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IRREDUCIBLE CHARACTERS OF $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ AND $GL(3, \mathbb{Z}/p^n\mathbb{Z})$

by

Qianglong Wen ©

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Abstract

Clifford Theory gives us a method to construct irreducible characters of a group G, by inducing up certain irreducible characters of subgroups H of G. In this thesis, we will apply Clifford Theory to construct three types of irreducible characters of groups $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ and $GL(3, \mathbb{Z}/p^n\mathbb{Z})$.

Since the method in the $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ case is similar to the one used in the $GL(3, \mathbb{Z}/p^n\mathbb{Z})$ case, this thesis focusses on constructing these three types of irreducible characters of $G = GL(3, \mathbb{Z}/p^n\mathbb{Z})$. We start with three different irreducible characters ϕ of K, a normal subgroup of G, and calculate the corresponding stabilizers of these characters. Then find some irreducible characters ψ of the stabilizers, which also satisfy the conditions in the Clifford's Theorem. Finally, we induce ψ up to G and ψ^G is an irreducible character of G.

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Contents

1

Introduction

Nowadays, people are more and more interested in the representations of $GL(n, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ are the p-adic integers. Since every continuous irreducible representation of $GL(n, \mathbb{Z}_p)$ comes from a representation of $GL(n, \mathbb{Z}_p/p^m\mathbb{Z}_p)$ and $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$, this thesis focuses on finding some irreducible characters of $GL(n, \mathbb{Z}/p^m\mathbb{Z})$.

Let G be a group and H be its subgroup. Suppose ϕ is an irreducible character of H, we know ϕ^G is a character of G. However, we can not tell directly whether ϕ^G is still irreducible. Clifford Theory gives us a method to determine when the induced character ϕ^G is still irreducible. So we can apply this theory to construct some irreducible characters of G. In this thesis, we will apply Clifford Theory to construct some irreducible characters of groups $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ and $GL(3, \mathbb{Z}/p^n\mathbb{Z})$.

The main idea of the Clifford's Theorem is as follows.

Let $N \lhd G$ be a normal subgroup of G. For any character ϕ of N, we can define

$$\phi^g : N \to C; \phi^g(n) = \phi(gng^{-1}), \forall g \in G, n \in N.$$

Then ϕ^g is also a character of N. Let $\phi \in Irr(N)$, denote $I_G(\phi) = \{g \in G \mid \phi^g = \phi\}$. If $\psi \in Irr(I_G(\phi))$, such that $[\psi_N, \phi] \neq 0$, then the Clifford's Theorem tells us that ψ^G is an irreducible character of G.

To apply Clifford's Theorem to the group $G = GL(n, \mathbb{Z}/p^m\mathbb{Z})$, we first need to choose an appropriate normal subgroup of G. As we will see in the following chapters, we have two cases, depending on whether m is even or odd. When $G = GL(n, \mathbb{Z}/p^{2m}\mathbb{Z})$, we pick $K_m = \{I + p^m A \mid A \in M_{n\times n}(\mathbb{Z}/p^m\mathbb{Z})\}$ as the normal subgroup N in the Clifford's Theorem. While $G = GL(n, \mathbb{Z}/p^{2m+1}\mathbb{Z})$, we will use the normal subgroup $K_{m+1} = \{I + p^{m+1}A \mid A \in M_{n\times n}(\mathbb{Z}/p^m\mathbb{Z})\}$.

Secondly, we also need some irreducible characters of the normal subgroups K_m of $GL(n, \mathbb{Z}/p^{2m}\mathbb{Z})$ and K_{m+1} of $GL(n, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ to start with. Since K_m and K_{m+1} are abelian, all their irreducible characters are of degree one, which are also group homomorphisms that are easy to construct. There are three kinds of irreducible characters in each case and they are defined on K_m and K_{m+1} similarly.

Thirdly, we will calculate the stabilizers $I_G(\phi)$ corresponding to different ϕ and G. With all the three kinds of irreducible characters we start with, the stabilizers can be computed directly and the calculations are also similar in the even and odd cases, no matter whether $n = 2$ or $n = 3$.

Finally, in order to construct irreducible characters of G, we have to find some irreducible characters ψ of the stabilizer $I_G(\phi)$, such that $[\psi_{K_m}, \phi] \neq 0$ or $[\psi_{K_{m+1}}, \phi] \neq 0$. Then by Clifford Theory, we know that $\psi^G \in Irr(G)$. In this step, we will see that the even case is much easier than the odd case. In the even case, we can actually extend ϕ of K_m to its stabilizer $I_G(\phi)$. While in the odd case, we need some subgroup H between K_{m+1} and $I_G(\phi)$ to construct an irreducible character ψ of $I_G(\phi)$ that satisfies $[\psi_{K_{m+1}}, \phi] \neq 0$. We will also notice that when $G = GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ or $G = GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$, with each character ϕ defined on G, we can only construct one kind of ψ that satisfies the condition of Clifford's Theorem. While in the $G = GL(3, \mathbb{Z}/p^n\mathbb{Z})$ case, there are more irreducible characters ψ corresponding to the same ϕ. Hence, there are more irreducible characters ψ^G of G.

Clifford's Theorem only gives us a method to construct irreducible characters of G, but we can not always tell the character values unless we know the character values of ψ of $I_G(\phi)$. In this thesis, we only care about the degrees of the irreducible characters of $GL(n, \mathbb{Z}/p^m\mathbb{Z})$. For $k < m$, it is clear that any irreducible character of $GL(n, \mathbb{Z}/p^k\mathbb{Z})$ can be lifted as an irreducible character of $GL(n, \mathbb{Z}/p^m\mathbb{Z})$. Hence, we only focus on finding irreducible characters of $GL(n, \mathbb{Z}/p^m\mathbb{Z})$ that do not come from $GL(n, \mathbb{Z}/p^k\mathbb{Z})$, for any $k < n$. We also want to know whether the degrees of the irreducible characters found in this thesis are complete. Since the complete degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ are already known, in [3], by comparing them with the ones found in this thesis, we know that we indeed find out all the possible degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$. However, we still do not know whether the ones we construct in the $GL(3, \mathbb{Z}/p^n\mathbb{Z})$ case are complete or not.

Although we only discuss $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ and $GL(3, \mathbb{Z}/p^n\mathbb{Z})$ in this thesis, the methods can be applied into the general $GL(n, \mathbb{Z}/p^m\mathbb{Z}), \forall n$. However, as n increases, it will become more complicated and more irreducible characters will appear.

In Chapter 3, we apply Clifford Theory to $GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ and $GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ separately to construct three types of irreducible characters. Then we will see that the degrees of the characters constructed in this chapter do not depend on whether n is even or odd. There are general formulas for the degrees of irreducible characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$.

Degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ constructed in Chapter 3 are :

$$p^{2m-1}(p+1), p^{2m-2}(p^2-1), p^{2m-1}(p-1).$$

Degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ constructed in Chapter 3 are:

$$p^{2m}(p+1), p^{2m-1}(p^2-1), p^{2m}(p-1).$$

And we will see that the general formulas for the degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ found by Clifford Theory are:

$$p^{n-1}(p+1), p^{n-1}(p-1), p^{n-2}(p^2-1),$$

which are all the possible degrees of irreducible characters that come from $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ directly.

In Chapter 4, we construct some irreducible characters of $GL(3, \mathbb{Z}/p^n\mathbb{Z})$. Again, we consider $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ and $GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ separately and find the corresponding irreducible characters. In this case, we can also apply Theorem 2.2.6 to construct some more irreducible characters. It is interesting that some the degrees we find can not be generalized.

Degrees of the irreducible characters of $GL(3,\mathbb{Z}/p^{2m}\mathbb{Z})$ constructed in Chapter 4 are :

$$p^{4m-2}(p^2+p+1), p^{4m-2}(p^2+p+1)p, p^{4m-2}(p^2+p+1)(p+1), p^{4m-2}(p^2+p+1)(p-1),$$

$$p^{4m-2}(p^2+p+1)p^{m-1}(p+1), p^{4m-2}(p^2+p+1)p^{m-1}(p-1), p^{4m-2}(p^2+p+1)p^{m-2}(p^2-1),$$

$$p^{4m-4}(p^3-1)(p+1), p^{4m-4}(p^3-1)(p+1)(p^m-p^{m-1}), p^{6m-3}(p-1)^2(p+1).$$

Degrees of the irreducible characters of $GL(3,\mathbb{Z}/p^{2m+1}\mathbb{Z})$ constructed in Chapter 4 are:

$$p^{4m}(p^2+p+1), p^{4m}(p^2+p+1)p, p^{4m}(p^2+p+1)(p+1), p^{4m}(p^2+p+1)(p-1),$$

$$p^{4m}(p^2+p+1)(p+1)p^{m-1}, p^{4m}(p^2+p+1)(p-1)p^{m-1}, p^{4m}(p^2+p+1)(p^2-1)p^{m-2},$$

$$p^{4m-2}(p^2-1)(p^2+p+1), p^{4m-2}(p^2-1)(p^2+p+1)(p^m-p^{m-1}), p^{6m}(p-1)^2(p+1).$$

For $GL(3,\mathbb{Z}/p^n\mathbb{Z})$, there are three kinds of regular irreducible characters with the following degrees:

$$p^{2n-2}(p^2+p+1), p^{2n-4}(p^3-1)(p+1), p^{3n-3}(p-1)^2(p+1).$$

However, there are also some irregular ones. We can not find a general formula for all the degrees of the above irreducible characters of $GL(3,\mathbb{Z}/p^n\mathbb{Z})$.

2

Preliminary

This chapter presents an overview of the character theory and Clifford Theory that is used throughout the remainder of this thesis. For results in 2.1 and 2.2, see [1]. This chapter also includes some useful results and lemmas that are used in the following two chapters. In this chapter, $H < G$ means H is a subgroup of G, and $H \lhd G$ means H is a normal subgroup of G.

2.1 Character Theory

2.1.1 Definition Let V be a finite-dimensional vector space over $\mathbb{C}$. A *representation* ρ of a group G is a group homomorphism $\rho : G \longrightarrow GL(V)$. And $dim(V)$ is also called the *dimension* of ρ, denoted by $dim(\rho)$.

We know that if we choose a basis of V, then $GL(V) \cong GL(n, \mathbb{C})$, where $n = dim(V)$. So it is equivalent to say that a group homomorphism $\rho : G \longrightarrow GL(n, \mathbb{C})$ is also a representation. In particular, a group homomorphism $\lambda : G \longrightarrow \mathbb{C}^{\times}$ is a representation.

2.1.2 Definition A subspace W of V is invariant under ρ if for each $w \in W$ and for all $g \in G, \rho(g)(w) \in W$. A representation $\rho : G \longrightarrow GL(V)$ is *irreducible* if there is no proper nonzero invariant subspace W of V under ρ.

We usually use character theory to determine whether a representation is irreducible.

2.1.3 Definition Let $\rho : G \to GL(n, \mathbb{C})$ be a representation of G. Then the character χ of G afforded by ρ is the function given by $\chi(g) = tr(\rho(g))$. χ is called *irreducible* if ρ is irreducible. And the degree of χ is defined by $deg(\chi) = dim(\rho) = \chi(1)$.

From now on, let $Irr(G)$ represent the set of all irreducible characters of the group G.

2.1.4 Proposition Let χ and ψ be characters of G. Define $\chi\psi$ on G by setting $(\chi\psi)(g) = \chi(g)\psi(g)$. Then $\chi\psi$ is also a character of G.

From the definitions above, it is clear that a $1-dimensional$ representation ρ is irreducible. Moreover, suppose χ is the character afforded by ρ, we have $\rho = \chi$. Namely, a $1-dimensional$ character is also a representation. We will use this fact in the next two chapters very often.

2.1.5 Definition Let $N < G$ be a subgroup and suppose that ϕ is a character of N. We say ϕ is extendible to G if $\exists\psi$, a character of G, such that $\psi_N = \phi$. We call ψ an extension of ϕ to G.

2.1.6 Definition Let ϕ and θ be characters of a group G. Then

$$[\phi, \theta] = \frac{1}{\mid G \mid} \sum_{g \in G} \phi(g)\overline{\theta(g)}$$

is the inner product of ϕ and θ.

2.1.7 Corollary Let λ and ψ be characters of G. Then $[\lambda, \psi] = [\psi, \lambda]$ is a nonnegative integer. Also λ is irreducible if and only if $[\lambda, \lambda] = 1$.

2.1.8 Definition Let $H < G$ be a subgroup and let ϕ be a character of H. Then ϕ^G, the induced character on G, is given by

$$\phi^G(g) = \frac{1}{\mid H \mid} \sum_{x \in G} \phi^\circ(xgx^{-1}),$$

where ϕ° is defined by $\phi^\circ(h) = \phi(h)$ if $h \in H$ and $\phi^\circ(y) = 0$ if $y \notin H$.

By the definition above, it is easy to calculate that

$$deg(\phi^G) = deg(\phi)\frac{\mid G \mid}{\mid H \mid}.$$

Also from the definition of induced character, we have the following proposition.

2.1.9 Proposition Let $H < K < G$ and suppose that ϕ is a character of H. Then $(\phi^K)^G = \phi^G$.

2.1.10 Lemma *(Frobenius Reciprocity)* Let $H < G$ and suppose that ϕ is a character on H and that θ is a character on G. Then

$$[\phi, \theta_H] = [\phi^G, \theta].$$

2.2 Clifford Theory

Let $H \lhd G$. If θ is a character of H and $g \in G$, we define $\theta^g : H \to \mathbb{C}$ by $\theta^g(h) = \theta(ghg^{-1})$. We say that θ^g is *conjugate* to θ in G.

2.2.1 Lemma Let $H \lhd G$ and let ϕ, θ be characters of H and $x, y \in G$. Then

(a) ϕ^x is a character;

(b) $(\phi^x)^y = \phi^{xy}$;

(c) $[\phi^x, \theta^y] = [\phi, \theta]$;

(d) $[\chi_H, \phi^x] = [\chi_H, \phi]$ for characters χ of G.

The Lemma follows from direct calculation.

2.2.2 Definition Let $H \lhd G$ and let $\theta \in Irr(H)$. Then

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is the inertia group of θ in G.

