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Impact Problems for Nonlinear Elastic Strings and Membranes with Variable Boundary Conditions

BY

Zhong Jian-Ling

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NAME OF AUTHOR: **Zhong Jian-Ling**

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(Signed)________________

Permanent Address:
Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: April 18, 1994

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Impact problems for nonlinear elastic strings and membranes with variable boundary conditions** submitted by **Zhong Jian-Ling** in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics.

R.J. Tait (Supervisor)

Y.S. Wong

M.G. Faulkner

Y. Lin

S. Shen

R.W. Ogden

Date: March, 28, 1994

I dedicate this thesis

to my parent

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Abstract

This thesis consists of two parts. In the first part, we consider the impact problem of a nonlinear elastic string with variable boundary conditions. Since there is no similarity solution for this case, we apply a perturbation method when the deformation has a small change from the initial equilibruim state. The perturbation solutions are compared with the numerical results obtained by a Godunov type scheme.

In the second part, we consider the impact problem of a circular elastic membrane with variable boundary condition. This problem is an extension of the impact problem with constant boundary condition which has been considered by Haddow, Wegner and Jiang. We rederive the nonlinear governing equations using the direct method of Green, Naghdi and Wainwright. The nonlinear results are compared with the results obtained from the linearised version equation used by Farrar.

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Chapter 1

Introduction

The purpose of this thesis is to consider certain quasilinear systems of hyperbolic partial differential equations which govern the propagation of nonlinear waves in isotropic hyperelastic solids; that is elastic solids which have a stored energy function and are isotropic in a natural reference state. We confine our attention to mechanical effects and consider certain basic problems concerning elastic strings and membranes. In general the system of equations may be written in the form

$$\frac{\partial \boldsymbol{u}(X,t)}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u}(X,t))}{\partial X} + \boldsymbol{B}(\boldsymbol{u}(X,t),X) = \boldsymbol{0}, \tag{1-1}$$

where $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^T$ is a vector in R^n and $(X,t) \in R \times R^+$ with X representing a spatial variable and t time. A superposed T denotes transpose. Similarly $\boldsymbol{H}$ and $\boldsymbol{B}$ are vectors in R^n with $\boldsymbol{H}$ at least once continuously differentiable in a suitable open set Ω of R^n. When $\boldsymbol{B} \equiv \boldsymbol{0}$ the system given by equation (1.1) is normally referred to as a system of conservation laws, and an excellent account of the properties of solutions and their applications to physical problems together with an historical account of investigations of such systems can be found in Smoller [24].

In matrix form system (1.1) may be written as

$$\frac{\partial \boldsymbol{u}(X,t)}{\partial t} + A(\boldsymbol{u}(X,t))\frac{\partial \boldsymbol{u}}{\partial X} + \boldsymbol{B}(\boldsymbol{u}(X,t),X) = \boldsymbol{0}, \tag{1-2}$$

where A is an $(n \times n)$ matrix with components

$$A_{ij} = \frac{\partial H_i}{\partial u_j}, \qquad i, j = 1, 2, ...n.$$

Following Whitham [31] we refer to the system (1.2) as hyperbolic if the eigenvalues of A are real and there is a full set of linearly independent left and right eigenvectors. If the eigenvalues are all real and distinct we say that the system is strictly or totally hyperbolic.

It has been recognised since the earliest attempts by Riemann [24] to solve problems in gas dynamics that, even if the initial and boundary conditions associated with a problem governed by a system of the form of equation (1.1) are smooth, the solution may be discontinuous, and this has led to the introduction of weak solutions. Much of the terminology developed for dealing with these gas dynamics problems, such as simple waves, shock solutions, Rankine-Hugoniot conditions and entropy conditions has carried over to general discussions and to the consideration of waves in solids. An account of progress in fluid and gas dynamics up to 1949 is contained in the classical book of Courant and Friedrichs [6]. A detailed account of further progress up to 1983 can be found in the book of Smoller [24], and a different approach may be found in the text of Whitham [31]. Concurrently with analytical attacks on problems governed by equations (1.1), (1.2) there has been substantial progress on numerical methods for dealing with such systems. A good general reference is provided by Sod [25].

We are concerned here with wave propagation in rubberlike solids. A good source of information on finite deformations is the book of Ogden [19]. Jeffrey [15] also treats quasilinear hyperbolic partial differential equations in a form suitable for application to solids. Additional references may be found in the books by Ciarlet [3] and by Engelbrecht [9].

Consider first of all the homogeneous form of equations (1.1), (1.2) when $B = 0$. The general theory of such systems is discussed in the paper by Lax [18], where, in attempting to distinguish physically relevant solutions of genuinely non-linear solutions, entropy inequalities were introduced, and will be discussed in subsequent chapters. An important problem for such a system is the Riemann problem where we require a solution of the system

$$\frac{\partial \boldsymbol{u}(X,t)}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u}(X,t))}{\partial X} = \boldsymbol{0}, \quad -\infty < X < \infty, \quad t > 0, \tag{1.3}$$

with

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_L, & X < 0, \\ \boldsymbol{u}_R, & X > 0, \end{cases} \tag{1-4}$$

where $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ are constant states. In general the solution to such a problem may be found as a combination of constant states, simple waves, and shock solutions. The nonlinearity then arises when such solutions are combined to form a complete solution. An extension of the Riemann problem arises in considering the impact loading or unloading of a nonlinear elastic string. The loading problem has been considered by [29] where a general form of the strain energy function and stress-stretch relation for the string is introduced. The unloading problem for a plucked hyperelastic string using the general form of strain energy function discussed by Ogden [19] has been considered by Wegner, Haddow and Tait [30]. In these cases it is assumed that equation (1.3) holds on $(X,t) \in (0,\infty) \times (0,\infty)$ and boundary initial condition of the type

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_R, \quad X > 0, \quad t = 0, \\ \boldsymbol{u}^* &= \boldsymbol{u}_T, \quad X = 0, \quad t > 0, \end{aligned} \tag{1-5}$$

are introduced where $\boldsymbol{u}_R$ denotes a constant state and $\boldsymbol{u}^*$ is an appropiate constant subset of $u(0,t)$. The constant initial/boundary conditions again allow the solution to be constructed as a combination of constant states, simple waves, shocks, and contact discontinuities. An interesting feature here is the effect of the choice of strain-energy function for the elastic material. This determines the stress-stretch relation and the eigenvalues of the matrix A corresponding to H in equation (1-3). For the elastic string problem there are, in general, four eigenvalues $\pm\Lambda_1$, $\pm\Lambda_2$, $\Lambda_1 > 0$, $\Lambda_2 > 0$. The eigenvalues determine wave speeds and as the stretch of the string changes the relative magnitudes of Λ_1 and Λ_2 also change so that at certain stretches the wave speeds may coincide (see Fig.1), and in fact change their order. The basic assumption made in obtaining the solutions described above is that the velocity at $X = 0$ remains constant.

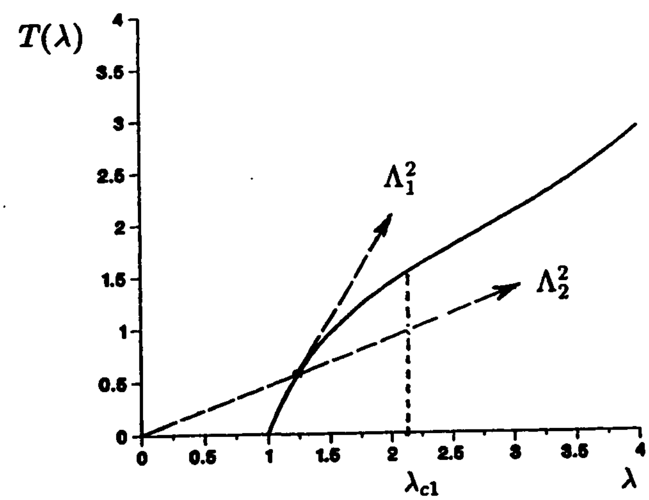


Fig.1

In attempting to explain experimental results, Dohrenwend, Drucker, Moore and Paul [8] included the elastic restoring force acting at $X = 0$, $t > 0$. As a result the boundary condition is no longer constant and the techniques outlined above no longer apply. We are then led to consider numerical methods. If no shocks occur characteristic theory can be used and otherwise it is necessary to consider a method suitable for dealing with discontinuous solutions. Difficulties with such methods are well known and we have chosen a Godunov type scheme based on an approximate Riemann solver suggested by Harten, Lax and Van Leer [14]. The method has been extended in order to produce improved shock resolution. If we consider the problem of a point mass M impacting an elastic string normally and symmetrically when the restoring force is taken into account, then if the effects of the mass and it's impacting velocity are not large a perturbation solution may be appropriate. We have investigated such solutions in the present thesis using an extension of a method suggested by Davison [7]. This method is not appropriate for all ranges of values of the eigenvalues Λ_1, Λ_2 and the method has been supplemented in certain cases by a technique first used by Collins [4,5] in dealing with elastic half space problems. The results obtained when the perturbation method applies may then be compared with the numerical results.

Next consider an example giving rise to the nonhomogeneous form of equations (1-1). If a circular nonlinear elastic membrane is impacted by a flat ended cylindrical object this form of the system (1-1) governs the resulting motion. For constant initial/boundary conditions the problem has been considered by Haddow, Wegner and Jiang [13]. Of course similarity solutions are no longer available. Paralleling the case for a string we consider the case when the restoring force is taken into account. Experimental results have been reported by Farrar [10] for this case and a linear analysis presented. Here we consider the nonlinear system and compare our results with the linear analysis and compare and contrast our results with those found in [10].

This thesis consists of eight chapters. In Chapter 2, we introduce some basic concepts about conservation laws such as the definition of a genuinely nonlinear characteristic field and the definition of a linearly degenerate characteristic field, the concepts of a weak solution of the initial value problem for a conservation law, shock waves and jump conditions, entropy conditions, and so on. We also consider simple wave solutions of a homogenous conservation law. The Riemann problem for a homogenous conservation law is also considered in this chapter.

In Chapter 3, the equations of motion and the compatibility conditions for the normal impact problem of a nonlinear elastic string are considered in the first section. These equations can be written as a system of four conservation laws. The initial conditions and the boundary conditions are set. In section 3.2, we consider the various strain energy functions such as the Ogden three parameter one [19], the Mooney-Rivlin, and the neo-Hookean . The corresponding stretch-stress relations are obtained for the incompressible hyperelastic string. In section 3.3, elementary waves are considered and, in section 3.4, we consider the solutions of the normal impact problem of a nonlinear elastic string with constant boundary values for the loading problem.

In the first section of Chapter 4, we consider perturbation solutions for the impact problem of the string with given small impact velocity under the assumption that $\Lambda_1 \neq \Lambda_2$. In section 4.2 we compare the perturbation solution with the solution for the constant boundary-value condition. In section 4.3, we apply the perturbation solution obtained in section 4.1 to the impact problem of a nonlinear elastic string.

In Chapter 5, we modify a method suggested by Collins [4,5] in dealing with a half space problem and in section 5.1, we obtain the approximating equations. In section 5.2, we consider the solution of the approximate system under the constant boundary value condition and compare these solutions with the corresponding solution of the original system. In section 5.3, we consider the solution of the approximate equations for the case of variable boundary value condition.

In the first section of Chapter 6, we will explain why it is essential that a finite difference scheme is in conservation form. We state the definitions of entropy function and entropy flux and state the entropy condition for a three-point finite difference scheme. In section 6.2, we discuss a scheme proposed by Godunov [25]. In section 6.3, we discuss Godunov type schemes based on approximate Riemann solvers and state a theorem proved by Harten, Lax and Van Leer [14] and describe their simplest Riemann solver containing only one intermediate state. In section 6.4, we extended the above simplest Riemann solver to a two-step Riemann solver. We have shown that the numerical solution obtained by the two-step Riemann solver satisfies the integral form of conservation law and the integral form of entropy condition. In section 6.5, we apply the Riemann solver suggested by Harten, Lax and Van Leer [14] and the two-step Rieman solver to the normal impact problem of a nonlinear elastic string. The numerical results obtained by the two-step Riemann solver have better shock resolution than those obtained by Harten's Riemann solver. We also compare the numerical solutions with the perturbation solutions when $\Lambda_1 \neq \Lambda_2$ and with the approximate solutions when $\Lambda_1 \approx \Lambda_2$.

In Chapter 7, we approximate the three-dimensional elastic sheet by a two-dimensional elastic membrane. Based on the general theory of Cosserat surfaces given by Green, Naghdi and Wainwright [11], we set up a membrane theory directly omitting thermomechanical effects. The nonlinear versions of the equations of motion are then obtained. These equations and the equations of compatibility can be combined into a system of five conservation laws. We also discuss the connection between the nonlinear equations and the linear equations used by Farrar [10].

In the last chapter, Chapter 8, we apply a characteristic method scheme suggested by Haddow, Wagner and Jiang [13] to the nonlinear version of the impact problem. A similar characteristic scheme is applied to the linearized version and the results are compared with those obtained by an extended Riemann solver. Interestingly, if we choose the parameters suitably for the nonlinear version, double-valued results can be obtained. These results are compared with those obtained by Roxburgh, Steigmann and Tait [20] for a corresponding static problem.

Chapter 2

Conservation Laws

2.1 Some Basic Facts about Conservation Laws

Let Ω be a fixed region in R^3 space, occupied by some physical substance with density $u(x,t)$, where x denotes the spatial coordinates and t denotes time. If $f(u)$ denotes the *flux* of the substance crossing unit area of $\partial\Omega$, $b(u,x,t)$ denotes a source term per unit volume of this substance, then one has

$$\frac{d}{dt}\int_{\Omega} u dv = -\int_{\partial\Omega} f \cdot n dS - \int_{\Omega} b dv, \qquad (2.1\text{-}1)$$

where dv is the element of volume of the region Ω, $\partial\Omega$ is the boundary of Ω, n is the outward normal vector to $\partial\Omega$, and dS is the element area of $\partial\Omega$. Assuming that all partial derivatives of u and f are continuous in the region Ω, then by applying the divergence theorem, we have

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} + \mathrm{Div} f + b\right) dv = 0, \qquad (2.1\text{-}2)$$

where Div denotes the divergence operator. Since Ω is arbitrary, we then have

$$\frac{\partial u}{\partial t} + \mathrm{Div} f + b = 0. \qquad (2.1\text{-}3)$$

If equation (2.1-3) is generalized to a system of equations we have a system of conservation laws

$$\frac{\partial u_i}{\partial t} + \mathrm{Div} f_i + b_i = 0, \qquad i = 1, 2, \cdots, n, \qquad (2.1\text{-}4)$$

where f_i is some nonlinear function of the u_j, $(j = 1, 2, \cdots, n)$.

If a single spatial variable x is considered, then (2.1-4) can be rewritten as

$$\boldsymbol{u}_{,t} + \boldsymbol{H}(\boldsymbol{u})_{,x} + \boldsymbol{B}(\boldsymbol{u}, x) = \boldsymbol{0}, \qquad (t > 0), \tag{2.1-5}$$

where

$$\begin{aligned} (\quad)_{,t} \equiv \frac{\partial(\quad)}{\partial t}, \qquad (\quad)_{,x} &\equiv \frac{\partial(\quad)}{\partial x}, \\ \boldsymbol{u} &= (u_1, u_2, \cdots, u_n)^T, \\ \boldsymbol{H}(\boldsymbol{u}) &= (H_1(\boldsymbol{u}), \cdots, H_n(\boldsymbol{u}))^T, \\ \boldsymbol{B}(\boldsymbol{u}, x) &= (B_1(\boldsymbol{u}, x), \cdots, B_n(\boldsymbol{u}, x))^T. \end{aligned}$$

The superscript T denotes the transpose. If $\boldsymbol{B}$ is not zero, then (2.1-5) is referred to as an *inhomogeneous system of conservation laws* or *conservation laws with source term.* If $\boldsymbol{B}$ is identically zero, then (2.1-5) is referred to as a *homogeneous system of conservation laws* or simply a *conservation law.*

We can rewrite (2.1-5) in the following matrix form

$$\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,x} + \boldsymbol{B}(\boldsymbol{u}, x) = \boldsymbol{0}, \tag{2.1-6}$$

where $A(\boldsymbol{u}) = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{u}}$ is the *Jacobian matrix* for $\boldsymbol{H}(\boldsymbol{u})$. The components of A are given by

$$A_{ij} = \frac{\partial H_i}{\partial u_j}, \qquad i, j = 1, 2, \cdots, n. \tag{2.1-7}$$

We assume for the moment that A has n distinct real eigenvalues

$$\Lambda_1(\boldsymbol{u}) < \Lambda_2(\boldsymbol{u}) < \cdots < \Lambda_n(\boldsymbol{u}), \tag{2.1-8}$$

with the corresponding linearly independent left eigenvectors $\boldsymbol{L}_i(\boldsymbol{u})$ and right eigenvectors $\boldsymbol{R}_i(\boldsymbol{u})$ which are defined by

$$\boldsymbol{L}_i(\boldsymbol{u})A(\boldsymbol{u}) = \Lambda_i(\boldsymbol{u})\boldsymbol{L}_i(\boldsymbol{u}), \qquad A(\boldsymbol{u})\boldsymbol{R}_i(\boldsymbol{u}) = \Lambda_i(\boldsymbol{u})\boldsymbol{R}_i(\boldsymbol{u}), \qquad (i = 1, 2. \cdots, n). \tag{2.1-9}$$

Following Lax [18] and Smoller[24], we define the i-characteristic field to be *genuinely nonlinear* if

$$(\mathrm{grad}\, \Lambda_i(\boldsymbol{u})) \cdot \boldsymbol{R}_i(\boldsymbol{u}) \neq 0, \qquad \text{for all} \quad \boldsymbol{u}, \tag{2.1-10}$$

and to be *linearly degenerate* if

$$(\mathrm{grad}\, \Lambda_i(\boldsymbol{u})) \cdot \boldsymbol{R}_i(\boldsymbol{u}) \equiv 0 \qquad \text{for all} \quad \boldsymbol{u}. \tag{2.1-11}$$

It is well known that the initial value problem for equation (2.1-5) may not have a differentiable solution for all time even for the smooth initial condtions $\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x)$. For example, if we consider the initial value problem for the following scalar homogeneous conservation law

$$\begin{aligned} u_{,t} + (\frac{u^n}{n})_{,x} = 0, \qquad (n > 1) \\ u(x, 0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} \end{aligned} \tag{2.1-12}$$

the geometric solution obtained by using characteristic methods is double -valued for $t > 0$ (see Fig.2.1-1). There is no regular differentiable solution for the initial value problem described by equations (2.1-12) for all $t > 0$. This example confirms that the initial value problem for the system (2.1-5) in general does not possess a differentiable solution for all time. It is necessary therefore to consider generalized solutions so that nondifferentiable functions or bounded measurable functions may be included as solutions.

Consider the initial value problem of (2.1-5) under the condition

$$u(x,0) = u_0(x), \tag{2.1-13}$$

and assume that $\phi(x,t)$ is any C^1 test function with compact support in $t \geq 0$. If $u(x,t)$ is a bounded measurable function and satisfies the following equation

$$\int\int_{t\geq 0} (u\frac{\partial\phi}{\partial t} + H(u)\frac{\partial\phi}{\partial x} - B(u,x)\phi)dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)dx = 0 \tag{2.1-14}$$

for any such test function $\phi(x,t)$, then $u(x,t)$ is said to be a *weak solution* of the initial value problem (2.1-5), (2.1-13).

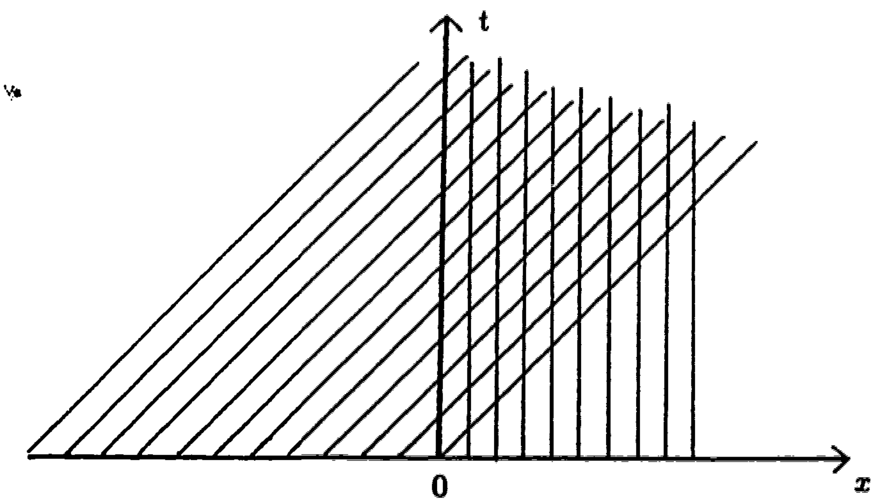


Fig.2.1-1

We consider what conditions the discontinuous solution should satisfy. Let C be a smooth curve across which u has a discontinuity and assume that u is smooth away from C and has well defined limits as (x,t) approaches points on C from either side. If the curve C is described by $x = x(t)$, we define $u_- = u(x(t)-0,t)$ and $u_+ = u(x(t)+0,t)$. Let P be any point on C,

R be a small circle centered at P and R_1, R_2 be the two parts of R separated by C. Assume also the test function ϕ is zero on ∂R and outside of R. Notice that u is smooth in R_1 and R_2. Using equation (2.1-5) we then have

$$\int\int_{R_i} (u\phi_{,t} + H\phi_{,x} - B\phi)dxdt = \int\int_{R_i} ((u\phi)_{,t} + (H\phi)_{,x})dxdt$$

and by applying Green's theorem, we have

$$\int\int_{R_i} ((u\phi)_{,t} + (H\phi)_{,x})dxdt = \int_{\partial R_i} \phi(-udx + Hdt).$$

Since $\phi = 0$ on ∂R , we have

$$\int_{R_1} \phi(-udx + Hdt) = \int_{P_1}^{P_2} \phi(-u_-dx + H(u_-)dt),$$
$$\int_{R_2} \phi(-udx + Hdt) = -\int_{P_1}^{P_2} \phi(-u_+dx + H(u_+)dt),$$

where P_1, P_2 are the intersection points of ∂R and C. Substituting the above results into equation (2.1-14), we get

$$\int_C \phi(-[u]dx + [H(u)]dt) = 0,$$

where $[u] = u_- - u_+$, $[H(u)] = H(u_-) - H(u_+)$. Since ϕ is arbitrary we have

$$V[u] = [H(u)] \tag{2.1-15}$$

where $V = \frac{dx}{dt}$ is the speed of the discontinuity. The condition given by equation (2.1-15) is called the ***jump condition*** or the ***Rankine-Hugoniot condition.***

We note that in general different conservation laws correspond to different jump conditions. However, a partial differential equation may correspond to more than one conservation law. For example, the equation $u_{,t} + u^{n-1}u_{,x} = 0$ gives rise to infinitely many conservation laws

$$\left(\frac{u^{k+1}}{k+1}\right)_{,t} + \left(\frac{u^{n+k}}{n+k}\right)_{,x} = 0 \qquad k = 0,1,2,3\cdots . \tag{2.1-16}$$

Therefore, if we admit discontinuous solutions, we cannot start from a differential equation which is not in conservation form. Instead, we must start from a physical conservation law. We also note that the weak solution satisfying equation (2.1-14) may not be unique. For example, the following initial-value problem

$$\begin{aligned} u_{,t} + \left(\frac{u^n}{n}\right)_{,x} = 0, \qquad (n > 1, \quad t > 0, \quad -\infty < x < \infty) \\ u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0 \end{cases} \end{aligned} \tag{2.1-17}$$

has a single-valued solution in the regions $x/t < 0$ and $x/t > 1$ but the values of u cannot be determined in the region $0 < x/t < 1$ by characteristic methods (see Fig.2.1-2).

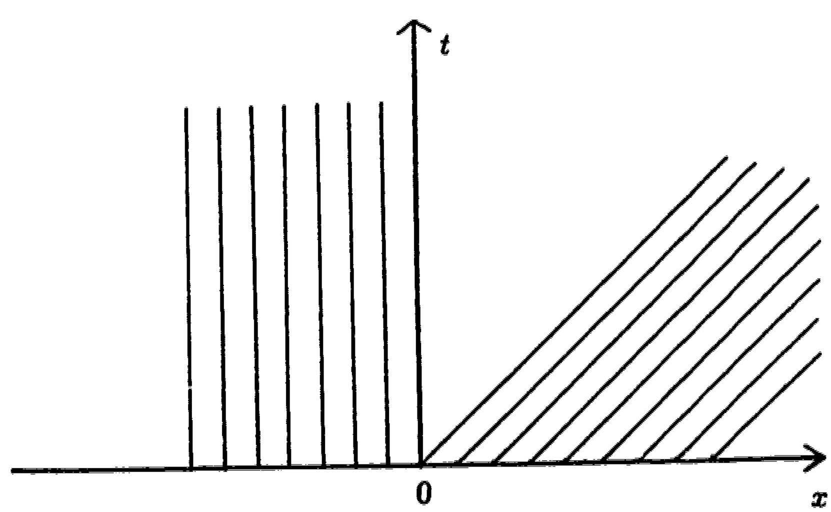


Fig.2.1-2

We can construct a solution in the region $0 < x/t < 1$ by introducing a discontinuous function which satisfies the jump condition across the line of discontinuity. We obtain

$$u_1(x,t) = \begin{cases} 0, & x < t/n, \\ 1, & x > t/n. \end{cases} \tag{2.1-18}$$

On the other hand, we can find a continuous solution

$$u_2(x,t) = \begin{cases} 0, & x < 0, \\ (\frac{x}{t})^{\frac{1}{n-1}}, & 0 < x < t, \\ 1, & x > t. \end{cases} \tag{2.1-19}$$

The problem now is how to choose an admissible weak solution. For this purpose, Lax proposed a criterion for selecting admissible discontinuous solutions. If the i-th characteristic field of system (2.1-5) is genuinely nonlinear, and the discontinuity under consideration has speed V, then the discontinuity is admissible in the sense of Lax if

$$\Lambda_i(u_+) < V < \Lambda_i(u_-), \quad \Lambda_{i-1}(u_-) < V < \Lambda_{i+1}(u_+). \tag{2.1-20}$$

for some $i = 1, 2, \cdots, n$. Such a discontinuity is called an *i-shock wave*. The conditions given by inequalities (2.1-20) are referred to as *entropy conditions* or *Lax shock conditions*. For the case of an homogeneous scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.1-21}$$

the entropy inequality reduces to

$$a(u_-) > V > a(u_+), \tag{2.1-22}$$

where $a(u) = f'(u)$. Clearly the solution described by equation (2.1-18) violates this condition and hence it is not an admissible solution of equation (2.1-17). Solution u_2 is an example of a centered simple wave; we will consider such solutions in the next section.

