Conditional Value-at-Risk hedging and its applications

by

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ABSTRACT

Imposing a constraint on the initial wealth may cause the perfect hedging impossible. In this case, the goal of an investor is to find a strategy that minimize the shortfall under a certain measure, which leads to the concept of partial hedging. In this thesis, the shortfall risk is measured by Conditional Value-at-Risk , a coherent risk measure. We investigate Conditional Value-at-Risk based partial hedging and its applications to equity linked life insurance contracts in different markets. First, we consider a Jump-Diffusion market model with transaction costs. A non-linear partial differential equation that an option value process inclusive of transaction costs should satisfy is provided. In addition, we give the closed-form expression of an European call option price in this market and derive the Conditional Value-at-Risk based partial hedging strategy for it. Our results are implemented to obtain target clients' survival probabilities and age of equity-linked life insurance contracts. Secondly, we deal with a defaultable Jump-Diffusion market. The minimal superhedging costs of claims with a zero recovery rate are calculated. Moreover, the Conditional Value-at-Risk minimization problem of such defaultable claims is solved successfully by converting it into a static optimization problem in the corresponding default free market. Furthermore, our method is implemented to derive minimal shortfall and optimal hedging strategies of defaultable equity-linked life insurance contracts whose payoffs are equal to the maximum of two risky assets conditioned by the occurrence of a default event. Thirdly, we take a deep look into the first continuous market model in mathematical finance – the Bachelier model. We introduce two modifications of such a model which are based on SDEs with absorption and reflection. They overcome the drawback of the Bachelier model that is stock prices can take negative values. Comparisons in aspects of perfect hedging price as well as Conditional Value-at-Risk based hedging among the standard Bachelier model, the modified Bachlier model and the Black-Scholes model are executed. In the last part, a risk measure called Range Value-at-Risk that contains Value-at-Risk and Conditional Value-at-Risk as two limiting cases is investigated. We solve the Range Value-at-Risk based partial hedging problem and describe its connections with Value-at-Risk as well as Conditional Value-at-Risk based hedging, which provides a more comprehensive picture about partial hedging. In addition, a numerical example is given to illustrate the application of our methodology in the area of mixed finance/insurance contracts in the market with long-range dependence.

PREFACE

This thesis is based on a collection of published and submitted for publication research articles.

Chapter 2 has been submitted for publication as Alexander Melnikov and Hongxi Wan, "CVaR-hedging and its applications on equity-linked life insurance contracts with transaction costs".

Chapter 3 has been published as Alexander Melnikov and Hongxi Wan, "CVaR hedging in defaultable Jump-Diffusion markets" *Springer Proceedings in Mathematics & Statistics (ISSN: 2194-1009) Operator Theory and Harmonic Analysis: Probability-Analytical Models, Methods and Applications.*

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Chapter 5 has been submitted for publication as Alexander Melnikov and Hongxi Wan, "On RVaR based optimal partial hedging".

In all joint papers, I was responsible for the proofs of results and their applications. Dr. Melnikov has advised on general approaches for the proofs.

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CHAPTER 1

Introduction

1.1 Outline

The focus of this thesis is to investigate Conditional Value-at-Risk based partial hedging and its applications to equity linked life insurance contracts in different markets, for instance, in markets with jumps, in markets with transaction costs, in markets with defaults and in markets with long-range-dependence. The dissertation is divided into five chapters.

Chapter 1 introduces the reader to concepts of risk measures, testing statistical hypotheses theory, the Neyman-Person lemma as well as the connection between the testing statistical hypotheses theory and the partial hedging problem.

Chapter 2 analyzes CVaR based partial hedging subject to a risk constraint in a Jump-Diffusion market model with transaction costs. A non-linear partial differential equation (PDE) that an option value process inclusive of transaction costs should satisfy is provided. In particular, the closed-form expression of an European call option price is given. Meanwhile, the CVaR based partial hedging strategy for a call option is derived explicitly. Both the CVaR hedging price and the weights of the hedging portfolio are based on an adjusted volatility. We obtain estimated values of expected total hedging errors and total transaction costs by a simulation method. Furthermore, our results are implemented to derive target clients' survival probabilities and age of equity-linked life insurance contracts.

In Chapter 3, we deal with a defaultable Jump-Diffusion market. The minimal superhedging costs of claims with a zero recovery rate are derived. Meanwhile, we investigate the CVaR minimization problem with an initial capital constraint which is converted into a static optimization problem in the corresponding default free market and the solution of such a problem is given with the help of the Neyman-Pearson lemma. Furthermore, our method is implemented to derive minimal values of CVaR and optimal hedging strategies of defaultable equity-linked life insurance contracts whose payoffs are equal to the maximum of two risky assets conditioned by the occurrence of a default event.

In Chapter 4, we take a deep look in to the Bachelier model. Mathematically, stock prices described by a classical Bachelier model are sums of a Brownian motion and an absolute continuous drift. Hence, stock prices can take negative values, and financially, it is not appropriate. Such a drawback is overcome by Samuelson who has proposed the exponential transformation and provided the so-called Geometrical Brownian motion. In this chapter, we introduce two additional modifications which are based on SDEs with absorption and reflection. We show that the model with reflection may admit arbitrage, but the model with an appropriate absorption leads to a better model. Comparisons regarding the option price among the standard Bachelier model, the Black-Scholes model and the modified Bachelier model with absorption at zero are executed. Moreover, our main findings are also devoted to the CVaR based partial hedging in the framework of these models.

In Chapter 5, in order to investigate connections between VaR and CVaR, the two most commonly employed risk criteria in financial institutions, we pay our attention to a tail risk measure called Range Value-at-Risk (RVaR) which belongs to a wider class of distortion risk measures and contains VaR and CVaR as important limiting cases. Explicit forms of such RVaR based optimal hedging strategies are derived. In addition, we provide a numerical example to demonstrate how to apply this more comprehensive methodology of partial hedging in the area of mixed finance/insurance contracts in the market with long-range dependence.

1.2 Risk measures

Let us consider a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{S}$ be the set of real-valued random variables denoting loss amounts (a negative value represents gains). A risk measure $\rho(\cdot)$ is a mapping from $\mathcal{S}$ to $\mathbb{R}$. We shall start with important properties of risk measures and would like to give their financial interpretations.

- A risk measure ρ is said to be **monotone** iff for all L_1, $L_2 \in \mathcal{S}$ such that $L_1 \leq L_2$, then it holds that

$$\rho(L_1) \leq \rho(L_2). \tag{1.1}$$

The financial interpretation of monotonicity is that if the final losses of the position L_2 are larger than another position L_1, then L_2 should be riskier than L_1.

- A risk functional ρ is called **positively homogeneous** iff for all $L \in \mathcal{S}$ and $s \in \mathbb{R}^+$, the following equality is satisfied

$$\rho(sL) = s\rho(L). \tag{1.2}$$

Such a property means the risk of a position increases in a linear way with the size of the position.

- A risk functional ρ is called **convex** iff for all L_1, $L_2 \in \mathcal{S}$ and $\lambda \in (0, 1)$, the following inequality holds

$$\rho(\lambda L_1 + (1 - \lambda)L_2) \leq \lambda\rho(L_1) + (1 - \lambda)\rho(L_2). \tag{1.3}$$

In addition, if the above inequality is strict for $L_1 \neq L_2$, then ρ is called strictly convex.

Convexity interprets the diversification effect of two positions. The position λL_1 and $(1 - \lambda)L_2$ may have offset effects on each other and hence the joint risk would be no more than the sum of weighted risk of holding L_1 and L_2 independently.

A similar property is **subadditivity**.

- A risk functional ρ is called **subadditive** iff for all L_1, $L_2 \in \mathcal{S}$, we have

$$\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2). \tag{1.4}$$

(1.4) also indicates the diversification effect of two positions. Indeed, if ρ is positively homogeneous and $\rho(0) < +\infty$, then ρ is subadditive iff ρ is convex.

The last property we would like to mention here is the **translation invariance**.

- A risk measure ρ is said to be **translation invariant** iff for all $L \in \mathcal{S}$ and $s \in \mathbb{R}$, the equation

$$\rho(L + s) = \rho(L) + s, \tag{1.5}$$

  is satisfied.

The financial explanation of (1.5) is that if the amount s of capital is reduced from the position (and hence the loss L is increased by the amount s), then the risk of the position is increased by the same amount.

An important class of risk measures is the **coherent risk measure** which according to the definition in Artzner et al. (1999) is a measure that satisfies **Monotonicity** (1.1), **Positively homogeneity** (1.2), **Subadditivity** (1.4) and **Translation invariance** (1.5).

In the financial industry, Value-at-Risk (VaR) is a commonly used risk measure which is defined as

$$VaR_\alpha(L) = \inf\{v \in \mathbb{R} : P(L > v) \leq 1 - \alpha\}, \tag{1.6}$$

where $\alpha \in (0, 1)$ is the risk level.

However, VaR has some weaknesses. For instance, it is just the upper-α quantile of the loss L and hence it does not capture the property of extreme losses that excess the α level. Also, VaR is not subadditive. As a consequence, in the consultative document by *Basel Committee on Banking Supervision, 2012*, it is recommended to substitute VaR with the metric Conditional Value-at-Risk (CVaR) to determine the required regulatory capital. Hence, in this thesis, we focus on the risk criterion CVaR and will discuss CVaR-based partial hedging strategies.

Definition 1.1. *(Rockafellar and Uryasev 2002) For a risk level $\alpha \in (0,1)$, $CVaR_\alpha$ of the loss L is defined as*

$$CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(L)ds. \tag{1.7}$$

$CVaR_\alpha$ is the mean of the α-tail of the loss distribution and it displays the severity of extreme losses. It is also called as Average VaR (AVaR) or Expected shortfall (ES).

Another prominent property of CVaR is that it is continuous with respect to α regardless of the underlying loss distribution, while VaR may not be continuous regarding to the risk level α and hence may have a jump in its value even if α changes by a small amount.

Proposition 1.2. *(Acerbi and Tasche 2002) For any real-valued random variable L satisfying $E(|L|) < \infty$, the mapping $\alpha \mapsto CVaR_\alpha(L)$ is continuous on $(0,1)$.*

Rockafellar and Uryasev (2002) have indicated that one can derive both VaR and $CVaR$ simultaneously by solving a one-dimensional convex optimization problem. Their results is summarized as the following theorem which contributes a lot to solve our CVaR-based partial hedging problem.

Theorem 1.3. *As a function of z, the function*

$$F_\alpha(L,z) = z + \frac{1}{1-\alpha}E\big((L-z)^+\big), \tag{1.8}$$

where $x^+ = \max(0,x)$, is finite and convex (hence continuous), and it satisfies

$$CVaR_\alpha(L) = \min_{z\in\mathbb{R}} F_\alpha(L,z), \tag{1.9}$$

$$VaR_\alpha(L) = \min\Big\{y : y \in argmin_{z\in\mathbb{R}} F_\alpha(L,z)\Big\}. \tag{1.10}$$

In particular, we have

$$VaR_\alpha(L) \in \text{argmin}_{z\in\mathbb{R}} F_\alpha(L,z), \tag{1.11}$$

$$CVaR_\alpha(L) = F_\alpha(L, VaR_\alpha(L)). \tag{1.12}$$

1.3 Testing statistical hypotheses theory and the generalized Neyman-Pearson lemma

Let $(\Omega, \mathcal{F})$ be a measurable space. Suppose that P is a probability measure in such a space and there are two families of probability measures $\mathbb{Q}^*$, $\mathbb{Q}$ such that any $Q^* \in \mathbb{Q}^*$ and $Q \in \mathbb{Q}$ are absolutely continuous with respect to P. Let us denote

$$Z^{Q^*} = \frac{dQ^*}{dP}, \quad Z^Q = \frac{dQ}{dP},$$

$$\mathbb{Z}_{\mathbb{Q}^*} = \{Z^{Q^*} : Q^* \in \mathbb{Q}^*\}, \tag{1.13}$$

and $E^{Q*}(\cdot)$, $E^Q(\cdot)$ are expectations under measures Q^*, Q respectively.

The problem of test theory is to discriminate the family $\mathbb{Q}^*$, and $\mathbb{Q}$. More specifically, we want to minimize the probability of accepting $\mathbb{Q}^*$ when it is false (probability of type-II-error) subject to the constraint that the probability of rejecting $\mathbb{Q}^*$ when it is true (probability of type-I-error) should be less than a given acceptable significance level $\alpha \in (0, 1)$.

To solve such a problem, let us introduce the concept of *randomized test* φ which is a random variable with values in $[0, 1]$. It can be interpreted as for a given event $\omega \in \Omega$, the hypothesis $\mathbb{Q}^*$ is rejected with the probability $\varphi(\omega)$. Hence, the probability of type-I-error is

$$E^{Q^*}(\varphi) = \int \varphi(\omega) Q^*(d\omega), \tag{1.14}$$

and the power of the test can be expressed as

$$E^Q(\varphi) = \int \varphi(\omega) Q(d\omega). \tag{1.15}$$

With the help of above notations, the problem of statistic hypothesis test is to search for a randomized test $\tilde{\varphi}$ that maximizes the smallest power $\inf_{Q \in \mathbb{Q}} E^Q(\varphi)$ over all randomized test of size no more than a significance level α: $\sup_{Q^* \in \mathbb{Q}^*} E^{Q^*}(\varphi) \leq \alpha$, i.e.,

$$\sup_{\varphi \in \mathcal{R}} \inf_{Q \in \mathbb{Q}} E^Q(\varphi), \tag{1.16}$$

where

$$\mathcal{R} = \{\varphi \in [0,1] : \sup_{Q^* \in \mathbb{Q}^*} E^{Q^*}(\varphi) \leq \alpha\}.$$

We start with the the special case that both $\mathbb{Q}$ and $\mathbb{Q}^*$ contain only one element, which corresponds to a simple hypothesis and hence the problem (1.16) is simplified as

$$\sup_{\varphi \in \mathcal{R}} E^Q(\varphi), \tag{1.17}$$

where

$$\mathcal{R} = \{\varphi \in [0,1] : E^{Q^*}(\varphi) \leq \alpha\}.$$

The form of the optimal randomized test to the problem (1.17) is given by the classical Neyman-Pearson lemma (see Föllmer and Leukert 2000).

Theorem 1.4. *The optimal randomized test $\tilde{\varphi}$ that solves the problem (1.17) for a level $\alpha \in (0,1)$ has the form*

$$\tilde{\varphi} = I_{\{\tilde{a} \cdot Z^{Q^*} < Z^Q\}} + \gamma I_{\{\tilde{a} \cdot Z^{Q^*} = Z^Q\}}, \tag{1.18}$$

where

$$\tilde{a} = \inf\{a \geq 0 : Q^*(a \cdot Z^{Q^*} < Z^Q) \leq \alpha\}, \tag{1.19}$$

and

$$\gamma = \frac{\alpha - Q^*(\tilde{a} \cdot Z^{Q^*} < Z^Q)}{Q^*(\tilde{a} \cdot Z^{Q^*} = Z^Q)}. \tag{1.20}$$

In a more general case, if $\mathbb{Z}_{\mathbb{Q}^*}$ is a compact set, the problem (1.16) is solved with the help of the generalized Neyman-Pearson lemma and such a problem has been discussed in Rudloff and Karatzas (2010). We summarize their main results as following for readers' convenience.

Theorem 1.5. *Let $\mathbb{Z}_{\mathbb{Q}^*}$ be a compact set. Denote the $\sigma-$algebra of all Borel sets of $\mathbb{Z}_{\mathbb{Q}^*}$ with $\mathfrak{B}$, the set of all finite measures on $(\mathbb{Z}_{\mathbb{Q}^*}, \mathfrak{B})$ with Λ_+ and the closure of the convex hull of densities Z_Q with respect to the norm topology in $\mathbb{L}^1$ with $\bar{co}\mathbb{Q}$. Then the optimal randomized test $\tilde{\varphi}$ of the problem (1.16) for any $\alpha \in (0,1)$ has the form*

$$\tilde{\varphi} = \begin{cases} 0, & Z^{\tilde{Q}} < \int_{\mathbb{Q}^*} Z^{Q^*} d\tilde{\lambda}, \\ 1, & Z^{\tilde{Q}} > \int_{\mathbb{Q}^*} Z^{Q^*} d\tilde{\lambda}, \end{cases} \quad (P-a.s.) \tag{1.21}$$

such that

$$E^{Q^*}(\tilde{\varphi}) = \alpha, \quad \tilde{\lambda} - a.s., \tag{1.22}$$

where the pair $(\tilde{Q}, \tilde{\lambda})$ *solves the problem*

$$\min_{Q \in \bar{co}\mathbb{Q}, \lambda \in \Lambda_+} \Big\{ E\big[(Z^Q - \int_{\mathbb{Q}^*} Z^{Q^*} d\lambda)^+\big] + \alpha\lambda(\mathbb{Z}_{\mathbb{Q}^*}) \Big\}. \tag{1.23}$$

Remark 1.6. *The optimal randomized test* $\tilde{\varphi}$ *in Theorem 1.5 can be rewritten as*

$$\tilde{\varphi} = I_{\{Z^{\tilde{Q}} > \int_{\mathbb{Q}^*} Z^{Q^*} d\tilde{\lambda}\}} + \gamma I_{\{Z^{\tilde{Q}} = \int_{\mathbb{Q}^*} Z^{Q^*} d\tilde{\lambda}\}}, \tag{1.24}$$

where the random variable γ *is chosen such that the condition (1.22) is satisfied.*

However, if the set $\mathbb{Z}_{\mathbb{Q}^*}$ is not compact, the problem (1.16) is solved based on a duality approach developed by Kramkov and Schachermayer (1999) and such a method is applied to option hedging in incomplete markets (see, Rudloff 2006 and Xu 2004). We describe the duality approach and its application in incomplete markets in the Appendix A.

1.4 Partial hedging

An European contingent claim H with maturity T is a nonnegative $\mathcal{F}_T$ measurable random variable which can be thought of as a contract or agreement that pays H at the maturity T. Option pricing and hedging are important topics in Mathematical Finance and according to the option pricing theory (see Black and Scholes 1973), in complete markets, for any contingent claim H, there exists a dynamic trading strategy that replicates the payoff of the claim. Such a duplication strategy is called the perfect hedging strategy. In addition, the fair price of H is the expectation of its discounted value with respect to the unique equivalent martingale measure. On the other hand, if a market is incomplete, given sufficient initial wealth, one can construct a replicating portfolio whose value at final time T, in any situation, is no less than the payoff of the claim H. Such a strategy is called a superhedging strategy. The minimal value required to construct the superhedging strategy is said to be the superhedging costs which turn out to be the supremium of expected values of the discounted payoff among all martingale measures and it is also the upper bound of the arbitrage-free price.

However, if the initial capital that can be invested to construct the hedging strategy is less than the fair price (in complete markets) or superhedging costs (in incomplete markets), only a partial hedging strategy can be applied and an investor would bear an intrinsic risk that cannot be hedged away completely. In this case, the aim is to find a hedging strategy that minimizes the losses due to the difference between the claim and the hedging portfolio at time T measured by a suitable risk criterion.

Let

$$S_t^0 = e^{rt}, \quad t \in [0, T],$$

be the value process of a riskless asset where the constant r is the risk free interest rate. And $S = (S_t)_{t \in [0,T]}$ denotes the price process of an underlying risky asset on a filtered complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \mathcal{F}_T$ and let $\mathbb{P}^*$ be the set containing all equivalent martingale measures with respect to P. We denote the Radon-Nikodym derivative of any equivalent measure Q with respect to P by $Z^Q = \frac{dQ}{dP}$ and the set $\mathbb{Z}_{\mathbb{P}^*} = \{Z^{P^*} : P^* \in \mathbb{P}^*\}$ contains densities of all equivalent martingale measures.

A $\mathbb{F}$-strategy is a $\mathbb{F}$-predictable process $\pi := (\pi_t^0, \pi_t^1)_{t \in [0,T]}$ such that

$$\int_0^T |\pi_t^0| dt < \infty, \int_0^T (\pi_t^1 S_t)^2 dt < \infty, P - a.s,$$

where π^0 and π^1 represent units hold in the risk-free and the risky asset correspondingly. Hence, the value process of the strategy π at time $t \in [0, T]$ is

$$V_t = \pi_t^0 S_t^0 + \pi_t^1 S_t. \tag{1.25}$$

In addition, for a given initial value $v \geq 0$, a trading strategy is called self-financing if its value process satisfies

$$V_t = v + \int_0^t \pi_u^0 dS_u^0 + \int_0^t \pi_u^1 dS_u, \ \forall t \in [0, T], \tag{1.26}$$

and it is called self-financing admissible if such a value process also satisfies

$$V_t \geq 0, \ \forall t \in [0, T].$$

We denote the set of all admissible self-financing strategies with an initial value v as $\mathcal{A}(v)$.

For a contingent claim H, the process

$$\tilde{U}_t = U_t e^{-rt} = \underset{P^* \in \mathbb{P}^*}{\text{ess sup}} E^{P^*}(e^{-rT} H | \mathcal{F}_t), t \in [0, T], \tag{1.27}$$

is a supermartingale with respect to any $P^* \in \mathbb{P}^*$ and represents the discounted value process of the minimal superhedging strategy of the claim H. We assume such superhedging costs are finite, i.e.,

$$U_0 = \sup_{\tilde{P^*} \in \mathbb{P}^*} E^{P^*}(e^{-rT} H) < \infty. \tag{1.28}$$

According to the optional decomposition theorem (see Kramkov 1996, El Karoui and Quenez 1995), there is an admissible strategy (U_0, π) and a discounted optional consumption process C with $C_0 = 0$ such that

$$\tilde{U}_t = U_0 + \int_0^t \pi_u^1 d\tilde{S}_u - C_t, \tag{1.29}$$

where $\tilde{S}_t = e^{-rt} S_t$ is the discounted value of the risky asset.

Remark 1.7. *In a complete market, the equivalent martingale measure is unique, and* (U_0, π) *is the replication strategy of* H*, i.e.,*

$$V_t e^{-rt} = E^{P^*}(e^{-rT} H | \mathcal{F}_t) = U_0 + \int_0^t \pi_u^1 d\tilde{S}_u. \tag{1.30}$$

If a hedger allocates capitals v_0 that are less than the minimum superhedging costs U_0, then there is a possibility of shortfall characterized by $L = H - V_T$. In this case, we look for a self-financing admissible strategy (v, π) with $v \le v_0 < U_0$ that minimizes the risk of losses measured by $\rho\big(L\big)$, i.e.,

$$\min_{(v,\pi) \in \mathcal{A}_0} \rho\big(H - V_T\big), \tag{1.31}$$

where $\mathcal{A}_0 = \{(v, \pi) | (v, \pi) \in \mathcal{A}(v), v \le v_0\}$ is the set of self-financing admissible strategies with the initial hedging capital no more than v_0.

Remark 1.8. *If* $\rho(\cdot)$ *is chosen such that*

$$\rho(L) = E\big[l((H - V_T)^+)\big], \tag{1.32}$$

where l is an increasing convex function defined on $[0, +\infty)$ and satisfies $l(0) = 0$. Then the problem (1.31) is the efficient hedging problem described in Föllmer and Leukert (2000).

For a monotonic risk measure ρ, Rudloff (2006) have indicated that the optimal partial hedging problem

$$\min_{(v,\pi)\in\mathcal{A}_0} \rho\big((H - V_T)^+\big) \tag{1.33}$$

can be converted to a static optimization problem of finding an optimal randomized test $\tilde{\varphi}$ that solves

$$\min_{\varphi\in\mathcal{R}} \rho\big((1-\varphi)H\big), \tag{1.34}$$

where

$$\mathcal{R} = \Big\{\varphi : \Omega \to [0,1] |\ \mathcal{F}_T - \text{measurable},\ \sup_{P^*\in\mathbb{P}^*} E^{P^*}\big(e^{-rT}H\varphi\big) \leq v_0\Big\}. \tag{1.35}$$

Theorem 1.9. *Assume the risk measure ρ is monotonic and let $\tilde{\varphi}$ be a solution of the problem (1.34). Then the admissible strategy $(v_0, \tilde{\pi})$ determined by the optional decomposition of the modified claim $\tilde{\varphi}H$ solves the optimal partial hedging problem (1.33) and it holds*

$$\min_{\varphi\in\mathcal{R}} \rho\big((1-\varphi)H\big) = \min_{(v,\pi)\in\mathcal{A}_0} \rho\big((H - V_T)^+\big). \tag{1.36}$$

Proof. For any admissible strategy (v, π) with $v \leq v_0$, let us define the success ratio of it as

$$\varphi = \varphi_{(v,\pi)} = I_{\{V_T \geq H\}} + \frac{V_T}{H} I_{\{V_T < H\}}. \tag{1.37}$$

It is clear that $\varphi H = V_T \wedge H$ and hence the shortfall can be rewritten as

$$(H - V_T)^+ = H - V_T \wedge H = (1 - \varphi)H. \tag{1.38}$$

For any $P^* \in \mathbb{P}^*$, according to the supermartingale property of the discounted value process, we have

$$E^{P^*}(e^{-rT}\varphi H) \leq E^{P^*}(e^{-rT}V_T) \leq v \leq v_0, \tag{1.39}$$

and as a consequence, we have $\varphi \in \mathcal{R}$, which implies

$$\rho\big((H - V_T)^+\big) = \rho\big((1-\varphi)H\big) \geq \rho\big((1-\tilde{\varphi})H\big), \tag{1.40}$$

since $\tilde{\varphi}$ is the optimal randomized test that solves (1.34).

On the other hand, let $\tilde{\pi}$ be the superhedging strategy of the modified claim $\tilde{\varphi}H$ determined by the optional decomposition theorem, i.e.,

$$\begin{aligned}\tilde{U}_t =& \operatorname*{ess\,sup}_{P^*\in\mathbb{P}^*} E^{P^*}(e^{-rT}\tilde{\varphi}H|\mathcal{F}_t)\\ =& \tilde{U}_0 + \int_0^t \tilde{\pi}_u^1 d\tilde{S}_u - \tilde{C}_t,\end{aligned} \tag{1.41}$$

where $\tilde{U}_0 = \sup_{P^*\in\mathbb{P}^*} E^{P^*}\big(e^{-rT}H\tilde{\varphi}\big) \leq v_0$.

The strategy $(\tilde{U}_0, \tilde{\pi})$ is admissible because of the following relationships

$$e^{-rt}\tilde{V}_t \geq e^{-rt}\tilde{V}_t - \tilde{C}_t = \operatorname*{ess\,sup}_{P^*\in\mathbb{P}^*} E^{P^*}(e^{-rT}\tilde{\varphi}H|\mathcal{F}_t) \geq 0. \tag{1.42}$$

In addition, its success ratio $\varphi_{(\tilde{U}_0,\tilde{\pi})}$ satisfies

$$\varphi_{(\tilde{U}_0,\tilde{\pi})}H = \tilde{V}_T \wedge H \geq \tilde{\varphi}H. \tag{1.43}$$

Since ρ is monotonic, we have

$$\rho\big((1-\varphi_{(\tilde{U}_0,\tilde{\pi})})H\big) \leq \rho\big((1-\tilde{\varphi})H\big). \tag{1.44}$$

Combing (1.40) and (1.44), we conclude that $(\tilde{U}_0, \tilde{\pi})$ is the optimal strategy and

$$\begin{aligned}&\min_{(v,\pi)\in\mathcal{A}_0} \rho\big((H-V_T)^+\big) = \rho\big((H-\tilde{V}_T)^+\big)\\ &= \rho\big((1-\varphi_{(\tilde{U}_0,\tilde{\pi})})H\big) = \rho\big((1-\tilde{\varphi})H\big)\\ &= \min_{\varphi\in\mathcal{R}} \rho\big((1-\varphi)H\big).\end{aligned} \tag{1.45}$$

Moreover, $\tilde{\varphi}$ coincides with the successful ratio of the optimal hedging strategy. □

As a consequence, the dynamic optimization problem (1.31) can be solved in two steps:

- Static optimization problem: Find the optimal randomized test $\tilde{\varphi}$ that solves (1.34);
- Replication problem: Find a superhedging strategy of the modified claim $\tilde{\varphi}H$.

Remark 1.10. *As mentioned in Rudloff (2006), when $\tilde{\varphi}$ can be solved with the help of the Neyman-Pearson lemma directly, one can see that $\tilde{U}_0 = v_0$ since the optimal test $\tilde{\varphi}$ attains the bound v_0 in (1.35).*

Example 1.11. *Let us take $\rho((H - V_T)^+) = E((H - V_T)^+)$ as an example and consider the case that $\mathbb{Z}_{\mathbb{P}^*} = \{P^*\}$ (a complete market). Theorem 1.9 shows that the problem*

$$\min_{(v,\pi)\in\mathcal{A}_0} E\big((H - V_T)^+\big), \tag{1.46}$$

is equivalent to

$$\max_{\varphi\in\mathcal{R}} E\big(\varphi H\big). \tag{1.47}$$

Define two measures Q and Q^* as

$$\frac{dQ}{dP} = \frac{H}{E(H)}, \quad \frac{dQ^*}{dP^*} = \frac{H}{E^{P^*}(H)}, \tag{1.48}$$

and then (1.47) becomes

$$\max_{\varphi\in\mathcal{R}} E^Q(\varphi), \tag{1.49}$$

subject to the constraint

$$E^{Q^*}(\varphi) \le \frac{e^{rT}v_0}{E^{P^*}(H)}. \tag{1.50}$$

With the help of Theorem 1.4, the optimal randomized test $\tilde{\varphi}$ is given by the Neyman-Pearson lemma and has the form

$$\tilde{\varphi} = I_{\{\frac{dP}{dP^*} > \tilde{a}\}} + \gamma I_{\{\frac{dP}{dP^*} = \tilde{a}\}}, \tag{1.51}$$

where

$$\tilde{a} = \inf\{a \ge 0 : E^{P^*}(HI_{\{\frac{dP}{dP^*} > a\}}) \le v_0 e^{rT}\}, \tag{1.52}$$

and

$$\gamma = \frac{v_0 e^{rT} - E^{P^*}(HI_{\{\frac{dP}{dP^*} > \tilde{a}\}})}{E^{P^*}(HI_{\{\frac{dP}{dP^*} = \tilde{a}\}})}. \tag{1.53}$$

Example 1.11 lists the solution of the efficient hedging problem for the special liner loss function $l(x) = x$ (see Föllmer and Leukert 2000) and plays an important role in following chapters to derive CVaR based hedging strategies.

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CHAPTER 2

CVaR-hedging and its applications to equity-linked life insurance contracts with transaction costs

2.1 Introduction

Since the famous paper of Black and Scholes (1973) , perfect hedging is a standard and powerful way to the pricing of options. However, when perfect hedging is impossible (for example, when the initial wealth is not sufficient), a partial hedging strategy that minimizes the shortfall risk should be considered and applied to the pricing of contracts. Quantile hedging and efficient hedging are most studied partial hedging methods. In Föllmer and Leukert (1999) (2000), they provided explicit solutions for the quantile hedging problem and the efficient hedging problem in complete markets by the Neyman-Pearson lemma. Another reason that makes partial hedging interesting is that although it has some downside risk, it indeed provides opportunities for companies to gain benefits. This is important for some financial institutions, such as insurance companies, since they exploit risk to make profits. Recent book of Melnikov and Nosrati (2017) discussed several partial hedging methods and their applications in pricing and hedging insurance contracts. In this chapter, a coherent risk criterion called Conditional Value-at-Risk (CVaR) is employed to measure the shortfall which provides information about the average loss that exceeds the Value-at-Risk (VaR) level.

The most developed theory of partial hedging deals with financial markets that are transaction costs free. However, transaction costs are common in the real world and in general cannot be ignored. There is a considerable amount of papers devoted to option pricing and hedging with transaction costs. Leland (1985) has indicated that with a modified volatility, one can construct a hedging strategy to replicate the payoff of an European call option almost surely as the length of the revision period tends to zero. The idea of Leland was to include the expected transaction costs in the costs of a duplication portfolio. Later, Hoggard et al. (1994) extended Leland's method to the pricing of standard options consist of a single asset. Mocioalca (2007) considered options on several assets and derived a non-linear PDE that a modified option value process should satisfy. Moreover, similar to the Leland's hedging volatility, a volatility adjustment is also introduced in the utility-based hedging strategy which was first designed by Hodges and Neuberger (1989). In addition, Merton (1990) started to study option pricing in a two-period Binomial market model with transaction costs. Boyle and Vorst (1992) extended Merton's analysis to several periods and constructed a hedging strategy for a call option. Results of Merton, Boyle and Vorst were unified on the paper by Melnikov and Petrachenko (2005). Recently, Melnikov and Tong (2014) considered quantile hedging with transaction costs. Leland's adjusted hedging volatility was utilized in their paper to rebalance the portfolio. Also, they discussed total hedging errors and total transaction costs of the quantile hedging method.

All of the above mentioned papers considered option pricing and hedging in the Black-Scholes model. However, growing number of evidences show that there are jumps in stock prices when some significant financial or political announcements published, so that pure diffusion models are not accurate enough to represent real life assets' dynamics and jump components should be taken into consideration. A Jump-Diffusion model for financial needs was proposed by Merton (1976), and now there is a long list of references on this subject. For instance, Cox and Ross (1976) provided ways to value options in markets with different jump components. Amin (1993) focused on the option valuation in discrete time. Mocioalca (2007) as well as Zhou et al. (2015) worked on option pricing in markets with only one risky asset following the Jump-Diffusion model and in the case that transaction costs existed.

Our main objective and contributions of this chapter is to implement a coherent risk measure named CVaR and extend partial hedging method in the complete Black-Scholes market to a market with jumps as well as transaction costs. This new market model is a more precise representation of the real life financial market and CVaR is recommended by the Basel committee as the risk measure applied in financial institutions. Hence, this chapter describes a more practical and comprehensive implementation of partial hedging. This chapter is organized as follows. In Section 2.2, we start with a transaction costs free Jump-Diffusion market. With the help of optimal CVaR-based hedging techniques developed by Melnikov and Smirnov (2012), the optimal hedging strategy that minimizes hedging costs while still satisfies a CVaR constraint for an European call option is derived explicitly. In Section 2.3, we take proportional transaction fees into consideration and show that the adjusted value of an option should satisfy a non-linear PDE. In particular, an explicit formula for the modified price of an call option is given and the CVaR hedging costs as well as the weights of the hedging portfolio of such a call option in this market are recalculated with an adjusted hedging volatility. Further, we investigated the estimated present values of total hedging errors and total transaction costs by a simulation method. In Section 2.4, a numerical example is given to illustrate the application of our CVaR-based partial hedging in finding target clients' survival probabilities and age for life insurance contracts. Section 2.5 gives a conclusion for the chapter.

2.2 CVaR hedging in the complete Jump-Diffusion market model

2.2.1 Model setup

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a standard stochastic basis. Consider a financial market with one riskless asset $(S_t^0)_{t\in[0,T]}$ and two risky assets, $(S_t^1)_{t\in[0,T]}$, $(S_t^2)_{t\in[0,T]}$, described by a two factor

Jump-Diffusion model:

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1,$$
$$dS_t^i = S_{t^-}^i(\mu_i dt + \sigma_i dW_t - \upsilon_i dN_t), \;\; i = 1, 2, \tag{2.1}$$

where $r \geq 0$ is the risk-free interest rate. Constants $\mu_i \in \mathbb{R}$, $\sigma_i > 0$, $\upsilon_i < 1, (i = 1, 2)$. W and N are the independent Wiener process and the Poisson process that generate the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, i.e., $\mathcal{F}_t = \sigma(W_s, N_s, s \leq t)$, and we assume $\mathcal{F} = \mathcal{F}_T$.

In the absence of transaction costs, the market (2.1) is complete if the following conditions are fulfilled (see Melnikov and Skornyakova 2005):

$$\frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\upsilon_1\sigma_2 - \upsilon_2\sigma_1} > 0, \quad \upsilon_1\sigma_2 - \upsilon_2\sigma_1 \neq 0.$$

Such a market admits an unique martingale measure P^* with the following local density:

$$Z_t = \frac{dP^*}{dP} \mid_{\mathcal{F}_t} = \exp\big[\alpha^* W_t - \frac{\alpha^{*2}}{2}t + (\lambda - \lambda^*)t + (\ln\lambda^* - \ln\lambda)N_t\big], \tag{2.2}$$

where λ is the intensity parameter of the Poisson process $(N_t)_{t\geq 0}$ under the measure P, and the pair (α^*, λ^*) satisfies:

$$\lambda^* = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\upsilon_1\sigma_2 - \upsilon_2\sigma_1}, \;\; \alpha^* = \frac{(\mu_1 - r)\upsilon_2 - (\mu_2 - r)\upsilon_1}{\upsilon_1\sigma_2 - \upsilon_2\sigma_1}. \tag{2.3}$$

According to the Girsanov theorem, $W_t^* = W_t - \alpha^* t$ and N_t are again the independent Wiener process and Poisson process (with intensity λ^*) under the martingale measure P^*. We denote the expectation under the measure P^* as $E^*()$.

The exponential representation of $S_t^i, \;\; i = 1, 2$ is:

$$S_t^i = S_0^i \exp\big(\sigma_i W_t + (\mu_i - \frac{1}{2}{\sigma_i}^2)t + N_t \ln(1 - \upsilon_i)\big)$$
$$= S_0^i \exp\big(\sigma_i W_t^* + (r + \upsilon_i\lambda^* - \frac{1}{2}{\sigma_i}^2)t + N_t \ln(1 - \upsilon_i)\big). \tag{2.4}$$

A $\mathbb{F}$-strategy is a $\mathbb{F}$-predictable process $\pi := (\pi_t^0, \pi_t^1, \pi_t^2)_{t\in[0,T]}$ such that

$$\int_0^T |\pi_t^0| dt < \infty, \;\; \int_0^T (\pi_t^i S_t^i)^2 dt < \infty, P - a.s \;\; (i = 1, 2).$$

At time $t \in [0, T]$, the value process corresponding to the strategy π is

$$V_t = \pi_t^0 S_t^0 + \pi_t^1 S_t^1 + \pi_t^2 S_t^2. \tag{2.5}$$

Moreover, for a given initial value $v \geq 0$, the trading strategy is called self-financing admissible if its value process satisfies

$$V_t = v + \int_0^t \pi_u^0 dS_u^0 + \int_0^t \pi_u^1 dS_u^1 + \int_0^t \pi_u^2 dS_u^2, \tag{2.6}$$

and

$$V_t \geq 0,\ \forall t \in [0, T].$$

We denote the set of all admissible self-financing strategies with an initial value v as $\mathcal{A}(v)$.

According to the option pricing theory, in the complete market (2.1), the fair price of any contingent claim $H(S_T^1, S_T^2)$ is defined as $V(0) = E^*\big(e^{-rT} H(S_T^1, S_T^2)\big)$. In addition, this claim can be hedged perfectly if the initial wealth is no less than $V(0)$. The self-financing replication strategy $\pi_t = (\pi_t^0, \pi_t^1, \pi_t^2)_{t\in[0,T]}$ for it can be determined from:

$$\pi_t^1 S_t^1 = \frac{\big(V_{S^1} S_t^1 \sigma_1 + V_{S^2} S_t^2 \sigma_2\big)\upsilon_2 + \big(V(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - V(S_{t^-}^1, S_{t^-}^2, t)\big)\sigma_2}{\sigma_1\upsilon_2 - \sigma_2\upsilon_1}, \tag{2.7}$$

$$\pi_t^2 S_t^2 = \frac{\big(V_{S^1} S_t^1 \sigma_1 + V_{S^2} S_t^2 \sigma_2\big)\upsilon_1 + \big(V(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - V(S_{t^-}^1, S_{t^-}^2, t)\big)\sigma_1}{\sigma_2\upsilon_1 - \sigma_1\upsilon_2}, \tag{2.8}$$

$$\pi_t^0 S_t^0 = V(S_t^1, S_t^2, t) - \pi_t^1 S_t^1 - \pi_t^2 S_t^2, \tag{2.9}$$

where $V(t) = V(S_t^1, S_t^2, t) = e^{-r(T-t)} E^*\big(H(S_T^1, S_T^2) \mid \mathcal{F}_t\big)$ is the value of $H(S_T^1, S_T^2)$ at time t, and

$$V_{S^1} = \frac{\partial V(S_t^1, S_t^2, t)}{\partial S^1}, \quad V_{S^2} = \frac{\partial V(S_t^1, S_t^2 . t)}{\partial S^2}.$$

Proof. Denote the value of the duplication strategy π of a claim $H(S_T^1, S_T^2)$ at time t as:

$$V_t^\pi = \pi_t^0 S_t^0 + \pi_t^1 S_t^1 + \pi_t^2 S_t^2. \tag{2.10}$$

According to the definition of the self-financing replication strategy, the value process of portfolio π should satisfy:

$$\begin{cases} dV_t^\pi = \pi_t^0 dS_t^0 + \pi_t^1 dS_t^1 + \pi_t^2 dS_t^2, \\ V_T^\pi = H(S_t^1, S_T^2). \end{cases} \tag{2.11}$$

Denote $W_t^* = W_t - \alpha^* t$, and $M_t = N_t - \lambda^* t$ (the martingale associated with the Poisson process), then dynamics of assets S_t^1 and S_t^2 can be represented as

$$dS_t^i = S_{t-}^i (rdt + \sigma_i dW_t^* - \upsilon_i dM_t).$$

Then, the first equation of (2.11) becomes:

$$dV_t^\pi = r(\pi_t^1 S_{t-}^1 + \pi_t^2 S_{t-}^2 + \pi_t^0 S_t^0)dt + (\pi_t^1 \sigma_1 S_{t-}^1 + \pi_t^2 \sigma_2 S_{t-}^2)dW_t^* - (\pi_t^1 \upsilon_1 S_{t-}^1 + \pi_t^2 \upsilon_2 S_{t-}^2)dM_t \quad (2.12)$$

On the other hand, by Itô formula, the value process $V(t)$ of $H(S_T^1, S_T^2)$ should satisfy:

$$\begin{aligned} dV(t) = \big[& V_t + \frac{1}{2} V_{S^1 S^1} {S_t^1}^2 {\sigma_1}^2 + \frac{1}{2} V_{S^2 S^2} {S_t^2}^2 {\sigma_2}^2 + V_{S^1 S^2} S_t^1 S_t^2 \sigma_1 \sigma_2 + r(V_{s^1} S_t^1 + V_{s^2} S_t^2) \\ & + \big(V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2), t) - V(S_{t-}^1, S_{t-}^2, t) + V_{s^1} S_t^1 \upsilon_1 + V_{s^2} S_t^2 \upsilon_2 \big) \lambda^* \big] dt \\ & + (V_{s^1} S_t^1 \sigma_1 + V_{s^2} S_t^2 \sigma_2) dW_t^* + \big(V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2), t) - V(S_{t-}^1, S_{t-}^2, t) \big) dM_t. \end{aligned} \quad (2.13)$$

Since π is the replication strategy, it satisfies $V_t^\pi = V(t)$, $\forall t \le T$. Comparing (2.12) and (2.13), we have

$$\begin{cases} \pi_t^1 \sigma_1 S_{t-}^1 + \pi_t^2 \sigma_2 S_{t-}^2 = V_{s^1} S_t^1 \sigma_1 + V_{s^2} S_t^2 \sigma_2, \\ \pi_t^1 \upsilon_1 S_{t-}^1 + \pi_t^2 \upsilon_2 S_{t-}^2 = -\big(V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2), t) - V(S_{t-}^1, S_{t-}^2, t) \big), \end{cases} \quad (2.14)$$

Solving the above linear system (2.14), we arrive to (2.7) and (2.8).