We also call $I_G(\theta)$ the stabilizer of θ in G. When $I_G(\theta) = G$, we say θ is stable under

G, or invariant in G.

2.2.3 Theorem ($Clifford$, [1]) Let $H \lhd G$, $\theta \in Irr(H)$, and $T = I_G(\theta)$. Let

$$A = \{\psi \in Irr(T) \mid [\psi_H, \theta] \neq 0\}, B = \{\chi \in Irr(G) \mid [\chi_H, \theta] \neq 0\}.$$

Then

(a) If $\psi \in A$, then ψ^G is irreducible;

(b) The map $\psi \mapsto \psi^G$ is a bijection of A onto B;

(c) If $\psi^G = \chi$, with $\psi \in A$, then ψ is the unique irreducible constituent of χ_T which lies in A;

(d) If $\psi^G = \chi$, with $\psi \in A$, then $[\psi_H, \theta] = [\chi_H, \theta]$.

In general, it is hard to tell whether the character of G induced from an irreducible character of $H < G$ is still irreducible. But this Theorem tells us when the induced character stays irreducible . So we can apply this theorem to construct some irreducible characters of G, from certain irreducible characters of the normal subgroup H. Part (a) of this theorem is used throughout the following two chapters.

2.2.4 Corollary Let $N \lhd G$ and $\theta \in Irr(N)$. Then $\theta^G \in Irr(G)$ if and only if $I_G(\theta) = N$.

The $I_G(\theta) = N \Rightarrow \theta^G \in Irr(G)$ direction follows immediately from (a) of last theorem and we will use this result very often in the next two chapters.

2.2.5 Corollary Let $N \lhd G$ and let $\chi \in Irr(G)$ and $\theta \in Irr(N)$ with $[\chi_N, \theta] \neq 0$. Then the following are equivalent:

(a) $\chi_N = e\theta$, with $e^2 =\mid G : N \mid$;

(b) χ vanishes on $G - N$ and θ is invariant in G;

(c) χ is the unique irreducible constituent of θ^G and θ is invariant in G.

2.2.6 Theorem ($Gallagher$, [1]) Let $N \lhd G$, $\chi \in Irr(G)$ be such that $\chi_N = \theta \in Irr(N)$. Then the characters $\beta\chi$ for $\beta \in Irr(G/N)$ are irreducible, distinct for distinct β, and are all of the irreducible constituents of θ^G.

Note that there is a projection $\pi : G \rightarrow G/N$. Thus, for any group representation ρ

of G/N, $\rho \circ \pi$ is a representation of G. And if ρ is irreducible, $\rho \circ \pi$ is also irreducible. As a result, we can consider the character $\beta \in Irr(G/N)$ as an irreducible character of G.Therefore, $\beta\chi$ above is well defined.

Consider set A in Theorem 2.2.3, we have

$$A = \{\psi \in Irr(T) \mid [\psi_H, \theta] \neq 0\} = \{\psi \in Irr(T) \mid [\psi, \theta^T] \neq 0\}.$$

In order to apply theorem 2.2.3 to construct irreducible characters of G, we need to induce up the characters in A. Theorem 2.2.6 tells us that, if we can actually extend θ to T, then by finding out all the irreducible characters of T/H, we can construct all the irreducible characters in A and, as a result, we will find more irreducible characters of G.

We will apply Theorem 2.2.6 in chapter 4.

2.2.7 Theorem Let $N \lhd G$ with G/N cyclic and let $\theta \in Irr(N)$ be invariant in G. Then θ is extendible to G.

By applying this theorem, we will come up with some crucial results. The following three lemmas are useful in the following two chapters to construct certain extensions of some characters of degree one.

2.2.8 Lemma Let G be a group, $N \lhd G, H < G$ and $G = NH$. Let $\phi \in Irr(N), \psi \in Irr(H)$ be such that $deg(\phi) = deg(\psi) = 1$. Assume $\phi_{N\cap H} = \psi_{N\cap H}$ and $\forall h \in H, \phi^h = \phi$. Then $\exists\theta \in Irr(G)$ such that $deg(\theta) = 1$ and $\theta_N = \phi$.

Proof:

Define

$$\theta : G \to \mathbb{C}^\times; \theta(nh) = \phi(n)\psi(h), \forall n \in N, h \in H.$$

Since ϕ and ψ are of degree one, they are also group homomorphisms. And since $\phi_{N\cap H} = \psi_{N\cap H}$, we know that θ is well-defined. In addition, $\forall n_1, n_2 \in N, h_1, h_2 \in H$, we have

$$\begin{aligned}\theta(n_1h_1n_2h_2) &= \theta(n_1h_1n_2h_1^{-1}h_1h_2)\\ &= \phi(n_1h_1n_2h_1^{-1})\psi(h_1h_2)\\ &= \phi(n_1)\phi(h_1n_2h_1^{-1})\psi(h_1)\psi(h_2)\\ &= \phi(n_1)\phi^{h_1}(n_2)\theta(h_1)\theta(h_2)\\ &= \phi(n_1)\phi(n_2)\theta(h_1)\theta(h_2)\\ &= \theta(n_1h_1)\theta(n_2h_2).\end{aligned}$$

Thus, θ is of degree one. And it is clear that $\theta_N = \phi$. ∎

2.2.9 Lemma Let G be a finite abelian group, let $N \lhd G$ and $\lambda \in Irr(N)$, then λ is extendible to G.

Proof:

Since G is a finite abelian group, it is a direct product of cyclic groups. Thus, we can find the subgroups $N_1, N_2, ..., N_m$ of G, such that $N_1/N, N_2/N_1, ..., N_m/N_{m-1}$ and G/N_m are all cyclic. Thus, by Theorem (1.10), λ can be extended to N_1. Call the extension λ_1. Since G is abelian, we have that any character of any subgroup of G is stable under G. Therefore λ_1 is stable under G, so is stable under N_2. Hence it is extended to N_2. So λ is extended to N_2. Keeping doing this, we know that finally λ will be extended to G. ∎

2.2.10 Lemma Let G be a group, $N \lhd G, S < G, S$ is abelian, and $G = NS$. Let $\phi \in Irr(N)$ be such that $deg(\phi) = 1$. Assume ϕ is stable under G, then ϕ is extendible to G.

Proof:

Let $\psi = \phi_{S\cap N}$, then $\psi \in Irr(S \cap N)$. Since S is abelian, we know $S \cap N \lhd S$. By Lemma 2.2.9, $\exists \theta \in Irr(S)$ such that $\theta_{S\cap N} = \psi = \phi_{S\cap N}$. Apply Lemma 2.2.8, we know that ϕ is extendible to G. ∎

Lemma 2.2.10 will be used a lot.

2.3 Useful results

In this section, we will calculate the orders of groups $GL(2,\mathbb{Z}/p^n\mathbb{Z}), GL(3,\mathbb{Z}/p^n\mathbb{Z})$ and some of their important subgroups.

In $GL(k,\mathbb{Z}/p^n\mathbb{Z})$, define $K = \{I + pA \mid A \in M_{k\times k}(\mathbb{Z}/p^{n-1}\mathbb{Z})\}$. Then $\mid K \mid=\mid M_{k\times k}(\mathbb{Z}/p^{n-1}\mathbb{Z}) \mid= p^{k^2(n-1)}$.

2.3.1 Proposition $\mid GL(k,\mathbb{Z}/p^n\mathbb{Z}) \mid= p^{k^2(n-1)} \prod_{t=1}^{k}(p^k - p^{t-1})$.

Proof:

Recall that there is a group homomorphism

$$\phi : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}; \quad \phi(a) = \overline{a}, \quad \forall a \in \mathbb{Z}/p^n\mathbb{Z}.$$

Thus, we can define

$$\psi : GL(k,\mathbb{Z}/p^n\mathbb{Z}) \to GL(k,\mathbb{Z}/p\mathbb{Z}); \psi(A) = \overline{A},$$

where $A \in GL(k,\mathbb{Z}/p^n\mathbb{Z})$ and $\overline{A}_{ij} = \phi(A_{ij})$. Then it is easy to check that ψ is a surjective group homomorphism. Moreover, $Ker(\psi) = K$. Hence, we have

$$GL(k,\mathbb{Z}/p^n\mathbb{Z})/K \cong GL(k,\mathbb{Z}/p\mathbb{Z})$$

$$\Rightarrow \mid GL(k,\mathbb{Z}/p^n\mathbb{Z}) \mid=\mid GL(k,\mathbb{Z}/p\mathbb{Z}) \mid\mid K \mid.$$

And since it is known that $\mid GL(k,\mathbb{Z}/p\mathbb{Z}) \mid= \prod_{t=1}^{k}(p^k - p^{t-1})$, the proposition follows. ■

2.3.2 Corollary $\mid GL(2,\mathbb{Z}/p^n\mathbb{Z}) \mid= (p^2 - p)(p^2 - 1)p^{4n-4}$, $\mid GL(3,\mathbb{Z}/p^n\mathbb{Z}) \mid= (p^3 - 1)(p^3 - p)(p^3 - p^2)p^{9n-9}$.

Next, we will calculate the orders of two important subgroups of $GL(2,\mathbb{Z}/p^n\mathbb{Z})$ and $GL(3,\mathbb{Z}/p^n\mathbb{Z})$.

Let $\varepsilon \in \mathbb{Z}/p\mathbb{Z}$ be such that $\sqrt{\varepsilon} \notin \mathbb{Z}/p\mathbb{Z}$, *i.e.* there is no $a \in \mathbb{Z}/p\mathbb{Z}$ such that $a^2 = \varepsilon$. Let

$$S' = \left\{ s' = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s' \in GL(2,\mathbb{Z}/p\mathbb{Z}) \right\}.$$

Then $S' < GL(2,\mathbb{Z}/p\mathbb{Z})$. Moreover, we can prove that

$$S' \cong (\mathbb{Z}/p\mathbb{Z}[\sqrt{\varepsilon}])^\times \Rightarrow \mid S' \mid= p^2 - 1.$$

The proof is exactly the same as the one in the $GL(3,\mathbb{Z}/p^n\mathbb{Z})$ case and we will talk about it later. Consider ε as an element of $\mathbb{Z}/p^n\mathbb{Z}$, *e.g.* $3 \in \mathbb{Z}/5\mathbb{Z}$, 3 is also an element of $\mathbb{Z}/25\mathbb{Z}$. Define

$$S = \left\{ s = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in G \right\}.$$

2.3.3 Proposition $\mid S \mid = (p^2 - 1)p^{2n-2}$.

Proof:

Let

$$\psi : GL(2,\mathbb{Z}/p^n\mathbb{Z}) \to GL(2,\mathbb{Z}/p\mathbb{Z})$$

be the projective group homomorphism defined in the proof of Proposition 2.3.1 in the case $k = 2$. Consider the restriction of ψ to S, then it is clear that ψ maps S onto S' and

$$ker\psi = \left\{ t = \begin{pmatrix} 1+px & py\varepsilon \\ py & 1+px \end{pmatrix} \mid t \in G \right\}.$$

Clearly, $\mid ker\psi \mid = p^{2n-2}$. And since $S/ker\psi \cong S'$, we have that

$$\mid S \mid = \mid S' \mid \mid ker\psi \mid = (p^2 - 1)p^{2n-2}.$$

2.3.4 Corollary Suppose $n > m$. Let $G = GL(2,\mathbb{Z}/p^n\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{2\times 2}(\mathbb{Z}/p^{n-m}\mathbb{Z})\}$ and $S = \{\begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in G\}$. Then $\mid K_m S \mid = p^{4n-2m-2}(p^2 - 1)$.

Proof:

It is clear that

$$K_m \cap S = \left\{ \begin{pmatrix} 1+p^m a & p^m b\varepsilon \\ p^m b & 1+p^m a \end{pmatrix} \mid a, b \in \mathbb{Z}/p^{n-m}\mathbb{Z} \right\},$$

so $\mid K_m \cap S \mid = (p^{n-m})^2$. Since we also have $\mid K_m \mid = (p^{n-m})^4$ and $\mid S \mid = (p^2 - 1)p^{2n-2}$, we can conclude that $\mid K_m S \mid = \frac{|K_m||S|}{|K_m \cap S|} = p^{6m-2}(p^2 - 1)$. ■

In particular, when $G = GL(2,\mathbb{Z}/p^{2m}\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{2\times 2}(\mathbb{Z}/p^m\mathbb{Z})$, we have $\mid K_m S \mid = p^{6m-2}(p^2 - 1)$; and if $G = GL(2,\mathbb{Z}/p^{2m+1}\mathbb{Z})$, we have $\mid K_m S \mid = p^{6m+2}(p^2 - 1)$. The subgroups S and $K_m S$ above play an important role in Chapter 3.

In the $GL(3,\mathbb{Z}/p^n\mathbb{Z})$ case, there is a similar subgroup and we will now talk about it.

Let $t^3 - ct^2 - bt - a$ be an irreducible polynomial in $\mathbb{Z}/p\mathbb{Z}[t]$. Then we have a field extension of $\mathbb{Z}/p\mathbb{Z}$ corresponding to the polynomial $t^3 - ct^2 - bt - a$. Call the field extension

$\mathbb{Z}/p\mathbb{Z}[\alpha]$, then $\alpha^3 - c\alpha^2 - b\alpha - a = 0$. We know that $\mathbb{Z}/p\mathbb{Z}[\alpha]$ is a 3 - dimensional linear space over $\mathbb{Z}/p\mathbb{Z}$, the basis is $\{1, \alpha, \alpha^2\}$.

