\partial x}u_1\zeta_1 + u_1\frac{\partial}{\partial x}(u_1\zeta_1)\right) - \left(\frac{\partial u_2}{\partial x}u_2\zeta_2 + u_2\frac{\partial}{\partial x}(u_2\zeta_2)\right)$$
$$- (1-\gamma)g\zeta_1\frac{\partial \zeta_1}{\partial x} - (1-\gamma)g\left(\zeta_1\frac{\partial \zeta_2}{\partial x} + \frac{\partial \zeta_1}{\partial x}\zeta_2\right) - g\zeta_2\frac{\partial \zeta_2}{\partial x}$$
$$- (1-\gamma)\left(\frac{\partial u_1}{\partial y}v_1\zeta_1 + u_1\frac{\partial}{\partial y}(v_1\zeta_1)\right) - \left(\frac{\partial u_2}{\partial y}v_2\zeta_2 + u_2\frac{\partial}{\partial y}(v_2\zeta_2)\right)$$

$$\Longrightarrow \frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{(1-\gamma)u_1^2\zeta_1 + u_2^2\zeta_2 + \frac{1}{2}(1-\gamma)g\zeta_1^2 + \frac{1}{2}g\zeta_2^2 + (1-\gamma)g\zeta_1\zeta_2\right\}$$
$$-\frac{\partial}{\partial y}\left\{(1-\gamma)u_1 v_1\zeta_1 + u_2 v_2\zeta_2\right\}. \qquad (7.3.16)$$

Equation (7.3.16) therefore verifies equations (7.3.1)-(7.3.4) for the coefficient $b \neq 0$ and $a = c = d = e = 0$.

To complete the verification of equations (7.3.1)-(7.3.4), assume that $a = 1$ and $b = c = d = e = 0$. Then, $\frac{\partial}{\partial t}$ of equation (7.3.2) results in

$$\frac{\partial P}{\partial t} = \left(u_2\frac{\partial u_2}{\partial t} + v_2\frac{\partial v_2}{\partial t}\right)\zeta_2 + \frac{1}{2}(u_2^2+v_2^2)\frac{\partial \zeta_2}{\partial t} + (1-\gamma)\left(u_1\frac{\partial u_1}{\partial t} + v_1\frac{\partial v_1}{\partial t}\right)\zeta_1$$
$$+\frac{1}{2}(1-\gamma)(u_1^2+v_1^2)\frac{\partial \zeta_1}{\partial t} + g\zeta_2\frac{\partial \zeta_2}{\partial t} + (1-\gamma)g\zeta_1\frac{\partial \zeta_1}{\partial t} + (1-\gamma)g\left(\frac{\partial \zeta_1}{\partial t}\zeta_2 + \zeta_1\frac{\partial \zeta_2}{\partial t}\right),$$

which simplifies to

$$\frac{\partial P}{\partial t} = \left(u_2\frac{\partial u_2}{\partial t} + v_2\frac{\partial v_2}{\partial t}\right)\zeta_2 + (1-\gamma)\left(u_1\frac{\partial u_1}{\partial t} + v_1\frac{\partial v_1}{\partial t}\right)\zeta_1 + \left(\frac{1}{2}(u_2^2+v_2^2) + g\zeta_2\right.$$
$$\left. + (1-\gamma)g\zeta_1\right)\frac{\partial \zeta_2}{\partial t} + \left(\frac{1}{2}(1-\gamma)(u_1^2+v_1^2) + (1-\gamma)g\zeta_1 + (1-\gamma)g\zeta_2\right)\frac{\partial \zeta_1}{\partial t}. \qquad (7.3.17)$$

Substitution of the six equations (7.3.5)-(7.3.10) into (7.3.17) removes the derivatives with respect to time to permit

$$\frac{\partial P}{\partial t} = \left[u_2\left(-u_2\frac{\partial u_2}{\partial x} - v_2\frac{\partial u_2}{\partial y} - (1-\gamma)g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right) + v_2\left(-u_2\frac{\partial v_2}{\partial x} - v_2\frac{\partial v_2}{\partial y}\right.\right.$$
$$\left.\left. - (1-\gamma)g\frac{\partial \zeta_1}{\partial y} - g\frac{\partial \zeta_2}{\partial y}\right)\right]\zeta_2 + (1-\gamma)\left[u_1\left(-u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial u_1}{\partial y} - g\frac{\partial \zeta_1}{\partial x} - g\frac{\partial \zeta_2}{\partial x}\right)\right.$$
$$\left. + v_1\left(-u_1\frac{\partial v_1}{\partial x} - v_1\frac{\partial v_1}{\partial y} - g\frac{\partial \zeta_1}{\partial y} - g\frac{\partial \zeta_2}{\partial y}\right)\right]\zeta_1$$
$$+\left[\frac{1}{2}(u_2^2+v_2^2) + g\zeta_2 + (1-\gamma)g\zeta_1\right]\left[-\frac{\partial}{\partial x}(u_2\zeta_2) - \frac{\partial}{\partial y}(v_2\zeta_2)\right]$$
$$+(1-\gamma)\left[\frac{1}{2}(u_1^2+v_1^2) + g\zeta_1 + g\zeta_2\right]\left[-\frac{\partial}{\partial x}(u_1\zeta_1) - \frac{\partial}{\partial y}(v_1\zeta_1)\right]. \qquad (7.3.18)$$

Reordering terms in the above expression (7.3.18) yields

$$\frac{\partial P}{\partial t} = -\left(u_2\frac{\partial u_2}{\partial x}u_2\zeta_2 + \frac{1}{2}u_2^2\frac{\partial}{\partial x}(u_2\zeta_2)\right) - \left(v_2\frac{\partial v_2}{\partial x}u_2\zeta_2 + \frac{1}{2}v_2^2\frac{\partial}{\partial x}(u_2\zeta_2)\right)$$
$$- \left(u_2\frac{\partial u_2}{\partial y}v_2\zeta_2 + \frac{1}{2}u_2^2\frac{\partial}{\partial y}(v_2\zeta_2)\right) - \left(v_2\frac{\partial v_2}{\partial y}v_2\zeta_2 + \frac{1}{2}v_2^2\frac{\partial}{\partial y}(v_2\zeta_2)\right)$$
$$-(1-\gamma)\left[u_1\frac{\partial u_1}{\partial x}u_1\zeta_1 + \frac{1}{2}u_1^2\frac{\partial}{\partial x}(u_1\zeta_1) + v_1\frac{\partial v_1}{\partial x}u_1\zeta_1 + \frac{1}{2}v_1^2\frac{\partial}{\partial x}(u_1\zeta_1)\right]$$
$$-(1-\gamma)\left[u_1\frac{\partial u_1}{\partial y}v_1\zeta_1 + \frac{1}{2}u_1^2\frac{\partial}{\partial y}(v_1\zeta_1) + v_1\frac{\partial v_1}{\partial y}v_1\zeta_1 + \frac{1}{2}v_1^2\frac{\partial}{\partial y}(v_1\zeta_1)\right]$$
$$-(1-\gamma)g\left[\frac{\partial \zeta_1}{\partial x}u_2\zeta_2 + \zeta_1\frac{\partial}{\partial x}(u_2\zeta_2) + \frac{\partial \zeta_1}{\partial x}u_1\zeta_1 + \zeta_1\frac{\partial}{\partial x}(u_1\zeta_1) + \frac{\partial \zeta_2}{\partial x}u_1\zeta_1\right.$$
$$\left.+\zeta_2\frac{\partial}{\partial x}(u_1\zeta_1)\right] - (1-\gamma)g\left[\frac{\partial \zeta_1}{\partial y}v_2\zeta_2 + \zeta_1\frac{\partial}{\partial y}(v_2\zeta_2) + \frac{\partial \zeta_1}{\partial y}v_1\zeta_1 + \zeta_1\frac{\partial}{\partial y}(v_1\zeta_1)\right.$$
$$\left.+\frac{\partial \zeta_2}{\partial y}v_1\zeta_1 + \zeta_2\frac{\partial}{\partial y}(v_1\zeta_1)\right] - g\left[\frac{\partial \zeta_2}{\partial x}u_2\zeta_2 + \zeta_2\frac{\partial}{\partial x}(u_2\zeta_2) + \frac{\partial \zeta_2}{\partial y}v_2\zeta_2\right.$$
$$\left.+\zeta_2\frac{\partial}{\partial y}(v_2\zeta_2)\right],$$

which simplifies using the product rule to

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{2}u_2^2u_2\zeta_2 + \frac{1}{2}v_2^2u_2\zeta_2\right) - \frac{\partial}{\partial y}\left(\frac{1}{2}u_2^2v_2\zeta_2 + \frac{1}{2}v_2^2v_2\zeta_2\right)$$
$$-(1-\gamma)\frac{\partial}{\partial x}\left(\frac{1}{2}u_1^2u_1\zeta_1 + \frac{1}{2}v_1^2u_1\zeta_1\right) - (1-\gamma)\frac{\partial}{\partial y}\left(\frac{1}{2}u_1^2v_1\zeta_1 + \frac{1}{2}v_1^2v_1\zeta_1\right)$$
$$-(1-\gamma)g\frac{\partial}{\partial x}\left(u_2\zeta_1\zeta_2 + u_1\zeta_1^2 + u_1\zeta_1\zeta_2\right) - (1-\gamma)g\frac{\partial}{\partial y}\left(v_2\zeta_1\zeta_2 + v_1\zeta_1^2\right.$$
$$\left.+v_1\zeta_1\zeta_2\right) - g\frac{\partial}{\partial x}\left(u_2\zeta_2^2\right) - g\frac{\partial}{\partial y}\left(v_2\zeta_2^2\right),$$

becoming, finally, expressed in the desired form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{1}{2}(u_2^2+v_2^2)u_2\zeta_2 + \frac{1}{2}(1-\gamma)(u_1^2+v_1^2)u_1\zeta_1\right.$$
$$\left.+ g\left[(1-\gamma)\left(u_1\zeta_1^2 + (u_1+u_2)\zeta_1\zeta_2\right) + u_2\zeta_2^2\right]\right\}$$
$$-\frac{\partial}{\partial y}\left\{\frac{1}{2}(u_2^2+v_2^2)v_2\zeta_2 + \frac{1}{2}(1-\gamma)(u_1^2+v_1^2)(v_1\zeta_1\right.$$
$$\left.+ g\left[(1-\gamma)\left(v_1\zeta_1^2 + (v_1+v_2)\zeta_1\zeta_2\right) + v_2\zeta_2^2\right]\right\}. \qquad (7.3.19)$$

Comparing the result (7.3.19) with the expressions for Q_a, R_a in formulas (7.3.3) and (7.3.4) show that the calculation, and hence the theorem, is complete. □

The statement of Theorem 7.5 is not as strong a property as desired: a result stating that the number of conservation laws associated with the three-dimensional shallow water equations is finite would be certainly of greater use. Such a result remains conjecture, since attempting to prove this statement by contradiction rests on theorems 7.4 and 7.5, and does not cover conserved quantities of the form $P(u_1, v_2, \zeta_1, \zeta_2)$, for example. As such, this problem remains unsolved.

Notwithstanding the clarification of this open problem, it has been shown in equation (7.3.2) of theorem 7.6, that the quantities of mass in each layer, horizontal momentum in two perpendicular directions, and energy, are conserved quantities of the flow and lead to five equations for the three-dimensional two-layer system. Thus, the six variables in equations (7.1.)-(7.6) are one equation short of being expressed as a closed system of conservation laws, and the numerical methods for hyperbolic conservation laws developed in previous chapters are not applicable in this case.

