University of Alberta

Stationary Cosmic Strings Near A Higher Dimensional Black Hole

by

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Chapter 1

Introduction

For some time now, there has been much attention directed to the possibility that spacetime may have extra dimensions. For example, superstring theory is naturally formulated in a spacetime with greater than four dimensions (for example, see [1]). However, the extra dimensions of superstring theory are generally compactified to being of a very small size, possibly of the order of the Planck length. In 1983, Rubakov and Shaposhnikov [2] examined the possibility that the universe could be a four dimensional brane embedded in a higher dimensional spacetime, with all the regular particles of the theory localized to the brane. A major attractive feature of such models is that they present the possibility to solve the hierarchy problem, that being the question of why is the quantum gravity scale of $M_{Pl} \sim 10^{19}$ GeV so much larger than the electroweak scale at $M_{ew} \sim 10^{2}$ GeV. Arkani-Hamed, Dimopoulos, and Dvali [3] have shown how this is possible by considering that the universe actually has $4+d$ dimensions, with the extra d dimensions being compact with a radius R, possibly much larger than the Planck length. In such a universe, the Newtonian gravitational potential at a close distance r ($r \ll R$) from a particle of mass M will be given according to Gauss's law as

$$\mathcal{V}(r \ll R) \sim \frac{M}{M_{Pl(4+d)}^{d+2}} \frac{1}{r^{d+1}}, \tag{1.1}$$

with $M_{Pl(4+d)}$ the $4+d$-dimensional quantum gravity scale, the Planck mass. Far away from the particle ($r \gg R$), the extra d dimensions will no longer represent a drop in the field, as R will already have been reached.

$$\mathcal{V}(r \gg R) \sim \frac{M}{M_{Pl(4+d)}^{d+2} R^d} \frac{1}{r}, \tag{1.2}$$

so we can see that for large r, the regular gravitational scale $M_{Pl(4)}$ is given as

$$M_{Pl(4)}^2 = M_{Pl(4+d)}^{d+2} R^d . \tag{1.3}$$

Thus, if we assume that there might be only one fundamental scale, then setting $M_{Pl(4+d)} = M_{ew}$, we can see how a very large M_{Pl} can result from this mechanism. An interesting result of this model is that it opens the possibility of creating black holes in particle colliders in the near future [4].

In this model, at distances $r \sim R$, we will begin to see deviations from Newton's gravitational law. There have been experiments carried out to examine consistency with Newton's gravitational law down to scales less than a millimeter, putting limits on the maximum size for extra dimensions in such a model [5, 6].

In this thesis, we will be considering black holes in a spacetime with more than (3+1) dimensions. Using compactified extra dimensions for such an analysis would most likely lead to very complicated equations, so to simplify things, we will assume that the extra dimensions are of a size much larger than the typical size of things we will be considering. This assumption allows us to consider that the extra dimensions of the theory are of an infinite size, and spacetime is flat in regions far from the black hole. In particular, most of our attention will be focused on the higher dimensional black holes found by Myers and Perry [7]. Myers-Perry black holes are similar to Kerr black holes in that they represent a stationary, asymptotically flat vacuum solution to the Einstein field equations, but they are generalized to an arbitrary number of dimensions. In a (3+1)-dimensional spacetime, the Myers-Perry metric reduces to the Kerr metric.

Chandrasekhar [8] once pointed out that the Kerr metric has many properties that have endowed it with an aura of the miraculous. Some of these properties include separability of the Hamilton-Jacobi equations of motion for a particle moving in the gravitational field, discovered by Carter [9], and the fact that the Kerr spacetime is Petrov type D [10], the same Petrov type as the non-rotating, Schwarzschild case. Later properties that were added to this were the separability of the d'Alembertian and the massive Klein-Gordon equations, [11, 12], and further the separability of the massless higher spin wave equations [13]. Also discovered was separability of the Dirac equation [14] and later the separability of the equilibrium equations for a stationary cosmic string near a Kerr black hole [15]. The property of separability of such equations in the Kerr metric is due to the existence of a rank two Killing tensor in that spacetime. A Killing tensor of rank i is any tensor which satisfies

$$K_{(\mu_1\mu_2\cdots\mu_i;\lambda)} = 0 \,. \tag{1.4}$$

In the case of $i = 1$, then the vector ξ_μ which satisfies $\xi_{(\mu;\lambda)} = 0$ is called a Killing vector, and it represents directly a symmetry in the metric $g_{\mu\nu}$, that being $\mathcal{L}_\xi g_{\mu\nu} = 0$, with $\mathcal{L}$ being the Lie derivative. For higher rank Killing tensors, there is no simple geometric interpretation as there is in the rank 1 case, but they still represent a symmetry of the spacetime (see [16] for a discussion). This symmetry is manifested by the existence of a conserved quantity K, given as

$$K = K_{\mu_1\mu_2\cdots\mu_i} P^{\mu_1} P^{\mu_2} \cdots P^{\mu_i} \,. \tag{1.5}$$

K is a conserved quantity for any particle with a momentum P^μ in geodesic motion. Note that the identity $g_{\mu\nu;\lambda} = 0$ guarantees that the metric can be considered a Killing tensor, with an associated conserved quantity $-m^2$, the mass squared of the particle. Given two Killing tensors $A_{\mu_1\mu_2\cdots\mu_i}$ and $B_{\mu_1\mu_2\cdots\mu_j}$, then defining $K_{\mu_1\mu_2\cdots\mu_{i+j}}$ as

$$K_{\mu_1\mu_2\cdots\mu_{i+j}} = A_{(\mu_1\mu_2\cdots\mu_i} B_{\mu_{i+1}\mu_{i+2}\cdots\mu_{i+j})} \,, \tag{1.6}$$

then $\mathbf{K}$ will be a Killing tensor, easily shown by the fact that both $\mathbf{A}$ and $\mathbf{B}$ satisfy (1.4). Further, any fixed linear combination of Killing tensors is also a Killing tensor. Any Killing tensor which can be constructed out of other Killing tensors and the metric in a combination of these two ways is said to be reducible, otherwise we call the Killing tensor irreducible.

To understand the importance of a Killing tensor, consider conserved quantities for particle motion in the Kerr geometry. Since the Kerr metric is stationary and axisymmetric, one has conserved quantities P_t and P_ϕ, the energy and z-component of angular momentum of the particle. Another constant of motion is the mass of the particle; however, this only gives us a total of three conserved quantities in a problem with four dimensional motion. Carter's discovery of the Killing tensor and associated conserved quantity in the Kerr metric allowed for the reduction of second order geodesic equations to first order equations for the geodesic motion of a test particle.

Recently, it has been demonstrated that the five-dimensional version of the higher-dimensional black hole metrics found by Myers and Perry shares many of the 'miraculous' properties that the Kerr metric has. Frolov and Stojković demonstrated the separability of the massless scalar field equation in the presence of the gravitational field [17], and the separation of the Hamilton-Jacobi equation for a particle moving in the gravitational field [18]. Also noted by De Smet was the fact that this metric has Petrov type $\underline{\underline{22}}$, the same Petrov type as if it were a non-rotating black hole, the Tangherlini-Schwarzschild case [19, 20]. Just as with the Kerr spacetime case, the separability of such equations is connected with the existence of a rank-two Killing tensor in the five-dimensional Myers-Perry metric [18]. This thesis will be based mostly on the paper [21], examining the 'miracle' of separation of the equilibrium equations for a cosmic string.

In the first part of this paper, we will begin by examining higher-dimensional black hole metrics, particularly the Myers-Perry metrics, with emphasis on the four-dimensional (Kerr) and five-dimensional cases, which are known to have rank-two Killing tensors. Then we will examine properties of topological defects, particularly on cosmic strings. The analysis of general properties of topological defects will be primarily done in flat spacetime with (3+1) dimensions, but it will be made clear how to generalize these results to an arbitrary number of dimensions in a curved background.

The main portion of the thesis will be considering stationary string configurations near a five dimensional Myers-Perry black hole. Stationary string configurations are of interest if we wish to consider final states of a string-black hole system, because it is reasonable to assume that many states will eventually settle down to an equilibrium configuration with the string stationary. For this analysis, we will be considering the cosmic string as a test object in the background black hole spacetime, and ignore possible effects that the string may have on the black hole.

In the last chapter, we will generalize our results to Myers-Perry spacetimes in an arbitrary number of dimensions. It will be demonstrated that some of the important properties of stationary cosmic strings hold in an arbitrary number of dimensions, even though it is possible that the spacetime lacks a Killing tensor. Also we will consider the 'friction' effect, demonstrated by [22]. The analysis done there showed a general result of the slowing of a rotating black hole through an interaction with any topological defects in the slow rotation limit. We demonstrate the explicit equations for this effect due to the interaction of a cosmic string with an arbitrarily rotating black hole in any number of dimensions. Finally, we consider the higher dimensional black hole spacetimes with a cosmological constant term found by Gibbons, Lü, Page and Pope [23]. We demonstrate in a special case that the Hamilton-Jacobi equation for a particle in free-fall and the massive Klein-Gordon equation allow separation of variables (see appendix D and [24]). We also show that the Hamilton-Jacobi equation for a stationary cosmic string in these metrics undergoes a similar separation of variables in this special case.

Throughout this paper, we will be working in a set of units where we have set the speed of light in vacuum $c = 1$, but we will explicitly leave in the n-dimensional gravitational coupling constant, denoted as $G^{(n)}$. We will use the sign convention for curvature defined in [25]. For sign conventions on the metric, we will be working in a metric with a signature $(-,+,+,+)$, or its higher dimensional generalization. We will denote indices for vectors and tensors in the bulk manifold with Greek letters $(\mu,\nu,... = 0,...,N)$ and indices for vectors and tensors in the spatial projection submanifold with lowercase Latin letters from the earlier part of the alphabet $(a,b,... = 1,...,N)$. We will denote indices on the induced geometry of an M-dimensional brane embedded in $N+1$ dimensional spacetime by capital Latin letters from the early part of the alphabet $(A,B,... = 0,...,M)$, and indices in the space normal to the brane as uppercase Latin letters from the later part of the alphabet $(R,S,... = M+1,...,N)$. Finally, we will also make use of indices $(i,j,k,...)$ for miscellaneous purposes, note that we will use the convention to sum all repeated indices except for this final type which will always have a summation symbol when a sum is desired. Explicit tensor calculations for this paper have been performed using the GRTensor software for Maple [26].

Chapter 2

Higher Dimensional Black Holes

2.1 Static Black Holes

In this thesis, we will be considering string configurations in the spacetime of a higher dimensional black hole, so first we will examine some properties of higher dimensional black holes, starting with the Tangherlini-Schwarzschild solution [27]. In general we may write the line element for any static, spherically symmetric configuration in an n-dimensional spacetime as

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2\,d\Omega_{n-2}^2\,, \tag{2.1.1}$$

with t being a timelike coordinate, r being a radial coordinate, f and g being arbitrary functions of r alone, and $d\Omega_n^2$ being the line element of a n-dimensional unit sphere. If we assume a vacuum, then the Einstein field equations give for this metric

$$f(r) = \frac{1}{g(r)} = \left(1 - \frac{\alpha}{r^{n-3}}\right)\,, \tag{2.1.2}$$

where α is a parameter with dimensionality [length]$^{n-3}$. The parameter α is connected to the black hole mass M, by

$$\alpha = \frac{16\pi G^{(n)}}{(n-2)A_{n-2}}M = \frac{8G^{(n)}\pi^{\frac{3-n}{2}}\,\Gamma(\frac{n-1}{2})}{n-2}M, \tag{2.1.3}$$

where A_n is the area of an n-dimensional unit sphere and Γ is the normal Gamma function. We can see that the horizon is located at $r^{n-3} = \alpha$, so $\alpha > 0$ is required to prevent naked singularities.

In three dimensional spacetime, (2.1.1) is a locally flat metric, consistent with the fact that in three dimensions both the Riemann tensor and the Ricci tensor have 6 algebraically independent components, so it turns out to be possible to express the Riemann tensor to be linear in the Ricci tensor [28]. Since vacuum implies the vanishing of the Ricci tensor, it must imply the vanishing of the Riemann tensor, and thus locally flat spacetime.

2.2 Stationary Myers-Perry Black Holes

Next we will examine black hole solutions in an n-dimensional spacetime which is stationary rather than static. This is the consideration of rotating black holes with a timelike symmetry. The group which represents rotations in $N(= n-1)$ spatial dimensions is $SO(N)$. For even N, $SO(N)$ has $N/2$ elements of the Cartan subalgebra, while for odd N, the Cartan subalgebra for $SO(N)$ has $(N-1)/2$ elements. So physically this means the rotation in an even number of spatial dimensions can be reduced into rotation within $N/2$ separate planes of rotation, each with a spin parameter a_i. If instead N is odd then you have $(N-1)/2$ spin parameters with an extra spatial dimension for which there is no spin parameter. We will define the spin parameter in a plane connected to the angular momentum in that plane, J_i, by

$$J_i = -\frac{2}{N-1} M a_i \, , \tag{2.2.1}$$

for both even and odd N. We define (2.2.1) to carry an additional negative sign because the sign convention adopted by Myers and Perry is opposite to that used in the Kerr metric. It will be convenient to denote p as the number of planes of rotation so we may say that i runs from 1 to p in all sums and products in this section, except where written explicitly.

For most of our analysis, we will be examining Myers-Perry metrics using Boyer-Lindquist coordinates. The coordinates are given as t being a time coordinate, r as a radial coordinate, ϕ_i is an angular planar coordinate associated with the plane that has spin parameter a_i, and μ_i is a direction cosine for the plane i. The μ_i are confined to the unit sphere with the restriction $\sum_i \mu_i^2 = 1$. Further, for an odd number of spatial dimensions, there is an additional direction cosine, μ for the extra spatial dimension, and the restriction is replaced by $\mu^2 + \sum_i \mu_i^2 = 1$. The value of ϕ_i ranges from 0 to 2π, and the values of $\phi_i = 0$ and $\phi_i = 2\pi$ are identified. For μ_i the range is from 0 to 1, and μ ranges from -1 to 1.

The solution for the metrics is found separately for even or odd number of spatial dimensions [7]. For even N

$$ds^2 = -\,dt^2 + \sum_i (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)$$

$$+\frac{\Pi L}{\Pi - \alpha r^2} dr^2 + \frac{\alpha r^2}{\Pi L}\left(dt + \sum_i a_i \mu_i^2 d\phi_i\right)^2 , \tag{2.2.2}$$

and for an odd number of spatial dimensions

$$ds^2 = -\,dt^2 + r^2 d\mu^2 + \sum_i (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)$$

$$+\frac{\Pi L}{\Pi - \alpha r} dr^2 + \frac{\alpha r}{\Pi L}\left(dt + \sum_i a_i \mu_i^2 d\phi_i\right)^2 . \tag{2.2.3}$$

In these metrics, Π and L are given by

$$\Pi = \prod_i (r^2 + a_i^2)\,, \qquad\qquad L = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} . \tag{2.2.4}$$

The restrictions on the μ_i and μ coordinates tell us that these coordinates are not independent, so it will sometimes be convenient to denote by θ_k independent coordinates. In even N we may define the coordinates θ_k $(k = 1, ..., p-1)$ by

$$\mu_i = \cos\theta_{p-i+1} \prod_{k=1}^{p-i} \sin\theta_k\,, \tag{2.2.5}$$

with the convention that $\theta_p = 0$ and if there is ever a product that has a upper limit less than the lower limit, then that product equals one. One can define a similar μ-θ relationship for odd N, but we will not need its specific form anywhere.

Sometimes we shall also make use of the notation $\omega_m = (\theta_j, \tilde{\phi}_i)$ $(m = 2, \ldots, N)$ for a total set of 'angular' coordinates.

These metrics are stationary, meaning that $\xi^\mu \partial_\mu = \partial_t$ is a Killing vector for the metric, called the *principal Killing vector*. The infinite red-shift surface of the black hole, which is the external boundary of the ergosphere, is given by $\xi^2 = 0$, which is equivalent to

$$\Pi L = \alpha r \text{ for } N \text{ odd}, \qquad \Pi L = \alpha r^2 \text{ for even } N. \tag{2.2.6}$$

The Myers-Perry black holes are axisymmetric, so that $\xi_i^\mu \partial_\mu = \partial_{\phi_i}$ are also Killing vectors associated with the planes of rotation. The surface gravity for these black holes is given by $\kappa = -\partial_r(\xi^2)/2$, which takes the form

$$\kappa = \left.\frac{\partial_r \Pi - 2\alpha r}{2\alpha r^2}\right|_{r=r_+} \text{ for } N \text{ even},$$

$$\kappa = \left.\frac{\partial_r \Pi - \alpha}{2\alpha r}\right|_{r=r_+} \text{ for } N \text{ odd}, \tag{2.2.7}$$

with r_+ being the location of the outer event horizon. The event horizon area $\mathcal{A}$ is given as

$$\mathcal{A} = A_{N-1} r_+ \alpha, \tag{2.2.8}$$

in both even and odd dimensional cases. In the next section we will discuss conditions for the existence of an event horizon, and how to locate them.

2.2.1 Singularities and Event Horizons in an Odd Number of Spatial Dimensions

Similar to the Kerr metric, the Myers-Perry metrics have curvature singularities. In an odd number of spatial dimensions it is possible to demonstrate that these singularities will occur whenever the factor $\alpha r/\Pi L$ diverges [7]. Suppose at least one of the a_i vanish, then Π contains a factor of r^2 causing a singularity at $r \to 0$. However, if all the spin parameters are non-vanishing then we must first note that we may use the restriction on μ and μ_i to express L as

$$L = \sum_i \frac{r^2 \mu_i^2}{r^2 + a_i^2} + \mu^2 \,. \tag{2.2.9}$$

So we may see that if $\mu = 0$, L will contain a factor r^2 to give a singularity as $r \to 0$. The singularity at $r = 0$, $\mu = 0$ is essentially similar to the ring singularity of the Kerr metric. In either case, we must expect an event horizon at $r = r_+ > 0$ to avoid a naked singularity. The horizons are located where

$$\Pi - \alpha r = 0, \tag{2.2.10}$$

and since Π is positive, α must be positive to prevent naked singularities.

If at least one spin parameter vanishes, then at $r = 0$ we have $\Pi - \alpha r = 0$ and $\frac{\partial(\Pi - \alpha r)}{\partial r} = -\alpha$. Since at large r we will have Π grow at least quadratically, then there must be one $r > 0$ that solves (2.2.10), and it is a unique solution because $\frac{\partial(\Pi - \alpha r)}{\partial r}$ is increasing. This is interesting because it means you can have arbitrarily large angular momentum with a given amount of mass simply by having at least one spin parameter vanish.

If no spin parameters vanish then we need more conditions. First, note that $\Pi - \alpha r|_{r=0} > 0$, and both Π and $\frac{\partial \Pi}{\partial r}$ are positive, increasing functions for $r > 0$. To find how many solutions (2.2.10) can have, we must first consider how many extrema $\Pi - \alpha r$ has. To find the extrema we consider $\frac{\partial(\Pi - \alpha r)}{\partial r} = 0$, which is equivalent to

$$\frac{\partial \Pi}{\partial r} = 2r \sum_i \prod_{j \neq i} (r^2 + a_j^2) = \alpha. \tag{2.2.11}$$

$\frac{\partial \Pi}{\partial r}$ is a monotonically increasing function with $\frac{\partial \Pi}{\partial r} = 0$ at $r = 0$, so this equation has a unique solution at $r = r_* > 0$. So $\Pi - \alpha r$ has just 1 extremum, and this extremum must be a minimum because for large r, Π will grow faster than αr. The existence of the horizon depends on $\Pi - \alpha r$ at $r = r_*$. We may summarize this situation as

$$\begin{array}{rl} \text{any } a_i = 0 & \text{one horizon} \\ \Pi - \alpha r\big|_{r=r_*} > 0 & \text{no horizons} \end{array}$$

$$\begin{aligned} \Pi - \alpha r\big|_{r=r_*} &= 0 \quad \text{one degenerate horizon} \\ \Pi - \alpha r\big|_{r=r_*} &< 0 \quad \text{two horizons.} \end{aligned} \tag{2.2.12}$$

In the four dimensional case we have $\alpha = 2M$ and $r_* = M$ so $\Pi - \alpha r|_{r=r_*} = a^2 - M^2$ thus we get what we expect. Namely that $a^2 < M^2$ gives two horizons, $a^2 = M^2$ gives a degenerate horizon, and $a^2 > M^2$ gives no horizons.

2.2.2 Singularities and Event Horizons in an Even Number of Spatial Dimensions

In an even number of spatial dimensions, the singularity and horizon conditions are somewhat more complicated. Singularities occur whenever $\alpha r^2/\Pi L$ diverges [7], so if at least two spin parameters vanish, then Π contains a factor r^4 to give a singularity as $r \to 0$. If instead there is exactly one vanishing spin parameter then Π contains a factor r^2, and expressing L as

$$L = \sum_i \frac{r^2 \mu_i^2}{r^2 + a_i^2}\,, \tag{2.2.13}$$

we can see that as long as the μ_i corresponding to the vanishing a_i is zero then there is a singularity as $r \to 0$. So it is a ring singularity similar to the odd N case. Finally, if all a_i are non-zero, then the spacetime is regular for all real values of r, with a coordinate singularity at $r = 0$. We may remove this singularity with the coordinate transformation $x = r^2$, and then it is possible to extend the metric to negative values of x. Let us consider the value of the smallest spin parameter, call it a_0. If there are two or more spin parameters which share the value a_0, then Π contains factors of $(x + a_0^2)$ and so as $x \to -a_0^2$ there will be a singularity. If instead the smallest spin parameter is unique $a_i = a_0$ then Π has only one factor $(x + a_0^2)$ and this will be cancelled by the similar factor in L unless the corresponding μ_i vanishes. Thus, we have a ring singularity in the subspace $\mu_i = 0$ as $x \to -a_0^2$.