Consider

$$1 \rightarrow \alpha, \alpha \rightarrow \alpha^2, \alpha^3 \rightarrow \alpha^3$$

as a linear transformation from $\mathbb{Z}/p\mathbb{Z}[\alpha]$ to $\mathbb{Z}/p\mathbb{Z}[\alpha]$. Then the corresponding matrix is

$$B = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Thus,

$$S' = \{s' = xI + yB + zB^2 \mid s' \in GL(3, Z/pZ)\} \cong (Z/pZ[\alpha])^\times \Rightarrow \mid S' \mid = p^3 - 1.$$

Consider B above as a matrix in $GL(3, \mathbb{Z}/p^n\mathbb{Z})$, then

$$S = \{s = xI + yB + zB^2 \mid x, y, z \in \mathbb{Z}/p^n\mathbb{Z}, s \in GL(3, \mathbb{Z}/p^n\mathbb{Z}\} < GL(3, \mathbb{Z}/p^n\mathbb{Z}).$$

2.3.5 Proposition Let S be the same as above. Then $\mid S \mid = (p^3 - 1)p^{3n-3}$.

Proof:

By the same argument as in (2.3.3), we know that $\mid S \mid = \mid S' \mid \mid ker\psi \mid$. In this case, $ker\psi = \{s = (1+px)I + pyB + pzB^2 \mid s \in GL(3, \mathbb{Z}/p^n\mathbb{Z})\}$. Clearly, $\mid ker\psi \mid = p^{3(n-1)}$, and the proposition follows. ■

2.3.6 Corollary Suppose $n > m$. Let $G = GL(3, \mathbb{Z}/p^n\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{3\times3}(\mathbb{Z}/p^{n-m}\mathbb{Z})\}$, and S be the same as above. Then $\mid K_m S \mid = (p^3 - 1)p^{9n-6m-3}$.

Proof:

By the same argument as in Corollary (2.3.4), note that in this case,

$$K_m \cap S = \{s = (1+p^m x)I + p^m yB + p^m zB^2 \mid x, y, z \in \mathbb{Z}/p^{n-m}\mathbb{Z}\} \Rightarrow \mid K_m \cap S \mid = (p^{n-m})^3.$$

And the corollary follows. ■

Again from the above corollary, when $G = GL(3, \mathbb{Z}/p^{2m}\mathbb{Z}), \mid K_m S \mid = p^{12m-3}(p^3 - 1)$; and if $G = GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z}), \mid K_m S \mid = p^{12m+6}(p^3 - 1)$. As we will see in Chapter 4, the above two subgroups are the stabilizers of the characters of K_m in $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ and K_{m+1} in $GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ respectively.

3

Characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$

The degrees of all the irreducible characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$, which do not come from $GL(2, \mathbb{Z}/p^k\mathbb{Z})$ for any $k < n$, are already known, in [3], they are:

$$p^{n-1}(p+1), p^{n-1}(p-1), p^{n-2}(p^2-1).$$

In this chapter, we will use Clifford Theory to find some irreducible characters with degrees $p^{n-1}(p+1), p^{n-1}(p-1)$ and $p^{n-2}(p^2-1)$. In certain cases, we can also easily find the character values, while in the other cases, we could only construct some irreducible characters without knowing their values.

In order to apply the Clifford Theory, we will discuss $GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ and $GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ separately. And we will find that the $GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ case is easier than the other. But finally we will find the characters we want and we will see that their degrees do not depend on whether n is even or odd. We will skip some details of calculating the stabilizers of some characters, because the calculations are similar to the ones in next chapter where we will give all the details.

Let p be a prime number, $p \neq 2$ and let m be a positive integer.

3.1 Characters of $GL(2, Z/p^{2m}\mathbb{Z})$

In this section, $G = GL(2, \mathbb{Z}/p^{2m}\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{2\times 2}(\mathbb{Z}/p^m\mathbb{Z})\}$. We will apply Clifford Theory to construct three kinds of irreducible characters of G. We start with a 1 - dimensional character ϕ of the normal subgroup K_m, calculate its stabilizer $I_G(\phi)$, extend ϕ to $\phi' \in Irr(I_G(\phi))$, then induce ϕ' up to G. By Theorem 2.2.3, we know that $\psi = (\phi')^G \in Irr(G)$. We begin with three different 1 - dimensional characters of K_m, so there are three cases.

3.1.1 The character of degree $p^{2m-1}(p+1)$

Since $p \neq 2$, we know that $(\mathbb{Z}/p^m\mathbb{Z})^\times$ is cyclic. Thus we can find $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ such that λ is an injective homomorphism.

Define

$$\phi : K_m \longrightarrow \mathbb{C}^\times; \phi(I + p^m A) = \lambda(1 + p^m A_{11}).$$

Then

$$\begin{aligned}
\phi[(I + p^m A)(I + p^m B)] &= \phi[I + p^m(A + B)] \\
&= \lambda[1 + p^m(A + B)_{11}] \\
&= \lambda(1 + p^m A_{11} + p^m B_{11}) \\
&= \lambda[(1 + p^m A_{11})(1 + p^m B_{11})] \\
&= \lambda(1 + p^m A_{11})\lambda(1 + p^m B_{11}) \\
&= \phi(I + p^m A)\phi(I + p^m B).
\end{aligned}$$

So ϕ is a group homomorphism. Clearly, ϕ is also a character of degree one.

Claim:

$$I_G(\phi) = T = \left\{ t = \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix}, t \in G \right\}, \mid T \mid = p^{6m-2}(p-1)^2.$$

Proof:

If $t = \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \in T$, then

$$(p^m t A t^{-1})_{11} = p^m a A_{11} a^{-1} = p^m A_{11}$$

$$\Rightarrow \phi(I+p^m A) = \phi^t(I+p^m A)$$

$$\Rightarrow \phi = \phi^t$$

$$\Rightarrow t \in T$$

$$\Rightarrow T \subseteq I_G(\phi).$$

On the other hand, suppose $B \in I_G(\phi)$, then

$$\phi^B(I+p^m A) = \phi(I+p^m BAB^{-1}) = \phi(I+p^m A).$$

By the definition of ϕ, we have

$$\lambda(1+p^m(BAB^{-1})_{11}) = \lambda(1+p^m A_{11}).$$

Since λ is injective, we have

$$1+p^m(BAB^{-1})_{11} = 1+p^m A_{11}$$

$$\Rightarrow p^m(BAB^{-1})_{11} = p^m A_{11}.$$

Denote

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B^{-1} = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix},$$

then

$$(BAB^{-1})_{11} = b'_{11}(b_{11}a_{11} + b_{12}a_{21}) + b'_{21}(b_{11}a_{12} + b_{12}a_{22}).$$

Since $p^m(BAB^{-1})_{11} = p^m A_{11}, \forall a_{ij} \in M_{2\times 2}(\mathbb{Z}/p^m\mathbb{Z})$, we have

$$(1) \quad p^m b'_{11} b_{11} = p^m;$$

$$(2) \quad p^m b'_{11} b_{12} = 0;$$

$$(3) \quad p^m b'_{21} b_{11} = 0.$$

By multiplying b_{11} on the both sides of (2), we get $p^m b'_{11} b_{11} b_{12} = 0$. By (1), we have

$$p^m b_{12} = 0$$

$$\Rightarrow b_{12} = p^m b.$$

Similarly from (3), we get that

$$b'_{21} = p^m d'.$$

Note that we also have $B^{-1} \in T$, thus we can replace B with B^{-1}. By the same argument above, we can conclude that

$$b_{21} = p^m d$$
$$\Rightarrow B \in T$$
$$\Rightarrow I_G(\phi) \subseteq T$$
$$\Rightarrow I_G(\phi) = T.$$

And it is clear that $\mid T \mid = (p^m)^2(p^{2m} - p^{2m-1})^2 = p^{6m-2}(p-1)^2$. ■

Define

$$\phi' : I_G(\phi) \to C^\times; \phi'(B) = \lambda(a), B = \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \in I_G(\phi).$$

Since $\forall B, B' \in I_G(\phi)$, we have $(BB')_{11} = B_{11}B'_{11}$. Therefore, ϕ' is a character of degree one, and so it is irreducible. Clearly, $\phi'_{K_m} = \phi$. Thus, $[\phi'_K, \phi] = 1 \neq 0$. By Theorem 2.2.3, we have

$$\psi = (\phi')^G \in Irr(G),$$

and

$$deg(\psi) = \frac{\mid G \mid}{\mid I_G(\phi) \mid} = p^{2m-1}(p+1).$$

In fact, as we will see in the $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ case, this method will work for any $GL(n, \mathbb{Z}/p^{2m}), \forall n$. Let $K_m = \{I + p^m A \mid A \in M_{n\times n}(\mathbb{Z}/p^m\mathbb{Z})\}$, λ be the same as above, define

$$\phi : K_m \to \mathbb{C}^\times; \phi(I + p^m A) = \lambda(1 + p^m A_{11}).$$

Then the stabilizer would be of the similar form and we can extend ϕ to it. Then we can induce the extension ϕ' up to G to construct an irreducible character of G. Moreover, the values of the extensions ϕ' are easily known, hence the character values of $(\phi')^G$ are also known.

Notice also that in this case, $I_G(\phi)/K_m$ is abelian, any irreducible character of $I_G(\phi)/K_m$ is of degree one. Thus, although we can still apply Theorem 2.2.6 to construct more irreducible characters of T that can substitute ϕ', the degree would not change. As a result, the degree of $(\phi')^G$ will stay the same. However, as in $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ case, $I_G(\phi)/K_m$ is not abelian anymore, we can actually find more irreducible characters of T

with different degrees, which also satisfy the condition in Theorem 2.3.3. Therefore, we can construct some more irreducible characters of $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ with different degrees. And it is easy to see that as n increases, more irreducible characters will be found by this method.

3.1.2 The character of degree $p^{2m-2}(p^2-1)$

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \to \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_m \to \mathbb{C}^\times; \phi(I + p^m A) = \lambda(p^m A_{21}).$$

Then

$$\begin{aligned}\phi[(I + p^m A)(I + p^m B)] &= \phi[(I + p^m (A + B)] \\ &= \lambda[p^m (A + B)_{21}] \\ &= \lambda(p^m A_{21})\lambda(p^m B_{21}) \\ &= \phi(I + p^m A)\phi(I + p^m B).\end{aligned}$$

Thus, ϕ is a character of degree one. And by the same calculation as in 4.1.2, we have

$$I_G(\phi) = \left\{ B = \begin{pmatrix} a & b \\ p^m c & a + p^m d \end{pmatrix} \mid a \in (\mathbb{Z}/p^{2m}\mathbb{Z})^\times; b \in \mathbb{Z}/p^{2m}\mathbb{Z}; c, d \in \mathbb{Z}/p^m\mathbb{Z} \right\},$$

and

$$I_G(\phi) = (p^{2m} - p^{2m-1})p^{2m}(p^m)^2 = p^{6m-1}(p-1).$$

Define

$$\phi' : I_G(\phi) \to \mathbb{C}^\times; \phi'(B) = \lambda(p^m c a^{-1}), B = \begin{pmatrix} a & b \\ p^m c & a + p^m d \end{pmatrix} \in G.$$

Let

$$B = \begin{pmatrix} a & b \\ p^m c & a + p^m d \end{pmatrix}, B' = \begin{pmatrix} a' & b' \\ p^m c' & a' + p^m d' \end{pmatrix},$$

then

$$BB' = \begin{pmatrix} aa' + p^m bc' & ab' + a'b + p^m bd' \\ p^m(a'c + ac') & aa' + p^m(b'c + ad' + a'd) \end{pmatrix}.$$

We have,

$$\begin{aligned}\phi'(BB') &= \lambda[p^m(ac'+a'c)(aa'+p^mbc')^{-1}]\\ &= \lambda[p^m(ac'+a'c)(aa')^{-1}]\\ &= \lambda(p^mca^{-1}+p^mc'a'^{-1})\\ &= \lambda(p^mca^{-1})\lambda(p^mc'a'^{-1})\\ &= \phi'(B)\phi'(B').\end{aligned}$$

Hence, ϕ' is a character of degree one. And for $I+p^mA \in K_m$, we have

$$\begin{aligned}\phi'(I+p^mA) &= \lambda[p^mA_{21}(1+p^mA_{11})^{-1}]\\ &= \lambda[p^mA_{21}(1-p^mA_{11})]\\ &= \lambda(p^mA_{21})\\ &= \phi(I+p^mA).\end{aligned}$$

Namely, $\phi'_{K_m} = \phi$. Thus, $[\phi'_K, \phi] = 1 \neq 0$. By Theorem 2.2.3, we have

$$\psi = (\phi')^G \in Irr(G),$$

and

$$deg(\psi) = \frac{\mid G \mid}{\mid I_G(\phi) \mid} = p^{2m-2}(p^2-1).$$

Again, this method can be generalized similarly to any $GL(n, \mathbb{Z}/p^{2m}\mathbb{Z}), \forall n$. And the character values can also be computed easily. In addition, when $n > 2$, $I_G(\phi)/K_m$ is not abelian anymore and as a result, more irreducible characters of $GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$ will come up.

3.1.3 The character of degree $p^{2m-1}(p-1)$

Let $\varepsilon \in \mathbb{Z}/p\mathbb{Z}$ be such that $\sqrt{\varepsilon}$ is not in $\mathbb{Z}/p\mathbb{Z}$. Consider ε as an element of $\mathbb{Z}/p^{2m}\mathbb{Z}$, let λ be the same as in 3.1.2. Let $B = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}/p^{2m}\mathbb{Z})$.
Define

$$\phi : K_m \to \mathbb{C}^\times; \phi(I+p^mA) = \lambda[tr(p^mAB)].$$

Then

$$\begin{aligned}\phi[(I+p^mA)(I+p^mC)] &= \phi[I+p^m(A+C)]\\ &= \lambda[tr(p^m(A+C)B]\\ &= \lambda[tr(p^mAB)]\lambda[tr(p^mCB)]\\ &= \phi(I+p^mA)\phi(I+p^mC).\end{aligned}$$

As a result, ϕ is a character of degree one.

Claim: $I_G(\phi) = K_mS$, where $S = \left\{ s = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in G \right\}$.

Proof: Let $s \in S$ and note that $sB = Bs$, we have

$$\begin{aligned}\phi^s(I+p^mA) &= \phi(I+p^msAs^{-1})\\ &= \lambda[tr(p^msAs^{-1}B]\\ &= \lambda[tr(p^msABs^{-1}]\\ &= \lambda[tr(p^mAB)]\\ &= \phi(I+p^mA).\end{aligned}$$

So s stabilizes ϕ, and we have $S \subseteq I_G(\phi)$. Hence, $K_mS \subseteq I_G(\phi)$.