2.2 Simple Waves

In this section, we will consider a system of homogeneous conservation laws

$$\boldsymbol{u}_{,t} + \boldsymbol{H}(\boldsymbol{u})_{,x} = \boldsymbol{0}, \qquad (-\infty < x < \infty, t > 0). \tag{2.2-1}$$

System (2.2-1) can be rewritten as a quasilinear hyperbolic system

$$\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,x} = \boldsymbol{0}, \tag{2.2-2}$$

where it is assumed that A has n real and distinct eigenvalues $\Lambda_1(\boldsymbol{u}) < \cdots < \Lambda_n(\boldsymbol{u})$ and that the corresponding left and right eigenvectors are linearly independent.

If the system (2.2-2) has a C^1 solution $\boldsymbol{u}$ depending on a single component of $\boldsymbol{u}$, say u_*, then

$$(u_{*,t}I + u_{*,x}A)\frac{d\boldsymbol{u}}{du_*} = \boldsymbol{0}. \tag{2.2-3}$$

If $\boldsymbol{u}$ is not identically zero, then one has

$$\frac{d\boldsymbol{u}}{du_*} = k\boldsymbol{R}, \qquad u_{*,t} + \tau u_{*,x} = 0, \tag{2.2-4}$$

where k is some constant and τ is an eigenvalue of A and $\boldsymbol{R}$ is the corresponding right eigenvector. By the second equation in (2.2-4), one has

$$\boldsymbol{u} = \boldsymbol{u}(u_*) = constant, \qquad \text{on} \quad \frac{dx}{dt} = \tau. \tag{2.2-5}$$

As u_* varies, the lines $\frac{dx}{dt} = \tau(u_*)$ sweep out a simple wave region. This idea is explained in a more precise way below.

Suppose $\boldsymbol{u} \in N \subset R^n$ on which $\boldsymbol{H}(\boldsymbol{u})$ is smooth. If $w : N \rightarrow R$ is a smooth function such that, for $\boldsymbol{u} \in N$,

$$\boldsymbol{R}_k \cdot \operatorname{grad} w(\boldsymbol{u}) = 0, \tag{2.2-6}$$

where $\boldsymbol{R}_k$ is a right eigenvector of A corresponding to the eigenvalue Λ_k, then the function w is called a *k-Riemann invariant* of the system (2.2-1). Since $\boldsymbol{R}_k$ defines a particular direction in R^n space, there are (n-1) k-Riemann invariants whose gradients are linearly independent in N (see Smoller [24]). Suppose $\boldsymbol{u}$ is C^1 in some domain D. If all k-Riemann invariants are constant in D then $\boldsymbol{u}$ is called a *k-simple wave* or a *k-rarefaction wave.*

Suppose now that D is a k-simple wave region with $w_i(\boldsymbol{u}) = c_i, i = 1, 2, \cdots, n-1,$ where c_i are constants and where $w_i(\boldsymbol{u})$ are the (n-1) k-Riemann invariants whose gradients are linearly independent. Then if $a_1, \cdots, a_{n-1}$ are (n-1) constants, we have

$$a_1 \operatorname{grad} w_1 + a_2 \operatorname{grad} w_2 + \cdots + a_{n-1} \operatorname{grad} w_{n-1} = \mathbf{0} \tag{2.2-7}$$

if and if only $a_1 = a_2 = \cdots = a_{n-1} = 0$. If we define a matrix M by

$$M = [(\operatorname{grad} w_1)^T, (\operatorname{grad} w_2)^T, \cdots, (\operatorname{grad} w_{n-1})^T], \tag{2.2-8}$$

where T denotes the transpose, then equation (2.2-7) can be written in a matrix form

$$M \cdot (a_1, a_2, \cdots, a_{n-1})^T = \mathbf{0}. \tag{2.2-9}$$

Since equation (2.2-9) requires $a_i = 0, i = 1, 2, \cdots, n-1,$ this implies that not all (n-1) by (n-1) submatrices in M are singular. Therefore, the (n-1) equations

$w_i(\boldsymbol{u}) = c_i$ with $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^T$ are equivalent to a system of the form

$$u_j = u_j(u_i), \qquad j = 1, 2, \cdots, n, \quad j \neq i, \quad \text{for some fixed} \quad i. \tag{2.2-10}$$

Thus a simple wave depends on a single component of $\boldsymbol{u}$.

Suppose $\boldsymbol{u} \in N$, then the intersection curve of the surfaces $w_i(\boldsymbol{u}) = c_i = constant$, $i = 1, 2, \cdots, n-1$, is the integral curve of $\boldsymbol{R}_k$ passing through $\boldsymbol{u}$. To see this, we let $\boldsymbol{v} = \boldsymbol{v}(s)$ denote such an integral curve. Then

$$\frac{dw(\boldsymbol{v}(s))}{ds} = \operatorname{grad} w \cdot \frac{d\boldsymbol{v}}{ds} = \operatorname{grad} w \cdot \boldsymbol{R}_k = 0, \tag{2.2-11}$$

where w is a k-Riemann invariant. On the other hand, if $\boldsymbol{v}(s)$ is a local curve along which all k-Riemann invariants are constant, then

$$\frac{dw(\boldsymbol{v}(s))}{ds} = \operatorname{grad} w \cdot \frac{d\boldsymbol{v}}{ds} = 0, \tag{2.2-12}$$

for every k-Riemann invariant w so that $\frac{d\boldsymbol{v}}{ds}$ is orthogonal to the (n-1) dimensional space spanned by $\operatorname{grad} w_i$, $i = 1, 2, \cdots, n-1$, and so lies in the span of $\boldsymbol{R}_k$, hence

$$\frac{d\boldsymbol{v}}{ds} = \alpha(s)\boldsymbol{R}_k(\boldsymbol{v}(s)). \tag{2.2-13}$$

Next, we will see that if $\boldsymbol{u}$ is a k-simple wave in some region D, then the k-characteristics are straight lines in D. First we note that

$$\boldsymbol{L}_k \cdot \frac{d\boldsymbol{u}}{dt} = \boldsymbol{L}_k \cdot \left(\frac{\partial \boldsymbol{u}}{\partial t} + \lambda_k \frac{\partial \boldsymbol{u}}{\partial x}\right) = 0$$

along $\frac{dx}{dt} = \lambda_k$. Secondly, we notice that $w_i, i = 1, 2, \cdots, n-1$, are all constant in D , then

$$\frac{dw_i}{dt} = \operatorname{grad} w_i \cdot \frac{d\boldsymbol{u}}{dt} = 0, \quad \text{on} \quad \frac{dx}{dt} = \lambda_k.$$

Thus

$$\begin{bmatrix} \boldsymbol{L}_k \\ \operatorname{grad} w_1 \\ \cdot \\ \cdot \\ \cdot \\ \operatorname{grad} w_{n-1} \end{bmatrix} \frac{d\boldsymbol{u}}{dt} = 0, \quad \text{on} \quad \frac{dx}{dt} = \lambda_k. \tag{2.2-14}$$

Since $\boldsymbol{L}_k \cdot \boldsymbol{R}_k \neq 0$, $\boldsymbol{L}_k$ has a component in the direction of $\boldsymbol{R}_k$ so that the matrix in (2.2-14) is nonsingular. Then one has

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \lambda_k \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0}, \quad \text{on} \quad \frac{dx}{dt} = \lambda_k, \tag{2.2-15}$$

and so $\boldsymbol{u}$ is constant along a k-characteristics and so the characteristics are straight lines, (See Fig.2.2-1).

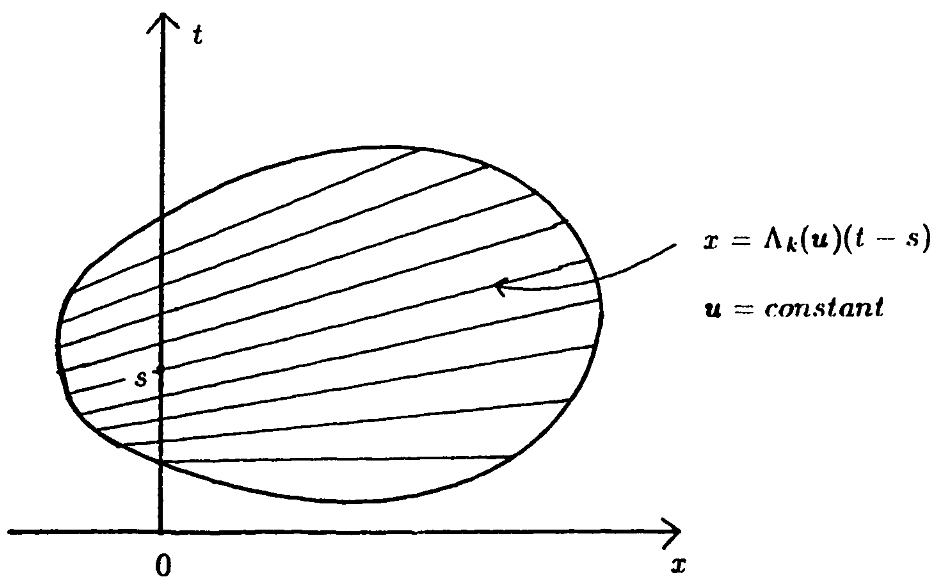


Fig.2.2-1

A particular class of simple waves depends only on the ratio $(x - x_0)/(t - t_0)$.

They are called ***centered simple waves*** or ***centered rarefaction waves***, where (x_0, t_0) is called the center of the wave.

If a simple wave solution is centered at the origin, then the solution is of the form

$$\boldsymbol{u}(x,t) = \boldsymbol{u}(x/t), \tag{2.2-16}$$

and a solution of this form is called a ***si· ıilarity solution.***

Now we will describe the states $\boldsymbol{u}$ which can be connected to a given constant state $\boldsymbol{u}_l$ on the right by a k-simple wave. Assume that the k^{th} characteristic field is genuinely nonlinear in N, then one can normalize $\boldsymbol{R}_k$ by $\operatorname{grad}\Lambda_k \cdot \boldsymbol{R}_k = 1$. Let $\boldsymbol{u}_l \in N$, then the following problem

$$\frac{d\boldsymbol{v}}{d\beta} = \boldsymbol{R}_k(\boldsymbol{v}(\beta)), \quad \boldsymbol{v}(\Lambda_k(\boldsymbol{u}_l)) = \boldsymbol{u}_l, \quad \beta > \Lambda_k(\boldsymbol{u}_l), \tag{2.2-17}$$

has a solution $\boldsymbol{v}(\beta)$ on the interval $\Lambda_k(\boldsymbol{u}_l) \le \beta \le \Lambda_k(\boldsymbol{u}_l) + a$, for sufficiently small a. We have

$$\frac{d}{d\beta}\Lambda_k(\boldsymbol{v}(\beta)) = \frac{d\boldsymbol{v}}{d\beta} \cdot \operatorname{grad}\Lambda_k = \boldsymbol{R}_k \cdot \operatorname{grad}\Lambda_k = 1.$$

Hence

$$\Lambda_k(\boldsymbol{v}(\beta)) = \beta. \tag{2.2-18}$$

Now we define $\boldsymbol{u}(x,t)$ by

$$\boldsymbol{u}(x,t) = \boldsymbol{v}(x/t), \qquad \Lambda_k(\boldsymbol{u}_l) \le x/t \le \Lambda_k(\boldsymbol{u}_l) + a.$$

If w is a k-Riemann invariant, and $\beta = x/t$, then

$$\frac{dw}{d\beta} = \operatorname{grad} w \cdot \frac{d\boldsymbol{u}}{d\beta} = \operatorname{grad} w \cdot \boldsymbol{R}_k = 0,$$

so w is constant for $\Lambda_k(\boldsymbol{u}_l) < x/t < \Lambda_k(\boldsymbol{u}_l) + a$. Thus $\boldsymbol{u}$ defines a simple wave for this region. Also, one has

$$\Lambda_k(\boldsymbol{v}(\beta)) = \beta > \Lambda_k(\boldsymbol{u}_l).$$

Thus Λ_k increases along the simple wave. Since $\boldsymbol{u}$ is smooth, then one has

$$\begin{aligned}
\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,x} &= -\frac{x}{t^2}\frac{d\boldsymbol{v}}{d\beta} + A(\boldsymbol{u})\frac{1}{t}\frac{d\boldsymbol{v}}{d\beta} \\
&= \frac{1}{t}(-\frac{x}{t}I + A)\boldsymbol{R}_k = 0,
\end{aligned}$$

since $x/t = \Lambda_k$. We have then constructed a one-parameter family $\boldsymbol{u}(\beta)$ which can be connected to $\boldsymbol{u}_l$ by a centered k-simple wave. If we introduce a new parameter ε by

$$\Lambda_k(\boldsymbol{u}) = \Lambda_k(\boldsymbol{u}_l) + \varepsilon, \tag{2.2-19}$$

and note that

$$w_i(\boldsymbol{u}) = w_i(\boldsymbol{u}_l), \quad i = 1, 2, \cdots, n-1, \tag{2.2-20}$$

then since $\operatorname{grad} w_i, i = 1, 2, \cdots, n-1$, and $\operatorname{grad} \Lambda_k$ are linearly independent, it follows by the implicit function theorem that, equations (2.2-19) and (2.2-20) have a unique solution for ε sufficiently small. By equation (2.2-19), one has $\frac{d\Lambda_k}{d\varepsilon} = 1$. This implies that $\varepsilon > 0$ so that $\Lambda_k(\boldsymbol{u}_l) < \Lambda_k(\boldsymbol{u}(\varepsilon))$. With some calculation, one can show that (see Smoller [24])

$$\frac{d\boldsymbol{u}}{d\varepsilon} = \boldsymbol{R}_k, \qquad \frac{d^2\boldsymbol{u}}{d\varepsilon^2} = \frac{d\boldsymbol{R}_k}{d\varepsilon}. \tag{2.2-21}$$

Next, we will introduce the notion of elementary waves. The elementary waves for the homogeneous conservation laws include simple waves and shock waves for the genuinely nonlinear fields, and contact discontinuities for the linearly degenerate field.

If $\boldsymbol{R}_i$ is the i-th right eigenvector of A in equation (2.2-1), we define the *i-simple wave curve,* $(i = 1, 2, \cdots, n)$ to be the integral curve of the vector field $\boldsymbol{R}_i(\boldsymbol{u})$ through the state $\boldsymbol{u}_0$.

If the i-th field is genuinely nonlinear, then a *centered i-simple wave* is a smooth solution which connects the left constant state $\boldsymbol{u}_0$ to the right constant state $\boldsymbol{u}_1$

$$\boldsymbol{u}(x,t) = \begin{cases} \boldsymbol{u}_0, & x/t \leq \Lambda_i(\boldsymbol{u}_0), \\ \boldsymbol{v}(x/t), & \Lambda_i(\boldsymbol{u}_0) \leq x/t \leq \Lambda_i(\boldsymbol{u}_1), \\ \boldsymbol{u}_1, & x/t \geq \Lambda_i(\boldsymbol{u}_1) \end{cases}$$
$$\Lambda_i(\boldsymbol{v}(x/t)) = x/t, \tag{2.2-22}$$

where $\boldsymbol{v}(x/t)$ can be connected to $\boldsymbol{u}_0$ by an integral curve of $\boldsymbol{R}_i$, with $\Lambda_i(\boldsymbol{v}) \geq \Lambda_i(\boldsymbol{u}_0)$.

If the i-th field is genuinely nonlinear, then a *centered i-shock wave* is a discontinuous solution $\boldsymbol{u}(x,t)$ satisfying the jump condition and the entropy inequalities

$$\boldsymbol{u}(x,t) = \begin{cases} \boldsymbol{u}_0, & x/t < V, \\ \boldsymbol{u}_1, & x/t > V, \end{cases}$$
$$(\boldsymbol{u}_1 - \boldsymbol{u}_0)V = \boldsymbol{H}(\boldsymbol{u}_1) - \boldsymbol{H}(\boldsymbol{u}_0), \quad \Lambda_i(\boldsymbol{u}_1) < V < \Lambda_i(\boldsymbol{u}_0). \tag{2.2-23}$$

If the i-th field is linearly degenerate, then we have a *contact discontinuity*

$$\boldsymbol{u}(x,t) = \begin{cases} \boldsymbol{u}_0, & x/t < V, \\ \boldsymbol{u}_1, & x/t > V, \end{cases}$$
$$V = \Lambda_i(\boldsymbol{u}_0) = \Lambda_i(\boldsymbol{u}_1), \tag{2.2-24}$$

where $\boldsymbol{u}_1$ can be connected to $\boldsymbol{u}_0$ by an integral curve of $\boldsymbol{R}_i$. In the next section, the solution of the Riemann problem will be considered using elementary waves.

2.3 Riemann Problem

The *Riemann Problem* for a homogeneous conservation law of the type considered here is defined as the following initial-value problem

$$\boldsymbol{u}_{,t} + \boldsymbol{H}(\boldsymbol{u})_{,x} = \boldsymbol{0}, \qquad -\infty < x < \infty, \quad t > 0, \tag{2.3-1}$$

$$\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_l, & x < 0, \\ \boldsymbol{u}_r, & x > 0, \end{cases} \tag{2.3-2}$$

where $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$ are constant vectors.

First we consider a system of two conservation laws, where $\boldsymbol{u} = (u_1, u_2)^T$, $\boldsymbol{H} = (H_1(\boldsymbol{u}), H_2(\boldsymbol{u}))^T$. For this case, the solution of the Riemann Problem (2.3-1) with arbitrary constant states $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$ is given by Smoller with the condition of genuine nonlinearity [23]. For the case without genuine nonlinearity, the Riemann problem of a system of two conservation laws with arbitrary initial constant states was solved by T.P. Liu [17]

As a particular example, Smoller solves the Riemann problem for the following *p-system*.

$$\boldsymbol{U}_{,t} + \boldsymbol{H}(\boldsymbol{U})_{,x} = \boldsymbol{0}. \quad t > 0, -\infty < x < \infty.$$

$$\boldsymbol{U} = (v,u)^T, \qquad \boldsymbol{H}(\boldsymbol{U}) = (-u, \quad p(v))^T. \tag{2.3-3}$$

where $p' < 0$, $p'' > 0$. The initial condition is

$$\boldsymbol{U}(x,0) = \begin{cases} \boldsymbol{U}_l = (v_l, u_l)^T, & x < 0, \\ \boldsymbol{U}_r = (v_r, u_r)^T, & x > 0. \end{cases} \tag{2.3-4}$$

If A is the Jacobian matrix for the p-system, then its components are given by

$$A_{11} = 0, \quad A_{12} = -1, \quad A_{21} = p'(v), \quad A_{22} = 0. \tag{2.3-5}$$

A has real and distinct eigenvalues

$$\Lambda_1 = -(-p'(v))^{1/2} < 0 < (-p'(v))^{1/2} = \Lambda_2. \tag{2.3-6}$$

With some calculation, we find the i-shock wave curves $S_i, (i = 1, 2,)$ are given by

$$\begin{aligned} S_1 : u - u_l &= -((v - v_l)(p(v_l) - p(v)))^{1/2}, \quad v_l > v, \\ S_2 : u - u_l &= -((v - v_l)(p(v_l) - p(v)))^{1/2}, \quad v_l < v. \end{aligned} \tag{2.3-7}$$

If $U_r = (v_r, u_r)^T$ lies on one of the curves $S_i, (i = 1, 2,)$ then it can be connected to U_l by a i-shock along which the shock speed V satisfies the jump conditions

$$V(v_r - v_l) = -(u_r - u_l), \qquad V(u_r - u_l) = p(v_r) - p(v_l). \tag{2.3-8}$$

The i-rarefaction curves $R_i, (i = 1, 2,)$ are given by

$$\begin{aligned} R_1 : u - u_l &= \int_{v_l}^{v} (-p'(y))^{1/2} dy, \quad v > v_l, \\ R_2 : u - u_l &= -\int_{v_l}^{v} (-p'(y))^{1/2} dy, \quad v < v_l. \end{aligned} \tag{2.3-9}$$

If U_r lies in the curve R_1, then it can be connected to U_l by a smooth centered simple wave $U(x/t) = (v(x/t), \quad u(x/t))^T$, where $v(x/t)$ is determined by

$$x/t = -(-p'(v(x/t)))^{1/2}, \quad \Lambda_1(U_l) < x/t < \Lambda_2(U_r). \tag{2.3-10}$$

Then $u(x/t)$ can be found from the first equation of (2.3-9). The case when U_r lies on the curve R_2 is similar.

It can be shown that the curves R_1 and S_1 have the same first and second derivatives at U_l. The same is true for R_2 and S_2. These curves divide the u-v plane into four regions as shown in Fig.2.3-1. If U_r lies in region 1, then we can find an intermediate state U_m which lies on the curve R_1 and can be connected to U_r from the right through a 2-shock, (see Fig.2.3-2). The cases for which U_r lies in regions 2,3 and 4 are shown in Fig.2.3-3, Fig.2.3-4 and Fig.2.3-5, respectively.

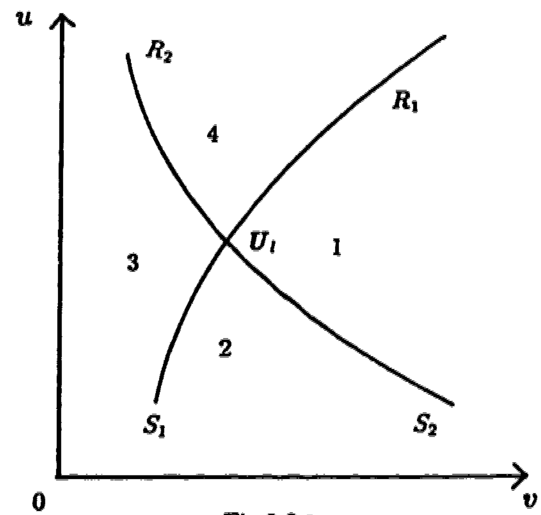


Fig.2.3-1

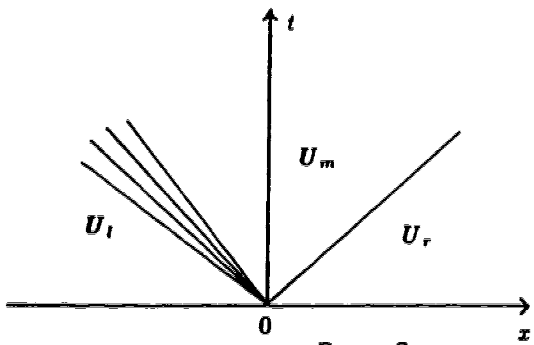
It is interesting to note that if U_r lies in region 4 and

$$u_0 \equiv \int_{v_l}^{\infty} (-p'(y))^{1/2} dy < \infty, \tag{2.3-11}$$

and if $u_r > u_l + 2u_0$, then the Riemann problem for the p-system has no solution. For the case of isentropic gas dynamics where

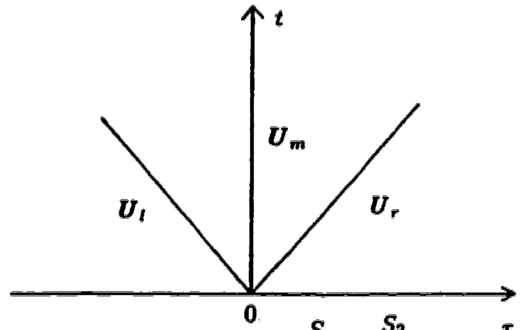
$$p(v) = \frac{k}{v^\gamma}, \quad \gamma > 1,$$

with k a positive constant, the integral in equation (2.3-11) is convergent. This



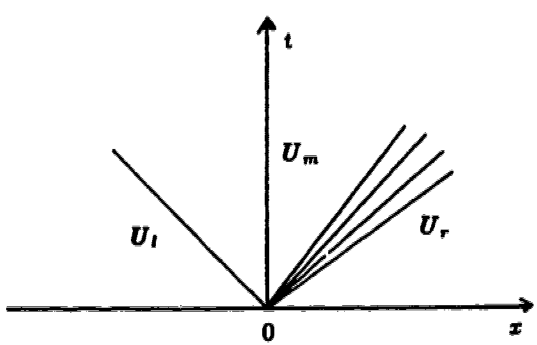
U_r in region 1, $U_l \xrightarrow{R_1} U_m \xrightarrow{S_2} U_r$

Fig.2.3- 2



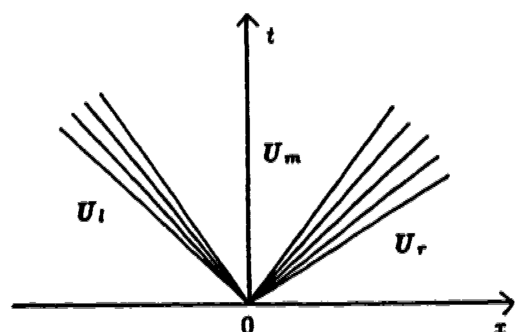
U_r in region 2, $U_l \xrightarrow{S_1} U_m \xrightarrow{S_2} U_r$

Fig.2.3-3



U_r in region 3, $U_l \xrightarrow{S_1} U_m \xrightarrow{R_2} U_r$

Fig.2.3-4



U_r in region 4, $U_l \xrightarrow{R_1} U_m \xrightarrow{R_2} U_r$

Fig.2.3-5

corresponds to the appearence of a vacuum (see Smoller [24]).

We will see in the next chapter that if there is no vertical motion, and if the Mooney-Rivlin stress-stretch relation is considered, then the governing equation for a nonlinear elastic string is reduced to the p-system. However, the integral in (2.3-11) is not convergent for this case, hence a vacuum will not appear. Therefore, the Riemann problem for this nonlinear elastic string can be solved for any given initial data.

For the system of n conservation laws, Lax proved the following fundamental result [18]: When u_l is close to u_r, every state u_l has a neighborhood such that, if u_r belongs to this neighborhood, the Riemann Problem (2.3-1) has a solution . This solution consists of $n+1$ constant states connected by centered shock waves, centered rarefaction waves, and centered contact discontinuities. There is exactly one solution of this kind, provided the intermediate states are restricted to lie in a neighborhood of u_l.

The Riemann problem plays an important role in the initial value problem for conservation laws. James Glimm, using the solution for the Riemann problem constructed by Lax as a building block, developed a method for solving any initial value problem for conservation laws with data $u(x,0) = u_0(x)$, as long as the total variation of $u_0(x)$ is small [18].

The first step of Glimm's method is to approximate $u_0(x)$ by a piecewise constant function $v_h(x,0)$

$$v_h(x,0) = m_j \qquad jh < x < (j+1)h, \quad j = 0, \pm 1, \cdots \tag{2.3-12}$$

where m_j is the average value of $u_0(x)$ over the interval (jh, (j+1)h) and $h = \Delta x$ is the element of length in the x direction.