We will consider event horizons located at positive values of x to prevent the appearance of a naked singularity. Later we will discuss possibilities for event horizons at negative x values. In terms of x, the equation for the location of the event horizon reads

$$\Pi(x) - \alpha x = 0, \tag{2.2.14}$$

note that the left side of (2.2.14) is nonnegative at $x = 0$, and is only zero if at least one spin parameter vanishes. If $\alpha \leq \sum_i \prod_{j \neq i} a_j^2$, then the left side of (2.2.14) has a nonnegative derivative at $x = 0$, and since both $\Pi(x)$ and $\frac{\partial \Pi}{\partial x}$ are positive, increasing functions for $x > 0$, and the left side of (2.2.14) is nonnegative at $x = 0$, we will not have a solution to (2.2.14).

If we assume $\alpha > \sum_i \prod_{j \neq i} a_j^2$, then we have a negative derivative to the left side of (2.2.14) at $x = 0$. Setting the derivative of the left side of (2.2.14) equal to zero to find extrema, we get

$$\frac{\partial \Pi}{\partial x} = \alpha, \tag{2.2.15}$$

which is equivalent to

$$\sum_i \prod_{j \neq i} (x + a_j^2) = \alpha. \tag{2.2.16}$$

For positive x, $\frac{\partial(\Pi(x) - \alpha x)}{\partial x}$ must be increasing, so we will get a unique minimum which will occur in the positive x domain if $\alpha > \sum_i \prod_{j \neq i} a_j^2$, and this minimum will be given by (2.2.16).

Assume we have a minimum which occurs at $x = x_* > 0$. Since for positive x, $\Pi(x)$ will grow at least quadratically, then if the minimum of the function

$$\Pi(x) - \alpha x|_{x = x_*}, \tag{2.2.17}$$

is a negative number, the dominance of $\Pi(x)$ at large x guarantees a solution to (2.2.14), and the fact that $\Pi(x) - \alpha x|_{x=0}$ is positive if no spin parameters vanish, we will have another solution. If (2.2.17) is a positive number, then there must be no solution to (2.2.14) at positive x, since the minimum value of the function is positive.

So to summarize this,

$\alpha \leq \sum_i \prod_{j \neq i} a_j^2$	no horizons
otherwise	
$\Pi(x) - \alpha x\|_{x=x_*} > 0$	no horizons

$$\begin{aligned} \Pi(x) - \alpha x|_{x=x_*} &= 0 && \text{one degenerate horizon} \\ \Pi(x) - \alpha x|_{x=x_*} &< 0 \text{ and at least one } a_i = 0 && \text{one horizon} \\ \Pi(x) - \alpha x|_{x=x_*} &< 0 \text{ and no } a_i = 0 && \text{two horizons.} \end{aligned} \tag{2.2.18}$$

There can only be one, two, or zero horizons. However, in the case with an even number of spatial dimensions the vanishing of one spin parameter does not guarantee the existence of a horizon since it is possible that $\alpha > \sum_i \prod_{j\neq i} a_j^2$ is not satisfied. If two or more spin parameters vanish though, then there will be a horizon, because $\alpha > \sum_i \prod_{j\neq i} a_j^2 = 0$ automatically, so $\Pi(x) - \alpha x$ has a negative first derivative at the origin, and since $\Pi(x) - \alpha x|_{x=0} = 0$ we will have one horizon at positive x.

In the case of all non-vanishing spin parameters, it is possible to consider avoiding a naked singularity by having an event horizon at a value of x between zero and $-a_0^2$, however if the black hole has a positive mass this will not be the case, and the horizons we have found are the only ones that will hide the singularity. We may demonstrate this by simply noting that for values of x between zero and $-a_0^2$, both $\Pi(x)$ and $-\alpha x$ are positive for a black hole of positive mass. Thus there will be no solutions of (2.2.14) for values of x less than zero that will hide the singularity at $-a_0^2$. It is possible to have black holes of negative mass that have an event horizon that hides the singularity as well, however these black holes will contain causality violating regions outside of the event horizon (see [7] for further discussion).

Considering this analysis in the particular case of five dimensions, we have

$$x_* = \frac{\alpha - a_1^2 - a_2^2}{2},$$

$$\Pi(x) - \alpha x|_{x=x_*} = -\frac{1}{4}(\alpha - a_1^2 - a_2^2)^2 + a_1^2 a_2^2 \,, \tag{2.2.19}$$

so we see that we must have $(\alpha - a_1^2 - a_2^2)^2 \geq 4\, a_1^2\, a_2^2$ to have an event horizon; i.e.

$$\alpha \geq (|a_1| + |a_2|)^2 \,. \tag{2.2.20}$$

We will have a naked singularity if the inequality is not satisfied, one degenerate horizon if both sides are equal, or one horizon if a_1 or a_2 vanishes, and we will have two horizons otherwise.

2.3 Gibbons-Lü-Page-Pope Black Holes

In this section, we will consider the metrics examined by Gibbons, Lü, Page, and Pope [23]. These metrics are a generalization of Myers-Perry metrics to include a cosmological constant term Λ. Similar to the Myers-Perry metric, the coordinates divide up spacetime into p planes of black hole rotation. Using Boyer-Lindquist coordinates, the metric in an even number of spatial dimensions is given as

$$ds^2 = -W(1-\Lambda r^2)dt^2 + \frac{\Pi L}{\Pi(1-\Lambda r^2) - \alpha r^2}dr^2 + \frac{\alpha r^2}{\Pi L}\left(dt - \sum_{i=1}^{p} \frac{a_i \mu_i^2}{1+\Lambda a_i^2} d\phi_i\right)^2$$

$$+\sum_{i=1}^{p} \frac{r^2+a_i^2}{1+\Lambda a_i^2}[d\mu_i^2 + \mu_i^2(d\phi_i - \Lambda a_i dt)^2] + \frac{\Lambda}{W(1-\Lambda r^2)}\left(\sum_{i=1}^{p} \frac{r^2+a_i^2}{1+\Lambda a_i^2}\mu_i d\mu_i\right)^2, \quad (2.3.1)$$

and for an odd number of spatial dimensions we have

$$ds^2 = -W(1-\Lambda r^2)dt^2 + \frac{\Pi L}{\Pi(1-\Lambda r^2) - \alpha r}dr^2 + \frac{\alpha r}{\Pi L}\left(dt - \sum_{i=1}^{p} \frac{a_i \mu_i^2}{1+\Lambda a_i^2} d\phi_i\right)^2 + r^2 d\mu^2$$

$$+\sum_{i=1}^{p} \frac{r^2+a_i^2}{1+\Lambda a_i^2}[d\mu_i^2 + \mu_i^2(d\phi_i - \Lambda a_i dt)^2] + \frac{\Lambda}{W(1-\Lambda r^2)}\left(\sum_{i=1}^{p} \frac{r^2+a_i^2}{1+\Lambda a_i^2}\mu_i d\mu_i + r^2 \mu d\mu\right)^2, \quad (2.3.2)$$

In these metrics, we have defined

$$\Pi = \prod_{i=1}^{p}(r^2+a_i^2), \qquad L = 1 - \sum_{i=1}^{p} \frac{a_i^2 \mu_i^2}{r^2+a_i^2}, \qquad W = \sum_{i=1}^{p} \frac{\mu_i^2}{1+\Lambda a_i^2}, \quad (2.3.3)$$

Π and L having the same definitions as before. Also as before, the μ are not independent but must obey the restriction $\sum_{i=1}^{p} \mu_i^2 = 1$ in an even number of spatial dimensions, and the restriction $\sum_{i=1}^{p} \mu_i^2 + \mu^2 = 1$ in odd N. These metrics solve the vacuum Einstein equations with cosmological constant

$$R_{\mu\nu} = N\Lambda g_{\mu\nu}\,, \tag{2.3.4}$$

so Λ is not equal to the usual cosmological constant but is scaled by the dimensions (the usual cosmological constant is defined so that $R_{\mu\nu} = \Lambda g_{\mu\nu}$). Note that this metric is axisymmetric and stationary, giving rise to Killing vectors $\xi^\mu \partial_\mu = \partial_t$ and $\xi_i^\mu \partial_\mu = \partial_{\phi_i}$ as before.

2.4 Other Higher Dimensional Black Holes

In a four dimensional spacetime, there are a number of well known uniqueness theorems that guarantee any vacuum, asymptotically flat, black hole solution will be uniquely characterized by its angular momentum and mass, and that the event horizon of such a black hole will have the topology of a sphere [29, 30, 31]. These theorems demonstrate that the Kerr solution is the only black hole solution we need if we are interested in studying asymptotically flat vacuum solutions. These theorems were proved in four dimensional spacetime, and we have seen that the Myers-Perry metrics are uniquely characterized by their mass and angular momentum, though with additional angular momentum parameters. It is natural to ask if these theorems will generalize to higher dimensions, that is to ask if the Myers-Perry metrics are all we need to discuss properties of higher dimensional black holes.

It is the case that for static black holes, the Tangherlini-Schwarzschild solution is unique, in that it is the only static vacuum black hole solution which is asymptotically flat [32]. Also, Cai and Galloway [33] have proved various theorems restricting the topologies of such higher dimensional black holes, but it is not so restrictive as to disallow all non-spherical topologies. Later, Emparan and Reall [34] demonstrated that there is no simple generalization to the uniqueness theorems by finding a five dimensional asymptotically flat, stationary solution to the vacuum Einstein equations that is not equivalent to a Myers-Perry solution. Their solution has a single angular momentum value J and an event horizon with a topology of $S^2 \times S^1$, meaning this is toroidal shaped, a 'black ring'. The angular momentum for the black ring is shown to obey a restriction of

$$\frac{J^2}{M^3} > 0.8437\frac{32G^{(5)}}{27\pi}, \tag{2.4.1}$$

meaning that the angular momentum of this five dimensional black hole has no upper bound, but it has a lower bound. We have seen that in the Myers-Perry situation, the angular momentum for a five dimensional black hole with a single spin parameter is bounded from above by $\alpha \geq a^2$, so using (2.1.3) and (2.2.1), we get

$$M = \frac{3\pi}{8G}\alpha, \qquad J = \frac{2}{3}Ma,$$

$$\frac{J^2}{M^3} < \frac{32G^{(5)}}{27\pi}, \tag{2.4.2}$$

so we can even see that there are black hole and black ring solutions with the same mass and angular momentum, demonstrating that there is no obvious generalization to the black hole uniqueness theorems. In this thesis, we will focus on the Myers-Perry metrics, as their four and five dimensional versions are known to have Killing tensors, although it is possible that other metrics have interesting properties similar to the ones we will demonstrate for Myers-Perry metrics.

Chapter 3

Topological Defects

3.1 Domain Walls

We will be interested in studying the configurations of strings in a Myers-Perry geometry, so now let us examine how topological defects occur and consider some of their properties. We will do this analysis in a spacetime with (3+1) dimensions, however it is easy to see how these results would generalize to an arbitrary number of dimensions. Topological defects are discussed in detail in the book by Vilenkin and Shellard [35] (also see [36] for a review). For illustrative purposes, we will begin by examining two dimensional topological defects known as domain walls. Suppose we have a scalar field φ obeying a potential $V(\varphi)$ given as

$$V(\varphi) = \frac{\beta}{4}(\varphi^2 - \eta^2)^2 , \tag{3.1.1}$$

for some given constants β, η. Note that this potential has a discrete $\varphi \to -\varphi$ symmetry. $V(\varphi)$ has two minima, at $\varphi = \pm\eta$, so certainly $\varphi(x) = \eta$ and $\varphi(x) = -\eta$ are ground state solutions to the classical equations of motion, distinguished by choice of boundary conditions. However, if we choose $\varphi(x) = \eta$ at $x \to \infty$ and $\varphi(x) = -\eta$ at $x \to -\infty$, then we can find the solution to the equations of motion as

$$\varphi(x) = \eta \tanh\left(\sqrt{\frac{\beta}{2}}\eta x\right) . \tag{3.1.2}$$

So the field starts out in the minimum, $\varphi = -\eta$, picks up out of the vacuum to $\varphi = 0$ at $x = 0$ and falls back into the other minimum at $\varphi = \eta$. Physically, one might assume that such boundary conditions occur by having different regions of the universe that were not yet in causal contact when the universe cooled to a temperature at which the symmetry was broken.

The stress-energy tensor for the scalar field φ is given by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\mathcal{L}\,, \tag{3.1.3}$$

where $\mathcal{L}$ is the Lagrangian density, $\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$. So we may find the stress energy tensor is given by

$$T^{\mu\nu} = \frac{\beta\eta^4}{2\cosh^4\left(\sqrt{\frac{\beta}{2}}\eta x\right)}\mathrm{diag}(1, 0, -1, -1)\,. \tag{3.1.4}$$

The interpretation is simple, a region of false vacuum is forced into existence by our boundary conditions, this region looks as a wall with thickness $\sqrt{\frac{2}{\beta}}\frac{1}{\eta}$, called a domain wall. The finite thickness is caused by a balance between the kinetic and potential terms, and the wall is stable due to continuity of the field φ.

These walls may occur whenever a discrete symmetry is broken, however, from a cosmological standpoint, domain walls probably don't exist. Since the energy density of walls is proportional to their area times the wall density, we expect the energy density of walls will drop as $\mathcal{R}^{-1}$, with $\mathcal{R}$ being the scale factor. Since the energy density of radiation drops like $\mathcal{R}^{-4}$ and the energy density of matter drops as $\mathcal{R}^{-3}$, we would expect domain walls to quickly dominate over matter and radiation. Also domain walls would lead to large fluctuations in the cosmic microwave background radiation, in contradiction with observations unless η is very small [37][38].

3.2 Cosmic Strings

Just as domain walls arise when a discrete symmetry is broken, cosmic strings are a one dimensional topological defect that arise when a $U(1)$ symmetry is broken. Suppose we have a field A_μ and a complex scalar field φ with a Lagrangian density of the form

$$\mathcal{L} = \sqrt{-g}\left\{-\frac{1}{2}D_\mu\varphi D^\mu\varphi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\beta}{4}(\varphi^*\varphi - \eta^2)^2\right\}, \tag{3.2.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi\,. \tag{3.2.2}$$

Where, as before, β and η are given constants, and e is a given coupling constant. This Lagrangian has the $U(1)$ gauge symmetry

$$\varphi \to \varphi\exp(iZ)\,, \qquad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu Z\,, \tag{3.2.3}$$

for any function of spacetime Z. In the vacuum state $|\varphi| = \eta$ the symmetry is broken, and there is a degenerate set of vacuums characterized by $\varphi = \eta\exp(i\omega)$, for a constant ω.

A stationary solution was first found by Nielson and Olsen [39], it has an asymptotic form of

$$\varphi \sim \eta\exp(i\mathcal{N}\phi)\,, \qquad A_\mu \sim \frac{1}{ie}\partial_\mu\ln\left(\frac{\varphi}{\eta}\right) \sim \frac{\mathcal{N}}{e}\delta^\phi_\mu\,, \tag{3.2.4}$$

with ϕ being the angular coordinate in the x-y plane, and $\mathcal{N}$ being an arbitrary integer, the winding number.

Consider a circle at a large distance from the origin in a solution with a winding number of 1. This circle maps one to one onto the vacuum $\varphi = \eta\exp(i\omega)$, but if we contract this circle continuously to a point, continuity of φ will demand that the solution pick up out of the vacuum somewhere. There will be a region of nonzero energy density, however it vanishes far away because (3.2.4) implies that $D_\mu\varphi \to 0$ and $F_{\mu\nu} \to 0$. This region of false vacuum will be a string lying along the z-axis, this is our cosmic string.

3.2.1 Cosmic String Dynamics

To study the dynamics of cosmic strings, it would be possible to analyze dynamics of the field theory; however it will make matters simpler if we consider approximating the cosmic string as an infinitely thin object and find an effective action for it. The effective action for a cosmic string is derived in [35]. We will highlight the steps of the derivation here. We begin with the action

$$S = \int \mathcal{L}\, d^4x = \int \left\{ -\frac{1}{2} D_\mu \varphi D^\mu \varphi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\beta}{4}(\varphi^* \varphi - \eta^2)^2 \right\} \sqrt{-g} d^4x \,. \quad (3.2.5)$$

Let us assume we have a string worldsheet $x^\mu(\zeta^A)$, parametrized by ζ^A $(A = (0,1))$ as the worldsheet coordinates. The worldsheet has tangent vectors $x^\mu{}_{,A}$ and an induced worldsheet metric given as

$$G_{AB} = g_{\mu\nu} x^\mu{}_{,A} x^\nu{}_{,B} \,, \quad (3.2.6)$$

and we assume that the metric G_{AB} has one spacelike and one timelike dimension. Next we introduce vectors normal to the string n^μ_R $(R = 2,3)$ with $n_R \cdot x_{,A} = 0$ and $n_R \cdot n_S = \delta_{RS}$. We also define the vectors to be complete, such that

$$g^{\mu\nu} = G^{AB} x^\mu{}_{,A} x^\nu{}_{,B} + \delta^{RS} n^\mu_R n^\nu_S \,. \quad (3.2.7)$$

The fact that these vectors are complete allows us to express any point near that string worldsheet y^μ using the coordinates ζ^A and ϱ^R as

$$y^\mu(\aleph) = x^\mu(\zeta^A) + \varrho^R n^\mu_R(\zeta^A) \,, \quad (3.2.8)$$

with $\aleph^\alpha = (\zeta^A, \varrho^R)$. These coordinates are well defined only if y^μ is closer to the string worldsheet than its curvature radius R. If we perform a coordinate change from y^μ to $\aleph^\alpha$, then $\sqrt{-g}$ transforms to

$$\sqrt{-g} \det \left(\frac{\partial y}{\partial \aleph} \right) = \sqrt{-\det M} \,, \quad (3.2.9)$$

where we have defined the new metric $M_{\alpha\beta}$ as

$$M_{\alpha\beta} = g_{\mu\nu} \frac{\partial y^\mu}{\partial \aleph^\alpha} \frac{\partial y^\nu}{\partial \aleph^\beta} = \mathrm{diag}(G_{AB}, \delta_{RS}) + O\left(\frac{\varrho}{R}\right) \,. \quad (3.2.10)$$

Defining $M = \det M$ and $G = \det G$, then to lowest order we have $\sqrt{-M} = \sqrt{-G}$. Also, to lowest order in ϱ/R the fields φ and A are independent of ζ^A so that we may integrate the action as

$$S = \int \left\{ -\frac{1}{2} D_\mu \varphi D^\mu \varphi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\beta}{4} (\varphi^* \varphi - \eta^2)^2 \right\} d^2\varrho \int \sqrt{-G} d^2\zeta \,. \quad (3.2.11)$$

The first integral over ϱ may be performed to get the string energy density $-\mu^*$. Unfortunately, except in special cases, it is not possible to find a closed form value for μ^*, however it will always be of order η^2. The inaccuracies of order (ϱ/R) are cut off by the exponential fall in the field strength and are reduced to order (δ/R) at large distances where δ is the string width. The final form for the effective string action is

$$S = -\mu^* \int \sqrt{-G} d^2\zeta \,. \quad (3.2.12)$$

The action functional (3.2.12) is called the Nambu-Goto action for a string. This action is generally covariant, and will apply in the context of general relativity in arbitrary dimensions. The stress-energy tensor is defined as

$$\sqrt{-g}\, T^{\mu\nu} = -2 \frac{\partial S}{\partial g_{\mu\nu}} \,. \quad (3.2.13)$$

Varying the string action (3.2.12) with respect to $g_{\mu\nu}$, we get the stress-energy tensor for the string

$$\sqrt{-g} T^{\mu\nu}(x) = -\mu^* \int d^2\zeta \delta^{(N+1)}(x - x(\zeta)) \sqrt{-G} G^{AB} x^\mu{}_{,A} x^\nu{}_{,B} \,, \quad (3.2.14)$$

where $x^\mu(\zeta^A)$ is the string configuration. Suppose we have a long string in flat (t, x, y, z) spacetime lying along the z-axis with an embedding of $\zeta^0 = t$, $\zeta^1 = z$. The stress-energy of this string is given as

$$T^{\mu\nu} = \mu^* \delta(x) \delta(y) \ \mathrm{diag}(1, 0, 0, -1) \,. \quad (3.2.15)$$

We can see that the string has a negative pressure equal in magnitude to its energy density.

Next, we will find the dynamical equations of string evolution. Variation of the action S with respect to the string coordinates $x^{\mu}(\zeta^A)$ gives one the Nambu-Goto equations of motion

$$\Box x^{\nu} + G^{AB}\Gamma^{\nu}{}_{\lambda\sigma}x^{\lambda}_{,A}x^{\sigma}_{,B} = 0\,, \tag{3.2.16}$$

where we have defined

$$\Box x^{\nu} = \frac{1}{\sqrt{-G}}\partial_A(\sqrt{-G}G^{AB}\partial_B x^{\nu}) \tag{3.2.17}$$

as the D'Alembertian operator on the induced string worldsheet metric G_{AB}, and $\Gamma^{\nu}{}_{\lambda\sigma}$ are the Christoffel symbols for the bulk metric $g_{\mu\nu}$

3.2.2 Cosmic Strings as a Gravitational Source

In linearized gravity, if we have a static distribution of matter with a stress-energy tensor

$$T^{\mu\nu} = \mathrm{diag}(\rho, P_x, P_y, P_z)\,, \tag{3.2.18}$$

the Newtonian gravitational potential $\mathcal{V}$ created by this matter will obey the Poisson equation

$$\nabla^2\mathcal{V} = 4\pi G^{(4)}(\rho + P_x + P_y + P_z)\,. \tag{3.2.19}$$

Consider a cosmic string lying in a straight line along the z-axis. Using (3.2.15) to be a source of gravity, we see that the Newtonian potential obeys $\nabla^2\mathcal{V} = 0$, so we would expect no Newtonian gravitational field.

