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NAME OF AUTHOR/*NOM DE L'AUTEUR* SENG-LUAN LEE

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NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* A. SHARMA

DATED/*DATÉ* SEPT 25, 1974 SIGNED/*SIGNÉ* [signature]

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ* 23, Tingkat Burmah,
PENANG, MALAYSIA.

THE UNIVERSITY OF ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled CARDINAL LACUNARY SPLINE INTERPOLATION submitted by SENG-LUAN LEE in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics.

....
Supervisor

(External Examiner)

....

A. Meir
....

S.D. Riemenschneider
....

D. Bickett
....

....
External Examiner

Date Sept. 20, 1974

ABSTRACT

CARDINAL LACUNARY SPLINE INTERPOLATION

Following the ideas of Lipow and Schoenberg on Cardinal Hermite Interpolation (C.H.I.P.), we study here a class of Cardinal Lacunary Interpolation Problems (C.L.I.P.). A special case for quintic splines has recently been treated by Meir and Sharma and a generalization of their results is contained in the work of Demko. The first Chapter deals in full with the statement of the problem and a complete solution to the problem of $(0,1,\ldots,r-2,r)$ interpolation by g-splines. When the data is of power growth, the interpolation problem has a unique solution. Chapter II is devoted to the problem of the zeros of the characteristic polynomials which arise in the study of C.L.I.P. These results generalize the theorem of Lipow and Schoenberg on the zeros of $\Pi_{n,r}(\lambda)$ and supplement it in some ways. In Chapter III we study Hankel determinants of Euler-Frobenius polynomials, their relation with the polynomials $\Pi_{n,r}(\lambda)$ and their generalizations. Chapter IV deals with a generalization of exponential Euler splines which arises in a natural way when we consider C.H.I.P. to λ^x, $\lambda \neq 1$. We apply these results in Chapter V to find the Fourier transform of B-splines for C.H.I.P. introduced by Schoenberg and Sharma in the case of (0,1) interpolation. The method can be applied to more general situations.

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O Govinda, I give unto you all
that which is yours.

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INTRODUCTION

Recently much attention has been given to interpolation and approximation by spline functions. The pioneering monograph of I. J. Schoenberg [10] of 1946, deals with what is now called the Cardinal interpolation problem (C.I.P.). The subject has been further investigated by him recently in [12] and [13]. Further references on C.I.P. can be found in the excellent monograph of I. J. Schoenberg [15].

Let n,r be positive integers with $n \geq 2r - 1$. The class $S_{n,r}$ of Cardinal splines of degree n with integer knots of multiplicity r consists of functions S(x) satisfying

(1) $S(x) \in C^{n-r}(-\infty,\infty)$

and

(2) S(x) is a polynomial of degree n in each of the intervals $[\nu, \nu+1]$ for all integers ν.

In [6] Lipow and Schoenberg and in [17] Schoenberg and Sharma studied the Cardinal Hermite interpolation problem (C.H.I.P.).

Problem (C.H.I.P.). Given r bi-infinite sequences of numbers

(3) $y^{(i)} = (y_\nu^{(i)})(i = 0, 1, \ldots, r-1) \ \forall$ integers ν,

find $S(x) \in S_{n,r}$ such that

(4) $S^{(i)}(\nu) = y_\nu^{(i)} (i = 0, 1, \ldots, r-1) \ \forall$ integers ν.

Recently Meir and Sharma [8] studied the problems of existence, uniqueness and convergence of quintic splines and quintic g-splines which together with their second derivatives interpolate a given function and its second derivative respectively at a finite number of equidistant knots. In the usual terminology we shall call this problem the (0,2) interpolation problem. More generally we shall say that S(x) belongs to the class $S^g_{n,r}$ of Cardinal g-splines of degree n corresponding to (0, 1, ..., r-2, r) interpolation problem when

(5) $S(x) \in C^{n-r-1}(-\infty,\infty)$,

(6) S(x) is a polynomial of degree n in each of the intervals $[\nu,\nu+1]$ for all ν,

and (7) $S^{(n-r+1)}(\nu+) = S^{(n-r+1)}(\nu-)(\nu = 0, \pm1, \pm2, \ldots)$.

In [19] B. Swartz and R. Varga have shown that the error bounds of Meir and Sharma are best possible in some sense. Using the technique of Meir and Sharma, S. Demko [2] has also extended their results to higher degree splines.

The main object of this investigation is to study the problem of Meir and Sharma for Cardinal splines of degree n, using the approach of Lipow and Schoenberg [6]. However we do not go into the convergence problem here. In the case of C.H.I.P. this approach consists, first of all, in finding the eigensplines of the space

(8) $\overset{\circ}{S}_{n,r} = \{S(x) \in S_{n,r} : S^{(\rho)}(\nu) = 0 \ \forall \text{ integers } \nu \ (\rho = 0,1,\ldots,r-1)\}$.

If $S(x) \in \overset{\circ}{S}_{n,r}$, $S(x) \not\equiv 0$ and if

(9) $$S(x+1) = \lambda\, S(x) \ \forall \text{ real } x,\ \lambda \neq 0,$$

then S(x) is called an eigenspline (E.S.) and λ is called the eigenvalue (E.V.). These eigensplines are used in the construction of fundamental splines $L_s(x)(s = 0, 1, \ldots, r-1)$ which satisfy

$$(10)\quad \begin{cases} L_s^{(\rho)}(\nu) = 0 \;\; \forall \text{ integer } \nu \neq 0 \quad (\rho = 0, 1, \ldots, r-1), \\ L_s^{(\rho)}(0) = 0 \;(\rho \neq s) \text{ and } L_s^{(s)}(0) = 1 . \end{cases}$$

The C.H.I.P. is essentially solved once these fundamental splines have been constructed.

Here, for the Cardinal lacunary interpolation problems (C.L.I.P.), the processes of finding the corresponding E.S. and of constructing the fundamental splines are basically the same as in C.H.I.P. The problem of finding the eigensplines reduces to the study of the zeros of some characteristic polynomials. More precisely let $P = \|\binom{i}{j}\|$ $(i,j = 0, 1, 2, \ldots)$ be the infinite matrix of the binomial coefficients so that the characteristic matrix is

$$(11)\qquad P - \lambda I = \| \tbinom{i}{j} - \lambda\, \delta_{ij} \| \; (i,j = 0, 1, 2, \ldots).$$

We shall denote by $P\begin{pmatrix} i_0, & i_1, & \ldots, & i_\nu \\ j_0, & j_1, & \ldots, & j_\nu \end{pmatrix} :\lambda$ the determinant of the sub-matrix obtained from (11) by deleting all the rows and columns except those labelled $\{i_0, i_1, \ldots, i_\nu\}$ and $\{j_0, j_1, \ldots, j_\nu\}$ respectively, and by $P\begin{pmatrix} i_0, & i_1, & \ldots, & i_\nu \\ j_0, & j_1, & \ldots, & j_\nu \end{pmatrix}$ the corresponding determinant obtained from P. In connection with the C.H.I.P. Lipow and Schoenberg [6] proved that the polynomial

$$(12)\qquad \Pi_{n,r}(x) = P\begin{pmatrix} r, & r+1, & \ldots, & n \\ 0, & 1, & \ldots, & n-r \end{pmatrix} :\lambda$$

is a reciprocal polynomial whose zeros are real, simple and of sign $(-1)^r$. They use the technique of reducing the problem to an eigenvalue problem and applying a powerful theorem of Gantmacher and Krein on eigenvalues of oscillating matrices. However this does not yield the interlacing property of the zeros of $\Pi_{n,r}(\lambda)$, a phenomenon that occurs when $r = 1$. This is due to G. Frobenius (see [14]).

Summary of Thesis. This work is divided into five chapters. The first chapter gives a full discussion of the special case of $(0,1,\ldots,r-2,r)$ interpolation by g-splines. The corresponding fundamental splines are constructed by a slight modification of the method of Lipow and Schoenberg [6].

In Chapter II we give another proof of the theorem of Lipow and Schoenberg. Furthermore we show that the zeros of $\Pi_{n,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ are indeed interlacing. Exploiting this interlacing property we are able to prove the corresponding results on the characteristic polynomials arising from C.L.I.P.

Chapter III is devoted to the study of the relations between the characteristic polynomials of C.L.I.P. and the Euler-Frobenius polynomials $\Pi_n(\lambda) \equiv \Pi_{n,1}(\lambda)$. In particular the Hankel determinant of $\Pi_n(\lambda)/n!$ is related to $\Pi_{n,r}(\lambda)$ of Lipow and Schoenberg.

In Chapter IV we study the analogue of the exponential Euler spline for the case of C.H.I.P. We call them exponential Hermite-Euler splines. The polynomial component of the exponential Hermite-Euler spline of degree n in the interval [0,1] turns out to be a linear combination of the polynomials $\{A_k(x;\lambda)\}_{k=n-r+1}^{n}$ (the exponential Euler polynomials, see Schoenberg [14]). We investigate the convergence as the degree of the spline tend to infinity, for the case $r = 2$. Also we make a brief mention of the analogue of exponential

Euler spline in relation to the problem of (0,2) interpolation by g-splines.

The exponential Euler splines introduced by Schoenberg [14] are extremely useful (see [15], [16]). To demonstrate the usefulness of the exponential Hermite-Euler splines introduced in Chapter IV, an application is made to compute the Fourier transforms of the B-splines and fundamental splines for C.H.I.P. This is the subject of Chapter V.

Appendix I is a table of examples of the characteristic polynomials in some special cases. Appendix II contains examples of B-splines for quintic g-splines and ordinary splines for (0,2) and (0,3) interpolation respectively.

CHAPTER I

STATEMENT OF C.L.I.P. AND A SPECIAL CASE

We shall begin with the statement of the Cardinal Lacunary Interpolation Problems (C.L.I.P.) to be considered in the thesis. In §2 we state the main results for a special case whose solution will be accomplished in the course of §§3 and 4.

1. Statement of problems.

Let n,r be positive integers such that $n \geq 2r-1$. Further, let p,q,k be non-negative integers such that

(1.1) $r = p+q,\ 1 \leq p \leq r,\ 0 \leq q \leq r-1,\ \text{and}\ p \leq k \leq n-r-q+1$.

We shall use the notation $\overline{k,q} = \{k,k+1,\ldots,k+q-1\}$ and $\overline{0,p};\overline{k,q} = \{0,1,\ldots,p-1,k,k+1,\ldots,k+q-1\}$.

If s,t are integers with $0 \leq t \leq r$ and $0 \leq s \leq n-r+1$, let $S_{n,r}^{\overline{s,t}}$ denote the class of Cardinal spline functions S(x) satisfying the following conditions:

(1.2) S(x) is a polynomial of degree n in each of the intervals $[\nu,\nu+1]$ $(\nu = 0,\pm1,\pm2,\ldots)$,

(1.3) $S(x) \in C^{s-1}(-\infty,\infty)$,

(1.4) $S^{(\rho)}(\nu+) = S^{(\rho)}(\nu-)\ (\rho = s+t,s+t+1,\ldots,n-r+t)\ \forall$ integers ν.

If t = 0, (1.3) and (1.4) together imply that $S_{n,r}^{\overline{s,o}} = S_{n,r}$.

If t = 1 and s = n-r then $S_{n,r}^{\overline{n-r,1}} = S_{n,r}^{g}$, the class of Cardinal g-splines for (0,1,...,r-2,r) interpolation defined in the Introduction.

We shall call the following problem C.L.I.P. of type $(\overline{0,p};\ \overline{k,q})$ in $\overline{S}^{s,t}_{n,r}$.

PROBLEM. Given positive integers p,q,k satisfying (1.1) and r bi-infinite sequences of numbers

(1.5) $y^{(i)} = (y^{(i)}_{\nu})$ $(i \in \overline{0,p};\ \overline{k,q})$

satisfying

(1.6) $y^{(i)}_{\nu} = O(|\nu|^{\gamma})$ $(i \in \overline{0,p};\ \overline{k,q})$

for some $\gamma > 0$, to find a function $S(x) \in \overline{S}^{s,t}_{n,r}$ such that

(1.7) $S^{(i)}(\nu) = y^{(i)}_{\nu}$ $(i \in \overline{0,p};\ \overline{k,q})$ $\forall$ integers ν.

Observe that when k = p or q = 0 the C.L.I.P. reduces to the C.H.I.P. In the case of a finite interval, instead of the whole real line $\mathcal{R}$, the corresponding problems of type (0,2) in $S_{5,2}$ and $S^{g}_{5,2}$ have been dealt with by Meir and Sharma [8], while the problem of type $(\overline{0,p},\ \overline{k,q})$ $(k = 2m-r-q)$ in $S_{2m-1,r}$ has been treated by S. Demko [2].

2. A special case.

We shall discuss in full the special case of $(0,1,\ldots,r-2,r)$ interpolation by Cardinal g-splines. In our terminology this corresponds to C.L.I.P. of type $(\overline{0,r-1};\ r)$ in $S^{g}_{n,r}$. We shall prove

Theorem 2.1. Given r bi-infinite sequences (1.5) satisfying (1.6) the C.L.I.P. of type $(\overline{0,r-1};r)$ has a unique solution $S(x) \in S^{g}_{2m-1,r}$ such that

(2.1) $S^{(\rho)}(x) = O(|x|^{\gamma})$ $(\rho \in \overline{0,r-1};r)$ as $|x| \to \infty$.

In §4 we shall construct the fundamental functions

$L_s(x) \equiv L^g_{2m-1,s}(x) \in S^g_{2m-1,r}(s \in \overline{0,r-1},r)$ which satisfy

$$\text{(2.2)} \quad \begin{cases} L_s^{(\rho)}(\nu) = 0 \ (\rho \in \overline{0,r-1};r), \ \nu = \pm1,\pm2,\pm3,\ldots , \\ L_s^{(\rho)}(0) = 0 \ (\rho \in \overline{0,r-1};r), \ \rho \neq s \text{ and } L_s^{(s)}(0) = 1 . \end{cases}$$

In terms of these fundamental functions the spline $S(x)$ in Theorem 2.1 is given by

$$\text{(2.3)} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} \left(\sum_{s\in \overline{0,r-1};r} y_\nu^{(s)} L_s(x-\nu) \right)$$

3. The characteristic polynomials.

Following Schoenberg, we set

$$\text{(3.1)} \quad \overset{\circ}{S}{}^g_{n,r} = \{S(x):S(x) \in S^g_{n,r}, S^{(\rho)}(\nu) = 0 \ (\rho \in \overline{0,r-1};r) \ \forall \text{ integers } \nu\} ,$$

and call $S(x) \in \overset{\circ}{S}{}^g_{n,r}$, $S(x) \not\equiv 0$, an eigenspline (E.S.) if it satisfies the functional relation

$$\text{(3.2)} \qquad S(x+1) = \lambda S(x) \quad \text{for all } x \in \mathbb{R},\ \lambda \neq 0 .$$

The number λ is called the eigenvalue (E.V.) of the eigenspline $S(x)$.

Lemma 3.1. The space $\overset{\circ}{S}{}^g_{n,r}$ is a linear space of dimension n-2r+1.

