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Topological Centers and Topologically Invariant Means Related to Locally Compact Groups

by

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ABSTRACT

In this thesis, we discuss two separate topics from the theory of harmonic analysis on locally compact groups. The first topic revolves around the topological centers of module actions induced by unitary representations while the second one deals with the set of topologically invariant means associated to an amenable representation.

Part I of this thesis is about the topological centers of bilinear maps induced by unitary representations. We give a characterization when the center is minimal in term of a factorization property. We give conditions which guarantee that the center is maximal. Various examples whose topological centers are maximal, minimal nor neither will be given. We also investigate the topological centers related to sub-representations, direct sums and tensor products.

In Part II we study of the set of topologically invariant means associated to an amenable representation. We construct topologically invariant means for an amenable representation by two different methods. A lower bound of the cardinality of the set of topologically invariant means will be given.

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Part I

Topological Centers

Chapter 1

Introduction and some preliminaries

In 1951, Arens initiated the study of extension of bilinear maps on normed space and introduced the concept of regularity of bilinear maps (see [1] and [2]). The study of Arens regularity of bilinear maps and the topological center problem has attracted some attention. In [41], Ulger showed that the Arens regularity of a bounded bilinear map can be characterized by its weakly compactness or its reflexiveness and simplified proofs of some old results. For more recent results, the reader is referred to [14] and [36]. On the other hand, special attention has been focused on the bilinear maps arisen from Banach algebras. See [34] and [13].

Our purpose in Part I of the thesis is to study a bounded bilinear map induced by a unitary representation π of a locally compact group G and the topological center problems related to it. Part I is organized as follows. In chapter 2, we introduce some notations in abstract harmonic analysis, defining the bounded bilinear map induced by a unitary representation π, giving some preliminary results. In chapter 3, we study cases under which the topological center is maximal, minimal nor neither. A characterization of the maximality of the topological center will be demonstrated and various examples will be given. In chapter 4, we investigate the topological centers of bilinear maps induced by direct sums, tensor products, subrepresentations of given representations.

A locally compact group is a group equipped with a locally compact Hausdorff topology such that the group operations are continuous with respect to that topology, i.e. both the multiplication $G \times G \to G$, $(x, y) \mapsto xy$ and the inverse $G \to G$, $x \mapsto x^{-1}$ are continuous. Let G be a locally compact group. It is well-known that G possesses a positive Radon measure m which is invariant under left translation, i.e. $m(E) = m(xE)$ for any Borel set $E \subseteq G$ and $x \in G$. We call such a measure a left Haar measure of the group G, denoted by m, dm, $dm(x)$ or simply dx. We remark that left Haar measure of the group G is unique up to a positive scalar multiple. Given a locally compact group G, we fix a left Haar measure once and forever.

We denote the modular function associated to the group G by Δ_G or simply Δ, which is a continuous group homomorphism $\Delta_G : G \to (0, \infty)$ from G into the multiplicative group of positive real numbers. We remark that the modular function has the following properties:

$$dm(xy_0) = \Delta(y_0)dm(x), \tag{1.1}$$

$$dm(x^{-1}) = \Delta(x^{-1})dm(x). \tag{1.2}$$

($x, y_0 \in G$; y_0 is regarded as a constant while x is regarded as a variable)

For $1 \leq p \leq \infty$, let $(L^p(G), \|\cdot\|_p)$ denotes the usual Banach space associated with G and m. For $p = 2$, $L^p(G)$ is a Hilbert space with the inner product $\langle f \,|\, g\rangle = \int f(x)\overline{g(x)}\,dx$. For a function $f : G \to \mathbb{C}$ and $x \in G$, we define the left translation of f by x by $l_x f : G \to \mathbb{C}$, $l_x f(y) = f(xy)$. Similarly, we define the right translation of f by x by $r_x f : G \to \mathbb{C}$, $r_x f(y) = f(yx)$. Sometime, we also denote $l_x f$ and $r_x f$ simply by ${}_x f$ and f_x respectively. We denote the space of all bounded complex-valued continuous functions on G by $\mathrm{CB}(G)$. If $f \in \mathrm{CB}(G)$, we identify f with its equivalence class in $L^\infty(G)$. With this identification, $\mathrm{CB}(G)$ is a unital C*-subalgebra of $L^\infty(G)$. Let $f \in \mathrm{CB}(G)$. If the map $G \to \mathrm{CB}(G)$, $x \mapsto l_x f$ is continuous with respect to the $\|\cdot\|_\infty$-norm topology, we say that f is left uniformly continuous. We denote the set of all left uniformly continuous functions by $\mathrm{LUC}(G)$. Similarly, we say that f is right uniformly continuous if the map $G \to \mathrm{CB}(G)$, $x \mapsto r_x f$ is continuous with respect to the $\|\cdot\|_\infty$-norm topology. We denote the set of all right uniformly continuous functions by $\mathrm{RUC}(G)$. We remark that $\mathrm{LUC}(G)$ and $\mathrm{RUC}(G)$ are unital C*-subalgebras of $\mathrm{CB}(G)$. Let $\mathrm{UCB}(G) = \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$. We say that f is a uniformly continuous function if and only if $f \in \mathrm{UCB}(G)$. It should be noted that in [21], $\mathrm{LUC}(G)$ (resp. $\mathrm{RUC}(G)$) is precisely the space of right (resp. left) uniformly continuous functions on G.

As well-known, the dual space of $\mathrm{LUC}(G)$, denoted by $\mathrm{LUC}(G)^*$, can be made into a Banach algebra as follows. Let $m, n \in \mathrm{LUC}(G)^*$, $f \in \mathrm{LUC}(G)$, $x \in G$. We define $m_l f : G \to \mathbb{C}$ by $m_l f(x) = \langle m, l_x f\rangle$. It is easy to check that $m_l f \in \mathrm{LUC}(G)$. Define $mn \in \mathrm{LUC}(G)^*$ by $\langle mn, f\rangle = \langle m, n_l f\rangle$. When equipped with the product $(m, n) \mapsto mn$, $\mathrm{LUC}(G)^*$ becomes a Banach algebra. The reader is referred to Lau [28] for more details. Let $\mathrm{M}(G)$ be the Banach algebra of all the regular complex Borel measures on G. Lau showed that $\mathrm{M}(G)$ can be embedded into $\mathrm{LUC}(G)^*$ as a closed subalgebra (see [28]). At later time, Ghahramani, Lau and Losert improved that result and proved the following lemma in [16].

Lemma 1.1. *The map* $\theta : M(G) \to LUC(G)^*$ *defined by* $\langle \theta(\mu), f \rangle = \int f(x)\, d\mu(x)$, $(\mu \in M(G), f \in LUC(G))$ *is an isometric algebra homomorphism. Moreover, we have:*

(a) $LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$, *and*

(b) $C_0(G)^\perp$ *is a closed two-sided ideal of* $LUC(G)^*$.

Let X be a Banach space and let G be a locally compact group. We say that X is a Banach G-module if G acts on X as bounded invertible operators with norm less than or equal to one such that the action is continuous with respect to the norm topology. More precisely, it means that there exists a map $X \times G \to X$ with the following properties:

- For each $\xi \in X$, $x, y \in G$, we have $\xi \cdot e = \xi$ and $(\xi \cdot x) \cdot y = \xi \cdot (xy)$.
- For each $x \in G$, the map $\xi \mapsto \xi \cdot x$ is a bounded, invertible linear operator on X with norm less than or equal to one.
- For each $\xi \in X$, the map $x \mapsto \xi \cdot x$ is continuous with respect to the norm topology.

We refer the reader [31] for details.

By a unitary representation π of a locally compact group G, we mean a group homomorphism $\pi : G \to \mathrm{B}(\mathcal{H})$ from G into the group of unitary operators acting on some Hilbert space $\mathcal{H}$ such that the map $x \mapsto \pi(x)$ is continuous with respect to the strong operator topology, i.e. for each $\xi \in \mathcal{H}$, the map $x \mapsto \pi(x)\xi$ is continuous. By integration, we obtain a non-degenerate *-representation, still denoted by π, of $L^1(G)$ on the Hilbert space $\mathcal{H}$, namely $f \mapsto \pi(f) = \int f(x)\pi(x)\,dx$.

Chapter 2

Topological Centers of a Left LUC$(G)^*$-Module Action

In this chapter, we associate a module action to a given unitary representation. The topological center of that module action will be defined and some preliminary results will be given.

2.1 Module Actions

Let $\pi : G \to \mathrm{B}(\mathcal{H})$ be a unitary representation of a locally compact group G. Bekka and Xu defined a unital C*-subalgebra of $\mathrm{B}(\mathcal{H})$ and a bilinear map as follows. See [4] and [42].

Define a map $\mathrm{B}(\mathcal{H}) \times G \to \mathrm{B}(\mathcal{H})$ by $T \cdot x = \pi(x^{-1})T\pi(x)$, then $\mathrm{B}(\mathcal{H})$ becomes a G-module. Define $\mathrm{UCB}(\pi) = \{T \in \mathrm{B}(\mathcal{H}) \mid \text{ The map } G \to \mathrm{B}(\mathcal{H}),\ x \mapsto T \cdot x \text{ is continuous in norm topology.}\}$, then $\mathrm{UCB}(\pi)$ is a unital C*-subalgebra of $\mathrm{B}(\mathcal{H})$. When the G-module action is restricted on $\mathrm{UCB}(\pi)$, $\mathrm{UCB}(\pi)$ becomes a Banach G-module.

Lemma 2.1. *The mapping $UCB(\pi)^* \times UCB(\pi) \to LUC(G)$, $(M, T) \mapsto MT$,* *defined by* $MT(x) = \langle M, T\cdot x\rangle$ *is bilinear and* $\|MT\|_\infty \le \|M\|\|T\|$.

Proof. Let $x \in G$, then $|MT(x)| = |\langle M, T\cdot x\rangle| \le \|M\|\|T\cdot x\| \le \|M\|\|T\|$, hence MT is a bounded function on G. Let (x_α) be a net in G such that $x_\alpha \to x \in G$, then $|MT(x_\alpha) - MT(x)| \le \|M\|\|T\cdot x_\alpha - T\cdot x\| \to 0$. Therefore, MT is a continuous function. Let $x_\alpha, x, y \in G$ such that $x_\alpha \to x$, then

$$\begin{aligned} & |_{x_\alpha}(MT)(y) -_x (MT)(y)| = |MT(x_\alpha y) - MT(xy)| \\ = \;& |M(T\cdot x_\alpha)(y) - M(T\cdot x)(y)| = |\langle \delta_y M, T\cdot x_\alpha - T\cdot x\rangle| \\ \le \;& \|\delta_y M\|\|T\cdot x_\alpha - T\cdot x\| \le \|M\|\|T\cdot x_\alpha - T\cdot x\| \to 0 \end{aligned}$$

uniformly about y, where $\delta_y M \in \mathrm{UCB}(\pi)^*$ is defined by $\langle \delta_y M, T\rangle = \langle M, T\cdot y\rangle$. Therefore $MT \in$LUC(G). The checking of the bilinearity of the mapping $(M,T) \mapsto MT$ is left to the reader. □

Next, we define a map $\mathrm{LUC}(G)^* \times \mathrm{UCB}(\pi)^* \to \mathrm{UCB}(\pi)^*$ by $(m, M) \mapsto mM$, where $\langle mM, T\rangle = \langle m, MT\rangle$, $T \in \mathrm{UCB}(\pi)$. It is routine to check that the map is a bounded bilinear map with $\|mM\| \le \|m\|\|M\|$.

Proposition 2.2. *With the mapping* $(m, M) \mapsto mM$ *defined above,* $\mathrm{UCB}(\pi)^*$ *becomes a left Banach* $\mathrm{LUC}(G)^*$*-module with* $\|mM\| \le \|m\|\|M\|$ *and* $\delta_e M = M$.

Proof. Let $T \in \mathrm{UCB}(\pi)$, $M \in \mathrm{UCB}(\pi)^*$, $x, y \in G$, then

$$\begin{aligned} & {}_x(MT)(y) = (MT)(xy) = \langle M,\, T\cdot xy\rangle \\ = \;& \langle M,\, (T\cdot x)\cdot y\rangle = M(T\cdot x)(y). \end{aligned}$$

Therefore ${}_x(MT) = M(T\cdot x)$. Next, for all $n \in \mathrm{LUC}(G)^*$,

$$\begin{aligned} & n_l(MT)(x) = \langle n,\, {}_x(MT)\rangle = \langle n,\, M(T\cdot x)\rangle \\ = \;& \langle nM,\, T\cdot x\rangle = (nM)(T)(x) \end{aligned}$$

and hence $n_l(MT) = (nM)T$. Therefore

$$\begin{aligned} & \langle (mn)M,\, T\rangle = \langle mn,\, MT\rangle \\ = \;& \langle m,\, n_l(MT)\rangle = \langle m,\, (nM)T\rangle = \langle m(nM),\, T\rangle, \end{aligned}$$

so $(mn)M = m(nM)$. Also

$$\begin{aligned} & \langle \delta_e M,\, T\rangle = \langle \delta_e,\, MT\rangle = (MT)(e) \\ = \;& \langle M,\, T\cdot e\rangle = \langle M,\, T\rangle, \end{aligned}$$

hence $\delta_e M = M$. □

Remark 2.3. The bilinear map $(m, M) \mapsto mM$ constructed above coincides with Arens' construction when G is discrete. We recall Arens' construction. Let X, Y, Z be Banach spaces and let $\theta : X \times Y \to Z$ be a bounded bilinear map. Define $\theta^* : Z^* \times X \to Y^*$, the adjoint of θ, by $\langle \theta^*(z', x), y\rangle = \langle z', \theta(x, y)\rangle$ $(x \in X,\, y \in Y,\, z' \in Z^*)$. The above process can be repeated and we define $\theta^{**} = (\theta^*)^* : Y^{**} \times Z^* \to X^*$ and $\theta^{***} = (\theta^{**})^* : X^{**} \times Y^{**} \to Z^{**}$. If G is a discrete group, both $\mathrm{UCB}(\pi)$ and $\mathrm{LUC}(G)$ have preduals. Namely, $\mathrm{UCB}(\pi) = \mathrm{B}(\mathcal{H}) = \mathrm{L}_1(\mathcal{H})^*$, where $\mathrm{L}_1(\mathcal{H})$, equipped with the trace-class norm, is the Banach space of all trace-class operators on the Hilbert space $\mathcal{H}$. (see [40] Chapter II, Section 1) and $\mathrm{LUC}(G) = l_\infty(G) = l_1(G)^*$. Define a G-module action on the space $\mathrm{L}_1(\mathcal{H})$ by $G \times \mathrm{L}_1(\mathcal{H}) \to \mathrm{L}_1(\mathcal{H})$, $(x, L) \mapsto x \cdot L = \pi(x)L\pi(x^{-1})$. By integration, we obtain a Banach $l_1(G)$-module $\mathrm{L}_1(\mathcal{H})$, namely $l_1(G) \times \mathrm{L}_1(\mathcal{H}) \to \mathrm{L}_1(\mathcal{H})$,

$$(f, L) \mapsto f \cdot L = \int f(x) x \cdot L\, dx = \sum_{x\in G} f(x) x \cdot L.$$

Define a bounded bilinear map $\theta : l_1(G) \times \mathrm{L}_1(\mathcal{H}) \to \mathrm{L}_1(\mathcal{H})$ by $\theta(f, L) = f \cdot L$.

Let $L \in \mathrm{L}_1(\mathcal{H})$, $T \in \mathrm{B}(\mathcal{H})$, $M \in \mathrm{B}(\mathcal{H})^*$, $x \in G$, then

$$\begin{aligned} &\langle \theta^*(T, \delta_x), L\rangle = \langle T, \theta(\delta_x, L)\rangle = \langle T, x \cdot L\rangle = \mathrm{tr}(T\pi(x)L\pi(x^{-1})) \\ = \ &\mathrm{tr}(\pi(x^{-1})T\pi(x)L) = \mathrm{tr}((T \cdot x)L) = \langle T \cdot x, L\rangle. \end{aligned}$$

Therefore $\theta^*(T, \delta_x) = T \cdot x$ and $\langle \theta^{**}(M, T), \delta_x\rangle = \langle M, \theta^*(T, \delta_x)\rangle = \langle M, T \cdot x\rangle = MT(x)$, hence $\theta^{**}(M, T) = MT$. Finally, $\langle \theta^{***}(m, M), T\rangle = \langle m, \theta^{**}(M, T)\rangle = \langle m, MT\rangle = \langle mM, T\rangle$. Therefore $\theta^{***}(m, M) = mM$.

2.2 Topological Centers

In this section, we define the notion of topological center. Using the notation defined in the previous section, it is obvious that for each fixed $M \in \mathrm{UCB}(\pi)^*$, the map $\mathrm{LUC}(G)^* \to \mathrm{UCB}(\pi)^*$, $m \mapsto mM$ is weak*-weak* continuous. However, it is false that for each $m \in \mathrm{LUC}(G)^*$, the map $\mathrm{UCB}(\pi)^* \to \mathrm{UCB}(\pi)^*$, $M \mapsto mM$ is weak*-weak* continuous. A natural question arises: For what m is the mapping $M \mapsto mM$ weak*-weak* continuous ? Therefore it makes sense to define

$$\mathrm{Z}(\pi) = \{m \in \mathrm{LUC}(G)^* \mid \text{ The map } \mathrm{UCB}(\pi)^* \to \mathrm{UCB}(\pi)^*,$$

$$M \to mM \text{ is weak*-weak* continuous.}\},$$

the topological center of the module action induced by π. Note that $\mathrm{Z}(\pi)$ contains $\mathrm{M}(G)$. Before proving this result, we would first state a proposition which characterizes $\mathrm{Z}(\pi)$.