On the other hand, let $C \in I_G(\phi)$, then $\phi^C = \phi$. Thus, $\forall I + p^mA \in K_m$, we have

$$\phi^C(I+p^mA) = \phi(I+p^mCAC^{-1}) = \phi(I+p^mA)$$

$$\Rightarrow \lambda(tr(p^mCAC^{-1}B)) = \lambda(tr(p^mAB)).$$

Since λ is injective, we have

$$tr(p^mCAC^{-1}B) = tr(p^mAB).$$

Therefore,

$$tr(A(p^m(B - C^{-1}BC)) = 0, \forall A.$$

Since the A above is an arbitrary matrix in $M_{2\times2}(\mathbb{Z}/p^m\mathbb{Z})$, we know $p^mBC = p^mCB$.

Denote

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

then

$$BC = \begin{pmatrix} \varepsilon c_{21} & \varepsilon c_{22} \\ c_{11} & c_{22} \end{pmatrix}, CB = \begin{pmatrix} c_{12} & \varepsilon c_{11} \\ c_{22} & \varepsilon c_{21} \end{pmatrix}.$$

Since $p^m BC = p^m CB$, we have

$$p^m c_{12} = p^m \varepsilon c_{21}, p^m c_{11} = p^m c_{22}.$$

Thus,

$$p^m C = p^m c_{11} I + p^m c_{21} B \Rightarrow C = c_{11} I + c_{21} B + p^m D,$$

for some $D \in M_{2\times 2}(\mathbb{Z}/p^m Z)$. Since $C \in I_G(\phi)$ is invertible, $C - p^m D = c_{11}I + c_{21}B$ is also invertible. Hence, $c_{11}I + c_{21}B \in S$. Let $A = D(c_{11}I + c_{21}B)^{-1}$, then

$$C = (I + p^m A)(c_{11}I + c_{21}B).$$

Therefore, $C \in K_m S$, which implies $I_G(\phi) \subseteq K_m S$. And the claim follows. ■

From 2.3.6, we have $| K_m S | = p^{6m-2}(p^2 - 1)$. Moreover, by Lemma 2.2.10, we can find $\phi' \in Irr(K_m S)$, such that $\phi'_{K_m} = \phi$. Thus, $[\phi'_{K_m}, \phi] = 1 \neq 0$. By Theorem 2.2.3, we have

$$\psi = (\phi')^G \in Irr(G),$$

and

$$deg(\psi) = \frac{| G |}{| I_G(\phi) |} = p^{2m-1}(p - 1).$$

This construction will also work for any $GL(n, \mathbb{Z}/p^{2m}\mathbb{Z}), \forall n$. However, as n increases, we should choose different matrix B. And ϕ is defined similarly, but the corresponding stabilizer would become a little more complicated to calculate. However, in this case, we only know the existence of the extension $\phi' \in Irr(K_m S)$, without knowing its value, so it is hard to tell the value of the induced character $\psi = (\phi')^G \in Irr(G)$.

3.2 Characters of $GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z})$

In this section, $G = GL(2, \mathbb{Z}/p^{2m+1}\mathbb{Z}), K_{m+1} = \{I + p^{m+1}A \mid A \in M_{2\times 2}(\mathbb{Z}/p^m\mathbb{Z})\}$. We start with certain irreducible characters of the normal subgroup K_{m+1}. Like in last section, there are also three cases and the three 1 - dimensional characters of K_{m+1} are defined similarly. Note that in this case $p^{2m+1} = 0$, and by a similar calculation, we can still find the corresponding stabilizer T. However, unlike the case in the last section, we cannot extend ϕ to T directly. But we are still able to find a character $\phi' \in Irr(T)$ such that $[\phi'_{K_{m+1}}, \phi] \neq 0$. Then by Theorem 2.2.3, we know that $\psi = (\phi')^G \in Irr(G)$.

In 3.2.3, we will also use the normal subgroup $K_m = \{I + p^m A \mid A \in M_{2\times 2}(\mathbb{Z}/p^{m+1}\mathbb{Z})\}$. And we have

$$\mid G \mid = p^{8m+1}(p-1)^2(p+1), \mid K_m \mid = p^{4m+4}, \mid K_{m+1} \mid = p^{4m}.$$

3.2.1 The character of degree $p^{2m}(p+1)$

Let $\lambda : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^\times \to \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_{m+1} \to \mathbb{C}^\times; \phi(I + p^{m+1}A) = \lambda(1 + p^{m+1}A_{11}).$$

Then ϕ is a character of degree one. By the same calculation as in 3.1.1, note that $p^{2m+1} = 0$, we have

$$T = I_G(\phi) = \left\{ t = \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \mid a, d \in (\mathbb{Z}/p^{2m+1}\mathbb{Z})^\times; b, c \in \mathbb{Z}/p^{m+1}Z \right\},$$

and

$$\mid T \mid = (p^{2m+1} - p^{2m})^2 (p^{m+1})^2 = p^{6m+2}(p-1)^2.$$

Let

$$N = \left\{ n = \begin{pmatrix} 1 + p^m a & p^{m+1} b \\ p^m c & 1 + p^m d \end{pmatrix} \mid a, c, d \in \mathbb{Z}/p^{m+1}\mathbb{Z}; b \in \mathbb{Z}/p^m\mathbb{Z}; \right\}.$$

Denote

$$t = \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix}, t^{-1} = \begin{pmatrix} a' & p^m b' \\ p^m c' & d' \end{pmatrix}, n = \begin{pmatrix} 1 + p^m x & p^{m+1} y \\ p^m z & 1 + p^m w \end{pmatrix},$$

Note that $p^{2m+1} = 0$ and $p^m ab' + p^m bd' = 0$, we have

$$(tnt^{-1})_{12} = p^{m+1}(p^{m-1}ab'x + ad'y + p^{m-1}bd'w).$$

Thus, $tnt^{-1} \in N$ and we have $N \lhd T$. Moreover, $\forall X, Y \in N, (XY)_{11} = X_{11}Y_{11}$. Thus,

$$\phi' : N \to \mathbb{C}^\times; \phi'(n) = \lambda(n_{11}), \forall n \in N$$

is a character of degree one and clearly, $\phi'_{K_{m+1}} = \phi$.

Let

$$H = \left\{ h = \begin{pmatrix} a & p^{m+1}b \\ p^m c & d \end{pmatrix} \mid b \in \mathbb{Z}/p^m\mathbb{Z}; c \in \mathbb{Z}/p^{m+1}\mathbb{Z}; a, d \in (\mathbb{Z}/p^{2m+1}\mathbb{Z})^\times \right\}.$$

Then H is a subgroup of T. We want to show $I_T(\phi') = H$.

Denote

$$A = \begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^m c & 1+p^m d \end{pmatrix}, h = \begin{pmatrix} x & p^{m+1}y \\ p^m z & w \end{pmatrix}, h^{-1} = \begin{pmatrix} x' & p^{m+1}y' \\ p^m z' & w' \end{pmatrix},$$

then $(hAh^{-1})_{11} = x(1 + p^m a)x' = 1 + p^m a$. Thus, h stabilizes ϕ' and hence we have $H \subseteq I_T(\phi')$.

On the other hand, consider

$$t = \begin{pmatrix} 1 & p^m \\ p^m & 1 \end{pmatrix}, t^{-1} = \begin{pmatrix} 1+p^{2m} & -p^m \\ -p^m & 1+p^{2m} \end{pmatrix}.$$

Then

$$(tAt^{-1})_{11} = 1 + p^m a + p^{2m} c \neq A_{11}.$$

As a result, $t \notin I_T(\phi') \Rightarrow I_T(\phi') \subsetneq T$. Note also that

$$\mid H \mid = (p^{2m+1} - p^{2m})^2 p^{m+1} p^m = p^{6m+1}(p-1)^2.$$

So $\frac{|T|}{|H|} = p$. And since $H \subseteq I_T(\phi') \subsetneq T$, we have $H = I_T(\phi')$.

Define

$$\theta : H \longrightarrow \mathbb{C}^\times; \theta(h) = \lambda(h_{11}), \forall h \in H.$$

Since $\forall h, h' \in H, (hh')_{11} = h_{11}h'_{11}$, θ is a character of degree one and $\theta_N = \phi'$. So $[\theta_N, \phi'] = 1$, hence $\theta^T \in Irr(T)$, and $deg(\theta^T) = \frac{|T|}{|H|} = p$. Note that we also have

$$[\theta^T, \theta^T] = [\theta^T_H, \theta] = 1 \neq 0,$$

thus, $\theta_{K_{m+1}}$ is an irreducible constituent of $(\theta^T_H)_{K_{m+1}}$. Since

$$\theta_{K_{m+1}} = (\theta_N)_{K_{m+1}} = \phi'_{K_{m+1}} = \phi, \quad (\theta^T_H)_{K_{m+1}} = \theta^T_{K_{m+1}},$$

we have $[\theta^T_{K_{m+1}}, \phi] \neq 0$. By Theorem 2.2.3, we know

$$(\theta^T)^G = \theta^G \in Irr(G),$$

and

$$deg(\theta^G) = \frac{\mid G \mid}{\mid H \mid} = p^{2m}(p+1).$$

3.2.2 The character of degree $p^{2m-1}(p^2-1)$

Let $\lambda : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_{m+1} \longrightarrow \mathbb{C}^\times; \phi(I + p^{m+1}) = \lambda(p^{m+1}A_{21}),$$

then ϕ is a character of degree one. By the same calculation as in 4.2.2, we have

$$T = I_G(\phi) = \left\{ B = \begin{pmatrix} a & b \\ p^m c & a + p^m d \end{pmatrix} \mid B \in G \right\}.$$

Let

$$N = \left\{ n = \begin{pmatrix} a & b \\ p^{m+1} c & a + p^m d \end{pmatrix} \mid n \in T \right\}.$$

Then $| N |= p^{6m+2}(p-1)$. Moveover, let

$$t = \begin{pmatrix} a & b \\ p^m c & a + p^m d \end{pmatrix}, t^{-1} = \begin{pmatrix} a' & b' \\ p^m c' & a' + p^m d' \end{pmatrix}, n = \begin{pmatrix} x & y \\ p^{m+1} z & x + p^m w \end{pmatrix},$$

note that $p^{2m+1} = 0$ and $p^m ca' + p^m c'a + p^{2m} dc' = 0$, we have

$$(tnt^{-1})_{21} = p^{m+1}(aa'z + p^{m-1}cc'y + p^{m-1}ac'w).$$

Thus, $tnt^{-1} \in T$ and we know that $N \lhd T$.

Define

$$\phi' : N \longrightarrow \mathbb{C}^\times; \phi'(n) = \lambda(p^{m+1}ca^{-1}), n = \begin{pmatrix} a & b \\ p^{m+1} c & a + p^m d \end{pmatrix} \in N.$$

Denote

$$n = \begin{pmatrix} a & b \\ p^{m+1} c & a + p^m d \end{pmatrix}, n' = \begin{pmatrix} a' & b' \\ p^{m+1} c' & a + p^m d' \end{pmatrix},$$

then

$$nn'_{11} = aa' + p^{m+1}bc', nn'_{21} = p^{m+1}(ac' + a'c).$$

Thus,

$$\begin{aligned} \phi'(nn') &= \lambda[p^{m+1}(ac' + a'c)(aa' + p^{m+1}bc')^{-1}] \\ &= \lambda[p^{m+1}(ac' + a'c)(aa')^{-1}] \\ &= \lambda(p^{m+1}ca^{-1} + p^{m+1}c'a'^{-1}) \\ &= \lambda(p^{m+1}ca^{-1})\lambda(p^{m+1}c'a'^{-1}) \\ &= \phi'(n)\phi'(n'). \end{aligned}$$

Hence, ϕ' is a character of degree one and $\phi'_{K_{m+1}} = \phi$.

Moreover, since $\frac{|T|}{|N|} = p$ and $t = \begin{pmatrix} 1 & 0 \\ p^m & 1 \end{pmatrix} \in T$ does not stabilize ϕ', we have

$$I_T(\phi') = N \Rightarrow \psi = (\phi')^T \in Irr(T).$$

Since $[\psi_N, \phi'] = [\psi, \psi] = 1 \neq 0$ and $\phi'_{K_{m+1}} = \phi$, we have $[\psi_{K_{m+1}}, \phi] \neq 0$. Thus,

$$\psi^G = [(\phi')^T]^G = (\phi')^G \in Irr(G),$$

and

$$deg(\psi^G) = \frac{|G|}{|N|} = p^{2m-1}(p^2 - 1).$$

3.2.3 The character of degree $p^{2m}(p-1)$

Let ε be a non square element in $\mathbb{Z}/p\mathbb{Z}$ and consider it as an element in $\mathbb{Z}/p^{2m+1}\mathbb{Z}$. Let λ be the same as above and let $B = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}/p^{2m+1})\mathbb{Z}$.

Define

$$\phi : K_{m+1} \to \mathbb{C}^{\times}; \phi(I + p^{m+1}A) = \lambda[tr(p^{m+1}AB)].$$

Since in K_{m+1}, $(I + p^{m+1}A)(I + p^{m+1}B) = I + p^{m+1}(A + B)$, we know that ϕ is a character of degree one. And by the same calculation as in **3.1.3**, note that $p^{2m+1} = 0$, we have $T = I_G(\phi) = K_m S$, where

$$S = \left\{ s = \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix} \mid s \in G \right\}.$$

And by Corollary 2.4.4, $| K_m S | = p^{6m+2}(p^2 - 1)$. Moreover, we can find $\psi \in Irr(T)$, such that

$$deg(\psi) = p, \quad [\psi_{K_{m+1}}, \phi] \neq 0 \Rightarrow \psi^G \in Irr(G),$$

and

$$deg(\psi^G) = p\frac{|G|}{|T|} = p^{2m}(p-1).$$

The idea to construct an irreducible character ψ of T such that $[\psi_{K_{m+1}}, \phi] \neq 0$ is exactly the same as the one used in 4.2.3 and the details will be given in next chapter.The main idea is as follows.

Let

$$K_1 = \{I + pA \mid A \in M_{2\times 2}(\mathbb{Z}/p^{2m}\mathbb{Z})\}, N = K_1 \cap S, N_m = K_m N, N_{m+1} = K_{m+1}.$$

Then $N_{m+1} \lhd T$. In particular, $N_{m+1} \lhd N_m$.