Proof. It is clear that the class of all polynomials $P(x)$ of degree n satisfying

$$\text{(3.3)} \qquad P^{(\rho)}(1) = P^{(\rho)}(0) = 0 \quad (\rho = 0,1,\ldots,r-2,r)$$

is a linear space of dimension $n-2r+1$. Thus we have only to show that every such $P(x)$ determines a unique spline $S(x) \in \overset{\circ}{S}{}^{g}_{n,r}$. To this end we set

$$(3.4) \qquad S(x) = P(x) + \sum_s c_1^{(s)}(x-1)_+^{n-s} + \sum_s c_1^{(s)}(x-2)_+^{(n-s)} + \ldots$$
$$+ \sum_s c_o^{(s)}(-x)_+^{n-x} + \sum_s c_{-1}^{(s)}(-x-1)_+^{n-s} + \ldots$$

where the summations are taken over all $s \in \overline{0,r-1};r$. Then $S(x) \in S^g_{n,r}$ and $S^{(\rho)}(0) = S^{(\rho)}(1) = 0$ $(\rho \in \overline{0,r-1};r)$. The conditions $S^{(\rho)}(2) = 0$ $(\rho \in \overline{0,r-1};r)$ give a non-homogeneous system of r equations in r unknowns $c_1^{(s)}$ $(s \in \overline{0,r-1};r)$ whose determinant is equal to $1!2!\ldots(r-2)!r!P\binom{n-r,n-r+2,\ \ldots\ ,n}{0\ ,\ 1\ ,\ldots,r-2,r}$ which is non-zero. Hence $c_1^{(s)}$ are uniquely determined in terms of $P(x)$. Suppose $c_k^{(s)}$ $(k = 1,2,\ldots,\nu-1)$ have been so determined. Then $c_\nu^{(s)}$ are uniquely determined in the same way by the conditions $S^{(\rho)}(\nu+1) = 0$ $(\rho \in \overline{0,r-1};r)$. By induction all the $c_k^{(s)}$ $(k = 1,2,\ldots,s \in \overline{0,r-1};r)$ are uniquely determined in terms of $P(x)$ by the conditions $S^{(s)}(\nu) = 0$ $(\nu = 2,3,\ldots,s \in \overline{0,r-1},r)$. Similarly the $c_k^{(s)}$ $(k = 0,-1,-2,\ldots,s \in \overline{0,r-1};r)$ are uniquely determined by the conditions $S^{(\rho)}(\nu) = 0$ $(\nu = -1,-2,\ldots,\rho \in \overline{0,r-1};r)$. □

<u>Eigenvalues.</u>

Suppose $S(x) \in \overset{\circ}{S}{}^{g}_{n,r}$ is an eigenspline satisfying (3.2). Then $S(x)$ is uniquely determined by its polynomial component $P(x)$ in $[0,1]$. It follows from (3.1) that

$$(3.5) \qquad P^{(\rho)}(1) = P^{(\rho)}(0) = 0 \quad (\rho \in \overline{0,r-1};r) ,$$

while (3.2) together with the continuity conditions on $S(x)$ give rise to the additional relations

$$(3.6)\qquad P^{(\rho)}(1) = \lambda P^{(\rho)}(0) \quad (\rho = r-1, r+1, \ldots, n-r, n-r+1)\ .$$

By (3.5) we can write

$$(3.7)\qquad P(x) = a_o x^n + a_1\binom{n}{1}x^{n-1} + \ldots + a_{n-r-1}\binom{n}{n-r-1}x^{r+1} + a_{n-r+1}\binom{n}{n-r+1}x^{r-1}\ .$$

Writing equations (3.5) and (3.6) upwards with increasing ρ, we obtain a homogeneous system of $(n-r+1)$ equations in $(n-r+1)$ unknowns $a_o, a_1, \ldots, a_{n-r-1}, a_{n-r+1}$. The determinant of the system is the $(n-2r+1)$-th degree polynomial in λ given by

$$(3.8)\qquad \Pi^g_{n,r}(\lambda) = P\left(\begin{matrix} r-1, & r+1, & \ldots & , & n \\ 0 & , 1 & , \ldots, & n-r-1, & n-r+1 \end{matrix};\lambda\right) =$$

$$\begin{vmatrix}
1 & \binom{r-1}{1}\ldots\binom{r-1}{r-2} & (1-\lambda) & 0 & 0 \ldots\ldots\ldots & 0 \\
1 & \binom{r+1}{1}\ldots\binom{r+1}{r-2} & \binom{r+1}{r-1} & \binom{r+1}{r} & (1-\lambda)\ldots\ldots\ldots & 0 \\
\vdots & \vdots & & & \vdots & \\
1 & \binom{n-r-1}{1} & \ldots\ldots\ldots\ldots & & (1-\lambda) & 0 \\
1 & \binom{n-r}{1} & \ldots\ldots\ldots\ldots & & \binom{n-r}{n-r-1} & 0 \\
1 & \binom{n-r+1}{1} & \ldots\ldots\ldots\ldots & & \binom{n-r+1}{n-r-1} & (1-\lambda) \\
1 & \binom{n-r+2}{1} & \ldots\ldots\ldots\ldots & & \binom{n-r+2}{n-r-1} & \binom{n-r+2}{n-r+1} \\
\vdots & \vdots & & & \vdots & \vdots \\
1 & \binom{n}{1} & \ldots\ldots\ldots\ldots & & \binom{n}{n-r-1} & \binom{n}{n-r+1}
\end{vmatrix}$$

Thus the E.V.'s of the E.S.'s of the null space $\overset{\circ}{S}{}^g_{n,r}$ are the zeros of $\Pi^g_{n,r}(\lambda)$. In § 7 we shall prove the following

<u>Theorem 3.2</u>. <u>Let $n = 2m-1$ be an odd integer such that $m \geq r+1$ $(r = 2,3,\ldots)$. The polynomial $\Pi^g_{2m-1,r}(\lambda)$ is a reciprocal polynomial of</u>

degree $d = 2m-2r$ with real simple zeros two of which are of sign $(-1)^{r-1}$ and $(d-2)$ of sign $(-1)^r$. (For proof see §7 p.).

Let us label the zeros of $\Pi^g_{2m-1,r}(\lambda)$ by $\{\lambda_j\}_{j=1}^d$ so that

(3.9) $$\lambda_d < -1 < \lambda_1 < 0 < \lambda_2 <\dots< \lambda_{m-r} < 1 < \lambda_{m-r-1} < \dots < \lambda_{d-1}$$

in case r is even (with obvious modifications when r is odd) and

(3.10) $$\lambda_1 . \lambda_d = \lambda_2 . \lambda_{d-1} = \quad = \lambda_{m-r}\lambda_{m-r+1} = 1 .$$

The eigensplines are constructed as follows. Let $P_j(x)$ be a solution of the system of equations (3.5) and (3.6) when $\lambda = \lambda_j$. Then set $S_j(x) = P_j(x)$ $(x \in [0,1])$ and extend $S_j(x)$ by the functional relation

(3.11) $$S_j(x+1) = \lambda_j S_j(x) \quad (x \in R).$$

It follows that for every integer ν we have

(3.12) $$S_j(x+\nu) = \lambda_j^\nu S_j(x) \quad (x \in R).$$

Lemma 3.3. Let $S_j(x)$ be an E.S. corresponding to λ_j. Then $S_j^{(r-1)}(0) \neq 0$.

Proof. Suppose $S_j^{(r-1)}(0) = 0$. Then its polynomial component

(3.13) $$P_j(x) = a_o x^{2m-1} + a_1\binom{2m-1}{1}x^{2m-2} + \dots + a_{2m-r-2}\binom{2m-1}{2m-r-2}x^{r+1}$$

satisfies the relations

(3.14) $$P_j^{(\rho)}(1) = 0 \quad (\rho = 0,1,\dots,r)$$

and

(3.15) $P_j^{(\rho)}(1) = \lambda_j P_j^{(\rho)}(0) \quad (\rho = r+1,\dots,2m-r-2,2m-r).$

Now (3.14) and (3.15) give a system of (2m-r) equations in (2m-r-1) unknowns a_k $(k = 0,1,\dots,2m-r-2)$. Since $P_j(x) \not\equiv 0$, it follows that λ_j is a zero of $\Pi_{2m-1,r+1}(\lambda)$ and at the same time a zero of $\Pi^{**}_{2m-2,r}(\lambda) = P\begin{pmatrix} r-1,r+1,\dots,2m-2 \\ 0\ ,\ 1\ ,\dots,2m-2-r \end{pmatrix}:\lambda\Big)$. But then (see §7) all the zeros of $\Pi^{**}_{2m-2,r}(\lambda)$ are of sign $(-1)^r$ except the zero $(-1)^{r+1}$, and all the zeros of $\Pi_{2m-1,r+1}(\lambda)$ are of sign $(-1)^{r+1}$. Since $\lambda_j \neq (-1)^{r+1}$ these are impossible. Hence $S_j^{(r-1)}(0) \neq 0$. □

Remarks. 1) The proof of lemma 3.3 also establishes the fact that the matrix whose determinant is $\Pi^g_{2m-1,r}(\lambda_j)$ has rank d-1. Thus the eigensplines $S_j(x)$ corresponding to the eigenvalue λ_j are uniquely determined up to a constant factor, and so there are exactly d distinct eigensplines in $\mathring{S}^g_{2m-1,r}$.

2) By a lemma of Schoenberg and Ziegler ([18], p. 424, lemma 2) it can be shown that these eigensplines can have at most a finite number of zeros in (0,1).

Since the λ_j $(j = 1,2,\dots,d)$ are distinct the relation (3.12) gives the following

Lemma 3.4. The eigensplines $\{S_j(x)\}_{j=1}^d$ are linearly independent and so form a basis of the vector space $\mathring{S}^g_{2m-1,r}$.

We shall normalise the eigensplines so that

(3.16) $S_j(x) > 0 \quad (x \in (0,\delta))$ for some $\delta > 0$,

and

(3.17) $S_j^{(r-1)}(0) = 1$.

Lemma 3.5. If $S(x) \in \overset{\circ}{S}{}^g_{2m-1,r}$ and $S(x) = O(|x|^\gamma)$ for some $\gamma > 0$, then $S(x) \equiv 0$.

Proof. By lemma 3.4 we can write $S(x) = \sum_{j=1}^{d} c_j S_j(x)$. Since $|\lambda_j| \neq 1$, the hypothesis of the lemma together with (3.12) implies that $c_j = 0$ $(j = 1,2,\ldots,d)$. Hence $S_j(x) \equiv 0$.

4. Fundamental splines.

Recall that the fundamental splines $L_s(x) \equiv L^g_{2m-1,s}(x) \in S^g_{n,r}(s \in \overline{0,r-1};r))$ should satisfy the following conditions

$$(4.1)\quad \begin{cases} L_s^{(\rho)}(\nu) = 0 \ (\rho \in \overline{0,r-1};r),\ \nu = \pm 1, \pm 2, \ldots, \\ L_s^{(\rho)}(0) = 0 \ (\rho \in \overline{0,r-1};r,\ \ \rho \neq s) \text{ and } L_s^{(s)}(0) = 1 . \end{cases}$$

We shall construct these fundamental functions in terms of the eigensplines obtained in §3. For this purpose let us consider the following functions

$$(4.2)\qquad F_1(x) = \sum_{j=1}^{m-r} c_j S_j(x) \quad ,$$

$$(4.3)\qquad F_2(x) = \sum_{j=m-r+1}^{2m-2r} d_j S_j(x) \quad ,$$

$$(4.4)\qquad P(x) = a_0 x^{2m-1} + a_1 x^{2m-2} + \ldots + a_{2m-r-2} x^{r+1} + a_{2m-r} x^{r-1} + \frac{x^s}{s!} \quad (s \in \overline{0,r-1};r) ,$$

and

$$(4.5) \qquad Q(x) = b_o x^{2m-1} + b_1 x^{2m-2} + \ldots + b_{2m-r-2} x^{r+1} + b_{2m-r} x^{r-1} + \frac{x^s}{s!}$$

$$(s \in \overline{0,r-1};r) \quad .$$

Observe that $F_i(x) \in \overset{\circ}{S}{}^g_{2m-1,r}$ $(i = 1,2)$ and $F_1(x) \in L_1[1,\infty)$ while $F_2(x) \in L_1(-\infty,-1]$. Also $P^{(\rho)}(0) = Q^{(\rho)}(0) = 0$ $(\rho \in \overline{0,r-1};r$ and $\rho \neq s)$ and $P^{(s)}(0) = Q^{(s)}(0) = 1$.

If we set

$$(4.6) \qquad P^{(\rho)}(0) = Q^{(\rho)}(0) \quad (\rho = r-1,r+1,\ldots,2m-r-2,2m-r) \ ,$$

$$(4.7) \qquad P^{(\rho)}(1) = \sum_{j=1}^{m-r} c_j S_j^{(\rho)}(1)$$

and

$$(4.8) \qquad Q^{(\rho)}(-1) = \sum_{j=m-r+1}^{2m-2r} d_j S_j^{(\rho)}(-1) \quad (\rho = 0,1,\ldots,2m-r-2,2m-r)$$

we obtain a non-homogeneous system of 6m-4r equations in 6m-4r unknowns $\{a_j,b_j : j=0,1,\ldots,2m-r-2,2m-r\}$, $\{c_j\}_{j=1}^{m-r}$ and $\{d_j\}_{j=m-r+1}^{2m-2r}$. We want to show that this system is non-singular. Observe that the constant terms of this system of equations consist of $\frac{1}{(s-\rho)!}$, $\frac{(-1)^{s-\rho}}{(s-\rho)!}$ which appear in the last two sets of equations (4.7) and (4.8) for $\rho = 0,1,\ldots,s$, and the rest are zeros. The corresponding homogeneous system corresponds to the system of equations given by (4.6), (4.7) and (4.8) in which the terms $\frac{x^s}{s!}$ are absent in $P(x)$ and $Q(x)$. If this system is singular we would obtain a spline $S(x) \in \overset{\circ}{S}{}^g_{n,r}$, $S(x) \not\equiv 0$, and $S(x) \in L_1(-\infty,\infty)$. This is impossible because of lemma 3.5. Hence the above system is non-singular. This means that we can find a unique spline $L_s(x) \in S^g_{2m-1,r}$ $(s \in \overline{0,r-1};r)$ satisfying (4.1). Furthermore $L_s(x) \in L_1(-\infty,\infty)$.

The symmetry relation

$$(4.9) \qquad L_s(x) = (-1)^s L_s(-x) \quad (s \in \overline{0,r-1};r)$$

follows from the following lemma whose proof is immediate.

Lemma 4.1. Let $\tilde{L}_s(x) = L_s(-x)$. Then $\tilde{L}_s(x) \in S^g_{n,r}$ and satisfies $\tilde{L}^{(\rho)}_s(\nu) = 0$ $(\rho \in \overline{0,r-1};r)$ $\forall$ integers $\nu \neq 0$, $\tilde{L}^{(\rho)}_s(0) = 0$ $(\rho \in \overline{0,r-1};r,$ $\rho \neq s)$ and $L^{(s)}_s(0) = (-1)^s$.

Lemma 4.2. There exists positive constants A and α depending on m and r such that

$$(4.10) \qquad |L^{(\rho)}_s(x)| \leq A\, e^{-\alpha|x|} \quad (\rho \in \overline{0,r-1};r) .$$

Proof. Recall that $L_s(x)$ is constructed so that

$$(4.11) \qquad L_s(x) = \sum_{j=1}^{m-r} c_j S_j(x) \quad (x \geq 1) .$$

Hence, in view of the relation (4.9), it is clearly enough to prove the same estimate for $S^{(\rho)}_j(x)$ $(j = 1,2,\ldots,m-r)$ for $x \geq 0$.

