University of Alberta

Diffraction Symmetries for Point Sets

by

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Abstract

This thesis is about the diffraction of discrete point sets.

Firstly we study a new topology and show some connections between the diffraction of a point set and its dynamical hull properties.

In the second part we show that the diffraction of Meyer sets always contains a relative dense set of Bragg peaks, which are aligned in a nice ordered way. Also we proved that the continuous part of the diffraction spectra is either empty or supported on a relatively dense set.

In the last part of the thesis we prove that the diffraction of the pinwheel substitution tiling is rotationally invariant.

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This thesis is typeset using LaTeX.

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Chapter 1

Introduction

This thesis is about diffraction of discrete point sets and discusses some problems from the mathematics of long-range aperiodic order.

Before the discovery of quasicrystals in the mid 1980's[1] it was believed that only crystals (fully periodic solids) could produce diffraction patterns consisting only of bright peaks, called Bragg peaks. Experiments in the last 20 years have demonstrated the existence of quasicrystals, yet many questions remain about their atomic structure.

Quasicrystals are solids which have diffraction consisting only of bright peaks (essentially no diffuse background) and which don't have the periodicity of crystals. A simple mathematical model of a quasicrystal is a set of points representing the positions of atoms in a real solid, which has the **Delone property** and a pure point diffraction measure. The Delone property is a natural one: a subset $\Lambda \subset \mathbb{R}^d$ is Delone if

- there exists a non-empty open set $U \subset \mathbb{R}^d$ such that $x, y \in \Lambda$ and $(x+U) \cap (y+U) \neq \emptyset$ implies $x = y$,
- there exists a compact set $K \subset \mathbb{R}^d$ such that $\Lambda + K = \mathbb{R}^d$.

These conditions represent the fact that atoms can't come arbitrarily close to each other, and it also requires that the solid "fills" the space (has no arbitrarily large holes).

In a diffraction experiment one gets information about the solid's structure. The diffraction pattern comes from self-interference of an incoming beam scattering from

[1]The quasicrystals were discovered in 1984 by Shechtman-Blech-Gratias-Cahn and independently in 1985 by Ishimasa-Nissen-Fukano, when they reported the discovery of solids with 5-fold symmetric pure point diffraction.

the atoms of the solid. If $\Lambda \subset \mathbb{R}^d$ is our real solid (which is necessary finite), we associate to it the measure

$$\delta_\Lambda := \sum_{x \in \Lambda} \delta_x \, .$$

In a more realistic setting in which there is more than one kind of atom we can give different points different scattering intensities by introducing a weighted function $c : \Lambda \to \mathbb{C}$ and look instead at $\sum_{x \in \Lambda} c(x)\delta_x$. However, for the introduction we take the simple equally-weighted case.

Then, the diffraction is just (a suitably normalized version of) the square of the Fourier transform of Λ:

$$I(q) := |\widehat{\delta_\Lambda}(q)|^2 = \sum_{x,y \in \Lambda} e^{i(x-y)q} \, .$$

When one tries to idealize this to Delone sets, one has to face the fact that the Fourier transform of δ_Λ doesn't make sense most of the time. To avoid this problem, we will use this method, introduced by A. Hof: we compute the intensity as the limit of the intensities of larger samples. Thus, the intensity becomes:

$$I = \lim_{R \to \infty} \frac{I_{\Lambda \cap B_R(0)}}{\text{vol}(B_R(0))} \, .$$

Of course we have to define in which topology we take this limit. I is a measure describing the intensity of the scattered beam, and it is called the *diffraction measure.* Any non-trivial Bragg peak in the diffraction is indication of some long-range alignment in the solid, while the diffuse background shows some disorder. If the diffraction pattern consists only of Bragg peaks it means that a lot of constructive interference must occur, thus we must have a very strong internal long-range order. We try to characterize the point sets with this property. We are also interested in a weaker form of long range order, namely the case when the diffraction pattern has infinitely many Bragg peaks (but is not necessarily consisting only of Bragg peaks).

The main difficulty one is facing when studying the diffraction is the fact that there are (infinitely) many different atomic structures which produce the same diffraction pattern. Thus, is it impossible to fully describe a point set knowing only its diffraction pattern.

Our goal is to construct mathematical models for the quasicrystals and characterize these models using only physically detectable properties.

In one-dimensional space, one can identify a Delone point set Λ, uniquely up to translation with a bi-sequence of numbers, representing the distances between consecutive atoms. Since we deal only with a finite number of atom types, we may assume that there are only finitely many local arrangements of the atoms and so these nearest neighbor distances can take values only in a finite set. We want Λ to have some long range order, thus we try to construct these bi-sequences following some rules. The most common method is to use substitution rules, the idea being that atoms combine to create some finite structures which then interact in the same way as the original atoms[2].

To understand better the substitution sequences let's look to the Thue-Morse sequence. We use the following substitution rules on two letters $\{a, b\}$: $a \xrightarrow{\Phi} ab$ and $b \xrightarrow{\Phi} ba$; or repeating it twice, $a \xrightarrow{\Phi} ab \xrightarrow{\Phi} abba$ and $b \xrightarrow{\Phi} ba \xrightarrow{\Phi} baab$. Starting from b and a and expanding b to the left and a to the right using Φ^2, we get a fixed bi-sequence

$$\cdots \; b \; b \; a \; b \; a \; a \; b \,|\, a \; b \; b \; a \; b \; a \; a \; \cdots$$

If we assign a unit length interval to each letter in the sequence, starting with the interval $[0, 1]$ at the initial letter a and $[-1, 0]$ at the initial letter b we can tile a real line $\mathbb{R}$.

···	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	···
···	b	b	a	b	a	a	b |	a	b	b	a	b	a	a	···

By defining Λ to be the set of endpoints of the a-type intervals, we get a Delone set in $\mathbb{R}^d$ called the **Thue Morse sequence**. Λ is fixed by Φ, i.e. if we group the symbols in pairs and replace $A := ab$ and $B := ba$ we get the Thue-Morse sequence again. This means that if we look at the tiling, double each tile and divide the supertiles as in the substitution rules ($a \xrightarrow{\Phi} ab$ and $b \xrightarrow{\Phi} ba$), then we get the same tiling.

A similar idea works in higher dimensional spaces. We look for a set of tiles which can be inflated by some factor λ so that each inflated tile can be cut into copies of the original tiles. Then, we can get a tiling of $\mathbb{R}^d$ just by repeatedly inflating and dividing.

One of the first and nicest examples of an aperiodic tiling in $\mathbb{R}^2$ was introduced by Penrose in the mid 70's. It is a tiling of the plane with two distinct tiles and can be created either by substitution or by following matching rules. Exactly as the first discovered quasicrystals, it has a 5-fold symmetric and pure point diffraction.

[2]In a way this is similar with the ice-flowers grown on a window: smaller ice-flowers combine together to create bigger ice-flowers. Substitution is a powerful mathematical tool, but there is no physical evidence to support its actual existence in the formation of quasicrystals.

Moreover, the two tiles can tile the space in infinitely many ways, all of them being aperiodic and locally indistinguishable.

Around 1970, Y. Meyer introduced in [24] a new concept into harmonic analysis: **harmonious sets**. Despite their originality, they were mainly ignored until the mid 90's, when their relevance to aperiodic crystals was shown ([27], [22]). Their importance to aperiodic order lies within their close connection to the cut and project method.

The cut and project method is a simple way of creating aperiodic structures. The resulting sets are pure point diffractive and are called *regular model sets*. Intuitively, they are the projections of a part of a higher dimensional lattice onto the ambient space[3]. Model sets are far easier to construct than substitution systems, but they suffer from being very hard to characterize.

The use of dynamical systems in the study of the internal order of discrete point sets in real spaces $\mathbb{R}^d$ has been remarkably effective. The basic idea, which probably has its roots in statistical mechanics, was explicitly formulated by Radin and Wolff in [37]. Let $\Lambda \subset \mathbb{R}^d$ be a point set. We will always assume that our point sets are **locally finite**, meaning that their intersections with compact subsets of $\mathbb{R}^d$ are finite (equivalently they are discrete and closed). The dynamical hull $\mathbb{X} = \mathbb{X}(\Lambda)$ of Λ is the closure of the $\mathbb{R}^d$-translation orbit of Λ in some suitable topology.

The commonly used topology, which is the one advocated in [37], declares that two point sets, Λ_1, Λ_2 are close if their restrictions to some large ball around 0 are close in the Hausdorff metric. The resulting space $\mathbb{X}$ is compact and $(\mathbb{R}^d, \mathbb{X})$ is a topological dynamical system. A variation of this topology is to require instead that the restrictions of the two sets to some large open ball around 0 are coincident after some small overall translation. If the sets have **finite local complexity** [4] (FLC) then the two topologies are the same.

The importance of the concept is that several fundamental geometrical properties of point sets have equally fundamental interpretations in terms of their dynamical hulls , notably *repetitivity* $\leftrightarrow$ *minimality* and *uniform cluster frequencies* $\leftrightarrow$ *unique ergodicity*. Some of the deepest results in the study of point sets and also in tiling theory have come by utilizing the machinery of dynamical systems through this connection.

We start by introducing basic definitions and notation for point sets and diffrac-

[3]While originally Penrose tilings were constructed using substitution rules, it was shown by N. de Bruijn that Penrose tilings can also be obtained using the cut & project method, thus explaining the nice long-range properties of this tiling.

[4]A set Ω has finite local complexity if, for each compact set K in $\mathbb{R}^d$, there are, up to translation, only finitely many classes of points that can appear in the form $\Omega \cap (a + K)$ as a runs over $\mathbb{R}^d$.

tion in Chapter 2. We are interested in identifying that part of the autocorrelation which creates the Bragg peaks, and in order to do this we can use the work of G. de Lamadrid and L. Argabright [21]. We explain their work in Section 2.2.

Meyer sets are Delone subsets of regular model sets. Thus the concept of Meyer set lies between that of Delone set and that of regular Model set. In contrast to model sets, Meyer sets have many characterizations, some of which are immediately accessible [27]. The most commonly used characterization is introduced by Lagarias [22] : $\Lambda \subset \mathbb{R}^d$ is Meyer if and only if:

- Λ is Delone,
- its set of "interatomic vectors", $\Lambda - \Lambda$ is a Delone set.

This simple condition is relatively easy to check, but, as we show in Chapter 4, it implies strong internal long range order: Λ is Meyer $\Rightarrow$ the diffraction of Λ has a relatively dense set of Bragg peaks.

In Chapter 4 we study the properties of the diffraction pattern for a (possibly weighted) Meyer set. The first part of the chapter is based on [41]; in this section we prove that the diffraction of an arbitrary Meyer set shows infinitely many Bragg peaks and that the Bragg peaks fill the space without arbitrarily large holes. Also, if the Meyer set is not pure point diffractive, the diffuse background also fills the space.

The result about Meyer set diffraction is a simple consequence of the following proposition, which is one of the results of the thesis:

Proposition 4.10 *Let S be a subset of a pure point diffractive set Λ with FLC. Then, each of the continuous and discrete diffraction spectra is either empty or supported on a relatively dense set. Moreover, if S is relatively dense, it has a relatively dense set of Bragg peaks.*

In the second part of Chapter 4 we study the positions at which the Bragg peaks of a Meyer set can appear. Imagine, as must always happen in real experiments, that there is a threshold of intensity below which Bragg peaks cannot be measured. In other words we look only at Bragg peaks whose intensity exceeds some value $v > 0$. We show that these "visible" Bragg peaks also verify the Meyer condition, and that, in the aperiodic case, no matter how small $v > 0$ is set, there are always non-visible Bragg peaks. In the last part of this section we prove that some of these properties persist even if we change the Meyer set by a set of sufficiently small density.

When using dynamical systems to study discrete point sets, we generally use the local Radin-Wolf topology. In Chapter 3, we study a new topology, introduced by M. Baake and R. V. Moody in [7], which is based on statistical coincidence and is connected with the diffraction measure. We prove the basic properties of this topology and study the hull $\mathbb{A}(\Lambda)$ of a point set Λ in this topology. If this topological space is compact, it becomes a dynamical system in a natural way and thus we can study this dynamical system to get information about the point set. We prove that in order for this to happen the set Λ must have pure point diffraction. Moreover, under the Meyer condition, compactness and pure pointedness are equivalent. We also show that for regular model sets, $\mathbb{A}(\Lambda)$ is isomorphic to the compact group from the cut & project scheme. These results were used in [5] to classify an large class of model sets, namely nonsingular, minimal regular model sets.

One special example of a substitution tiling is the pinwheel tiling. It is a tiling of the plane by copies of a rectangular triangle with sides 1,2 and $\sqrt{5}$ by a substitution rule, and it was the first example of such tiling in which the tiles appear in infinitely many orientations. It have been proved by Radin [33, 36] that the diffraction of the pinwheel tiling is circularly symmetric. His prove uses the Dworkin argument which provides a deep but not fully understood connection between the diffraction of a point set and the spectral measure of the local dynamical system. In Chapter 5 we provide a new proof for this result, by a direct computation of the autocorrelation measure.

The circular invariance of the pinwheel diffraction shows that the Bragg spectrum consists of exactly one peak at the origin. Thus the pinwheel tiling doesn't have long range order in the standard sense, while the rotationally invariance of the diffuse background shows that there is still some kind of order. We still don't know anything more about the nature of the diffuse background of its diffraction.

This thesis consists of 4 different papers, which have been slightly modified to bring them into a coherent work for this thesis. These papers are:

- R. V. Moody and N. Strungaru, *Point Sets and Dynamical Systems in the Autocorrelation Topology*, Canad. Math. Bull. Vol. 47 (1), 82-99, 2004.
- N. Strungaru, *Almost periodic measures and long-range order in Meyer sets*, Discrete and Computational Geometry vol. 33(3), 483-505, 2005.
- N. Strungaru, *Bragg spectra of a Meyer Set*, preprint, 2004.
- R. V. Moody, D. Postnikoff and N. Strungaru *Circular Symmetry of Pinwheel Diffraction*, preprint 2004, to appear in Annales Henri Poincare.

Chapter 2

Preliminaries

In this Chapter we collect together some of the basic definitions that are commonly used in the mathematics of discrete aperiodic point sets, and some of the less known facts about almost periodic functions and measures. Although the most important setting for applications is $\mathbb{R}^d$ (more particularly $\mathbb{R}^2, \mathbb{R}^3$), most of the concepts work in the setting of locally compact abelian groups. The reader might wish to skip this section at first, referring to it only to pick up definitions as they are used in the thesis.

2.1 Definitions and notation

For the entire thesis G will be a locally compact abelian group, which we treat additively, and θ will denote its Haar measure. We will work in this general setting as much as we can. For some applications we will need to restrict to the case $G = \mathbb{R}^d$.

We say that a set $S \subset G$ is **locally finite** if $S \cap K$ is finite for all compact sets $K \subset G$.

Definition 2.1 Let $\Lambda \subset G$ be a locally finite set.

- For $K \subset G$ a compact set, Λ is **K-relatively dense** if for all $x \in G$, $(x + K) \cap \Lambda \neq \emptyset$.
- For a neighbourhood V of $\{0\}$, Λ is **V-uniformly discrete** if for all $x \in G$ we have $(x + V) \cap (\Lambda \backslash \{x\}) = \emptyset$.
- Λ is **weakly-uniformly discrete** if for every compact K in G there exists a constant c_K such that for any $t \in G$

$$\sharp(\Lambda \cap (t + K)) \leq c_K \,,$$

where $\sharp$ means cardinality.

- For K a compact set and V a neighbourhood of 0, Λ is a (K, V)-**Delone set** if Λ is K-relatively dense and V-uniformly discrete.

Remark 2.2 When we don't need the parameters we say only uniformly discrete, relatively dense, or Delone set.

Definition 2.3 For two compact sets $A, K \subset G$ we define the K**-boundary of** A by:

$$\partial^K A = ((K + A)\backslash A^\circ) \cup ((-K + \overline{G\backslash A}) \cap A).[1]$$

Intuitively, $\partial^K A$ represents the points of G "within K" of the boundary of A.

Definition 2.4 A sequence $\mathcal{A} = \{A_n\}_n$ of compact sets $A_n \subset G$ is called a **van Hove sequence** [2] if for all compact sets $K \subset G$ we have:

$$\lim_{n\to\infty} \frac{\theta(\partial^K(A_n))}{\theta(A_n)} = 0\,.$$

Intuitively the van Hove condition says that the surface to bulk ratio of the A_n tends to 0 as n tends to infinity.

In $\mathbb{R}^d$ we generally use $A_n = B_n(0)$, the set of ball of radius n centered at origin, as van Hove sequence.

For a measure[3] μ on G we denote by $\tilde{\mu}$ the measure defined by:

$$\tilde{\mu}(E) = \overline{\mu(-E)}\,,$$

for all measurable sets $E \subset G$.

Given a locally finite set $S \subset G$ we define δ_S by:

$$\delta_S = \sum_{x\in S} \delta_x\,,$$

where δ_x is the normalized point measure at x.

---

[1] Note that for sets $X, Y \subset G$,

$$X \pm Y := \{x \pm y \mid x \in X, y \in Y\}\,,$$

$$X\backslash Y := \{x \in X \mid x \notin Y\}\,.$$

[2] See Appendix A for more about van Hove sequences.

[3] For the definitions and basic properties of measures one can consult [12].

Definition 2.5 A sequence of measures $\{\mu_n\}_n$ **converges vaguely** to μ if, for all $f \in \mathbb{C}_c(C)$, $\{\mu_n(f)\}_n \stackrel{n\to\infty}{\longrightarrow} \mu(f)$.

Definition 2.6 Given a locally finite point set S and a van Hove sequence $\{A_n\}_n$, we say that S has a well defined **autocorrelation** with respect to $\{A_n\}_n$, if the sequence:

$$\eta_n = \frac{\delta_{S\cap A_n} * \tilde{\delta}_{S\cap A_n}}{\theta(A_n)} = \frac{1}{\theta(A_n)} \sum_{x,y\in(S\cap A_n)} \delta_{x-y},$$

converges vaguely to a measure η. We call this measure the **autocorrelation** of S.

Remark 2.7 The net η_n has always a cluster point [4]. Thus, given a van Hove sequence, the autocorrelation always exists with respect to a subsequence.

We use the notation: $C_c(G) = \{f : G \to \mathbb{C} \mid f$ is continuous and has compact support$\}$. Let $\mathcal{K}_2(G)$ be the subspace of $C_c(G)$ spanned by $\{f * g \mid f, g \in C_c(G)\}$.

Definition 2.8 Given a function $g \in L^1(G)$ we can define a new function $\widehat{g}$ on the dual group $\widehat{G}$ by:

$$\widehat{g}(\chi) = \int_G g(t)\chi(t)d\theta(t).$$

$\widehat{g}$ is called the **Fourier transform** of g.

Definition 2.9 A measure μ on a locally compact abelian group G is called **Fourier transformable** if and only if there exists a measure $\widehat{\mu}$ on the dual group $\widehat{G}$, called the **Fourier transform** of μ, such that

$$< \widehat{\mu}, g >=< \mu, \widehat{g} >,$$

for all $g \in \mathcal{K}_2(G)$, where $\widehat{g}$ denotes the Fourier transform of the function g.

The basic properties of Fourier transformable measures can be found in ([9], Chapter 1) or ([21], Chapters 10-11).

Definition 2.10 μ a measure on G is called **translation bounded** if for every compact set $K \subset G$ there exists a constant C so that:

$$|\mu|(x + K) \leq C \quad \forall x \in G,$$

where $|\mu|$ denotes the variation norm. We denote by $\mathcal{M}^\infty(\mathbf{G})$ the set of translation bounded measures on G.

Definition 2.11 A continuous function g is called **positive definite** if the inequality:

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{\alpha}_j\alpha_k g(x_k - x_j) \geq 0\,,$$

holds for all subsets $\{x_1, x_2, ..., x_m\} \subset G$ and all sequences $(\alpha_1, \alpha_2, ..., \alpha_m)$ of complex numbers.

Definition 2.12 A measure $\mu \in \mathcal{M}^\infty(G)$ is called **positive definite** if, for all $f \in C_c(G)$, $\mu * f * \tilde{f}$ is a positive definite function, where $\tilde{f}$ means the function $\tilde{f}(x) = \overline{f}(-x)$.

Proposition 2.13 ([9] ,[21]) *Let μ be a positive definite measure. Then*

i) $\mu \in \mathcal{M}^\infty(G)$,

ii) μ *is Fourier transformable.*

The autocorrelation of a point set is a positive definite measure.

Definition 2.14 Let S be a locally finite set. Suppose that its autocorrelation η exists with respect to some van Hove sequence $\mathcal{A}$. Then we call $\widehat{\eta}$ the **diffraction measure** (or pattern) of S (with respect to $\mathcal{A}$).

S is called **pure point diffractive** if the diffraction pattern of Λ is a pure point measure.

Definition 2.15 Let A be a Delone set in G and suppose that its autocorrelation η exists. Let $B = \{x \in \widehat{G} \mid \widehat{\eta}(\{x\}) \neq 0\}$. B is called the set of **Bragg peaks** of A.

Definition 2.16 A standard *cut and project scheme* consists of a direct product $G \times H$ of G and a locally compact abelian group H, and a lattice L in $G \times H$ such that with respect to the natural projections $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ we have:

i) π_1 restricted to L is $1-1$,

ii) $\pi_2(L)$ is dense in H.

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \tilde{L} & & \end{array} \quad . \qquad (2.1)$$

We denote the cut and project scheme by $(G \times H, L)$.

Let θ_H denote the Haar measure of H.

Definition 2.17 A subset $\Lambda \subset G$ is called a *model set* if there exists a cut and project scheme $(G \times H, L)$ and a relatively compact subset W of H with non-empty interior such that:

$$\Lambda = u + \Lambda(W) := \{u + \pi_1(x) \mid x \in L, \pi_2(x) \in W\}, \text{ for some } u \in G.$$

In this case W is called *the window* of the model set.

If, in addition, $\theta_H(\partial W) = 0$ and W is the closure of its interior, we say that Λ is a *regular model set.*

Proposition 2.18 Let $(G \times H, L)$ be a cut and project scheme, and let $W \subset H$ be a compact set with non-empty interior. Then $\Lambda(W)$ is Delone. In particular, since $\Lambda(W') \pm \Lambda(W'') \subset \Lambda(W' \pm W'')$ for any $W', W'' \subset H$, the set

$$\Lambda(W) \pm \Lambda(W) \pm \Lambda(W) \pm ... \pm \Lambda(W)$$

is Delone for any or all choices of sign.

Since in many applications we work simultaneously with both G and $\widehat{G}$, we write x to refer to an element from G, and when we write χ we refer to an element from $\widehat{G}$.

Definition 2.19 For a subset $\Lambda \subset \mathbb{R}^d$ and $\epsilon > 0$ we define:

$$\Lambda^\epsilon = \{\chi \in \widehat{\mathbb{R}^d} \mid |\chi(x) - 1| \leq \epsilon \text{ for all } x \in \Lambda\}.$$

We call Λ^ϵ the *ϵ-dual* of Λ.

Definition 2.20 A relatively dense subset $\Lambda \subset \mathbb{R}^d$ is called a *Meyer set* if $\Lambda - \Lambda := \{x - y \mid x, y \in \Lambda\}$ is uniformly discrete.

Proposition 2.21 [27] *If Λ is a relatively dense subset of $\mathbb{R}^d$ then the following are equivalent:*

i) Λ is Meyer set,

ii) Λ is a subset of a model set,

iii) For any $0 < \epsilon$, the ϵ-dual set Λ^ϵ is relatively dense, *iv) For some $0 < \epsilon < 1/2$, the ϵ-dual set Λ^ϵ is relatively dense.*

Whenever we need the Meyer condition we have to restrict ourselves to the case of $G = \mathbb{R}^d$ since only here we have the full characterization of Meyer sets. Recently, using the structure theorem for compactly generated locally compact abelian groups, it was proved in [5] that some of the equivalent definitions for Meyer sets are still equivalent in these more general groups.

Definition 2.22 A weighted comb is a measure $\mu = \sum_{x\in S} c_x\delta_x$ where S is a discrete point set and $c_x \in \mathbb{C}$, with $\{c_x\}_{x\in S}$ bounded. (i.e. μ is a complex valued pure point measure). We say that the weighted comb μ is *supported on a Meyer set* if we can write $\mu = \sum_{x\in S} c_x\delta_x$ with S Meyer.

Remark 2.23 The measure μ is a weighted comb supported on a Meyer set if and only if $\mathrm{supp}(\mu)$ is a subset of a Meyer set.

2.2 Almost periodic measures

For the entire section G is a $\sigma-$compact, locally compact abelian group, and θ is its Haar measure.

Given a locally finite point set S, we are interested mainly in its pure point diffraction spectrum. Thus, if η is the autocorrelation and $\widehat{\eta}$ its Fourier transform, we are interested in $(\widehat{\eta})_{pp}$ and $(\widehat{\eta})_c$. The theory of almost periodic measures [21] can be used, so one gets a unique decomposition $\eta = \eta_S + \eta_0$, into the strong and null weakly almost periodic components, such that

$$(\widehat{\eta})_{pp} = (\widehat{\eta_S})\,,$$

$$(\widehat{\eta})_c = (\widehat{\eta_0})\,.$$

We explain below the decomposition $\eta = \eta_S + \eta_0$.

Restrict for a moment to $G = \mathbb{R}^d$. We can write

$$(\widehat{\eta})_{pp} = \sum_{x\in A} c_x\delta_x\,,$$

where A is a countable set. Let η_S be the part of the autocorrelation which is mapped by Fourier transform into $(\widehat{\eta})_{pp}$. Then, using the inverse Fourier transform, we should get:

$$\eta_S \sim \sum_{x\in A} c_x e^{2\pi<x,.>}\,.$$

The problem here is to determine in what sense is the second sum convergent. Anyhow, this is similar with the Bohr approximation of almost periodic functions, so the answer should be similar. In [21] the authors proved that η_S is almost periodic in a sense that we describe below, and they studied both the measures η_S and $\eta_0 = \eta - \eta_S$.

We use the following notations:

$$C(G) = \{f : G \to \mathbb{C} \mid f \text{ continuous}\},$$

$$C_B(G) = \{f \in C(G) \mid f \text{ bounded}\},$$

$$C_U(G) = \{f \in C_B(G) \mid f \text{ uniformly continuous}\},$$

$$C_0(G) = \{f \in C_U(G) \mid f \text{ vanishing at } \infty\}.$$

For each $x \in G$ we define the left translation $\tau_x : C_U(G) \to C_U(G)$ by:

$$(\tau_x f)(y) = f(-x + y) .$$

Definition 2.24 We define the **weak topology** on $C_U(G)$ as the topology defined by its dual space. We refer to the Banach topology as the **strong topology**.