Proposition 2.4. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group G and let $UCB(\pi)$ and $Z(\pi)$ be defined as above. For each $m \in LUC(G)^*$, the following are equivalent:*

1. *For each $T \in UCB(\pi)$, the map $Tm : UCB(\pi)^* \to \mathbb{C}$ defined by $\langle M\, , Tm\rangle = \langle mM\, , T\rangle$ ($M \in UCB(\pi)^*$) lies in $UCB(\pi)$.*

2. *$m \in Z(\pi)$.*

3. *The map $UCB(\pi)^* \to UCB(\pi)^*$, $M \mapsto mM$ is weak*-weak* continuous on all bounded parts of $UCB(\pi)^*$.*

Proof. The equivalence of (1) and (2) are clear and the implication of (2)⇒(3) is trivial. Now we prove that (3)⇒(1). Suppose that the mapping $M \mapsto mM$ is weak*-weak* continuous on all bounded parts of $\mathrm{UCB}(\pi)^*$. Let $T \in \mathrm{UCB}(\pi)$ be fixed, then the linear functional $Tm \in \mathrm{UCB}(\pi)^{**}$ is $\sigma(\mathrm{UCB}(\pi)^*, \mathrm{UCB}(\pi))$ continuous on any bounded part of $\mathrm{UCB}(\pi)^*$. By [10] [V.5.6], Tm is a $\sigma(\mathrm{UCB}(\pi)^*, \mathrm{UCB}(\pi))$ continuous linear functional on $\mathrm{UCB}(\pi)^*$ and hence $Tm \in \mathrm{UCB}(\pi)$ by [8] P.125 Theorem 1.3. □

Lemma 2.5. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group G. Then $M(G) \subseteq Z(\pi)$, where $M(G)$ is regarded as a subspace of $LUC(G)^*$ as in lemma 1.1*

Proof. Define a map $\mathrm{UCB}(\pi) \times \mathrm{LUC}(G)^* \to \mathrm{UCB}(\pi)^{**}$ by $(T, m) \mapsto Tm$, where $\langle M, Tm\rangle = \langle mM, T\rangle$, $M \in \mathrm{UCB}(\pi)^*$. The map is clearly bilinear and $\|Tm\| \le \|T\|\|m\|$. Let $m \in \mathrm{M}(G)$ with $m \neq 0$, $T \in \mathrm{UCB}(\pi)$ and $\epsilon > 0$. First, we assume that the support of m, denoted by $K = \mathrm{supp}(m)$ is compact. Choose a finite partition $\{E_i \,|i = 1, 2, \ldots, n\}$ of K consisting of Borel sets E_i such that $\|T\cdot x - T\cdot y\| < \epsilon/|m|(K)$ whenever $x, y \in E_i$. This is possible by the uniform continuity of the map $x \mapsto T\cdot x$ on the compact set K. For each

i, fix $x_i \in E_i$. Let $M \in \mathrm{UCB}(\pi)^*$, then

$$
\begin{aligned}
&|\langle M\,,\, Tm - \sum_{i=1}^{n} m(E_i) T\cdot x_i\rangle| = |\langle m\,,\, MT\rangle - \sum_{i=1}^{n} m(E_i)\langle M\,,\, T\cdot x_i\rangle| \\
= \;& |\int_K MT(x)\, dm(x) - \sum_{i=1}^{n} m(E_i)\langle M\,,\, T\cdot x_i\rangle| \\
= \;& |\sum_{i=1}^{n}\int_{E_i}\langle M\,,\, T\cdot x - T\cdot x_i\rangle\, dm(x)| \le \sum_{i=1}^{n}\|M\|\frac{\epsilon}{|m|(K)}|m|(E_i) = \epsilon\|M\|.
\end{aligned}
$$

Therefore $\|Tm - \sum_{i=1}^{n} m(E_i)T\cdot x_i\| \le \epsilon$. Since Tm can be approximated by a sequence of elements in $\mathrm{UCB}(\pi)$ with respect to the norm topology, $Tm \in \mathrm{UCB}(\pi)$. If the support of the measure m is not compact, we can choose a sequence of measures $(\mu_n)_n$ with compact supports such that $\|m-\mu_n\| \to 0$. Then $\|Tm - T\mu_n\| \le \|T\|\|m-\mu_n\| \to 0$, so $Tm \in \mathrm{UCB}(\pi)$. By proposition 2.4 , $m \in \mathrm{Z}(\pi)$. □

Proposition 2.6. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group G and let UCB(π) and Z(π) be defined as above. Then the topological center Z(π) is a Banach subalgebra of LUC(G)* containing M(G).*

Proof. By proposition 2.4 (1) and the previous lemma, $\mathrm{Z}(\pi)$ is a subalgebra of $\mathrm{LUC}(G)^*$ containing $\mathrm{M}(G)$. Let (m_k) be a sequence in $\mathrm{Z}(\pi)$ such that $m_k \to m \in \mathrm{LUC}(G)^*$ with respect to the norm topology. Let (M_α) be a bounded net in $\mathrm{UCB}(\pi)^*$ such that $M_\alpha \to M \in \mathrm{UCB}(\pi)^*$ with respect to the weak* topology. Let $T \in \mathrm{UCB}(\pi)$ and let $\epsilon > 0$. Choose $K > 0$ such that $\|M_\alpha\| \le K$ for all α. Fix k such that $\|m_k - m\| < \epsilon/(K(\|T\|+1))$, then

$$
\begin{aligned}
&|\langle mM_\alpha, T\rangle - \langle mM, T\rangle| \\
\le \;& |\langle m_k(M_\alpha - M), T\rangle| + |\langle (m-m_k)M_\alpha, T\rangle| + |\langle (m_k - m)M, T\rangle| \\
\le \;& |\langle m_k(M_\alpha - M), T\rangle| + 2\epsilon.
\end{aligned}
$$

Consequently $\limsup_\alpha |\langle mM_\alpha, T\rangle - \langle mM, T\rangle| \le 2\epsilon$ and hence $\langle mM_\alpha, T\rangle \to \langle mM, T\rangle$, i.e. $m \in \mathrm{Z}(\pi)$ by proposition 2.4. Therefore $\mathrm{Z}(\pi)$ is closed. □

Chapter 3

Minimality and Maximality of the Topological Centers

In this chapter we study two extreme cases about topological centers, namely, minimality and maximality of topological centers. An example whose topological center is neither minimal nor maximal will be demonstrated.

3.1 Minimality of the Topological Centers

In this section, we state a theorem which characterizes the minimality of a topological center (i.e. $\mathrm{Z}(\pi) = \mathrm{M}(G)$) in terms of a factorization property. It follows immediately that the topological center of the module action induced by the left regular representation of any locally compact group is always minimal. Lastly, we give an example that the topological center of a countable direct sum of finite dimensional representations is minimal.

Before stating the main theorem, we need a few lemmas. The first lemma may be well known, however, we include a proof for completeness.

Lemma 3.1. *Let G be a locally compact group and let K_1, K_2 be two disjoint compact subsets of G, then there exists a compact, symmetric neighborhood U of e such that K_1U and K_2U are disjoint.*

Proof. First we claim that for any $y \in K_2$, there exist open neighborhoods U_y, V_y of e such that $K_1U_y \cap yV_y = \emptyset$. Let $y \in K_2$ be given. For each $x \in K_1$, there exist open neighborhoods S_x and T_x of e such that $xS_xS_x \cap yT_x = \emptyset$. Note that $\{xS_x \,|\, x \in K_1\}$ covers K_1, so we may select a finite subcover $\{x_iS_{x_i} \,|\, i = 1, 2, \ldots, n\}$. Define $U_y = \bigcap_{i=1}^n S_{x_i}$, $V_y = \bigcap_{i=1}^n T_{x_i}$, then U_y and V_y are open neighborhoods of e. We show that $K_1U_y \cap yV_y = \emptyset$. Let $x \in K_1$, then $x \in x_iS_{x_i}$ for some i. Consequently $xU_y \cap yV_y \subseteq x_iS_{x_i}S_{x_i} \cap yT_{x_i} = \emptyset$ and hence $K_1U_y \cap yV_y = \emptyset$.

By the above claim, for each $y \in K_2$, we may choose open neighborhoods U_y and V_y for e such that $K_1U_y \cap yV_yV_y = \emptyset$. Note that $\{yV_y \,|\, y \in K_2\}$ covers K_2, so we may select a finite subcover $\{y_iV_{y_i} \,|\, i = 1, 2, \ldots, m\}$. Define open neighborhoods $U = \bigcap_{i=1}^m U_{y_i}$, $V = \bigcap_{i=1}^m V_{y_i}$ of e. Let $y \in K_2$, then $y \in y_iV_{y_i}$ for some i. Therefore $K_1U \cap yV \subseteq K_1U_{y_i} \cap y_iV_{y_i}V_{y_i} = \emptyset$ and hence $K_1U \cap K_2V = \emptyset$. We finish the proof by choosing a compact, symmetric neighborhood of e contained in $U \cap V$. □

Lemma 3.2. *Let G be a locally compact, non-compact group and let K_1 and K_2 be two disjoint compact subsets of G. Then there exists a compact, symmetric neighborhood U of e and a sequence (x_n) in G such that:*

1. *$K_1Ux_i \cap K_1Ux_j = \emptyset$ whenever $i \neq j$,*
2. *$K_2Ux_i \cap K_2Ux_j = \emptyset$ whenever $i \neq j$,*
3. *$K_1Ux_i \cap K_2Ux_j = \emptyset$ for any i, j.*

Moreover the set $\{x_n \,|\, n \in \mathbb{N}\}$ is closed but not compact.

Proof. Without loss of generality, we assume that K_1 and K_2 are non-empty. By lemma 3.1, we choose a compact, symmetric neighborhood U of e such that $K_1U \cap K_2U = \emptyset$ and construct a sequence (x_n) inductively. Let $x_1 = e$. Suppose that $x_1, x_2, \ldots, x_n$ have been choosen such that:

$K_1Ux_i \cap K_1Ux_j = \emptyset$ whenever $i \neq j$, and $K_2Ux_i \cap K_2Ux_j = \emptyset$ whenever $i \neq j$, and

$K_1Ux_i \cap K_2Ux_j = \emptyset$ for any $i, j \in \{1, 2, \ldots n\}$.

We assert that there exists $y \in G$ such that:

$K_1Ux_i \cap K_1Uy = \emptyset$ for $i = 1, 2, \ldots, n$ and

$K_2Ux_i \cap K_2Uy = \emptyset$ for $i = 1, 2, \ldots, n$ and

$K_1Ux_i \cap K_2Uy = \emptyset$ for $i = 1, 2, \ldots, n$ and

$K_1Uy \cap K_2Ux_i = \emptyset$ for $i = 1, 2, \ldots, n$.

Suppose the contrary that the assertion is false, then for any $y \in G$, we have

$$y \in \bigcup_{i=1}^{n}[U^{-1}K_1^{-1}K_1Ux_i \cup U^{-1}K_2^{-1}K_2Ux_i \cup U^{-1}K_2^{-1}K_1Ux_i \cup U^{-1}K_1^{-1}K_2Ux_i]$$

and hence

$$G = \bigcup_{i=1}^{n}[U^{-1}K_1^{-1}K_1Ux_i \cup U^{-1}K_2^{-1}K_2Ux_i \cup U^{-1}K_2^{-1}K_1Ux_i \cup U^{-1}K_1^{-1}K_2Ux_i],$$

which is a contradiction since the set on the right is compact. Choose $x_{n+1} = y$, where $y \in G$ is any element which satisfies the above condition. By induction, we obtain a sequence (x_n) in G. Clearly (i) and (ii) are satisfied by our construction. For (iii), if $i = j$, $K_1Ux_i \cap K_2Ux_i = (K_1U \cap K_2U)x_i = \emptyset$. If $i \neq j$, $K_1Ux_i \cap K_2Ux_j = \emptyset$ by our construction. We show that $\{x_n \mid n \in \mathbb{N}\}$ is a closed, non-compact subset of G. Suppose the contrary that $\{x_n \mid n \in \mathbb{N}\}$ is compact. For the net $(x_n)_n$, it has a subnet $(x_{n_\alpha})_\alpha$ which converges to a point, say x_k in $\{x_n \,|\, n \in \mathbb{N}\}$. Choose α_0 such that $x_{n_\alpha} \in Ux_k$ whenever $\alpha \succeq \alpha_0$. Choose α_1 such that $n_\alpha \geq k+1$ whenever $\alpha \succeq \alpha_1$. Choose α_2 such that $\alpha_2 \succeq \alpha_1$ and $\alpha_2 \succeq \alpha_0$, then $x_{n_{\alpha_2}} \in Ux_k$. Clearly $x_{n_{\alpha_2}} \in Ux_{n_{\alpha_2}}$. However $n_{\alpha_2} \geq k+1$ implies that $x_{n_{\alpha_2}} \neq x_k$, contradicting to $Ux_k \cap Ux_{n_{\alpha_2}} = \emptyset$. Let $(x_\alpha)_\alpha$ be a net in $\{x_n \,|\, n \in \mathbb{N}\}$ which converges to $x \in G$. We assert that there exists α_0 such that $x_\alpha = x_{\alpha_0}$ whenever $\alpha \succeq \alpha_0$. In that case, $x_\alpha \to x_{\alpha_0} \in \{x_n \,|\, n \in \mathbb{N}\}$. Suppose the contrary. Choose an open neighborhood V of e such that $VV^{-1} \subseteq U$. Choose α_0 such that $x_\alpha \in Vx$ whenever $\alpha \succeq \alpha_0$. By assumption, there exist $\alpha_1, \alpha_2 \succeq \alpha_0$ such that $x_{\alpha_1} \neq x_{\alpha_2}$. Note that $x_{\alpha_1} \in Vx$ and $x_{\alpha_2} \in Vx$, so $x_{\alpha_1}x_{\alpha_2}^{-1} \in Vxx^{-1}V^{-1} \subseteq U$ and hence $x_{\alpha_1} \in Ux_{\alpha_2}$, which is a contradiction. □

Lemma 3.3. *Let G be a locally compact, non-compact group and let K_1,K_2 be two disjoint, compact subsets of G. Let U, (x_n) be the compact, symmetric neighborhood of e and the sequence in G respectively as in the previous lemma. Then there exists $f \in LUC(G)\backslash C_0(G)$, $0 \leq f \leq 1$, such that $f = 1$ on $\bigcup_{n=1}^\infty K_1x_n$ and f vanishes outside $\bigcup_{n=1}^\infty K_1Ux_n$. In particular, $f = 0$ on $\bigcup_{n=1}^\infty K_2Ux_n$.*

Proof. By Urysohn Lemma, we choose $g \in C_0(G)$, $0 \leq g \leq 1$, such that $g = 1$ on K_1 and g vanishes outside K_1U. Define $f : G \to [0, +\infty]$ by $f(x) = \sum_{n=1}^\infty g(xx_n^{-1})$. Note that for each $x \in G$, there exists at most one n such that $g(xx_n^{-1}) \neq 0$. For, if $m \neq n$ but $g(xx_n^{-1}) \neq 0$ and $g(xx_m^{-1}) \neq 0$, then $xx_n^{-1} \in K_1U$ and $xx_m^{-1} \in K_1U$, hence $x \in K_1Ux_n \cap K_1Ux_m$ which is impossible. It is immediate that $0 \leq f \leq 1$ and f is a Borel function. Let $(t_\alpha)_\alpha$ be a net in G such that $t_\alpha \to t \in G$. Let $s \in G$ be arbitrary. Let $\epsilon > 0$. Choose α_0 such that $\|_{t_\alpha}g -_t g\| < \epsilon$ whenever $\alpha \succeq \alpha_0$. Let $\alpha \succeq \alpha_0$. Note that there are at most two integers n such that the term $|g(t_\alpha sx_n^{-1}) - g(tsx_n^{-1})|$ is non-zero. Moreover, for such non-zero terms, we have $|g(t_\alpha sx_n^{-1}) - g(tsx_n^{-1})| \leq \|_{t_\alpha}g -_t g\| < \epsilon$. Therefore

$|{}_{t_\alpha}f(s) - {}_t f(s)| = |f(t_\alpha s) - f(ts)| \leq \sum_{n=1}^{\infty} |g(t_\alpha s x_n^{-1}) - g(ts x_n^{-1})| < 2\epsilon$, so $f \in \mathrm{LUC}(G)$. By the construction, it is clear that $f = 1$ on $\bigcup_{n=1}^{\infty} K_1 x_n$ and $f = 0$ on $(\bigcup_{n=1}^{\infty} K_1 U x_n)^c$. In particular, $f \notin \mathrm{C}_0(G)$. Since $\bigcup_{n=1}^{\infty} K_1 U x_n$ and $\bigcup_{n=1}^{\infty} K_2 U x_n$ are disjoint, $f = 0$ on $\bigcup_{n=1}^{\infty} K_2 U x_n$. □

Remark 3.4. A similar technique was used by Granirer and Lau. We refer the reader to [18, lemma 4].

Lemma 3.5. *Let G be a locally compact group. Given $\mu \in M(G)$ and $f \in LUC(G)$, we define $f \cdot \mu(x) = \int f(yx)\, d\mu(y)$, then $f \cdot \mu \in LUC(G)$. Moreover, $\|f \cdot \mu\|_\infty \leq \|f\|_\infty \|\mu\|$.*

Proof. Let $M = \|\mu\| + 1$. Let $\epsilon > 0$, then there exists an open neighborhood U of e such that $|f(x) - f(y)| < \epsilon/M$ whenever $xy^{-1} \in U$. First, we assume that the support of μ, denoted by K, is compact. For each $y \in K$, there exists a symmetric open neighborhood V_y of e such that $V_y V_y V_y \subseteq y^{-1} U y$. Note that $\{yV_y \,|\, y \in K\}$ is an open covering of K, so we may choose a finite subcover $\{y_i V_{y_i} \,|\, i = 1, 2, \ldots, n\}$. Define $V = \bigcap_{i=1}^{n} V_{y_i}$. Let $x_1, x_2 \in G$ such that $x_1 x_2^{-1} \in V$. Let $y \in K$ be arbitrary, then $y \in y_i V_{y_i}$ for some i. Therefore $(yx_1)(yx_2)^{-1} = y x_1 x_2^{-1} y^{-1} \in y_i V_{y_i} V_{y_i} (V_{y_i})^{-1} y_i^{-1} = y_i V_{y_i} V_{y_i} V_{y_i} y_i^{-1} \subseteq U$ and hence $|f(yx_1) - f(yx_2)| < \epsilon/M$. Consequently

$$|f \cdot \mu(x_1) - f \cdot \mu(x_2)| \leq \int_K |f(yx_1) - f(yx_2)|\, d|\mu|(y) \leq \epsilon \|\mu\|/M < \epsilon.$$

This proves that $f \cdot \mu \in \mathrm{LUC}(G)$. It is clear that $\|f \cdot \mu\|_\infty \leq \|f\|_\infty \|\mu\|$ for general $\mu \in \mathrm{M}(G)$, $f \in \mathrm{LUC}(G)$. Lastly, if the support of μ is not compact, we may, by inner regularity of μ, choose a sequence (μ_n) in $\mathrm{M}(G)$, with $\mathrm{supp}(\mu_n)$ compact and $\|\mu_n - \mu\| \to 0$, then $\|f \cdot \mu - f \cdot \mu_n\| \leq \|f\|_\infty \|\mu_n - \mu\| \to 0$. As $f \cdot \mu_n \in \mathrm{LUC}(G)$ and $\mathrm{LUC}(G)$ is a closed subspace of $L_\infty(G)$, $f \cdot \mu \in \mathrm{LUC}(G)$. □

Now, we are ready to state the main theorem in this paper which characterizes the minimality of the topological center $Z(\pi)$ in terms of a factorization property. The forward implication of that theorem is inspired by [14, Theorem 3.1] .