The second step is to construct an exact solution for (2.3-1) with the initial value $u = u_h(x,0)$. This solution is obtained by solving a series of Riemann problems of (2.3-1)

$$u(x,0) = \begin{cases} m_{j-1}, & (x < jh), \\ m_j, & (x > jh), \end{cases} \qquad j = 0, \pm 1, \cdots \tag{2.3-13}$$

The solutions for these Riemann problems can be obtained by the above theorem given by Lax since m_{j-1} and m_j are close by the assumption that $u_0(x)$ has small total variation. If λ is an upper bound for $|\lambda_k|$, then the solutions of the neighboring Riemann problems do not intersect as long as

$$t < \frac{h}{2\lambda}. \tag{2.3-14}$$

The third step is to take $u_h(x, \frac{h}{2\lambda})$ as a new initial value function, where $u_h(x,t)$ is the approximate solution constructed by step one and step two, then we approximate this function by a piecewise constant value function. Let $\alpha_1, \alpha_2, \cdots,$ be a sequence of random numbers uniformly distributed in the interval [0,1]. Glimm used the following approximation

$$v_h(x, h/2\lambda) = u_h(jh + \alpha_1 h, \quad h/2\lambda), \qquad jh < x < (j+1)h, \quad j = 0, \pm 1, \cdots \tag{2.3-15}$$

Step four is to repeat the above steps by taking

$$\begin{aligned} & v_h(x, nh/2\lambda) = u_h(jh + \alpha_n, nh/2\lambda), \\ & jh < x < (j+1)h, \quad j = 0, \pm 1, \cdots \qquad n = 2, 3, \cdots \end{aligned} \tag{2.3-16}$$

Let $u_h(x,t)$ be an approximate solution constructed by the above method. Glimm showed the following fundamental results:

1) For any given real number ε, one can choose δ so small that if the

total variation of $u_0(x)$ is less then δ, then for any t, the total variation of $u_h(x,t)$ along any line in the x direction is less then ε.

2) There is a subsequence of $u_h(x,t)$ which converges in the L_1 norm with respect to x, uniformly in t, to a limit $u(x,t)$.

3) For almost all choices of the random sequence α_n, this limit $u(x,t)$ is a weak solution of (2.3-1).

These results and the solution for the Riemann problem constructed by Lax are the basis for the subsequent works on systems of conservation laws.

The solution for the Riemann problem also plays an important role in building numerical schemes. For example, Glimm's method can be used directly to develop a finite difference scheme which is often refered to as *Glimm's scheme* or *the random choice method.* Another example is *Godunov's scheme* which is also based on the solution of Riemann problem, and will be discussed more fully in Chaper 6.

Chapter 3

Governing Equations and Elementary Waves for the Normal Impact Problem of a Nonlinear Elastic String

3.1 *Governing Equations*

In this chapter we consider the problem of a stretched nonlinear elastic string subjected to a normal impact. We assume that the string consists of homogeneous incompressible material, and that it is perfectly flexible so that the shearing force and the bending moment can be ignored. Since a rubber string is a poor heat conductor, thermal effects are ignored and a purely mechanical theory is considered.

We consider a fixed rectangular Cartesian coordinate system OXY and assume that the undeformed string lies along the X axis, with $-L \leq X \leq L$. We take this configuration as the reference configuration and denote the position of a material point of the string by X.

The material point at X moves to the position $\boldsymbol{x}$ in the deformed state, where

$$\boldsymbol{x} = (x_1(X,t),\ x_2(X,t))^T, \tag{3.1-1}$$

and we assume that the Ox_1x_2 axes coincide with OXY.

The string is then stretched to its initial position: $x_1(X,0) = \frac{L_1 X}{L}$, $x_2(X,0) = 0$, where $L_1 \geq L$, and subjected to a normal impact at $X = 0$. Since the problem is symmetric, we need only consider the string in the interval $0 \leq X \leq L$.

Let $S(\boldsymbol{x},t)$ denote the arc length measured from $\boldsymbol{x} = \boldsymbol{x}(0,t)$ in the deformed configuration and define the stretch $\lambda(X,t)$ by

$$\lambda(X,t) = \frac{\partial S}{\partial X}. \tag{3.1-2}$$

Define also $\theta = \theta(X,t)$ to be the angle that the tangent to the string makes with the positive X axis, and let u and v denote the components of velocity in the directions x_1 and x_2 respectively so that $u = \frac{\partial x_1}{\partial t}$, $v = \frac{\partial x_2}{\partial t}$. $T(\lambda)$ denotes the nominal tension of the string.

We then have the following compatibility relations:

$$\frac{\partial(\lambda\cos\theta)}{\partial t} = \frac{\partial u}{\partial X}, \qquad \frac{\partial(\lambda\sin\theta)}{\partial t} = \frac{\partial v}{\partial X}. \tag{3.1-3}$$

If body forces are neglected, the equations of motion are given by

$$\frac{\partial(T\cos\theta)}{\partial X} = \rho\frac{\partial u}{\partial t}, \quad \frac{\partial(T\sin\theta)}{\partial X} = \rho\frac{\partial v}{\partial t}, \tag{3.1-4}$$

where ρ denotes the constant density of the string in the undeformed configuration.

We may rewrite equations (3.1-3) and (3.1-4) in the nondimensional conservation form

$$\boldsymbol{u}_{,t} + \boldsymbol{H}(\boldsymbol{u})_{,X} = \boldsymbol{0}, \quad 0 \le X < 1, \quad t > 0, \tag{3.1-5}$$

where

$$\begin{aligned} \boldsymbol{u} &= (\lambda\cos\theta,\ \lambda\sin\theta, u, v)^T, \\ \boldsymbol{H}(\boldsymbol{u}) &= -\big(u,\ v,\ T(\lambda)\cos\theta,\ T(\lambda)\sin\theta\big)^T, \end{aligned} \tag{3.1-6}$$

with the superposed T denoting the transpose. The nondimensional form is obtained by setting

$$\widehat{X} = X/L, \quad \widehat{t} = ct/L, \quad \widehat{u} = u/c, \quad \widehat{v} = v/c, \quad \widehat{T} = T/\mu, \tag{3.1-7}$$

in equations (3.1-3) and (3.1-4) and dropping the hats. Additional details of the derivation may be found in the papers [2] and [29]. We have taken

$$c^2 = \mu/\rho, \tag{3.1-8}$$

where μ the shear modulus for infinitesimal deformation from the undeformed state.

Initial conditions are taken as

$$\lambda(X,0) = \lambda_0, \quad \theta(X,0) = u(X,0) = v(X,0) = 0, \quad 0 \leq X \leq 1, \tag{3.1-9}$$

where $\lambda_0 = L_1/L \geq 1$.

If restoring forces are ignored, the boundary conditions are:

$$\begin{aligned} &u(0,t) = 0, \qquad v(0,t) = q, \\ &u(1,t) = v(1,t) = 0, \qquad t > 0, \end{aligned} \tag{3.1-10}$$

and if the restoring forces are taken into account, the boundary conditions are

$$\begin{aligned} &u(0,t) = 0, \quad \frac{dv}{dt}(0,t) = 2\alpha T\{\lambda(0,t)\}\sin\theta(0,t), \\ &u(1,t) = v(1,t) = 0, \qquad t > 0, \\ &v(0,0^+) = q, \end{aligned} \tag{3.1-11}$$

where

$$\alpha = \frac{\rho A L}{M}, \tag{3.1-12}$$

with A denoting the cross sectional area of the string and M the impacting mass.

If equation (3.1-5) is written in matrix form the nonzero entries of the matrix A in the equation

$$\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,X} = 0, \tag{3.1-13}$$

are

$$
\begin{aligned}
A_{13} &= A_{24} = -1 \\
-A_{31} &= T' \cos^2\theta + T\sin^2\theta/\lambda, \\
-A_{32} &= -A_{41} = (T' - T/\lambda)\sin\theta\cos\theta, \\
-A_{42} &= T'\sin^2\theta + T\cos^2\theta/\lambda
\end{aligned}
\tag{3.1-14}
$$

where $' \equiv \frac{d}{d\lambda}$. The eigenvalues of A are

$$
\Lambda_{\pm 1} = \pm (T')^{1/2}, \quad \Lambda_{\pm 2} = \pm (T/\lambda)^{1/2},
\tag{3.1-15}
$$

with corresponding right and left eigenvectors

$$
\begin{aligned}
\boldsymbol{R}^{(\pm 1)} &= (\lambda\cos\theta, \quad \lambda\sin\theta, \quad \mp\Lambda_1\lambda\cos\theta, \quad \mp\Lambda_1\lambda\sin\theta)^T, \\
\boldsymbol{R}^{(\pm 2)} &= (\lambda\sin\theta, \quad -\lambda\cos\theta, \quad \mp\Lambda_2\lambda\sin\theta, \quad \pm\Lambda_2\lambda\cos\theta)^T, \\
\boldsymbol{L}^{(\pm 1)} &= (\Lambda_1\lambda\cos\theta, \quad \Lambda_1\lambda\sin\theta, \quad \mp\lambda\cos\theta, \quad \mp\lambda\sin\theta), \\
\boldsymbol{L}^{(\pm 2)} &= (\Lambda_2\lambda\sin\theta, \quad -\Lambda_2\lambda\cos\theta, \quad \mp\lambda\sin\theta, \quad \pm\lambda\cos\theta).
\end{aligned}
\tag{3.1-16}
$$

3.2 *Constitutive Relations*

If $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the position vector of an arbitrary given material point in the undeformed state, $\boldsymbol{x}(\boldsymbol{X}) = (x_1, x_2, x_3)$ denotes the corresponding position vector of the same material point in the deformed state, then the deformation gradient tensor $\mathcal{F}$ is defined by

$$
\mathcal{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}},
\tag{3.2-1}
$$

with components given by

$$
F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \qquad i = 1,2,3, \quad \alpha = 1,2,3.
\tag{3.2-2}
$$

If, for an elastic material, there exist a scalar function of the deformation gradient tensor $\mathcal{F}$, say $W(\mathcal{F})$, such that

$$\dot{W} = tr(\mathcal{H}(\mathcal{F})\dot{\mathcal{F}}), \tag{3.2-3}$$

where $\mathcal{H}(\mathcal{F})$ is the response function (see Ogden [19]), then this material is called a Green elastic or a hyperelastic material. The scalar function W is called a strain-energy function or an elastic potential energy function.

For an incompressible hyperelastic material, Ogden [19] proposed the following form of strain-energy function

$$W = \sum_{i=1}^{3} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3), \tag{3.2-4}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. In equation (3.2-4), if we take $\alpha_1 = 2$, $\alpha_2 = -2$, $\mu_1 = \mu\alpha$, $\mu_2 = -\mu(1-\alpha)$, $\mu_3 = 0$, then we have the Mooney-Rivlin strain -energy function:

$$W = \frac{\mu}{2}(\alpha(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (1-\alpha)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)) \tag{3.2-5}$$

and if we take $\alpha = 1$, we have the neo-Hookean strain-energy function

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3). \tag{3.2-6}$$

For the normal impact problem of the incompressible hyperelastic string, we assume that $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$, where λ is the uniaxial stretch. For this case, equation (3.2-4) reduces to

$$W = \sum_{i=1}^{3} \frac{\mu_i}{\alpha_i}(\lambda^{\alpha_i} + 2\lambda^{-\frac{\alpha_i}{2}} - 3). \tag{3.2-7}$$

The nondimensional nominal stress-stretch relation is given by

$$T(\lambda) = \frac{1}{\mu}\frac{dW}{d\lambda} \tag{3.2-8}$$

so that

$$T(\lambda) = \frac{1}{\mu}\sum_{i=1}^{3}\mu_i(\lambda^{\alpha_i - 1} - \lambda^{-\frac{\alpha_i}{2}-1}). \tag{3.2-9}$$

We use the values given by Ogden

$$\mu_1 = 1.491\mu, \qquad \mu_2 = 0.003\mu, \qquad \mu_3 = -0.0237\mu,$$

$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0.$$

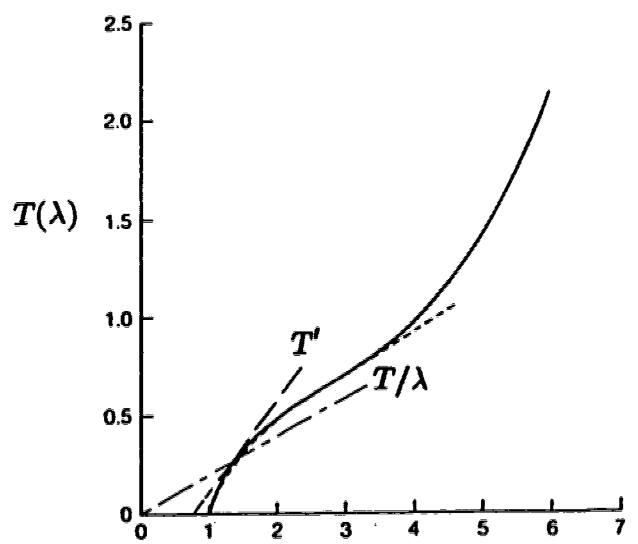


Fig.3.2-1

Graph of nominal stress $T(\lambda)$ as a function of the stretch λ for the stress/stretch relation (3.2-9) (———) and for a Mooney–Rivlin material (- - - - -) with $\alpha = 0.6$.

The graph of $T(\lambda)$ as a function of λ is an $S-$ shaped curve with an inflection point at $\lambda_i = 2.6403$. The eigenvalues Λ_1 and Λ_2 coincide at $\lambda_{c1} = 2.1267$, and $\lambda_{c2} = 3.1674$.

The Mooney-Rivlin nominal stress-stretch relation is given by:

$$T(\lambda) = (\alpha + (1-\alpha)/\lambda)(\lambda - 1/\lambda^2). \tag{3.2-10}$$

In Fig.3.2-1, we plot the graph of $T(\lambda)$ as a function of λ by using equations (3.2-9) and (3.2-10) with $\alpha = 0.6$. The curves are approximately the same for $1.0 \le \lambda \le 3.5$. For further consideration of the impact problem of the elastic string, we will adopt Ogden's 3-term formula (3.2-9).

3.3 Elementary Waves

If we denote

$$u_1 = \lambda\cos\theta, \quad u_2 = \lambda\sin\theta, \quad u_3 = u, \quad u_4 = v,$$

then the eigenvalues of the Jacobian matrix A in equation (3.1-13) are

$$\Lambda_{\pm 1} = \pm\Lambda_1 = \pm(T')^{1/2}, \quad \Lambda_{\pm 2} = \pm\Lambda_2 = \pm(T/\lambda)^{1/2}, \quad \lambda = (u_1^2 + u_2^2)^{1/2} \tag{3.3-1}$$

and the corresponding right eigenvectors are

$$\begin{aligned} \boldsymbol{R}^{(\pm 1)} &= (u_1, u_2, \mp\Lambda_1 u_1, \mp\Lambda_1 u_2)^T, \\ \boldsymbol{R}^{(\pm 2)} &= (u_2, -u_1, \mp\Lambda_2 u_2, \pm\Lambda_2 u_1)^T. \end{aligned} \tag{3.3-2}$$

Then

$$\begin{aligned} \operatorname{grad}\Lambda_{\pm 1} \cdot \boldsymbol{R}^{(\pm 1)} &= \pm(\Lambda_1)'\lambda = \frac{T''\lambda}{2\Lambda_1}, \\ \operatorname{grad}\Lambda_{\pm 2} \cdot \boldsymbol{R}^{(\pm 2)} &= 0, \qquad \text{for all } \boldsymbol{u} \end{aligned} \tag{3.3-3}$$

and it follows that the characteristic fields related to $\pm\Lambda_2$ are linearly degenerate, while the fields related to $\pm\Lambda_1$ are more complicated. If either a neo-Hookean or a

Mooney-Rivlin stress-stretch relation is considered, then the fields related to $\pm\Lambda_1$ are genuinely nonlinear since in these two cases $\frac{T''\lambda}{2\Lambda_1} > 0$ for all $\boldsymbol{u}$. However, if we adopt the three terms stress-stretch formula suggested by Ogden, then the $\pm\Lambda_1$ fields can no longer be considered as genuinely nonlinear for all λ since $T'' = 0$ at $\lambda = \lambda_i$. We note that λ_i is the only inflection point of the function $T(\lambda)$, so if $1 \leq \lambda < \lambda_i$ or $\lambda > \lambda_i$, the $\pm\Lambda_1$ fields are genuinely nonlinear.

For future reference, we consider some known facts on the behaviour of solutions of the constant boundary condition problem.

Consider first the simple wave solution for the Λ_1 field. If $T''(\lambda) \neq 0$, we normalize the right eigenvector $\boldsymbol{R}^{(+1)}$ by

$$\boldsymbol{r}_1 = \frac{1}{\Lambda_1'\lambda}\boldsymbol{R}^{(+1)} = \frac{1}{\Lambda_1'\lambda}(u_1, u_2, -\Lambda_1 u_1, -\Lambda_1 u_2)^T \tag{3.3-4}$$

so that $\operatorname{grad}\Lambda_1 \cdot \boldsymbol{r}_1 = 1$.

Suppose now $\boldsymbol{u}_0$ is a constant state; we will construct a solution of the form $\boldsymbol{u} = \boldsymbol{u}(\beta)$ which can be connected to the constant state on the right. Consider the following problem

$$\frac{d\boldsymbol{u}(\beta)}{d\beta} = \boldsymbol{r}_1(\boldsymbol{u}(\beta)), \qquad \boldsymbol{u}(\Lambda_1(\boldsymbol{u}_0)) = \boldsymbol{u}_0, \quad \beta < \Lambda_1(\boldsymbol{u}_0), \tag{3.3-5}$$

where β is a parameter. The differential equation can be written in component form

$$\frac{du_1(\beta)}{d\beta} = \frac{u_1}{\Lambda_1'\lambda}, \quad \frac{du_2(\beta)}{d\beta} = \frac{u_2}{\Lambda_1'\lambda}, \quad \frac{du_3(\beta)}{d\beta} = \frac{-\Lambda_1 u_1}{\Lambda_1'\lambda}, \quad \frac{du_4(\beta)}{d\beta} = \frac{-\Lambda_1 u_2}{\Lambda_1'\lambda}. \tag{3.3-6}$$

By the first two of equations (3.3-6), we have

$$\frac{du_2}{du_1} = \frac{u_2}{u_1} \tag{3.3-7}$$

and if we recall that $u_1 = \lambda \cos\theta, u_2 = \lambda \sin\theta$, it then follows from equation (3.3-7) that

$$\theta = \theta_0 = constant. \tag{3.3-8}$$

By the first and the third of equations (3.3-6), we have

$$\frac{du_3}{du_1} = -\Lambda_1. \tag{3.3-9}$$

Since $u_3 = u$, equations (3.3-9) and (3.3-8) give

$$u + \int^{\lambda} \Lambda_1(p)dp \cos\theta_0 = constant. \tag{3.3-10}$$

Similarly, the second and the fourth equations of (3.3-6) give

$$\frac{du_4}{du_2} = -\Lambda_1, \tag{3.3-11}$$

and since $u_4 = v$, we have

$$v + \int^{\lambda} \Lambda_1(p)dp \sin\theta_0 = constant. \tag{3.3-12}$$

We note that

$$\frac{d\Lambda_1(\boldsymbol{u}(\beta))}{d\beta} = \operatorname{grad}\Lambda_1 \cdot \frac{d\boldsymbol{u}}{d\beta} = \operatorname{grad}\Lambda_1 \cdot \boldsymbol{r}_1 = 1 \tag{3.3-13}$$

so we have

$$\Lambda_1(\boldsymbol{u}(\beta)) = \beta \tag{3.3-14}$$

since $\boldsymbol{u}(\Lambda_1(\boldsymbol{u}_0)) = \boldsymbol{u}_0$.

Since $\Lambda_1 = (T'(\lambda))^{1/2}$, we have

$$T'(\lambda(\beta)) = \beta^2, \tag{3.3-15}$$

and since we require $\beta < \Lambda_1(\boldsymbol{u}_0)$, it follows that $T'(\lambda(\beta)) < T'(\lambda_0)$. Hence if $\lambda(\beta) > \lambda_0$ so that we have a loading problem, then we require that $T'(\lambda)$ is decreasing in the interval considered, and this implies that $1 \leq \lambda < \lambda_i$. If $\lambda(\beta) < \lambda_0$, we have an unloading problem and we require that $T'(\lambda)$ is an increasing function and this implies that $\lambda > \lambda_i$. In these intervals $T'(\lambda)$ is a monotone function of λ and so the inverse function exists, so we can solve for $\lambda(\beta)$ using equation (3.3-15) as

$$\lambda(\beta) = (T')^{-1}(\beta^2). \tag{3.3-16}$$

Then u and v can be found by using (3.3-10), (3.3-12) and (3.3-16).

Suppose now w is any 1-Riemann invariant, then

$$\frac{dw}{d\beta} = \operatorname{grad} w \cdot \frac{d\boldsymbol{u}}{d\beta} = \operatorname{grad} w \cdot \boldsymbol{r}_1 = 0$$

and so $\boldsymbol{u}(\beta)$ constructed by the above method is a simple wave for the Λ_1 field. Setting $\beta = \frac{X-X_0}{t-t_0}$ we easily verify that $\boldsymbol{u}(\frac{X-X_0}{t-t_0})$ is a solution for (3.1-13) since

$$\begin{aligned} \boldsymbol{u}_{,t} + A\boldsymbol{u}_{,X} &= \frac{1}{t-t_0}(A - \frac{X-X_0}{t-t_0}I)\boldsymbol{u}_{,\beta} \\ &= \frac{1}{t-t_0}(A - \Lambda_1(\boldsymbol{u}(\beta))I)\boldsymbol{r}_1(\boldsymbol{u}(\beta)) = \boldsymbol{0}. \end{aligned} \tag{3.3-17}$$

The special choice $\beta = x/t$ gives a similarity solution, which is referred to as a centered simple wave.

Next, we consider the shock wave solution for the Λ_1 field, which should satisfy the jump condition, and the entropy inequality.

The jump conditions are

$$V[\lambda\cos\theta] = [-u], \quad V[\lambda\sin\theta] = [-v], \quad V[u] = [-T\cos\theta], \quad V[v] = [-T\sin\theta]. \tag{3.3-18}$$

Eliminating $[u]$ and $[v]$ in equations (3.3-18) we have

$$V^2[\lambda \cos\theta] = [T \cos\theta], \quad V^2[\lambda \sin\theta] = [T \sin\theta] \tag{3.3-19}$$

and by eliminating V in the above equations, we have

$$[T/\lambda] \sin[\theta] = 0. \tag{3.3-20}$$

It follows that

$$\text{either} \quad [T/\lambda] = 0, \qquad \text{or} \quad \sin[\theta] = 0, \quad \text{or both.} \tag{3.3-21}$$

Case 1: If $[\lambda] \neq 0$, $[T/\lambda] \neq 0$, $[\theta] = 0$, then

$$V^2 = V_L^2 = [T]/[\lambda], \tag{3.3-22}$$

where $V_L(\lambda^+, \lambda^-) = (\frac{T(\lambda^+) - T(\lambda^-)}{\lambda^+ - \lambda^-})^{1/2}$ is the velocity for a λ discontinuity and λ^-, λ^+ are the values of λ immediately behind and ahead of a shock.

Case 2: If $[\lambda] \neq 0$, $[T/\lambda] = 0$, $[\theta] = 0$, then

$$V^2 = V_L^2 = [T]/[\lambda] = T(\lambda^-)/\lambda^- = T(\lambda^+)/\lambda^+, \tag{3.3-23}$$

Case 3: If $[\lambda] \neq 0$, $[T/\lambda] = 0$, $[\theta] \neq 0$, then

$$V^2 = T(\lambda^-)/\lambda^- = T(\lambda^+)/\lambda^+ = [T]/[\lambda]. \tag{3.3-24}$$

Case 4: If $[\lambda] = 0$, $[T/\lambda] = 0$, $[\theta] \neq 0$, then

$$V^2 = V_T^2 = T(\lambda)/\lambda, \tag{3.3-25}$$

where $V_T = (T(\lambda)/\lambda)^{1/2}$ is the velocity for a θ discontinuity.

The above four cases can be reclassified as two cases

Case I : $[\lambda] \neq 0$ (case 1 , case 2 and case 3.)

$$V_L^2 = \frac{[T]}{[\lambda]}.$$

Case II: $[\theta] \neq 0$ (case 3 and case 4.)

$$V_T^2 = \frac{T(\lambda)}{\lambda}.$$

Since the Λ_1 field is genuinely nonlinear for the cases $\lambda < \lambda_i$ or $\lambda > \lambda_i$ we have the following centered shock solution

Region 1: $X/t > V$

$$\lambda = \lambda_0, \quad \theta = \theta_0, \quad u = u_0, \quad v = v_0.$$

Region 2: $V > X/t > 0$

$$\lambda = \lambda_1, \quad \theta = \theta_0, \quad u = u_0 - V(\lambda_1 - \lambda_0)\cos\theta_0, \quad v = v_0 - V(\lambda_1 - \lambda_0)\sin\theta_0$$

where

$$V = V_L(\lambda_0, \lambda_1), \qquad \Lambda_1(\lambda_0) < V < \Lambda_1(\lambda_1). \tag{3.3-26}$$

If $\lambda_0 < \lambda_1$, then the inequality in (3.3-26) requires $\lambda_i < \lambda_0$ and if $\lambda_0 > \lambda_1$ we require $\lambda_0 < \lambda_i$. For the loading problem we must have $\lambda_0 < \lambda_1$ in the genuinely non-linear case.

Finally, since the Λ_2 field is linearly degenerate, we have the following contact discontinuity solution

Region 1: $X/t > V$

$$\lambda = \lambda_0, \quad \theta = \theta_0, \quad u = u_0, \quad v = v_0.$$

Region 2: $V > X/t > 0$

$$\lambda = \lambda_0, \qquad \theta = \theta_1,$$
$$u = u_0 - (T(\lambda_0)\lambda_0)^{1/2}(\cos\theta_1 - \cos\theta_0),$$
$$v = v_0 - (T(\lambda_0)\lambda_0)^{1/2}(\sin\theta_1 - \sin\theta_0), \tag{3.3-27}$$

where

$$V = V_T = \Lambda_2(\lambda_0).$$

As we mentioned before, in general the Λ_1 characteristic field is not genuinely non-linear over its entire range. A general elementary wave solution of the planar motion of an elastic string was given by Michael Shearer [22] who also solved the corresponding Riemann problem using an extension of a method of Wendroff for two by two systems that fail to be genuinely nonlinear [28]. We employ this idea in summarising the cases in section 3.4 below.

Interestingly, if there is no vertical motion, then $v = \theta = 0$ and the system (3.1-5) is reduced to

$$\lambda_{,t} - u_{,X} = 0, \qquad u_{,t} - T(\lambda)_{,X} = 0. \tag{3.3-28}$$

If the Mooney-Rivlin stress-stretch relation is considered, then $T'(\lambda) > 0$ and $T''(\lambda) < 0$ so that equations (3.3-28) are in fact a p-system considered earlier in Chapter 2.