Suppose $\nu \leq x < \nu+1$. By (3.12) we have

$$(4.12) \qquad S^{(\rho)}_j(x) = \lambda^{\nu}_j S^{(\rho)}_j(x-\nu)$$

and so $|S^{(\rho)}_j(x)| \leq B|\lambda_j|^{\nu} \leq B|\lambda_j|^{-1}|\lambda_j|^x$ $(j = 1,2,\ldots,m-r)$, where $B = \sup_{0\leq x\leq 1} S^{(\rho)}_j(x)$. Hence

$$(4.13) \qquad |S^{(\rho)}_j(x)| \leq B \frac{\exp\{-x \log|\lambda_j|^{-1}\}}{|\lambda_j|} \quad (j = 1,2,\ldots,m-r).$$

If we set $\alpha = \min_{j=1,2,\ldots,m-r} \log|\lambda_j|^{-1}$, we have

(4.14) $$|S_j^{(\rho)}(x)| \le B \frac{\exp(-\alpha x)}{|\lambda_j|} \quad (j = 1,2,\ldots,m-r),\ x \ge 0 .$$

Hence (4.10) follows from (4.11) and (4.14). □

Proof of Theorem 2.1.

We shall show that if

(4.15) $$y_\nu^{(\rho)} = O(|\nu|^\gamma) \quad (\rho \in \overline{0,r-1};r) \text{ for some } \gamma > 0,$$

then

(4.16) $$S^{(\rho)}(x) = \sum_{-\infty}^{\infty} \left(\sum_{s \in \overline{0,r-1,r}} y_\nu^{(s)} L_s^{(\rho)}(x-\nu) \right) \quad (\rho \in \overline{0,r-1};r)$$

converges locally uniformly and

(4.17) $$S^{(\rho)}(x) = O(|x|^\gamma) \quad (\rho \in \overline{0,r-1};r) \text{ as } |x| \to \infty.$$

The proof of (4.16) and (4.17) is the same as in [13] p. 415 with obvious modifications. □

CHAPTER II

PROOF OF THEOREM 3.2 AND SOME GENERALISATIONS

In §3 the zeros of $\Pi^{g}_{2m-1,r}(\lambda)$ are used in the construction of the eigensplines for the C.L.I.P. of type $(\overline{0,r-1};r)$ in $S^{g}_{2m-1,r}$. This chapter is devoted to the proof of theorem 3.2. We shall also discuss the corresponding results for some general cases. These results generalise the theorem of Lipow and Schoenberg [6] and also our theorem 3.2.

5. The characteristic polynomials for general C.L.I.P..

Let n be a positive integer such that $n \geq 2r-1$ and p,q,k be non-negative integers satisfying (1.1). Further, let s,t be non-negative integers with

(5.1) $\quad k+q \leq s \leq n-r+1 \;, \; 0 \leq t < r \;.$

Following the same argument as in §3, we see that for the C.L.I.P. of type $(\overline{0,p};\ \overline{k,q})$ in $S^{\overline{s,t}}_{n,r}$ the E.V.'s of the E.S.'s belonging to the null space

(5.2) $$\overset{\circ}{S}{}^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}) = \{S(x) \in S^{\overline{s,t}}_{n,r} : S^{(\rho)}(\nu) = 0$$

$$\text{for } \rho \in \overline{0,p};\ \overline{k,q} \text{ and } \nu = 0,\pm 1,\pm 2,\ldots\}$$

are the zeros of the characteristic polynomials $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda)$ where

(5.3) $$\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda) = P\binom{r-t,r-t+1,\ldots,n-s-t,n-s+1,\ \ldots\ ,n}{0\ ,\ 1\ ,\ \ldots\ ,n-k-q,n-k+1,\ldots,n-p}:\lambda) =$$

$$
\begin{array}{l}
n-s-r+1\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\
s-k-q\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\
q\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\
k-p\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \\
p\left\{\begin{array}{l}\\ \\ \\ \end{array}\right.
\end{array}
\left|\begin{array}{cccccccc}
1 & \binom{r-t}{1} & \cdots (1-\lambda) & 0 & \cdots & & & 0 \\
\vdots & \vdots & & \ddots & & & & \vdots \\
1 & \binom{n-s-t}{1} & \cdots & (1-\lambda) & 0 & \cdots & & 0 \\
1 & \binom{n-s+1}{1} & \cdots & \cdots & (1-\lambda) & 0 & \cdots & 0 \\
\vdots & \vdots & & & & \ddots & & \vdots \\
1 & \binom{n-k-q}{1} & \cdots & \cdots & & (1-\lambda) & 0 \cdots & 0 \\
1 & \binom{n-k-q+1}{1} & \cdots & \cdots & & \binom{n-k-q+1}{n-k-q} & 0 \cdots & 0 \\
\vdots & \vdots & & & & \vdots & \vdots & \vdots \\
1 & \binom{n-k+1}{1} & \cdots & \cdots & & \binom{n-k+1}{n-k-q} & (1-\lambda) & 0 \cdots 0 \\
\vdots & \vdots & & & & \vdots & \vdots & \ddots \\
1 & \binom{n-p}{1} & \cdots & \cdots & & \binom{n-p}{n-k-q} & \binom{n-p}{n-k+1} & \cdots (1-\lambda) \\
\vdots & \vdots & & & & \vdots & \vdots & \vdots \\
1 & \binom{n}{1} \cdots & \binom{n}{r-t} \cdots & \binom{n}{n-s-t} \cdots & \binom{n}{n-s+1} \cdots & \binom{n}{n-k-q} & \binom{n}{n-k+1} \cdots & \binom{n}{n-p}
\end{array}\right|
$$

$$\underbrace{\qquad}_{r-t}\quad\underbrace{\qquad\qquad}_{n-s-r+1}\quad\underbrace{\quad}_{t}\quad\underbrace{\qquad\qquad}_{s-k-q}\quad\underbrace{\qquad\qquad}_{k-p}$$

We shall write

$$
(5.3a)\qquad \begin{cases} \overline{\Pi^{s,o}_{n,r}}(\overline{0,p};\ \overline{k,q}:\lambda) \equiv \Pi_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda) \\ \\ \overline{\Pi^{s,t}_{n,r}}(\overline{0,p};\ \overline{k,0};\lambda) \equiv \overline{\Pi^{s,t}_{n,r}}(\lambda) \ . \end{cases}
$$

Observe that λ occurs in $(n-2r+1)$ rows and $(n-2r+1)$ columns, and it is clear that $\overline{\Pi^{s,t}_{n,r}}(\overline{0,p};\ \overline{k,q}:\lambda)$ is a polynomial in λ of degree $d = n-2r+1$. By the same reason as for $\Pi^{g}_{n,r}(\lambda)$, the polynomials $\overline{\Pi^{s,t}_{n,r}}(\overline{0,p};\ \overline{k,q}:\lambda)$ are reciprocal polynomials. Some particular cases are

$$
(5.4)\quad \begin{cases} \Pi^{\overline{s,o}}_{n,r}(\overline{0,p};\ \overline{k,0}:\lambda) = \Pi_{n,r}(\lambda)\ , \\[2ex] \Pi^{\overline{n-r,1}}_{n,r}(\overline{0,r-1},r:\lambda) = \Pi^{g}_{n,r}(\lambda)\ . \end{cases}
$$

For the sake of convenience we shall also write

$$
(5.6)\qquad \Pi^{\overline{n-r,1}}_{n,r}(\overline{0,p};\overline{k,0}:\lambda) = \Pi^{**}_{n,r}(\lambda)
$$

and

$$
(5.7)\qquad \Pi^{\overline{s,o}}_{n,r}(\overline{0,r-1};r:\lambda) = \Pi^{*}_{n,r}(\lambda)\ .
$$

The relation

$$
(5.8)\qquad (n-r+1)\Pi^{*}_{n,r}(\lambda) = r\ \Pi^{**}_{n,r}(\lambda)
$$

is a particular case of the following lemma.

<u>Lemma 5.1</u>. <u>Let $n \geq 2r-1$ and $p,q,k,s,t,$ satisfy (1.1) and (5.1)</u>. <u>Then</u>

$$
(5.9)\qquad \Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda) = C\Pi^{\overline{n-q-k+1,q}}_{n,r}(\overline{0,r-t},\overline{n-s-t+1,t}:\lambda)\ ,
$$

$$
\text{where } C = \frac{\binom{n}{n-p}\binom{n}{n-p-1}\cdots\binom{n}{n-k+1}\binom{n}{n-k-q}\cdots\binom{n}{1}\binom{n}{0}}{\binom{n}{n-r+t}\binom{n}{n-r+t-1}\cdots\binom{n}{s+t}\binom{n}{s-1}\cdots\binom{n}{1}\binom{n}{0}}\ .
$$

<u>Proof</u>. By taking transpose and the interchanging rows and columns in the determinant (5.3) we can write $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda) =$

$$
\left|\begin{array}{ccccccccc}
\binom{n}{n-p} & \binom{n-1}{n-p} & \cdots & (1-\lambda) & 0 & 0 & \cdots & 0 & 0 \\
\binom{n}{n-p-1} & \binom{n-1}{n-p-1} & \cdots & \cdots & (1-\lambda) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & & & & & \vdots & \vdots \\
\binom{n}{n-k+1} & \binom{n-1}{n-k+1} & & \cdots & \cdots & & & \vdots & \vdots \\
\binom{n}{n-k-q} & \binom{n-1}{n-k-q} & & \cdots & \cdots & & & & \\
 & & & & & & & 0 & 0 \\
\vdots & \vdots & & & & & & (1-\lambda) & 0 \\
\vdots & \vdots & & & & & & \binom{r-t+1}{r-t-1} & (1-\lambda) \\
\vdots & \vdots & & & & & & \binom{r-t+1}{r-t-1} & \binom{r-t}{r-t-1} \\
 & & & & & & & \vdots & \vdots \\
\binom{n}{1} & \binom{n-1}{1} & \cdots & & \binom{n-s+1}{1} & \binom{n-s-t}{1} & \cdots & \binom{r-t+1}{1} & \binom{r-t}{1} \\
1 & 1 & \cdots & & 1 & 1 & \cdots & 1 & 1
\end{array}\right|
$$

If we factor out $\binom{n}{n-p}$, $\binom{n-1}{n-p-1}, \ldots, \binom{n}{n-k+1}$, $\binom{n}{n-k-q}, \ldots, \binom{n}{1}$, $\binom{n}{0}$ from the 1st, 2nd and so forth until the last row respectively, and apply the relation

$$\binom{k}{\ell} / \binom{m}{\ell} = \binom{m-\ell}{m-k} / \binom{m}{m-k} \quad (m \geq k) ,$$

then factor out $1/\binom{n}{n-r+t}\binom{n}{n-r+t-1}\cdots\binom{n}{s+t+1}\binom{n}{s+t}\binom{n}{s-1}\cdots\binom{n}{1}\binom{n}{0}$ from the columns we obtain (5.9). □

6. A determinantal identity.

For convenience of reference we now state a determinantal identity and a theorem of S. Karlin which will be used in the sequel.

Consider the following $n \times (n+2)$ matrix

$$(6.1)\qquad \left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 & f_1^{(1)} & f_1^{(2)} & \cdots & f_1^{(n-2)} \\ a_2 & b_2 & c_2 & d_2 & f_2^{(1)} & f_2^{(2)} & \cdots & f_2^{(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_n & b_n & c_n & d_n & f_n^{(1)} & f_n^{(2)} & \cdots & f_n^{(n-2)} \end{matrix} \right\|$$

Let $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}, \underset{\sim}{f}^{(1)}, \ldots, \underset{\sim}{f}^{(n-2)}$ denote the column vectors of (6.1) and $D(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f}) \equiv D(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f}^{(1)}, \ldots, \underset{\sim}{f}^{(n-2)})$ denote the minors of (6.1) formed from the columns $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f}^{(1)}, \ldots, \underset{\sim}{f}^{(n-2)}$. Then an identity of S. Karlin ([4], p. 7) states that

$$(6.2)\qquad \begin{vmatrix} D(\underset{\sim}{a},\underset{\sim}{c},\underset{\sim}{f}) & D(\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{f}) \\ D(\underset{\sim}{a},\underset{\sim}{d},\underset{\sim}{f}) & D(\underset{\sim}{b},\underset{\sim}{d},\underset{\sim}{f}) \end{vmatrix} = D(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f})D(\underset{\sim}{c},\underset{\sim}{d},\underset{\sim}{f}) .$$

<u>Theorem 6.1 ([4], p. 85).</u> <u>Let $A = \|a_{ij}\|$ $(i,j = 0,1,\ldots,n)$ be a triangular matrix, i.e. $a_{ij}=0$ for $j > i$. If $A\binom{i,i+1,\ldots,i+p}{0,\ 1\ ,\ldots,\ p} > 0$ for $0 \le i \le n-p$, $0 \le p \le n$, then A is totally positive. Furthermore</u>

$$A\begin{pmatrix} i_0, i_1, \ldots, i_\nu \\ j_o, j_1, \ldots, j_\nu \end{pmatrix} > 0 \text{ for } 0 \leq \begin{pmatrix} i_o < i_1 < \ldots < i_\nu \\ j_o < j_1 < \ldots < j_\nu \end{pmatrix} \leq n$$

whenever $j_k \leq i_k$ $(k = 0,1,\ldots,\nu;\ \nu = 0,1,\ldots,n)$.

In the case of the matrix $P = \| \binom{i}{j} \|$ $(i,j = 0,1,2,\ldots)$, it is easy to show by induction that (see [6], p. 284)

(6.3) $P\binom{i, i+1, \ldots, i+p}{0,\ 1\ , \ldots,\ p} = 1 \qquad (i,p = 0,1,2,\ldots)$.

Consequently theorem 6.1 gives the following

Lemma 6.2. The matrix $P = \| \binom{i}{j} \|$ $(i,j = 0,1,2,\ldots)$ is totally positive. Furthermore

(6.4) $P\begin{pmatrix} i_o, i_1, \ldots, i_\nu \\ j_o, j_1, \ldots, j_\nu \end{pmatrix} > 0 \text{ for } 0 \leq \begin{pmatrix} i_o < i_1 < \ldots < i_\nu \\ j_o < j_1 < \ldots < j_\nu \end{pmatrix}$

whenever $j_k \leq i_k$ $(k = 0,1,\ldots,\nu)$.

We deduce from (5.3) and lemma 6.2 that

(6.5) $\overline{\Pi^{s,t}_{n,r}}(\overline{0,p};\ \overline{k,q};0) > 0$.

7. Proof of Theorem 3.2.

First we establish some identities for the characteristic polynomials.

Lemma 7.1. Let $r = 1,2,\ldots$. For $n \geq 2r+1$ we have

(7.1) $r\Pi_{n,r+1}(\lambda)\Pi_{n-2,r-1}(\lambda) = n\{\Pi_{n-1,r}(\lambda)\}^2 - (n-r)\Pi_{n-2,r}(\lambda)\Pi_{n,r}(\lambda)$.