Theorem 3.6. *Let G be a locally compact, non-compact group and let $\pi : G \to B(\mathcal{H})$ be a continuous unitary representation. Let $\mathcal{F} = \{MT \mid M \in UCB(\pi)^*, T \in UCB(\pi)\}$, then the following are equivalent:*
(a) The linear span of $\mathcal{F}$ is norm dense in $LUC(G)$.
(b) $Z(\pi) = M(G)$.

Proof. We prove that (a)⇒(b). Suppose that the condition in (a) holds. Let $\mathcal{Z}$ be the topological center of $\mathrm{LUC}(G)^*$, i.e. $\mathcal{Z}$ is the subset of $\mathrm{LUC}(G)^*$ which consists of all m such that the map $\mathrm{LUC}(G)^* \to \mathrm{LUC}(G)^*$, $n \mapsto mn$ is weak*-weak* continuous. Recall that $\mathcal{Z} = \mathrm{M}(G)$ by [28] and we already know that $\mathrm{M}(G) \subseteq \mathrm{Z}(\pi)$, so we will finish the proof once we show that $\mathrm{Z}(\pi) \subseteq \mathcal{Z}$. Let $m \in \mathrm{Z}(\pi)$. To prove that $m \in \mathcal{Z}$, it suffices that the map $\mathrm{LUC}(G)^* \to \mathrm{LUC}(G)^*$, $n \mapsto mn$ is weak*-weak* continuous on all bounded parts of $\mathrm{LUC}(G)^*$ (see [28]). Let (n_α) be a bounded net in $\mathrm{LUC}(G)^*$ such that $n_\alpha \to n \in \mathrm{LUC}(G)^*$ with respect to the weak*-topology. Let $f \in \mathrm{LUC}(G)$. First, we assume that $f \in$ span $\mathcal{F}$. Write

$$f = \sum_{i=1}^{k} M_i T_i,$$

where $M_i \in \mathrm{UCB}(\pi)^*$ and $T_i \in \mathrm{UCB}(\pi)$, then

$$
\begin{aligned}
\langle mn_\alpha, f\rangle &= \sum_{i=1}^{k}\langle mn_\alpha, M_iT_i\rangle \\
&= \sum_{i=1}^{k}\langle m, (n_\alpha)_l(M_iT_i)\rangle = \sum_{i=1}^{k}\langle m, (n_\alpha M_i)T_i\rangle \\
&= \sum_{i=1}^{k}\langle m(n_\alpha M_i), T_i\rangle \to \sum_{i=1}^{k}\langle m(nM_i), T_i\rangle \\
&= \sum_{i=1}^{k}\langle m, (nM_i)T_i\rangle = \sum_{i=1}^{k}\langle m, n_l(M_iT_i)\rangle \\
&= \langle mn, f\rangle.
\end{aligned}
$$

Then we drop the assumption that $f \in$ span $\mathcal{F}$. Let $\epsilon > 0$. Choose $K > 0$ such that $\|n_\alpha\| \leq K$ and $\|m\| \leq K$. Choose $f_0 \in$ span $\mathcal{F}$ such that $\|f - f_0\|_\infty < \epsilon/(K^2)$. Since $\langle mn_\alpha, f_0\rangle \to \langle mn, f_0\rangle$, there exists α_0 such that $|\langle mn_\alpha, f_0\rangle - \langle mn, f_0\rangle| < \epsilon$ whenever $\alpha \succeq \alpha_0$. For any $\alpha \succeq \alpha_0$,

$$
\begin{aligned}
& |\langle mn_\alpha, f\rangle - \langle mn, f\rangle| \\
\leq\ & |\langle mn_\alpha, f\rangle - \langle mn_\alpha, f_0\rangle| + |\langle mn_\alpha, f_0\rangle - \langle mn, f_0\rangle| + |\langle mn, f_0\rangle - \langle mn, f\rangle| \\
\leq\ & 3\epsilon.
\end{aligned}
$$

Therefore $m \in \mathcal{Z}$.

We prove the direction (b)$\Rightarrow$(a) by contradiction. Suppose the contrary that the closed linear span of $\mathcal{F} \neq$ LUC(G). Pick $m \in$ LUC$(G)^*$ such that $m \neq 0$ but m vanishes on the closed linear span of $\mathcal{F}$. Note that for any $M \in$ UCB$(\pi)^*$, $mM = 0$. In particular $m \in$ Z(π). By lemma 1.1, we may write $m = m_1 + m_2$ where $m_1 \in$ M(G) and $m_2 \in$ C$_0(G)^\perp$. If $m_2 \neq 0$, we obtain a contradiction immediately since $m_2 = m - m_1 \in$ Z(π). Suppose that $m_2 = 0$. We denote $m = m_1 = \mu \in$ M(G). Now we try to produce another

$m'' \in \mathrm{Z}(\pi)$ with $m'' \neq 0$ and $m'' \in \mathrm{C}_0(G)^{\perp}$, then we will arrive a contradiction. Note that for $M \in \mathrm{UCB}(\pi)^*$, $T \in \mathrm{UCB}(\pi)$, $x, y \in G$, we have

$$\begin{aligned} & (\delta_x M)T(y) = \langle \delta_x M, T \cdot y \rangle = \langle \delta_x, M(T \cdot y) \rangle \\ = \; & (M \cdot (T \cdot y))(x) = \langle M, (T \cdot y) \cdot x \rangle = \langle M, T \cdot (yx) \rangle \\ = \; & (MT)_x(y) \end{aligned}$$

and hence $(MT)_x = (\delta_x M)T$. Therefore

$$\int MT(yx)\, d\mu(y) = \langle \mu, (\delta_x M)T \rangle = 0,$$

for each $x \in G$.

First we consider the case that μ is a signed-measure. By Jordan decomposition theorem, we have $\mu = \mu^+ - \mu^-$. Furthermore, we assume that $\mathrm{supp}(\mu^+) \subseteq K_1$ and $\mathrm{supp}(\mu^-) \subseteq K_2$, where K_1 and K_2 are two disjoint compact subsets of G. Choose a compact, symmetric neighborhood U of e, a sequence (x_n) in G, $f \in \mathrm{LUC}(G) \backslash \mathrm{C}_0(G)$ as in lemma 6.2 and lemma 3.3. By lemma 6.3, $f \cdot \mu \in \mathrm{LUC}(G)$. We assert that $f \cdot \mu \notin \mathrm{C}_0(G)$. Note that the constant function $1 \in \mathcal{F}$. (For, let $T = \mathrm{id}_H$ and choose $M \in \mathrm{UCB}(\pi)^*$ such that $\langle M, T \rangle = 1$), so $\mu(G) = \langle \mu, 1 \rangle = 0$. Therefore $\mu^+(G) = \mu^-(G) \neq 0$. We prove that $f \cdot \mu^+ = \mu^+(G)$ on $\{x_n \,|\, n \in \mathbb{N}\}$ and $f \cdot \mu^- = 0$ on $\{x_n \,|\, n \in \mathbb{N}\}$. Let $y \in K_1$ and $x \in \{x_n \,|\, n \in \mathbb{N}\}$, then $yx \in \bigcup_{n=1}^{\infty} K_1 x_n$, so $f(yx) = 1$. Consequently,

$$f \cdot \mu^+(x) = \int_{K_1} f(yx)\, d\mu^+(y) = \mu^+(K_1) = \mu^+(G)$$

and hence $f \cdot \mu^+ = \mu^+(G)$ on $\{x_n \,|\, n \in \mathbb{N}\}$. If $y \in K_2$, $x \in \{x_n \,|\, n \in \mathbb{N}\}$, then $yx \in \bigcup_{n=1}^{\infty} K_2 x_n \subseteq \bigcup_{n=1}^{\infty} K_2 U x_n$, so $f(yx) = 0$ and consequently $f \cdot \mu^-(x) = \int_{K_2} f(yx)\, d\mu^-(y) = 0$. Therefore $f \cdot \mu = f \cdot \mu^+ - f \cdot \mu^- = \mu^+(G)$ on the non-compact set $\{x_n \,|\, n \in \mathbb{N}\}$. In particular $f \cdot \mu \notin \mathrm{C}_0(G)$. Note that in our case, $\|f\|_\infty = 1$.

Then we consider the case that μ is a general signed measure. Denote $\lambda = \|\mu\| > 0$, then $\mu^+(G) = \mu^-(G) = \lambda/2$. By inner regularity of μ^+, μ^-, we may choose positive measures μ_0^+,μ_0^- with $\mathrm{supp}(\mu_0^+) \subseteq K_1$, $\mathrm{supp}(\mu_0^-) \subseteq K_2$, K_1, K_2 being compact and $0 \le \mu_0^+ \le \mu^+$, $0 \le \mu_0^- \le \mu^-$, $\|\mu^+ - \mu_0^+\| < \lambda/100$, $\|\mu^- - \mu_0^-\| < \lambda/100$. Since μ^+ and μ^- are mutually singular, K_1,K_2 can be chosen such that $K_1 \bigcap K_2 = \emptyset$. By the previous argument, there exists $f \in \mathrm{LUC}(G)$ with $\|f\|_\infty = 1$ such that $f \cdot (\mu_0^+ - \mu_0^-) = \mu_0^+(G) > \lambda/2 - \lambda/100 = 49\lambda/100$ on a non-compact set $\{x_n \,|\, n \in \mathbb{N}\}$. Set $\mu_0 = \mu_0^+ - \mu_0^-$, then $\|\mu - \mu_0\| < \lambda/50$, so $\|f \cdot \mu - f \cdot \mu_0\|_\infty \le \|f\|_\infty \|\mu - \mu_0\| < \lambda/50$. Therefore $|f \cdot \mu(x)| > 49\lambda/100 - \lambda/50$ for any $x \in \{x_n \,|\, n \in \mathbb{N}\}$ hence $f \cdot \mu \notin \mathrm{C}_0(G)$. If $\mu \in \mathrm{M}(G)$ is a complex measure, we may write $\mu = \mu_1 + i\mu_2$ for some finite signed measures μ_1, μ_2. Note that at least one of μ_1, μ_2 is non-zero. Choose $f \in \mathrm{LUC}(G)$ as before, according to the non-zero measure μ_i, then $f \cdot \mu = (f \cdot \mu_1) + i(f \cdot \mu_2)$. Note that both $f \cdot \mu_1$, $f \cdot \mu_2$ are real-valued and at least one of them is not in $\mathrm{C}_0(G)$, so $f \cdot \mu \notin \mathrm{C}_0(G)$.

We conclude that there exists $f_0 \in \mathrm{LUC}(G)$ such that $f_0 {\cdot} \mu \in \mathrm{LUC}(G) \backslash \mathrm{C}_0(G)$. By Hahn Banach theorem, there exists $m' \in \mathrm{LUC}(G)^*$ such that $m'(f_0 \cdot \mu) \neq 0$ while $m' = 0$ on $\mathrm{C}_0(G)$. Define $m'' \in \mathrm{LUC}(G)^*$ by $\langle m'', f\rangle = \langle m', f {\cdot} \mu\rangle$. Clearly if $f \in \mathrm{C}_0(G)$, then $f \cdot \mu \in \mathrm{C}_0(G)$, so $m'' \in \mathrm{C}_0(G)^\perp$. $m''(f_0) = \langle m', f_0 \cdot \mu\rangle \neq 0$, so $m'' \neq 0$ and in particular $m'' \notin \mathrm{M}(G)$. If $M \in \mathrm{UCB}(\pi)^*, T \in \mathrm{UCB}(\pi)$, then $\langle m'', MT\rangle = \langle m', (MT) \cdot \mu\rangle$. However $(MT) \cdot \mu(x) = \int MT(yx)\, d\mu(y) = 0$, so $\langle m'', MT\rangle = 0$. Consequently, $m''M = 0$ for all $M \in \mathrm{UCB}(\pi)^*$. In particular

$m'' \in \mathrm{Z}(\pi)\backslash\mathrm{M}(G)$. □

We notice that in our main theorem, if the closed linear span of $\mathcal{F}$ is strictly contained in LUC(G), the $m \in \mathrm{Z}(\pi)\backslash\mathrm{M}(G)$ constructed has the property that $mM = 0$ for any $M \in \mathrm{UCB}(\pi)^*$. It is interesting to ask: Is it possible to find such a m other than that form ? In the following, we give a sufficient condition which guarantees the existence of $m \in \mathrm{Z}(\pi)\backslash\mathrm{M}(G)$ with the property $mM = M$ for all $M \in \mathrm{UCB}(\pi)^*$.
We recall some facts about Stone-Čech Compactification. Let Ω be a Tychonoff space (i.e. T1 and completely regular). The Stone-Čech compactification $\beta\Omega$ of Ω is defined as the Gelfand spectrum of CB(Ω), the commutative, unital C*-algebra of all bounded, continuous, complex-valued functions defined on Ω. Note that $\beta\Omega$ has the following properties:

1. $\beta\Omega$ is compact.

2. The identity map $\iota : \Omega \to \beta\Omega$ is a topological embedding, i.e. $\iota(\Omega)$ is dense in $\beta\Omega$ and the map $\iota : \Omega \to \iota(\Omega)$ is a homeomorphism.

The reader is referred to [8] P.137-138 for more detail. We remark that a locally compact Hausdorff space is a Tychonoff space. We identify Ω with $\iota(\Omega)$ and simply write ω for $\iota(\omega)$. The following lemma is probably well-known. However, we cannot find a proof from standard textbooks, so we include a proof here for completeness.

Lemma 3.7. *Let Ω be a locally compact Hausdorff space. Let $f \in CB(\Omega)$, then $f \in C_0(\Omega)$ if and only if $\widehat{f}(\omega) = 0$ for any $\omega \in \beta\Omega\backslash\Omega$. ($\widehat{f}$ denotes the Gelfand transform of f.)*

Proof. Let $f \in \mathrm{C}_0(\Omega)$. We prove by contradiction. Suppose that there exists $\omega_0 \in \beta\Omega\backslash\Omega$ such that $\widehat{f}(\omega_0) \neq 0$. Since Ω is dense in $\beta\Omega$, we may choose a net

$(\omega_\alpha)_\alpha$ is Ω such that $\omega_\alpha \to \omega_0$. Fix $\epsilon_0 > 0$ such that $|\widehat{f}(\omega_0)| > \epsilon_0$. By passing to a subnet, we may assume, without lose of generality, that $|\widehat{f}(\omega_\alpha)| > \epsilon_0$ for all α. Let $K = \{\omega \in \Omega \,|\, |\widehat{f}(\omega)| \geq \epsilon_0\}$ which is compact. Choose a subnet $(\omega_{\alpha'})$ of (ω_α) such that $(\omega_{\alpha'})$ converges to some $\omega_0' \in K$. However, $\omega_{\alpha'} \to \omega_0$ and $\omega_0 \neq \omega_0'$, which is a contradiction. Conversely, let $f \in \mathrm{CB}(\Omega)$ such that $\widehat{f}(\omega) = 0$ for each $\omega \in \beta\Omega \backslash \Omega$. Let $\epsilon > 0$ and define $K = \{\omega \in \Omega \,|\, |\widehat{f}(\omega)| \geq \epsilon\}$. Let $(\omega_\alpha)_\alpha$ be a net in K. By regarding $(\omega_\alpha)_\alpha$ as a net in $\beta\Omega$ and by the compactness of $\beta\Omega$, there exists a subnet $(\omega_{\alpha'})$ of (ω_α), and $\omega_0 \in \beta\Omega$ such that $\omega_{\alpha'} \to \omega_0$. Observe that $|\widehat{f}(\omega_0)| = \lim_{\alpha'} |\widehat{f}(\omega_{\alpha'})| \geq \epsilon$, so $\omega_0 \in K$. Therefore K is compact and hence $f \in \mathrm{C}_0(\Omega)$. □

Proposition 3.8. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group. Let $N = \{x \in G \mid T\cdot x = T \text{ for any } T \in UCB(\pi)\}$, the kernel of the G-module action induced by π, which is a closed normal subgroup of G. If N is non-compact, there exists $m \in Z(\pi)\backslash M(G)$ such that $mM = M$ for any $M \in UCB(\pi)^*$. In particular, $M(G)$ is strictly contained in $Z(\pi)$.*

Proof. Regard N as a locally compact Hausdorff topological space. As N is non-compact, we can select $\omega_0 \in \beta N \backslash N$ and define a character $\widetilde{m}$ on $\mathrm{CB}(N)^*$ by $\langle \widetilde{m}, f\rangle = \widehat{f}(\omega_0)$, where we identify the two C*-algebras $\mathrm{CB}(N)$, $\mathrm{C}(\beta N)$ via the Gelfand transform $f \mapsto \widehat{f}$. By the previous lemma, $\langle \widetilde{m}, f\rangle = 0$, for any $f \in \mathrm{C}_0(N)$. Define $m \in \mathrm{LUC}(G)^*$ by $\langle m, f\rangle = \langle \widetilde{m}, f|_N\rangle$. Let $f \in \mathrm{C}_c(G)$, then clearly $f|_N \in \mathrm{C}_c(N)$. Therefore $\langle m, f\rangle = \langle \widetilde{m}, f|_N\rangle = 0$, i.e. $m \in \mathrm{C}_0(G)^\perp$. Denote the identity functions on G and on N by 1_G and 1_N respectively, then $\langle m, 1_G\rangle = \langle \widetilde{m}, 1_N\rangle = 1$, hence $m \neq 0$. Let $M \in \mathrm{UCB}(\pi)^*$, $T \in \mathrm{UCB}(\pi)$ and $x \in N$, then $MT(x) = \langle M, T\cdot x\rangle = \langle M, T\rangle$. Therefore $MT|_N = \langle M, T\rangle 1_N$ and hence $\langle mM, T\rangle = \langle M, T\rangle\langle \widetilde{m}, 1_N\rangle = \langle M, T\rangle$, i.e. $mM = M$. □

We also remark the following observation.