Moreover, since the discounted value process is also a martingale under P^*, so the drift term of (2.13) should be equal to $rV(t)$, which yields

$$\begin{aligned} & V_t + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 V_{S^i S^j} S_t^i S_t^j \sigma_i \sigma_j + (r + \upsilon_1 \lambda^*) V_{s^1} S_t^1 + (r + \upsilon_2 \lambda^*) V_{s^2} S_t^2 \\ & + \big(V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2), t) - V(S_{t-}^1, S_{t-}^2, t) \big) \lambda^* - rV = 0. \end{aligned} \quad (2.15)$$

□

Costs of perfect hedging, however, are often too high for investors, and hence the partial hedging that allows investors to spend a smaller amount of initial capital while still control the hedging loss under a certain level is more commonly implemented. In this chapter, we focus on CVaR-based partial hedging since CVaR is an advanced and widely applied risk criterion in financial institutions.

2.2.2 Optimal CVaR hedging strategy

We assume that a hedger is exposed to a future obligation $H = H(S_T^1, S_T^2)$ at maturity time T. Meanwhile, the hedger constructs a self-financing hedging portfolio π and hence $L = H - V_T$ can be seen as a $\mathcal{F}_T$-measurable random variable that characterizes the hedging loss. The CVaR of the loss L at a confidence level $\alpha \in (0, 1)$ is defined as:

$$CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(L)ds,$$

where $\alpha \in (0, 1)$ is the risk level and $VaR_\alpha = \inf\{s \in R : P(L \leq s) > \alpha\}$. $CVaR_\alpha(L)$ represents the expected loss of a hedging strategy given that the loss exceeds its upper α quantile.

We start with the market (2.1) excluding transaction fees and consider the problem of deriving the optimal hedging strategy that minimizes hedging costs while keeps the hedging loss less than or equal to a constraint, i.e.,

$$\begin{cases} \min_{(\tilde{V}_0,\xi)\in\mathcal{A}} \tilde{V}_0, \\ \text{subject to } \; CVaR_\alpha(L) \leq C, \end{cases} \tag{2.16}$$

where $\tilde{V}_0$ represents initial hedging costs, $\mathcal{A}$ is the set containing all self-financing admissible strategies and C is a fixed CVaR constraint.

Melnikov and Smirnov (2012) provided a semi-explicit solution of the problem (2.16) by using the Neyman-Pearson lemma. The main result of their paper is summarized as following:

A. The optimal hedging strategy of the problem (2.16) is a perfect hedging of a modified contingent claim $(H - \hat{z})^+[1 - \varphi(\hat{z})]$ if conditions

$$E(H) > C(1-\alpha), \quad E((H-C)^+) > 0, \tag{2.17}$$

hold true, where $\varphi(z)$ is defined as:

$$\varphi(z) = I_{\{\frac{dP^*}{dP} > a(z)\}} + \Gamma(z) I_{\{\frac{dP^*}{dP} = a(z)\}}, \tag{2.18}$$

$$a(z) = \inf\left\{a \geq 0 : E[(H-z)^+ I_{\{\frac{dP^*}{dP} > a(z)\}}] \leq (C-z)(1-\alpha)\right\}, \tag{2.19}$$

$$\Gamma(z) = \frac{(C-z)(1-\alpha) - E\left[(H-z)^+ I_{\{\frac{dP^*}{dP} > a(z)\}}\right]}{E\left[(H-z)^+ I_{\{\frac{dP^*}{dP} = a(z)\}}\right]}, \tag{2.20}$$

and $\hat{z}$ is the solution of

$$\min_{z\in[0,C]} E^*\big[e^{-rT}(H-z)^+(1-\varphi(z))\big]. \tag{2.21}$$

B. The optimal hedging strategy of the problem (2.16) is a passive strategy if conditions (2.17) are not satisfied, that is to say do not hedge at all.

Note that the explicit form of the modified contingent claim would depend on the option and the financial model of underlying assets. Melnikov and Smirnov (2012) provided the optimal hedging strategy for a call option in the Black-Scholes model. We extend their methodology to the market (2.1) and the explicit expression of the optimal CVaR hedging strategy for a call option is given as following:

Theorem 2.1. *Consider an European call option $H=(S_T^1-K)^+$ in the market (2.1). Under a risk restriction $CVaR_\alpha(L)\le C$ and assume conditions (2.17) are satisfied, the optimal CVaR hedging strategy and its capital are as follows:*

Case (a), $\alpha^*<0$

(a.1) The optimal CVaR hedging strategy π^* is given by the perfect hedging (2.7)-(2.9) of the modified contingent claim $H^*=\big(S_T^1-K(\hat{z})\big)^+ I_{\{S_T^1\ge m(\hat{z})b^{*N_T}\}}$, where $m(z)$ is the unique solution of the equation

$$\begin{aligned}\sum_{n\in\overline{A}(m,z)}&\Big[S_0^1(1-\upsilon_1)^n e^{\mu_1 T}\big(\Phi(\Lambda_2(n)+\sigma_1\sqrt{T})-\Phi(\Lambda_1(m,n)+\sigma_1\sqrt{T})\big)\\&-K(z)\big(\Phi(\Lambda_2(n))-\Phi(\Lambda_1(m,n))\big)\Big]p_{n,T}=(C-z)(1-\alpha),\end{aligned} \tag{2.22}$$

and $\hat{z}$ is the solution of

$$\begin{aligned}&\min_{z\in[0,C]}\sum_{A(m(z),z)} p_{n,T}^* CB\big(s_{0,n}^1,K(z),\sigma_1,T\big)\\&+\sum_{\overline{A}(m(z),z)} p_{n,T}^*\Big[CB\big(s_{0,n}^1,m(z)b^{*n},\sigma_1,T\big)+e^{-rT}\big(m(z)b^{*n}-K(z)\big)\Phi\big(\Lambda_1^*(m(z),n)\big)\Big].\end{aligned} \tag{2.23}$$

(a.2) The initial hedging costs $\tilde{V}_0$ *are*

$$\sum_{A(\hat{m},\hat{z})} p^*_{n,T} CB\big(s^1_{0,n}, K(\hat{z}), \sigma_1, T\big)$$
$$+ \sum_{\overline{A}(\hat{m},\hat{z})} p^*_{n,T}\Big[CB\big(s^1_{0,n}, \hat{m}b^{*n}, \sigma_1, T\big) + e^{-rT}\big(\hat{m}b^{*n} - K(\hat{z})\big)\Phi\big(\Lambda^*_1(\hat{m},n)\big)\Big]. \tag{2.24}$$

Case (b), $\alpha^* > 0$

(b.1) The optimal CVaR hedging strategy is given by the perfect hedging (2.7)-(2.9) of the modified contingent claim $H^* = \big(S^1_T - K(\hat{z})\big)^+ I_{\{S^1_T \le m(\hat{z})b^{*N_T}\}}$, *where* $m(z)$ *is the unique solution of the equation*

$$\sum_{A(m,z)} p_{n,T}\Big[S^1_0(1-\upsilon_1)^n e^{\mu_1 T}\Phi\big(\Lambda_2(n) + \sigma_1\sqrt{T}\big) - K(z)\Phi\big(\Lambda_2(n)\big)\Big]$$
$$+ \sum_{\overline{A}(m,z)} p_{n,T}\Big[S^1_0(1-\upsilon_1)^n e^{\mu_1 T}\Phi\big(\Lambda_1(m,n) + \sigma_1\sqrt{T}\big) - K(z)\Phi\big(\Lambda_1(m,n)\big)\Big]$$
$$= (C - z)(1-\alpha), \tag{2.25}$$

and $\hat{z}$ *is the solution of*

$$\min_{z\in[0,C]} \sum_{\overline{A}(m(z),z)} p^*_{n,T}\Big[CB\big(s^1_{0,n}, K(z), \sigma_1, T\big) - CB\big(s^1_{0,n}, m(z)b^{*n}, \sigma_1, T\big)$$
$$+ e^{-rT}\big(K(z) - m(z)b^{*n}\big)\Phi\big(\Lambda^*_1(m(z),n)\big)\Big]. \tag{2.26}$$

(b.2) The initial hedging costs $\tilde{V}_0$ *are*

$$\sum_{\overline{A}(\hat{m},\hat{z})} p^*_{n,T}\Big[CB\big(s^1_{0,n}, K(\hat{z}), \sigma_1, T\big) - CB\big(s^1_{0,n}, \hat{m}b^{*n}, \sigma_1, T\big)$$
$$+ e^{-rT}\big(K(\hat{z}) - \hat{m}b^{*n}\big)\Phi\big(\Lambda^*_1(\hat{m},n)\big)\Big], \tag{2.27}$$

where $\Phi(x)$ *is the distribution function of a standard normal random variable.* $CB(S_t, K, \sigma, T-t)$ *denotes the Black-Scholes formula, i.e.,*

$$CB(S_t, K, \sigma, T-t) = S_t\Phi\big(b_+(S_t,K,\sigma)\big) - Ke^{-r(T-t)}\Phi\big(b_-(S_t,K,\sigma)\big),$$

and

$$b_{\pm}(S_t, K, \sigma) = \frac{\ln(\frac{S_t}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad K(z) = K + z,$$

$$s^1_{t,n} = S^1_t v_{n,T-t} = S^1_t (1-\upsilon_1)^n \exp(\upsilon_1 \lambda^*(T-t)), \quad b^* = (1-\upsilon_1)(\frac{\lambda}{\lambda^*})^{\frac{\sigma_1}{\alpha^*}},$$

$$p^*_{n,t} = \exp(-\lambda^* t)\frac{(\lambda^* t)^n}{n!}, \quad p_{n,t} = \exp(-\lambda t)\frac{(\lambda t)^n}{n!},$$

$$\Lambda_1(m,n) = \frac{\ln \frac{S^1_0(1-\upsilon_1)^n}{m b^{*n}} + (\mu_1 - \frac{\sigma_1{}^2}{2})T}{\sigma_1\sqrt{T}}, \ \Lambda_2(n) = \frac{\ln \frac{S^1_0(1-\upsilon_1)^n}{K(z)} + (\mu_1 - \frac{\sigma_1{}^2}{2})T}{\sigma_1\sqrt{T}},$$

$$\Lambda^*_1(m,n) = \frac{\ln \frac{s^1_{0,n}}{m b^{*n}} + (r - \frac{\sigma_1{}^2}{2})T}{\sigma_1\sqrt{T}}, \quad \hat{m} = m(\hat{z}),$$

$$A(m,z) = \{n : \Lambda_1(m,n) \geq \Lambda_2(n)\} = \{n : m b^{*n} \leq K(z)\}.$$

Proof. The density of the unique martingale measure in transaction costs free market can be represented in terms of S^1_T and N_T. i.e.,

$$\begin{aligned}
\frac{dP^*}{dP} &= \exp(\alpha^* W_T - \frac{\alpha^{*2}}{2}t + (\lambda - \lambda^*)T + (\ln\lambda^* - \ln\lambda)N_T) \\
&= \left[S^1_0 \exp\left(\sigma_1 W_T + (\mu_1 - \frac{\sigma_1{}^2}{2})T\right)(1-\upsilon_1)^{N_T}\right]^{\frac{\alpha^*}{\sigma_1}} \\
&\times S_0^{1-\frac{\alpha^*}{\sigma_1}} \exp\left(-\frac{\alpha^*\mu_1}{\sigma_1}T + \frac{\sigma_1\alpha^*}{2}T - \frac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T\right)\left(\frac{\lambda^*}{\lambda(1-\upsilon)^{\frac{\alpha^*}{\sigma_1}}}\right)^{N_T}, \\
&= g(S^1_T)^{\frac{\alpha^*}{\sigma_1}} b^{N_T}
\end{aligned}$$

where $b = \frac{\lambda^*}{\lambda(1-\upsilon)^{\frac{\alpha^*}{\sigma_1}}}$ and $g = S_0^{1-\frac{\alpha^*}{\sigma_1}} \exp(-\frac{\alpha^*\mu_1}{\sigma_1}T + \frac{\sigma_1\alpha^*}{2}T - \frac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T)$.

Case 1. $\alpha^* < 0$

In this case, the set $\{\frac{dP^*}{dP} > a\}$ can be rewritten as

$$\{\frac{dP^*}{dP} > a\} = \{g(S^1_T)^{\frac{\alpha^*}{\sigma_1}} b^{N_T} > a\} = \{S^1_T < m b^{*N_T}\},$$

$$P(\frac{dP^*}{dP} = a) = P^*(\frac{dP^*}{dP} = a) = 0,$$

where $b^* = b^{-\frac{\sigma_1}{\alpha^*}}$. m is a constant to be determined.

And hence, we have

$$\varphi(z) = I_{\{S_T^1 < m(z) b^{*N_T}\}},$$

$$m(z) = \sup\{m > 0 : E[(H-z)^+ I_{\{S_T^1 < mb^{*N_T}\}}] \leq (C-z)(1-\alpha)\}.$$

Since $z \geq 0$, we have $(H-z)^+ = \big(S_T^1 - K(z)\big)^+$, where $K(z) = K + z$ and

$$E[(H-z)^+ I_{\{S_T^1 < mb^{*N_T}\}}] = E[(S_T^1 - K(z)) I_{\{S_T^1 < mb^{*N_T}\}} I_{\{S_T^1 \geq K(z)\}}].$$

Considering the evolution of S_T^1 under the measure P and conditioning on each set $\{N_T = n\}, n = 0, 1, 2 \ldots$, we get

$$\begin{aligned}
\{S_T^1 < mb^{*n}\} &= \{S_0^1 (1-\upsilon_1)^n \exp(\sigma_1 W_T + (\mu_1 - \frac{\sigma_1^2}{2})T) < mb^{*n}\} \\
&= \{Y > \Lambda_1(m,n)\}, \\
\{S_T^1 \geq K(z)\} &= \{Y \leq \Lambda_2(n)\},
\end{aligned}$$

where $Y = -\frac{W_T}{\sqrt{T}} \sim N(0,1)$ and

$$\Lambda_1(m,n) = \frac{\ln \frac{S_0^1(1-\upsilon_1)^n}{mb^{*n}} + (\mu_1 - \frac{{\sigma_1}^2}{2})T}{\sigma_1 \sqrt{T}}, \;\; \Lambda_2(n) = \frac{\ln \frac{S_0^1(1-\upsilon_1)^n}{K(z)} + (\mu_1 - \frac{{\sigma_1}^2}{2})T}{\sigma_1 \sqrt{T}}.$$

We need to compare the size of $\Lambda_1(m,n)$ and $\Lambda_2(n)$. Let us denote

$$A(m,z) = \{n : \Lambda_1(m,n) \geq \Lambda_2(n)\} = \{n : mb^{*n} \leq K(z)\}.$$

Then, $E[(S_T^1 - K(z)) I_{\{S_T^1 < mb^{*n}\}} I_{\{S_T^1 \geq K(z)\}}]$ can be calculated as

$$\begin{aligned}
&E[(S_T^1 - K(z)) I_{\{S_T^1 < mb^{*n}\}} I_{\{S_T^1 \geq K(z)\}}] \\
&= E[(S_T^1 - K(z)) I_{\{Y > \Lambda_1(m,n)\}} I_{\{Y \leq \Lambda_2(n)\}}] \\
&= \begin{cases} 0 & for \;\; n \in A(m,z), \\ f(n) & for \;\; n \in \overline{A}(m,z), \end{cases}
\end{aligned}$$

where,

$$\begin{aligned}
f(n) &= E[(S_T^1 - K(z))I_{\{\Lambda_1(m,n)<Y\leq\Lambda_2(n)\}}] \\
&= E[S_T^1 I_{\{Y\leq\Lambda_2(n)\}}] - E[S_T^1 I_{\{Y\leq\Lambda_1(m,n)\}}] - K(z)P\Big(\Lambda_1(m,n) < Y \leq \Lambda_2(n)\Big) \\
&= S_0^1(1-\upsilon_1)^n e^{(\mu_1 - \frac{{\sigma_1}^2}{2})T}\big[E(e^{\sigma_1 W_T} I_{\{Y\leq\Lambda_2(n)\}}) - E(e^{\sigma_1 W_T} I_{\{Y\leq\Lambda_1(m,n)\}})\big] \\
&- K(z)\big[\Phi(\Lambda_2(n)) - \Phi(\Lambda_1(m,n))\big]
\end{aligned}$$

According to the Multi-asset theorem (see Melnikov and Skornyakova 2005) for the $k=1$ case, we have

$$E(e^{-Z}I_{\{X<x\}}) = \exp(\frac{{\sigma_z}^2}{2} - \mu_z)\Phi\big(\frac{x-\mu_X+cov(Z,X)}{\sigma_X}\big),$$

where $X \sim N(\mu_X, {\sigma_X}^2), Z \sim N(\mu_Z, {\sigma_Z}^2)$.

And consequently, we arrive to

$$\begin{aligned}
f(n) &= S_0^1(1-\upsilon_1)^n e^{\mu_1 T}[\Phi(\Lambda_2(n)+\sigma_1\sqrt{T}) - \Phi(\Lambda_1(m,n)+\sigma_1\sqrt{T})] \\
&- K(z)[\Phi(\Lambda_2(n)) - \Phi(\Lambda_1(m,n))].
\end{aligned}$$

Hence, we prove that

$$\begin{aligned}
&E[(H-z)^+ I_{\{S_T^1<mb^{*N_T}\}}] \\
&= \sum_{n=0}^{\infty} P(N_T=n)E[(S_T^1-K(z))I_{\{Y>\Lambda_1(m,n)\}}I_{\{Y\leq\Lambda_2(n)\}} \mid N_T=n] \\
&= \sum_{\overline{A}(m,z)} f(n)p_{n,T}. \qquad (2.28)
\end{aligned}$$

Since equation (2.28) is an increasing function about m, and

$$E[(H-z)^+ I_{\{S_T^1<0b^{*N_T}\}}] = 0,\ \ E((H-z)^+) > (C-z)(1-\alpha),$$

$m(z)$ is the solution for

$$\sum_{\overline{A}(m,z)} f(n)p_{n,T} = (C-z)(1-\alpha).$$

Let us calculate the term $e^{-rT}E^*((S_T^1-K(z))^+ I_{\{S_T^1>m(z)b^{*N_T}\}})$.

Considering the evolution of S_T^1 under the measure P^* and conditioning on each set $\{N_T = n\}, n = 0, 1, 2, \ldots,$ we have

$$\{S_T^1 \geq m(z)b^{*n}\} = \{S_0^1(1-v_1)^n e^{v_1\lambda^* T}\exp(\sigma_1 W_T^* + (r - \frac{{\sigma_1}^2}{2})T) \geq m(z)b^{*n}\}$$

$$= \{Y^* \leq \Lambda_1^*(m(z), n)\},$$

$$\{S_T^1 \geq K(z)\} = \{Y^* \leq \Lambda_2^*(n)\},$$

where $Y^* = -\frac{W_T^*}{\sqrt{T}} \sim N(0,1)$ under the measure P^*, and

$$\Lambda_1^*(m(z), n) = \frac{\ln \frac{s_{0,n}^1}{m(z)b^{*n}} + (r - \frac{{\sigma_1}^2}{2})T}{\sigma_1\sqrt{T}}, \ \Lambda_2^*(n) = \frac{\ln \frac{s_{0,n}^1}{K(z)} + (r - \frac{{\sigma_1}^2}{2})T}{\sigma_1\sqrt{T}}.$$

Then, we have

$$e^{-rT}E^*\big[(S_T^1 - K(z))^+ I_{\{S_T^1 > m(z)b^{*n}\}}\big] = e^{-rT}E^*\big[(S_T^1 - K(z)) I_{\{Y^* \leq \Lambda_1^*(m(z),n)\} \cap \{Y^* \leq \Lambda_2^*(n)\}}\big]$$

$$= \begin{cases} e^{-rT}E^*\big[(S_T^1 - K(z)) I_{\{Y^* \leq \Lambda_2^*(n)\}}\big], & for \ n \in A(m(z), z), \\ e^{-rT}E^*\big[(S_T^1 - K(z)) I_{\{Y^* \leq \Lambda_1^*(m(z),n)\}}\big], & for \ n \in \overline{A}(m(z), z), \end{cases}$$

which, again according to the multi-asset theorem can be rewritten as

$$= \begin{cases} s_{0,n}^1 \Phi\big(\Lambda_2^*(n) + \sigma_1\sqrt{T}\big) - K(z)e^{-rT}\Phi\big(\Lambda_2^*(n)\big), & for \ n \in A(m(z), z), \\ s_{0,n}^1 \Phi\big(\Lambda_1^*(m(z), n) + \sigma_1\sqrt{T}\big) - K(z)e^{-rT}\Phi\big(\Lambda_1^*(m(z), n)\big), & for \ n \in \overline{A}(m(z), z). \end{cases}$$

$$= \begin{cases} CB\big(s_{0,n}^1, K(z), \sigma_1, T, \big) & for \ n \in A(m(z), z), \\ CB\big(s_{0,n}^1, m(z)b^{*n}, \sigma_1, T\big) + e^{-rT}\big(m(z)b^{*n} - K(z)\big)\Phi\big(\Lambda_1^*(m(z), n)\big), & for \ n \in \overline{A}(m(z), z). \end{cases}$$

And hence, we have

$$\begin{aligned} & e^{-rT}E^*\big((S_T^1 - K(z))^+ I_{\{S_T^1 > m(z)b^{*N_T}\}}\big) \\ & = \sum_{n=0}^{\infty} p_{n,T}^* e^{-rT} E^*\big[(S_T^1 - K(z)) I_{\{Y^* \leq \Lambda_1^*(m(z),n)\}} I_{\{Y^* < \Lambda_2^*(n)\}} \mid N_T = n\big] \\ & = \sum_{A(m(z),z)} p_{n,T}^* CB\big(s_{0,n}^1, K(z), \sigma_1, T\big) \\ & + \sum_{\overline{A}(m(z),z)} p_{n,T}^*\big[CB(s_{0,n}^1, m(z)b^{*n}, \sigma_1, T) + e^{-rT}(m(z)b^{*n} - K(z))\Phi(\Lambda_1^*(m(z), n))\big]. \end{aligned} \tag{2.29}$$

As a consequence, $\hat{z}$ is a point of minimum of the function (2.29) over the interval $z \in [0, C]$.

Case 2. $\alpha^* > 0$

In this case, we have

$$\{\frac{dP^*}{dP} > a\} = \{g(S_T^1)^{\frac{\alpha^*}{\sigma_1}} b^{N_T} > a\} = \{S_T^1 > m{b^*}^{N_T}\},$$

and hence, (2.18)-(2.20) becomes

$$\begin{aligned}
&\varphi(z) = I_{\{S_T^1 > m(z){b^*}^{N_T}\}},\\
&m(z) = \inf\{m > 0 : E[(H-z)^+ I_{\{S_T^1 > m{b^*}^{N_T}\}}] \leq (C-z)(1-\alpha)\}.
\end{aligned}$$

As a consequence,

$$E[(H-z)^+ I_{\{S_T^1 > m{b^*}^{N_T}\}}] = E[(S_T^1 - K(z)) I_{\{S_T^1 > m{b^*}^{N_T}\}} I_{\{S_T^1 \geq K(z)\}}].$$

Conditioning on each set $\{N_T = n\}, n = 0, 1, 2 \ldots$, we get

$$\{S_T^1 > m{b^*}^n\} = \{Y < \Lambda_1(m,n)\},\ \{S_T^1 \geq K(z)\} = \{Y \leq \Lambda_2(n)\}.$$

Thus, we have

$$\begin{aligned}
E[(H-z)^+ I_{\{S_T^1 > m{b^*}^n\}}] &= E[(S_T^1 - K(z)) I_{\{Y < \Lambda_1(m,n)\}} I_{\{Y \leq \Lambda_2(n)\}}]\\
&= \begin{cases} E((S_T^1 - K(z)) I_{\{Y < \Lambda_2(n)\}}), & for\ \ n \in A(m,z),\\ E((S_T^1 - K(z)) I_{\{Y < \Lambda_1(m,n)\}}), & for\ \ n \in \overline{A}(m,z), \end{cases}
\end{aligned}$$

$$= \begin{cases} S_0^1(1-\upsilon_1)^n e^{\mu_1 T} \Phi(\Lambda_2(n) + \sigma_1\sqrt{T}) - K(z)\Phi(\Lambda_2(n)) & for\ \ n \in A(m,z),\\ S_0^1(1-\upsilon_1)^n e^{\mu_1 T} \Phi(\Lambda_1(m,n) + \sigma_1\sqrt{T}) - K(z)\Phi(\Lambda_1(m,n)) & for\ \ n \in \overline{A}(m,z). \end{cases} \tag{2.30}$$

Finally, by (2.30), we arrive to

$$\begin{aligned}
&E[(H-z)^+ 1_{\{S_T^1 > m{b^*}^{N_T}\}}]\\
&= \sum_{n=0}^{\infty} P(N_T = n) E_P[(S_T^1 - K(z)) I_{\{Y < \Lambda_1(m,n)\}} 1_{\{Y \leq \Lambda_2(n)\}} \mid N_T = n]\\
&= \sum_{A(m,z)} p_{n,T}[S_0^1(1-\upsilon)^n e^{\mu_1 T} \Phi(\Lambda_2(n) + \sigma_1\sqrt{T}) - K(z)\Phi(\Lambda_2(n))]\\
&+ \sum_{\overline{A}(m,z)} p_{n,T}[S_0^1(1-\upsilon)^n e^{\mu_1 T} \Phi(\Lambda_1(m,n) + \sigma_1\sqrt{T}) - K(z)\Phi(\Lambda_1(m,n))].
\end{aligned} \tag{2.31}$$

And hence $m(z)$ is the solution of (2.31) $= (C-z)(1-\alpha)$.

Similarly, under the measure P^* and conditioning on each set $\{N_T = n\}, n = 0, 1, 2, \ldots,$ we have

$$\{S_T^1 \le m(z)b^{*n}\} = \{Y^* \ge \Lambda_1^*(m(z), n)\},\ \{S_T^1 \ge K(z)\} = \{Y^* \le \Lambda_2^*(n)\}.$$

Hence, we arrive to

$$\begin{aligned}
&e^{-rT}E^*((S_T^1 - K(z))^+ I_{\{S_T^1 \le m(z)b^{*n}\}}) \\
&= e^{-rT}E^*((S_T^1 - K(z)) I_{\{Y^* \ge \Lambda_1^*(m(z),n)\}} I_{\{Y^* \le \Lambda_2^*(n)\}}) \\
&= \begin{cases} 0, & for\ \ n \in A(m(z), z), \\ e^{-rT}E^*((S_T^1 - K(z)) I_{\{\Lambda_1^*(m(z),n) \le Y^* \le \Lambda_2^*(n)\}}), & for\ \ n \in \overline{A}(m(z), z). \end{cases}
\end{aligned}$$

Moreover, by some calculation, we arrive to

$$\begin{aligned}
&e^{-rT}E^*((S_T^1 - K(z)) I_{\{\Lambda_1^*(m(z),n) \le Y^* \le \Lambda_2^*(n)\}}) \\
&= e^{-rT}E^*(S_T^1 I_{\{\Lambda_1^*(m(z),n) \le Y^* \le \Lambda_2^*(n)\}}) - e^{-rT}K(z)[\Phi(\Lambda_2^*(n)) - \Phi(\Lambda_1^*(m(z), n))] \\
&= s_{0,n}^1[\Phi(\Lambda_2^*(n) + \sigma_1\sqrt{T}) - \Phi(\Lambda_1^*(m(z), n) + \sigma_1\sqrt{T})] - e^{-rT}K(z)[\Phi(\Lambda_2^*(n)) - \Phi(\Lambda_1^*(m(z), n)) \\
&= CB(s_{0,n}^1, K(z), \sigma_1, T) - CB(s_{0,n}^1, m(z)b^{*n}, \sigma_1, T) + e^{-rT}[K(z) - m(z)b^{*n}]\Phi(\Lambda_1^*(m(z), n)).
\end{aligned}$$

and consequently, we derive that

$$\begin{aligned}
&e^{-rT}E^*((S_T^1 - K(z))^+ I_{\{S_T^1 \le m(z)b^{*N_T}\}}) \\
&= \sum_{n=0}^{\infty} p_{n,T}^* e^{-rT}E^*((S_T^1 - K(z)) I_{\{Y^* \ge \Lambda_1^*(n)\}} I_{\{Y^* \le \Lambda_2^*(n)\}}) \\
&= \sum_{\overline{A}(m(z),n)} p_{n,T}^* \Big[CB(s_{0,n}^1, K(z), \sigma_1, T) - CB(s_{0,n}^1, m(z)b^{*n}, \sigma_1, T) \\
&+ e^{-rT}[K(z) - m(z)b^{*n}]\Phi(\Lambda_1^*(m(z), n))\Big]. \qquad (2.32)
\end{aligned}$$

Then, $\hat{z}$ is a point of minimum of function (2.32) over the interval $z \in [0, C]$. □

2.3 CVaR hedging in the Jump-Diffusion market with transaction costs

2.3.1 Option pricing and hedging in the market (2.1) with transaction costs

If transaction costs are taken into account, the continuous replication policy (2.7)-(2.9) would incur infinite amounts of transaction costs. In this case, hedges should be rebalanced discretely. Let us assume that buying and selling stocks need to pay transaction fees which are proportional to trading volumes, i.e.,$\sum_{i=1}^{2} k|\delta\hat{\triangle}_t^i|S_t^i$ at time t, where $|\delta\hat{\triangle}_t^i|$ represents shares of the trading risky asset S^i and k represents a fixed proportion of transaction fees. Following Leland (1985), we assume that trades can only be executed at certain points of time $\{t_0, t_1, \ldots, t_M\},\ t_M = T$. The time interval δt between successive rehedgings is assumed to be fixed and is much smaller than the time to expiration.

Intuitively, transaction costs affect prices of options. Leland has indicated that if one applies delta hedging with an augmented volatility $\hat{\sigma} = \sigma\sqrt{1 + 2k\sqrt{2/\pi}/\sigma\sqrt{\delta t}}$ where σ is the volatility of the asset in the Black-Scholes model, payoffs of an European call option can be replicated almost surely as $\delta t \to 0$ and the price of a call option inclusive of transaction costs is given by the Black-Scholes formula with the modified volatility $\hat{\sigma}$. The main idea of Leland was to include the expected amounts of transaction costs to the basic Black-Scholes option price. Here, we would like to use the same idea as Leland and define the modified value of a claim at time t by the value of an adjusted delta hedging portfolio at that point.

Lemma 2.2. *In the market (2.1) including transaction costs, the modified value process $\overline{V}(t) = \overline{V}(S_t^1, S_t^2, t)$ of an option $H(S_T^1, S_T^2)$ satisfies the following non-linear PDE (we drop dependence on time in order to abbreviate the notation):*

$$\overline{V}_t + (r + \lambda^*\upsilon_1)\overline{V}_{S^1}S_t^1 + (r + \lambda^*\upsilon_2)\overline{V}_{S^2}S_t^2 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}$$
$$+ \lambda^*[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)] - r\overline{V} + k\Theta = 0, \qquad (2.33)$$

subject to the boundary condition $\overline{V}(S_T^1, S_T^2, T) = H(S_T^1, S_T^2)$*, where*

$$\overline{V}_t = \frac{\partial \overline{V}(S_t^1, S_t^2, t)}{\partial t}, \overline{V}_{S^i} = \frac{\partial \overline{V}(S_t^1, S_t^2, t)}{\partial S^i}, \overline{V}_{S^i S^j} = \frac{\partial \overline{V}(S_t^1, S_t^2, t)}{\partial S^i S^j},$$

$$\Theta = (1-\lambda^*\delta t)\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\eta_i S_t^i \delta t^{-\frac{1}{2}} + \lambda^* \sum_{i=1}^{2}\Big[\sqrt{\frac{2}{\pi}}\eta_i e^{-\frac{\zeta_i^2}{2\eta_i{}^2\delta t}}\delta t^{\frac{1}{2}} + \zeta_i\big(1-2\Phi(-\frac{\zeta_i}{\eta_i \delta t^{\frac{1}{2}}})\big)\Big]S_t^i,$$

$$\zeta_i = \sum_{j=1}^{2} -\overline{V}_{S^i S^j}(t) S_t^j \upsilon_j, \quad \eta_i = |\sum_{j=1}^{2}\overline{V}_{S^i S^j}(t)\sigma_j S_t^j|, \quad (i=1,2).$$

Proof. We shall set up the model in a discrete time framework. The time interval between two transactions is assumed to be fixed and equal to δt.

The number of jumps during the time interval $[t, t+\delta t)$ satisfies:

$$N_{t+\delta t} - N_t = \begin{cases} 0 & \textit{with probability } 1-\lambda\delta t + o(\delta t), \\ 1 & \textit{with probability } \lambda\delta t + o(\delta t), \\ \textit{others} & \textit{with probability } o(\delta t). \end{cases}$$

When δt is small, it is reasonable to assume there is at most 1 jump during the time interval and the underlying assets follow the model described by Cox and Ross (1976), i.e.,

$$\frac{\delta S_t^i}{S_t^i} = \mu_i \delta t + \sigma_i \delta W_t, \text{if jump dose not happen,}$$
$$\frac{\delta S_t^i}{S_t^i} = \mu_i \delta t + \sigma_i \delta W_t - \upsilon_i, \text{if one jump occurs,}$$

where $\delta S_t^i = S_{t+\delta t}^i - S_t^i$ represents the small change in the stock price during the time interval $t - t + \delta t$.

Denote the option value inclusive of transaction costs as $\overline{V}(t) = \overline{V}(S_t^1, S_t^2, t)$. Consider a portfolio with $\hat{\triangle}_t^i$ shares of stock S^i, $(i = 1, 2)$ and a short position in the option at time t. Its value is:

$$\Pi_t = \hat{\triangle}_t^1 S_t^1 + \hat{\triangle}_t^2 S_t^2 - \overline{V}(t).$$

The change in the value of the portfolio from t to $t + \delta t$ is:

$$\delta\Pi_t = \hat{\triangle}_t^1 \delta S_t^1 + \hat{\triangle}_t^2 \delta S_t^2 - \delta\overline{V}(t) - k|\delta\hat{\triangle}_t^1| S_{t+\delta t}^1 - k|\delta\hat{\triangle}_t^2| S_{t+\delta t}^2,$$

where $\delta\hat{\triangle}_t^i = \hat{\triangle}_{t+\delta t}^i - \hat{\triangle}_t^i$.

Also, let us expand $\overline{V}(t)$ using Itô formula, such that

$$\delta\overline{V}(t) = (\overline{V}_t + \overline{V}_{S^1}\mu_1 S_t^1 + \overline{V}_{S^2}\mu_2 S_t^2)\delta t + (\overline{V}_{S^1}\sigma_1 S_t^1 + \overline{V}_{S^2}\sigma_2 S_t^2)\delta W_t$$
$$+ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t + [\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)]\delta N_t. \tag{2.34}$$

Since we apply the Leland delta hedging method, shares of asset i are

$$\hat{\triangle}_t^i = \frac{\partial\overline{V}(S_t^1, S_t^2, t)}{\partial S^i} = \overline{V}_{S^i}, \quad (i = 1, 2).$$

Apply Taylor extension to $\delta\hat{\triangle}_t^i$, we obtain for the leading order

$$\delta\hat{\triangle}_t^1 = \overline{V}_{S^1S^1}\delta S_t^1 + \overline{V}_{S^1S^2}\delta S_t^2,$$

$$\delta\hat{\triangle}_t^2 = \overline{V}_{S^2S^1}\delta S_t^1 + \overline{V}_{S^2S^2}\delta S_t^2.$$

Let us consider cases that there is no jump and one jump separately.

a. No jump—$\delta N_T = 0$

In this case, the portfolio value changes by the amount

$$\begin{aligned}
\delta\Pi_t &= \overline{V}_{S^1}S_t^1(\mu_1\delta t + \sigma_1\delta W_t) + \overline{V}_{S^2}S_t^2(\mu_2\delta t + \sigma_2\delta W_t) \\
&- \big[(\overline{V}_t + \overline{V}_{S^1}\mu_1 S_t^1 + \overline{V}_{S^2}\mu_2 S_t^2)\delta t + (\overline{V}_{S^1}\sigma_1 S_t^1 + \overline{V}_{S^2}\sigma_2 S_t^2)\delta W_t \\
&+ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t\big] - k|\delta\hat{\triangle}_t^1|S_{t+\delta t}^1 - k|\delta\hat{\triangle}_t^2|S_{t+\delta t}^2 \\
&= -\overline{V}_t\delta t - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t - k|\delta\hat{\triangle}_t^1|S_{t+\delta t}^1 - k|\delta\hat{\triangle}_t^2|S_{t+\delta t}^2.
\end{aligned} \tag{2.35}$$

b.One jump—$\delta N_T = 1$

In this case, the portfolio value changes by the amount

$$\begin{aligned}
\delta\Pi_t &= \overline{V}_{S^1}S_t^1(\mu_1\delta t + \sigma_1\delta W_t - \upsilon_1) + \overline{V}_{S^2}S_t^2(\mu_2\delta t + \sigma_2\delta W_t - \upsilon_2) \\
&- \big[(\overline{V}_t + \overline{V}_{S^1}\mu_1 S_t^1 + \overline{V}_{S^2}\mu_2 S_t^2)\delta t + (\overline{V}_{S^1}\sigma_1 S_t^1 + \overline{V}_{S^2}\sigma_2 S_t^2)\delta W_t \\
&+ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t\big] + \big[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)\big] \\
&- k|\delta \triangle_t^1| S_{t+\delta t}^1 - k|\delta \triangle_t^2| S_{t+\delta t}^2 \\
&= -\overline{V}_t\delta t - \overline{V}_{S^1}S_t^1\upsilon_1 - \overline{V}_{S^2}S_t^2\upsilon_2 - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t \\
&- \big[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)\big] - k|\delta\hat{\triangle}_t^1|S_{t+\delta t}^1 - k|\delta\hat{\triangle}_t^2|S_{t+\delta t}^2. \qquad (2.36)
\end{aligned}$$

In Hoggard et al. (1994), they assumed the return of the portfolio was equal to the risk free rate under the measure P, that is, $E(\delta\Pi_t) = r\Pi_t\delta t$. This is because in the Black-Scholes model, delta hedging eliminates all systematic risk. However, in the Jump-Diffusion case, risk caused by jumps is not eliminated. Notice that the expected return of risky assets and the risk free asset are both equal to r under measure P^* which means investors are risk neutral and the expected return for the portfolio under P^* should also be r. As a consequence, in this chapter, different from Hoggard et al., we assume $E^*(\delta\Pi_t) = r\Pi_t\delta t$.

As the next step, let use consider the expected value of transaction costs. In Leland (1985), an important step in his proof is that he assumed that $\delta S/S$ was normally distributed with mean zero and variance $\sigma\sqrt{\delta t}$, then according to the expectation of the absolute value of a normal random variable, i.e., $E(|\delta W_t|) = \sqrt{2/\pi}\delta t$, the expected transaction costs during δt were $k \mid \overline{V}_{SS} \mid {S_t}^2\sqrt{2/\pi}\sigma\sqrt{\delta t}$. Similarly, we would like to derive the expression of $E(|X|)$, where

$X \sim N(\mu_x, \sigma_x^2)$. i.e.,

$$\begin{aligned}
E(|X|) &= \int_{-\infty}^{0} -xn(x)dx + \int_{0}^{\infty} xn(x)dx. \\
&= -\int_{-\infty}^{0} (x-\mu_x)\frac{1}{\sqrt{2\pi{\sigma_x}^2}}e^{-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}dx + \int_{0}^{\infty}(x-\mu_x)\frac{1}{\sqrt{2\pi{\sigma_x}^2}}e^{-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}dx \\
&- \int_{-\infty}^{0} \mu_x\frac{1}{\sqrt{2\pi{\sigma_x}^2}}e^{-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}dx + \int_{0}^{\infty}\mu_x\frac{1}{\sqrt{2\pi{\sigma_x}^2}}e^{-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}dx \\
&= \frac{\sigma_x}{\sqrt{2\pi}}(e^{-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}|_{-\infty}^{0}) - \frac{\sigma_x}{\sqrt{2\pi}}(e^{\frac{(x-\mu_x)^2}{2{\sigma_x}^2}}|_{0}^{\infty}) + \mu_x(P(X\geq 0) - P(X\leq 0)) \\
&= \sqrt{\frac{2}{\pi}}\sigma_x e^{-\frac{\mu^2}{2\sigma_x^2}} + \mu_x\big(1-2\Phi(-\frac{\mu_x}{\sigma_x})\big). \qquad (2.37)
\end{aligned}$$

a. $\delta N_T = 0$

The expected transaction costs (disregarding the higher power of δt) are

$$k\sum_{i=1}^{2}E^*(|\sum_{j=1}^{2}\overline{V}_{S^iS^j}\delta S_t^j|)S_t^i = k\sum_{i=1}^{2}E^*(|\sum_{j=1}^{2}\overline{V}_{S^iS^j}\sigma_j S_t^j\delta t^{1/2}\epsilon|)S_t^i,$$

where $\epsilon \sim N(0,1)$.

Note that $\sum_{j=1}^{2}\overline{V}_{S^iS^j}\sigma_j S_t^j\delta t^{\frac{1}{2}}\epsilon \sim N(0, {\eta_i}^2\delta t)$, where $\eta_i = |\sum_{j=1}^{2}(\overline{V}_{S^iS^j}\sigma_j S_t^j)|$, and consequently, by (2.37), the expected transaction costs can be written as

$$k\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\eta_i S_t^i\delta t^{\frac{1}{2}}. \qquad (2.38)$$

b. $\delta N_T = 1$.