Let $r = 2,3,\ldots$. For $n \geq 2r+1$

(7.2) $\Pi_{n,r-1}(\lambda)\Pi_{n-1,r}(\lambda) = \Pi^*_{n,r}(\lambda)\Pi_{n-1,r-1}(\lambda) - \Pi_{n,r}(\lambda)\Pi^*_{n-1,r-1}(\lambda)$

and

$$\Pi_{n,r-1}(\lambda)\Pi_{n,r+1}(\lambda) = \Pi^{*}_{n,r}(\lambda)\Pi^{**}_{n,r}(\lambda) - \Pi^{g}_{n,r}(\lambda)\Pi_{n,r}(\lambda). \tag{7.3}$$

Proof. To prove (7.1) let us consider the following (n-r+1) × (n-r+1) matrix.

$$\left\| \begin{array}{cccccccc} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & (1-\lambda) & 0 & \cdots & 0 & 0 \\ 1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & (1-\lambda) & \cdots & 0 & 0 \\ \vdots & & & & & & \ddots & & \\ 1 & \binom{n-r-1}{1} & \cdots & \binom{n-r-1}{r-1} & \binom{n-r-1}{r} & \binom{n-r-1}{r+1} & \cdots & (1-\lambda) & 0 \\ 1 & \binom{n-r}{1} & \cdots & \binom{n-r}{r-1} & \binom{n-r}{r} & \binom{n-r}{r+1} & \cdots & \binom{n-r}{n-r-1} & (1-\lambda) \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \binom{n}{1} & \cdots & \binom{n}{r-1} & \binom{n}{r} & \binom{n}{r+1} & \cdots & \binom{n}{n-r-1} & \binom{n}{n-r} \end{array} \right\| \tag{7.4}$$

Let us denote the first and the last columns of (7.4) respectively by $\underset{\sim}{c}$ and $\underset{\sim}{d}$ and its ν-th column by $\underset{\sim}{f}^{(\nu-1)}$ for $\nu = 2,3,\ldots,n-r$. Set $\underset{\sim}{a} = (1,0,\ldots,0)^T$ and $\underset{\sim}{b} = (0,\ldots,0,1)^T$. Then after some simplifications we have $D(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f}) = (-1)^{n-r-1}\binom{n-1}{r}\Pi_{n-2,r}(\lambda)$, $D(\underset{\sim}{a},\underset{\sim}{c},\underset{\sim}{f}) = \Pi_{n,r+1}(\lambda)$, $D(\underset{\sim}{a},\underset{\sim}{d},\underset{\sim}{f}) = (-1)^{n-r-1}\binom{n}{r}\Pi_{n-1,r}(\lambda)$, $D(\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{f}) = (-1)^{n-r}\Pi_{n-1,r}(\lambda)$, $D(\underset{\sim}{b},\underset{\sim}{d},\underset{\sim}{f}) = -\binom{n-1}{r-1}\Pi_{n-2,r-1}(\lambda)$ and $D(\underset{\sim}{c},\underset{\sim}{d},\underset{\sim}{f}) = (-1)^{n-r-1}\Pi_{n,r}(\lambda)$.

Substituting these into (6.2) we obtain (7.1).

In order to prove (7.2) we again set $\underset{\sim}{a} = (1,0,\ldots,0)^T$, $\underset{\sim}{b} = (0,0,\ldots,0,1)^T$ and let $\underset{\sim}{f}^{(\nu)}$ denote the ν-th column of (7.4) for $\nu = 1,2,\ldots,n-r-1$, with $\underset{\sim}{c}$, $\underset{\sim}{d}$ representing the last two columns. Then

$D(\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{f}) = (-1)^{n-r-1}\Pi_{n-1,r+1}(\lambda)$, $D(\underset{\sim}{a},\underset{\sim}{c},\underset{\sim}{f}) = (-1)^{n-r-1}\Pi_{n,r+1}(\lambda)$, $D(\underset{\sim}{a},\underset{\sim}{d},\underset{\sim}{f}) = (-1)^{n-r-1}\Pi^*_{n,r+1}(\lambda)$, $D(\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{f}) = -\Pi_{n-1,r}(\lambda)$, $D(\underset{\sim}{b},\underset{\sim}{d},\underset{\sim}{f}) = -\Pi^*_{n-1,r}(\lambda)$ and $D(\underset{\sim}{c},\underset{\sim}{d},\underset{\sim}{f}) = \Pi_{n,r}(\lambda)$.

Replacing r by (r-1) in the above determinants, and using the identity (6.2) we obtain (7.2).

In order to prove (7.3) we use the same vectors as in the proof of (7.2) except that $\underset{\sim}{b}$ is replaced by the vector $(0,1,0,\ldots,0)^T$. Then (7.3) is obtained by replacing r by r-1. □

<u>Lemma 7.2</u>. <u>If $n \geq 2r-1$, then</u>

(7.5) $\Pi_{n,r}(\lambda) > 0 \quad \text{for} \quad (-1)^{r+1}\lambda \geq 0 .$

<u>Proof</u>. A straightforward computation shows that

(7.6) $\Pi_{n,r}(\lambda) = a_o[(-1)^{r+1}\lambda]^{n-2r+1} + a_1[(-1)^{r+1}\lambda]^{n-2r} + \ldots + a_{n-2r+1}$,

where

(7.7) $a_k = \sum P\begin{pmatrix} r+\nu_1, r+\nu_2, \ldots, r+\nu_k, n-r+1, \ldots, & n \\ 0 \quad , \quad 1 \quad , \ldots, r-1, r+\nu_1, r+\nu_2, \ldots, & r+\nu_k \end{pmatrix}$

$(k = 1,2,\ldots,n-2r+1)$ the summation is over all the $\binom{n-2r+1}{k}$ choices of $\{\nu_1,\nu_2,\ldots,\nu_k\}$ from $\{0,1,\ldots,n-2r\}$, and

(7.8) $a_o = P\begin{pmatrix} n-r+1, \ldots\ldots, & n \\ 0 \quad , 1, \ldots, & r-1 \end{pmatrix}$.

Hence (7.5) follows from lemma 6.2. □

<u>Theorem 7.3</u>. <u>If $r \geq 1$, then for $n \geq 2r+1$ the zeros $\Pi_{n,r}(\lambda)$ are real, simple, of sign $(-1)^r$ and interlace with the zeros of $\Pi_{n-1,r}(\lambda)$.</u>

Proof. The proof follows by induction on n. We shall give the proof only for the case when r is even. The case when r is odd can be treated in the same way. We shall denote the zeros of $\Pi_{n,r}(\lambda)$ by $\lambda_i^{(n)}$.

If we set $n = 2r+1$ in (7.1) and take into account that $\Pi_{2r+1,r+1}(\lambda) \equiv \Pi_{2r-1,r}(\lambda) \equiv 1$, we obtain

$$(7.9) \qquad (r+1)\Pi_{2r+1,r}(\lambda) = (2r+1)\{\Pi_{2r,r}(\lambda)\}^2 - r\Pi_{2r-1,r-1}(\lambda) .$$

Now $\Pi_{2r,r}(\lambda)$ is a reciprocal polynomial of degree 1 which is positive for $\lambda \leq 0$. Hence $\Pi_{2r,r}(1) = 0$, and it follows that $(r+1)\Pi_{2r+1,r}(1) = -r\Pi_{2r-1,r-1}(1) < 0$ by lemma 7.2. But then $\Pi_{2r+1,r}(0) > 0$ and therefore $\Pi_{2r+1,r}(\lambda)$ has a zero in $(0,1)$. Since it is a reciprocal polynomial it also has a zero in $(1,\infty)$. Thus the zeros of $\Pi_{2r+1,r}(\lambda)$ interlace with the zero of $\Pi_{2r,r}(\lambda)$.

Let us suppose that the assertion has been proved for $\Pi_{n-2,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$. More precisely, let us suppose that the zeros of $\Pi_{n-2,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ satisfy the relation

$$(7.10) \qquad 0 < \lambda_1^{(n-1)} < \lambda_1^{(n-2)} < \lambda_2^{(n-1)} < \ldots < \lambda_{d-2}^{(n-2)} < \lambda_{d-1}^{(n-1)} \quad (d = n-2r+1).$$

It follows from (7.1) that

$$(7.11) \qquad -(n-r)\Pi_{n,r}(\lambda_i^{(n-1)})\Pi_{n-2,r}(\lambda_i^{(n-1)}) =$$

$$r\Pi_{n,r+1}(\lambda_i^{(n-1)})\Pi_{n-2,r-1}(\lambda_i^{(n-1)}) \qquad (i = 1,2,\ldots,d-1).$$

Since r is even, lemma 7.2 says that the R.H.S. of (7.11) is positive for $i = 1,2,\ldots,d-1$. Hence

$$(7.12) \qquad \Pi_{n,r}(\lambda_i^{(n-1)})\Pi_{n-2,r}(\lambda_i^{(n-1)}) < 0 \quad (i = 1,2,\ldots,d-1) .$$

It follows from (7.10) that $\Pi_{n-r}(\lambda_i^{(n-1)})$ have alternating signs for $i = 1,2,\ldots,d-1$. Hence $\Pi_{n,r}(\lambda)$ has at least one zero in each of the intervals $(\lambda_i^{(n-1)},\lambda_{i+1}^{(n-1)})(i = 1,2,\ldots,d-2)$. By inductive hypothesis and the fact that $\Pi_{n-2,r}(0) > 0$, we have $\Pi_{n-2,r}(\lambda_1^{(n-1)}) > 0$. It follows that $\Pi_{n,r}(\lambda_1^{(n-1)}) < 0$. But then $\Pi_{n,r}(0) > 0$. Hence $\Pi_{n,r}(\lambda)$ has a zero in $(0,\lambda_1^{(n-1)})$ and since it is a reciprocal it must also have a zero in $(\lambda_{d-1}^{(n-1)}, \infty)$. The assertion is proved by induction. □

Theorem 7.4. If $r \geq 2$, then for $n \geq 2r+1$, $\Pi^*_{n,r}(\lambda)(\equiv\Pi_{n,r}(\overline{0,r-1};r:\lambda))$ has real simple zeros, one of which is $(-1)^{r-1}$. The remaining $d-1$ zeros are of sign $(-1)^r$ and interlace with the zeros of $\Pi_{n,r}(\lambda)$.

Proof. We shall again assume that r is even, and let us suppose that the zeros of $\Pi_{n-1,r}(\lambda)$ and $\Pi_{n,r}(\lambda)$ are given by

(7.13) $$0 < \lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_{d-1}^{(n-1)} < \lambda_d^{(n)}.$$

We can certainly assume this by theorem 7.3. Now (7.2) of lemma 7.1 implies that

(7.14) $$\Pi_{n,r-1}(\lambda_i^{(n)})\Pi_{n-1,r}(\lambda_i^{(n)}) = \Pi^*_{n,r}(\lambda_i^{(n)})\Pi_{n-1,r-1}(\lambda_i^{(n)}),$$

$(i = 1,2,\ldots,d)$.

Since $\Pi_{n,r-1}(\lambda_i^{(n)})$ and $\Pi_{n-1,r-1}(\lambda_i^{(n)})$ are positive for all $i = 1,\ldots,d$, it follows from (7.14) that

(7.15) $$\text{Sgn}\ \Pi_{n-1,r}(\lambda_i^{(n)}) = \text{Sgn}\ \Pi^*_{n,r}(\lambda_i^{(n)}) .$$

By the same argument as in the proof of theorem 7.3, $\Pi^*_{n,r}(\lambda)$

has exactly one zero in each of the intervals $(\lambda_i^{(n)}, \lambda_{i+1}^{(n)})$, $(i = 1,2,\ldots,d-1)$. It cannot have a zero in $(0,\lambda_1^{(n)})$ or $(\lambda_d^{(n)},\infty)$. For if it has a zero in $(0, \lambda_1^{(n)})$ it must also have a zero in $(\lambda_d^{(n)},\infty)$ and vice versa, which is impossible since it is only of degree d. Since $\Pi^*_{n,r}(\lambda)$ is a reciprocal equation the remaining zero must be -1. The theorem is thus established. □

Theorem 7.5. If $r \geq 2$, then for $n \geq 2r+1$, $\Pi^g_{n,r}(\lambda)$ $(\equiv \Pi^{n-r,1}_{n,r}(\overline{0,r-1};r:\lambda))$ has real zeros two of which are of sign $(-1)^{r-1}$ and $(d-2)$ of sign $(-1)^r$. If n is odd the zeros are simple and are separated by the zeros of $\Pi^*_{n,r}(\lambda)$. If n is even $(-1)^{r-1}$ is a double zero and the remaining $(d-2)$ zeros are simple and interlace with the $(d-1)$ zeros of $\Pi^*_{n,r}(\lambda)$ of sign $(-1)^r$.

Proof. Again, we shall assume that r is even.

(i) n is odd

Let us denote by $\{\mu_i^{(n)}\}_{i=1}^d$, with $\mu_1^{(n)} = 1$, the zeros of $\Pi^*_{n,r}(\lambda)$. By theorem 7.4, we can write

$$(7.16) \qquad \mu_1^{(n)} < \lambda_1^{(n)} < \mu_2^{(n)} < \ldots < \mu_d^{(n)} < \lambda_d^{(n)}.$$

From (7.3) of lemma 7.1, we have

$$(7.17) \qquad \Pi^g_{n,r}(\mu_i^{(n)})\Pi_{n,r}(\mu_i^{(n)}) = -\Pi_{n,r-1}(\mu_i^{(n)})\Pi_{n,r+1}(\mu_i^{(n)})\,(i=1,2,\ldots,d).$$

Since $\mu_i^{(n)} > 0$ for $i = 2,3,\ldots,d$, it follows, using the usual argument, that $\Pi^g_{n,r}(\lambda)$ has at least one zero in each of the intervals $(\mu_i^{(n)},\mu_{i+1}^{(n)})$, $(i = 2,3,\ldots,d-1)$. Thus $\Pi^g_{n,r}(\lambda)$ has at least $(d-2)$ distinct positive zeros.

We shall show that $\Pi^g_{n,r}(\lambda)$ possesses two distinct negative zeros. Let us observe, first of all, that since $\Pi_{n,r+1}(0)$ and $\Pi_{n,r-1}(0)$ are both positive, it follows that $\mathrm{Sgn}\, \Pi_{n,r+1}(-1) = (-1)^{(n-2r-1)/2}$ and $\mathrm{Sgn}\, \Pi_{n,r-1}(-1) = (-1)^{(n-2r+3)/2}$. Hence, from (7.17) and the fact that $\Pi_{n,r}(-1)$ is positive, it follows that $\Pi^g_{n,r}(-1)$ is negative. But then $\Pi^g_{n,r}(0)$ is positive, and therefore $\Pi^g_{n,r}(\lambda)$ must have a zero in $(-1,0)$. Since it is a reciprocal polynomial it must also have a zero in $(-\infty,-1)$. Thus we have shown that $\Pi^g_{n,r}(\lambda)$ has at least two distinct negative zeros and at least $(d-2)$ distinct positive zeros. Since its degree is d we conclude that it has exactly two distinct negative and $(d-2)$ distinct positive zeros which interlace with the zeros of $\Pi^*_{n,r}(\lambda)$.

(ii) <u>n is even</u>

In this case $\Pi_{n,r-1}(-1) = \Pi_{n,r+1}(-1) = \Pi^*_{n,r}(-1) = 0$. Since $\Pi_{n,r}(-1)$ is positive, it follows from (7.3) that (-1) is a zero of $\Pi^g_{n,r}(\lambda)$. By the same argument as case (i) we see that $\Pi^g_{n,r}(\lambda)$ has exactly $(d-2)$ distinct positive zeros which interlace with the $(d-1)$ positive zeros of $\Pi^*_{n,r}(\lambda)$. Since $\Pi^g_{n,r}(\lambda)$ is a reciprocal polynomial the remaining zero must be -1. Thus -1 is a zero of multiplicity 2. Hence theorem 7.5 is established. □

Theorem 3.2 follows easily from theorem 7.5.