Definition 2.25 The map $\mu \to \{\mu * f\}_{\{f \in C_c(G)\}}$ is an embedding of $\mathcal{M}^\infty(G)$ in $[C_U(G)]^{C_c(G)}$. Giving $[C_U(G)]^{C_c(G)}$the usual product topology, the induced topology on $\mathcal{M}^\infty(G)$ is called the **product topology**. We will also refer to this topology as the **strong topology**. The **weak topology** is defined by the dual space of $\mathcal{M}^\infty(G)$.

Remark 2.26 The product topology defines a structure of locally convex topological vector space on $\mathcal{M}^\infty(G)$. A fundamental system of semi-norms is given by $\{\| \cdot \|_f\}_{f \in C_c(G)}$, where :

$$\| \mu \|_f := \| \mu * f \|_\infty .$$

Definition 2.27 Let μ be a translation bounded measure on G. Let $D_\mu = \{\delta_x * \mu\}_{x \in G}$ and $C_\mu =$ the closed [4] convex hull of D_μ.

We say that μ is **amenable** (see [16][5] or page 52 of [21]) if and only if C_μ contains exactly one scalar multiple μ_0 of the Haar measure θ.

[4] A theorem by Mazuro and Bourgin [23] says that in a locally convex topological vector space a convex set is closed if and only if it is weakly closed. Thus the closure in this definition is the same in both strong and weak topologies.

[5] Eberlein uses a more general setting and the concept of ergodicity. In the particular case of $\mathcal{M}^\infty(G)$, the Definition 2.27 of an ergodic element becomes the one we use for the amenability.

In this case we write:

$$\mu_0 = M(\mu)\theta\,,$$

and call $M(\mu)$ the **mean** of μ.

We say that $f \in C_U(G)$ is **amenable** if and only if the measure $f d\theta$ is amenable. If this happens we define: $M(f) = M(f d\theta)$.

Remark 2.28 i) f is amenable if and only if C_f, the closed convex hull of $\{\delta_x * f\}_{x\in G}$ contains exactly one constant function. In this case this constant is $M(f)$.

ii) This definition is difficult to use. In [16] it is proved that for any amenable function the averaged integral exists [6], if one computes the average over some particular sequences. We prove in the Appendix that the average integral can be computed over van Hove sequences.

It is easy to see that, if it is well defined, the average integral is constant on C_f. Thus, if a function is amenable, the mean is the average integral.

Definition 2.29 $f \in C_U(G)$ is called **strongly almost periodic** if C_f is compact in the strong topology. f is called **weakly almost periodic** if C_f is compact in the weak topology. f is called **null weakly almost periodic** if it is weakly almost periodic, and $|f|$ is amenable[7] with $M(|f|) = 0$.

We denote by SAP(G), WAP(G) and WAP$_0(G)$ the spaces of strongly, weakly, and null weakly almost periodic functions on G, respectively.

Remark 2.30 In a Banach space a closed set is compact if and only if its closed convex hull is compact ([17], [30]). Thus the previous definition is equivalent to the usual one for almost periodic functions, namely that the closure of D_f is compact. We prefer to use the one with C_f because we can use the same set for both weak and strong topology.

These definitions extend to translation bounded measures.

[6]We say that the average integral of $f \in C_U(G)$ exists with respect to the averaging sequence $\mathcal{A} = \{A_n\}_n$ if the following limit exists:

$$\lim_{n\to\infty} \frac{\int_{A_n} f(t)d\theta(t)}{\theta(A_n)}\,.$$

[7]Eberlein showed in [16] that f weakly almost periodic implies $|f|$ is amenable, thus one can ignore this requirement in the definition of null weakly almost periodic function.

Definition 2.31 A measure $\mu \in \mathcal{M}^\infty(G)$ is called **strongly almost periodic** if C_μ is compact in the product topology and μ is called **weakly almost periodic** if C_μ is compact in the weak topology. We denote by $\mathcal{SAP}(G)$ and $\mathcal{WAP}(G)$ the spaces of strongly and respectively weakly almost periodic measures on G. A translation bounded measure μ over a locally compact abelian group G is called **null weakly almost periodic** if and only if for each $g \in C_c(G), g * \mu$ is a null weakly almost periodic function ($g * \mu$ is weakly almost periodic and $M(|g * \mu|) = 0$). The corresponding space of measures is denoted by $\mathcal{WAP}_0(G)$.

Remark 2.32 ([21] Corollary 5.4 and Corollary 5.5) For these properties we can talk about a correspondence between measures and functions. More precisely, if P is the property of being strongly, weakly, null weakly almost periodic or amenable, then the following are true:

i) $f \in C_U(G)$ has property P if and only if $f d\theta \in \mathcal{M}^\infty(G)$ has property P,

ii) $\mu \in \mathcal{M}^\infty(G)$ has property P if and only if $f * \mu$ has property P for every $f \in C_c(G)$.

Remark 2.33 Using Remark 2.32 we can see that any null weakly almost periodic measure is in fact a weakly almost periodic one.

Definition 2.34 For $K \subset G$ a compact set with non-empty interior we define a norm on $\mathcal{M}^\infty(G)$ by

$$\| \mu \|_K := \sup_{x \in G} |\mu|(x+K) .$$

We define the **norm topology** as the topology defined by this norm.

Definition 2.35 A measure $\mu \in \mathcal{M}^\infty(G)$ is called **norm almost periodic** if D_μ is precompact in the norm topology.

We will make use of the following results:

Proposition 2.36 *Let μ be a transformable measure on G, with $\widehat{\mu}$ translation bounded and Fourier transformable. Then $\mu \in \mathcal{WAP}(G)$.*

Proof: We apply ([21], Theorem 11.1) to the inverse Fourier transform of μ. □

Proposition 2.37 ([21], Theorem 7.2 and Theorem 8.1) *Let $\mu \in \mathcal{WAP}(G)$. Then μ can be written uniquely in the form*

$$\mu = \mu_S + \mu_0 ,$$

with $\mu_S \in \mathcal{SAP}(G), \mu_0 \in \mathcal{WAP}_0(G)$.

Proposition 2.38 ([21], Theorem 7.2) *Let $\mu \in \mathcal{WAP}(G)$ be a positive measure. Then μ_S is positive.*

Proposition 2.39 ([21], Theorem 11.2) *Let μ be a transformable measure and let $\widehat{\mu}$ be translation bounded and Fourier transformable. Then*

$$(\widehat{\mu})_{pp} = (\widehat{\mu_S}),$$

$$(\widehat{\mu})_c = (\widehat{\mu_0}).$$

Corollary 2.40 *Let μ be a transformable measure and let $\widehat{\mu}$ be translation bounded. Then μ is a pure point measure if and only if $\widehat{\mu} \in \mathcal{SAP}(G)$ and μ is a continuous measure if and only if $\widehat{\mu} \in \mathcal{WAP}_0(G)$.*

Remark 2.41 Suppose that $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is Fourier transformable. Then $\widehat{\mu}$ is Fourier transformable and

$$\widehat{\widehat{\mu}} = \tilde{\mu}.$$

To see this, one may use the following argument suggested by M. Baake. $\widehat{\widehat{\mu}}$ is well defined in the tempered distribution sense and $\widehat{\widehat{\mu}} = \tilde{\mu}$, when viewed as tempered distributions. Thus $\langle \tilde{\mu}, g \rangle = \langle \widehat{\mu}, \widehat{g} \rangle \; \forall g \in \mathcal{S}(\mathbb{R}^d)$.

Any function in $C_c(\mathbb{R}^d)$ can be approximated by a function in the Schwartz class $S(\mathbb{R}^d)$, so one gets:

$$\langle \tilde{\mu}, g \rangle = \langle \widehat{\mu}, \widehat{g} \rangle \; \forall g \in \mathcal{K}_2(\mathbb{R}^d).$$

Chapter 3

Point Sets and Dynamical Systems in the Autocorrelation Topology

3.1 Introduction

As we mentioned in the Introduction, we can get information about a discrete point set Λ by using a topology on the set of discrete point sets and studying the hull of Λ in this topology. Usually we use the *local* Radin-Wolf topology, which is based on local configurations.

In [4] was shown that pure point diffraction is a result of the existence of many **ϵ-almost-periods** for every positive ϵ, that is, translations t that almost perfectly match up Λ with itself in an average or statistical sense:

$$\lim_{R\to\infty} \frac{\sharp((t+\Lambda) \bigtriangleup \Lambda) \cap B_R(0)}{\mathrm{vol}(B_R(0))} < \epsilon\,,$$

where $\bigtriangleup$ is the symmetric difference operator.

Now this suggests quite a different notion of closeness which reflects a low average discrepancy between the two sets or, to put is another way, high statistical coincidence. This can be supplemented to include small translations: two point sets are close if after a small translation they are statistically almost the same. This is the **autocorrelation topology**. We can again form the dynamical hull of a point set Λ, say $\mathbb{A} = \mathbb{A}(\Lambda)$.

There is no reason to expect $\mathbb{X}$ and $\mathbb{A}$ to be in any way related, and indeed this is in general what happens. But it is a striking fact that it is the local topology that captures the fundamental geometric properties of the set and the autocorrelation that holds the keys to the diffractive properties. Since most of the famous examples of aperiodic point sets have very beautiful local structure and are also pure point diffractive, it comes as no surprise that for these examples $\mathbb{X}$ and $\mathbb{A}$ are related, namely $\mathbb{A}$ is a factor of $\mathbb{X}$. In fact this result holds for all Λ which are regular generic model sets. In the final section of this Chapter we prove that for a regular model set, $\mathbb{A}(\Lambda)$ is isomorphic to the "torus" $\mathbb{T}$ of its cut and project scheme, thus laying down the connection to the paper of Schlottmann [40] which shows the existence of a mapping $\mathbb{X} \longrightarrow \mathbb{T}$.

This Chapter is about the topologies arising by statistical coincidence. The first part is about statistical coincidence alone (no translations included) and centres on a completeness result for locally finite sets in this topology. The second part adds in translations and leads to some results on $\mathbb{A}$ (which is actually an abelian group), when it is compact, and when it is pure point diffractive.

The results from this Chapter do not depend very much on the special properties of $\mathbb{R}^d$ other than it is a σ-compact locally compact abelian group. Thus we work in the more general context of a σ-compact locally compact abelian group G (written additively) and its Haar measure θ, unique up to a positive factor. Autocorrelation depends on averaging over something and for that purpose we fix once and for all an averaging sequence $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ satisfying

i) each A_n is a compact subset of G;

ii) for all n, $A_n \subset A^\circ_{n+1}$;

iii) $\bigcup_{n\in\mathbb{N}} A_n = G$;

iv) the van Hove condition [1].

Since $G = \bigcup_{n\in\mathbb{N}} A^\circ_{n+1}$, we see that for any compact subset $K \subset G$, there is a finite cover of it using sets from $\mathcal{A}$, and then $K \subset A_n$ for some n. In particular, for any $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ so that $A_m + K \subset A_n$.

3.2 $(\mathcal{D}, d)$ as a complete metric space

Definition: Let $\Lambda, \Lambda' \subset G$ be two locally finite sets. Define

$$d(\Lambda,\Lambda') := \limsup_{n\to\infty} \frac{\sharp((\Lambda \bigtriangleup \Lambda') \cap A_n)}{\theta(A_n)}\,. \tag{3.1}$$

[1] See Definition 2.4. We do not consider here the question of the existence of such a sequence. For compactly generated locally compact abelian groups one can use the structure theorem to explicitly construct such sequences.

This is a pseudometric. We obtain a metric by defining the equivalence relation

$$\Lambda \equiv \Lambda' \Leftrightarrow d(\Lambda, \Lambda') = 0$$

and factoring d through it:

$$\mathcal{D} := \{\Lambda \subset G \mid \Lambda \quad \text{locally finite}\}/\equiv \quad \text{and} \quad d : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}_{\geq 0}\,. \tag{3.2}$$

Proposition 3.1 *$(\mathcal{D}, d)$ is a complete metric space.*

Proof: Let $\{\Lambda_m\}$ be a sequence of locally finite subsets of G which form a Cauchy sequence when regarded in $\mathcal{D}$. We will construct a locally finite subset Λ of G to which this sequence converges when considered in $\mathcal{D}$.

(**First Case:** $\lim_{n\to\infty} \theta(A_n) = \infty$)

We can pick a subsequence $\{\Lambda_{k_m}\}$ such that $d(\Lambda_{k_m}, \Lambda_{k_{m+1}}) \leq 4^{-m}$ for each $m \geq 0$. Since

$$d(\Lambda_{k_m}, \Lambda_{k_{m+1}}) = \limsup_{n\to\infty} \frac{\sharp((\Lambda_{k_m} \bigtriangleup \Lambda_{k_{m+1}}) \cap A_n)}{\theta(A_n)} \leq \frac{1}{4^m}$$

there exists $n_m > 0$ such that for all $n \geq n_m$ we have

$$\frac{\sharp((\Lambda_{k_m} \bigtriangleup \Lambda_{k_{m+1}}) \cap A_n)}{\theta(A_n)} \leq \frac{1}{2^m}\,.$$

We may assume that the sequence n_m is increasing (since we can replace each n_m with any larger natural number).

We define now

$$\Lambda'_1 = \Lambda_{n_1}$$

and inductively

$$\Lambda'_{m+1} = \Lambda_{n_{m+1}} \bigtriangleup \left((\Lambda'_m \bigtriangleup \Lambda_{n_{m+1}}) \cap A_{n_m}\right)\,.$$

In fact

$$\begin{aligned} \Lambda'_{m+1} \cap A_{n_m} &= \Lambda'_m \cap A_{n_m} \quad \text{and} \\ \Lambda'_{m+1} \cap (G \backslash A_{n_m}) &= \Lambda_{n_{m+1}} \cap (G \backslash A_{n_m})\,. \end{aligned}$$

Since $\lim_{n\to\infty} \theta(A_n) = \infty$ we have : $\Lambda'_{m+1} \equiv \Lambda_{n_{m+1}}$. By construction we have $\sharp((\Lambda'_m \bigtriangleup \Lambda'_{m+1}) \cap A_n)/\theta(A_n) \leq 2^{-m}$ for each m and n.

Let $1 \leq k < l$ be integers and n be arbitrary. Then

$$\frac{\sharp((\Lambda'_k \bigtriangleup \Lambda'_l) \cap A_n)}{\theta(A_n)} = \frac{\sharp(\bigtriangleup_{i=k}^{l-1}(\Lambda'_i \bigtriangleup \Lambda'_{i+1}) \cap A_n)}{\theta(A_n)} \leq \frac{\sharp(\cup_{i=k}^{l-1}(\Lambda'_i \bigtriangleup \Lambda'_{i+1}) \cap A_n)}{\theta(A_n)}$$
$$\leq \frac{\sum_{i=k}^{l-1} \sharp((\Lambda'_i \bigtriangleup \Lambda'_{i+1}) \cap A_n)}{\theta(A_n)} \leq \sum_{i=k}^{l-1} \frac{1}{2^i} \leq \frac{1}{2^{k-1}}. \tag{3.3}$$

Let n be arbitrary and let $l(n) := 2 + \lfloor \log_2 \theta(A_n) \rfloor$ where $\lfloor \quad \rfloor$ means the integer part. Let $m, k \geq l(n)$. Then by (3.3):

$$\frac{\sharp((\Lambda'_m \Delta \Lambda'_k) \cap A_n)}{\theta(A_n)} \leq \frac{1}{2^{\min\{m,k\}-1}} \leq \frac{1}{2^{l(n)-1}} < \frac{1}{\theta(A_n)}$$

since by the definition of $l(n)$ we have $2^{l(n)-1} > \theta(A_n)$. Hence for each $m, k \geq l(n)$ we have:

$$\begin{aligned} \sharp((\Lambda'_m \bigtriangleup \Lambda'_k) \cap A_n) < 1 &\Rightarrow \sharp((\Lambda'_m \Delta \Lambda'_k) \cap A_n) = 0 \\ &\Rightarrow (\Lambda'_m \Delta \Lambda'_k) \cap A_n = \emptyset \Rightarrow \Lambda'_m \cap A_n = \Lambda'_k \cap A_n. \end{aligned} \tag{3.4}$$

We are now able to define a new set Λ by

$$\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n \tag{3.5}$$

for all n. This is well defined since for $n < n'$, $l(n) \leq l(n')$ and hence by (3.4) we have

$$\Lambda'_{l(n)} \cap A_n = \Lambda'_{l(n')} \cap A_n = (\Lambda'_{l(n')} \cap A_{n'}) \cap A_n.$$

Now, Λ is our required limit. First of all we note that for any compact $K \subset G$, $K \subset A_n$ for some n. Now $\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n$ and $\Lambda'_{l(n)}$ is made up from subsets of $\Lambda_1, \ldots, \Lambda_{l(n)}$. In turn, each of these contains only finitely many points from K since each $\Lambda_k \in \mathcal{D}$. Thus $\Lambda \cap K$ is finite, showing that $\Lambda \in \mathcal{D}$.

Second, we prove that $d(\Lambda, \Lambda'_m) \leq 2^{-(m-1)}$ for each m. Let $n \in \mathbb{N}$ be arbitrary, and let $k \geq \max\{m, l(n)\}$. Then by (3.4)

$$\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n = \Lambda'_k \cap A_n.$$

Hence

$$\frac{\sharp((\Lambda'_m \Delta \Lambda) \cap A_n)}{\theta(A_n)} = \frac{\sharp((\Lambda'_m \Delta \Lambda'_k) \cap A_n)}{\theta(A_n)} \leq \frac{1}{2^{m-1}}$$

because of (3.3) and

$$d(\Lambda, \Lambda'_m) = \limsup_{n\to\infty} \frac{\sharp((\Lambda \,\Delta\, \Lambda'_m) \cap A_n)}{\theta(A_n)} \leq \limsup_{n\to\infty} \frac{1}{2^{m-1}} = \frac{1}{2^{m-1}},$$

showing that

$$\lim_{m\to\infty} \Lambda'_m = \Lambda .$$

However, $\Lambda'_m \equiv \Lambda_{k_m}$ by construction. Hence:

$$\lim_{m\to\infty} \Lambda_{k_m} = \Lambda .$$

So we started with an arbitrary Cauchy sequence and we proved that this has a converging subsequence. This proves that our space is complete.

(**Second Case:** $\lim_{n\to\infty} \theta(A_n) = c < \infty$) Let $\{\Lambda_m\}$ be a Cauchy sequence in $\mathcal{D}$. $\{\Lambda_m\}$ is a Cauchy sequence, hence there exists a m_0 so that $\forall m, l > m_0$ we have $d(\Lambda_m, \Lambda_n) < (2c)^{-1}$.

Let now $m, l > m_0$ be arbitrary. Since

$$\limsup_{n\to\infty} \frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\theta(A_n)} < \frac{1}{2c}$$

there exists an n_0 such that for all $n > n_0$ we have :

$$\frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\theta(A_n)} < \frac{1}{c} .$$

But the sequence $\{\theta(A_n)\}$ is increasing and convergent to c , hence $\theta(A_n) \leq c$ for all n. This implies that:

$$\frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{c} \leq \frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\theta(A_n)} < \frac{1}{c} .$$

It follows that $\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n) < 1$, so

$$\Lambda_m \cap A_n = \Lambda_l \cap A_n \,, \forall n > n_0 .$$

Finally $\Lambda_m = \Lambda_l$ so $\Lambda_m \equiv \Lambda_l$, the sequence is constant from m_0 on, and hence it is convergent. □

Remark 3.2 Note that if we have n'_m an increasing sequence of natural numbers with the property that $n'_m \geq n_m \forall m$, then in the previous proof we can replace $\{n_m\}_m$ by $\{n'_m\}_m$. We will use this fact in the following results.

Remark 3.3 In the second case of the proof of Proposition 3.1 (when the measure of G is finite), we have proved that in fact d induces the discrete topology on $\mathcal{D}$. In this case all the results of the next section become trivial.

Remark 3.4 Since n_m is increasing we have the following description of Λ'_m:

$$\begin{aligned}\Lambda'_m \cap A_{n_1} &= \Lambda_{k_1} \cap A_{n_1} \\ \Lambda'_m \cap (A_{n_i} \backslash A_{n_{i-1}}) &= \Lambda_{k_i} \cap (A_{n_i} \backslash A_{n_{i-1}}),\ 2 \le i \le m \\ \Lambda'_m \cap (G \backslash A_{n_m}) &= \Lambda_{k_m} \cap (G \backslash A_{n_m})\end{aligned}$$

and hence the following description of Λ:

$$\begin{aligned}\Lambda \cap A_{n_1} &= \Lambda_{k_1} \cap A_{n_1} \\ \Lambda \cap (A_{n_i} \backslash A_{n_{i-1}}) &= \Lambda_{k_i} \cap (A_{n_i} \backslash A_{n_{i-1}}),\ i \ge 2\,.\end{aligned}$$

Remark 3.5 Neither the pseudometric d nor the metric d inherited from it is necessarily G-invariant. Invariance has to be derived from the van Hove property of our sequence. However, the van Hove property is a statement about boundary to bulk ratios in terms of measure, whereas the metric is involved with actual counting of points. Only when the points actually "eat up volume" is it possible to link the two ideas. Later, when we introduce uniform discreteness we will be able to do this and then obtain G-invariance on the smaller spaces $\mathcal{D}_V$ (see Corollary 3.14).

With the notation from the proof of Proposition 3.1 we have $\lim_{n\to\infty} \Lambda'_n = \Lambda$ in the local topology. However, in general there is no connection between these two topologies, as the following example shows.

Example 3.6 Let $\Lambda_m = \mathbb{Z} \backslash \{-m, -m+1, \dots, m\}$ and let $A_n := [-n, n]$. Then in the local topology

$$\lim_{n\to\infty} \Lambda_n = \emptyset\,,$$

whereas in the autocorrelation topology we have

$$\lim_{n\to\infty} \Lambda_n = \mathbb{Z}\,.$$

More generally let $\{\Lambda_m\}$ be any sequence of locally finite subsets of G and Λ any other locally finite set subset of G. Let A_n be any van Hove sequence with the property that

$$\begin{aligned} \lim_{n\to\infty} \theta(A_n) &= \infty, \\ \cup_{n\in\mathbb{N}} A_n &= G. \end{aligned}$$

Define $\{\Lambda'_m\}$ by

$$\begin{aligned} \Lambda'_m \cap A_m &:= \Lambda \cap A_m \\ \Lambda'_m \cap (G\backslash A_m) &:= \Lambda_m \cap (G\backslash A_m). \end{aligned}$$

(so we replace the points Λ_m which inside A_m by those of Λ). Then we have $\lim\{\Lambda'_m\} = \lim\{\Lambda_m\}$ in the autocorrelation topology (assuming that the limit exists), but in the local topology $\lim\{\Lambda'_m\} = \Lambda$.

3.3 Stable geometric properties under convergence

As above, G is a σ-compact locally compact abelian group, $\mathcal{A} = \{A_n\}$ is a fixed van Hove sequence, and d is the metric defined by this van Hove sequence on $\mathcal{D}$.

If $\theta(G) < \infty$ all the results in this section are trivial since, as we have pointed out above, the metric then induces the discrete topology. For this reason in all the proofs we study only the case $\theta(G) = \infty$. In particular

$$\lim_{n\to\infty} \theta(A_n) = \infty.$$

Definition 3.7 Let $\{\Lambda_\alpha\}_\alpha \subset G$ be a family of locally finite sets. We say that this family is:

equi-uniformly discrete if there exists a neighbourhood V of $\{0\}$ such that Λ_α is V- uniformly discrete for all α.

equi-relatively dense if there exists a compact set K such that Λ_α is K-relatively dense for all α.

equi-weakly-uniformly discrete if for any compact K in G there exists a constant c_K such that for all α and for all $t \in G$, $\sharp(\Lambda_\alpha \cap (t+K)) \leq c_K$

equi-Delone if the family is equi-relatively dense and equi-uniformly discrete.

Remark 3.8 If a family is V -uniformly discrete then it is W-uniformly discrete for some neighbourhood W of $\{0\}$ with compact closure. If we have a family of V-equi-uniformly discrete sets then we can chose the same W for the entire family.

Lemma 3.9 *Let $A, K \subset G$ with $0 \in K$ and K compact. Let V be a compact neighbourhood of 0 in G with $V = -V$. Then*

$$V + \partial^K(A) \subset \partial^{V+K}(A)$$

(see Definition 2.3 for the notation).

Proof: Let $x \in \partial^K(A)$, $v \in V$. We need to show that $v + x \in \partial^{V+K}(A)$.

Suppose that $x \in (K + A)\backslash A^\circ$. Then $v + x \in V + K + A$, and if $v + x \notin A^\circ$, we have what we wish. If $v + x \in A^\circ \subset A$, then from $x \in G\backslash A^\circ = \overline{G\backslash A}$,

$$v + x \in (V + \overline{G\backslash A}) \cap A \subset \partial^{V+K}(A)\,,$$

as required.

On the other hand, if $x \in (-K + \overline{G\backslash A}) \cap A$ then $v + x \in V - K + \overline{G\backslash A}$, and if $v + x \in A$ we have what we need. If $v + x \notin A$ then from $x \in A$ we have $v + x \in V + A \subset V + K + A$, so

$$v + x \in (V + K + A)\backslash A^\circ \subset \partial^{V+K}(A)\,.$$

□

Definition 3.10 We say that a set Λ has a certain $\mathcal{A}$- statistical property if we can find a set Λ' which has that property and $d(\Lambda, \Lambda') = 0$.

Proposition 3.11 *Let $\Lambda \subset G$ be statistically relatively dense and statistically uniformly discrete. Then Λ is a statistically Delone set.*

Proof: Λ is statistically relatively dense means that there exists $B \subset G$ and a compact K such that B is K-relatively dense and $d(\Lambda, B) = 0$. Λ is statistically uniformly discrete means that there exists $C \subset G$ and a neighbourhood of zero V such that C is V-uniformly discrete and $d(\Lambda, C) = 0$. Without loss of generality we may assume that V has compact closure.

Let $\mathcal{P} = \{E \mid C \subset E \subset B \cup C \text{ and } E \text{ is } V-\text{uniformly discrete}\}$ and order it by inclusion. Since $C \in \mathcal{P}$, $P \neq \emptyset$.

Let $\mathcal{T} \subset \mathcal{P}$ be non-empty and totally ordered. Let $M = \cup\{E \mid E \in \mathcal{T}\}$. Obviously $C \subset M \subset B \cup C$. Suppose by contradiction that M is not V-uniformly discrete. Then there exists $x \in M$ so that

$$(x+V) \cap (M \backslash \{x\}) \neq \emptyset .$$

Let $y \in ((x+V) \cap (M \backslash \{x\})$. Since $x, y \in M$, there exists $E_1, E_2 \in \mathcal{T}$ such that $x \in E_1$ and $y \in E_2$. But $\mathcal{T}$ is totally ordered, so $E_1 \subset E_2$ or $E_2 \subset E_1$. So we can find $E \in \mathcal{T}$ such that

$$x, y \in E \Rightarrow y \in ((x+V) \cap (E \backslash \{x\})) .$$

Hence E is not V-uniformly discrete, contradicting the fact that $E \in \mathcal{T}$.