Step 1, we extend ϕ of $K_{m+1}$to ϕ' of N_{m+1}. The existence of ϕ' is guaranteed by Lemma 2.2.10. Moreover, we can also show that ϕ' is stable under T.

Step 2, find an appropriate subgroup H between N_{m+1} and N_m such that we can extend ϕ' of N_{m+1} to θ of H. In addition, $I_{N_m}(\theta) = H$. Then apply Corollary 2.2.4, we know that $\psi' = \theta^{N_m} \in Irr(N_m)$.

Step 3, we will show that $\psi'_{N_{m+1}} = p^i \phi'$ for some integer i. In this case, $i = 1$, and we will see in the next chapter that $i = 3$. Moreover, ψ' vanishes on $N_m - N_{m+1}$. Since ϕ' is stable under T, we know ψ' is also stable under T.

Step 4, since T/N_m is cyclic and $\psi' \in Irr(N_m)$ is stable under T, by Theorem 2.2.8, we can extend ψ' to ψ of T. And this ψ is what we want.

All the three constructions can be applied to $GL(n, \mathbb{Z}/p^{2m+1}\mathbb{Z}), \forall n$. But it will become more involved as n increases and more irreducible characters will appear.

From the characters constructed in 3.1 and 3.2, we find that their degrees do not depend on $2m$ or $2m+1$. For any n, there exist characters of $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ with degrees: $p^{n-1}(p+1), p^{n-1}(p-1)$, and $p^{n-2}(p^2-1)$.

4

Characters of $GL(3, \mathbb{Z}/p^n\mathbb{Z})$

In this chapter, we will apply Clifford Theory to construct some irreducible characters of $GL(3, \mathbb{Z}/p^n\mathbb{Z})$. Like in the $GL(2, \mathbb{Z}/p^n\mathbb{Z})$ case, we discuss $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$ and $GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$ separately. Still, the even case is easier. We will use the similar method as in the last chapter and we can find some irreducible characters whose degrees do not depend on whether n is even or not. However, in $4.1.1, 4.1.2, 4.2.1, 4.2.2$, by Theorem 2.2.6, we can construct some more irreducible characters, the degrees of which would depend on the number n.

Let p be a prime number, $p \neq 2$. Let $m \in \mathbb{Z}$, $m \geq 1$.

4.1 Characters of $GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$

Let

$$G = GL(3, \mathbb{Z}/p^{2m}\mathbb{Z}), K_m = \{I + p^m A \mid A \in M_{3\times 3}(\mathbb{Z}/p^m\mathbb{Z})\}.$$

Then

$$\mid G \mid = p^{9(2m-1)}(p^3 - 1)(p^3 - p)(p^3 - p^2), \mid K_m \mid = p^{9m}.$$

In this section, we use the same method as in 3.1. Start with a 1-dimensional character ϕ of the normal subgroup K_m, find its stabilizer T, extend ϕ to character ϕ' of T, then the induced character $(\phi')^G \in Irr(G)$. In the $GL(2, \mathbb{Z}/p^m\mathbb{Z})$ case, there is only one kind of irreducible character. But in this case, since T/K_m is not abelian, we can construct some more irreducible characters by applying Theorem 2.2.6.

4.1.1 The character of degree $p^{4m-2}(p^2+p+1)$

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \to \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_m \to \mathbb{C}^\times; \phi(I + p^m A) = \lambda(1 + p^m A_{11}).$$

Then ϕ is a character of degree one.

Claim:

$$I_G(\phi) = T = \left\{ t = \begin{pmatrix} a & p^m b & p^m c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix}, t \in G \right\}, | T |= p^{14m-4}(p-1)^3(p+1).$$

Proof:

It is easy to check that if $t = \begin{pmatrix} a & p^m b & p^m c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix} \in T$, then

$$\begin{aligned} (p^m t A t^{-1})_{11} &= p^m a A_{11} a^{-1} = p^m A_{11} \\ &\Rightarrow \phi(I + p^m A) = \phi^t(I + p^m A) \\ &\Rightarrow \phi = \phi^t \\ &\Rightarrow t \in T \\ &\Rightarrow T \subseteq I_G(\phi). \end{aligned}$$

On the other hand, suppose $B \in I_G(\phi)$, then

$$\phi^B(I + p^m A) = \phi(I + p^m BAB^{-1}) = \phi(I + p^m A).$$

By the definition of ϕ, we have

$$\lambda(1 + p^m(BAB^{-1})_{11}) = \lambda(1 + p^m A_{11}).$$

Since λ is injective, we have

$$1 + p^m(BAB^{-1})_{11} = 1 + p^m A_{11}$$

$$\Rightarrow p^m(BAB^{-1})_{11} = p^m A_{11}.$$

Denote

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, B^{-1} = \begin{pmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \end{pmatrix},$$

then

$$(BAB^{-1})_{11} = b'_{11}(b_{11}a_{11}+b_{12}a_{21}+b_{13}a_{31})+b'_{21}(b_{11}a_{12}+b_{12}a_{22}+b_{13}a_{32})+b'_{31}(b_{11}a_{13}+b_{12}a_{23}+b_{13}a_{33}).$$

Since $p^m(BAB^{-1})_{11} = p^m A_{11}, \forall a_{ij} \in M_{3\times3}(\mathbb{Z}/p^m\mathbb{Z})$, we have

$$(1) \quad p^m b'_{11} b_{11} = p^m;$$

$$(2) \quad p^m b'_{11} b_{12} = 0;$$

$$(3) \quad p^m b'_{11} b_{13} = 0;$$

$$(4) \quad p^m b'_{21} b_{11} = 0;$$

$$(5) \quad p^m b'_{31} b_{11} = 0.$$

Multiply b_{11} on the both sides of (2), we get $p^m b'_{11} b_{11} b_{12} = 0$. By (1), we have

$$p^m b_{12} = 0$$

$$\Rightarrow b_{12} = p^m b.$$

Similarly from $(3), (4), (5)$, we get that

$$b_{13} = p^m c, b'_{21} = p^m d', b'_{31} = p^m e'.$$

Note that we also have $B^{-1} \in T$, thus we can replace B with B^{-1}, by the same argument above, we can conclude that

$$b_{21} = p^m d, b_{31} = p^m e$$

$$\Rightarrow B \in T$$

$$\Rightarrow I_G(\phi) \subseteq T$$

$$\Rightarrow I_G(\phi) = T.$$

And it is clear that $\mid T \mid = (p^{2m} - p^{2m-1})(p^m)^4 \mid GL(2, \mathbb{Z}/p^m\mathbb{Z}) \mid = p^{14m-4}(p-1)^3(p+1)$.

■

Since $\forall A, B \in T, (AB)_{11} = A_{11}B_{11}$, we have $\phi' : T \to \mathbb{C}^{\times}$, defined by

$$\phi'(t) = \lambda(a), t = \begin{pmatrix} a & p^m b & p^m c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix} \in G,$$

is a group homomorphism, so is a character of degree one. Clearly, $\phi'_{k_m} = \phi$. Thus $[\phi'_{K_m}, \phi] = [\phi, \phi] = 1 \neq 0$. By Theorem 2.2.3, we know that

$$\psi = (\phi')^G \in Irr(G),$$

and

$$deg(\psi) = \frac{|\, G \,|}{|\, T \,|} = p^{4m-2}(p^2 + p + 1).$$

Note that

$$(*) \quad T/K_m \cong \left\{ A = \begin{pmatrix} a & 0 & 0 \\ 0 & x & y \\ 0 & z & w \end{pmatrix}, A \in GL(3, \mathbb{Z}/p^m\mathbb{Z}) \right\} \cong (\mathbb{Z}/p^m\mathbb{Z})^{\times} \times GL(2, \mathbb{Z}/p^m\mathbb{Z}).$$

By Theorem 2.2.6, we know that $\forall \beta \in Irr(T/K_m)$, $\beta\phi'$ is an irreducible constituent of ϕ^T. Thus,

$$[\beta\phi', \phi^T] = [(\beta\phi')_{k_m}, \phi] \neq 0 \Rightarrow (\beta\phi')^G \in Irr(G).$$

And we also have

$$deg[(\beta\phi')^G] = deg(\beta\phi')\frac{|\, G \,|}{|\, T \,|} = deg(\beta)p^{4m-2}(p^2 + p + 1).$$

By $(*)$, we know that any $\gamma \in Irr[GL(2, \mathbb{Z}/p^m\mathbb{Z})]$ can be lifted as an irreducible character of T. And since we know the degrees of the irreducible characters of $GL(2, \mathbb{Z}/p^m\mathbb{Z})$ are

$$p, p+1, p-1, p^{m-1}(p+1), p^{m-1}(p-1), p^{m-2}(p^2-1),$$

we can find some new irreducible characters of G with the following degrees:

$$p^{4m-2}(p^2+p+1)p, p^{4m-2}(p^2+p+1)(p+1), p^{4m-2}(p^2+p+1)(p-1),$$

$$p^{4m-2}(p^2+p+1)p^{m-1}(p+1), p^{4m-2}(p^2+p+1)p^{m-1}(p-1), p^{4m-2}(p^2+p+1)p^{m-2}(p^2-1).$$

4.1.2 The character of degree $p^{4m-4}(p^3-1)(p+1)$

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_m \longrightarrow \mathbb{C}^\times; \phi(I + p^m A) = \lambda(p^m A_{31}).$$

Then ϕ is a character of degree one.

Claim:

$$I_G(\phi) = T = \left\{ t = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix} \mid t \in G \right\}, | T | = p^{14m-2}(p-1)^2.$$

Proof:

It is easy to check that if $t = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix} \in T$, then

$$Col_1 t^{-1} = \begin{pmatrix} a^{-1} \\ p^m b' \\ p^m c' \end{pmatrix}, Row_3(p^m tA) = \begin{pmatrix} p^m a A_{31} & p^m a A_{32} & p^m a A_{33} \end{pmatrix}.$$

Since $p^{2m} = 0$, we have

$$(p^m t A t^{-1})_{31} = p^m a A_{31} a^{-1} = p^m A_{31}.$$

Thus, $\phi(I + p^m A) = \phi^t(I + p^m A), \forall I + p^m A \in K_m$. Hence, t stabilizes ϕ and we have $T \subseteq I_G(\phi)$.

On the other hand, suppose $B \in I_G(\phi)$, then

$$\phi^B(I + p^m A) = \phi(I + p^m BAB^{-1}) = \phi(I + p^m A).$$

By the definition of ϕ, we have

$$\lambda(p^m(BAB^{-1})_{31}) = \lambda(p^m A_{31}).$$

Since λ is injective, we have

$$p^m(BAB^{-1})_{31} = p^m A_{31}.$$

Denote

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, B^{-1} = \begin{pmatrix} b'_{11} & b'_{12} & b'_{13} \\ b'_{21} & b'_{22} & b'_{23} \\ b'_{31} & b'_{32} & b'_{33} \end{pmatrix},$$

then

$$(BAB^{-1})_{31} = b'_{11}(b_{31}a_{11}+b_{32}a_{21}+b_{33}a_{31})+b'_{21}(b_{31}a_{12}+b_{32}a_{22}+b_{33}a_{32})+b'_{31}(b_{31}a_{13}+b_{32}a_{23}+b_{33}a_{33}).$$

Since $p^m(BAB^{-1})_{11} = p^m A_{11}, \forall a_{ij} \in M_{3\times3}(\mathbb{Z}/p^m\mathbb{Z})$, we have

$$(1) \quad p^m b'_{11} b_{33} = p^m;$$

$$(2) \quad p^m b'_{11} b_{31} = 0;$$

$$(3) \quad p^m b'_{11} b_{32} = 0;$$

$$(4) \quad p^m b'_{21} b_{33} = 0.$$

By multiplying b_{33} on the both sides of (2), we get

$$p^m b'_{11} b_{33} b_{31} = 0.$$

By (1), we have

$$p^m b_{31} = 0$$

$$\Rightarrow b_{31} = p^m c.$$

Similarly from $(3), (4)$, we get that

$$b_{32} = p^m d, b'_{21} = p^m b'.$$

Since$B^{-1} \in T$, we can replace B with B^{-1}, by the same argument as above, we can conclude that

$$b_{21} = p^m b.$$

Note that we also have

$$(5) \quad p^m(BAB^{-1})_{11} = p^m b_{11} b'_{11} = p^m.$$

Let $(1) - (5)$, we have

$$(6) \quad p^m b'_{11}(b_{33} - b_{11}) = 0.$$

Multiply b_{33} on the both sides of (6) and by (1), we have

$$b_{33} - b_{11} = p^m e.$$

Thus,

$$b_{33} = b_{11} + p^m e, B = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix} \in T.$$

So $T = I_G(\phi)$. And $\mid T \mid = (p^{2m} - p^{2m-1})^2 (p^m)^3 (p^{2m})^3 = p^{14m-2}(p-1)^2$. ■

Define

$$\phi' : T \to \mathbb{C}^\times ; \phi' \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix} = \lambda(p^m c a^{-1}).$$

Let

$$t = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix}, t' = \begin{pmatrix} a' & x' & y' \\ p^m b' & z' & w' \\ p^m c' & p^m d' & a' + p^m e' \end{pmatrix},$$

then

$$(tt')_{11} = aa' + p^m(xb' + yc'), (tt')_{31} = p^m(ac' + a'c).$$

So $(tt')_{11}(tt')_{31} = p^m ca^{-1} + p^m c' a'^{-1}$. As a result, ϕ' is a group homomorphism, and so is a character of degree one. Moreover, since $p^m c(1 + p^m a)^{-1} = p^m c$, we have $\phi'_{k_m} = \phi$. Thus $[\phi'_{K_m}, \phi] = [\phi, \phi] = 1 \neq 0$. By Theorem 2.2.3, we have that

$$\psi = (\phi')^G \in Irr(G), deg(\psi) = \frac{\mid G \mid}{\mid T \mid} = p^{4m-4}(p^3 - 1)(p + 1).$$

We will turn next to apply Theorem 2.2.6 to construct one more irreducible character of G.

Note that

$$T/K_m \cong H = \left\{ A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & a \end{pmatrix} \mid A \in GL(3, Z/p^m Z) \right\}.$$

Let $N = \left\{ n = \begin{pmatrix} a & x & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix} \mid n \in H \right\}$, then $N \lhd H$.