The exact solution for the metric near such a cosmic string with finite thickness has been found and analyzed by Gott [40], the solution inside the string is given as

$$ds^2 = -dt^2 + r_0^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2\,. \tag{3.2.20}$$

The coordinates t and z take on any real values, ϕ is an angle bounded from 0 to 2π and θ is an angle ranging from 0 to a maximum $\theta_M < \pi$. The constant r_0 is defined by $2\pi\sin\theta_M r_0$ being the string circumference. The spacetime outside of the string looks as

$$ds^2 = -dt^2 + \frac{dr^2}{\cos^2\theta_M} + r^2 d\phi^2 + dz^2\,, \tag{3.2.21}$$

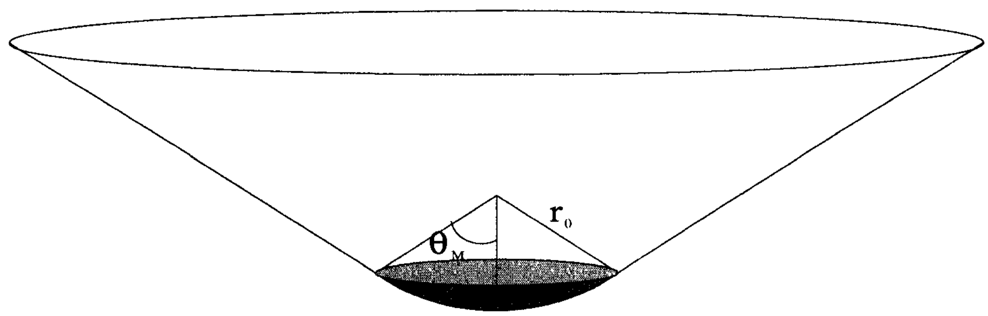

Figure 3.1: Embedding of metric (3.2.23) in flat 3-space

with t, z, and ϕ being the same coordinates as the interior ones, and r being a radial coordinate bounded below by $r_0 \sin\theta_M$ (at which point you enter the string interior).

The non-zero components of the stress energy tensor needed to generate this spacetime can be found by use of the field equations to be

$$T^{tt} = -T^{zz} = \frac{1}{8\pi G^{(4)} r_0^2}, \qquad (3.2.22)$$

inside, and zero everywhere outside. We can see that we have $\rho = -P_z$, just as we expect. One can perform a coordinate transformation $r' = \frac{r}{\cos\theta_M}$, $\phi' = \phi\cos\theta_M$ to bring this metric into a locally flat form, as one might expect from the Newtonian potential. However, the locally flat angle coordinate ϕ' runs from 0 to $2\pi\cos\theta_M$, corresponding to an angle deficit of $2\pi(1-\cos\theta_M)$.

To examine how an embedding of this spacetime would look, let us consider slices of constant t and z, so the metric takes the form

$$ds^2 = r_0^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad \text{(inside)},$$

$$ds^2 = \frac{dr^2}{\cos^2\theta_M} + r^2 d\phi^2 \qquad \text{(outside)}. \qquad (3.2.23)$$

The inside metric is a part of a sphere with radius r_0, extending from $\theta = 0$ to $\theta = \theta_M$, while the outside metric is a cone tilted at an angle θ_M. These two metrics match up at the border of the string exterior.

The angle θ_M is related to the mass per unit length of the string μ^* as

$$\cos\theta_M = 1 - 4G^{(4)}\mu^*. \qquad (3.2.24)$$

If θ_M is greater that $\frac{\pi}{2}$, then the cone of (3.2.23) turns back on itself and r is bounded from above by $r_0 \sin\theta_M$ instead of below. If we assume that the external universe to the string is to be large, then we must have $G^{(4)}\mu^* < \frac{1}{4}$, which is the same as $\mu^* < 3.37\text{x}10^{-26}\text{Kg/m}$. However, we will see that the cosmic microwave background gives a much stricter bound on the value of μ^* for a string.

3.2.3 Cosmic Strings as Cosmological Objects

Unlike domain walls, strings are not "bad" from a cosmological point of view. The energy density of long strings scales as $\mathcal{R}^{-4}$, the same as radiation. We have noted that the spacetime around a cosmic string is locally flat, but with an angle deficit of $8\pi G^{(4)}\mu^*$. This angle deficit allows cosmic strings to act as gravitational lenses, and cosmic strings moving with velocity v will result in a small Doppler shift of the cosmic microwave background radiation of

$$\frac{\delta T}{T} = 8\pi G^{(4)}\mu^* v\,. \qquad (3.2.25)$$

So, based on observation, we may say that any strings that exist today must have $G^{(4)}\mu^* < 10^{-6}$ ($\mu^* < 1.42\text{x}10^{-32}$ Kg/m) [41].

Although a cosmic string will have no gravitational effect on matter that is stationary relative to it, as a cosmic string moves past a distribution of matter it results in a 'string wake', disrupting the matter around it. There has been much analysis of such models of galaxy formation, for example see [42]-[44].

Analysis of the cosmic microwave background anisotropy has excluded some proposed scenarios in which cosmic strings are the main source of anisotropy, in favor of the models where inflation is the main source. However, there have been models proposed where a mixture of cosmic strings and inflation provide the anisotropy (see [45]-[47]).

We will also note that in addition to domain walls and cosmic strings, there are also zero dimensional topological defects known as 'monopoles', due to the fact that they act as magnetic monopoles in the gauge field. There are also three dimensional topological defects referred to as textures. Textures are not localized in space but get their energy density from field gradients rather than a core of false vacuum. In higher dimensional spacetimes there exist general M-dimensional topological defects. Collectively, topological defects are known as branes.

There has been much interest on motion of cosmic strings and other topological defects near non-rotating gravitating bodies [48]-[53]. As well there has been analysis done on scattering and capture of cosmic strings moving near a rotating black hole [54, 55]. We will study the interactions of stationary cosmic strings with higher dimensional rotating black holes. This is a rare situation in which there are extended strongly gravitating bodies interacting where the geometrical symmetries of the system allow for a quite complete analysis. In our analysis of stationary strings interacting with higher dimensional rotating black holes we discover a number of interesting properties. We find separability of the string equations in the five dimensional case, and the existence of principal Killing string solutions.

Chapter 4

Strings in a Five Dimensional Spacetime

4.1 Stationary Strings in Stationary Spacetimes

In this chapter, we will examine stationary strings in a Myers-Perry metric, so first we will consider general properties of stationary metrics. Stationary spacetimes have a Killing vector field $\xi^\mu\partial_\mu = \partial_t$ that is timelike somewhere, usually at infinity. A general metric of a stationary spacetime M can be written in the form

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = -F(dt + A_a dx^a)^2 + H_{ab}dx^a dx^b\,, \tag{4.1.1}$$

with $F = -\xi^2$, A_a, and H_{ab} being functions of the spatial coordinates x^a $(a, b = 1\ldots N)$. We assume that the spacetime is a foliation of Killing trajectories and denote $\mathcal{M} = \mathsf{M}/\,G_1$ as the factor space, where G_1 is the symmetry group representing time translation along ξ. We can define a tensor

$$H_{\mu\nu} = g_{\mu\nu} + \frac{\xi_\mu\xi_\nu}{F}\,, \tag{4.1.2}$$

and this tensor will be a projector onto the factor space $\mathcal{M}$ with the metric H_{ab}. Any tensor $\mathbf{T}$ which is stationary ($\mathcal{L}_\xi\mathbf{T} = 0$, where $\mathcal{L}$ is the Lie derivative) can be projected onto this factor space by use of the projector operator **H** as follows,

$$T^{\alpha\cdots\beta}_{\gamma\cdots\delta} = H^{\alpha}_{\mu} H^{\rho}_{\gamma} \cdots H^{\beta}_{\nu} H^{\sigma}_{\delta} T^{\mu\cdots\nu}_{\rho\cdots\sigma} \,. \tag{4.1.3}$$

It has also been shown by Geroch [56] that we may consistently define the covariant derivative in the metric **H** as

$$T^{\mu\cdots}{}_{\cdots\nu\,:\,\lambda} = H^{\mu}_{\alpha} \cdots H^{\beta}_{\nu} H^{\sigma}_{\lambda} T^{\alpha\cdots}{}_{\cdots\beta\,;\,\sigma} \,. \tag{4.1.4}$$

Finally, if we have a stationary Killing tensor of rank i in the spacetime given as $K_{\mu_1\cdots\mu_i}$ obeying Killing's equation (1.4), then its projection $K_{a_1\cdots a_i}$ will be a Killing tensor in the metric **H**, meaning

$$K_{(a_1\cdots a_i\,:\,b)} = 0 \,. \tag{4.1.5}$$

Let us define H^{ab} to be the matrix inverse of H_{ab}, so that

$$H_{ab} H^{bc} = \delta^{c}_{a} \,, \tag{4.1.6}$$

then we have the contravariant components of the metric g given by

$$g^{tt} = -F^{-1} + F^{-2} H^{ab} \xi_a \xi_b \,, \qquad g^{ta} = F^{-1} H^{ab} \xi_b, \qquad g^{ab} = H^{ab} \,. \tag{4.1.7}$$

We call a two dimensional surface Σ a stationary surface if ξ is tangent to it. A stationary surface is formed by a one dimensional family of the Killing trajectories for the field ξ. Suppose we have a string with a configuration given by $x^a(\sigma)$ in the factor space $\mathcal{M}$ and that this string is stationary. Its worldsheet Σ will be given by propagation of $x^a(\sigma)$ forward in time along ξ. Let us choose coordinates ζ^A ($A = 0, 1$) on Σ with $\zeta^0 = t$ being the affine parameter along Killing trajectories and let $\zeta^1 = \sigma$. The induced metric G_{AB} on Σ is

$$d\gamma^2 = G_{AB}\, d\zeta^A\, d\zeta^B = -F\,(dt + \mathcal{A}\, d\sigma)^2 + \mathcal{H}\, d\sigma^2 \,, \tag{4.1.8}$$

where $\mathcal{A}$ and $\mathcal{H}$ are defined as

$$\mathcal{A} = A_a \frac{dx^a}{d\sigma}, \qquad \mathcal{H} = H_{ab} \frac{dx^a}{d\sigma} \frac{dx^b}{d\sigma}. \tag{4.1.9}$$

We will consider the string with mass density μ^* to be a test object in the external black hole metric, the black hole having a mass M. Certainly such an assumption will be good for timescales much less than $\tau = M/\mu^*$. The equation of motion for a test string worldsheet can be found as an extremum of the Nambu-Goto action (3.2.12). For a stationary string (4.1.8) tells us that the action reduces to

$$S = -I\Delta t, \tag{4.1.10}$$

with

$$I = \mu^* \int d\sigma \sqrt{h_{ab} \frac{dx^a}{d\sigma} \frac{dx^b}{d\sigma}}, \qquad h_{ab} = F H_{ab}. \tag{4.1.11}$$

Extremizing the action S is equivalent to extremizing the value of I, telling us that the line $x^a(\sigma)$ representing the stationary string is a geodesic in the metric $h_{ab} = FH_{ab}$. This means that an equilibrium string configuration has extremized its length in the metric $\mathbf{h}$, which is different from the spatial projection metric by a redshift factor F.

4.2 Properties of Five Dimensional Myers-Perry Geometry

We will be considering stationary string configurations in a five dimensional Myers-Perry spacetime. The metric as given by (2.2.2) is

$$\begin{aligned} ds^2 = &-dt^2 + (r^2 + a_1^2)(d\mu_1^2 + \mu_1^2 d\phi_1^2) + (r^2 + a_2^2)(d\mu_2^2 + \mu_2^2 d\phi_2^2) \\ &+ \frac{\Pi L}{\Pi - \alpha r^2} dr^2 + \frac{\alpha r^2}{\Pi L}(dt + a_1 \mu_1^2 d\phi_1 + a_2 \mu_2^2 d\phi_2)^2, \end{aligned} \tag{4.2.1}$$

with

$$\Pi = (r^2 + a^2)(r^2 + b^2), \qquad L = 1 - \frac{a_1^2 \mu_1^2}{r^2 + a_1^2} - \frac{a_2^2 \mu_2^2}{r^2 + a_2^2}. \tag{4.2.2}$$

To simplify the future equations, we will replace the coordinate r by the coordinate $x = r^2$. We will also transform from the dependent μ_i values to the independent θ coordinates. The μ_i's must obey the condition

$$\mu_1^2 + \mu_2^2 = 1\,, \qquad 0 < \mu_i < 1\,, \tag{4.2.3}$$

so this will be satisfied if we let θ be an angle from 0 to $\pi/2$, with the definition

$$\mu_1 = \sin\theta, \qquad \mu_2 = \cos\theta\,. \tag{4.2.4}$$

Further, for simplicity, we will make the defintions

$$a = a_1\,, \quad b = a_2\,, \quad \phi = \phi_1\,, \quad \psi = \phi_2\,,$$

$$\rho^2 = x + a^2\cos^2\theta + b^2\sin^2\theta\,, \quad \Delta = (x+a^2)(x+b^2) - x\alpha\,. \tag{4.2.5}$$

So the metric takes the form

$$ds^2 = -dt^2 + (x+a^2)\sin^2\theta d\phi^2 + (x+b^2)\cos^2\theta d\psi^2 + \frac{\rho^2}{4\Delta}dx^2 + \rho^2 d\theta^2$$

$$+\frac{\alpha}{\rho^2}\left[dt + a\sin^2\theta d\phi + b\cos^2\theta d\psi\right]^2\,. \tag{4.2.6}$$

The infinite red-shift surface of the black hole is defined by the equation $\xi^2 = 0$, which is given as $x = \alpha - a^2\cos^2\theta - b^2\sin^2\theta$, equivalent to $\rho^2 = \alpha$. The event horizon of the black hole is located at $x = x_+$ where

$$x_\pm = \frac{1}{2}\left[\alpha - a^2 - b^2 \pm \sqrt{(\alpha - a^2 - b^2)^2 - 4a^2b^2}\right]\,, \tag{4.2.7}$$

so we may see that in order for an event horizon to exist, the parameters of the black hole will have to obey the restriction $\alpha \geq (|a| + |b|)^2$, consistent with what we demonstrated previously (2.2.20). The surface gravity κ (2.2.7) is given in the five dimensional case as

$$\kappa = \left.\frac{\partial_x\Delta}{\alpha\sqrt{x}}\right|_{x=x_+}\,. \tag{4.2.8}$$

As well as the principal Killing vector $\xi = \partial_t$, the metric has additional Killing vectors by axial symmetry of the planes of rotation. These Killing vectors can be expressed as $\xi_\phi = \partial_\phi$ and $\xi_\psi = \partial_\psi$. It has also been shown that the five dimensional Myers-Perry metric has a second rank Killing tensor $K^{\mu\nu}$ [17]

$$K^{\mu\nu} = -(a^2\cos^2\theta + b^2\sin^2\theta)(g^{\mu\nu} + \delta^\mu_t\delta^\nu_t) + \frac{\delta^\mu_\phi\delta^\nu_\phi}{\sin^2\theta} + \frac{\delta^\mu_\psi\delta^\nu_\psi}{\cos^2\theta} + \delta^\mu_\theta\delta^\nu_\theta\,. \tag{4.2.9}$$

Finally, if we consider the degenerate case $a = b$ the metric has increased symmetry, we may understand this symmetry by considering 'cartesianised' coordinates on the planes of black hole rotation, defined as

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$

$$z = r\cos\theta\cos\psi \qquad w = r\cos\theta\sin\psi\,. \tag{4.2.10}$$

If we then define 'rotation vectors' as

$$J_1^\mu\partial_\mu = x\,\partial_z - z\,\partial_x \qquad J_2^\mu\partial_\mu = x\,\partial_w - w\,\partial_x$$

$$J_3^\mu\partial_\mu = y\,\partial_z - z\,\partial_y \qquad J_4^\mu\partial_\mu = y\,\partial_w - w\,\partial_y\,, \tag{4.2.11}$$

direct calculations show that these vectors J are not Killing vectors, so in this sense the black hole spacetime does not have total rotational symmetry, however the vectors $\xi_{(1)} = J_1 + J_4$ and $\xi_{(2)} = J_2 - J_3$ are Killing vectors. These vectors represent a symmetry of rotating the planes of rotation into each other. In Boyer-Lindquist coordinates, these vectors take the form

$$\xi^\mu_{(1)}\partial_\mu = -\cos(\phi-\psi)\partial_\theta + \cot\theta\sin(\phi-\psi)\partial_\phi + \tan\theta\sin(\phi-\psi)\partial_\psi\,, \tag{4.2.12}$$

and

$$\xi^\mu_{(2)}\partial_\mu = \sin(\phi-\psi)\partial_\theta + \cot\theta\cos(\phi-\psi)\partial_\phi + \tan\theta\cos(\phi-\psi)\partial_\psi\,, \tag{4.2.13}$$

equivalent to the Killing vectors previously found by [18].

One can check that in this degenerate case, the Killing tensor obeys the relationship

$$K^{\mu\nu} = \xi^\mu_\phi\,\xi^\nu_\phi + \xi^\mu_\psi\,\xi^\nu_\psi - 2\,\xi^{(\mu}_\phi\,\xi^{\nu)}_\psi + \xi^\mu_{(1)}\,\xi^\nu_{(1)} + \xi^\mu_{(2)}\,\xi^\nu_{(2)} - \xi^\mu\,\xi^\nu - g^{\mu\nu}\,, \qquad (4.2.14)$$

demonstrating that the Killing tensor is reducible in the $a = b$ case. This property is related to the enlarged symmetry in the degenerate case[1].

4.3 Stationary Strings Near a Five Dimensional Black Hole

Expressing the metric (4.2.6) in the form (4.1.1), we can find

$$F = -\xi^2 = \frac{\rho^2 - \alpha}{\rho^2}\,, \qquad (4.3.1)$$

$$A_\phi = \frac{\alpha\, a \sin^2\theta}{\alpha - \rho^2} \qquad A_\psi = \frac{\alpha\, b \cos^2\theta}{\alpha - \rho^2}\,, \qquad (4.3.2)$$

$$H_{xx} = \frac{\rho^2}{4\Delta}\,, \quad H_{\theta\theta} = \rho^2\,, \quad H_{\phi\psi} = \frac{\alpha\, ab \sin^2\theta\cos^2\theta}{\rho^2 - \alpha}$$

$$H_{\phi\phi} = \left(x + a^2 + \frac{\alpha a^2 \sin^2\theta}{\rho^2 - \alpha}\right)\sin^2\theta\,, \quad H_{\psi\psi} = \left(x + b^2 + \frac{\alpha b^2\cos^2\theta}{\rho^2 - \alpha}\right)\cos^2\theta\,, \qquad (4.3.3)$$

with all other H_{ab} and A_a zero. Letting h^{ab} be defined as the matrix inverse of h_{ab} (defined in (4.1.11), so that $h^{ab} = \frac{1}{F} H^{ab}$, we find that the non-zero components of h^{ab} are

$$h^{\phi\phi} = \frac{1}{\rho^2 - \alpha}\left[\frac{1}{\sin^2\theta} - \frac{(a^2 - b^2)(x + b^2) + b^2\alpha}{\Delta}\right],$$

$$h^{\psi\psi} = \frac{1}{\rho^2 - \alpha}\left[\frac{1}{\cos^2\theta} + \frac{(a^2 - b^2)(x + a^2) - a^2\alpha}{\Delta}\right], \qquad (4.3.4)$$

$$h^{\phi\psi} = -\frac{ab\alpha}{(\rho^2 - \alpha)\,\Delta}\,, \qquad h^{xx} = 4\frac{\Delta}{\rho^2 - \alpha}\,, \qquad h^{\theta\theta} = \frac{1}{\rho^2 - \alpha}\,.$$

We have seen that stationary string configurations will be geodesic in the metric $\mathbf{h}$, so we will use Hamilton-Jacobi methods to find these geodesics (see for example,

[1]The author would like to thank D. N. Page for this point.

[25]). The Hamilton-Jacobi equation is

$$\frac{\partial I}{\partial \sigma} + \frac{1}{2} h^{ab} \frac{\partial I}{\partial x^a} \frac{\partial I}{\partial x^b} = 0 \,. \tag{4.3.5}$$

By analogy with particles in geodesic motion, we may consider the 'momentum' of the stationary string to be given as

$$P_a = \frac{\partial I}{\partial x^a} \,. \tag{4.3.6}$$

Note that we raise and lower the index of P with the metric $\mathbf{h}$, we also have $P^a = dx^a/d\sigma$. The existence of the Killing vectors $\xi^a_\phi \partial_a = \partial_\phi$, $\xi^a_\psi \partial_a = \partial_\psi$ guarantees that $P_\phi = \partial_\phi I$ and $P_\psi = \partial_\psi I$ are constants, we will denote them by Φ and Ψ, respectively. We will show that in the metric $\mathbf{h}$ (4.3.4) the equation (4.3.5) allows separation of variables of the form

$$I = -\frac{1}{2} m^2 \sigma + \Phi \phi + \Psi \psi + I_\theta + I_x \,, \tag{4.3.7}$$

with I_θ and I_x being functions of θ and x respectively. The constant $m^2 = h^{ab} P_a P_b$ depends on the choice of the length parameter σ. After derivation of the equations for a stationary string configuration we put $m = 1$ and use a proper length in the metric $\mathbf{h}$ as σ. Substituting this expression of the action into (4.3.5) and separating variables, one obtains

$$\left(\frac{dI_\theta}{d\theta}\right)^2 - m^2(a^2\cos^2\theta + b^2\sin^2\theta) + \frac{1}{\sin^2\theta}\Phi^2 + \frac{1}{\cos^2\theta}\Psi^2 = K \,, \tag{4.3.8}$$

and

$$\begin{aligned} -K = 4\Delta \left(\frac{dI_x}{dx}\right)^2 + m^2(\alpha - x) - \Delta^{-1}\alpha(a\Psi + b\Phi)^2 \\ + \Delta^{-1}(a^2 - b^2)\left[\Psi^2(x + a^2) - \Phi^2(x + b^2)\right] \,. \end{aligned} \tag{4.3.9}$$

Here K is a separation constant. These equations are similar to the the separated equations for particle motion in a five dimensional Myers-Perry metric [17], with three differences. First, the energy E has been set to zero, second the mass squared m^2 has been changed to $-m^2$ and third, there is an extra term of $m^2\alpha$ in the x-

equation. The extra term of $m^2\alpha$ is due to the redshift factor $\sqrt{F}$ appearing in equation (4.1.11).