Corollary 3.9. *Using the above notation, if G is non-compact and the kernel N of the G-module action is non-trivial, i.e. $N \neq \{e\}$, then the factorization property in the main theorem fails to hold, hence $M(G)$ is properly contained in $Z(\pi)$.*

Proof. Suppose that there exists $x \in N$ with $x \neq e$. Clearly for each $M \in \mathrm{UCB}(\pi)^*$ and $T \in \mathrm{UCB}(\pi)$, $MT(x) = MT(e)$. Consequently, $f(x) = f(e)$ for each f in the closed linear span of $\mathcal{F}$. However, by Urysohn lemma, there exists $g \in \mathrm{C}_c(G) \subseteq \mathrm{LUC}(G)$ such that $g(x) \neq g(e)$ and hence $g \in \mathrm{LUC}(G)\backslash$ the closed linear span of $\mathcal{F}$. □

Now we apply our main theorem to some examples.

Corollary 3.10. *Let λ be the left regular representation of a locally compact group G. Then $\{MT \mid M \in UCB(\lambda)^*, T \in UCB(\lambda)\} = LUC(G)$, hence $Z(\lambda) = M(G)$.*

Proof. Let $f \in \mathrm{LUC}(G)$ be given. Define $T_f : L_2(G) \to L_2(G)$ be the multiplication operator induced by f, i.e. $T_f(g) = gf$ $(g \in L_2(G))$. We recall that the map $f \mapsto T_f$ is an isometric embedding of $\mathrm{LUC}(G)$ into $\mathrm{B}(L_2(G))$. Note that for any $x \in G$, $T_f \cdot x = T_{{}_x f}$. Therefore if (x_α) is a net in G converging to $x \in G$, we have $\|T_f \cdot x_\alpha - T_f \cdot x\| = \|{}_{x_\alpha}f - {}_x f\| \to 0$, hence $T_f \in \mathrm{UCB}(\lambda)$. For $y \in G$, we let $\delta_y \in \mathrm{LUC}(G)^*$ be the evaluation at y. We regard $\mathrm{LUC}(G)$ as a subspace of $\mathrm{UCB}(\lambda)$ and let $M_y \in \mathrm{UCB}(\lambda)^*$ be any Hahn-Banach extension of δ_y. If $x \in G$, then $(M_y T_f)(x) = \langle M_y, T_f \cdot x\rangle = \langle M_y, T_{{}_x f}\rangle = \langle \delta_y, {}_x f\rangle = f(xy) = f_y(x)$. Therefore $M_y T_f = f_y$. In particular, $M_e T_f = f$. It follows that, by the main theorem, $\mathrm{Z}(\lambda) = \mathrm{M}(G)$ if G is non-compact. If G is compact, we always have $\mathrm{Z}(\lambda) = \mathrm{M}(G)$ since $\mathrm{LUC}(G)^* = \mathrm{C}(G)^* = \mathrm{M}(G)$. □

Example 3.11. Let $G = \mathbb{Z}$ be the discrete group of integers. For each $n \in \mathbb{N}$, let $q_n : \mathbb{Z} \to \mathbb{Z}_n$ be the canonical quotient map and let $\widetilde{\lambda_n} : \mathbb{Z}_n \to \mathrm{B}(l_2(\mathbb{Z}_n))$ be the left regular representation. Let $\lambda_n = \widetilde{\lambda_n} \circ q_n$ and define $\pi = \bigoplus_{n=1}^{\infty} \lambda_n$. Then $\mathrm{Z}(\pi) = \mathrm{M}(G)$.

Proof. We identify $l_2(\mathbb{Z}_n) \cong \mathbb{C}^n$ and let $e_k^{(n)}$ $(k = 1, 2, \ldots, n)$, be the canonical orthonormal base of $l_2(\mathbb{Z}_n)$. Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} l_2(\mathbb{Z}_n)$, then $\{e_k^{(n)} \mid n \in \mathbb{N}$ and $k = 1, 2, \ldots, n\}$ is an orthonormal base of $\mathcal{H}$. For each $x \in G$, $\pi(x)e_k^{(n)} = e_{[k+x]_n}^{(n)}$, where $[k+x]_n$ is the unique integer in $\{1, 2, \ldots, n\}$ such that $[k+x]_n \equiv k + x \pmod{n}$. First, we claim that for any subset $A \subseteq \mathbb{N} \cup \{0\} \subseteq G$, $\chi_A \in \{MT \mid M \in \mathrm{UCB}(\pi)^*, T \in \mathrm{UCB}(\pi)\}$. Let $A \subseteq \mathbb{N} \cup \{0\}$ be given. Define $T \in \mathrm{B}(\mathcal{H})$ by setting

$$T(e_k^{(n)}) = \begin{cases} e_k^{(n)} & \text{, if } n \text{ is even, } k \in \{1, 2, \ldots, n/2\} \text{ and } k - 1 \in A \\ 0 & \text{, otherwise} \end{cases}$$

For each $n \in \mathbb{N}$, we define $M_n = e_1^{(2n)} \otimes e_1^{(2n)} \in \mathrm{L}_1(\mathcal{H}) \subseteq \mathrm{B}(\mathcal{H})^*$. (If $\xi_1, \xi_2 \in \mathcal{H}$, we define a rank-one operator $\xi_1 \otimes \xi_2$ on $\mathcal{H}$ by $\xi_1 \otimes \xi_2(\eta) = \langle \eta, \xi_2 \rangle \xi_1$.) Choose a subnet $(M_{n_\alpha})_\alpha$ of $(M_n)_n$ such that $M_{n_\alpha} \to M \in \mathrm{B}(\mathcal{H})^*$ with respect to the weak* topology. Let $x \in G$. Fix $n_0 \in \mathbb{N}$ such that $n_0 > |x|$. Note that

$$\begin{aligned} & M_nT(x) = \mathrm{tr}(M_nT \cdot x) \\ = \ & \mathrm{tr}(M_n\pi(x^{-1})T\pi(x)) = \mathrm{tr}(\pi(x)M_n\pi(x^{-1})T) \\ = \ & \mathrm{tr}(e_{[1+x]_{2n}}^{(2n)} \otimes e_{[1+x]_{2n}}^{(2n)} T) = \langle Te_{[1+x]_{2n}}^{(2n)} \mid e_{[1+x]_{2n}}^{(2n)} \rangle. \end{aligned}$$

Consider two cases.

Case I: Suppose that $x \in A$. Let $n \geq n_0$, then $1 \leq 1 + x \leq 2n$, so $[1 + x]_{2n} = 1 + x$. As $[1 + x]_{2n} - 1 = x \in A$, $Te_{[1+x]_{2n}}^{(2n)} = e_{[1+x]_{2n}}^{(2n)}$. Therefore

$$MT(x) = \lim_{\alpha} \langle M_{n_\alpha}, T \cdot x \rangle = \lim_{n \to \infty} \langle M_n, T \cdot x \rangle = 1$$

.

Case II: Suppose that $x \notin A$. Let $n \geq n_0$. If $x \geq 0$, we still have $1 \leq 1+x \leq 2n$, so $[1+x]_{2n} - 1 = x \notin A$ and consequently $Te^{(2n)}_{[1+x]_{2n}} = 0$. If $x < 0$, we have $-n < 1-n < 1+x \leq 0$, so $[1+x]_{2n} = 2n + (1+x) \in \{n+1, n+2, \ldots, 2n\}$ and consequently $Te^{(2n)}_{[1+x]_{2n}} = 0$. Therefore

$$MT(x) = \lim_{n\to\infty} M_n T(x) = 0.$$

This shows that $\chi_A = MT$.

Then we prove that for any $A \subseteq \{n \in \mathbb{Z} \,|\, n < 0\} \subseteq G$, $\chi_A \in \{MT \,|\, M \in \mathrm{UCB}(\pi)^*, T \in \mathrm{UCB}(\pi)\}$. Let such set A be given. Define $T \in \mathrm{B}(\mathcal{H})$ by setting:

$$T(e^n_k) = \begin{cases} e^{(n)}_k & \text{, if } n \text{ is even, } k \in \{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n\} \text{ and } k-1-n \in A \\ 0 & \text{, otherwise} \end{cases}$$

For each $n \in \mathbb{N}$, define $M_n = e^{(2n)}_1 \otimes e^{(2n)}_1 \in \mathrm{L}_1(\mathcal{H}) \subseteq \mathrm{B}(\mathcal{H})^*$. Choose a subnet $(M_{n_\alpha})_\alpha$ of $(M_n)_n$ such that $M_{n_\alpha} \to M \in \mathrm{B}(\mathcal{H})^*$ with respect to the weak* topology. Let $x \in G$ and fix $n_0 \in \mathbb{N}$ such that $n_0 > |x|$. We consider two cases.

Case I: Suppose that $x \in A$. Let $n > n_0$, then $M_n T(x) = \langle Te^{(2n)}_{[1+x]_{2n}} \,|\, e^{(2n)}_{[1+x]_{2n}} \rangle$. As $[1+x]_{2n} = (1+x) + 2n \in \{n+1, n+2, \ldots, 2n\}$, $[1+x]_{2n} - 1 - 2n = x \in A$. Consequently, $Te^{(2n)}_{[1+x]_{2n}} = e^{(2n)}_{[1+x]_{2n}}$ so $M_n T(x) = 1$. Therefore

$$MT(x) = \lim_\alpha M_{n_\alpha} T(x) = \lim_n M_n T(x) = 1.$$

Case II: Suppose that $x \notin A$. Let $n > n_0$. If $x < 0$, we have $[1+x]_{2n} - 1 - n = x \notin A$, so $Te^{(2n)}_{[1+x]_{2n}} = 0$. If $x \geq 0$, we have $[1+x]_{2n} = 1+x \notin \{n+1, n+2, \ldots, 2n\}$. Therefore $T(e^{(2n)}_{[1+x]_{2n}}) = 0$ and hence $M_n T(x) = 0$. Consequently

$$MT(x) = \lim_{n\to\infty} M_n T(x) = 0,$$

so $\chi_A = MT$. It is now clear that for any $B \subseteq G$, χ_B lies in the linear span of $\{MT \,|\, M \in \mathrm{UCB}(\pi)^*, T \in \mathrm{UCB}(\pi)\}$. Therefore the linear span of $\{MT \,|\, M \in \mathrm{UCB}(\pi)^*, T \in \mathrm{UCB}(\pi)\}$ contains all the simple functions and consequently it is dense in $l_\infty(G) = \mathrm{LUC}(G)$. By our characterization theorem, $\mathrm{Z}(\pi) = \mathrm{M}(G)$. □

Remark 3.12. In this example, each representation λ_n is finite dimensional and by proposition 3.13.1 below, $\mathrm{Z}(\lambda_n) = \mathrm{LUC}(G)^*$. However, $\mathrm{Z}(\bigoplus_{n=1}^{\infty} \lambda_n)$ is minimal, i.e. equals to $\mathrm{M}(G)$.

3.2 Maximality of the topological centers

In this section, we give two sufficient conditions, each of which will guarantee that the topological center is maximal, i.e. $\mathrm{Z}(\pi) = \mathrm{LUC}(G)^*$. An example whose topological center is maximal is also demonstrated.

Proposition 3.13. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group G. If at least one of the following conditions is satisfied, the topological center $Z(\pi)$ is maximal, i.e. $Z(\pi) = LUC(G)^*$.*

1. $\dim(\pi) < \infty$.
2. *For each $\epsilon > 0$, we define $N(\pi, \epsilon) = N_\epsilon = \{x \in G \,|\, \|T \cdot x - T\| < \epsilon\|T\| \text{ for any } T \in UCB(\pi)\}$. Suppose that for each $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_m \in G$ satisfying that: For each $x \in G$, there exist $i \in \{1, 2, \ldots, m\}$ and $y \in N_\epsilon$ such that $x = x_i y$.*

Proof. Suppose that the first condition holds. Let $(M_\alpha)_\alpha$ be a net in $\mathrm{UCB}(\pi)^*$ such that $M_\alpha \to M \in \mathrm{UCB}(\pi)^*$ with respect to the weak* topology. Since

$\dim(\pi) < \infty$, $\mathrm{UCB}(\pi)^*$ is a finite dimensional vector space and all locally convex topologies coincide. Therefore for any $m \in \mathrm{LUC}(G)^*$ and $T \in \mathrm{UCB}(\pi)$, $|\langle mM_\alpha, T\rangle - \langle mM, T\rangle| = |\langle m, (M_\alpha - M)T\rangle| \leq \|m\|\|M_\alpha - M\|\|T\| \to 0$, hence $m \in \mathrm{Z}(\pi)$.

Suppose that the second condition holds. We assert that for any bounded net $(M_\alpha)_\alpha$ in $\mathrm{UCB}(\pi)^*$, $M \in \mathrm{UCB}(\pi)^*$, $T \in \mathrm{UCB}(\pi)$, if $M_\alpha \to M$ with respect to the weak*-topology, then $\|M_\alpha T - MT\|_\infty \to 0$. Without loss of generality, we assume that $\|M_\alpha\| \leq 1$, $\|M\| \leq 1$, $\|T\| \leq 1$. Let $\epsilon > 0$ be arbitrary. Choose $x_1, x_2, \ldots, x_m \in G$ as in the assumption, then we obtain a partition $\{A_1, A_2, \ldots, A_m\}$ of G with the property that for any $x \in A_i$, there exists $y \in N_\epsilon$ such that $x = x_i y$. Let $x \in A_i$ and write $x = x_i y$ for some $y \in N_\epsilon$, then $\|T \cdot x - T \cdot x_i\| = \|(T \cdot x_i) \cdot y - T \cdot x_i\| < \epsilon\|T \cdot x_i\| \leq \epsilon$. Therefore $|MT(x) - MT(x_i)| = |\langle M, T \cdot x - T \cdot x_i\rangle| \leq \|M\|\|T \cdot x - T \cdot x_i\| < \epsilon$. Similarly, we have $|M_\alpha T(x) - M_\alpha T(x_i)| < \epsilon$. Define $\lambda_i = MT(x_i)$ and $\lambda_i^\alpha = M_\alpha T(x_i)$. Since $M_\alpha T \to MT$ pointwisely, we may choose α_0 such that $|M_\alpha T(x_i) - MT(x_i)| < \epsilon$ whenever $i \in \{1, 2 \ldots, m\}$ and $\alpha \succeq \alpha_0$, i.e. $|\lambda_i - \lambda_i^\alpha| < \epsilon$. By the above discussion, it is clear that $\|MT - \sum_{i=1}^m \lambda_i \chi_{A_i}\|_\infty \leq \epsilon$ and $\|M_\alpha T - \sum_{i=1}^m \lambda_i^\alpha \chi_{A_i}\|_\infty \leq \epsilon$. Therefore

$$\begin{aligned}
& \|M_\alpha T - MT\|_\infty \\
\leq\ & \|M_\alpha T - \sum_{i=1}^m \lambda_i^\alpha \chi_{A_i}\|_\infty + \|\sum_{i=1}^m \lambda_i^\alpha \chi_{A_i} - \sum_{i=1}^m \lambda_i \chi_{A_i}\|_\infty + \|\sum_{i=1}^m \lambda_i \chi_{A_i} - MT\|_\infty \\
\leq\ & 3\epsilon
\end{aligned}$$

whenever $\alpha \succeq \alpha_0$ and hence $\|M_\alpha T - MT\|_\infty \to 0$. Let $m \in \mathrm{LUC}(G)^*$. In order to show $m \in \mathrm{Z}(\pi)$, by proposition 2.4 it suffices that the map $M \mapsto m \cdot M$ is weak*-weak* continuous on all the bounded part of $\mathrm{UCB}(\pi)^*$. Let $(M_\alpha)_\alpha$ be a bounded net in $\mathrm{UCB}(\pi)$ and suppose that $M_\alpha \to M$ with respect to the weak* topology. Let $T \in \mathrm{UCB}(\pi)$, then $|\langle mM_\alpha, T\rangle - \langle mM, T\rangle| \leq \|m\|\|M_\alpha T - MT\| \to 0$. Therefore $m \in \mathrm{Z}(\pi)$. □

Remark 3.14. Let $N = \{x \in G \,|\, T \cdot x = T \text{ for any } T \in \mathrm{UCB}(\pi)\}$. If $|G/N| < \infty$, the second condition will be satisfied and hence the topological center $\mathrm{Z}(\pi)$ is maximal. For, suppose that $|G/N| = m$. We pick an element x_i from each N-coset, then for each $\epsilon > 0$, $x_1, x_2, \ldots, x_m$ and N clearly satisfy the second condition since $N \subseteq N_\epsilon$.

Identify the quotient group $\mathbb{R}/\mathbb{Z}$ with $[0,1)$ in a canonical way. Let $\alpha \in [0,1)\backslash\mathbb{Q}$. It is an easy exercise to check that the subgroup of $\mathbb{R}/\mathbb{Z}$ generated by α, namely $\{n\alpha \,|\, n \in \mathbb{Z}\}$ is dense in $[0,1)$.