Chapter Summary

An investigation of the conservation laws which are admitted by the shallow-water equations has been completed in this chapter. This analysis was motivated by the existence of an infinite number of conservation equations for the two-dimensional single layer system, as shown in Whitham (1974). These results are valid for flow for which the effects of the Coriolis Force are negligible, and the bottom boundary is horizontal.

In the two-dimensional, two-layer case examined in Section 7.1, the four equations for shallow-water flow (7.1.1)-(7.1.4), were found to admit a finite number of conservation equations. The conserved quantities found were all multinomial real-valued functions $P : \mathbb{R}^4 \to \mathbb{R}$ of the form

$$P(u_1, u_2, \zeta_1, \zeta_2) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{ij}(u_1, u_2)\zeta_1^i \zeta_2^j,$$

for functions p_{ij} which are C^2 in each of the variables u_1 and u_2. The resulting six equations in conservation form represent: conservation of mass and velocity (in each layer), horizontal momentum, and energy.

In the three-dimensional, one-layer case investigated in Section 7.2, the three equations for shallow-water flow are shown to admit only four conservation equations with conserved quantities $P : \mathbb{R}^3 \rightarrow \mathbb{R}$ of the general form $P(u, v, \zeta)$ which are also C^2 in each variable. The conserved quantities represent mass, momentum (in the x and y directions) and energy.

For the full three-dimensional and two-layer situation described in Section 7.3, it was found that the generalized mass, momentum and energy detailed in the previous sections resulted in five conserved quantities. Although it is thought that only a finite number of C^2 conserved quantities exist for this system, this idea remains unproved. Therefore, the five relevant equations are insufficient to close the six-variable system, and allow it to be restated as a system of hyperbolic conservation laws. Hence, the numerical method contained in Chapter 6 is not applicable in this situation.

Chapter 8

Conclusions

A summary of the results of this thesis is contained in this chapter. Where it is applicable, the results are discussed in context with existing knowledge so that the new contributions may be more easily observed. Ideas for future research are interspersed throughout the discussion instead of being tabulated separately, so that they can be viewed in context.

Gravity currents, examples of which are prevalent in nature and the laboratory, may be created from sudden releases of fixed volumes of dense fluid into larger quiescent volumes of lighter fluid. The modelling of this time-dependent motion has been achieved through the use of shallow-water methods, resulting in analytic and numerical methods of solution (see for example, Rottman and Simpson, 1983). These methods are limited by their requirement to rely on the specification of a head or front condition, which must be determined prior to the development of the gravity current. To the best of the author's knowledge, a general front condition which encompasses non-horizontal bottom topography, friction, entrainment, turbulence, and volume changes does not exist. This thesis describes a model, based on shallow-water theory, which overcomes the limitation of a front condition by a consideration of the equations of motion stated as a system of conservation laws for which a gravity current front may be considered as a vertical discontinuity or shock.

The model equations derived are for two homogeneous layers of stably-stratified incompressible Newtonian fluid overlying a rigid bottom boundary and beneath a semi-infinite quiescent region. The layers are depicted in Figure 2.1, and the dimensional equations of motion consist of equation (2.1.2) and (2.1.6) for a rotating fluid with the f-plane approximation. For small aspect-ratio flows the hydrostatic approximation and shallow-water approximation were made. The resulting six equations in six unknowns, simplified through use of the boundary conditions, are stated as equations (2.2.17)-(2.2.21) with either the free surface condition (2.2.22) or rigid lid condition (2.2.23). This analysis is similar to that found in Pedlosky (1987), and is not new. However, a comprehensive derivation which specifies the assumptions and scalings is not generally found in connection with gravity currents. Therefore, this development of the model equations is useful in this context.

A new addition to the existing theory for two-layer gravity currents resulting from initial releases of dense fluid was stated in Chapter 2, Section 3. This is in the proposition of a nonlinear forcing term (2.3.2) for the lower layer. The addition of this forcing term was motivated by observations (Benjamin, 1968) that the nature of a gravity current does not change appreciably within the region of small bottom slopes from the horizontal ($\lessapprox \pm 5$ degrees) and that drag due to bottom friction generally acts to slow propagation speed (Middleton, 1966). The forcing term was introduced in a way that it is consistent with shallow-water theory, that is, independent of height within the layer. The form of the layer forcing is similar to Chézy's basal drag law (Whitham, 1974), and allows for spatial variation along the length of the lower layer gravity current. Motivated by physical observation (Huppert and Simpson, 1980) a spatial variation within the forcing term is included in the form of a truncation function which decreases the strength of the drag with increasing distance behind the front.

The addition of the forcing term is done in a fairly heuristic and intuitive manner, and is still a current topic of research (Montgomery and Moodie, 1999a). A more direct link to physical motivation, for example through experiments and parameter estimation, is required before it can be regarded as a fully justified addition to the theory. The front of a gravity current is generally characterized as a region of entrainment and large vertical accelerations for which the assumptions made in deriving the model equations do not apply (Simpson, 1997). However, the successes of describing gravity current behaviour by assuming shallow-water theory throughout the current (Rottman and Simpson, 1983, Bonnecaze *et al.*, 1993) show that such an approach is valid. The addition of a new term consistent with shallow-water theory merely expands upon these successes. The forcing term can be described as a way to capture non-shallow-water effects at the head of a gravity current in a way that is consistent with shallow-water theory.

Some special cases of the model equations have been portrayed. The equations in three spatial dimensions for a thick upper layer with a free surface are given by the three equations (2.4.4)-(2.4.6) in three variables. In axisymmetric form, these equations are stated in polar coordinates as (2.4.19)-(2.4.21). In two spatial dimensions, horizontal and vertical, the general equations for two layers with a free surface are stated as the system (2.4.29) of four equations in four variables. This is further simplified to two equations in two variables for: the weakly strafitied case (2.4.38), a thin lower layer system (2.4.61), thin upper layer system (2.4.81), and rigid lid upper boundary condition (2.4.89). Recasting of these first order systems of partial differential equations in conservation form was also completed. These equations have been used previously, for example by Lawrence (1990), Rottman and Simpson (1983), Bonnecaze *et al.*(1993), Baines (1995), and Montgomery and Moodie (1998a, 1999a). However, they are derived in Chapter 2 with the lower layer forcing term and variable bottom height. The development of these equations is useful as it is a comprehensive equation development which is not found elsewhere. This should be of assistance to researchers trying to relate general models to specific cases.

A review of most of the theoretical concepts used in the thesis is collected in Chapter 3. This list includes a definition of a first order hyperbolic system of partial differential equations in one or more spatial variables (3.1.7), and how such a system may be written in conservation form (3.1.10). By stating a first order system in conservation form, the class of solutions to an initial boundary value problem (IBVP) grows to include weak solutions which may have discontinuities. Rankine-Hugoniot jump conditions are used to select the correct solution at a jump discontinuity (John, 1982), and these are generalized to include possible discontinuities in the forcing terms, yielding a new jump condition (3.2.11). This jump condition is examined using several simple variations on the Burgers' Equation to show its usefulness in solving IBVPs.

To examine well-posedness of IBVPs, several recent results (Kreiss and Lorenz, 1989 and Godlewski and Raviart, 1996) are discussed and framed in notation which is applicable to the model equations from Chapter 2. The method of linearization and localizaton is useful in examining nonlinear hyperbolic systems since it allows the suitability of the boundary conditions to be assessed through a statement such as Lemma 3.3.

A thorough analysis of the mathematical properties of the shallow-water models for two-dimensional gravity currents is contained in Chapter 4. The equations for two layers with a free surface have been shown previously to be hyperbolic in certain cases (Montgomery and Moodie, 1998a). Of particular usefulness is the result that for small flow velocities, such as those near a vertical barrier in the fluid, the equations were shown to be strictly hyperbolic if, from (4.1.8), the inequality $(\zeta_1+\zeta_2)^2 > 4\gamma\zeta_1\zeta_2$ holds. This is satisfied if $\gamma < 0.5$, which is clearly a valid assumption for layers which are close in density. For the weak-stratification equations, an exact expression of hyperbolicity is given by (4.1.21), which reduces for zero endflow to a simple Froude Number requirement stated dimensionally as (4.1.23). Analysis of the rigid-lid equations yielded the same result. All of these relations characterizing regions in which the equations are hyperbolic are new results, and complete the analysis stated previously for the thin layer models (Montgomery and Moodie 1998a). Such knowledge of hyperbolicity is critical in choosing an appropriate numerical scheme to solve the equations.

Under the assumption that the model equations for two-dimensional flow are hyperbolic, the jump conditions derived in Chapter 3 were applied resulting in several expressions discussed in Chapter 4, Section 2. In these cases, the front of an advancing gravity current was assumed to be a simple discontinuity in the height of the lower layer, arising from a discontinuity existing in the initial geometry. Although the general two-layer system with a free surface could not be solved completely (4.2.28), a small γ asymptotic expansion solution was obtained in (4.2.35). An exact result in the small $\gamma \to 0$ limit is given by (4.2.79) in the special case that the upper layer is assumed to be quiescent. Similar manipulations yielded jump conditions for the weak-stratification case (4.2.92), the thin lower layer (4.2.101) and thin upper layer situation (4.2.109), as well as the rigid-lid equations (4.2.119).

The importance of all of the jump condition expressions is that they all have the general form $u_2^2 = f(h_1, h_2, h_B, \gamma)$, which is a generalization of previous results which do not include variations in bottom height. The equations represent a purely theoretical prediction arising from the shallow-water equations expressed in conservation form, and are independent of experiments and the forcing term parameter, C_f. This analysis is new, and provides both results and methodology to the field, although there is still a need to compare these results with both pre-existing methods of modelling gravity currents and experiments.

The final section of results from Chapter 4 shows the well-posedness of the IBVP for sudden releases via the method of localization. All of the equations for two-dimensional flow are shown to be well-posed whenever the equations are strictly hyperbolic. This is an important theoretical result, which provides a strong theoretical basis for the initial release problem.

For gravity currents in three dimensions, Chapter 5 contains the pertinent results. Sufficient conditions for the two-layer equations are described in Theorem 5.1, where they are described precisely. The single layer equations give a simpler result (5.1.41) which guarantees hyperbolicity of the equations for all conditions of flow in which the layer thickness is strictly positive. By stating the single-layer equations in conservation form through polar coordinates, an asymptotic jump condition (5.2.35) is obtained for small deviations from the axisymmetric case.

The result includes rotation, but is limited to the special case of constant bottom height. This condition is new, and generalization to include changes in bottom height is worthy of future study. A comparison to experimental and numerically generated results would also be useful to confirm the practical usefulness of equation (5.2.35).

The generalized relaxation method, described in Chapter 6, was shown to be a useful tool for computing solutions to the IBVPs for systems of nonlinear hyperbolic conservation laws. New generalizations to the finite difference relaxation method developed by Jin and Xin (1995) are discussed in Section 6.1, and allow the method to be applied with boundary conditions, spatially dependent flux functions, and forcing terms on the right hand side of the equations which are independent of derivatives in the system variables. The method is shown to resolve discontinuities well, without oscillations, and several calculations investigating resolution and parameter limits are portrayed in Figures 6.1 to 6.4.

The relaxation method has been applied to many of the two-dimensional model equations to calculate solutions to the initial release problems by Montgomery and Moodie (1998a,b and 1999a,b). In section 2 of Chapter 6, some similar results are portrayed to highlight some of the important features of the numerical solutions to the model equations. The weak stratification and rigid lid are both shown to yield good approximation to the two-layer equations, although for small values of γ the weak stratification equations yield a quantitatively better approximation, as seen in Figure 6.8. The usefulness of the jump condition expressions at the front is also demonstrated in Figure 6.9, where an experimental front condition of the form $u_2^2 = Fr\zeta_2$ is compared. Varying functions for the bottom topography are shown to have important effects both in the shape of the gravity current (Figure 6.10) and the front speed (Figure 6.11). All of these results would benefit from further analysis and comparison to experimental results to permit a quantitative description of the usefulness of the numerical scheme.