The fact that separation of variables works in the metric $\mathbf{h}$ is connected with the existence of a rank two Killing tensor $\tilde{K}_{ab}$ in the metric. Assuming that the separation constant K is the conserved quantity connected with this Killing tensor, we may examine (4.3.8) to see that $\tilde{K}^{ab}$ must take the form

$$\tilde{K}^{ab} = \delta^a_\theta \delta^b_\theta - h^{ab}(a^2\cos^2\theta + b^2\sin^2\theta) + \frac{\delta^a_\phi\delta^b_\phi}{\sin^2\theta} + \frac{\delta^a_\psi\delta^b_\psi}{\cos^2\theta}, \tag{4.3.10}$$

then the existence of a constant $K = \tilde{K}^{ab}P_a P_b$ will be guaranteed. Direct calculations show that $\tilde{K}^{ab}$ is a Killing tensor in the metric h_{ab}.

Rearranging the separated equations (4.3.8-4.3.9), we get

$$\frac{\partial I_\theta}{\partial\theta} = \varsigma_\theta\sqrt{\Theta} \qquad \frac{\partial I_x}{\partial x} = \varsigma_x\frac{\sqrt{\chi}}{2\Delta}, \tag{4.3.11}$$

with Θ and χ given as

$$\Theta = K + m^2(a^2\cos^2\theta + b^2\sin^2\theta) - \frac{\Phi^2}{\sin^2\theta} - \frac{\Psi^2}{\cos^2\theta}, \tag{4.3.12}$$

$$\chi = \Delta\left[m^2(x-\alpha) - K\right] + \alpha(b\Phi + a\Psi)^2 + (a^2 - b^2)\left[\Phi^2(x+b^2) - \Psi^2(x+a^2)\right]. \tag{4.3.13}$$

The two sign functions $\varsigma_\theta = \pm 1$ and $\varsigma_x = \pm 1$ in the equations can be chosen independently. In each equation the change of sign occurs when the expression on the right hand side vanishes.

We can now write the Hamilton-Jacobi action as

$$I = -\frac{1}{2}m^2\sigma + \Phi\phi + \Psi\psi + \varsigma_\theta\int\sqrt{\Theta}d\theta + \varsigma_x\int\frac{\sqrt{\chi}}{2\Delta}dx. \tag{4.3.14}$$

By setting the derivatives of I with respect to K, m^2, Φ, Ψ equal to zero, we get integral equations for stationary string configurations

$$\int \varsigma_\theta \frac{d\theta}{\sqrt{\Theta}} = \int \varsigma_x \frac{dx}{2\sqrt{X}}, \tag{4.3.15}$$

$$\sigma = \int \varsigma_\theta \frac{1}{\sqrt{\Theta}}(a^2\cos^2\theta + b^2\sin^2\theta)d\theta + \int \varsigma_x \frac{x-\alpha}{2\sqrt{X}}dx\,, \tag{4.3.16}$$

$$\phi = \int \varsigma_\theta \frac{\Phi}{\sin^2\theta\sqrt{\Theta}}d\theta - \int \varsigma_x \frac{\Phi[(a^2-b^2)(x+b^2)+\alpha b^2]+\alpha ab\Psi}{2\Delta\sqrt{X}}dx\,, \tag{4.3.17}$$

$$\psi = \int \varsigma_\theta \frac{\Psi}{\cos^2\theta\sqrt{\Theta}}d\theta - \int \varsigma_x \frac{\Psi[(b^2-a^2)(x+a^2)+\alpha a^2]+\alpha ab\Phi}{2\Delta\sqrt{X}}dx\,. \tag{4.3.18}$$

By differentiating these equations with respect to σ, it is possible to write these equations in a first order differential form

$$(\rho^2-\alpha)\,\dot{x} = \varsigma_x 2\sqrt{X}\,, \tag{4.3.19}$$

$$(\rho^2-\alpha)\,\dot{\theta} = \varsigma_\theta\sqrt{\Theta}\,, \tag{4.3.20}$$

$$(\rho^2-\alpha)\,\dot{\phi} = \frac{\Phi}{\sin^2\theta} + \frac{\Phi[(b^2-a^2)(x+b^2)-\alpha b^2]-\alpha ab\Psi}{\Delta}\,, \tag{4.3.21}$$

$$(\rho^2-\alpha)\,\dot{\psi} = \frac{\Psi}{\cos^2\theta} + \frac{\Psi[(a^2-b^2)(x+a^2)-\alpha a^2]-\alpha ab\Phi}{\Delta}\,, \tag{4.3.22}$$

with a dot denoting differentiation with respect to σ. Now that we have the configuration equations, m is no longer needed. We now set $m=1$, so that σ is the proper length in the metric $\mathbf{h}$.

The normalization condition $h_{ab}P^aP^b = h_{ab}\dot{x}^a\dot{x}^b = 1$ and the integrals of motion Φ, Ψ, and K suffice to determine the 4 initial data for $\dot{x}^a$. So K, Φ, Ψ, and the initial data $x^a(0)$ uniquely specify a string configuration. The axial symmetry of the problem means that two of these quantities $\phi(0)$ and $\psi(0)$ are cyclic. Thus a stationary string is completely determined by Φ, Ψ, and K and $x(0)$ and $\theta(0)$. Unfortunately, it is not likely that there will exist closed form solutions for the string equations (4.3.19-4.3.22) in the most general case, however we have found some specific solutions in appendix A.

4.4 Properties of String Equations

4.4.1 Radial Equation and Asymptotic Behavior

We discuss first properties of the radial equation (4.3.19). Consider χ given by (4.3.13) as a function of x for given values of other parameters. The allowed configurations must only occur where $\chi \geq 0$, $\chi = 0$ gives radial turning points. In the limit of large x, χ has a leading term x^3, so it is possible for configurations to extend to infinity. Examining (4.3.19) in the region $x \to \infty$, we get

$$\dot{x} \sim 2\sqrt{x}\,. \tag{4.4.1}$$

So we may say that in the region of large x, we have $\sqrt{x} = r \sim \sigma$. Considering θ to be expanded in powers of $1/r$ as $\theta = \theta_0 + \theta_1/r + O(1/r^2)$, we have for (4.3.20)

$$\frac{d\theta}{dr} \sim \frac{\sqrt{\Theta(\theta_0)}}{r^2} + O\left(\frac{1}{r^3}\right), \tag{4.4.2}$$

Thus, we have $\theta \sim \theta_0 - \frac{\sqrt{\Theta(\theta_0)}}{r}$. The last equations (4.3.21)-(4.3.22) become

$$\frac{d\phi}{dr} \sim \frac{\Phi}{r^2 \sin^2\theta_0} + O\left(\frac{1}{r^3}\right), \tag{4.4.3}$$

$$\frac{d\psi}{dr} \sim \frac{\Psi}{r^2 \cos^2\theta_0} + O\left(\frac{1}{r^3}\right), \tag{4.4.4}$$

and these equations may be integrated simply. So we may say that in the limit of $x \to \infty$, string configurations which extend to infinity look like

$$\sigma \sim \sqrt{x} = r\,, \quad \theta \sim \theta_0 - \frac{\sqrt{\Theta(\theta_0)}}{r}\,,$$

$$\phi \sim \phi_0 - \frac{\Phi}{r\sin^2\theta_0}\,, \quad \psi \sim \psi_0 - \frac{\Psi}{r\cos^2\theta_0}\,. \tag{4.4.5}$$

4.4.2 Properties of the θ Equation

Similar to the x equation, configurations are only allowable where $\Theta \geq 0$ and turning points in the θ equation occur when $\Theta = 0$. Since Θ has negative definite terms of the form $(\Phi^2/\sin^2\theta)$ and $(\Psi^2/\cos^2\theta)$ we see that the string can extend into the subspace $\theta = 0$ only if $\Phi = 0$, and can reach the subspace $\theta = \frac{\pi}{2}$ only if $\Psi = 0$.

Consider a special type of configuration when the string is aligned so that θ remains constant $\theta = \theta_0$. This configuration can occur when

$$\Theta(\theta_0) = \frac{d\Theta}{d\theta}(\theta_0) = 0\,. \tag{4.4.6}$$

This is equivalent to

$$K + a^2\cos^2\theta_0 + b^2\sin^2\theta_0 - \frac{\Phi^2}{\sin^2\theta_0} - \frac{\Psi^2}{\cos^2\theta_0} = 0\,, \tag{4.4.7}$$

$$\frac{\Phi^2}{\sin^4\theta_0} - \frac{\Psi^2}{\cos^4\theta_0} - (a^2 - b^2) = 0\,. \tag{4.4.8}$$

The configuration will be entirely in the $\theta = \frac{\pi}{2}$ plane only if $\Psi = 0$ and $K = \Phi^2 - b^2$. Similarly, it is entirely in the $\theta = 0$ plane if $\Phi = 0$ and $K = \Psi^2 - a^2$.

4.4.3 No Stable Bounded Configurations

Let us consider strings that lie in any bounded configuration outside of the event horizon at $x = x_+$, possibly a circular configuration or one that appears as an ellipse. Radial turning points are met at $\chi = 0$, and so any such configuration must occur in a region where χ is positive and bounded at either end by points of $\chi = 0$. Let us represent these two points as $x = x_1$ and $x = x_2$, with $x_1 < x_2$. For the configuration to be bounded, we must have that χ be positive in the region between these turning points. Note from (4.3.13) that χ is a cubic in x which grows to positive infinity in the limit $x \to \infty$. If we use the value of x_+ from (4.2.7) in (4.3.13) for χ, we get a value of

$$\chi(x_+) = x_+\alpha^2\left(\frac{a\Phi}{x_+ + a^2} + \frac{b\Psi}{x_+ + b^2}\right)^2\,, \tag{4.4.9}$$

demonstrating a positive definite value. Thus for a bounded configuration to exist, χ must be positive at x_+, zero at x_1, positive for x in the range $x_1 < x < x_2$, zero at x_2, and positive at large x. Therefore $\chi(x)$ must have at least four zeros. However, χ is only a third order polynomial in x, and so it has at most three zeros. Thus, there are no stable bounded configurations. Note that in principle a circular solution to the configuration equations could exist at a value of x obeying the relations

$$\chi = 0, \qquad \frac{\partial \chi}{\partial x} = 0\,, \tag{4.4.10}$$

however, since χ would be positive for values of x on either side of the x value that solves this, the configuration would not be stable.

4.4.4 A String in the Brane

In a brane world model we have a (3+1)-dimensional universe that exists on a brane in a bulk spacetime with large extra dimensions. The usual matter of the theory (fermions, bosons, and gauge fields) is confined to exist on the brane, while gravity is allowed to propagate into the bulk. Cosmic strings which are composed of the matter fields must be confined to the brane as well. Let us discuss the properties of such strings.

In the presence of a (3+1)-brane a stationary black hole can have only one parameter of the rotation, due to a friction effect [22]. We let $b = 0$ and let the brane equation be described as ψ =const. Since the string must be confined to the brane, we must have $\dot{\psi} = 0$, so equation (4.3.22) tells us this means $\Psi = 0$. Using the radial coordinate $r = \sqrt{x}$ again, the remaining string equations (4.3.19)-(4.3.21) become

$$(r^2 + a^2\cos^2\theta - \alpha)\dot{r} = \varsigma_x\sqrt{(r^2 - K - \alpha)(r^2 + a^2 - \alpha) + a^2\Phi^2}\,, \tag{4.4.11}$$

$$(r^2 + a^2\cos^2\theta - \alpha)\dot{\theta} = \varsigma_\theta\sqrt{K + a^2\cos^2\theta - \frac{\Phi^2}{\sin^2\theta}}\,, \tag{4.4.12}$$

$$\dot{\phi} = \frac{\Phi}{(r^2 + a^2 - \alpha)\sin^2\theta}\,. \tag{4.4.13}$$

To simplify these equations for analysis, we will assume that we have a string lying

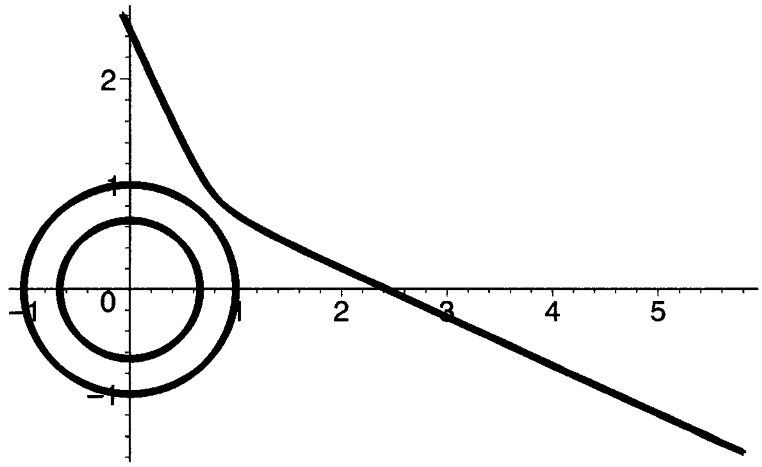

Figure 4.1: Plot of (4.4.16) using normalized units $\alpha = 1$ with $a = 0.75$ and $\Phi = 1$, displayed in 'cartesianised' coordinates $x = r\cos\phi$, $y = r\sin\phi$. Also plotted are the event horizon at $r = 1 - a^2$ and the infinite red-shift surface at $r = 1$.

in an equatorial configuration $\theta = \pi/2$. We have seen in the previous subsections that such a configuration demands $K = \Phi^2$, so we are left with

$$\dot{r} = \varsigma_x \sqrt{\frac{(r^2 - \alpha + a^2 - \Phi^2)}{r^2 - \alpha}}, \tag{4.4.14}$$

$$\dot{\phi} = \frac{\Phi}{r^2 - \alpha + a^2}. \tag{4.4.15}$$

Meaning that the r-ϕ relation for this configuration must satisfy

$$\frac{d\phi}{dr} = \varsigma_x \frac{\Phi}{r^2 - \alpha + a^2} \sqrt{\frac{r^2 - \alpha}{r^2 - \alpha + a^2 - \Phi^2}}. \tag{4.4.16}$$

We see that we have $dr/d\phi = 0$ at $r^2 = \alpha - a^2 + \Phi^2$, and at $r^2 = \alpha - a^2$. The first one of these is a turning point where the string has minimal distance to the black hole, while the second one occurs at the event horizon on the black hole, this is not a radial turning point but an occurrence of infinite winding of the string at the event horizon, related with the failure of Boyer-Lindquist coordinates at the event horizon. This equation also has a singular point at $r^2 = \alpha$, where the string would cross the infinite red-shift surface. If we have a string configuration with $\Phi^2 > a^2$, then the string encounters its turning point at $r^2 = \alpha + \Phi^2 - a^2$ outside of $r^2 = \alpha$

and the infinite red-shift surface is never encountered. In the case $\Phi^2 < a^2$, then for the region between $r^2 = \alpha$ and $r^2 = \alpha - a^2 + \Phi^2$, the right side of equation (4.4.16) becomes complex, and we will see later that such solutions give unphysical results. Finally, in the special case $\Phi^2 = a^2$, (4.4.16) has no singularities anywhere outside of the event horizon.

To further examine properties of these string configurations, let us consider the geometry of the string worldsheet of these cosmic string solutions. Using the conditions $b = 0$, $\theta = \pi/2$, ψ =const, and (4.4.16), we can find the induced metric on the string worldsheet to be

$$ds^2 = \left[\frac{r^2\iota^2 - \alpha\Phi^2 a^2}{\iota^2(\iota - \Phi^2)}\right] dr^2 - dt^2 + \frac{\alpha}{r^2}\left[dt + \frac{a\Phi}{\iota}\sqrt{\frac{r^2-\alpha}{\iota - \Phi^2}}dr\right]^2 , \tag{4.4.17}$$

where we have defined $\iota = r^2 - \alpha + a^2$ for brevity. The determinant of this metric is found to be

$$g = -\frac{r^2 - \alpha}{(r^2 + a^2 - \alpha - \Phi^2)} , \tag{4.4.18}$$

and the curvature scalar for this string worldsheet metric is

$$R = \frac{2\alpha}{r^4(r^2-\alpha)^2}\left[3(r^2-\alpha)^2 + (4r^2 - 3\alpha)(a^2 - \Phi^2)\right] . \tag{4.4.19}$$

We can see that this metric can have curvature singularities only at $r^2 = \alpha$ and $r = 0$.

Examining our three cases from before, we saw that for the case $\Phi^2 > a^2$, the string has its turning point outside of $r^2 = \alpha$, so this string is regular everywhere. The case of $\Phi^2 < a^2$ has its turning point located at a value of r^2 less than α, so this string worldsheet has a singularity at $r^2 = \alpha$. Also we note that the metric determinant given by (4.4.18) changes sign at this singularity, indicating that the metric has changed from having one spacelike and one timelike dimension to being totally spacelike. We can say that such a totally spacelike string is an unphysical solution. The final special case $\Phi^2 = a^2$ has a metric determinant of $g = -1$ and a curvature given simply as $R = 6\alpha/r^4$. This metric has a singularity only at $r = 0$, the black hole singularity.

The uniqueness of the $a^2 = \Phi^2$ solution, being the only solution to cross the infinite red-shift surface without singularity, is a general feature. It is demonstrated in appendix B that the only minimal surface to cross the infinite red-shift surface and remain regular there is a special class of solutions called principal Killing strings.

4.5 Principal Killing Surfaces

4.5.1 Principal Null Congruences

In the Kerr metric, it has been shown that the only stationary string that can cross the infinite red-shift surface into the ergosphere and remain regular is a string which has a worldsheet generated by the principal null vector and the timelike Killing vector [57]. Such a surface is called a principal Killing surface. We will begin this section with a discussion of the principal null vectors in a five dimensional Myers-Perry spacetime.

The *principal null vectors* are defined as solutions of the equation

$$l_{\pm[\alpha}C_{\beta]\gamma\delta\epsilon}l^{\gamma}_{\pm}l^{\delta}_{\pm} = 0\,, \tag{4.5.1}$$

with $C_{\beta\gamma\delta\epsilon}$ is the Weyl tensor. In the five dimensional Myers-Perry metric, $l_{\pm}$ takes the form [7, 17]

$$l^{\mu}_{\pm}\partial_{\mu} = \frac{(x+a^2)(x+b^2)}{\Delta}\left[\partial_t - \frac{a}{x+a^2}\partial_\phi - \frac{b}{x+b^2}\partial_\psi\right] \pm 2\sqrt{x}\,\partial_x\,. \tag{4.5.2}$$

These vectors obey the equation $l^{\mu}_{\pm}l^{\nu}_{\pm\,;\mu} = 0$ meaning that the integral lines of the principal null vectors are geodesic in our metric. By analogy with similar congruences in the four dimensional Kerr geometry, we call the congruences generated by $l^{\mu}_{\pm}$ principal null congruences. It is interesting to note that in the four dimensional case the principal null congruence was shear-free, guaranteed by the Goldberg-Sachs theorem (demonstrated in [58]), however, it has been found that the congruence generated by these vectors has a non-vanishing shear in five dimensions [17].

It is possible to define a convenient basis by accompanying the two null vectors

l_+ and l_- by the vectors m, $\bar{m}$ and k defined as

$$m^\mu \partial_\mu = \frac{1}{\rho\sqrt{2}} \left\{ \partial_\theta + \frac{i \sin\theta \cos\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \left[(b^2 - a^2)\partial_t + \frac{a}{\sin^2\theta}\partial_\phi - \frac{b}{\cos^2\theta}\partial_\psi \right] \right\} , \tag{4.5.3}$$

$$k^\mu \partial_\mu = \frac{1}{\sqrt{x}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \left(ab\partial_t - b\partial_\phi - a\partial_\psi \right) , \tag{4.5.4}$$

and $\bar{m}$ denotes complex conjugation of m. The primary motivation for choosing vectors like this is that $\xi_{\mu\,;\nu}$ takes the form

$$\xi_{\mu\,;\nu} = -\frac{\Delta F_{,x}}{\rho^2\sqrt{x}} l_{+[\mu} l_{-\nu]} - \frac{2i\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}(1-F)}{\rho^2} m_{[\mu}\bar{m}_{\nu]} , \tag{4.5.5}$$

with $F = -\xi^2$ the red-shift factor as before. Further, the vectors obey the conditions

$$\begin{gathered} (m\cdot m) = (\bar{m}\cdot\bar{m}) = 0\,, \quad (m\cdot\bar{m}) = 1\,, \quad (l_\pm\cdot m) = 0\,, \\ (k\cdot k) = 1\,, \quad (k\cdot m) = (k\cdot\bar{m}) = (k\cdot l_\pm) = 0\,, \\ (l_+\cdot l_-) = -2x\rho^2/\Delta\,, \quad (l_\pm\cdot\xi) = -1\,. \end{gathered} \tag{4.5.6}$$

We also have

$$\xi_{\mu\,;\rho} l^\rho_\pm = \pm\sqrt{x} F_{,x} l_{\pm\mu} , \tag{4.5.7}$$

one can confirm this either by direct calculation, or by comparison with equation (4.5.5). Due to the property of being an eigenvector of the first derivative of the principal Killing vector, we call $l_\pm$ *Killing null vectors* of the five dimensional Myers-Perry metric. We can see that in the five dimensional case, the Killing null vectors and the principal null vectors are equivalent.

4.5.2 Principal Killing Surfaces

Next we will examine stationary surfaces generated by the Killing null vector and the principal Killing vector ξ. Our goal is to demonstrate that such surfaces are stationary solutions to the Nambu-Goto equations. For definiteness, we will use the ingoing null vector, which is linearly independent from ξ at the future event horizon and denote it $l = l_-$. The time symmetry implies that

$$\mathcal{L}_\xi l = [\xi, l] = 0\,, \tag{4.5.8}$$

where $\mathcal{L}$ is a Lie derivative. In general the Frobenius theorem states that given two vectors u and v which satisfy the condition

$$[u, v] = f(x^\mu)\, u + g(x^\mu)\, v\,, \tag{4.5.9}$$

with f and g being any functions of spacetime, then there will exist a surface $x^\mu(\zeta)$ which has tangent vectors u and v (for example see [59]). Further in the special case $f = g = 0$, there is a choice of embedding coordinates $\zeta^A (A = 0, 1)$, such that we have $x^\mu{}_{,0} = u^\mu$ and $x^\mu{}_{,1} = v^\mu$. Thus (4.5.8) gives us that there exists a two dimensional surface Σ given by $x^\mu = x^\mu(\zeta)$, $(\zeta^A = (w, \lambda))$ such that $x^\mu{}_{,w} = \xi^\mu$, $x^\mu{}_{,\lambda} = -l^\mu$. We call such a surface, generated by l and ξ, a *principal Killing surface.* The coordinate w is the same as the time as defined by the Killing vector ξ, and λ is an affine parameter along the principal null geodesic.