Define

$$\theta : N \to \mathbb{C}^\times ; \theta \begin{pmatrix} a & x & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix} = \sigma(xa^{-1}),$$

where $\sigma : (\mathbb{Z}/p^m\mathbb{Z})^+ \to \mathbb{C}^\times$ is an injective group homomorphism. Let

$$n = \begin{pmatrix} a & x & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix}, n' = \begin{pmatrix} a' & x' & y' \\ 0 & a' & z' \\ 0 & 0 & a' \end{pmatrix},$$

then

$$(nn')_{11} = aa', (nn')_{12} = ax' + a'x.$$

So $(nn')_{12}(nn')_{11}^{-1} = xa^{-1} + x'a'^{-1}$. Hence we have θ is a character of degree one.

Let

$$h = \begin{pmatrix} a_0 & x_0 & y_0 \\ 0 & b_0 & z_0 \\ 0 & 0 & a_0 \end{pmatrix}, h^{-1} = \begin{pmatrix} a_0^{-1} & x_0' & y_0' \\ 0 & b_0^{-1} & z_0' \\ 0 & 0 & a_0^{-1} \end{pmatrix}, n = \begin{pmatrix} a & x & y \\ 0 & a & z \\ 0 & 0 & a \end{pmatrix},$$

then

$$(hnh^{-1})_{11} = a_0 a a_0^{-1} = a, (hnh^{-1})_{12} = a(a_0 x_0' + b_0^{-1} x_0) + a_0 b_0^{-1} x = a_0 b_0^{-1} x.$$

Since

$$\begin{aligned} h \in I_H(\theta) &\Longleftrightarrow \theta^h = \theta \\ &\Longleftrightarrow \theta(n) = \theta^h(n) \quad \forall n \in N \\ &\Longleftrightarrow \theta(n) = \theta(hnh^{-1}) \\ &\Longleftrightarrow \sigma(xa^{-1}) = \sigma(a_0 b_0^{-1} x a^{-1}) \\ &\Longleftrightarrow xa^{-1} = a_0 b_0^{-1} x a^{-1} \quad \textit{since} \quad \sigma \quad \textit{is} \quad \textit{injective} \\ &\Longleftrightarrow a_0 b_0^{-1} x = x \\ &\Longleftrightarrow a_0 = b_0 \\ &\Longleftrightarrow h \in N. \end{aligned}$$

Thus, $I_H(\sigma) = N$. By Corollary 2.2.4, we have

$$\beta = \sigma^H \in Irr(H),$$

and

$$deg(\beta) = \frac{|H|}{|N|} = p^m - p^{m-1}.$$

Then by Theorem 2.2.6, we have

$$[(\phi'\beta)_{K_m}, \phi] \neq 0$$

$$\Rightarrow \chi = (\phi'\beta)^G \in Irr(G),$$

and

$$deg(\chi) = deg(\beta)\frac{|G|}{|T|} = p^{4m-4}(p^3-1)(p+1)(p^m - p^{m-1}).$$

4.1.3 The character of degree $p^{6m-3}(p-1)^2(p+1)$

Let

$$B = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \in GL(3, \mathbb{Z}/p^{2m}\mathbb{Z})$$

be the same as we discussed in 2.3. Let

$$S = \{s = xI + yB + zB^2 \mid s \in GL(3, \mathbb{Z}/p^{2m}\mathbb{Z}\} \subseteq GL(3, \mathbb{Z}/p^{2m}\mathbb{Z}),$$

then by Corollary 2.3.6, we have $\mid S \mid= (p^3-1)p^{6m-3}$.

Let $\lambda : (\mathbb{Z}/p^{2m}\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_m \rightarrow \mathbb{C}^\times; \phi(I + p^m A) = \lambda(tr(p^m AB)).$$

Then

$$\begin{aligned}\phi[(I + p^m A)(I + p^m C)] &= \phi[I + p^m(A + C)] \\ &= \lambda[tr(p^m(A + C)B] \\ &= \lambda[tr(p^m AB)]\lambda[tr(p^m CB)] \\ &= \phi(I + p^m A)\phi(I + p^m C).\end{aligned}$$

So ϕ is a character of degree one.

Claim: Denote $T = I_G(\phi)$, we have $T = K_m S$ and $\mid T \mid= p^{12m-3}(p^3 - 1)$.

Proof:

Let $s \in S$, since S is abelian and $B \in S$, we have $sB = Bs$.

Thus,

$$\begin{aligned}\phi^s(I + p^m A) &= \phi(I + p^m sAs^{-1}) \\ &= \lambda(tr(p^m sAs^{-1}B)) \\ &= \lambda(tr(p^m sABs^{-1})) \\ &= \lambda(tr(p^m AB)) \\ &= \phi(I + p^m A).\end{aligned}$$

So $\phi^s = \phi$, $s \in T$ and we have $S \subseteq T$. Clearly, $K_m \subseteq T$, therefore $K_m S \subseteq T$.

On the other hand, let $C \in T$, then $\phi^C = \phi$. Thus,

$$\phi^C(I + p^m A) = \phi(I + p^m CAC^{-1}) = \phi(I + p^m A), \forall I + p^m A \in K_m.$$

By the definition of ϕ, we have

$$\lambda(tr(p^m CAC^{-1}B)) = \lambda(tr(p^m AB)).$$

Since λ is injective, this yields

$$\begin{aligned} tr(p^m CAC^{-1}B) = tr(p^m AB) &\Rightarrow tr(Ap^m B) = tr(Ap^m C^{-1}BC) \\ &\Rightarrow tr(A(p^m(B - C^{-1}BC)) = 0. \end{aligned}$$

Since the A above is an arbitrary matrix in $M_{3\times 3}(\mathbb{Z}/p^m\mathbb{Z})$, we conclude that

$$p^m(B - C^{-1}BC) = 0$$

$$\Rightarrow p^m BC = p^m CB.$$

Denote

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

then

$$BC = \begin{pmatrix} ac_{31} & ac_{32} & ac_{33} \\ c_{11} + bc_{31} & c_{12} + bc_{32} & c_{13} + bc_{33} \\ c_{21} + cc_{31} & c_{22} + cc_{32} & c_23 + cc_{33} \end{pmatrix}, CB = \begin{pmatrix} c_{12} & c_{13} & ac_{11} + bc_{12} + cc_{13} \\ c_{22} & c_{23} & ac_{21} + bc_{22} + cc_{23} \\ c_{32} & c_{33} & ac_{31} + bc_{32} + cc_{33} \end{pmatrix}.$$

Since $p^m BC = p^m CB$, we have

$$\begin{aligned} p^m c_{12} &= p^m ac_{31}, \\ p^m c_{22} &= p^m(c_{11} + bc_{31}), \\ p^m c_{32} &= p^m(c_{21} + cc_{31}), \\ p^m c_{13} &= p^m(ac_{21} + acc_{31}), \\ p^m c_{23} &= p^m(ac_{31} + bc_{21} + bcc_{31}), \\ p^m c_{33} &= p^m(c_{11} + bc_{31} + cc_{21} + c^2 c_{31}). \end{aligned}$$

Thus,

$$\begin{aligned} p^m C &= \begin{pmatrix} p^m c_{11} & p^m ac_{31} & p^m(ac_{21} + acc_{31}) \\ p^m c_{21} & p^m(c_{11} + bc_{31}) & p^m(ac_{31} + bc_{21} + bcc_{31}) \\ p^m c_{31} & p^m(c_{21} + cc_{31}) & p^m(c_{11} + bc_{31} + cc_{21} + c^2 c_{31}) \end{pmatrix} \\ &= p^m c_{11} I + p^m c_{21} B + p^m c_{31} B^2. \end{aligned}$$

Hence,

$$C = c_{11}I + c_{21}B + c_{31}B^2 + p^m D, \quad D \in M_{3\times 3}(Z/p^m Z).$$

Since $C \in T$ is invertible, $C - p^m D = c_{11}I + c_{21}B + c_{31}B^2$ is also invertible. Hence, $c_{11}I + c_{21}B + c_{31}B^2 \in S$. Let $A = D(c_{11}I + c_{21}B + c_{31}B^2)^{-1}$, then

$$C = (I + p^m A)(c_{11}I + c_{21}B + c_{31}B^2).$$

So $C \in K_m S$, and we have $T \subseteq K_m S$. Therefore, $T = K_m S$. And by Corollary 2.3.6, we have $\mid T \mid = p^{12m-3}(p^3 - 1)$. ■

By Lemma 2.2.10, we can find $\psi \in Irr(K_m S)$ such that $\psi_{K_m} = \phi$. Therefore, $[\psi_{K_m}, \phi] = 1 \neq 0$. By Theorem 2.2.3, we have

$$\chi = \psi^G \in Irr(G),$$

and

$$deg(\chi) = \frac{\mid G \mid}{\mid T \mid} = p^{6m-3}(p-1)^2(p+1).$$

Since

$$T/K_m = K_m S/S \cong \{xI + yB + zB^2\} < GL(3, \mathbb{Z}/p^m\mathbb{Z})$$

is abelian, any $\beta \in Irr(T/K_m)$ is of degree one. So $deg(\psi\beta) = deg(\psi)$ and we can not find other irreducible characters of G with different degrees.

4.2 Characters of $GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$

Let $G = GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$,

$K_m = \{I + p^m A \mid A \in M_{3\times 3}(\mathbb{Z}/p^{m+1}\mathbb{Z})\}, K_{m+1} = \{I + p^{m+1}A \mid A \in M_{3\times 3}(\mathbb{Z}/p^m\mathbb{Z})\}.$

Then

$$\mid G \mid = p^{18m}(p^3-1)(p^3-p)(p^3-p^2), \mid K_{m+1} \mid = p^{9m}, \mid K_m \mid = p^{9m+9}.$$

In this section, we start with a 1 - dimensional character ϕ of the normal subgroup K_{m+1}, find its stabilizer T, then find $\phi' \in Irr(T)$ such that $[\phi'_{K_{m+1}}, \phi] \neq 0$. Then apply Theorem 2.2.3 to find the corresponding irreducible characters of G. Like in the last section, we can still apply Theorem 2.2.6 to construct some more irreducible characters of G.

4.2.1 The character of degree $p^{4m}(p^2+p+1)$

Let $\lambda : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^\times \to \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_{m+1} \to \mathbb{C}^\times; \phi(I + p^{m+1}A) = \lambda(1 + p^{m+1}A_{11}).$$

By the same argument as in 4.1.1 and note that in this case $p^{2m+1} = 0$, we can conclude that ϕ is a character of degree one. Moreover, by the similar method in 4.1.1 to calculate the stabilizer of ϕ, we have

$$I_G(\phi) = T = \left\{ t = \begin{pmatrix} a & p^m b & p^m c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix} \mid t \in G \right\}, \mid T \mid = p^{14m+5}(p-1)^3(p+1).$$

Let

$$N = \left\{ n = \begin{pmatrix} 1+p^m a & p^{m+1} b & p^{m+1} c \\ p^m d & 1+p^m x & p^m y \\ p^m e & p^m z & 1+p^m w \end{pmatrix} \mid n \in G \right\}.$$

Then $K_{m+1} \lhd N \lhd T$ and we can extend ϕ to N as follows.

Define

$$\phi' : N \to \mathbb{C}^\times; \phi' \begin{pmatrix} 1+p^m a & p^{m+1} b & p^{m+1} c \\ p^m d & 1+p^m x & p^m y \\ p^m e & p^m z & 1+p^m w \end{pmatrix} = \lambda(1 + p^m a).$$

Note that $\forall A, B \in N$, we have $(AB)_{11} = A_{11}B_{11}$. Thus,ϕ' is a character of degree one. Clearly, $\phi'_{K_{m+1}} = \phi$.

Let

$$n = \begin{pmatrix} 1+p^m n_{11} & p^{m+1} n_{12} & p^{m+1} n_{13} \\ p^m n_{21} & 1+p^m n_{22} & p^m n_{23} \\ p^m n_{31} & p^m n_{32} & 1+p^m n_{33} \end{pmatrix} \in N,$$

$$t = \begin{pmatrix} a & p^m b & p^m c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix}, t^{-1} = \begin{pmatrix} a' & p^m b' & p^m c' \\ p^m d' & x' & y' \\ p^m e' & z' & w' \end{pmatrix} \in T.$$

Then

$$(tnt^{-1})_{11} = aa' + p^{2m}bd' + p^{2m}ce' + p^m aa'n_{11} + p^{2m}bn_{21}a' + p^{2m}cn_{31}a'.$$

Note that

$$(tt^{-1})_{11} = aa' + p^{2m}bd' + p^{2m}ce' = 1, p^m aa' = p^m(tt^{-1})_{11} = p^m,$$

we get

$$(tnt^{-1})_{11} = 1 + p^m n_{11} + p^{2m} b n_{21} a' + p^{2m} c n_{31} a'.$$

Since λ is injective, a' is invertible and n_{21} and n_{31} can be any number in $\mathbb{Z}/p^{2m+1}\mathbb{Z}$, we have

$$\begin{aligned}
t \in I_T(\phi') &\iff \phi'(tnt^{-1}) = \phi'(n) \quad \forall n \in N \\
&\iff \lambda[(tnt_{11}^{-1}] = \lambda(1 + p^m n_{11}) \\
&\iff (tnt^{-1})_{11} = 1 + p^m n_{11} \\
&\iff 1 + p^m n_{11} + p^{2m} b n_{21} a' + p^{2m} c n_{31} a' = 1 + p^m n_{11} \\
&\iff p^{2m} b n_{21} + p^{2m} c n_{31} = 0 \quad \forall n_{21}, n_{31} \in \mathbb{Z}/p^{2m+1}\mathbb{Z} \\
&\iff p^{2m} b = 0, p^{2m} c = 0 \\
&\iff t = \begin{pmatrix} a & p^{m+1}b & p^{m+1}c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix}.
\end{aligned}$$

Thus,

$$I_T(\phi') = H = \left\{ h = \begin{pmatrix} a & p^{m+1}b & p^{m+1}c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix}, h \in T \right\}.$$

Define

$$\theta : H \to \mathbb{C}^\times; \theta \begin{pmatrix} a & p^{m+1}b & p^{m+1}c \\ p^m d & x & y \\ p^m e & z & w \end{pmatrix} = \lambda(a).$$

Since $\forall h, h' \in H, (hh')_{11} = h_{11}h'_{11}$, we know θ is a character of degree one. And clearly, $\theta_N = \phi'$. Therefore, $[\theta_N, \phi'] = 1$. By Theorem 2.2.3, we have

$$\theta' = \theta^T \in Irr(T), deg(\theta') = \frac{\mid T \mid}{\mid H \mid} = p^2.$$

Since $[\theta', \theta'] = [\theta'_H, \theta] = 1 \neq 0$ and $\theta_{K_{m+1}} = (\theta_N)_{K_{m+1}} = \phi'_{K_{m+1}} = \phi$, we have $[\theta'_{K_{m+1}}, \phi] \neq 0$. By Theorem 2.2.3 again, we have

$$(\theta')^G = \theta^G = \psi \in Irr(G),$$

and

$$deg(\psi) = \frac{\mid G \mid}{\mid H \mid} = p^{4m}(p^2 + p + 1).$$

We just constructed an irreducible character ψ of G, and will turn next to apply Theorem 2.2.6 to find some more irreducible characters.