By Zorn's Lemma we know that there exists a maximal element $Z \in \mathcal{P}$. In particular Z is V-uniformly discrete. We prove that Z is K'-relatively dense , where $K' = K + \overline{V}$ is compact.

Suppose by contradiction that Z is not K'-relatively dense. Then there exists $x \in G$ such that

$$(x+K') \cap Z = \emptyset .$$

Since B is K-relatively dense, there exists $y \in (x+K) \cap B$.

Let $N = Z \cup \{y\}$. Then $y \notin Z$ and Z is maximal in $\mathcal{P}$ implies that $N \notin \mathcal{P}$. But $C \subset Z \subset N \subset B \cup C$ and $N \notin \mathcal{P}$ implies that N is not V-uniformly discrete.

Hence there exists $z \in (y+V) \cap (N \backslash \{y\})$, from which $z \in Z$; and also $z \in (y+V)$ and $y \in (x+K)$ from which $z \in (x+K+V) \cap Z \subset (x+K') \cap Z = \emptyset$. This contradiction proves that Z is K' relatively discrete.

Now

$$C \subset Z \subset B \cup C \Rightarrow 0 = d(C,\Lambda) \leq d(Z,\Lambda) \leq d(B \cup C,\Lambda) \leq d(B,\Lambda) + d(C,\Lambda) = 0 .$$

Hence $d(Z,\Lambda) = 0$. □

Lemma 3.12 *Given an arbitrary compact set K, we can construct $\{n_m\}$ in Proposition 3.1 such that:*

$$\lim_{k\to\infty} \frac{\sum_{m=1}^{k} \theta(\partial^K(A_{n_m}))}{\theta(A_{n_k})} = 0 .$$

Proof:

For this to be true it is enough to have:

$$\theta(\partial^K(A_{n_m})) < \theta(A_{n_m}) \text{ for all } m ,$$

and

$$k^2\theta(A_{n_m}) < \theta(A_{n_k}) \text{ for all } m < k\,.$$

We have to prove two things:

i) the two conditions imply the result of the lemma

ii) we can chose m_n in the proof of 3.1 to satisfy these conditions.

i)

$$\begin{aligned}
\frac{\sum_{m=1}^{k}\theta(\partial^K(A_{n_m}))}{\theta(A_{n_k})} &= \frac{\theta(\partial^K(A_{n_k}))+\sum_{m=1}^{k-1}\theta(\partial^K(A_{n_m}))}{\theta(A_{n_k})} \\
&\leq \frac{\theta(\partial^K(A_{n_k}))+\sum_{m=1}^{k-1}\theta((A_{n_m}))}{\theta(A_{n_k})} \\
&\leq \frac{\theta(\partial^K(A_{n_k}))}{\theta(A_{n_k})}+\frac{\sum_{m=1}^{k-1}(1/k^2)\theta((A_{n_k}))}{\theta(A_{n_k})} \\
&\leq \frac{\theta(\partial^K(A_{n_k}))}{\theta(A_{n_k})}+\frac{k-1}{k^2}\,.
\end{aligned} \tag{3.6}$$

Since both terms on the right side of the inequality go to zero we get the result.
ii) The key for this is the fact that in the proof of Proposition 3.1, as long as $n_m > n_{m-1}$, we can replace each n_m by any larger number. Since

$$\lim_{n\to\infty}\frac{\theta(\partial^K(A_n))}{\theta(A_n)}=0\,,$$

there exists a j such that $\theta(\partial^K(A_n)) < \theta(A_n)$ for all $n > j$. By taking $n_1 > j$ the first condition is satisfied.

Proceeding inductively, let $C(k) := \max_{1\leq m\leq k-1} m^2\theta(A_{n_m})$. At the beginning of the section we showed that we can assume $\lim_{n\to\infty}\theta(A_n)=\infty$. Find $n(k)$ so that $n \geq n(k)$ implies $\theta(A_n) > C(k)$. Choose any $n_k \geq n(k)$. Then for all $m < k$, $\theta(A_{n_k}) > C(k) \geq k^2\theta(A_{n_m})$. The second condition is satisfied. □

Proposition 3.13 *Let $\{\Lambda_m\}$ be a convergent sequence of locally finite sets.*

i) If Λ_n are equi-uniformly discrete then the limit is statistically uniformly discrete (with the same V).

ii) If all Λ_n are equi-Delone sets then the limit is statistically Delone set.

iii) If all Λ_n are equi-relatively dense then the limit is statistically relatively dense.

Proof:

i) Choose V in the definition of the uniform discreteness so that its closure is compact and $V = -V$. Let $K = \overline{V + V}$ and let $\{n_m\}$ be as in the previous lemma. We may also assume that $A_{n_m} + K + K \subset A_{n_{m+1}}$. Let $K' = \overline{V}$. Let Λ be the set constructed in Proposition 3.1 with this $\{n_m\}$ and let

$$B := \Lambda \cap \bigcup_{m \in \mathbb{N}} \partial^{K'}(A_{n_m}) .$$

We prove that $\Lambda \backslash B$ is V-uniformly discrete and B has density zero.

If $x \in \Lambda \backslash B$ then there exists some m such that $x \in A_{n_m} \backslash A_{n_{m-1}}$. Then from the construction of B, $(x+V) \cap \Lambda \subset (A_{n_m} \backslash A_{n_{m-1}}) \cap \Lambda \subset \Lambda_{n_m}$, which itself is V-uniformly discrete. This shows that $\Lambda \backslash B$ is V-uniformly discrete.

On the other hand, for $x \in \Lambda \cap \partial^{K'}(A_{n_m})$ we have, by Lemma 3.9, $x + V \subset \partial^K(A_{n_m})$. We show now that each set $x + V$ contains at most two points from Λ.

Let r be minimal such that $(x + V) \cap (A_{n_r} \backslash A_{n_{r-1}}) \neq \emptyset$. Let $y \in (x + V) \cap (A_{n_r} \backslash A_{n_{r-1}})$. Then $y \in x + V$. Since $V = -V$ we get $x \in y + V$, so

$$x + V \subset y + V + V \subset y + K \subset A_{n_r} + K \subset A_{n_{r+1}} .$$

Thus $x + V \subset A_{n_{r+1}}$.

We show now that $(x + V) \cap A_{n_{r-1}} = \emptyset$:

Suppose by contradiction that $(x + V) \cap A_{n_{r-1}} \neq \emptyset$. From the minimality of r we get that

$$(x + V) \cap (A_{n_{r-1}} \backslash A_{n_{r-2}}) = \emptyset .$$

Thus $(x + V) \cap A_{n_{r-1}} \subset (x + V) \cap A_{n_{r-2}}$, so $(x + V) \cap A_{n_{r-2}} \neq \emptyset$.

Let $y \in (x + V) \cap A_{n_{r-2}}$. As above,

$$x + V \subset y + K \subset A_{n_{r-2}} + K \subset A_{n_{r-1}} ,$$

contrary to $x + V \cap (A_{n_r} \backslash A_{n_{r-1}}) \neq \emptyset$.

Now, since $x + V \subset A_{n_{m+1}}$ and $(x + V) \cap A_{n_{m-1}} = \emptyset$ we get that $x + V \subset (A_{n_{m+1}} \backslash A_{n_{m-1}})$, thus $x + V$ can meet only $(A_{n_{m+1}} \backslash A_{n_m})$ and $A_{n_m} \backslash A_{n_{m-1}}$.

Since each set $x + V$ contains at most two points from Λ we get

$$\theta(V) \sharp (\Lambda \cap \partial^{K'}(A_{n_m})) \leq 2\, \theta(\partial^K(A_{n_m})) .$$

Now the previous lemma gives $d(B, \emptyset) = 0$.

ii) We know from a) that Λ is statistically uniformly discrete. We prove now that it is statistically relatively dense. Let K be given by the equi-relative density. We can assume that $0 \in K$ and $K = -K$. Let $K'' := K + K$.

Let $\{n_m\}$ be as in the previous lemma. We can also ask that $A_{n_m}+K'' \subset A_{n_{m+1}}$. Let Λ be the set constructed in Proposition 3.1 with this $\{n_m\}$ and set

$$B := \bigcup_{m=1}^{\infty} (\Lambda_{n_m} \cup \Lambda_{n_{m+1}}) \cap \partial^{K''}(A_{n_m}) .$$

In the same way as above we can prove that $\Lambda \cup B$ is K-relatively dense and B has density zero.

iii) Let K be defined by the relative density . Let V be a compact neighbourhood of $\{0\}$. Let $K' := K+\overline{V}$. We make the same construction as in b). The only problem is that B may not have density zero.

As in Proposition 3.11 we construct B' a maximal V-uniformly discrete subset of B. Then B' has density zero and, exactly as in Proposition3.11, $\Lambda \cup B'$ is K'-relatively dense. □

$\mathcal{D}_V$ be the set of equivalence classes of V-uniformly discrete subsets of G. We let d_V denote the restriction of the d both to the set of V-uniformly discrete subsets of G and to their equivalence classes $\mathcal{D}_V$. Restriction to $\mathcal{D}_V$ brings with it the property of G-invariance which we will need in the next section.

Corollary 3.14 *Let $V = -V$ be a compact symmetric neighbourhood of $\{0\}$ in G. Then*

i) d_V is a G-invariant on the set of V-uniformly discrete subsets of G;

ii) $\mathcal{D}_V$ is complete and G-invariant.

Proof: i) Let Λ, Λ' be V-uniformly discrete sets and let $t \in G$. Let $W = -W$ be a compact symmetric neighbourhood of $\{0\}$ satisfying $W + W \subset V$. Then for all $x, y \in \Lambda$ with $x \neq y$, $(x + W) \cap (y + W) = \emptyset$. Now

$$\begin{aligned} d(t+\Lambda, t+\Lambda') &= \limsup_{n\to\infty} \frac{\sharp(((t+\Lambda) \bigtriangleup (t+\Lambda')) \cap A_n)}{\theta(A_n)} \\ &= \limsup_{n\to\infty} \frac{\sharp((\Lambda \bigtriangleup \Lambda') \cap (-t+A_n))}{\theta(A_n)} . \end{aligned}$$

Comparing this with $d(\Lambda, \Lambda')$ we see that the difference is due to $(-t+A_n)\backslash A_n$ and $A_n\backslash(-t+A_n)$ both of which are in $\partial^K(A_n)$ for $K := \{0, t, -t\}$; and in magnitude the difference is bounded by the sum of

$$\limsup_{n\to\infty} \frac{\sharp(\Lambda \cap \partial^K(A_n))}{\theta(A_n)}$$

and the corresponding value for Λ'. However, for each $x \in \Lambda \cap \partial^K(A_n)$, $x + W \subset \partial^{W+K}(A_n)$, by Lemma 3.9, and so, taking into account the V-uniformness of Λ,

$$\sharp(\Lambda \cap \partial^K(A_n)) \leq \frac{\theta(\partial^{W+K}(A_n))}{\theta(W)}.$$

There is a similar expression for Λ'. Now the van Hove property shows that the limits are 0, and so $d_V(\Lambda, \Lambda') = d_V(t + \Lambda, t + \Lambda')$ as required.
ii) The set of V-uniformly discrete subsets of G is G-invariant, and by a) so is the pseudo-metric d_V on it. Thus d_V induces a G-invariant metric on $\mathcal{D}_V$. Proposition 3.13 (and its proof) show that $\mathcal{D}_V$ is complete. □

Remark 3.15 i) Let $\Lambda = \mathbb{Z} \setminus \bigcup_{n=1}^{\infty} \{2^n, 2^n+1, ..., 2^n+n\}$. Then Λ is not relatively dense, but $d(\Lambda, \mathbb{Z}) = 0$.

ii) Let $\Lambda' = \mathbb{Z} \cup \bigcup_{n=1}^{\infty} \{2^n + \frac{1}{n}\}$. Then Λ' is not uniformly discrete , but $d(\Lambda', \mathbb{Z}) = 0$.

iii) Let now $\Lambda'' = \Lambda' \setminus \bigcup_{n=1}^{\infty} \{2^n + 1, ..., 2^n + n\}$. Then Λ'' is neither relatively dense or uniformly discrete , but $d(\Lambda'', \mathbb{Z}) = 0$.

3.4 The autocorrelation group $\mathbb{A}(\Lambda)$

Let $\Lambda \subset G$ be any Delone set with FLC.

Definition 3.16 We define a pseudo-metric on G: $d_\Lambda(t, t') = d(t + \Lambda, t' + \Lambda)$.

d_Λ is a G-invariant pseudo-metric (see Corollary 3.14). The interest in this pseudo-metric stems from its connection with the autocorrelation of Λ. For $t \in G$,

$$\eta(t) := \lim_{n\to\infty} \frac{\sharp(\Lambda \cap (t + \Lambda) \cap A_n)}{\theta(A_n)}$$

is the t-**autocorrelation coefficient** of Λ, and

$$\eta := \sum \eta(t)\delta_t$$

is the **autocorrelation (measure)**[2]. If the autocorrelation exists, then in fact for all $t \in G$,

$$d_\Lambda(t, 0) = 2(\eta(0) - \eta(t)).$$

[2]This definition is equivalent to the original one since Λ has FLC.

For more on this, see [7].

Note that d_Λ is not in general a metric on G: for $t, t' \in G$,

$$d_\Lambda(t,t') = 0 \Leftrightarrow d_\Lambda(t-t',0) = 0 \Leftrightarrow d(t-t'+\Lambda, +\Lambda) = 0$$

that is, $t - t'$ is a **statistical period** of Λ.

Definition 3.17 For each open neighbourhood V of 0 and each $\epsilon > 0$ define

$$U(V,\epsilon) := \{(x,y) \in G \times G \mid \exists v \in V \,\text{such that}\, d_\Lambda(-v+x,y) < \epsilon\}.$$

The set of all of these $U(V,\epsilon)$ form a fundamental set of entourages for a uniformity $\mathcal{U}$ on G. Moreover, since each $U(V,\epsilon)$ is G-invariant, we obtain in this way a new topological group structure on G, called the **mixed topology** of G.

Let $\mathbb{A} = \mathbb{A}(\Lambda)$ denote the completion of G in this new topology, which is a new topological group called the **autocorrelation completion** of G.

For each $y \in G$ and each $U \in \mathcal{U}$ define $U[y] := \{x \in G \mid (x,y) \in U\}$.

Definition 3.18 For each $\epsilon > 0$, define the **ϵ-almost periods** of Λ: $P_\epsilon := \{t \in G \mid d_\Lambda(t,0) < \epsilon\}$.

For each $\epsilon > 0$ and V a neighbourhood of $\{0\}$ we have:

$$U(V,\epsilon)[0] = P_\epsilon + V.$$

Remark 3.19 Let $\epsilon_0 := 2d(\Lambda,\emptyset)$. Then for all $\epsilon > \epsilon_0$, $P_\epsilon = G$, and if V is a neighbourhood of $\{0\}$ then $U(V,\epsilon)[0] = G$.

Recall that a uniform space X is said to be **precompact** if and only if its Hausdorff completion $\widehat{X}$ is compact or, equivalently, for each entourage U of X there exists finite cover of X with U-small sets ([10],Theorem 4.2.3).

Lemma 3.20 *Let V be an open neighbourhood of $\{0\}$ with compact closure in the standard topology of G and let $\epsilon > 0$. Then the following are equivalent:*

i) $U(V,\epsilon)[0]$ is precompact in the mixed topology,

ii) for all $0 < \epsilon' < \epsilon$ there exists K a compact set in G with the standard topology so that

$$P_\epsilon \subset P_{\epsilon'} + K.$$

Proof: Suppose that $U(V,\epsilon)[0]$ is precompact and let $0<\epsilon'<\epsilon$. Cover $U(V,\epsilon)[0]$ by finitely many translations of $U(V,\epsilon')[0]$. Then using the previous remark there exist $t_1, ..., t_n$ such that:

$$P_\epsilon + V \subset \bigcup_{i=1}^{n}(t_i + P_{\epsilon'} + V)\,.$$

Since V has compact closure, $K := \overline{\bigcup_{i=1}^{n} t_i + V}$ is compact. Hence:

$$P_\epsilon \subset P_\epsilon + V \subset \bigcup_{i=1}^{n}(t_i + P_{\epsilon'} + V) \subset P_{\epsilon'} + K\,.$$

Conversely, let U' be an open neighbourhood of $\{0\}$ for G in the mixed topology. We need to cover $U(V,\epsilon)[0]$ with finitely many translates of U'. For this purpose we can assume that $U' = U(V',\epsilon')[0]$ for some open neighbourhood V' of $\{0\}$ and some $\epsilon' < \epsilon$. By assumption there exists compact K in the standard topology so that $P_\epsilon \subset P_{\epsilon'} + K$. Then

$$U(V,\epsilon)[0] = P_\epsilon + V \subset P_{\epsilon'} + K + V \subset P_{\epsilon'} + K + \overline{V}\,.$$

Since $K+\overline{V}$ is compact there exist $t_1, ..., t_n$ such that $K + \overline{V} \subset \bigcup_{i=1}^{n}(t_i + V')$, so we obtain

$$U(V,\epsilon)[0] \subset P_{\epsilon'} + K + \overline{V} \subset \bigcup_{i=1}^{n}(t_i + P_{\epsilon'} + V') = \bigcup_{i=1}^{n}(t_i + U(V',\epsilon')[0]) \subset \bigcup_{i=1}^{n}(t_i + U')\,.$$

This proves that $U(V,\epsilon)[0]$ is precompact. □

Proposition 3.21 *$\mathbb{A}$ is compact if and only if for all $\epsilon > 0$, P_ϵ is relatively dense in G (in the standard topology).*

Proof: Suppose that $\mathbb{A}$ is compact. Let $\epsilon > 0$. Choose $\epsilon' > \max\{\epsilon, \epsilon_0\}$. Since $\mathbb{A}$ is compact, G is precompact. Let V be an arbitrary open neighbourhood of $\{0\}$ with compact closure. Then $U(V,\epsilon')[0] = G$ is precompact hence there exists K, compact in G such that

$$G = P_{\epsilon'} \subset P_\epsilon + K\,.$$

Hence P_ϵ is relatively dense.

Conversely, fix any $\epsilon > \epsilon_0$. Let $0 < \epsilon' < \epsilon$. Since $P_{\epsilon'}$ is relatively dense in G then there exists K compact such that $P_\epsilon \subset P_{\epsilon'} + K$. Hence for any V open neighbourhood of $\{0\}$ with compact closure we have by Lemma 3.20 that $G = U(V,\epsilon)[0]$ is precompact. □

Corollary 3.22 *Let G be a σ-compact locally compact abelian group. Let $\Lambda \subset G$ be a locally finite with a well-defined $\mathcal{A}$-autocorrelation. Assume that $\Lambda - \Lambda$ is uniformly discrete. Then the following are equivalent:*

i) *P_ϵ is relatively dense for all $\epsilon > 0$;*

ii) *Λ is pure point diffractive;*

iii) *$\mathbb{A}(\Lambda)$ is compact.*

Proof: Proposition 3.21 proves the equivalence of **i)** and **iii)**. For the equivalence of **i)** and **iii)** see [7], Theorem 5. □

Proposition 3.23 *Let Λ be a Delone subset of the locally compact abelian group G. The following are equivalent:*

i) *$\mathbb{A}$ is locally compact;*

ii) *There exists an $\epsilon > 0$ such that for all $0 < \epsilon' < \epsilon$ there exists compact K with $P_\epsilon \subset P_{\epsilon'} + K$.*

Proof: Suppose that $\mathbb{A}$ is locally compact. Let $\varphi : G \to \mathbb{A}$ be the uniformly continuous map which defines the completion.

Let U' be a compact neighbourhood of $\{0\}$ in $\mathbb{A}$. Then we can find $\epsilon > 0$ and V an open neighbourhood of $\{0\}$ in G such that $\varphi(U(V, \epsilon))[0] \subset U'$. Then $U(V, \epsilon)[0]$ is precompact, so we can apply Lemma 3.20.

Conversely, let V be an open neighbourhood of $\{0\} \in G$ with compact closure. Again by Lemma 3.20, $U(V, \epsilon)$ is precompact. □

Remark 3.24 The completion mapping $\varphi : G \to \mathbb{A}$ provides a natural G-action on $\mathbb{A}$. If $\mathbb{A}$ is compact we have a dynamical system, both topologically and measure theoretically (using Haar measures). Compact or not, the action of G on $\mathbb{A}$ is **minimal** in the sense that every G-orbit is dense in $\mathbb{A}$.

As pointed out in the introduction, Λ has an associated local dynamical hull obtained from the closure of its G-orbit in the local topology. In general, one should not expect any nice relationship between $\mathbb{X}$ and $\mathbb{A}$. However, for model sets, there is a strong connection between the two, as we shall see in Section 3.5.

In the case that G is a real space $\mathbb{R}^d$, the use of the Hausdorff metric d_H on subsets of $\mathbb{R}^d$ allows a simple reformulation of some of the results above. Note that for $A \subset B \subset \mathbb{R}^d$,

$$d_H(A, B) < \infty \Longleftrightarrow B \subset A + K \text{ for some compact set } K \subset \mathbb{R}^d .$$

Now the following are obvious:

Corollary 3.25 *The following are equivalent in* $\mathbb{R}^d$*:*

i) $\mathbb{A}$ *is locally compact;*

ii) There exists an $\epsilon > 0$ *so that for all* $0 < \epsilon' < \epsilon, d_H(P_\epsilon, P_{\epsilon'}) < \infty$.

iii) There exists an $\epsilon > 0$ *such that for all* $0 < \epsilon', \epsilon'' < \epsilon, d_H(P_{\epsilon'}, P_{\epsilon''}) < \infty$.

Corollary 3.26 *The following are equivalent in* $\mathbb{R}^d$*:*

i) $\mathbb{A}$ *is compact;*

ii) for all $\epsilon > 0, d_H(P_\epsilon, \mathbb{R}^d) < \infty$*;*

iii) for all $0 < \epsilon, \epsilon', d_H(P_\epsilon, P_{\epsilon'}) < \infty$.

3.5 Regular model sets

Let $(G, H, \widetilde{L})$ be a cut and project scheme:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array} \tag{3.7}$$

(see Definition 2.16 and following for definitions)

We let $L := \pi_1(\widetilde{L})$ and $^* : L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1)^{-1}|_L$. By hypothesis, the group $\mathbb{T} := (G \times H)/\widetilde{L} = \{(t, t^*)| t \in L\}$ is compact. The obvious G-action on $\mathbb{T}$ makes it into a (minimal, see below) dynamical system.

Let $\Lambda = \Lambda(W)$ be a regular model set. We will assume that for $u \in H$, $u + W = W$ if and only if $u = 0$ [39] [3]. The regular model set Λ is **generic** if $\partial W \cap L^* = \emptyset$.

[3]This condition is called irredundancy. By slightly generalizing the definition of model sets to sets Λ of the form $u + \Lambda(W^\circ) \subset \Lambda \subset u + \Lambda(\overline{W^\circ})$ we can replace H by a quotient of it so as to create a cut and project scheme for Λ which satisfies irredundancy. See [7] for details.

Regular model sets are always Delone sets [26, 25] and have well-defined autocorrelations. In particular we can consider the autocorrelation group $\mathbb{A}(\Lambda)$. A key point is that $\mathbb{A}$ and $\mathbb{T}$ are isomorphic, so in fact $\mathbb{T}$ for a regular model set has a very natural interpretation - namely the completion of the orbit of Λ under the autocorrelation topology.

Proposition 3.27 *Let G be a compactly generated locally compact abelian group and let Λ be a regular model set of the cut and project scheme* (2.1). *Then $\mathbb{A}(\Lambda) \simeq \mathbb{T}$ and the isomorphism is also a G-mapping.*

Proof: There is no loss in assuming that $\Lambda = \{t \in L | t^* \in W\}$. The action of G on $\mathbb{T} = (G \times H)/\widetilde{L}$ is defined by $x + (t + \widetilde{L}) = x + t + \widetilde{L}$, and it is easy to see that the image of G in $\mathbb{T}$ under this map is dense. So $\mathbb{T}$ and $\mathbb{A}$ are the *completions* of G under the respective topologies on G induced by the G-orbits of $\{0\}$ in these two groups. It suffices to show that these topologies on G coincide. For the $\mathbb{T}$-topology, $x \in G$ is close to zero if and only if there is a small open neighbourhood V of G and a pair (t, t^*), where $t \in L$ so that $x - t \in V$ and t^* is close to $\{0\} \in H$.

On the other hand, $x \in G$ is close to zero in the $\mathbb{A}$-topology if and only if there is a small open neighbourhood V of G, and a small $\epsilon > 0$, and a $t \in G$ so that $x - t \in V$ and $t \in P_\epsilon$. Such a t necessarily lies in $\Lambda - \Lambda \subset L$. So we need to show that for $t \in L$, t^* is close to zero in H if and only if $t \in P_\epsilon$ for some small ϵ.

By uniform distribution (see [25, 39])

$$\begin{aligned} d_\Lambda(t,0) &= \lim_{n\to\infty} \frac{1}{\theta(A_n)} \sum_{x\in(\Lambda\backslash(t+\Lambda))\cap A_n} 1 + \lim_{n\to\infty} \frac{1}{\theta(A_n)} \sum_{x\in((t+\Lambda)\backslash\Lambda)\cap A_n} 1 \\ &= \theta_H(W\backslash(t^*+W)) + \theta_H((t^*+W)\backslash W) \\ &= \theta_H(W\backslash(t^*+W)) + \theta_H(W\backslash(-t^*+W)), \end{aligned}$$

since the second term converges to the autocorrelation $d(t + \Lambda, \Lambda) = d_\Lambda(t, 0)$. It remains to prove that $\theta_H(W\backslash(t^* + W))$ converges to 0 if and only if t^* converges to 0.

Now, for all $u \in H$,

$$\theta_H(W\backslash(u+W)) = \theta_H(W) - (\mathbf{1_W} * \widetilde{\mathbf{1_W}})(\mathbf{u}),$$

where $\mathbf{1_W}$ is the indicator function for the set W and $\widetilde{\{\}}$ changes the sign of the argument. This is uniformly continuous in u (for this result on convolutions see [38], Chapter 1) and so disposes of the 'if' part.