Lemma 3.15. *Identify the quotient group $\mathbb{R}/\mathbb{Z}$ with $[0,1)$. Given sufficiently small $\epsilon > 0$, $\alpha \in [0,1)\backslash\mathbb{Q}$ (where α is regarded as an element in $\mathbb{R}/\mathbb{Z}$) , we define $\mathbb{Z}_0 = \{n \in \mathbb{Z} \,|\, n\alpha \in (0,\epsilon)\}$, then*

1. *$\mathbb{Z}_0$ is an infinite set, and*

2. *There exists $k \in \mathbb{N}$ such that $|m - n| \leq k$ whenever $m, n \in \mathbb{Z}_0$ are two successive elements in $\mathbb{Z}_0$.*

Proof. Since α is irrational, the map $n \mapsto n\alpha \in \mathbb{R}/\mathbb{Z}$ is injective. Since $\{n\alpha \,|\, n \in \mathbb{Z}\}$ is dense in $[0,1)$, there exist infinitely many $n \in \mathbb{Z}$ such that $n\alpha \in (0,\epsilon)$. This proves the first part. Next, we choose $n_0 \in \mathbb{Z}$ such that $0 < n_0\alpha < \epsilon/2$. Denote $\beta = n_0\alpha$ and define $\mathbb{Z}_1 = \{n \in \mathbb{Z} \,|\, n\beta \in (0,\epsilon)\}$. Let n_1 be the smallest positive integer such that $n_1\beta < 1 < (n_1+1)\beta$. Let m, n be two successive elements in $\mathbb{Z}_1$ with $m < n$. If $m\beta \in (0, \epsilon - \beta)$, we clearly have $m\beta + \beta \in (0,\epsilon)$ so $n = m + 1$. If $m\beta \in [\epsilon - \beta, \epsilon)$, then $m\beta + n_1\beta \simeq m\beta + n_1\beta - 1 > (\epsilon - \beta) - \beta > 0$. Note that $m\beta + n_1\beta \simeq m\beta + (n_1\beta - 1) < m\beta < \epsilon$. Therefore $(m + n_1)\beta \in (0,\epsilon)$ and hence $n \leq m + n_1$. Lastly, if $n \in \mathbb{Z}_1$, then

$n\beta \in (0,\epsilon)$, $nn_0\alpha \in (0,\epsilon)$, so $nn_0 \in \mathbb{Z}_0$. Consequently $n_0\mathbb{Z}_1 \subseteq \mathbb{Z}_0$. Take $k = n_0 n_1$, then $|m-n| \le k$ for any two successive elements $m, n \in \mathbb{Z}_0$. $\square$

Example 3.16. Let $\mathbb{Z}$ be the usual discrete group of integers. Choose $\theta \in [0, 2\pi)$ such that $\theta/(2\pi)$ is irrational. Let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $\mathcal{H} = \mathbb{C}^2$ and let $\pi : \mathbb{Z} \to \mathrm{B}(\mathcal{H}) \simeq \mathrm{M}_2(\mathbb{C})$ be defined by $\pi(n) = A^n$. Let $\widetilde{\pi} = \bigoplus_{n=1}^{\infty} \pi$ be the countable direct sum of π. Let $\widetilde{\mathcal{H}} = \bigoplus_{n=1}^{\infty} \mathcal{H}$. Although $\mathrm{N}_{\widetilde{\pi}} = \mathrm{N}_{\pi} = \{0\}$ and $\dim(\widetilde{\pi}) = \infty$, we still have $\mathrm{Z}(\widetilde{\pi}) = l^{\infty}(G)^*$.

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$\left\| \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - I \right\| < \epsilon/2,$$

whenever $\phi \in (-\delta, \delta) + 2\pi\mathbb{Z}$. Therefore

$$\left\| \bigoplus_{n=1}^{\infty} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - \widetilde{I} \right\| = \left\| \bigoplus_{n=1}^{\infty} \left[\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - I \right] \right\|$$

$$= \left\| \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} - I \right\| < \epsilon/2,$$

where $\widetilde{I}$ is the identity operator on $\widetilde{\mathcal{H}}$. Define $\mathrm{N}_{\epsilon} = \{n \in \mathbb{Z} \,|\, \|\widetilde{\pi}(-n)T\widetilde{\pi}(n) - T\| \le \epsilon\|T\|$, for any $T \in \mathrm{B}(\widetilde{\mathcal{H}})\}$. Set $\mathbb{Z}_0 = \{n \in \mathbb{Z} \,|\, n\theta/(2\pi) \in (0, \delta/(2\pi)) + \mathbb{Z}\}$. If $n \in \mathbb{Z}_0$, then for any $T \in \mathrm{B}(\widetilde{\mathcal{H}})$,

$$\begin{aligned} & \|\widetilde{\pi}(-n)T\widetilde{\pi}(n) - T\| \\ \le\ & \|\widetilde{\pi}(-n)T\widetilde{\pi}(n) - T\widetilde{\pi}(n)\| + \|T\widetilde{\pi}(n) - T\| \\ \le\ & \|\widetilde{\pi}(-n) - \widetilde{I}\|\|T\| + \|\widetilde{\pi}(n) - \widetilde{I}\|\|T\| \\ \le\ & \epsilon\|T\|, \end{aligned}$$

by observing that $\widetilde{\pi}(n) = \bigoplus \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$, $\widetilde{\pi}(-n) = \bigoplus \begin{pmatrix} \cos(-n\theta) & -\sin(-n\theta) \\ \sin(-n\theta) & \cos(-n\theta) \end{pmatrix}$ with $n\theta \in (0, \delta) + 2\pi\mathbb{Z}$ and $-n\theta \in (-\delta, 0) + 2\pi\mathbb{Z}$. Therefore $\mathbb{Z}_0 \subseteq \mathrm{N}_\epsilon$. By the previous lemma, $\mathbb{Z}_0$ is an infinite set such that the distance between two successive elements in bounded. As N_ϵ is a superset of $\mathbb{Z}_0$, N_ϵ has the same property. Now it clear that condition (2) in the proposition 3.13 is fulfilled, so $\mathrm{Z}(\widetilde{\pi}) = l^\infty(\mathbb{Z})^*$. □

3.3 An example that $\mathbf{M}(G) \subsetneq \mathbf{Z}(\pi) \subsetneq \mathbf{LUC}(G)^*$

In this section, we give an example that the topological center $\mathrm{Z}(\pi)$ is neither minimal nor maximal.

Example 3.17. Let $G = \mathbb{Z} \times \mathbb{Z}$ and let $q : G \to \mathbb{Z}$ be the canonical quotient map defined by $q(i, j) = j$. Let $\lambda : \mathbb{Z} \to \mathrm{B}(l_2(\mathbb{Z}))$ be the left regular representation and let $\pi = \lambda \circ q$, then $\mathrm{M}(G) \subsetneq \mathrm{Z}(\pi) \subsetneq \mathrm{LUC}(G)^*$.

Proof. Since the kernel of the G-action induced by π is non-trivial, $\mathrm{M}(G)$ is properly contained in $\mathrm{Z}(\pi)$. Let $\mathcal{H} = l_2(\mathbb{Z})$ and let $\{e_k \mid k \in \mathbb{Z}\}$ be the canonical base of $\mathcal{H}$. We show that $\mathrm{Z}(\pi)$ is not maximal. For each $i \in \mathbb{N}$, let $n_i = \delta_{(0,i)} \in l_1(G)$ and let $n \in l_\infty(G)^*$ be any weak*-cluster point of the net (n_i). We assert that $n \notin \mathrm{Z}(\pi)$. For each $j \in \mathbb{N}$, define $M_j = e_{-j} \otimes e_{-j} \in \mathrm{L}^1(\mathcal{H}) \hookrightarrow \mathrm{B}(\mathcal{H})^*$. Let $T_0 \in \mathrm{B}(\mathcal{H})$ be defined by

$$T_0(e_k) = \begin{cases} e_k & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0 \end{cases}$$

Let $T \in \mathrm{B}(\mathcal{H})$. We observe that

$$\begin{aligned}
& \langle n_i M_j, T\rangle = \langle n_i, M_j T\rangle = (M_j T)((0,i)) = \langle M_j, \pi((0,-i))T\pi((0,i))\rangle \\
= \ & \mathrm{tr}\ (M_j \pi((0,-i))T\pi((0,i))) = \mathrm{tr}\ (\pi((0,i))M_j\pi((0,-i))T) \\
= \ & \mathrm{tr}\ (e_{i-j} \otimes e_{i-j} T) = \langle e_{i-j} \otimes e_{i-j}, T\rangle,
\end{aligned}$$

and hence $n_i M_j = e_{i-j} \otimes e_{i-j}$. Let M be an arbitrary weak*-cluster point of the net (M_j), then we have

$$\begin{aligned}
& \langle nM, T_0\rangle = \lim_{i\to\infty} \langle n_i M, T_0\rangle \\
= \ & \lim_{i\to\infty} \lim_{j\to\infty} \langle n_i M_j, T_0\rangle = \lim_{i\to\infty} \lim_{j\to\infty} \langle e_{i-j} \otimes e_{i-j}, T_0\rangle \\
= \ & \lim_{i\to\infty} \lim_{j\to\infty} \langle T_0 e_{i-j} \,|\, e_{i-j}\rangle = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \lim_{j\to\infty} \langle nM_j, T_0\rangle = \lim_{j\to\infty} \lim_{i\to\infty} \langle n_i M_j, T_0\rangle \\
= \ & \lim_{j\to\infty} \lim_{i\to\infty} \langle e_{i-j} \,|\, e_{i-j}\rangle = 1.
\end{aligned}$$

Let M_{j_α} be a subnet of (M_j) such that $\lim_\alpha M_{j_\alpha} = M$ with respect to the weak*-topology. Now it is clear that $nM_{j_\alpha} \not\to nM$ with respect to the weak*-topology and hence $n \notin \mathrm{Z}(\pi)$. $\square$

Chapter 4

Direct Sums and Tensor Products of Unitary Representations and Their Topological Centers

In this chapter, we investigate the relations between the topological centers of sub-representations, finite direct sums, tensor products with that of the underlying representations. We prove that if π_1 is a sub-representation of π_2, we always have $\mathrm{Z}(\pi_2) \subseteq \mathrm{Z}(\pi_1)$. We also show that for an arbitrary unitary representation π, and $n \in \mathbb{N}$, the finite direct sum $\bigoplus_{i=1}^n \pi = n\pi$ and the original representation π have the same topological centers. Lastly, we give a condition which guarantees that $\mathrm{Z}(\pi_1 \otimes \pi_2) = \mathrm{M}(G)$.

Lemma 4.1. *Let $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ be unitary representations of G. Suppose that π_1 is a subrepresentation of π_2. Let $P : \mathcal{H}_2 \to \mathcal{H}_1$ be the canonical projection. For each $T \in B(\mathcal{H}_2)$, we define $T' \in B(\mathcal{H}_1)$ by $T' = P \circ T|_{\mathcal{H}_1}$. If*

$T \in UCB(\pi_2)$, then $T' \in UCB(\pi_1)$. Moreover, the map $T \mapsto T'$ is surjective.

Proof. Let $x \in G$. Notice that

$$\begin{aligned} & T' \cdot x = \pi_1(x^{-1})T'\pi_1(x) = \pi_1(x^{-1})P(T|_{\mathcal{H}_1})\pi_1(x) \\ = \quad & P\pi_2(x^{-1})T[\pi_2(x)|_{\mathcal{H}_1}] = P[T \cdot x]|_{\mathcal{H}_1}. \end{aligned}$$

Let (x_α) be a net in G such that $x_\alpha \to x \in G$, then

$$\|T' \cdot x_\alpha - T' \cdot x\|_{\mathrm{B}(\mathcal{H}_1)} \leq \|P\| \|T \cdot x_\alpha - T \cdot x\|_{\mathrm{B}(\mathcal{H}_2)} \to 0,$$

i.e. $T' \in \mathrm{UCB}(\pi_1)$. Given $T_0 \in \mathrm{UCB}(\pi_1)$, we define $T = T_0 \circ P \in \mathrm{B}(\mathcal{H}_2)$. It is clear that $T \in \mathrm{UCB}(\pi_2)$ with $T' = T_0$, so the map $T \mapsto T'$ is surjective. □

Lemma 4.2. *Using the above notation, if $M \in UCB(\pi_1)^*$, we define $\widetilde{M} \in UCB(\pi_2)^*$ by $\langle \widetilde{M}, T \rangle = \langle M, T' \rangle$. Then $\widetilde{M}T = MT'$ as an element in $LUC(G)$ for any $T \in UCB(\pi_2)$.*

Proof. Let $x \in G$, then

$$\widetilde{M}T(x) = \langle \widetilde{M}, T \cdot x \rangle = \langle M, P \circ (T \cdot x)|_{\mathcal{H}_1} \rangle = \langle M, T' \cdot x \rangle = MT'(x)$$

. □

We now state a proposition relating the topological centers of a sub-representation and the original representation.

Proposition 4.3. *Let $(\pi_1, \mathcal{H}_1)$,$(\pi_2, \mathcal{H}_2)$ be unitary representations of a locally compact group G. If π_1 is a subrepresentation of π_2, then $Z(\pi_2) \subseteq Z(\pi_1)$.*

Proof. Let $m \in Z(\pi_2)$. Let (M_α) be a net in $\mathrm{UCB}(\pi_1)^*$ such that $M_\alpha \to M \in \mathrm{UCB}(\pi_1)^*$ with respect to the $\sigma(\mathrm{UCB}(\pi_1)^*, \mathrm{UCB}(\pi_1))$- topology. Let $T \in \mathrm{UCB}(\pi_2)$, then $\langle \widetilde{M_\alpha}, T \rangle = \langle M_\alpha, T' \rangle \to \langle M, T' \rangle = \langle \widetilde{M}, T \rangle$, hence $\widetilde{M_\alpha} \to$

$\widetilde{M}$ with respect to the $\sigma(\mathrm{UCB}(\pi_2)^*, \mathrm{UCB}(\pi_2))$- topology. Let $T_0 \in \mathrm{UCB}(\pi_1)$. Choose $T \in \mathrm{UCB}(\pi_2)$ such that $T' = T_0$, then

$$\begin{aligned} & \langle mM_\alpha,\, T_0\rangle = \langle m,\, M_\alpha T_0\rangle = \langle m,\, M_\alpha T'\rangle \\ = & \langle m,\, \widetilde{M_\alpha} T\rangle = \langle m\widetilde{M_\alpha},\, T\rangle \to \langle m\widetilde{M},\, T\rangle \\ = & \langle m,\, \widetilde{M}T\rangle = \langle m,\, MT'\rangle = \langle m,\, MT_0\rangle \\ = & \langle mM,\, T_0\rangle. \end{aligned}$$

Hence $mM_\alpha \to mM$ with respect to the $\sigma(\mathrm{UCB}(\pi_1)^*, \mathrm{UCB}(\pi_1))$-topology, i.e. $m \in Z(\pi_1)$. □

Next, we consider direct sum of unitary representations. Let π be a unitary representation of G and let $\pi' = \bigoplus_\alpha \pi$ be the direct sum of α (a cardinal) copies of π. It is interesting to ask: How are $Z(\pi)$ and $Z(\pi')$ related ? Since π is a subrepresentation of π', we have $Z(\pi') \subseteq Z(\pi)$ by the previous proposition. In fact, if α is finite, we can say more. Before stating and proving the proposition, we first introduce some notations. Let $\mathcal{H}$ be the underlying Hilbert space for π and let $\mathcal{H}' = \bigoplus_n \mathcal{H}$, the direct sum of n copies of $\mathcal{H}$. In order to avoid confusion, we let $\mathcal{H}_1 = \mathcal{H}_2 = \ldots = \mathcal{H}_n = \mathcal{H}$ and write $\mathcal{H}' = \bigoplus_{i=1}^n \mathcal{H}_i$. For each $i \in \{1, 2, \ldots, n\}$, we let $P_i : \mathcal{H}' \to \mathcal{H}_i$ be the canonical projection and let $I_i : \mathcal{H}_i \to \mathcal{H}'$ be the canonical injection. Given $T \in \mathrm{B}(\mathcal{H}')$, we associate n^2 operators on $\mathcal{H}$ (here we identify $\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \ldots \simeq \mathcal{H}_n \simeq \mathcal{H}$) as follow:
For $i, j \in \{1, 2, \ldots, n\}$, we define $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ by $T_{ij} = P_i \circ T \circ I_j$. We call $\{T_{ij} \,|\, 1 \le i \le n, 1 \le j \le n\}$ the components of T. Conversely, given n^2 bounded linear operators $T_{ij} \in \mathrm{B}(\mathcal{H})$, we can associate $T \in \mathrm{B}(\mathcal{H}')$ by

$$T = \sum_{1 \le i \le n, 1 \le j \le n} I_i \circ T_{ij} \circ P_j.$$

We remark that the above processes of decomposition and composition are converse to each other. More precisely, given $T \in \mathrm{B}(\mathcal{H}')$, we first decompose it and obtain $T_{ij} \in \mathrm{B}(\mathcal{H})$, then use these n^2 operators T_{ij} to construct $\widetilde{T} \in \mathrm{B}(\mathcal{H}')$. It can be verified that $T = \widetilde{T}$. On the other hand, given n^2 operators T_{ij} on $\mathcal{H}$, we compose them and obtain $\widetilde{T} \in \mathrm{B}(\mathcal{H}')$. It can be shown that $\widetilde{T}_{ij} = T_{ij}$. If $x \in G$ and $T \in \mathrm{B}(\mathcal{H}')$ or $T \in \mathrm{B}(\mathcal{H})$, we denote $\pi'(x^{-1})T\pi'(x)$ and $\pi(x^{-1})T\pi(x)$ by the same symbol $T \cdot x$. We then prove few lemmas.