The expected transaction costs (disregarding the higher power of δt) are

$$k\sum_{i=1}^{2}E^*(|\sum_{j=1}^{2}\overline{V}_{S^iS^j}\delta S_t^j|)S_t^i = k\sum_{i=1}^{2}E^*(|\sum_{j=1}^{2}\overline{V}_{S^iS^j}S_t^j(\sigma_j\delta t^{\frac{1}{2}}\epsilon\ -\upsilon_j)|)S_t^i. \qquad (2.39)$$

Also, we have $\sum_{j=1}^{2}\overline{V}_{S^iS^j}S_t^j(\sigma_j\delta t^{\frac{1}{2}}\epsilon\ -\upsilon_j) \sim N(\zeta_i, {\eta_i}^2\delta t)$ where $\zeta_i = \sum_{j=1}^{2}-\overline{V}_{S^iS^j}S_t^j\upsilon_j$.

Consequently, the expected transaction costs are rewritten as

$$k\sum_{i=1}^{2}[\sqrt{\frac{2}{\pi}}\eta_i e^{-\frac{\zeta^2}{2{\eta_i}^2\delta t}}\delta t^{\frac{1}{2}} + \zeta_i(1-2\Phi(-\frac{\zeta_i}{\eta_i\delta t^{\frac{1}{2}}}))]S_t^i. \qquad (2.40)$$

Combine (2.35)-(2.40), we arrived to

$$
\begin{aligned}
& E^*(\delta\Pi_t) \\
&= E^*(\delta\Pi_t \mid \delta N_T = 0)P^*(\delta N_T = 0) + E_{P^*}(\delta\Pi_t \mid \delta N_T = 1)P^*(\delta N_T = 1) \\
&= (1-\lambda^*\delta t)\big[-\overline{V}_t\delta t - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t - k\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\eta_i S_t^i \delta t^{\frac{1}{2}}\big] \\
&+ \lambda^*\delta t\Big[-\overline{V}_t\delta t - \overline{V}_{S^1}S_t^1\upsilon_1 - \overline{V}_{S^2}S_t^2\upsilon_2 - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t \\
&- \big[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)\big] \\
&- k\sum_{i=1}^{2}\big[\sqrt{\frac{2}{\pi}}\eta_i e^{-\frac{\zeta^2}{2\eta_i{}^2\delta t}}\delta t^{\frac{1}{2}} + \zeta_i(1-2\Phi(-\frac{\zeta_i}{\eta_i\delta t^{\frac{1}{2}}}))\big]S_t^i\Big] \\
&= -\overline{V}_t\delta t - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t - \lambda^*\delta t\overline{V}_{S^1}S_t^1\upsilon_1 - \lambda^*\delta t\overline{V}_{S^2}S_t^2\upsilon_2 \\
&- \lambda^*\delta t\Big[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)\Big] \\
&- k\Big\{(1-\lambda^*\delta t)\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\eta_i S_t^i\delta t^{\frac{1}{2}} + \lambda^*\delta t\sum_{i=1}^{2}[\sqrt{\frac{2}{\pi}}\eta_i e^{-\frac{\zeta^2}{2\eta_i{}^2\delta t}}\delta t^{\frac{1}{2}} + \zeta_i(1-2\Phi(-\frac{\zeta_i}{\eta_i\delta t^{\frac{1}{2}}}))]S_t^i\Big\} \\
&= -\overline{V}_t\delta t - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t - \lambda^*\delta t\overline{V}_{S^1}S_t^1\upsilon_1 - \lambda^*\delta t\overline{V}_{S^2}S_t^2\upsilon_2 \\
&- \lambda^*\delta t\big[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)\big] - k\Theta\delta t, \qquad (2.41)
\end{aligned}
$$

where $\Theta = (1-\lambda^*\delta t)\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\eta_i S_t^i\delta t^{-\frac{1}{2}} + \lambda^*\sum_{i=1}^{2}[\sqrt{\frac{2}{\pi}}\eta_i e^{-\frac{\zeta^2}{2\eta_i{}^2\delta t}}\delta t^{\frac{1}{2}} + \zeta_i(1-2\Phi(-\frac{\zeta_i}{\eta_i\delta t^{\frac{1}{2}}}))]S_t^i$.

Finally, the equation $E^*(\delta\Pi_t) = r\Pi_t\delta t$ can be rewritten as

$$
\begin{aligned}
& -\overline{V}_t\delta t - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^iS^j}\delta t - \lambda^*\delta t\overline{V}_{S^1}S_t^1\upsilon_1 - \lambda^*\delta t\overline{V}_{S^2}S_t^2\upsilon_2 - \\
& \lambda^*\delta t[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)] - k\Theta\delta t \\
& = r\big(\overline{V}_{S^1}S_t^1 + \overline{V}_{S^2}S_t^2 - \overline{V}(t)\big)\delta t \qquad (2.42)
\end{aligned}
$$

Rearranging the above equation we arrive to

$$\overline{V}_t + (r+\lambda^* \upsilon_1)\overline{V}_{S^1} S_t^1 + (r+\lambda^* \upsilon_2)\overline{V}_{S^2} S_t^2 + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j \overline{V}_{S^i S^j}$$

$$+ \lambda^*[\overline{V}(S_{t^-}^1(1-\upsilon_1), S_{t^-}^2(1-\upsilon_2), t) - \overline{V}(S_{t^-}^1, S_{t^-}^2, t)] - r\overline{V}(t) + k\Theta = 0,$$

which is the equation (2.33).

In the special case that the option consists of only one asset (assume S^1), we have

$$\overline{V}_{S^2} = \overline{V}_{S^1S^2} = \overline{V}_{S^2S^1} = \overline{V}_{S^2S^2} = 0,$$

$$\eta_1 = |\overline{V}_{S^1S^1}|\sigma_1 S_t^1, \quad \zeta_1 = -\overline{V}_{S^1S^1} S_t^1 \upsilon_1, \quad \eta_2 = 0, \quad \zeta_2 = 0,$$

and hence Θ can be simplified as

$$\begin{aligned}
\Theta &= (1-\lambda^*\delta t)\sqrt{\frac{2}{\pi}}|\overline{V}_{S^1S^1}|\sigma_1 {S_t^1}^2 \delta t^{-\frac{1}{2}} + \lambda^*\sqrt{\frac{2}{\pi}} e^{-\frac{{\upsilon_1}^2}{2{\sigma_1}^2\delta t}}|\overline{V}_{S^1S^1}|\sigma_1 {S_t^1}^2 \delta t^{\frac{1}{2}} \\
&+ \overline{V}_{S^1S^1}{S_t^1}^2\upsilon_1\lambda^*(2\Phi(\frac{\overline{V}_{S^1S^1}\upsilon_1}{|\overline{V}_{S^1S^1}|\sigma_1\delta t^{\frac{1}{2}}}) - 1) \\
&= |\overline{V}_{S^1S^1}|\sigma_1 {S_t^1}^2[\sqrt{\frac{2}{\pi\delta t}}(1-\lambda^*\delta t + \lambda^*\delta t e^{-\frac{{\upsilon_1}^2}{2{\sigma_1}^2\delta t}}) \\
&+ \frac{\upsilon_1}{\sigma_1}\lambda^* sign(\overline{V}_{S^1S^1})(2\Phi(sign(\overline{V}_{S^1S^1})\frac{\upsilon_1}{\sigma_1\delta t^{\frac{1}{2}}}) - 1)] \\
&= |\overline{V}_{S^1S^1}|\sigma_1 {S_t^1}^2[\sqrt{\frac{2}{\pi\delta t}}(1-\lambda^*\delta t + \lambda^*\delta t e^{-\frac{{\upsilon_1}^2}{2{\sigma_1}^2\delta t}}) + \frac{\upsilon_1}{\sigma_1}\lambda^*(2\Phi(\frac{\upsilon_1}{\sigma_1\delta t^{\frac{1}{2}}}) - 1)] \\
&= |\overline{V}_{S^1S^1}|\sigma_1 {S_t^1}^2\theta_1, \qquad (2.43)
\end{aligned}$$

where $\theta_1 = \sqrt{\frac{2}{\pi\delta t}}(1-\lambda^*\delta t + \lambda^*\delta t e^{-\frac{{\upsilon_1}^2}{2{\sigma_1}^2\delta t}}) + \frac{\upsilon_1}{\sigma_1}\lambda^*(2\Phi(\frac{\upsilon_1}{\sigma_1\delta t^{\frac{1}{2}}}) - 1)$.

Consequently, equation (2.42) is reduced to

$$\begin{aligned}
&\overline{V}_t + (r+\lambda^*\upsilon_1)\overline{V}_{S^1}S_t^1 + \frac{1}{2}{\sigma_1}^2{S_t^1}^2\overline{V}_{S^1S^1} + \lambda^*[\overline{V}(S_{t^-}^1(1-\upsilon_1), t) - \overline{V}(S_{t^-}^1, t)] \\
&- r\overline{V}(t) + k\overline{V}_{S^1S^1}\sigma_1{S_t^1}^2 sign(\overline{V}_{S^1S^1})\theta_1 = 0 \rightarrow \\
&\overline{V}_t + (r+\lambda^*\upsilon_1)\overline{V}_{S^1}S_t^1 + \frac{1}{2}\hat{\sigma}_1^2{S_t^1}^2\overline{V}_{S^1S^1} + \lambda^*[\overline{V}(S_{t^-}^1(1-\upsilon_1), t) - \overline{V}(S_{t^-}^1, t)] \\
&- r\overline{V}(t) = 0, \qquad (2.44)
\end{aligned}$$

where $\hat{\sigma}_1^2 = \sigma_1^2(1 + sign(\overline{V}_{S^1S^1})\frac{2k}{\sigma_1}\theta_1)$. □

Remark 2.3. *Because of transaction fees, the market (2.1) is no longer complete and in this case, as indicated in Dewynne et al. (1994), one can define the value of an option according to the hedging strategy that is applied. In our paper, we implement Leland adjusted delta hedging method.*

Remark 2.4. *Similar non-linear PDE also exists in the Black-Scholes model including transaction costs (see Dewynne et al. 1994). In particular, in Leland single asset model, such a non-linear PDE can be reduced to a liner PDE. However, in the two assets case, (2.33) can never be reduced to a linear PDE and can only be solved numerically. Also, similar to the conclusion in Zakamulin (2008) based on the Black-Scholes model, with two underlying assets, it may not be reasonable to explain the option hedging strategy as hedging with an adjusted volatility as Leland.*

In our case, we focus on hedging a call option whose payoff only depends on the first risky asset, i.e., $H(S_T^1, S_T^2) = (S_T^1 - K)^+$, so (2.33) can be reduced to:

$$\overline{V}_t + (r + \lambda^* \upsilon_1)\overline{V}_{S^1} S_t^1 + \frac{1}{2}\hat{\sigma}_1^2 {S_t^1}^2 \overline{V}_{S^1 S^1} + \lambda^*[\overline{V}(S_{t^-}^1(1-\upsilon_1), t) - \overline{V}(S_{t^-}^1, t)] - r\overline{V} = 0, \tag{2.45}$$

subject to the condition $\bar{V}(T) = H(S_T^1)$, where

$$\hat{\sigma}_1^2 = \sigma_1^2(1 + sign(\overline{V}_{S^1 S^1})\frac{2k}{\sigma_1}\theta_1),$$

$$\theta_1 = \sqrt{\frac{2}{\pi \delta t}}\big(1 - \lambda^* \delta t + \lambda^* \delta t e^{-\frac{{\upsilon_1}^2}{2{\sigma_1}^2 \delta t}}\big) + \frac{\upsilon_1}{\sigma_1}\lambda^*\big(2\Phi(\frac{\upsilon_1}{\sigma_1 \delta t^{\frac{1}{2}}}) - 1\big).$$

On the other hand, in a Jump-Diffusion model excluding transaction fees, according to the Itô formula and martingale property of the discounted value process, the option value $V(t) = E^*(e^{-r(T-t)}H(S_T^1) \mid \mathcal{F}_t)$ satisfies :

$$V_t + (r + \lambda^* \upsilon_1)V_{S^1} S_t^1 + \frac{1}{2}{\sigma_1}^2 {S_t^1}^2 V_{S^1 S^1} + \lambda^*[V(S_{t^-}^1(1-\upsilon_1), t) - V(S_{t^-}^1, t)] - rV = 0, \tag{2.46}$$

subject to the condition $V(T) = H(S_T^1)$, where $V_t = \frac{\partial V(t)}{\partial t}$.

Formulas (2.45) and (2.46) indicate that the modified value of an option that consists of only one asset (S^1 or S^2) and whose sign of gamma is a constant has the same form as that in the corresponding complete market, but such a value process is based on an adjusted volatility. It is worth to mention that, unlike Leland's adjusted volatility which depends on the transaction costs rate k and the revision period δt, our adjusted volatility $\hat{\sigma}_1$ also depends on the jump size υ_1 and Poisson intensity λ^*.

Remark 2.5. *If $\lambda^* = 0$ (without the jump), $\hat{\sigma}_1$ is just the Leland's adjusted volatility. Meanwhile, if $k = 0$ (without transaction costs), (2.45) is consistent with (2.46).*

Proposition 2.6. *In the market (2.1), the modified price inclusive of transaction costs of a call option $H = (S_T^1 - K)^+$ is*

$$\overline{V}(0) = \sum_{n=0}^{\infty} p^*_{n,T} CB(s^1_{0,n}, K, \hat{\sigma}_1, T), \tag{2.47}$$

where $\hat{\sigma}_1^2 = \sigma_1^2(1 + \frac{2k}{\sigma_1}\theta_1)$.

Proof. $V(0) = e^{-rT}E^*\big((S_T^1 - K)^+\big) = \sum_{n=0}^{\infty} p^*_{n,T} CB(s^1_{0,n}, K, \sigma_1, T)$ is the fair price of a call option in the transaction costs free complete market (2.1). Substituting σ_1 in the Black-Scholes formula with $\hat{\sigma}_1$ gives (2.47). □

2.3.2 CVaR hedging method in the market (2.1) with transaction costs

Theorem 2.1 provides the modified claim as well as the initial hedging costs of CVaR partial hedging in the complete market (2.1). However, with transaction fees, the CVaR hedging costs of the call option should be recalculated with the adjusted hedging volatility $\hat{\sigma}_1$.

Proposition 2.7. *In the market (2.1) including transaction costs, at each revision point $t_m,\ m = 0, 1, \ldots, M-1$, the CVaR price (value of the CVaR hedging portfolio) X_{t_m} of the call option $H = (S_T^1 - K)^+$ is defined by:*

(a) For $\alpha^* < 0$

$$\sum_{A(M_{t_m},\hat{z})} p^*_{n,T_m} CB\big(s^1_{t_m,n}, K(\hat{z}), \hat{\sigma}_1, T_m\big) + \sum_{\overline{A}(M_{t_m},\hat{z})} p^*_{n,T_m} \Big[CB\big(s^1_{t_m,n}, M_{t_m}b^{*n}, \hat{\sigma}_1, T_m\big)$$
$$+ e^{-rT_m}\big(M_{t_m}b^{*n} - K(\hat{z})\big)\Phi\big(\hat{\Lambda}^*_-(M_{t_m}b^{*n}, n, T_m)\big)\Big]; \tag{2.48}$$

(b) For $\alpha^* > 0$

$$\sum_{\overline{A}(M_{t_m},\hat{z})} p^*_{n,T_m}\Big[CB\big(s^1_{t_m,n}, K(\hat{z}), \hat{\sigma}_1, T_m\big) - CB\big(s^1_{t_m,n}, M_{t_m}b^{*n}, \hat{\sigma}_1, T_m\big)$$
$$+ e^{-rT_m}\big(K(\hat{z}) - M_{t_m}b^{*n}\big)\Phi\big(\hat{\Lambda}^*_-(M_{t_m}b^{*n}, n, T_m)\big)\Big], \tag{2.49}$$

where

$$T_m = T - t_m, \quad M_t = \hat{m}b^{*N_t}, \quad \hat{\Lambda}^*_\pm(x,n,t) = \frac{\ln\frac{s^1_{T-t,n}}{x} + (r \pm \frac{\hat{\sigma}_1^2}{2})t}{\hat{\sigma}_1\sqrt{t}}.$$

In addition, according to the fact

$$\frac{\partial CB(S_t, K, \sigma, T-t)}{\partial S_t} = \Phi\big(\frac{(\ln\frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\big)$$

and equations (2.48)-(2.49), over the hedging period $t_m - t_{m+1}$, shares of stock S^1, i.e., $\hat{\Delta}^1_{t_m} = \partial X_{t_m}/\partial S^1$, in the hedging portfolio are

(a) For $\alpha^* < 0$

$$\sum_{A(M_{t_m},\hat{z})} p^*_{n,T_m}v_{n,T_m}\Phi\Big(\hat{\Lambda}^*_+(K(\hat{z}), n, T_m)\Big) + \sum_{\overline{A}(M_{t_m},\hat{z})}\Big[\Phi\Big(\hat{\Lambda}^*_+(M_{t_m}b^{*n}, n, T_m)\Big)$$
$$+ e^{-rT_m}\frac{M_{t_m}b^{*n} - K(\hat{z})}{s^1_{t_m,n}\hat{\sigma}_1\sqrt{T_m}}\phi\Big(\hat{\Lambda}^*_-(M_{t_m}b^{*n}, n, T_m)\Big)\Big]p^*_{n,T_m}v_{n,T_m}; \tag{2.50}$$

(b) For $\alpha^* > 0$

$$\sum_{\overline{A}(M_{t_m},\hat{z})} p^*_{n,T_m}v_{n,T_m}\Big[\Phi\Big(\hat{\Lambda}^*_+(K(\hat{z}), n, T_m)\Big) - \Phi\Big(\hat{\Lambda}^*_+(M_{t_m}b^{*n}, n, T_m)\Big)$$
$$+ e^{-rT_m}\frac{K(\hat{z}) - M_{t_m}b^{*n}}{s^1_{t_m,n}\hat{\sigma}_1\sqrt{T_m}}\phi\Big(\hat{\Lambda}^*_-(M_{t_m}b^{*n}, n, T_m)\Big)\Big], \tag{2.51}$$

where $\phi(x)$ is the density function of a standard normal random variable.

Amounts $\hat{B}_{t_m}$ invested in the risk free asset during this period are given by

$$\hat{B}_{t_m} = X_{t_m} - \hat{\Delta}^1_{t_m}S^1_{t_m}. \tag{2.52}$$

2.3.3 Total hedging errors and total transaction costs of CVaR hedging method

At each revision point t_m ($m = 0, .., M-1$), the hedging portfolio includes $\hat{\Delta}^1_{t_m}$ shares of the risky asset and $\hat{B}_{t_m}$ amounts of the risk free bond, while at the next point t_{m+1}, the hedging strategy should be rebalanced such that it consists $\hat{\Delta}^1_{t_{m+1}}$ units of the risky asset, and $\hat{B}_{t_{m+1}}$ amounts of the risk free asset. Toft (1996) showed that the hedging position cannot be self-adjusted. Moreover, he defined the hedging errors at time t_{m+1} $(m = 0, .., M-1)$ as differences between the value of the hedging portfolio before and after rebalancing, i.e.,

$$\begin{aligned} HE_{t_{m+1}} &= \hat{\Delta}^1_{t_m} S^1_{t_{m+1}} + e^{r\delta t} \hat{B}_{t_m} - (\hat{\Delta}^1_{t_{m+1}} S^1_{t_{m+1}} + \hat{B}_{t_{m+1}}) \\ &= \hat{\Delta}^1_{t_m} S^1_{t_{m+1}} + e^{r\delta t} \hat{B}_{t_m} - X_{t_{m+1}}. \end{aligned} \tag{2.53}$$

Such amounts represent benefits from rehedging at time t_{m+1} and the present value of the total hedging errors during the whole hedging period are

$$HE = \sum_{m=0}^{M-1} e^{-rt_{m+1}} HE_{t_{m+1}}. \tag{2.54}$$

On the other hand, at each revision point t_{m+1} $(m = 0, .., M-1)$, the hedger needs to pay proportional transaction fees which equal to $TC_{t_{m+1}} = kS^1_{t_{m+1}} \mid \hat{\Delta}^1_{t_{m+1}} - \hat{\Delta}^1_{t_m} \mid$ and the present value of the total transaction costs during the whole contract period are

$$TC = \sum_{m=0}^{M-1} e^{-rt_{m+1}} TC_{t_{m+1}}. \tag{2.55}$$

Hedging errors are cash inflows from the CVaR hedging strategy, while transaction costs are cash outflows, and hence differences between HE and TC can be considered as the net income for the hedger during the whole CVaR hedging period. Toft (1996) provided explicit formulas of expected total hedging errors and total transaction costs for Leland delta hedging strategy of a call option. Later on, Melnikov and Tong (2014) derived closed-form expressions of HE and TC for the quantile hedging method with Leland's adjusted volatility. Both of them indicated that with Leland's volatility, hedging errors would offset transaction costs and hence the present value of net cash flows during the whole hedging period would be nearly zero if $\delta t \to 0$. Above authors considered the Black-Scholes model, however, in our case, since the form of $\hat{\Delta}^1_{t_m}$ are intricate, the closed-form expression of the present value of expected total hedging errors and total transaction costs would be too complicated. Instead, we would like to implement the time-based simulation method that described in Boyle and Hardy (1997) to investigate estimations of HE and TC. Note that in Boyle and Hardy (1997), hedges were based on the original volatility instead of the Leland's adjusted volatility. In our case, for comparison, we obtain estimated values of HE and TC for delta hedging with the original volatility σ_1 and with the adjusted volatility $\hat{\sigma}_1$ separately.

First, according to the exponential representation of the risky asset, i.e.,

$$S^1_{t_{m+1}} = S^1_{t_m} \exp\big[\sigma_1(W_{t_{m+1}} - W_{t_m}) + (\mu_1 - \frac{\sigma_1^2}{2})\delta t + (N_{t_{m+1}} - N_{t_m})\ln(1-\upsilon_1)\big], \tag{2.56}$$

where $W_{t_{m+1}} - W_{t_m} \overset{iid}{\sim} N(0, \delta t)$, and $N_{t_{m+1}} - N_{t_m} \overset{iid}{\sim} Poisson(\lambda \delta t)$, $m = 0, 1, \ldots, M-1$, we generate sequences of values of independent Normal as well as Poisson random variables and consequently, a series of stock price at each revision point is simulated. Then, by equations (2.54)-(2.55), total hedging errors and total transaction costs can be computed. Repeat such a process for N=2000 times and calculate average values of HE and TC. Such average values are estimations of present values of expected total hedging errors and total transaction costs.

In Melnikov and Skornyakova (2005), they considered the financial indices of Russell 2000 (RUT-I) and the Dow Jones Industrial Average (DJIA) as S^1 and S^2. They estimated (μ_i, σ_i) $(i = 1, 2)$ for those two risky assets using daily observations. In our paper, we assume underlying risky assets with same drifts and volatilities as Melnikov and Skornyakova, but with additional jump components. Parameters are listed as following:

$$\mu_1 = 0.0481,\ \sigma_1 = 0.2232, \quad \mu_2 = 0.0417,\ \sigma_2 = 0.2089,\ S^1_0 = S^2_0 = 100,$$

$$\upsilon_1 = -0.05,\ \upsilon_2 = -0.1,\ \lambda = 0.1,\ r = 0.03.$$

We assume there are 26 bi-weeks, 52 weeks, and 252 business days in one year. An investor would like to construct CVaR hedging for a call option $(S^1_T - K)^+$ with the strike price $K = 1.1S^1_0$. One way transaction costs rate is $k = 0.5\%$.

For a fixed risk constraint $C = 5$, Table 2.1 shows estimated present values of total hedging errors and total transaction costs for CVaR hedging of call options with different maturities and with various revision periods based on the adjusted volatility $\hat{\sigma}_1$. Table 2.2 displays corresponding results for hedging with the original volatility σ_1. It is observed that the total hedging errors implied by $\hat{\sigma}_1$ are positive and are not negligible, while, for the original volatility σ_1, they are very

Table 2.1 Estimated present values of total hedging errors and total transaction costs with the adjusted volatility $\hat{\sigma}_1$. C=5

Maturity T(years)	Revision period	CVaR price	HE	TC	HE-TC
	Biweekly	5.46	0.808	0.7688	0.0392
T=1	Weekly	5.7489	1.0306	1.0588	-0.0282
	Daily	6.8641	2.2084	2.2208	-0.0124
	Biweekly	14.6754	1.4408	1.3929	0.0479
T=3	Weekly	15.1997	1.8731	1.9117	-0.0386
	Daily	17.1948	3.9429	3.9721	-0.0292
	Biweekly	21.8367	1.6432	1.7092	-0.066
T=5	Weekly	22.488	2.3572	2.3839	-0.0267
	Daily	34.958	4.9192	4.9291	-0.0099
	Biweekly	35.6998	2.0476	2.083	-0.0354
T=10	Weekly	36.5054	2.9219	2.93	-0.0081
	Daily	39.5508	5.9992	5.9938	0.0054
	Biweekly	46.291	2.1415	2.1668	-0.0253
T=15	Weekly	47.1447	3.0194	3.0096	0.0098
	Daily	50.3664	6.2761	6.2833	-0.0073

small for all maturities and decrease in absolute values if hedges happen more frequently. Also, for a fixed maturity and a fixed revision period, total transaction costs are smaller if one hedges with the adjusted hedging volatility. Both transaction costs and hedging errors from $\hat{\sigma}_1$ increase as the revision period or the maturity time increase because more hedges occur and the differences between them are nearly zero especially under the daily rebablancing condition. However, the differences between total hedging errors and total transaction costs implied by σ_1 are significant and increase with respect to rebanlancing frequency. Those observations are consistent with results in Toft (1996) as well as Melnikov and Tong (2014) which pointed out that with a carefully chosen hedging volatility, hedging errors generate by it would almost offset total transaction costs during the whole hedging period.

Simulation results also indicate that the modified price of an option is close to the sum of its fair price in the transaction costs free market and the total transaction costs which confirms that our modified option value adjusts the fair price by including the total transaction fees.

Table 2.2 Estimated present values of total hedging errors and total transaction costs with the original volatility σ_1. C=5

Maturity T(years)	Revision period	CVaR price	HE	TC	HE-TC
	Biweekly	4.7298	-0.0298	0.8092	-0.839
T=1	Weekly	4.7298	0.0177	1.104	-1.0863
	Daily	4.7298	-0.0041	2.4556	-2.4597
	Biweekly	13.3329	0.0305	1.4584	-1.427
T=3	Weekly	13.3329	0.0182	2.0448	-2.0266
	Daily	13.3329	-0.0018	4.4939	-4.4957
	Biweekly	20.1652	-0.0288	1.7894	-1.8182
T=5	Weekly	20.1652	-0.0108	2.5308	-2.5416
	Daily	20.1652	-0.0029	5.5339	-5.5368
	Biweekly	33.6293	-0.022	2.1664	-2.1884
T=10	Weekly	33.6293	0.0215	3.0789	-3.0574
	Daily	33.6293	0.0053	6.7483	-6.743
	Biweekly	44.0962	-0.0259	2.2768	-2.3027
T=15	Weekly	44.0962	0.0081	3.2585	-3.2504
	Daily	44.0962	0.0035	6.9676	-6.9641

Table 2.3 Estimated present values of total hedging errors and total transaction costs with adjusted volatility $\hat{\sigma}_1$ for different levels of CVaR constraint. T=1

Revision period	$CVaR_{0.95} \leq 5$			$CVaR_{0.95} \leq 7.5$			$CVaR_{0.95} \leq 10$		
	HE	TC	$HE-TC$	HE	TC	$HE-TC$	HE	TC	$HE-TC$
Biweekly	0.808	0.7688	0.0392	0.7154	0.7505	-0.0351	0.6769	0.7228	-0.0459
Weekly	1.0306	1.0588	-0.0282	1.0291	1.0622	-0.0331	0.9786	1.0173	-0.0387
Daily	2.2084	2.2208	-0.0124	2.13	2.1519	-0.0219	2.0485	2.065	-0.0165

Table 2.4 Estimated present values of total hedging errors and total transaction costs with original volatility σ_1 for different levels of CVaR constraint. T=1

Revision period	$CVaR_{0.95} \leq 5$			$CVaR_{0.95} \leq 7.5$			$CVaR_{0.95} \leq 10$		
	HE	TC	$HE-TC$	HE	TC	$HE-TC$	HE	TC	$HE-TC$
Biweekly	-0.0298	0.8092	-0.839	-0.0064	0.7919	-0.7983	-0.0166	0.7383	-0.7549
Weekly	0.0177	1.104	-1.0863	0.0169	1.0956	-1.0787	0.0281	1.0077	-0.9796
Daily	-0.0041	2.4556	-2.4597	0.0042	2.3388	-2.3346	0.021	2.255	-2.234

Furthermore, we would like to investigate effects of the CVaR constraint on HE and TC. For a fixed time to maturity $T = 1$, Table 2.3 displays total hedging errors and total transaction costs for CVaR hedging with an adjusted volatility while Table 2.4 shows corresponding results for hedging with σ_1. One thing we would like to point out is that, for the same revision period, values of $HE - TC$ implied by σ_1 would increase if the risk constraint increases. This can be explained by the fact that on the one hand, for all levels of the risk constraint, HE are almost zero if one hedges with σ_1, while on the other hand, TC would decrease as the risk constraint increases, because the value of the hedging portfolio would decrease so do the transaction fees paid to rebalance it. But there is no clear relationship between the risk constraint and values of $HE - TC$ implied by $\hat{\sigma}_1$ since both HE and TC decrease if the risk constraint C increases.

2.4 CVaR-hedging of equity-linked life insurance contracts

One important application of partial hedging is to deal with the pricing of equity-linked life insurance contracts, an innovative area of life insurance. In Melnikov and Skornyakova (2005) as well as Kirch and Melnikov (2005), authors discussed the application of quantile hedging and efficient hedging on life insurance contracts. Here, we would like to implement CVaR-based partial hedging to this area.

Following actuarial traditions, let a random variable $T(x)$ on an "actuarial" probability space $(\Omega, \widetilde{\mathcal{F}}, \widetilde{P})$ denote the remaining life time of a person of current age x and ${}_Tp_x = \widetilde{P}(T(x) > T)$ be the survival probability for the next T years of an insured. Since usually the insurance risk reflected by the insured mortality and the financial market risk have no effect on each other, we would take a natural assumption that $(\Omega, \mathcal{F}, P)$ and $(\Omega, \widetilde{\mathcal{F}}, \widetilde{P})$ can be treated as independent. Consider a pure endowment contract with a fixed guarantee K which will pay an insured $\overline{H} = \max\{S_T^1, K\}$ at time T if the insured is still alive. Notice that, $\overline{H} = \max\{S_T^1, K\} = K + (S_T^1 - K)^+$, the payoff of the contract is determined by a call option plus a constant guarantee, so it is sufficient to only consider the embedded call option $H = (S_T^1 - K)^+$ for our purpose.

Let us start with the transaction costs free and complete market (2.1). Since the mortality risk is essentially independent from the financial market, the premium for such a contract is defined as (see Brennan and Schwartz 1976):

$$ {}_TU_x = E^*(e^{-rT}H)E^{\widetilde{P}}(I_{\{T(x)>T\}}) = {}_Tp_xV(0), \tag{2.57} $$

where x is the insured's age, T is the maturity time of the contract and $V(0)$ is the fair price of the embedded call option. Notice that ${}_TU_x < V(0)$, which means the premium that the insurance company can collect would be less than the initial wealth needed to hedge perfectly. Therefore, only a partial hedging strategy can be constructed. Here, we assume the insurance company would implement the CVaR-based partial hedging strategy that is described in Section 2.2 and accept some financial risk. The CVaR hedging costs $\tilde{V}_0$ can be derived from the equation (2.24) or (2.27) which intuitively should be equal to the premium of the contract, i.e., ${}_TU_x = \tilde{V}_0$. Hence, we have

$$ {}_TU_x = {}_Tp_xV(0) = \tilde{V}_0, \quad {}_Tp_x = \frac{\tilde{V}_0}{V(0)}. \tag{2.58} $$

The equation (2.58) is called a balance equation. It can be used to determine the survival probability of insureds for which the contract is suitable corresponding to a specified level of risk. Then, according to the most recently published United States 2015 Life Table (National Vital Statistics Reports volume 67, Number 7), target clients' age can also be found.

However, if transaction costs exist, both the price of the embedded call option and CVaR partial hedging costs should be based on the adjusted volatility. As a consequence, the balance equation in this case becomes:

$$ {}_Tp_x = \frac{X_0}{\overline{V}(0)}, \tag{2.59} $$

where $\overline{V}(0)$ is given by (2.47) and X_0 is given by (2.48) or (2.49).

For comparison, we calculate the survival probabilities and age of target clients for contracts with different maturities and under different risk constraints in the market (2.1) with and without transaction costs separately. Table 2.5 shows results in the market with transaction costs, while Table 2.6 displays results in the complete market. We assume rebalances occur daily. As expected, for contracts with the same maturity, target clients for the insurance company are older for a

Table 2.5 Survival probabilities and age of insured in the market with transaction costs

	$CVaR_{0.95} \leq 5$		$CVaR_{0.95} \leq 10$	
	${}_Tp_x$	age	${}_Tp_x$	age
T=3	0.9078	75	0.8237	82
T=5	0.9359	64	0.8762	72
T=10	0.9633	45	0.9284	53
T=15	0.9749	31	0.9507	41

Table 2.6 Survival probabilities and age of insured in the complete market

	$CVaR_{0.95} \leq 5$		$CVaR_{0.95} \leq 10$	
	${}_Tp_x$	age	${}_Tp_x$	age
T=3	0.8806	78	0.7741	84
T=5	0.9166	67	0.8398	75
T=10	0.9516	48	0.9058	57
T=15	0.9665	36	0.9343	44

higher CVaR constraint, that is, the increment of financial risk is compensated by the decrements of insurance risk. Also, insurance company can trade long term contracts among younger clients. It is worth to mention that, for contracts with the same maturity and under the same risk constraint, target clients' survival probability is higher if transaction costs exist, which may be explained by the fact that, with transaction fees, CVaR hedging costs increase more significantly compared to the adjusted price of the call option, in other words, $X_0 - \widetilde{V}_0 \geq \bar{V}(0) - V(0)$.

We further investigate effects of revision frequency and the level of risk constraint on the target clients' survival probability. Results are displayed in Figure 2.1 which indicates that for a fixed contract maturity $T = 5$ and under the same risk threshold, target clients' survival probability of an insurance contract increases as the revision frequency increases. This is because the CVaR hedging price inclusive of transaction costs increase to a larger extend if hedges occur more frequently and the increasing trend is more significant than that of the perfect hedging price which is the denominator in the balance equation (2.59). Also, we notice that differences among survival probabilities for various reversion periods are more obvious if the CVaR constraint is larger, since the CVaR price would decrease if C increases and therefore the effect of transaction costs (which can be measured by TC/X_0) would be more significant.

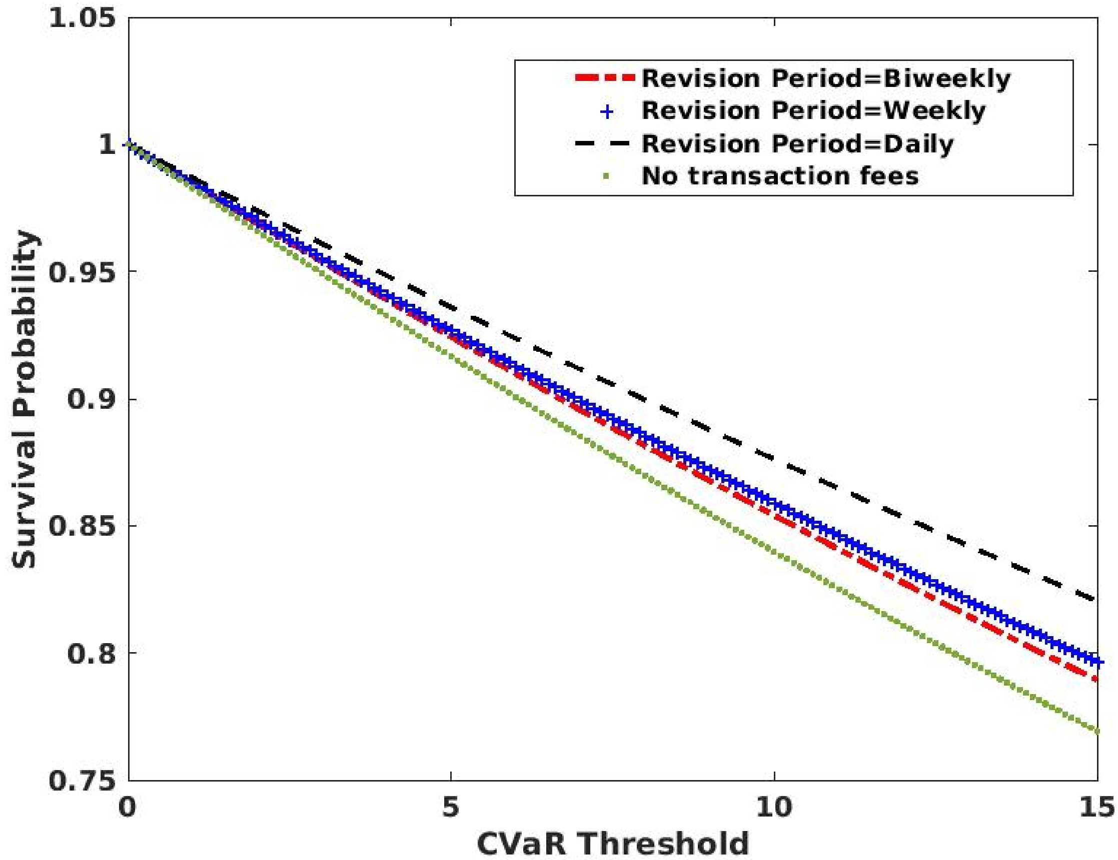

Figure 2.1 Survival Probability vs CVaR for Life insurance contracts for different revision frequency, T=5.

2.5 Conclusion

This chapter considers option pricing and CVaR partial hedging in the Jump-Diffusion financial market model with transaction costs. According to results presented in Melnikov and Smirnov (2012), we first derived a closed-form solution of optimal CVaR hedging strategy for a call option in the transaction costs free market (2.1). Then, we take transaction fees into consideration and prove that the modified option value process is a solution of a non-linear PDE which can only be solved numerically if options consist of two assets. However, for options based on a single asset and whose sign of gamma is a constant, their modified prices can be explained as pricing with an adjusted volatility which is consistent with the conclusion in Toft (1996). Note that, unlike the Leland's adjusted volatility in the Black-Scholes model, our adjusted volatility also depends on the jump size and Poisson intensity. In particular, an explicit formula of a call option price is given. In addition, the CVaR hedging costs as well as the weights of the hedging portfolio are then based on such an adjusted volatility $\hat{\sigma}_1$. Furthermore, estimated values of total hedging errors as well as total transaction costs are obtained by a simulation method and results indicate that, with such an adjusted hedging volatility $\hat{\sigma}_1$, differences between them are nearly zero which means total hedging errors generated by this carefully chosen hedging volatility can almost offset transaction costs. Finally, CVaR hedging is implemented into the area of equity-linked life insurance contracts to find target clients' survival probabilities and age. Our observations show that, with transaction costs, insurance companies can trade the same contract with younger clients.

For future studies, there are several possible directions worth exploring. First, other options can be considered. For instance, Melnikov and Skornyakova (2005) discussed quantile hedging for the option named change of assets, i.e, $(S_T^1 - S_T^2)^+$, which we can also apply CVaR hedging method to. Another interesting extension is to investigate CVaR hedging method in more complicated financial models. Since our research also focuses on the application of partial hedging to equity-linked life insurance contracts, which usually have long maturities (5, 10 or even 20 years), it is reasonable to consider a financial market with long-range dependence which is presented with the help of the fractional Brownian motion and its mixture with the standard Brownian Motion (MFBM). Bratyk and Mishura (2008) investigated quantile hedging in such a model. We can further study CVaR hedging in this financial market.

2.6 References

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CHAPTER 3

CVaR hedging in defaultable Jump-Diffusion markets

3.1 Introduction

Hedging and risk management are crucial topics in financial mathematics. In a complete market, any contingent claim can be replicated given sufficient initial wealth. However, if a market is incomplete, the initial costs of superhedging are often too high. In this case, a hedger usually allocates initial capitals that are less than the superhedging costs while accepts the possibility of shortfall. Such a hedging strategy is called partial hedging. Föllmer and Leukert are pioneers in this filed. They studied quantile hedging and efficient hedging (see Föllmer and Leukert 1999; 2000) in seimimartingale financial market models. In their papers, explicit solutions in complete markets were provided with the help of the classical Neyman-Pearson lemma while solutions in incomplete markets were given according to the convex duality approach. In this chapter, we consider the partial hedging problem in defaultable markets which may be incomplete. There is a list of references in this area. For instance, Nakano (2011) solved problems of optimal quantile hedging and efficient hedging with a linear loss function for claims with a single default time. Later, Melnikov and Nosrati (2015) focused on the efficient hedging problem with more general loss functions for claims with several independent default times and provided a closed form solution in the special case that recovery rates were zeros. Here, we would discuss partial hedging by employing a coherent risk measure called Conditional Value-at-Risk (CVaR) which provides information about the average loss that exceeds the Value-at-Risk (VaR). Melnikov and Smirnov (2012) studied partial hedging with such a measure and provided a semi-explicit solution in complete markets.

Aforementioned papers dealt with option pricing and hedging in Brownian market models. However, growing number of evidences show that pure diffusion models are not accurate enough to represent real life assets' dynamics. It is not rare to observe jumps in stock prices when some significant financial or political announcements are published. In order to address this drawback, a Jump-Diffusion model was proposed by Merton (1976). Melnikov and Skornyakova (2005), Kirch and Melnikov (2005) discussed quantile hedging and efficient hedging problems respectively in a two factor Jump-Diffusion model. However, to our knowledge, a Jump-Diffusion market model with defaults has not been well studied and CVaR hedging problems have not been discussed in this literature.

Our main objective in this chapter is to derive a hedging strategy that minimizes CVaR of the hedging loss subject to a constraint on the initial wealth in a Jump-Diffusion defaultable market. This chapter is organized as follows. In Section 3.2, we introduce our financial model. Several useful properties regarding the default time are listed. Most importantly, similar to the discussion about martingales in a defaultable Brownian market in Bielecki and Rutkowski (2004), we derive densities of martingale measures in our model. Furthermore, utilizing properties of equivalent martingale measures, we show that the minimal superhedging costs of a defaultable claim with a zero recovery rate coincide with the perfect hedging costs of the corresponding non-defaultable claim. In Section 3.3, the explicit form of the optimal CVaR hedging strategy in our incomplete market is derived. We prove that the CVaR minimization problem of a zero-recovery rate defaultable claim can be converted to a problem of finding the optimal randomized test in the default free complete market and the optimal strategy is then given by the option decomposition of a modified claim. In Section 3.4, a numerical example is provided to illustrate the implementation of our method in life insurance contracts that have stochastic guarantees. Section 3.5 gives a conclusion for the chapter.