Conversely, let $\{u_i\}$ be a net in H for which $\{\theta_H(W\backslash(u_i+W))\}$ converges to 0. The u_i eventually lie in $W-W$ which is compact, so we may assume that in fact the u_i converge, say to u_0. Then $\theta_H(\overset{\circ}{W}\backslash(u_0+W))=0$ and $\overset{\circ}{W}\backslash(u_0+W)$ is open, so $\overset{\circ}{W}\backslash(u_0+W)=\emptyset$. Thus $\overset{\circ}{W}\subset u_0+W$, so $W\subset u_0+W$. A similar argument leads to the reverse inclusion. Then, by our assumptions above, $u_0=0$. □

Corollary 3.28 *For any regular generic model set there is a G-invariant surjective continuous mapping $\mathbb{X}(\Lambda)\longrightarrow\mathbb{A}(\Lambda)$. Furthermore, this mapping is $1-1$ almost everywhere with respect to the Haar measure on $\mathbb{A}(\Lambda)$.*

Proof: By [40] there is a unique G-invariant continuous surjective mapping $\mathbb{X}(\Lambda)\longrightarrow\mathbb{T}(\Lambda)$ which maps Λ to $\{0\}$ in $\mathbb{T}$, and it is $1-1$ $\mathbb{T}$-almost everywhere. □

Remark 3.29 Corollary 3.28 in effect characterizes the regular model sets amongst the relatively dense sets Λ satisfying the Meyer property $\Lambda-\Lambda$ is uniformly discrete [5].

Chapter 4

Diffraction of Meyer sets

In this Chapter we study the diffraction of arbitrary Meyer sets and the similarities between the diffraction patterns of (subsets of) model sets and lattices.

Just as the regular model sets, Meyer sets can be obtained by the cut & project method. Meyer sets require weaker conditions on the window, thus they are more general and easier to characterize than model sets, but also have less long-range properties.

Lattices are highly ordered periodic point sets and we should see this order in the diffraction experiments. It is a well known result, which follows from Poisson's summation formula, that the diffraction of a periodic crystal is a fully-periodic collection of Bragg peaks:

Proposition 4.1 *Let $L \subset \mathbb{R}^d$ be a lattice and let L^* be the dual lattice. Then the diffraction pattern of L is $C\delta_{L^*}$, for some constant C.*

Proposition 4.2 *Let $L \subset \mathbb{R}^d$ be a lattice and let $F \subset \mathbb{R}^d$ be finite. Then, the diffraction pattern of $F + L$ is supported on the dual lattice L^*. In particular, $F + L$ is pure point diffractive.*

Since regular model sets are the projection of a part of a lattice using a nice window, they should preserve some of the long range order of the lattices. Indeed, they have as nice diffraction patterns:

Proposition 4.3 ([39], [7]) *If Λ is a regular model set then Λ is pure point diffractive (with respect to any averaging sequence $\mathcal{A}$).*

Given a discrete point set Λ which shows long range order, if we look to an arbitrarily subset $S \subset \Lambda$ we expect S to preserve only some of its order. In the case of lattice subsets, the following result has been proved by Michael Baake:

Proposition 4.4 [3] *Let L be a lattice in $\mathbb{R}^d$ and let $S \subset L$. Let L^* be the dual lattice of L. Then the diffraction pattern of S is L^*-invariant. In particular each of the continuous and discrete spectrum is either zero or supported on a relatively dense set.*

In this Chapter we prove that similar results hold for Meyer sets. More exactly we prove the following Propositions:

Proposition 4.15 *Let M be a Meyer set and suppose that its autocorrelation η exists with respect to an averaging sequence $\mathcal{A}$. Then the set of Bragg peaks (relative to $\mathcal{A}$) is relatively dense. Moreover, if M is not pure point diffractive, it has a relatively dense support for the continuous spectrum as well.*

Corollary 4.37 *Let Λ be a Meyer set in $\mathbb{R}^d$. Then, there exists a constant $C > 0$ such that, for every $\epsilon > 0$ the pure point diffraction measure of $\Lambda, (\widehat{\eta})_{pp}$, is ϵ-invariant under translations by the ϵ-dual set $\Delta^{\epsilon/C}$; where $\Delta = \Lambda - \Lambda$.*

Proposition 4.41 *Let Λ be a Meyer set in $\mathbb{R}^d$. Then for any $0 < a < \widehat{\eta}(\{0\}), I(a)$ the set of Bragg peaks with intensity at least a, is a Meyer set.*[1]

One can see Proposition 4.41 as a necessary diffraction condition for a point set to be a Meyer set.

We also prove in Section 4.2.2 that for Meyer sets, unless we are in the periodic case, the set of Bragg peaks is strictly larger than the set of visible Bragg peaks.

In Section 4.1.3 we study the existence of Bragg peaks in a diffraction experiment. The well known Riemann-Lebesgue lemma tells us that the Fourier transform of any absolutely continuous measure must vanish at infinity. This gives a necessary condition for the autocorrelation of a point set to have absolutely continuous diffraction spectrum.

In the case of translation bounded measures, a necessary and sufficient condition for a measure to have a continuous Fourier transform is given in [21]. The condition (namely null-weak almost periodicity) is mentioned below, but generally is not easy to check. In Section 4.1.3 we provide an equivalent asymptotic condition for the case of positive translation bounded measures. In particular we get a simple necessary and sufficient condition on asymptotic behavior of the autocorrelation coefficients for the diffraction to be a continuous measure (i.e. no Bragg peaks)[2]. We also provide a necessary and sufficient condition for no Bragg peaks to appear in the

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[1]For a subset of a lattice, it follows easily from Proposition 4.4 that the set of Bragg peaks consists of finitely many translates of the dual lattice.

[2]This result is interesting only in the case of diffraction from (possible complex valued) weighted combs with non-negative autocorrelation. The diffraction pattern of a Delone set has always a Bragg peak in the origin.

diffraction pattern, when the autocorrelation is not positive, but it is supported on an uniformly discrete set (in particular for the autocorrelations of a complex weighted comb supported on a Meyer set).

Although physical applications are concerned with $\mathbb{R}^d$ (when $d = 2, 3$), much of the theory that we develop is valid for (compactly generated) σ-compact, locally compact abelian groups, and we use this setting as long as we can.

4.1 Almost periodic measures and long-range order in Meyer sets

4.1.1 The diffraction of Meyer Sets

For most of this part we work with an arbitrarily locally compact abelian group G. For the main applications of the results, we will restrict to the case $G = \mathbb{R}^d$, since only here do we have the complete characterization of Meyer sets. We will work with respect to a fixed, but arbitrary van Hove sequence $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$.

Lemma 4.5 *Let M be a Meyer set in $\mathbb{R}^d$. Then M is subset of a regular model set.*

Proof: M Meyer set implies that M is a subset of a model set [27]. Let H be the internal space of this model set and W be its window. Let U be an open set in H such that $\overline{U}$ is compact and $\theta_H(\partial U) = 0$.

$\overline{W}$ is compact implies $\overline{W} \subset \bigcup_{i=1}^{n}(t_i + U)$ for some $t_1, ..., t_n \in H$. Let $V := \bigcup_{i=1}^{n}(t_i + U)$, then V is open , has compact closure and $\theta_H(\partial V) = 0$.

Let

$$\Lambda = \Lambda(V) \,,$$

in the notation of [27]. Then $M \subset \Lambda$ and Λ is a regular model set. □

Proposition 4.6 *Let Λ be a pure point diffractive set in G with finite local complexity, and let M be a subset of Λ. Suppose that both their autocorrelations η_M and η_Λ exist with respect to a $\mathcal{A}$. Let*

$$\eta_M = (\eta_M)_S + (\eta_M)_0 \,,$$

be the decomposition into the strongly and null-weakly almost periodic parts. Then $(\eta_M)_S, (\eta_M)_0$ are pure point measures.

Proof: Since Λ is pure point diffractive, $\eta_\Lambda = (\eta_\Lambda)_S$.

Since $\eta_M \geq 0$ we get by Proposition 2.38 that $(\eta_M)_S \geq 0$.

Now:

$$M \subset \Lambda \implies \eta_M \leq \eta_\Lambda \implies \eta_\Lambda - \eta_M \geq 0 \implies (\eta_\Lambda - \eta_M)_S \geq 0\,,$$

$$(\eta_\Lambda - \eta_M)_S \geq 0,\ \eta_\Lambda = (\eta_\Lambda)_S \implies (\eta_M)_S \leq \eta_\Lambda\,.$$

So we get:

$$0 \leq (\eta_M)_S \leq \eta_\Lambda \quad \text{hence} \quad \mathrm{supp}((\eta_M)_S) \subset \mathrm{supp}(\eta_\Lambda) \subset \Lambda - \Lambda\,.$$

In particular $(\eta_M)_S$ is a pure point measure.

$$\begin{aligned}(\eta_M)_0 &= \eta_M - (\eta_M)_S \Rightarrow \mathrm{supp}((\eta_M)_0) \subset \mathrm{supp}((\eta_M)_S) \cup \mathrm{supp}(\eta_M)\\ &\subset (\Lambda - \Lambda) \cup (M - M) = \Lambda - \Lambda\,.\end{aligned}$$

□

Proposition 4.7 *Let $0 \neq f \in C_U(G)$ be a strongly almost periodic function and let $A = \mathrm{supp}(f)$. Then G can be covered by finitely many translates of A.*

Proof: $f \neq 0 \Rightarrow f(x) \neq 0$ for some $x \in G$. Since $\tau_{-x}f$ is strongly almost periodic and $\mathrm{supp}(\tau_x f) = x + A$, without loss of generality we may assume that $f(0) \neq 0$. Let $0 < \epsilon < |f(0)|$.

For $g \in C_U(G)$ we define $B_\epsilon(g) = \{h \in C_U(G) \mid \| g - h \|_\infty < \epsilon\}$. $B_\epsilon(g)$ is an open set. $D_f = \{\tau_x f\}_{x\in G}$ is a subset of C_f which implies that D_f has compact closure. It is easy to see that

$$\overline{D_f} \subset \bigcup_{x\in G} B_\epsilon(\tau_x f)\,.$$

Since $\overline{D_f}$ is compact, there exists $x_1, ..., x_n \in G$ such that

$$\overline{D_f} \subset \bigcup_{i=1}^{n} B_\epsilon(\tau_{x_i} f)\,.$$

So for any fixed $x \in G$ there is an i such that $\tau_x f \in B_\epsilon(\tau_{x_i} f)$. Thus, we get:

$$\| \tau_x f - \tau_{x_i} f \|_\infty < \epsilon \text{ and } |\tau_x f(x_i) - \tau_{x_i} f(x_i)| < \epsilon < |f(0)|\,.$$

Consequently,

$$|f(-x+x_i) - f(0)| < f(0), \text{ hence } f(-x+x_i) \neq 0, \text{ hence } -x+x_i \in A, \text{ so } x \in x_i - A\,.$$

Since $x \in G$ was arbitrary, we get $G \subset \bigcup_{i=1}^{n}(x_i - A)$, thus $G = -G \subset \bigcup_{i=1}^{n}(-x_i + A)$.

□

Proposition 4.8 *The support of any non-trivial strongly almost periodic measure on G is relatively dense.*

Proof: Let μ be a non-trivial strongly almost periodic measure on G. Let $A = \text{supp}(\mu)$. Let $f \in C_c(G)$ be a non-negative function such that $f \neq 0$, and let $K = \text{supp}(f)$.

If $0 \neq f * \mu(x) = \int_G f(x-y)d\mu(y) = \int_A f(x-y)d\mu(y)$ then there exists $y \in A$ such that $f(x-y) \neq 0$, so $x - y \in K$, hence $x \in K + A$.

Thus we get that $f * \mu \equiv 0$ on $G \backslash (K + A)$. Since $K + A$ is closed we get that:

$$\text{supp}(f * \mu) \subset A + K\,.$$

Since μ is strongly almost periodic we get that $f * \mu$ is a strongly almost periodic function by Remark 2.32. Applying the previous proposition we get that there exist $x_1, ..., x_n \in G$ so that

$$G \subset \bigcup_{i=1}^{n}(x_i + \text{supp}(f * \mu)) \subset \bigcup_{i=1}^{n}(x_i + A + K) = \bigcup_{i=1}^{n}(x_i + K) + A\,,$$

proving that A is relatively dense. □

Remark 4.9 Let S be a Delone set and suppose that its autocorrelation η exists with respect to $\mathcal{A}$. Since η is positive and positive definite, we know by Proposition 2.38 that η_S is positive and positive definite. Thus each of η and η_S are Fourier transformable and also $\widehat{\eta}$ and $\widehat{\eta_S}$ are Fourier transformable. Taking the difference we get also that η_0 is Fourier transformable and also $\widehat{\eta_0}$ is Fourier transformable.

Thus we can apply the Corollary 2.40 for η, η_S and η_0.

Proposition 4.10 *Let S be a subset of a pure point diffractive set Λ with FLC, and suppose that both their autocorrelations η and η_Λ exist with respect to $\mathcal{A}$. Then each of* $\text{supp}(\widehat{\eta})_{pp}$ *and* $\text{supp}(\widehat{\eta})_c$ *is either relatively dense or empty.*

Proof: We know from Proposition 4.6 that η_S and η_0 are pure point measures. Thus, from Corollary 2.40, we get that $(\widehat{\eta})_{pp}$ and $(\widehat{\eta})_c$ are strongly almost periodic measures. The result follows now from Proposition 4.8. □

Proposition 4.11 *Let S be a set in G which has finite local complexity, and suppose that S is pure point diffractive. Then S has a relatively dense set of Bragg peaks.*

Proof: Let η be the autocorrelation of S and let $\widehat{\eta}$ be its Fourier transform. Since η is pure point we get that $\widehat{\eta}$ is strongly almost periodic. We know that $\widehat{\eta}$ is also pure point. Then, the set of Bragg peaks is dense in its support. Applying Proposition 4.8 we get that the set of Bragg peaks is relatively dense. $\square$

Lemma 4.12 *Let A be a relatively dense set in G and let B be another subset of G such that $A \subset B$ and B has finite local complexity. Then there exists a finite set F such that:*

$$B \subset A + F .$$

Proof: Let K be compact such that $G = A + K$, and define $F = (B - A) \cap K$. F is finite since $F \subset (B - B) \cap K$.

Let $y \in B$. Since $G = A + K$ we get $y = x + z$ with $x \in A$ and $z \in K$.

But $z = y - x \in B - A$, hence $z \in F$. This proves that: $B \subset A + F$. $\square$

Lemma 4.13 *Let M be a relatively dense subset of a set Λ with FLC, and suppose that both their autocorrelations η and η_Λ exist with respect to $\mathcal{A}$. Then there exists a finite set F such that:*

$$\eta_\Lambda \leq \sum_{x,y \in F} \tau_{(x-y)} \eta .$$

In particular, if Λ is pure point diffractive, $\eta_S \neq 0$.

Proof: From Lemma 4.12 we get that there exists a finite set F with $\Lambda \subset F + M$.

Let $t \in M - M$. Then :

$$\begin{aligned}
\frac{\sharp((\Lambda \cap (t + \Lambda)) \cap A_n)}{\theta(A_n)} &\leq \frac{\sharp(((M + F) \cap (t + M + F)) \cap A_n)}{\theta(A_n)} \\
&\leq \frac{\sharp(\bigcup_{x,y \in F}((M + y) \cap (t + M + x)) \cap A_n)}{\theta(A_n)} \\
&\leq \frac{\sum_{x,y \in F} \sharp(((M + y) \cap (t + M + x)) \cap A_n)}{\theta(A_n)} \\
&= \sum_{x,y \in F} \frac{\sharp(((M + y) \cap (t + M + x)) \cap A_n)}{\theta(A_n)} ,
\end{aligned} \tag{4.1}$$

where θ is Haar measure in G.

Now letting $n \to \infty$ we get:

$$\eta_\Lambda \leq \sum_{x,y \in F} \tau_{(x-y)} \eta \,.$$

For the last claim we see that if Λ is pure point diffractive then

$$\eta_\Lambda = (\eta_\Lambda)_S \leq \left(\sum_{x,y \in F} \tau_{(x-y)} \eta \right)_S = \sum_{x,y \in F} (\tau_{(x-y)} \eta)_S = \sum_{x,y \in F} \tau_{(x-y)} \eta_S \,.$$

Since Λ is relatively dense we get that $\eta_\Lambda(\{0\}) \neq 0$, hence $\eta_S \neq 0$. □

Lemma 4.14 *Let Λ be a pure point diffractive set with FLC and let M be a subset of Λ. Suppose that both autocorrelations η_Λ and η_M exist with respect to $\mathcal{A}$. Then each of the pure point and continuous diffraction spectra for M is either empty or supported on a relatively dense set. Moreover, if M is relatively dense, it has a relatively dense set of Bragg peaks.*

Proof: The proof is a consequence of Proposition 4.10 and Lemma 4.13. □

Proposition 4.15 *Let M be a Meyer set and suppose that its autocorrelation η exists with respect to $\mathcal{A}$. Then the set of Bragg peaks is relatively dense. Moreover, if M is not pure point diffractive, it has a relatively dense support for the continuous spectrum as well.*

Proof: Since M is Meyer we get that M is a subset of a regular model set (Lemma 4.5). Since the autocorrelation of a regular model set exists with respect to any van Hove sequence [25] and the regular Model sets are pure point diffractive ([39] or [7]), the Proposition is an easy consequence of Lemma 4.14. □

Remark 4.16 In Section 4.2.2 we show that there is a connection between the set of Bragg peaks for a Meyer set and the ϵ-dual sets.

Proposition 4.17 *Let M be a Meyer set. Let D be a Delone set such that $M \subset D$ and suppose that the autocorrelation η_D exists with respect to $\mathcal{A}$. Then D has an infinite set of Bragg peaks.*

Proof: Suppose by way of contradiction that D has only finitely many Bragg peaks.

We take any regular model set Λ such that $M \subset \Lambda$. Then there exists F finite for which $\Lambda \subset M + F \subset D + F$. Just as in Corollary 4.13 we get :

$$\eta_\Lambda \leq \sum_{x,y \in F} \tau_{(x-y)} \eta_D \,.$$

Looking at the strongly almost periodic parts we get:

$$\eta_\Lambda \leq \sum_{x,y\in F} \tau_{(x-y)}(\eta_D)_S \,.$$

But η_Λ is a non-trivial pure point measure, hence $((\eta_D)_S)_{pp} \neq 0$.

Since D has only finitely many Bragg peaks, $(\widehat{\eta_D})_{pp} = \sum_{z\in B} c_z\delta_z$ for some finite set B. Thus $\widetilde{(\eta_D)_S} = \widehat{\widehat{(\eta_D)_S}} = \widehat{(\widehat{\eta_D})_{pp}} = \sum_{z\in B} c_z e^{-2\pi i<z,.>}$ is a continuous measure. But this contradicts the fact that $((\eta_D)_S)_{pp} \neq 0$. □

Remark 4.18 We used the intermediate regular model set Λ to prove that $((\eta_D)_S)_{pp} \neq 0$ because we don't know whether the autocorrelation of M exists with respect to $\mathcal{A}$. One could also use the fact that for any point set C and any van Hove sequence there exists a subsequence with respect to which the autocorrelation of C exists [4].

Definition 4.19 Let D be a locally finite subset of G. We say that D has **$\mathcal{A}$-zero density** if

$$\lim_{n\to\infty} \sup \frac{\sharp(D \cap A_n)}{\theta(A_n)} = 0 \,.$$

We say that two sets B and C are **$\mathcal{A}$-statistically the same** if $B \triangle C$ has $\mathcal{A}$-zero density.

We say that B is an **$\mathcal{A}$-statistical Meyer set** if there exists a Meyer set M such that B and M are $\mathcal{A}$-statistically the same.

Remark 4.20

i) If two sets are $\mathcal{A}$-statistically the same, and the autocorrelation of one can be computed with respect to $\mathcal{A}$, then the autocorrelation of the other can be computed with respect to $\mathcal{A}$ and the autocorrelations are equal.

ii) For a given set B we can find a set C which is $\mathcal{A}$-statistically the same as B and contains a Meyer subset if and only if B contains an $\mathcal{A}$-statistical Meyer subset.

iii) For more properties of the statistical equality, see Chapter 3.

Corollary 4.21 *Let S be a Delone set, whose autocorrelation exists with respect to the van Hove sequence $\mathcal{A}$. If S contains a $\mathcal{A}$-statistical Meyer set, S has an infinite set of Bragg peaks.*

Proof: From the previous remark we know that we can find a set B which contains a Meyer set and has the same autocorrelation as S. Applying Proposition 4.17 we are done. □

4.1.2 The diffraction of weighted combs with Meyer set support

Throughout this section we work with a real weighted comb:

$$\omega = \sum_{x \in M} \omega(\{x\})\delta_x \,,$$

where M is a Meyer set. We require that ω is a translation bounded measure, or equivalently that the set $\{\omega(\{x\}) \mid x \in M\}$ is bounded. Also, for the entire section we consider a fixed, but arbitrary van Hove sequence $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$.

We say that the autocorrelation of ω exists with respect to $\mathcal{A}$ if

$$\eta(z) = \lim_{n \to \infty} \frac{1}{\theta(A_n)} \sum_{\substack{S \cap A_n \\ x-y=z}} \omega(\{x\})\overline{\omega(\{y\})} \,,$$

exists for all $z \in M - M$. In this case we define:

$$\eta = \sum_{z \in M-M} \eta(z)\delta_z \,,$$

and we call it the **autocorrelation** of ω. [3]

We will further assume that the autocorrelation exists with respect to $\mathcal{A}$, and that the autocorrelation η_M of M exists with respect to $\mathcal{A}$. This is possible since, if we pick an arbitrary van Hove sequence, the autocorrelation of ω exists for a subsequence [4] and we can repeat the argument for M.

We assume for the entire section that $\widehat{\eta}$ is Fourier transformable. Then $(\widehat{\eta})_{pp}$ and $(\widehat{\eta})_c$ are also Fourier transformable ([21], Theorem 11.2).

Proposition 4.22 *Let $\hat{\eta}$ be Fourier transform of the autocorrelation and B the set of Bragg peaks . Let also $\{D_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^d$ be an arbitrary van Hove sequence.*

Then the following are equivalent:

i) B is non-empty;

ii) $\hat{\eta}$ is not continuous;

iii) $\lim_{k\to\infty} \frac{|\eta|(D_k)}{\theta(D_k)} \neq 0$ (i.e. either the limit doesn't exist or it exists and is not zero);

[3] For the properties of the autocorrelation one can see [7]. There, it is also shown that in the case of equal weighted point sets, this definition is equivalent with the one we use.

iv) B is relatively dense ;

v) B is an infinite set.

Proof: **iv)**$\Rightarrow$ **v)** $\Rightarrow$ **i)** $\Leftrightarrow$ **ii)** are trivial.

ii) $\Leftrightarrow$ **iii)** follows from Proposition 4.30, proved in the next section.

ii) $\Rightarrow$ **iv)**

Let C be such that $|\omega(\{x\})| < C$ for all $x \in M$. Then $-C\delta_M \leq \omega \leq C\delta_M$.

It is easy to see that

$$-C^2\eta_M \leq \eta \leq C^2\eta_M \,.$$

Since $(\eta_M)_S$ is a pure point measure, by Proposition 4.6 and Proposition 2.38 we get that η_S is a pure point measure.
Now, since $\hat{\eta}$ is not continuous, we obtain $\hat{\eta}_{pp} \neq 0$. But we know that $\hat{\eta}_{pp} = \widehat{\eta_S}$, so $\hat{\eta}_{pp}$ has a pure point Fourier transform, hence is strongly almost periodic.

Applying Proposition 4.8 we get that B is relatively dense. □

Remark 4.23 A similar argument shows that if there exists $r > 0$ for which $\omega(\{x\}) > r$ for all $x \in M$ then the support of the pure point part of the diffraction pattern of ω is relatively dense. Moreover, in this case, η is positive and positive definite, thus $\widehat{\eta}$ and $(\widehat{\eta})_{pp}$ are Fourier transformable.

4.1.3 Translation bounded measures with continuous Fourier Transform

Let μ be a positive translation bounded and transformable measure on G. In this section we will provide a necessary and sufficient condition for $\widehat{\mu}$ to be continuous. The idea behind the proof is that whenever $f \in C_c(G)$ and μ are positive, we can ignore the absolute value in the definition of null-weak almost periodicity.

We want the mean of $f * \mu$ to be zero. The key to the result is that $(f * \mu)\mid_A$ is different from $f * (\mu \mid_A)$ only on the K-boundary of A, for some compact K. The integral of the second function over G is just $\mu(A) \int_G f$. Thus, in this case, the average integral of $f * \mu$ is just the average of μ multiplied with $\int_G f$.

Proposition 4.24 ([21], Corollary 11.1) *Let μ be a translation bounded transformable measure on G and suppose that $\widehat{\mu}$ is transformable and translation bounded. Then $\widehat{\mu}$ is a continuous measure if and only if $\mu \in \mathcal{WAP}_0(\mathrm{G})$.*

□

Definition 4.25 For $g \in C_c(G)$ let

$$\theta(g) := \int_G g(x) d\theta(x) .$$

Proposition 4.26 *Let μ be a positive translation bounded measure on G and let $g \in C_c(G)$ be positive. Let $K = -\text{supp}(g) \cup \text{supp}(g)$ and let $A, B \subset G$ be two sets such that $A + K \subset B$. Then*

i) $(\mu\mid_A) * g \leq (\mu * g)\mid_B$,

ii) $(\mu * g)\mid_A \leq (\mu\mid_B) * g$.

Proof: i) Let $y \in G$. $(\mu\mid_A) * g(y) = \int_G g(y-x) d(\mu\mid_A)(x) = \int_A g(y-x) d\mu(x)$.
If $y \notin A + K$ then $(\mu\mid_A) * g(y) = 0$ and $(\mu * g)\mid_B \geq 0$. Thus **i)** holds.
If $y \in A + K$ then $y \in B$, so

$$(\mu * g)\mid_B (y) = \mu * g(y) = \int_G g(y-x) d\mu(x) \geq \int_A g(y-x) d\mu(x) ,$$

and again **i)** holds.
ii) Let $y \in G$. If $y \notin A$ then $(\mu * g)\mid_A (y) = 0 \leq (\mu\mid_B) * g$.
If $y \in A$ then $y + K \subset B$. Then

$$\begin{aligned}(\mu\mid_B) * g(y) &= \int_G g(y-x) d(\mu\mid_B)(x) = \int_{y-\text{supp}(g)} g(y-x) d(\mu\mid_B)(x) \\ &= \int_{y+K} g(y-x) d\mu(x) ,\end{aligned}$$

the last equality following the fact that $y + K \subset B$. Since $g(y-x) \neq 0$ only if $x \in y + K$ we get

$$(\mu\mid_B) * g(y) = \int_G g(y-x) d\mu(x) \geq (\mu * g)\mid_A (y) .$$

□

Proposition 4.27 *If μ is positive, translation bounded and transformable and $g \in C_c(G)$ is positive, then $g * \mu$ is amenable and*

$$M(g * \mu) = \theta(g) \lim_{n\to\infty} \frac{\mu(A_n)}{\theta(A_n)}$$

for any sequence $\{A_n\}_n$ which has the **M**-*property* [4]. *In particular this is true if $\{A_n\}_n$ is a van Hove sequence.*

[4]See Appendix A for definition

Proof: Since μ is transformable we know that μ is a weakly almost periodic measure ([21] Theorem 11.1). Hence $\mu * g$ is an amenable function ([21] Corollary 5.4) and by Proposition A.6 we have

$$M(g * \mu) = \lim_{n\to\infty} \frac{\int_{A'_n} g * \mu(x) d\theta}{\theta(A'_n)}, \tag{4.2}$$

for any sequence $\{A'_n\}_n$ which has the Følner property. In particular we obtain that the limit exists.