The metric on Σ is of the form

$$d\gamma^2 = G_{AB} d\zeta^A d\zeta^B = \xi^2 dw^2 - 2(\xi \cdot l) dw d\lambda + l^2 d\lambda^2\,. \tag{4.5.10}$$

Using the metric, one may calculate

$$\xi^2 = -F = \frac{\alpha}{\rho^2} - 1\,, \quad (\xi \cdot l) = -1\,, \quad l^2 = 0\,, \tag{4.5.11}$$

so that the metric and inverse metric induced on Σ are given by

$$G_{AB} d\zeta^A d\zeta^B = -F dw^2 + 2 dw d\lambda\,, \qquad G^{AB} \partial_A \partial_B = 2\partial_w \partial_\lambda + F \partial_\lambda^2\,, \tag{4.5.12}$$

with $F = 1 - \alpha/\rho^2$. We introduce vectors normal to the principal Killing surface, denoted as n^μ_R $(R = 2, 3, 4)$. These vectors obey the conditions

$$g_{\mu\nu} n^\mu_R n^\nu_S = \delta_{RS}\,, \qquad g_{\mu\nu} x^\mu{}_{,A} n^\nu_R = 0\,, \tag{4.5.13}$$

and these five vectors are complete, in the sense that

$$g^{\mu\nu} = G^{AB} x^{\mu}{}_{,A} x^{\nu}{}_{,B} + \delta^{RS} n^{\mu}_{R} n^{\nu}_{S} \,. \tag{4.5.14}$$

The principal Killing surface is minimal. Using the equations of embedding in coordinate form it is possible to directly verify that it satisfies the Nambu-Goto equations. We will prove that it is minimal by making calculations in a covariant form. This will be useful in establishing the uniqueness theorem in appendix B. First we write the Nambu-Goto equations (3.2.16) as

$$G^{AB}\, x^{\mu}{}_{,A\,,B} - G^{AB} \gamma^{C}{}_{AB}\, x^{\mu}{}_{,C} + G^{AB} \Gamma^{\mu}{}_{\nu\sigma}\, x^{\nu}{}_{,A} x^{\sigma}{}_{,B} = 0 \,, \tag{4.5.15}$$

where $\gamma^{C}{}_{AB}$ is the connection associated with the metric G_{AB}. Contracting this equation with $n_{R\mu}$ and using the fact that $n_R \cdot x_{,C} = 0$ (4.5.13), we have

$$G^{AB} n_{R\mu} x^{\mu}{}_{,B\,,A} + G^{AB} n_{R\mu} x^{\nu}{}_{,A} \Gamma^{\mu}{}_{\nu\sigma}\, x^{\sigma}{}_{,B} = 0 \,. \tag{4.5.16}$$

Next, we may replace $x^{\mu}{}_{,B\,,A}$ with the directional derivative form $x^{\nu}{}_{,A} \partial_{\nu} x^{\mu}{}_{,B}$, we get

$$\begin{aligned} G^{AB} n_{R\mu} (x^{\nu}{}_{,A} \partial_{\nu} x^{\mu}{}_{,B} + x^{\nu}{}_{,A} \Gamma^{\mu}{}_{\nu\sigma}\, x^{\sigma}{}_{,B}) &= 0 \,, \\ G^{AB} n_{R\mu} x^{\nu}{}_{,A} (x^{\mu}{}_{,B})_{;\nu} &= 0 \,, \\ G^{AB} \Omega_{RAB} &= 0 \,, \end{aligned} \tag{4.5.17}$$

where we have used the second fundamental form, defined on Σ as

$$\Omega_{RAB} = g_{\mu\rho} n^{\rho}_{R} x^{\nu}{}_{,A} (x^{\mu}{}_{,B}\,)_{;\nu} \,. \tag{4.5.18}$$

Thus, Σ is a minimal surface when the trace of the second fundamental form $\Omega_R \equiv G^{AB} \Omega_{RAB}$ vanishes. We will define the vector z as

$$z^{\mu} = G^{AB} x^{\rho}_{,A}\, (x^{\mu}{}_{,B}\,)\,_{;\rho} \,, \tag{4.5.19}$$

Then we may express Ω_R as

$$\Omega_R = G^{AB}\Omega_{RAB} = (n_R \cdot z)\,. \tag{4.5.20}$$

Using the indiced metric, z has the form

$$z^\mu = -\xi^\rho l^\mu{}_{;\rho} - l^\rho \xi^\mu{}_{;\rho} + F l^\rho l^\mu{}_{;\rho}\,. \tag{4.5.21}$$

The last term in this equation vanishes because the principal null congruences are geodesic, while we may use the relation $[\xi, l] = 0$ to get

$$z^\mu = -2\xi^\mu{}_{;\rho} l^\rho\,. \tag{4.5.22}$$

The fact that l is an eigenvalue of $\xi_{\mu;\nu}$ (4.5.7) allows us to express z as

$$z^\mu = 2\sqrt{x} F_{,x} l^\mu\,. \tag{4.5.23}$$

Since we have defined the vectors n to be orthogonal to l, this means that $z \cdot n_R = 0$ and thus $\Omega_R = 0$. Thus, we have shown that the trace of the fundamental form vanishes, therefore Σ is a minimal surface. In appendix A we give the explicit solution to the principal Killing surface. In appendix B, we show that this Σ is the only stationary surface to cross the infinite red-shift surface without singularity.

4.5.3 The Principal Killing String as a Two Dimensional Black Hole

Up until now we have been working in Boyer-Lindquist coordinates, however, for the purpose of analyzing principal Killing string surfaces we will now transform into ingoing Eddington-Finkelstein coordinates, which are regular at the future event horizon. The coordinate transformation is given by

$$dv = dt + \frac{(x+a^2)(x+b^2)}{2\Delta\sqrt{x}}dx\,,$$

$$d\tilde{\phi} = d\phi - \frac{(x+b^2)a}{2\Delta\sqrt{x}}dx\,, \tag{4.5.24}$$

$$d\tilde{\psi} = d\psi - \frac{(x+a^2)b}{2\Delta\sqrt{x}}dx \,.$$

In these coordinates, our metric takes the form

$$ds^2 = -dv^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\tilde{\phi}^2 + (r^2+b^2)\cos^2\theta d\tilde{\psi}^2$$

$$+\frac{\alpha}{\rho^2}\left(dv + a\sin^2\theta d\tilde{\phi} + b\cos^2\theta d\tilde{\psi}\right)^2 + 2dr\left(dv + a\sin^2\theta d\tilde{\phi} + b\cos^2\theta d\tilde{\psi}\right) \,. \quad (4.5.25)$$

Note that we have reintroduced the radial coordinate r instead of $x = r^2$. Another convenience of choosing these coordinates is that the ingoing principal null vector $l = l_-$ now takes the simple form $l^\mu \partial_\mu = -\partial_r$. The original Killing vectors ∂_t, ∂_ϕ and ∂_ψ take the form ∂_v, $\partial_{\tilde{\phi}}$ and $\partial_{\tilde{\psi}}$, respectively.

The principal Killing string, constructed from the vectors ξ and l has a simple form in these coordinates

$$\theta = \theta_0 \,, \quad \tilde{\phi} = \tilde{\phi}_0 \,, \quad \tilde{\psi} = \tilde{\psi}_0 \,, \qquad (4.5.26)$$

all as constants. We use coordinates of $\zeta^0 = v$ and $\zeta^1 = r$ as coordinates on Σ. The induced metric in these coordinates is

$$d\gamma^2 = -Fdv^2 + 2drdv \,, \quad F = 1 - \frac{\alpha}{r^2 + a^2\cos^2\theta_0 + b^2\sin^2\theta_0} \,. \qquad (4.5.27)$$

This is a metric of a two dimensional black hole with an event horizon located at

$$r^2 + a^2\cos^2\theta_0 + b^2\sin^2\theta_0 = \alpha \,. \qquad (4.5.28)$$

The surface gravity for this two dimensional black hole is $\kappa_{(2)} = \frac{1}{2}F_{,r}$ evaluated at the horizon $F = 0$. We get

$$\kappa_{(2)} = \frac{\sqrt{\alpha - a^2\cos^2\theta_0 - b^2\sin^2\theta_0}}{\alpha} \,. \qquad (4.5.29)$$

For comparison, we restate the five dimensional surface gravity $\kappa_{(5)}$ from (4.2.8) in an explicit form

$$\kappa_{(5)} = \sqrt{2}\frac{\sqrt{C^2 - 4a^2b^2}}{\alpha\sqrt{C + \sqrt{C^2 - 4a^2b^2}}} \,, \qquad (4.5.30)$$

where

$$C = \alpha - a^2 - b^2 \,. \tag{4.5.31}$$

Next we will demonstrate that the surface gravity of the two dimensional string hole $\kappa_{(2)}$ is always at least as large as the surface gravity $\kappa_{(5)}$ of the bulk black hole. Begin by defining

$$f(r) = \frac{2r^2 - C}{\alpha r} \,, \tag{4.5.32}$$

and since $C > 0$ is a necessary condition for the existence of an event horizon, f is a monotonically increasing function. f is defined such that

$$\kappa_{(5)} = f(r_+) \,, \tag{4.5.33}$$

where r_+ is the location of the event horizon, given as

$$r_+^2 = \frac{1}{2}(C + \sqrt{C^2 - 4a^2b^2}) \,. \tag{4.5.34}$$

Since $r_+ \leq C^{1/2}$ one has

$$\kappa_{(5)} = f(r_+) \leq f(C^{1/2}) = \frac{C^{1/2}}{\alpha} \leq \kappa_{(2)} \,. \tag{4.5.35}$$

This relation means that the surface gravity of the two dimensional string black hole is always at least as large as the surface gravity of the five dimensional black hole. The equality of $\kappa_{(2)}$ with $\kappa_{(5)}$ only occurs when

$$a^2 \sin^2\theta_0 + b^2 \cos^2\theta_0 = 0 \,. \tag{4.5.36}$$

This means that for $\kappa_{(2)} = \kappa_{(5)}$, we must have either $a = b = 0$ so that the black hole is non-rotating, or one of the rotation parameters, say b, vanishes and the string is in the $\theta = 0$ plane, orthogonal to the plane with a non-zero rotation parameter. Finally, we note that for

$$a^2 \cos^2\theta_0 + b^2 \sin^2\theta_0 > 0 \,, \tag{4.5.37}$$

the metric of the two dimensional black hole remains regular at $r = 0$.

Chapter 5

Results in Higher Dimensions

5.1 Principal Killing Strings in Higher Dimensional Myers-Perry Metrics

5.1.1 Myers-Perry Metrics in Ingoing Coordinates

Some of the results we have found can be generalized to Myers-Perry metrics in an arbitrary number of dimensions. We will restate the metrics for higher dimensional Myers-Perry metrics here for convenience. For an even number of spatial dimensions (N) we have $p = N/2$ planes of rotation and a metric

$$\begin{aligned} ds^2 = &- dt^2 + \sum_{i=1}^{p}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2\, d\phi_i^2) + \frac{\Pi L}{\Pi - \alpha r^2} dr^2 \\ &+ \frac{\alpha r^2}{\Pi L}\left(dt + \sum_{i=1}^{p} a_i\, \mu_i^2\, d\phi_i\right)^2 , \end{aligned} \tag{5.1.1}$$

and for an odd number of spatial dimensions there are $p = (N-1)/2$ separate planes and a metric

$$\begin{aligned} ds^2 = &- dt^2 + r^2 d\mu^2 + \sum_{i=1}^{p}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2\, d\phi_i^2) + \frac{\Pi L}{\Pi - \alpha r} dr^2 \\ &+ \frac{\alpha r}{\Pi L}\left(dt + \sum_{i=1}^{p} a_i\, \mu_i^2\, d\phi_i\right)^2 , \end{aligned} \tag{5.1.2}$$

where Π and L given by

$$\Pi = \prod_{i=1}^{p}(r^2 + a_i^2)\,, \quad L = 1 - \sum_{i=1}^{p} \frac{a_i^2\,\mu_i^2}{r^2 + a_i^2}\,. \tag{5.1.3}$$

These metrics have a principal Killing vector $\xi^\mu\partial_\mu = \partial_t$ as well as axial Killing vectors $\xi_i^\mu\partial_\mu = \partial_{\phi_i}$.

Let us define $l_\pm$ in these metrics as

$$l_\pm^\mu\partial_\mu = \Xi\left(\partial_t - \sum_{i=1}^{p}\frac{a_i}{r^2 + a_i^2}\partial_{\phi_i}\right) \pm \partial_r\,. \tag{5.1.4}$$

where

$$\Xi = \frac{\Pi}{\Pi - \alpha r}\,, \qquad \text{for odd } N,$$

$$\Xi = \frac{\Pi}{\Pi - \alpha r^2}\,, \qquad \text{for even } N. \tag{5.1.5}$$

We will demonstrate that $l_\pm$ are the Killing null vectors for the Myers-Perry metrics and that the integral lines of $l_\pm$ are geodesic. To demonstrate this we will introduce ingoing $(-)$ and outgoing $(+)$ Eddington-Finkelstein coordinates as

$$dv_\pm = dt \mp \Xi dr\,, \tag{5.1.6}$$

$$d\tilde{\phi}_{\pm i} = d\phi_i \pm \frac{\Xi a_i}{r^2 + a_i^2}dr\,. \tag{5.1.7}$$

One can use either the upper or the lower sign, but once chosen must be consistently used throughout the next sections. In these coordinates $l_\pm$ has components

$$l_\pm^\mu\partial_\mu = \pm\partial_r\,, \quad l_{\pm\mu}dx^\mu = -\left[dv_\pm + \sum_{i=1}^{p}\mu_i^2\,a_i\,d\tilde{\phi}_{\pm i}\right]\,. \tag{5.1.8}$$

For odd N the metric takes the form

$$ds^2 = -dv_\pm^2 + \sum_{i=1}^{p}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\tilde{\phi}_{\pm i}^2) + \frac{\alpha r}{\Pi L}(l_{\pm\mu}dx^\mu)^2 \pm 2dr(l_{\pm\mu}dx^\mu) + r^2 d\mu^2\,. \tag{5.1.9}$$

For even N we have a similar metric, simply by removing the term $r^2 d\mu^2$ and replacing $\alpha r \to \alpha r^2$ we get the metric for even N. The principal Killing vector ξ

and the axial Killing vectors ξ_i are now given as $\xi^\mu\partial_\mu = \partial_{v_\pm}$, $\xi_i^\mu\partial_\mu = \partial_{\tilde{\phi}_{\pm i}}$

5.1.2 Properties of Killing Null Vectors in Higher Dimensions

Now we will demonstrate that the vectors given by (5.1.8) are geodesic. To show this we must examine

$$l_\pm^\mu l^\nu_{\pm\,;\mu} = l_\pm^\mu l^\nu_{\pm\,,\mu} + \Gamma^\nu{}_{\alpha\beta} l_\pm^\alpha l_\pm^\beta \,. \tag{5.1.10}$$

Using (5.1.8), we see that the only component of $l_\pm^\mu$ is given as $l_\pm^\mu = \pm\delta_r^\mu$, so we have

$$l_\pm^\mu l^\nu_{\pm\,;\mu} = \Gamma^\nu{}_{r\,r} = \frac{1}{2} g^{\nu\,\delta}(2g_{r\,\delta\,,r} - g_{r\,r\,,\delta}) \,. \tag{5.1.11}$$

By examining of the metric (5.1.9) we can see that $g_{r\,r} = 0$ and $g_{r\,\delta\,,r} = 0$. So we have that $l_\pm^\mu l^\nu_{\pm\,;\mu} = 0$, thus the integral lines of $l_\pm$ are geodesic.

Next we will demonstrate the $l_\pm$ are the Killing null vectors of the Myers-Perry metric. We begin by considering the product $\xi_{\mu\,;\nu} l_\pm^\nu$

$$\xi_{\mu\,;\nu} l_\pm^\nu = \xi_{\mu\,,\nu} l_\pm^\nu - \Gamma^\sigma{}_{\mu\,\nu}\xi_\sigma l_\pm^\nu \,. \tag{5.1.12}$$

In the Eddington-Finkelstein coordinates we have $l_\pm^\nu = \pm\delta_r^\nu$ and $\xi^\mu = \delta_v^\mu$, so this reduces to

$$\xi_{\mu\,;\nu} l_\pm^\nu = \pm g_{\mu\,v\,,r} \mp \frac{1}{2}(g_{r\,v\,,\mu} + g_{\mu\,v\,,r} - g_{r\,\mu\,,v}) \,. \tag{5.1.13}$$

The time symmetry guarantees that $g_{r\,\mu\,,v} = 0$, and examining the metric (5.1.9) gives us $g_{r\,v\,,\mu} = 0$. The remaining term reads

$$\xi_{\mu\,;\nu} l_\pm^\nu = \pm\frac{1}{2} g_{\mu\,v\,,r} \,. \tag{5.1.14}$$

Finally, we note that $g_{\mu\,v\,,r}$ has only one contribution from the metric, the term in the metric proportional to $(l_{\pm\mu} dx^\mu)^2$, and $l_{\pm\mu}$ is independent of r. Since $l_{\pm v} = -1$, we may write that

$$g_{\mu\,v\,,r} = -g_{v\,v\,,r} l_{\pm\mu} \,. \tag{5.1.15}$$

This gives us the result finishing the proof,

$$\xi_{\mu;\nu} l_{\pm}^{\nu} = \pm \frac{1}{2} \partial_r F \, l_{\pm\mu} \,, \tag{5.1.16}$$

where $F = -g_{v\,v} = -\xi^2$.

Note that the time symmetry of the system guarantees the relationship $[\xi, l_\pm] = 0$, so it is possible to consider Killing surfaces generated by the vectors ξ and $l_\pm$.

5.1.3 Principal Killing Strings

Consider a principal Killing surface, defined in a manner analogous to previously, as a surface spanned by the principal Killing vector ξ and the Killing null vector $l_\pm$ (we will consider strings generated by either of the Killing null vectors in the analysis here). The equation for the worldsheet configuration is given as

$$x^0 = v, \quad x^1 = r \,, \tag{5.1.17}$$

with all the angular coordinates $\tilde{\phi}_i$ and θ_i as constants. This means that our tangent vectors to the string are given as $x^{\mu}{}_{,v} = \xi^\mu$ and $x^{\mu}{}_{,r} = \pm l_{\pm}^{\mu}$, as expected. The Nambu-Goto equations (3.2.16), contracted with $g_{\mu\nu}$ for convenience, are

$$g_{\mu\nu} \Box x^\nu + G^{AB} g_{\mu\nu} \Gamma^{\nu}{}_{\alpha\beta} x^{\alpha}{}_{,A} x^{\beta}{}_{,B} = 0 \,, \tag{5.1.18}$$

with

$$\Box x^\nu = \frac{1}{\sqrt{-G}} \partial_A (\sqrt{-G} G^{AB} \partial_B x^\nu) \,. \tag{5.1.19}$$

The induced metric on the string worldsheet is given as

$$G^{AB} \partial_A \partial_B = F \partial_r^2 \mp 2 \partial_r \partial_v \,, \quad \sqrt{-G} = 1 \,. \tag{5.1.20}$$

The first term in this equation reads as

$$g_{\mu\nu} \Box x^\nu = g_{\mu\nu} \left[\partial_r (F \partial_r x^\nu) \mp 2 \partial_r \partial_v x^\nu \right] = \pm g_{\mu r} \partial_r F \,, \tag{5.1.21}$$

and the second term reads

$$G^{AB} g_{\mu\nu} \Gamma^{\nu}{}_{\alpha\beta} x^{\alpha}{}_{,A} x^{\beta}{}_{,B} = g_{\mu\nu} \Gamma^{\nu}{}_{\alpha\beta} \left[F x^{\alpha}_{,r} x^{\beta}_{,r} \mp 2 x^{\alpha}_{,r} x^{\beta}_{,v} \right] = g_{\mu\nu} \left(F \Gamma^{\nu}{}_{rr} - 2\Gamma^{\nu}{}_{rv} \right), \tag{5.1.22}$$

$$G^{AB} g_{\mu\nu} \Gamma^{\nu}{}_{\alpha\beta} x^{\alpha}{}_{,A} x^{\beta}{}_{,B} = \frac{1}{2} F \left(2\, g_{\mu r,r} - g_{rr,\mu} \right) - \left(g_{\mu r,v} + g_{\mu v,r} - g_{vr,\mu} \right) \tag{5.1.23}$$

The metric given as (5.1.9) implies that

$$g_{\mu r,r} = g_{rr} = g_{\mu r,v} = g_{vr,\mu} = 0\,. \tag{5.1.24}$$

So, we have

$$g_{\mu\nu} \Box x^{\nu} + G^{AB} g_{\mu\nu} \Gamma^{\nu}{}_{\alpha\beta} x^{\alpha}{}_{,A} x^{\beta}{}_{,B} = \pm g_{\mu r}\, \partial_r F - g_{\mu v,r}\,. \tag{5.1.25}$$

Equation (5.1.15) tells us the second term is equal to $-g_{vv,r}\, l_{\pm\mu} = \partial_r F\, l_{\pm\mu}$, while the form of $l_{\pm}$ tells us that $l_{\pm\mu} = \pm g_{\mu r}$. Thus we have that (5.1.25) vanishes and so this string worldsheet is a solution to the Nambu-Goto equations of motion.