Addition of the nonlinear forcing term in the lower layer is examined for three truncation functions. For a constant forcing, the effect is quite different from forcing which is focussed on the front of the gravity current (Figure 6.13). Although the forcing terms permit long-time downslope gravity currents to achieve a steady-state (see also Montgomery and Moodie, 1999a), a quantitative comparison of the effects of the forcing term with experimental results is necessary to fully justify the inclusion of such a term in the model equations. However, the implementation of a front-based drag term yields qualitative results which are similar to experiments such as the small decelerations observed in front speed (Middleton, 1966).

Some results for the three-dimensional single layer case are portrayed in Chapter 6, Section 3. Several calculations were completed for the initial release of a fixed volume at rest which then spreads out under the influence of gravity. These new results show that the relaxation method captures the accelerative affects of constant bottom slope, as well as Coriolis accelerations. These results are somewhat preliminary, and would benefit from further analysis to investigate the regions of lateral instability due to rotation, as well as the effects of variable bottom slopes, a continuous volume source, and vertical boundaries. Comparison with experimentally obtained results would also be quite useful to further verify both the appropriateness of the model equations, and the predictive capabilities of the numerical method.

The one-layer shallow-water equations in one spatial dimension admit an infinite number of conservation equations. In Chapter 7, the previously mentioned methodology of Whitham (1974) was employed to find that the various shallow-water model equations only have associated with them a finite number of polynomial conservation equations, corresponding to the conserved quantities of mass, momentum and energy. For the two-layer case in two dimensions, the conserved quantities and associated fluxes are given for velocity (7.1.72), momentum (7.1.74), and energy (7.1.85)-(7.1.86). Higher order polynomial conserved quantities for this case were shown not to exist, as discussed by Theorem 7.4. In the three dimensional case, the one-layer equations are shown to also only have a finite number of conserved quantities, listed in Theorem 7.5. The three dimensional, two-layer case is slightly different from the previous ones in that the simpler results are generalized to describe the conserved quantities relating to mass, momentum and energy (Theorem 7.6). A more general result that these equations only permit a finite number of conservation laws still remains conjecture. Such a result would be useful since the relaxation method is applicable only to systems of conservation laws, and thus should not be used for any system which cannot be expressed as a closed system of conservation laws.

The results concerning the number of associated conservation laws to the different systems of equations are new, and are of theoretical concern in their own right. The only direct relation to the modelling of gravity currents is a complete characterization of the types of equations in conservation form which exist. In addition, the general methodology of Whitham (1974) is shown to be quite useful, and may be of use as a general method to characterize the possible ways to rewrite systems of first order partial differential equations in conservation form.

The shallow-water model constructed in this thesis has been investigated both theoretically and numerically. The hyperbolic nature of the equations allows a gravity current front to be approximated as a vertical discontinuity for which a nonlinear forcing term may be introduced in a way that is consistent with shallow-water assumptions. The model permits the effects of variable bottom topography and volume changes within the current to be investigated numerically without reliance on an experimentally imposed front condition. This advance is quite useful in generalizing present methods of prediction of the properties and behaviour of gravity currents.

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Appendices

Appendix 1

To solve the cubic equation (3.3.5), assume $\zeta > 0$, and write the equation as

$$0 = \lambda^3 - 3\varpi\zeta^{-1}\lambda^2 + (3\varpi\zeta^{-1} - \omega^2\zeta)\lambda + (\omega^2\varpi - \varpi^3\zeta^{-3}). \qquad \text{(A1.1)}$$

Transform (A1.1) via

$$x = \varpi\zeta^{-1} \text{ and } y = \omega^2\zeta \qquad \text{(A1.2)}$$

to get the equation

$$0 = \lambda^3 - 3x\lambda^2 + (3x^2 - y)\lambda + (xy - x^3) \qquad \text{(A1.3)}$$

noting that from definition (A1.2) the variable y is positive since $\omega^2 > 0$ and $\zeta > 0$.

Comparison of (A1.3) with the general form

$$0 = \lambda^3 + p\lambda^2 + q\lambda + r, \qquad \text{(A1.4)}$$

with $p = -3x$, $q = sx^2 - y$, and $r = xy - x^3$ gives 3 real unequal roots to (A1.4) (see, for example the CRC Handbook 1970 p.129/130) if

$$\frac{b^2}{4} + \frac{a^3}{27} < 0, \qquad \text{(A1.5)}$$

where

$$a = \frac{1}{3}(3q - p^2) = \frac{1}{3}(9x^2 - 3y - (-3x)^2) = -y$$

and

$$\begin{aligned} b &= \frac{1}{27}(2p^3 - 9pq + 27r) \\ &= \frac{1}{27}[2(-3x)^3 - 9(-3x)(3x^2 - y) + 27(xy - x^3)] \\ &= \frac{1}{27}[-54x^3 + 81x^3 - 27xy + 27xy - 27x^3] \\ &= 0. \end{aligned}$$

Therefore, (A1.5) holds precisely when $y > 0$. Since this was initially assumed to be so, it follows that equation (A1.1) has three real unequal roots.

The roots of (A1.1) are given by

$$\begin{aligned}\lambda_0 &= A+B+x\\ \lambda_1 &= -\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\sqrt{-3}+x\\ \lambda_2 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3}+x,\end{aligned} \tag{A1.6}$$

where in (A1.6) the symbols A, and B are defined as

$$A=\left(-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right)^{\frac{1}{3}}=\left(i\sqrt{\frac{y^3}{27}}\right)^{\frac{1}{3}}=-i\sqrt{\frac{y}{3}}$$
$$B=\left(-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}\right)^{\frac{1}{3}}=\left(-i\sqrt{\frac{y^3}{27}}\right)^{\frac{1}{3}}=i\sqrt{\frac{y}{3}}.$$

Thus the solutions (A1.6) simplify to

$$\begin{aligned}\lambda_0 &= x\\ \lambda_1 &= -(0)+(-i\sqrt{\frac{y}{3}})(i\sqrt{3})+x=\sqrt{y}+x\\ \lambda_2 &= -(0)+(i\sqrt{\frac{y}{3}})(i\sqrt{3})+x=-\sqrt{y}+x.\end{aligned} \tag{A1.7}$$

Substituting for the original variables from (A1.2) into the solution (A1.7) gives the three roots

$$\lambda_0=\frac{\overline{\omega}}{\zeta},\quad \lambda_1=\frac{\overline{\omega}}{\zeta}+\sqrt{\zeta\omega^2},\quad \lambda_2=\frac{\overline{\omega}}{\zeta}-\sqrt{\zeta\omega^2}. \tag{A1.8}$$

Roots of Equation (4.2.78)

In a similar manner to the previous method, equation (4.2.78), which is a cubic equation in the u_2^2 may be solved. A comparison with (A1.4) allows (4.2.78) to be stated with $\lambda = u_2^2$, and

$$p=\frac{2}{\gamma(2-\gamma)}\left\{2(1-\gamma)(2-\gamma)h_1^+-(4-\gamma)(\zeta_2+h_B)\right\},$$

$$q = \frac{4}{\gamma^2(2-\gamma)}\Big\{ [(1-\gamma)h_1^+ - (\zeta_2 + h_B)]\,[(1-\gamma)h_1^+ - 3\zeta_2 + (2\gamma^2 - 4\gamma - 1)h_B] - (1-\gamma)(\zeta_2 + \gamma h_B)^2\Big\},$$

$$r = \frac{8}{\gamma^2(2-\gamma)}(\zeta_2 + h_B)[(1-\gamma)h_1^+ - (\zeta_2 + h_B)]^2.$$

Using this p, q, and r, notation is introduced as

$$a = \frac{1}{3}(3q - p^2) \text{ and } b = \frac{1}{27}(2p^3 - 9pq + 27r).$$

The solution of equation (4.2.78) is then given by

$$A + B - \frac{p}{3}, \text{ and } -\left(\frac{A+B}{2}\right) \pm \left(\frac{A-B}{2}\right)\sqrt{-3} - \frac{p}{3}, \tag{A1.9}$$

where A and B are given by

$$A = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

and

$$B = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}.$$

The choice of the physical root from (A1.9) is made by comparison with the $\gamma \to 0$ limit given by equation (4.2.79).

Roots of Equation (5.1.27)

Equation (5.1.27) is in the reduced form

$$x^3 + px + q = 0, \tag{A1.10}$$

where p and q are given by

$$p = \frac{1}{2}[2\varpi^2 + \omega^2\gamma^{-1}(\zeta_1 + \zeta_2)], \tag{A1.11}$$

and

$$q = -\frac{1}{2}\varpi\omega^2\gamma^{-1}(\zeta_1 - \zeta_2). \tag{A1.12}$$

The roots x_1, x_2, and x_3 of (A1.10) are given, in general by

$$x_1 = A + B, \quad x_{2,3} = -\frac{A+B}{2} \pm i\frac{A-B}{2}\sqrt{3}, \tag{A1.13}$$

where

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}},$$

and

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \tag{A1.14}$$

Calculating Q from (A1.11) and (A1.12) gives

$$\begin{aligned}
Q &= \left(\frac{1}{3}\right)^3 \left(-\frac{1}{2}\right)^3 \left[(2\varpi^2 + \omega^2\gamma^{-1}(\zeta_1 + \zeta_2)\right]^3 \\
&\quad + \left(\frac{1}{2}\right)^2 \left(-\frac{1}{2}\right)^2 \varpi^2\omega^4\gamma^{-2}(\zeta_2 - \zeta_2)^2 \\
&= \left(\frac{1}{2}\right)^3 \left\{\frac{1}{2}\varpi^2\omega^4\gamma^{-2}(\zeta_1 - \zeta_2)^2 - \left(\frac{1}{3}\right)^3 \left[8\varpi^2 + 12\varpi^4\gamma^{-1}(\zeta_1 + \zeta_2)\right.\right. \\
&\quad \left.\left. + 6\varpi^2\omega^4\gamma^{-2}(\zeta_1 + \zeta_2)^2 + \omega^6\gamma^{-3}(\zeta_1 + \zeta_2)^3\right]\right\}.
\end{aligned}$$

Collecting powers in ϖ then gives this as

$$\begin{aligned}
Q &= -\frac{1}{8}\left\{\frac{8}{27}\varpi^6 + \frac{4}{9}\varpi^4\omega^2\gamma^{-1}(\zeta_1 + \zeta_2) + \varpi^2\omega^4\gamma^{-2}\left[\frac{2}{9}(\zeta_1^2 + 2\zeta_1\zeta_2 + \zeta_2^2)\right.\right. \\
&\quad \left.\left. - \frac{1}{2}(\zeta_1^2 - 2\zeta_1\zeta_2 + \zeta_2^2)\right] + \frac{1}{27}\omega^6\gamma^{-3}(\zeta_1 + \zeta_2)^3\right\} \\
&= -\frac{1}{8}\left\{\frac{8}{27}\varpi^6 + \frac{4}{9}\varpi^4\omega^2\gamma^{-1}(\zeta_1 + \zeta_2) + \varpi^2\omega^4\gamma^{-2}\left(-\frac{5}{18}\zeta_1^2 + \frac{13}{9}\zeta_1\zeta_2 - \frac{5}{18}\zeta_2^2\right)\right. \\
&\quad \left. + \frac{1}{27}\omega^6\gamma^{-3}(\zeta_1 + \zeta_2)^3\right\} \\
&= \left(-\frac{1}{8}\right)\left(\frac{1}{9}\right)\left\{\frac{8}{3}\varpi^6 + 4\varpi^4\omega^2\gamma^{-1}(\zeta_1 + \zeta_2) + \frac{1}{2}\varpi^2\omega^4\gamma^{-2}(-5\zeta_1^2 + 26\zeta_1\zeta_2 - 5\zeta_2^2)\right. \\
&\quad \left. + \frac{1}{3}\omega^6\gamma^{-3}(\zeta_1 + \zeta_2)^3\right\}.
\end{aligned} \tag{A1.15}$$