3.4 Solutions of the Constant Boundary Condition Problem

Consider the constant boundary condition problem of system (3.1-5) with conditions (3.1-9) and (3.1-10). The solutions of such problem were given by Wegner, Haddow and Tait [29] up to reflection for various cases. These solutions hold for large deformations. In this section, we will consider some cases in detail and outline the solutions for the other cases briefly for later reference. We consider only the loading problem here.

Consider first the λ waves.

Case (A): $1 \leq \lambda_0 < \lambda_m \leq \lambda_i$, where λ_0 is the initial stretch, λ_m is the maximum stretch, and λ_i is the inflection point where $T''(\lambda_i) = 0$. Since for this case $\Lambda_1(\lambda_m) < \Lambda_1(\lambda) < \Lambda_1(\lambda_0)$ for $1 \leq \lambda_0 \leq \lambda \leq \lambda_m \leq \lambda_i$, then, by the analysis of simple wave solution in section 3.3, we have the solution as follows

$$\begin{aligned} \lambda = \lambda_0, &\qquad X/t > \Lambda_1(\lambda_0). \\ \Lambda_1(\lambda) = X/t, &\qquad \Lambda_1(\lambda_0) > X/t > \Lambda_1(\lambda_m), \\ \lambda = \lambda_m, &\qquad \Lambda_1(\lambda_m) > X/t \geq 0. \end{aligned} \tag{3.4-1}$$

Case (B): $1 \leq \lambda_0 < \lambda_i < \lambda_m$. For this case, we define λ_T (see Fig.3.4-1) by

$$\frac{T(\lambda_T) - T(\lambda)}{\lambda_T - \lambda} = T'(\lambda), \qquad 1 \leq \lambda < \lambda_i. \tag{3.4-2}$$

We can easily show that $\lambda_T > \lambda_i$. If not, then $\lambda_T < \lambda_i$ and $\lambda_T \neq \lambda$. We may suppose that $\lambda_T > \lambda$. Since $T''(\lambda) < 0$ for $\lambda < \lambda_i$, we have $T'(\lambda_T) < \frac{T(\lambda_T)-T(\lambda)}{\lambda_T-\lambda} < T'(\lambda)$ and this is a contradiction with equation (3.4-2). Thus we must have $\lambda_T > \lambda_i$.

We define λ_* (see Fig.3.4-2) by

$$\frac{T(\lambda_*) - T(\lambda)}{\lambda_* - \lambda} = T'(\lambda_*), \qquad \lambda > \lambda_i. \tag{3.4-3}$$

Clearly we have $\lambda_* < \lambda_i$.

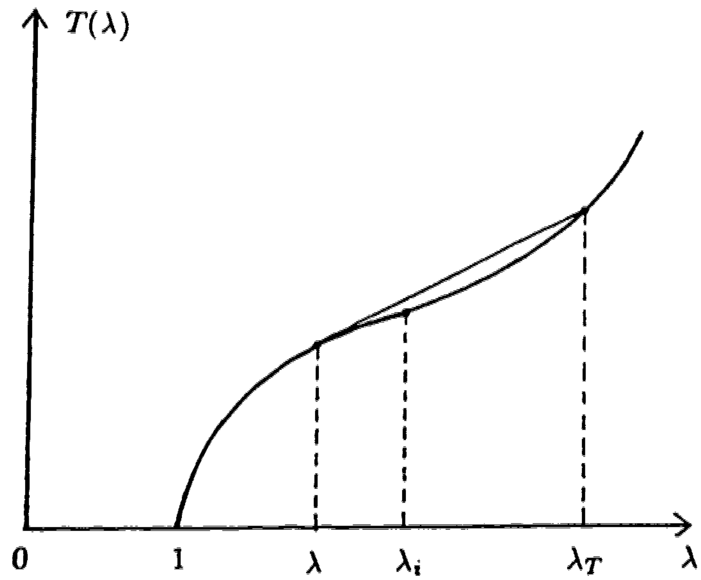


Fig.3.4-1

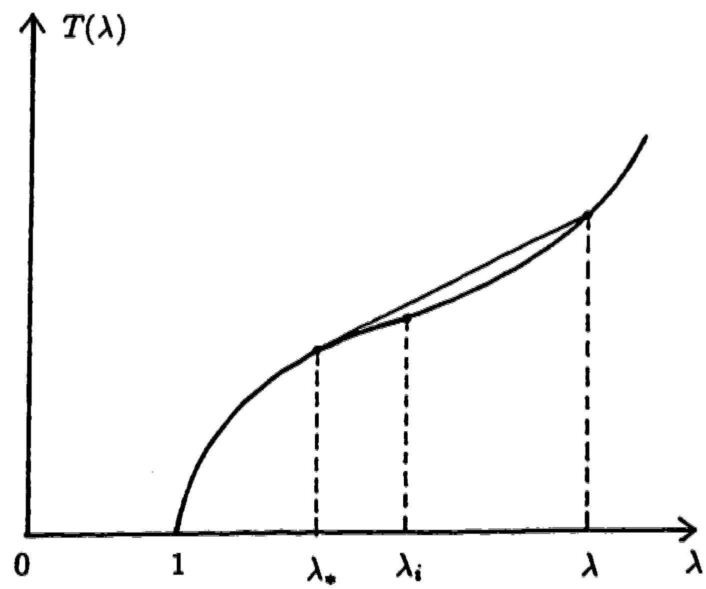


Fig.3.4-2

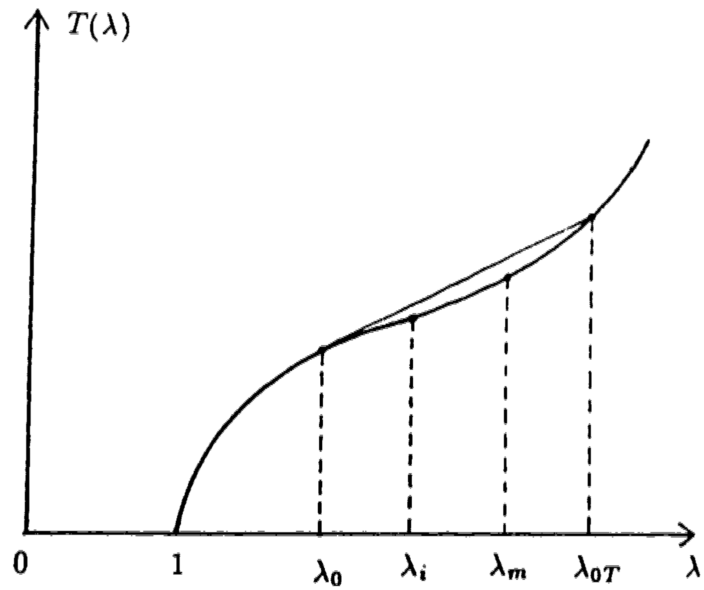


Fig.3.4-3

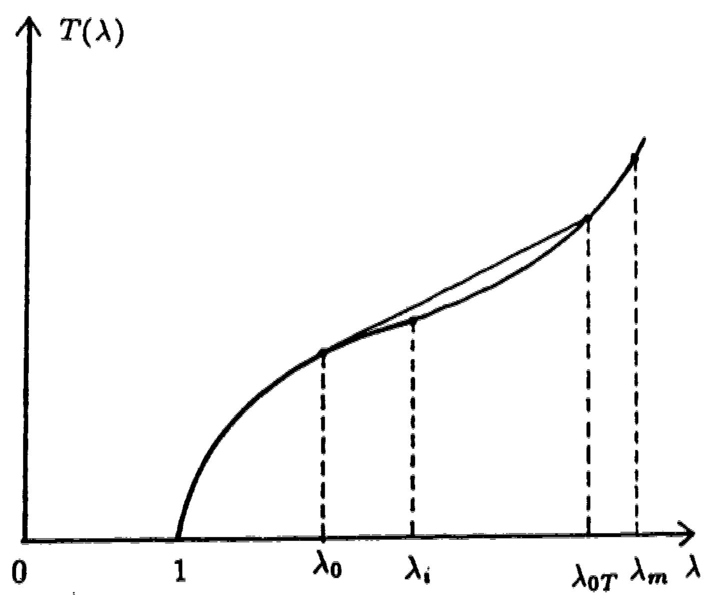
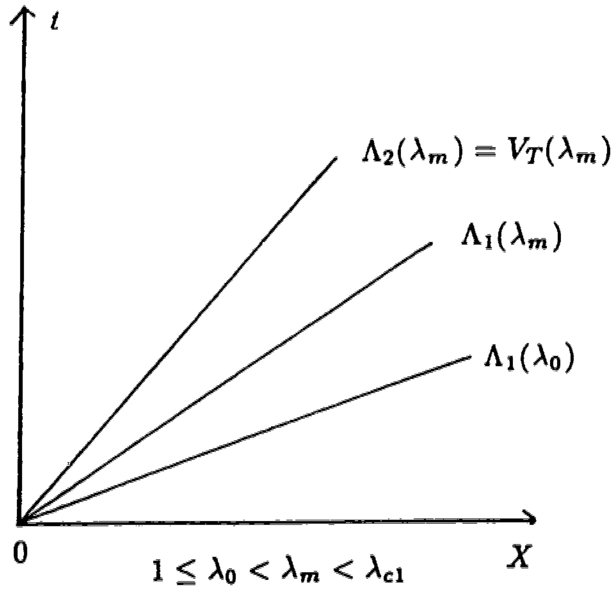
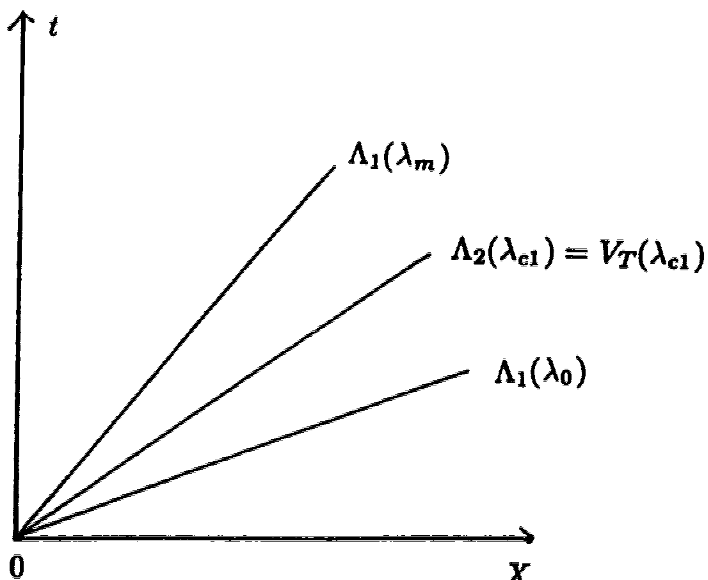


Fig.3.4-4



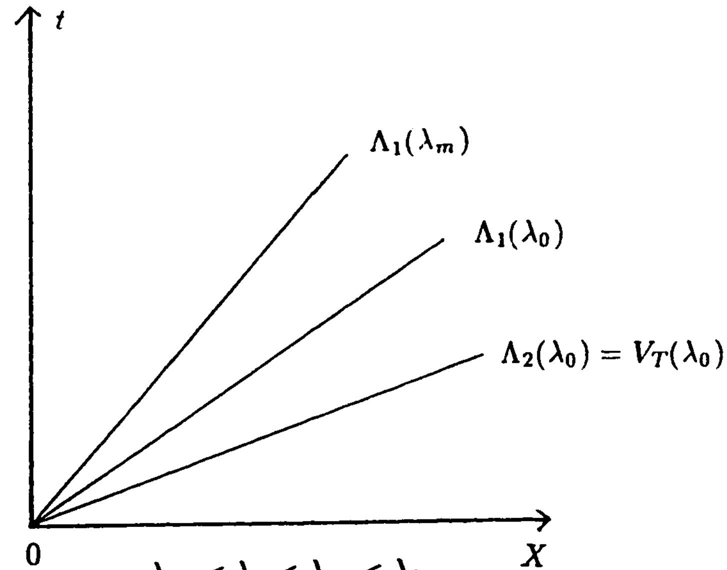
$1 \leq \lambda_0 < \lambda_m < \lambda_{c1}$

Fig.3.4-5



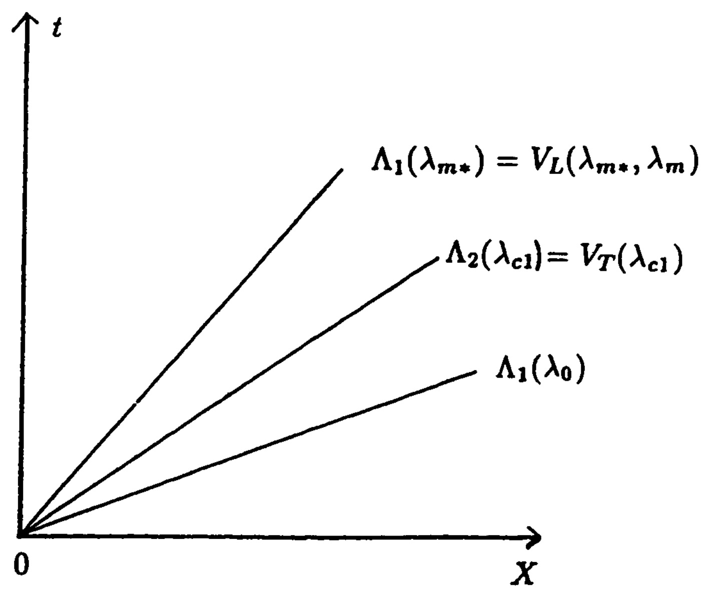
$1 \leq \lambda_0 < \lambda_{c1} < \lambda_m < \lambda_i$

Fig.3.4-6



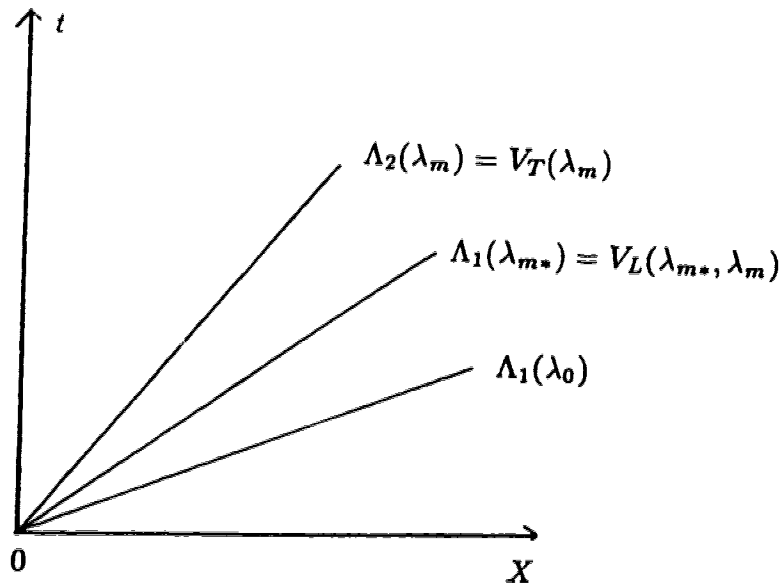
$\lambda_{c1} < \lambda_0 < \lambda_m < \lambda_i$

Fig.3.4-7



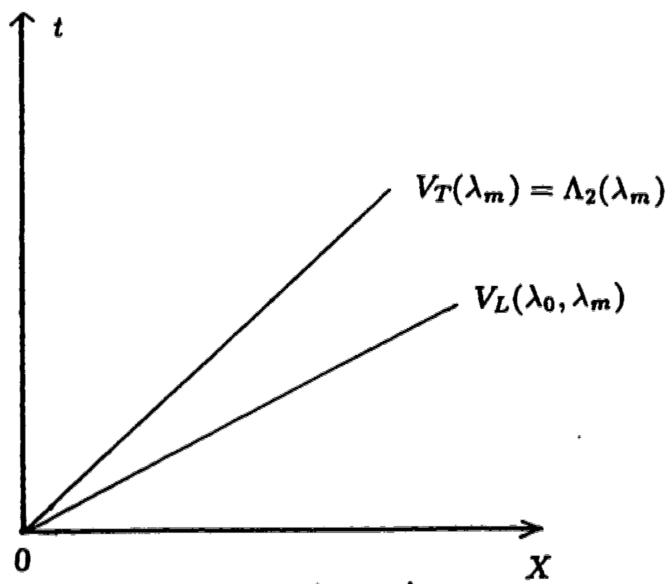
$1 \leq \lambda_0 < \lambda_{c1} < \lambda_i < \lambda_m < \lambda_{c1T} < \lambda_{0T}$

Fig.3.4-8



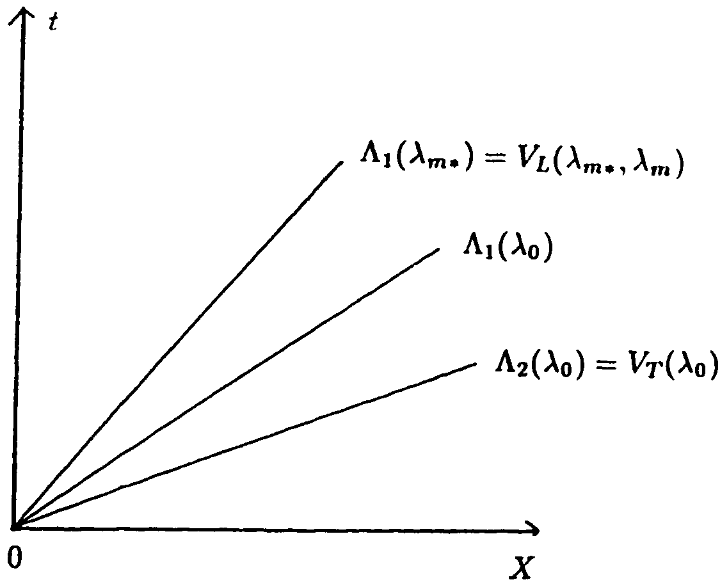
$1 \leq \lambda_0 < \lambda_{c1} < \lambda_i < \lambda_{c1T} < \lambda_m < \lambda_{0T}$

Fig.3.4-9



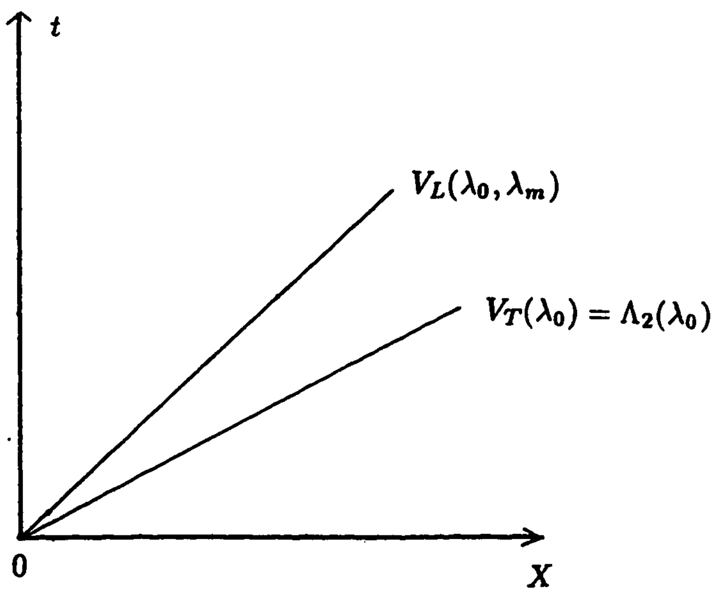
$1 \leq \lambda_0 < \lambda_i < \lambda_{0T} < \lambda_m$

Fig.3.4-10



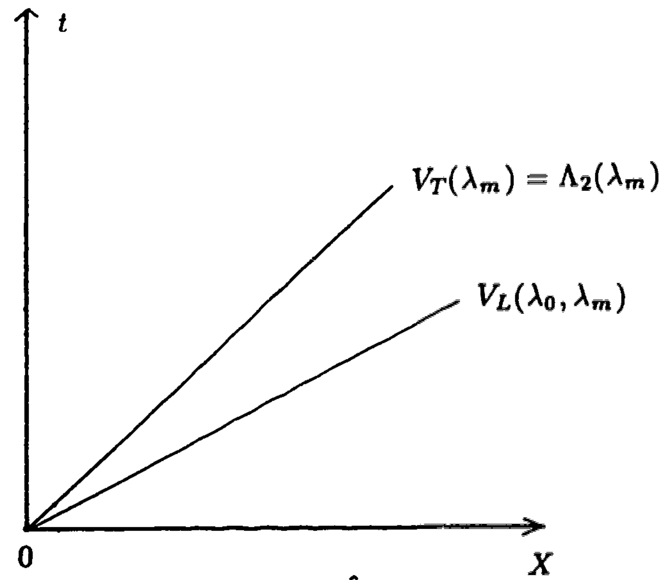
$\lambda_{c1} < \lambda_0 < \lambda_i < \lambda_m < \lambda_{0T}$

Fig.3.4-11



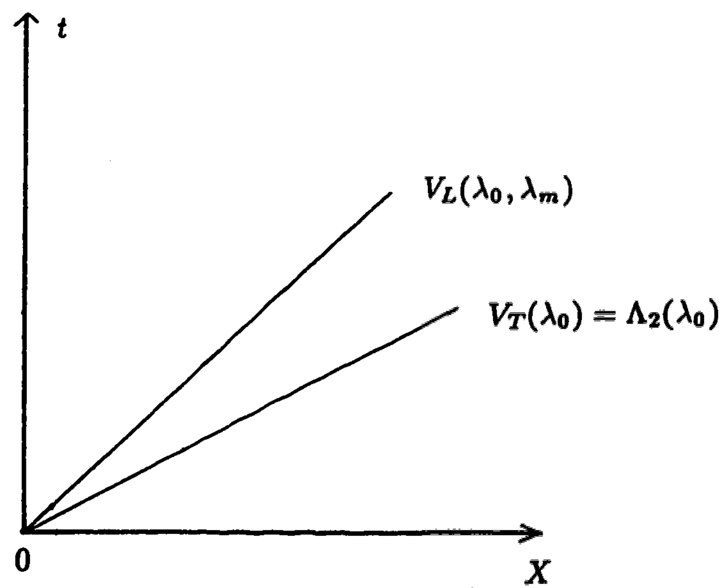
$\lambda_{c1} < \lambda_0 < \lambda_i < \lambda_{0T} < \lambda_m < \hat{\lambda}_0$

Fig.3.4-12



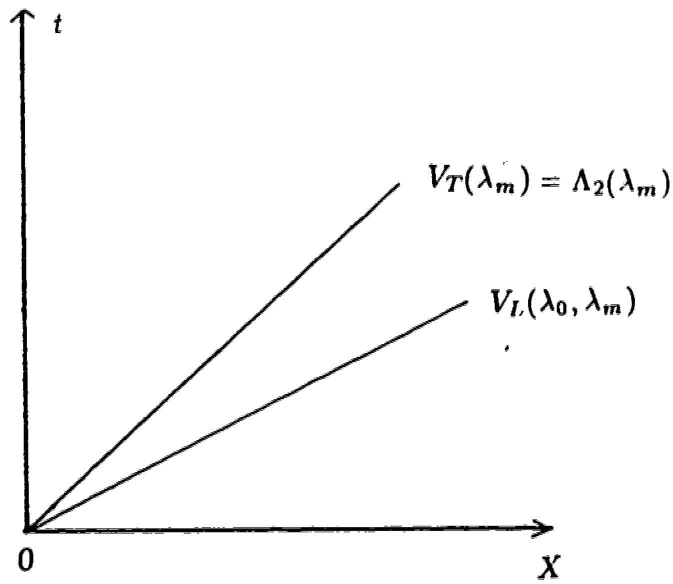
$\lambda_{c1} < \lambda_0 < \lambda_i < \hat{\lambda}_0 < \lambda_m$

Fig.3.4-13



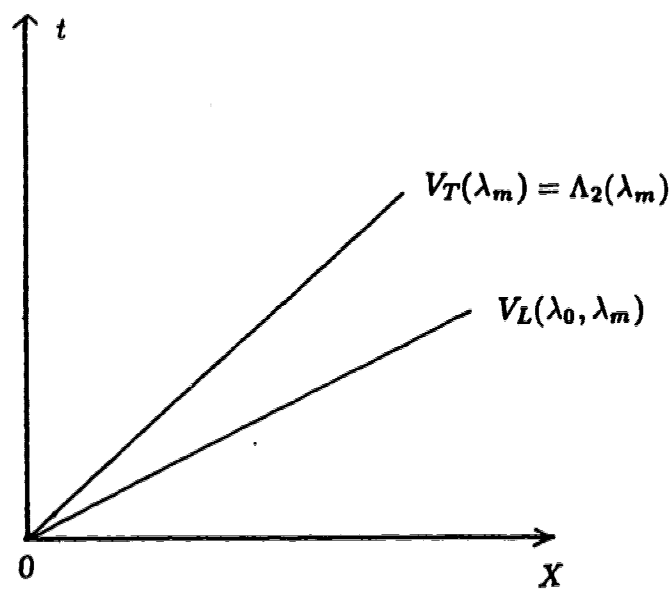
$\lambda_i < \lambda_0 < \lambda_{c2}, \quad \lambda_0 < \lambda_m < \hat{\lambda}_0.$

Fig.3.4-14



$\lambda_i < \lambda_0 < \lambda_{c2}, \quad \lambda_0 < \hat{\lambda}_0 < \lambda_m.$

Fig.3.4-15



$\lambda_i < \lambda_{c2} < \lambda_0 < \lambda_m$

Fig.3.4-16

If $\lambda_m < \lambda_{0T}$ (see Fig.3.4-3), then the solution is :

$$\begin{aligned} \lambda &= \lambda_0, & X/t &> \Lambda_1(\lambda_0), \\ \Lambda_1(\lambda) &= X/t, & \Lambda_1(\lambda_0) &> X/t > \Lambda_1(\lambda_{m*}), \\ \lambda &= \lambda_m, & \Lambda_1(\lambda_{m*}) &> X/t \geq 0. \end{aligned} \tag{3.4-4}$$

If $\lambda_m > \lambda_{0T}$ (see Fig.3.4-4), then the final state $\lambda = \lambda_m$ can be connected directly to the initial state λ_0 by a shock. The solution is

$$\begin{aligned} \lambda &= \lambda_0, & X/t &> V_L(\lambda_0, \lambda_m), \\ \lambda &= \lambda_m, & V_L(\lambda_0, \lambda_m) &> X/t \geq 0. \end{aligned} \tag{3.4-5}$$

Case (C): $\lambda_i \leq \lambda_0 < \lambda_M$. The solution is

$$\begin{aligned} \lambda &= \lambda_0, & X/t &> V_L(\lambda_0, \lambda_m), \\ \lambda &= \lambda_m, & V_L(\lambda_0, \lambda_m) &> X/t \geq 0. \end{aligned} \tag{3.4-6}$$

Next we will fit in the θ waves.