Since $N \lhd K_m$ and $I_{K_m}(\phi') = I_T(\phi') \cap K_m = H \cap N = N$, by Corollary 2.2.4, we have

$$\alpha = (\phi')^{K_m} \in Irr(K_m), deg(\alpha) = \frac{| K_m |}{| N |} = p^2.$$

Since $\phi'_{K_{m+1}} = \phi$ and ϕ is stable under K_m, $\forall k \in K_{m+1}$, we have

$$\begin{aligned}
\alpha(k) &= \phi'(k) \\
&= \frac{1}{| N |} \sum_{x \in K_m} (\phi')^{\circ}(xkx^{-1}) \\
&= \frac{1}{| N |} \sum_{x \in K_m} \phi^{x}(k) \\
&= \frac{1}{| N |} \sum_{x \in K_m} \phi(k) \\
&= p^2 \phi(k).
\end{aligned}$$

Therefore, $\alpha_{K_{m+1}} = p^2 \phi$.

Claim 1: α is extendible to T and the extension is θ'. Namely, $\theta'_{K_m} = \alpha$.

Proof:

Since α is irreducible and $deg(\theta) = deg(\theta'_{K_m}) = deg(\alpha) = p^2$, it suffices to show $[\theta'_{K_m}, \alpha] \neq 0$. And note that $[\theta'_{K_m}, \alpha] = [(\theta'_{K_m})_N, \phi'] = [\theta'_N, \phi']$, it suffices to show $\theta'_N, \phi'] \neq 0$. Indeed, since $[\theta'_H, \theta] = 1 \neq 0$, we know that θ is an irreducible constituent of θ'_H. Hence, $\phi' = \theta_N$ is an irreducible constituent of $(\theta'_H)_N = \theta'_N$. Therefore, $[\theta'_N, \phi'] \neq 0$. Hence, $[\theta'_{K_m}, \alpha] \neq 0$. And the claim follows. ■

Claim 2: $I_G(\alpha) = T$.

Proof:

Since $\theta'_{K_m} = \alpha$ and $\theta' = \theta^T$ is a character of T, $\forall t \in T, k \in K_m$, we have

$$\alpha^t(k) = (\theta')^t(k) = \theta'(tkt^{-1}) = \theta'(k) = \alpha(k).$$

Thus, $T \subseteq I_G(\alpha)$.

On the other hand, note that $\alpha_{K_{m+1}} = p^2\phi$, we know $I_G(\alpha) \subseteq I_G(\phi)$. And since $I_G(\phi) = T$, we have $I_G(\alpha) \subseteq T$. The claim follows. ■

By Theorem 2.2.6, $\forall \beta \in Irr(T/K_m), \beta\theta^T$ is an irreducible constituent of α^T. Then by

Theorem 2.2.3, we have

$$(\beta\theta^T)^G \in Irr(G),$$

and

$$deg(\beta\theta^T) = deg(\beta)deg((\theta^T)^G) = deg(\beta)deg(\theta^G).$$

Since

$$T/K_m \cong \left\{ A = \begin{pmatrix} a & 0 & 0 \\ 0 & x & y \\ 0 & z & w \end{pmatrix}, A \in GL(3, Z/p^m Z) \right\} \cong (Z/p^m Z)^\times \times GL(2, Z/p^m Z),$$

any $\beta \in Irr(GL(2,\mathbb{Z}/p^m\mathbb{Z}))$ can be viewed as an element in $Irr(T/K_m)$. Since we already know the degrees of the irreducible characters of $GL(2,\mathbb{Z}/p^m\mathbb{Z})$, we can find new irreducible characters of G with the following degrees:

$$p^{4m}(p^2+p+1)p, p^{4m}(p^2+p+1)(p+1), p^{4m}(p^2+p+1)(p-1),$$

$$p^{4m}(p^2+p+1)(p+1)p^{m-1}, p^{4m}(p^2+p+1)(p-1)p^{m-1}, p^{4m}(p^2+p+1)(p^2-1)p^{m-2}.$$

4.2.2 The character of degree $p^{4m-2}(p^2-1)(p^2+p+1)$

Let $\lambda : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^+ \longrightarrow \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_{m+1} \longrightarrow \mathbb{C}^\times ; \phi(I + p^{m+1}A) = \lambda(p^{m+1}A_{31}).$$

Since $\forall k, k' \in K_{m+1}, (kk')_{31} = k_{31} + k'_{31}$, we know that ϕ is a character of degree one. And by the same calculation as in 4.1.2, note that $p^{2m+1} = 0$, we have

$$T = I_G(\phi) = \left\{ t = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^m c & p^m d & a + p^m e \end{pmatrix} \mid t \in G \right\},$$

and

$$| T |= (p^{m+1})^4 (p^{2m+1} - p^{2m})^2 (p^{2m+1})^3 = p^{14m+7}(p-1)^2.$$

Let

$$N = \left\{ n = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^{m+1} c & p^{m+1} d & a + p^m e \end{pmatrix} \mid n \in T \right\}.$$

Then by the same argument as in 3.2.2, we know that $N \lhd T$. And $\mid N \mid = (p^m)^2(p^{m+1})^2(p^{2m+1} - p^{2m})^2(p^{2m+1})^3 = p^{14m+5}(p-1)^2$.

Define

$$\phi' : N \to \mathbb{C}^\times; \phi' \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^{m+1}c & p^{m+1}d & a + p^m e \end{pmatrix} = \lambda(p^{m+1}ca^{-1}).$$

Then by the same argument as in 4.1.2, we can check that ϕ' is a character of degree one. And since $p^{m+1}c(1 + p^{m+1}a)^{-1} = p^{m+1}c$, we have $\phi'_{K_{m+1}} = \phi$.

Claim 1: $I_T(\phi') = N$.

Proof:

Let

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p^m & 0 & 1 \end{pmatrix}, t^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p^m & 0 & 1 \end{pmatrix}, n = \begin{pmatrix} a & x & y \\ p^m b & z & w \\ p^{m+1}c & p^{m+1}d & a + p^m e \end{pmatrix},$$

then

$$(tnt_{11}^{-1}) = a - p^m y, (tnt^{-1})_{31} = p^{m+1}c - p^{2m}y - p^{2m}e.$$

Since λ is injective and by the definition of ϕ', we know that

$$t \in I_T(\phi') \iff (tnt^{-1})_{31}[(tnt^{-1})_{11}]^{-1} = p^{m+1}ca^{-1}, \forall n \in N.$$

However, when $y = 0, e = 1$, the above equality fails. Namely, t does not stabilize ϕ'.

Similarly, we can show that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p^m & 1 \end{pmatrix}$ does not stabilize ϕ', either.

Since

$$T/N \cong N' = \left\{ n' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p^m x & p^m y & 1 \end{pmatrix} \mid n' \in GL(3, \mathbb{Z}/p^{m+1}\mathbb{Z}) \right\},$$

$I_T(\phi')/N$ should be a subgroup of N'. However, from we just showed, we know that $I_T(\phi')/N$ must be the identity subgroup of N'. As a result, $I_T(\phi') = N$. ■

By Corollary 2.2.4, we have

$$\psi = (\phi')^T \in Irr(T), deg(\psi) = \frac{\mid T \mid}{\mid N \mid} = p^2.$$

Also, since$[\psi_N, \phi'] = [\psi, \psi] = 1 \neq 0$, we can conclude that $[\psi_{K_{m+1}}, \phi'_{K_{m+1}}] = [\psi_{K_{m+1}}, \phi] \neq 0$. By Theorem 2.2.3, we have

$$\psi^G \in Irr(G), deg(\psi^G) = deg(\psi)\frac{\mid G \mid}{\mid T \mid} = p^{4m-2}(p^2-1)(p^2+p+1).$$

We have already constructed an irreducible character of G, we will turn next to apply Theorem 2.2.6 to construct one more irreducible character.

Consider

$$N_0 = N \cap K_m = \left\{ \begin{pmatrix} 1+p^m a & p^m b & p^m c \\ p^m d & 1+p^m x & p^m y \\ p^{m+1} e & p^{m+1} z & 1+p^m w \end{pmatrix} \right\}.$$

Then $\mid N_0 \mid= (p^{m+1})^7 (p^m)^2 = p^{9m+7}$. Thus,

$$\mid NK_m \mid = \frac{\mid N \mid\mid K_m \mid}{\mid N_0 \mid} = \frac{p^{14m+5}(p-1)^2 p^{9m+9}}{p^{9m+7}} = p^{14m+7}(p-1)^2 = \mid T \mid.$$

Since $N \subseteq T, K_m \subseteq T$, we have $NK_m = T$.

Define

$$\theta' : N_0 \longrightarrow \mathbb{C}^\times; \theta' \begin{pmatrix} 1+p^m a & p^m b & p^m c \\ p^m d & 1+p^m x & p^m y \\ p^{m+1} e & p^{m+1} z & 1+p^m w \end{pmatrix} = \lambda(p^{m+1}e).$$

Since $p^{m+1}e = p^{m+1}e(1+p^m a)^{-1}$, we know that $\theta' = \phi'_{N_0}$. Hence, θ' is a character of degree one. It is clear that we also have $\theta'_{K_{m+1}} = \phi$. Moreover,

$$I_{K_m}(\theta') = K_m \cap I_T(\phi') = K_m \cap N = N_0.$$

Thus, by Corollary 2.4.4, we have

$$\theta = (\theta')^{K_m} \in Irr(K_m),$$

and

$$deg(\theta) = \frac{\mid K_m \mid}{\mid N_0 \mid} = p^2 = deg(\psi), \theta_{k_{m+1}} = p^2\phi.$$

Claim 2: θ is extendible to T and the extension is ψ. Namely, $\psi_{K_m} = \theta$.

Proof:

It suffices to show $[\psi_{K_m}, \theta] = [\psi_{N_0}, \theta'] \neq 0$ because we know θ is irreducible and $deg(\psi_{K_m}) = deg(\psi) = deg(\theta) = p^2$. Since $[\psi, \psi] = [\psi_N, \phi'] = 1$, i.e. ϕ' is an irreducible constituent of ψ_N, we have $\phi'_{N_0} = \theta'$ is an irreducible constituent of ψ_{N_0}. Therefore, $[\psi_{N_0}, \theta'] = [\psi_{K_m}, \theta] \neq 0$. And the claim follows. ■

Claim 3: $I_G(\theta) = T$.

Proof:

By claim 2, since θ is extendible to T, we know that $T \subseteq I_G(\theta)$.

On the other hand, since $\theta'_{K_{m+1}} = \phi$ and ϕ is stable under $K_m, \forall k \in K_{m+1}$, we have

$$\begin{aligned}\theta(k) &= \theta^{K_m}(k)\\ &= \frac{1}{\mid N_0 \mid} \sum_{x \in K_m} (\theta')^{\circ}(xkx^{-1})\\ &= \frac{1}{\mid N_0 \mid} \sum_{x \in K_m} \phi^x(k)\\ &= \frac{1}{\mid N_0 \mid} \sum_{x \in K_m} \phi(k)\\ &= p^2\phi(k).\end{aligned}$$

Thus, $\theta_{k_{m+1}} = p^2\phi$. Therefore, $I_G(\theta) \subseteq I_G(\phi)$. Note that $I_G(\phi) = T$, the claim follows. ■

Up to now, we have $\theta \in Irr(K_m), I_G(\theta) = T$ and θ is extendible to T. The extension is ψ. Thus, we can apply Theorem 2.2.6.

Since

$$T/K_m \cong \left\{ \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & a \end{pmatrix} \right\} = H,$$

and we have already known, in last section, that $\exists \beta \in Irr(H)$ such that $deg(\beta) = p^m - p^{m-1}$. Then by Theorem 2.2.3, 2.2.6, we have

$$[(\psi\beta)_{K_m}, \phi] \neq 0 \Rightarrow \chi = (\psi\beta)^G \in Irr(G),$$

and

$$deg(\chi) = deg(\beta)\frac{\mid G \mid}{\mid T \mid} = p^{4m-2}(p^2-1)(p^2+p+1)(p^m - p^{m-1}).$$

4.2.3 The character of degree $p^{6m}(p-1)^2(p+1)$

Let

$$B = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \in GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z})$$

be the same as we discussed in 2.3. Let

$$S = \{s = xI + yB + zB^2 \mid s \in GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z}\} \subseteq GL(3, \mathbb{Z}/p^{2m+1}\mathbb{Z}),$$

then by Corollary 2.3.6, we have $\mid S \mid = p^{6m}(p^3 - 1)$.

Let $\lambda : (\mathbb{Z}/p^{2m+1}\mathbb{Z})^+ \to \mathbb{C}^\times$ be an injective homomorphism.