5.1.4 Strings in Higher Dimensions as Two Dimensional Black Holes

In Eddington-Finkelstein coordinates, the worldsheet configuration of a principal Killing string is given as

$$\mu_i = \text{const}\,, \quad \tilde{\phi}_{\pm i} = \text{const}\,. \tag{5.1.26}$$

We will examine ingoing principal Killing strings, with $l = l_{-}$. Using coordinates $\zeta^0 = v$ and $\zeta^1 = r$ for coordinates on the principal Killing surface defined such that $x^{\mu}{}_{,v} = \xi^{\mu}$ and $x^{\mu}{}_{,r} = -l^{\mu}$, we have an induced surface metric of

$$d\gamma^2 = -F dv^2 + 2 dr dv\,. \tag{5.1.27}$$

This is a metric of a two dimensional black hole with an event horizon at $F = 0$, the infinite red-shift surface of the bulk black hole. For odd N one has

$$F = -\xi^2 = -g_{v\,v} = 1 - \frac{\alpha r}{\Pi L}\,. \tag{5.1.28}$$

So, in the case of odd N, the surface gravity for this two dimensional black hole is given as

$$\kappa_{(2)} = \frac{1}{2}\partial_r F\,\Big|_{F=0} = \frac{\partial_r(\Pi L)\alpha\, r - \alpha\,\Pi L}{2\Pi^2 L^2}\Bigg|_{\Pi L=\alpha r} = \frac{\partial_r(\Pi L) - \alpha}{2\alpha r}\Bigg|_{\Pi L=\alpha r}\,. \tag{5.1.29}$$

If we have an odd number of spatial dimensions, the surface gravity of the bulk n-dimensional black hole is given by (2.2.7)

$$\kappa_{(n)} = \frac{\partial_r \Pi - \alpha}{2\alpha r}\Bigg|_{\Pi=\alpha r}\,. \tag{5.1.30}$$

For even N, we instead have

$$F = 1 - \frac{\alpha r^2}{\Pi L}\,, \tag{5.1.31}$$

and the surface gravity for the two dimensional black hole is given as

$$\kappa_{(2)} = \frac{1}{2}\partial_r F\,\Big|_{F=0} = \frac{\partial_r(\Pi L)\alpha\, r^2 - 2\alpha r\,\Pi L}{2\Pi^2 L^2}\Bigg|_{\Pi L=\alpha r^2} = \frac{\partial_r(\Pi L) - 2\alpha r}{2\alpha r^2}\Bigg|_{\Pi L=\alpha r^2}\,. \tag{5.1.32}$$

In an even number of spatial dimensions, the bulk n-dimensional black hole has a surface gravity given as (2.2.7)

$$\kappa_{(n)} = \frac{\partial_r \Pi - 2\alpha r}{2\alpha r^2}\Bigg|_{\Pi=\alpha r^2}\,. \tag{5.1.33}$$

In both cases L is given by

$$L = 1 - \sum_{i=1}^{p} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}\,, \tag{5.1.34}$$

so we can see that in the case of either N even or odd, if the strings angular position satisfies the conditions

$$\mu_i^2 a_i^2 = 0\,, \tag{5.1.35}$$

for each i, then L is identically equal to one, and so $\kappa_{(2)} = \kappa_{(n)}$ for this string.

5.2 Friction Effect of a String with a Black Hole

The general analysis of [22] shows that there exists a friction effect between a black hole and a brane attached to it that may cause the black hole to lose some bulk components of its rotation. The analysis done there was for general branes and in the limit that the black hole has slow rotation. In this section we will analyze the friction effect between a string and a higher dimensional black hole with arbitrary rotation. We will do the calculations for one string segment attached to the black hole, but for the final result we will assume that we have two such string segments, the second one being an inverse image of the first one ($\tilde{\phi} \to \tilde{\phi} + \pi$ and additionally $\mu \to -\mu$ if N is odd). We assume such a configuration to guarantee that the black hole will remain at rest during the process.

In this section, we use the angular coordinates mentioned earlier in chapter two. We will replace the dependent coordinates μ_i and μ with the independent coordinates θ_i, and let ω_m represent the total set of angular coordinates $(\theta_i, \tilde{\phi}_j)$.

Suppose there exists a distribution of matter outside the black hole, with a stress-energy tensor $T_{\mu\nu}$. The fluxes of energy and angular momenta of this matter through a surface r =const are given as

$$\Delta E = -\int T_\mu{}^\nu \, \xi^\mu d\varpi_\nu\,, \quad \Delta J_i = \int T_\mu{}^\nu \, \xi_i^\mu d\varpi_\nu\,, \tag{5.2.1}$$

where ξ and ξ_i are the Killing vectors of time translation and rotational symmetry. The infinitesimal element of surface area $d\varpi_\mu$ is

$$d\varpi_\mu = r_{,\mu}\sqrt{-g}\, dv\, d\omega^{N-1}\,, \tag{5.2.2}$$

with

$$d\omega^{N-1} = \prod_{i=2}^{N} d\omega_i\,. \tag{5.2.3}$$

For a stationary configuration the rate of energy and angular momentum fluxes through the matter distribution will be constant, and given by

$$\dot{E} \equiv \frac{dE}{dv} = -\int d\omega^{N-1}\,\sqrt{-g}\,T_{\mu}{}^{\nu}\,\xi^{\mu} r_{,\nu}\,, \tag{5.2.4}$$

$$\dot{J}_i \equiv \frac{dJ_i}{dv} = \int d\omega^{N-1}\,\sqrt{-g}\,T_{\mu}{}^{\nu}\,\xi_i^{\mu} r_{,\nu}\,. \tag{5.2.5}$$

We will assume that we have a situation with an ingoing principal Killing string captured in the black hole event horizon and calculate the energy and angular momenta transfer through the string. The stress-energy tensor of a string with a configuration given by $x^{\mu} = x^{\mu}(\zeta^A)$ is given by equation (3.2.14)

$$\sqrt{-g}T^{\mu\nu} = -\mu^* \int d^2\zeta \delta^{(N+1)}(x - x(\zeta))\, t^{\mu\nu}\,,$$

$$t^{\mu\nu} = \sqrt{-G}\, G^{AB} x^{\mu}{}_{,A} x^{\nu}{}_{,B}\,, \tag{5.2.6}$$

with μ^* being the tension of the string. Using the ingoing principal Killing string generated by ξ and $l = l_-$, we have

$$t^{\mu\nu} = F\, l^{\mu} l^{\nu} - l^{\mu}\xi^{\nu} - \xi^{\mu} l^{\nu}\,. \tag{5.2.7}$$

We have in particular

$$t_{\mu}{}^{\nu}\xi^{\mu} = F l^{\nu}(\xi \cdot l) - \xi^{\nu}(\xi \cdot l) - l^{\nu}(\xi^2)\,,$$

$$t_{\mu}{}^{\nu}\xi_i^{\mu} = F l^{\nu}(\xi_i \cdot l) - \xi^{\nu}(\xi_i \cdot l) - l^{\nu}(\xi \cdot \xi_i)\,. \tag{5.2.8}$$

We also have

$$\xi \cdot l = -1\,, \qquad \xi^2 = -F\,, \qquad \xi_i \cdot l = -\mu_i^2 a_i\,, \qquad \xi \cdot \xi_i = (1 - F)\mu_i^2 a_i\,, \tag{5.2.9}$$

which may be used to find that

$$t_{\mu}{}^{\nu}\xi^{\mu} = \xi^{\nu}\,, \qquad t_{\mu}{}^{\nu}\xi_i^{\mu} = a_i \mu_i^2 (\xi^{\nu} - l^{\nu})\,. \tag{5.2.10}$$

Using these relationships for our original integrals (5.2.4-5.2.5), we get

$$\dot{E} = 0\,, \qquad \dot{J}_i = -\mu^* a_i \mu_i^2\,. \tag{5.2.11}$$

We note that the flux is independent of r, agreeing with conservation of energy and angular momentum. The loss of angular momentum by the black hole is the negative of the flux of angular momentum through the string,

$$\dot{J}_i^{BH} = -\dot{J}_i\,. \tag{5.2.12}$$

Using our original definition of the spin parameters a_i (2.2.1), and taking into account the previous assumption that there would be two string segments attached to the black hole, we get the relationship for the loss of angular momentum from the black hole;

$$\dot{J}_i^{BH} = -\mu_i^2 (N-1)\frac{\mu^*}{M} J_i^{BH}\,. \tag{5.2.13}$$

This result agrees with the special four dimensional case of the Kerr black hole obtained in [22].

Note that this is a first-order effect on the rotation parameters and mass of the black hole. After the rotation parameter has changed by an apprecable amount, the shape of the string will be different due to the modified rotation parameters. For small $\mu^* r_+/M$ this process will be slow. One can describe it as quasistationary, that is as a slow change from one stationary configuration into another. Dynamics of this process can be considered as an evolution in the space of rotation parameters. The friction effect between the string and the black hole will continue until a stationary configuration is reached with $\dot{J}_i^{BH} = 0$ for each plane of rotation i. (5.2.13) shows that for $\dot{J}_i^{BH} = 0$, we must have either $J_i^{BH} = 0$, or $\mu_i = 0$, meaning that there will be a slowing of the black holes rotation parameters until the strings angular position is 'orthogonal' to any planes of black hole rotation which have a non-zero rotation parameter. That is, the final equilibrium state must be one which has $a_i^2 \mu_i^2 = 0$ for each i. Comparing to (5.1.35) we see that the final equilibrium state will be one which balances the surface gravity of the n-dimensional black hole with the surface gravity of the two dimensional string black hole. Thus, we may interpret this as the friction effect tends to a thermal equilibrium of the black hole-string system.

To estimate the timescale of this interaction we will consider a special case which allows a complete analytical solution. Consider that we have a black hole with only one nonzero rotation parameter a and the cosmic string lies in the plane corresponding to this parameter, that is $\mu_i = 1$ for that i and $\mu_i = 0$ for all other planes. This is analogous to the string lying in the equatorial plane in the Kerr case. The symmetry of the problem guarantees that μ_i will remain constant so we do not have to consider dynamical changing of the strings angular position. Assuming that μ^* is small compared to M/r_+ so that we may assume the process is quasistationary and adiabatic, the surface area $\mathcal{A}$ of the black hole will be constant. The surface area is given from (2.2.8) as

$$\mathcal{A} = A_{N-1} r_+ \alpha = \frac{16\pi G^{(N+1)}}{N-1} M r_+ \,, \tag{5.2.14}$$

with the black hole event horizon r_+ given for a black hole with one nonvanishing spin parameter by (2.2.10) or (2.2.14)

$$\alpha = r_+^{N-4}(r_+^2 + a^2)\,. \tag{5.2.15}$$

Using the dynamical equation for J (5.2.13), the fact that area is constant (5.2.14), and the definition for the spin parameter (2.2.1), we can get the equation for the rate at which the horizon radius changes

$$\dot{r}_+ = \frac{32\pi G^{(N+1)} \mu^*}{(N-1)A_{N-1}} \frac{a^2}{r_+^{N-5}(r_+^2 + a^2)^2}\,. \tag{5.2.16}$$

We can see that r_+ will increase while J and a decrease, until the process stops when J and a vanish. Constant black hole area and increasing r_+ also implies that the black hole mass M will decrease. We will define the dimensionless parameter $\beta \equiv a^2/r_+^2$, we may write (5.2.15) with (5.2.14) as

$$\beta = \frac{\mathcal{A}}{A_{N-1} r_+^{N-1}} - 1\,. \tag{5.2.17}$$

We can see that r_+ will increase monotically until reaching the final value of

$$r_f = \left(\frac{\mathcal{A}}{A_{N-1}}\right)^{1/(N-1)}, \tag{5.2.18}$$

at which point $\beta = 0$ and the process stops. At this point, the mass of the black hole will reach its final value

$$M_f = \frac{\mathcal{A}(N-1)}{16\pi G^{(N+1)} r_f}. \tag{5.2.19}$$

Differentiating (5.2.17) and using (5.2.16), we can find that β evolves as

$$\dot{\beta} = -\frac{2(N-1)\mu^*}{M_f}\frac{\beta}{(1+\beta)^{1/(N-1)}}. \tag{5.2.20}$$

We can solve this equation analytically, the solution is given as

$$\ln\beta + \frac{\beta F\left(\left[1,1,\frac{N-2}{N-1}\right],[2,2],-\beta\right)}{N-1} = -\frac{2(N-1)\mu^*}{M_f}t + C, \tag{5.2.21}$$

with C as an integration constant. The hypergeometric function is for our purposes given as

$$F\left(\left[1,1,\frac{N-2}{N-1}\right],[2,2],-\beta\right) = \frac{1}{(N-1)\Gamma(\frac{N-2}{N-1})}\sum_{k=0}^{\infty}\frac{(-\beta)^k\Gamma(\frac{N-2}{N-1}+k)}{(k+1)!(k+1)}. \tag{5.2.22}$$

In the limit of $t \to \infty$, we have $\beta \to 0$ and so the logarithm is the leading term on the left. Therefore, asymptotically we have

$$\beta \sim a^2 \sim \exp(-2t/T), \tag{5.2.23}$$

with $T = M_f/((N-1)\mu^*)$ as the characteristic time. We can see that the timescale T is the same as what one would expect from simply examining equation (5.2.13).

5.3 Separability of String Equations in Gibbons-Lü-Page-Pope Spacetimes

A recent paper [24] demonstrated that the Gibbons-Lü-Page-Pope metrics mentioned earlier have a separation of variables in the even-spatial-dimension case with all rotation parameters equal, $a_i = a$. For this special case the Hamilton-Jacobi equations for a particle in free-fall and the massive Klein-Gordon equations undergo a separation of variables (we state the results in appendix D). Here we will demonstrate that the Hamilton-Jacobi equations for a stationary string also have a similar separation in that special case.

Since separation has only been demonstrated for a Gibbons-Lü-Page-Pope spacetime with an even number of spatial dimensions, we will restate only that metric here

$$ds^2 = -W(1-\Lambda r^2)dt^2 + \frac{\Pi L}{\Pi(1-\Lambda r^2) - \alpha r^2}dr^2 + \frac{\alpha r^2}{\Pi L}\left(dt - \sum_{i=1}^{p}\frac{a_i\mu_i^2}{1+\Lambda a_i^2}d\phi_i\right)^2$$

$$+\sum_{i=1}^{p}\frac{r^2+a_i^2}{1+\Lambda a_i^2}[d\mu_i^2+\mu_i^2(d\phi_i-\Lambda a_i dt)^2]+\frac{\Lambda}{W(1-\Lambda r^2)}\left(\sum_{i=1}^{p}\frac{r^2+a_i^2}{1+\Lambda a_i^2}\mu_i d\mu_i\right)^2, \quad (5.3.1)$$

where, as before,

$$\Pi = \prod_{i=1}^{p}(r^2+a_i^2), \qquad L = 1-\sum_{i=1}^{p}\frac{a_i^2\mu_i^2}{r^2+a_i^2}, \qquad W = \sum_{i=1}^{p}\frac{\mu_i^2}{1+\Lambda a_i^2}. \qquad (5.3.2)$$

One may examine the metric to see some difficulties that may occur in an attempt to separate the Hamilton-Jacobi action for a string. First of all, the coefficient W appears in some inconvenient places from a separation perspective. In the $\Lambda = 0$ case, $W = 1$, so this was not a problem. The other major difficulty is that there is a new term at the end of expression (5.3.1) which vanishes in the $\Lambda = 0$ case. However, if we assume that all the spin parameters of the black hole are the same $a_i = a$, then the final term simply becomes proportional to $\sum_{i=1}^{p}\mu_i d\mu_i$. If we differentiate the restriction $\sum_{i=1}^{p}\mu_i^2 = 1$ then we get $\sum_{i=1}^{p}\mu_i d\mu_i = 0$. So we may see that the metric simplifies to

$$ds^2 = -\frac{1-\Lambda r^2}{1+\Lambda a^2}dt^2 + \frac{\Pi r^2}{(r^2+a^2)(\Pi(1-\Lambda r^2)-\alpha r^2)}dr^2$$

$$+\frac{\alpha(r^2+a^2)}{\Pi}\left(dt - \frac{a}{1+\Lambda a^2}\sum_{i=1}^{p}\mu_i^2 d\phi_i\right)^2 + \frac{r^2+a^2}{1+\Lambda a^2}\sum_{i=1}^{p}[d\mu_i^2 + \mu_i^2(d\phi_i - \Lambda a dt)^2]\,. \tag{5.3.3}$$

In order to get the equations for stationary string configurations, we first need the inverse metric $h^{ab} = H^{ab}/F$. For the metric (5.3.3), we have

$$F = -g_{tt} = \frac{1-\Lambda r^2 - (r^2+a^2)\Lambda^2 a^2}{1+\Lambda a^2} - \frac{\alpha(r^2+a^2)}{\Pi}\,, \tag{5.3.4}$$

and equation (4.1.7) gives us that to find the components H^{ab}, we need only find the inverse metric $g^{\mu\nu}$. The components of the inverse metric that we need are given as

$$g^{rr} = \frac{(\Pi(1-\Lambda r^2)-\alpha r^2)(r^2+a^2)}{\Pi r^2}\,,$$

$$g^{\phi_i\phi_j} = \frac{1+\Lambda a^2}{(r^2+a^2)\mu_i^2}\delta^{ij} + Q\,, \qquad g^{\theta_i\theta_j} = \frac{1+\Lambda a^2}{(r^2+a^2)\prod_{k=1}^{i-1}\sin^2\theta_k}\delta^{ij}\,, \tag{5.3.5}$$

where the μ_i are defined in terms of the θ_i as before (2.2.5)

$$\mu_i = \cos\theta_{p-i+1}\prod_{j=1}^{p-i}\sin\theta_j\,, \tag{5.3.6}$$

keeping in mind the earlier definition $\theta_p = 0$ and that we use the convention that in any product which has an upper limit smaller than the lower limit then the product equals one. We have also defined

$$Q = -\frac{2\alpha^2 r^2 a^2(1+\Lambda a^2)(1+\Lambda r^2+\Lambda a^2)}{\Pi(1-\Lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)(r^2+a^2)} - \frac{\alpha a^2(1+\Lambda r^2+2\Lambda a^2)}{\Pi(r^2+a^2)(1-\Lambda r^2)}\,. \tag{5.3.7}$$

To find stationary string configurations we must find geodesics in the metric **h**. We begin as before with the Hamilton-Jacobi equation

$$\frac{\partial I}{\partial\sigma} + \frac{1}{2}h^{ab}\frac{\partial I}{\partial x^a}\frac{\partial I}{\partial x^b} = 0\,. \tag{5.3.8}$$

This metric allows a separation of variables in the action of the form

$$I = -\frac{1}{2}m^2\sigma + \sum_{i=1}^{p} \Phi_i\phi_i + \sum_{i=1}^{p-1} I_{\theta_i}(\theta_i) + I_r(r)\,, \tag{5.3.9}$$

where we have defined Φ_i as the conserved quantities associated with the Killing vectors $\xi_i = \partial_{\phi_i}$. Using our metric (5.3.5) in the Hamilton-Jacobi equation, we find a separation of the form

$$F\left(\frac{r^2+a^2}{1+\Lambda a^2}\right)m^2 - \frac{(\Pi(1-\Lambda r^2)-\alpha r^2)}{r^2(r^2+a^2)^{p-2}(1+\Lambda a^2)}\left(\frac{dI_r}{dr}\right)^2 - \frac{(r^2+a^2)Q}{1+\Lambda a^2}\sum_{i,j=1}^{p}\Phi_i\Phi_j = K_1\,,$$

$$\sum_{i=1}^{p}\frac{\Phi_i^2}{\mu_i^2} + \sum_{i=1}^{p-1}\frac{1}{\prod_{k=1}^{i-1}\sin^2\theta_k}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_1\,. \tag{5.3.10}$$

We have called the separation constant K_1 in anticipation of a series of separation constants that happen when the θ_i coordinates are separated. The explicit separation of the I_{θ_i} functions is performed in Appendix C, but the result that we get is

$$I_r = \varsigma_r\int\sqrt{R}dr\,, \qquad I_{\theta_i} = \varsigma_{\theta_i}\int\sqrt{\Theta_i}d\theta_i\,,$$

$$R = \frac{r^2(r^2+a^2)^{p-1}}{\Pi(1-\Lambda r^2)-\alpha r^2}\left(Fm^2 - Q\sum_{i,j=1}^{p}\Phi_i\Phi_j - \frac{1+\Lambda a^2}{r^2+a^2}K_1\right)\,, \tag{5.3.11}$$

$$\Theta_i = K_i - \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i} - \frac{K_{i+1}}{\sin^2\theta_i}\,, \qquad i = 1, ..., p-1\,.$$

We have $p-1$ separation constants $K_i > 0$ and we denote $K_p = \Phi_1^2$ as the final constant, so that we don't have to write out the equation for Θ_{p-1} separately. Also we have defined the sign functions ς_r and ς_{θ_i} to be all independently ± 1.