Lemma 4.4. *Using the above notations and let $T \in B(\mathcal{H}')$, $T_{ij} \in B(\mathcal{H})$ the components of T, then for each $x \in G$, $T \cdot x$ has components $T_{ij} \cdot x$, i.e. $(T \cdot x)_{ij} = T_{ij} \cdot x$.*

Proof. Note that

$$T \cdot x = \pi'(x^{-1})T\pi'(x) = \sum_{i,j} \pi'(x^{-1}) \circ I_i \circ T_{ij} \circ P_j \circ \pi'(x).$$

But for each i, j,

$$\pi'(x^{-1}) \circ I_i \circ T_{ij} \circ P_j \circ \pi'(x) = I_i \circ \pi(x^{-1}) \circ T_{ij} \circ \pi(x) \circ P_j = I_i \circ (T_{ij} \cdot x) \circ P_j.$$

Therefore $T \cdot x = \sum_{i,j} I_i \circ (T_{ij} \cdot x) \circ P_j$ and hence $(T \cdot x)_{ij} = T_{ij} \cdot x$. □

Lemma 4.5. *Let $T \in B(\mathcal{H}')$ with components $T_{ij} \in B(\mathcal{H})$, then $T \in UCB(\pi')$ if and only if for each i,j, $T_{ij} \in UCB(\pi)$.*

Proof. Suppose that $T \in \mathrm{UCB}(\pi')$. Let $x, y \in G$. By the previous lemma, for any i, j,

$$\begin{aligned} & \|T_{ij} \cdot x - T_{ij} \cdot y\| = \|(T \cdot x)_{ij} - (T \cdot y)_{ij}\| \\ = \ & \|P_i \circ (T \cdot x - T \cdot y) \circ I_j\| \\ \leq \ & \|P_i\| \|T \cdot x - T \cdot y\| \|I_j\| \to 0 \end{aligned}$$

as $x \to y$. Therefore $T_{ij} \in \mathrm{UCB}(\pi)$.

Conversely, suppose that for each i, j, $T_{ij} \in \mathrm{UCB}(\pi)$. Let $x \in G$, then

$$\begin{aligned} & (I_i \circ T_{ij} \circ P_j) \cdot x = \pi'(x^{-1}) \circ (I_i \circ T_{ij} \circ P_j) \circ \pi'(x) \\ = \ & I_i \circ \pi(x^{-1}) \circ T_{ij} \circ \pi(x) \circ P_j = I_i \circ (T_{ij} \cdot x) \circ P_j, \end{aligned}$$

so for any $x, y \in G$,

$$\begin{aligned} & \|T \cdot x - T \cdot y\| \\ = & \|\sum((I_i \circ T_{ij} \circ P_j) \cdot x - (I_i \circ T_{ij} \circ P_j) \cdot y)\| \\ \leq & \sum \|I_i\| \|T_{ij} \cdot x - T_{ij} \cdot y\| \|P_j\| \to 0 \end{aligned}$$

as $x \to y$. Therefore $T \in \mathrm{UCB}(\pi')$. □

Then we consider decomposition of elements in $\mathrm{B}(\mathcal{H}')^*$. We continue to use the notations defined in above. Given $M \in \mathrm{B}(\mathcal{H}')^*$, we associate n^2 elements in $\mathrm{B}(\mathcal{H})^*$ as follow:
For each $i, j \in \{1, 2, \ldots, n\}$, we define $M_{ij} \in \mathrm{B}(\mathcal{H})^*$ by $\langle M_{ij}, T \rangle = \langle M, I_i \circ T \circ P_j \rangle$. We call M_{ij} the components of M. We remark that if $M \in \mathrm{UCB}(\pi')^*$, then $M_{ij} \in \mathrm{UCB}(\pi)^*$.

Lemma 4.6. *Let $M \in B(\mathcal{H}')^*$ and $T \in B(\mathcal{H}')$ with components M_{ij} and T_{ij} respectively, then $\langle M, T \rangle = \sum_{ij} \langle M_{ij}, T_{ij} \rangle$.*

Proof. $\langle M, T \rangle = \sum_{ij} \langle M, I_i \circ T_{ij} \circ P_j \rangle = \sum_{ij} \langle M_{ij}, T_{ij} \rangle$. □

Lemma 4.7. *Let $M \in UCB(\pi')^*$, $T \in UCB(\pi')$ with components M_{ij},T_{ij} respectively, then $MT = \sum_{ij} M_{ij} T_{ij}$.*

Proof. Let $x \in G$, then $MT(x) = \langle M, T \cdot x \rangle = \sum_{ij} \langle M_{ij}, (T \cdot x)_{ij} \rangle = \sum_{ij} \langle M_{ij}, T_{ij} \cdot x \rangle = \sum_{ij} M_{ij} T_{ij}(x)$. □

We also need a lemma which deals with weak*-convergence.

Lemma 4.8. *Let $(M^\alpha)_\alpha$ be a net in $UCB(\pi')^*$, $M \in UCB(\pi')^*$. Let M^α_{ij}, M_{ij} be the components of M^α and M respectively, then the following are equivalent:*
(a) $M^\alpha \to M$ with respect to $\sigma(UCB(\pi')^*, UCB(\pi'))$-topology,
(b) For each i,j, $M^\alpha_{ij} \to M_{ij}$ with respect to $\sigma(UCB(\pi)^*, UCB(\pi))$-topology.

Proof. Let $(M^\alpha)_\alpha$ be a net in UCB$(\pi')^*$ such that $M^\alpha \to M$ with respect to the weak*-topology. Let $i, j \in \{1, 2, \ldots, n\}$ and let $T \in$ UCB(π), then

$$\begin{aligned} \langle M_{ij}^\alpha, T\rangle &= \langle M^\alpha, I_i \circ T \circ P_j\rangle \\ \to \quad &\langle M, I_i \circ T \circ P_j\rangle = \langle M_{ij}, T\rangle. \end{aligned}$$

Therefore $M_{ij}^\alpha \to M_{ij}$ with respect to the weak*-topology. Conversely, let (M^α) be a net in UCB$(\pi')^*$, $M \in$ UCB$(\pi')^*$ such that for each i,j, $M_{ij}^\alpha \to M_{ij}$ with respect to the weak*-topology. Let $T \in$ UCB(π'), then $\langle M^\alpha, T\rangle = \sum_{ij}\langle M_{ij}^\alpha, T_{ij}\rangle \to \sum_{ij}\langle M_{ij}, T_{ij}\rangle = \langle M, T\rangle$. Therefore $M^\alpha \to M$ with respect to the weak*-topology. □

Now we are able to state and prove the following proposition.

Proposition 4.9. *Let π be a unitary representation of a locally compact group G and let $\pi' = \bigoplus_n \pi$ be the direct sum of n copies of π ($n \in \mathbb{N}$), then $Z(\pi) = Z(\pi')$.*

Proof. Since π is a sub-representation of π', we have $Z(\pi') \subseteq Z(\pi)$. Therefore, it suffices to show the reversed inclusion. Let $m \in Z(\pi)$. Let (M^α) be a net in UCB$(\pi')^*$ such that $M^\alpha \to M \in$ UCB$(\pi')^*$ with respect to the weak* topology. Let $T \in$ UCB(π'), then $\langle mM^\alpha, T\rangle = \langle m, M^\alpha T\rangle = \langle m, \sum_{ij} M_{ij}^\alpha T_{ij}\rangle = \sum_{ij}\langle mM_{ij}^\alpha, T_{ij}\rangle$. By the previous lemma, $M_{ij}^\alpha \to M_{ij}$ with respect to the weak*-topology for each i,j. Since $m \in Z(\pi)$, $mM_{ij}^\alpha \to mM_{ij}$ with respect to the weak*-topology. Therefore, $\sum_{ij}\langle mM_{ij}^\alpha, T_{ij}\rangle \to \sum_{ij}\langle mM_{ij}, T_{ij}\rangle = \sum_{ij}\langle m, M_{ij}T_{ij}\rangle = \langle m, MT\rangle = \langle mM, T\rangle$, hence $m \in Z(\pi')$. □

In the following, we consider the tensor product of two unitary representations. Let $\pi_1 : G \to \mathrm{B}(\mathcal{H}_1)$, $\pi_2 : G \to \mathrm{B}(\mathcal{H}_2)$ be unitary representations of a locally compact group G. We denote the inner tensor product of π_1 and π_2 by

$\pi_1 \otimes \pi_2$, i.e. $\pi_1 \otimes \pi_2 : G \to \mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ defined by $\pi_1 \otimes \pi_2(x) = \pi_1(x) \otimes \pi_2(x)$, $x \in G$.

Lemma 4.10. *Using the above notations, if $T_1 \in UCB(\pi_1)$ and $T_2 \in UCB(\pi_2)$, then $T_1 \otimes T_2 \in UCB(\pi_1 \otimes \pi_2)$.*

Proof. Let $T_1 \in \mathrm{UCB}(\pi_1)$ and $T_2 \in \mathrm{UCB}(\pi_2)$ and let $x \in G$. Note that $(T_1 \otimes T_2) \cdot x = (\pi_1(x^{-1}) \otimes \pi_2(x^{-1}))(T_1 \otimes T_2)(\pi_1(x) \otimes \pi_2(x)) = (\pi_1(x^{-1}) T_1 \pi_1(x)) \otimes (\pi_2(x^{-1}) T_2 \pi_2(x)) = (T_1 \cdot x) \otimes (T_2 \cdot x)$. Therefore, if (x_α) is a net in G which converges to $x \in G$, then $\|(T_1 \otimes T_2) \cdot x_\alpha - (T_1 \otimes T_2) \cdot x\| = \|(T_1 \cdot x_\alpha) \otimes (T_2 \cdot x_\alpha) - (T_1 \cdot x) \otimes (T_2 \cdot x)\| \leq \|(T_1 \cdot x_\alpha) \otimes (T_2 \cdot x_\alpha) - (T_1 \cdot x_\alpha) \otimes (T_2 \cdot x)\| + \|(T_1 \cdot x_\alpha) \otimes (T_2 \cdot x) - (T_1 \cdot x) \otimes (T_2 \cdot x)\| \leq \|T_1 \cdot x_\alpha\| \|T_2 \cdot x_\alpha - T_2 \cdot x\| + \|T_2 \cdot x\| \|T_1 \cdot x_\alpha - T_1 \cdot x\| \leq \|T_1\| \|T_2 \cdot x_\alpha - T_2 \cdot x\| + \|T_2\| \|T_1 \cdot x_\alpha - T_1 \cdot x\| \to 0$ □

Proposition 4.11. *Using the above notations, if there exists $i \in \{1, 2\}$ such that* $\mathrm{span}\{MT \mid M \in UCB(\pi_i)^*, T \in UCB(\pi_i)\}$ *is norm dense in $LUC(G)$, then $Z(\pi_1 \otimes \pi_2) = M(G)$.*

Proof. Let $\mathcal{F} = \{MT \mid M \in \mathrm{UCB}(\pi_i)^*, T \in \mathrm{UCB}(\pi_i)\}$ and $\mathcal{F}' = \{M'T' \mid M' \in \mathrm{UCB}(\pi_1 \otimes \pi_2)^*, T' \in \mathrm{UCB}(\pi_1 \otimes \pi_2)\}$. We try to show that $\mathcal{F} \subseteq \mathcal{F}'$. We simplify our notation and assume that $i = 1$. The case that $i = 2$ can be proved in exactly the same way. Let $M_1 \in \mathrm{UCB}(\pi_1)^*$ and $T_1 \in \mathrm{UCB}(\pi_1)$. Choose $M_2 \in \mathrm{UCB}(\pi_2)$, such that $\langle M_2, I_2 \rangle = 1$, where I_2 is the identity operator on the Hilbert space $\mathcal{H}_2$. Note that the map $\mathrm{UCB}(\pi_1) \times \mathbb{C} I_2 \to \mathbb{C}$, $(T, \lambda I_2) \mapsto \langle M_1, T \rangle \langle M_2, \lambda I_2 \rangle = \lambda \langle M_1, T \rangle$ is bounded bilinear, so it induces a bounded linear functional $M' : \mathrm{UCB}(\pi_1) \otimes \mathbb{C} I_2 \to \mathbb{C}$. We extend M' (still denoted by M') and obtain a bounded linear functional on $\mathrm{UCB}(\pi_1 \otimes \pi_2)$ by Hahn-Banach Theorem. Define $T' = T_1 \otimes I_2$, then $T' \in \mathrm{UCB}(\pi_1 \otimes \pi_2)$ by the previous lemma. If $x \in G$, then $M'T'(x) = \langle M', T' \cdot x \rangle = \langle M', (T_1 \cdot x) \otimes I_2 \rangle =$

$\langle M_1, T_1 \cdot x \rangle = M_1 T_1(x)$. Therefore the linear span of $\mathcal{F}'$ is norm dense in LUC(G). If G is non-compact, $Z(\pi_1 \otimes \pi_2) = M(G)$ by 3.6. If G is compact, it is automatic that $Z(\pi_1 \otimes \pi_2) = M(G)$ $\square$

Part II

Topologically Invariant Means for Amenable Representations

Chapter 5

On the Set of Topologically Invariant Means

This part of the thesis is devoted to the studies of the set of topologically invariant means associated to a given amenable representation. The studies of amenability can be dated back to 1904 when Lebesgue asked whether the Lebesgue integral is still uniquely defined if the countable additivity is replaced by just finite additivity. In the classical period, mathematicians mainly concerned about the studies of finitely additive, invariant measures. At that time, Banach-Tarski Theorem was discovered and the old notion of amenability of a group was formulated by von Neumann. Later, Day revolutionized the subject and gave the modern definition of amenability of a group G.

Once the notion of amenability has been properly defined, a natural question arises "How large is the set of left invariant means?". Day [9] and Granirer [17] initiated the studies of the cardinality of the set of invariant means. Chou [6] showed that for a discrete infinite amenable group G, the cardinality of the set of all left invariant means on $l^\infty(G)$ is $2^{2^{|G|}}$. Later, Lau and Paterson [32] generalized this result and proved that for a noncompact, amenable locally compact group G the cardinality of the set of topologically left invariant means on $L^\infty(G)$ is $2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a covering of G by compact sets. Also see Yang [44], Miao [35], Hu [23] for recent developments.

In 1990, Bekka [4] generalized the classical notion of amenability and gave the definition of amenable representations. It is natural to ask the cardinality problem paralleling to the classical version.

We attempt to solve such a problem in two different ways. In the first method, we mainly focus on the left regular representation of an amenable [IN]-group. We modify Bekka's construction and give a "canonical extension" for each topologically left invariant mean on $L^\infty(G)$. Consequently, we are able to estimate a lower bound of the set of topologically left invariant means on $\mathrm{B}(\mathcal{H})$. In the second method, we apply Day's Fixed Point Theorem (which is inspired by Lau and Paterson's paper [32]) to construct topologically invariant means on $\mathrm{B}(\mathcal{H})$.

In part II, unless otherwise specified, G always denotes a locally compact group equipped with a fixed left Haar measure dx. Let m be a linear functional on $L^\infty(G)$. m is said to be a mean if it satisfies any two of the conditions: $\|m\| = 1$, $m(1) = 1$, $m(\phi) \geq 0$ whenever $\phi \in L^\infty(G)$ with $\phi \geq 0$. It is well-known that any two of the above conditions imply the others. If moreover, m satisfies the condition $m({}_x\phi) = m(\phi)$ for each $x \in G$ and $\phi \in L^\infty(G)$, m is said to be a left invariant mean. The notions of left uniform continuity, right uniform continuity and uniform continuity are defined as in Part I. Moreover, we still denote the space of left uniformly continuous functions, right uniformly

continuous functions and uniformly continuous functions by LUC(G), RUC(G) and UCB(G) respectively. A locally compact group G is amenable if there exists a left invariant mean on $L^\infty(G)$. It is well known that G is amenable if there exists a left invariant mean on any one of the function spaces: LUC(G), RUC(G), UCB(G). We refer the reader to [19] for detail. In 1990, Bekka [4] generalized the notion of amenability and defined the notion of amenable representations. Let $\pi : G \to \mathrm{B}(\mathcal{H})$ be a continuous unitary representation. The representation π is said to be amenable if there exists a state $M \in \mathrm{B}(\mathcal{H})^*$ such that $M(\pi(x)T\pi(x^{-1})) = M(T)$ for any $x \in G$ and any $T \in \mathrm{B}(\mathcal{H})$. In this case, M is called a G-invariant mean on $\mathrm{B}(\mathcal{H})$ for the representation π. Let $m \in L^\infty(G)^*$ be a mean. We say that m is a topologically left invariant mean if $m(f * \phi) = m(\phi)$ for any $\phi \in L^\infty(G)$ and $f \in L^1(G)$ satisfying $f \geq 0$ and $\int f(x)\,dx$=1. The operator version was defined by Bekka [4] as follow:
Let $\pi : G \to \mathrm{B}(\mathcal{H})$ be a unitary representation. Let $T \in \mathrm{B}(\mathcal{H})$ and let $f \in L^1(G)$. Define $f \cdot T \in \mathrm{B}(\mathcal{H})$ in weak sense by the formula:

$$f \cdot T = \int f(x)\pi(x)T\pi(x^{-1})\,dx.$$

More precisely, $f \cdot T$ is defined as the unique bounded linear operator on $\mathcal{H}$ such that for any $\xi, \eta \in \mathcal{H}$, one has

$$\langle f \cdot T(\xi) \,|\, \eta\rangle = \int f(x)\langle \pi(x)T\pi(x^{-1})\xi \,|\, \eta\rangle\,dx.$$

As in part I, we define UCB(π) to be the set consisting all $T \in \mathrm{B}(\mathcal{H})$ such that the mapping $G \to \mathrm{B}(\mathcal{H})$, $x \mapsto x \cdot T$ is norm continuous. Note that UCB(π) is a unital C*-subalgebra of $\mathrm{B}(\mathcal{H})$ containing all the compact operators. As pointed out by Bekka [4], UCB(π) is a non-commutative analog of UCB(G). By Cohen factorization theorem, we actually have $\mathrm{UCB}(\pi) = L^1(G) \cdot \mathrm{B}(\mathcal{H}) = \{f \cdot T \,|\, f \in L^1(G), T \in \mathrm{B}(\mathcal{H})\}$. Let $\pi : G \to \mathrm{B}(\mathcal{H})$ be a unitary representation.