3.2 Model set up and preliminaries

Let $(\Omega, \mathcal{G}, P)$ be a standard probability space. Consider a financial market with the terminal time $T \in (0, \infty)$ consisting one riskless asset $(S_t^0)_{t\in[0,T]}$ and two risky assets, $(S_t^1)_{t\in[0,T]}$, $(S_t^2)_{t\in[0,T]}$, described by a two factor Jump-Diffusion model:

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \quad S_0^0 = 1, \\ dS_t^i &= S_{t-}^i(\mu_i dt + \sigma_i dW_t - \upsilon_i dN_t), \quad S_0^i > 0, \quad i = 1, 2, \end{aligned} \tag{3.1}$$

where $r \geq 0$ is the risk-free interest rate. Constants $\mu_i \in R$, $\sigma_i > 0$, $\upsilon_i < 1$ are drifts, volatilities and jump parameters. W and N are independent Wiener process and Poisson process with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t = \sigma(N_s, W_s; s \leq t)$. Here, we assume $\sigma_1 > \sigma_2$ (otherwise, we can define the stock with a higher volatility as S^1) and the intensity of the Poisson process is a nonnegative constant λ.

In addition, let a positive random variable τ denote the default time such that $P(\tau = 0) = 0$ and $P(\tau > t) > 0, \ \forall t \geq 0$. The default indicator process is defined as

$$H_t = I_{\{\tau \leq t\}}, \, t \geq 0,$$

and the corresponding filtration generated by it is $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$, $\mathcal{H}_t = \sigma(H_s; s \leq t)$.

Let us specify that τ is independent of W and N. Moreover, let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be a joined filtration, i.e, $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. For simplicity, we assume $\mathcal{G} = \mathcal{G}_T$. It is worth to mention that since $\mathbb{H}$ and $\mathbb{F}$ are independent, any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale (see, Bielecki and Rutkowski 2004 Section 6.1.1).

In this chapter, credit risk is modeled with the help of a hazard process, that is, the survival probability is defined by

$$P(\tau > t) = e^{-\Gamma_t}, \tag{3.2}$$

where $(\Gamma_t)_{t\geq 0}$ is the hazard process such that $\Gamma_t = \int_0^t \beta_u du$ and β_u is a nonnegative deterministic function called the hazard rate of the random time τ.

Let us summarize some crucial results regarding the default time and the default indicator process from Bielecki and Rutkowski (2004) for the reader convenience:

1. The process

$$M_t = H_t - \int_0^{t\wedge\tau} \beta_u ds = H_t - \int_0^t \beta_u (1-H_u) du, \tag{3.3}$$

is both a $\mathbb{H}$-martingale and a $\mathbb{G}$-martingale; (Proposition 5.1.3);

2. The process $Q_t = (1-H_t)e^{\Gamma_t}$, $t \geq 0$ follows a $\mathbb{G}$-martingale with the dynamic $dQ_t = -Q_{t-}dM_t$. Moreover, for any bounded $\mathbb{F}$-martingale m the product Qm and the quadratic covariation $[Q,m]$ are $\mathbb{G}$-martingales; (Lemma 5.1.7);

3. For any $\mathcal{G}_t$-measurable random variable Y, there is a $\mathcal{F}_t$-measurable random variable

$$\tilde{Y} = \frac{E(I_{\{\tau>t\}}Y|\mathcal{F}_t)}{P(\tau > t)}, \tag{3.4}$$

such that $YI_{\{\tau>t\}} = \tilde{Y}I_{\{\tau>t\}}, P-a.s.$ (Lemma 5.1.2).

Note that the market (3.1) with default may be incomplete due to the additional random source τ, so that there is a set of martingale measures. Let us start with the structure of martingales in our setting.

Theorem 3.1. *For a $\mathcal{G}$-measurable integrable random variable X_T, we define a $\mathbb{G}$-measurable martingale $X_t = E(X_T|\mathcal{G}_t),\ t \in [0,T]$. Then, it admits the following representation:*

$$X_t = X_0 + \int_0^t \xi_u^W dW_u + \int_0^t \xi_u^N d\hat{N}_u + \int_0^t \xi_u^M dM_u + \int_0^t \xi_u^{MN} d[M,\hat{N}]_u, \tag{3.5}$$

where $\hat{N}_t = N_t - \lambda t$ is the compensated Poisson process and $\xi^W,\ \xi^N,\ \xi^{MN},\ \xi^M$ are $\mathbb{G}$-predictable processes.

Proof. Since $\mathcal{G}_T{=}\mathcal{H}_T \vee \mathcal{F}_T$, according to Bielecki and Rutkowski (2004), it is sufficient to consider a random variable $X_T = (1-H_s)Y'$ for some fixed $s \leq T$ and some $\mathcal{F}_T$-measurable random variable Y'.

We introduce a notation $\bar{Y} = e^{-\Gamma_s}Y'$ and hence

$$X_T = (1-H_s)Y' = (1-H_s)e^{\Gamma_s}\bar{Y} = Q_s\bar{Y}.$$

Let us define a $\mathbb{F}$-martingale m:

$$m_t = E(\bar{Y}|\mathcal{F}_t) = E(\bar{Y}) + \int_0^t \xi_u dW_u + \int_0^t \zeta_u d\hat{N}_u,$$

where ξ and ζ are $\mathbb{F}$-predictable processes. The second equality is because of the martingale representation property in the filtration $\mathbb{F}$.

Since Q is a process with finite variation, we have

$$[Q,m]_t = \sum_{u\le t}\Delta Q_u \Delta m_u = \sum_{u\le t} -Q_{u-}\zeta_u \Delta M_u \Delta \hat{N}_u = -\int_0^t Q_{u-}\zeta_u d[M,\hat{N}]_u.$$

Obviously, $m_T = \bar{Y}$ and according to the integration by parts formula, we get

$$\begin{aligned} X_T = Q_s m_T &= Q_0 m_0 + \int_0^T Q_{u-} dm_u + \int_0^T m_{u-} I_{[0,s]}(u) dQ_u + [Q,m]_s \\ &= E(\bar{Y}) + \int_0^T Q_{u-}\xi_u dW_u + \int_0^T Q_{u-}\zeta_u d\hat{N}_u - \int_0^T m_{u-} I_{[0,s]}(u) Q_{u-} dM_u \\ &- \int_0^T \zeta_u Q_{u-} I_{[0,s]}(u) d[M,\hat{N}]_u. \end{aligned}$$

With the choice $\xi^W_u = Q_{u-}\xi_u$, $\xi^N_u = Q_{u-}\zeta_u$, $\xi^M_t = -m_{u-}I_{[0,s]}(u)Q_{u-}$, $\xi^{MN} = -\zeta_u Q_{u-} I_{[0,s]}(u)$, (3.5) is proved. $\square$

A simple modification of Theorem 3.1 implies that the Radon-Nikodym density of a martingale measure $\tilde{P}$ which is equivalent to P on $(\Omega,\mathcal{G})$ has the form

$$\hat{Z}_t = \frac{d\tilde{P}}{dP}|_{\mathcal{G}_t} = 1 + \int_0^t Z_{u-}(\theta_u dW_u + \psi_u d\hat{N}_u + k_u dM_u + \gamma_u d[M,\hat{N}]_u), \tag{3.6}$$

where θ, ψ, k, γ are $\mathbb{G}$-predictable processes.

In addition, the solution of (3.6) is

$$\hat{Z}_t = \mathcal{E}_t\Big(\int_0^\cdot \theta_u dW_u + \int_0^\cdot \psi_u d\hat{N}_u + \int_0^\cdot k_u dM_u + \int_0^\cdot \gamma_u d[M,\hat{N}]_u\Big), \tag{3.7}$$

where $\mathcal{E}_t(\cdot)$ is the Doléans exponential.

Note that, in order for $\hat{Z}$ to be a positive martingale, we have to impose the following restrictions:

(a) $k_t > -1$, $\psi_t > -1$, $k_t + \psi_t + \gamma_t > -1, \forall t \in [0,T]$;

(b) $E(\hat{Z}_t) = 1$, $\forall t \in [0,T]$.

With the help of Girsanov theorem, we conclude that processes

$$W^*_t = W_t - \int_0^t \theta_u du, \;\; \tilde{N}_t = N_t - \int_0^t (1+\psi_u)\lambda du,$$

are martingales under the measure $\tilde{P}$ defined by (3.6).

Furthermore, we have to derive processes θ, ψ, k and γ. Let us consider discounted value processes $\tilde{S}_t^i = e^{-rt} S_t^i$, $(i = 1, 2)$ which can be rewritten as

$$\begin{aligned} d\tilde{S}_t^i &= \tilde{S}_{t-}^i \big[(\mu_i - r)dt + \sigma_i dW_t - v_i dN_t\big] \\ &= \tilde{S}_{t-}^i \big[(\mu_i - r + \sigma_i \theta_t - v_i(1 + \psi_t)\lambda)dt + \sigma_i dW_t^* - v_i d\tilde{N}_t\big], \ (i = 1, 2). \end{aligned} \tag{3.8}$$

They are martingales under the measure $\tilde{P}$ if drift terms vanish, i.e.,

$$\begin{cases} \mu_1 - r + \sigma_1 \theta_t - v_1(1 + \psi_t)\lambda = 0, \\ \mu_2 - r + \sigma_2 \theta_t - v_2(1 + \psi_t)\lambda = 0, \end{cases} \tag{3.9}$$

which implies

$$\theta = \frac{(\mu_1 - r)v_2 - (\mu_2 - r)v_1}{\sigma_2 v_1 - \sigma_1 v_2}, \quad \psi = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{(\sigma_2 v_1 - \sigma_1 v_2)\lambda} - 1. \tag{3.10}$$

However, processes k and γ cannot be determined uniquely and hence the market (3.1) with default is incomplete. We denote $\mathbb{P}^*$ as the set containing all martingale measures and $\mathbb{Z}^*$ is the set of Radon-Nikodym densities of martingale measures.

In particular, since $\hat{N}$ and M are purely discontinuous martingales, their quadratic covariation is $[M, \hat{N}]_t = \sum\limits_{u \leq t} \Delta M_u \Delta \hat{N}_u = \sum\limits_{u \leq t} \Delta H_u \Delta N_u$ and hence $[M, \hat{N}]_t I_{\{\tau > T\}}$ is 0 for all $t \in [0, T]$. Therefore, we arrive to

$$\begin{aligned} \hat{Z}_T I_{\{\tau > T\}} &= \mathcal{E}_T(\theta W + \psi \hat{N} + \int_0^\cdot k_u dM_u) I_{\{\tau > T\}} \\ &= \mathcal{E}_T(\theta W) \mathcal{E}_T(\psi \hat{N} + \int_0^\cdot k_u dM_u) I_{\{\tau > T\}} \\ &= \mathcal{E}_T(\theta W) \mathcal{E}_T(\psi \hat{N}) \mathcal{E}_T(\int_0^\cdot k_u dM_u) I_{\{\tau > T\}} \\ &= \exp\Big(\theta W_T - \frac{\theta^2}{2} T + (\lambda - \lambda^*)T + (\ln \lambda^* - \ln \lambda) N_T\Big) \\ &\quad \exp\Big(\int_0^T \ln(1 + k_u) dH_u - \int_0^{T \wedge \tau} k_u \beta_u du\Big) I_{\{\tau > T\}} = Z_T^* Z_T^k I_{\{\tau > T\}}, \end{aligned} \tag{3.11}$$

where

$$\lambda^* = (1 + \psi)\lambda = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{(\sigma_2 v_1 - \sigma_1 v_2)},$$

$$Z_T^* = \exp\big(\theta W_T - \frac{\theta^2}{2}T + (\lambda - \lambda^*)T + (\ln\lambda^* - \ln\lambda)N_T\big),$$

$$Z_T^k = \exp\big(-\int_0^T k_u\beta_u du\big).$$

The second and third equalities are both due to the multiplication rule of Doléans exponential and the fact $[W,M]_t = 0$, $[W,\hat{N}]_t = 0$, $[\hat{N},M]_t I_{\{\tau>T\}} = 0$, $\forall t \leq T$.

Remark 3.2. *The probability measure P^* defined by $\frac{dP^*}{dP} = Z_T^*$ that satisfies $\psi > -1$ and $\sigma_2 \upsilon_1 - \sigma_1 \upsilon_2 \neq 0$ is the unique martingale measure in the non-defaultable market (3.1) on the probability space $(\Omega, \mathcal{F}_T, P)$. Moreover, W^* and N are independent Wiener process and Poisson process (with the intensity λ^*) under this measure . We denote $E^*(\cdot)$ as the expectation under the measure P^*.*

With notations introduced above, $S_t^i, \;\; i = 1,2$, can be rewritten as

$$\begin{aligned} S_t^i &= S_0^i \exp\big(\sigma_i W_t + (\mu_i - \frac{1}{2}{\sigma_i}^2)t + N_t \ln(1-\upsilon_i)\big) \\ &= S_0^i \exp\big(\sigma_i W_t^* + (r + \upsilon_i\lambda^* - \frac{1}{2}{\sigma_i}^2)t + N_t \ln(1-\upsilon_i)\big). \end{aligned} \tag{3.12}$$

A $\mathbb{G}$-strategy is a $\mathbb{G}$-predictable process $\pi := (\pi_t^0, \pi_t^1, \pi_t^2)_{t\in[0,T]}$ such that

$$\int_0^T |\pi_t^0| dt < \infty, \int_0^T (\pi_t^i S_t^i)^2 dt < \infty, P-a.s \;\; (i = 1,2),$$

and the value process corresponding to the strategy π at time $t \in [0,T]$ is

$$V_t = \pi_t^0 S_t^0 + \pi_t^1 S_t^1 + \pi_t^2 S_t^2. \tag{3.13}$$

In addition, for a given initial value $v \geq 0$, a trading strategy is called self-financing admissible if its value process satisfies

$$V_t = v + \int_0^t \pi_u^0 dS_u^0 + \int_0^t \pi_u^1 dS_u^1 + \int_0^t \pi_u^2 dS_u^2,$$

and

$$V_t \geq 0, \; \forall t \in [0,T]. \tag{3.14}$$

We denote the set of all admissible self-financing strategies with an initial value v as $\mathcal{A}(v)$.

Remark 3.2 indicates that the default free market (3.1) is complete with the filtration $\mathbb{F}$ and hence the fair price of any contingent claim with payoff C which is nonnegative $\mathcal{F}_T$-measurable is defined as $E^*(e^{-rT}C)$ and there is a self-financing strategy π that duplicates it which is determined by:

$$\pi_t^1 S_t^1 = \frac{(V_{S^1} S_t^1 \sigma_1 + V_{S^2} S_t^2 \sigma_2)\upsilon_2 + (V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2, t)) - V(S_{t-}^1, S_{t-}^2, t))\sigma_2}{\sigma_1 \upsilon_2 - \sigma_2 \upsilon_1}, \quad (3.15)$$

$$\pi_t^2 S_t^2 = \frac{(V_{S^1} S_t^1 \sigma_1 + V_{S^2} S_t^2 \sigma_2)\upsilon_1 + (V(S_{t-}^1(1-\upsilon_1), S_{t-}^2(1-\upsilon_2, t)) - V(S_{t-}^1, S_{t-}^2, t))\sigma_1}{\sigma_2 \upsilon_1 - \sigma_1 \upsilon_2}, \quad (3.16)$$

$$\pi_t^0 S_t^0 = V(t) - \pi_t^1 S_t^1 - \pi_t^2 S_t^2, \quad (3.17)$$

where $V(t) = V(S_t^1, S_t^2, t) = e^{-r(T-t)} E^*(C \mid \mathcal{F}_t)$ is the value of C at time t and

$$V_{S^1} = \frac{\partial V(S_t^1, S_t^2, t)}{\partial S^1}, V_{S^2} = \frac{\partial V(S_t^1, S_t^2, t)}{\partial S^2}.$$

(see the proof for (2.7)-(2.9)).

However, we would like to investigate the hedging problem of a defaultable claim $C^0 = CI_{\{\tau > T\}}$ in the enlarged filtration $\mathbb{G}$ described before. As we know from the option pricing theory in incomplete markets, the minimal initial superhedging costs of such a claim is defined as

$$U_0 = \sup_{\tilde{P} \in \mathbb{P}^*} E^{\tilde{P}}(e^{-rT} C^0). \quad (3.18)$$

Let us denote

$$\tilde{U}_t = U_t e^{-rt} = \underset{\tilde{P} \in \mathbb{P}^*}{\text{ess sup}} E^{\tilde{P}}(e^{-rT} C^0 | \mathcal{G}_t), \quad t \in [0, T], \quad (3.19)$$

which is a supermartingale with respect to any $\tilde{P} \in \mathbb{P}^*$ and represents the discounted value process of the minimal superhedging strategy of the claim C^0.

According to the optional decomposition theorem (see El Karoui and Quenez 1995, Kramkov 1996), there is an admissible strategy (U_0, π) and a discounted optional consumption process D with $D_0 = 0$ such that

$$\tilde{U}_t = U_0 + \int_0^t \pi_u^1 d\tilde{S}_u^1 + \int_0^t \pi_u^2 d\tilde{S}_u^2 - D_t. \quad (3.20)$$

In our setting, the minimal superhedging costs U_0 can be derived explicitly.

Lemma 3.3. *Assume $E^*(C) < +\infty$. In the market (3.1) with default, the discounted superhedging value process of C^0 satisfies*

$$\tilde{U}_t = ess \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(e^{-rT}C^0|\mathcal{G}_t\big) = E^*\big(e^{-rT}C|\mathcal{F}_t\big)I_{\{\tau>t\}}, \quad \in [0,T]. \tag{3.21}$$

In particular, $U_0 = \tilde{U}_0 = E^*(e^{-rT}C)$, i.e., the minimal superhedging costs equal to the fair price of the contingent claim C in the non-defaultable market.

Proof. For a given martingale measure $\tilde{P}$ with a density $\hat{Z}_T$, we know that

$$\tilde{U}_t = \text{ess}\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}[e^{-rT}CI_{\{\tau>T\}}|\mathcal{G}_t] = \text{ess}\sup_{\hat{Z}\in\mathbb{Z}^*} \frac{E[e^{-rT}\hat{Z}_T CI_{\{\tau>T\}}|\mathcal{G}_t]}{\hat{Z}_t}.$$

Moreover, by (3.11), we have

$$E[\hat{Z}_T CI_{\{\tau>T\}}|\mathcal{G}_t] = e^{-\int_0^t k_u\beta_u du}E[Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{G}_t].$$

Furthermore, according to (3.4), the $\mathcal{F}_t$-measurable random variable

$$\begin{aligned} &E\Big[E[Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{G}_t]I_{\{\tau>t\}}|\mathcal{F}_t\Big]e^{\int_0^t \beta_u du} \\ &= E\Big(Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{F}_t\Big)e^{\int_0^t \beta_u du} \end{aligned}$$

satisfies

$$E\big(Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{F}_t\big)e^{\int_0^t \beta_u du}I_{\{\tau>t\}} = E\big(Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{G}_t\big)I_{\{\tau>t\}}.$$

Thus, $\tilde{U}_t$ can be rewritten as

$$\begin{aligned} \tilde{U}_t &= \text{ess}\sup_{\hat{Z}\in\mathbb{Z}^*} \frac{e^{-\int_0^t k_u\beta_u du}E\big(e^{-rT}Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{F}_t\big)e^{\int_0^t \beta_u du}I_{\{\tau>t\}}}{\hat{Z}_t} \\ &= \text{ess}\sup_{\hat{Z}\in\mathbb{Z}^*} \frac{E(e^{-rT}Z_T^* e^{-\int_t^T k_u\beta_u du}CI_{\{\tau>T\}}|\mathcal{F}_t)}{Z_t^*}e^{\int_0^t \beta_u du}I_{\{\tau>t\}}. \end{aligned}$$

In particular, choosing k constant and $k \searrow -1$, we have

$$\tilde{U}_t \geq \lim_{k\searrow -1} \frac{e^{-\int_t^T k\beta_u du}E(e^{-rT}Z_T^* CI_{\{\tau>T\}}|\mathcal{F}_t)}{Z_t^*}e^{\int_0^t \beta_u du}I_{\{\tau>t\}} = E^*(e^{-rT}C|\mathcal{F}_t)I_{\{\tau>t\}}, \tag{3.22}$$

where the second equality is due to the independence of τ and $\mathcal{F}_T$.

On the other hand, since $k_t > -1$, we get

$$\tilde{U}_t \le \frac{E(e^{-rT} Z_T^* e^{\int_t^T \beta_u du} C I_{\{\tau>T\}} | \mathcal{F}_t)}{Z_t^*} e^{\int_0^t \beta_u du} I_{\{\tau>t\}} = E^*\big(e^{-rT} C | \mathcal{F}_t\big) I_{\{\tau>t\}}. \tag{3.23}$$

Combing (3.22) and (3.23), we arrive to

$$\tilde{U}_t = E^*\big(e^{-rT} C | \mathcal{F}_t\big) I_{\{\tau>t\}}. \tag{3.24}$$

□

3.3 CVaR hedging

If a hedger allocates less capitals than the minimum superhedging costs U_0, there is a possibility of shortfall characterized by $L = C^0 - V_T$, where V_T is the value of a hedging portfolio at T.

Our goal is to find a self-financing admissible strategy with an initial budget constraint $v_0 < U_0$ that minimizes hedging losses under the measure CVaR, i.e.,

$$\min_{(v,\pi)\in\mathcal{A}_0} CVaR_\alpha(L), \tag{3.25}$$

where $\mathcal{A}_0 = \{(v,\pi) | (v,\pi) \in \mathcal{A}(v), v \le v_0\}$ is the set of self-financing admissible strategies with the initial hedging capital no more than v_0.

With the help of the alternative representation of CVaR (1.3), the problem (3.25) becomes

$$\min_{(v,\pi)\in\mathcal{A}_0} CVaR_\alpha(L) = \min_{(v,\pi)\in\mathcal{A}_0} \min_{z\in\mathbb{R}} \Big[z + \frac{1}{1-\alpha} E\big((C^0 - V_T - z)^+\big)\Big]. \tag{3.26}$$

Melnikov and Smirnov (2012) have indicated that we can interchange the order of two minimization problems:

$$\begin{aligned}\min_{(v,\pi)\in\mathcal{A}_0} CVaR_\alpha(L) &= \min_{(v,\pi)\in\mathcal{A}_0} \min_{z\in\mathbb{R}} \Big[z + \frac{1}{1-\alpha} E\big((C^0 - V_T - z)^+\big)\Big] \\ &= \min_{z\in\mathbb{R}} \Big[z + \frac{1}{1-\alpha} \min_{(v,\pi)\in\mathcal{A}_0} E\big((C^0 - V_T - z)^+\big)\Big]. \end{aligned} \tag{3.27}$$

and if the inner minimization problem in (3.27) for each z is solved, then the initial problem (3.27) is reduced to a one-dimensional optimization problem over z.

Thus, for a fixed z, let us consider the problem

$$\min_{(v,\pi)\in\mathcal{A}_0} E\big((C^0 - V_T - z)^+\big). \tag{3.28}$$

We focus on $z \geq 0$, because z is corresponding to the VaR_α of the hedging loss and it is nonnegative when α is close to 1. Consequently, we have

$$(C^0 - V_T - z)^+ = ((C^0 - z)^+ - V_T)^+ = (C^0(z) - V_T)^+,$$

where $C^0(z) = (C^0 - z)^+ = (C - z)^+ I_{\{\tau>T\}} = C(z)I_{\{\tau>T\}}$ with $C(z) = (C - z)^+$.

Obviously, $C^0(z)$ is a $\mathcal{G}_T$-measurable non-negative random variable, so it can be treated as a contingent claim and thus the problem (3.28) is equivalent to an optimal efficient hedging problem of the contingent claim $C^0(z)$. Föllmer and Leukert (2000) studied this kind of problem (also see Chapter 1.4) and proved that if a random variable φ' solves

$$\max_{\varphi\in\mathcal{R}} E\big(\varphi C^0(z)\big), \tag{3.29}$$

where $\mathcal{R} = \{\varphi : \Omega \to [0,1] |\ \mathcal{G}_T - \text{measurable}, \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(e^{-rT}C^0(z)\varphi\big) \leq v_0\}$, then the optimal hedging strategy π is obtained from the optional decomposition (3.20) for the modified claim $\varphi' C^0(z)$. However, usually the optimal randomized test does not admit an explicit form in incomplete markets. To address such a difficulty, we transfer the optimization problem (3.29) which is in the enlarged filed $\mathcal{G}_T$ into a problem in the filed $\mathcal{F}_T$.

Lemma 3.4. *If a $\mathcal{F}_T$-measurable random variable $\tilde{\varphi} \in \tilde{\mathcal{R}}$ solves the problem*

$$\max_{\varphi\in\tilde{\mathcal{R}}} E\big(\varphi C(z)\big), \tag{3.30}$$

where $\tilde{\mathcal{R}} = \{\varphi : \Omega \to [0,1] |\ \mathcal{F}_T - \text{measurable}, E^*\big(e^{-rT}C(z)\varphi\big) \leq v_0\}$, then $\varphi' = \tilde{\varphi} I_{\{\tau>T\}}$ is the solution of the problem (3.29).

Proof. Suppose $\tilde{\varphi}$ is the solution for (3.30). Let us define $\varphi' = \tilde{\varphi} I_{\{\tau>T\}}$. Obviously $\varphi' \in [0,1]$ and is $\mathcal{G}_T$-measurable.

Also, by Lemma 3.3, we know that

$$\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(e^{-rT}C^0(z)\varphi'\big) = \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(e^{-rT}C(z)\tilde{\varphi} I_{\{\tau>T\}}\big) = E^*\big(e^{-rT}C(z)\tilde{\varphi}\big) \leq v_0,$$

and hence we have $\varphi' \in \mathcal{R}$.

On the other hand, for any other $\varphi \in \mathcal{R}$, define $\varphi_0 = E(I_{\{\tau>T\}}\varphi|\mathcal{F}_T)e^{\int_0^T \beta_u du}$ and by (3.4), we conclude that $\varphi I_{\{\tau>T\}} = \varphi_0 I_{\{\tau>T\}}$. Thus, $E^*(e^{-rT}C(z)\varphi_0) = \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}(e^{-rT}C^0(z)\varphi) \leq v_0$ which indicates that $\varphi_0 \in \tilde{\mathcal{R}}$.

Furthermore,

$$
\begin{aligned}
E\big(\varphi C^0(z)\big) &= E\big(\varphi C(z) I_{\{\tau>T\}}\big) = E\big(\varphi_0 C(z) I_{\{\tau>T\}}\big) \\
&= E\big(\varphi_0 C(z)\big) E(I_{\{\tau>T\}}) \leq E\big(\tilde{\varphi} C(z)\big) E\big(I_{\{\tau>T\}}\big) = E\big(\varphi' C^0(z)\big),
\end{aligned}
\tag{3.31}
$$

where the last line is due to the independence of $\mathcal{F}_T$ and τ and the optimal property of $\tilde{\varphi}$.

The inequality (3.31) indicates that φ' is the solution of (3.29) and hence this Lemma is proved. □

The solution of the problem (3.30) is given with the help of the classical Neyman-Pearson lemma (see Föllmer and Leukert 2000) such that the optimal randomized test has the form

$$
\varphi^*(z) = I_{\{\tilde{a}(z) < Z_T^{*\,-1}\}} + \Gamma(z) I_{\{\tilde{a}(z) = Z_T^{*\,-1}\}} \tag{3.32}
$$

$$
\tilde{a}(z) = \inf\{a \geq 0, E^*(C(z) I_{\{a < Z_T^{*\,-1}\}}) \leq v_0 e^{rT}\}, \tag{3.33}
$$

$$
\Gamma(z) = \frac{v_0 e^{rT} - E^*(C(z) I_{\{\tilde{a}(z) < Z_T^{*\,-1}\}})}{E^*(C(z) I_{\{\tilde{a}(z) = Z_T^{*\,-1}\}})}. \tag{3.34}
$$

Here, $\Gamma(z) = 0$ if $P(\tilde{a}(z) = Z_T^{*\,-1}) = 0$ and such a condition is satisfied in our setting.

Theorem 3.5. *A. The optimal hedging strategy* $(v_0, \hat{\pi})$ *of the problem (3.25) is given by the optional decomposition of the modified contingent claim* $\varphi^*(\hat{z})C(\hat{z})I_{\{\tau>T\}}$*, where* $\varphi^*(z)$ *is given by (3.32)-(3.34) and* $\hat{z}$ *is the point of minimum of the function*

$$
d(z) = z + \frac{1}{1-\alpha} \cdot E\left[C(z)(1-\varphi^*(z)) I_{\{\tau>T\}}\right],
$$

in the interval $[0, z^*]$*, where* z^* *is the solution of the equation*

$$
\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(e^{-rT}C^0(z)\big) = E^*\big(e^{-rT}C(z)\big) = v_0.
$$

In addition, the value $d(\hat{z})$ is the minimal CVaR.

B. The optimal hedging strategy $\hat{\pi} = (\hat{\pi}_t^0, \hat{\pi}_t^1, \hat{\pi}_t^2)_{t\in[0,T]}$ satisfies

$$\hat{\pi}_t^i = {\pi'}_t^i I_{\{\tau\geq t\}}, \ \ (i = 1, 2);$$

and

$$\hat{\pi}_t^0 = \begin{cases} {\pi'}_t^0, & \text{for } t \in [0, \tau \wedge T], \\ V'_\tau e^{-r\tau}, & \text{for } t \in (\tau, T], \end{cases}$$

where $\pi' = ({\pi}_t'^0, {\pi}_t'^1, {\pi}_t'^2)_{t\in[0,T]}$ is the duplication strategy of the nondefaultable claim $\varphi^*(\hat{z})C(\hat{z})$ derived from (3.15)-(3.17) and

$$V'_t = \pi_t'^0 S_t^0 + \pi_t'^1 S_t^1 + \pi_t'^2 S_t^2, \ \ t \in [0, T],$$

is the value process of such a strategy.

Proof. For a fixed z, from Lemma 3.4 and equations (3.32)-(3.34), we know that the optimal hedging strategy of the efficient hedging problem (3.29) is a superhedging strategy of the modified claim $\varphi^*(z)C(z)I_{\{\tau>T\}}$ and hence the shortfall is

$$\begin{aligned} (C^0(z) - V_T)^+ &= C^0(z) - V_T \wedge C^0(z) \\ &= C^0(z) - \varphi^*(z)C(z)I_{\{\tau>T\}} \\ &= \big(1 - \varphi^*(z)\big)C(z)I_{\{\tau>T\}}. \end{aligned}$$

Thereby,

$$CVaR(L) = \min_{z\in\mathbb{R}}\Big[z + \frac{1}{1-\alpha}E\Big(C(z)\big(1 - \varphi^*(z)\big)I_{\{\tau>T\}}\Big)\Big].$$

With the notation

$$d(z) = z + \frac{1}{1-\alpha}E\Big(C(z)\big(1 - \varphi^*(z)\big)I_{\{\tau>T\}}\Big),$$

it is clear that $CVaR(L) = d(\hat{z})$, where $\hat{z}$ is the point of minimum of the function $d(z)$.

Furthermore, applying Lemma 3.3, the discounted value process of the superhedging strategy of the modified claim $\varphi^*(\hat{z})C(\hat{z})I_{\{\tau>T\}}$ is

$$\begin{aligned} \tilde{U}_t &= E^*\big(e^{-rT}\varphi^*(\hat{z})C(\hat{z})|\mathcal{F}_t\big)I_{\{\tau>t\}} \\ &= E^*\big(e^{-rT}\varphi^*(\hat{z})C(\hat{z})|\mathcal{F}_t\big) - E^*\big(e^{-rT}\varphi^*(\hat{z})C(\hat{z})|\mathcal{F}_t\big)I_{\{\tau\leq t\}}. \end{aligned} \tag{3.35}$$

Let us define

$$V_t' e^{-rt} = E^*(e^{-rT}\varphi^*(\hat{z})C(\hat{z})|\mathcal{F}_t),$$

which is the discounted value process for $\varphi^*(\hat{z})C(\hat{z})$ in the default free market and this non-defaultable claim has a replication strategy π' such that

$$V_t' e^{-rt} = v_0 + \int_0^t \pi_u'^1 d\tilde{S}_u^1 + \int_0^t \pi_u'^2 d\tilde{S}_u^2,$$

where π'^i, $i = 1, 2$ are $\mathbb{F}$ predictable.

Applying the integration by parts formula for $E^*(e^{-rT}\varphi^*(\hat{z})C(\hat{z})|\mathcal{F}_t)I_{\{\tau\leq t\}} = V_t' e^{-rt}H_t$ and because H_t has finite variation, we get

$$V_t' e^{-rt} H_t = \int_0^t H_{u-}\pi_u'^1 d\tilde{S}_u^1 + \int_0^t H_{u-}\pi_u'^2 d\tilde{S}_u^2 + \int_0^t V_u' e^{-ru} dH_u.$$

Notice that $H_{u-} = \lim_{t\nearrow u} I_{\{\tau\leq t\}} = I_{\{\tau<u\}}$ and consequently we have

$$\tilde{U}_t = v_0 + \int_0^t \pi_u'^1 I_{\{\tau\geq u\}} d\tilde{S}_u^1 + \int_0^t \pi_u'^2 I_{\{\tau\geq u\}} d\tilde{S}_u^2 - V_\tau' e^{-r\tau} H_t. \tag{3.36}$$

Comparing (3.36) with the form of option decomposition, we conclude

$$\hat{\pi}_t^i = \pi_t'^i I_{\{\tau\geq t\}}, \ \ (i = 1, 2) \ \ \text{and} \ \ D_t = V_\tau' e^{-r\tau} H_t. \tag{3.37}$$

Note that, for $\tau \geq t$, we have

$$\hat{\pi}_t^0 S_t^0 = (V_t' - \pi_t'^1 S_t^1 - \pi_t'^2 S_t^2) = \pi_t'^0 S_t^0,$$

which implies $\hat{\pi}_t^0 = \pi_t'^0$, while for $\tau < t$, we get

$$\hat{\pi}_t^0 = V_\tau' e^{-r\tau}.$$

The proof is completed. □

Remark 3.6. *The reason why we focus on $z \leq z^*$ is that, with the initial capital v_0, for $z \geq z^*$, $C^0(z)$ can be hedged perfectly, which means $\min_{(\pi,v)\in\mathcal{A}_0} E\big((C^0(z) - V_T)^+\big) = 0$ and $d(z) = z$. $d(z)$ is increasing after z^* and hence it achieves its minimum during $[0, z^*]$.*

Remark 3.7. *The optimal hedging strategy can be explained as constructing the perfect hedging of the modified claim $\varphi^*(\hat{z})C(\hat{z})$ during $[0, \tau]$. However, if a default event happens, the investor should hold zero positions in risky assets and deposit all cash into the risk free account. Also he can consume up to that amount.*

3.4 Applications to equity-linked life insurance contracts

One important application of partial hedging is to deal with pricing and hedging of equity-linked life insurance contracts. In this section, we will calculate the minimal CVaR and provide the optimal CVaR hedging strategy of defaultable equity linked life insurance contracts.

Following actuarial traditions, let a random variable $T(x)$ on an "actuarial" probability space $(\Omega, \mathcal{G}_2, P_2)$ denote the remaining life time of a person of current age x and ${}_Tp_x = P_2(T(x) > T)$ be the survival probability for the next T years of an insured. There are two sources of risk. The first one is market risk associated with underline assets prices as well as the default time and another one is insurance risk reflected by the insured mortality. Since usually the insurance risk and the financial market risk have no effect on each other, we would take a natural assumption that $(\Omega, \mathcal{G}, P)$ and $(\Omega, \mathcal{G}_2, P_2)$ are independent.

Instead of simple call or put options, we consider a pure endowment life insurance contract with payoff $C^0 = \max(S_T^1, S_T^2)I_{\{\tau>T\}}$ provided that an insured is alive at T. This kind of contract has a flexible guarantee S^2 and a potential for future gains associated with S^1. It is popularly traded in insurance companies. According to Brennan and Schwartz (1976), the premium for such a contract is defined as

$${}_TU_x = \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}(e^{-rT}C^0)E^{P_2}(I_{\{T(x)>T\}}) = {}_Tp_xU_0, \tag{3.38}$$

where x is the insured's age, T is the maturity time of the contract and U_0 is the minimal superhedging costs of the defaultable claim C^0. Notice that ${}_TU_x < U_0$, which means the premium that the insurance company collects would be less than the minimal superhedging costs and therefore only a partial hedging strategy can be constructed. In our setting, we assume that the insurance company constructs the optimal hedging strategy described in Section 3.3 with the premium $v_0 = {}_Tp_xU_0$.

Theorem 3.8. *A. The minimal superhedging costs U_0 of the defaultable claim $C^0 = \max(S^1_T, S^2_T)I_{\{\tau>T\}}$ is*

$$U_0 = \sum_{n=0}^{\infty} p^*_{n,T}\Big[s^1_{0,n}\Phi(\Lambda_1(n)+\sigma_1\sqrt{T}) + s^2_{0,n}\Phi(-\Lambda_1(n)-\sigma_2\sqrt{T})\Big]. \tag{3.39}$$

B. The optimal CVaR hedging strategy for $C^0 = \max(S^1_T, S^2_T)I_{\{\tau>T\}}$ is given by the perfect hedging of the modified contingent claim $(\max(S^1_T, S^2_T)-\hat{z})^+ I_{\{Z^{*\,-1}_T > \tilde{a}(\hat{z})\}}$ during $[0,\tau]$ while holding zero positions in risky assets after the default, where $\tilde{a}(z)$ is the unique solution of $e^{-rT}E^*(C(z)I_{\{a<Z^{*\,-1}_T\}}) = v_0$, that is

$$\sum_{n=0}^{\infty} p^*_{n,T}\Big[s^1_{0,n}\Phi^2\big(\Lambda_{1,3}(z,n)+\sigma_1\sqrt{T}, \Lambda_2(a,n)-\sigma_1 sign(\theta)\sqrt{T}, \Sigma_1\big)$$
$$+ s^2_{0,n}\Phi^3\big(-\Lambda_1(n)-\sigma_2\sqrt{T}, \Lambda_2(a,n)-\sigma_2 sign(\theta)\sqrt{T}, \Lambda_4(z,n)+\sigma_2\sqrt{T}, \Sigma_2\big)$$
$$- ze^{-rT}(\Phi^2\big(\Lambda_{1,3}(z,n), \Lambda_2(a,n), \Sigma_1\big) + \Phi^3\big(-\Lambda_1(n), \Lambda_2(a,n), \Lambda_4(z,n), \Sigma_2\big)\Big] = v_0. \tag{3.40}$$

$\hat{z}$ is the point of minimum of the function

$$d(z) = z + \frac{1}{1-\alpha}\cdot f(z) \tag{3.41}$$

in the interval $[0, z^*]$, where $f(z) = E[C(z)(1-\varphi^*(z))I_{\{\tau>T\}}]$, i.e.,

$$f(z) = e^{-\int_0^T \beta_u du}\sum_{n=0}^{\infty} p_{n,T}\big[S^1_0(1-v_1)^n e^{\mu_1 T}\Phi^2(\Lambda_{5,6}(n)+\sigma_1\sqrt{T}, \Lambda_7(n)+\sigma_1 sign(\theta)\sqrt{T}, \Sigma_3)$$
$$+ S^2_0(1-v_2)^n e^{\mu_2 T}\Phi^3(-\Lambda_6(n)-\sigma_2\sqrt{T}, \Lambda_7(n)+\sigma_2 sign(\theta)\sqrt{T}, \Lambda_8(n)+\sigma_2\sqrt{T}, \Sigma_4)$$
$$- z\big(\Phi^2(\Lambda_{5,6}(n), \Lambda_7(n), \Sigma_3) + \Phi^3(-\Lambda_6(n), \Lambda_7(n), \Lambda_8(n), \Sigma_4)\big)\big]. \tag{3.42}$$

Meanwhile, $d(\hat{z})$ is the corresponding value of minimal CVaR.