Next we wish to get the equations of configuration from the action

$$I = -\frac{1}{2}m^2\sigma + \sum_{i=1}^{p}\Phi_i\phi_i + \sum_{i=1}^{p-1}\varsigma_{\theta_i}\int\sqrt{\Theta_i}d\theta_i + \varsigma_r\int\sqrt{R}dr\,. \tag{5.3.12}$$

We begin as before by setting the derivatives of I with respect to the parameters

m^2, Φ_i, and K_i equal to zero to get

$$\sigma = \varsigma_r \int \frac{F\,dr}{\sqrt{\mathcal{R}}}\,, \tag{5.3.13}$$

$$\varsigma_r \int \frac{1+\Lambda a^2}{r^2+a^2}\frac{dr}{\sqrt{\mathcal{R}}} = \varsigma_{\theta_1}\int \frac{d\theta_1}{\sqrt{\Theta_1}}\,, \tag{5.3.14}$$

$$\varsigma_{\theta_{i-1}} \int \frac{d\theta_{i-1}}{\sin^2\theta_{i-1}\sqrt{\Theta_{i-1}}} = \varsigma_{\theta_i}\int \frac{d\theta_i}{\sqrt{\Theta_i}}\,, \qquad i = 2, ..., p-1\,, \tag{5.3.15}$$

$$\phi_1 = \varsigma_{\theta_{p-1}} \int \frac{\Phi_1}{\sin^2\theta_{p-1}}\frac{d\theta_{p-1}}{\sqrt{\Theta_{p-1}}} + \varsigma_r \int Q\sum_{j=1}^{p}\Phi_j\frac{dr}{\sqrt{\mathcal{R}}}\,, \tag{5.3.16}$$

$$\phi_i = \varsigma_{\theta_{p-i+1}} \int \frac{\Phi_i}{\cos^2\theta_{p-i+1}}\frac{d\theta_{p-i+1}}{\sqrt{\Theta_{p-i+1}}} + \varsigma_r \int Q\sum_{j=1}^{p}\Phi_j\frac{dr}{\sqrt{\mathcal{R}}}\,, \qquad i = 2, ..., p\,, \tag{5.3.17}$$

where we have defined

$$\mathcal{R} = \left(\frac{\Pi(1-\Lambda r^2) - \alpha r^2}{r^2(r^2+a^2)^{p-1}}\right)^2 R\,. \tag{5.3.18}$$

We may differentate these equations and combine them to get the first order equations of motion

$$F\,\dot{r} = \varsigma_r\sqrt{\mathcal{R}}\,, \tag{5.3.19}$$

$$F\,\dot{\theta}_i = \varsigma_{\theta_i}\left(\frac{1+\Lambda a^2}{r^2+a^2}\right)\frac{\sqrt{\Theta_i}}{\prod_{k=1}^{i-1}\sin^2\theta_k}\,, \qquad i = 1, ..., p-1\,, \tag{5.3.20}$$

$$F\,\dot{\phi}_i = \left(\frac{1+\Lambda a^2}{r^2+a^2}\right)\frac{\Phi_i}{\mu_i^2} + Q\sum_{j=1}^{p}\Phi_j\,, \qquad i = 1, ..., p\,. \tag{5.3.21}$$

Note that in the final equation we use the μ_i notation instead of the θ coordinates.

Chapter 6

Discussion

In this thesis, we have examined properties of stationary string configurations in higher dimensional black hole spacetimes, in particular Myers-Perry black holes and their generalization with a cosmological constant. We have demonstrated that the configurations for an equilibrium string in such a stationary geometry obey the geodesic equations in a metric conformal to the spatial projection of the bulk metric, with a conformal factor $-\xi^2$. For the five dimensional Myers-Perry metric we have found that the Hamilton-Jacobi equations allow a separation of variables. The first order equations of possible stationary string configurations have been found, and some particular solutions are demonstrated. It has been shown that there is a set of special solutions, referred to as the principal Killing strings, which share a particular property. The property of uniqueness that these string solutions share is that they are the only stationary string solutions which can cross the infinite red-shift surface of the black hole, without developing any singularity. These solutions pass through the ergosphere and cross into the black hole event horizon remaining timelike and regular.

It is likely that the 'miracle' of separation of variables is something unique to the four and five dimensional Myers-Perry cases. Examining the metric for the six dimensional case, one encounters a problem with off-diagonal terms in the metric. In four and five dimensions all off diagonal components of the metric were connected with cyclic coordinates, so that they did not get in the way of separation of variables. In the six dimensional case, the existence of two angular θ coordinates leads to off diagonal components of the metric which are not connected with a Killing vector. However, it has been demonstrated that the special case of a Myers-Perry metric with only one non-vanishing spin parameter does allow separation of the massless scalar field equations, in any number of dimensions [60]. Also, the possibility has been demonstrated that the Hamilton-Jacobi equations for free test particle motion and the Klein-Gordon equation may separate in the special case of a set of spin parameters equal and the rest zero [61]. This was found in any number of dimensions and it would be interesting to see if the stationary string equations also separate in these cases. However even though separation of variables might not work in the most general case, we have found that some of the results we have do generalize to the higher dimensional Myers-Perry cases. In particular we have demonstrated that the principal Killing strings, generated by the principal null vector and the principal Killing vector, are solutions to the Nambu-Goto equations of motion in any number of dimensions.

We studied the worldsheet for a principal Killing string, and examined the induced geometry on such a worldsheet. We have seen that the induced geometry on a principal Killing surface is that of a two dimensional black hole with an event horizon located where the surface crosses the infinite red-shift surface. Further, it has been demonstrated in five dimensions that the surface gravity of the two dimensional black hole is always at least as large as the surface gravity of the bulk five dimensional black hole. It may be possible to argue that the stability of the principal Killing string is connected with the stability of the induced black hole. In the case of the Kerr-Newman black hole, it has been shown that the two dimensional black hole in the induced string geometry can be obtained as solutions of two dimensional dilaton gravity [57]. A possible future project would be to see if such an analysis is possible in the five dimensional and higher cases.

It has been demonstrated in [22] that, in general, there is a friction effect between rotating black holes and branes attached to them. This effect slows the rotation of the black hole in any planes of rotation that do not preserve the symmetry of the brane under rotation of that plane. We have demonstrated this effect for the particular case of principal Killing strings attached to the black hole, and have found that the friction effect will slow the rotation of the black hole in any planes of rotation that the string is not orthogonal to. This is consistent with the result of [22] in the four dimensional case. The friction effect occurs on a timescale proportional to $\tau = M/\mu^*$, so for light test strings this will be a long time period. On timescales of the order of τ, the black hole's rotation parameters will have changed by an apprecable amount, and then we will have to take into account dynamical effects of the string. Certainly in the future an analysis of this effect might be interesting. However, we may still comment on the possible final configurations of the string. It is reasonable to assume that the state of the system after a long period of time would be stationary, and that such a final configuration would have reached a state such that $\dot{J}_i^{BH} = 0$ for all of the black hole planes of rotation. In such a case, a principal Killing string will be orthogonal to all of the black hole planes of rotation that have a non-zero rotation parameter and a principal Killing string will cross the event horizon at a location where the event horizon coincides with the infinite red-shift surface. In this case, the surface gravity of the rotating black hole will be equal to the surface gravity of the two dimensional black hole induced on the string worldsheet. So it is possible to consider that the friction effect between the rotating black hole and the two dimensional black hole is a 'thermalization' process, reaching equilibrium when the two black holes are at the same temperature.

Finally, we have demonstrated separation of variables in the Gibbons-Lü-Page-Pope metrics. The Hamilton-Jacobi equation for geodesic motion of a particle and the Klein-Gordon equation separated in a special case of an even number of spatial dimensions with all spin parameters equal. We also demonstrated that the Hamilton-Jacobi equation for stationary string configurations separates in this special case. One may consider that there might exist a coordinate transformation to find a separation in a more general case, but it may also be possible that the spacetime lacks sufficient symmetry to allow separation in more general cases. We have also found first order stationary string configuration equations for the special case. In the future, one may examine these equations and determine some properties, however lack of time has prevented such an analysis here.

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Appendix A

String Solutions in Five Dimensions

A.1 Equal Rotation Myers-Perry Case

Consider the degenerate case $a = b$. In this case, due to the enlarged symmetry we have additional Killing vectors in the metric $\mathbf{h}$, taking the same form as those of the bulk five dimensional metric (4.2.12)

$$\xi^a_{(1)}\partial_a = -\cos(\phi-\psi)\partial_\theta + \cot(\theta)\sin(\phi-\psi)\partial_\phi + \tan(\theta)\sin(\phi-\psi)\partial_\psi\,,$$

$$\xi^a_{(2)}\partial_a = \sin(\phi-\psi)\partial_\theta + \cot(\theta)\cos(\phi-\psi)\partial_\phi + \tan(\theta)\cos(\phi-\psi)\partial_\psi\,. \qquad \text{(A.1.1)}$$

Also, as with the bulk metric, the Killing tensor $\tilde{\mathbf{K}}$ (4.3.10) in the metric $\mathbf{h}$ is now reducible

$$\tilde{K}^{ab} = \xi^a_\phi\,\xi^b_\phi + \xi^a_\psi\,\xi^b_\psi - 2\,\xi^{(a}_\phi\,\xi^{b)}_\psi + \xi^a_{(1)}\,\xi^b_{(1)} + \xi^a_{(2)}\,\xi^b_{(2)} - h^{ab}\,. \qquad \text{(A.1.2)}$$

In this degenerate case, the string configuration equations now take the simpler form

$$(x + a^2 - \alpha)\,\dot{x} = \varsigma_x 2\sqrt{\Delta(x-\alpha-K) + \alpha a^2(\Phi+\Psi)^2}\,, \qquad \text{(A.1.3)}$$

$$(x + a^2 - \alpha)\,\dot{\theta} = \varsigma_\theta\sqrt{K + a^2 - \frac{\Phi^2}{\sin^2\theta} - \frac{\Psi^2}{\cos^2\theta}}\,, \qquad \text{(A.1.4)}$$

$$(x + a^2 - \alpha)\,\dot{\phi} = \frac{\Phi}{\sin^2\theta} - \frac{\alpha\, a^2\,(\Psi + \Phi)}{(x + a^2)^2 - x\alpha}\,, \tag{A.1.5}$$

$$(x + a^2 - \alpha)\,\dot{\psi} = \frac{\Psi}{\cos^2\theta} - \frac{\alpha\, a^2\,(\Psi + \Phi)}{(x + a^2)^2 - x\alpha}\,. \tag{A.1.6}$$

Note that there is no θ dependence in the $\dot{x}$ equation, so that this equation is a separable one

$$\int d\sigma = \frac{1}{2}\int \frac{(x + a^2 - \alpha)dx}{\sqrt{(x + a^2)^2(x - \alpha - K) - \alpha(x^2 - x\alpha - xK + a^2(\Phi + \Psi)^2)}}\,. \tag{A.1.7}$$

If we denote p_1, p_2, and p_3 as the roots of the denominator, so that

$$(x + a^2)^2(x - \alpha - K) - \alpha(x^2 - x\alpha - xK + a^2(\Phi + \Psi)^2)$$

$$= (x - p_1)(x - p_2)(x - p_3)\,, \tag{A.1.8}$$

then we may integrate the equation explicitly

$$\sigma = \sqrt{p_3 - p_1}\,E\left(\arcsin\left(\sqrt{\frac{x - p_1}{p_2 - p_1}}\right), \sqrt{\frac{p_2 - p_1}{p_3 - p_1}}\right)$$

$$+\frac{p_3 + a^2 - \alpha}{\sqrt{p_3 - p_1}}F\left(\arcsin\left(\sqrt{\frac{x - p_1}{p_2 - p_1}}\right), \sqrt{\frac{p_2 - p_1}{p_3 - p_1}}\right)\,, \tag{A.1.9}$$

where the elliptic functions are defined in the notation of [62] as

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2\sin^2 z}\; dz, \qquad F(\phi, k) = \int_0^{\phi} \frac{dz}{\sqrt{1 - k^2\sin^2 z}}\,. \tag{A.1.10}$$

Next, we may divide (4.3.19) by (4.3.20) and separate to get

$$\int \frac{dx}{2\sqrt{X}} = \int \frac{d\theta}{\sqrt{\Theta}}\,. \tag{A.1.11}$$

The left hand side can be integrated to get

$$\int \frac{dx}{2\sqrt{X}} = \frac{F\left(\arcsin\left(\sqrt{\frac{x - p_1}{p_2 - p_1}}\right), \sqrt{\frac{p_2 - p_1}{p_3 - p_1}}\right)}{\sqrt{p_3 - p_1}}\,, \tag{A.1.12}$$

with p_1, p_2, and p_3 the same roots defined in (A.1.8). For the right hand side, the substitution $y = \tan^2\theta$ changes this integral to

$$\int \frac{d\theta}{\sqrt{\Theta}} = \int \frac{dy}{2(1+y)\sqrt{-\Psi^2 y^2 + (K + a^2 - \Phi^2 - \Psi^2)y - \Phi^2}}, \tag{A.1.13}$$

so we may integrate this equation directly to find

$$\frac{F\left(\arcsin\left(\sqrt{\frac{x-p_1}{p_2-p_1}}\right), \sqrt{\frac{p_2-p_1}{p_3-p_1}}\right)}{\sqrt{p_3-p_1}}$$

$$= \frac{\tan^{-1}\left(\frac{(K+a^2)(\sin^2\theta-\cos^2\theta)+\Psi^2-\Phi^2}{2\sqrt{K+a^2}\sqrt{(K+a^2)\sin^2\theta\cos^2\theta-\Phi^2\cos^2\theta-\Psi^2\sin^2\theta}}\right)}{\sqrt{K+a^2}}. \tag{A.1.14}$$

Now we have implicit solutions for $x(\sigma)$ and $\theta(\sigma)$. To get information on ϕ and ψ into this, we can make use of our Killing vector $\xi_{(2)}$

$$\xi_{(2)} \cdot P = \sin(\phi-\psi)P_\theta + \cot\theta\cos(\phi-\psi)P_\phi + \tan\theta\cos(\phi-\psi)P_\psi = Q\,, \tag{A.1.15}$$

with $Q = \xi_{(2)} \cdot P$ being a constant associated with the Killing vector. We can see the components of the momentum vector as defined in (4.3.6) are

$$P_\phi = \Phi \qquad P_\psi = \Psi$$

$$P_\theta = \frac{\partial I}{\partial\theta} = \sqrt{\Theta} = \sqrt{K + a^2 - \frac{\Phi^2}{\sin^2\theta} - \frac{\Psi^2}{\cos^2\theta}}. \tag{A.1.16}$$

We may set the value of Q by considering an initial point $\phi = \psi$, $\theta = \theta_0$

$$\sin(\phi-\psi)\sqrt{K + a^2 - \frac{\Phi^2}{\sin^2\theta} - \frac{\Psi^2}{\cos^2\theta}} + \cos(\phi-\psi)(\Phi\cot\theta + \Psi\tan\theta)$$

$$= \cot\theta_0\,\Phi + \tan\theta_0\,\Psi\,. \tag{A.1.17}$$

(A.1.9) gives us the relationship between x and σ, (A.1.14) gives us a relationship for θ as a function of x, and (A.1.17) gives us how ϕ and ψ vary with θ. Note that since ϕ and ψ are still coupled in the form $\phi - \psi$, this is not a complete solution. One would have to integrate (A.1.5) and (A.1.6) to get a full solution, but since the solutions for x involve elliptic functions integrating these equations would most likely prove difficult.

A.2 θ-x Relationship for String Configurations

It is always possible to divide (4.3.19) by (4.3.20) to get

$$\int \frac{d\theta}{\sqrt{\Theta}} = \int \frac{dx}{2\sqrt{\chi}}, \tag{A.2.1}$$

which, upon letting $u = \tan^2\theta$, we can see that

$$\int \frac{d\theta}{\sqrt{\Theta}} = \frac{1}{2}\int \frac{du}{\sqrt{(u+1)W(u)}}, \tag{A.2.2}$$

where we have defined the polynomial $W(u)$ as

$$W(u) = -\Psi^2 u^3 + (K + b^2 - 2\Psi^2 - \Phi^2)u^2 + (K + a^2 - \Psi^2 - 2\Phi^2)u - \Phi^2\,. \tag{A.2.3}$$

If we allow q_1, q_2, and q_3 to be the roots of W, such that

$$W(u) = -\Psi^2(u - q_1)(u - q_2)(u - q_3)\,, \tag{A.2.4}$$

and we denote p_1, p_2, and p_3 to be roots of χ, as

$$\chi = (x - p_1)(x - p_2)(x - p_3)\,, \tag{A.2.5}$$

similar to (A.1.8), then it is possible in the general case to integrate (A.2.1) to get

$$\frac{-F\left(\arcsin\left(\frac{\sqrt{\sec^2\theta(q_3-q_1)}}{\tan^2\theta-q_1}\right), \sqrt{\frac{(q_3+1)(q_1-q_2)}{(q_2+1)(q_3-q_1)}}\right)}{\sqrt{(q_2+1)(q_1-q_3)\Psi^2}} = \frac{F\left(\arcsin\left(\sqrt{\frac{x-p_1}{p_2-p_1}}\right), \sqrt{\frac{p_2-p_1}{p_3-p_1}}\right)}{\sqrt{p_3-p_1}}\,. \tag{A.2.6}$$

A.3 Solution for the Principal Killing Surface

Consider a principal Killing string in the five dimensional Myers-Perry case. The form of the principal null vectors (4.5.2) tells us

$$P = \frac{(x+a^2)(x+b^2)}{\Delta}\xi - l\,, \tag{A.3.1}$$

is a spacelike vector tangent to the string's worldsheet, and further that

$$\dot{x}^a = P^a\,. \tag{A.3.2}$$

Reading this equation off in components, we then get for the string configuration equations

$$\begin{aligned} \dot{x} &= \mp 2\sqrt{x}\,, & \dot{\theta} &= 0\,, \\ \dot{\phi} &= \frac{a(x+b^2)}{\Delta}\,, & \dot{\psi} &= \frac{b(x+a^2)}{\Delta}\,. \end{aligned} \tag{A.3.3}$$

We may say that $\theta = \theta_0$, a constant, and comparing these equations with (4.3.19)-(4.3.22) we may see that the constants Φ, Ψ and K take the form

$$\Phi = a\sin^2\theta_0\,, \quad \Psi = b\cos^2\theta_0\,, \quad K = (a^2-b^2)(\sin^2\theta_0 - \cos^2\theta_0)\,. \tag{A.3.4}$$

By integrating the equations (A.3.3), we can get

$$x = r^2 = \sigma\,, \tag{A.3.5}$$

$$\phi = \frac{a}{2(r_+^2 - r_-^2)}\left[\frac{r_+^2+b^2}{r_+}\ln\left(\frac{r-r_+}{r+r_+}\right) - \frac{r_-^2+b^2}{r_-}\ln\left(\frac{r-r_-}{r+r_-}\right)\right] + \phi_0\,, \tag{A.3.6}$$

$$\psi = \frac{b}{2(r_+^2 - r_-^2)}\left[\frac{r_+^2+a^2}{r_+}\ln\left(\frac{r-r_+}{r+r_+}\right) - \frac{r_-^2+a^2}{r_-}\ln\left(\frac{r-r_-}{r+r_-}\right)\right] + \psi_0\,, \tag{A.3.7}$$

with ϕ_0 and ψ_0 being initial data for the string, and $r_\pm$ being the horizon locations, defined in (4.2.7). We see that for $a \neq 0$, the value of ϕ diverges in the limit $r \to r_+$, meaning that there is an infinite amount of string winding in this plane as it approaches the event horizon. For $b \neq 0$, there is a similar effect in the ψ plane. These effects are connected with the failure of Boyer-Lindquist coordinates near the event horizon, this was discussed to explain the singularity in (4.4.16).

Appendix B

Uniqueness property

In this section, we will demonstrate the uniqueness property of principal Killing string solutions in the five dimensional Myers-Perry case. That is, these solutions are the only stationary string solutions to cross the infinite red-shift surface and remain regular.

We begin by considering that we have a stationary surface, with coordinates $\zeta^A = (v, \lambda)$ on its surface. The strings worldsheet in spacetime is given by $x^\mu(\zeta^A)$. We will define v to coincide with the Killing time, so that $dx^\mu/dv = \xi^\mu$. We still have the freedom $v \to v + f(\lambda)$ for any function f, and this freedom does not effect dx^μ/dv. Let us use this freedom so that $L^\mu = -dx^\mu/d\lambda$ is a null vector. The metric takes the form

$$d\gamma^2 = \xi^2 dv^2 - 2(\xi \cdot L) dv d\lambda \,. \tag{B.1}$$

We still have scaling freedom in λ as $\lambda \to g(\lambda)$, which will not effect L being null. Since $\xi \cdot L$ can only depend on λ, we can use this freedom to set $\xi \cdot L = -1$. So we have for the metric

$$d\gamma^2 = -F dv^2 + 2 dv d\lambda \,, \qquad F = -\xi^2 \,, \tag{B.2}$$

and an inverse metric

$$G^{AB} \partial_A \partial_B = 2\partial_v \partial_r + F \partial_r^2 \,. \tag{B.3}$$

Let us denote the set of vectors normal to the surface as n_R. We have the relationship for the trace of the second fundamental form (4.5.20),

$$\Omega_R = n_R \cdot z\,, \qquad z^\mu = G^{AB} x^\rho{}_{,A} \, (x^\mu{}_{,B})_{;\rho}\,, \tag{B.4}$$

$$z^\mu = -2L^\rho \xi^\mu{}_{;\rho} + F L^\rho L^\mu{}_{;\rho}\,. \tag{B.5}$$

Antisymmetry of $\xi_{\mu;\nu}$ and the fact that $L^2 = 0$ guarantees that $L \cdot z = 0$. If we assume that our stationary surface is a minimal surface, it means that we have

$$\Omega_R = G^{AB} \Omega_{RAB} = 0\,, \tag{B.6}$$

$$\Omega^2 = \delta^{RS} \Omega_R \Omega_S = \delta^{RS} (n^\mu_R \, z_\mu)(n^\nu_S \, z_\nu) = \delta^{RS} n^\mu_R n^\nu_S (z_\mu z_\nu) = 0\,. \tag{B.7}$$

The completeness relation (4.5.14) gives us

$$\Omega^2 = (g^{\mu\nu} - G^{AB} x^\mu{}_{,A} x^\nu{}_{,B}) z_\mu z_\nu = 0\,. \tag{B.8}$$

Using the form of (B.3), we get for the second term

$$G^{AB} x^\mu{}_{,A} x^\nu{}_{,B} z_\mu z_\nu = [2(\xi \cdot z)(L \cdot z) + F(L \cdot z)^2]\,, \tag{B.9}$$

which vanishes since $L \cdot z = 0$. Thus if our surface is to be a minimal one, z must be null, $g_{\mu\nu} z^\mu z^\nu = 0$. z being null and orthogonal to L, which is also null, implies that z and L are parallel, that being

$$z^\mu = -2L^\rho \xi^\mu{}_{;\rho} + F\, L^\rho L^\mu{}_{;\rho} = q L^\mu\,. \tag{B.10}$$

To find q, we will multiply this relationship by ξ_μ to get

$$q = 2\xi_\mu L^\rho \xi^\mu{}_{;\rho} - F\, \xi_\mu L^\rho L^\mu{}_{;\rho}\,, \tag{B.11}$$

$$q = L^\rho (\xi^2)_{;\rho} - F\, L^\rho (\xi \cdot L)_{;\rho} + F\, L^\rho L^\mu \xi_{\mu;\rho} = \frac{dF}{d\lambda}\,. \tag{B.12}$$

Using this value of q in (B.10), we see

$$2\xi^\mu{}_{;\rho} L^\rho = F\, L^\rho L^\mu{}_{;\rho} - \frac{dF}{d\lambda} L^\mu\,. \tag{B.13}$$

Examining this relation at the infinite red-shift surface $F = 0$, we can see that L is a real eigenvector of $\xi_{\mu;\nu}$. Comparison with (4.5.5) shows that this means it must be the case that $L = l_+$ or $L = l_-$ at $F = 0$. For definiteness, we will assume $L = l_-$ at the infinite red-shift surface. Next, if we consider regions close to the infinite red-shift surface, then we have

$$L = (1+\lambda)l_- + \mu\bar{m} + \bar{\mu}m + \nu k\,, \tag{B.14}$$

where we will consider λ, μ, and ν to be terms much smaller than 1, and thus will drop any multiple of these two in this analysis. The term proportional to l_+ does not appear since $L \cdot L = 0$. The forms of the vectors m and k (4.5.3-4.5.4), gives us the following

$$(m \cdot \xi) = \frac{i \sin\theta\, \cos\theta}{\rho\sqrt{2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}(a^2 - b^2)\,, \tag{B.15}$$

$$(k \cdot \xi) = -\frac{ab}{r\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\,. \tag{B.16}$$

We also have $L \cdot \xi = -1$. Using L in (B.14), we obtain

$$\lambda = \frac{1}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\left[\frac{i(a^2-b^2)\sin\theta\cos\theta}{\rho\sqrt{2}}(\bar{\mu} - \mu) - \frac{ab}{r}\nu\right]\,, \tag{B.17}$$

Contracting (B.13) with m_μ, we may use the relationships

$$m_\mu L^\rho \xi^\mu{}_{;\rho} = \frac{i\mu(1-F)\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\rho^2}\,, \qquad m_\mu L^\mu = \mu\,, \tag{B.18}$$

$$m_\mu L^\rho L^\mu{}_{;\rho} = l_-^\rho \mu_{,\rho} - \frac{\mu}{\rho^2}\left(2i\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + r\right)\,, \tag{B.19}$$

to get

$$F\, l_-^\rho \mu_{,\rho} = -F\frac{d\mu}{dr} = -\Omega\mu\,, \tag{B.20}$$

where we have defined

$$\Omega = -\frac{2i\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + rF}{\rho^2} - \frac{dF}{dr}\,. \tag{B.21}$$

We will define the tortoise coordinate r^* as

$$\frac{dr}{dr^*} = F\,, \tag{B.22}$$

so that the infinite red-shift surface lies at $r^* \to -\infty$, then we can solve for μ as

$$\mu = \mu_0\, e^{\Omega r^*}\,. \tag{B.23}$$

However, $Re(\Omega)$ is negative definite in a region near the infinite red-shift surface, so we must have $\mu_0 = 0$ to get a solution which is regular at the infinite red-shift surface.