Class (A): $1 \leq \lambda_0 < \lambda_m \leq \lambda_i$. If $1 \leq \lambda_0 < \lambda_m < \lambda_{c1}$, the solution is given by the following

Region 1: $X/t > \Lambda_1(\lambda_0)$

$$\lambda = \lambda_0, \quad \theta = 0, \quad u = 0, \quad v = 0,$$

Region 2: $\Lambda_1(\lambda_0) \geq X/t > \Lambda_1(\lambda_m)$

$$\Lambda_1(\lambda) = X/t, \quad \theta = 0, \quad u = -I(\lambda), \quad v = 0,$$

Region 3: $\Lambda_1(\lambda_m) \geq X/t > \Lambda_2(\lambda_m)$

$$\lambda = \lambda_m, \quad \theta = 0, \quad u = -I(\lambda_m), \quad v = 0,$$

Region 4: $\Lambda_2(\lambda_m) \geq X/t \geq 0$

$$\lambda = \lambda_m, \quad \theta = \theta_m, \quad u = 0, \quad v = q. \tag{3.4-7}$$

The regions of this solution are shown in Fig.3.4-5.

In equations (3.4-7) $I(\lambda) = \int_{\lambda_0}^{\lambda} \Lambda_1(p)dp$, and λ_m and θ_m are determined by

$$q^2 = I(\lambda_m)(2\lambda_m\Lambda_2(\lambda_m) - I(\lambda_m)), \qquad \sin\theta_m = \frac{-q}{\lambda_m V_T(\lambda_m)},$$

where the above relations can be obtained from equations (3.3-18).

If $1 \leq \lambda_0 < \lambda_{c1} < \lambda_m$, the solution is (see also Fig.3.4-6)

Region 1: $X/t > \Lambda_1(\lambda_0)$

$$\lambda = \lambda_0, \theta = 0, u = 0, v = 0,$$

Region 2: $\Lambda_1(\lambda_0) > X/t > \Lambda_2(\lambda_{c1})$

$$\Lambda_1(\lambda) = X/t, \theta = 0, u = -I(\lambda), v = 0,$$

Region 3: $\Lambda_2(\lambda_{c1}) \geq X/t > \Lambda_1(\lambda_m)$

$$\begin{aligned} \Lambda_1(\lambda) &= X/t, \quad \theta = \theta_m, \\ u &= \cos\theta_m \int_{\lambda}^{\lambda_m} \Lambda_1(p)dp, \\ v &= q + \sin\theta_m \int_{\lambda}^{\lambda_m} \Lambda_1(p)dp, \end{aligned}$$

Region 4: $\Lambda_1(\lambda_m) \geq X/t \geq 0$

$$\lambda = \lambda_m, \quad \theta = \theta_m, \quad u = 0, \quad v = q, \tag{3.4-8}$$

where

$$\left(\int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp + K\right)^2 = q^2 + K^2,$$
$$\sin\theta_m = -q/K,$$
$$K = \lambda_{c1}\Lambda_2(\lambda_{c1}) - \int_{\lambda_0}^{\lambda_{c1}} \Lambda_1(p)dp.$$

If $\lambda_{c1} \leq \lambda_0 < \lambda_m < \lambda_i$, the solution for this case is (see also Fig.3.4-7)

Region 1: $X/t > \Lambda_2(\lambda_0)$

$$\lambda = \lambda_0, \quad \theta = 0, \quad u = 0, \quad v = 0,$$

Region 2: $\Lambda_2(\lambda_0) \geq X/t > \Lambda_1(\lambda_0)$

$$\lambda = \lambda_0, \quad \theta = \theta_m, \quad u = u_2, \quad v = v_2,$$

Region 3: $\Lambda_1(\lambda_0) \geq X/t > \Lambda_1(\lambda_m)$

$$\Lambda_1(\lambda) = X/t, \qquad \theta = \theta_m,$$
$$u = u_2 - I(\lambda)\cos\theta_m,$$
$$v = v_2 - I(\lambda)\sin\theta_m,$$

Region 4: $\Lambda_1(\lambda_m) \geq X/t \geq 0$

$$\lambda = \lambda_m, \quad \theta = \theta_m, \quad u = 0, \quad v = q, \tag{3.4-9}$$

where

$$u_2 = I(\lambda_m)\cos\theta_m, \qquad v_2 = q + I(\lambda_m)\sin\theta_m.$$
$$q^2 = I(\lambda_m)(2\lambda_0\Lambda_2(\lambda_0) + I(\lambda_m)), \quad \sin\theta_m = \frac{-q}{\lambda_0\Lambda_2(\lambda_0) + I(\lambda_m)}.$$

Class (B): $1 \le \lambda_0 < \lambda_i < \lambda_m$. If $1 \le \lambda_0 < \lambda_{c1} < \lambda_i < \lambda_m < \lambda_{c1T} < \lambda_{0T}$, the solution for λ starts from the initial constant state $\lambda = \lambda_0$ then connects to a simple wave region $\Lambda_1(\lambda) = X/t$, $\Lambda_1(\lambda_{m*}) < X/t < \Lambda_1(\lambda_0)$ and the value λ_{m*} can be connected to the final state $\lambda = \lambda_m$ by a shock since $\Lambda_1(\lambda_{m*}) = V_L(\lambda_{m*}, \lambda_m)$. The solution for θ starts from $\theta = 0$ and jumps to its final value $\theta = \theta_m$ on the line $X = \Lambda_2(\lambda_{c1})t$ (see Fig.3.4-8). If $1 \le \lambda_0 < \lambda_{c1} < \lambda_i < \lambda_{c1T} < \lambda_m < \lambda_{0T}$, the solution for λ starts from the initial constant state $\lambda = \lambda_0$ then connects to a simple wave region $\Lambda_1(\lambda) = X/t$, $\Lambda_1(\lambda_{m*}) < X/t < \Lambda_1(\lambda_0)$ and the value λ_{m*} can be connected to the final state $\lambda = \lambda_m$ by a shock since $\Lambda_1(\lambda_{m*}) = V_L(\lambda_{m*}, \lambda_m)$. The solution of θ starts from $\theta = 0$ and jumps to its final value $\theta = \theta_m$ on the line $X = \Lambda_2(\lambda_m)t$ (see Fig.3.4-9). If $1 \le \lambda_0 < \lambda_{c1} < \lambda_i < \lambda_{0T} < \lambda_m$, the solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant states, the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_m) = \Lambda_2(\lambda_m)$. Since $V_T(\lambda_m) < V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-10). If $\lambda_{c1} < \lambda_0 < \lambda_i < \lambda_m < \lambda_{0T}$, the solution for λ starts from the initial constant state $\lambda = \lambda_0$ then connects to a simple wave region $\Lambda_1(\lambda) = X/t$, $\Lambda_1(\lambda_{m*}) < X/t < \Lambda_1(\lambda_0)$ and the value λ_{m*} can be connected to the final state $\lambda = \lambda_m$ by a shock since $\Lambda_1(\lambda_{m*}) = V_L(\lambda_{m*}, \lambda_m)$. The solution of θ starts from $\theta = 0$ and jumps to its final value $\theta = \theta_m$ on the line $X = \Lambda_2(\lambda_0)t$ (see Fig.3.4-11). If $\lambda_{c1} < \lambda_0 < \lambda_i < \lambda_{0T} < \lambda_m < \hat{\lambda}_0$, where $\hat{\lambda}_0$ is defined by

$$\frac{T(\hat{\lambda}_0)}{\hat{\lambda}_0} = \frac{T(\lambda_0)}{\lambda_0}, \qquad \lambda_{c1} < \lambda_0 < \lambda_i.$$

The solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant states, the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_0) = \Lambda_2(\lambda_0)$. Since $V_T(\lambda_0) > V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-12). If $\lambda_{c1} < \lambda_0 < \lambda_i < \hat{\lambda}_0 < \lambda_m$, the solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant states, the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_m) = \Lambda_2(\lambda_m)$. Since $V_T(\lambda_m) < V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-13).

Class (C): $\lambda_i < \lambda_0 < \lambda_m$. If $\lambda_i < \lambda_0 < \lambda_{c2}$ and $\lambda_0 < \lambda_m < \hat{\lambda}_0$, the solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant states, the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_0) = \Lambda_2(\lambda_0)$. Since $V_T(\lambda_0) > V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-14). If $\lambda_i < \lambda_0 < \lambda_{c2}$ and $\lambda_0 < \hat{\lambda}_0 < \lambda_m$, the solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant state, the the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_m) = \Lambda_2(\lambda_m)$. Since $V_T(\lambda_m) < V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-15). The final case $\lambda_i < \lambda_{c2} < \lambda_0 < \lambda_m$, the solution for λ consists of two constant states, the initial state λ_0 is connected to the final state λ_m by a shock with speed $V_L(\lambda_0, \lambda_m)$. The solution for θ also consists of two constant state, the the state $\theta = 0$ is connected to the final state θ_m by a contact discontinuity with speed $V_T(\lambda_m) = \Lambda_2(\lambda_m)$. Since $V_T(\lambda_m) < V_L(\lambda_0, \lambda_m)$, there are three regions for this solution (see Fig.3.4-16).

Chapter 4

Perturbation Solutions for the Impact Problem of a Nonlinear Elastic String.

4.1 Solution for the Case with Given Variable Boundary Condition

Consider equation (3.1-13) described in Chapter 3, namely

$$\boldsymbol{u}_{,t} + A\boldsymbol{u}_{,X} = 0 \tag{4.1-1}$$

under the conditions

$$\begin{aligned} \boldsymbol{u}(X,0) &= (\lambda_0, 0, 0, 0)^T = \boldsymbol{u}_0, \\ u(0,t) &= 0, \qquad t \geq 0, \\ v(0,t) &= \varepsilon f(t), \quad t \geq 0, \end{aligned} \tag{4.1-2}$$

where $f(t)$ is a given continuous function for $t > 0$ and $f(t) = 0$ for $t \leq 0$. ε is a small positive real number. In equation (4.1-1), $\boldsymbol{u} = (\lambda \cos\theta, \lambda \sin\theta, u, v)^T$ and the components of A are given in the last chapter. If $f(t)$ is a constant when $t > 0$, then the problem is reduced to the constant boundary condition problem discussed in Chapter 3. However, if $f(t)$ is not a constant when $t > 0$, then there is no similarity solution and the problem must be solved by a different method. In this chapter, we will apply a perturbation method to the above problem. We will see later that this method fails when $\Lambda_1 = \Lambda_2$, where $\Lambda_1 = (T')^{1/2}$ and $\Lambda_2 = (T/\lambda)^{1/2}$ are the eigenvalues of the matrix A. The case for $\Lambda_1 = \Lambda_2$ will be considered separately in the next chapter.

We begin by introducing two new independent variables s_1 and s_2 as follows

$$X_{,s_2} - \Lambda_1 t_{,s_2} = 0, \quad X(s,s) = 0,$$
$$X_{,s_1} - \Lambda_2 t_{,s_1} = 0, \quad t(s,s) = s. \tag{4.1-3}$$

Equations (4.1-3) have the following interpretation. If $s_2 = constant$, then X and t are functions of s_1 only so that

$$\frac{dX}{dt} = \frac{X_{,s_1}}{t_{,s_1}} = \Lambda_2 \tag{4.1-4}$$

and the family of curves $s_2 = constant$ is described by $\frac{dX}{dt} = \Lambda_2$. Similarly, the family of curves $s_1 = constant$ is described by $\frac{dX}{dt} = \Lambda_1$.

Writing $\boldsymbol{u}(X,t) = \boldsymbol{U}(s_1, s_2)$, we have

$$\boldsymbol{U}_{,s_i} = \boldsymbol{u}_{,t} t_{,s_i} + \boldsymbol{u}_{,X} X_{,s_i}, \qquad (i = 1,2), \tag{4.1-5}$$

and it follows from equations (4.1-5) that

$$\boldsymbol{u}_{,t} = \frac{X_{,s_2}\boldsymbol{U}_{,s_1} - X_{,s_1}\boldsymbol{U}_{,s_2}}{X_{,s_2}t_{,s_1} - X_{,s_1}t_{,s_2}},$$
$$\boldsymbol{u}_{,X} = \frac{t_{,s_2}\boldsymbol{U}_{,s_1} - t_{,s_1}\boldsymbol{U}_{,s_2}}{X_{,s_2}t_{,s_1} - X_{,s_1}t_{,s_2}}. \tag{4.1-5}$$

Substituting (4.1-3) into (4.1-5), we have

$$\boldsymbol{u}_{,t} = \frac{\Lambda_1 t_{,s_2}\boldsymbol{U}_{,s_1} - \Lambda_2 t_{,s_1}\boldsymbol{U}_{,s_2}}{(\Lambda_1 - \Lambda_2)t_{,s_1}t_{,s_2}},$$
$$\boldsymbol{u}_{,X} = \frac{t_{,s_2}\boldsymbol{U}_{,s_1} - t_{,s_1}\boldsymbol{U}_{,s_2}}{(\Lambda_2 - \Lambda_1)t_{,s_1}t_{,s_2}} \tag{4.1-6}$$

and equations (4.1-6) break down when $\Lambda_1 = \Lambda_2$. We assume in this chapter that $\Lambda_1 \neq \Lambda_2$.

By substituting (4.1-6) into (4.1-1), we have

$$t_{,s_2}(A - \Lambda_1 I)\boldsymbol{U}_{,s_1} - t_{,s_1}(A - \Lambda_2 I)\boldsymbol{U}_{,s_2} = 0, \tag{4.1-7}$$

where suffixes are used to denote partial derivatives and I denotes the identity matrix. If we assume that quantities of interest can be expanded as a power series in terms of a small parameter ε, then

$$t = \sum_{k=0}^{\infty} t_k(s_1, s_2)\varepsilon^k, \qquad x = \sum_{k=0}^{\infty} x_k(s_1, s_2)\varepsilon^k,$$

$$\boldsymbol{U} = \sum_{k=0}^{\infty} \boldsymbol{U}_k(s_1, s_2)\varepsilon^k, \quad \boldsymbol{U}_0 = \boldsymbol{u}_0, \tag{4.1-8}$$

and

$$A = \sum_{k=0}^{\infty} A_k\varepsilon^k, \quad \Lambda_i = \sum_{k=0}^{\infty} \Lambda_{ik}\varepsilon^k. \tag{4.1-9}$$

In equation (4.1-9) the matrix A and the eigenvalues Λ_i depend on $\boldsymbol{U}$ and the resulting coefficients A_k, Λ_{ik} are then dependent on the coefficients $\boldsymbol{U}_k$. Clearly

$$A_0 = A(\boldsymbol{U}_0), \quad \Lambda_{i0} = \Lambda_i(\boldsymbol{U}_0), \quad i = \pm 1, \pm 2. \tag{4.1-10}$$

In the case described by equations (4.1-1) and (4.1-2) we use the following notations

$$\boldsymbol{U}(s_1, s_2) = (\lambda(s_1, s_2)\cos\theta(s_1, s_2),\ \lambda(s_1, s_2)\sin\theta(s_1, s_2),\ u(s_1, s_2),\ v(s_1, s_2))^T.$$

The appropriate expansions are

$$\lambda(s_1,s_2) = \lambda_0 + \varepsilon\lambda_1(s_1,s_2) + \varepsilon^2\lambda_2(s_1,s_2) + \cdots ,$$
$$\theta(s_1,s_2) = \varepsilon\theta_1(s_1,s_2) + \varepsilon^2\theta_2(s_1,s_2) + \cdots ,$$
$$u(s_1,s_2) = \varepsilon u_1(s_1,s_2) + \varepsilon^2 u_2(s_1,s_2) + \cdots ,$$
$$v(s_1,s_2) = \varepsilon v_1(s_1,s_2) + \varepsilon^2 v_2(s_1,s_2) + \cdots ,$$
$$(4.1\text{-}11)$$

so that

$$\begin{aligned}\lambda\cos\theta &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2(\lambda_2 - \lambda_0\tfrac{\theta_1^2}{2}) + \ldots,\\ \lambda\sin\theta &= \varepsilon\lambda_0\theta_1 + \varepsilon^2(\lambda_0\theta_2 + \lambda_1\theta_1) + \ldots,\end{aligned} \qquad (4.1\text{-}12)$$

and consequently

$$\boldsymbol{U}_0 = (\lambda_0, 0, 0, 0)^T,$$
$$\boldsymbol{U}_1 = (\lambda_1, \lambda_0\theta_1, u_1, v_1)^T,$$
$$\boldsymbol{U}_2 = (\lambda_2 - \lambda_0\theta_1^2/2, \lambda_0\theta_2 + \lambda_1\theta_1, u_2, v_2)^T,$$
$$\Lambda_{10} = (T_0')^{1/2}, \quad \Lambda_{11} = \lambda_1 T_0''/2(T_0')^{1/2},$$
$$\Lambda_{12} = T_0''\lambda_2/2(T_0')^{1/2} + \lambda_1^2\{\frac{2T_0'''}{(T_0')^{1/2}} - \frac{(T_0'')^2}{(T_0')^{3/2}}\}/8,$$
$$\Lambda_{20} = (T_0/\lambda_0)^{1/2}, \quad \Lambda_{21} = \frac{\lambda_1}{2}\,(\frac{T_0}{\lambda_0})^{1/2}(\frac{T_0'}{T_0} - \frac{1}{\lambda_0}),$$
$$\Lambda_{22} = (\frac{T_0}{\lambda_0})^{1/2}\,\{\frac{\lambda_2}{2}\,(\frac{T_0'}{T_0} - \frac{1}{\lambda_0}) + \frac{\lambda_1^2}{8}\,[(\frac{2T_0''}{T_0} - \frac{4T_0'}{\lambda_0 T_0} + \frac{4}{\lambda_0^2}) - (\frac{T_0'}{T_0} - \frac{1}{\lambda_0})^2]\},$$
$$(4.1\text{-}13)$$

where

$$T_0^{(n)} = \frac{d^n}{d\lambda^n}\,T(\lambda)|_{\lambda=\lambda_0}.$$

The required terms in the expansion of the matrix A are

$$
\begin{aligned}
&(A_0)_{13} = (A_0)_{24} = -1, \quad (A_0)_{31} = -(\Lambda_{10})^2, \quad (A_0)_{42} = -(\Lambda_{20})^2, \\
&(A_1)_{31} = -2\Lambda_{10}\Lambda_{11}, \quad (A_1)_{42} = -2\Lambda_{20}\Lambda_{21}, \\
&(A_1)_{32} = (A_1)_{41} = -\{(\Lambda_{10})^2 - (\Lambda_{20})^2\}\theta_1, \\
&(A_2)_{31} = (\Lambda_{11})^2 + 2\Lambda_{10}\Lambda_{12} - \theta_1^2\{(\Lambda_{10})^2 - (\Lambda_{20})^2\}, \\
&(A_2)_{42} = (\Lambda_{21})^2 + 2\Lambda_{20}\Lambda_{22} + \theta_1^2\{(\Lambda_{10})^2 - (\Lambda_{20})^2\}, \\
&(A_2)_{32} = (A_2)_{41} = -2\theta_1\{\Lambda_{10}\Lambda_{11} - \Lambda_{20}\Lambda_{21}\} - (\theta_2 - \frac{\theta_1}{2})^2((\Lambda_{10})^2 - (\Lambda_{20})^2),
\end{aligned}
\tag{4.1-14}
$$

with other entries to this order being zero. The appropriate left and right eigenvectors evaluated at $\boldsymbol{U}_0$ are easily obtained from equation (3.1-16).

Equation (4.1-2) can be written in terms of the variables s_1 and s_2. Ahead of the fastest characteristic through the origin is the undisturbed region where $\boldsymbol{U} = \boldsymbol{u}_0 = (\lambda_0, 0, 0, 0)^T$ and so if $\Lambda_{10} > \Lambda_{20}$ we have

$$\boldsymbol{U}(0, s_2) = \boldsymbol{u}_0 = (\lambda_0, 0, 0, 0)^T, \qquad (s_2 < 0). \tag{4.1-15}$$

However, if $\Lambda_{10} < \Lambda_{20}$ then we have:

$$\boldsymbol{U}(s_1, 0) = \boldsymbol{u}_0 = (\lambda_0, 0, 0, 0)^T, \qquad (s_1 < 0). \tag{4.1-16}$$

The second and third equations in (4.1-2) can be written as:

$$
\begin{aligned}
u(s,s) &= 0, \qquad s \geq 0, \\
v(s,s) &= \varepsilon f(s), \quad s \geq 0.
\end{aligned}
\tag{4.1-17}
$$

Substituting these expressions into equations (4.1-3) and (4.1-8) gives a sequence of approximations. The lowest order approximation gives the constant state and constant characteristics. We note that $\boldsymbol{U}_0 = \boldsymbol{u}_0$ satisfies equation (4.1-7) automatically. To order ε^0 we have

$$X_{0,s_2} - \Lambda_{10} t_{0,s_2} = 0, \quad X_0(s,s) = 0,$$
$$X_{0,s_1} - \Lambda_{20} t_{0,s_1} = 0, \quad t_0(s,s) = s, \tag{4.1-18}$$

and by equation (4.1-18), we can conclude that

$$X_0 - \Lambda_{10} t_0 = -\Lambda_{10} s_1,$$
$$X_0 - \Lambda_{20} t_0 = -\Lambda_{20} s_2, \tag{4.1-19}$$

and so we can solve for X_0, t_0 from equation (4.1-19). To order ε^0 the solutions are:

$$\boldsymbol{U}_0 = \boldsymbol{u}_0, \quad X_0 = \frac{\Lambda_{10}\Lambda_{20}}{\Lambda_{10} - \Lambda_{20}}(s_1 - s_2), \quad t_0 = \frac{\Lambda_{10}s_1 - \Lambda_{20}s_2}{\Lambda_{10} - \Lambda_{20}} \,. \tag{4.1-20}$$

For the n^{th} approximation with $n \geq 1$, equations (4.1-3) and (4.1-7) give

$$t_{0,s_2}(A_0 - \Lambda_{10} I)\boldsymbol{U}_{n,s_1} - t_{0,s_1}(A_0 - \Lambda_{20} I)\boldsymbol{U}_{n,s_2} = \boldsymbol{F}_n, \tag{4.1-21}$$

$$X_{n,s_2} - \sum_{r=0}^{n} \Lambda_{1r} t_{r,s_2} = 0,$$
$$t_{n,s_1} - \sum_{r=0}^{n} \Lambda_{2r} t_{r,s_1} = 0, \tag{4.1-22}$$

where

$$\boldsymbol{F}_n = \sum_{k=1}^{n-1} \Big\{ \sum_{r=0}^{k} t_{r,s_2}(A_{k-r} - \Lambda_{1\;k-r} I)\boldsymbol{U}_{n-k,s_1}$$
$$- \sum_{r=0}^{k} t_{r,s_1}(A_{k-r} - \Lambda_{2\;k-r} I)\boldsymbol{U}_{n-k,s_2} \Big\}. \tag{4.1-23}$$

Since $\boldsymbol{U}_0$ is constant the term for $k = n$ drops out of equation (4.1-23) and $\boldsymbol{F}_n$ depends only on approximations of order $< n$.

If $\boldsymbol{L}_i, \boldsymbol{R}_i$ denote the left and right eigenvectors of A corresponding to eigenvalues Λ_i we have

$$\boldsymbol{L}_i \cdot \boldsymbol{R}_j = 0, \quad i \neq j. \tag{4.1-24}$$

In particular, we write the solution of equation (4.1-21) in terms of these eigenvectors evaluated at $\boldsymbol{U}_0$. We then renumber the right eigenvectors with $\boldsymbol{R}_1 = \boldsymbol{R}_1(\boldsymbol{U}_0)$, $\boldsymbol{R}_2 = \boldsymbol{R}_2(\boldsymbol{U}_0)$, $\boldsymbol{R}_3 = \boldsymbol{R}_{-1}(\boldsymbol{U}_0)$, $\boldsymbol{R}_4 = \boldsymbol{R}_{-2}(\boldsymbol{U}_0)$, and similarly for the left eigenvectors. With $\boldsymbol{F}_n$ known we may write

$$\boldsymbol{U}_n = \sum_{i=1}^{4} P_i^{(n)}(s_1, s_2)\boldsymbol{R}_i, \quad \boldsymbol{F}_n = \sum_{i=1}^{4} F_i^{(n)}(s_1, s_2)\boldsymbol{R}_i. \tag{4.1-25}$$

Multiplying equation (4.1-21) by each $\boldsymbol{L}_i$, $i = 1, 2, 3, 4$ in turn gives

$$\begin{aligned}
t_{0,s_1}(\Lambda_{10} - \Lambda_{20})P_{1,s_2}^{(n)} &= -F_1^{(n)}, \\
t_{0,s_2}(\Lambda_{10} - \Lambda_{20})P_{2,s_1}^{(n)} &= -F_2^{(n)}, \\
2\Lambda_{10}t_{0,s_2}P_{3,s_1}^{(n)} - t_{0,s_1}(\Lambda_{10} + \Lambda_{20})P_{3,s_2}^{(n)} &= -F_3^{(n)}, \\
(\Lambda_{10} + \Lambda_{20})t_{0,s_2}P_{4,s_1}^{(n)} - 2\Lambda_{20}t_{0,s_2}P_{4,s_2}^{(n)} &= -F_4^{(n)}.
\end{aligned} \tag{4.1-26}$$

The last two of equations (4.1-26) can be brought to a more compact form by introducing "backward" characteristics

$$\xi = \mu\left(s_1 - \frac{\nu s_2}{\Lambda_{10}}\right), \quad \eta = \mu\left(\frac{\nu s_1}{\Lambda_{20}} - s_2\right) \tag{4.1-27}$$

where

$$\mu = \frac{\Lambda_{10} + \Lambda_{20}}{\Lambda_{10} - \Lambda_{20}}, \quad \nu = \frac{2\Lambda_{10}\Lambda_{20}}{\Lambda_{10} + \Lambda_{20}}.$$

Then we have

$$\Lambda_{10}P_{1,s_2}^{(n)} = -F_1^{(n)}, \quad \Lambda_{20}P_{2,s_1}^{(n)} = F_2^{(n)}, \quad \Lambda_{10}P_{3,\xi}^{(n)} = -F_3^{(n)}, \quad \Lambda_{20}P_{4,\eta}^{(n)} = F_4^{(n)}. \tag{4.1-28}$$

The problem then is essentially reduced to quadratures once appropriate initial and boundary conditions are imposed. The inclusion of the variables ξ, η causes the problem to become complicated after a number of iterations and we implement the procedure for our problem to order ε^2 below.