The parameter z^* is determined from $E^*(e^{-rt}C(z)) = v_0$, that is

$$\sum_{n=0}^{\infty} p^*_{n,T}\Big[s^1_{0,n}\Phi(\Lambda_{1,3}(z,n)+\sigma_1\sqrt{T}) + s^2_{0,n}\Phi^2(-\Lambda_1(n)-\sigma_2\sqrt{T}, \Lambda_4(z,n)+\sigma_2\sqrt{T}, \Sigma_5)$$
$$- ze^{-rT}\big(\Phi(\Lambda_{1,3}(z,n)) + \Phi^2(-\Lambda_1(n), \Lambda_4(z,n), \Sigma_5)\big)\Big] = v_0. \tag{3.43}$$

Here, $\Phi(\cdot)$ is the distribution function of a standard normal random variable and $\Phi^J(\cdot,\Sigma)$ denotes the distribution function of J jointly normally distributed random variables with zero means, unit variances and the correlation matrix Σ and

$$s^i_{t,n} = S^i_t e^{v_i \lambda^*(T-t)}(1-v_i)^n,$$

$$\Lambda_1(n) = \frac{\ln(\frac{s^1_{0,n}}{s^2_{0,n}}) + \frac{\sigma_2^2-\sigma_1^2}{2}T}{(\sigma_1-\sigma_2)\sqrt{T}}, \quad \Lambda_2(a,n) = \frac{-\ln(aZ^*_t) - (\frac{\theta^2}{2}+\lambda-\lambda^*)T - (\ln\lambda^* - \ln\lambda)n}{|\theta|\sqrt{T}},$$

$$\Lambda_3(z,n) = \frac{\ln\frac{s^1_{0,n}}{z} + (r-\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}, \quad \Lambda_4(z,n) = \frac{\ln\frac{s^2_{0,n}}{z} + (r-\frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}},$$

$$\Lambda_5(n) = \frac{\ln\frac{S^1_0(1-v_1)^n}{z} + (\mu_1-\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}, \quad \Lambda_6(n) = \frac{\ln\frac{S^1_0(1-v_1)^n}{S^2_0(1-v_2)^n} + (\mu_1-\mu_2+\frac{\sigma_2^2-\sigma_1^2}{2})T}{(\sigma_1-\sigma_2)\sqrt{T}},$$

$$\Lambda_7(n) = \frac{\ln\tilde{a}(z) - \frac{\theta^2}{2}T + (\lambda-\lambda^*)T + (\ln\lambda^*-\ln\lambda)n}{|\theta|\sqrt{T}}, \quad \Lambda_8(n) = \frac{\ln\frac{S^2_0(1-v_2)^n}{z} + (\mu_2-\frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}},$$

$$\Lambda_{1,3}(z,n) = \min\{\Lambda_1(n),\Lambda_3(z,n)\}, \quad \Lambda_{5,6}(n) = \min\{\Lambda_5(n),\Lambda_6(n)\},$$

$$p_{n,T} = \exp(-\lambda T)\frac{(\lambda T)^n}{n!}, \quad p^*_{n,T} = \exp(-\lambda^* T)\frac{(\lambda^* T)^n}{n!},$$

$$\Sigma_1 = \begin{pmatrix} 1 & -sign(\theta) \\ -sign(\theta) & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & sign(\theta) & -1 \\ sign(\theta) & 1 & -sign(\theta) \\ -1 & -sign(\theta) & 1 \end{pmatrix},$$

$$\Sigma_3 = \begin{pmatrix} 1 & sign(\theta) \\ sign(\theta) & 1 \end{pmatrix}, \quad \Sigma_4 = \begin{pmatrix} 1 & -sign(\theta) & -1 \\ -sign(\theta) & 1 & sign(\theta) \\ -1 & sign(\theta) & 1 \end{pmatrix},$$

$$\Sigma_5 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Proof. By Lemma 3.3, the minimal superhedging costs of $C^0 = \max(S^1_T, S^2_T)I_{\{\tau>T\}}$ are

$$\begin{aligned} U_0 &= \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}(e^{-rT}CI_{\{\tau>T\}}) = E^*(e^{-rT}C) \\ &= e^{-rT}\big[E^*(S^1_T I_{\{S^1_T\ge S^2_T\}}) + E^*(S^2_T I_{\{S^2_T>S^1_T\}})\big] \\ &= e^{-rT}\sum_{n=0}^{\infty}\Big[\big(E^*(S^1_T I_{\{S^1_T\ge S^2_T\}}|N_T=n) + E^*(S^2_T I_{\{S^2_T>S^2_T\}}|N_T=n)\big)P^*(N_T=n)\Big]. \end{aligned} \tag{3.44}$$

Conditioning on $\{N_T = n\}$, we have

$$\{S_T^1 \geq S_T^2\} = \{S_0^1(1-v_1)^n e^{v_1\lambda^* T}\exp[\sigma_1 W_T^* + (r - \frac{\sigma_1^2}{2})T] \geq S_0^2(1-v_2)^n e^{v_2\lambda^* T}\exp[\sigma_2 W_T^* + (r - \frac{\sigma_2^2}{2})T]\}$$

$$= \{Y_1 \leq \Lambda_1(n)\},$$

where $Y_1 = -\frac{W_T^*}{\sqrt{T}} \sim N(0,1)$, $\Lambda_1(n) = \frac{\ln(\frac{S_0^1(1-v_1)^n e^{v_1\lambda^* T}}{S_0^2(1-v_2)^n e^{v_2\lambda^* T}}) + \frac{\sigma_2^2 - \sigma_1^2}{2}T}{(\sigma_1 - \sigma_2)\sqrt{T}}$.

$$\{S_T^2 > S_T^1\} = \{Y_1 > \Lambda_1(n)\} = \{-Y_1 < -\Lambda_1(n)\} = \{Y_2 < -\Lambda_1(n)\},$$

where $Y_2 = \frac{W_T^*}{\sqrt{T}} \sim N(0,1)$.

According to the multi-asset theorem (see Melnikov and Romanyuk 2008) equation (3.44) can be rewritten as

$$U_0 = e^{-rT}\sum_{n=0}^{\infty} p_{n,T}^*[s_{0,n}^1 e^{(r-\frac{\sigma_1^2}{2})T} E^*(e^{\sigma_1 W_T^*} I_{\{Y_1 \leq \Lambda_1(n)\}}) + s_{0,n}^2 e^{(r-\frac{\sigma_2^2}{2})T} E^*(e^{\sigma_2 W_T^*} I_{\{Y_2 < -\Lambda_1(n)\}})]$$

$$= \sum_{n=0}^{\infty} p_{n,T}^*[s_{0,n}^1 \Phi(\Lambda_1(n) + \sigma_1\sqrt{T}) + s_{0,n}^2 \Phi(-\Lambda_1(n) - \sigma_2\sqrt{T})],$$

where $p_{n,T}^* = P^*(N_T = n) = \exp(-\lambda^* T)\frac{(\lambda^* T)^n}{n!}$, and hence part A of this Theorem is proved.

Let us start to prove the part B. Firstly, for a fixed z, we have

$$E^*(C(z) I_{\{a < Z_T^{*-1}\}}) = E^*((S_T^1 - z)^+ I_{\{S_T^1 \geq S_T^2\}} I_{\{a < Z_T^{*-1}\}}) + E^*((S_T^2 - z)^+ I_{\{S_T^1 < S_T^2\}} I_{\{a < Z_T^{*-1}\}})$$

$$= E^*((S_T^1 - z) I_{\{S_T^1 > z\}} I_{\{S_T^1 \geq S_T^2\}} I_{\{a < Z_T^{*-1}\}}) + E^*((S_T^2 - z) I_{\{S_T^2 > z\}} I_{\{S_T^1 < S_T^2\}} I_{\{a < Z_T^{*-1}\}}).$$

Conditioning on $\{N_T = n\}$, we get

$$\{Z_T^{*-1} > a\} = \{\exp(-\theta W_T^* - \frac{\theta^2}{2}T - (\lambda - \lambda^*)T - (\ln\lambda^* - \ln\lambda)n) > a\}$$

$$= \{Y_3 < \Lambda_2(a,n)\},$$

where $Y_3 = sign(\theta)\frac{W_T^*}{\sqrt{T}} \sim N(0,1)$, $\quad \Lambda_2(a,n) = \frac{-\ln a Z_t^* - (\frac{\theta^2}{2} + (\lambda - \lambda^*) + (\ln\lambda^* - \ln\lambda)n)T}{|\theta|\sqrt{T}}$.

$$\{S_T^1 > z\} = \{S_0^1(1-v_1)^n e^{v_1\lambda^* T}\exp[\sigma_1 W_T^* + (r - \frac{\sigma_1^2}{2})T] > z\} = \{Y_1 < \Lambda_3(z,n)\},$$

where $\Lambda_3(z,n) = \frac{\ln\frac{s_{0,n}^1}{z} + (r - \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

$$\{S_T^2 > z\} = \{S_0^2(1-v_2)^n e^{v_2\lambda^* T}\exp[\sigma_2 W_T^* + (r - \frac{\sigma_2^2}{2})T] > z\} = \{Y_1 < \Lambda_4(z,n)\},$$

where $\Lambda_4(z,n) = \frac{\ln \frac{s^2_{0,n}}{z} + (r - \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}}$.

And thus, we have

$$E^*(C(z)I_{\{a<Z^*_T-1\}}) = \sum_{n=0}^{\infty} E^*(C(z)I_{\{a<Z^*_T-1\}}|N_T = n)P^*(N_T = n)$$

$$= \sum_{n=0}^{\infty} p^*_{n,T}\big[E^*((s^1_{0,n}\exp[\sigma_1 W^*_T + (r - \frac{\sigma_1^2}{2})T] - z)I_{\{Y_1<\Lambda_1(n)\}}I_{\{Y_3<\Lambda_2(a,n)\}}I_{\{Y_1<\Lambda_3(z,n)\}})$$

$$+ E^*((s^2_{0,n}\exp[\sigma_1 W^*_T + (r - \frac{\sigma_1^2}{2})T] - z)I_{\{Y_2<-\Lambda_1(n)\}}I_{\{Y_3<\Lambda_2(a,n)\}}I_{\{Y_1<\Lambda_4(z,n)\}})\big]$$

$$= \sum_{n=0}^{\infty} p^*_{n,T}\big[E^*((s^1_{0,n}\exp[\sigma_1 W^*_T + (r - \frac{\sigma_1^2}{2})T] - z)I_{\{Y_1<\Lambda_{1,3}(z,n)\}}I_{\{Y_3<\Lambda_2(a,n)\}})$$

$$+ E^*((s^2_{0,n}\exp[\sigma_2 W^*_T + (r - \frac{\sigma_2^2}{2})T] - z)I_{\{Y_2<-\Lambda_1(n)\}}I_{\{Y_3<\Lambda^0_2(a,n)\}}I_{\{Y_1<\Lambda_4(z,n)\}})\big], \tag{3.45}$$

where $\Lambda_{1,3}(z,n) = \min\{\Lambda_1(n), \Lambda_3(z,n)\}$.

Again, according to the multi-asset theorem, (3.45) becomes

$$= \sum_{n=0}^{\infty} p^*_{n,T}\big[s^1_{0,n}e^{(r-\frac{\sigma_1^2}{2})T}E^*(e^{\sigma_1 W^*_T}I_{\{Y_1<\Lambda_{1,3}(z,n)\}}I_{\{Y_3<\Lambda_2(a,n)\}}) - zP^*(Y_1 < \Lambda_{1,3}(z,n), Y_3 < \Lambda_2(a,n))$$

$$+ s^2_{0,n}e^{(r-\frac{\sigma_2^2}{2})T}E^*(e^{\sigma_2 W^*_T}I_{\{Y_2<-\Lambda_1(n)\}}I_{\{Y_3<\Lambda_2(a,n)\}}I_{\{Y_1<\Lambda_4(z,n)\}}))$$

$$- zP^*(Y_2 < -\Lambda_1(n), Y_3 < \Lambda_2(a,n), Y_1 < \Lambda_4(z,n))\big]$$

$$= \sum_{n=0}^{\infty} p^*_{n,T}\big[s^1_{0,n}e^{rT}\Phi^2(\Lambda_{1,3}(z,n) + \sigma_1\sqrt{T}, \Lambda_2(a,n) - \sigma_1 sign(\theta)\sqrt{T}, \Sigma_1) - z\Phi^2(\Lambda_{1,3}(z,n), \Lambda_2(a,n), \Sigma_1)$$

$$+ s^2_{0,n}e^{rT}\Phi^3(-\Lambda_1(n) - \sigma_2\sqrt{T}, \Lambda_2(a,n) - \sigma_2 sign(\theta)\sqrt{T}, \Lambda_4(z,n) + \sigma_2\sqrt{T}, \Sigma_2)$$

$$- z\Phi^3(-\Lambda_1(n), \Lambda_2(a,n), \Lambda_4(z,n), \Sigma_2)\big],$$

where the correlation matrix Σ_1, Σ_2 are

$$\Sigma_1 = \begin{pmatrix} 1 & -sign(\theta) \\ -sign(\theta) & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & sign(\theta) & -1 \\ sign(\theta) & 1 & -sign(\theta) \\ -1 & -sign(\theta) & 1 \end{pmatrix}.$$

Thus, the equation $E^*(e^{-rT}C(z)I_{\{a<Z^*_T-1\}}) = v_0$ is (3.40) and for a fixed $z \in [0, z^*]$ and $0 \le a_1 < a_2$, we have

$$E^*\big(C(z)I_{\{a_1<Z^*_T-1\}}\big) \ge E^*\big(C(z)I_{\{a_2<Z^*_T-1\}}\big). \tag{3.46}$$

However, the equality holds if and only if $C(z)I_{\{a_1<Z_T^{*-1}\}} = C(z)I_{\{a_2<Z_T^{*-1}\}}, a.s.$, i.e.,

$$P^*\big(C(z)I_{\{a_1<Z_T^{*-1}\}} = C(z)I_{\{a_2<Z_T^{*-1}\}}\big) = 1 \to P^*\big(C(z)I_{\{a_1<Z_T^{*-1}\leq a_2\}} = 0\big) = 1,$$

which is not true. Hence $E^*(e^{-rT}C(z)I_{\{a<Z_T^{*-1}\}})$ is a strictly decreasing function regarding to a.

On the other hand, for $z \in [0, z^*]$, we have

$$E^*\big(C(z)I_{\{0<Z_T^{*-1}\}}\big) = E^*\big(C(z)\big) \geq v_0, \text{(because of the initial value constraint)} \tag{3.47}$$

and

$$\lim_{a\to\infty} E^*\big(C(z)I_{\{a<Z_T^{*-1}\}}\big) = 0. \tag{3.48}$$

Consequently, we can conclude that there is a root for (3.40) and it is unique.

As for $E(C(z)I_{\{\tilde{a}(z)\geq Z_T^{*-1}\}}I_{\{\tau>T\}})$, we have

$$\begin{aligned}
&E(C(z)I_{\{\tilde{a}(z)\geq Z_T^{*-1}\}}I_{\{\tau>T\}}) = E(C(z)I_{\{\tilde{a}(z)\geq Z_T^{*-1}\}})e^{-\int_0^T \beta_u du} \\
&= e^{-\int_0^T \beta_u du}\big[E((S_T^1 - z)I_{\{S_T^1\geq S_T^2\}}I_{\{S_T^1>z\}}I_{\{\tilde{a}(z)\geq Z_T^{*-1}\}}) \\
&+ E((S_T^2 - z)I_{\{S_T^2>S_T^1\}}I_{\{S_T^2>z\}}I_{\{\tilde{a}(z)\geq Z_T^{*-1}\}})\big].
\end{aligned} \tag{3.49}$$

Conditioning on $\{N_T = n\}$, $n = 0, 1_{...}$, and under the measure P, we can represent $\{S_T^1 \geq S_T^2\}$, $\{S_T^1 > z\}$,$\{\tilde{a}(z) \geq Z_T^{*-1}\}$, $\{S_T^2 > S_T^1\}$, $\{S_T^2 > z\}$ as follows:

$$\{S_T^1 > z\} = \{S_0^1(1-v_1)^n \exp(\sigma_1 W_T + (\mu_1 - \frac{\sigma_1^2}{2})T) > z\} = \{Y_4 < \Lambda_5(n)\},$$

where $Y_4 = -\frac{W_T}{\sqrt{T}} \sim N(0,1)$, and $\Lambda_5(n) = \frac{\ln\frac{S_0^1(1-v_1)^n}{z} + (\mu_1 - \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

$$\begin{aligned}
\{S_T^1 \geq S_T^2\} &= \{S_0^1(1-v_1)^n \exp(\sigma_1 W_T + (\mu_1 - \frac{\sigma_1^2}{2})T) \geq S_0^2(1-v_2)^n \exp(\sigma_2 W_T + (\mu_2 - \frac{\sigma_2^2}{2})T)\} \\
&= \{Y_4 \leq \Lambda_6(n)\},
\end{aligned}$$

where $\Lambda_6(n) = \frac{\ln\frac{S_0^1(1-v_1)^n}{S_0^2(1-v_2)^n} + (\mu_1 - \mu_2 + \frac{\sigma_2^2 - \sigma_1^2}{2})T}{(\sigma_1 - \sigma_2)\sqrt{T}}$.

$$\{S_T^1 < S_T^2\} = \{-Y_4 < -\Lambda_6(n)\} = \{Y_5 < -\Lambda_6(n)\},$$

where $Y_5 = \frac{W_T}{\sqrt{T}} \sim N(0,1)$.

$$\{\tilde{a}(z) \geq Z_T^{*-1}\} = \{\exp(-\theta W_T + \frac{\theta^2}{2}T - (\lambda - \lambda^*)T - (\ln\lambda^* - \ln\lambda)n) < \tilde{a}(z)\}$$
$$= \{Y_6 \leq \Lambda_7(n)\},$$

where $Y_6 = -sign(\theta)\frac{W_T}{\sqrt{T}} \sim N(0,1)$ and $\Lambda_7(n) = \frac{\ln\tilde{a}(z) - \frac{\theta^2}{2}T + (\lambda-\lambda^*)T + (\ln\lambda^* - \ln\lambda)n}{|\theta|\sqrt{T}}$.

$$\{S_T^2 > z\} = \{S_0^2(1-v_2)^n \exp(\sigma_2 W_T + (\mu_2 - \frac{\sigma_2^2}{2})T) > z\} = \{Y_4 < \Lambda_8(n)\},$$

where $\Lambda_8(n) = \frac{\ln\frac{S_0^2(1-v_2)^n}{z} + (\mu_2 - \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}}$.

Thus, the equation (3.49) can be rewritten as

$$= e^{-\int_0^T \beta_u du}\sum_{n=0}^{\infty} p_{T,n}\Big[E((S_T^1 - z)I_{\{Y_4<\Lambda_5(n)\}}I_{\{Y_4\leq\Lambda_6(n)\}}I_{\{Y_6\leq\Lambda_7(n)\}})$$
$$+ E((S_T^2 - z)I_{\{Y_5<-\Lambda_6(n)\}}I_{\{Y_6\leq\Lambda_7(n)\}}I_{\{Y_4\leq\Lambda_8(n)\}})\Big]$$
$$= e^{-\int_0^T \beta_u du}\sum_{n=0}^{\infty} p_{T,n}\Big[S_0^1(1-v_1)^n e^{\mu_1 T}\Phi^2(\Lambda_{5,6}(n) + \sigma_1\sqrt{T}, \Lambda_7(n) + \sigma_1 sign(\theta)\sqrt{T}, \Sigma_3)$$
$$- z\Phi^2(\Lambda_{5,6}(n), \Lambda_7(n), \Sigma_3)$$
$$+ S_0^2(1-v_2)^n e^{\mu_2 T}\Phi^3(-\Lambda_6(n) - \sigma_2\sqrt{T}, \Lambda_7(n) + \sigma_2 sign(\theta)\sqrt{T}, \Lambda_8(n) + \sigma_2\sqrt{T}, \Sigma_4)$$
$$- z\Phi^3(-\Lambda_6(n), \Lambda_7(n), \Lambda_8(n), \Sigma_4)\Big],$$

where $p_{n,T} = \exp(-\lambda T)\frac{(\lambda T)^n}{n!}$, $\Lambda_{5,6} = \min\{\Lambda_5(n), \Lambda_6(n)\}$, and

$$\Sigma_3 = \begin{pmatrix} 1 & sign(\theta) \\ sign(\theta) & 1 \end{pmatrix}, \quad \Sigma_4 = \begin{pmatrix} 1 & -sign(\theta) & -1 \\ -sign(\theta) & 1 & sign(\theta) \\ -1 & sign(\theta) & 1 \end{pmatrix}.$$

Consequently, the equation (3.41) is established.

According to Lemma 3.3, we have

$$\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}(e^{-rT}C^0(z)) = \sup_{\tilde{P}\in\in\mathbb{P}^*} E^{\tilde{P}}(e^{-rT}C(z)I_{\{\tau>T\}}) = E^*(e^{-rT}C(z))$$
$$= e^{-rT}[E^*((S^1_T-z)I_{\{S^1_T>z\}}I_{\{S^1_T\geq S^2_T\}}) + E^*((S^2_T-z)I_{\{S^2_T>z\}}I_{\{S^2_T>S^1_T\}})]$$
$$= e^{-rT}\sum_{n=0}^{\infty}p^*_{n,T}\Big[s^1_{0,n}e^{(r-\frac{\sigma_1^2}{2})T}E^*(e^{\sigma_1 W^*_T}I_{\{Y_1\leq\Lambda_1(n)\}}I_{\{Y_1<\Lambda_3(z,n)\}}) - zP^*(Y_1\leq\Lambda_1(n), Y_1<\Lambda_3(z,n))$$
$$+ s^2_{0,n}e^{(r-\frac{\sigma_2^2}{2})T}E^*(e^{\sigma_2 W^*_T}I_{\{Y_2<-\Lambda_1(n)\}}I_{\{Y_1<\Lambda_4(z,n)\}}) - zP^*(Y_2<-\Lambda_1(n), Y_1<\Lambda_4(z,n))\Big]$$
$$= \sum_{n=0}^{\infty}p^*_{n,T}\Big[s^1_{0,n}\Phi(\Lambda_{1,3}(z,n)+\sigma_1\sqrt{T}) - ze^{-rT}\Phi(\Lambda_{1,3}(z,n))$$
$$+ s^2_{0,n}\Phi^2(-\Lambda_1(n)-\sigma_2\sqrt{T}, \Lambda_4(z,n)+\sigma_2\sqrt{T}, \Sigma_5) - ze^{-rT}\Phi^2(-\Lambda_1(n), \Lambda_4(z,n), \Sigma_5)\Big],$$

where $\Sigma_5 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Since $C(z) = (C-z)^+ = \big(\max(S^1_T, S^2_T) - z\big)^+$ is decreasing regarding to z and hence $E^*(e^{-rT}C(z_1)) \geq E^*(e^{-rT}C(z_2))$ for any $0 \leq z_1 < z_2$.

The inequality is strict, because we have

$$E^*(e^{-rT}C(z_1)) = E^*(e^{-rT}C(z_2)) \to C(z_1) = C(z_2), \quad a.s.$$
$$\to (C-z_1)^+I_{\{C>z_2\}} = (C-z_2)^+I_{\{C>z_2\}}, \quad a.s.$$
$$\to (z_2-z_1)I_{\{C>z_2\}} = 0, \quad a.s. \to P^*\big(C>z_2\big) = 0,$$

which is not true because of $C = \max(S^1_T, S^2_T)^+ \in (0,\infty)$ and the probability that C is bigger than a given constant is positive. Hence, $E^*(e^{-rT}C(z))$ is strictly decreasing regarding to z.

Moreover, we have

$$E^*\big(C(0)\big) = E^*\big(C\big) > v_0, \quad \lim_{z\to\infty} E^*\big(C(z)\big) = 0,$$

and consequently, we conclude that there is a root for the equation $E^*(e^{-rT}C(z)) = v_0$ which is (3.43) and it is unique. □

Remark 3.9. *Note that distribution functions* $\Phi^2(x, y, \Sigma_2^{\pm})$ *and* $\Phi^2(x, y, z, \Sigma_3^{\pm})$ *can be expressed in terms of* $\Phi(\cdot)$*:*

$$\Phi^2(x, y, \Sigma_2^{+}) = \Phi\big(min(x, y)\big); \tag{3.50}$$

$$\Phi^2(x, y, \Sigma_2^{-}) = \begin{cases} \Phi(x) - \Phi(-y), & if \;\; x > -y, \\ 0, & otherwise; \end{cases} \tag{3.51}$$

$$\Phi^3(x, y, z, \Sigma_3^{+}) = \begin{cases} \Phi\big(min(x, y)\big) - \Phi(-z), & if \;\; min(x, y) > -z, \\ 0, & otherwise; \end{cases} \tag{3.52}$$

$$\Phi^3(x, y, z, \Sigma_3^{-}) = \begin{cases} \Phi(x) - \Phi(max(-y, -z)), & if \;\; x > max(-y, -z), \\ 0, & otherwise; \end{cases} \tag{3.53}$$

where

$$\Sigma_2^{\pm} = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad \Sigma_3^{\pm} = \begin{pmatrix} 1 & \pm 1 & -1 \\ \pm 1 & 1 & \mp 1 \\ -1 & \mp 1 & 1 \end{pmatrix}.$$

Example 3.10. *We consider the Russell 2000 (RUT-I) and the Dow Jones Industrial Avergae (DJIA) as* S^1 *and* S^2*. Melnikov and Skornyakova (2005) estimated* (μ_i, σ_i) $(i = 1, 2)$ *for those two risky assets using daily observations and assume* $\upsilon_1 = \upsilon_2 = 0$*, i.e., the Black-Scholes model. In our case, we assume underlying risky assets with same drifts and volatilities as Melnikov and Skornyakova (2005) but with non-zero jump components. Values of parameters are given as following:*

$$\mu_1 = 0.0481, \;\; \sigma_1 = 0.2232, \quad \mu_2 = 0.0417, \;\; \sigma_2 = 0.2089, \;\; S_0^1 = S_0^2 = 100,$$

$$\upsilon_1 = -0.05, \;\; \upsilon_2 = -0.1, \;\; \lambda = 0.1, \;\; r = 0.$$

We assume the hazard rate β is constant. The insured's age, contract maturities and levels of β would differ in this example. According to the most recently published United States 2015 Life Table (National Vital Statistics Reports volume 67, Number 7), the survival probability ${}_Tp_x$ of a given insured can be found and then for a fixed confidence level $\alpha = 0.95$, utilizing results in the

Theorem 3.8, we can derive the minimal CVaR that can be achieved with the hedging capital ${}_Tp_xU_0$. Results are displayed in the Table 3.1. It is observed that the minimal CVaR increases as the insured's age or the contract maturity increases. This is because for an older client and a long term contract, the survival probability of the insured would decrease so as the premium which leads to an increase of hedging losses. Moreover, for a given insured and a contract maturity time, the minimal CVaR decreases as the hazard rate increases. In addition, the decreasing trend is more significant for long term contracts. This can be explained by the fact that there is a higher chance that contracts would mature with a zero payoff if the hazard rate increases and for long term contracts, there is an even higher possibility that a default event would occur during contract periods, which reduces the shortfall risk further.

Table 3.1 Minimal $CVaR_{0.95}$ for contracts with different maturities, insured's age and levels of β

	Age=20			Age=30			Age=40		
	T=10	T=15	T=20	T=10	T=15	T=20	T=10	T=15	T=20
${}_Tp_x$	0.9902	0.9839	0.9764	0.9860	0.9758	0.9609	0.97462	0.9509	0.9166
$\beta = 0$	0.996	1.6431	2.4136	1.4318	2.4745	4.017	2.5966	5.0477	8.6448
$\beta = 0.01$	0.9958	1.6378	2.3884	1.4316	2.4698	3.9978	2.5964	5.045	8.6394
$\beta = 0.015$	0.9956	1.6337	2.3684	1.4314	2.4662	3.9818	2.5963	5.0427	8.6335
$\beta = 0.02$	0.9954	1.6283	2.3411	1.4312	2.4613	3.9596	2.5961	5.0395	8.624

3.5 Conclusion

This chapter focuses on the problem of CVaR based partial hedging in a defaultable Jump-Diffusion model. We first provide the set of martingale measures in this incomplete market which admit the special form Z^*Z^k conditioned on $\{\tau > T\}$ and then by their properties, the minimal superhedging costs of a defautable claim with a zero recovery rate are given that coincide with the initial wealth required for perfectly hedging the non-defaultable claim. Most importantly, we prove that the optimal CVaR hedging problem in the defaultable market can be converted to a problem of finding an optimal randomized test in the corresponding default free market. The hedging strategy can be explained as constructing the perfect hedging of a modified nondefaultable claim during $[0, \tau]$ while investing nothing in risky assets and depositing all cash into the saving account after the default. Furthermore, our method is applied to the area of life insurance. Numerical results show that for a given insured and a contract maturity time the minimal CVaR decreases as the hazard rate increases.

3.6 References

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CHAPTER 4

CVaR hedging in the Bachelier model and its modifications

4.1 Introduction

Bachelier was the pioneer who studied the continuous time process called now as the Brownian motion and who developed the first option pricing theory (see, Bachelier 1900). Since then, many scientists are inspired by the work of Bachelier developing the theory of stochastic processes and its applications in different areas including Mathematical Finance (see, Taqqu 2001). Speaking about the impact of Bachelier, Kolmogorov (1931) wrote "Here we note only that Bachelier's contributions are by no means mathematically rigorous". However, the main drawback of the standard Bachelier model is that stock prices can turn negative which is unappealing in the financial literature. That is why Samuelson (1965) proposed the exponential transformation of the Bachelier model and introduced the so-called Geometrical Brownian motion (the Black-Scholes model).

In this work, we investigate two more approaches to modify the Bachelier model. The first one is by considering a SDE with absorption. Going in this way, we introduce a stopping time which is the first time the stock price hits zero and once it becomes zero it stays there forever. This adjustment ensures that stock prices are always non-negative and also resolves a problem with limiting behavior of volatility and time to maturity pertaining to the classical Bachelier model. In Goldenberg (1991), he considered arithmetic Brownian motion absorbed at zero and provided the option pricing formula of a call option with the help of transition density for such an absorbed process. Instead, in this chapter, we derive the price of a call option by a straightforward application of the reflection principle of a Brownian motion which is more intuitive and convenient. The second modification method describes the evolution of the risky asset as a solution of a SDE with a reflection boundary at zero. SDEs with reflection were introduced by Skorokhod (1961). He also proved the existence and uniqueness of the solution for those equations under some assumptions (see Pilipenko 2014). In summary, such a stock price process is nonnegatively defined and when it hits zero, it would be compensated by a process l which is non-decreasing and only increases at points when the stock price is zero. We show that such a modified Bachelier market with reflection may admit arbitrage and hence it is not an appropriate financial model. Thus, in this chapter, our attention is paid to the standard Bachelier model, the Black-Scholes model and the modified Bachelier model with absorption at zero (called below as the modified Bachelier model).

Option pricing is one of the main research areas of Mathematical Finance. The famous Bachelier formula and the Black-Scholes formula describe fair prices of call options in the standard Bachelier model and the Black-Scholes model correspondingly. Schachermayer and Teichmann (2008) have indicated that for small values of volatility and short contract periods, those two formulas give close price values. We extend their research and prove that price differences between an at-the-money call option in the modified Bacherlier model and another one with the same strike price and the time to maturity in the Black-Scholes model are even smaller.

Also, option pricing theory points out that a contingent claim can be perfectly hedged if the initial hedging capital is no less than its fair price (in complete markets) or its minimal superhedging price (in incomplete markets). However, with insufficient wealth, a hedger can only construct a partial hedging strategy while accepts the possibility of shortfall. Föllmer and Leukert (1999), (2000) are pioneers in the field of partial hedging. They studied quantile hedging and efficient hedging in seimimartingale financial market models. They derived explicit solutions in complete markets by using the classical Neyman-Pearson lemma while solutions in incomplete markets were given with the help of the convex duality approach. Although the Bachelier model and its modifications attract a certain interest in the filed of general option pricing, these modifications remain a relatively unexplored area for partial hedging. A related article is Glazyrina and Melnikov (2020) where the problem of quantile hedging in the Bachelier model with a stopping time was discussed. In this chapter, we employ the Conditional Value-at-Risk (CVaR) to measure hedging losses. Such a measure provides information about the average losses that exceeds the Value-at-Risk (VaR) level and is widely used in financial institutions. Melnikov and Smirnov (2012) studied CVaR hedging in complete markets and they provided the derivation of the optimal CVaR hedging strategy as well as its illustrations in the Black-Scholes model. Inspired by them, we derive explicit solutions for the problem of optimal CVaR hedging in both the standard Bachelier model and the modified Bachelier model.

The main aim of this chapter is to introduce new modifications of the Bachelier model and compare them in aspects of the option pricing and the CVaR hedging. This chapter is organized as follows. In Section 4.2, basic properties of the standard Bachelier model and the Black-Scholes model are described. In addition, we introduce two modified models: the one with absorption and another one with reflection. The meaning behind those two modifications is that once the stock price reaches zero, the company issuing such a stock either go bankrupt and hence the stock price becomes zero forever (which corresponds to the modified Bachelier model with absorption) or the company takes actions to improve its financial situation quickly and effectively to make the stock price bounce back (which corresponds to the SDE with reflection). In Section 4.3, firstly, we derive the set containing all martingale measures in the modified Bachelier model with absorption. Although this market may be incomplete, the arbitrage free price of a call option can be uniquely determined and an explicit form of such a price is derived. Then, we compare fair prices of at-the-money call options in the standard Bachelier model, the Black-Scholes model and the modified Bachelier model. In Section 4.4, we discuss CVaR hedging problem in the classical Bachelier model which is complete while, in Section 4.5, CVaR hedging in the incomplete modified

Bachelier model is investigated. In Section 4.6, numerical examples are provided to illustrate results derived in previous sections. Section 4.7 gives a conclusion for the chapter.

4.2 The Bachelier model and its modifications

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The first financial model that describes the evolution of a stock price with the help of a Brownian motion is called the Bachelier model. There are two assets $(S_t^0, S_t)_{0\leq t\leq T}$ in such a model, where the bank account $S_t^0 = 1,\ \forall t \in [0, T]$, (i.e, we assume the interest rate $r = 0$) and the stock price process is described as

$$S_t = S_0 + \mu t + \sigma W_t,\ \ S_0 > 0,\ \ \ t \in [0, T]. \tag{4.1}$$

Here, constants $\sigma > 0$, $\mu > 0$ are called the volatility and the drift. The process $(W_t)_{t\in[0,T]}$ is a Brownian motion with its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]},\ \ \mathcal{F}_t = \sigma(W_s, s \leq t)$. We assume $\mathcal{F} = \mathcal{F}_T$.

It is well known that the stock price process (4.1) is a martingale with respect to a unique martingale measure P^* defined by the relation

$$Z_t^* = \frac{dP^*}{dP}\ |_{\mathcal{F}_t}= \exp\big(- \frac{\mu}{\sigma} W_t - \frac{1}{2}(\frac{\mu}{\sigma})^2 t\big). \tag{4.2}$$

Moreover, by the Girsanov theorem, the process $W_t^* = W_t + \frac{\mu}{\sigma} t$ is a Brownian motion under the measure P^*.

However, the drawback of the standard Bachelier model is that the stock price can take negative values. The famous Geometrical Brownian motion (the Black-Scholes model) proposed by Samuelson (1965) overcomes such a drawback. The price process of the risky asset in this market is given by

$$S_t^{BS} = S_0 \exp\big((\mu^{BS} - \frac{(\sigma^{BS})^2}{2})t + \sigma^{BS} W_t\big),\ \ \ S_0 > 0,\ \ \ t \in [0, T]. \tag{4.3}$$

Here, constants $\sigma^{BS} > 0$, μ^{BS} denote the volatility and the drift in the Black-Scholes model. The density of the unique martingale measure P^*_{BS} in the market (4.3) is

$$Z_t^{BS} = \frac{dP^*_{BS}}{dP}\ |_{\mathcal{F}_t}= \exp\big(- \frac{\mu^{BS}}{\sigma^{BS}} W_t - \frac{1}{2}(\frac{\mu^{BS}}{\sigma^{BS}})^2 t\big). \tag{4.4}$$

Now, we would like to introduce two additional modifications that adjust stock prices to be nonnegative and thereby may provide better models to fit real market data than the standard Bachelier model.

The first intuitive way to make the price process (4.1) nonnegative is to introduce a stopping time τ and consider the risky asset with the evolution

$$S_{t\wedge\tau} = S_0 + \mu(t\wedge\tau) + \sigma W_{t\wedge\tau}, \quad S_0 > 0, \quad t \in [0,T], \tag{4.5}$$

where τ is defined as follows:

$$\tau = \inf\{t : S_t = 0\}. \tag{4.6}$$

Such a modified stock price process is always nonnegative and once the price hits zero, it stays at zero forever. It can be seen as a special case of a process with absorption at the lower boundary zero. In the modified Bachelier model with absorption, we consider the stopped stock price process $\{S_{t\wedge\tau}\}_{t\in[0,T]}$ in the filtration $\mathbb{F}$ that is generated by the Brownian motion as before.

The stopping time τ might be treated as a default time of the company issuing the stock. Let us investigate properties of such a stopping time.

According to Le Gall (2016), for any constants c and $a > 0$, define a stopping time

$$U_a = \inf\{t : W_t + ct = a\}.$$

The density function of U_a is

$$f_{U_a}(t) = \frac{a}{\sigma\sqrt{2\pi t^3}} \exp\big(-\frac{1}{2t}(a-ct)^2\big). \tag{4.7}$$

Integrating this density, we obtain the probability that the stopping time U_a is finite:

$$P(U_a < \infty) = \begin{cases} 1, & \text{if } \ c \geq 0, \\ e^{2ca}, & \text{if } \ c < 0. \end{cases} \tag{4.8}$$

Furthermore, the relation (4.6) can be rewritten as

$$\begin{aligned} \inf\{t : S_t = 0\} &= \inf\{t : S_0 + \mu t + \sigma W_t = 0\} \\ &= \inf\{t : \frac{\mu}{\sigma}t + W_t = -\frac{S_0}{\sigma}\} \\ &= \inf\{t : -\frac{\mu}{\sigma}t - W_t = \frac{S_0}{\sigma}\}. \end{aligned}$$

Substituting $a = \frac{S_0}{\sigma}$ and $c = -\frac{\mu}{\sigma} < 0$ into (4.7), (4.8), we find the following expression of the density function of τ:

$$f_\tau(t) = \frac{S_0}{\sigma\sqrt{2\pi t^3}} \exp\big(-\frac{1}{2t}(\frac{S_0}{\sigma} + \frac{\mu}{\sigma}t)^2\big).$$

Therefore, the probability that the stock price will attain zero during a finite time horizon is

$$P(\tau < \infty) = \exp\big(-\frac{2\mu S_0}{\sigma^2}\big). \tag{4.9}$$

The equation (4.9) indicates that for a small σ, the possibility that the stock price will drop to zero within a fixed time interval is quite low, and hence the modified Bachelier model with absorption is very close to the standard Bachelier model. However, such a possibility increases as σ increases, so that the model (4.5) provides a better fit to stock prices by avoiding negative values in this case. In section 4.6, we would discuss properties of the model (4.5) in the case that σ is large and in the case it is small respectively.

The second method we may utilize is to consider a stock price process with reflection at zero which is described by the SDE:

$$dS_t^* = \mu dt + \sigma dW_t + dl_t, \quad S_0^* = S_0 > 0, \tag{4.10}$$

where

1) l is nondecreasing and $l_0 = 0$;

2) $\int_0^t I_{\{S_s^* > 0\}} dl_s = 0, \quad \forall t \in [0, T]$;

3) $S_t^* \geq 0, \quad \forall t \in [0, T]$.

It turns out that there is a unique pair of continuous adapted processes (S^*, l) that solves (4.10). More specifically, the continuous process l_t which only increases at points $S_t^* = 0$ equals the local time of S^* at zero (see, for instance, Pilipenko 2014 Theorem 1.3.1), i.e.,

$$\begin{aligned} l_t &= \int_0^t \lim_{\epsilon \to 0+} (2\epsilon)^{-1} I_{[-\epsilon,\epsilon]}(S_s^*) d < S^*, S^* >_s \\ &= \int_0^t \delta(S_s^*) \sigma^2 ds. \end{aligned} \tag{4.11}$$

Here, we use the notation $\delta(S_s^*) = \lim_{\epsilon \to 0+} (2\epsilon)^{-1} I_{[-\epsilon,\epsilon]}(S_s^*)$ and the second equality is due to the property:

$$< S^*, S^* >_s = < S^{*C}, S^{*C} >_s = < \sigma W, \sigma W >_s = \sigma^2 s,$$

where S^{*C} is the continuous martingale part of S^*.

The model (4.10) indeed overcomes the shortcoming of the Bachelier model. However, no an equivalent martingale measure exists in this market, which indicates the existence of arbitrage. To show this, we suppose that there is an equivalent martingale measure Q with respect to P in the market (4.10). Then, according to the property of the filtration generated by the Brownian motion, the density of Q can be represented as the solution of the equation:

$$dZ_t^Q = \phi_t Z_t^Q dW_t, \quad Z_0^Q = 1,$$

for a predictable process $\{\phi_t\}_{t \in [0,T]}$.

In addition, by the Girsanov theorem, the process $W_t^Q = W_t - \int_0^t \phi_s ds$ is a Brownian motion under Q.

The SDE (4.10) can be rewritten as

$$\begin{aligned} dS_t^* &= \mu dt + \sigma dW_t + dl_t, \\ &= (\mu + \sigma \phi_t + \delta(S_t^*)\sigma^2)dt + \sigma dW_t^Q. \end{aligned}$$

It is a martingale under the measure Q if and only if $\mu + \sigma\phi_t + \delta(S_t^*)\sigma^2 = 0, \ \forall t \in [0,T]$, and therefore the stock price process is

$$S_t^* = \sigma W_t^Q, \quad t \in [0,T].$$

On one hand side, such a stock price S_t^* satisfies

$$P(S_t^* < 0) = 0, \quad \forall t \in [0,T].$$

On the other hand, we have

$$Q(S_t^* < 0) = Q(\sigma W_t^Q < 0) = 0.5, \quad \forall t \in [0,T],$$

which contradicts the assumption that Q and P are equivalent. Thus, there is no an equivalent martingale measure in the market (4.10).

Since the market (4.10) admits arbitrage, in the following, we only consider models (4.1), (4.3), (4.5), and call the Bachelier model with absorption (4.5) as the modified Bachelier model.

4.3 Comparison of models through perfect hedging prices

It is well known that markets (4.1) and (4.3) are complete with the unique martingale measure given by (4.2) and (4.4) correspondingly. Let us investigate martingale measures in the market (4.5).

By the martingale property in the filtration $\mathbb{F}$, the density of any martingale measure $\tilde{P}$ in the market (4.5) can be written as

$$dZ_t^{\tilde{P}} = Z_t^{\tilde{P}} \theta_t dW_t, \quad Z_0^{\tilde{P}} = 1, \tag{4.12}$$

where θ is a predictable process.

According to the Girsanov theorem the process

$$\tilde{W}_t = W_t - \int_0^t \theta_s ds,$$

is a Brownian motion under the measure $\tilde{P}$.

The dynamic of the risky asset (4.5) can be rewritten as

$$\begin{aligned} dS_{t\wedge\tau} &= \mu I_{\{\tau \geq t\}} dt + \sigma I_{\{\tau \geq t\}} dW_t, \\ &= \mu I_{\{\tau \geq t\}} dt + \sigma I_{\{\tau \geq t\}} (d\tilde{W}_t + \theta_t dt) \\ &= (\mu + \sigma\theta_t) I_{\{\tau \geq t\}} dt + \sigma I_{\{\tau \geq t\}} d\tilde{W}_t. \end{aligned}$$

Such a process is a martingale under $\tilde{P}$ if and only if

$$(\mu + \sigma\theta_t) I_{\{\tau \geq t\}} = 0, \tag{4.13}$$

which implies that conditioning on $\{\tau \geq t\}$, we have

$$\theta_t = -\frac{\mu}{\sigma}. \tag{4.14}$$

However, on the set $\{\tau < t\}$, θ_t can be arbitrary which means the martingale measure is not unique and hence the market (4.5) is not complete. Let us denote the set containing all martingale measures in the modified Bachelier market as $\mathbb{P}^*$. Note that P^* defined by (4.2) belongs to $\mathbb{P}^*$, if we choose $\theta_t = -\frac{\mu}{\sigma}$ on the set $\{\tau < t\}$.

For simplicity of notations, let us use $\tilde{S}_t$ to denote S_t in the model (4.1) and to denote $S_{t\wedge\tau}$ in the model (4.5).

A strategy π is a $\mathbb{F}$-predictable process $\pi := (\pi_t^0, \pi_t^1)_{t\in[0,T]}$ such that

$$\int_0^T |\pi_t^0| dt < \infty, \quad \int_0^T (\pi_t^1 \tilde{S}_t)^2 dt < \infty, \quad P - a.s.$$

At time $t \in [0, T]$, the value process corresponding to the strategy π is

$$V_t = \pi_t^0 + \pi_t^1 \tilde{S}_t.$$

For a given initial value $v \geq 0$, the trading strategy is called self-financing if its value process satisfies

$$V_t = v + \int_0^t \pi_u^1 d\tilde{S}_u,$$

and is called admissible, if such a process also satisfies

$$V_t \geq 0, \; \forall t \in [0, T].$$

We denote the set of all admissible self-financing strategies with an initial value v as $\mathcal{A}(v)$.