8. Some generalisations

In this section we shall discuss some generalisations of the results in §7. More precisely we shall discuss the zeros of the reciprocal polynomials $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda)$ under certain restrictions on the integers k, q and t. These results can be accomplished by using the same techniques employed in §7.

The following theorems generalise theorem 7.3.

Theorem 8.1. If $r \geq 1$, then for $n \geq 2r+1$ the zeros of $\Pi_{n,r}(\overline{0,p};\ \overline{n-r-q+1,q}:\lambda)$ are real, simple, of sign $(-1)^p$ and interlace with the zeros of $\Pi_{n+1,r+1}(\overline{0,p};\ \overline{n-r-q,q+1}:\lambda)$ (respectively $\Pi_{n-1,r}(\overline{0,p};\overline{n-r-q,q}:\lambda)$).

Theorem 8.2 Let p, q, k be integers satisfying (1.1), s,t be integers with $0 \leq t < r < k+q \leq s \leq n-r$ and suppose that q,t are both even. Then the zeros of the polynomials $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda)$, $\Pi^{\overline{s,t}}_{n-1,r}(\overline{0,p};\ \overline{k,q}:\lambda)$ and $\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,p};\ \overline{k-1,q}:\lambda)$ are real, simple and of sign $(-1)^p$. Furthermore the zeros of $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\ \overline{k,q}:\lambda)$ interlace with the zeros of $\Pi^{\overline{s,t}}_{n-1,r}(\overline{0,p};\ \overline{k,q}:\lambda)$ (respectively $\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,p};\ \overline{k-1,q}:\lambda)$).

When q = 0, theorem 8.1 not only reduces to theorem 7.3 but also says that the zeros of $\Pi_{n,r}(\lambda)$ and $\Pi_{n+1,r+1}(\overline{0,r};\ \overline{n-r,1};\lambda)$ are interlacing. Similarly when t = q = 0, theorem 8.2 gives theorem 7.3.

Theorems 8.1 and 8.2 are proved in the same way as theorem 7.3 using respectively the identities

$$\begin{aligned}(8.1)\quad &(n+1-p)\Pi_{n+1,r+1}(\overline{0,p};\overline{n-r-q,q+1}:\lambda)\Pi_{n-1,r}(\overline{0,p};\overline{n-r-q,q}:\lambda)\\ &-(n+1)\Pi_{n,r+1}(\overline{0,p};\overline{n-r-q-1,q+1}:\lambda)\Pi_{n,r}(\overline{0,p};\overline{n-r-q+1,q}:\lambda)\\ &=(r+q+1)\Pi_{n-1,r}(\overline{0,p-1};\overline{n-r-q-1,q+1}:\lambda)\quad \Pi_{n+1,r+1}(\overline{0;p+1};\overline{n-r-q+1,q}:\lambda)\\ &(n \geq 2r+1,\quad r = 1,2,\ldots)\ ,\end{aligned}$$

and

$$(8.2)\qquad (n-p)\overline{\Pi}^{s,t}_{n,r}(\overline{0,p};\overline{k,q}:\lambda)\overline{\Pi}^{s-1,t}_{n-2,r}(\overline{0,p};\overline{k-1,q}:\lambda)$$

$$= n\overline{\Pi}^{s,t}_{n-1,r}(\overline{0,p};\overline{k,q}:\lambda)\overline{\Pi}^{s-1,t}_{n-1,r}(\overline{0,p};\overline{k-1,q}:\lambda)$$

$$-\ (r-t)\overline{\Pi}^{s,t}_{n,r+1}(\overline{0,p+1};\overline{k,q}:\lambda)\overline{\Pi}^{s-1,t}_{n-2,r-1}(\overline{0,p-1};\overline{k-1,q}:\lambda)$$

$$(0 \le t < r < k+q \le s \le n-r).$$

Identities (8.1) and (8.2) can be derived from (6.2).

Theorem 7.4 admits the following generalisation which can be proved in the same way.

Theorem 8.3 Let n, r, s, t be integers with $0 \le t < r-1 \le s-2 \le n-r-2$, and suppose t is even. Then $\overline{\Pi}^{s,t}_{n,r}(\overline{0,r-1};r:\lambda)$ has real simple zeros, one of which is $(-1)^{r-1}$. The remaining (d-1) zeros are of sign $(-1)^r$ and interlace with the zeros of $\overline{\Pi}^{s,t}_{n,r}(\lambda)$.

Using theorem 8.3 we shall prove a more general theorem, viz.

Theorem 8.4 Let n, r, k, s, t be integers with $0 \le t < r-1 \le k \le s-1 \le n-r-1$, and suppose that t is even. Then $\overline{\Pi}^{s,t}_{n,r}(\overline{0,r-1};k:\lambda)$ has real simple zeros, (k-r+1) of which are of sign $(-1)^{r-1}$ and (d-k+r-1) of sign $(-1)^r$. Furthermore the zeros of $\overline{\Pi}^{s,t}_{n,r}(\overline{0,r-1};k:\lambda)$ and $\overline{\Pi}^{s,t}_{n,r}(\overline{0,r-1};k+1:\lambda)$ are separated for k=r-1,r,...,s-2.

Before proving the theorem, we first establish the following

Lemma 8.5 Let n, r, k, s, t be integers such that $0 \le t < r-1 \le k-1 \le s-3 \le n-r-3$. Then

$$(8.3)\quad \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k:\lambda)\,\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};k-2;k:\lambda)$$

$$-\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k+1:\lambda)\,\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{k-2,2}:\lambda)$$

$$=\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k-1:\lambda)\,\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{k-1,2}:\lambda)$$

$$(8.4)\quad \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k:\lambda)\,\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};k-1;k+1:\lambda)$$

$$-\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k+1:\lambda)\,\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{k-1,2}:\lambda)$$

$$=\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};k-1:\lambda)\,\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{k,2}:\lambda)$$

<u>where</u>

$$(8.5)\quad \Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};k-1;k+1:\lambda) =$$

$$P\begin{pmatrix} r-t+1,\ldots,n-s-t,n-s+1,\ldots\ldots, & n & \\ & & :\lambda \\ 0\quad,\ldots,n-k-2,n-k,n-k+2,\ldots,n-r+1 & & \end{pmatrix}.$$

<u>Proof</u>. Consider the following matrix of order $(n-r+1)\times(n-r+2)$

$$
(8.6)\qquad
\left\|\begin{array}{cccccccccc}
1 & \binom{r-t}{1} & \cdots & \binom{r-t}{r-t-1} & (1-\lambda) & 0 & 0 & \cdots & \cdots & 0 \\
1 & \binom{r-t+1}{1} & \cdots & \cdots & \cdots & (1-\lambda) & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & & & \ddots & & & \vdots \\
1 & \binom{n-s-t}{1} & \cdots & \cdots & \cdots & \cdots & (1-\lambda) & 0 \cdots 0 & \cdots 0 & \cdots 0 \\
1 & \binom{n-s+1}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & (1-\lambda) & 0 & \cdots 0 \\
\vdots & \vdots & & & & & & & \ddots & \vdots \\
1 & \binom{n-r+1}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & (1-\lambda) \\
\vdots & \vdots & & & & & & & & \vdots \\
1 & \binom{n}{1} \cdots & \binom{n}{n-k-2} & \binom{n}{n-k-1} & \binom{n}{n-k} & \binom{n}{n-k+1} & \binom{n}{n-k+2} & \cdots & \cdots & \binom{n}{n-r+1}
\end{array}\right\|
$$

$$
\begin{array}{cccccccccc}
\| & \| & \| & \| & \| & \| & \| & & & \| \\
\underset{\sim}{f}^{(1)} & \underset{\sim}{f}^{(2)} \cdots & \underset{\sim}{f}^{(n-k-1)} & \underset{\sim}{a} & \underset{\sim}{b} & \underset{\sim}{c} & \underset{\sim}{f}^{(n-k)} & \cdots & & \underset{\sim}{f}^{(n-r-1)}
\end{array}
$$

Let $\underset{\sim}{f}^{(i)}$ denote the i-th column $(i = 1,2,\ldots,n-k-1)$, $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ denote the $(n-k)$, $(n-k+1)$ and $(n-k+2)$-th columns respectively and $\underset{\sim}{f}^{(\nu-3)}$ denote the ν-th column $(\nu = n-k+3,\ldots,n-r+2)$ of (8.6) as shown above. If we set $\underset{\sim}{d} = (0,\ldots,0,1)^T$ we obtain (8.3) from (6.2), and if we take $\underset{\sim}{d} = (1,0,\ldots,0)^T$ we get (8.4). □

<u>Proof of theorem 8.4</u> The proof follows by induction on k. Observe that since $\Pi_{n,r}^{\overline{s,t}}(\overline{0,r-1};r-1:\lambda) \equiv \Pi_{n,r}^{\overline{s,t}}(\lambda)$, theorems 8.2 and 8.3 assert that theorem 8.4 is true for $k = r-1$ and r.

Let us suppose that the assertion is true for $k = r-1,\ldots,\ell$. W. l. o. g. we may assume that r is even, and let us denote the zeros of $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell:\lambda)$ by $\{\gamma^{(\ell)}_i\}^d_{i=1}$ so that

$$(8.7)\qquad \gamma^{(\ell)}_1 < \gamma^{(\ell-1)}_1 < \gamma^{(\ell)}_2 < \ldots < \gamma^{(\ell-1)}_{\ell-r} < \gamma^{(\ell)}_{\ell-r+1} < 0 < \gamma^{(\ell-1)}_{\ell-r+1}$$
$$< \gamma^{(\ell)}_{\ell-r+2} < \ldots < \gamma^{(\ell)}_d < \gamma^{(\ell-1)}_d .$$

It follows from (8.3) and (8.4) that

$$(8.8)\qquad \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ \ell+1:\ \gamma^{(\ell)}_i)\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{\ell-2,2}:\gamma^{(\ell)}_i)$$
$$= -\Pi^{\overline{s,t}}_{nrr}(\overline{0,r-1};\ell-1:\gamma^{(\ell)}_i)\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{\ell-1,2}:\ \gamma^{(\ell)}_i)$$

and

$$(8.9)\qquad \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\gamma^{(\ell)}_i)\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{\ell-1,2}:\gamma^{(\ell)}_i)$$
$$= -\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell-1:\gamma^{(\ell)}_i)\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{\ell,2}:\gamma^{(\ell)}_i)$$

$$(i = 1,2,\ldots,d\ ,\quad r \le \ell \le j-2)\ .$$

Since $\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{\ell-1,2}:\gamma^{(\ell)}_i)$ and $\Pi^{\overline{s,t}}_{n,r+1}(\overline{0,r-1};\overline{\ell,2}:\gamma^{(\ell)}_i)$ are positive for $i = \ell-r+2,\ldots,d$, and $\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{\ell-2,2}:\gamma^{(\ell)}_i)$ and $\Pi^{\overline{s-1,t}}_{n-1,r}(\overline{0,r-2};\overline{\ell-1,2}:\gamma^{(\ell)}_i)$ are positive for $i = 1,2,\ldots,\ell-r+1$, it follows that

$$(8.10)\qquad \mathrm{Sgn}\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\gamma^{(\ell)}_i) = -\mathrm{Sgn}\ \Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell-1:\gamma^{(\ell)}_i)$$

$$(i = 1,2,\ldots,d).$$

Hence $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\lambda)$ has exactly one zero in each of the intervals $(\gamma^{(\ell)}_{i}, \gamma^{(\ell)}_{i+1})$, $(i = 1,2,\ldots,d-1)$. Clearly (8.7) shows that $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell-1:\gamma^{(\ell)}_{\ell-r+1}) > 0$, since $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell-1:0) > 0$. Hence $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\gamma^{(\ell)}_{\ell-r+1}) < 0$. But then $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:0) > 0$. Hence the zero of $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\lambda)$ between $(\gamma^{(\ell)}_{\ell-r+1}, \gamma^{(\ell)}_{\ell-r+2})$ actually lies in $(\gamma^{(\ell)}_{\ell-r+1}, 0)$. Since $\Pi^{\overline{s,t}}_{n,r}(\overline{0,r-1};\ell+1:\lambda)$ is a reciprocal polynomial it must also have a zero in $(-\infty,\gamma^{(\ell)}_{1})$. The assertion follows by induction.

An analogous method gives the following generalisation of theorem 7.5.

Theorem 8.6 Let $r \geq 2$, and let $n = 2m-1$ be an odd integer such that $m \geq r+1$. Then for each integer k with $r-1 \leq k \leq n-r-1$, $\Pi^{\overline{n-r,1}}_{n,r}(\overline{0,r-1};k:\lambda)$ has real simple zeros, $(k-r+2)$ of which are of sign $(-1)^{r-1}$ and $(d-k+r-2)$ of sign $(-1)^{r}$. Furthermore, the zeros of $\Pi^{\overline{n-r,1}}_{n,r}(\overline{0,r-1};k:\lambda)$ and $\Pi^{\overline{n-r,1}}_{n,r}(\overline{0,r-1};k+1:\lambda)$ are separated for $k = r-1,r,\ldots,n-r-2$.

9. Discussion of the C.L.I.P.

From the results of §8 we see that if $n = 2m-1$ is odd, the zeros of the polynomial $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda)$ are real, simple and not equal to ± 1 for each of the following cases:

(9.1) $t = 0$, $k = n-r-q+1$, $n \geq 2r+1$,

(9.2) t and q are both even and $0 \leq t < r < k+q \leq s \leq n-r+1$, (9.3) t even, $q = 1$ and $0 \le t < r-1 < k \le s-1 \le n-r$ with $(k-r)$ odd,

(9.4) $t = 1$, $q = 1$, $s = n-r$ and $0 < r-1 < k \le n-r-1$ with $(k-r)$ even.

The method of Chapter I leads to the following theorem.

Theorem 9.1 Let p,q satisfy (1.1) and suppose that p, q, k, s, t satisfy one of the conditions (9.1)-(9.4). Then given r bi-infinite sequences (1.5) satisfying (1.6) the C.L.I.P. of type $(\overline{0,p};\overline{k,q})$ has a unique solution $S(x) \in S^{\overline{s,t}}_{2m-1,r}$ such that

(9.5) $S^{(\rho)}(x) = O(|x|^{\gamma}) \quad (\rho \in \overline{0,p};\overline{k,q})$

as $|x| \to \infty$.

If one of the zeros of the characteristic polynomial is 1 (or -1) the corresponding C.L.I.P. does not have a unique solution. Indeed if $S_1(x)$ is an E.S. corresponding to the E.V. 1 (or -1), then $S_1(x) + S(x)$ is a solution of the C.L.I.P. safisfying 9.5, if $S(x)$ is a solution of the same problem satisfying (9.5).

Remark. We are concerned here only with a particular class of C.L.I.P., namely problems of type $(\overline{0,p};\overline{k,q})$ in $S^{\overline{s,t}}_{n,r}$ in which only one gap is allowed in the interpolating conditions. Even in this particular class we do not completely consider all the cases. Conspicuously absent is the case when q and t are both odd. The only problem of this type which we have considered is given by condition (9.4).