If $n = 1$, by (4.1-23), we have $\boldsymbol{F}_1 = 0$, so by (4.1-28), we have

$$P_{1,s_2}^{(1)} = 0, \quad P_{2,s_1}^{(1)} = 0, \quad P_{3,\xi}^{(1)} = 0, \quad P_{4,\eta}^{(1)} = 0, \tag{4.1-29}$$

and from the first two equation of (4.1-29), we have

$$P_1^{(1)} = P_1^{(1)}(s_1), \quad P_2^{(1)} = P_2^{(1)}(s_2). \tag{4.1-30}$$

If $\Lambda_{10} > \Lambda_{20}$ then we have

$$\boldsymbol{U}(0, s_2) = \boldsymbol{u}_0 = (\lambda_0, 0, 0, 0)^T, \qquad (s_2 \le 0), \tag{4.1-15}$$

and to order ε the solution of equation (4.1-15) is then given by

$$\boldsymbol{U}_1(0, s_2) = 0, \quad (s_2 \le 0). \tag{4.1-31}$$

From the third and fourth equations of (4.1-29), using (4.1-31) we have

$$\begin{aligned} 2\Lambda_{20}P_{3,s_1}^{(1)} - (\Lambda_{10} + \Lambda_{20})P_{3,s_2}^{(n)} &= 0, \\ P_3^{(1)}(0, s_2) = 0, &\qquad (s_2 \le 0) \end{aligned} \tag{4.1-32}$$

and

$$(\Lambda_{10} + \Lambda_{20})P^{(1)}_{4,s_1} - 2\Lambda_{20}P^{(1)}_{4,s_2} = 0.$$
$$P_4^{(1)}(0, s_2) = 0, \qquad (s_2 \leq 0). \tag{4.1-33}$$

The solution of (4.1-32) and (4.1-33) is found to be

$$P_3^{(1)}(s_1, s_2) = 0, \qquad P_4^{(1)}(s_1, s_2) = 0 \tag{4.1-34}$$

and so

$$\boldsymbol{U}_1 = P_1^{(1)}(s_1)\boldsymbol{R}_1 + P_2^{(1)}(s_2)\boldsymbol{R}_2 \tag{4.1-35}$$

or

$$\boldsymbol{U}_1 = (\lambda_0 P_1^{(1)}(s_1), -\lambda_0 P_2^{(1)}(s_2), -\lambda_0\Lambda_{10}P_1^{(1)}(s_1), \lambda_0\Lambda_{20}P_2^{(1)}(s_2))^T. \tag{4.1-36}$$

Recall that

$$\boldsymbol{U}_1 = (\lambda_1, \quad \lambda_0\theta_1, \quad u_1, \quad v_1)^T, \tag{4.1-37}$$

and since to order ε we have

$$u_1(s, s) = 0, \qquad s \geq 0,$$
$$v_1(s, s) = f(s), \quad s \geq 0, \tag{4.1-38}$$

it follows from equations (4.1-31) and (4.1-38) that

$$P_1^{(1)}(s_1) \equiv 0, \qquad P_2^{(1)} = \frac{f(s_2)}{\lambda_0\Lambda_{20}} \tag{4.1-39}$$

and consequently

$$\boldsymbol{U}_1 = (0, \quad -\frac{f(s_2)}{\Lambda_{20}}, \quad 0, \quad f(s_2))^T,$$

or alternatively

$$\boldsymbol{U}_1 = \frac{f(s_2)}{\lambda_0\Lambda_{20}}\boldsymbol{R}_2, \tag{4.1-40}$$

and

$$\lambda_1 \equiv 0, \quad \theta_1 = -\frac{f(s_2)}{\lambda_0\Lambda_{20}}, \quad u_1 = 0, \quad v_1 = f(s_2). \tag{4.1-41}$$

Since $\lambda_1 \equiv 0$, the ε order of (4.1-3) is given by

$$\begin{aligned} X_{1,s_2} - \Lambda_{10}t_{1,s_2} &= 0, \quad X_1(s,s) = 0, \\ X_{1,s_1} - \Lambda_{20}t_{1,s_1} &= 0, \quad t_1(s,s) = 0, \end{aligned} \tag{4.1-42}$$

and the solution of equation (4.1-42) is

$$\begin{aligned} X_1 - \Lambda_{10}t_1 &= 0, \\ X_1 - \Lambda_{20}t_1 &= 0. \end{aligned} \tag{4.1-43}$$

Since $\Lambda_{10} \neq \Lambda_{20}$, equation (4.1-43) gives

$$X_1 \equiv 0, \qquad t_1 \equiv 0. \tag{4.1-44}$$

Next, we will consider the solution to order ε^2. Since $t_1 \equiv 0$ and $\boldsymbol{U}_1$ is a function of s_2 only, then by equation (4.1-23) we have

$$\boldsymbol{F}_2 = -t_{0,s_1}(A_1 - \Lambda_{21}I)\boldsymbol{U}_{1,s_2}. \tag{4.1-45}$$

Since $\lambda_1 = 0$ then using equations (4.1-13) we have $\Lambda_{21} = 0$ and substituting $A_1, \boldsymbol{U}_1$ into equation (4.1-45), we get

$$\boldsymbol{F}_2 = -t_{0,s_1}(0, 0, -\frac{(\Lambda_{20}^2 - \Lambda_{10}^2)\theta_1 f'(s_2)}{\Lambda_{20}}, \quad 0)^T, \tag{4.1-46}$$

where $\theta_1 = -\frac{f(s_2)}{\lambda_0\Lambda_{20}}$. Using the second equation of (4.1-25), we have

$$F_i^{(2)} = \frac{\boldsymbol{L}_i \cdot \boldsymbol{F}_2}{\boldsymbol{L}_i \cdot \boldsymbol{R}_i}, \qquad (\text{no sum}, i = 1,2,3,4,) \tag{4.1-47}$$

after some calculation, we obtain

$$\begin{aligned} F_1^{(2)} &= \frac{t_{0,s_1}(\Lambda_{10}^2 - \Lambda_{20}^2) f(s_2) f'(s_2)}{2\lambda_0^2 \Lambda_{10} \Lambda_{20}^2}, \\ F_2^{(2)} &= 0, \\ F_3^{(2)} &= \frac{t_{0,s_1}(\Lambda_{10}^2 - \Lambda_{20}^2) f(s_2) f'(s_2)}{2\lambda_0^2 \Lambda_{10} \Lambda_{20}}, \\ F_4^{(2)} &= 0, \end{aligned} \tag{4.1-48}$$

so by equations (4.1-26) we have

$$\begin{aligned} P_{1,s_2}^{(2)} &= -\frac{(\Lambda_{10} + \Lambda_{20}) f(s_2) f'(s_2)}{2\lambda_0^2 \Lambda_{10} \Lambda_{20}^2}, \\ P_{2,s_1}^{(2)} &= 0, \\ P_{3,s_1}^{(2)} + \frac{\Lambda_{10} + \Lambda_{20}}{2\Lambda_{20}} P_{3,s_2}^{(2)} &= \frac{(\Lambda_{20}^2 - \Lambda_{10}^2) f(s_2) f'(s_2)}{4\lambda_0^2 \Lambda_{10} \Lambda_{20}^3}, \\ P_{4,s_1}^{(2)} + \frac{2\Lambda_{10}}{\lambda_{10} + \Lambda_{20}} P_{4,s_2}^{(2)} &= 0. \end{aligned} \tag{4.1-49}$$

The first two equations give

$$\begin{aligned} P_1^{(2)} &= -\frac{(\Lambda_{10} + \Lambda_{20}) f^2(s_2)}{4\lambda_0^2 \Lambda_{10} \Lambda_{20}^2} + g(s_1), \\ P_2^{(2)} &= h(s_2), \end{aligned} \tag{4.1-50}$$

where $g(s_1)$ and $h(s_2)$ are functions to be determined. Since to order ε^2 equation (4.1-15) gives

$$P_i^{(2)}(0, s_2) = 0, \qquad (s_2 \le 0, \quad i = 1,2,3,4), \tag{4.1-51}$$

then the third and fourth equations of (4.1-49) and (4.1-51) give

$$P_3^{(2)} = \frac{(\Lambda_{20} - \Lambda_{10}) f^2(s_2)}{4\lambda_0^2 \Lambda_{10} \Lambda_{20}^2}, \qquad P_4^{(2)} = 0. \tag{4.1-52}$$

Using equation (4.1-25) and the fact that $P_4^{(2)} = 0$, we have

$$\boldsymbol{U}_2 = (\lambda_0(P_1^{(2)} + P_3^{(2)}), -\lambda_0 P_2^{(2)}, -\Lambda_{10}\lambda_0(P_1^{(2)} - P_3^{(2)}), \lambda_0\Lambda_{20} h(s_2))^T. \tag{4.1-53}$$

Since $\lambda_1 = 0$, $\theta_1 = \frac{f(s_2)}{\lambda_0 \Lambda_{20}}$ then using the third equation in (4.1-13), we get

$$\boldsymbol{U}_2 = (\lambda_2 - \frac{f^2(s_2)}{2\lambda_0 \Lambda_{20}^2}, \lambda_0 \theta_2, u_2, v_2)^T. \tag{4.1-54}$$

The ε^2 order of (4.1-17) is

$$u_2(s,s) = 0, \qquad v_2(s,s) = 0, \qquad (s \geq 0) \tag{4.1-55}$$

then by equations (4.1-53), (4.1-54) and (4.1-55) we have

$$h(s) = 0, \qquad P_1^{(2)}(s,s) = P_3^{(2)}(s,s), \qquad (s \geq 0) \tag{4.1-56}$$

so that

$$g(s_1) = \frac{f^2(s_1)}{2\lambda_0^2 \Lambda_{10} \Lambda_{20}}. \tag{4.1-57}$$

The solution to order ε^2 of equation (4.1-9) is then

$$\begin{aligned} v_2 &= 0, \qquad \theta_2 = 0 \\ u_2 &= \frac{f^2(s_2) - f^2(s_1)}{2\lambda_0 \Lambda_{20}} \\ \lambda_2 &= \frac{f^2(s_1)}{2\lambda_0 \Lambda_{10} \Lambda_{20}} \\ \boldsymbol{U}_2 &= P_1^{(2)} \boldsymbol{R}_1 + P_3^{(2)} \boldsymbol{R}_3 \end{aligned} \tag{4.1-58}$$

where $P_1^{(2)}, P_3^{(2)}$ are given by

$$P_1^{(2)} = \{f^2(s_1) - \frac{(\Lambda_{10} + \Lambda_{20})}{2\Lambda_{20}} f^2(s_2)\}/2\lambda_0^2\Lambda_{10}\Lambda_{20},$$
$$P_3^{(2)} = -(\Lambda_{10} - \Lambda_{20})f^2(s_2)/4\lambda_0^2\Lambda_{10}\Lambda_{20}^2. \tag{4.1-59}$$

If we define

$$a = -T_0''\Lambda_{20}/2\Lambda_{10}(\Lambda_{10} - \Lambda_{20}),$$
$$b = \Lambda_{10}(\Lambda_{10} + \Lambda_{20})/2\lambda_0\Lambda_{20}, \tag{4.1-60}$$

then to order ε^2 the solution of equation (4.1-3) can be written as

$$X_{2,s_2} - \Lambda_{10}t_{2,s_2} = a\lambda_2, \quad X_2(s,s) = 0,$$
$$X_{2,s_1} - \Lambda_{20}t_{2,s_1} = b\lambda_2, \quad t_2(s,s) = 0, \tag{4.1-61}$$

where λ_2 is given by equation (4.1-58). We can solve equation (4.1-61) and find that

$$X_2 = K\{a\Lambda_{20}f^2(s_1)(s_1 - s_2) + b\Lambda_{10}\int_{s_2}^{s_1} f^2(p)dp\}$$
$$t_2 = K\{af^2(s_1)(s_1 - s_2) + b\int_{s_2}^{s_1} f^2(p)dp\} \tag{4.1-62}$$

where $K = 1/2\lambda_0\Lambda_{10}\Lambda_{20}(\Lambda_{10} - \Lambda_{20})$ and a and b are given by (4.1-60).

Next, we consider the equation for the characteristics. First we consider the equation for $s_1 = constant$. Since

$$X - \Lambda_{10}t = (X_0 + \varepsilon X_1 + \cdots) - \Lambda_{10}(t_0 + \varepsilon t_1 + \cdots) \tag{4.1-63}$$

and $X_0 - \Lambda_{10}t_0 = -\Lambda_{10}s_1, \quad X_1 = t_1 = 0,$ then

$$X - \Lambda_{10}t = -\Lambda_{10}s_1 + \varepsilon^2(X_2 - \Lambda_{10}t_2) + \cdots . \tag{4.1-64}$$

Substituting equation (4.1-62) into equation (4.1-64), we have

$$X - \Lambda_{10}t = -\Lambda_{10}s_1 + \varepsilon^2 \frac{af^2(s_1)}{2\lambda_0\Lambda_{10}\Lambda_{20}}(s_2 - s_1) + \cdots \qquad (4.1\text{-}65)$$

and since

$$X = X_0 + \varepsilon^2 X_2 + \cdots = \frac{\Lambda_{10}\Lambda_{20}(s_1 - s_2)}{\Lambda_{10} - \Lambda_{20}} + \varepsilon^2 X_2 + \cdots \qquad (4.1\text{-}66)$$

we have

$$s_1 - s_2 = \frac{\Lambda_{10} - \Lambda_{20}}{\Lambda_{10}\Lambda_{20}}(X - \varepsilon^2 X_2 - \cdots). \qquad (4.1\text{-}67)$$

On substituting equation (4.1-67) into equation (4.1-65) and recalling that a is given by equation (4.1-60), we obtain the approximate equation for the s_1 characteristics

$$X\left\{1 + \frac{\varepsilon^2 a(\Lambda_{10} - \Lambda_{20})f^2(s_1)}{2\lambda_0(\Lambda_{10}\Lambda_{20})^2}\right\} = \Lambda_{10}(t - s_1). \qquad (4.1\text{-}68)$$

Now consider the equation for the s_2 characteristic. Since

$$X - \Lambda_{20}t = (X_0 + \varepsilon X_1 + \cdots) - \Lambda_{20}(t_0 + \varepsilon t_1 + \cdots) \qquad (4.1\text{-}69)$$

and $X_0 - \Lambda_{20}t_0 = -\Lambda_{20}s_2$, $X_1 = t_1 = 0$, and we have $X_2 - \Lambda_{20}t_2 = \frac{b}{2\lambda_0\Lambda_{10}\Lambda_{20}}\int_{s_2}^{s_1} f^2(p)dp$, then by equation (4.1-69), we have

$$X - \Lambda_{20}t = -\Lambda_{20}s_2 + \frac{\varepsilon^2 b}{2\lambda_0\Lambda_{10}\Lambda_{20}}\int_{s_2}^{s_1} f^2(p)dp. \qquad (4.1\text{-}70)$$

Neglecting the higher order terms, equations (4.1-67) and (4.1-70) give

$$X - \frac{\varepsilon^2(\Lambda_{10} + \Lambda_{20})}{4\lambda_0^2\Lambda_{20}^2}\int_{s_2}^{s_2 + \frac{\Lambda_{10} - \Lambda_{20}}{\Lambda_{10}\Lambda_{20}}X} f^2(p)dp = \Lambda_{20}(t - s_2). \qquad (4.1\text{-}71)$$

The above results are obtained for the case $\Lambda_{10} > \Lambda_{20}$. For the case $\Lambda_{10} < \Lambda_{20}$, the condition (4.1-15) should be replaced by (4.1-16). From a similar analysis, the results for $\Lambda_{10} < \Lambda_{20}$ are exactly parallel to those for the case $\Lambda_{10} > \Lambda_{20}$.

To summarise, the approximation to order ε is given by

$$\lambda = \lambda_0, \quad \theta = -\frac{\varepsilon}{\Lambda_{20}} f(t - \frac{X}{\Lambda_{20}}), \quad u = 0, \quad v = \varepsilon f(t - \frac{X}{\Lambda_{20}}), \tag{4.1-72}$$

with the correction at order ε^2. We have

$$\begin{aligned} \lambda &= \lambda_0 + \frac{\varepsilon^2 f^2(s_1)}{2\lambda_0 \Lambda_{10} \Lambda_{20}}, & \theta &= -\frac{\varepsilon}{\lambda_0 \Lambda_{20}} f(s_2), \\ u &= -\frac{\varepsilon^2}{2\lambda_0 \Lambda_{20}} (f^2(s_1) - f^2(s_2)), & v &= \varepsilon f(s_2), \end{aligned} \tag{4.1-73}$$

where the characteristics are taken as

$$\begin{aligned} X \left\{1 + \frac{\varepsilon^2 a(\Lambda_{10} - \Lambda_{20}) f^2(s_1)}{2\lambda_0 (\Lambda_{10}\Lambda_{20})^2} \right\} &= \Lambda_{10}(t - s_1), \\ X - \frac{\varepsilon^2 (\Lambda_{10} + \Lambda_{20})}{4\lambda_0^2 \Lambda_{20}^2} \int_{s_2}^{s_2 + \frac{\Lambda_{10} - \Lambda_{20}}{\Lambda_{10}\Lambda_{20}} X} f^2(p) dp &= \Lambda_{20}(t - s_2). \end{aligned} \tag{4.1-74}$$

In the next section, we will compare the perturbation solution with the solution for the case of constant boundary conditions.

4.2 Comparison of the perturbation solution with the exact solution for the case of constant boundary conditions

As we mentioned earlier, the normal impact problem for a nonlinear elastic string can be solved in closed form as long as the boundary values are constant. We refer to these solutions as exact solutions (see Wegner, Haddow and Tait [29]).

In order to compare the perturbation solution obtained in section 4.1 with the results in [29], we will consider equation (4.1-1)

$$\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,X} = \boldsymbol{0} \tag{4.2-1}$$

under the conditions:

$$\begin{aligned} \boldsymbol{u}(X,0) = \boldsymbol{u}_0, \qquad u(0,t) = 0, \\ u(L,t) = v(L,t) = 0, \qquad t > 0, \end{aligned} \tag{4.2-2}$$

and

$$v(0,t) = \begin{cases} \varepsilon F(t), & 0 \le t \le d, \\ q = \varepsilon v_0, & t > d, \end{cases} \tag{4.2-3}$$

with $F(t)$ a smooth monotone increasing function on $(0,d)$ with $F(0) = 0$, $F(d) = v_0$. We may take, for example, $F(t) = \frac{v_0 t}{d}$. This preliminary form allows a transition region and permits a comparison of the perturbation solution with the exact solution in the constant boundary condition case $d = 0$.

Let $\lambda_m(X,t)$ denote the maximum stretch at any point of the string and $\theta_m(X,t)$ denote the corresponding angle of inclination. In the following, we will compare the perturbation solution with the exact one, case by case. We refer to Fig.3.2-1 for the interpretations of $\lambda_{c1}, \lambda_{c2}$ and λ_i used below.

Consider first the case $1 \le \lambda_0 < \lambda_m < \lambda_{c1}$, so that $\Lambda_{10} > \Lambda_{20}$, $T_0'' < 0$ and thus $a > 0$. The exact solution for the case $d = 0$ is given as follows (see also Fig.3.4-1)

Region 1: $X/t \ge \Lambda_{10}$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0.$$

Region 2: $\Lambda_{10} \ge X/t \ge \Lambda_{1m}$,

$$X/t = \Lambda_1(\lambda), \qquad \theta = v = 0, \qquad u = -I(\lambda).$$

Region 3: $\Lambda_{1m} \geq X/t \geq \Lambda_{2m}$,

$$\lambda = \lambda_m, \qquad \theta = v = 0, \qquad u = -I(\lambda_m).$$

Region 4: $\Lambda_{2m} > X/t \geq 0$,

$$\lambda = \lambda_m, \qquad \theta = \theta_m, \quad u = 0, \quad v = q, \tag{4.2-4}$$

where $\Lambda_{im} = \Lambda_i(\lambda_m)$, $i = 1,2$, and $I(\lambda) = \int_{\lambda_0}^{\lambda} \Lambda_1(p)dp$. λ_m and θ_m can be determined from the equations

$$q^2 = \int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp\{2\lambda_m \Lambda_2(\lambda_m) - \int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp\}, \tag{4.2-5}$$

and

$$\sin\theta_m = -q/\lambda_m \Lambda_2(\lambda_m). \tag{4.2-6}$$

Consider the corresponding perturbation solution given by equations (4.1-73) and (4.1-74). For the case $d > 0$, the solution for λ starts from an initial state $\lambda = \lambda_0$, this state is then connected to a simpl. ave region by a straight line $l_1 : X = \Lambda_{10}t$. For small ε, the simple wave region is bounded by the lines $X = 0$, l_1 and $l_2 : X = \Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^3 \Lambda_{20}})(t - d)$. The final state $\lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_{10} \Lambda_{20}}$ is connected to the simple wave region through the line l_2. As $d \to 0^+$, the simple wave degenerates to a simple expansion fan through the origin.

We notice that θ has a constant value on the curves $s_2 = constant$. Differentiating equation $(4.1 - 74)_{(2)}$ with respect to t shows that for $t \geq d$, the constant s_2 curves have the same slope :

$$\frac{dx}{dt} = \Lambda_{20}\ \{1 + cq^2\}, \tag{4.2-7}$$

where

$$c = \frac{(\Lambda_{10}^2 - \Lambda_{20}^2)}{4\lambda_0^2 \Lambda_{10} \Lambda_{20}^3} \ .$$

For $0 \leq t < d$, the slope of an s_2 curve is less than that given by (4.2-7). However, as X, t increase, they approach the same limit value given by (4.2-7). On the other hand, differentiating equation $(4.1-74)_{(2)}$ with respect to s_2 shows that these curves do not intersect, at least for ε small. As a result, as $d \to 0^+$, the region where θ varies degenerates to a line with slope given by equation (4.2-7). A shock is not formed but a line of discontinuity across which θ jumps from 0 to θ_m. If ε is small, we can write down the perturbation solution for the case $d \to 0^+$ as follows

Region 1: $X/t \geq \Lambda_{10}$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0$$

Region 2: $\Lambda_{10} \geq X/t \geq \Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^3 \lambda_{20}})$,

$$X/t = \Lambda_{10}(1 + \frac{T_0''(\lambda - \lambda_0)}{2\Lambda_{10}^2}), \qquad \theta = v = 0, \quad u = -\Lambda_{10}(\lambda - \lambda_0),$$

Region 3: $\Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^3 \lambda_{20}}) \geq X/t > \Lambda_{20}(1 + cq^2)$,

$$\lambda = \lambda_m, \qquad \theta = v = 0, \qquad u = -\Lambda_{10}(\lambda_m - \lambda_0),$$

Region 4: $\Lambda_{20}(1 + cq^2) \geq X/t \geq 0$,

$$\lambda = \lambda_m, \qquad \theta = \theta_m = -\frac{q}{\lambda_0 \Lambda_{20}}, \quad u = 0, \quad v = q, \tag{4.2-8}$$

where

$$\lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_{10} \lambda_{20}}, \qquad \theta_m = -\frac{q}{\lambda_0 \Lambda_{20}}. \tag{4.2-9}$$

If we apply the expressions given earlier, it can be seen that (4.2-9) agrees with (4.2-5) and (4.2-6) to order ε^2. We also notice that, by using (4.2-9) we find that

$$\Lambda_{1m} = \Lambda_{10} + \frac{T_0''(\lambda_m - \lambda_0)}{2\Lambda_{10}} + \cdots \approx \Lambda_{10} + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^2 \Lambda_{20}} \tag{4.2-10}$$

and

$$\Lambda_{2m} = \Lambda_{20} + \frac{(T_0'\lambda_0 - T_0)(\lambda_m - \lambda_0)}{2\Lambda_{20}} + \cdots \approx \Lambda_{20}(1 + cq^2). \tag{4.2-11}$$

We can see that the corresponding slope in the perturbation solution agrees with the exact one to order ε^2. Therefore, by the above analysis, we conclude that the perturbation solution agrees with the exact solution to order ε^2 for the case $1 \le \lambda_0 < \lambda_m < \lambda_{c1}$.

Now we consider the case $\lambda_{c1} \le \lambda_0 < \lambda_m < \lambda_i$. For this case, we have $\Lambda_{20} > \Lambda_{10}, a < 0$. The exact solution for the case $d = 0$ is given as follows (see also Fig.3.4-3),

Region 1: $X/t \ge \Lambda_{20}$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0.$$

Region 2: $\Lambda_{20} \ge X/t \ge \Lambda_{10}$,

$$\lambda = \lambda_0, \qquad \theta = \theta_m, \quad u = I(\lambda_m)\cos\theta_m, \quad v = q + I(\lambda_m)\sin\theta_m.$$

Region 3: $\Lambda_{10} \ge X/t \ge \Lambda_{1m}$,

$$X/t = \Lambda_1(\lambda), \qquad \theta = \theta_m,$$
$$u = (I(\lambda_m) - I(\lambda))\cos\theta_m,$$
$$v = q + (I(\lambda_m) - I(\lambda))\sin\theta_m.$$

Region 4: $\Lambda_{1m} > X/t \geq 0$,

$$\lambda = \lambda_m, \qquad \theta = \theta_m, \quad u = 0, \quad v = q, \tag{4.2-12}$$

where λ_m, θ_m are determined by

$$q^2 = \int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp\{2\lambda_0\Lambda_{20} + \int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp\}, \tag{4.2-13}$$

and

$$\sin\theta_m = -q/\big(\lambda_0\Lambda_{20} + \int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp\big). \tag{4.2-14}$$

The corresponding approximation for $d \to 0^+$ is

Region 1: $X/t \geq \Lambda_{20}$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0.$$

Region 2: $\Lambda_{20} \geq X/t \geq \Lambda_{10}$,

$$\lambda = \lambda_0, \quad \theta = \theta_m, \quad u = \frac{q^2}{2\lambda_0\Lambda_{20}}, \quad v = q,$$

Region 3: $\Lambda_{10} \geq X/t > \Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0\Lambda_{10}^3\lambda_{20}})$,

$$\begin{aligned} X/t &= \Lambda_{10}(1 + \frac{T_0''(\lambda - \lambda_0)}{2\Lambda_{10}^2}), \\ \theta &= \theta_m, \quad v = q, \\ u &= -\Lambda_{10}(\lambda - \lambda_0) + \frac{q^2}{2\lambda_0\Lambda_{20}}, \end{aligned}$$

Region 4: $\Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0\Lambda_{10}^3\lambda_{20}}) \geq X/t > 0$,

$$\lambda = \lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0\Lambda_{10}\Lambda_{20}}, \quad \theta = \theta_m = -\frac{q}{\lambda_0\Lambda_{20}}, \quad u = 0, \quad v = q. \tag{4.2-15}$$

It can be shown that the exact solution and the approximate solution are the same to order ε^2.

The next case to be considered is $\lambda_0 < \lambda_i < \lambda_m$. For this case the above approximation is not sufficiently accurate and this can be seen by considering the constant boundary condition case. It is only necessary to discuss the λ values. The initial λ value $\lambda = \lambda_0$ is connected to a final λ value $\lambda = \lambda_m$ by, in general, an expansion fan which contracts to a shock on its trailing edge, see [29]. Equation $(4.1-74)_{(1)}$ on the other hand predicts only an expansion fan as $d \to 0^+$. The difficulty arises due to the change in sign of the second derivative of T. We exclude this case.

Now we consider the case $\lambda_i < \lambda_0 < \lambda_m < \lambda_{c2}$. For this case $T_0'' > 0$, $\Lambda_{10} < \Lambda_{20}$, so that $a < 0$.