According to the option pricing theory, in complete markets (eg, the model (4.1)), for a given claim H, there is an admissible strategy that duplicates H, i.e.,

$$V_t = E^M(H|\mathcal{F}_t) = v + \int_0^t \pi_u^1 d\tilde{S}_u, \tag{4.15}$$

for some predictable process π^1. And V_t is the unique price of the claim H at time t. Here, $E^M(\cdot)$ represents the expectation under the unique martingale measure in such a market.

It is well known that the fair price of an European call option $(S_T - K)^+$ in the complete market (4.1) is given by the Bachelier formula (see, for example, Schachermayer and Teichmann 2008):

$$V_0^B = E^*\big((S_T - K)^+\big) = (S_0 - K)\Phi(\frac{S_0 - K}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\phi(\frac{S_0 - K}{\sigma\sqrt{T}}), \tag{4.16}$$

where $E^*(\cdot)$ is the expectation under the unique martingale measure P^* in the market (4.1).

And the fair price of an European call option $(S_T^{BS}-K)^+$ in the market (4.3) is given by the Black-Scholes formula:

$$V_0^{BS} = E^{P^*_{BS}}\big((S_T^{BS}-K)^+\big) = S_0\Phi(d_+) - K\Phi(d_-), \tag{4.17}$$

where $d_\pm = \frac{\ln(\frac{S_0}{K})\pm\frac{(\sigma^{BS})^2}{2}T}{\sigma^{BS}\sqrt{T}}$ and $E^{P^*_{BS}}(\cdot)$ is the expectation under the measure P^*_{BS}.

However, in incomplete markets (eg, the model (4.5)), not every contingent claim admits a unique arbitrage free price and can be duplicated. In this case, we define

$$U_t = \operatorname*{ess\ sup}_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}(H|\mathcal{F}_t), \quad t\in[0,T], \tag{4.18}$$

which is a supermartingale with respect to any $\tilde{P}\in\mathbb{P}^*$ and represents the value process of the minimal superhedging strategy of the claim H. According to the optional decomposition theorem (see El Karoui and Quenez 1995, Kramkov 1996), there is an admissible strategy (v,π) and a nonnegative consumption process C with $C_0=0$ such that

$$U_t = v + \int_0^t \pi_u^1 d\tilde{S}_u - C_t. \tag{4.19}$$

Such a strategy π is then called the superhedging strategy.

Now, let us investigate superhedging costs of a call option in the modified model (4.5).

Theorem 4.1. *The superhedging costs of an European call option $(S_{T\wedge\tau}-K)^+$ in the market (4.5) is*

$$\begin{aligned} V_0^{MB} &= \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big((S_{T\wedge\tau}-K)^+\big) \\ &= (S_0-K)\Phi(\frac{S_0-K}{\sigma\sqrt{T}}) + (S_0+K)\Phi(\frac{-S_0-K}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\Big[\phi(\frac{S_0-K}{\sigma\sqrt{T}}) - \phi(\frac{S_0+K}{\sigma\sqrt{T}})\Big], \end{aligned} \tag{4.20}$$

where $K>0$ is the strike price and T is the time to maturity of the option. $\Phi(\cdot)$, $\phi(\cdot)$ represent the cumulative distribution function and the density function of a standard normal random variable.

Proof. For any martingale measure $\tilde{P} \in \mathbb{P}^*$, we have

$$\begin{aligned}
E^{\tilde{P}}\big((S_{T\wedge\tau} - K)^+\big) &= E^{\tilde{P}}\big((S_T - K)^+ I_{\{\tau > T\}}\big) + E^{\tilde{P}}\big((S_\tau - K)^+ I_{\{\tau \le T\}}\big) \\
&= E^{\tilde{P}}\big((S_T - K)^+ I_{\{\tau > T\}}\big) \\
&= E\big(Z_T^{\tilde{P}}(S_T - K)^+ I_{\{\tau > T\}}\big) \\
&= E\Big(\exp\big(-\frac{\mu}{\sigma}W_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T\big)(S_T - K)^+ I_{\{\tau > T\}}\Big) \\
&= E^*\big((S_T - K)^+ I_{\{\tau > T\}}\big) \\
&= E^*\big((S_T - K)^+\big) - E^*\big((S_T - K)^+ I_{\{\tau \le T\}}\big). \qquad (4.21)
\end{aligned}$$

The above equation indicates that the price of the call option would not be affected by the choice of the martingale measure. In other words, the arbitrage free price of a call option is unique and equals to $E^*\big((S_{T\wedge\tau} - K)^+\big)$.

Moreover, according to the reflection principle of the Brownian motion, the following equation holds

$$\begin{aligned}
E^*\Big((S_T - K)^+ I_{\{\tau \le T\}}\Big) &= E^*\Big((S_0 + \sigma W_T^* - K)^+ I_{\{\tau \le T\}}\Big) \\
&= E^*\Big((-S_0 + \sigma W_T^* - K)^+\Big). \qquad (4.22)
\end{aligned}$$

By (4.16), (4.21) and (4.22), for any $\tilde{P} \in \mathbb{P}^*$, we arrive to

$$\begin{aligned}
E^{\tilde{P}}\big((S_{T\wedge\tau} - K)^+\big) &= (S_0 - K(z))\Phi(\frac{S_0 - K}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\phi(\frac{S_0 - K}{\sigma\sqrt{T}}) \\
&\quad - (-S_0 - K)\Phi(\frac{-S_0 - K}{\sigma\sqrt{T}}) - \sigma\sqrt{T}\phi(\frac{-S_0 - K}{\sigma\sqrt{T}}) \\
&= (S_0 - K)\Phi(\frac{S_0 - K}{\sigma\sqrt{T}}) + (S_0 + K)\Phi(\frac{-S_0 - K}{\sigma\sqrt{T}}) \\
&\quad + \sigma\sqrt{T}\Big[\phi(\frac{S_0 - K}{\sigma\sqrt{T}}) - \phi(\frac{S_0 + K}{\sigma\sqrt{T}})\Big],
\end{aligned}$$

and hence this theorem is proved. □

Remark 4.2. *If $\sigma\sqrt{T} \to 0$ or S_0 is very large, both $\Phi(\frac{-S_0 - K}{\sigma\sqrt{T}})$ and $\phi(\frac{S_0 + K}{\sigma\sqrt{T}})$ in the equation (4.20) tend to zero which implies that the price of a call option in the modified Bachelier model is close to the price of a call option that has the same strike and time to maturity in the Bachelier model.*

Theorem 4.1 indicates that even though the market (4.5) is incomplete, the value of a call option is uniquely determined and hence we can call such a value as the fair price.

Schachermayer and Teichmann (2008) showed that for fixed $\sigma > 0, T > 0$ and with relationships $\frac{\sigma}{S_0} = \sigma^{BS}$, $S_0 = K$, the difference between V_0^B and V_0^{BS} satisfies the following inequality:

$$0 \leq V_0^B - V_0^{BS} \leq \frac{S_0}{24\sqrt{2\pi}} {\sigma^{BS}}^3 T^{\frac{3}{2}}. \tag{4.23}$$

One step further, we would like to compare perfect hedging prices of at-the-money call options in the modified Bachelier model and in the Black-Scholes model.

Theorem 4.3. *A. Let $\frac{\sigma}{S_0} = \sigma^{BS}$. For fixed $\sigma > 0$, $T > 0$, fair prices of at-the-money European call options with the same maturity in the market (4.5) and (4.3) satisfy the relation*

$$\begin{aligned} V_0^{MB} - V_0^{BS} \leq \frac{S_0}{\sqrt{2\pi}} \Big[&\frac{(\sigma^{BS}\sqrt{T})^3}{24} - (\sigma^{BS}\sqrt{T}) \exp(-\frac{2}{(\sigma^{BS}\sqrt{T})^2}) \\ &+ 2\sqrt{2}(\arctan(-\frac{\sqrt{2}}{\sigma^{BS}\sqrt{T}}) + \frac{\pi}{2}) \Big]. \end{aligned} \tag{4.24}$$

B. Moreover, the absolute value of differences between V_0^B and V_0^{BS} are bigger than that between V_0^{MB} and V_0^{BS}, i.e.,

$$|V_0^{MB} - V_0^{BS}| \leq |V_0^B - V_0^{BS}|.$$

In other words, the modified Bachelier formula (4.20) provides a closer fit to the Black-Scholes formula (4.17).

Proof. For $S_0 = K$, according to (4.20) and (4.17), we have

$$V_0^{MS} = 2S_0\Phi(-\frac{2}{\sigma^{BS}\sqrt{T}}) + S_0\sigma^{BS}\sqrt{T}\Big[\phi(0) - \phi(\frac{2}{\sigma^{BS}\sqrt{T}})\Big].$$

$$V_0^{BS} = S_0\Big[\Phi(\frac{\sigma^{BS}\sqrt{T}}{2}) - \Phi(-\frac{\sigma^{BS}\sqrt{T}}{2})\Big].$$

Hence, differences between them are

$$
\begin{aligned}
V_0^{MS} - V_0^{BS} &= S_0\Big(\frac{x}{\sqrt{2\pi}} + 2\Phi(-\frac{2}{x}) - x\phi(\frac{2}{x}) - \Phi(\frac{x}{2}) + \Phi(-\frac{x}{2})\Big) \mid_{x=\sigma^{BS}\sqrt{T}} \\
&= \frac{S_0}{\sqrt{2\pi}}\Big(\int_{-\frac{x}{2}}^{\frac{x}{2}} (1-\exp(-\frac{y^2}{2}))dy + 2\int_{-\infty}^{-\frac{2}{x}} \exp(-\frac{y^2}{2})dy - x\exp(-\frac{2}{x^2})\Big) \mid_{x=\sigma^{BS}\sqrt{T}} \\
&\leq \frac{S_0}{\sqrt{2\pi}}\Big(\int_{-\frac{x}{2}}^{\frac{x}{2}} \frac{y^2}{2}dy + 2\int_{-\infty}^{-\frac{2}{x}} \frac{1}{1+\frac{y^2}{2}}dy - x\exp(-\frac{2}{x^2})\Big) \mid_{x=\sigma^{BS}\sqrt{T}} \\
&= \frac{S_0}{\sqrt{2\pi}}\Big(\frac{x^3}{24} + 2\sqrt{2}(\arctan(-\frac{\sqrt{2}}{x}) + \frac{\pi}{2}) - x\exp(-\frac{2}{x^2})\Big) \mid_{x=\sigma^{BS}\sqrt{T}} .
\end{aligned}
$$

The above inequality is due to the fact $e^y \geq 1+y, \forall y \in \mathbb{R}$ as well as $e^y \leq \frac{1}{1-y}$, for $y < 1$ and, in the last equation, we utilize the relation $(\arctan(y))' = \frac{1}{1+y^2}$.

Let us denote

$$
\begin{aligned}
\Delta_1 &= V_0^B - V_0^{BS} = S_0\big(\frac{x}{\sqrt{2\pi}} - \Phi(\frac{x}{2}) + \Phi(-\frac{x}{2})\big) \mid_{x=\sigma^{BS}\sqrt{T}} \geq 0, \\
\Delta_2 &= -E^*\big((S_T - K(z))^+ 1_{\{\tau \leq T\}}\big) = S_0\big(2\Phi(-\frac{2}{x}) - x\phi(\frac{2}{x})\big) \mid_{x=\sigma^{BS}\sqrt{T}} \leq 0.
\end{aligned}
$$

With above notations, we have $V_0^{MB} - V_0^{BS} = V_0^B + \Delta_2 - V_0^{BS} = \Delta_1 + \Delta_2$.

Consider the function

$$
\begin{aligned}
f(x) &= -\Delta_2 - 2\Delta_1 \\
&= S_0\Big(x\phi(\frac{2}{x}) - 2\Phi(-\frac{2}{x}) - 2\big(\frac{x}{\sqrt{2\pi}} - \Phi(\frac{x}{2}) + \Phi(-\frac{x}{2})\big)\Big),
\end{aligned}
$$

and we have $f(x) \to 0$ as $x \to 0$.

Moreover, by taking derivative, we get the following inequality

$$
f'(x) = S_0\big(\phi(\frac{2}{x}) + 2\phi(\frac{x}{2}) - 2\phi(0)\big) \leq 0, \quad x > 0,
$$

and hence $f(x) \leq 0$.

Consequently, we arrive to the relation

$$
0 \leq -\Delta_2 \leq 2\Delta_1,
$$

which implies

$$-\Delta_1 \leq -\Delta_1 - \Delta_2 \leq \Delta_1;$$

$$|\Delta_1 + \Delta_2| \leq \Delta_1;$$

$$|V_0^{MB} - V_0^{BS}| \leq |V_0^B - V_0^{BS}|.$$

The proof is completed. □

Remark 4.4. *We only compare prices of at-the-money call options, because Schachermayer and Teichmann (2008) have mentioned that for the case $S_0 \neq K$, as $\sigma^{BS}\sqrt{T} \to 0$, the Bachelier price and the Black-Scholes price tend to $(S_0 - K)^+$ of order higher than $(\sigma^{BS}\sqrt{T})^n$ for every $n \geq 1$ and the functional dependence of the prices on $\sigma^{BS}\sqrt{T}$ is not analytical.*

4.4 CVaR hedging in the standard Bachelier model

Perfect hedging requires sufficient initial wealth and this is one reason for the implementation of partial hedging when the initial capital is not enough. We assume that a hedger is exposed to a future obligation H at the maturity time T. Meanwhile, the hedger constructs a hedging portfolio π with the initial capital v_0 that is less than the perfect hedging costs. In this case, perfect hedging is impossible and $L = H - V_T$ is a $\mathcal{F}_T$-measurable random variable that characterizes hedging losses.

In this chapter, we would like to employ CVaR to measure the hedging loss, that is

$$CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(L)ds, \tag{4.25}$$

where $\alpha \in (0,1)$ is the risk level and $VaR_\alpha(L) = \inf\{s \in \mathbb{R} : P(L \leq s) \geq \alpha\}$ is the α-quantile of L.

$CVaR_\alpha(L)$ measures the expected loss of a hedging strategy given that the loss exceeds its α-quantile. According to Rockafellar and Uryasev (2002), CVaR can be represented as

$$CVaR_\alpha(L) = \inf\big\{z + \frac{1}{1-\alpha}E((L-z)^+) : \ z \in \mathbb{R}\big\}, \tag{4.26}$$

and they also indicated that $VaR_\alpha(L)$ satisfies

$$VaR_\alpha(L) = \min\Big\{y|y \in \underset{z\in\mathbb{R}}{\operatorname{argmin}}\Big(z + \frac{1}{1-\alpha}E\big((L-z)^+\big)\Big)\Big\}.$$

The purpose of the hedger is to minimize the CVaR among all admissible strategies with the initial wealth no more than v_0, i.e.,

$$\min_{(v,\pi)\in\mathcal{A}_0} CVaR_\alpha(L), \tag{4.27}$$

where $\mathcal{A}_0 = \{(v,\pi)|(v,\pi) \in \mathcal{A}(v), v \le v_0\}$.

Melnikov and Smirnov (2012) discussed such a problem in complete markets and provided a semi-explicit solution using the Neyman-Pearson Lemma. The main result of their paper is summarized as following:

In a complete market, the optimal CVaR hedging strategy for a claim H is a perfect hedge of the modified contingent claim $(H-\hat{z})^+\varphi^*(\hat{z})$, where

$$\varphi^*(z) = I_{\{\tilde{a}(z)<Z_T^{M^{-1}}\}} + \Gamma(z)I_{\{\tilde{a}(z)=Z_T^{M^{-1}}\}}, \tag{4.28}$$

$$\tilde{a}(z) = \inf\{a \ge 0, E^M\big((H-z)^+I_{\{a<Z_T^{M^{-1}}\}}\big) \le v_0\}, \tag{4.29}$$

$$\Gamma(z) = \frac{v_0 - E^M\big((H-z)^+I_{\{\tilde{a}(z)<Z_T^{M^{-1}}\}}\big)}{E^M\big((H-z)^+I_{\{\tilde{a}(z)=Z_T^{M^{-1}}\}}\big)}, \tag{4.30}$$

where Z^M is the density of the unique martingale measure in such a market. In particular, $\Gamma(z) = 0$ if $P(Z_T^{M^{-1}} = \tilde{a}(z)) = 0$.

And the parameter $\hat{z}$ is the point of minimum of the function

$$d(z) = \begin{cases} z + \frac{1}{1-\alpha}\cdot E\big[(H-z)^+(1-\varphi^*(z))\big], & 0 \le z < z^*, \\ z, & z = z^*. \end{cases} \tag{4.31}$$

where z^* is the solution of the equation

$$E^M\big((H-z)^+\big) = v_0. \tag{4.32}$$

Applying the above result to the complete market (4.1), the optimal CVaR hedging strategy of a call option is given in the following theorem.

Theorem 4.5. *A. Consider an European call option $(S_T - K)^+$ in the Bachelier market (4.1). Under an initial capital constraint $v_0 < V_0^B$, the optimal CVaR hedging strategy is a perfect hedge for the modified claim $(S_T - K(\hat{z}))^+ I_{\{S_T > b(\hat{z})\}}$, where, for a given $z \in [0, z^*)$, $b(z)$ is the unique solution of the the equation $E^*\big[(S_T - K(z))^+ I_{\{S_T > b\}}\big] = v_0$, i.e.,*

$$\big(S_0 - K(z)\big)\Phi\Big(\frac{S_0 - b}{\sigma\sqrt{T}}\Big) + \sigma\sqrt{T}\phi\Big(\frac{S_0 - b}{\sigma\sqrt{T}}\Big) = v_0, \tag{4.33}$$

that satisfies $b(z) \geq K(z)$ and $\hat{z}$ is the solution of

$$\min_{z\in[0,z^*]} d(z) = \begin{cases} z + \frac{1}{1-\alpha}\kappa_1(z), & 0 \leq z < z^*, \\ z^*, & z = z^*, \end{cases} \tag{4.34}$$

where, $\kappa_1(z) = E\big[(S_T - K(z))^+ I_{\{S_T \leq b(z)\}}\big]$, i.e.,

$$\begin{aligned}\kappa_1(z) = &\big(S_0 + \mu T - K(z)\big)\Big[\Phi\Big(\frac{b(z) - S_0 - \mu T}{\sigma\sqrt{T}}\Big) - \Phi\Big(\frac{K(z) - S_0 - \mu T}{\sigma\sqrt{T}}\Big)\Big] \\ &+ \sigma\sqrt{T}\Big[\phi\Big(\frac{K(z) - \mu T - S_0}{\sigma\sqrt{T}}\Big) - \phi\Big(\frac{b(z) - \mu T - S_0}{\sigma\sqrt{T}}\Big)\Big].\end{aligned} \tag{4.35}$$

And $d(\hat{z})$ is the value of the minimal CVaR.

The parameter z^* is determined from $E^*\big[(S_T - K(z))^+\big] = v_0$, i.e.,

$$\big(S_0 - K(z)\big)\Phi\Big(\frac{S_0 - K(z)}{\sigma\sqrt{T}}\Big) + \sigma\sqrt{T}\phi\Big(\frac{S_0 - K(z)}{\sigma\sqrt{T}}\Big) = v_0, \tag{4.36}$$

where $K(z) = K + z$.

In particular, $b(z^*) = -\infty$.

B. The value of the optimal CVaR hedging strategy at time $t < T$ is

$$V_t = \big(S_t - K(\hat{z})\big)\Phi\Big(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\Big) + \sigma\sqrt{T-t}\phi\Big(\frac{b(\hat{z}) - S_t}{\sigma\sqrt{T-t}}\Big). \tag{4.37}$$

Moreover, components of the optimal hedging strategy are

$$\pi_t^0 = -K(\hat{z})\Phi\Big(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\Big) + \Big(S_t\sigma\sqrt{T-t} - \frac{b(\hat{z}) - K(\hat{z})}{\sigma\sqrt{T-t}}\Big)\phi\Big(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\Big), \tag{4.38}$$

$$\pi_t^1 = \Phi\Big(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\Big) + \frac{b(\hat{z}) - K(\hat{z})}{\sigma\sqrt{T-t}}\phi\Big(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\Big). \tag{4.39}$$

Proof. Let us rewrite the density $Z_T^* = \frac{dP}{dP^*}$ as follows:

$$\begin{aligned}\frac{dP}{dP^*} &= \exp\big(\frac{\mu}{\sigma}W_T^* - \frac{1}{2}(\frac{\mu}{\sigma})^2 T\big)\\ &= \exp(\frac{\mu}{\sigma^2}(S_0 + \sigma W_T^*) - \frac{1}{2}(\frac{\mu}{\sigma})^2 T - \frac{\mu}{\sigma^2}S_0)\\ &= \exp(\frac{\mu}{\sigma^2}S_T)\text{const},\end{aligned}$$

where $\text{const} = \exp(-\frac{1}{2}(\frac{\mu}{\sigma})^2 T - \frac{\mu}{\sigma^2}S_0)$ is positive.

Hence, the set $\{\frac{dP}{dP^*} > a\}$ can be rewritten as

$$\{\frac{dP}{dP^*} > a\} = \{\exp(\frac{\mu}{\sigma^2}S_T)\text{const} > a\} = \{S_T > b\},$$

for some constant b.

Moreover, $P\Big(\frac{dP}{dP^*} = a\Big) = 0$ is satisfied because of the continuous distribution. Thereby, $\Gamma(z) = 0$.

Applying above results to (4.28)-(4.30), we arrive to

$$\varphi^*(z) = I_{\{S_T > b(z)\}}, \tag{4.40}$$

$$b(z) = \inf\{b \in \mathbb{R}, E^*\big((H-z)^+ I_{\{S_T>b\}}\big) \leq v_0\}. \tag{4.41}$$

Also, for $z \geq 0$, we have $((S_T - K)^+ - z)^+ = (S_T - K(z))^+$, where $K(z) = K + z$.

Note that, in our case, the infimum in (4.41) is always attained since we deal with an atomless measure and thus we need to find b that solves $E^*\big((S_T - K(z))^+ I_{\{S_T>b\}}\big) = v_0$.

Consider the case $z \in [0, z^*)$, since otherwise we have $d(z^*) = z^*$.

If $b < K(z)$, we have

$$E^*\big((S_T - K(z))^+ I_{\{S_T>b\}}\big) = E^*\big((S_T - K(z))^+\big) > E^*\big((S_T - K(z^*))^+\big) = v_0.$$

Hence, the root $b(z)$ of $E^*\big((S_T - K(z))^+ I_{\{S_T>b\}}\big) = v_0$ is in the interval $[K(z), \infty)$.

Further, in this interval, we have

$$E^*\big((S_T - K(z))^+ I_{\{S_T>b\}}\big) = E^*\big((S_0 + \sigma W_T^* - K(z))I_{\{S_T>b\}}\big)$$

$$= \big(S_0 - K(z)\big)E^*(I_{\{S_0+\sigma W_T^*>b\}}) + \sigma E^*(W_T^* I_{\{S_0+\sigma W_T^*>b\}})$$

$$= \big(S_0 - K(z)\big)\Phi(\frac{S_0 - b}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\int_{\frac{b-S_0}{\sigma\sqrt{T}}}^{\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= \big(S_0 - K(z)\big)\Phi(\frac{S_0 - b}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\phi(\frac{b - S_0}{\sigma\sqrt{T}}).$$

For $b \geq K(z)$, the above expectation is strictly decreasing regarding to b, and thereby $b(z)$ is the unique root of the equation (4.33).

Now, let us deal with $E\big[\big(S_T - K(z)\big)^+\big(1 - \varphi^*(z)\big)\big] = E\big[\big(S_T - K(z)\big)^+ I_{\{S_T \leq b(z)\}}\big]$. We find that

$$E\big[(S_T - K(z))^+ I_{\{S_T\leq b(z)\}}\big] = E\big[\big(S_T - K(z)\big)I_{\{K(z)<S_T\leq b(z)\}}\big]$$

$$= E\big[(S_0 + \mu T + \sigma W_T - K(z))I_{\{K(z)<S_T\leq b(z)\}}\big]$$

$$= \big(S_0 + \mu T - K(z)\big)E\big(I_{\{\frac{K(z)-\mu T-S_0}{\sigma}<W_T\leq\frac{b(z)-S_0-\mu T}{\sigma}\}}\big)$$

$$+ \sigma E\big(W_T I_{\{\frac{K(z)-\mu T-S_0}{\sigma}<W_T\leq\frac{b(z)-S_0-\mu T}{\sigma}\}}\big)$$

$$= (S_0 + \mu T - K(z))\Big[\Phi(\frac{b(z) - S_0 - \mu T}{\sigma\sqrt{T}}) - \Phi(\frac{K(z) - S_0 - \mu T}{\sigma\sqrt{T}})\Big]$$

$$+ \sigma\sqrt{T}\Big[\phi(\frac{K(z) - S_0 - \mu T}{\sigma\sqrt{T}}) - \phi(\frac{b(z) - S_0 - \mu T}{\sigma\sqrt{T}})\Big].$$

Hence, we conclude that the function $d(z)$ admits the form (4.34). And the parameter $\hat{z}$ is the point of minimum of $d(z)$ over the interval $[0, z^*]$, where z^* is the root of the equation $E^*\Big((S_T - K(z))^+\Big) = v_0$.

The value of $E^*\Big((S_T - K(z))^+\Big)$ can be given by substituting K in (4.16) with $K(z)$, and hence we arrive to the equation (4.36).

In particular, $E^*\Big((S_T - K(z^*))^+\Big) = E^*\Big((S_T - K(z^*))^+ I_{\{S_T>-\infty\}}\Big) = v_0$, which implies $b(z^*) = -\infty$.

At time $t < T$ the value of the optimal hedging strategy is

$$V_t = E^*\big((S_T - K(\hat{z}))I_{\{S_T > b(\hat{z})\}}\big|\mathcal{F}_t\big) = E^*\big((S_t + \sigma W^*_{T-t} - K(\hat{z}))I_{\{S_t+\sigma W^*_{T-t} > b(\hat{z})\}}\big)$$
$$= \big(S_t - K(\hat{z})\big)E^*(I_{\{S_t+\sigma W^*_{T-t} > b(\hat{z})\}}) + \sigma E^*(W^*_{T-t}I_{\{S_t+\sigma W^*_{T-t} > b(\hat{z})\}})$$
$$= \big(S_t - K(\hat{z})\big)\Phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}) + \sigma\sqrt{T-t}\phi(\frac{b(\hat{z}) - S_t}{\sigma\sqrt{T-t}}).$$

Assume (π^0, π^1) is the duplication strategy of the modified claim, such that

$$dV_t = \pi^1_t dS_t,$$

and by Itô's formula, we have

$$dV_t = \frac{\partial V_t}{\partial S}dS_t + \big(\frac{\partial V_t}{\partial t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S^2}\big)dt.$$

Taking into account both these equations we find components of the optimal hedging strategy satisfy

$$\pi^1_t = \frac{\partial V_t}{\partial S} = \Phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}) + \frac{S_t - K(\hat{z})}{\sigma\sqrt{T-t}}\phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}})$$
$$- \frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}\phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}})$$
$$= \Phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}) + \frac{b(\hat{z}) - K(\hat{z})}{\sigma\sqrt{T-t}}\phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}),$$

and

$$\pi^0_t = V_t - S_t\pi^1_t = -K(\hat{z})\Phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}) + \big(S_t\sigma\sqrt{T-t} - \frac{b(\hat{z}) - K(\hat{z})}{\sigma\sqrt{T-t}}\big)\phi(\frac{S_t - b(\hat{z})}{\sigma\sqrt{T-t}}).$$

The proof of theorem 4.5 is completed. □

4.5 CVaR hedging in the modified Bachelier model

In the modified Bachelier model (4.5) which may be incomplete, we have to discuss CVaR hedging problem of a call option $H = (S_{T\wedge\tau} - K)^+$ in more detail.

According to (4.26), the CVaR minimization problem can be rewritten as

$$\min_{(v,\pi)\in\mathcal{A}_0} CVaR_\alpha(L) = \min_{(v,\pi)\in\mathcal{A}_0}\min_{z\in\mathbb{R}}\big\{z + \frac{1}{1-\alpha}E\big((H - V_T - z)^+\big)\big\}$$
$$= \min_{z\in\mathbb{R}}\big\{z + \frac{1}{1-\alpha}\min_{(v,\pi)\in\mathcal{A}_0}E\big((H - V_T - z)^+\big)\big\}. \tag{4.42}$$

Since $V_T \geq 0$, we have $(H - V_T - z)^+ = ((H - z)^+ - V_T)^+$. Furthermore, $H(z) = (H - z)^+$ is non-negative, so that it can be considered as a contingent claim. Hence, for a fixed z the inner minimization problem in (4.42) is an efficient hedging problem of the claim $H(z)$. It is worth to mention that we can focus on $z \in [0, z^*]$, where z^* is the solution of the equation

$$\sup_{\tilde{P} \in \mathbb{P}^*} E^{\tilde{P}}\big(H(z)\big) = v_0.$$

This is because, on the one hand, for $z > z^*$, v_0 is bigger than the minimal superhedging price of the claim $H(z)$. In other words, $\min\limits_{(v,\pi) \in \mathcal{A}_0} E((H(z) - V_T)^+) = 0$ and hence,

$$z + \frac{1}{1-\alpha} \min_{(v,\pi) \in \mathcal{A}_0} E((H - V_T - z)^+) = z > z^* \quad \text{for} \quad z > z^*,$$

which is increasing after z^*. Consequently, the infimum of (4.42) would not be attained in the interval $(z^*, +\infty)$.

On the other hand, because z is corresponding to the VaR_α of the hedging loss L, it is nonnegative when α is close to 1.

Since we only consider $z \geq 0$, it is true that $H(z) = ((S_{T \wedge \tau} - K)^+ - z)^+ = (S_{T \wedge \tau} - K(z))^+$ and the efficient hedging problem for it is

$$\min_{(v,\pi) \in \mathcal{A}_0} E\big[\big((S_{T \wedge \tau} - K(z))^+ - V_T\big)^+\big]. \tag{4.43}$$

Föllmer and Leukert (2000) studied the efficient hedging problem for general European claims. According to their results, if a random variable $\varphi' \in \mathcal{R}$ solves

$$\max_{\varphi \in \mathcal{R}} E\big[\varphi\big(S_{T \wedge \tau} - K(z)\big)^+\big], \tag{4.44}$$

where $\mathcal{R} = \{\varphi : \Omega \to [0,1] |\ \mathcal{F}_T - \text{measurable}, \sup\limits_{\tilde{P} \in \mathbb{P}^*} E^{\tilde{P}}((S_{T \wedge \tau} - K(z))^+ \varphi) \leq v_0\}$, then the optimal hedging strategy π is obtained from the optional decomposition of the modified claim $\varphi'(S_{T \wedge \tau} - K(z))^+$.

In summary, the problem of finding the optimal CVaR hedging strategy of a call option in the market (4.5) can be solved in three steps:

1) For a fixed $z \in [0, z^*]$, derive the optimal randomized test $\varphi'(z)$ for (4.44), in particular, $\varphi'(z^*) = 1$;

2) Derive the point of minimum $\hat{z}$ of the function

$$d_1(z) = z + \frac{1}{1-\alpha} \cdot E\big[(S_{T\wedge\tau} - K(z))^+(1 - \varphi'(z))\big],$$

on the interval $[0, z^*]$;

3) The optimal hedging strategy is given by optional decomposition (4.19) of the modified claim $H(\hat{z})\varphi'(\hat{z})$.

Usually it is hard to provide an explicit form of the randomized test $\varphi'(z)$ in incomplete markets. However, in our case

$$\begin{aligned}
\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(\varphi(S_{T\wedge\tau} - K)^+\big) &= \sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big(\varphi(S_T - K(z))^+ I_{\{\tau > T\}}\big) \\
&= \sup_{\tilde{P}\in\mathbb{P}^*} E\big(Z_T^{\tilde{P}}\varphi(S_T - K(z))^+ I_{\{\tau > T\}}\big) \\
&= E\Big(\exp\big(-\frac{\mu}{\sigma}W_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T\big)\varphi(S_T - K(z))^+ I_{\{\tau > T\}}\Big) \\
&= E^*\big(\varphi(S_T - K(z))^+ I_{\{\tau > T\}}\big) \\
&= E^*\big(\varphi(S_{T\wedge\tau} - K(z))^+\big),
\end{aligned}$$

and hence $\mathcal{R}$ can be rewritten as

$$\mathcal{R} = \{\varphi : \Omega \to [0,1] | \ \mathcal{F}_T - \text{measurable}, \quad E^*[(S_{T\wedge\tau} - K(z))^+\varphi] \le v_0\}.$$

The problem (4.44) becomes a problem of finding the optimal randomized test of a simple hypothesis. Such an optimal randomized test $\varphi'(z)$ can be derived with the help of the classical Neyman-Pearson lemma (see Föllmer and Leukert 2000) and again is given by (4.28)-(4.30) with $(H - z)^+ = (S_{T\wedge\tau} - K(z))^+$ and $Z_T^M = Z_T^*$ defined by (4.2).

Theorem 4.6. *A. Consider an European call option $(S_{T\wedge\tau} - K)^+$ in the modified Bachelier market (4.5). Under an initial capital constraint $v_0 < V_0^{MB}$, the optimal CVaR hedging strategy is given by the optional decomposition of the modified claim $(S_{T\wedge\tau} - K(\hat{z}))^+ I_{\{S_T > \tilde{b}(\hat{z})\}}$, where, for a* *fixed* $z \in [0, z^*)$, $\tilde{b}(z)$ *is the unique solution of the the equation* $E^*\big[(S_{T\wedge\tau} - K(z))^+ I_{\{S_T > b\}}\big] = v_0$, *i.e.,*

$$\big(S_0 - K(z)\big)\Phi(\frac{S_0 - b}{\sigma\sqrt{T}}) + \big(S_0 + K(z)\big)\Phi(\frac{-S_0 - b}{\sigma\sqrt{T}}) + \sigma\sqrt{T}\Big[\phi(\frac{S_0 - b}{\sigma\sqrt{T}}) - \phi(\frac{S_0 + b}{\sigma\sqrt{T}})\Big] = v_0, \quad (4.45)$$

that satisfies $\tilde{b}(z) \geq K(z)$ *and* $\hat{z}$ *is the solution of*

$$\min_{z\in[0,z^*]} d_1(z) = \begin{cases} z + \frac{1}{1-\alpha}\kappa_2(z), & 0 \leq z < z^*, \\ z^*, & z = z^*, \end{cases} \quad (4.46)$$

where $\kappa_2(z) = E\big[(S_{T\wedge\tau} - K(z))^+ I_{\{S_T \leq \tilde{b}(z)\}}\big]$ *that is*

$$\begin{aligned} \kappa_2(z) = {} & \big(S_0 + \mu T - K(z)\big)\Big[\Phi(\frac{\tilde{b}(z) - S_0 - \mu T}{\sigma\sqrt{T}}) - \Phi(\frac{K(z) - S_0 - \mu T}{\sigma\sqrt{T}})\Big] \\ & + \sigma\sqrt{T}\Big[\phi(\frac{K(z) - S_0 - \mu T}{\sigma\sqrt{T}}) - \phi(\frac{\tilde{b}(z) - S_0 - \mu T}{\sigma\sqrt{T}})\Big] \\ & + (S_0 - \mu T + K(z))\exp(-\frac{2S_0\mu}{\sigma^2})\Big[\Phi(\frac{\tilde{b}(z) + S_0 - \mu T}{\sigma\sqrt{T}}) - \Phi(\frac{K(z) + S_0 - \mu T}{\sigma\sqrt{T}})\Big] \\ & - \sigma\sqrt{T}\exp(-\frac{2S_0\mu}{\sigma^2})\Big[\phi(\frac{K(z) + S_0 - \mu T}{\sigma\sqrt{T}}) - \phi(\frac{\tilde{b}(z) + S_0 - \mu T}{\sigma\sqrt{T}})\Big]. \end{aligned} \quad (4.47)$$

And $d_1(\hat{z})$ *is the value of the minimal CVaR.*

The parameter z^* *is the root of the equation* $\sup_{\tilde{P}\in\mathbb{P}^*} E^{\tilde{P}}\big[(S_{T\wedge\tau} - K(z))^+\big] = v_0$, *i.e.,*

$$\begin{aligned} & \big(S_0 - K(z)\big)\Phi(\frac{S_0 - K(z)}{\sigma\sqrt{T}}) + \big(S_0 + K(z)\big)\Phi(\frac{-S_0 - K(z)}{\sigma\sqrt{T}}) \\ & + \sigma\sqrt{T}\Big[\phi(\frac{S_0 - K(z)}{\sigma\sqrt{T}}) - \phi(\frac{S_0 + K(z)}{\sigma\sqrt{T}})\Big] = v_0. \end{aligned} \quad (4.48)$$

In particular, $\tilde{b}(z^*) = -\infty$.

B. The value of the optimal CVaR hedging portfolio at time $t < T$ *is*

$$\tilde{V}_t = \begin{cases} \big(S_t - K(\hat{z})\big)\Phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \big(S_t + K(\hat{z})\big)\Phi(-\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\ +\sigma\sqrt{(T-t)}\Big[\phi(\frac{\tilde{b}(\hat{z}) - S_t}{\sigma\sqrt{(T-t)}}) - \phi(\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big], & t \in [0, \tau\wedge T); \\ 0, & t \in [\tau, T). \end{cases} \quad (4.49)$$

Moreover, weights in the hedging portfolio satisfy

$$\tilde{\pi}_t^0 = \begin{cases} K(\hat{z})\Big[\Phi(-\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) - \Phi(\frac{S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big] \\ +\sigma\sqrt{(T-t)}\Big[\phi(\frac{\tilde{b}(\hat{z})-S_t}{\sigma\sqrt{(T-t)}}) - \phi(\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big] \\ -\frac{S_t\left(\tilde{b}(\hat{z})-K(\hat{z})\right)}{\sigma\sqrt{(T-t)}}\Big[\phi(\frac{S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \phi(\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big], \quad t \in [0, \tau\wedge T); \\ 0, \quad t \in [\tau, T). \end{cases} \tag{4.50}$$

$$\tilde{\pi}_t^1 = \begin{cases} \Phi(\frac{S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \Phi(\frac{-S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\ +\frac{\tilde{b}(\hat{z})-K(\hat{z})}{\sigma\sqrt{(T-t)}}\Big[\phi(\frac{S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \phi(\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big], \quad t \in [0, \tau\wedge T); \\ 0, \quad t \in [\tau, T). \end{cases} \tag{4.51}$$

Proof. Similar to the discussion in the proof of Theorem 4.5, for a fixed $0 \le z < z^*$, if $b < K(z)$, we have

$$\begin{aligned} E^*\big((S_{T\wedge\tau} - K(z))^+ I_{\{\frac{dP}{dP^*}>a\}}\big) &= E^*\big((S_{T\wedge\tau} - K(z))^+ I_{\{S_T>b\}}\big) \\ &= E^*\big((S_T - K(z)) I_{\{S_T>b\}} I_{\{S_T>K(z)\}} I_{\{\tau>T\}}\big) \\ &= E^*\big((S_T - K(z)) I_{\{S_T>K(z)\}} I_{\{\tau>T\}}\big) \\ &= E^*\big((S_{T\wedge\tau} - K(z))^+\big) > E^*\big((S_{T\wedge\tau} - K(z^*))^+\big) = v_0. \end{aligned}$$

Thus, the root of the equation $E^*\big(I_{\{S_T>b\}}(S_{T\wedge\tau} - K(z))^+\big) = v_0$ is in the interval $[K(z), \infty)$.

In addition, $E^*\big(I_{\{S_T>b\}}(S_{T\wedge\tau}-K(z))^+\big)$ can be calculated as

$$
\begin{aligned}
E^*\big(I_{\{S_T>b\}}(S_{T\wedge\tau}-K(z))^+\big) &= E^*\Big[(S_T-K(z))I_{\{S_T>b\}}I_{\{\tau>T\}}\Big] \\
&= E^*\Big[(S_0+\sigma W_T^*-K(z))I_{\{S_0+\sigma W_T^*>b\}}I_{\{\tau>T\}}\Big] \\
&= \big(S_0-K(z)\big)P^*\Big(S_0+\sigma W_T^*>b, \min_{0\le t\le T} S_0+\sigma W_t^*>0\Big) \\
&+ \sigma E^*\Big(W_T^* I_{\{S_0+\sigma W_T^*>b\}} I_{\{\min\limits_{0\le t\le T} S_0+\sigma W_t^*>0\}}\Big) \\
&= \big(S_0-K(z)\big)\int_{\frac{b-S_0}{\sigma}}^{\infty}\int_{-\frac{S_0}{\sigma}}^{0\wedge x}\frac{2(x-2y)}{T\sqrt{2\pi T}}e^{-\frac{(2y-x)^2}{2T}}dydx \\
&+ \sigma\int_{\frac{b-S_0}{\sigma}}^{\infty} x\int_{-\frac{S_0}{\sigma}}^{0\wedge x}\frac{2(x-2y)}{T\sqrt{2\pi T}}e^{-\frac{(2y-x)^2}{2T}}dydx \\
&= \big(S_0-K(z)\big)\int_{\frac{b-S_0}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi T}}\Big(e^{-\frac{x^2}{2T}}-e^{-\frac{(x+\frac{2S_0}{\sigma})^2}{2T}}\Big)dx \\
&+ \sigma\int_{\frac{b-S_0}{\sigma}}^{\infty} x\frac{1}{\sqrt{2\pi T}}\Big(e^{-\frac{x^2}{2T}}-e^{-\frac{(x+\frac{2S_0}{\sigma})^2}{2T}}\Big)dx \\
&= \big(S_0-K(z)\big)\Phi(\frac{S_0-b}{\sigma\sqrt{T}})+\big(S_0+K(z)\big)\Phi(-\frac{S_0+b}{\sigma\sqrt{T}}) \\
&+ \sigma\sqrt{T}\Big[\phi(\frac{b-S_0}{\sigma\sqrt{T}})-\phi(\frac{S_0+b}{\sigma\sqrt{T}})\Big],
\end{aligned}
$$

where we apply the joint density of a driftless Brownian motion and its minimum (see, Privault 2014):

$$
f_{W_T^*,\min\limits_{0\le t\le T} W_t^*}(x,y) = I_{\{x\ge y\}}I_{\{y\le 0\}}\frac{2(x-2y)}{T\sqrt{2\pi T}}\exp\Big\{-\frac{(2y-x)^2}{2T}\Big\}.
$$

Therefore, $\tilde{b}(z)$ is the solution of the equation (4.45).