Define

$$\phi : K_{m+1} \to \mathbb{C}^\times; \phi(I + p^{m+1}A) = \lambda[(tr(p^{m+1}AB)].$$

Since $(I + p^{m+1}A)(I + p^{m+1}B) = I + p^{m+1}(A + B)$, we know that ϕ is a character of degree one. By the same calculation as in 4.1.3, we have

$$T = I_G(\phi) = K_m S, \mid T \mid = p^{12m+6}(p^3 - 1).$$

Let

$$K_1 = \{I + pA, A \in M_{3\times 3}(\mathbb{Z}/p^{2m}\mathbb{Z})\}, N = K_1 \cap S, N_m = K_m N, N_{m+1} = K_{m+1}N.$$

Claim 1: $N_m \lhd T, N_{m+1} \lhd T, \mid N \mid = p^{6m}, \mid N_m \mid = p^{12m+6}, \mid N_{m+1} \mid = p^{12m}.$

Proof:

By the definition of N, we have

$$N = \{(1 + px)I + pyB + pzB^2, x, y, z \in \mathbb{Z}/p^{2m}\mathbb{Z}\}.$$

As a result, $\mid N \mid = (p^{2m})^3 = p^{6m}$.

Since S is abelian and $K_m \lhd T = K_m S$, we know that T/K_m is abelian. Clearly, $K_m \subset N_m \subseteq T$, thus $N_m \lhd T$. Note that $K_m \subset K_1$, we have

$$K_m \cap N = K_m \cap S = \{(1 + p^m x)I + p^m yB + p^m zB, x, y, z \in \mathbb{Z}/p^{m+1}\mathbb{Z}\}.$$

Thus, $\mid K_m \cap N \mid = (p^{m+1})^3 = p^{3m+3}$. And $\mid N_m \mid = \frac{|K_m||N|}{|K_m \cap N|} = p^{12m+6}$.

Since $N \cap K_{m+1} = K_{m+1} \cap S = \{(1 + p^m x)I + p^m yB + p^m zB, x, y, z \in \mathbb{Z}/p^m\mathbb{Z}\}$, we know $\mid N \cap K_{m+1} \mid = p^{6m}$. Thus, $\mid N_{m+1} \mid = \frac{|K_{m+1}||N|}{|K_{m+1} \cap N|} = p^{12m}$.

To prove $N_{m+1} \lhd T$, let $I + p^m A \in K_m, I + pC \in N$, then

$$(I + p^m A)(I + pC)(I - p^m A + p^{2m} A) = I + pC + p^{m+1}(AC - CA)$$
$$= (I + pC)[I + p^m(1 + pC)^{-1}(AC - CA)].$$

Namely, $\forall k \in K_m, kNk^{-1} \subseteq N_{m+1}$. Since $K_{m+1} \lhd K_m$, we know $\forall k \in K_m, kN_{m+1}k^{-1} \subseteq N_{m+1}$. Note that $N \subset S$, S is abelian, and $K_{m+1} \lhd G$, we have $\forall s \in S, sN_{m+1}s^{-1} \subseteq N_{m+1}$. Hence, $N_{m+1} \lhd T$. ■

Since N is abelian, by Lemma 2.2.10 we know that $\phi \in Irr(K_{m+1})$ can be extended to N_{m+1}, i.e. $\exists \phi' \in Irr(N_{m+1})$, such that $\phi'_{K_{m+1}} = \phi$.

Claim 2: $\exists\, \theta \in Irr(N_m)$ such that $\theta_{N_{m+1}} = p^3\phi'$, and θ vanishes on $N_m - N_{m+1}$.

Claim 3: ϕ' is stable under T.

Suppose the two claims above are correct, then the θ in Claim 2 is stable under T. Note that $T = K_m S = N_m S$, we have

$$T/N_m = N_m S/N_m \cong S/(N_m \cap S) = S/(K_1 \cap S)$$

is cyclic, thus by Theorem 2.2.7, θ is extended to T, i.e. we can find $\theta' \in Irr(T)$ such that $\theta'_{N_m} = \theta$. Therefore,

$$[\theta'_{K_{m+1}}, \phi] = [\theta_{K_{m+1}}, \phi] = [p^3\phi'_{K_{m+1}}, \phi] = [p^3\phi, \phi] \neq 0.$$

By Theorem 2.2.3, we know that

$$\chi = (\theta')^G \in Irr(G),$$

and

$$deg(\chi) = p^3\frac{|\,G\,|}{|\,T\,|} = p^{6m}(p-1)^2(p+1).$$

Proof of Claim 2:

Let

$$N_1 = \left\{ \begin{pmatrix} 1+p^m a_{11} & p^{m+1}a_{12} & p^{m+1}a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^{m+1}a_{23} \\ p^{m+1}a_{31} & p^{m+1}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}/p^{2m+1}\mathbb{Z} \right\}.$$

Then

$$K_{m+1} \lhd N_1 \lhd K_m, N_1 \cap N_{m+1} = K_{m+1}, |\, N_1 \,| = p^{9m+3}.$$

Let $H = N_{m+1}N_1$, then $H \lhd N_m$ and $|\, H\,| = \frac{|N_{m+1}||N_1|}{|K_{m+1}|} = p^{12m+3}$.

Define

$$\phi_1 : N_1 \to \mathbb{C}^\times; \phi_1(n) = \lambda(tr(B(n-I))), n \in N_1.$$

Let

$$n = \begin{pmatrix} 1+p^m a_{11} & p^{m+1}a_{12} & p^{m+1}a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^{m+1}a_{23} \\ p^{m+1}a_{31} & p^{m+1}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix}, n' = \begin{pmatrix} 1+p^m a'_{11} & p^{m+1}a'_{12} & p^{m+1}a'_{13} \\ p^m a'_{21} & 1+p^m a'_{22} & p^{m+1}a'_{23} \\ p^{m+1}a'_{31} & p^{m+1}a'_{32} & 1+p^{m+1}a'_{33} \end{pmatrix},$$

then

$$nn' = \begin{pmatrix} * & p^{m+1}(a_{12}+a'_{12}) & * \\ * & * & p^{m+1}(a_{23}+a'_{23}) \\ p^{m+1}(a_{31}+a'_{31}) & p^{m+1}(a_{32}+a'_{32}) & 1+p^{m+1}(a_{33}+a'_{33}) \end{pmatrix}.$$

Thus,

$$tr[B(n-I)] = p^{m+1}a_{12} + p^{m+1}a_{23} + p^{m+1}(aa_{31} + ba_{32} + ca_{33}),$$

and

$$tr[B(nn'-I)] = tr[B(n-I)] + tr[B(n'-I)].$$

Hence, ϕ_1 is a character of degree one and $(\phi_1)_{K_{m+1}} = \phi = \phi'_{K_{m+1}}$.

Moreover, $\forall s \in S \cap K_1$, since S is abelian and $B \in S$, we know $sB = Bs$. Denote $n = I + p^m n_1 \in N_1$, we have

$$\begin{aligned} \phi_1^s(n) &= \lambda[tr(B(sns^{-1} - I))] \\ &= \lambda[tr(Bp^m sn_1 s^{-1}] \\ &= \lambda[tr(sBp^m n_1 s^{-1}] \\ &= \lambda[tr(Bp^m n_1)] \\ &= \phi_1(n). \end{aligned}$$

Notice that $H = N_1 N_{m+1} = N_1 K_{m+1}(S \cap K_1) = N_1(S \cap K_1)$, and we just showed ϕ_1 is stable under $S \cap K_1$. Therefore, ϕ_1 is stable under H. Namely, $H \subseteq I_{N_m}(\phi_1)$.

On the other hand, we can actually show that $H = I_{N_m}(\phi_1)$. Indeed, let

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p^m & 0 & 1 \end{pmatrix}, k^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p^m & 0 & 1 \end{pmatrix}, n = \begin{pmatrix} 1+p^m a_{11} & p^{m+1}a_{12} & p^{m+1}a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^{m+1}a_{23} \\ p^{m+1}a_{31} & p^{m+1}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix},$$

then

$$knk^{-1} = \begin{pmatrix} 1+p^m a_{11} & p^{m+1}a_{12} & p^{m+1}a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^{m+1}a_{23} \\ p^{2m}a_{11} + p^{m+1}a_{31} & p^{m+1}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix}.$$

Thus,

$$\begin{aligned} tr[B(knk^{-1}-1)] &= p^{m+1}a_{12}+p^{m+1}a_{23}+p^{m+1}(aa_{31}+ba_{32}+ca_{33})+p^{2m}aa_{11} \\ &\neq p^{m+1}a_{12}+p^{m+1}a_{23}+p^{m+1}(aa_{31}+ba_{32}+ca_{33}) \\ &= tr[B(n-I)]. \end{aligned}$$

As a result,

$$\phi_1^k(n) = \phi_1(knk^{-1}) \neq \phi_1(n).$$

Namely, k does not stabilize ϕ_1. Similarly, we can show that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p^m & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+p^m \end{pmatrix}$$

do not stabilize ϕ_1, either. Hence, we know that $\frac{|N_m|}{|I_{N_m}(\phi_1)|}$ is at least p^3. Since we already have $H \subseteq I_{N_m}(\phi_1)$ and note that $\frac{|N_m|}{|H|} = p^3$, we can conclude that $H = I_{N_m}(\phi_1)$.

Define

$$\phi_1' : H \to \mathbb{C}^\times; \phi_1'(n_{m+1}n_1) = \phi'(n_{m+1})\phi_1(n_1), n_{m+1} \in N_{m+1}, n_1 \in N_1.$$

Since $H = N_1 N_{m+1}, N_1 \cap N_{m+1} = K_{m+1}$ and $(\phi_1)_{K_{m+1}} = \phi'_{K_{m+1}} = \phi$, by Lemma 2.2.8, we know that ϕ_1' is a well-defined character of degree one. And clearly, $(\phi_1')_{N_1} = \phi_1'$. Therefore, $I_{N_m}(\phi_1') \subseteq I_{N_m}(\phi_1) = H$. Hence, $I_{N_m}(\phi_1') = H$. By Corollary 2.2.4, we have

$$\theta = (\phi_1')^{N_m} \in Irr(N_m).$$

By claim 3, we know that ϕ' is stable under T, in particular, ϕ' is stable under $N_m \subset T$. Thus, $\forall n \in N_{m+1}$, we have

$$\begin{aligned} \theta(n) &= \frac{1}{\mid H \mid} \sum_{x \in N_m} (\phi_1')^\circ(xnx^{-1}) \\ &= \frac{1}{\mid H \mid} \sum_{x \in N_m} (\phi')^x(n) \\ &= \frac{1}{\mid H \mid} \sum_{x \in N_m} \phi'(n) \\ &= p^3\phi'(n). \end{aligned}$$

Therefore, $\theta_{N_{m+1}} = p^3\phi'$. Apply Corollary 2.2.5, claim 2 follows. ■

Proof of Claim 3:

Note that $K_{m+1}S = N_{m+1}S$ and S is abelian, by lemma 2.2.10, ϕ' can be extended to $K_{m+1}S$, and hence ϕ' is stable under $K_{m+1}S$. Thus, it suffices to show that ϕ' is stable under K_m. In the proof of Claim 2, we know that ϕ' is extended to ϕ'_1 of $H = N_{m+1}N_1$. Thus, ϕ' is stable under $H = N_{m+1}N_1$. In particular, it is stable under N_1. In fact, we can replace the N_1 with

$$N_2 = \left\{ \begin{pmatrix} 1+p^{m+1}a_{11} & p^m a_{12} & p^m a_{13} \\ p^{m+1}a_{21} & 1+p^{m+1}a_{22} & p^m a_{23} \\ p^{m+1}a_{31} & p^{m+1}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}/p^{2m+1}\mathbb{Z} \right\},$$

or

$$N_3 = \left\{ \begin{pmatrix} 1+p^{m+1}a_{11} & p^{m+1} a_{12} & p^{m+1} a_{13} \\ p^{m+1}a_{21} & 1+p^{m+1}a_{22} & p^{m+1} a_{23} \\ p^{m}a_{31} & p^{m}a_{32} & 1+p^{m+1}a_{33} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}/p^{2m+1}\mathbb{Z} \right\}.$$

This is because both

$$\phi_2 : N_2 \longrightarrow \mathbb{C}^\times; \phi_2(n_2) = \lambda(tr(B(n_2 - I))), n_2 \in N_2,$$

and

$$\phi_3 : N_3 \longrightarrow \mathbb{C}^\times; \phi_3(n_3) = \lambda(tr(B(n_3 - I))), n_3 \in N_3,$$

are well-defined characters of degree one. And they are both stable under N_{m+1}. Also,

$$N_2 \cap N_{m+1} = N_3 \cap N_{m+1} = K_{m+1}, (\phi_2)_{K_{m+1}} = (\phi_3)_{K_{m+1}} = \phi'_{K_{m+1}} = \phi.$$

Therefore, ϕ' can also be extended to $H_2 = N_{m+1}N_2$ and $H_3 = N_{m+1}N_3$, so it is stable under H_2 and H_3. In particular, ϕ' is stable under $N_1N_2N_3 \lhd K_m$.

Let

$$N_4 = \left\{ \begin{pmatrix} 1+p^m a & 0 & 0 \\ 0 & 1+p^m a & 0 \\ 0 & 0 & 1+p^m a \end{pmatrix} \mid a \in \mathbb{Z}/p^{m+1}\mathbb{Z} \right\},$$

since N_4 is in the center of G, we have $\forall n \in N_4$, $(\phi')^n = \phi'$. Thus, we can conclude that ϕ' is stable under $N_1N_2N_3N_4 \lhd K_m$. But

$$\mid N_1N_2N_3N_4 \mid = \mid K_m \mid \Rightarrow N_1N_2N_3N_4 = K_m.$$

Another way to see $N_1N_2N_3N_4 = K_m$ is that, by the definition of N_1 and N_2, N_1N_2 will generate all the matrices in K_m of the form

$$\begin{pmatrix} 1+p^m a_{11} & p^m a_{12} & p^m a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^m a_{23} \\ p^{m+1} a_{31} & p^{m+1} a_{32} & 1+p^{m+1} a_{33} \end{pmatrix}.$$

By the definition of N_3, all the matrices in K_m of the form

$$\begin{pmatrix} 1+p^m a_{11} & p^m a_{12} & p^m a_{13} \\ p^m a_{21} & 1+p^m a_{22} & p^m a_{23} \\ p^m a_{31} & p^m a_{32} & 1+p^{m+1} a_{33} \end{pmatrix}$$

will be generated in $N_1N_2N_3$. By multiplying N_4, we will have all the elements of K_m. Therefore, $N_1N_2N_3N_4 = K_m$. And Claim 2 follows. ■

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