CHAPTER III

HANKEL DETERMINANTS OF EULER-FROBENIUS POLYNOMIALS

The characteristic polynomials $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda)$ play a fundamental role in the corresponding C.L.I.P. Since much is known about the Euler-Frobenius polynomial $\Pi_n(\lambda)$, it is natural to search for relations between $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda)$ and $\Pi_n(\lambda)$. It turns out that $\Pi_{n,r}(\lambda)$ is related to a Hankel determinant of $\Pi_n(\lambda)$. We shall also state without proof an expression for the Hankel determinant of exponential Euler polynomials. This relation also occurs in an interpolation problem with cardinal splines, but we shall not dwell upon it here.

10. A relation between $\Pi_{n,r}(\lambda)$ and $\Pi_n(\lambda)$.

Let us denote by $H_{r+1}(a_n)$ the Hankel determinant of order $(r+1) \times (r+1)$ given by

$$(10.1) \qquad H_{r+1}(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r-1} \\ \vdots & \vdots & & \vdots \\ a_{n-r} & a_{n-r-1} & \cdots & a_{n-2r} \end{vmatrix}$$

Determinants of this type in which the entries a_n are orthogonal polynomials have been studied by several mathematicians (see [1] and [3]). If $a_n = \Pi_n(\lambda)/n!$ we have the following relations.

<u>Theorem 10.1. Let n and r be non-negative integers such that $n \geq 2r+1$. Then</u>

$$(10.2)\quad H_{r+1}\left(\frac{\Pi_n(\lambda)}{n!}\right) = \frac{(-1)^{[\frac{r+1}{2}]}\, 2!3!\ldots r!(1-\lambda)^{r(n-r)}\Pi_{n,r+1}(\lambda)}{n!(n-1)!\ldots(n-r)!} .$$

The proof of theorem 10.1 depends on the following lemma, which is a particular case of a more general identity on symmetric determinants (see [9], p. 372).

<u>Lemma 10.2. For $n \geq 2r+1$ we have</u>

$$(10.3)\qquad H_r(a_n)\, H_r(a_{n-2}) - [H_r(a_{n-1})]^2 = H_{r-1}(a_{n-2})H_{r+1}(a_n) .$$

<u>Proof of Lemma.</u> The lemma is easily established using the determinantal identity (6.2) on the following vectors: c and d are the first and last columns of (10.1) respectively, $\underset{\sim}{f}^{(\nu)}$ the $(\nu+1)$-th column $(\nu = 1,2,\ldots,r-1)$ and $\underset{\sim}{a} = (1,0,\ldots,0)^T$, $\underset{\sim}{b} = (0,0,\ldots,0,1)^T$.

<u>Proof of Theorem 10.1.</u> The proof will be carried out by induction on r. First of all, let us observe that

$$H_1\left(\frac{\Pi_n(\lambda)}{n!}\right) = \frac{\Pi_n(\lambda)}{n!} \quad \text{for all } n \geq 1$$

and by (7.1) of lemma 7.1, with r = 2, we have

$$H_2\left(\frac{\Pi_n(\lambda)}{n!}\right) = \frac{(-1)(1-\lambda)^{n-1}\Pi_{n,2}(\lambda)}{n!(n-1)!} \quad \text{for all } n \geq 3 .$$

Let us suppose that (10.2) holds for $H_k(\Pi_n(\lambda)/n!)(k = 0,1,\ldots,r)$ and we shall prove that (10.2) also holds for $H_{r+1}(\Pi_n(\lambda)/n!)$.

Using (10.3) after some calculations, we have, for $n \geq 2r+1$,

$$(10.4)\quad H_{r+1}\left(\frac{\Pi_n(\lambda)}{n!}\right)(-1)^{[\frac{r-1}{2}]}\Pi_{n-2,r-1}(\lambda)$$

$$= -\frac{2!3!\ldots(r-2)!(r-1)!(1-\lambda)^{r(n-r)}}{n!(n-1)!\ldots(n-r)!}\{n(\Pi_{n-1,r}(\lambda))^2$$

$$-(n-r)\Pi_{n,r}(\lambda)_{n-2,r}(\lambda)\},$$

from which we obtain (10.2), with the help of (7.1). □

11. Some relations between $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q};\lambda)$ and $\Pi_n(\lambda)$.

Relations similar to (10.2) involving other polynomials $\Pi^{\overline{s,t}}_{n,r}(\overline{0,p};\overline{k,q}:\lambda)$ can be derived by the same technique. Set

$$(11.1)\quad H_{r+1}(\overline{0,r};k:a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} & a_{n-r-k} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} & a_{n-r-k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-r} & a_{n-r-1} & \cdots & a_{n-2r+1} & a_{n-2r-k} \end{vmatrix}$$

and

$$(11.2)\quad H^g_{r+1}(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} & a_{n-r-1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} & a_{n-r-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-r+1} & a_{n-r} & \cdots & a_{n-2r+2} & a_{n-2r} \\ a_{n-r-1} & a_{n-r-2} & \cdots & a_{n-2r} & a_{n-2r-2} \end{vmatrix},$$

then using a method similar to that of Theorem 10.1 we have

Theorem 11.1. Let n,r be positive integers. Then

$$(11.3)\qquad H_{r+1}(\overline{0,r};k:\frac{\Pi_n(\lambda)}{n!}) = \frac{(-1)^{[\frac{r+1}{2}]}\,1!2!\ldots r(1-\lambda)^{r(n-r)-k}}{n!(n-1)!\ldots(n-r+1)!(n-r-k)!}\Pi_{n,r+1}(\overline{0,r};r+k:\lambda)$$

$$(n \geq 2r+k+1,\ k \geq 0)$$

$$(11.4)\qquad H^g_{r+1}(\frac{\Pi_n(\lambda)}{n!}) = \frac{(-1)^{[\frac{r+1}{2}]}\,1!2!\ldots(r-1)!(r+1)!(1-\lambda)^{r(n-r)-2}}{n!(n-1)!\ldots(n-r+1)!(n-r-1)!} \times$$

$$\times\ \Pi^g_{n,r+1}(\lambda)\quad (n \geq 2r+3)\ .$$

where the polynomials $\Pi^g_{n,r+1}(\lambda)$ are given by (3.8) and the polynomials $\Pi_{n,r+1}(\overline{0,r},r+k:\lambda)$ are given by (5.3) and (5.3a).

12. Hankel determinant of exponential Euler polynomials.

Using the same technique, a more general form of the relation (10.2) can be obtained in another direction. More precisely, if $A_n(x;\lambda)$ are the exponential Euler polynomials (see §13), then for $r = 0,1,2,\ldots,$ $n \geq 2r+1$,

$$(12.1)\qquad H_{r+1}(\frac{A_n(x;\lambda)}{n!}) = \frac{(-1)^{[\frac{r+1}{2}]+(r+1)n}\,1!2!\ldots r!}{n!(n-1)!\ldots(n-r)!} \times \left\{\frac{\Delta_{n,r+1}(x;\lambda)}{(1-\lambda)^{n-r}}\right\},$$

where

$$\Delta_{n,r}(x;\lambda) = \begin{vmatrix} 1 & \binom{r}{1}\cdots\cdots\binom{r}{r-1}(1-\lambda) & 0 & 0\cdots\cdots\cdots 0 \\ 1 & \binom{r+1}{1}\cdots\cdots\cdots\cdots\binom{r+1}{r} & (1-\lambda) & 0\cdots\cdots\cdots 0 \\ \vdots & & & \ddots \qquad \vdots \\ 1 & \binom{n}{1}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \binom{n}{n-1}\ (1-\lambda) \\ x^{n-r+1} & \binom{n-r+1}{1}x^{n-r}\cdots\cdots\cdots & 1 & 0\cdots\cdots\cdots\cdots 0 \\ x^{n-r+2} & \binom{n-r+2}{1}x^{n-r+1}\cdots\cdots\cdots\cdots & 1 & 0\cdots\cdots\cdots 0 \\ \vdots & \vdots & & \vdots \\ x^{n} & \binom{n}{1}\ x^{n-1}\cdots\cdots\cdots\cdots\cdots & & \binom{n}{n-1}x\quad 1 \end{vmatrix} \tag{12.2}$$

In view of the relation $A_n(0;\lambda) = \Pi_n(\lambda)/(\lambda-1)^n$ (see §13), it is clear that (10.2) is a particular case of (12.1) when $x = 0$. A full discussion of the relation (12.1) and the corresponding interpolation problems will be discussed elsewhere in a joint paper with A. Sharma and J. Tzimbalario.

CHAPTER IV

EXPONENTIAL HERMITE EULER SPLINES

In a very interesting paper [14] Schoenberg studied the Cardinal splines that interpolate the functions λ^x at the integers, where λ is a complex number (see also [15]). He called them the exponential Euler splines, because they are "periodic extensions" of the exponential Euler polynomials $A_n(x;\lambda)$ generated by the relations

$$\frac{\lambda-1}{\lambda-e^z} e^{xz} = \sum_{n=0}^{\infty} \frac{A_n(x;\lambda)}{n!} z^n .$$

This chapter deals with Cardinal splines that interpolate the function λ^x as well as its consecutive derivatives. The problem is to find a spline $S_{n,r}(x) \in S_{n,r}$ such that $S_{n,r}^{(\rho)}(\nu) = (\log \lambda)^{\rho} \lambda^{\nu}$ for all integers ν $(\rho = 0,1,\ldots,r-1)$.

13. The polynomial $A_{n,r,s}(x;\lambda)$.

Let n,r be positive integers such that $n \geq 2r-1$ and let $s = 0,1,\ldots,r-1$ be a fixed integer. Let us define the polynomials $A_{n,r,s}(x;\lambda)$ for $0 \leq x \leq 1$ as follows:

(13.1) $(-1)^s A_{n,r,s}(x;\lambda) =$

$$\begin{vmatrix} \frac{A_n(x;\lambda)}{n!} & \frac{A_n(0;\lambda)}{n!} & \cdots & \frac{A_{n-s+1}(0;\lambda)}{(n-s+1)!} & \frac{A_{n-s-1}(0;\lambda)}{(n-s-1)!}, & \cdots & \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} \\ \frac{A_{n-1}(x;\lambda)}{(n-1)!} & \frac{A_{n-1}(0;\lambda)}{(n-1)!} & \cdots & \frac{A_{n-s}(0;\lambda)}{(n-s)!} & \frac{A_{n-s-2}(0;\lambda)}{(n-s-2)!} & \cdots & \frac{A_{n-r}(0;\lambda)}{(n-r)!} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{A_{n-r+1}(x;\lambda)}{(n-r+1)!} & \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} & \cdots & \frac{A_{n-r-s+2}(0;\lambda)}{(n-r-s+2)!} & \frac{A_{n-r-s}(0;\lambda)}{(n-r-s)!} & \cdots & \frac{A_{n-2r+2}(0;\lambda)}{(n-2r+2)!} \end{vmatrix}$$

where the exponential Euler polynomials $A_n(x;\lambda) = x^n + \binom{n}{1}a_1(\lambda)x^{n-1} + \cdots + a_n(\lambda)$, and $A_n(0;\lambda)/n! = a_n(\lambda)/n! = \Pi_n(\lambda)/n!(\lambda-1)^n$ (see [14]). It follows from (10.2) and (13.1) that

$$(13.2)\quad \begin{cases} A^{(s)}_{n,r,s}(0;\lambda) = H_r\left(\frac{A_n(0;\lambda)}{n!}\right) = \\ (-1)^{[\frac{r}{2}]+(r-1)(n-r+1)} \times \frac{1!2!\cdots\cdots(r-1)!\,\Pi_{n,r}(\lambda)}{n!(n-1)!\cdots(n-r+1)!(\lambda-1)^{n-r+1}} . \end{cases}$$

Using the relations $A_n'(x;\lambda)/n! = A_{n-1}(x;\lambda)/(n-1)!$, and $A_n^{(\rho)}(1;\lambda) = \lambda A_n^{(\rho)}(0;\lambda)$ $(\rho = 0,1,\ldots,n-1)$, it is easy to check that $A_{n,r,s}(x;\lambda)$ satisfy the relations

$$(13.3)\quad A^{(\rho)}_{n,r,s}(1;\lambda) = \lambda A^{(\rho)}_{n,r,s}(0;\lambda) \quad (\rho = 0,1,\ldots,n-r),$$

$$(13.4)\quad \begin{cases} A^{(\rho)}_{n,r,s}(1;\lambda) = A^{(\rho)}_{n,r,s}(0;\lambda) = 0 \ (\rho = 0,1,\ldots,r-1, \rho \neq s), \text{ and} \\ A^{(s)}_{n,r,s}(0;\lambda)/H_r\left(\frac{A_n(0;\lambda)}{n!}\right) = 1, \end{cases}$$

provided $\lambda \neq 1$ and λ is not a zero of $\Pi_{n,r}(\lambda)$, an assumption which we shall impose throughout this chapter.

Observe that when $r = 1$ (in which case s can take only the value zero), $A_{n,1,0}(x;\lambda) = A_n(x;\lambda)$.

14. The spline $S_{n,r,s}(x;\lambda)$.

Let us define a function $S_{n,r,s}(x;\lambda)$ such that

$$(14.1)\quad S_{n,r,s}(x;\lambda) = A_{n,r,s}(x;\lambda)/H_r\left(\frac{A_n(0;\lambda)}{n!}\right) \ (0 \leq x \leq 1), \text{ and}$$

$S_{n,r,s}(x+1;\lambda) = \lambda S_{n,r,s}(x;\lambda)$ for all real x.

It follows from (13.3) and (13.4) that $S_{n,r,s}(x;\lambda) \in C^{n-r}(-\infty,\infty)$, and

$$(14.2)\quad \begin{cases} S^{(\rho)}_{n,r,s}(\nu:\lambda) = 0 \ (\rho = 0,1,\ldots,r-1, \rho \neq s) \text{ and} \\ S^{(s)}_{n,r,s}(\nu:\lambda) = \lambda^{\nu} \quad (\nu = 0,\pm1,\pm2,\ldots). \end{cases}$$

so that it is a Cardinal spline with integer knots of multiplicity r, and belonging to the class $\mathcal{S}^{(s)}_{n,r} = \{S(x) \in S_{n,r}: \ S^{(\rho)}(\nu) = 0 \ \forall$ integer ν, $\rho = 0,1,\ldots,r-1;\ \rho \neq s\}$.

Now let us set

$$(14.3)\quad S_{n,r}(x,\lambda) = \sum_{s=0}^{r-1} (\log \lambda)^s \, S_{n,r,s}(x;\lambda) \quad (x \in R).$$

The following theorem is an easy consequence of (14.2).

Theorem 14.1. *The splines $S_{n,r}(x;\lambda) \in S_{n,r}$ and satisfy the following intepolatory conditions:*

(14.4) $$S^{(\rho)}_{n,r}(\nu;\lambda) = (\log \lambda)^{\rho} \lambda^{\nu} \quad (\rho = 0,1,\ldots,r-1, \text{ and } \forall \text{ integers } \nu).$$

15. Convergence of exponential Hermite-Euler splines when $r = 2$.

When $r = 1$, $S_{n,1}(x;\lambda) \equiv S_n(x;\lambda)$ are the exponential Euler splines considered by Schoenberg [14] who proved that $\lim_{n\to\infty} S_n(x;\lambda) = \lambda^x$ uniformly for all x belonging to a finite interval, when λ is a non-negative complex number. When $r = 2$ we have the following

Theorem 15.1. *If λ is not a positive complex number, then*

(15.1) $$\lim_{n\to\infty} S^{(\rho)}_{n,2}(x;\lambda) = (\log \lambda)^{\rho} \lambda^{x} \quad (\rho = 0,1)$$

uniformly for x belonging to a finite interval.