From equation (A1.15) it can be seen that $Q < 0$ if

$$\begin{aligned}
&\frac{16}{3}\varpi^6 + 8\varpi^4\omega^2\gamma^{-1}(\zeta_1 + \zeta_2) + \frac{2}{3}\omega^6\gamma^{-3}(\zeta_1 + \zeta_2)^3 \\
&\qquad + \frac{1}{2}\varpi^2\omega^4\gamma^{-2}[-5(\zeta_1 - \zeta_2)^2 + 16\zeta_1\zeta_2] > 0,
\end{aligned}$$

which occurs precisely when

$$\frac{16}{3}\varpi^6 + 8\varpi^4\omega^2\gamma^{-1}(\zeta_1+\zeta_2) + \frac{2}{3}\omega^6\gamma^{-3}(\zeta_1+\zeta_2)^3 > \frac{1}{2}\varpi^2\omega^4\gamma^{-2}[5(\zeta_1-\zeta_2)^2 + 16\zeta_1\zeta_2]. \tag{A1.16}$$

In the case $Q < 0$, that is the situation in which (A1.16) holds, there are three different real roots (A1.13). These may be expressed via the trigonometric solutions

$$x_1 = 2\sqrt{-\frac{p}{3}}\cos(\theta/3), \quad x_{2,3} = -2\sqrt{-\frac{p}{3}}\cos[(\theta \pm \pi)/3], \tag{A1.17}$$

where the angle θ is defined by

$$\cos\theta = \frac{-q}{2\sqrt{-(p/3)^3}}. \tag{A1.18}$$

Substituting (A1.11) and (A1.12) into (A1.18) gives $\cos\theta$ as

$$\cos\theta = \frac{\frac{1}{2}\varpi\omega^2\gamma^{-1}(\zeta_1-\zeta_2)}{2\sqrt{\frac{1}{6^3}[2\varpi^2+\omega^2\gamma^{-1}(\zeta_1+\zeta_2)]^3}} = \sqrt{\frac{27}{2}}\frac{\varpi\omega^2\gamma^{-1}(\zeta_1-\zeta_2)}{[2\varpi^2+\omega^2\gamma^{-1}(\zeta_1+\zeta_2)]^3}. \tag{A1.19}$$

In general, for $p < 0$, the relative signs of the roots (A1.17) are ordered as $x_3 < 0 \le x_2 < x_3$. These correspond to the roots (A1.13).

Appendix 2

Derivation of the weak stratification equations

Differentiation of equation (2.4.35) with respect to time is completed in the following steps.

$$\begin{aligned}\frac{\mathrm{d}Q(t)}{\mathrm{d}t} &= \frac{\partial(1-h_2)}{\partial t}u_1 + (1-h_2)\frac{\partial u_1}{\partial t} + \frac{\partial(h_2-h_B)}{\partial t}u_2 + (h_2-h_B)\frac{\partial u_2}{\partial t}\\ &= -\frac{\partial(h_2-h_B)}{\partial t}u_1 + (1-h_2)\frac{\partial u_1}{\partial t} + u_2\frac{\partial(h_2-h_B)}{\partial t} + (h_2-h_B)\frac{\partial u_2}{\partial t}.\end{aligned}$$

The derivatives with respect to time in the right hand side of the last equation are converted into spatial derivatives via equations (2.4.25), (2.4.31), and (2.4.32).

$$\begin{aligned}\frac{\mathrm{d}Q(t)}{\mathrm{d}t} &= (u_2-u_1)\frac{\partial(h_2-h_B)}{\partial t} + (1-h_2)\frac{\partial u_1}{\partial t} + (h_2-h_B)\frac{\partial u_2}{\partial t}\\ &= -(u_2-u_1)\frac{\partial}{\partial x}[(h_2-h_B)u_2] + (1-h_2)[-\frac{\partial}{\partial x}(\frac{1}{2}u_1^2+\eta)]\\ &\quad + (h_2-h_B)[-\frac{\partial}{\partial x}(\frac{1}{2}u_2^2+h_2-h_B+\eta) - \overline{C}_f]\end{aligned}$$

The notation $\overline{C}_f = \kappa C_f \frac{u_2^2}{h_2-h_B} T$ has been used in the last equation. Further simplification now yields the following series of equations, with equation (2.4.35) used in the second step.

$$\begin{aligned}
-\frac{\mathrm{d}Q(t)}{\mathrm{d}t} &= (u_2-u_1)\frac{\partial}{\partial x}[(h_2-h_B)u_2] + (1-h_2)\frac{\partial}{\partial x}(\frac{1}{2}u_1^2+\eta) \\
&\quad + (h_2-h_B)\frac{\partial}{\partial x}(\frac{1}{2}u_2^2+h_2-h_B+\eta) + (h_2-h_B)\overline{C}_f \\
&= u_2\left[\frac{\partial(h_2-h_B)}{\partial x} + (h_2-h_B)\frac{\partial u_2}{\partial x}\right] - u_1\frac{\partial}{\partial x}[Q-(1-h_2))u_1] \\
&\quad + (1-h_2)(u_1\frac{\partial u_1}{\partial x} + \frac{\partial \eta}{\partial x}) + (h_2-h_B)(u_2\frac{\partial u_2}{\partial x} + \frac{\partial(h_2-h_B)}{\partial x} + \frac{\partial \eta}{\partial x}) \\
&\quad + (h_2-h_B)\overline{C}_f \\
&= u_2^2\frac{\partial(h_2-h_B)}{\partial x} + 2(h_2-h_B)u_2\frac{\partial u_2}{\partial x} + 2u_1(1-h_2)\frac{\partial u_1}{\partial x} + u_1^2\frac{\partial(1-h_2)}{\partial x} \\
&\quad + (1-h_2)\frac{\partial \eta}{\partial x} + (h_2-h_B)\frac{\partial \eta}{\partial x} + (h_2-h_B)\left(\frac{\partial(h_2-h_B)}{\partial x} + \overline{C}_f\right) \\
&= \frac{\partial}{\partial x}\left[(h_2-h_B)u_2^2 + (1-h_2)u_1^2\right] + (1-h_B)\frac{\partial \eta}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}(h_2-h_B)^2 \\
&\quad + (h_2-h_B)\overline{C}_f \\
&= \frac{\partial}{\partial x}\left[(h_2-h_B)u_2^2 + (1-h_2)u_1^2 + \frac{1}{2}(h_2-h_B)^2\right] + (1-h_B)\frac{\partial \eta}{\partial x} \\
&\quad + \frac{\partial(1-h_B)}{\partial x}\eta - \frac{\partial(1-h_B)}{\partial x}\eta + (h_2-h_B)\overline{C}_f \\
&= \frac{\partial}{\partial x}\left[(h_2-h_B)u_2^2 + (1-h_2)u_1^2 + \frac{1}{2}(h_2-h_B)^2 + (1-h_B)\eta\right] - \\
&\quad - \eta\frac{\partial(1-h_B)}{\partial x} + (h_2-h_B)\overline{C}_f \\
&= \frac{\partial}{\partial x}\left[(h_2-h_B)u_2^2 + (1-h_2)u_1^2 + \frac{1}{2}(h_2-h_B)^2 + (1-h_B)\eta\right] \\
&\quad + \eta\frac{\mathrm{d}h_B}{\mathrm{d}x} + \kappa C_f u_2^2 T. \qquad \text{(A2.1)}
\end{aligned}$$

In the special case that $C_f = 0$, $h_B = 0$, and $Q = 0$, equation (A2.1) simplifies to

$$\frac{\partial}{\partial x}\left[h_2u_2^2 + (1-h_2)u_1^2 + \frac{1}{2}h_2^2 + \eta\right] = 0. \qquad \text{(A2.2)}$$

Equation A2.2 may be integrated to give another algebraic condition in addition to (2.4.35),

$$\eta = -\left[h_2u_2^2 + (1-h_2)u_1^2 + \frac{1}{2}h_2^2\right]. \qquad \text{(A2.3)}$$

To derive equation (2.4.36), the identity (2.4.35) is used to calculate the associated partial derivatives. Equation (2.4.25) is also used to remove temporal derivatives of h_2. The first step is

$$\begin{aligned}
\frac{\partial}{\partial t}(u_2 - u_1) &= \frac{\partial u_2}{\partial t} - \frac{\partial}{\partial t}\left(\frac{Q-(h_2-h_B)u_2}{1-h_2}\right) \\
&= \frac{\partial u_2}{\partial t} - \frac{1}{(1-h_2)}\frac{\partial}{\partial t}[Q-(h_2-h_B)u_2] \\
&\quad + \frac{[Q-(h_2-h_B)u_2]}{(1-h)^2}\frac{\partial}{\partial t}(1-h_2) \\
&= \frac{\partial u_2}{\partial t} - \frac{1}{(1-h_2)}[\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\partial}{\partial t}[(h_2-h_B)u_2]] - \frac{[Q-(h_2-h_B)u_2]}{(1-h_2)^2}\frac{\partial h_2}{\partial t} \\
&= \frac{\partial u_2}{\partial t} - \frac{Q'}{1-h_2} + \frac{1}{(1-h_2)}[\frac{\partial(h_2-h_B)}{\partial t}u_2 + (h_2-h_B)\frac{\partial u_2}{\partial t}] \\
&\quad + \frac{[Q-(h_2-h_B)u_2]}{(1-h_2)^2}\frac{\partial}{\partial x}[(h_2-h_B)u_2] \\
&= (1+\frac{h_2-h_B}{1-h_2})\frac{\partial u_2}{\partial t} - \frac{Q'}{1-h_2} \\
&\quad + [\frac{-u_2}{1-h_2} + \frac{Q-(h_2-h_B)u_2}{(1-h_2)^2}]\frac{\partial}{\partial x}[(h_2-h_B)u_2] \\
&= \left(\frac{1-h_B}{1-h_2}\right)\frac{\partial u_2}{\partial t} - \frac{Q'}{1-h_2} + \left[\frac{Q-(1-h_B)u_2}{(1-h_2)^2}\right]\frac{\partial}{\partial x}[(h_2-h_B)u_2]
\end{aligned} \tag{A2.4}$$

Similarly to the derivation of (A2.4), another term is derived using identity (2.4.35) exclusively.

$$\begin{aligned}
\frac{\partial u_1}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{Q-(h_2-h_B)u_2}{1-h_2}\right] \\
&= \frac{1}{(1-h_2)^2}\left[-\frac{\partial}{\partial x}[(h_2-h_B)u_2](1-h_2) - [Q-(h_2-h_B)u_2]\frac{\partial}{\partial x}(1-h_2)\right] \\
&= -\frac{1}{1-h_2}\frac{\partial}{\partial x}[(h_2-h_B)u_2] + \frac{[Q-(h_2-h_B)u_2]}{(1-h_2)^2}\frac{\partial}{\partial x}(h_2-h_B+h_B).
\end{aligned} \tag{A2.5}$$

Equations (A2.4) and (A2.5) may now be substituted into the difference of equations (2.4.31) and (2.4.32) to remove η. The resulting manipulations follow.