Let $K = -\mathrm{supp}(g) \cup \mathrm{supp}(g)$. From Proposition 4.26 we have:

$$\int_{A_n^{K-}} g * \mu(x) d\theta \leq \int_G g * (\mu\,|_{A_n})(x) d\theta \leq \int_{A_n^{K+}} g * \mu(x) d\theta\,.$$

Since everything is positive, by the Tonelli Theorem we can change the order of integration. Thus we get:

$$\begin{aligned}&\int_G g * (\mu\,|_{A_n})(x) d\theta(x) = \int_G \Big(\int_G g(x-y) d(\mu\,|_{A_n})(y)\Big) d\theta(x) \\ &= \int_G \int_G g(x-y) d\theta(x) d(\mu\,|_{A_n})(y) = \int_G \theta(g) d(\mu\,|_{A_n})(y) = \theta(g)\mu(A_n)\,.\end{aligned} \tag{4.3}$$

So we get:

$$\int_{A_n^{K-}} g * \mu(x) d\theta \leq \theta(g)\mu(A_n) \leq \int_{A_n^{K+}} g * \mu(x) d\theta\,.$$

Dividing by $\theta(A_n)$ we get:

$$\frac{\int_{A_n^{K-}} g * \mu(x) d\theta}{\theta(A_n^{K-})} \frac{\theta(A_n^{K-})}{\theta(A_n)} \leq \theta(g) \frac{\mu(A_n)}{\theta(A_n)} \leq \frac{\int_{A_n^{K+}} g * \mu(x) d\theta}{\theta(A_n^{K-})} \frac{\theta(A_n^{K-})}{\theta(A_n)}\,.$$

Using (4.2) and the definition of M-sequences we get that the first and last terms converge to $M(g * \mu)$, so:

$$M(g * \mu) = \theta(g) \lim_{n\to\infty} \frac{\mu(A_n)}{\theta(A_n)}\,.$$

□

Corollary 4.28 *If μ is a positive transformable measure on G and $\{A_n\}_n$ has the* **M**-*property then:*

$$M(\mu) = \lim_{n\to\infty} \frac{\mu(A_n)}{\theta(A_n)}\,.$$

Proof: In ([21], Corollary 5.4) it is proved that if $g \in C_c(G)$ is positive with $\theta(g) = 1$, then

$$M(\mu) = M(\mu * g).$$

Hence the result follows from Proposition 4.27. □

Corollary 4.29 *If μ is a positive, translation bounded and transformable measure, and $\{A_n\}$ has the* **M***-property (in particular if $\{A_n\}$ is a van Hove sequence), then the following are equivalent:*

i) $\mu \in \mathcal{WAP}_0(\mathrm{G})$,

ii) μ *has continuous Fourier transform,*

iii) $\lim_{n\to\infty} \frac{\mu(A_n)}{\theta(A_n)} = 0$,

iv) $M(\mu) = 0$.

Proof: We already stated that **i)** $\Leftrightarrow$ **ii)** and **iii)** $\Leftrightarrow$ **iv)**.

i) $\Rightarrow$ **iii)**: Let $g \in C_c(G)$ be a positive function, not identical to zero. Then $\theta(g) > 0$. By Definition 2.31 and Proposition 4.27:

$$0 = M(|g * \mu|) = M(g * \mu) = \theta(g) \lim_{n\to\infty} \frac{\mu(A_n)}{\theta(A_n)}.$$

Dividing by $\theta(g) \neq 0$ we obtain **iii)**.

iii) $\Rightarrow$ **i)** : Let $g \in C_c(G)$ be arbitrary. It is easy to see that:

$$M(|g * \mu|) \leq M(|g| * \mu).$$

Applying Proposition 4.27 to $|g| \in C_c(G)$ we obtain $M(|g| * \mu) = 0$. Since $M(|g * \mu|) \geq 0$ we are done. □

We use $|\mu|$ for the variation measure of μ.

Proposition 4.30 *Let μ be a transformable, translation bounded measure and let $\widehat{\mu}$ be translation bounded. Let $\{A_n\}$ have the* **M***-property (in particular let $\{A_n\}$ be any van Hove sequence).*

Consider the following statements:

i) $\mu \in \mathcal{WAP}_0(\mathrm{G})$,

ii) μ *has continuous Fourier transform,* *iii)* $|\mu|$ *has continuous Fourier transform,*

iv) $|\mu| \in \mathcal{WAP}_0(\mathrm{G})$,

v) $\lim_{n\to\infty} \frac{|\mu|(A_n)}{\theta(A_n)} = 0$,

vi) $M(|\mu|) = 0$.

Then always iii) $\Leftrightarrow$ *iv)* $\Leftrightarrow$ *v)* $\Leftrightarrow$ *vi)* $\Rightarrow$ *i)* $\Leftrightarrow$ *ii).*
Moreover, if $\mathrm{supp}(\mu)$ *is uniformly discrete, all statements are equivalent.*

Proof: The equivalence **iii)** $\Leftrightarrow$ **iv)** $\Leftrightarrow$ **v)** $\Leftrightarrow$ **vi)** is obvious from Corollary 4.29, and **i)** $\Leftrightarrow$ **ii)** follows from Proposition 4.24.
iii) $\Rightarrow$ **i)** Let $g \in C_c(G)$ be arbitrary. It is easy to see that: $|g * \mu| \leq |g| * |\mu|$. Thus $M(|g*\mu|) \leq M(|g|*|\mu|)$. Using **iii)** we obtain: $M(|g*\mu|) = 0$. Thus $\mu \in \mathcal{WAP}_0(\mathrm{G})$.

We prove now, under the assumption $\mathrm{supp}(\mu)$ is uniformly discrete, that **i)** implies **vi)**. Let $S := \mathrm{supp}(\mu)$ and let U be open such that $(S-S) \cap U = \{0\}$. Let V be an open neighbourhood of 0 such that $V - V \subset U$. Since $S = \mathrm{supp}(\mu)$ is uniformly discrete then $\mu = \sum_{s\in S} c_s\delta_s$ for some $c_s \in \mathbb{R}$, and it is easy to see that $|\mu| = \sum_{s\in S}|c_s|\delta_s$.

Let $g \in C_c(G)$ be arbitrary so that $K := \mathrm{supp}(g) \subset V$. We prove that $|g * \mu| = |g| * |\mu|$:

$$g * \mu(x) = \int_G g(x-y)d\mu(y) = \sum_{y\in S} g(x-y)\mu(\{y\})$$

$$g * \mu(x) = \int_G g(x-y)d\mu(y) = \sum_{y\in S} g(x-y)\mu(\{y\}) .$$

If $y \notin x - K$ then $x - y \notin K$, hence $g(x-y) = 0$. Thus

$$g * \mu(x) = \sum_{y\in S\cap(x+K)} g(x-y)\mu(\{y\}) .$$

By the choice of V we have $\sharp(S \cap (x-K)) \leq 1$, whence

$$\left|\sum_{y\in S\cap(x-K)} g(x-y)\mu(\{y\})\right| = \sum_{y\in S\cap(x-K)} |g(x-y)\mu(\{y\})| .$$

and

$$|g * \mu(x)| = \sum_{y\in S\cap(x-K)} |g(x-y)\mu(y)| = \sum_{y\in S\cap(x-K)} |g(x-y)||\mu(y)| = |g| * |\mu|(x) .$$

Now assume **i)**. By Definition 2.31 we have $M(|g*\mu|)=0$. Thus $0=M(|g*\mu|)=M(|g|*|\mu|)$. Then by Proposition 4.27 and Corollary 4.28 we get $0=M(|\mu|)$. □

A natural question now is if we can replace the condition of uniform discreteness by a weaker one. The natural condition to think of is weak uniform discreteness. The following is an example of a null weakly almost periodic measure which doesn't satisfy the fifth condition of the Proposition 4.30:

Example 4.31 Let $\mu=\sum_{n\in\mathbb{Z}\setminus\{0\}}(\delta_n-\delta_{n+1/n})\in\mathcal{M}^\infty(\mathbb{R})$. Let $A_n=[-n,n]$.

Then $\mu\in\mathcal{WAP}_0(\mathbb{R})$, but

$$\lim_{n\to\infty}\frac{|\mu|(A_n)}{\theta(A_n)}=\lim_{n\to\infty}\frac{\sum_{s\in\mathbb{Z}^*\cap A_n}|\mu(s)|}{\theta(A_n)}=2\,.$$

Proof: The only thing which is not trivial is that $\mu\in\mathcal{WAP}_0(\mathbb{R})$, so we concentrate on this.

First we prove that for all $g\in C_c(\mathbb{R})$, $g*\mu$ is a function vanishing at ∞.

Let k be a positive integer such that $\text{supp}(g)\subset[-k,k]$. Let $\epsilon>0$ be arbitrary.

$g\in K(\mathbb{R})$ implies that g is uniformly continuous. Thus there exists a $\delta>0$ such that $|x-y|<\delta$ implies $|g(x)-g(y)|<\epsilon/(2k+2)$.

Let $n_0>0$ be such that $1/n_0<\delta$. Let $|n|>n_0$. Let $x\in\mathbb{R}$. Then

$$|(\delta_n-\delta_{n+1/n})*g(x)|<\epsilon/(2k+2)\,.$$

Moreover, $\text{supp}\,((\delta_n-\delta_{n+1/n})*g)\subset[-n-k-1/n,k-n]\subset[-n-k-1,k-n]$.

Let $|y|>n_0+k+1$. Then $(\delta_n-\delta_{n+1/n})*g(y)\neq0$ only if $y\in[-n-k-1,k-n]$, thus only if $n\in[-y-k-1,k-y]$. But this implies that there are at most $2k+2$ terms of the type $(\delta_n-\delta_{n+1/n})*g(y)$ we have to consider in $\mu*g(y)$. Also $n\in[-y-k-1,k-y]$ implies $|n|\geq|y|-k-1>n_0$. Thus, for any $n\in[-y-k-1,k-y]$, we have $\left|(\delta_n-\delta_{n+1/n})*g(y)\right|<\epsilon/(2k+2)$. Thus, for all y with $|y|>n_0+k+1$, we have $|\mu*g(y)|<\epsilon$.

Now since $\mu*g$ is vanishing at ∞ we get $\mu*g$ is null weakly almost periodic [16]. Since this is true for any $g\in C_c(\mathbb{R})$, by Remark 2.32 we get $\mu\in\mathcal{WAP}_0(\mathbb{R})$. □

Remark 4.32 For any pure point measure μ, then following implications are true:

$$M(|\mu|)=0\Rightarrow\mu\in\mathcal{WAP}_0(\mathrm{G})\Rightarrow\mathrm{M}(\mu)=0\,.$$

We saw that if μ is positive all three are equivalent. We saw that the first two are equivalent under the assumption of uniform discreteness, but not equivalent for weakly uniform discreteness. We construct an example of pure point measure μ with uniformly discrete support so that $M(\mu) = 0$, but $\mu \notin \mathcal{WAP}_0(G)$:

Let

$$\mu = \sum_{n \in \mathbb{Z}} (-1)^n \delta_n .$$

It is obvious that μ transformable (because it is periodic) and $M(\mu) = 0$.

Now, since supp $(\mu) = \mathbb{Z}$ we know

$$\mu \in \mathcal{WAP}_0(\mathrm{G}) \Leftrightarrow \mathrm{M}(|\mu|) = 0 .$$

But $M(|\mu|) = 1$. In fact, $\mu \in \mathcal{SAP}(G)$, with

$$\widehat{\mu} = \delta_{\frac{1}{2}(2\mathbb{Z}+1)} .$$

4.2 The Bragg Spectrum of a Meyer Set

4.2.1 Preliminaries

For an $\chi \in \mathbb{R}^d$ we denote by λ_χ the element of $\widehat{\mathbb{R}^d}$ defined by:

$$\lambda_\chi(x) = e^{-2\pi i <\chi, x>} .$$

Proposition 4.33 *[20] If $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is Fourier transformable and $\mathcal{A}$ is a van Hove sequence of cubes, then for any $\chi \in \mathbb{R}^d$ we have:*

$$\widehat{\mu}(\{\chi\}) = \lim_{n \to \infty} \frac{\int_{A_n} \lambda_{-\chi}(x) d\mu(x)}{\theta(A_n)} .$$

Corollary 4.34 *If $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is Fourier transformable and non-negative then for any $\chi \in \mathbb{R}^d$ we have:*

$$|\widehat{\mu}(\{\chi\})| \leq \widehat{\mu}(\{0\}) .$$

Proof:

$$|\widehat{\mu}(\{\chi\})| = \lim_{n \to \infty} \left| \frac{\int_{A_n} \lambda_{-\chi}(x) d\mu(x)}{\theta(A_n)} \right| \leq \lim_{n \to \infty} \frac{\int_{A_n} |\lambda_\chi(x)| d\mu(x)}{\theta(A_n)} = \widehat{\mu}(\{0\}) .$$

□

Remark 4.35 Under the conditions of the previous corollary we have: $(\widehat{\mu})_{pp} \neq 0 \Leftrightarrow \widehat{\mu}(\{0\}) \neq 0 \Leftrightarrow \lim_{n \to \infty} \frac{\mu(A_n)}{\theta(A_n)} \neq 0$, which was also proved in Proposition 4.30.

4.2.2 Translation bounded measures with Meyer set support

For this section, we consider a Meyer set $\Delta \subset \mathbb{R}^d$. We let $\eta \in \mathcal{M}^\infty(\mathbb{R}^d)$ be a Fourier transformable measure with $\mathrm{supp}(\eta) \subset \Delta$. η can take complex values. $\mathcal{A} = \{A_n\}_n$ will represent a van Hove sequence of cubes and θ will denote the Lebesgue measure in $\mathbb{R}^d$.

For most of the applications, η is the autocorrelation of a (weighted comb supported on a) Meyer set.

Proposition 4.36 *There exists a constant $C > 0$ such that for all $0 < \epsilon, \chi \in \mathbb{R}^d$ and for all $\psi \in \Delta^{\epsilon/C}$ we have*

$$|\widehat{\eta}(\{\psi + \chi\}) - \widehat{\eta}(\{\chi\})| < \epsilon \,.$$

Proof: Let

$$C' = \limsup_{n\to\infty} \frac{|\eta|(A_n)}{\theta(A_n)} \,.$$

Then $0 \leq C' < \infty$, since one can use $K = [0,1]^d$, the tiling $\mathbb{R}^d = \mathbb{Z}^d + K$, and the fact that η is translation bounded to get $C' \leq C_K = \sup_{x\in\mathbb{R}^d} |\eta|(x+K)$. (If η is positive, then $C' = \widehat{\eta}(\{0\})$.)

Let $C > C'$ be arbitrary but fixed.

We know from Proposition 4.33 that the Fourier-Bohr coefficients of $\widehat{\eta}$ are given by:

$$\widehat{\eta}(\{\chi\}) = \lim_{n\to\infty} \frac{\int_{A_n} \lambda_{-\chi}(x) d\eta(x)}{\theta(A_n)} \,.$$

Then, for any $\epsilon > 0$, any $\psi \in \Delta^{\epsilon/C}, \chi \in \mathbb{R}^d$ and $x \in \Delta$ we get:

$$|\lambda_{-\psi-\chi}(x) - \lambda_{-\chi}(x)| < \epsilon/C \,.$$

Using the fact that $\mathrm{supp}(\eta) \subset \Delta$ we get

$$\begin{aligned} |\widehat{\eta}(\{\psi + \chi\}) \; -\widehat{\eta}(\{\chi\})| &\leq \lim_{n\to\infty} \tfrac{\int_{A_n} |\lambda_{-\psi-\chi}(x) - \lambda_{-\chi}(x)| \, d|\eta|(x)}{\theta(A_n)} \\ &\leq \tfrac{\int_{A_n} (\epsilon/C) \, d|\eta|(x)}{\theta(A_n)} < \epsilon \,. \end{aligned}$$

□

Corollary 4.37 *Let Λ be a Meyer set in $\mathbb{R}^d$, and let $\Delta = \Lambda - \Lambda$.*

i) *Let η be an autocorrelation of Λ. Then there exists a constant $C > 0$ such that for all $0 < \epsilon$, $(\widehat{\eta})_{pp}$ is ϵ-invariant under translations by $\Delta^{\epsilon/C}$ i.e. when we translate by any element of $\Delta^{\epsilon/C}$ we get an error smaller than ϵ.*

ii) *Let μ be a weighted comb with the support inside Λ and let η be its autocorrelation. Then there exists a constant $C > 0$ such that for all $0 < \epsilon$, $(\widehat{\eta})_{pp}$ is ϵ-invariant under translations by $\Delta^{\epsilon/C}$.*

Remark 4.38 i) For each $0 < \epsilon < 2/C$, $\Delta^{\epsilon/C}$ is a model set[27].

ii) Corollary 4.37 and Corollary 4.4 show that there are similarities between the diffraction of lattice subsets and diffraction of subsets of model sets.

The result from Corollary 4.4, for the pure point part only, can be obtained as a corollary to Proposition 4.36, using the fact that, for a lattice $L \subset \mathbb{R}^d$ with dual lattice L^* and for $\epsilon < 1/2$, we have $L^\epsilon = L^*$.

We denote $I_{sup}(\eta) = \sup_{\chi \in \widehat{\mathbb{R}^d}} \{\widehat{\eta}(\{\chi\})\}$.

Remark 4.39 i) $I_{sup}(\eta) \geq 0$. Note that since $\widehat{\eta}_{pp}$ is concentrated on a countable set, there exists $\chi \in \widehat{\mathbb{R}^d}$ such that $\widehat{\eta}(\{\chi\}) = 0$.

ii) If $\eta \geq 0$ then we know that

$$|\widehat{\eta}(\{\chi\})| \leq \widehat{\eta}(0),$$

so

$$I_{sup}(\eta) = \widehat{\eta}(\{0\}) = \limsup_{n \to \infty} \frac{\eta(A_n)}{\theta(A_n)} = M(\eta).$$

For the rest of the section we suppose that η is positive definite. C will denote the constant given by Proposition 4.36.

For any $a \in \mathbb{R}$ we define

$$I(a) = \{\chi \in \mathbb{R}^d \mid \widehat{\eta}(\{\chi\}) \geq a\}.$$

Remark 4.40 We are interested in $I(a)$ only when $0 < a \leq I_{sup}(\eta)$. For a fixed a we refer to this set as the set of *a-visible Bragg peaks.*

Proposition 4.41 i) *For any $\epsilon > 0$ we have:*

$$I(a) \pm \Delta^{\epsilon/C} \subset I(a - \epsilon).$$

ii) If η is also positive, and $I_{sup}(\eta) > 0$, then for any $0 < a < I_{sup}(\eta)$, $I(a)$ is a Meyer set.

Proof:

i) Follows trivially from Proposition 4.36, since $\Delta^{\epsilon/C} = -\Delta^{\epsilon/C}$.

ii) Let $\epsilon > 0$ be such that $\epsilon < a$ and $\epsilon < I_{sup}(\eta) - a$.

We prove that

$$\Delta^{\epsilon/C} \subset I(a) \subset \Delta^{\epsilon/C} + F\,,$$

for some finite set F.

Let $A := \Delta^{\epsilon/C}$.

$\mathbf{A} \subset \mathbf{I(a)}$: Since $0 \in I(I_{sup}(\eta))$ we have $A \subset I(I_{sup}(\eta)) + A \subset I(I_{sup}(\eta) - \epsilon)$. Since $a < I_{sup}(\eta) - \epsilon$ we get $I(I_{sup}(\eta) - \epsilon) \subset I(a)$, thus $A \subset I(a)$.

$\mathbf{I(a)} \subset \mathbf{F} + \mathbf{A}$: Let $b = a - \epsilon > 0$. We know that $I(a) - A \subset I(b)$. Since A is relatively dense, then there exists a compact set K such that $A + K = \mathbb{R}^d$. We have $\widehat{\eta}(K) < \infty$ (for example from the fact that $\widehat{\eta}$ is translation bounded).

Let $F = K \cap I(b)$. Then $\widehat{\eta}(K) \geq \sum_{\chi \in F} \widehat{\eta}(\{\chi\}) \geq b\,\mathrm{card}(F)$. Since $b \neq 0$ and $\widehat{\eta}(K) < \infty$ we get that F is finite.

Let $\chi \in I(a)$. Since $A + K = \mathbb{R}^d$ we can write $\chi = \alpha + k$ with $\alpha \in A, k \in K$. Then $k = \chi - \alpha \in (I(a) - A) \subset I(b)$, thus $k \in (K \cap I(b)) = F$. So $\chi \in A + F$. Thus $I(a) \subset A + F$.

Since $\Delta^{\epsilon/C} \subset I(a) \subset \Delta^{\epsilon/C} + F$ and $\Delta^{\epsilon/C}$ is a model set, it follows at once that $I(a)$ is a Meyer set. □

Corollary 4.42 *Let Λ be a Meyer set in $\mathbb{R}^d$.*

i) Let η be an autocorrelation of Λ. Then, for any $0 < a < \widehat{\eta}(\{0\})$, the set of Bragg peaks with intensity at least a is a Meyer set.

ii) Let μ be a weighted comb with the support inside Λ and let η be its autocorrelation. If $\widehat{\eta}$ is translation bounded and transformable, then for any $a > 0$ the set of Bragg peaks with intensity at least a is either empty or a Meyer set.

Remark 4.43 A necessary (but not sufficient) diffraction condition for a point set to be a Meyer set is that $I_{sup}(\eta) > 0$ and for any $0 < a < I_{sup}(\eta)$, $I(a)$ is a Meyer set.

Remark 4.44 If in Proposition 4.41 ii) η is not positive, we can still prove that for any $0 < a < I_{sup}(\eta)$ there exists a finite set $F \subset \widehat{\mathbb{R}^d}$ and a $\chi \in \widehat{\mathbb{R}^d}$ such that $\chi + \Delta^{\epsilon/C} \subset I(a) \subset \Delta^{\epsilon/C} + F$.

One known result is the following, due to Cordoba. It says that if the Fourier transform of δ_Δ is pure point and Δ is uniformly discrete, then we are in the crystal case. Since the original proof is very long one would like to get a simpler one.

Proposition 4.45 [13] *Suppose that the point sets $\Lambda_1, ..., \Lambda_n$ are pairwise disjoint and $\Delta = \bigcup_{i=1}^{n} \Lambda_i$ is uniformly discrete. Let*

$$\mu = \sum_{i=1}^{n} c_i \delta_{\Lambda_i} ,$$

for some (different) complex numbers $c_1, ..., c_n$.

If $\widehat{\mu}$ is a translation bounded pure point measure, then each Λ_i is a finite disjoint union of translates of lattices.

We can prove the following:

Proposition 4.46 *Suppose that η is positive and positive definite and supported on a Meyer set Δ. Suppose that the set of Bragg peaks is $I(a)$ for some $a > 0$ (i.e., all the intensities of the Bragg peaks are above some positive value). Then there exists a lattice L, with dual lattice L^*, such that Δ is a subset of finitely many translations of L and the set of Bragg peaks is a subset of finitely many translates of L^*.*

Proof: For $1/2 > \epsilon > 0$ small enough, by Proposition 4.36 and Proposition 4.41 we get $\Delta^\epsilon \subset I(a)$ and $I(a) + \Delta^\epsilon \subset B$, where B is the supporting set for the Bragg peaks. But $B = I(a)$, thus:

$$I(a) + \Delta^\epsilon \subset I(a) .$$

Using the fact that $\Delta^\epsilon \subset I(a)$ we get that:

$$\Delta^\epsilon + \Delta^\epsilon + ... + \Delta^\epsilon \subset I(a) ,$$

with n terms in the sum.

Thus L^* the group generated by Δ^ϵ is in $I(a)$ so it is discrete. Since Δ^ϵ is relatively dense we get that L^* is a lattice. Let L be the dual lattice.

Since $\Delta^\epsilon \subset L^*$ we get [27] that $L^{*\epsilon} \subset (\Delta^\epsilon)^\epsilon$. But since $\epsilon < 1/2$ we get that $L^{*\epsilon} = L$. This implies that $L \subset (\Delta^\epsilon)^\epsilon$. Then by applying Lemma 4.12 we get a finite set F such that:

$$(\Delta^\epsilon)^\epsilon \subset L + F .$$

Thus

$$\Delta \subset (\Delta^\epsilon)^\epsilon \subset L + F .$$

Also applying the same lemma to $\Delta^\epsilon \subset I(a)$ and using $\Delta^\epsilon \subset L^*$ one gets the last part of the proposition. □

Corollary 4.47 *Let Λ be a Meyer set in $\mathbb{R}^d$ and let η be an autocorrelation of Λ. Suppose that the set of Bragg peaks is $I(a)$ for some $a > 0$. Then there exists a lattice L and a finite set F such that $\Lambda \subset L + F$.*

Proof: We know that $\text{supp}(\eta) \subset \Delta = \Lambda - \Lambda$. Since Δ is a Meyer set we get that there exists a lattice L and a finite set F' such that $\Delta \subset L + F'$. Now for an element $x \in \Lambda$ we have $\Lambda - x \subset \Lambda - \Lambda \subset L + F'$. Thus

$$\Lambda \subset L + F\,,$$

where $F = F' + x$. □

Remark 4.48 In a diffraction experiment one is limited in one's ability to see the Bragg peaks by the physical sensitivity of the equipment. One cannot expect to see all the Bragg peaks in general, only the Bragg peaks with the intensity above a certain value. The previous proposition says that if in the diffraction of a Meyer set you see all the Bragg peaks then you are in the crystal case (i.e. the set is inside finitely many translates of a lattice).

By applying the previous result to $\widehat{\eta}$ one gets the following:

Proposition 4.49 *Suppose that η is positive and positive definite and there exists $a > 0$ such that for all $x \in \mathbb{R}^d$ we have $\eta(\{x\}) \notin (0, a)$ (i.e., $(\eta)_{pp}$ doesn't take values arbitrarily close to 0). If $\widehat{\eta}$ is supported on a Meyer set then both $\widehat{\eta}$ and $(\eta)_{pp}$ are supported on subsets of finitely many translates of lattices.*

Remark 4.50 If one compares this result with Proposition 4.45, we see that we ask less about η (is not necessary η to be pure point or the pure point part to take only finitely many values) but we ask that $\widehat{\eta}$ is pure point with Meyer set support, which is a very strong requirement. Thus this proposition is weaker than Cordoba's result.

Also we can prove the following:

Proposition 4.51 *Suppose that the point sets $\Lambda_1, ..., \Lambda_n$ are pairwise disjoint and $\Lambda = \bigcup_{i=1}^{n} \Lambda_i$ is a Meyer set. Let*

$$\mu = \sum_{i=1}^{n} c_i \delta_{\Lambda_i}\,,$$

for some (different) complex numbers $c_1, ..., c_n$.