An element M in $\mathrm{B}(\mathcal{H})^*$ (resp. $\mathrm{UCB}(\pi)^*$) is called a topologically invariant mean on $\mathrm{B}(\mathcal{H})$ (resp. $\mathrm{UCB}(\pi)$) if M satisfies:

(i) M is a state,

(ii) $M(f \cdot T) = M(T)$ for any $T \in \mathrm{B}(\mathcal{H})$ (resp. $T \in \mathrm{UCB}(\pi)$) and $f \in L^1(G)$ satisfying $f \geq 0$ and $\int f(x)\, dx = 1$.

Chapter 6

G-Invariant Means and Topologically Invariant Means

In this section, we try to give a lower bound on the size of the set of all topologically invariant means on B($\mathcal{H}$) by using two different methods. The first method works well for the left regular representation of amenable [IN] groups while the second method works for general unitary representation of amenable locally compact groups.

Lemma 6.1. *Let $\lambda : G \to B(\mathcal{H})$ be the left regular representation, where $\mathcal{H} = L^2(G)$. Given $f \in \mathcal{H}$, we define a rank-one trace-class operator $f \otimes f$ by $f \otimes f(g) = \langle g \,|\, f \rangle f$. Given $\phi \in L^\infty(G)$, we define $T_\phi \in B(\mathcal{H})$ by $T_\phi(f) = \phi f$ ($f \in \mathcal{H}$). Let $\mathcal{U}$ be the set of all open neighborhoods of e, directed under the reversed set inclusion $\supseteq$. For each $U \in \mathcal{U}$, we choose $f_U \in Cc(G)$ such that $f_U \geq 0$, $\int f_U^2 = 1$ and $supp(f_U) \subseteq U$. We denote the net $(f_U)_{U \in \mathcal{U}}$ simply by (f_α). Let $T \in B(\mathcal{H})$. Define $\varphi_T^\alpha(x) = tr(T\lambda(x)(f_\alpha \otimes f_\alpha)\lambda(x^{-1}))$, then for each $\phi \in RUC(G)$, $\varphi_{T_\phi}^\alpha \to \phi$ uniformly.*

Proof. Let $\phi \in$ RUC(G) be given. Let $\epsilon > 0$. Choose an open neighborhood U

of e such that $|\phi(xy) - \phi(x)| < \epsilon$ whenever $x \in G$ and $y \in U$. Denote $\alpha_0 = U$. Let $x \in G$, then

$$\begin{aligned} \varphi^{\alpha}_{T_\phi}(x) &= \operatorname{tr}(T_\phi \lambda(x)(f_\alpha \otimes f_\alpha)\lambda(x^{-1})) = \operatorname{tr}(\lambda(x^{-1})T_\phi\lambda(x)(f_\alpha \otimes f_\alpha)) \\ &= \langle \lambda(x^{-1})T_\phi \lambda(x) f_\alpha \otimes f_\alpha(f_\alpha) \,|\, f_\alpha\rangle = \langle T_\phi \, {}_{x^{-1}}(f_\alpha) \,|\, {}_{x^{-1}}(f_\alpha)\rangle \\ &= \int \phi(y) f_\alpha^2(x^{-1}y)dy = \int \phi(xy) f_\alpha^2(y)dy \end{aligned}$$

Therefore, $|\varphi^{\alpha}_{T_\phi}(x) - \phi(x)| = |\int_U [\phi(xy) - \phi(x)] f_\alpha^2(y)dy| \leq \epsilon$ whenever $\alpha \succeq \alpha_0$ and $x \in G$. □

Lemma 6.2. *Let G be a locally compact group and let $\lambda : G \to B(\mathcal{H})$ be the left regular representation, where $\mathcal{H} = L^2(G)$. Given a topologically left invariant mean m on $L^\infty(G)$, we can associate a topologically invariant mean M on $B(\mathcal{H})$ in such a way that $M(T_\phi) = m(\phi)$ for each $\phi \in RUC(G)$, where $T_\phi \in B(\mathcal{H})$ is defined by $T_\phi(f) = \phi f$.*

Proof. Let (f_α) be the net defined in lemma 6.1. Let $D = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ and let B be the closed unit ball of $\mathrm{B}(\mathcal{H})$. Given $T \in \mathrm{B}(\mathcal{H})$, we define $\varphi^\alpha_T \in \mathrm{CB}(G)$ as in lemma 6.1. It is routine to check that $\|\varphi^\alpha_T\|_\infty \leq \|T\|$. Let m be a topologically left invariant mean on $L^\infty(G)$. Define $\theta_\alpha \in D^B$ by $\theta_\alpha(T) = m(\varphi^\alpha_T)$. By Tychonoff theorem, we choose a converging subnet of (θ_α) which converges to some $\theta \in D^B$. We still denote such a subnet by (θ_α). Define $M : \mathrm{B}(\mathcal{H}) \to \mathbb{C}$ by

$$M(T) = \begin{cases} \|T\|\theta(T/\|T\|), \text{ if } T \neq 0 \\ 0, \text{ if } T = 0 \end{cases}$$

Note that $M(T) = \lim_\alpha m(\varphi^\alpha_T)$. By observing that for any $T_1, T_2 \in \mathrm{B}(\mathcal{H})$, $k \in \mathbb{C}$, $\varphi^\alpha_{kT_1+T_2} = k\varphi^\alpha_{T_1} + \varphi^\alpha_{T_2}$, it follows that M is linear. Note that if $T \in \mathrm{B}(\mathcal{H})$ is positive, $\varphi^\alpha_T \geq 0$. Therefore, if $T \in \mathrm{B}(\mathcal{H})$ is positive, $M(T) \geq 0$. If I is the identity operator on $\mathcal{H}$, φ_I^α is the constant function 1. It follows that $M(I) = 1$ and hence M is a mean on $\mathrm{B}(\mathcal{H})$. Denote $S_\alpha = f_\alpha \otimes f_\alpha$. Observe that for each $x \in G$, $g \in L^1(G)$ with $g \geq 0$ and $\int g(x)\,dx = 1$, we have

$$\begin{aligned} g * \varphi_T^\alpha(x) &= \int g(y)\varphi_T^\alpha(y^{-1}x)\ dy = \int g(y)\ \mathrm{tr}(\lambda(x^{-1}y)T\lambda(y^{-1}x)S_\alpha)\ dy \\ &= \mathrm{tr}[\lambda(x^{-1})\int g(y)\lambda(y)T\lambda(y^{-1})\ dy\lambda(x)S_\alpha] = \mathrm{tr}[\lambda(x^{-1})g \cdot T\lambda(x)S_\alpha] \\ &= \varphi_{g\cdot T}^\alpha(x), \end{aligned}$$

i.e. $g * \varphi_T^\alpha = \varphi_{g\cdot T}^\alpha$. For $T \in \mathrm{B}(\mathcal{H})$, we have

$$\begin{aligned} M(g \cdot T) &= \lim_\alpha m(\varphi_{g\cdot T}^\alpha) \\ &= \lim_\alpha m(g * \varphi_T^\alpha) \\ &= \lim_\alpha m(\varphi_T^\alpha) \\ &= M(T). \end{aligned}$$

Therefore M is a topologically invariant mean. Let $\phi \in \mathrm{RUC}(G)$. By lemma 6.1, $\varphi_{T_\phi}^\alpha \to \phi$ uniformly, hence $M(T_\phi) = \lim_\alpha m(\varphi_{T_\phi}^\alpha) = m(\phi)$. □

We need few lemmas about quotient groups and uniformly continuous functions. Let $\phi : G \to \mathbb{C}$ be a bounded continuous function. We remark that $\phi \in \mathrm{LUC}(G)$ if and only if for each $\epsilon > 0$, there exists an open neighborhood U of e such that $|\phi(yx) - \phi(x)| < \epsilon$ whenever $x \in G$ and $y \in U$. Similarly $\phi \in \mathrm{RUC}(G)$ if and only if for each $\epsilon > 0$, there exists an open neighborhood U of e such that $|\phi(xy) - \phi(x)| < \epsilon$ whenever $x \in G$ and $y \in U$.

Lemma 6.3. *Let G be a locally compact group and let K be a compact normal subgroup of G. Let m_K be the normalized Haar measure on K and regard m_K as a measure on G in a canonical way. Given $\phi \in LUC(G)$, we define $\phi'(x) = \int_K \phi(tx)\,dm_K(t)$, $(x \in G)$, then $\phi' \in LUC(G)$ and ϕ' is constant on each K coset.*

Proof. Let $\phi \in \mathrm{LUC}(G)$. Let $\epsilon > 0$. Choose an open neighborhood U_0 of e such that $|\phi(yx) - \phi(x)| < \epsilon$ whenever $y \in U_0$ and $x \in G$. By the continuity of the map $G \to \mathrm{CB}(G)$, $t \mapsto_t \phi$, for each $t \in K$, there exists an open neighborhood V_t of t such that $\|_{t'}\phi -_t \phi\| < \epsilon$ whenever $t' \in V_t$. By compactness of K, we select $t_1, t_2, \ldots, t_n \in K$ such that $K \subseteq \cup_{j=1}^n V_{t_j}$. Define $U = \cap_{j=1}^n t_j^{-1} U_0 t_j$. Let $y \in U$, $x \in G$, $t \in K$. Choose j such that $t \in V_{t_j}$, then

$$\begin{aligned} |_t\phi(yx) -_t \phi(x)| &\leq |_{t_j}\phi(yx) -_{t_j} \phi(x)| + 2\|_{t_j}\phi -_t \phi\| \\ &\leq |\phi(t_j y t_j^{-1} t_j x) - \phi(t_j x)| + 2\epsilon \\ &\leq 3\epsilon \end{aligned}$$

by noticing that $t_j y t_j^{-1} \in U_0$. Therefore $|\phi'(yx) - \phi'(x)| \leq \int_K |_t\phi(yx) -_t \phi(x)|\, dm_K(t) \leq 3\epsilon$. Clearly ϕ' is bounded and continuous and hence $\phi' \in \mathrm{LUC}(G)$. Let $x \in G$, $k \in K$, then $\phi'(kx) = \int_K \phi(tkx)\, dm_K(t) = \int_K \phi(tx)\, dm_K(t) = \phi'(x)$ Therefore ϕ' is constant on each K coset. □

Lemma 6.4. *Let K be a compact normal subgroup of a locally compact group G and let $Q : G \to G/K$ be the canonical quotient map. Define $\Psi : CB(G/K) \to CB(G)$ by $\Psi(\phi) = \phi \circ Q$, then:*
(i)$\Psi(LUC(G/K)) = \{\phi \in LUC(G) \,|\, \phi$ is constant on each K coset.$\}$,
(ii)$\Psi(RUC(G/K)) = \{\phi \in RUC(G) \,|\, \phi$ is constant on each K coset.$\}$.

Proof. We prove (i) only. (ii) can be proved in a similar way. Let $\phi \in \mathrm{LUC}(G/K)$. Let $\epsilon > 0$. Choose an open neighborhood $\mathcal{U}$ of $\dot{e} = eK \in G/K$ such that $|\phi(\dot{y}\dot{x}) - \phi(\dot{x})| < \epsilon$ whenever $\dot{y} \in \mathcal{U}$ and $\dot{x} \in G/K$. Let $U = Q^{-1}(\mathcal{U})$ which is an open neighborhood of $e \in G$. If $y \in U$, $x \in G$, we have: $|\Psi(\phi)(yx) - \Psi(\phi)(x)| = |\phi(\dot{y}\dot{x}) - \phi(\dot{x})| < \epsilon$, so $\Psi(\phi) \in \mathrm{LUC}(G)$. It is also obvious that $\Psi(\phi)$ is constant on each K coset. Conversely, suppose that $\varphi \in \mathrm{LUC}(G)$ such that φ is constant on each K coset. Define $\phi : G/K \to \mathbb{C}$

by $\phi(\dot{x}) = \varphi(x)$, $(x \in G)$. Note that ϕ is bounded. Let $\epsilon > 0$, then there exists an open neighborhood U of $e \in G$ such that $|\varphi(yx) - \varphi(x)| < \epsilon$ whenever $y \in U$ and $x \in G$. Let $\mathcal{U} = Q(U)$ which is an open neighborhood of $\dot{e} \in G/K$. Let $\xi \in \mathcal{U}$, $\eta \in G/K$ be given. Choose $y \in U$, $x \in G$ such that $Q(y) = \xi$ and $Q(x) = \eta$, then $|\phi(\xi\eta) - \phi(\eta)| = |\phi \circ Q(yx) - \phi \circ Q(x)| = |\varphi(yx) - \varphi(x)| < \epsilon$. Therefore $\phi \in \mathrm{LUC}(G/K)$ with $\Psi(\phi) = \varphi$. □

Let G be a locally compact group and let $A \subseteq G$. We say that A is invariant (under conjugation) if for each $x \in G$, $A = xAx^{-1}$, where $xAx^{-1} = \{xyx^{-1} \,|\, y \in A\}$. We say that G is a [SIN] group, denoted by $G \in$ [SIN], if the set of all invariant compact neighborhoods of the identity $e \in G$ forms a local base of e. We remark that a locally compact group G is a [SIN] group if and only if $\mathrm{LUC}(G) = \mathrm{RUC}(G)$. We say that a locally compact group G is an [IN] group, denoted by $G \in$ [IN] if there exists a compact invariant neighborhood U of e. Let $G \in$ [IN]. Let K be the intersection of all compact invariant neighborhoods of e. We remark that K is a compact, normal subgroup of G and $G/K \in$ [SIN]. See [20] for detail.

Lemma 6.5. *Let G be an amenable [IN] group and let m_1, m_2 be topologically left invariant means on $L^\infty(G)$. Suppose that $m_1 \neq m_2$, then there exists $\varphi \in UCB(G)$ such that $m_1(\varphi) \neq m_2(\varphi)$.*

Proof. Let K be the intersection of all compact invariant neighborhoods of e. Let $Q : G \to G/K$ be the canonical quotient map. Define $\Psi : \mathrm{CB}(G/K) \to \mathrm{CB}(G)$ by $\Psi(\phi) = \phi \circ Q$. Choose $\varphi_0 \in L^\infty(G)$ such that $m_1(\varphi_0) \neq m_2(\varphi_0)$. Choose $f \in L^1(G)$ with $f \geq 0$ and $\int f(x)\,dx = 1$, then $m_1(f * \varphi_0) = m_1(\varphi_0) \neq m_2(\varphi_0) = m_2(f * \varphi_0)$. Note that $f * \varphi_0 \in \mathrm{LUC}(G)$, so we may assume, without loss of generality that $\varphi_0 \in \mathrm{LUC}(G)$. Define $\varphi(x) = \int_K \varphi_0(tx)\,dm_K(t)$, then by lemma 6.3, $\varphi \in \mathrm{LUC}(G)$ and φ is constant on each K coset. By lemma 6.4, $\varphi = \Psi(\phi)$ for some $\phi \in \mathrm{LUC}(G/K)$. Since $G/K \in$ [SIN], we have $\phi \in \mathrm{LUC}(G/K) = \mathrm{UCB}(G/K)$, so $\varphi \in \mathrm{UCB}(G)$ by lemma 6.4 again. Regard $\varphi = \int_K {}_t\varphi_0 \, dm_K(t)$ as a vector valued integral. Note that since the map $t \mapsto_t \varphi_0$ is continuous with respect to the norm topology, the integral can be approximated by a Riemann integral. Therefore, $m_1(\varphi) = \int_K m_1({}_t\varphi_0) \, dm_K(t) = \int_K m_1(\varphi_0) \, dm_K(t) = m_1(\varphi_0)$. By the same argument, $m_2(\varphi) = m_2(\varphi_0)$, hence $m_1(\varphi) \neq m_2(\varphi)$. □

Theorem 6.6. *Let G be an amenable [IN] group. Let $\lambda : G \to B(\mathcal{H})$ be the left regular representation of G on the Hilbert space $\mathcal{H} = L^2(G)$. For each $\phi \in L^\infty(G)$, we define $T_\phi \in B(\mathcal{H})$ by $T_\phi(f) = \phi f$, where $f \in \mathcal{H}$. For each topologically left invariant mean m on $L^\infty(G)$, we can associate a topologically invariant mean M_m on $B(\mathcal{H})$ such that $M_m(T_\phi) = m(\phi)$ for each $\phi \in RUC(G)$. In particular if $m_1 \neq m_2$, then $M_{m_1} \neq M_{m_2}$.*

Proof. We define M_m according to lemma 6.2. Let m_1, m_2 be topologically left invariant means on $L^\infty(G)$ with $m_1 \neq m_2$. By the previous lemma, we may choose $\phi \in \mathrm{UCB}(G)$ such that $m_1(\phi) \neq m_2(\phi)$. Therefore $M_{m_1}(T_\phi) \neq M_{m_2}(T_\phi)$. In particular $M_{m_1} \neq M_{m_2}$ □

Corollary 6.7. *Let G be a non-compact amenable [IN] group and let $\lambda : G \to B(\mathcal{H})$ be the left regular representation of G on the Hilbert space $\mathcal{H} = L^2(G)$. Then the cardinality of the set of topologically invariant means for λ is at least $2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a covering of G by compact sets.*

Proof. By [32], the cardinality of the set of topologically left invariant means on $L^\infty(G)$ is exactly $2^{2^{d(G)}}$. By the previous theorem, the result follows immediately. □

Remark 6.8. We say that any topologically invariant mean M on $\mathrm{B}(\mathcal{H})$ satisfying $M(T_\phi) = m(\phi)$ a canonical extension of m.