The exact solution, for the case $d = 0$, consists of three regions (see also Fig.3.4-10)

Region 1: $X/t > \Lambda_{20}$

$$\lambda = \lambda_0, \qquad \theta = u = v = 0,$$

Region 2: $\Lambda_{20} > X/t > V_L(\lambda_0, \lambda_m)$

$$\begin{aligned} &\lambda = \lambda_0, \quad \theta = \theta_m, \\ &u = \Lambda_{20}\lambda_0(1 - \cos\theta_m), \quad v = -\Lambda_{20}\lambda_0 \sin\theta_m, \end{aligned}$$

Region 3: $V_L(\lambda_0, \lambda_m) \geq X/t \geq 0$

$$\lambda = \lambda_m, \quad \theta = \theta_m, \quad u = 0, \quad v = q \tag{4.2-16}$$

where λ_m, θ_m may be found from

$$\{(T(\lambda_m) - T(\lambda_0))(\lambda_m - \lambda_0)\}^{1/2} = \{v_0^2 + (\lambda_0 \Lambda_{20})^2\}^{1/2} - \lambda_0 \Lambda_{20}, \quad (4.2\text{-}17)$$

$$\sin\theta_m = -q/\{V_L(\lambda_0, \lambda_m)(\lambda_m - \lambda_0) + \lambda_0 \Lambda_{20}\}, \quad (4.2\text{-}18)$$

and

$$V_L(\lambda_0, \lambda) = \left(\frac{T(\lambda) - T(\lambda_0)}{\lambda - \lambda_0}\right)^{1/2}, \quad V_L(\lambda_0, \lambda) < \Lambda_{20}. \quad (4.2\text{-}19)$$

Consider the corresponding perturbation solution for $d > 0$. We consider first the s_1 curves on which $\lambda = constant$. By the first equation of (4.1-74), for $s_1 \leq 0$, the s_1 curves are described by $X = \Lambda_{10}(t - s_1)$. For $0 \leq s_1 < d$ with ε small, s_1 curves are described by $X = \Lambda_{10}(1 + \frac{T_0'' F^2(s_1)}{4\lambda_0 \Lambda_{10}^3 \Lambda_{20}})(t - s_1)$ and for $s_1 \geq d$, we have $X = \Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^3 \Lambda_{20}})(t - s_1)$. Since $T_0'' > 0$, it can be seen that as $d \to 0^+$, there is a region, bounded by $X = \Lambda_{10} t$ and $X = \Lambda_{10}(1 + \frac{T_0'' q^2}{4\lambda_0 \Lambda_{10}^3 \Lambda_{20}})t$, where λ has three possible values. This implies that a shock forms immediately. We can modify the perturbation solution as follows. Since the λ shock speed V_L for this case is given by (4.2-19) and since $\lambda - \lambda_0 = \frac{\varepsilon^2 f^2(s_1)}{2\lambda_0 \Lambda_{10} \Lambda_{20}}$ for small ε, we have

$$T(\lambda) = T_0 + T_0'(\lambda - \lambda_0) + \frac{T_0''}{2}(\lambda - \lambda_0)^2 + \cdots \quad (4.2\text{-}20)$$

so that

$$\begin{aligned} V_L(\lambda_0, \lambda) &= (T_0' + T_0''(\lambda - \lambda_0)/2 + \cdots)^{1/2} \\ &= \Lambda_{10} + \frac{T_0''}{4\Lambda_{10}}(\lambda - \lambda_0) + \cdots \\ &= \Lambda_{10} + \frac{T_0'' \varepsilon^2 f^2(s_1)}{8\lambda_0 \Lambda_{10}^2 \Lambda_{20}} + \cdots \end{aligned} \quad (4.2\text{-}21)$$

and so to order ε^2

$$V_L(\lambda_0, \lambda) = \frac{m_- + m_+}{2} \tag{4.2-22}$$

where m_-, m_+ are the slopes of s_1 curves on both sides of the shock path. The s_1 characteristics run into the λ shock path from both sides and the perturbation solution for the case $d \to 0^+$ can be written down as follows:

Region 1: $X/t > \Lambda_{20}$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0,$$

Region 2: $\Lambda_{20} > X/t > (\Lambda_{10} + \frac{T_0'' q^2}{8\lambda_0 \Lambda_{10}^2 \Lambda_{20}})$,

$$\lambda = \lambda_0, \quad \theta = \theta_m = -\frac{q}{\lambda_0 \Lambda_{20}},$$
$$u = \frac{q^2}{2\Lambda_{20}\lambda_0}, \quad v = q,$$

Region 3: $(\Lambda_{10} + \frac{T_0'' q^2}{8\lambda_0 \Lambda_{10}^2 \Lambda_{20}}) \geq X/t \geq 0$,

$$\lambda = \lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_{10} \Lambda_{20}}, \quad \theta = \theta_m = -\frac{q}{\lambda_0 \Lambda_{20}}, \quad u = 0, \quad v = q. \tag{4.2-23}$$

It can be shown that the modified perturbation solution agrees with the exact one to order ε^2.

We consider finally the case when $\lambda_{c2} < \lambda_0 < \lambda_m$. For this case, $T_0'' > 0$, $\Lambda_{10} > \Lambda_{20}$ and the exact solution for the case $d = 0$ is given as follows (see also Fig.3.4-12),

Region 1: $X/t > V_L(\lambda_0, \lambda_m)$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0,$$

Region 2: $V_L(\lambda_0, \lambda_m) > X/t > \Lambda_{2m}$,

$$\lambda = \lambda_m, \quad \theta = 0,$$
$$u = -V_L(\lambda_0, \lambda_m)(\lambda_m - \lambda_0), \quad v = 0,$$

Region 3: $\Lambda_{2m} \geq X/t \geq 0$,

$$\lambda = \lambda_m, \quad \theta = \theta_m, \quad u = 0, \quad v = q, \tag{4.2-24}$$

where $V_L(\lambda_0, \lambda_m)$ is given by (4.2-19). λ_m, θ_m are determined by

$$-V_L(\lambda_0, \lambda_m)(\lambda_m - \lambda_0) = \lambda_m \Lambda_{2m}(\cos\theta_m - 1),$$
$$\sin\theta_m = -\frac{q}{\lambda_m \Lambda_{2m}}. \tag{4.2-25}$$

The corresponding approximation as $d \to 0^+$ is given by:

Region 1: $X/t > (\Lambda_{10} + \frac{T_0'' q^2}{8\lambda_0 \Lambda_{10}^2 \Lambda_{20}})$,

$$\lambda = \lambda_0, \qquad \theta = u = v = 0,$$

Region 2: $(\Lambda_{10} + \frac{T_0'' q^2}{8\lambda_0 \Lambda_{10}^2 \Lambda_{20}}) > X/t > \Lambda_{20}(1 + cq^2)$,

$$\lambda = \lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_{10} \Lambda_{20}}, \quad \theta = v = 0, \quad u = -\Lambda_{10}(\lambda_m - \lambda_0),$$

Region 3: $\Lambda_{20}(1 + cq^2) \geq X/t \geq 0$,

$$\lambda = \lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_{10} \Lambda_{20}}, \quad \theta = \theta_m = -\frac{q}{\lambda_0 \Lambda_{20}}, \quad u = 0, \quad v = q. \tag{4.2-26}$$

It can be shown that the exact solution and the perturbation solution agree to order ε^2.

In the next section, we will apply the perturbation solution to the normal impact problem of the nonlinear elastic string.

4.3 Application of the perturbation solution to the impact problem of a nonlinear elastic string

With the restoring force taken into consideration, the impact problem of a nonlinear elastic string may be described by equations (4.2-1), (4.2-2) and the following conditions as $d \to 0^+$,

$$\begin{aligned} v(0,t) &= \varepsilon F(t), && 0 \le t \le d, \\ \frac{dv(0,t)}{dt} &= 2\alpha T\{\lambda(0,t)\}\sin\{\theta(0,t)\}, && t > d, \end{aligned} \tag{4.3-1}$$

with $F(t)$ as before a monotone increasing continuous function in $(0,d)$. We also require $F(0) = 0$, $\varepsilon F(d) = q$. Equations (4.3-1) can be rewritten in terms of the variables s_1, s_2 as

$$\begin{aligned} v(s,s) &= \varepsilon F(s), && 0 \le s \le d, \\ \frac{dv(s,s)}{ds} &= 2\alpha T\{\lambda(s,s)\}\sin\{\theta(s,s)\}, && s > d. \end{aligned} \tag{4.3-2}$$

To order ε the second equation of (4.3-2) gives

$$\frac{dv_1(s,s)}{ds} = 2\alpha T_0\theta_1(s,s) \tag{4.3-3}$$

where, from equation (4.1-41)

$$\theta_1 = -\frac{v_1}{\lambda_0\Lambda_{20}}. \tag{4.3-4}$$

It then follows that

$$\frac{dv_1}{ds} = -2\alpha\Lambda_{20}v_1. \tag{4.3-5}$$

Using equations (4.3-5), (4.3-2) and noting that $v_1(d) = v_0$, we have

$$v_1(s,s) = \begin{cases} F(s), & 0 \leq s \leq d, \\ v_0 \exp\{-2\alpha\Lambda_{20}(s-d)\}, & s > d. \end{cases} \tag{4.3-6}$$

If we define

$$f(t) = \begin{cases} F(t), & 0 \leq t \leq d, \\ v_0 \exp\{-2\alpha\Lambda_{20}(t-d)\}, & t > d, \end{cases} \tag{4.3-7}$$

we can apply the perturbation solution obtained in section 4.1 to this problem.

First we consider the case $1 \leq \lambda_0 < \lambda_m < \lambda_{c1}$, from the discussion above it is clear that we may allow $d \to 0^+$. The solution for λ starts from the constant state in the region $X/t \geq \Lambda_{10}$ and this constant state is connected by an expansion fan through the origin. The λ value increases from λ_0 to $\lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0\Lambda_{10}\Lambda_{20}}$, or

$$\lambda = \lambda_0 + \frac{\varepsilon^2 F^2(s_1)}{2\lambda_0\Lambda_{10}\Lambda_{20}}, \qquad (0 \leq s_1 \leq d). \tag{4.3-8}$$

The expansion fan is followed by a contracting simple wave for λ

$$\lambda = \lambda_0 + \frac{q^2}{2\lambda_0\Lambda_{10}\Lambda_{20}} \exp\{-4\alpha\Lambda_{20}(s_1-d)\}, \quad (s_1 \geq d). \tag{4.3-9}$$

In general this leads to shock formation. Differentiating equation $(4.1-74)_{(1)}$ with respect to s_1, $s_1 \geq d$, and minimising the breakdown time it follows that the shock forms on $s_1 = d$ and as $d \to 0^+$ this implies the shock forms on the upper edge of the expansion fan at

$$x_* = \Omega/2\alpha T_0, \quad t_* = \Omega(1 + 1/2\Omega), \tag{4.3-10}$$

where $\Omega = \lambda_0(\Lambda_{10}\Lambda_{20})^3/q^2 a(\Lambda_{10} - \Lambda_{20})$. Since $t_* = O(1/\varepsilon^2)$ one would normally expect reflection to occur before shock formation. The approximation for θ

is linear but the curves of constant θ, $s_2 =$ constant, are not straight lines. The behaviour is best handled numerically as we do later. It is worth noting that as $d \to 0^+$, θ has a discontinuity along the curve $s_2 = 0^+$ described by $X = \Lambda_{20}(1 + cq^2)t$ with $\theta = 0$ for $s_2 < 0$ and $\theta = -q/\lambda_0\Lambda_{20}$ on $s_2 = 0^+$, so that the jump in value is independent of α. For nonzero values of θ the curves of constant θ have an initial speed greater than Λ_{20} and this decreases to Λ_{20} for large X, t. The expansion wave for λ precedes the θ discontinuity and of course $|\theta|$ decays exponentially along $X = 0$.

Graphs of λ, θ as functions of X, at fixed time are shown in Fig.6.5-1 and Fig.6.5-2 in Chapter 6. The values obtained using the present perturbation technique are given together with the values obtained from the direct numerical method outlined in Chapter 6.

The case $\lambda_{c1} < \lambda_0 < \lambda_m < \lambda_i$ is essentially the same as in the previous case except that $\Lambda_{20} > \Lambda_{10}$ so that the θ discontinuity precedes the λ expansion fan.

The case $\lambda_0 < \lambda_i < \lambda_m$ is excluded as before since the approximation is not accurate enough to deal with the case in which $T'' \approx 0$.

For the case $\lambda_i < \lambda_0 < \lambda_m < \lambda_{c2}$, $\Lambda_{10} < \Lambda_{20}$, $T_0'' > 0$ and the λ solution as $d \to 0^+$ consists of a steady initial state $\lambda = \lambda_0$ bounded by a shock then followed by a simple expansion wave. The shock wave velocity is given to order ε^2, by

$$V_L(\lambda) = \{\frac{T(\lambda) - T(\lambda_0)}{(\lambda - \lambda_0)}\}^{1/2} = \Lambda_{10}\,\{1 - \frac{a\varepsilon^2 f^2(s_1)(\Lambda_{10} - \Lambda_{20})}{4\lambda_0(\Lambda_{10}\Lambda_{20})^2}\}. \qquad (4.3\text{-}11)$$

Thus, at a given point on the shock the velocity $V_L(\lambda)$ is the average of the velocities of the characteristics on each side, and the shock velocity slows along the shock path to the final value Λ_{10}, the shock strength being then zero. The actual shock path is not a straight line; it can be found by integrating the equation (4.3-10) using equations $(4.1-74)_{(1)}$ and $(4.3-7)$. The solution for θ starts from the undisturbed state $\theta = 0$ connected by the line of discontinuity $X = \Lambda_{20}t$ where θ jumps from zero to θ_m. Along a curve $s_2 =$ constant, θ is constant and the slope $\frac{dx}{dt}$ is initially, at $X = 0$, smaller than Λ_{20} and increases to Λ_{20} for large X. Of course, the θ discontinuity precedes the λ shock. Graphs of λ, θ as functions of X are shown in Fig.6.4-3 and Fig.6.5-4 in Chapter 6.

For the final case $\lambda_{c2} < \lambda_0 < \lambda_m$, $\Lambda_{10} > \Lambda_{20}$, $T_0^{''} > 0$ and the solution is similar to the above case except that the θ discontinuity is behind the λ shock.

Chapter 5

Normal Impact of the Nonlinear Elastic String for the Case $\Lambda_1 \simeq \Lambda_2$

5.1 Approximate Equations

As we mentioned in chapter 4, the perturbation method employed in that chapter fails when $\Lambda_1 \simeq \Lambda_2$ and we have to proceed differently. In this chapter, we will modify a method suggested by Collins in dealing with a half space problem [4,5].

Consider the system given by

$$\boldsymbol{u}_{,t} + \boldsymbol{H}(\boldsymbol{u})_{,X} = \boldsymbol{0}. \tag{5.1-1}$$

As before equation (5.1-1) may be written as

$$\boldsymbol{u}_{,t} + A(\boldsymbol{u})\boldsymbol{u}_{,X} = \boldsymbol{0}, \tag{5.1-2}$$

where $A(\boldsymbol{u}) = \frac{\partial \boldsymbol{H}(\boldsymbol{u})}{\partial \boldsymbol{u}}$. The eigenvalues of A and the corresponding eigenvectors are given in Chapter 3.

On multiplying equation (5.1-2) on the left by $\boldsymbol{L}^{(-2)}$, we have

$$\lambda \sin\theta \, \frac{\partial}{\partial \tau} \, (\lambda\Lambda_2 \cos\theta + u) - \lambda \cos\theta \, \frac{\partial}{\partial \tau}(\lambda\Lambda_2 \sin\theta + v) = 0, \tag{5.1-3}$$

where

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} - \Lambda_2(\lambda) \, \frac{\partial}{\partial X} \, . \tag{5.1-4}$$

Next we multiply equation (5.1-2) on the left by $\boldsymbol{L}^{(-1)}$ and using the first two equations of system (5.1-1) to replace $u_{,X}$ and $v_{,X}$ and noting that

$$\frac{d\Lambda_2}{d\lambda} = (\Lambda_1^2 - \Lambda_2^2)/2\lambda\Lambda_2, \tag{5.1-5}$$

we have

$$\lambda\cos\theta\,\frac{\partial}{\partial\tau}(\lambda\Lambda_2\cos\theta+u)+\lambda\sin\theta\,\frac{\partial}{\partial\tau}(\lambda\Lambda_2\sin\theta+v)=\frac{\lambda(\Lambda_1^2-\Lambda_2^2)}{2\Lambda_2}\,\frac{\partial\lambda}{\partial\hat{\tau}}\,, \quad (5.1\text{-}6)$$

where

$$\frac{\partial}{\partial\hat{\tau}}=\frac{\partial}{\partial t}+\Lambda_2(\lambda)\,\frac{\partial}{\partial X}\,. \quad (5.1\text{-}7)$$

Since $\Lambda_1\simeq\Lambda_2$, if λ varies slowly in the direction $\frac{dX}{dt}=\Lambda_2(\lambda)$, we may neglect the right hand side of equation (5.1-6). We then have

$$\frac{\partial}{\partial\tau}(\lambda\Lambda_2\cos\theta+u)=0,\qquad \frac{\partial}{\partial\tau}(\lambda\Lambda_2\sin\theta+v)=0. \quad (5.1\text{-}8)$$

If the initial state is constant we then have

$$\begin{aligned}&\lambda\Lambda_2(\lambda)\cos\theta+u=\lambda_0\Lambda_{20},\\&\lambda\Lambda_2(\lambda)\sin\theta+v=0.\end{aligned} \quad (5.1\text{-}9)$$

By using (5.1-9) and the first two of equations (5.1-1), we have

$$\begin{aligned}\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial X}\,(\lambda\Lambda_2(\lambda))&=0,\\ \frac{\partial\theta}{\partial t}+\Lambda_2(\lambda)\,\frac{\partial\theta}{\partial X}&=0.\end{aligned} \quad (5.1\text{-}10)$$

The first equation of (5.1-10) is already in conservation form. If we rewrite the second equation in conservation form, we obtain a system of two conservation laws. Of course there is more than one way to do this. For example, we may rewrite equations (5.1-10) in the following form

$$\boldsymbol{v}_{,t}+\boldsymbol{f}_{,X}=\boldsymbol{0} \quad (5.1\text{-}11)$$

where $v = (\lambda, \lambda\theta^p)^T$, $f = (\lambda\Lambda_2, \lambda\Lambda_2\theta^p)^T$, $p > 0$. The corresponding jump condition is

$$V \begin{bmatrix} \lambda \\ \lambda\theta^p \end{bmatrix} = \begin{bmatrix} \lambda\Lambda_2(\lambda) \\ \lambda\theta^p\Lambda_2(\lambda) \end{bmatrix} , \tag{5.1-12}$$

where $[\varphi] = \varphi^- - \varphi^+$ is the jump in φ across the shock with φ^-, φ^+ the values immediately behind and ahead of it. V is the shock velocity. For the present case the possibilities are

$$\begin{aligned} &[\theta] \neq 0, \quad [\lambda] = 0, \quad V = V_T(\lambda) = \Lambda_2(\lambda), \\ &[\theta] = 0, \quad [\lambda] \neq 0, \quad \widehat{V}_L(\lambda^+, \lambda^-) = \frac{[\lambda\Lambda_2(\lambda)]}{[\lambda]} . \end{aligned} \tag{5.1-13}$$

The path of possible discontinuity in θ is precisely that obtained for the full set of equations (5.1-1). If we denote by V_L the longitudinal shock velocity for equations (5.1-1) then

$$V_L = \frac{[T]}{[\lambda]} , \tag{5.1-14}$$

and to terms of order ε^2, noting that $\Lambda_2'(\lambda_{ci}) = 0 \quad (i = 1, 2)$,

$$V_L = \widehat{V}_L = \Lambda_2(\lambda_{ci}) + \frac{T''(\lambda_{ci})}{4\Lambda_2(\lambda_{ci})} (\lambda^+ - \lambda_{ci} + \lambda^- - \lambda_{ci}). \tag{5.1-15}$$

Therefore, system (5.1-11), together with equations (5.1-9), are approximations of the system (5.1-1).

5.2 Exact Solution of the Approximate System For Constant Boundary Conditions

For the purpose of simplicity, we take $p = 1$ in system (5.1-11) and rewrite it in the following form

$$\boldsymbol{v}_{,t} + B\boldsymbol{v}_{,X} = \boldsymbol{0} \tag{5.2-1}$$

where $v = (\lambda, \lambda\theta)^T$ and the components of the matrix B are

$$B_{11} = (\lambda\Lambda_2(\lambda))', \qquad B_{12} = 0, \qquad B_{21} = \lambda\theta\Lambda_2'(\lambda), \qquad B_{22} = \Lambda_2(\lambda). \tag{5.2-2}$$

The eigenvalues of B are

$$\alpha_1(\lambda) = (\lambda\Lambda_2(\lambda))', \qquad \alpha_2(\lambda) = \Lambda_2(\lambda). \tag{5.2-3}$$

The corresponding right eigenvectors are

$$r_1 = (1, \theta)^T, \qquad r_2 = (0, 1)^T. \tag{5.2-4}$$

Since

$$\begin{aligned} &\operatorname{grad}\alpha_1 \cdot r_1 = (\lambda\Lambda_2)'' \neq 0, \qquad (\lambda \approx \lambda_{ci}, \quad i = 1, 2,), \\ &\operatorname{grad}\alpha_2 \cdot r_2 = 0, \end{aligned} \tag{5.2-5}$$

the α_1 characteristic field is genuinely nonlinear so that there are simple waves and shock waves for this field. On the other hand, the α_2 characteristic field is linearly degenerate; hence there is a contact discontinuity for this field. By using these elementary waves, one can find the exact solution of the approximate system (5.2-1) for the case in which the constant boundary conditions are given by

$$u(0,t) = 0, \qquad v(0,t) = q = \varepsilon v_0, \quad (t > 0). \tag{5.2-6}$$

Let λ_0 denote the initial stretch and λ_m the final stretch and let θ_m denote the angle of inclination corresponding to λ_m. Then, by using equations (5.1-9), we find

$$\lambda_m T(\lambda_m) = \lambda_0 T(\lambda_0) + q^2, \qquad \tan\theta_m = -\frac{q}{\lambda_0\Lambda_{20}}. \tag{5.2-7}$$

Since $\lambda T(\lambda)$ is a increasing function of λ, then by equation (5.2-7), one has

$$\lambda_m > \lambda_0. \tag{5.2-8}$$

Using equation (5.2-7) one can solve for λ_m and θ_m to get the boundary conditions

$$\lambda(0,t) = \lambda_m, \qquad \theta = \theta_m, \qquad (t > 0). \tag{5.2-9}$$

The initial conditions are

$$\lambda(X,0) = \lambda_0, \qquad \theta(X,0) = 0, \qquad (X > 0). \tag{5.2-10}$$

We consider a number of cases, with λ close to λ_{c1} in cases 1, 2, 3 and close to λ_{c2} in cases 4, 5.

Case 1. $\lambda_0 < \lambda_m < \lambda_{c1}$.

The exact solution for the approximate system is

Region 1: $X/t > \alpha_1(\lambda_0)$.

$$\lambda = \lambda_0, \qquad \theta = 0.$$

Region 2: $\alpha_1(\lambda_0) > X/t > \alpha_1(\lambda_m)$.

$$\alpha_1(\lambda) = X/t, \qquad \theta = 0.$$

Region 3: $\alpha_1(\lambda_m) > X/t > \Lambda_2(\lambda_m)$.

$$\lambda = \lambda_m, \qquad \theta = 0.$$

Region 4: $\Lambda_2(\lambda_m) > X/t \geq 0$.

$$\lambda = \lambda_m, \qquad \theta = \theta_m. \tag{5.2-11}$$

Case 2. $\lambda_{c1} < \lambda_0 < \lambda_m$.

The solution is

Region 1: $X/t > \Lambda_2(\lambda_0)$.

$$\lambda = \lambda_0, \qquad \theta = 0.$$

Region 2: $\Lambda_2(\lambda_0) > X/t > \alpha_1(\lambda_0)$.

$$\lambda = \lambda_0, \qquad \theta = \theta_m.$$

Region 3: $\alpha_1(\lambda_0) > X/t > \alpha_1(\lambda_m)$.

$$\alpha_1(\lambda) = X/t, \qquad \theta = \theta_m.$$

Region 4: $\alpha_1(\lambda_m) > X/t \geq 0$.

$$\lambda = \lambda_m, \qquad \theta = \theta_m. \tag{5.2-12}$$

Case 3. $\lambda_0 < \lambda_{c1} < \lambda_m$.

The solution for this case is

Region 1: $X/t > \alpha_1(\lambda_0)$.

$$\lambda = \lambda_0, \qquad \theta = 0.$$

Region 2: $\alpha_1(\lambda_0) > X/t > \Lambda_2(\lambda_{c1})$.

$$\alpha_1(\lambda) = X/t, \quad \lambda_0 < \lambda < \lambda_{c1}, \quad \theta = 0.$$

Region 3: $\Lambda_2(\lambda_{c1}) > X/t > \alpha_1(\lambda_m)$.

$$\alpha_1(\lambda) = X/t, \quad \lambda_{c1} < \lambda < \lambda_m, \quad \theta = \theta_m.$$

Region 4: $\alpha_1(\lambda_m) > X/t \geq 0$.

$$\lambda = \lambda_m, \qquad \theta = \theta_m. \tag{5.2-13}$$

Case 4. $\lambda_0 < \lambda_m < \lambda_{c2}$ or $\lambda_0 < \lambda_{c2} < \lambda_m < \lambda_*$.

Here, λ_* satisfies $\Lambda_2(\lambda_*) = \Lambda_2(\lambda_0)$, where λ_0 is close to but less than λ_{c2}.

Region 1: $X/t > \Lambda_2(\lambda_0)$.

$$\lambda = \lambda_0, \qquad \theta = 0.$$

Region 2: $\Lambda_2(\lambda_0) > X/t > \widehat{V}_L(\lambda_0, \lambda_m)$.

$$\lambda = \lambda_0, \qquad \theta = \theta_m.$$

Region 3: $\widehat{V}_L(\lambda_0, \lambda_m) > X/t \geq 0$.

$$\lambda = \lambda_m, \qquad \theta = \theta_m. \tag{5.2-14}$$

Case 5. $\lambda_0 < \lambda_{c2} < \lambda_* < \lambda_m$ or $\lambda_{c2} < \lambda_0 < \lambda_m$.

Here, λ_* has the same meaning as in case 4.