$$\frac{\partial}{\partial t}(u_2-u_1) + u_2\frac{\partial u_2}{\partial x} + \frac{\partial h_2 - h_B}{\partial x} - u_1\frac{\partial u_1}{\partial x} = -\frac{\mathrm{d}h_B}{\mathrm{d}x} - \overline{C}_f$$

$$
\begin{aligned}
\Longrightarrow & \left(\frac{1-h_B}{1-h_2}\right)\frac{\partial u_2}{\partial t} + \left[\frac{Q-(1-h_B)u_2}{(1-h_2)^2}\right]\frac{\partial}{\partial x}[(h_2-h_B)u_2] - \frac{Q'}{1-h_2} + u_2\frac{\partial u_2}{\partial x} \\
& + \frac{\partial(h_2-h_B)}{\partial x} - \left[\frac{Q-(h_2-h_B)u_2}{1-h_2}\right]\left[\frac{-1}{1-h_2}\frac{\partial}{\partial x}[(h_2-h_B)u_2]\right. \\
& \left. + \frac{[Q-(h_2-h_B)u_2]}{(1-h_2)^2}\frac{\partial}{\partial x}(h_2-h_B) + \frac{[Q-(h_2-h_B)u_2]}{(1-h_2)^2}\frac{\mathrm{d}h_B}{\mathrm{d}x}\right] \\
& = -\frac{\mathrm{d}h_B}{\mathrm{d}x} - \overline{C}_f
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow & \left(\frac{1-h_B}{1-h_2}\right)\frac{\partial u_2}{\partial t} + \left[\frac{Q-(1-h_B)u_2+Q-(h_2-h_B)u_2}{(1-h_2)^2}\right]\frac{\partial}{\partial x}[(h_2-h_B)u_2] \\
& + \left[1-\frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^3}\right]\frac{\partial}{\partial x}(h_2-h_B) + u_2\frac{\partial u_2}{\partial x} \\
& = \frac{Q'}{1-h_2} - \frac{\mathrm{d}h_B}{\mathrm{d}x} + \frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^3}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \overline{C}_f
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow & (1-h_B)\frac{\partial u_2}{\partial t} + \left[\frac{2Q-(1+h_2-2h_B)u_2}{1-h_2}\right]\frac{\partial}{\partial x}[(h_2-h_B)u_2] + (1-h_2)u_2\frac{\partial u_2}{\partial x} \\
& + \left[1-h_2-\frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^2}\right]\frac{\partial}{\partial x}(h_2-h_B) \\
& = Q' - \left[1-h_2-\frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^2}\right]\frac{\mathrm{d}h_B}{\mathrm{d}x} - (1-h_2)\overline{C}_f
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow & (1-h_B)\frac{\partial u_2}{\partial t} + \left\{\left[\frac{2Q-(1+h_2-2h_B)u_2}{1-h_2}\right](h_2-h_B) + (1-h_2)u_2\right\}\frac{\partial u_2}{\partial x} \\
& + \left\{\left[\frac{2Q-(1+h_2-2h_B)u_2}{1-h_2}\right]u_2\right. \\
& \left. + \left[\frac{(1-h_2)^3-[Q-(h_2-h_B)u_2]^2}{(1-h_2)^2}\right]\right\}\frac{\partial}{\partial x}(h_2-h_B) \\
& = Q' - \left[1-h_2-\frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^2}\right]\frac{\mathrm{d}h_B}{\mathrm{d}x} - (1-h_2)\overline{C}_f
\end{aligned}
$$

$$\Longrightarrow (1-h_B)\frac{\partial u_2}{\partial t} + \left\{\frac{2Q(h_2-h_B)+[(1-h_2)^2-(1+h_2-2h_B)(h_2-h_B)]u_2}{1-h_2}\right\}\frac{\partial u_2}{\partial x} + \frac{1}{(1-h_2)^2}\Big\{2Q(1-h_2)u_2-(1+h_2-2h_B)(1-h_2)u_2^2 + (1-h_2)^3-[Q-(h_2-h_B)u_2]^2\Big\}\frac{\partial}{\partial x}(h_2-h_B) = Q' - \left[1-h_2-\frac{[Q-(h_2-h_B)u_2]^2}{(1-h_2)^2}\right]\frac{\mathrm{d}h_B}{\mathrm{d}x} - (1-h_2)\overline{C}_f. \tag{A2.6}$$

In the special case $Q=0$, $h_B=0$, and $\overline{C}_f=0$, equation (A2.6) simplifies to

$$\frac{\partial u_2}{\partial t} + \left\{\frac{0+[(1-h_2)^2-(1+h_2)h_2]u_2}{1-h_2}\right\}\frac{\partial u_2}{\partial x} + \frac{1}{(1-h_2)^2}\{0-(1+h_2)(1-h_2)u_2^2+(1-h_2)^3-(-h_2u_2)^2\}\frac{\partial h_2}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial u_2}{\partial t} + \frac{[1-2h_2+h_2^2-h_2-h_2^2]u_2}{1-h_2}\frac{\partial u_2}{\partial x} + \frac{1}{(1-h_2)^2}[(1-h_2)^3-(1-h_2^2+h_2^2)u_2^2]\frac{\partial h_2}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial u_2}{\partial t} + \left(\frac{1-3h_2}{1-h_2}\right)u_2\frac{\partial u_2}{\partial x} + \left(\frac{(1-h_2)^3-u_2^2}{(1-h_2)^2}\right)\frac{\partial h_2}{\partial x} = 0. \tag{A2.7}$$

Rewriting equation (A2.3) using (2.4.35) for u_1,

$$\eta = -\frac{1}{2}h_2^2 - \frac{h_2u_2^2}{1-h},$$

allows (A2.7) to be written as

$$\frac{\partial u_2}{\partial t} + (u_2 + \frac{\partial \eta}{\partial u_2})\frac{\partial u_2}{\partial x} + (1+\frac{\partial \eta}{\partial h_2})\frac{\partial h_2}{\partial x} = 0,$$

or the conservation form

$$\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u_2^2 + \eta + h_2) = 0. \tag{A2.8}$$

The last equation is in the form stated in Montgomery and Moodie (1998a).

Derivation of the rigid lid equations

In a similar manner to the above calculations, the system form for the rigid lid equations (2.4.82) and (2.4.84) derived in section 2.4.5 may be completed using equation (2.4.86). First, equation (2.4.86) is used to calculate $\frac{\partial u_1}{\partial t}$ and $\frac{\partial u_1}{\partial x}$, where h_1, it must be recalled, is considered to be a constant.

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t}\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)$$

$$\Longrightarrow \frac{\partial u_1}{\partial t} = \frac{Q' - u_2\frac{\partial h_2}{\partial t} - (h_2-h_B)\frac{\partial u_2}{\partial t}}{h_1-h_2} + [Q-(h_2-h_B)u_2]\left(\frac{-1}{(h_1-h_2)^2}\right)\left(-\frac{\partial h_2}{\partial t}\right)$$

$$\Longrightarrow \frac{\partial u_1}{\partial t} = \frac{Q'}{h_1-h_2} - \frac{h_2-h_B}{h_1-h_2}\frac{\partial u_2}{\partial t} + \left(\frac{Q-(h_2-h_B)u_2}{(h_1-h_2)^2} - \frac{u_2}{h_1-h_2}\right)\frac{\partial h_2}{\partial t}$$

$$\Longrightarrow \frac{\partial u_1}{\partial t} = \frac{Q'}{h_1-h_2} - \frac{h_2-h_B}{h_1-h_2}\frac{\partial u_2}{\partial t} + \left(\frac{Q-(h_2-h_B)u_2-(h_1-h_2)u_2}{(h_1-h_2)^2}\right)\frac{\partial h_2}{\partial t}$$

$$\Longrightarrow \frac{\partial u_1}{\partial t} = \frac{Q'}{h_1-h_2} - \left(\frac{h_2-h_B}{h_1-h_2}\right)\frac{\partial u_2}{\partial t} + \left(\frac{Q-(h_1-h_B)u_2}{(h_1-h_2)^2}\right)\frac{\partial h_2}{\partial t}. \qquad \text{(A2.9)}$$

Similarly,

$$\frac{\partial u_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)$$

$$\Longrightarrow \frac{\partial u_1}{\partial x} = \frac{0 - u_2\frac{\partial}{\partial x}(h_2-h_B) - (h_2-h_B)\frac{\partial u_2}{\partial x}}{h_1-h_2} + [Q-(h_2-h_B)u_2]\left(\frac{-1}{(h_1-h_2)^2}\right)\left(-\frac{\partial h_2}{\partial x}\right)$$

$$\Longrightarrow \frac{\partial u_1}{\partial x} = -\left(\frac{h_2-h_B}{h_1-h_2}\right)\frac{\partial u_2}{\partial x} + \left(\frac{Q-(h_2-h_B)u_2}{(h_1-h_2)^2} - \frac{u_2}{h_1-h_2}\right)\frac{\partial h_2}{\partial x} + \frac{u_2}{h_1-h_2}\frac{\mathrm{d}h_B}{\mathrm{d}x} \qquad \text{(A2.10)}$$

Using the expressions (A2.9) and (A2.10) allows equation (2.4.84) to be rearranged using the same notation $\overline{C}_f = \kappa C_f \frac{u_2^2}{h_2 - h_B} T$ (as introduced previously) is used. Substitution into equation (2.4.84) of equation (2.4.86) for u_1 gives

$$\frac{\partial u_2}{\partial t} - (1-\gamma)\frac{\partial u_1}{\partial t} + u_2\frac{\partial u_2}{\partial x} - (1-\gamma)\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)\frac{\partial u_1}{\partial x} + \frac{\partial h_2}{\partial x} = -\overline{C}_f,$$

into which expressions (A2.9) and (A2.10) are now introduced in the following steps, and equation (2.4.82) used to remove $\dfrac{\partial h_2}{\partial t}$.