As for $E\big((S_{T\wedge\tau}-K(z))^+(1-\varphi'(z))\big)=E\big((S_{T\wedge\tau}-K(z))^+I_{\{S_T\le\tilde b(z)\}}\big)$, we have

$$
\begin{aligned}
&E\big((S_{T\wedge\tau}-K(z))^+I_{\{S_T\le\tilde b(z)\}}\big)=E((S_T-K(z))^+I_{\{\tau>T\}}I_{\{S_T\le\tilde b(z)\}})\\
&=E\big[(S_T-K(z))I_{\{\tau>T\}}I_{\{K(z)<S_T\le\tilde b(z)\}}\big]\\
&=E\big[(S_0+\mu T+\sigma W_T-K(z))I_{\{\frac{K(z)-S_0}{\sigma}<\frac{\mu}{\sigma}T+W_T\le\frac{\tilde b(z)-S_0}{\sigma}\}}I_{\{\min\limits_{0\le t\le T}S_0+\mu t+\sigma W_t>0\}}\big]\\
&=\big(S_0-K(z)\big)P\Big(\frac{K(z)-S_0}{\sigma}<W_T^*\le\frac{\tilde b(z)-S_0}{\sigma},\min_{0\le t\le T}W_t^*>-\frac{S_0}{\sigma}\Big)\\
&+\sigma E\big(W_T^*I_{\{\frac{K(z)-S_0}{\sigma}<W_T^*\le\frac{\tilde b(z)-S_0}{\sigma}\}}I_{\{\min\limits_{0\le t\le T}W_t^*>-\frac{S_0}{\sigma}\}}\big)\\
&=\big(S_0+\mu T-K(z)\big)\Big[\Phi(\frac{\tilde b(z)-S_0-\mu T}{\sigma\sqrt{T}})-\Phi(\frac{K(z)-S_0-\mu T}{\sigma\sqrt{T}})\Big]\\
&+\big(S_0-\mu T+K(z)\big)\exp(-\frac{2S_0\mu}{\sigma^2})\Big[\Phi(\frac{\tilde b(z)+S_0-\mu T}{\sigma\sqrt{T}})-\Phi(\frac{K(z)+S_0-\mu T}{\sigma\sqrt{T}})\Big]\\
&-\sigma\sqrt{T}\exp(-\frac{2S_0\mu}{\sigma^2})\Big[\phi(\frac{K(z)+S_0-\mu T}{\sigma\sqrt{T}})-\phi(\frac{\tilde b(z)+S_0-\mu T}{\sigma\sqrt{T}})\Big]\\
&+\sigma\sqrt{T}\Big[\phi(\frac{K(z)-S_0-\mu T}{\sigma\sqrt{T}})-\phi(\frac{\tilde b(z)-S_0-\mu T}{\sigma\sqrt{T}})\Big],
\end{aligned}
$$

where in the last equation we utilize the joint density of a drifted Brownian motion with mean $\frac{\mu}{\sigma}t$ and its minimum (see, Privault 2014):

$$
f_{W_T^*,\min\limits_{0\le t\le T}W_t^*}(x,y)=I_{\{x\ge y\}}I_{\{y\le 0\}}\frac{2(x-2y)}{T\sqrt{2\pi T}}\exp\big\{-(\frac{\mu}{\sigma})^2\frac{T}{2}+\frac{\mu}{\sigma}x-\frac{(2y-x)^2}{2T}\big\}.
$$

Thus, the function $d_1(z)$ admits the form (4.46).

Also, according to Theorem 4.1, we know that

$$
\begin{aligned}
&\sup_{\tilde P\in\mathbb{P}^*}E^{\tilde P}((S_{T\wedge\tau}-K(z))^+)=E^*((S_{T\wedge\tau}-K(z))^+)\\
&=(S_0-K(z))\Phi(\frac{S_0-K(z)}{\sigma\sqrt{T}})+\sigma\sqrt{T}\phi(\frac{S_0-K(z)}{\sigma\sqrt{T}})\\
&-(-S_0-K(z))\Phi(\frac{-S_0-K(z)}{\sigma\sqrt{T}})-\sigma\sqrt{T}\phi(\frac{-S_0-K(z)}{\sigma\sqrt{T}})\\
&=(S_0-K(z))\Phi(\frac{S_0-K(z)}{\sigma\sqrt{T}})+(S_0+K(z))\Phi(\frac{-S_0-K(z)}{\sigma\sqrt{T}})\\
&+\sigma\sqrt{T}\Big[\phi(\frac{S_0-K(z)}{\sigma\sqrt{T}})-\phi(\frac{S_0+K(z)}{\sigma\sqrt{T}})\Big],
\end{aligned}
$$

and thus z^* is the root of the equation (4.48).

As for the value of CVaR hedging strategy at time $t < T$, we have, for any $\tilde{P} \in \mathbb{P}^*$,

$$
\begin{aligned}
& E^{\tilde{P}}\big((S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big) \\
&= \frac{E\big(\tilde{Z}_T (S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big)}{\tilde{Z}_t} \\
&= \frac{E\big(\tilde{Z}_T (S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big) I_{\{\tau > t\}}}{\tilde{Z}_t} \\
&= \frac{E\big(Z^*_T (S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big) I_{\{\tau > t\}}}{Z^*_t} \\
&= E^*\big((S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big) I_{\{\tau > t\}} \\
&= E^*\big((S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big).
\end{aligned}
$$

Therefore, the arbitrage free value of the modified claim is unique and hence it can be replicated, i.e., $\exists \tilde{\pi}$ and a consumption process $C_t = 0, \ \forall t \in [0, T]$, such that

$$
U_t = E^*\big((S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}} | \mathcal{F}_t\big) = \tilde{\pi}^0_t + \tilde{\pi}^1_t S_{t\wedge\tau} = \tilde{V}_t,
$$

and

$$
d\tilde{V}_t = \tilde{\pi}^1_t dS_{t\wedge\tau} = \tilde{\pi}^1_t I_{\{\tau \geq t\}} dS_t.
$$

In addition, by some calculations, we have

$$
\begin{aligned}
& E^*\big((S_T - K(\hat{z}))I_{\{S_T > \tilde{b}(\hat{z})\}} I_{\{\tau > T\}}|\mathcal{F}_t\big) \\
&= E^*\big[(S_t + \sigma(W_T^* - W_t^*) - K(\hat{z}))I_{\{S_t+\sigma(W_T^*-W_t^*)>\tilde{b}(\hat{z})\}} I_{\{\min\limits_{s\in[0,T]} S_s > 0\}}|\mathcal{F}_t\big] \\
&= E^*\big[(S_t + \sigma(W_T^* - W_t^*) - K(\hat{z}))I_{\{S_t+\sigma(W_T^*-W_t^*)>\tilde{b}(\hat{z})\}} I_{\{\min\limits_{s\in[0,t]} S_s > 0\}} I_{\{\min\limits_{s\in[t,T]} S_s > 0\}}|\mathcal{F}_t\big] \\
&= E^*\big[(S_t + \sigma W_{T-t}^* - K(\hat{z}))I_{\{S_t+\sigma W_{T-t}^*>\tilde{b}(\hat{z})\}} I_{\{\min\limits_{s\in[t,T]} \sigma W_{s-t}^* > -S_t\}}\big] I_{\{\tau>t\}} \\
&= \Bigg[\big(S_t - K(\hat{z})\big)\int_{\frac{\tilde{b}(\hat{z})-S_t}{\sigma}}^{\infty}\int_{-\frac{S_t}{\sigma}}^{0\wedge x} \frac{2(x-2y)}{T\sqrt{2\pi(T-t)}} e^{-\frac{(2y-x)^2}{2(T-t)}}\,dydx \\
&+ \sigma\int_{\frac{\tilde{b}(\hat{z})-S_t}{\sigma}}^{\infty} x\int_{-\frac{S_t}{\sigma}}^{0\wedge x} \frac{2(x-2y)}{(T-t)\sqrt{2\pi(T-t)}} e^{-\frac{(2y-x)^2}{2(T-t)}}\,dydx\Bigg] I_{\{\tau>t\}} \\
&= \Bigg[\big(S_t - K(\hat{z})\big)\int_{\frac{\tilde{b}(\hat{z})-S_t}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}}\Big(e^{-\frac{x^2}{2(T-t)}} - e^{-\frac{(x+\frac{2S_t}{\sigma})^2}{2(T-t)}}\Big)dx \\
&+ \sigma\int_{\frac{\tilde{b}(\hat{z})-S_t}{\sigma}}^{\infty} x\frac{1}{\sqrt{2\pi T}}\Big(e^{-\frac{x^2}{2(T-t)}} - e^{-\frac{(x+\frac{2S_t}{\sigma})^2}{2T}}\Big)dx\Bigg] I_{\{\tau>t\}} \\
&= \Bigg[\big(S_t - K(\hat{z})\big)\Phi\Big(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}\Big) + \big(S_t + K(\hat{z})\big)\Phi\Big(-\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}\Big) \\
&+ \sigma\sqrt{(T-t)}\Big[\phi\Big(\frac{\tilde{b}(\hat{z}) - S_t}{\sigma\sqrt{(T-t)}}\Big) - \phi\Big(\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}\Big)\Big]\Bigg] I_{\{\tau>t\}}.
\end{aligned}
$$

Thus, the value of the CVaR hedging strategy at time t is

$$
\tilde{V}_t = \begin{cases}
\big(S_t - K(\hat{z})\big)\Phi(\frac{S_t-\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \big(S_t + K(\hat{z})\big)\Phi(-\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\
+\sigma\sqrt{(T-t)}\Big[\phi(\frac{\tilde{b}(\hat{z})-S_t}{\sigma\sqrt{(T-t)}}) - \phi(\frac{S_t+\tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\Big], \quad t \in [0, \tau\wedge T); \\
0, \quad t \in [\tau, T).
\end{cases}
$$

For $t < \tau$, unites of the stock in the hedging portfolio are

$$\begin{aligned}
\tilde{\pi}_t^1 = \frac{d\tilde{V}_t}{dS} &= \Phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \Phi(\frac{-S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \frac{S_t - K(\hat{z})}{\sigma\sqrt{(T-t)}}\phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\
&- \frac{S_t + K(\hat{z})}{\sigma\sqrt{(T-t)}}\phi(\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) - \frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}\phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\
&+ \frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}\phi(\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\
&= \Phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \Phi(\frac{-S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) \\
&+ \frac{\tilde{b}(\hat{z}) - K(\hat{z})}{\sigma\sqrt{(T-t)}}\big[\phi(\frac{S_t - \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}}) + \phi(\frac{S_t + \tilde{b}(\hat{z})}{\sigma\sqrt{(T-t)}})\big],
\end{aligned}$$

and values of the riskless asset in the portfolio are given by

$$\tilde{\pi}_t^0 = \tilde{V}_t - \tilde{\pi}_t^1 S_t.$$

Meanwhile, for $\tau \le t$, the option is worthless and hence it is hedged with a void strategy, i.e.,

$$\tilde{\pi}_t^0 = 0, \quad \tilde{\pi}_t^1 = 0.$$

□

4.6 Illustrative numerical examples

In this section, numerical examples are provided to compare models (4.1), (4.3), (4.5) in aspects of perfect option pricing as well as CVaR based partial hedging.

We assume models (4.1), (4.3), (4.5) with following parameters:

$$S_0 = 100, \quad \mu^{BS} S_0 = \mu = 4.$$

As we have mentioned in Section 4.2, σ directly affects the property of the stopping time τ, and thus in the following, we would assume different levels of the volatility σ and the time to maturity T to investigate their impacts. Through the rest of this chapter, we assume $\sigma^{BS} S_0 = \sigma$ and $K = S_0$.

Table 4.1 Fair prices of at-the-money call options in the standard Bachelier model, the Black-Scholes model and the modified Bachelier model.

		Bahelier (1)	Modified Bachelier (2)	Black-Scholes (3)	difference (1)-(3)	difference (2)-(3)
T=1/12	σ=2.4	0.2764	0.2764	0.2764	$5.5279\cdot 10^{-7}$	$5.5279\cdot 10^{-7}$
T=10	σ=2.4	3.0278	3.0278	3.0270	$7.2651\cdot 10^{-4}$	$7.2651\cdot 10^{-4}$
T=1/12	σ=10	1.1516	1.1516	1.1516	$3.9986\cdot 10^{-5}$	$3.9986\cdot 10^{-5}$
T=10	σ=10	12.6157	12.6157	12.5633	0.0524	0.0524
T=1/12	σ=30	3.4549	3.4549	3.4539	0.0011	0.0011
T=10	σ=30	37.8470	37.2471	36.4744	1.3726	0.7727

Table 4.1 lists perfect hedging costs of call options in those three models. It is observed that for small values of σ and T, both $V_0^B - V_0^{BS}$ and $V_0^{MB} - V_0^{BS}$ are negligible. This conclusion is consistent with the result in Schachermayer and Teichmann (2008) which has indicated that the Bachelier option pricing formula (4.16) coincides closely with the Black-Scholes formula (4.17) if $\sigma^{BS}\sqrt{T}$ is small. Moreover, the Bachelier price V_0^B and the price in the modified Bachelier model V_0^{MB} are the same. This is because for such small σ and T, the possibility that the stock price would attach zero is extremely low. However, as σ and T increase, both $V_0^B - V_0^{BS}$ and $V_0^{MB} - V_0^{BS}$ become noticeable and such differences are smaller between the modified Bachelier model and the Black-Scholes model, which is supported by the Theorem 4.3. Meanwhile, in this case, V_0^{MB} is lower than V_0^B, since even if the stock price hits zero, it still can bounce back in the standard Bachelier model, while in the modified Bachelier market, once the stock price hits 0, it stays there forever, which lowers the probability of payouts.

Moreover, Figure 4.1 displays prices of at-the-money call options as functions of $\sigma^{BS}\sqrt{T}$ in those three models and Figure 4.2 shows differences between V_0^B and V_0^{BS} as well as differences between V_0^{MB} and V_0^{BS} as functions of $\sigma^{BS}\sqrt{T}$. Again, they satisfy the relation $|V_0^{MB} - V_0^{BS}| \leq |V_0^B - V_0^{BS}|$. Meanwhile, according to the equation (4.23), V_0^B is always lower

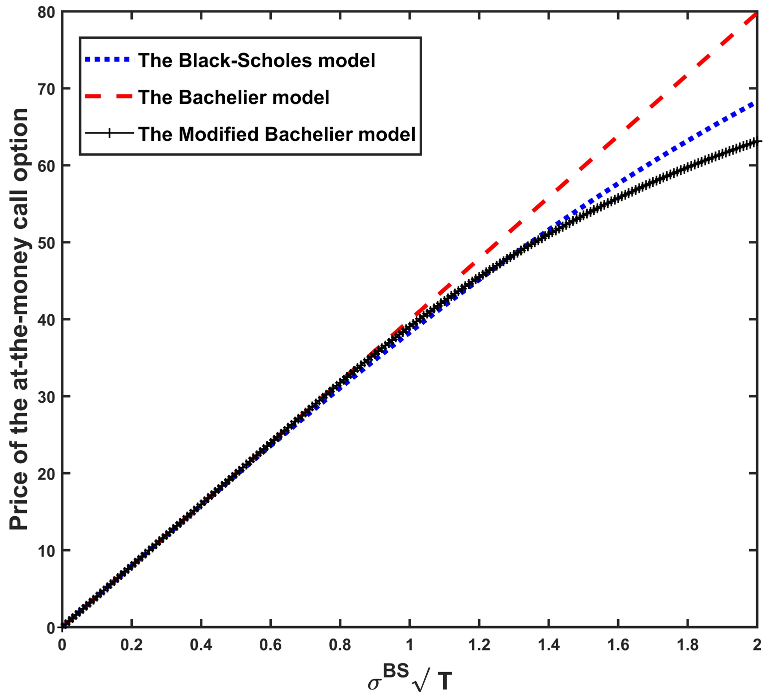

Figure 4.1 Comparison through fair price.

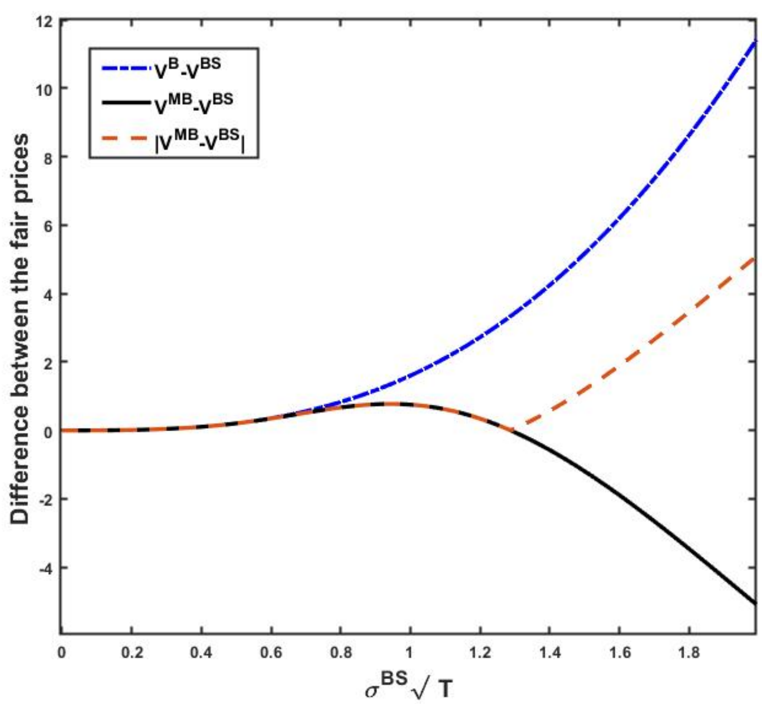

Figure 4.2 Difference between fair prices.

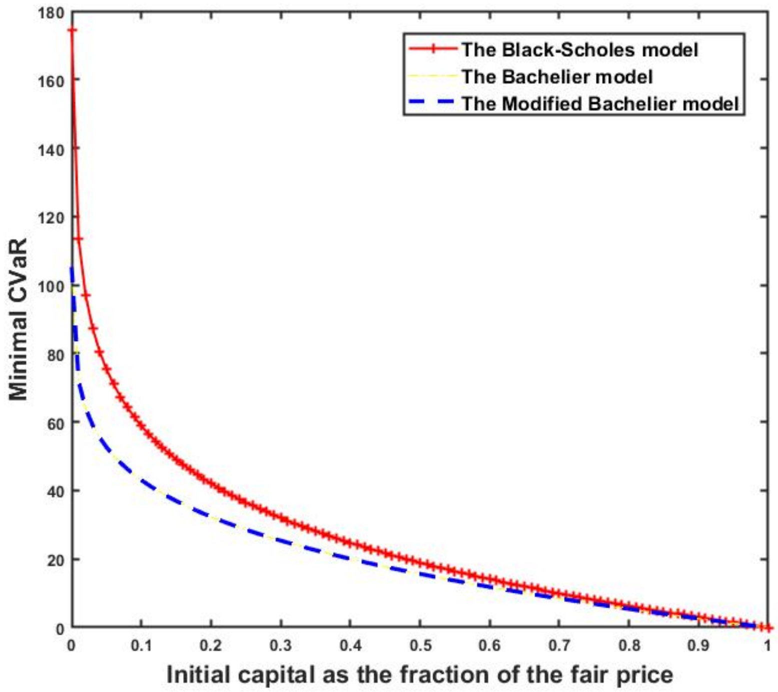

Figure 4.3 Minimal CVaR for varying levels of initial wealth $T = 10,\ \sigma = 10$.

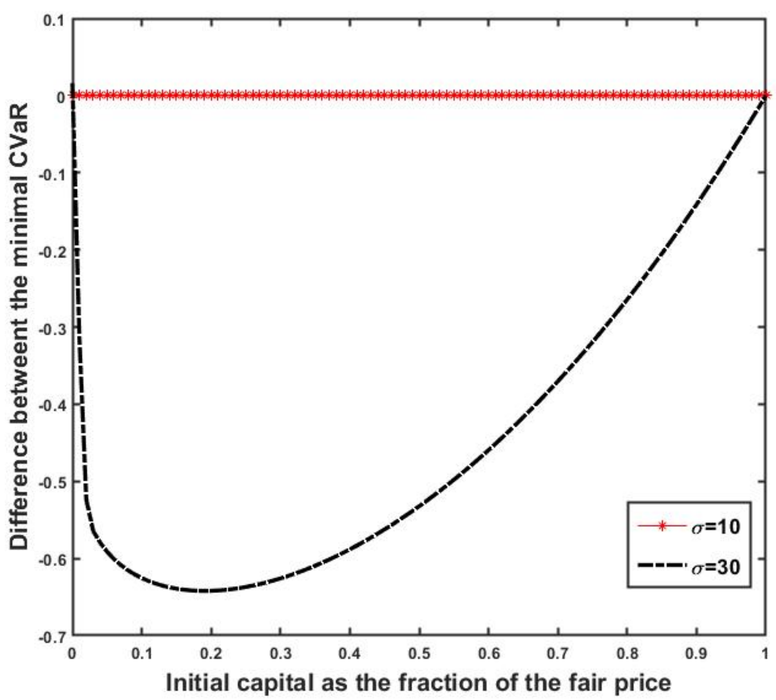

Figure 4.4 Difference between the minimal CVaR in the Bachelier model and the modified Bachelier model.

than V_0^{BS}, however, as the Figure 4.2 shows, V_0^{MB} can be lower than or higher than V_0^{BS} depending on the size of $\sigma^{BS}\sqrt{T}$. When it is large, we have the relation $V_0^{MB} < V_0^{BS}$, which is because of the high chance that the payoff in the modified Bachelier model is 0. Here, we only show the part $\sigma^{BS}\sqrt{T} \leq 2$ that is the usual case in reality.

Furthermore, let us compare models (4.1), (4.3), (4.5) through the CVaR partial hedging. For fixed $\alpha = 0.95$, $T = 10$, and $\sigma = 10$, Figure 4.3 displays the minimal CVaR for different initial wealth (as a fraction of the fair price) in those three models. It is observed that the minimal CVaR is larger in the Black-Scholes model. In order to compare the standard and the modified Bachelier model more clearly, we display the difference between minimal CVaR in those two models in Figure 4.4. Assume $T = 10$, for a small σ ($\sigma = 10$), minimal values of CVaR are the same in those two models. But, when σ is large ($\sigma = 30$), the minimal CVaR is smaller in the standard Bachelier model if the same fraction of the initial wealth is allocated. The smaller value of the optimal CVaR can be explained by the fact that the fair price in the modified Bachelier model is lower than that in the standard Bachelier model (for $\sigma = 30$, T=10, we have $V_0^{MB} = 37.24$, $V_0^{B} = 37.84$), and thus the initial hedging capital is higher in the later model. On the other hand, if the same amount of initial capital is invested, for example $v_0 = 28$, the minimal CVaR in the standard Bachelier model is 21.6580 and it is larger than that in the modified Bachelier model which is 20.8808.

4.7 Conclusion

The drawback of the standard Bachelier model is that stock prices can take negative values. The Black-Scholes model overcomes such a shortcoming due to its exponential property. Our main objective of this part is to provide two additional modifications and compare those models from points of view of the perfect hedging price as well as the CVaR hedging strategy. The first modified model is the one with absorption and another one is the model with reflection. However, there is no an equivalent martingale measure in the market model with reflection and thus we only focus on the standard Bachelier model, the Black-Scholes model and the modified Bachelier model with absorption. Comparisons of at-the-money call options' fair prices among those three models are implemented. Results indicate that when the volatility and the time to maturity are small, fair prices in those three models are quite close, while as the volatility and the time to maturity increase, differences among them are no longer negligible. An important finding is that differences of prices between the standard Bachelier model and the Black-Scholes model are larger than that between the modified Bachelier model and the Black-Scholes model. Moreover, the partial hedging problem is also investigated in this paper by employing a coherent risk measure called CVaR. Explicit forms of the optimal CVaR hedging strategy are provided with the help of the Neyman-Pearson Lemma and conclusions in Melnikov and Smirnov (2012).

For future research, one can extend approaches in this chapter to modify the class of Levy processes which gain attentions in the last two decades and are exploited in Mathematical Finance to model stock prices in the form of their exponent (see, for instance, Cont and Tankov 2004).

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CHAPTER 5

On RVaR based optimal partial hedging

5.1 Introduction

Control of risk is a main and permanent research topic of modern mathematical finance. It gives a perfect motivation for theoretical developments and is vitally important for practice. According to the option pricing theory, any contingent claim can be hedged perfectly if the initial wealth is no less than the fair price of the claim in complete markets or if it is no less than the minimal superhedging costs in incomplete markets. However, perfect hedging costs are usually too high to be of practical interest in most cases and that is why the partial hedging comes out. During such hedging processes, investors allocate initial capitals less than perfect hedging costs and would like to construct strategies that minimize their shortfall under certain risk measures. Another reason that makes the partial hedging interesting is that although it has some downside risk, it brings opportunities to gain benefits. Some financial institutions such as insurance companies indeed exploit risk to make profits. Föllmer and Leukert are pioneers in the filed of optimal partial hedging. They studied quantile hedging (see Föllmer and Leukert 1999) and efficient hedging (see Föllmer and Leukert 2000) in seimimartingale financial market models. Moreover, partial hedging problems have been discussed in more sophisticated markets. For example, Spivak and Cvitanić (1999) investigated quantile hedging in markets with partial information and in markets with large investors. Nakano (2011) solved problems of optimal quantile hedging and efficient hedging with linear loss functions for claims with a single default time. The book of Melnikov and Nosrati

(2017) discussed several partial hedging methods and their applications in pricing and hedging of insurance contracts.

One important component that investors need to consider when they construct a partial hedging is the risk criterion employed. The theory of risk measures was discussed in Artzner et al. (1999) and was applied to pricing and hedging of contingent claims by Xu (2006), where the associated optimal portfolio is determined by minimizing a convex measure of risk. We also note some other works: Bernard et al. (2015) for Law-Invariance risk measures, Madan and Schoutens (2016) for risk measures generated by distortion functions. In financial institutions, Value-at-risk (VaR) and Conditional Value-at-Risk (CVaR) are most commonly used risk measures and there is a long list of references regarding them. For instance, Melnikov and Smirnov (2012) studied partial hedging with the measure CVaR where the semi-explicit solution of the optimal CVaR hedging problem in complete markets was given. Cong et al. (2013) discussed VaR based optimal hedging while, in Cong et al. (2014), the CVaR based optimal hedging problem was solved without the restriction regarding the completeness of markets. In this part, we would like to implement a more general tail risk measure named Range Value-at-Risk (RVaR) which was applied to hedging problems by Cont et al. (2010) and Embrechts et al. (2018). Such a measure includes VaR and CVaR as two limiting cases and hence it provides insights into connections between these two other measures and it is more customized since investors can set their risk appetite by choosing two risk level parameters α as well as β. Meanwhile, Cont et al. (2010) have indicated that a risk measure can not be both robust and coherent, but RVaR constitutes a tradeoff between the sensitivity of CVaR and the robustness of VaR nonetheless.

The main objective of this chapter is to derive an optimal hedging strategy that minimizes the RVaR of a hedger's risk exposure subject to an initial wealth constraint and compare it with VaR and CVaR based optimal hedging strategies to investigate relationships among them. Similar to the procedure in Cong et al. (2013) and Cong et al. (2014), the optimization problem is solved in two steps. First, we search for an optimal partition between the hedged loss which is the part to be hedged with the initial capital and the retained loss that can not be hedged. Then, a hedging strategy of the hedged loss should be constructed. It is worth to emphasize that the method in this chapter can be applied to any arbitrage free market even if it is incomplete and hence our methodology has some advantages since as was mentioned in Föllmer and Leukert (1999) that deriving explicit solutions of optimal partial hedging in incomplete markets was extremely hard. In addition, the structure of our optimal partial hedging is independent of the dynamic of the underlying asset and such a strategy can be easily derived even though the financial model is complex. Furthermore, as an application, we construct optimal RVaR hedging strategies of life insurance contracts in Mixed Fractional Brownian motion (MFBM) markets which have the property of a long-range dependence. Melnikov and Mishura (2011) investigated the quantile hedging problem in such markets and pointed out that explicit solutions of partial hedging problems were rather difficult to be derived because of the complicated structure of the martingale measure. Such a difficulty is overcome by our method. In our opinion, MFBM markets are not used yet in Equity-linked life insurance, but they may bring better modelling and pricing properties in actuarial calculations.

The rest of the chapter is organized as follows. In Section 5.2, we provide the definition of RVaR and introduce some properties of it . In Section 5.3, we begin with the formalization of our RVaR based hedging problem and then we derive two explicit solutions of it depending on the size of the initial wealth. Most importantly, in Section 5.3.2, we show that VaR based hedging and CVaR based hedging can be seen as two special cases of RVaR hedging. In Section 5.4, a numerical example is provided to explain our method and to describe how it can be implemented to hedge life insurance contracts in sophisticated markets. Section 5.5 concludes the chapter.

5.2 Range Value-at-Risk

Let $(\Omega, \mathcal{F}, P)$ be an probability space and $\mathcal{S}$ be the set of real-valued random variables denoting loss amounts (a negative value represents gains). A risk measure $\rho(x)$ is a mapping from $\mathcal{S}$ to $\mathbb{R}$. For instance, VaR and CVaR are commonly used risk measures in the financial industry which are defined as

$$VaR_\alpha(L) = \inf\{v \in \mathbb{R} : P(L > v) \leq 1 - \alpha\}, \tag{5.1}$$

and

$$CVaR_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_s(L)ds, \tag{5.2}$$

where $\alpha \in (0, 1)$ is the risk level and the random variable $L \in \mathcal{S}$.

In this chapter, we would like to investigate a measure called Range Value-at-Risk (RVaR) that can be seen as a bridge between VaR and CVaR.

The RVaR at the level (α, β) such that $0 < \alpha \leq \alpha + \beta < 1$ is defined as

$$RVaR_{\alpha,\beta}(L) = \begin{cases} \frac{1}{\beta} \int_\alpha^{\alpha+\beta} VaR_s(L)ds, & if \;\; \beta > 0, \\ VaR_\alpha(L), & if \;\; \beta = 0, \end{cases} \tag{5.3}$$

which is the average value of VaR among specific risk levels $[\alpha, \alpha + \beta]$.

As we have introduced in Section 1.2, a risk measure is called coherent if it satisfies **Monotonicity** (1.1), **Positively homogeneity** (1.2), **Subadditivity** (1.4) and **Translation invariance** (1.5). CVaR is a coherent risk measure, however, Embrechts et al. (2018) have proved that both VaR and RVaR only satisfy monotonicity, positively homogeneity and translation invariance while they do not satisfy subadditivity. Instead, for all L_1, $L_2 \in \mathcal{S}$, the following inequalities hold true:

$$VaR_{\alpha_1+\alpha_2}(L_1 + L_2) \leq VaR_{\alpha_1}(L_1) + VaR_{\alpha_2}(L_2), \tag{5.4}$$

$$RVaR_{\tilde{\alpha}+\tilde{\beta}}(L_1 + L_2) \leq RVaR_{\alpha_1,\beta_1}(L_1) + RVaR_{\alpha_2,\beta_2}(L_2), \tag{5.5}$$

where $\tilde{\alpha} = \alpha_1 + \alpha_2$ and $\tilde{\beta} = \max\{\beta_1, \beta_2\}$. (5.4) and (5.5) are called the special form of subadditivity.

Desired properties of RVaR are that it is robust and it is a general tail risk measure which includes VaR as well as CVaR as two limiting cases, i.e.,

$$RVaR_{\alpha,0}(L) = VaR_\alpha(L) = \lim_{\beta \to 0^+} RVaR_{\alpha,\beta}(L), \tag{5.6}$$

$$RVaR_{\alpha,1-\alpha}(L) = CVaR_\alpha(L). \tag{5.7}$$

Additionally, all three measures belong to a wide class of distortion risk measures, such that

$$\rho(L) = \int_0^1 VaR_s(L)dh(s), \tag{5.8}$$

where $h(s)$ called a distortion function is a non-decreasing and left-continuous function from $[0,1]$ to $[0,1]$ satisfying $h(0) = 0$ and $h(1) = 1$. Specifically, for $\alpha \in (0,1)$, $\beta \in [0,1)$, the distortion function of $RVaR_{\alpha,\beta}$ is

$$h_{\alpha,\beta}(s) = \begin{cases} \min\left\{I_{\{s>\alpha\}}\frac{s-\alpha}{\beta}, 1\right\}, & if \;\; \beta > 0, \\ I_{\{s>\alpha\}}, & if \;\; \beta = 0. \end{cases} \tag{5.9}$$

As we know that while VaR and CVaR have some common properties, they are still quite different and consequently investigating RVaR which is a bridge between them helps us to understand their connections and to gain a more integrated picture regarding risk criteria.

5.3 Optimal RVaR based hedging and connection with CVaR and VaR hedging

5.3.1 Optimal RVaR based hedging

Assume that a financial market is arbitrage free but does not have to be complete. In this case, arbitrage free price of a claim may not be unique and a popular option pricing approach in incomplete markets is the utility based indifference pricing (UBIP). The idea behind this pricing method is that all investors' risk appetite can be fully described by an utility function and all investors are presumed to maximize their expected utility of wealth. Then, the utility based indifference price of a claim is the amount that makes no difference to investors' expected utility no matter whether they include the claim into their portfolio or not. One important property of utility based indifference price is that it does not depend on the completeness of the market and in the special case that a market is complete, it coincides with the fair price of that claim (see Henderson and Hobson 2009 for more properties regarding UBIP).

Suppose a hedger is exposed to a future obligation X at time T. Option pricing theorem points out that a claim can be hedged perfectly if the initial hedging capital is no less than its minimal super-hedging price (or fair price in complete markets). However, if the initial hedging capital $\tilde{v}_0$ is not enough, the hedger is exposed to some downside risk. The main goal of this part is to find an optimal hedging strategy that achieves the minimal risk under the measure RVaR subject to an initial wealth constraint.

We start with a partition of the contingent claim X such that $X = f(X) + R_f(X)$. $f(X)$ named hedged loss function represents the part that would be hedged with the initial capital $\tilde{v}_0$ and $R_f(X)$ called the retained loss function is the payout the hedger retains. Thus, the problem of optimal hedging can be solved in two steps: first, find the optimal partition $f^*(X)$, $R_{f^*}(X)$ satisfying the initial capital constraint and some assumptions regarding hedged loss functions; second, hedge the payout $f^*(X)$ perfectly. Cong et al. (2013, 2014) considered VaR and CVaR hedging problems with some restrictive assumptions regarding the hedged loss functions. Assumptions they imposed are

(a) Not globally over-hedged: $f(x) \leq x$ for all $x \geq 0$;

(b) Not locally over-hedged: $f(x_2) - f(x_1) \leq x_2 - x_1$ for all $0 \leq x_1 \leq x_2$;

(c) Nonnegativity of the hedged loss: $f(x) \geq 0$ for all $x \geq 0$.

Assumption (a) ensures that the hedged loss would be bounded from above by the original risk to be hedged. Assumption (b) indicates that the increment of the hedged part should be no more than the increment of the risk itself. Although adding the assumption (b) makes hedged loss functions less general, it indeed brings advantages. For instance, as we will show later, the resulting optimal hedged loss function does not depend on the underlying market model if we have the assumption (b). Assumption (c) is commonly imposed that ensures hedging strategies take positive values and Cong et al. (2014) mentioned that without the assumption (c) the optimal partial hedging problem might be ill-posed.

Hence, the admissible set $\mathcal{D}$ of the hedged loss function is represented as

$$\mathcal{D} = \big\{0 \leq f(x) \leq x : \;\; R_f(x) = x - f(x) \tag{5.10}$$
$$\text{is a non-decreasing and left continuous function}\big\}.$$

Based on the above considerations, the RVaR hedging problem in our setting is

$$\begin{cases} \min\limits_{f \in \mathcal{D}} RVaR_{\alpha,\beta}(R_f(X)), \\ s.t\ \Pi\big(f(X)\big) \le \tilde{v}_0 < \Pi\big(X\big), \end{cases} \tag{5.11}$$

where $\Pi(X)$ denotes the utility indifference price of X at time 0.

Depending on the size of the initial capital $\tilde{v}_0$, we represent optimal hedged loss functions of the problem (5.11) in Theorem 5.1 and Theorem 5.4 correspondingly.

Theorem 5.1. *If $\Pi\big(XI_{\{X \le VaR_{\alpha+\beta}(X)\}}\big) \le \tilde{v}_0$, the optimal hedged loss function satisfies*

$$f^*(x) = xI_{\{x \le v_{\alpha+\beta}\}}, \tag{5.12}$$

where $v_{\alpha+\beta} = VaR_{\alpha+\beta}(X)$.

Moreover, the minimal value of RVaR is zero, i.e.,

$$RVaR_{\alpha,\beta}(R_{f^*}(X)) = 0.$$

Proof. For $f^*(x)$ defined as (5.12) , the retained loss function is

$$R_{f^*}(x) = x - f^*(x) = x - xI_{\{x \le v_{\alpha+\beta}\}} = xI_{\{x > v_{\alpha+\beta}\}},$$

which is non-decreasing and left continuous regarding to x and hence the hedged loss function $f^*(x) \in \mathcal{D}$ and $f^*(X)$ satisfies the wealth constraint.

Notice that, for any $s \le \alpha + \beta$, we have

$$\begin{aligned} P\Big(XI_{\{X > v_{\alpha+\beta}\}} > 0\Big) &\le P\Big(X > v_{\alpha+\beta}\Big) \\ &\le \big(1 - (\alpha+\beta)\big) \le 1 - s, \end{aligned}$$

and consequently, we get

$$VaR_s\Big(XI_{\{X > v_{\alpha+\beta}\}}\Big) = 0, \quad \forall s \le \alpha + \beta,$$

which implies

$$\begin{aligned} RVaR_{\alpha,\beta}\big(R_{f^*}(X)\big) &= RVaR_{\alpha,\beta}\big(XI_{\{X > v_{\alpha+\beta}\}}\big) \\ &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s\big(XI_{\{X > v_{\alpha+\beta}\}}\big) ds \\ &= 0. \end{aligned}$$

Furthermore, for any $f \in \mathcal{D}$, we have $RVaR_{\alpha+\beta}\big(R_f(X)\big) \geq 0 = RVaR_{\alpha+\beta}\big(R_{f^*}(X)\big)$ and hence we conclude that f^* is the optimal hedged loss function. $\square$

Let us now consider the second case: $\tilde{v}_0 < \Pi\big(XI_{\{X \leq VaR_{\alpha+\beta}(X)\}}\big)$. To begin with, we need to recall the definition of stop loss ordering between two random variables X_1 and X_2:

Suppose X_1 and X_2 are two random variables with finite means under a probability measure P. We say X_1 is smaller than X_2 in stop-loss order under P, if

$$E\big((X_1 - m)^+\big) \leq E\big((X_2 - m)^+\big), \quad \forall m \in \mathbb{R}. \tag{5.13}$$

Such a relationship is denoted as $X_1 \leq_{sl}^{P} X_2$.

Lemma 5.2. *For a given random variable X and any function $f \in \mathcal{D}$, let*

$$g_f(x) = \min\big\{(x-d)^+, \bar{u}\big\} I_{\{x \leq v_{\alpha+\beta}\}}, \tag{5.14}$$

where $v_\alpha = VaR_\alpha(X)$, $d = v_\alpha - f(v_\alpha)$ and $\bar{u}$ is chosen such that

$$RVaR_{\alpha,\beta}(R_f(X)) = RVaR_{\alpha,\beta}(R_{g_f}(X)). \tag{5.15}$$

Then, we have

1) g_f is well defined and $g_f \in \mathcal{D}$;

2) $g_f(X)$ is smaller than $f(X)$ in stop-loss order under the measure P, i.e., $g_f(X) \leq_{sl}^{P} f(X)$.

Proof. First of all, we note that $RVaR_{\alpha,\beta}(R_{g_f}(X))$ is continuous and non-increasing as a function of $\bar{u}$ and for $\bar{u} = 0$, we have

$$R_{g_f}(X) = X \geq R_f(X).$$

Thereby, the following inequality holds:

$$RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big) \geq RVaR_{\alpha,\beta}\big(R_f(X)\big).$$

Moreover, for $\bar{u} = v_{\alpha+\beta} + f(v_\alpha) - v_\alpha$, we get

$$(x-d)^+ I_{\{x \leq v_{\alpha+\beta}\}} \leq (v_{\alpha+\beta} - d)^+ = \bar{u}, \tag{5.16}$$

which implies

$$g_f(x) = (x-d)^+ I_{\{x \le v_{\alpha+\beta}\}}.$$

In this case, we have following inequalities

$$\begin{aligned}
RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big) &= RVaR_{\alpha,\beta}\big(X - (X-d)^+ I_{\{x \le v_{\alpha+\beta}\}}\big) \\
&= RVaR_{\alpha,\beta}\big((X - (X-d)^+) I_{\{X \le v_{\alpha+\beta}\}} + X I_{\{X > v_{\alpha+\beta}\}}\big) \\
&\le RVaR_{0,\beta}\big((X - (X-d)^+) I_{\{X \le v_{\alpha+\beta}\}}\big) + RVaR_{\alpha,\beta}\big(X I_{\{X > v_{\alpha+\beta}\}}\big) \\
&= RVaR_{0,\beta}\big(\min\{X, d\} I_{\{X \le v_{\alpha+\beta}\}}\big) \\
&\le d, \qquad (5.17)
\end{aligned}$$

where the first inequality is due to (5.5) and the last inequality is because of the monotonicity of RVaR.

On the other hand, since $R_f(x)$ is nondecreasing and left continuous regarding to x, by Theorem 1 in Dhaene et al. (2002), we have $R_f(VaR_\alpha(X)) = VaR_\alpha(R_f(X))$ and hence

$$\begin{aligned}
RVaR_{\alpha,\beta}\big(R_f(H)\big) &\ge VaR_\alpha(R_f(X)) = R_f(VaR_\alpha(X)) \\
&= VaR_\alpha(X) - f(VaR_\alpha(X)) \\
&= d. \qquad (5.18)
\end{aligned}$$

Combining (5.17) and (5.18), we arrive to $RVaR_{\alpha,\beta}\big(R_f(X)\big) \ge RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big)$ when $\bar{u} = v_{\alpha+\beta} - d$ and hence there is a $\bar{u} \in [0, v_{\alpha+\beta} - d]$ that solves the equation $RVaR_{\alpha,\beta}(R_f(X)) = RVaR_{\alpha,\beta}(R_{g_f}(X))$.

Meanwhile, by the definition of $g_f(x)$, $R_{g_f}(x)$ is non-decreasing and left continuous as a function of x and therefore we have proved $g_f(x) \in \mathcal{D}$.

In order to show that $g_f(X) \le_{sl}^P f(X)$, let us introduce a new random variable u_α^β that is uniformly distributed on $[\alpha, \alpha+\beta]$ and is independent of all other random variables involved in this chapter.