Now, if we contract (B.13) instead with k_μ, we may use the relationships

$$k_\mu L^\rho \xi^\mu{}_{;\rho} = 0\,, \qquad k_\mu L^\mu = \nu\,, \tag{B.24}$$

$$k_\mu L^\rho L^\mu{}_{;\rho} = -\frac{\nu}{r} + l_-^\rho \nu_{,\rho}\,, \tag{B.25}$$

to get

$$F\, l_-^\rho \nu_{,\rho} = -F\,\frac{d\nu}{dr} = -\Upsilon \nu\,, \tag{B.26}$$

where we have defined

$$\Upsilon = -\frac{dF}{dr} - \frac{F}{r}\,. \tag{B.27}$$

Using the tortoise coordinate (B.22) again, we may solve this as

$$\nu = \nu_0\, e^{\Upsilon r^*}\,. \tag{B.28}$$

Just as before, $Re(\Upsilon)$ is negative definite in a region near the infinite red-shift surface, so we must have $\nu_0 = 0$ so that our solution is regular at the infinite red-shift surface.

Thus, we have shown that μ and ν in (B.14) must vanish identically in some finite region about the infinite red-shift surface. By analytic continuation, L must be equivalent to l_- everywhere else as well, so the only regular minimal stationary surface which crosses the infinite red-shift surface is a principal Killing surface.

Appendix C

Angular Equation Separation for a String

Here we demonstrate the separation of the I_θ part in the Gibbons-Lü-Page-Pope metric explicitly (5.3.10). Begin with the equation to separate written out with θ dependence explicit,

$$\sum_{i=1}^{p} \frac{\Phi_i^2}{\cos^2\theta_{p-i+1}\prod_{j=1}^{p-i}\sin^2\theta_j} + \sum_{i=1}^{p-1} \frac{1}{\prod_{k=1}^{i-1}\sin^2\theta_k}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_1 \,. \tag{C.1}$$

Note that this equation guarantees that $K_1 > 0$. Next, to make the separation simpler, we renumber the index on the first sum as $i \to p - i + 1$, keeping the sum running from 1 to p

$$\sum_{i=1}^{p} \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i\prod_{j=1}^{i-1}\sin^2\theta_j} + \sum_{i=1}^{p-1} \frac{1}{\prod_{k=1}^{i-1}\sin^2\theta_k}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_1 \,. \tag{C.2}$$

Now we are ready to begin separation, first by peeling off the bottom term from both sums,

$$\frac{\Phi_n^2}{\cos^2\theta_1} + \left(\frac{dI_{\theta_1}}{d\theta_1}\right)^2 + \sum_{i=2}^{p} \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i\prod_{j=1}^{i-1}\sin^2\theta_j} + \sum_{i=2}^{p-1} \frac{1}{\prod_{k=1}^{i-1}\sin^2\theta_k}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_1 \,. \tag{C.3}$$

Note that every term in the sums now has $\sin^2\theta_1$ in its denominator. Next, multiply across by $\sin^2\theta_1$ and rearrange terms to get

$$K_1 \sin^2\theta_1 - \left(\frac{dI_{\theta_1}}{d\theta_1}\right)^2 \sin^2\theta_1 - \frac{\Phi_p^2 \sin^2\theta_1}{\cos^2\theta_1} = \sum_{i=2}^{p} \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i \prod_{j=2}^{i-1}\sin^2\theta_j} + \sum_{i=2}^{p-1} \frac{(dI_{\theta_i}/d\theta_i)^2}{\prod_{k=2}^{i-1}\sin^2\theta_k}\,. \tag{C.4}$$

Here we see that the left hand side is a function of θ_1 only, while the right hand side is not a function of θ_1 anywhere. So we may state that both sides are equal to a constant we will call K_2. We now have

$$\left(\frac{dI_{\theta_1}}{d\theta_1}\right)^2 = K_1 - \frac{K_2}{\sin^2\theta_1} - \frac{\Phi_p^2}{\cos^2\theta_1}\,,$$

$$\sum_{i=2}^{p} \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i \prod_{j=2}^{i-1}\sin^2\theta_j} + \sum_{i=2}^{p-1} \frac{1}{\prod_{k=2}^{i-1}\sin^2\theta_k}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_2\,. \tag{C.5}$$

The second equation has the same form as (C.2), thus we may see that this separation will continue with an inductive form of

$$\left(\frac{dI_{\theta_k}}{d\theta_k}\right)^2 = K_k - \frac{K_{k+1}}{\sin^2\theta_k} - \frac{\Phi_{p-k+1}^2}{\cos^2\theta_k}\,, \tag{C.6}$$

$$\sum_{i=k+1}^{p} \frac{\Phi_{p-i+1}^2}{\cos^2\theta_i \prod_{j=k+1}^{i-1}\sin^2\theta_j} + \sum_{i=k+1}^{p-1} \frac{1}{\prod_{j=k+1}^{i-1}\sin^2\theta_j}\left(\frac{dI_{\theta_i}}{d\theta_i}\right)^2 = K_{k+1}\,, \tag{C.7}$$

with all $K_i > 0$ and this pattern is valid for all $k = (1, ..., p-2)$. For the final step $k = p-2$ in equation (C.7), we may finish expanding the sums to get

$$\left(\frac{dI_{\theta_{p-1}}}{d\theta_{p-1}}\right)^2 = K_{p-1} - \frac{\Phi_1^2}{\sin^2\theta_{p-1}} - \frac{\Phi_2^2}{\cos^2\theta_{p-1}}\,. \tag{C.8}$$

We see that if we use the notation $K_p \equiv \Phi_1^2$, then all steps have the same form of (C.6). This finishes the proof of separation.

Appendix D

Gibbons-Lü-Page-Pope Separability

D.1 Hamilton-Jacobi Equation

Here we will discuss the separability of the Hamilton-Jacobi equation for particles in geodesic motion and the separability of the Klein-Gordon equation for a scalar field in the Gibbons-Lü-Page-Pope spacetime (see also [24]).

The Hamilton-Jacobi equation to find geodesics for a free-falling particle is given by

$$-\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{D.1.1}$$

where S is the action associated with the particle and λ is an affine parameter along the worldline of the particle. The nonzero components of the inverse metric in an even number of spatial dimensions when all rotation parameters are equal are given as

$$g^{tt} = V - \frac{\alpha^2 r^2 (r^2 + a^2)}{\Pi(1 - \Lambda r^2)^2 (\Pi(1 - \Lambda r^2) - \alpha r^2)},$$

$$g^{t\phi_i} = \Lambda a V - \frac{\alpha^2 a r^2 (1 + \Lambda a^2)}{\Pi(1 - \Lambda r^2)^2 (\Pi(1 + \Lambda a^2) - \alpha r^2)} - \frac{\alpha a}{\Pi(1 - \lambda r^2)},$$

$$g^{rr} = \frac{(\Pi(1 - \Lambda r^2) - \alpha r^2)(r^2 + a^2)}{\Pi r^2},$$

$$g^{\phi_i\phi_j} = \frac{1+\Lambda a^2}{(r^2+a^2)\mu_i^2}\delta^{ij} + Q\,, \qquad g^{\theta_i\theta_j} = \frac{1+\Lambda a^2}{(r^2+a^2)\prod_{k=1}^{i-1}\sin^2\theta_k}\delta^{ij}\,, \tag{D.1.2}$$

where V and Q are defined to be

$$V = -\frac{1+\Lambda a^2}{1-\Lambda r^2} - \frac{\alpha(r^2+a^2)}{\Pi(1-\Lambda r^2)^2}\,,$$

$$Q = -\frac{2\alpha^2 r^2 a^2(1+\Lambda a^2)(1+\Lambda r^2+\Lambda a^2)}{\Pi(1-\Lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)(r^2+a^2)} - \frac{\alpha a^2(1+\Lambda r^2+2\Lambda a^2)}{\Pi(r^2+a^2)(1-\Lambda r^2)}\,. \tag{D.1.3}$$

and $\Pi = (r^2+a^2)^p$, as before.

This metric allows separation of variables. Let

$$S = \frac{1}{2}m^2\lambda - Et + \sum_{i=1}^{p}\Phi_i\phi_i + S_r(r) + \sum_{i=1}^{p-1} S_{\theta_i}(\theta_i)\,. \tag{D.1.4}$$

t and ϕ_i are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and the conserved quantities associated with rotation in the ϕ_i planes are the corresponding angular momenta Φ_i.

Using this form of the action, the Hamilton-Jacobi equation separates as

$$\begin{aligned}
-K_1^2(1+\Lambda a^2) &= m^2(r^2+a^2) + V(r^2+a^2)\left[E - \Lambda a\sum_{i=1}^{p}\Phi_i\right]^2 \\
&+ \frac{\alpha^2 r^2}{(r^2+a^2)^{p-2}(1-\Lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)}\left[E + \frac{a(1+\Lambda a^2)}{r^2+a^2}\sum_{i=1}^{p}\Phi_i\right]^2 \\
&- \frac{2\alpha^2 E^2 r^2}{(r^2+a^2)^{p-2}(1-\Lambda r^2)(\Pi(1-\Lambda r^2)-\alpha r^2)} + (r^2+a^2)Q\sum_{i,j=1}^{p}\Phi_i\Phi_j \\
&+ \frac{2\alpha a E}{(r^2+a^2)^{p-1}(1-\Lambda r^2)}\sum_{i=1}^{p}\Phi_i + \frac{\Pi(1-\Lambda r^2)-\alpha r^2}{r^2(r^2+a^2)^{p-2}}\left[\frac{dS_r}{dr}\right]^2,
\end{aligned} \tag{D.1.5}$$

where K_1^2 is a positive separation constant, and we get for the θ dependence

$$K_1^2 = \sum_{i=1}^{p}\left[\frac{\Phi_i^2}{\left(\prod_{k=1}^{p-i}\sin^2\theta_k\right)\cos^2\theta_{p-i+1}}\right] + \sum_{i=1}^{p-1}\frac{1}{\prod_{k=1}^{i-1}\sin^2\theta_k}\left(\frac{dS_{\theta_i}}{d\theta_i}\right)^2\,. \tag{D.1.6}$$

To show complete separation of the Hamilton-Jacobi equation we must finish separation of the θ equation. The separation is essentially equivalent to that done in appendix C for stationary string equations, having the inductive form for $k = 1, ..., p-2$:

$$K_k^2 \sin^2\theta_k - \frac{\Phi_{p-k+1}^2 \sin^2\theta_k}{\cos^2\theta_k} - \sin^2\theta_k \left(\frac{dS_{\theta_k}}{d\theta_k}\right)^2 = K_{k+1}^2\,,$$

$$K_{k+1}^2 = \sum_{i=k+1}^{p} \frac{\Phi_{p-i+1}^2}{\left(\prod_{j=k+1}^{i-1}\sin^2\theta_j\right)\cos^2\theta_i} + \sum_{i=k+1}^{p-1} \frac{1}{\prod_{j=k+1}^{i-1}\sin^2\theta_j}\left(\frac{dS_{\theta_i}}{d\theta_i}\right)^2\,, \qquad \text{(D.1.7)}$$

and the final step of

$$K_{p-1}^2 = \frac{\Phi_2^2}{\cos^2\theta_{p-1}} + \frac{\Phi_1^2}{\sin^2\theta_{p-1}} + \left(\frac{dS_{\theta_{p-1}}}{d\theta_{p-1}}\right)^2. \qquad \text{(D.1.8)}$$

Next, to derive equations of motion, we will use the action in its separated form

$$S = \frac{1}{2}m^2\lambda - Et + \sum_{i=1}^{p}\Phi_i\phi_i + \varsigma_r \int \sqrt{R(r)}dr + \sum_{i=1}^{p-1}\varsigma_{\theta_i}\int\sqrt{\Theta_i(\theta_i)}d\theta_i \qquad \text{(D.1.9)}$$

with

$$\Theta_k = K_k^2 - \frac{K_{k+1}^2}{\sin^2\theta_k} - \frac{\Phi_{p-k+1}^2}{\cos^2\theta_k}\,, \qquad k = 1, ..., p-1\,, \qquad \text{(D.1.10)}$$

$$\begin{aligned} R = &-K_1^2\frac{(1+\Lambda a^2)(r^2+a^2)^{p-2}r^2}{\Pi(1-\Lambda r^2)-\alpha r^2} - \frac{(r^2+a^2)^{p-1}r^2\,V}{\Pi(1-\Lambda r^2)-\alpha r^2}\left[E - \Lambda a\sum_{i=1}^{p}\Phi_i\right]^2 \\ &-m^2\frac{(r^2+a^2)^{p-1}\,r^2}{\Pi(1-\Lambda r^2)-\alpha r^2} - \frac{2\alpha r^2 aE}{(\Pi(1-\Lambda r^2)-\alpha r^2)(r^2+a^2)}\sum_{i=1}^{p}\Phi_i \\ &-\frac{2\alpha^2 r^4 E^2}{(1-\Lambda r^2)(\Pi(1-\Lambda r^2)-\alpha r^2)^2} - \frac{Q(r^2+a^2)^{p-1}r^2}{(\Pi(1-\Lambda r^2)-\alpha r^2)}\sum_{i,j=1}^{p}\Phi_i\Phi_j \qquad \text{(D.1.11)} \\ &-\frac{\alpha^2 r^4}{(1-\Lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)^2}\left[E + \frac{a(1+\Lambda a^2)}{r^2+a^2}\sum_{i=1}^{p}\Phi_i\right]^2, \end{aligned}$$

with Q and V defined in (D.1.3). We also denote $K_p^2 = \Phi_1^2$ so that the inductive definition given above applies for Θ_{p-1}.

To obtain the equations of motion, we differentiate S with respect to the parameters m^2, E, Φ_i, K_j^2 and set these derivatives to zero, giving us integral equations as before. We can then arrive at first order equations by the usual procedure, in particular for the r and θ_i equations, we get

$$\dot{r} = \varsigma_r \frac{(\Pi(1-\Lambda r^2) - \alpha r^2)\sqrt{R}}{(r^2+a^2)^{p-1} r^2}$$

$$\dot{\theta}_i = \varsigma_{\theta_i} \frac{(1+\Lambda a^2)\sqrt{\Theta_i}}{(r^2+a^2)\prod_{j=1}^{i-1}\sin^2\theta_j}\,. \tag{D.1.12}$$

Allowed orbits may only exist where $R > 0$, and radial turning points occur where $R = 0$. Analyzing the $\Lambda = 0$ case for $r \to \infty$, R has an asymptotic form of $E^2 - m^2$. Thus we can say that for $E^2 < m^2$, we cannot have unbounded orbits, whereas for $E^2 > m^2$, such orbits are possible, exactly as one would expect. In the generic case $\Lambda \neq 0$, R asymptotically behaves as $\frac{m^2}{\Lambda r^2}$. Thus, for $\Lambda < 0$ only bound orbits are possible, but if $\Lambda > 0$ then unbounded orbits may be possible.

D.2 The Scalar Field Equation

Next we will demonstrate the the Klein-Gordon scalar field equation also has a separation of variables in the Gibbons-Lü-Page-Pope metrics when we assume an even number of spatial dimensions with all $a_i = a$. We begin with the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = m^2\varphi\,. \tag{D.2.1}$$

For the metric determinant, we have

$$g = -BA\,, \qquad B = \frac{r^2(r^2+a^2)^{2p-2}}{(1+\lambda a^2)^{2p}}\,, \qquad A = \prod_{j=1}^{p-1}\sin^{4p-4j-2}\theta_j\cos^2\theta_j\,. \tag{D.2.2}$$

The metric allows a multiplicative separation for φ as

$$\varphi = e^{-iEt}e^{i\sum_i \Phi_i\phi_i}\varphi_r(r)\varphi_\theta(\theta_k)\,. \tag{D.2.3}$$

Then the r-θ part of the Klein-Gordon equation separates as

$$m^2\varphi_r = -\frac{2\alpha a E}{\Pi(1-\Lambda r^2)}\sum_{i=1}^{p}\Phi_i\varphi_r + \frac{1}{\sqrt{B}}\frac{d}{dr}\left(\sqrt{B}\frac{\Pi(1-\Lambda r^2-\alpha r^2)}{(r^2+a^2)^{p-1}r^2}\frac{d\varphi_r}{dr}\right)$$

$$-\frac{\alpha^2 r^2(r^2+a^2)}{\Pi(1-\lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)}\left[E+\frac{a(1+\Lambda a^2)}{r^2+a^2}\sum_{i=1}^{p}\Phi_i\right]^2\varphi_r + \frac{1+\Lambda a^2}{r^2+a^2}K_1\varphi_r$$

$$+\frac{2\alpha^2 r^2 E^2(r^2+a^2)}{\Pi(1-\Lambda r^2)^2(\Pi(1-\Lambda r^2)-\alpha r^2)}\varphi_r - Q\sum_{ij=1}^{p}\Phi_i\Phi_j\varphi_r - V\left[E-\Lambda a\sum_{i=1}^{p}\Phi_i\right]^2\varphi_r\,, \tag{D.2.4}$$

$$K_1 = \frac{1}{\varphi_\theta}\sum_{i=1}^{p}\left[-\frac{\Phi_i^2}{\mu_i^2}\right] + \sum_{i=1}^{p-1}\frac{1}{\varphi_\theta\sqrt{A}}\partial_{\theta_i}\left(\sqrt{A}g^{\theta_i\theta_i}\frac{\partial\varphi_\theta}{\partial\theta_i}\right)\,, \tag{D.2.5}$$

where the constant to separate θ from r is K_1. The θ part will also completely separate as

$$\varphi_\theta = \prod_{k=1}^{p-1}\varphi_{\theta_k}(\theta_k)\,. \tag{D.2.6}$$

The θ separation then steps down inductively as

$$K_1 = \sum_{i=1}^{k-1}C_i + \frac{K_k}{\prod_{j=1}^{k-1}\sin^2\theta_j}\,, \qquad k = 1,...,p-1\,, \tag{D.2.7}$$

where

$$C_i = \frac{1}{\varphi_{\theta_i}\cos\theta_i\sin^{2p-2i-1}\theta_i\prod_{k=1}^{i-1}\sin^2\theta_k}\frac{d}{d\theta_i}\left(\cos\theta_i\sin^{2p-2i-1}\theta_i\frac{d\varphi_{\theta_i}}{d\theta_i}\right) \tag{D.2.8}$$

$$-\frac{\Phi_{p-i+1}^2}{\cos^2\theta_i\prod_{j=1}^{i-1}\sin^2\theta_j}\,.$$

Then we have the complete separation of the θ_i dependence as

$$K_k = \frac{K_{k+1}}{\sin^2\theta_k} - \frac{\Phi_{p-k+1}^2}{\cos^2\theta_k} + \frac{1}{\varphi_{\theta_k}\cos\theta_k\sin^{2p-2k-1}\theta_k}\frac{d}{d\theta_k}\left(\cos\theta_k\sin\theta_k\frac{d\varphi_{\theta_k}}{d\theta_k}\right)\,, \tag{D.2.9}$$

Here there is a set of constants K_i separating out the θ equations. Finally, we use the convention $K_p \equiv -\Phi_1^2$. This completes separation of the Klein-Gordon equation.