$$\begin{aligned}
&\frac{\partial u_2}{\partial t} + (1-\gamma)\left[-\frac{Q'}{h_1-h_2} + \left(\frac{h_2-h_B}{h_1-h_2}\right)\frac{\partial u_2}{\partial t} - \left(\frac{Q-(h_1-h_B)u_2}{(h_1-h_2)^2}\right)\frac{\partial h_2}{\partial t}\right] \\
&\quad + u_2\frac{\partial u_2}{\partial x} + (1-\gamma)\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)\left(\frac{h_2-h_B}{h_1-h_2}\right)\frac{\partial u_2}{\partial x} \\
&\quad - (1-\gamma)\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)\left(\frac{Q-(h_2-h_B)u_2}{(h_1-h_2)^2} - \frac{u_2}{h_1-h_2}\right)\frac{\partial h_2}{\partial x} \\
&\quad - (1-\gamma)\left(\frac{Q-(h_2-h_B)u_2}{h_1-h_2}\right)\frac{u_2}{h_1-h_2}\frac{\mathrm{d}h_B}{\mathrm{d}x} + \frac{\partial h_2}{\partial x} = -\overline{C}_f
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow &\left(1 + \frac{(1-\gamma)(h_2-h_B)}{h_1-h_2}\right)\frac{\partial u_2}{\partial t} - \left(\frac{(1-\gamma)[Q-(h_1-h_B)u_2]}{(h_1-h_2)^2}\right)\times \\
&\times \frac{\partial}{\partial x}[-(h_2-h_B)u_2] + \left(u_2 + \frac{(1-\gamma)(h_2-h_B)[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2}\right)\frac{\partial u_2}{\partial x} \\
&+ \left(1 - \frac{(1-\gamma)[Q-(h_2-h_B)u_2]^2}{(h_1-h_2)^3} + \frac{(1-\gamma)u_2[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2}\right)\frac{\partial h_2}{\partial x} \\
&= \frac{(1-\gamma)Q'}{h_1-h_2} + \frac{(1-\gamma)u_2[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2}\frac{\mathrm{d}h_B}{\mathrm{d}x} - \overline{C}_f,
\end{aligned}$$

which becomes

$$\begin{aligned}
&\left(\frac{h_1-h_2+(1-\gamma)(h_2-h_B)}{h_1-h_2}\right)\frac{\partial u_2}{\partial t} + \left(u_2 + \frac{(1-\gamma)(h_2-h_B)[Q-(h_1-h_B)u_2]}{(h_1-h_2)^2}\right. \\
&+ \left.\frac{(1-\gamma)(h_2-h_B)[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2}\right)\frac{\partial u_2}{\partial x} + \left(1 + \frac{(1-\gamma)u_2[Q-(h_1-h_B)u_2]}{(h_1-h_2)^2}\right. \\
&+ \left.\frac{(1-\gamma)u_2[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2} - \frac{(1-\gamma)[Q-(h_2-h_B)u_2]^2}{(h_1-h_2)^3}\right)\frac{\partial h_2}{\partial x} = \frac{(1-\gamma)Q'}{h_1-h_2} \\
&+ \left(\frac{(1-\gamma)u_2[Q-(h_2-h_B)u_2]}{(h_1-h_2)^2} + \frac{(1-\gamma)u_2[Q-(h_1-h_B)u_2]}{(h_1-h_2)^2}\right)\frac{\mathrm{d}h_B}{\mathrm{d}x} - \overline{C}_f.
\end{aligned}$$

Multiplying by $h_1 - h_2$ gives this equation as

$$[h_1 - \gamma h_2 - (1-\gamma)h_B]\frac{\partial u_2}{\partial t} + ((h_1 - h_2)u_2 + \frac{(1-\gamma)(h_2 - h_B)[2Q - (h_1 + h_2 - 2h_B)u_2]}{h_1 - h_2}\Big) \frac{\partial u_2}{\partial x} + (h_1 - h_2 + \frac{(1-\gamma)u_2[2Q - (h_1 + h_2 - 2h_B)u_2]}{h_1 - h_2} - \frac{(1-\gamma)[Q - (h_2 - h_B)u_2]^2}{(h_1 - h_2)^2}\Big) \frac{\partial h_2}{\partial x}$$
$$= (1-\gamma)Q' + \left(\frac{(1-\gamma)u_2[2Q - (h_1 + h_2 - 2h_B)u_2]}{h_1 - h_2}\right)\frac{\mathrm{d}h_B}{\mathrm{d}x} - (h_1 - h_2)\overline{C}_f. \tag{A2.11}$$

The simplification $h_B = 0$, $Q = 0$, $C_f = 0$ gives equation (A2.11) as

$$(h_1 - \gamma h_2)\frac{\partial u_2}{\partial t} + \left((h_1 - h_2)u_2 - \frac{(1-\gamma)h_2 u_2 (h_1 + h_2)}{h_1 - h_2}\right)\frac{\partial u_2}{\partial x} + \left(h_1 - h_2 - \frac{(1-\gamma)u_2^2(h_1 + h_2)}{h_1 - h_2} - \frac{(1-\gamma)h_2^2 u_2^2}{(h_1 - h_2)^2}\right)\frac{\partial h_2}{\partial x} = 0$$

$$\Longrightarrow (h_1 - \gamma h_2)\frac{\partial u_2}{\partial t} + \left(\frac{(h_1 - h_2)^2 - (1-\gamma)(h_1 h_2 + h_2^2)}{h_1 - h_2}\right) u_2 \frac{\partial u_2}{\partial x} + \left(\frac{(h_1 - h_2)^3 - (1-\gamma)u_2^2[(h_1 - h_2)(h_1 + h_2) + h_2^2]}{(h_1 - h_2)^2}\right)\frac{\partial h_2}{\partial x} = 0$$

$$\Longrightarrow (h_1 - \gamma h_2)\frac{\partial u_2}{\partial t} + \left(\frac{h_1^2 - 2h_1 h_2 + h_2^2 - h_1 h_2 - h_2^2 + \gamma(h_1 + h_2)h_2}{h_1 - h_2}\right) u_2 \frac{\partial u_2}{\partial x} + \left(\frac{(h_1 - h_2)^3 - (1-\gamma)u_2^2(h_1^2 - h_2^2 + h_2^2)}{(h_1 - h_2)^2}\right)\frac{\partial h_2}{\partial x} = 0,$$

which becomes

$$\frac{\partial u_2}{\partial t} + \left(\frac{h_1^2 - 3h_1 h_2 + \gamma(h_1 + h_2)h_2}{(h_1 - \gamma h_2)(h_1 - h_2)}\right) u_2 \frac{\partial u_2}{\partial x} + \left(\frac{(h_1 - h_2)^3 - (1-\gamma)h_1^2 u_2^2}{(h_1 - \gamma h_2)(h_1 - h_2)^2}\right)\frac{\partial h_2}{\partial x} = 0. \tag{A2.12}$$

It should be observed that equation (A2.12) reduces to equation (A2.7) with the simplification $h_1 = 1$ and in the limit as $\gamma \to 0$.

Appendix 3

Roots of equation (4.1.6)

Equation (4.1.5) simplifies upon the change of variable $\lambda = \eta + (u_1 + u_2)/2$ as follows.

$$
\begin{aligned}
[\gamma(\lambda - u_2)^2 - \zeta_2][\gamma(\lambda - u_1)^2 - \zeta_1] &= (1-\gamma)\zeta_1\zeta_2 \\
\Rightarrow [\gamma(\eta + \tfrac{1}{2}u_1 - \tfrac{1}{2}u_2)^2 - \zeta_2][\gamma(\eta - \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2)^2 - \zeta_1] &= (1-\gamma)\zeta_1\zeta_2 \\
\Rightarrow [\gamma(\eta + \overline{u}_2) - \zeta_2][\gamma(\eta - \overline{u}_2) - \zeta_1] &= (1-\gamma)\zeta_1\zeta_2,
\end{aligned}
\tag{A3.1}
$$

where the notation $\overline{u}_2 = \frac{1}{2}u_1 - \frac{1}{2}u_2$ has been introduced in equation (A3.1). Multiplying out equation (A3.1) gives a fourth order polynomial in η,

$$
\begin{aligned}
\gamma^2(\eta + \overline{u}_2)^2(\eta - \overline{u}_2)^2 - \gamma\zeta_1(\eta + \overline{u}_2) - \gamma\zeta_2(\eta - \overline{u}_2) + \zeta_1\zeta_2 &= (1-\gamma)\zeta_1\zeta_2 \\
\Rightarrow (\eta^2 - \overline{u}_2^2)^2 - \gamma^{-1}\zeta_1(\eta + \overline{u}_2) - \gamma^{-1}\zeta_2(\eta - \overline{u}_2) + \gamma^{-1}\zeta_1\zeta_2 &= 0 \\
\Rightarrow \eta^4 - 2\overline{u}_2^2\eta^2 - \gamma^{-1}(\zeta_1 + \zeta_2)\eta + \overline{u}_2^4 - \gamma^{-1}\overline{u}_2(\zeta_1 - \zeta_2) + \gamma^{-1}\zeta_1\zeta_2 &= 0.
\end{aligned}
\tag{A3.2}
$$

Equation (A3.2) may now be solved by the standard methods such as the Descartes-Euler Solution (see Korn & Korn 1968 p. 23/24). The solution depends on the roots of the cubic equation for z

$$
\begin{aligned}
z^3 + \frac{1}{2}(-2\overline{u}_2^2)z^2 + \frac{1}{16}[(-2\overline{u}_2^2)^2 - 4(\overline{u}_2^4 - \gamma^{-1}\overline{u}_2(\zeta_1 - \zeta_2) + \gamma^{-1}\zeta_1\zeta_2)]z& \\
- \frac{1}{64}[-\gamma^{-1}(\zeta_1 + \zeta_2)]^2 &= 0
\end{aligned}
$$

$$
\Rightarrow z^3 - \overline{u}_2^2 z^2 + \frac{1}{4}\gamma^{-1}[\overline{u}_2(\zeta_1 - \zeta_2) - \zeta_1\zeta_2]z - \frac{1}{64}\gamma^{-2}(\zeta_1 + \zeta_2)^2 = 0. \tag{A3.3}
$$

The roots of equation (A3.3) may be expressed as

$$
\begin{aligned}
z_1 &= A + B + \frac{\overline{u}_2^2}{3} \\
z_2 &= -\frac{A+B}{2} + \mathrm{i}\frac{A-B}{2}\sqrt{3} + \frac{\overline{u}_2^2}{3} \\
z_3 &= -\frac{A+B}{2} - \mathrm{i}\frac{A-B}{2}\sqrt{3} + \frac{\overline{u}_2^2}{3},
\end{aligned}
\tag{A3.4}
$$

where

$$
A = \left[-\frac{C}{2} + \sqrt{D}\right]^{\frac{1}{3}}, \text{ and } B = \left[-\frac{C}{2} - \sqrt{D}\right]^{\frac{1}{3}}.
$$

The expressions C and D above are given by

$$C = \frac{-\overline{u}_2^2}{3}[2(\frac{-\overline{u}_2^2}{3})^2 - \frac{\gamma^{-1}}{4}\overline{u}_2(\zeta_1 - \zeta_2) + \frac{\gamma^{-1}}{4}\zeta_1\zeta_2] - \frac{\gamma^{-2}}{64}(\zeta_1 + \zeta_2)^2$$
$$= \frac{-\overline{u}_2^2}{3}[\frac{2}{9}\overline{u}_2^4 - \frac{\gamma^{-1}}{4}\overline{u}_2(\zeta_1 - \zeta_2) - \frac{\gamma^{-1}}{4}\zeta_1\zeta_2] - \frac{\gamma^{-2}}{64}(\zeta_1 + \zeta_2)^2$$
$$= -\frac{2}{27}\overline{u}_2^6 + \frac{\gamma^{-1}}{12}\overline{u}_2^3(\zeta_1 - \zeta_2) - \frac{\gamma^{-1}}{12}\overline{u}_2^2\zeta_1\zeta_2 - \frac{\gamma^{-2}}{64}(\zeta_1 + \zeta_2)^2,$$

and

$$D = [-\frac{E^2}{3} + F]^3 + [(\frac{E}{3})^3 - \frac{EF}{6} + \frac{G}{2}]^2$$
$$= -\frac{E^6}{3^3} + \frac{E^4F}{3} - E^2F^2 + F^3 + \frac{E^6}{3^6} - \frac{E^4F}{3^2} + \frac{E^3}{G}3^3 - \frac{EFG}{6} + \frac{E^2F^2}{6^2} + \frac{G^2}{4}$$
$$= -\frac{26}{3^6}E^6 + \frac{2}{3^2}E^4F + \frac{1}{3^3}E^3G - \frac{35}{6^6}E^2F^2 + F^3 - \frac{1}{6}EFG + \frac{1}{4}G^2,$$

where E, F, and G are the coefficients in (A3.3),

$$E = -\overline{u}_2^2$$
$$F = \frac{1}{4}\gamma^{-1}[\overline{u}_2(\zeta_1 - \zeta_2) - \zeta_1\zeta_2]$$
$$G = \frac{1}{64}\gamma^{-2}(\zeta_1 + \zeta_2)^2.$$