If

$$g_f\big(VaR_{u_\alpha^\beta}(X)\big) \le_{sl}^P f\big(VaR_{u_\alpha^\beta}(X)\big), \qquad (5.19)$$

holds true, we have, for any $m \in \mathbb{R}$, that

$$
\begin{aligned}
E\Big[\big(g_f(X) - m\big)^+\Big] &= \int_0^1 \big(g_f(VaR_s(X)) - m\big)^+ ds \\
&= \int_0^\alpha \big(g_f(VaR_s(X)) - m\big)^+ ds + \int_\alpha^{\alpha+\beta} \big(g_f(VaR_s(X)) - m\big)^+ ds \\
&+ \int_{\alpha+\beta}^1 \big(g_f(VaR_s(X)) - m\big)^+ ds \\
&= \int_0^\alpha \big(g_f(VaR_s(X)) - m\big)^+ ds + \beta E\Big[\big(g_f(VaR_{u_\alpha^\beta}(X)) - m\big)^+\Big] + \int_{\alpha+\beta}^1 \big(-m\big)^+ ds \\
&\leq \int_0^\alpha \big(f(VaR_s(X)) - m\big)^+ ds + \beta E\Big[\big(f(VaR_{u_\alpha^\beta}(X)) - m\big)^+\Big] \\
&+ \int_{\alpha+\beta}^1 \big(f(VaR_s(X)) - m\big)^+ ds \\
&= E\Big[\big(f(X) - m\big)^+\Big],
\end{aligned}
$$

where the inequality is due to the non-decreasing property of $R_f(x)$ which leads to

$$VaR_s(X) - f(VaR_s(X)) \leq VaR_\alpha(X) - f(VaR_\alpha(X)),$$

for $s \leq \alpha$ and hence

$$g_f\big(VaR_s(X)\big) \leq \big(VaR_s(X) - VaR_\alpha(X) + f(VaR_\alpha(X))\big)^+ \leq f\big(VaR_s(X)\big).$$

Thus, in order to prove $g_f(X) \leq_{sl}^P f(X)$, it is sufficient to show (5.19).

Rolski et al. (1999) indicated that for two random variables X_1 and X_2 with finite means, a sufficient condition for the stop-loss order $X_1 \leq_{sl}^P X_2$ is as follows:

(i) $E(X_1) \leq E(X_2)$, and

(ii) there exists $t_0 \in \mathbb{R}$ such that $P(X_1 \leq t) \leq P(X_2 \leq t)$ for $t < t_0$ while $P(X_1 \leq t) \geq P(X_2 \leq t)$ for $t > t_0$.

Let us now put that $X_1 = g_f(VaR_{u_\alpha^\beta}(X))$, $X_2 = f(VaR_{u_\alpha^\beta}(X))$ and prove that these random variables satisfy above two conditions.

According to Theorem 1 in Dhaene et al. (2002), we have

$$\begin{aligned}
RVaR_{\alpha,\beta}\big(R_f(X)\big) &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s\big(R_f(X)\big)ds \\
&= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} R_f\big(VaR_s(X)\big)ds \\
&= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s(X) - f\big(VaR_s(X)\big)ds \\
&= RVaR_{\alpha,\beta}(X) - E\Big[f\big(VaR_{u_{\alpha}^{\beta}}(X)\big)\Big].
\end{aligned}$$

Similarly, it is true that

$$RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big) = RVaR_{\alpha,\beta}(X) - E\Big[g_f\big(VaR_{u_{\alpha}^{\beta}}(X)\big)\Big].$$

Since $RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big) = RVaR_{\alpha,\beta}\big(R_f(X)\big)$, we conclude that

$$E\Big[g_f\big(VaR_{u_{\alpha}^{\beta}}(X)\big)\Big] = E\Big[f\big(VaR_{u_{\alpha}^{\beta}}(X)\big)\Big], \tag{5.20}$$

which is the desired property (i).

Notice that $VaR_{u_{\alpha}^{\beta}}(X) \geq VaR_{\alpha}(X)$ and therefore we have

$$VaR_{u_{\alpha}^{\beta}}(X) - f(VaR_{u_{\alpha}^{\beta}}(X)) \geq VaR_{\alpha}(X) - f(VaR_{\alpha}(X)),$$

which implies that, for any $t < \bar{u}$,

$$\begin{aligned}
P\Big(g_f(VaR_{u_{\alpha}^{\beta}}(X)) \leq t\Big) &= P\Big(VaR_{u_{\alpha}^{\beta}}(X) - VaR_{\alpha}(X) + f\big(VaR_{\alpha}(X)\big) \leq t\Big) \\
&\leq P\Big(f(VaR_{u_{\alpha}^{\beta}}(X)) \leq t\Big).
\end{aligned} \tag{5.21}$$

On the other hand, for $t > \bar{u}$, we have

$$P\Big(g_f(VaR_{u_{\alpha}^{\beta}}(X)) \leq t\Big) = 1 \geq P\Big(f(VaR_{u_{\alpha}^{\beta}}(X)) \leq t\Big). \tag{5.22}$$

Combining (5.21) and (5.22), we arrive to the property (ii).

As a consequence, we have proved that $g_f(VaR_{u_{\alpha}^{\beta}}(X)) \leq_{sl}^{P} f(VaR_{u_{\alpha}^{\beta}}(X))$ which implies $g_f(X) \leq_{sl}^{P} f(X)$. □

Remark 5.3. *Let us denote $u = d + \bar{u}$. Since $\bar{u} \in [0, v_{\alpha+\beta} - d]$, and $d = v_\alpha - f(v_\alpha) \geq 0$, we have $d \leq u \leq v_{\alpha+\beta}$. Consequently, $g_f(x)$ can be rewritten as*

$$\begin{aligned} g_f(x) &= \min\big\{(x-d)^+, u-d\big\} I_{\{x \leq v_{\alpha+\beta}\}} \\ &= \Big[(x-d)^+ - (x-u)^+\Big] I_{\{x \leq v_{\alpha+\beta}\}}, \end{aligned} \tag{5.23}$$

where $0 \leq d \leq v_\alpha$ and $d \leq u \leq v_{\alpha+\beta}$.

According to Cong et al. (2014), the utility based indifference price preserves the stop-loss order, i.e., $\Pi(X_1) \leq \Pi(X_2)$, if $X_1 \leq_{sl}^P X_2$ and therefore Lemma 5.2 indicates that for any $f \in \mathcal{D}$, there is a $g_f \in \mathcal{D}$ such that

$$RVaR_{\alpha,\beta}\big(R_f(X)\big) = RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big);$$

$$\Pi\big(g_f(X)\big) \leq \Pi\big(f(X)\big).$$

Thereby, we can focus on hedged loss functions with the form (5.23). Let us formulate another theorem about the problem (5.11).

Theorem 5.4. *If $\tilde{v}_0 < \Pi\big(XI_{\{X \leq v_{\alpha+\beta}\}}\big)$, the optimal hedged loss function that solves the RVaR minimization problem (5.11) is*

$$f^*(x) = \Big[(x-d^*)^+ - (x-u^*)^+\Big] I_{\{x \leq v_{\alpha+\beta}\}}, \tag{5.24}$$

where (d^*, u^*) is the solution to the following 2-dimensional optimization problem

$$\begin{cases} \min\limits_{0 \leq d \leq v_\alpha, d \leq u \leq v_{\alpha+\beta}} d + \frac{1}{\beta}\int_\alpha^{\alpha+\beta} (VaR_s(X) - u)^+ ds, \\ s.t.\ \Pi\Big(\big[(X-d)^+ - (X-u)^+\big] I_{\{X \leq v_{\alpha+\beta}\}}\Big) \leq \tilde{v}_0. \end{cases} \tag{5.25}$$

Proof. By Remark 5.3, we know that the optimal hedged loss function admits the form (5.23) and hence the objective function in (5.11) becomes

$$\begin{aligned} RVaR_{\alpha+\beta}\big(R_{g_f}(X)\big) &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s\big(R_{g_f}(X)\big)ds \\ &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} R_{g_f}\big(VaR_s(X)\big)ds \\ &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s(X) - g_f(VaR_s(X))ds \\ &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s(X) - (VaR_s(X) - d) + (VaR_s(X) - u)^+ds \\ &= d + \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} (VaR_s(X) - u)^+ds, \end{aligned}$$

such that $d \in [0, v_\alpha]$ and $u \in [d, v_{\alpha+\beta}]$.

Hence, (5.11) is rewritten as (5.25) and Theorem 5.4 is proved. □

Theorem 5.4 shows that the optimal strategy is to long a knock-out call option on the payout X with a strike price d^* and a barrier $v_{\alpha+\beta}$ while short another knock-out call option with a higher strike price u^* but with the same barrier. If those knock-out call options are available in the market, our optimal partial hedging is to buy (sell) and hold those options. Otherwise, it is to dynamically hedge those knock-out call options with the initial capital $\tilde{v}_0$. It is worth to mention that both Theorem 5.1 and Theorem 5.4 indicate that the optimal strategy does not cover the extreme loss that exceeds $v_{\alpha+\beta}$. This is because we only focus on the risk between levels $[\alpha, \alpha+\beta]$.

5.3.2 Connection among RVaR, CVaR and VaR optimal hedging

As discussed in Section 5.2, VaR and CVaR can be seen as two limiting cases of RVaR. In this subsection, we want to discuss connections among optimal RVaR hedging, optimal VaR hedging and optimal CVaR hedging.

By the definition of CVaR, we have

$$CVaR_\alpha(X) = RVaR_{\alpha,1-\alpha}(X), \tag{5.26}$$

i.e., $\beta = 1 - \alpha$ and hence we get

$$I_{\{X \le v_{\alpha+\beta}\}} = 1; \quad P - a.s.$$

According to (5.24), the optimal hedged loss function satisfies the form

$$\begin{aligned} f^*(x) &= \big((x-d)^+ - (x-u)^+\big) I_{\{x \le v_{\alpha+\beta}\}} \\ &= (x-d)^+ - (x-u)^+. \end{aligned} \tag{5.27}$$

Cong et al. (2014) considered the optimal CVaR hedging and showed the optimal hedged loss function f^*_{CVaR} admitted the form

$$f^*_{CVaR}(x) = (x-d)^+ - (x-u)^+, \tag{5.28}$$

which is consistent with (5.27). Therefore, the optimal CVaR hedging can be seen as a special case of the RVaR based hedging.

As for VaR, we have

$$VaR_\alpha(X) = RVaR_{\alpha,0}(X),$$

i.e., $\beta = 0$. In this case, (5.23) becomes

$$g_f(x) = \min\big\{(x-d)^+, \bar{u}\big\} I_{\{x \le v_\alpha\}}.$$

On the one hand, we know that

$$\begin{aligned} RVaR_{\alpha,0}\big(R_f(X)\big) &= VaR_\alpha\big(R_f(X)\big) \\ &= VaR_\alpha(X) - f\big(VaR_\alpha(X)\big). \end{aligned} \tag{5.29}$$

On the other hand, we have

$$\begin{aligned} RVaR_{\alpha,0}\big(R_{g_f}(X)\big) &= VaR_\alpha\big(R_{g_f}(X)\big) \\ &= VaR_\alpha(X) - g_f\big(VaR_\alpha(X)\big) \\ &= VaR_\alpha(X) - \min\big\{(VaR_\alpha(X) - d)^+, \bar{u}\big\} \\ &= VaR_\alpha(X) - \min\big\{f\big(VaR_\alpha(X)\big), \bar{u}\big\}. \end{aligned} \tag{5.30}$$

Since $\bar{u}$ is derived from the equation

$$RVaR_{\alpha,0}\big(R_{g_f}(X)\big) = RVaR_{\alpha,0}\big(R_f(X)\big),$$

comparing (5.29), (5.30), we have $\bar{u} \geq f\big(VaR_\alpha(X)\big)$, while, from the previous discussions, we know $\bar{u} \leq VaR_{\alpha+\beta}(X) + f\big(VaR_\alpha(X)\big) - VaR_\alpha(X)$. The right hand side is $f\big(VaR_\alpha(X)\big)$ when $\beta = 0$, and hence $\bar{u} = f\big(VaR_\alpha(X)\big)$.

Meanwhile, the following relationship

$$\Big(x + f\big(VaR_\alpha(X)\big) - VaR_\alpha(X)\Big)^+ I_{\{x \leq VaR_\alpha(X)\}} \leq f\big(VaR_\alpha(X)\big) = \bar{u},$$

implies that the optimal hedged loss function has the form

$$\begin{aligned} f^*(x) &= \min\big\{(x-d)^+, \bar{u}\big\} I_{\{x \leq v_\alpha\}} \\ &= (x-d)^+ I_{\{x \leq v_\alpha\}}. \end{aligned} \tag{5.31}$$

This result is consistent with the conclusion in Cong et al. (2013) that the optimal hedged loss function f^*_{VaR} of the VaR based hedging problem has the form

$$f^*_{VaR}(x) = (x-d)^+ I_{\{x \leq v_\alpha\}}. \tag{5.32}$$

and thereby the optimal VaR hedging can be seen as a special case of the RVaR hedging.

5.4 Application to equity-linked life insurance contracts

One important application of partial hedging is to deal with pricing and hedging of equity-linked life insurance contracts. Well-known papers are Melnikov and Skornyakova (2005), Kirch and Melnikov (2005), where authors discussed the implementation of quantile hedging and efficient hedging in this area. In this section, we focus on the RVaR based optimal hedging of life insurance contracts and would compare RVaR, CVaR and VaR hedging results.

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. Consider a financial market with a terminal time $T \in (0, \infty)$ consisting two assets. One is the riskless asset

$$S^0_t = e^{rt}, \quad t \in [0, T], \tag{5.33}$$

where r is the constant risk-free interest rate. Another one is the risky asset, $(S_t)_{t\in[0,T]}$, described by a Mixed Fractional Brownian motion (MFBM) model:

$$S_t = S_0 \exp(\mu t + \sigma_1 B_t + \sigma_2 B_t^H), \quad S_0 > 0, \tag{5.34}$$

where B is a Brownian motion (BM) independent of a Fraction Brownian motion B^H (FBM). Here H is a real number in $(0,1)$, called the Hurst parameter. The constant μ is the drift, $\sigma_1 > 0$ is the volatility of the Brownian motion B and $\sigma_2 > 0$ is the volatility of B^H.

The reason why we assume the risky asset follows MFBM model is that, for $H > \frac{1}{2}$, this model has the property of long range dependence. Since Equity-linked life insurance contracts usually have long term maturities, the factor of long-term dependence should be included in the list of key factors having a certain influence on pricing and hedging.

Let us denote the mixed process $M_t^\sigma = B_t + \sigma B_t^H$, $t \in [0,T]$, where $\sigma = \frac{\sigma_2}{\sigma_1}$. It is well known that except for $H = \frac{1}{2}$, B^H is neither a Markov process nor a semimartingale and thus the mixed process is also not a semimartingale with respect to the filtration generated by the BM and the FBM. However, Cheridito (2001) has proved that in the case $H \in (\frac{3}{4}, 1)$, the process M^σ is a semimartingale with respect to its natural filtration: $\mathbb{F}^M = \{\mathcal{F}_t^M \quad 0 < t < T\}$,
$\mathcal{F}_t^M = \sigma\{M_u^\sigma, \quad 0 \le u \le t\}$. Moreover, it is equivalent in measure to a Brownian motion. In this chapter, we consider the filtration $\mathbb{F}^M$ and assume $H \in (\frac{3}{4}, 1)$, $\mathcal{F} = \mathcal{F}_T^M$ and hence there is a Brownian motion $\{W_t\}_{t\in[0,T]}$ and a unique real-valued Volterral kernal $r_\sigma \in L_2([0,T]^2)$ such that the following relationship holds:

$$M_t^\sigma = W_t + \int_0^t \int_0^s r_\sigma(s,u) dW_u ds, \quad t \in [0,T], \tag{5.35}$$

where r_σ is the unique solution of the equation:

$$\sigma^2 H(2H-1)(t-s)^{2H-2} = r_\sigma(t,s) + \int_0^s r_\sigma(t,x) r_\sigma(s,x) dx, \quad 0 \le s < t \le T.$$

The MFBM market is complete and the unique martingale measure P^* is defined by the relation (see, Melnikov and Mishura 2011):

$$Z_t^* = \frac{dP^*}{dP}|_{\mathcal{F}_t^M} = \exp\Big\{ - \int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u)dW_u\Big)dW_s \tag{5.36}$$
$$- \frac{1}{2}\int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u)dW_u\Big)^2 ds\Big\}.$$

Note that, in this complete market, the utility indifference price of a claim is equivalent to its fair price, i.e., $\Pi(X) = e^{-rT}E^*(X)$, where $E^*(\cdot)$ is the expectation under the martingale measure P^*.

According to the Girsanov theorem, the process

$$\tilde{W}_t = W_t + (\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2})t + \int_0^t \int_0^s r_\sigma(s,u)dW_u ds, \tag{5.37}$$

is a Wiener process under the measure P^*.

With the help of (5.35) and (5.37) the stock price process can be rewritten as

$$S_t = S_0 \exp\Big\{\mu t + \sigma_1\big(W_t + \int_0^t \int_0^s r_\sigma(s,u)dW_u ds\big)\Big\} \tag{5.38}$$
$$= S_0 \exp\big\{\sigma_1 \tilde{W}_t + (r - \frac{\sigma_1^2}{2})t\big\}.$$

For equity linked life insurance contracts, there are two sources of risk: market risk associated with the underlying asset price and insurance risk reflected by the insureds' mortality and hence, besides characteristics of the financial market, we also need to describe mortality properties of insureds. Following actuarial traditions, let a random variable $T(x)$ on an "actuarial" probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$ denote the remaining life time of a person of current age x and ${}_Tp_x = \tilde{P}(T(x) > T)$ be the survival probability for the next T years of the insured. Since usually the insurance risk and the financial market risk have no effect on each other, we would take a natural assumption that $(\Omega, \mathcal{F}, P)$ and $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$ can be treated as independent.

Let us consider a pure endowment life insurance contract with the payoff $X = \max(S_T, K)$ provided that an insured is alive at T, where K is a constant guarantee amount. Since the mortality risk is essentially independent of the financial market, according to Brennan and

Schwartz (1976), the premium of such a contract is defined as

$$ {}_TU_x = E^*(e^{-rT}X)E^{\tilde{P}}(I_{\{T(x)>T\}}) = {}_Tp_xV_0, \tag{5.39} $$

where x is the insured's age, T is the maturity time of the contract and V_0 is perfect hedging costs of the claim X. According to the most recently published United States 2015 Life Table (National Vital Statistics Reports volume 67, Number 7), the survival probability ${}_Tp_x$ of a given insured can be found. Obviously, ${}_Tp_xV_0 < V_0$ and thus a perfect hedge of the option is impossible and a partial hedging strategy should be constructed. With the help of methodologies in Section 5.3, we derive the minimal value of RVaR that can be achieved with the initial wealth $\tilde{v}_0 = {}_Tp_xV_0$.

Proposition 5. The fair price of the claim $X = \max\{S_T, K\}$ is

$$ \begin{aligned} V_0 &= e^{-rT}E^*(X) \\ &= S_0\Phi\big(\Lambda_+(K)\big) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big), \end{aligned} \tag{5.40} $$

where Φ is the distribution function of a standard normal random variable and

$$ \Lambda_\pm(x) = \frac{\ln\frac{S_0}{x} + (r \pm \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}. $$

Proof. In the complete MFBM market, the utility indifference price of the claim $X = \max\{S_T, K\}$ is equal to its fair price, i.e.,

$$ \Pi(X) = e^{-rT}E^*(X) = e^{-rT}\big[E^*(S_TI_{\{S_T>K\}}) + E^*(KI_{\{S_T\le K\}})\big]. $$

Note that

$$ \begin{aligned} \{S_T > K\} &= \{S_0e^{\sigma_1\tilde{W}_T + (r-\frac{\sigma_1^2}{2})T} > K\} \\ &= \{\tilde{W}_T > \frac{\ln\frac{K}{S_0} - (r-\frac{\sigma_1^2}{2})T}{\sigma_1}\} \\ &= \{Z_1 < \Lambda_-(K)\}, \end{aligned} \tag{5.41} $$

where $Z_1 = -\frac{\tilde{W}_T}{\sqrt{T}} \sim N(0,1)$ under the measure P^* and $\Lambda_-(K) = \frac{\ln\frac{S_0}{K} + (r-\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

With the help of the "Multi-asset Theorem" (see Melnikov and Romanyuk 2008), the price of the claim can be rewritten as:

$$
\begin{aligned}
\Pi(X) &= e^{-rT}\big[E^*(S_0 e^{\sigma_1 \tilde{W}_T + (r-\frac{\sigma_1^2}{2})T} I_{\{Z_1<\Lambda_-(K)\}}) + KP^*\big(Z_1 \geq \Lambda_-(K)\big)\big] \\
&= S_0 e^{-\frac{\sigma_1^2}{2}T} E^*(e^{\sigma_1 \tilde{W}_T} I_{\{Z_1<\Lambda_-(K)\}}) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big) \\
&= S_0 \Phi\big(\Lambda_+(K)\big) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big),
\end{aligned}
$$

where $\Lambda_+(K) = \Lambda_-(K) + \sigma_1\sqrt{T} = \frac{\ln\frac{S_0}{K} + (r+\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$. □

In addition, applying Theorem 5.1 and Theorem 5.4 to a claim $X = \max\{S_T, K\}$, we arrive to the following results.

Theorem 5.5. *(a) If $\tilde{v}_0 \geq V^{\alpha+\beta} = e^{-rT}E^*(XI_{\{X\leq v_{\alpha+\beta}\}})$, the optimal hedging strategy is a perfect hedge of the claim $XI_{\{X\leq v_{\alpha+\beta}\}}$ and the minimal value of risk under the measure $RVaR_{\alpha,\beta}$ is 0.*

(b) If $\tilde{v}_0 < V^{\alpha+\beta}$, the optimal hedging strategy is a perfect hedge of the claim

$$\big[(X-d^*)^+ - (X-u^*)^+\big] I_{\{X\leq v_{\alpha+\beta}\}},$$

such that (d^*, u^*) are points of minimum of the system

$$
\begin{cases}
\min\limits_{0\leq d\leq v_\alpha, d\leq u\leq v_{\alpha+\beta}} d + \frac{1}{\beta}\int_\alpha^{\alpha+\beta}(v_s-u)^+ ds, \\
s.t.\ C(d) - C(u) \leq \tilde{v}_0,
\end{cases}
\tag{5.42}
$$

where

$$
\begin{aligned}
V^{\alpha+\beta} &= e^{-rT}E^*(XI_{\{X\leq v_{\alpha+\beta}\}}) \\
&= S_0\Phi^{(2)}\big(\Lambda_+(K), -\Lambda_+(v_{\alpha+\beta}), -1\big) + e^{-rT}K\Phi\big(-\Lambda_-(K)\big),
\end{aligned}
$$

$$\sigma_M^2 = \sigma_1^2 T + \sigma_2^2 T^{2H},$$

$$
v_s = \begin{cases}
K, & \text{if } \Phi\Big(\frac{\ln\frac{S_0}{K} + \mu T}{\sigma_M}\Big) \leq 1-s, \\
S_0 e^{\mu T - \sigma_M z_{1-s}}, & \text{otherwise};
\end{cases}
$$

$$\Lambda_{d,K} = \min\{\Lambda_-(d), \Lambda_-(K)\},\ \ \Lambda_{u,K} = \min\{\Lambda_-(u), \Lambda_-(K)\},$$

$$C(x) = S_0\Phi^{(2)}\big(\Lambda_{x,K} + \sigma_1\sqrt{T}, -\Lambda_+(v_{\alpha+\beta}), -1\big) - xe^{-rT}\Phi^{(2)}\big(\Lambda_{x,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big)$$
$$+ e^{-rT}(K-x)^+\Phi\big(-\Lambda_-(K)\big),$$

Here, z_{1-s} is the $(1-s)$ quantile of a standard normal random variable. $\Phi^{(2)}(z_1, z_2, \rho)$ denotes the cumulative distribution function of two jointly normally distributed random variables (Z_1, Z_2) with zero means, unit variances and the correlation ρ.

Proof. By the definition of VaR, we have

$$v_s = VaR_s(X) = \inf\big\{v : P(\max\{S_T, K\} > v) \le 1-s\big\}, \;\; s \in (0,1).$$

Note that, if $v < K$, we get

$$P(\max\{S_T, K\} > v) = 1,$$

and hence v_s satisfies $v_s \ge K$.

For $v \ge K$, the above probability can be calculated as

$$\begin{aligned} P(\max\{S_T, K\} > v) &= P\big(S_T > v\big) \\ &= P\big(S_0 e^{\mu T + \sigma_1 M_T^\sigma} > v\big) \\ &= P\big(\sigma_1 M_T^\sigma > \ln\frac{v}{S_0} - \mu T\big) \\ &= \Phi(\frac{\ln\frac{S_0}{v} + \mu T}{\sigma_M}), \end{aligned} \tag{5.43}$$

where in the last equation we use the fact $\sigma_1 M_T^\sigma \sim N(0, \sigma_M^2)$ with $\sigma_M^2 = \sigma_1^2 T + \sigma_2^2 T^{2H}$, under the measure P.

If

$$\Phi(\frac{\ln\frac{S_0}{K} + \mu T}{\sigma_M}) \le 1-s,$$

we conclude $v_s = K$. Otherwise, since the equation (5.43) is a decreasing function regarding v, v_s is the unique solution of the equation

$$\Phi(\frac{\ln\frac{S_0}{v} + \mu T}{\sigma_M}) = 1-s,$$

in the interval $[K, +\infty)$, which can be rewritten as

$$v_s = S_0 e^{\mu T - \sigma_M z_{1-s}},$$

where z_{1-s} is the $1-s$ quantile of a standard normal random variable.

Let us start with the calculation of $V^{\alpha+\beta}$:

$$\begin{aligned} V^{\alpha+\beta} &= \Pi\big(XI_{\{X \le v_{\alpha+\beta}\}}\big) = e^{-rT} E^*(XI_{\{X \le v_{\alpha+\beta}\}}) \\ &= e^{-rT}\big[E^*(S_T I_{\{S_T \ge K\}} I_{\{S_T \le v_{\alpha+\beta}\}}) + E^*(K I_{\{S_T < K\}} I_{\{K \le v_{\alpha+\beta}\}})\big] \\ &= e^{-rT}\big[E^*(S_T I_{\{S_T \ge K\}} I_{\{S_T \le v_{\alpha+\beta}\}}) + E^*(K I_{\{S_T < K\}})\big], \end{aligned}$$

where in the last equation we utilize the fact that $K \le v_{\alpha+\beta}$.

As for $\{S_T \le v_{\alpha+\beta}\}$, we have

$$\begin{aligned} \{S_T \le v_{\alpha+\beta}\} &= \{S_0 e^{\sigma_1 \tilde{W}_T + (r - \frac{\sigma_1^2}{2})T} \le v_{\alpha+\beta}\} \\ &= \{\tilde{W}_T \le \frac{\ln \frac{v_{\alpha+\beta}}{S_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1}\} \\ &= \{Z_2 \le -\Lambda_-(v_{\alpha+\beta})\}, \end{aligned}$$

where $Z_2 = \frac{\tilde{W}_T}{\sqrt{T}} \sim N(0,1)$ under the measure P^* and $\Lambda_-(v_{\alpha+\beta}) = \frac{\ln \frac{S_0}{v_{\alpha+\beta}} + (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}$.

Hence, with the help of multi-asset theorem, we arrive to

$$\begin{aligned} V^{\alpha+\beta} &= e^{-rT}\big[S_0 e^{(r - \frac{\sigma_1^2}{2})T} E^*(e^{\sigma_1 \tilde{W}_T} I_{\{Z_1 \le \Lambda_-(K)\}} I_{\{Z_2 \le -\Lambda_-(v_{\alpha+\beta})\}}) \\ &\quad + K P^*\big(Z_1 \ge \Lambda_-(K)\big)\big] \\ &\quad = S_0 \Phi^{(2)}\big(\Lambda_+(K), -\Lambda_+(v_{\alpha+\beta}), -1\big) + K e^{-rT} \Phi\big(-\Lambda_-(K)\big), \end{aligned}$$

where $\Lambda_+(v_{\alpha+\beta}) = \frac{\ln \frac{S_0}{v_{\alpha+\beta}} + (r + \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}$.

Now, let us move to

$\Pi\Big(\big[(X-d)^+ - (X-u)^+\big] I_{\{X \le v_{\alpha+\beta}\}}\Big) = E^*\Big(e^{-rT}\big[(X-d)^+ - (X-u)^+\big] I_{\{X \le v_{\alpha+\beta}\}}\Big).$

Similar to the derivation of (5.41), we have

$$\{S_T > d\} = \{Z_1 < \Lambda_-(d)\},$$

where $\Lambda_-(d) = \frac{\ln \frac{S_0}{d} + (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}$.

And therefore

$$
\begin{aligned}
&E^*\big((\max\{S_T, K\} - d)^+ I_{\{X \le v_{\alpha+\beta}\}}\big) \\
&= E^*\big((S_T - d)^+ I_{\{S_T > K\}} I_{\{S_T \le v_{\alpha+\beta}\}}\big) + E^*\big((K-d)^+ I_{\{S_T \le K\}} I_{\{K \le v_{\alpha+\beta}\}}\big) \\
&= E^*\big((S_T - d) I_{\{S_T > d\}} I_{\{S_T > K\}} I_{\{S_T \le v_{\alpha+\beta}\}}\big) + E^*\big((K-d)^+ I_{\{S_T \le K\}}\big) \\
&= E^*\big(S_T I_{\{Z_1 < \Lambda_{d,K}\}} I_{\{Z_2 \le -\Lambda_-(v_{\alpha+\beta})\}}\big) - dP^*\big(Z_1 < \Lambda_{d,K}, Z_2 \le -\Lambda_-(v_{\alpha+\beta})\big) \\
&+ (K-d)^+ P^*\big(S_T \le K\big) \\
&= S_0 e^{(r - \frac{\sigma_1^2}{2})T} E^*\big(e^{\sigma_1 \sqrt{T} Z_2} I_{\{Z_1 < \Lambda_{d,K}\}} I_{\{Z_2 \le -\Lambda_-(v_{\alpha+\beta})\}}\big) - d\Phi^{(2)}\big(\Lambda_{d,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big) \\
&+ (K-d)^+ \Big(1 - \Phi\big(\Lambda_-(K)\big)\Big) \\
&= S_0 e^{rT} \Phi^{(2)}\big(\Lambda_{d,K} + \sigma_1 \sqrt{T}, -\Lambda_+(v_{\alpha+\beta}), -1\big) - d\Phi^{(2)}\big(\Lambda_{d,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big) \\
&+ (K-d)^+ \Phi\big(-\Lambda_-(K)\big),
\end{aligned}
$$

where $\Lambda_{d,K} = \min\{\Lambda_-(d), \Lambda_-(K)\}$, and $\Lambda_+(v_{\alpha+\beta}) = \frac{\ln \frac{S_0}{v_{\alpha+\beta}} + (r + \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}$.

With similar calculations, it is clear that

$$
\begin{aligned}
&E^*\big((\max\{S_T, K\} - u)^+ I_{\{X \le v_{\alpha+\beta}\}}\big) \\
&= S_0 e^{rT} \Phi^{(2)}\big(\Lambda_{u,K} + \sigma_1 \sqrt{T}, -\Lambda_+(v_{\alpha+\beta}), -1\big) - u\Phi^{(2)}\big(\Lambda_{u,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big) \\
&+ (K-u)^+ \Phi\big(-\Lambda_-(K)\big),
\end{aligned}
$$

where $\Lambda_{u,K} = \min\{\Lambda_-(u), \Lambda_-(K)\}$.

Consequently, we arrive to

$$
\begin{aligned}
&E^*\Big(e^{-rT}\big[(X-d)^+ - (X-u)^+\big] I_{\{X \le v_{\alpha+\beta}\}}\Big) \\
&= e^{-rT}\Big[E^*\big((\max\{S_T, K\} - d)^+ I_{\{X \le v_{\alpha+\beta}\}}\big) - E^*\big((\max\{S_T, K\} - u)^+ I_{\{X \le v_{\alpha+\beta}\}}\big)\Big] \\
&= C(d) - C(u),
\end{aligned}
$$

where

$$C(x) = S_0\Phi^{(2)}\big(\Lambda_{x,K} + \sigma_1\sqrt{T}, -\Lambda_+(v_{\alpha+\beta}), -1\big) - xe^{-rT}\Phi^{(2)}\big(\Lambda_{x,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big)$$
$$+ e^{-rT}(K-x)^+\Phi\big(-\Lambda_-(K)\big).$$

The proof is completed. □

We illustrate our results with the following numerical example.

Example: Assume a MFBM market with following parameters:

$$\mu = 0.1,\ \ \sigma_1 = 0.15,\ \ \sigma_2 = 0.1,\ \ S_0 = 100,\ \ H = 0.8,\ \ r = 0.05.$$

An insured at age 40 buys a 10 years contract $X = \max\{S_T, K\}$ with the guarantee amount $K = 110$. By the United States 2015 Life Table, the survival probability of the insured is ${}_{10}p_{40} = 0.97462$ and hence according to equations (5.39) and (5.40) the premium that the insurance company can receive is ${}_TU_x = 0.97462 * 104.16 = 101.521$. For a fixed $\alpha = 0.95$, Table 5.1 shows optimal hedged loss functions and corresponding minimal values of RVaR for different levels of β.

Table 5.1 Optimal hedged loss and minimal RVaR for different levels of β

	$f^*(X)$	$RVaR_{0.95,\beta}$
$\beta = 10^{-6}$	$(X-4.3584)^+I_{\{X\le 1641.8\}}$	4.3584
$\beta = 0.025$	$(X-4.35847)^+I_{\{X\le 2105.5\}}$	4.358
β=0.04	$(X-4.3588)^+I_{\{X\le 2811.6\}}$	4.3588
β=0.05	$(X-4.3588)^+$	4.3625

Meanwhile, subject to the same initial capital constraint, we provide optimal hedged loss functions of VaR based hedging and CVaR based hedging correspondingly.

For VaR based hedging at the level 95%, with the help of the method in Cong et al. (2013), we derive the optimal hedged loss function which is

$$f^*_{VaR}(X) = (X-4.3584)^+I_{\{X\le 1641.8\}}. \tag{5.44}$$

For CVaR based hedging at the level 95%, according to the method in Cong et al. (2014), we arrive to the conclusion that the optimal hedged loss function is

$$f^*_{CVaR}(X) = (X - 4.3588)^+. \tag{5.45}$$

Comparing (5.44) and (5.45) with results in Table 5.1, it is not hard to find that, as $\beta \to 0$, the optimal hedged loss function of RVaR hedging coincides with $f^*_{VaR}(X)$, while in another extreme case $\beta = 1 - \alpha$, it coincides with $f^*_{CVaR}(X)$. Such results are consistent with conclusions in Section 5.3.2. Moreover, for a small value of β, the optimal RVaR hedging is close to the optimal VaR based hedging while for a big value of β, it is closer to the optimal CVaR hedging, which indicates that the measure RVaR is a bridge between VaR and CVaR.

5.5 Conclusion

In this chapter, explicit forms of optimal hedging strategies that minimize RVaR of a hedger's risk exposure subject to an initial wealth constraint are derived. We show that for sufficiently large hedging budget $\big(\Pi(X) > \tilde{v}_0 \geq \Pi(XI_{\{X<v_{\alpha+\beta}\}})\big)$ the optimal strategy is to hedge the entire risk up to the level $v_{\alpha+\beta}$. On the other hand, if $\tilde{v}_0 < \Pi(XI_{\{X<v_{\alpha+\beta}\}})$, the optimal strategy is to hedge a bull call spread on the claim itself with a knock out barrier $v_{\alpha+\beta}$. In both cases, it is optimal not to hedge at all on the set $\{X > v_{\alpha+\beta}\}$. Our RVaR based partial hedging method has some advantages. First, it can be applied in incomplete markets and the solution can be easily derived. Furthermore, the optimal strategy is model independent. Most importantly, we demonstrate that CVaR hedging and VaR hedging can be seen as two limiting cases of RVaR hedging and hence our RVaR hedging method is more general and it is more customized since investors can set their risk appetite by choosing two risk level parameters α and β. Finally, a numerical example is provided to explain how such a method can be implemented to the area of life insurance even if the financial model is sophisticated.

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APPENDIX A

An application of the duality method in partial hedging in incomplete markets

In this appendix, we suppose $r = 0$ to simplify the notations.

In incomplete markets, the set of densities of equivalent martingale measures is never compact, and hence the solution of

$$\min_{(v,\pi)\in\mathcal{A}_0} E\big((H - V_T)^+\big), \tag{A.1}$$

where

$$\mathcal{A}_0 = \{(v,\pi)|(v,\pi) \in \mathcal{A}(v), v \le v_0\},$$

can not be derived by the generalized Neyman-Pearson lemma directly. Such a problem is solve by Rudloff (2006) as well as Xu (2004) and we would like to summarize their main results here.

Let $\mathcal{V}(x)$ be the set of admissible self-financing value processes with initial capital $x > 0$

$$\mathcal{V}(x) = \Big\{V : V_t = x + \int_0^t \pi_s^1 dS_s \ge 0, \quad t \in [0,T]\Big\}, \tag{A.2}$$

and the set of all contingent claims that can be super-replicated by some admissible self-financing strategies with initial capital x is denote as

$$\mathcal{C}(x) = \big\{g \in L^0(\Omega,\mathcal{F},P) : \ 0 \le g \le V_T \ \text{ for some } \ V \in \mathcal{V}(x)\big\}, \tag{A.3}$$

where $L^0(\Omega, F, P)$ is the set of all random variables on $(\Omega,\mathcal{F},P)$.

Define the state dependent utility function U: $\mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^+$ as

$$U(x,\omega) = H(\omega) - (H(\omega) - x)^+ = H(\omega) \wedge x, \tag{A.4}$$

and the primal problem for $x > 0$ is

$$u(x) = \sup_{V \in \mathcal{V}(x)} E\big[U(V_T(\omega), \omega)\big]$$

$$= \sup_{g \in \mathcal{C}(x)} E\big[U(g(\omega), \omega)\big]$$

$$= \sup_{g \in \mathcal{C}(x)} E\big[H \wedge g\big]. \tag{A.5}$$

As in Xu (2004), we define the dual space as a set of processes Y such that

$$\mathcal{Y}(y) = \big\{Y \geq 0 : \quad Y_0 = y \text{ and } VY \text{ is a } P\text{-supermartingale } \forall V \in \mathcal{V}(1)\big\} \tag{A.6}$$

and the dual extended set $\mathcal{D}$ of random variables h is

$$\mathcal{D}(y) = \big\{h \in L^0(\Omega, \mathcal{F}, P) : \; 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\big\}. \tag{A.7}$$

Let us consider the stochastic conjugate function $W : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^+$ such that

$$W(y, \omega) = \sup_{x \geq 0} \big\{U(x, \omega) - xy\big\}. \tag{A.8}$$

By the property of U (A.4) and the fact that $W(0, \omega) \geq U(0, \omega)$, we arrive to

$$W(y, \omega) = (1 - y)^+ H(\omega). \tag{A.9}$$

Consider the following dual problem

$$w(y) = \inf_{Y \in \mathcal{Y}(y)} E[W(Y_T(\omega), \omega)]$$

$$= \inf_{h \in \mathcal{D}(y)} E[W(h_T(\omega), \omega)]$$

$$= \inf_{h \in \mathcal{D}(y)} E[(1 - h)^+ H]. \tag{A.10}$$

The utility function $U(\cdot, \omega)$ and the value function u are concave, continuous and increasing, while the functions $W(\cdot, \omega)$ and w are convex, continuous and decreasing. For a fixed ω, we defined the function U: $\mathbb{R}^+ \to \mathbb{R}^+$ by $U(g) = U(g(\omega), \omega)$ for $g \in \mathcal{C}(x)$ and $\partial U(g)$ is the subdifferential of U at g. $W(h) : \mathbb{R}^+ \to \mathbb{R}^+$ for $h \in \mathcal{D}(y)$ and $\partial W(h)$ are defined analogously. Then the following duality theorem holds true.

Theorem A.1. *(1) For $x > 0$ and $y > 0$ an optimal solution $\tilde{g}(x) \in \mathcal{C}(x)$ of the prime problem (A.5) exists and an optimal solution $\tilde{h}(y) \in \mathcal{D}(y)$ of the dual problem (A.10) exists.*

(2) The value functions u and w satisfy

$$w(y) = \sup_{x>0}\{u(x) - xy\}, \quad \forall y > 0,$$
$$u(y) = \inf_{y>0}\{w(x) + xy\}, \quad \forall x > 0. \tag{A.11}$$

(3) For $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Then, the relationships

$$E(\tilde{g}\tilde{h}) = xy \quad and$$
$$\tilde{h} \in \partial U(\tilde{g}) \;\; or \;\; \tilde{g} \in -\partial W(\tilde{g}) \;\; P - a.s. \tag{A.12}$$

hold true iff $\tilde{g}$ solves (A.5) and $\tilde{h}$ solves (A.10).

In addition, the structure of a primal solution with respect to a dual solution is given by the following theorem.

Theorem A.2. *Let $x > 0$ and $y > 0$ such that $y \in \partial u(x)$. Let $\tilde{h}$ be an optimal solution to the dual problem (A.10). Then the optimal solution $\tilde{g}$ of (A.5) satisfies*

$$\tilde{g} = \big(I_{\{0\le\tilde{h}<1\}} + \delta I_{\{\tilde{h}=1\}}\big)H$$

and

$$E(\tilde{g}\tilde{h}) = xy,$$

where δ is a $[0,1]$ valued random variable.

With the choice $x = v_0$, and $\tilde{\varphi} = I_{\{0\le\tilde{h}<1\}} + \delta I_{\{\tilde{h}=1\}}$, the optimal solution $\tilde{g}(v_0)$ of (A.5) can be represented as $\tilde{g}(v_0) = \tilde{\varphi}H$.

Finally, the optimal strategy of the problem (A.1) is provided in the following theorem.

Theorem A.3. *The optimal strategy (v_0, π) of the problem (A.1) is a superhedging strategy for the modified claim $\tilde{\varphi}H$ where*

$$\tilde{\varphi} = I_{\{0\le\tilde{h}<1\}} + \delta I_{\{\tilde{h}=1\}}, \tag{A.13}$$

such that

$$E(\tilde{\varphi} H \tilde{h}) = v_0 \tilde{y}, \tag{A.14}$$

where $\tilde{y} \in \partial u(v_0)$ *and* $\tilde{h} \in \partial \mathcal{D}(\tilde{y})$ *solves*

$$\inf_{h \in \mathcal{D}(\tilde{y})} E((1-h)^+ H). \tag{A.15}$$