We shall show that (15.1) is an easy consequence of the corresponding results on $S_{n,2,s}(x;\lambda)$. Let us define two functions

(15.2) $$\alpha(x) = \lambda^{x} e^{-\pi i x} \frac{\sin \pi x}{\pi},$$

(15.3) $$\beta(x) = \lambda^{x} - (\log \lambda)\, \alpha(x).$$

Theorem 15.1 clearly follows from the following

Theorem 15.2. *Let $\lambda = |\lambda| e^{i\alpha}$. If $0 < \alpha < 2\pi$, the following relations hold uniformly for all x belonging to a finite interval:*

(15.4) $$\lim_{n\to\infty} S^{(\rho)}_{n,2,0}(x;\lambda) = \beta^{(\rho)}(x) \quad (\rho = 0,1),$$

and

(15.5) $$\lim_{n\to\infty} S^{(\rho)}_{n,2,1}(x;\lambda) = \alpha^{(\rho)}(x) \quad (\rho = 0,1).$$

Lemma 15.3. Let $\lambda = |\lambda|\, e^{i\alpha}$ ~~and~~ $\lambda_k = \log|\lambda| + i(\alpha+2\pi k)$. The following relations hold uniformly for all x in $[0,1]$:

(15.6) $$\lim_{n\to\infty} \lambda_o^{n+1} \frac{A_n(x;\lambda)}{n!} = (\lambda-1)\lambda^{-1}\lambda^x \quad (-\pi < \alpha \le \pi),$$

(15.7) $$\lim_{n\to\infty} \lambda_1^{n+1} \left\{\frac{A_{n-1}(x;\lambda)}{(n-1)!} - \frac{\lambda_o A_n(x;\lambda)}{n!}\right\} = (\lambda-1)\lambda^{-1}\lambda^x e^{2\pi i x}(2\pi i)$$

$$(-\pi < \alpha < 0)$$

(15.8) $$\lim_{n\to\infty} \lambda_{-1}^{n+1} \left\{\frac{A_{n-1}(x;\lambda)}{(n-1)!} - \frac{\lambda_o A_n(x;\lambda)}{n!}\right\} = (\lambda-1)\lambda^{-1}\lambda^x e^{-2\pi i x}(-2\pi i)$$

$$(0 < \alpha \le \pi).$$

Proof. Using the expansion (see [14] p. 399)

(15.9) $$\frac{A_n(x;\lambda)}{n!} = (\lambda-1)\lambda^{-1}\lambda^x \sum_{k=-\infty}^{\infty} e^{2\pi k i x}/\lambda_k^{n+1} \quad (0 \le x \le 1),$$

we have

(15.10) $$\lambda_o^{n+1}\frac{A_n(x;\lambda)}{n!} = (\lambda-1)\lambda^{-1}\lambda^x \sum_k e^{2\pi k i x}(\lambda_o/\lambda_k)^{n+1}.$$

Since $|\lambda_o| < |\lambda_k|$ $\forall\, k \ne 0$, (15.6) follows from (15.10).

Also from (15.9) we have

(15.11) $$\lambda_1^{n+1}\left\{\frac{A_{n-1}(x;\lambda)}{(n-1)!} - \frac{\lambda_o A_n(x;\lambda)}{n!}\right\} = (\lambda-1)\lambda^{-1}\lambda^x \times$$
$$\times \sum_{k\ne 0} (\lambda_k - \lambda_o)\left(\frac{\lambda_1}{\lambda_k}\right)^{n+1} e^{2\pi k i x}.$$

If $-\pi < \alpha < 0$, $|\lambda_1| < |\lambda_k|$ $\forall\, k \ne 0,1$, and (15.7) follows from (15.11). The limit (15.8) is proved in the same way. □

Proof of Theorem 15.2. We shall prove only the relation

(15.12) $$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \beta(x).$$

The rest are proved in the same way.

We can write $\lambda_o^{n+1}\lambda_{-1}^{n+1}A_{n,2,0}(x;\lambda) =$

$$= \begin{vmatrix} \lambda_o^{n+1}\frac{A_n(x;\lambda)}{n!} & \lambda_o^{n+1}\frac{A_{n-1}(0;\lambda)}{(n-1)!} \\ \lambda_{-1}^{n+1}\{\frac{A_{n-1}(x;\lambda)}{(n-1)!} - \frac{\lambda_o A_n(x;\lambda)}{n!}\} & \lambda_{-1}^{n+1}\{\frac{A_{n-2}(0;\lambda)}{(n-2)!} - \frac{\lambda_o A_{n-1}(0;\lambda)}{(n-1)!}\} \end{vmatrix}$$

If $0 < \alpha \leq \pi$, it then follows from (15.6) and (15.8) that

$$(15.13) \quad \lambda_o^{n+1}\lambda_{-1}^{n+1}A_{n,2,0}(x;\lambda) \to (\lambda-1)^2\lambda^{-2}\lambda^x(\lambda_{-1}-\lambda_o)\,(\lambda_{-1}-\lambda_o e^{-2\pi ix}).$$

Hence

$$(15.14)\begin{cases} \lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \dfrac{\lambda^x(\lambda_{-1}-\lambda_o e^{-2\pi ix})}{(\lambda_{-1}-\lambda_o)} = \\ = \lambda^x\{1 - \dfrac{(\log\lambda)(1-e^{-2\pi ix})}{2\pi i}\} \quad (0 < \alpha \leq \pi). \end{cases}$$

Similarly

$$(15.15)\begin{cases} \lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x\{1 + \dfrac{(\log\lambda)(1-e^{2\pi ix})}{2\pi i}\} = \\ = \lambda^x e^{2\pi ix}\{1 - \dfrac{(\log\lambda + 2\pi i)(1 - e^{-2\pi ix})}{2\pi i}\} \quad (-\pi < \alpha < 0). \end{cases}$$

Combining (15.14) and (15.15) we obtain

$$(15.16) \quad \lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x\{1 - \frac{(\log\lambda)(1-e^{-2\pi ix})}{2\pi i}\}$$

when $\lambda = |\lambda|e^{i\alpha}$ for $0 < \alpha < 2\pi$, from which (15.12) follows. □

Remarks. For the convergence of the exponential Hermite-Euler splines we deal only with the case $r = 2$. The general results can perhaps be obtained by the same method.

16. Exponential Euler g-splines.

The techniques of the preceding sections can be applied without difficulty to study C.L.I.P. for the function λ^x. For instance, let us consider the problem of finding a spline $S^g_{n,2}(x) \in S^g_{n,2}$ such that $S^{g(\rho)}_{n,2}(\nu) = (\log \lambda)^\rho \lambda^\nu$ ($\rho = 0,2$, and $\forall$ integers ν).

In this case we introduce the polynomials

$$(16.1)\qquad \begin{cases} A^g_{n,2,0}(x;\lambda) = \begin{vmatrix} \dfrac{A_n(x;\lambda)}{n!} & \dfrac{A_{n-2}(0;\lambda)}{(n-2)!} \\[2ex] \dfrac{A_{n-2}(x;\lambda)}{(n-2)!} & \dfrac{A_{n-4}(0;\lambda)}{(n-4)!} \end{vmatrix}, \\[6ex] A^g_{n,2,2}(x;\lambda) = -\begin{vmatrix} \dfrac{A_n(x;\lambda)}{n!} & \dfrac{A_n(0;\lambda)}{n!} \\[2ex] \dfrac{A_{n-2}(x;\lambda)}{(n-2)!} & \dfrac{A_{n-2}(0;\lambda)}{(n-2)!} \end{vmatrix}, \end{cases}$$

and define the splines

$$(16.2)\qquad S^g_{n,2,s}(x;\lambda) = A^g_{n,2,s}(x;\lambda)/H^g_2\left(\frac{A_n(0;\lambda)}{n!}\right)$$

for $x \in [0,1]$, and $S^g_{n,2,s}(x+1;\lambda) = \lambda S^g_{n,2,s}(x;\lambda)$ for all $x \in R$ ($s=0,2$). Then $S^g_{n,2,s}(x;\lambda) \in S^g_{n,r}$ ($s = 0,2$), and we have

<u>Theorem 16.1</u>. <u>The spline functions</u>

$$(16.3)\qquad S^g_{n,2}(x;\lambda) = S^g_{n,2,0}(x;\lambda) + (\log \lambda)^2\, S^g_{n,2,2}(x;\lambda)$$

belong to $S^g_{n,2}$ and satisfy the following interpolating conditions:

(16.4) $$S^{g(\rho)}_{n,2}(\nu;\lambda) = (\log\lambda)^\rho \lambda^\nu \quad (\rho = 0,2, \text{ and } \forall \text{ integers } \nu).$$

CHAPTER V

FOURIER TRANSFORMS OF B-SPLINES AND FUNDAMENTAL SPLINES

Fourier transforms of B-splines and fundamental splines provide a powerful tool in the C.I.P. (see [7], [10], [12], [15]). As an application of the exponential Hermite-Euler splines, we shall compute the Fourier transforms of B-splines and fundamental splines belonging to $S_{n,2}$ for (0,1) interpolation. The same method can be applied, with suitable modifications to the general C.I.P.

17. Fourier transforms of $L_{2m-1,2,s}(x)$.

Let us recall that if a function f(x) satisfies the condition

(17.1) $f^{(\rho)}(x) = O(|x|^{\gamma})(\rho = 0,1)$ as $|x| \to \infty$, $\gamma > 0$,

then there exists a unique spline $S_{2m-1}(x) \in S_{2m-1,2}$ such that

(17.2) $S^{(\rho)}_{2m-1}(\nu) = f^{(\rho)}(\nu)$ $(\rho = 0,1)$ $\forall$ integers ν

and $S_{2m-1}(x)$ is given by the Hermite interpolation formula

(17.3) $$S_{2m-1}(x) = \sum_{\nu=-\infty}^{\infty} f(\nu)\, L_{2m-1,2,0}(x-\nu) + \sum_{\nu=-\infty}^{\infty} f'(\nu)\, L_{2m-1,2,1}(x-\nu) ,$$

where $L_{2m-1,2,s}(x)$ $(s = 0,1)$ is uniquely determined by the conditions

(17.4) $L^{(\rho)}_{2m-1,2,s}(\nu) = 0$ $\forall$ integers $\nu \neq 0$ $(\rho = 0,1)$,

(17.5) $L^{(\rho)}_{2m-1,2,s}(0) = 0$ $(\rho \neq s)$ and $L^{(s)}_{2m-1,2,s}(0) = 1$.

<u>Theorem 17.1. The Fourier integral representations of the fundamental splines are given by</u>

$$(17.6)\quad \begin{cases} L_{2m-1,2,0}(x) = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\frac{\left[\left(\frac{2\sin u/2}{u}\right)^{2m}\rho_{2m-2}(u) - \left(\frac{2\sin u/2}{u}\right)^{2m-1}\rho_{2m-1}(u)\right]}{H_2(\rho_{2m}(u))}\,du, \end{cases}$$

$$(17.7)\quad \begin{cases} L_{2m-1,2,1}(x) = \\ = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{iux}\frac{\left[\left(\frac{2\sin u/2}{u}\right)^{2m}\rho_{2m-1}(u) - \left(\frac{2\sin u/2}{u}\right)^{2m-1}\rho_{2m}(u)\right]}{(2\sin u/2)H_2(\rho_{2m}(u))}\,du, \end{cases}$$

<u>where (using the notation of Schoenberg [12], p. 178)</u>

$$(17.8)\quad \rho_n(u) = (2\sin u/2)^n \sum_{j=-\infty}^{\infty} 1/(u+2\pi j)^n .$$

<u>Proof</u>. Let $S_{2m-1,s}(x;e^{iu})$ $(s = 0,1)$ be the exponential Hermite-Euler splines of Chapter IV with $\lambda = e^{iu}$ $(0 < u < 2\pi)$. Following Schoenberg [16], it is easy to show that the functions

$$(17.9)\quad \frac{1}{2\pi}\int_0^{2\pi} S_{2m-1,2,s}(x;e^{iu})\,du \quad (s = 0,1)$$

are spline functions belonging to $S_{2m-1,2}$. Furthermore the functions (17.9) satisfy conditions (17.4) and (17.5) in view of the interpolating conditions $S^{(s)}_{2m-1,2,s}(\nu;e^{iu}) = e^{i\nu u}$ $(s = 0,1, \nu = 0,\pm1,\pm2 \ldots)$, and $S^{(\rho)}_{2m-1,2,s}(\nu;e^{iu}) = 0$ $(\rho \neq s,\ \nu = 0,\pm1,\pm2,\ldots)$. Hence from uniqueness

(17.10) $$L_{2m-1,2,s}(x) = \frac{1}{2\pi}\int_0^{2\pi} S_{2m-1,2,s}(x;e^{iu})\,du \quad (s = 0,1).$$

From (14.1) we have for $0 < x < 1$

(17.11) $$S_{2m-1,2,0}(x;e^{iu}) = \frac{A_{2m-1,2,0}(x;e^{iu})}{A_{2m-1,2,0}(0;e^{iu})}.$$

Using (13.1) and (15.9) we can write

(17.12) $$e^{-iux} S_{2m-1,2,0}(x;e^{iu}) = \frac{\Delta_{2m-1}(x;u)}{\Delta_{2m-1}(0;u)}$$

where

(17.13) $$\Delta_{2m-1}(x;u) = \begin{vmatrix} \sum_{k=-\infty}^{\infty} e^{2\pi kix}/(u+2\pi k)^{2m} & \sum_{j=-\infty}^{\infty} 1/(u+2\pi j)^{2m-1} \\ \sum_{k=-\infty}^{\infty} e^{2\pi kix}/(u+2\pi k)^{2m-1} & \sum_{j=-\infty}^{\infty} 1/(u+2\pi j)^{2m-2} \end{vmatrix}$$

Clearly $\Delta_{2m-1}(x;u)$ is a 2π-periodic function of u. From (17.10), (17.12) and (17.13), on interchanging the order of integration and summation (can be justified by Fubini theorem) we obtain

(17.14) $$L_{2m-1,2,0}(x) =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} \frac{e^{i(u+2\pi k)x}}{\Delta_{2m-1}(0;u)} \begin{vmatrix} 1/(u+2\pi k)^{2m} & \sum_{j=-\infty}^{\infty} 1/(u+2\pi j)^{2m-1} \\ 1/(u+2\pi k)^{2m-1} & \sum_{j=-\infty}^{\infty} 1/(u+2\pi j)^{2m-2} \end{vmatrix} du;$$

Since $\Delta_{2m-1}(0;u)$ is a 2π-periodic function of u, by a change of variable, we obtain

(17.15) $L_{2m-1,2,0}(x) =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iux}\left[\frac{1}{u^{2m}}\sum_{j=-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-2}} - \frac{1}{u^{2m-1}}\sum_{j=-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-1}}\right]}{\Delta_{2m-1}(0;u)}\,du \quad .$$

Mutatis mutandis, we have

(17.16) $L_{2m-1,2,1}(x) =$

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iux}\left[\frac{1}{u^{2m}}\sum_{j=-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-1}} - \frac{1}{u^{2m-1}}\sum_{j=-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m}}\right]}{\Delta_{2m-1}(0;u)}\,du \quad .$$

Straightforward calculation using (17.8) shows that

(17.17) $$\Delta_{2m-1}(0;u) = (2\sin {}^{u}/2)^{-4m+2}\left[\rho_{2m}(u)\,\rho_{2m-2}(u) - \rho^{2}_{2m-1}(u)\right]$$

$$= (2\sin {}^{u}/2)^{-4m+2}\, H_2(\rho_{2m}(u)) \;,$$

and (17.6), (17.7) follow from (17.15) and (17.16) respectively. □

18. Fourier Transforms of B-splines.

The B-splines for C.H.I.P. have been studied by I. J. Schoenberg and A. Sharma [17]. For the case when $r = 2$ these B-splines $N_s(x)$ $(s = 0,1)$ are defined by

(18.1) $$N_s(x) = \sum_{\nu=-(m-2)}^{(m-2)} c_\nu\, L_{2m-1,2,s}(x-\nu) \,,$$

where c_ν are the coefficients of the Euler-Frobenius polynomials $\Pi_{2m-1,2}(\lambda)$ given by

(18.2) $$\Pi_{2m-1,2}(\lambda) = c_{-(m-2)} + c_{-(m-3)}\lambda + \ldots + c_o\lambda^{m-2} + \ldots + c_{(m-2)}\lambda^{2m-4} \quad (c_o > 0) .$$

For the B-splines $N_s(x)$ $(s = 0,1)$ we have the following representation theorem.