If $\widehat{\mu}$ is a translation bounded pure point measure, then each Λ_i is a finite disjoint union of translates of the same lattice.

Proof: Since $\widehat{\mu}$ is pure point we get that μ is a strongly almost periodic measure [21]. μ strongly almost periodic and supported on a Meyer set implies μ norm almost periodic [7].

If one uses an $0 < \epsilon < C$, where C is the smallest absolute value of the differences $|c_i - c_j|$ with $i \neq j$ and a compact set V with non-empty interior such that $\Lambda - \Lambda$ is $V - V$ uniformly discrete to define the norm topology in [7], one gets that every ϵ norm almost period is actually a period for μ.

Thus the set L of periods for μ is relatively dense, so it is a lattice.

Since

$$\mu = \sum_{i=1}^{n} c_i \delta_{\Lambda_i} ,$$

for different complex numbers $c_1, ..., c_n$ is L-periodic we get that each Λ_i is L-periodic, thus consists of finitely many (since Λ_i is uniformly discrete) translates of L. □

We have proved the Cordoba's result under a stronger assumption, namely that either μ or $\widehat{\mu}$ is Meyer set supported.

4.2.3 (ϵ, m) dual characters

Recall that in Proposition 4.15 we have proved that the diffraction of a Meyer set Λ always has a relatively dense set of Bragg peaks. How robust is this?

We now show that as long as Λ is not too badly disturbed the property survives. Our idea is to study what happens when Λ is replaced by a subset of it augmented with a locally finite set. The main results are Proposition 4.71 and Proposition 4.74 in Section 4.2.4. Here in Section 4.2.3 we lay down the technical requirements for proving these.

For the entire section, $\eta \in \mathcal{M}^\infty(\mathbb{R}^d)$ is a positive and positive definite measure. By Bochner's theorem, η is Fourier transformable and $\widehat{\eta}$ is a positive and positive definite measure. In particular $\widehat{\eta} \in \mathcal{M}^\infty(\mathbb{R}^d)$.

Also we know that $(\widehat{\eta})_{pp} \neq 0$ if and only if $\widehat{\eta}(\{0\}) \neq 0$. Thus for the entire section we assume that $\widehat{\eta}(\{0\}) \neq 0$.

$\mathcal{A}$ will represent again a van Hove sequence of cubes.

Definition 4.52 For any $\chi \in \widehat{\mathbb{R}^d}$ and $0 < \epsilon < 2$ we define:

$$D(\chi, \epsilon) = \{x \in \mathbb{R}^d \mid 1 - \mathrm{Re}(\lambda_\chi(x)) < \epsilon\} .$$

Remark 4.53

$$2(1 - \mathrm{Re}(\lambda_\chi(x))) = |1 - \lambda_\chi(x)|^2 .$$

This shows that for $\Lambda \subset \widehat{\mathbb{R}^d} = \mathbb{R}^d$ and $\epsilon > 0$ we have:

$$\bigcap_{\chi \in \Lambda} D(\chi, \epsilon) = \Lambda^{\sqrt{2\epsilon}}.$$

Definition 4.54 For $\eta \in \mathcal{M}^\infty(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^d$, we define:

$$d(\Lambda; \eta) = \liminf_{n\to\infty} \frac{1}{\widehat{\eta}(\{0\})} \frac{\eta(\Lambda \cap A_n)}{\theta(A_n)}.$$

Remark 4.55 i) $d(\mathbb{R}^d; \eta) = 1$,

ii) For any $\Lambda \subset \mathbb{R}^d$ we have $0 \leq d(\Lambda; \eta) \leq 1$,

iii)

$$\limsup_{n\to\infty} \frac{\eta(A_n \backslash \Lambda)}{\theta(A_n)} = \widehat{\eta}(\{0\})(1 - d(\Lambda; \eta)).$$

Definition 4.56 For $\chi \in \widehat{\mathbb{R}^d}$ and $0 < \epsilon < 2$ we say that χ is a (ϵ, m) **character on** η if

$$m \leq d(D(\chi, \epsilon); \eta).$$

Remark 4.57 For $\Lambda = supp(\eta)$, $\chi \in \Lambda^{\epsilon^2/2}$, we have

$$d(D(\chi, \epsilon); \eta) = 1.$$

In particular χ is an (ϵ, m) character on η for any $0 \leq m \leq 1$.

Lemma 4.58 *For any $\chi \in \mathbb{R}^d$ we have :*

$$\widehat{\eta}(\{0\}) - \widehat{\eta}(\{\chi\}) = \lim_{n\to\infty} \frac{\int_{A_n}(1 - \mathrm{Re}(\lambda_\chi(x))d\eta(x)}{\theta(A_n)} \geq 0.$$

Proof:

$$\widehat{\eta}(\{0\}) = \lim_{n\to\infty} \frac{\int_{A_n} 1 d\eta(x)}{\theta(A_n)},$$

$$\widehat{\eta}(\{\chi\}) = \lim_{n\to\infty} \frac{\int_{A_n} \lambda_\chi(x) d\eta(x)}{\theta(A_n)}$$

Now η is positive and positive definite. Thus:

$$\widehat{\eta}(\{\chi\}) = \mathrm{Re}(\widehat{\eta}(\{\chi\})) = \lim_{n\to\infty} \frac{\int_{A_n} \mathrm{Re}(\lambda_\chi(x))d\eta(x)}{\theta(A_n)}.$$

The inequality follows from the fact that $\mathrm{Re}(\lambda_\chi(x))) \leq 1$. □

Lemma 4.59 *For any $\chi \in \mathbb{R}^d$ and $0 < \epsilon < 2$, we have :*

i) $\widehat{\eta}(\{0\}) - \widehat{\eta}(\{\chi\}) \leq \epsilon d(D(\chi,\epsilon);\eta)\widehat{\eta}(\{0\}) + 2\widehat{\eta}(\{0\}) - 2d(D(\chi,\epsilon);\eta)\widehat{\eta}(\{0\})$.

ii) $\widehat{\eta}(\{0\}) - \widehat{\eta}(\{\chi\}) \geq \epsilon\widehat{\eta}(\{0\}) - \epsilon d(D(\chi,\epsilon);\eta)\widehat{\eta}(\{0\})$.

iii) $|\widehat{\eta}(\{\psi\}) - \widehat{\eta}(\{\psi+\chi\})| \leq \sqrt{2\epsilon d(D(\chi,\epsilon);\eta)\widehat{\eta}(\{0\}) + 2\widehat{\eta}(\{0\}) - 2d(D(\chi,\epsilon);\eta)\widehat{\eta}(\{0\})}$.

Proof:

Let χ, ϵ be fixed and let $\Lambda = D(\chi,\epsilon)$ and $d_0 = d(\Lambda;\eta)$.

i)

$$\begin{aligned}
\widehat{\eta}(\{0\}) - \widehat{\eta}(\{\chi\}) &= \lim_{n\to\infty} \frac{\int_{A_n}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&= \lim_{n\to\infty} \frac{\int_{A_n\cap\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x) + \int_{A_n\backslash\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&= \liminf_{n\to\infty} \frac{\int_{A_n\cap\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&+ \limsup_{n\to\infty} \frac{\int_{A_n\backslash\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&\leq \liminf_{n\to\infty} \frac{\epsilon\eta(A_n\cap\Lambda)}{\theta(A_n)} + \limsup_{n\to\infty} \frac{2\eta(A_n\backslash\Lambda)}{\theta(A_n)} \\
&\leq \epsilon d_0\widehat{\eta}(\{0\}) + 2(1-d_0)\widehat{\eta}(\{0\}) .
\end{aligned} \tag{4.4}$$

ii)

$$\begin{aligned}
\widehat{\eta}(\{0\}) - \widehat{\eta}(\{\chi\}) &= \lim_{n\to\infty} \frac{\int_{A_n}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&= \lim_{n\to\infty} \frac{\int_{A_n\cap\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x) + \int_{A_n\backslash\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&= \limsup_{n\to\infty} \frac{\int_{A_n\cap\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x) + \int_{A_n\backslash\Lambda}(1-\mathrm{Re}(\lambda_\chi(x)))d\eta(x)}{\theta(A_n)} \\
&\geq \limsup_{n\to\infty} \frac{\epsilon\eta(A_n\backslash\Lambda)}{\theta(A_n)} \\
&= \epsilon\widehat{\eta}(\{0\}) - \epsilon d_0\widehat{\eta}(\{0\}) .
\end{aligned} \tag{4.5}$$

iii)

$$
\begin{aligned}
|\widehat{\eta}(\{\psi\}) - \widehat{\eta}(\{\psi+\chi\})| &= \left| \lim_{n\to\infty} \frac{\int_{A_n}(\lambda_\psi(x) - \lambda_{\psi+\chi}(x))d\eta(x)}{\theta(A_n)} \right| \\
&\le \lim_{n\to\infty} \frac{\int_{A_n}|\lambda_\psi(x) - \lambda_{\psi+\chi}(x)|d\eta(x)}{\theta(A_n)} \\
&= \lim_{n\to\infty} \frac{\int_{A_n}|1 - \lambda_\chi(x)|d\eta(x)}{\theta(A_n)} \\
&\le \sqrt{2\epsilon} d_0 \widehat{\eta}(\{0\}) + 2\widehat{\eta}(\{0\}) - 2d_0\widehat{\eta}(\{0\}) ,
\end{aligned}
\tag{4.6}
$$

the last inequality being obtained as in **i)**. □

Remark 4.60 If one tries to work with $|1 - \lambda_\chi(x)| < \epsilon$, which is natural because we want to use the ϵ−dual sets, i) is trivial, but ii) cannot be proved so easily. The reason for ii) to be true is that we can eliminate the absolute value using Lemma 4.58.

Corollary 4.61 *Let $\chi \in \mathbb{R}^d, \epsilon > 0$ and $a > 0$ be given. Then, we have:*

i) Suppose that

$$(2d(D(\chi,\epsilon);\eta) - \epsilon d(D(\chi,\epsilon);\eta) - 1)\widehat{\eta}(\{0\}) \ge a\,.$$

Then $\chi \in I(a)$.In particular if for an $\epsilon > 0$ we have $d(D(\chi,\epsilon);\eta) \ge \frac{1}{2-\epsilon}$ then $\widehat{\eta}(\{\chi\}) \neq 0$.

ii) If $\chi \in I(a)$ then :

$$(\epsilon d(D(\chi,\epsilon);\eta) - \epsilon + 1)\widehat{\eta}(\{0\}) \ge a\,.$$

In particular :

$$d(D(\chi,\epsilon);\eta) \ge \frac{a/\widehat{\eta}(\{0\}) + \epsilon - 1}{\epsilon}\,.$$

Proof:

Using Lemma 4.59, we get lower and upper estimates for $\widehat{\eta}(\{\chi\}$ and we use the fact that $\chi \in I(a)$ if and only if $\widehat{\eta}(\{\chi\}) \ge a$. □

Corollary 4.62 *Let $\chi \in \mathbb{R}^d$ and $a > 0$.*

i) Suppose that χ is an (ϵ, m) character on η and that

$$(2m - \epsilon m - 1)\widehat{\eta}(\{0\}) \geq a\,.$$

Then $\chi \in I(a)$.

ii) Suppose that $\chi \in I(a)$. Let $2 > \epsilon > 1 - \frac{a}{\widehat{\eta}(\{0\})}$. Then for any

$$0 \leq m \leq \frac{a/\widehat{\eta}(\{0\}) + \epsilon - 1}{\epsilon}\,,$$

χ is an (ϵ, m) character on η.

Corollary 4.63 *Let (ϵ, m) be fixed and let*

$$B(\epsilon; m) = \{\chi \in \mathbb{R}^d \mid \chi \, \text{is an} \, (\epsilon, m) \, \text{character on} \, \eta\}\,.$$

Then $(\widehat{\eta})_{pp}$ is $(m\sqrt{2\epsilon} + 2 - 2m)-$ invariant under $B(\epsilon; m)$.

Corollary 4.64 *Suppose that there exists Λ a Meyer set such that $m = d(\Lambda; \eta) > 1/2$. Then, for any $0 < \epsilon < \frac{2m-1}{m}$, the set $(\Lambda)^{\sqrt{2\epsilon}}$ is a subset of supp$((\widehat{\eta})_{pp})$.*

Proof:

From $m > 1/2$ and $0 < \epsilon < \frac{2m-1}{m}$ we get that $m \geq \frac{1}{2-\epsilon}$. Thus, since $\Lambda^{\sqrt{2\epsilon}} \subset D(\chi, \epsilon)$ we get that

$$d(D(\chi, \epsilon); \eta) \geq d(\Lambda; \eta) \geq m\,,$$

for all $\chi \in \Lambda^{\sqrt{2\epsilon}}$.

The claim follows now from Corollary 4.61. □

Remark 4.65 It is easy to see that all the results in Section 4.2.3 are trivial in locally compact abelian groups for subsets of model sets (one can use that the ϵ-duals of model sets are relatively dense [27] and the fact that for any $\epsilon > 0$ and $A \subset B$ we have $B^\epsilon \subset A^\epsilon$).

4.2.4 Sets with a large Meyer subset

For this entire section M is a subset of a Meyer set, S is an arbitrary locally finite set and $\Lambda = M \cup S$. We also assume that $S \cap M = \emptyset$. Let $\Delta = M - M$, η_Λ be the autocorrelation of Λ and η_M be the autocorrelation of M respectively, which are assumed to exist with respect to our van Hove $\mathcal{A}$ of cubes.

Remark 4.66 Let N be a Meyer set so that $M \subset N$. Then $\Delta \subset N - N$, which is a Meyer set. Thus $(N-N)^\epsilon$ is relatively dense for all $\epsilon > 0$ and $(N-N)^\epsilon \subset \Delta^\epsilon$. In particular Δ^ϵ is relatively dense for all $\epsilon > 0$.

Lemma 4.67

$$d(\Delta; \eta_\Lambda) \geq \frac{\widehat{\eta_M}(\{0\})}{\widehat{\eta_\Lambda}(\{0\})}.$$

Proof:

$$\begin{aligned} d(\Delta; \eta_\Lambda) &= \liminf_{n\to\infty} \frac{1}{\widehat{\eta_\Lambda}(\{0\})} \frac{\eta_\Lambda(\Delta \cap A_n)}{\theta(A_n)} \geq \liminf_{n\to\infty} \frac{1}{\widehat{\eta_\Lambda}(\{0\})} \frac{\eta_M(\Delta \cap A_n)}{\theta(A_n)} \\ &= \liminf_{n\to\infty} \frac{1}{\widehat{\eta_\Lambda}(\{0\})} \frac{\eta_M(A_n)}{\theta(A_n)} = \frac{\widehat{\eta_M}(\{0\})}{\widehat{\eta_\Lambda}(\{0\})}. \end{aligned}$$

□

Proposition 4.68 *For any $0 < \epsilon < 2$ and $\chi \in \Delta^{\sqrt{2\epsilon}}$, χ is an $\epsilon, (\frac{\widehat{\eta_M}(\{0\})}{\widehat{\eta_\Lambda}(\{0\})})$ character on η_Λ. In particular:*

$$\widehat{\eta_\Lambda}(\{0\}) - \widehat{\eta_\Lambda}(\{\chi\}) \leq 2\widehat{\eta_\Lambda}(\{0\}) - (2-\epsilon)\widehat{\eta_M}(\{0\}).$$

Proof: Since $\chi \in \Delta^{\sqrt{2\epsilon}}$, we get $\Delta \subset D(\chi, \epsilon)$, thus

$$d(D(\chi,\epsilon); \eta_\Lambda) \geq d(\Delta; \eta_\Lambda) \geq \frac{\widehat{\eta_M}(\{0\})}{\widehat{\eta_\Lambda}(\{0\})}. \tag{4.7}$$

From Lemma 4.59 i) we get:

$$\widehat{\eta_\Lambda}(\{0\}) - \widehat{\eta_\Lambda}(\{\chi\}) \leq 2\widehat{\eta_\Lambda}(\{0\}) - (2-\epsilon) d(D(\chi,\epsilon); \eta_\Lambda)\widehat{\eta_\Lambda}(\{0\}).$$

From (4.7) we get

$$-(2-\epsilon) d(D(\chi,\epsilon); \eta_\Lambda)\widehat{\eta_\Lambda}(\{0\}) \leq -(2-\epsilon)\widehat{\eta_M}(\{0\}),$$

thus the desired result. □

Corollary 4.69 *For any $0 < \epsilon < 2$ and $\chi \in \Delta^{\sqrt{2\epsilon}}$, we have:*

$$\widehat{\eta_\Lambda}(\{\chi\}) \geq (2-\epsilon)\widehat{\eta_M}(\{0\}) - \widehat{\eta_\Lambda}(\{0\}).$$

Definition 4.70 We say that S is **small with respect to** M, if

$$2\widehat{\eta_M}(\{0\}) > \widehat{\eta_\Lambda}(\{0\} \,,$$

that is the intensity of the Bragg peak at the origin for $M \cup S$ is less than twice the intensity of the Bragg peak at the origin for M.

Let B be the set of Bragg peaks of Λ.

Proposition 4.71 *If S is small with respect to M, then there exists an ϵ_0 such that for all $\epsilon < \epsilon_0$ we have $\Delta^{\sqrt{2\epsilon}} \subset B$.*

In particular, $\Lambda = M \cup S$ has a relatively dense set of Bragg peaks.

Proof:

Since $2\widehat{\eta_M}(\{0\}) > \widehat{\eta_\Lambda}(\{0\}$ we can find an ϵ_0 such that

$$(2-\epsilon_0)\widehat{\eta_M}(\{0\}) - \widehat{\eta_\Lambda}(\{0\}) > 0\,.$$

The first part of the Proposition follows now from Corollary 4.69. □

Definition 4.72 We say that a locally finite set T is **uniformly distributed** if the limit:

$$\text{dens}(x+T) = \lim_{n\to\infty} \frac{\sharp((x+T)\cap A_n)}{\theta(A_n)}\,,$$

exists uniformly and independently in x .

Proposition 4.73 *If S and M are uniformly distributed, then S is small with respect to M if and only if*

$$\text{dens}(M) > (1+\sqrt{2})\text{dens}(S)\,.$$

Proof: Since M and S are uniformly distributed Λ is also uniformly distributed. Then, by [20], we have:

$$\widehat{\eta_\Lambda}(\{0\}) = (\text{dens}(\Lambda))^2\,,$$

$$\widehat{\eta_M}(\{0\}) = (\text{dens}(M))^2\,.$$

Thus, S is small with respect to M if and only if $\text{dens}(\Lambda) < \sqrt{2}\text{dens}(M)$. Since $\text{dens}(\Lambda) = \text{dens}(S) + \text{dens}(M)$ we get that S is small with respect to Mif and only if

$$\text{dens}(S) < (\sqrt{2}-1)\text{dens}(M)\,.$$

Proposition 4.74 *Let Λ and N be locally finite sets, with N Meyer set. Assume that both $\Lambda \backslash N$ and $\Lambda \cap N$ are uniformly distributed and that:*

$$\text{dens}(\Lambda \bigtriangleup N) < \frac{1}{\sqrt{2}+2}\text{dens}(\Lambda)\,.$$

Then Λ has a relatively dense set of Bragg peaks.

Proof: Let $M = \Lambda \cap N$ and $S = \Lambda \backslash N$. Then

$$\Lambda = M \dot{\cup} S\,,$$

and

$$\begin{aligned} \text{dens}(M) = \text{dens}(\Lambda) - \text{dens}(S) &> [(\sqrt{2}+2)\text{dens}(\Lambda \Delta N)] - \text{dens}(S) \\ &\geq [(\sqrt{2}+2)\text{dens}(S)] - \text{dens}(S) = (1+\sqrt{2})\text{dens}(S)\,. \end{aligned} \tag{4.8}$$

Thus, M is a subset of a Meyer set and S is small with respect to M. □

Chapter 5

Circular Symmetry of Pinwheel Diffraction

5.1 The Pinwheel Tiling as a Substitution

The pinwheel tiling was first conceived as a substitution tiling by John H. Conway. Charles Radin later developed the matching rules that determine the same structure [35]. It is an aperiodic tiling of the plane by $1 : 2 : \sqrt{5}$ right triangles and may be constructed by iterating the following substitution rule:

The substitution consists of a standard type of inflation and subdivision rule, but also requires a second step: a rotation through the angle $\omega := -\arctan \frac{1}{2}$ that aligns the new central triangle with the original tile. This extra step is necessary if we require the pinwheel substitution to have a fixed point.

What makes the pinwheel tiling interesting is that it exhibits tiles of infinitely many orientations, and hence is composed of infinitely many types of tiles in the sense of translational symmetry. Thus the well-developed theory of tiles and point sets which are of finite local (translational) complexity breaks down. In particular, the diffraction of the pinwheel tiling (say of its vertices or of control points, one from each tile) is still unknown, even qualitatively.

Progress has been especially hampered by the fact that the number of orientations of the pinwheel grows only linearly in the number of substitutions while the number of tiles is growing exponentially. Thus images derived from computation of the diffraction or autocorrelation turn out to be totally unrepresentative of what is actually happening in the limit.

However, it has been established [33, 36] that the diffraction of the pinwheel tiling is circularly symmetric. This was done by studying the associated dynamical

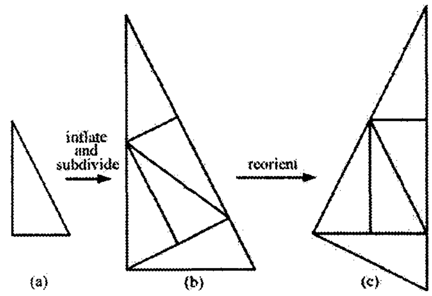

Figure 5.1: The pinwheel substitution

system and its unique invariant probability measure μ (see Section 5.6) and then looking at the spectrum of the unitary action of $\mathbb{R}^2$ on $L^2(X, \mu)$. The relationship between the dynamical and and diffraction spectra is given by a piece of formalism called Dworkin's argument [15]. In the case of the pinwheel tiling, the dynamical spectrum is circularly symmetric [33], and using the Dworkin argument one can pass this information over to the diffraction side. In Section 5.6 we offer another short way to prove the circular symmetry based on the circular symmetry of the associated dynamical system, this time without passing through the Dworkin argument.

The main purpose of this Chapter is to offer a proof of the circular symmetry by explicitly working with the autocorrelation and showing that it, and hence also its Fourier transform (which is *by definition* the diffraction), converges to a circularly symmetric measure. This process is longer than the argument using the dynamical system. However, it studies the substitution and the corresponding evolving autocorrelation in a very direct and detailed way. It is hoped that the insights of this setting will shed light on the remaining problem of determining the radial autocorrelation, which still remains wide open.

5.2 Control Points and the Pinwheel Tiling as a Point Substitution

It is convenient from the point of view of notation and calculation to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Thus let Γ_0 be the $1 : 2 : \sqrt{5}$ right triangle with vertices $\frac{-1-i}{2}$, $\frac{1-i}{2}$, and $\frac{-1+3i}{2}$. The pinwheel tiling Γ is obtained by iteratively applying the substitution seen in Figure 5.1 to Γ_0 infinitely many times. Γ_n denotes the set of 5^n tiles obtained by iterating the substitution n times. Tiles that differ from Γ_0 by a Euclidean motion are said to have positive chirality; those that differ by a Euclidean motion and a reflection possess negative chirality.

Let γ be any tile in Γ. Let x be the vertex at the right angle of γ and let y be the other terminal vertex of the short leg. We define the orientation of γ to be $\theta_\alpha := y - x = e^{i\alpha}$, where α is the angle that the short leg makes to the positive x-axis. Note that θ_α is an element of $U(1) := \{x \mid x \in \mathbb{C}, |x| = 1\}$, the group of rotations around 0 in the plane.

Following [35] we locate a specific distinguished point within a pinwheel tile: if x and y are the vertices defined above and z is the remaining vertex, then the distinguished point of a tile γ is located at $\frac{x+2y+z}{4}$. Notice that 0 is the control point of Γ_0 and that 0 occupies the same relative position with respect to the vertices in every supertile Γ_n. It is this property that determines this choice of control points, for it allows us to replace the tiling substitution by a point substitution (see Definition 5.2).

All of the information of a tile γ is encapsulated in its distinguished point, orientation, and chirality, which motivates the following definition:

Definition 5.1 *Let γ be any tile in Γ. The* **control point** *of γ is a triple (x, θ_α, χ) consisting of the distinguished point of γ, the orientation of γ, and the chirality of γ (± 1) respectively.*

The set of all control points in Γ is denoted by Λ and the set of the control points of Γ_n is Λ_n. By Λ^+, Λ^- we mean the subsets of Λ comprised of the control points of positive and negative chirality, respectively.

Definition 5.2 *The* **pinwheel substitution** *is given by:*

$$(x\,,\theta_\alpha\,,\chi) \mapsto \begin{cases} (\sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega}\,,\chi) \\ (\theta_{\alpha+\omega-\chi\omega+\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,\chi) \\ (2\theta_{\alpha+\omega-\chi\omega+\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,-\chi) \\ (\theta_{\alpha+\omega-\chi\omega+\pi} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,-\chi) \\ (\theta_{\alpha+\omega-\chi\omega-\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega-\frac{\chi\pi}{2}}\,,-\chi)\,. \end{cases}$$

By infinitely iterating the above substitution on the set starting with the single element $\Lambda_0 = (0\,,\theta_0\,,1)$ we generate Λ. Note that we arbitrarily started with a tile of positive chirality; we could just as easily have used a negative chirality tile. If we repeated the above arguments for the tile $\overline{\Gamma_0}$ (which is the mirror image of Γ_0 in the x-axis), we would obtain a tiling that is a mirror image of the pinwheel tiling. This process would also involve the creation of a mirror substitution. We will use the mirror iterates $V_n := \overline{\Lambda_n}$ in Section 5.4.2.

5.3 Uniform Distribution of Orientations

Since there is an exact copy of Λ_k in Λ_{k+1} we can define two sequences of angles $\{\alpha_i\}_{i=1}^{\infty}, \{\beta_i\}_{i=1}^{\infty} \subseteq [0, 2\pi)$ such that for any k, $\theta_{\alpha_1}, \ldots, \theta_{\alpha_{m_k}}$ are the orientations of the $\chi = 1$ points in Λ_k and $\theta_{\beta_1}, \ldots, \theta_{\beta_{n_k}}$ are the orientations of the $\chi = -1$ points. $m_k := \frac{5^k+(-1)^k}{2}$, $n_k := \frac{5^k-(-1)^k}{2}$ are the number of chirality 1, -1 points in Λ_k respectively.

We fix such a sequence for the rest of the Chapter.

A key property of the pinwheel tiling is the uniform distribution of the orientations of the tiles [34], in other words the uniform distribution of the two sequences that we have just defined. For the convenience of the reader we provide a short proof of this.