We introduce the second approach. Let $\pi : G \to \mathrm{B}(\mathcal{H})$ be a continuous unitary representation of a locally compact group G. Let $\mathrm{L}^1(\mathcal{H})$ be the Banach space of all trace-class operators on $\mathcal{H}$, equipped with the trace-norm norm $\|\cdot\|_1$. If $L \in \mathrm{L}^1(\mathcal{H})$ and $x \in G$, we define $L \cdot x = \pi(x^{-1})L\pi(x)$. Under the map $(L, x) \mapsto L \cdot x$, $\mathrm{L}^1(\mathcal{H})$ becomes a right Banach G-module, i.e.:
(i) $L \cdot e = L$ and for any $x, y \in G$, $(L \cdot x) \cdot y = L \cdot (xy)$,
(ii) For each $x \in G$, the map $L \mapsto L \cdot x$ is a bounded invertible linear operator on $L^1(\mathcal{H})$ with $\|L \cdot x\|_1 \leq \|L\|_1$,
(iii) For each $L \in L^1(\mathcal{H})$, the map $x \mapsto L \cdot x$ is continuous with respect to the $\|\cdot\|_1$-topology.
Dualize the above G-action, we obtain a left G-module action on $\mathrm{B}(\mathcal{H})$, $(x, T) \mapsto x \cdot T$, where $\langle x \cdot T, L\rangle = \langle T, L \cdot x\rangle$, $(L \in \mathrm{L}^1(\mathcal{H}),\ x \in G,\ T \in \mathrm{B}(\mathcal{H}))$. Note that

$$\begin{aligned}\langle T, L \cdot x\rangle &= \mathrm{tr}(TL \cdot x) = \mathrm{tr}(T\pi(x^{-1})L\pi(x)) \\ &= \mathrm{tr}(\pi(x)T\pi(x^{-1})L) = \mathrm{tr}(x \cdot TL)\end{aligned}$$

Therefore $x \cdot T = \pi(x)T\pi(x^{-1})$. In general, for a fixed $T \in \mathrm{B}(\mathcal{H})$, the map $x \mapsto x \cdot T$ is not continuous with respect to the norm topology. We restrict our attention on the C*-subalgebra $\mathrm{UCB}(\pi)$ which consists of all $T \in \mathrm{B}(\mathcal{H})$ such that the map $x \mapsto x \cdot T$ is continuous with respect to the norm topology. Note that $\mathrm{UCB}(\pi)$ is a left Banach G-module. We further dualize the G-module action and obtain a right G-module action on $\mathrm{UCB}(\pi)^*$: $(M, x) \mapsto M \cdot x$, where $\langle M \cdot x, T\rangle = \langle M, x \cdot T\rangle$, $(T \in \mathrm{UCB}(\pi))$. It is clear that:
(i)For each $x \in G$, the map $M \mapsto M \cdot x$ is weak*-weak* continuous,
(ii)For each $M \in \mathrm{UCB}(\pi)^*$, the map $x \mapsto M \cdot x$ is weak*-continuous.

We need a lemma about G-invariant means on UCB(π) and topologically invariant means on B($\mathcal{H}$). Recall that if $f \in L^1(G)$ and $T \in$ B($\mathcal{H}$), we define $f \cdot T = \int_G f(x)x \cdot T \, dx$.

Lemma 6.9. *Let $f, g \in L^1(G)$, $x \in G$, $T \in B(\mathcal{H})$. Then:*
(i) $({}_x f) \cdot T = (x^{-1}) \cdot (f \cdot T)$, where ${}_x f(y) = f(xy)$.
(ii)$g \cdot (f \cdot T) = (g * f) \cdot T$.
(iii) If $f \geq 0$ and $T \geq 0$, then $f \cdot T \geq 0$.

Proof. (i)$({}_x f) \cdot T = \int f(xy)\pi(y)T\pi(y^{-1}) \, dy = \int f(y)\pi(x^{-1})\pi(y)T\pi(y^{-1})\pi(x) \, dy$
$= \pi(x^{-1})(f \cdot T)\pi(x) = (x^{-1}) \cdot (f \cdot T)$

(ii)$g \cdot (f \cdot T) = \int g(y)y \cdot (f \cdot T) \, dy = \int g(y)({}_{y^{-1}} f \cdot T) \, dy$
$= \iint g(y)f(y^{-1}x)\pi(x)T\pi(x^{-1}) \, dxdy = \int g * f(x)\pi(x)T\pi(x^{-1})dx$
$= (g * f) \cdot T.$

(iii) Suppose that $f \geq 0$ and $T \geq 0$. Let $\eta \in \mathcal{H}$, then $\langle f \cdot T(\eta) \,|\, \eta \rangle$
$= \int f(y)\langle T\pi(y^{-1})\eta \,|\, \pi(y^{-1})\eta \rangle \geq 0$. Therefore $f \cdot T \geq 0$. □

Lemma 6.10. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Let $UCB(\pi)$ be the C^*-subalgebra of $B(\mathcal{H})$ defined in above. Then the map $M \mapsto M|_{UCB(\pi)}$ is a bijection from the set of all topologically invariant means on $B(\mathcal{H})$ onto the set of all G invariant means on $UCB(\pi)$.*

Proof. Let M be a G-invariant mean on UCB(π). We assert that for each fixed $T \in$ B($\mathcal{H}$), there exists $k \in \mathbb{C}$ such that $\langle M, f \cdot T \rangle = k \int f(x) \, dx$ for all $f \in L^1(G)$. Let $T \in$ B($\mathcal{H}$) be fixed. Without loss of generality, we assume that $T \geq 0$. Define $\theta \in L^1(G)^*$ by $\theta(f) = \langle M, f \cdot T \rangle$. By the previous lemma, $\theta|_{C_c(G)}$ is a positive, left translation invariant linear functional, so $\theta|_{C_c(G)}$ is induced by a left Haar measure. There exists $k > 0$ such that $\theta(f) = k \int f(x)\,dx$, $f \in C_c(G)$. As $C_c(G)$ is $\|\cdot\|_1$ dense in $L^1(G)$, it follows that $\langle M, f \cdot T\rangle = \theta(f) = k \int f(x)\,dx$, for any $f \in L^1(G)$. Now fix $f_0 \in L^1(G)$ with $f_0 \geq 0$ and $\int f_0(x)\,dx = 1$. Define $M' \in \mathrm{B}(\mathcal{H})^*$ by $\langle M', T\rangle = \langle M, f_0 \cdot T\rangle$. By the previous lemma (iii), M' is positive and $\langle M', I\rangle = \langle M, f_0 \cdot I\rangle = \langle M, I\rangle = 1$. Let $g \in L^1(G)$ with $g \geq 0$ and $\int g(x)\,dx = 1$, then

$$
\begin{aligned}
\langle M', g \cdot T\rangle &= \langle M, (f_0 \cdot (g \cdot T))\rangle \\
&= \langle M, (f_0 * g) \cdot T\rangle \\
&= \langle M, f_0 \cdot T\rangle, \text{ since } \int f_0 * g(x)\,dx = \int f_0(x)\,dx \\
&= \langle M', T\rangle.
\end{aligned}
$$

Therefore M' is a topologically invariant mean on $\mathrm{B}(\mathcal{H})$. Consider the map $M \mapsto M|_{\mathrm{UCB}(\pi)}$ from the set of all topologically invariant means on $\mathrm{B}(\mathcal{H})$ to the set of all G-invariant means on $\mathrm{UCB}(\pi)$. Let M_1, M_2 be two topologically invariant means on $\mathrm{B}(\mathcal{H})$. If $M_1 \neq M_2$, there exists $T \in \mathrm{B}(\mathcal{H})$ such that $\langle M_1, T\rangle \neq \langle M_2, T\rangle$. Fix $f \in L^1(G)$ with $f \geq 0$ and $\int f(x)\,dx = 1$, then
$\langle M_1|_{\mathrm{UCB}(\pi)}, f \cdot T\rangle = \langle M_1, f \cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, f \cdot T\rangle$
$= \langle M_2|_{\mathrm{UCB}(\pi)}, f \cdot T\rangle$, so $M_1|_{\mathrm{UCB}(\pi)} \neq M_2|_{\mathrm{UCB}(\pi)}$. Therefore the map is injective. On the other hand, given a G-invariant mean N on $\mathrm{UCB}(\pi)$, we fixed $f_0 \in L^1(G)$ with $f_0 \geq 0$ and $\int f_0(x)\,dx = 1$ and define $\langle M, T\rangle = \langle N, f_0 \cdot T\rangle$, $T \in \mathrm{B}(\mathcal{H})$. Note that M is a topologically invariant mean on $\mathrm{B}(\mathcal{H})$. Let

$S \in \mathrm{UCB}(\pi)$, then $S = g \cdot T$ for some $g \in L^1(G)$ and $T \in \mathrm{B}(\mathcal{H})$. Observe that

$$\begin{aligned} \langle M, S \rangle &= \langle N, f_0 \cdot (g \cdot T) \rangle \\ &= \langle N, (f_0 * g) \cdot T \rangle \\ &= \langle N, g \cdot T \rangle, \text{ since } \int f_0 * g(x)\,dx = \int g(x)\,dx \\ &= N(S) \end{aligned}$$

This shows that $M|_{\mathrm{UCB}(\pi)} = N$ and hence the map is surjective. □

Remark 6.11. The lemma in above is an analog of [39] Proposition 1.7.

Theorem 6.12. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of an amenable locally compact group. Suppose that $\mathcal{H}$ has a family of mutually orthogonal invariant subspaces $\{\mathcal{H}_i \,|\, i \in \Lambda\}$, then for each $i \in \Lambda$, there exists a topologically invariant mean M_i on $B(\mathcal{H})$ such that $\{M_i \,|\, i \in \Lambda\}$ are linearly independent.*

Proof. For each $i \in \Lambda$, pick $\xi_i \in \mathcal{H}_i$ with $\|\xi_i\| = 1$ and define $L_i = \xi_i \otimes \xi_i \in \mathcal{L}^1(\mathcal{H})$, then $L_i \geq 0$ and $\|L_i\|_1 = 1$. Let $\mathcal{C}_i$ be the convex hull of $\{L_i \cdot x \,|\, x \in G\}$ which is invariant under the right G-action. Let $\widehat{\mathcal{C}_i} = \{\widehat{L}|_{\mathrm{UCB}(\pi)} \,|\, L \in \mathcal{C}_i\}$, where $\widehat{L}$ denotes the canonical image of L under the canonical embedding $\mathcal{L}^1(\mathcal{H}) \hookrightarrow \mathrm{B}(\mathcal{H})^*$. Note that $\widehat{\mathcal{C}_i}$ is a bounded subset of $\mathrm{UCB}(\pi)^*$, so its $\sigma(\mathrm{UCB}(\pi)^*, \mathrm{UCB}(\pi))$-closure, denoted by $\mathcal{K}_i$, is weak* compact. Note that when equipped with weak* topology, $\mathrm{UCB}(\pi)^*$ is a locally convex space and each $\mathcal{K}_i$ is a compact, convex subset of $\mathrm{UCB}(\pi)^*$. G acts affinely on $\mathcal{K}_i$ by $(M, x) \mapsto M \cdot x$, $(M \in \mathcal{K}_i,\ x \in G)$ and the action is clearly separately continuous. By Day's fixed point theorem, there exists $M_i \in \mathcal{K}_i$ such that $M_i \cdot x = M_i$ for each $x \in G$, hence M_i is a left invariant mean on $\mathrm{UCB}(\pi)$. By lemma 6.10, M_i is extended in a unique way to a topologically invariant mean on $\mathrm{B}(\mathcal{H})$. We denote that topologically invariant mean by the same symbol M_i. We assert that the means in $\{M_i \,|\, i \in \Lambda\}$ are linearly independent. Let $M_1, M_2, \ldots, M_n \in \{M_i \,|\, i \in \Lambda\}$ and let $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$. Suppose that $\sum_{i=1}^n \alpha_i M_i = 0$. Let $P_i : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}_i$. Since P_i commutes with $\pi(x)$, it follows that $x \cdot P_i = P_i$ and therefore $P_i \in \mathrm{UCB}(\pi)$. Note that $\langle M_i, P_j \rangle = \delta_{ij}$. For, choose a net $(L_{i,\beta})_\beta$ in $\mathcal{C}_i$ such that $\widehat{L_{i,\beta}}|_{\mathrm{UCB}(\pi)} \to M_i$ with respect to the weak*-topology. Then $\langle M_i, P_j \rangle = \lim_\beta \langle \widehat{L_{i,\beta}}, P_j \rangle = \delta_{ij}$. It follows that $\alpha_j = \langle \sum_{i=1}^n \alpha_i M_i, P_j \rangle = 0$ and hence $M_1, M_2, \ldots, M_n$ are linearly independent. □

Corollary 6.13. *Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of an amenable locally compact group G. In order that there exists a unique topologically invariant mean M on $B(\mathcal{H})$, it is necessary that π is irreducible.*

We recall that a locally compact group G is a Moore group, denoted by $G \in [\text{Moore}]$, if every irreducible representation of G is finite dimensional. All abelian groups and all compact groups are Moore groups and Moore groups are amenable. In fact, the latter conclusion can be proved easily by Bekka's theory. For, if G is a Moore group and $\pi : G \to \mathrm{B}(\mathcal{H})$ is an irreducible representation, then π is amenable since $\dim(\mathcal{H}) < \infty$ ([4] Theorem 1.3). Now all irreducible representations of G are amenable, so G is amenable by [4] Corollary 5.5.

Proposition 6.14. *Let G be a Moore group. If $\pi : G \to B(\mathcal{H})$ is an irreducible representation, there exists a unique G-invariant/topologically invariant mean on $B(\mathcal{H})$.*

Proof. We remark that if $\dim(\mathcal{H}) < \infty$, a G-invariant mean M on $\mathrm{B}(\mathcal{H})$ is automatically a topologically invariant mean. For, suppose that M is a G-invariant mean on $\mathrm{B}(\mathcal{H})$. Suppose that $\dim(\mathcal{H}) < \infty$, then $\mathrm{B}(\mathcal{H})^* \simeq \mathrm{L}^1(\mathcal{H})$. Write $M = \sum_{i=1}^n \alpha_i \xi_i \otimes \xi_i$, for some $\alpha_i \in \mathbb{C}$, $\xi_i \in \mathcal{H}$ with $\|\xi_i\| = 1$, i.e.

$\langle M, T\rangle = \mathrm{tr}(T\sum_{i=1}^n \alpha_i\xi_i\otimes\xi_i)$, $(T\in \mathrm{B}(\mathcal{H}))$. Let $T\in \mathrm{B}(\mathcal{H})$ and let $f\in L^1(G)$ with $f\geq 0$ and $\int f(x)\,dx = 1$. Then

$$\begin{aligned}\langle M, f\cdot T\rangle &= \sum_{i=1}^n \alpha_i \ \mathrm{tr}(f\cdot T\xi_i\otimes\xi_i) = \sum_{i=1}^n \alpha_i\langle f\cdot T\xi_i\,|\,\xi_i\rangle \\ &= \sum_{i=1}^n \alpha_i \int f(x)\langle x\cdot T(\xi_i)\,|\,\xi_i\rangle\,dx = \int f(x)\ \mathrm{tr}[(x\cdot T)\sum_{i=1}^n \alpha_i\xi_i\otimes\xi_i]\,dx \\ &= \int f(x)\langle M, x\cdot T\rangle\,dx = \int f(x)\langle M, T\rangle\,dx = \langle M, T\rangle.\end{aligned}$$

Let $\pi : G\to \mathrm{B}(\mathcal{H})$ be an irreducible representation. Let M_1, M_2 be two G-invariant means on $\mathrm{B}(\mathcal{H})$. Let $T_0\in \mathrm{B}(\mathcal{H})$. Define $E = \mathrm{B}(\mathcal{H})$ and regard it as a locally convex space. Let $\mathcal{K}$ be the closed convex hull of $\{x\cdot T_0\,|\,x\in G\}$. Note that G acts on $\mathcal{K}$ affinely by $(x, T)\mapsto x\cdot T$ and the action is separately continuous. By Day's fixed point theorem, there exists $T\in\mathcal{K}$ such that $x\cdot T = T$ for any $x\in G$, i.e. $\pi(x)T = T\pi(x)$. Since π is irreducible, $T = \lambda I$ for some $\lambda\in\mathbb{C}$ by Schur's lemma. Choose a net (S_α) in $\mathcal{K}$ such that $S_\alpha\to T$, then $\lambda = \langle M_1, \lambda I\rangle = \lim_\alpha\langle M_1, S_\alpha\rangle = \langle M_1, T_0\rangle$. By the same reason, $\lambda = \langle M_2, T_0\rangle$. Therefore $M_1 = M_2$. $\square$

Let $\lambda : G\to \mathrm{B}(\mathcal{H})$ be the left regular representation of a locally compact group G on the Hilbert space $\mathcal{H} = L^2(G)$. Given a topologically left invariant mean m on $L^\infty(G)$, we obtain a topologically invariant mean M on $\mathrm{B}(\mathcal{H})$ via the canonical extension by passing to a suitable converging subnet. However, the original net may have more than one cluster point, so in general, topologically invariant mean arisen from canonical extension is not unique. In the following, we give such an example.

Example 6.15. Let $\mathbb{T}$ be the circle group equipped with the normalized Haar measure. Let $\lambda : \mathbb{T}\to \mathrm{B}(\mathcal{H})$ be the left regular representation of $\mathbb{T}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$. For each $n\in\mathbb{Z}$, we associate a character $\chi_n : \mathbb{T}\to\mathbb{C}$

by $\chi_n(z) = z^n$ $(z \in \mathbb{T})$. We remark that $\{\chi_n \,|\, n \in \mathbb{Z}\}$ is an orthonormal base of $\mathcal{H}$ and for each $n \in \mathbb{Z}$, $\mathbb{C}\chi_n$ is an invariant subspace of $\mathcal{H}$. By theorem 6.12, there exists a family $\{M_n \,|\, n \in \mathbb{Z}\}$ of topologically invariant means which are linearly independent. By tracing the proof of theorem 6.12, we note that for each $x \in \mathbb{T}$, $n \in \mathbb{Z}$, we have: $x \cdot (\chi_n \otimes \chi_n) = \chi_n \otimes \chi_n$. Therefore $M_n = \chi_n \otimes \chi_n \in L^1(\mathcal{H})$. Let $\phi \in L^\infty(\mathbb{T})$ and let $T_\phi \in \mathrm{B}(\mathcal{H})$ be defined by $T_\phi(f) = \phi f$, then

$$\begin{aligned} \langle M_n, T_\phi \rangle &= \mathrm{tr}(M_n T_\phi) = \langle T_\phi \chi_n \,|\, \chi_n \rangle \\ = &\int \phi(x) |\chi_n(x)|^2 \, dx = \int \phi(x) \, dx = \langle m, \phi \rangle, \end{aligned}$$

where m is the unique topologically left invariant mean on $L^\infty(\mathbb{T})$. Therefore M_n is an extension of m.

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