The roots of (A3.2) and hence (A3.1) may now be given as the combinations

$$\pm\sqrt{z_1} \pm \sqrt{z_2} \pm \sqrt{z_3}$$

with the signs chosen so that

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3} = -\frac{F}{8}.$$

Expansion Solution of equation (4.1.6)

Substitution of the small parameter expansion (4.1.7) into the characteristic polynomial (4.1.6) is completed as follows.

$$(\lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)})^4 - 2\varepsilon(u_1^{(1)} + u_2^{(1)} + \varepsilon u_1^{(2)} + \varepsilon u_2^{(2)})(\lambda^{(0)} + \varepsilon\lambda^{(1)})^3$$
$$+ [\varepsilon^2(u_1^{(1)} + u_2^{(1)})^2 + 2\varepsilon^2 u_1^{(1)}u_2^{(1)} - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)} + \varepsilon\zeta_1^{(1)} + \varepsilon\zeta_2^{(1)} + \varepsilon^2\zeta_1^{(2)}$$
$$+ \varepsilon^2\zeta_2^{(2)})](\lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)})^2 + 2[\gamma^{-1}\varepsilon(u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})$$
$$+ \gamma^{-1}\varepsilon^2(u_1^{(2)}\zeta_2^{(0)} + u_1^{(1)}\zeta_2^{(1)} + u_2^{(1)}\zeta_1^{(1)} + u_2^{(2)}\zeta_1^{(0)})](\lambda^{(0} + \varepsilon\lambda^{(1)})$$
$$- \gamma^{-1}\varepsilon^2(u_1^{(1)^2}\zeta_2^{(0)} + u_2^{(1)^2}\zeta_1^{(0)}) + \gamma^{-1}[\zeta_1^{(0)}\zeta_2^{(0)} + \varepsilon(\zeta_1^{(0)}\zeta_2^{(1)} + \zeta_1^{(1)}\zeta_2^{(0)})$$
$$+ \varepsilon^2(\zeta_1^{(0)}\zeta_2^{(2)} + \zeta_1^{(1)}\zeta_2^{(1)} + \zeta_1^{(2)}\zeta_2^{(0)})] = O(\varepsilon^3). \qquad \text{(A3.5)}$$

Equation (A3.5) simplifies further by multiplying some of the terms and removing $O(\varepsilon^3)$ coefficients. An intermediate equation is

$$[\lambda^{(0)^2} + 2\varepsilon\lambda^{(0)}\lambda^{(1)} + \varepsilon^2(\lambda^{(1)^2} + 2\lambda^{(0)}\lambda^{(2)})]^2 - 2\varepsilon(u_1^{(1)} + u_2^{(1)} + \varepsilon u_1^{(2)} + \varepsilon u_2^{(2)})\times$$
$$\times(\lambda^{(0)^3} + 3\varepsilon\lambda^{(0)^2}\lambda^{(1)}) + [\varepsilon^2(u_1^{(1)} + u_2^{(1)})^2 + 2\varepsilon^2 u_1^{(1)}u_2^{(1)} - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)}$$
$$+ \varepsilon\zeta_1^{(1)} + \varepsilon\zeta_2^{(1)} + \varepsilon^2\zeta_1^{(2)} + \varepsilon^2\zeta_2^{(2)})][\lambda^{(0)^2} + 2\varepsilon\lambda^{(0)}\lambda^{(1)} + \varepsilon^2(\lambda^{(1)^2} + 2\lambda^{(0)}\lambda^{(2)})]$$
$$+ 2\gamma^{-1}\varepsilon[(u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)}) + \varepsilon(u_1^{(2)}\zeta_2^{(0)} + u_1^{(1)}\zeta_2^{(1)} + u_2^{(1)}\zeta_1^{(1)} + u_2^{(2)}\zeta_1^{(0)})]\times$$
$$\times\lambda^{(0)} + 2\gamma^{-1}\varepsilon^2(u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(1)} - \gamma^{-1}\varepsilon^2(u_1^{(1)^2}\zeta_2^{(0)} + u_2^{(1)^2}\zeta_1^{(0)})$$
$$+ \gamma^{-1}[\zeta_1^{(0)}\zeta_2^{(0)} + \varepsilon(\zeta_1^{(0)}\zeta_2^{(1)} + \zeta_1^{(1)}\zeta_2^{(0)}) + \varepsilon^2(\zeta_1^{(0)}\zeta_2^{(2)} + \zeta_1^{(1)}\zeta_2^{(1)}$$
$$+ \zeta_1^{(2)}\zeta_2^{(0)})] = O(\varepsilon^3),$$

which expands further to

$$\lambda^{(0)^4} + 4\varepsilon\lambda^{(0)^3}\lambda^{(1)} + \varepsilon^2(6\lambda^{(0)^2}\lambda^{(1)^2} + 4\lambda^{(0)^3}\lambda^{(2)}) - 2\varepsilon(u_1^{(1)} + u_2^{(1)})\lambda^{(0)^3}$$
$$- 2\varepsilon^2[(u_1^{(2)} + u_2^{(2)})\lambda^{(0)^3} + 3(u_1^{(1)} + u_2^{(1)})\lambda^{(0)^2}\lambda^{(1)}] - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})\lambda^{(0)^2}$$
$$+ \varepsilon[(\zeta_1^{(1)} + \zeta_2^{(1)})\lambda^{(0)^2} - 2\gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})\lambda^{(0)}\lambda^{(1)}] + \varepsilon^2[(u_1^{(1)^2} + 4u_1^{(1)}u_2^{(1)}$$
$$+ u_2^{(1)^2})\lambda^{(0)^2} + 2(\zeta_1^{(1)} + \zeta_2^{(1)})\lambda^{(0)}\lambda^{(1)} - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})(\lambda^{(1)^2} + 2\lambda^{(0)}\lambda^{(2)})]$$
$$+ 2\gamma^{-1}\varepsilon(u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(0)} + 2\gamma^{-1}\varepsilon^2[(u_1^{(2)}\zeta_2^{(0)} + u_1^{(1)}\zeta_2^{(1)} + u_2^{(1)}\zeta_1^{(1)}$$
$$+ u_2^{(2)}\zeta_1^{(0)})\lambda^{(0)} + (u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(1)}] + \gamma^{-1}\zeta_1^{(0)}\zeta_2^{(0)} + \varepsilon\gamma^{-1}(\zeta_1^{(0)}\zeta_2^{(1)}$$
$$+ \zeta_1^{(1)}\zeta_2^{(0)}) + \varepsilon^2[\zeta_1^{(0)}\zeta_2^{(2)} + \zeta_1^{(1)}\zeta_2^{(1)} + \zeta_1^{(2)}\zeta_2^{(0)}$$
$$- \gamma^{-1}(u_1^{(1)^2}\zeta_2^{(0)} + u_2^{(1)^2}\zeta_1^{(0)})] = O(\varepsilon^3). \qquad \text{(A3.6)}$$

From equation (A3.6), the various coefficients of the orders of ε may be singled out to separate the equation into various order problems. The order 1 problem corresponding to the coefficient of ε^0 is written as

$$\lambda^{(0)^4} - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})\lambda^{(0)^2} + \gamma^{-1}\zeta_1^{(0)}\zeta_2^{(0)} = 0. \qquad \text{(A3.7)}$$

Equation (A3.7) has a solution given by the quadratic formula as

$$\lambda^{(0)^2} = \frac{1}{2}\gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)}) \pm \frac{1}{2}\gamma^{-1}\sqrt{(\zeta_1^{(0)} + \zeta_2^{(0)})^2 - 4\gamma\zeta_1^{(0)}\zeta_2^{(0)}}. \qquad \text{(A3.8)}$$

There are two real, positive and distinct solutions of (A3.8) if and only if

$$(\zeta_1^{(0)} + \zeta_2^{(0)})^2 > 4\gamma\zeta_1^{(0)}\zeta_2^{(0)},$$

or

$$(1-2\gamma)(\zeta_1^{(0)}+\zeta_2^{(0)})^2 > -2\gamma(\zeta_1^{(0)^2}+\zeta_2^{(0)^2}).$$

A sufficient condition for (A3.8) to have four distinct real roots is therefore

$$1-2\gamma>0, \text{ or } \gamma<\frac{1}{2}. \tag{A3.9}$$

Assuming that (A3.9) holds, which is often the case in most physical or laboratory applications where $0\le\gamma\le 2$, the four solutions of (A3.8) can be given by

$$\lambda_{1,2}^{(0)}=\lambda_{+,-}^{(0)}=\left[\frac{1}{2}\gamma^{-1}(\zeta_1^{(0)}+\zeta_2^{(0)})\pm\frac{1}{2}\gamma^{-1}\sqrt{(\zeta_1^{(0)}+\zeta_2^{(0)})^2-4\gamma\zeta_1^{(0)}\zeta_2^{(0)}}\right]^{\frac{1}{2}}, \tag{A3.10}$$

as well as

$$\lambda_3^{(0)}=-\lambda_2^{(0)},\quad \lambda_4^{(0)}=-\lambda_1^{(0)}, \tag{A3.11}$$

which are ordered such that $\lambda_1^{(0)}>\lambda_2^{(0)}>0>\lambda_3^{(0)}>\lambda_4^{(0)}$.

The next condition from (A3.6) is the $O(\varepsilon)$ problem, which may be seen to be

$$4\lambda^{(0)^3}\lambda^{(1)}-2(u_1^{(1)}+u_2^{(1)})\lambda^{(0)^3}+(\zeta_1^{(1)}+\zeta_2^{(1)})\lambda^{(0)^2}-2\gamma^{-1}(\zeta_1^{(0)}+\zeta_2^{(0)})\lambda^{(0)}\lambda^{(1)} \\ +2\gamma^{-1}(u_1^{(1)}\zeta_2^{(0)}+u_2^{(1)}\zeta_1^{(0)})\lambda^{(0)}+\gamma^{-1}(\zeta_1^{(0)}\zeta_2^{(1)}+\zeta_1^{(1)}\zeta_2^{(0)})=0. \tag{A3.12}$$

Equations (A3.12) is linear in $\lambda^{(1)}$, so it may be solved uniquely as

$$\lambda^{(1)}=\{2(u_1^{(1)}+u_2^{(1)})\lambda^{(0)^3}-(\zeta_1^{(1)}+\zeta_2^{(1)})\lambda^{(0)^2}-2\gamma^{-1}(u_1^{(1)}\zeta_2^{(0)}+u_2^{(1)}\zeta_1^{(0)})\lambda^{(0)} \\ -\gamma^{-1}(\zeta_1^{(0)}\zeta_2^{(1)}+\zeta_1^{(1)}\zeta_2^{(0)})\}/\{4\lambda^{(0)^3}-2\gamma^{-1}(\zeta_1^{(0)}+\zeta_2^{(0)})\lambda^{(0)}\}. \tag{A3.13}$$

The fact that the denominator in (A3.13) is nonzero may be seen from its simplification to

$$4\lambda^{(0)}\left[\lambda^{(0)^2}-\frac{1}{2}\gamma^{-1}(\zeta_1^{(0)}+\zeta_2^{(0)})\right]=\pm 2\gamma^{-1}\lambda^{(0)}\sqrt{(\zeta_1^{(0)}+\zeta_2^{(0)})^2-4\gamma\zeta_1^{(0)}\zeta_2^{(0)}}\neq 0,$$

from substitution of equation (A3.8) for $\lambda^{(0)^2}$. Equation (A3.13) may now be used to provide the $O(\varepsilon)$ correction term to $\lambda^{(0)}$ through the four corrections $\lambda_{1,2,3,4}^{(1)}$ corresponding to the four values for $\lambda^{(0)}$ in (A3.10) and (A3.11).