Recall that a sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ is **uniformly distributed on $U(1)$** if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(z_n) = \int_{U(1)} f(z) d\lambda^{U(1)}(z) = \lambda^{U(1)}(f)$$

for all $f : U(1) \to \mathbb{C}$ continuous ($\lambda^{U(1)}$ is normalized Haar measure on $U(1)$). We say that a sequence $\{\gamma_j\}_{j=1}^{\infty} \subset [0, 2\pi)$ is **uniformly distributed modulo 2π** if $\{e^{i\gamma_j}\}_{j=1}^{\infty}$ is uniformly distributed on $U(1)$.

Define

$$M(t) := \begin{pmatrix} e^{it0} + e^{it\pi} & 2e^{it(2\omega-\pi)} + e^{it(2\omega+\frac{\pi}{2})} \\ 2e^{it(\pi)} + e^{it(-\frac{\pi}{2})} & e^{it(2\omega)} + e^{it(2\omega-\pi)} \end{pmatrix} \tag{5.1}$$

for all $t \in \mathbb{Z}$.

If the indices 1 (respectively 2) refer to the tiles of positive (respectively negative) chirality then $(M(1))_{jk}$ is the sum of the orientations of the type j tiles obtained after applying the pinwheel substitution to a single type k tile with orientation 1. Then

$$(M(t))^k = \begin{pmatrix} \sum_{j=1}^{m_k} e^{it\alpha_j} & \sum_{j=1}^{n_k} e^{it(2k\omega-\beta_j)} \\ \sum_{j=1}^{n_k} e^{it\beta_j} & \sum_{j=1}^{m_k} e^{it(2k\omega-\alpha_j)} \end{pmatrix}$$

follows immediately from the definition of $M(t)$ and the pinwheel substitution. [1]

[1]This matrix is similar to the matrix used in [34]. The primary difference comes from the fact that Radin rotates Λ_n at every step so that, considered as one big triangle, it has orientation θ_0. We must use the above matrix in place of Radin's because of our requirement that we work with a fixed point substitution. Also, Radin's type 1 tile corresponds to what we have chosen to be our type 2 tile and vice versa.

Theorem 5.3 (Radin) *$\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are uniformly distributed.*

Proof: By the well-known Weyl criterion a sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ is uniformly distributed if and only if for all $t \in \mathbb{Z} \setminus \{0\}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (z_n)^t = 0 .$$

Since $\lim_{k\to\infty} \frac{m_k}{5^k} = \lim_{k\to\infty} \frac{n_k}{5^k} = \frac{1}{2}$, it will be enough to prove that, for all $t \neq 0, 1 \leq i, j \leq 2$

$$\lim_{k\to\infty} \frac{((M(t))^k)_{ij}}{5^k} = 0 .$$

Let $t \neq 0$ be arbitrary but fixed. Let A be the matrix defined by $A_{ij} = |(M(t))_{ij}|$. Then $0 < A \leq \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ in an entrywise sense, with the additional restriction $A \neq \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Also $|((M(t))^k)_{ij}| \leq (A^k)_{ij}$ for all k, i, j. Let λ be the Perron-Frobenius eigenvalue of A. Then, by the Perron-Frobenius Theorem $\lambda < 5$, since 5 is the PF eigenvalue of $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Furthermore there exists a constant c such that for all $n > 0$, $\frac{(A^n)_{ij}}{\lambda^n} \leq c$. Then

$$\left| \frac{((M(t))^k)_{ij}}{5^k} \right| \leq \frac{(A^k)_{ij}}{5^k} = \frac{(A^k)_{ij}}{\lambda^k} \cdot \left(\frac{\lambda}{5}\right)^k \leq c \cdot \left(\frac{\lambda}{5}\right)^k \stackrel{k\to 0}{\longrightarrow} 0 ,$$

since $\frac{\lambda}{5} < 1$. Hence, the desired result follows. □

5.4 Autocorrelation of the Pinwheel Tiling

5.4.1 Introduction to the Autocorrelation and Some Notation

Let $\mathcal{M}(\mathbb{R}^2)$ denote the space of all regular Borel measures on $\mathbb{R}^2$.

Let $\mathcal{P} := \mathbb{C} \setminus \{0\}$, the punctured complex plane and let $\mathcal{M}^*_{pp}(\mathcal{P}) := \{ \sum_{k=1}^{n} c_k \delta_{a_k} | c_k \in \mathbb{R}, n \geq 0, a_k \in \mathcal{P} \}$ be the span of all real finitely supported measures on $\mathcal{P}$.

For any $\alpha \in [0, 2\pi)$, we let $R(\alpha) : \mathcal{P} \to \mathcal{P}$ be rotation through angle α: $R(\alpha)(z) := e^{i\alpha} z$. Let σ be the operation of reflection in the x-axis: $\sigma(z) = \overline{z}$.

Both these types of operations extend to functions and to measures; in particular $R(\alpha)$ and σ act naturally on $\mathcal{M}^*_{pp}(\mathcal{P})$: $R(\alpha)(\sum_{k=1}^{n} c_k \delta_{a_k}) = \sum_{k=1}^{n} c_k \delta_{R(\alpha)(a_k)}$, $\sigma(\sum_{k=1}^{n} c_k \delta_{a_k}) = \sum_{k=1}^{n} c_k \delta_{\overline{a_k}}$

Let $\lambda^{\mathbb{R}^2}$ be Lebesgue measure on $\mathbb{R}^2$, $\lambda^{U(1)}$ the normalized Haar measure on $U(1)$, and δ_z be the delta measure supported at $z \in \mathbb{R}^2$.

Remark 5.4 When we use the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ as our averaging van Hove sequence, the autocorrelation of Λ is the vague limit $\eta := \lim_{n\to\infty} \eta_n$, if it exists [2], where

$$\eta_n := \frac{1}{5^n} \sum_{x,y \in (\Lambda \cap \Gamma_n)} \delta_{x-y} \, .$$

Remark 5.5 In Section 5.5 we will see that we can in fact use any van Hove sequence to compute the autocorrelation.

5.4.2 Substitution Formulation for Measures

The primary objective of this section is to verify that η is circularly symmetric. The pinwheel substitution of Definition 5.2 involves complication-causing reflections that we prefer to avoid. Imagine that each tile of some finite part of the pinwheel tiling carries some measure and we are interested in the total sum ν of these measures. We break this total measure into two pieces ν^+ and ν^-, with ν^+ carrying the total measure of the positive chirality tiles and ν^- carrying the measure of the negative chirality tiles *after they have been reflected in the x-axis.* Thus $\nu = \nu^+ + \sigma\nu^-$, but rather than this sum we work with the matrix

$$\begin{pmatrix} \nu^+ \\ \nu^- \end{pmatrix} .$$

In this scheme all measures lie on tiles of positive chirality and the process of reflection can then be relegated to a single operation at the very end to that brings the second measure into the correct position. Figure 5.2 illustrates the formalism as it appears in the substitution process.

Once we have established our formalism, we will use it to generate the η_n. By letting $n \to \infty$, we achieve the desired result.

[2] Throughout this section, we will almost exclusively understand Λ to represent only the locations of the control points, which are points of $\mathbb{C} \simeq \mathbb{R}^2$.

Definition 5.6 *Let $\Omega, \Phi : (\mathcal{M}^*_{pp}(\mathcal{P}))^2 \to (\mathcal{M}^*_{pp}(\mathcal{P}))^2$ be linear maps defined by:*

$$\Omega\begin{pmatrix}\mu\\ \nu\end{pmatrix} := \begin{pmatrix}R(-\omega) & 0\\ 0 & R(\omega)\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}$$

$$\Phi\begin{pmatrix}\mu\\ \nu\end{pmatrix} := \frac{1}{5}\begin{pmatrix}R(0)+R(\pi) & 2R(\pi)+R(-\frac{\pi}{2})\\ 2R(-\pi)+R(\frac{\pi}{2}) & R(-0)+R(-\pi)\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}$$

$$\Theta\begin{pmatrix}\mu\\ \nu\end{pmatrix} := \begin{pmatrix} id & \sigma\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \mu + \sigma\nu$$

Definition 5.7 *For any $k \geq h \geq 1$, define the linear map $\Psi^k_h : (\mathcal{M}^*_{pp}(\mathcal{P}))^2 \to (\mathcal{M}^*_{pp}(\mathcal{P}))^2$ by:*

$$\Psi^k_h\begin{pmatrix}\mu\\ \nu\end{pmatrix} := \Omega^{-k}\Phi\Omega^k\Omega^{-(k-1)}\Phi\Omega^{k-1}\ldots\Omega^{-h}\Phi\Omega^h\begin{pmatrix}\mu\\ \nu\end{pmatrix}.$$

If we define Ψ^m_h to be the identity map whenever $m < h$, then we have $\Psi^k_h\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \Psi^k_l\Psi^{l-1}_h\begin{pmatrix}\mu\\ \nu\end{pmatrix}$ for $k \geq l \geq h \geq 1$. This decomposition of Ψ^k_h will feature in several induction arguments.

Proposition 5.8 *For any $k \geq h \geq 1$,*

$$\Psi^k_h\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \frac{1}{5^{k-(h-1)}}\begin{pmatrix}\sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j) & \sum_{j=1}^{n_{k-(h-1)}} R(2h\omega+\beta_j)\\ \sum_{j=1}^{n_{k-(h-1)}} R(-2h\omega-\beta_j) & \sum_{j=1}^{m_{k-(h-1)}} R(-\alpha_j)\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}.$$

Proof (by induction on k): Fix an arbitrary $h \geq 1$ for the remainder of this proof.
$k = h$:

$$\Psi^h_h\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \Omega^{-h}\Phi\Omega^h\begin{pmatrix}\mu\\ \nu\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}R(0)+R(\pi) & 2R(2h\omega+\pi)+R(2h\omega-\frac{\pi}{2})\\ 2R(-2h\omega-\pi)+R(-2h\omega+\frac{\pi}{2}) & R(-0)+R(-\pi)\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}.$$

Induction step: $\Psi^{k+1}_h\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \Psi^{k+1}_{k+1}\Psi^k_h\begin{pmatrix}\mu\\ \nu\end{pmatrix}$. We know what Ψ^{k+1}_{k+1} looks like from our base case above, and we have Ψ^k_h by the induction hypothesis. Because of the symmetry of the Ψ matrices, it is sufficient to consider $(\Psi^{k+1}_h)_{11}$ and $(\Psi^{k+1}_h)_{12}$:

$$(\Psi_h^{k+1})_{11} = (R(0)+R(\pi)) \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j)$$

$$+(2R(2(k+1)\omega+\pi) + R(2(k+1)\omega - \frac{\pi}{2}) \sum_{j=1}^{n_{k-(h-1)}} R(-2h\omega - \beta_j)$$

$$= \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j) + \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j+\pi) + \sum_{j=1}^{n_{k-(h-1)}} 2R(2(k-(h-1))\omega - \beta_j + \pi)$$

$$+ \sum_{j=1}^{n_{k-(h-1)}} R(2(k-(h-1))\omega - \beta_j - \frac{\pi}{2}) = \sum_{j=1}^{m_{(k+1)-(h-1)}} R(\alpha_j)\,.$$

For help visualizing this argument, see Figure 5.2. The argument for $(\Psi_h^{k+1})_{12}$ is similar. □

Now that we understand Ψ_h^k in terms of our sequences of angles, we can put the uniform distribution result to good use.

Proposition 5.9 *For any $u \in \mathcal{P}$ and uniformly distributed sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ we have:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} R(z_n)\delta_u = \lambda^{U(1)} \otimes \delta_{|u|}\,,$$

where the limit is in the vague topology.

Proof. Note that the product of measures above refers to the product decomposition $\mathcal{P} = U(1) \times \mathbb{R}_{>0}$.

Let f be any continuous compactly supported $\mathbb{C}$-valued function on $\mathcal{P}$. We are required to show that

$$\lim_{N\to\infty} \left\langle \frac{1}{N} \sum_{n=1}^{N} R(z_n)\delta_u, f \right\rangle = \langle \lambda^{U(1)} \otimes \delta_{|u|}, f \rangle\,.$$

We have

$$\left\langle \frac{1}{N} \sum_{n=1}^{N} R(z_n)\delta_u, f \right\rangle = \frac{1}{N} \sum_{n=1}^{N} \langle \delta_u, R(z_n)^{-1} f \rangle = \frac{1}{N} \sum_{n=1}^{N} \int_{\mathcal{P}} f(z_n x)\, d\delta_u(x)$$

$$= \frac{1}{N} \sum_{n=1}^{N} f(z_n u) \overset{N\to\infty}{\longrightarrow} \int_{U(1)} f(zu) dz = \int_{U(1)} f(z\,|u|) dz$$

$$= \int_{U(1)\times\mathbb{R}_{>0}} f(zr)\, d\lambda^{U(1)}(z) \otimes \delta_{|u|}(r) = \langle \lambda^{U(1)} \otimes \delta_{|u|}, f \rangle$$

□

Remark 5.10 Note that measures of the form $\lambda^{U(1)} \otimes \sigma$, where σ is a positive measure on $K \subset (0,\infty)$, are not what one may intuitively think from the perspective of usual Lebesgue measure on $\mathbb{R}^2$. For example, consider that $\|\lambda^{U(1)} \otimes \sigma\| = \lambda^{U(1)}(U(1))\sigma(K) = \sigma(K)$. This is independent of where K lies in $(0,\infty)$. The Lebesgue measure of $B_K = U(1) \times K \subset \mathbb{R}^2$ (see Equation (5.4)) is its area, and hence depends on where K is located.

Definition 5.11 *Let $P : \mathcal{P} \to (0,\infty)$ be defined by $P(z) := |z|$. Then P determines a linear map (also denoted by P) from $\mathcal{M}^*_{pp}(\mathcal{P})$ to $\mathcal{M}^\infty((0,\infty))$ by $P(\sum_{k=1}^n c_k\delta_{a_k}) := \sum_{k=1}^n c_k\delta_{P(a_k)}$.*

Corollary 5.12 *For all $\mu, \nu \in \mathcal{M}^*_{pp}(\mathcal{P})$,*

$$\Psi_h^k \begin{pmatrix} \mu \\ \nu \end{pmatrix} \stackrel{k\to\infty}{\longrightarrow} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes (P(\mu+\nu)) \\ \lambda^{U(1)} \otimes (P(\mu+\nu)) \end{pmatrix}$$

in the vague topology.

Proof. The combination of Propositions 5.8 and 5.9 yields

$$\Psi_h^k \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} \stackrel{k\to\infty}{\longrightarrow} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes \delta_{|x|} + \lambda^{U(1)} \otimes \delta_{|y|} \\ \lambda^{U(1)} \otimes \delta_{|x|} + \lambda^{U(1)} \otimes \delta_{|y|} \end{pmatrix}$$

for any $h \geq 1$. Since μ, ν are finite linear combinations of deltas, the desired result follows by linearity. □

5.4.3 The Autocorrelation on the nth Iterate

We recall that Λ_n consists of 5 isometrical copies of Λ_{n-1}. Let $n \geq 1$. We define

$$\begin{aligned} D_n &:= \left\{(x,y) \in \Lambda \times \Lambda \;\middle|\; x,y \in \Lambda_n \textit{ and are in different copies of } \Lambda_{n-1}\right\}, \\ C_n &:= \left\{(x,y) \in \Lambda \times \Lambda \;\middle|\; x,y \in \Lambda_n, x \neq y, \textit{ and are in the same copy of } \Lambda_{n-1}\right\}. \end{aligned} \tag{5.2}$$

Then

$$\eta_n = \delta_0 + \frac{1}{5^n} \sum_{x,y \in C_n} \delta_{x-y} + \frac{1}{5^n} \sum_{x,y \in D_n} \delta_{x-y}. \tag{5.3}$$

Because the minimum distance between pinwheel control points is $\frac{1}{\sqrt{5}}$, there exists $\frac{1}{\sqrt{5}} > r > 0$ such that $\eta_n|_{B_r(0)} = \delta_0$. For such an r, $\lim_{n\to\infty} \eta_n|_{B_r(0)} = \delta_0$. In other words, 0 is separated from the rest of the support of η.

For any K bounded in $(0, \infty)$, we define

$$B_K := \{a \in \mathcal{P} \mid |a| \in K\} = P^{-1}(K), \tag{5.4}$$

the K-**corona** around 0, whose intersection with the positive x axis is K.

For any $\mu, \nu \in \mathcal{M}_{pp}^*(\mathcal{P})$, we have:

i) $\mu(B_K) = P(\mu)(K)$,

ii) $\Psi_h^k \begin{pmatrix} \mu \\ \nu \end{pmatrix} (B_K) = \frac{1}{5^{k-(h-1)}} \begin{pmatrix} m_{k-(h-1)}\mu(B_K) + n_{k-(h-1)}\nu(B_K) \\ n_{k-(h-1)}\mu(B_K) + m_{k-(h-1)}\nu(B_K) \end{pmatrix}$.

It is immediate that:

Lemma 5.13 *For all $\nu, \nu' \geq 0$, we have*

$$\Psi_h^k \begin{pmatrix} \nu \\ \nu' \end{pmatrix} (B_K) \leq \begin{pmatrix} \nu(B_K) + \nu'(B_K) \\ \nu(B_K) + \nu'(B_K) \end{pmatrix}.$$

□

Definition 5.14

i) $\rho_n := \frac{1}{5^n} \sum_{x,y \in D_n} \delta_{x-y}$.

ii) η_n^+, η_n^- *are defined recursively as follows:*

$\eta_1^+ = \eta_1^- := 0$,

$\begin{pmatrix} \eta_n^+ \\ \eta_n^- \end{pmatrix} := \Psi_{n-1}^{n-1} \begin{pmatrix} \eta_{n-1}^+ + \rho_{n-1} \\ \eta_{n-1}^- \end{pmatrix}$.

By a standard induction argument, we get

$$\begin{pmatrix} \eta_n^+ \\ \eta_n^- \end{pmatrix} = \sum_{k=1}^{n-1} \Psi_{n-k}^{n-1} \begin{pmatrix} \rho_{n-k} \\ 0 \end{pmatrix} \quad \text{for all } n \geq 2. \tag{5.5}$$

Proposition 5.15 *For any $n \geq 1$ we have*

$$\eta_n = \delta_0 + \eta_n^+ + \sigma\eta_n^- + \rho_n. \tag{5.6}$$

Proof: To see that equation (5.6) holds, by (5.3) we must prove that

$$\eta^+_{n+1} + \sigma\eta^-_{n+1} = \frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y}\,.$$

We prove this by induction. Figure 5.2 may help clarify the following argument.
$n=0$: $C_1 = \emptyset$; $\eta^+_1 = 0, \eta^-_1 = 0$, which gives us our desired equality.
Induction step: Λ_{n+1} consists of the union of the five disjoint copies of Λ_n resulting from the application of the mappings $f_1, \ldots, f_5$ upon Λ_n. Here f_1, f_2 are direct isometries of $\mathbb{C}$, while f_3, f_4, f_5 are opposite (i.e., chirality reversing) isometries of $\mathbb{C}$ (note that the translation and reflection components of these isometries depend on n, while the rotation components are independent of n). Then,

$$\begin{aligned} C_{n+1} &= \Big\{(x,y) \in \Lambda\times\Lambda \;\Big|\; \exists 1 \le i \le 5 \textit{ and } (a,b) \in \Lambda_n\times\Lambda_n \textit{ with } a \ne b \\ &\qquad \textit{such that } (x,y) = (f_i(a), f_i(b))\Big\}\ \text{, so} \end{aligned}$$

The translation part of f_i cancels when we take differences:

$$\begin{aligned} \frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y} &= \frac{1}{5}(R(0)+R(\pi))\Big(\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\ne y}} \delta_{x-y}\Big) \\ &\quad + \frac{1}{5}R(n\omega)\left(2R(\pi) + R(-\frac{\pi}{2})\right)\sigma R(-n\omega)\Big(\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\ne y}} \delta_{x-y}\Big)\,. \end{aligned}$$

Now, by the induction hypothesis:

$$\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\ne y}} \delta_{x-y} = \eta_n - \delta_0 \overset{(3.2)}{=} \eta^+_n + \rho_n + \sigma\eta^-_n\,.$$

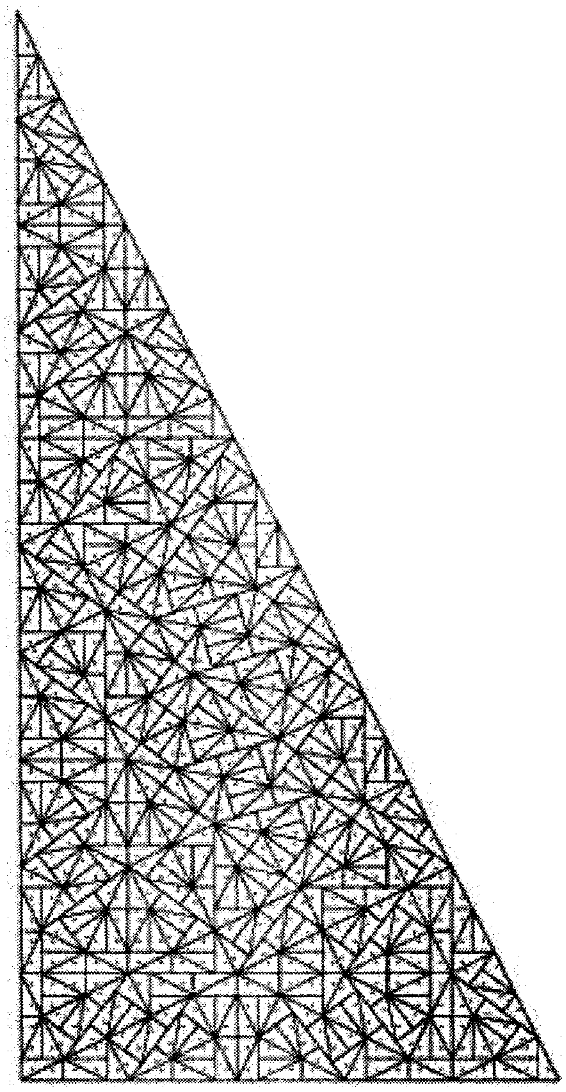

Figure 5.2: Successive powers of Ψ build up sums of rotation operators which we ultimately apply to measures. Rather than carrying two chiralities in one plane, we prefer to work with one chirality (namely, $+1$) in two planes. These two planes are represented here by the half planes above and below the x-axis respectively. It is to be understood that the full chiral picture is obtained by reflecting the lower half plane onto the upper through the x-axis. The rotations involved are indicated here by the orientation of the triangles which we see being built up by the substitution. After reflection, the upper and lower pictures fit together to give the full substitution.

Therefore,

$$\begin{aligned}\frac{1}{5^{n+1}}\sum_{(x,y)\in C_{n+1}}\delta_{x-y} &= \frac{1}{5}(R(0)+R(\pi))(\eta_n^+ + \rho_n + \sigma\eta_n^-)\\ &+\frac{1}{5}R(n\omega)(2R(\pi)+R(-\frac{\pi}{2}))\sigma R(-n\omega)(\eta_n^+ + \rho_n + \sigma\eta_n^-)\\ &= \frac{1}{5}\Big((R(0)+R(\pi))(\eta_n^+ + \rho_n) + \sigma(R(-0)+R(-\pi))\eta_n^-\\ &+\sigma(2R(-2n\omega-\pi)+R(-2n\omega+\frac{\pi}{2}))(\eta_n^+ + \rho_n)\\ &+(2R(2n\omega+\pi)+R(2n\omega-\frac{\pi}{2}))\eta_n^-\Big) = \eta_{n+1}^+ + \sigma\eta_{n+1}^-\end{aligned}$$

by Definition 5.14. □

$$\frac{1}{5^{n+1}}\sum_{(x,y)\in C_{n+1}}\delta_{x-y} = \frac{1}{5}\sum_{i=1}^{5}\frac{1}{5^n}\sum_{\substack{x,y\in\Lambda_n\\ x\neq y}}\delta_{f_i(x)-f_i(y)}\,.$$

5.4.4 Convergence and Circular Symmetry

Definition 5.16 $\mu_{n,K} := P(\eta_n|_{B_K}) \in \mathcal{M}^\infty((0,\infty))$ *for* $K \subset (0,\infty)$ *bounded.*

Proposition 5.17 *Let* $K \subseteq (0,\infty)$ *be any bounded set. Then* $\{\mu_{n,K}\}_{n=1}^\infty$ *converges in the total variation norm topology to a pure point measure.*

Proof: By Definition 5.16 and Proposition 5.15, we get

$$\mu_{n+1,K} = P(\eta_{n+1}|_{B_K}) = P(\eta_{n+1}^+|_{B_K}) + P(\eta_{n+1}^-|_{B_K}) + P(\rho_{n+1}|_{B_K}).$$

Note that $P(\eta_{n+1}^-) = P(\sigma\eta_{n+1}^-)$.

By a remark following Equation (5.4), if $K' \subseteq K$ is any set we have

$$\begin{pmatrix}\eta_{n+1}^+(B_{K'})\\ \eta_{n+1}^-(B_{K'})\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\eta_n^+(B_{K'}) + 2\rho_n(B_{K'}) + 3\eta_n^-(B_{K'})\\ 3\eta_n^+(B_{K'}) + 3\rho_n(B_{K'}) + 2\eta_n^-(B_{K'})\end{pmatrix},$$

whence

$$\begin{aligned}&P(\eta_{n+1}^+)(K') + P(\eta_{n+1}^-)(K') = \eta_{n+1}^+(B_{K'}) + \eta_{n+1}^-(B_{K'})\\ &\quad = \eta_n^+(B_{K'}) + \eta_n^-(B_{K'}) + \rho_n(B_{K'}) = \eta_n(B_{K'}) = P(\eta_n)(K')\,.\end{aligned}$$

Thus, for all $K' \subseteq K$ we have $P(\eta_{n+1}^+)(K') + P(\eta_{n+1}^-)(K') = P(\eta_n)(K')$.

Hence,

$$P(\eta^+_{n+1}|_{B_K}) + P(\eta^-_{n+1}|_{B_K}) = P(\eta^+_{n+1})|_K + P(\eta^-_{n+1})|_K$$
$$= P(\eta_n)|_K = P(\eta_n|_{B_K}) = \mu_{n,K}\,. \tag{5.7}$$

So we get

$$\mu_{n+1,K} = \mu_{n,K} + P(\rho_{n+1}|_{B_K})\,.$$

Therefore, $\mu_{n+1,K} \geq \mu_{n,K}$ and

$$\|\mu_{n+1,K} - \mu_{n,K}\| = \rho_{n+1}(B_K) = \frac{1}{5^{n+1}} \sharp\big\{(x,y) \in D_{n+1} \;\big|\; |x-y| \in K\big\}.$$

Because x, y must be in different copies of Λ_n, x must be in the B_K-boundary of one of those copies and $y \in x + B_K$. Let $c := \max_{a \in \mathbb{C}}(\sharp\{\Lambda \cap (a + B_K)\})$, a finite quantity because the minimum distance between control points is $\frac{2}{\sqrt{5}}$. Then,

$$\sharp\big\{(x,y) \in D_{n+1} \;\big|\; |x-y| \in K\big\} \leq c \cdot \sum_{j=1}^{5} \sharp\big\{x \in \Lambda \cap \partial^{B_K}(f_j\Gamma_n)\big\}\,.$$

When we inflate Γ_n we have $\lambda^{\mathbb{R}^2}(\partial^{B_K}\Gamma_{n+1}) \simeq \sqrt{5}\lambda^{\mathbb{R}^2}(\partial^{B_K}\Gamma_n)$, since the linear scaling is by $\sqrt{5}$.