<u>Theorem 18.1.</u> <u>The Fourier integral representations of the B-splines are given by</u>

(18.3) $$N_o(x) = \frac{(2m-1)!(2m-2)!(-1)^m}{2\pi} \times$$

$$\times \int_{-\infty}^{\infty} e^{iux}\left[\frac{(2 \sin {}^u/2)^2}{u^{2m}} \rho_{2m-2}(u) - \frac{(2 \sin {}^u/2)}{u^{2m-1}} \rho_{2m-1}(u)\right] du.$$

(18.4) $$N_1(x) = \frac{(2m-1)!(2m-2)!(-1)^m i}{2\pi} \times$$

$$\times \int_{-\infty}^{\infty} e^{iux}\left[\frac{(2 \sin {}^u/2)}{u^{2m}} \rho_{2m-1}(u) - \frac{1}{u^{2m-1}} \rho_{2m}(u)\right] du.$$

<u>Proof.</u> Using (17.15) and (18.1) we have

(18.5) $$N_o(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\left(\sum_{-(m-2)}^{(m-2)} c_\nu e^{-i\nu u}\right) \times$$

$$\times \frac{\left[\frac{1}{u^{2m}} \sum_{-\infty}^{\infty} \frac{1}{(u+2\pi j)^{2m-2}} - \frac{1}{u^{2m-1}} \sum_{-\infty}^{\infty} \frac{1}{(u+2\pi j)^{2m-1}}\right]}{\Delta_{2m-1}(0;u)} du ,$$

whence

$$(18.6)\qquad N_o(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\, e^{-(m-2)iu}\, \Pi_{2m-1,2}(e^{iu}) \times$$

$$\times \frac{\left[\frac{1}{u^{2m}}\sum_{-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-2}} - \frac{1}{u^{2m-1}}\sum_{-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-1}}\right]}{\Delta_{2m-1}(0;u)}\, du\ .$$

From (10.2) and (15.9) we have after some easy calculations

$$(18.7)\qquad \begin{cases} \Pi_{2m-1,2}(e^{iu}) = -\,(2m-1)!\,(2m-2)!\,(e^{iu}-1)^{2m-2}\, H_2\left(\dfrac{A_{2m-1}(0;e^{iu})}{(2m-1)!}\right) = \\ \qquad = (2m-1)!\,(2m-2)!\,\dfrac{(e^{iu}-1)^{2m}}{e^{2iu}}\,\Delta_{2m-1}(0;u)\ . \end{cases}$$

It follows from (18.6) and (18.7), after some calculations, that

$$(18.8)\qquad N_o(x) = \frac{(2m-1)!\,(2m-2)!\,(-1)^m}{2\pi} \times$$

$$\times \int_{-\infty}^{\infty} e^{iux}\,(2\sin {}^{u}/2)^{2m}\left[\frac{1}{u^{2m}}\sum_{-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-2}} - \frac{1}{u^{2m-1}}\sum_{-\infty}^{\infty}\frac{1}{(u+2\pi j)^{2m-1}}\right] du,$$

from which we obtain (18.3).

Similarly starting with (17.16), we prove (18.4). □

Remark. The Fourier transforms of the fundamental splines for C.L.I.P. can also be computed by the same method. For instance if we denote by $L^g_{2m-1,2,s}(s)$ $(s = 0,2)$ the fundamental splines of (0,2) interpolation by cardinal g-splines (r = 2 ; §2), then using the exponential Euler g-splines of §16 we obtain the following representations.

(18.9) $$L^g_{2m-1,2,0}(x) =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \frac{\left[\left(\frac{2 \sin {}^u/_2}{u}\right)^{2m} \rho_{2m-4}(u) - \left(\frac{2 \sin {}^u/_2}{u}\right)^{2m-2} \rho_{2m-2}(u)\right]}{\left[\rho_{2m}(u)\rho_{2m-4}(u) - [\rho_{2m-2}(u)]^2\right]} \, du$$

and

(18.10) $$L^g_{2m-1,2,2}(x) =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \frac{\left[\left(\frac{2 \sin {}^u/_2}{u}\right)^{2m} \rho_{2m-2}(u) - \left(\frac{2 \sin {}^u/_2}{u}\right)^{2m-2} \rho_{2m}(u)\right]}{(2 \sin {}^u/_2)^2\left[\rho_{2m}(u)\rho_{2m-4}(u) - [\rho_{2m-2}(u)]^2\right]} \, du.$$

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Appendix I

TABLE OF EXAMPLES

Type of C.L.I.P.	Null Spaces	Characteristic Polynomials
(0,2) I.P. in $S^g_{5,2}$	$\overset{\circ}{S}{}^g_{5,2}$	$\Pi^g_{5,2}(\lambda) = 7\lambda^2+16\lambda+7$ has 2 negative zeros
(0,2) I.P. in $S^g_{7,2}$	$\overset{\circ}{S}{}^g_{7,2}$	$\Pi^g_{7,2}(\lambda) = 11\lambda^4-104\lambda^3-234\lambda^2-104\lambda+11$ has 2 negative and 2 positive zeros
(0,1,3) I.P. in $S^g_{7,3}$	$\overset{\circ}{S}{}^g_{7,3}$	$\Pi^g_{7,3}(\lambda) = 2(7\lambda^2-20\lambda+7)$ has 2 positive zeros
(0,3) I.P. in $S_{7,2}$	$\overset{\circ}{S}{}^{\overline{s,o}}_{7,2}(\overline{0,1};\overline{3,1})$	$\Pi_{7,2}(\overline{0,1};\overline{3,1};\lambda) = 3\lambda^4-8\lambda^3-153\lambda^2-8\lambda+3$ has 2 negative and 2 positive zeros
(0,5) I.P. in $S_{7,2}$	$\overset{\circ}{S}{}^{\overline{s,o}}_{7,2}(\overline{0,1};\overline{5,1})$	$\Pi_{7,2}(\overline{0,1};\overline{5,1}:) = 5\lambda^4+176\lambda^3+478\lambda^2+176\lambda+5$ has 4 negative zeros
(0,2,3) I.P. in $S_{7,3}$	$\overset{\circ}{S}{}^{\overline{s,o}}_{7,3}(\overline{0,1};\overline{2,2})$	$\Pi_{7,3}(\overline{0,1};\overline{2,2}:\lambda) = 3\lambda^3+8\lambda+3$ has 2 negative zeros
(0,3,4) I.P. in $S_{7,3}$	$\overset{\circ}{S}{}^{\overline{s,o}}_{7,3}(\overline{0,1};\overline{3,2})$	$\Pi_{7,3}(\overline{0,1};\overline{3,2}:\lambda) = 6(\lambda^2+5\lambda+1)$ has 2 negative zeros
(0,1,2) I.P. in $S^{\overline{3,2}}_{7,3}$	$\overset{\circ}{S}{}^{\overline{3,2}}_{7,3}(\overline{0,3};\overline{k,0})$	$\Pi^{\overline{3,2}}_{7,3}(\lambda) = 50(\lambda^2+5\lambda+1)$ has 2 negative zeros

Appendix II

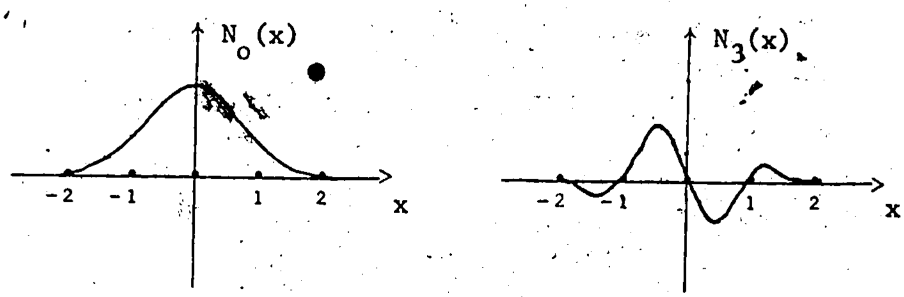
EXAMPLES OF B-SPLINES

1. B-Splines for (0,3) - Interpolation by $S_{5,2}$.

(1.1) The characteristic polynomial is $\Pi_{5,2}(0;3:\lambda) = 3\lambda^2+14\lambda+3$ with eigenvalues $\lambda_1 = (-7+2\sqrt{10})/3$, $\lambda_2 = (-7-2\sqrt{10})/3$.

(1.2) The spline $N_o(x) = 8(1-x)_+^5 - 2(2-x)_+^5 - 10(1-x)_+^4 + 5(2-x)_+^4$, $(x\geq 0)$, $N_o(x) = N_o(-x)$, $(x<0)$ satisfies $N_o(-1) = 3$, $N_o(0) = 14$, $N_o(1) = 3$, $N_o^{(3)}(-1) = N_o^{(3)}(0) = N_o^{(3)}(1) = 0$, and has support in $(-2,2)$.

(1.3) The spline $N_3(x) = -\frac{10}{12}(1-x)_+^5 - \frac{1}{12}(2-x)_+^5 + \frac{26}{12}(1-x)_+^4 + \frac{1}{12}(2-x)_+^4$, $(x\geq 0)$, $N_3(x) = -N_3(-x)$, $(x<0)$ satisfies $N_3(-1) = N_3(0) = N_3(1) = 0$, $N_3^{(3)}(-1) = 3$, $N_3^{(3)}(0) = 14$, $N_3^{(3)}(1) = 3$, and has support in $(-2,2)$.



(1.4) The fundamental function $L_o(x)$ satisfying $L_o(\nu) = \begin{cases} 0 & \text{if } \nu\neq 0 \\ 1 & \text{if } \nu=0 \end{cases}$

$L_o^{(3)}(\nu) = 0 \ \forall \nu$, is given by

$$L_o(x) = \frac{\sqrt{10}}{40} \sum_{n=-\infty}^{\infty} \lambda_1^{|n|} N_o(x-n) .$$

The fundamental function $L_3(x)$ satisfying $L_3^{(3)}(\nu) = \begin{cases} 0 & \text{if } \nu \neq 0 \\ 1 & \text{if } \nu = 0 \end{cases}$

$L_3(\nu) = 0$, $\forall\ \nu$, is given by

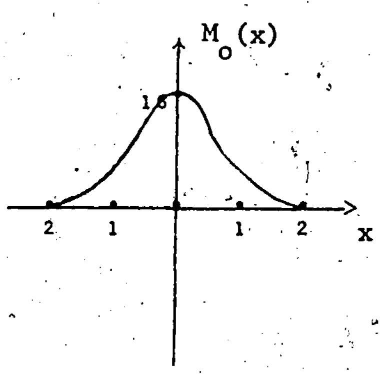
$$L_3(x) = \frac{\sqrt{10}}{40} \sum_{-\infty}^{\infty} \lambda_1^{|n|} N_3(x-n) \quad .$$

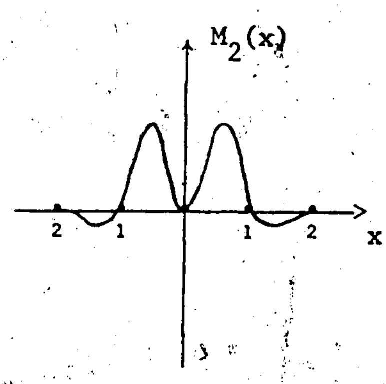
2. <u>B-Splines for (0,2) - Interpolation by</u> $S^g_{5,2}$.

(2.1) The characteristic polynomial is $\Pi^g_{5,2}(\lambda) = 7\lambda^2+16\lambda+7$ with eigenvalues $\gamma_1 = (-8 + \sqrt{15})/7$, $\gamma_2 = (-8 - \sqrt{15})/7$.

(2.2) The spline $M_o(x) = -$[illegible]$2)^5_+ + 12(x+1)^5_+ - 18x^6_+ + 12(x-1)^5_+ - 3(x-2)^5_+ + 10(x+2)^3_+ + 20(x+1)^3_+ - 60x^3_+ + 20(x-1)^3_+ + 10(x-2)^3_+$ satisfies $M_o(-1) = 7$, $M_o(0) = 16$, $M_o(1) = 7$, $M_o^{(2)}(-1) = M_o^{(2)}(0) = M_o^{(2)}(1) = 0$, and has support in $(-2,2)$.

(2.3) The spline $M_2(x) = \frac{1}{2}(x+2)^5_+ + (x+1)^5_+ - 3x^5_+ + (x-1)^5_+ + \frac{1}{2}(x-2)^5_+ - \frac{1}{2}(x+2)^3_+ - 13(x+1)^3_+ - 33x^3_+ - 13(x-1)^3_+ - \frac{1}{2}(x-2)^5_+$ satisfies $M_2^{(2)}(-1) = 7$, $M_2^{(2)}(0) = 16$, $M_2^{(2)}(1) = 7$, $M_2(-1) = M_2(0) = M_2(1) = 0$, and has support in $(-2,2)$.





(2.4) The fundamental spline $\Lambda_o(x)$ satisfying $\Lambda_o(\nu) = \begin{cases} 0 & \text{if } \nu \neq 0 \\ 1 & \text{if } \nu = 0 \end{cases}$

and $\Lambda_o^{(2)}(\nu) = 0 \;\; \forall \nu$, is given by

$$\Lambda_o(x) = \frac{\sqrt{15}}{30} \sum_{-\infty}^{\infty} \gamma_1^{|n|} M_o(x-n) \quad .$$

The fundamental spline $\Lambda_2(x)$ satisfying $\Lambda_2^{(2)}(\nu) = \begin{cases} 0 & \text{if } \nu \neq 0 \\ 1 & \text{if } \nu = 0 \end{cases}$

and $\Lambda_2(\nu) = 0 \;\; \forall \nu$, is given by

$$\Lambda_2(x) = \frac{\sqrt{15}}{30} \sum_{-\infty}^{\infty} \gamma_1^{|n|} M_2(x-n) \quad .$$