The third equation which may be extracted from (A3.6) is the $O(\varepsilon^2)$ problem, which can written as

$$\begin{aligned}
&6\lambda^{(0)^2}\lambda^{(1)^2} + 4\lambda^{(0)^3}\lambda^{(2)} - 2[(u_1^{(2)} + u_2^{(2)})\lambda^{(0)^3} + 3(u_1^{(1)} + u_2^{(1)})\lambda^{(0)^2}\lambda^{(1)}] \\
&\quad + (u_1^{(1)^2} + 4u_1^{(1)}u_2^{(1)} + u_2^{(1)^2})\lambda^{(0)^2} + 2(\zeta_1^{(1)} + \zeta_2^{(1)})\lambda^{(0)}\lambda^{(1)} \\
&\quad - \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})(\lambda^{(1)^2} + 2\lambda^{(0)}\lambda^{(2)}) + 2\gamma^{-1}[(u_1^{(2)}\zeta_2^{(0)} + u_1^{(1)}\zeta_2^{(1)} + u_2^{(1)}\zeta_1^{(1)} \\
&\quad + u_2^{(1)}\zeta_1^{(0)})\lambda^{(0)} + (u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(1)}] + \zeta_1^{(0)}\zeta_2^{(2)} + \zeta_1^{(1)}\zeta_2^{(1)} + \zeta_1^{(2)}\zeta_2^{(0)} \\
&\quad - \gamma^{-1}(u_1^{(1)^2}\zeta_2^{(0)} + u_2^{(1)^2}\zeta_1^{(0)}) = 0. \qquad \text{(A3.14)}
\end{aligned}$$

Equation (A3.14) is also linear in the unknown variable $\lambda^{(2)}$, and may be therefore solved uniquely as

$$\begin{aligned}
\lambda^{(2)} = \{&-6\lambda^{(0)^2}\lambda^{(1)^2} + 2[(u_1^{(2)} + u_2^{(2)})\lambda^{(0)^3} + 3(u_1^{(1)} + u_2^{(1)})\lambda^{(0)^2}\lambda^{(1)}] \\
&- (u_1^{(1)^2} + 4u_1^{(1)}u_2^{(1)} + u_2^{(1)^2})\lambda^{(0)^2} - 2(\zeta_1^{(1)} + \zeta_2^{(1)})\lambda^{(0)}\lambda^{(1)} \\
&+ \gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})\lambda^{(1)^2} - 2\gamma^{-1}[(u_1^{(2)}\zeta_2^{(0)} + u_1^{(1)}\zeta_2^{(1)} + u_2^{(1)}\zeta_1^{(1)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(0)} \\
&+ (u_1^{(1)}\zeta_2^{(0)} + u_2^{(1)}\zeta_1^{(0)})\lambda^{(1)}] - \zeta_1^{(0)}\zeta_2^{(2)} - \zeta_1^{(1)}\zeta_2^{(1)} - \zeta_1^{(2)}\zeta_2^{(0)} \\
&+ \gamma^{-1}(u_1^{(1)^2}\zeta_2^{(0)} + u_2^{(1)^2}\zeta_1^{(0)})\}/\{4\lambda^{(0)^3} - 2\gamma^{-1}(\zeta_1^{(0)} + \zeta_2^{(0)})\lambda^{(0)}\}. \qquad \text{(A3.15)}
\end{aligned}$$

It can be noted that the denominator of (A3.15) is identical to the denominator of (A3.13) which is nonzero. Hence, four second-order correction terms to the eigenvalues λ can be calculated from the expression (A3.15) to obtain $\lambda_{1,2,3,4}^{(2)}$ by substituting in the appropriate solutions $\lambda_i^{(1)}$ and $\lambda_i^{(0)}$ for $i = 1, 2, 3, 4$.

Appendix 4

The change of variable

$$\mathbf{u}(x,t) = \mathbf{P}(x,t)\widetilde{\mathbf{u}}(x,t), \qquad \text{(A4.1)}$$

for n vectors $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ and $n \times n$ matrix $\mathbf{P}$, may be substituted into equation (3.3.1) as follows.

$$\frac{\partial}{\partial t}(\mathbf{P}\widetilde{\mathbf{u}}) + \mathbf{A}\frac{\partial}{\partial x}(\mathbf{P}\widetilde{\mathbf{u}}) = \mathbf{b},$$

$$\Longrightarrow \frac{\partial \mathbf{P}}{\partial t}\widetilde{\mathbf{u}} + \mathbf{P}\frac{\partial \widetilde{\mathbf{u}}}{\partial t} + \mathbf{A}\left(\frac{\partial \mathbf{P}}{\partial x}\widetilde{\mathbf{u}} + \mathbf{P}\frac{\partial \widetilde{\mathbf{u}}}{\partial x}\right) = \mathbf{b},$$

$$\Longrightarrow \mathbf{P}\frac{\partial \widetilde{\mathbf{u}}}{\partial t} + \mathbf{AP}\frac{\partial \widetilde{\mathbf{u}}}{\partial x} = \mathbf{b} - \left(\frac{\partial \mathbf{P}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{P}}{\partial x}\right)\widetilde{\mathbf{u}},$$

$$\Longrightarrow \frac{\partial \widetilde{\mathbf{u}}}{\partial t} + \mathbf{P}^{-1}\mathbf{AP}\frac{\partial \widetilde{\mathbf{u}}}{\partial x} = \widetilde{\mathbf{b}} + \mathbf{C}(x,t)\widetilde{\mathbf{u}}, \tag{A4.2}$$

where

$$\widetilde{\mathbf{b}}(x,t) = \mathbf{P}^{-1}\mathbf{b}, \tag{A4.3}$$

and

$$\mathbf{C}(x,t) = -\mathbf{P}^{-1}\left(\frac{\partial \mathbf{P}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{P}}{\partial x}\right). \tag{A4.4}$$

The two other calculations from Section 3.3 involve various manipulations using matrix mulitiplication. To confirm equation (3.3.11), consider a $m \times n$ matrix $\mathbf{A}$ with components a_{ij} and a n-vector $\mathbf{x}$ written as a column, and choose an integer p such that the vector $\mathbf{x}$ has components denoted by $\mathbf{x} = (x_1, \dots, x_p, x_{p+1}, \dots, x_n)^T$. Then the product of the matrix $\mathbf{A}$ and the vector $\mathbf{x}$ may be calculated as

$$\begin{aligned}
\mathbf{Ax} &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p \end{bmatrix} + \begin{bmatrix} a_{1p+1}x_{p+1} + \cdots + a_{1n}x_n \\ \vdots \\ a_{mp+1}x_{p+1} + \cdots + a_{mn}x_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} + \begin{bmatrix} a_{1p+1} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{mp+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_{p+1} \\ \vdots \\ x_n \end{bmatrix} \\
&= \mathbf{A}_+\mathbf{x}_+ + \mathbf{A}_-\mathbf{x}_-,
\end{aligned} \tag{A4.5}$$

where the notation from Section 3.3 is used in the last line of equation (A4.5), such that $\mathbf{A}_+$ is a $m \times p$ matrix, and $\mathbf{A}_-$ is a $m \times (n-p)$ matrix.

Now consider a $n \times n$ matrix $\mathbf{B}$ with components b_{ij}. The product matrix $(\mathbf{AB})_+$ is a $m \times p$ matrix which may be observed from the $m \times n$ matrix product

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a}_{1\rightarrow} \cdot \mathbf{b}_{\downarrow 1} & \cdots & \mathbf{a}_{1\rightarrow} \cdot \mathbf{b}_{\downarrow n} \\ \vdots & \cdots & \vdots \\ \mathbf{a}_{m\rightarrow} \cdot \mathbf{b}_{\downarrow 1} & \cdots & \mathbf{a}_{m\rightarrow} \cdot \mathbf{b}_{\downarrow n} \end{bmatrix}.
\end{aligned} \tag{A4.6}$$

From (A4.6) it can be seen that simplifies $(\mathbf{AB})_+$ to $\mathbf{AB}_+$ as shown:

$$\begin{aligned}(\mathbf{AB})_+ &= \begin{bmatrix} \mathbf{a}_{1\rightarrow}\cdot\mathbf{b}_{\downarrow 1} & \cdots & \mathbf{a}_{1\rightarrow}\cdot\mathbf{b}_{\downarrow p} \\ \vdots & \cdots & \vdots \\ \mathbf{a}_{m\rightarrow}\cdot\mathbf{b}_{\downarrow 1} & \cdots & \mathbf{a}_{m\rightarrow}\cdot\mathbf{b}_{\downarrow p} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1p} & \cdots & b_{1n} \\ \vdots & \cdots & \vdots \\ b_{np} & \cdots & b_{nn} \end{bmatrix} \\ &= \mathbf{AB}_+. \end{aligned} \tag{A4.7}$$

Appendix 5

The figures in Chapter 6 were produced by reading lists of data into the program MATLAB, version 5.3.0.10183 (R11). To allow reconstruction of plots, some of the commands necessary for their creation are included here.

For line plots, the data is saved as separate files, for example, the files "x.dat", "y1.dat", and "y2.dat", may contain long lists of data. These are entered into MATLAB as vectors through the commands,

```
> load x.dat
> load y1.dat
> load y2.dat
```

for example, where the MATLAB prompt > is included.

The data vectors can be plotted with the command

```
> plot(x,y1,'-',x,y2,':')
```

which creates two line plots, the x list on the horizontal axis, and y1, y2 on the vertical axis, with the two lines drawn with different symbols. The scale of the diagram may be altered by changing the horizontal and vertical axes by a command such as

```
> axis([-3 3 0 0.2])
```

which gives the domain as $[-3, 3]$ and the range $[0, 0.2]$. Additional commands which further specify the figure by creating a legend, and axis labels with normal TeX fonts. These are, for example,

```
> legend('\alpha=1,','\alpha=100');
> xlabel('x')
> ylabel('\zeta')
```

To create contour plots, similar methods may be employed. A data list (single column) may be read in as

```
> h=load('h10.dat');
```

which needs to be reworked as a square matrix. This may be done through the commands

```
> n=sqrt(length(h))
> H=reshape(h,n,n)
```

With the matrix H defined, the x and y coordinates must be defined via the command

```
> [x,y]=meshgrid(0:0.1:10,0:0.1:10);
```

which creates a matrix of spacing as indicated. Care should be taken so that the matrices x,y, and H are of the same size. Once this is done, the plot may be created via

```
> [A,B]=contour(x,y,H,5)
```

which has the handle [A,B] assigned to it, and will draw 5 contours. This is so that the diagram may be labelled through the command

```
> clabel(A,B)
```

Labelling a vector allows the contour plot to be plotted at a specified level, for example,

```
> v=[0.1 0.1]
> contour(x,y,H,v)
```

To place an additional vector plot, similar commands as the above should be done for the square matrices containing the velocity data. With a grid x1,y1, and data u,v, These may be overlaid on the contour plot via the commands

```
> hold on
> quiver(x,y,u,v)
> hold off
```

Similar commands which specify labels and sizes are used in the 3D plots. One additional command which was of some use was the

```
> axis square
```

command which gives a 1:1 aspect ratio for the diagram.

Curriculum Vitae

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Scholarships, Honours and Awards

1999 NSERC Postdoctoral Fellowship
1998 C.D. (December 1)
Graduate Student Teaching Award, UA
1993 NSERC Postgraduate Scholarship (PGSB) at UA
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Publications

1999 with Moodie, T.B., Shock speeds for discontinuous solutions to the two-layer shallow-water equations, stated as a nonlinear hyperbolic system of forced conservation laws. European Journal of Applied Mathematics (submitted).

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