Therefore, $\exists$ a constant c' depending only on K such that

$$\rho_{n+1}(B_K) \leq c' \left(\frac{1}{\sqrt{5}}\right)^{n+1}\,. \tag{5.8}$$

Then $\|\mu_{m,K} - \mu_{n,K}\| \leq c' \sum_{j=n+1}^{m} (\frac{1}{\sqrt{5}})^j$ shows that $\{\mu_{n,K}\}_n$ is Cauchy in the total variation norm. By a comment following Proposition 3 of [7], $\{\mu_{n,K}\}_n$ converges in the total variation norm topology to a pure point measure. □

Definition 5.18 $\mu_K := \lim_{n\to\infty} \mu_{n,K}$ *is a pure point measure on* $(0, \infty)$.

Proposition 5.19 $\eta_n|_{B_K} \longrightarrow \lambda^{U(1)} \otimes \mu_K$ *in the vague topology.*

Proof: Let $\eta_K = \lambda^{U(1)} \otimes \mu_K$. Let U be any neighbourhood of 0 in the vague topology. Then $\exists\, V$, a neighbourhood of 0, such that $V + V + V + V + V + V \subseteq U$. Also, we may assume that $V = -V$. Since the total variation topology is stronger than the vague topology, there exists $\epsilon > 0$ such that whenever $\|\nu\| < \epsilon$ then $\nu \in V$.

Because $\mu_{n,K} \xrightarrow{\|\cdot\|} \mu_K$, there exists N such that for all $n > N$, we have $\|\mu_{n,K} - \mu_K\| < \epsilon$. This gives us $\|\lambda^{U(1)} \otimes \mu_{n,K} - \lambda^{U(1)} \otimes \mu_K\| < \epsilon$, and hence, $\lambda^{U(1)} \otimes \mu_{n,K} - \eta_K \in V$ for all $n > N$.

(5.8) says that $\rho_n(B_K) \le c'\left(\frac{1}{\sqrt{5}}\right)^n$, so $\exists\, M \ge N+1$ such that

$$\sum_{k=M}^{m} \rho_k(B_K) < \epsilon \ \text{ for all}\, m \ge M. \tag{5.9}$$

We know by (5.5) that

$$\begin{pmatrix} \eta_n^+|_{B_K} \\ \eta_n^-|_{B_K} \end{pmatrix} = \sum_{k=1}^{n-1} \Psi_{n-k}^{n-1} \begin{pmatrix} \rho_{n-k}|_{B_K} \\ 0 \end{pmatrix}. \tag{5.10}$$

Splitting the above sum yields

$$\begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} = \sum_{k=1}^{n} \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k}|_{B_K} \\ 0 \end{pmatrix},$$

and using the triangle inequality gets us

$$\left\| \begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \right\| \le \sum_{k=1}^{n} \left| \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k} \\ 0 \end{pmatrix} \right| (B_K)$$

$$= \sum_{k=1}^{n} \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k} \\ 0 \end{pmatrix} (B_K),$$

where $\left\| \begin{pmatrix} \nu \\ \nu' \end{pmatrix} \right\| := \begin{pmatrix} \|\nu\| \\ \|\nu'\| \end{pmatrix}$ and $\|\cdot\|$ is the total variation norm.

Thus, by Lemma 5.13 and (5.9),

$$\left| \begin{pmatrix} \eta_{n+M}^+ \\ \eta_{n+M}^- \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+ \\ \eta_M^- \end{pmatrix} \right| (B_K) < \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix}. \tag{5.11}$$

We also know that $\lambda^{U(1)} \otimes \mu_{M-1} - \eta_K \in V$. From Corollary 5.12 and the fact that M is fixed, we know

$$\Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \xrightarrow{n\to\infty} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes P(\eta_M^+|_{B_K} + \eta_M^-|_{B_K}) \\ \lambda^{U(1)} \otimes P(\eta_M^+|_{B_K} + \eta_M^-|_{B_K}) \end{pmatrix}$$

and so, by (5.7), we get that

$$\Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \xrightarrow{n\to\infty} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix}.$$

Therefore, $\exists N'$ such that

$$\Psi_M^{n+M-1}\begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix} \in \begin{pmatrix} V \\ V \end{pmatrix} \quad \text{for all}\, n \geq N',$$

and by (5.11), we have

$$\begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1}\begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \in \begin{pmatrix} V \\ V \end{pmatrix} \quad \text{for all}\, n \geq 0\,.$$

Combining these, we get

$$\begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix} \in \begin{pmatrix} V - V \\ V - V \end{pmatrix} \quad \text{for all}\, n \geq N'\,.$$

Finally, from Proposition 5.15

$$\begin{aligned} \eta_{n+M}|_{B_K} - \eta_K &= (\eta_{n+M}^+|_{B_K} - \frac{1}{2}\lambda^{U(1)} \otimes \mu_{M-1}) + (\eta_{n+M}^-|_{B_K} - \frac{1}{2}\lambda^{U(1)} \otimes \mu_{M-1}) \\ &\quad + (\lambda^{U(1)} \otimes \mu_{M-1} - \lambda^{U(1)} \otimes \mu_K) + \rho_{n+M}|_{B_K} \end{aligned}$$

and then (5.9) gives us $\|\rho_{n+M}|_{B_K}\| < \epsilon$ whence $\rho_{n+M}|_{B_K} \in V$, by our choice of ϵ. Thus

$$\eta_{n+M}|_{B_K} - \eta_K \in V - V + V - V + V + V \subseteq U\,,$$

and therefore $\eta_n|_{B_K} - \eta_K \in U$ for all $n > N' + M$. □

5.4.5 Autocorrelation Conclusions

It is easy to see that if $K \subseteq K'$ then $\mu_{K'}|_K = \mu_K$. This allows the following definition:

Definition 5.20 *μ is the pure point measure on $(0, \infty)$ defined by $\mu|_K = \mu_K$ for all K bounded in $(0, \infty)$.*

From Proposition 5.19 and the fact that $\lim_{n\to\infty} \eta_n|_{B_r(0)} = \delta_0$ for some sufficiently small $r > 0$, we get that $\eta_n|_{\{0\}\cup B_K} \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu|_K$ for all $K \subseteq (0, \infty)$ bounded. This final remark sets us up for the main theorem.

Theorem 5.21 *The autocorrelation of Λ, η, exists with respect to $\{\Gamma\}_{n=1}^{\infty}$ and $\eta = \delta_0 + \lambda^{U(1)} \otimes \mu$.*

Proof. Suppose that f is an arbitrary real valued continuous function of compact support. Then, $\mathrm{supp}(f) \subseteq \{0\} \cup B_K$ for some bounded $K \subset (0, \infty)$.

From Proposition 5.19 we have

$$\eta_n\big|_{\{0\}\cup B_K} \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu_K\,,$$

which means

$$\eta_n\big|_{\{0\}\cup B_K}(f) \longrightarrow (\delta_0 + \lambda^{U(1)} \otimes \mu_K)(f).$$

Because $\mathrm{supp}(f) \subseteq \{0\} \cup B_K$, this gives us

$$\eta_n(f) \longrightarrow (\delta_0 + \lambda^{U(1)} \otimes \mu)(f)\,,$$

and finally, by the definition of vague convergence,

$$\eta_n \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu\,.$$

□

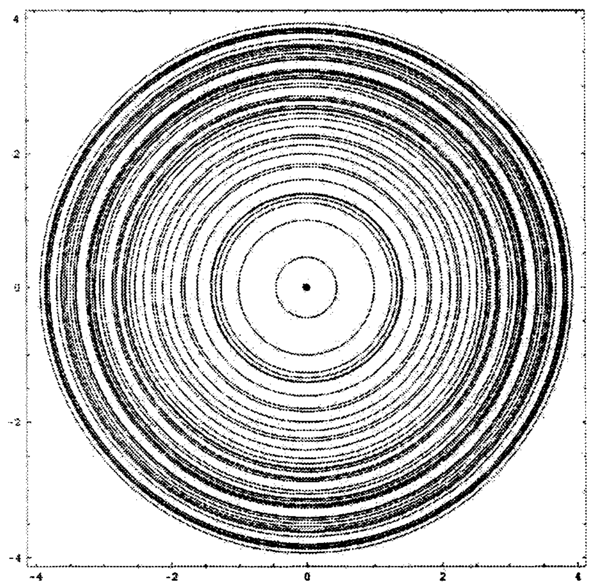

Figure 5.3: Part of the support of the pinwheel autocorrelation measure η

Our understanding of the autocorrelation of the pinwheel tiling only lacks knowledge regarding the pure point measure μ, and hence about the radii and heights of the circles. In [32], Charles Radin suggests that the support of μ has a self-similar structure. While we were not able to exploit this observation, it may prove useful to future pinwheel enthusiasts.

5.4.6 Diffraction

It is easy to check that for $R(\alpha) \in U(1)$, $f \in C_c(\mathbb{R}^2)$ we have: $(R(\alpha)\widehat{f})(k) = \widehat{R(\alpha)f}(k)$ and hence, if ν is a Fourier transformable measure and $R(\alpha)\nu = \nu$ then $\widehat{\nu}(R(\alpha)f) = \widehat{\nu}(f)$. Since the pinwheel autocorrelation is fully circularly symmetric, so is the diffraction.

From this, we can see that the diffraction may only have a pure point part at the origin. We show that this is indeed the case.

Proposition 5.22 $\widehat{\eta}_{pp} = (dens(\Lambda))^2\delta_0$.

Proof: This result follows from Theorem 2.2 in [20]. We have

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Gamma_n)} \int 1\, d(\sum_{x\in\Lambda_n} \delta_x) = \lim_{n\to\infty} \frac{\sharp(\Lambda_n)}{\mathrm{vol}(\Gamma_n)} = \mathrm{dens}(\Lambda)\,.$$

Then, by the result mentioned above, $\widehat{\eta}(\{0\}) = (dens(\Lambda))^2$. □

5.5 van Hove sequences

In the construction of the autocorrelation and the diffraction we have assumed that the averaging sequence is the ascending chain of super triangles Γ_n created by the substitution process itself. In this section we prove that we get the same autocorrelation (and hence diffraction) for any van Hove sequence $\mathcal{A} = \{A_m\}$.

Proposition 5.23 *Let $\mathcal{A} = \{A_m\}$ be any van Hove sequence in $\mathbb{R}^2$ and let*

$$\eta_{A_m} := \frac{1}{\mathrm{vol}\,(A_m)} \sum_{x,y\in\Lambda\cap A_m} \delta_{x-y} \tag{5.12}$$

be the averaged autocorrelation of A_m. Then $\eta_{A_m} \to \eta$ in the vague topology.

Proof: It suffices to take one fixed, but otherwise arbitrary, continuous function f on $\mathbb{R}^2$ of compact support and show that $\eta_{A_m}(f) \to \eta(f)$. We assume $f \neq 0$.

We know that $\eta_n \to \eta = \lambda^{U(1)} \otimes \mu$ in the vague topology. Since $\lambda^{U(1)} \otimes \mu = \sigma(\lambda^{U(1)} \otimes \mu)$, we also have that $\sigma\eta_n \to \eta$.

Let B be a closed ball around 0 which contains $\mathrm{supp}(f)$. Since $R_\theta(\eta) = \eta$ for all θ, it follows that $R_\theta\eta_n(f) \to \eta(f)$ for each θ. Since the mapping $(\theta, x) \mapsto f(R_\theta x)$ is continuous and $U(1) \times B$ is compact we see that the convergence to $\eta(f)$ is uniform in θ. In the same way $R_\theta(\sigma\eta_n) \to \eta$ uniformly for all θ.

Consider any of the triangles Γ_n and in particular its inner boundary of width equal to the diameter of the ball B. Let $p(\Gamma_n)$ be the perimeter of Γ_n. Since Λ is a Delone set, there is a positive constant c_B so that the number of points of Λ inside this inner boundary is bounded above by $c_B p(\Gamma_n)$ for all n, no matter where or in what orientation the triangle Γ_n is placed in $\mathbb{R}^2$. Let n_B be the maximum number of points of Λ inside any translate of B.

Combining all this information, for any $\epsilon > 0$ we can choose $N = N(\epsilon, f) > 0$ so that for all θ and for all $k \geq 0$ we have simultaneously

- $|R_\theta \eta_N(f) - \eta(f)| < \epsilon$
- $|R_\theta \sigma \eta_N(f) - \eta(f)| < \epsilon$
- $\dfrac{c_B p(\Gamma_N) n_B}{\text{vol}\,(\Gamma_N)} ||f||_\infty < \epsilon\,.$

Let A be any region of $\mathbb{R}^2$ precisely tiled by a subset of the super-tiles Γ_N in the total tiling Γ of $\mathbb{R}^2$. Then

$$A = \bigcup_{i=1}^{M} T_i \Gamma_N$$

where the T_i are composed of Euclidean isometries, and since autocorrelations are unaffected by translations,

$$\left| \frac{1}{M} \sum_{i=1}^{M} \eta_{T_i \Gamma_N}(f) - \eta(f) \right| \leq \frac{1}{M} \sum_{i=1}^{M} |\eta_{T_i \Gamma_N}(f) - \eta(f)| \leq \epsilon\,. \tag{5.13}$$

The averaged autocorrelation of η_A of A is

$$\eta_A(f) = \frac{1}{M} \sum_{i=1}^{M} \eta_{T_i \Gamma_N}(f) + \frac{1}{\text{vol}\,A} \sum_{(x,y) \in D(A,N)} f(x-y) \tag{5.14}$$

where $D(A, N)$ is the set of all pairs $(x, y) \in (A \cap \Lambda) \times (A \cap \Lambda)$ where the two components come from different copies of the tile Γ_N in its tiling of A.

Since $x - y \in B$ is necessary for (x, y) to make any contribution to the sum, x is restricted to the inner B-boundary of the tile it belongs to and y is restricted to the ball $x + B$. Thus

$$\left| \frac{1}{\text{vol}\,A} \sum_{(x,y) \in D(A,N)} \delta_{x-y}(f) \right| < \frac{M c_B p(\Gamma_N) n_B ||f||_\infty}{M \text{vol}(\Gamma_N)} < \epsilon\,. \tag{5.15}$$

Thus

$$|(\eta_A - \eta)(f)| < 2\epsilon\,. \tag{5.16}$$

Now consider the van Hove sequence $\{A_m\}$. Let K be any closed disk centered on 0 containing Γ_N and $\overline{\Gamma_N}$. Let $A(m)$ be that part of A_m which is composed of complete Γ_N - tiles taken from the full tiling Γ. Then $A_m \backslash A(m) \subset \partial^K(A_m)$.

Now the point is that because of the van Hove property the boundary can contain only a number of points of Λ that is bounded by $c_K \text{vol}(\partial^K A_m)$ for some positive constant c_K that is independent of m. Thus

$$\begin{aligned} \left|\eta_{A_m}(f) - \frac{\text{vol}\, A(m)}{\text{vol}\, A_m}\eta_{A(m)}(f)\right| &= \left|\frac{1}{\text{vol}\, A_m}\sum \delta_{x,y}(f)\right| \\ &\leq \frac{c_1}{\text{vol}\, A_m}\text{vol}\,(\partial^K A_m)||f||_\infty \end{aligned} \tag{5.17}$$

where the sum is over all $x, y \in \Gamma \cap A_m$ in which at least one of x or y is in $\partial^K(A_m)$. The van Hove property shows that $\eta_{A(m)} - \eta_{A_m} \to 0$ as $m \to \infty$.

Combining this with equation (5.16) we see that $|\eta(f) - \eta_{A_m}(f)| < 3\epsilon$ for all $m >> N = N(\epsilon, f)$. As ϵ and f are arbitrary, we have proven the proposition. □

5.6 Further Remarks

We offer here another proof (worked out with Michael Baake) of circular symmetry, this time based on the theory of dynamical systems. Consider the space X of all tilings that are locallly indistinguishable from the pinwheel tiling that we have constructed. These are the tilings each of whose patches is a copy, under the rotation-translation group of the full Euclidean group of isometries of the plane, to a patch of the given pinwheel tiling, and vica-versa. This is evidently closed under the rotation-translation group and in particular under the translation group of the plane. We give this space the standard topology [37]. The uniform distribution of orientations allows us to conclude that X is the closure of the *translation* orbit of any of its tilings – it is minimal.

The substitution together with the uniform distribution of orientation allows one to see quite easily that patch frequencies are uniform, that is, the limit defining the frequency, or density, of a patch of tiles is approached uniformly, independent of the position or orientation of the patch. It follows from this that the autocorrelation is identical for all elements of X, and so this measure must be circularly symmetric.

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Appendix A

Averaging Sequences

The setting for all averaging processes in this Appendix is that of van Hove sequences. In the Section 4.1.3 we needed various results which require particular specialized properties of a sequence to prove. In this Appendix we discuss these properties, particularly the property **M**, and show that all van Hove sequences satisfy it. One can find these results in ([44], Appendix 3). We provide the proofs here because the discussion in [44] is in a more general context and also, some of the Propositions there are mentioned without proofs.

Definition A.1 Let $\{A_n\}_n$ be a sequence of measurable subsets of G. We say that the sequence has the **Følner** property if

$$\lim_n \frac{\theta(A_n \ \triangle \ (x + A_n))}{\theta(A_n)} = 0 \text{ for all } x \in G\,.$$

Definition A.2 Let $\{A_n\}_n$ be a sequence of subsets of G. For any compact $K \subset G$ we define:

$$A_n^{K-} = A_n \setminus ((-K + \overline{G \setminus A_n}) \cap A_n)\,,$$
$$A_n^{K+} = A_n + (\{0\} \cup K)\,.$$

Remark A.3 One has $A_n^{K-} = A_n \setminus \partial^K(A_n)$ and $A_n^{K+} = A_n \cup \partial^K(A_n)$.

Definition A.4 Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets in G. We say that this sequence has the property **M** if :

i) $\{A_n\}_n$ has the Følner property.

ii) $\{A_n^{K-}, A_n^{K+}\}$ have the Følner property for all compact K,

iii) $\lim_{n\to\infty}(\theta(A_n^{K-})/\theta(A_n)) = \lim_{n\to\infty}(\theta(A_n^{K+})/\theta(A_n)) = 1$.

We begin with the following lemma:

Lemma A.5 [2] *Every weakly almost periodic function is amenable. Furthermore, if $\{f_n\}_n$ is a sequence of non-negative functions in $L^1(G)$ such that:*

i) $\int_G f_n(x)dx = 1 \, \forall n$,

ii) $\lim_n \| \delta_x * f_n - f_n \|_1 = 0$ *for all $x \in G$,*

then for any weakly almost periodic function g on G we have

$$M(f) = \lim_n \int_G f_n(x)g(x)dx \, .$$

□

Proposition A.6 *Let $\{A_n\}_n$ be a sequence of subsets of G which has the Følner property . Let*

$$f_n = \frac{1}{\theta(A_n)} \mathbf{1}_{A_n} \, ,$$

where $\mathbf{1}_{.}$ is the characteristic function of the set. Then $\{f_n\}$ verifies the conditions from the previous lemma. In particular, for any weakly almost periodic function g on G, we have:

$$M(g) = \lim_n \frac{\int_{A_n} g(x)dx}{\theta(A_n)} \, .$$

Proof: It is easy to see that :

$$|\delta_x * f_n - f_n| = \frac{1}{\theta(A_n)} \mathbf{1}_{A_n \Delta (x+A_n)} \, .$$

Then the proof follows from the definition of the Følner property. □

Next proposition shows that any van Hove sequence has the Følner property. For this we show in fact that:

$$A_n \Delta (x + A_n) \subset \partial^{\{x,-x\}}(A_n) \, .$$

Proposition A.7 *Let $\{A_n\}_n$ be a van Hove sequence in G. Then $\{A_n\}_n$ has the Følner property.*

Proof: Let $x \in G$. Let $K' = \{x, -x\}$. We prove that:

$$A_n \triangle (x + A_n) \subset \partial^{K'}(A_n).$$

Let $y \in A_n \triangle (x + A_n)$.
First case $y \in A_n, y \notin (x + A_n)$.

Since $y \notin (x + A_n)$ then $y - x \notin A_n$ so $y - x \in \overline{G \setminus A_n}$. Since $x \in K' = -K'$ we get:

$$y \in -K' + \overline{G \setminus A_n}.$$

We know $y \in A_n$. Thus $y \in \partial^{K'}(A_n)$, hence we are done.
Second case $y \notin A_n, y \in (x + A_n)$.

$y \in (x + A_n)$ implies $y \in K' + A_n$. Since $y \notin A_n$ then $y \notin \text{Int}(A_n)$. Hence $y \in \partial^{K'}(A_n)$. □

In the next two Propositions we show in fact that when we compute the mean by averaging over van Hove sequences, we can ignore what happens in the $K-$boundary of the sequence.

Proposition A.8 *Let $\{A_n\}_n$ be a van Hove sequence in G. Let K be an arbitrary compact set and let*

$$B_n = A_n \setminus ((-K + \overline{G \setminus A_n}) \cap A_n),$$
$$C_n = (\{0\} \cup K) + A_n.$$

Then B_n, C_n are van Hove sequences.

Proof:
i) The proof for B_n: Let K' be a compact subset of G. Let $K'' = (\{0\} \cup K) + (\{0\} \cup K')$. K'' is compact. Let $x \in \partial^{K'}(B_n)$. Then $x \in K' + B_n \setminus \text{Int}(B_n)$ or $x \in (-K' + \overline{G \setminus B_n}) \cap B_n$.
First case $x \in (K' + B_n) \setminus \text{Int}(B_n)$. Then $x \in (K' + (A_n \setminus ((-K + \overline{G \setminus A_n}) \cap A_n))) \setminus \text{Int}(B_n)$, hence $x \in K' + A_n$.
If $x \notin \text{Int}(A_n)$ we get $x \in \partial^{K'}(A_n)$, so $x \in \partial^{K''}(A_n)$.
If $x \in \text{Int}(A_n)$, since $x \notin \text{Int}(B_n)$ we get $x \notin \text{Int}(A_n \setminus ((-K + \overline{G \setminus A_n}) \cap A_n))$, so $x \notin \text{Int}(A_n \cap (G \setminus ((-K + \overline{G \setminus A_n}) \cap A_n)))$ hence

$$x \notin \text{Int}(A_n) \cap \text{Int}((G \setminus ((-K + \overline{G \setminus A_n}) \cap A_n))).$$

This implies $x \notin \text{Int}((G \setminus ((-K + \overline{G \setminus A_n}) \cap A_n)))$, so $x \notin (G \setminus \overline{((-K + \overline{G \setminus A_n}) \cap A_n)})$. Hence

$$x \in \overline{((-K + \overline{G \setminus A_n}) \cap A_n))} \subset ((-K + \overline{G \setminus A_n})) \cap \overline{A_n} \subset \overline{G \setminus A_n})).$$

Thus , we get $x \in \partial^K(A_n) \subset \partial^{K''}(A_n)$.
Second case $x \in (-K' + \overline{G \setminus B_n}) \cap B_n$ implies $x \in B_n \Rightarrow x \in A_n$. We also know

$$x \in (-K' + \overline{G \setminus B_n}) = (-K' + \overline{G \setminus (A_n \setminus ((-K + \overline{G \setminus A_n}) \cap A_n))}$$
$$= -K' + \overline{(G \setminus A_n) \cup (-K + \overline{(G \setminus A_n)})} = -K' + (\overline{(G \setminus A_n)} + (-K \cup \{0\}))\,.$$

Hence $x \in \partial^{K''}(A_n)$.

This proves that B_n is a van Hove sequence.

ii) The proof for C_n: Let $K' \subset G$ be compact. Let $K'' = (K \cup \{0\}) + (K' \cup \{0\})$. K'' is compact. Let $x \in \partial^{K'}(C_n)$, then $x \in (K' + C_n) \setminus \mathrm{Int}(C_n)$ or $x \in (-K' + \overline{G \setminus C_n}) \cap C_n$
First case $x \in (K' + C_n) \setminus \mathrm{Int}(C_n)$. We get $x \in K' + C_n$ hence $x \in K' + (K \cup \{0\}) + A_n \subset K'' + A_n$.
Since $A_n \subset C_n$ we obtain $\mathrm{Int}(A_n) \subset \mathrm{Int}(C_n)$ so $x \notin \mathrm{Int}(A_n)$. Hence $x \in \partial^{K''}(A_n)$.
Second case $x \in (-K' + \overline{G \setminus C_n}) \cap C_n$, then $x \in C_n$ so $x \in K'' + A_n$.
If $x \notin \mathrm{Int}(A_n)$ we are done.
If $x \in \mathrm{Int}(A_n)$, we know that $x \in (-K' + \overline{G \setminus C_n})$ so we get $x \in (-K' + \overline{G \setminus A_n})$, hence $x \in \partial^{K''}(A_n)$.

□

The previous proposition says that if $\{A_n\}_n$ is a van Hove sequence then $\{A_n^{K+}\}_n$ and $\{A_n^{K-}\}_n$ are van Hove sequences.

Proposition A.9 *Let $\{A_n\}_n$ be a van Hove sequence in G and $K \subset G$ any compact set. Then*

i) $A_n^{K-} + K \subset A_n$,

ii) $\lim_{n\to\infty} \frac{\theta(A_n^{K-})}{\theta(A_n)} = \lim_{n\to\infty} \frac{\theta(A_n^{K+})}{\theta(A_n)} = 1$:

Proof:

i) Let $x \in A_n^{K-}, y \in K$ and $z = x + y$. Suppose that $z \notin A_n$ then $z \in G \setminus A_n$ so $x = z - y \in (-K + \overline{G \setminus A_n})$, a contradiction.

ii) We have $A_n^{K-} \subset A_n \subset A_n^{K-} \cup \partial^K(A_n)$, and similarly $A_n \subset A_n^{K+} \subset A_n \cup \partial^{K\cup\{0\}}(A_n)$, with the second inclusion following from the fact that

$$x \in A_n^{K-}, x \notin A_n \text{ implies } x \notin \mathrm{Int}(A_n) \text{ hence } x \in \partial^{K\cup\{0\}}(A_n)\,.$$

Now the proof follows from the definition of van Hove sequences.

□

Corollary A.10 *Let $\{A_n\}_n$ be a van Hove sequence in G. Then $\{A_n\}$ has the property* **M**.