Region 1: $X/t > \widehat{V}_L(\lambda_0, \lambda_m)$.

$$\lambda = \lambda_0, \qquad \theta = 0.$$

Regoin 2: $\widehat{V}_L(\lambda_0, \lambda_m) > X/t > \Lambda_2(\lambda_m)$.

$$\lambda = \lambda_m, \qquad \theta = 0.$$

Region 3 $\Lambda_2(\lambda_m) > X/t \geq 0$.

$$\lambda = \lambda_m, \qquad \theta = \theta_m. \tag{5.2-15}$$

Once the values of λ and θ are found, one can find the values of u and v from equations (5.1-9) and (5.2-7). For example, the values of u and v in case 3 can be found as follows: in region 1, $u = v = 0$; in region 2, $u = \lambda_0\Lambda_{20} - \lambda\Lambda_2(\lambda)$, $v = 0$; in region 3, $u = \lambda_0\Lambda_{20} - \lambda\Lambda_2(\lambda)\cos\theta_m$, $v = -\lambda\Lambda_2(\lambda)\sin\theta_m$; in region 4, $u = 0, v = q$. Then we can compare the exact solutions of the approximate system with the exact solutions of the original system for the case of constant boundary values. For example, the exact solution of the original system for the case $\lambda_0 < \lambda_{c1} < \lambda_m$ was obtained by Wegner Haddow and Tait [29]

$$
\begin{aligned}
&\lambda = \lambda_0,\, \theta = 0,\, u = 0,\, v = 0, \qquad \text{for} \quad X/t > \Lambda_1(\lambda_0).\\
&\Lambda_1(\lambda) = X/t, \theta = 0, u = -I(\lambda), v = 0, \qquad \text{for} \quad \Lambda_1(\lambda_0) > X/t > \Lambda_2(\lambda_{c1}).\\
&\Lambda_1(\lambda) = X/t, \theta = \theta_m, u = \cos\theta_m \int_\lambda^{\lambda_m} \Lambda_1(p)dp,\\
&\quad v = q + \sin\theta_m \int_\lambda^{\lambda_m} \Lambda_1(p)dp, \qquad \text{for} \quad \Lambda_2(\lambda_{c1}) \geq X/t > \Lambda_1(\lambda_m).\\
&\lambda = \lambda_m, \theta = \theta_m, u = 0, v = q, \qquad \text{for} \quad \Lambda_1(\lambda_m) \geq X/t \geq 0,
\end{aligned}
\tag{5.2-16}
$$

where

$$
\begin{aligned}
\left(\int_{\lambda_0}^{\lambda_m} \Lambda_1(p)dp + K\right)^2 &= q^2 + K^2,\\
\sin\theta_m &= -q/K,\\
K &= \lambda_{c1}\Lambda_2(\lambda_{c1}) - \int_{\lambda_0}^{\lambda_{c1}} \Lambda_1(p)dp,\\
I(\lambda) &= \int_{\lambda_0}^{\lambda} \Lambda_1(p)dp.
\end{aligned}
\tag{5.2-17}
$$

Equations (5.2-17) can be obtained from equations (3.3-18).

Since $(\lambda\Lambda_2)' - \Lambda_1 = \frac{(\Lambda_1-\Lambda_2)^2}{2\Lambda_2}$ the slopes in the corresponding regions of the two solutions are the same to terms $O(\Lambda_1 - \Lambda_2)^2$. On expanding equations

(5.2-7), (5.2-17) correct to terms $O(q^4)$, both give

$$\lambda_m = \lambda_0 + \frac{q^2}{2\lambda_0 \Lambda_1^2(\lambda_{c1})}. \tag{5.2-18}$$

5.3 Solution for the Case of Variable Boundary Conditions

Consider the approximate equations

$$\begin{aligned} \lambda\Lambda_2(\lambda)\cos\theta + u &= \lambda_0\Lambda_{20}, \\ \lambda\Lambda_2(\lambda)\sin\theta + v &= 0, \\ \frac{\partial\lambda}{\partial t} + \frac{\partial}{\partial X}\left(\lambda\Lambda_2(\lambda)\right) &= 0, \\ \frac{\partial\theta}{\partial t} + \Lambda_2(\lambda)\,\frac{\partial\theta}{\partial X} &= 0, \end{aligned} \tag{5.3-1}$$

subject to the conditions

$$\begin{aligned} &u(0,t) = 0, \quad \frac{dv(0,t)}{dt} = 2\alpha T\big(\lambda(0,t)\big)\sin\theta(0,t), \quad t > 0, \\ &\lambda(X,0) = \lambda_0, \quad \theta(X,0) = 0, \quad X \geq 0, \\ &u(X,0) = v(X,0) = 0, \quad X > 0, \quad u(0,0) = 0. \end{aligned} \tag{5.3-2}$$

We set

$$v(0,0^+) = \varepsilon v_0 = q, \quad \lambda(0,0^+) = \lambda_m, \quad \theta(0,0^+) = \theta_m, \tag{5.3-3}$$

where λ_m, θ_m are given by

$$\lambda_m T(\lambda_m) = \lambda_0 T(\lambda_0) + q^2, \qquad \tan\theta_m = -\frac{q}{\lambda_0\Lambda_{20}}. \tag{5.3-4}$$

Equations (5.3-4) are obtained from the first two equations in (5.3-1) using conditions (5.3-2) and (5.3-3).

Substituting from equations (5.3-1) into equations (5.3-2) leads to

$$\frac{d\lambda(0,t)}{dt} = \frac{-4\alpha\Lambda_2(\lambda T - \lambda_0 T_0)}{\lambda(\Lambda_1^2 + \Lambda_2^2)}, \tag{5.3-5}$$

where the right hand side is evaluated on $X = 0,\ t > 0$. This equation may be integrated to give

$$t = \frac{1}{4\alpha}\int_{\lambda}^{\lambda_m} \frac{\lambda(\Lambda_1^2(\lambda) + \Lambda_2^2(\lambda))}{\Lambda_2(\lambda)(\lambda T(\lambda) - \lambda_0 T_0)}\, d\lambda, \tag{5.3-6}$$

so that $\lambda(0,t)$ decays monotonically from λ_m but does not decrease to λ_0 in a finite time. Fig. 5.3-1 shows the graph of $\lambda(0,t)$ as a function of time.

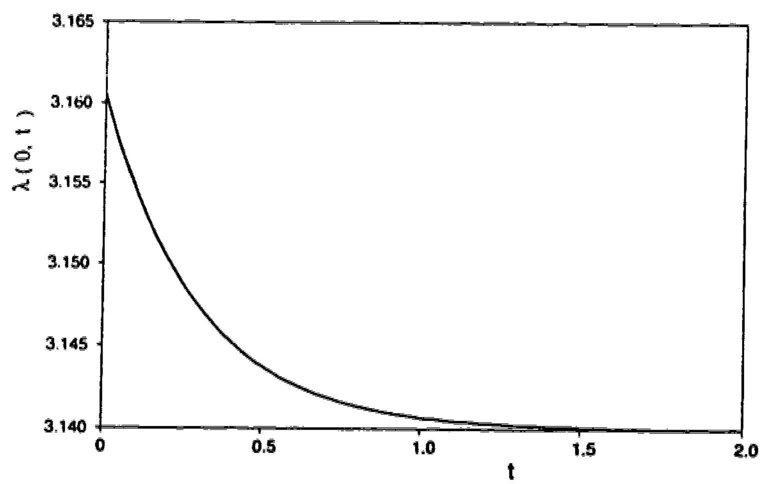


Fig.5.3-1 $\lambda_0 = 3.14,\ q = 0.3,\ \alpha = 1.0.$

Consider first the solution of λ for the case $\lambda \approx \lambda_{c1}$. We introduce a transition region [0,d] on the t axis by

$$\lambda = F(t), \qquad 0 \le t \le d;$$

$$\lambda = \lambda(0, t-d), \qquad t \ge d, \qquad \lambda(0, 0_+) = \lambda_m, \tag{5.3-7}$$

where $F(t)$ is a monotone increasing function with $F(0) = \lambda_0$ and $F(d) = \lambda_m$. $\lambda(0,t)$ is determined by equation (5.3-6). The solution for λ under condition (5.3-7) can be found by the method of characteristics. The solution for λ under condition (5.3-6) can then be found by letting $d \to 0^+$. It consists of an initial steady state $\lambda = \lambda_0$ followed by an expansion fan through the origin, where λ increases from λ_0 to λ_m, and this is followed by a contracting simple wave region. A shock will eventually form.

The initial breakdown time is obtained by minimising t_* for $s > 0$ where

$$t_* = s + \frac{(\lambda\Lambda_2)'}{(\lambda\Lambda_2)''\dot{\lambda}} \tag{5.3-8}$$

where $\dot{} = \frac{d}{ds}$. Since, referring to equation (5.3-5), this is of order q^{-2} the shock will not occur before reflection. Fig.5.3-2 shows the λ characteristics in the $X-t$ plane for the case $\lambda \approx \lambda_{c1}$ before a λ shock occurs. Since $\lambda(0,t)$ is known $\theta(0,t)$ is given by

$$\sin\theta(0,t) = -\frac{\sqrt{\lambda T - \lambda_0 T_0}}{\lambda\Lambda_2(\lambda)}, \quad \lambda = \lambda(0,t), \tag{5.3-9}$$

and $\theta(0,t)$ increases monotonically from $-\theta_m$ to zero. Since $\lambda(X,t)$ is known everywhere and θ is constant along curves $\frac{dX}{dt} = \Lambda_2\{\lambda(X,t)\}$, θ can be determined. The precise description is best carried out numerically but we give a brief qualitative discussion. For the case $\lambda_0 < \lambda_{c1} < \lambda_m$, the value θ_m, from the transition region argument, is carried out along the straight line $X = \Lambda_2(\lambda_{c1})t$ with θ jumping from 0 to θ_m. Similarly, the value of θ corresponding to $\lambda(0,t) = \lambda_{c1}$ is carried out along a parallel straight line. For $\lambda_{c1} < \lambda(0,t) < \lambda_m$ the corresponding curve for θ constant moves into a region of increasing λ since $(\lambda\Lambda_2)' < \Lambda_2$ if $\lambda > \lambda_{c1}$. Thus the slope $\frac{dX}{dt}$ decreases until the curve crosses $X = (\lambda\Lambda_2)'(\lambda_m)t$ where it moves into a region of decreasing λ so that the slope increases to a limiting value $\Lambda_2(\lambda_{c1})$. For values $\lambda_0 < \lambda(0,t) < \lambda_{c1}$, $(\lambda\Lambda_2)' > \Lambda_2$ so that the curve of constant θ, starting on the t axis moves to lower values of λ so that the slope $\frac{dX}{dt} = \lambda_2(\lambda)$ decreases and tends to a limiting value $\Lambda_2(\lambda_0)$. Shock incidence will affect this description but as pointed out above this in general will not happen before reflection has occurred. Fig.5.3-3 shows the λ and θ characteristics in the $X-t$ plane for the case $\lambda_0 < \lambda_{c1} < \lambda_m$ before a λ shock occurs. The other cases when $\lambda \simeq \lambda_{c1}$ can be studied in a similar way (see Fig.5.3-4 and Fig.5.3-5).

If λ varies in the neighborhood of λ_{c2} the sign of T'' changes. We then have

$$\begin{gathered} T>0, \quad T'>0, \quad T''>0, \\ (\lambda\Lambda_2)' = \Lambda_2 + \lambda\Lambda_2' = \frac{\Lambda_1^2+\Lambda_2^2}{2\Lambda_2} > 0, \\ (\lambda\Lambda_2)'' = \frac{2TT'' - (\Lambda_1^2-\Lambda_2^2)^2}{4\lambda\Lambda_2^3}, \end{gathered} \tag{5.3-10}$$

so that with $\Lambda_1 \simeq \Lambda_2$, $(\lambda\Lambda_2)'' > 0$ and the graph of $(\lambda\Lambda_2)$ as a function of λ is concave up. Again introducing a transition case as before and allowing $d \to 0^+$ shows that a λ shock is initiated immediately in the loading phase. Assuming that the first two equations in (5.3-1) hold and that the values of λ and θ on $X=0$ are given by equations (5.3-6), (5.3-9), with values $\lambda(0,0^+) = \lambda_m$, $\theta(0,0^+) = \theta_m$ as before. Since $\lambda(0,t)$ decreases for t increasing the straight line λ characteristics now form an expanding simple wave behind the shock. If the last two equations in (5.3-1) are written in conservation form by multiplying the second equation by λ we have the shock condition

$$V\begin{bmatrix}\lambda\\ \theta\end{bmatrix} = \begin{bmatrix}\lambda\Lambda_2(\lambda)\\ \lambda\theta\Lambda_2(\lambda)\end{bmatrix}, \tag{5.3-11}$$

where V is the shock velocity and the two possible solutions of V are given by equation (5.1-13).

A straightforward calculation shows that

$$(\lambda\Lambda_2)'(\lambda_0) < \widehat{V}_L(\lambda_0, \lambda) < (\lambda\Lambda_2(\lambda))', \quad \lambda_0 < \lambda < \lambda_m. \tag{5.3-12}$$

We thus have a stable shock moving into a region of constant $\lambda = \lambda_0$, and since $\lambda(0,t)$ decreases with t the shock slows as X, t increase eventually slowing to a speed $(\lambda\Lambda_2)'(\lambda_0)$ with the shock strength decaying to zero. $\lambda(X,t)$ is then known everywhere once the shock path is calculated and this is best done numerically. Fig.5.3-6 shows the λ shock path and the λ characteristics in the $X-t$ plane for the case $\lambda \approx \lambda_{c2}$. The variation of the shock path from a straight line is of necessity small and we indicate the deviation from the straight line in Fig. 5.3-7.

With $\lambda(X,t)$ known $\theta(X,t)$ may now be found from the last equation in (5.3-1). Since

$$(\lambda\Lambda_2)' = \Lambda_2 + \frac{\Lambda_1^2 - \Lambda_2^2}{2\Lambda_2} \tag{5.3-13}$$

it follows that

$$0 < (\lambda\Lambda_2(\lambda))' < \Lambda_2(\lambda), \qquad \lambda < \lambda_{c2},$$
$$0 < \Lambda_2(\lambda) < (\lambda\Lambda_2(\lambda))', \quad \lambda_{c2} < \lambda. \tag{5.3-15}$$

Consider the case where $\lambda_0 < \lambda_m < \lambda_{c2}$. Since $\Lambda_2(\lambda_m) > \widehat{V}_L(\lambda_0, \lambda_m)$ the curve $\theta(x,t) = \theta_m$ must precede the shock. Since θ is continuous across a λ shock this implies the curve is continued as a straight line of slope $\Lambda_2(\lambda_0)$ (see Fig.5.3-8). Across this line θ is discontinuous, jumping from the initial value

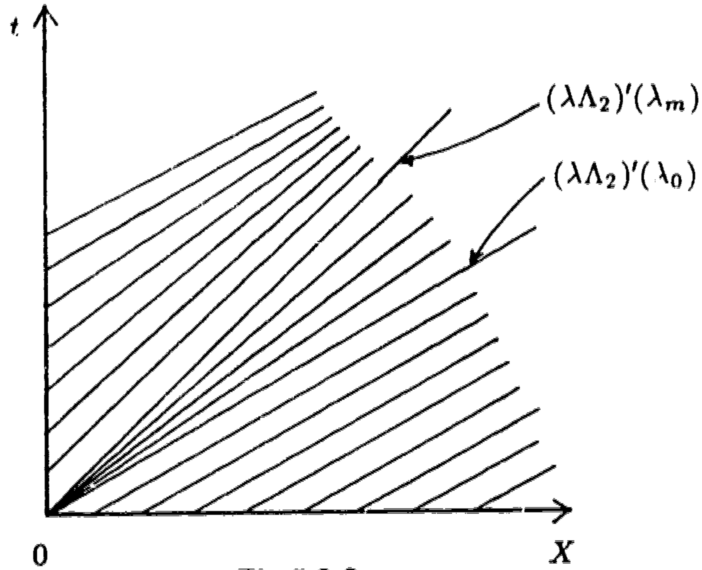
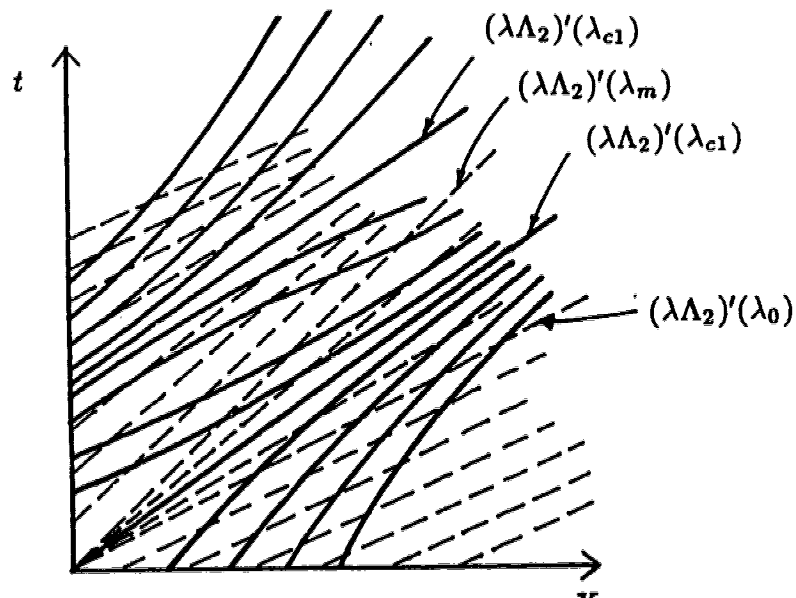


Fig.5.3-2



$\lambda_0 < \lambda_{c1} < \lambda_m$

Fig.5.3-3

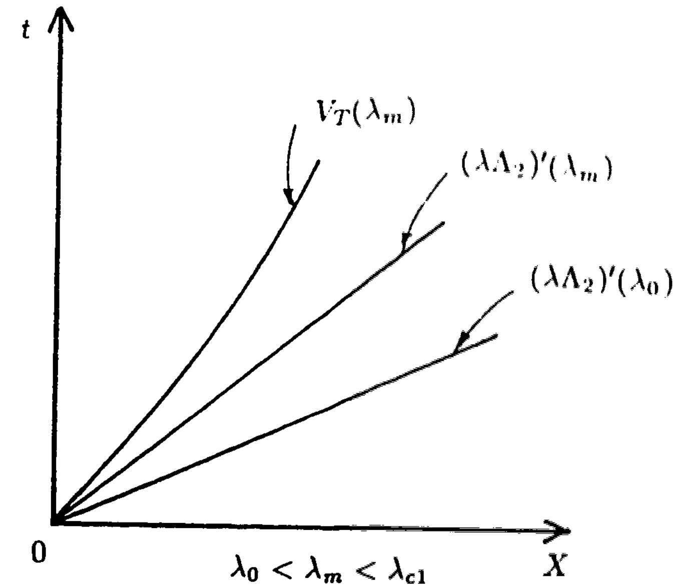


Fig.5.3-4

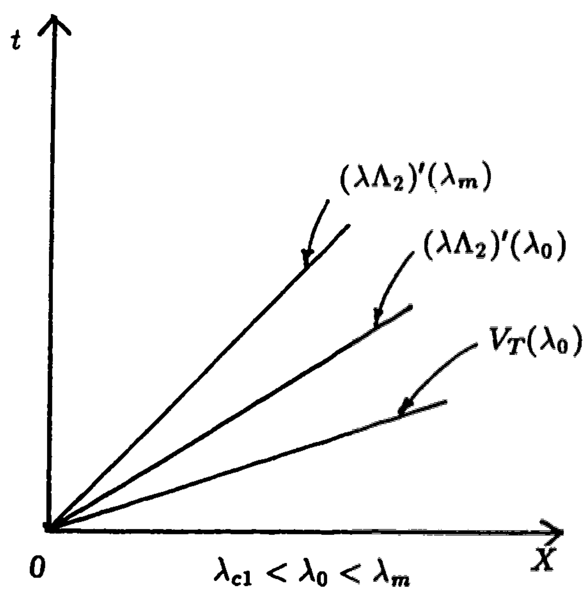


Fig.5.3-5

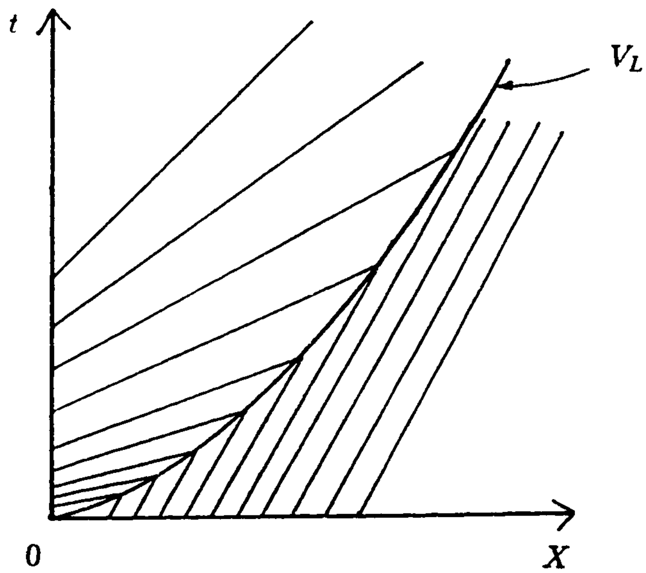


Fig.5.3-6

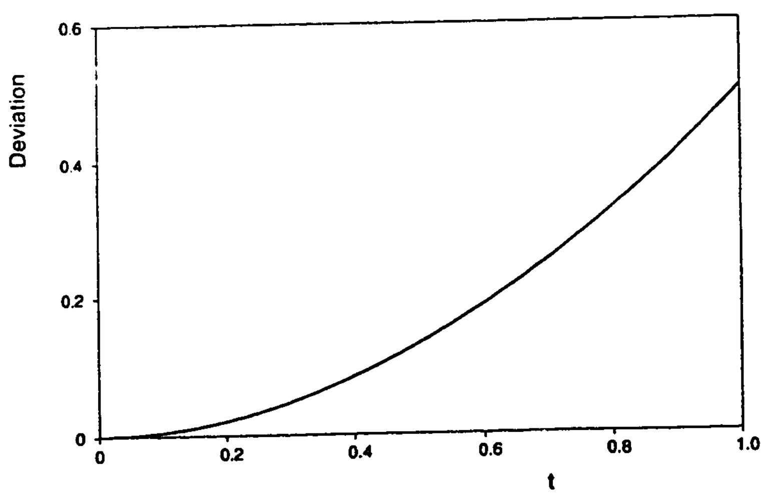


Fig.5.3-7

The quantity $(X_{V_m} - X_s) \cdot 10^3$ plotted as a function of time t, where X_s denotes the λ shock path. $V_m = (V_L)_{max}$, where $V_L = [\lambda\Lambda_2]/[\lambda]$ and X_{V_m} is given by $X = V_m t$, for $\lambda_0 = 3.14, q = 1.0, \alpha = 1.0.$

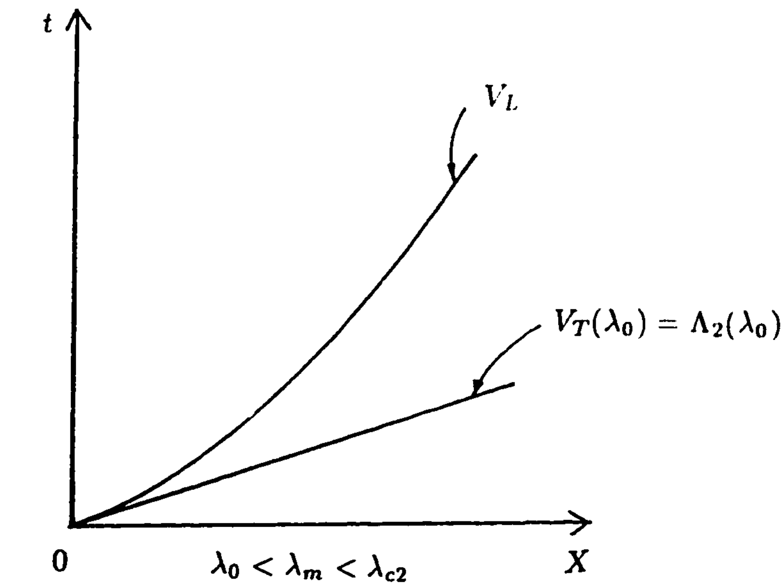


Fig.5.3-8

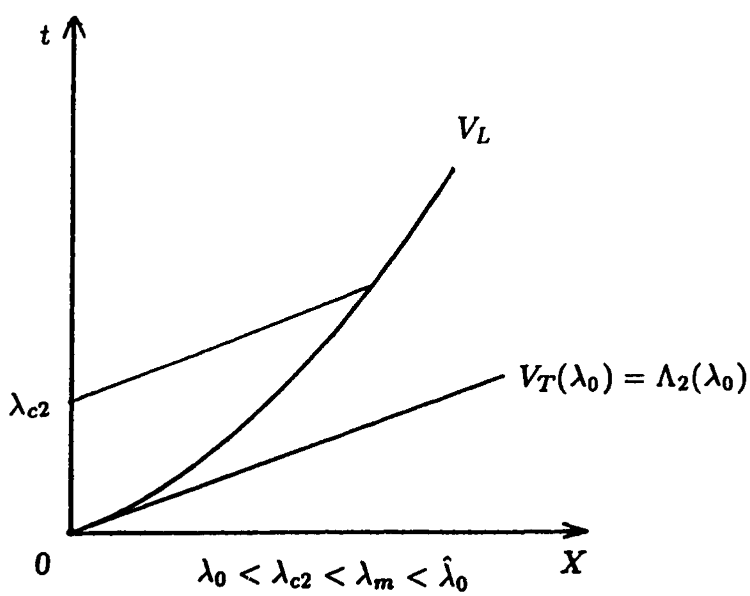


Fig.5.3-9

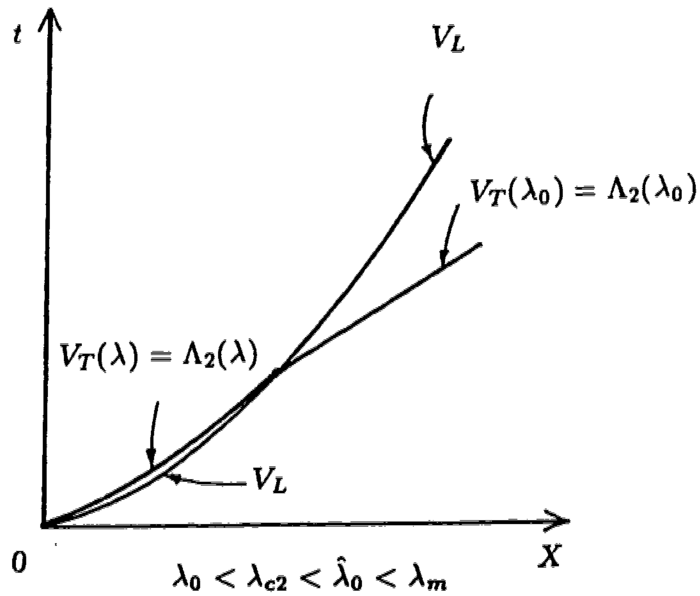


Fig.5.3-10

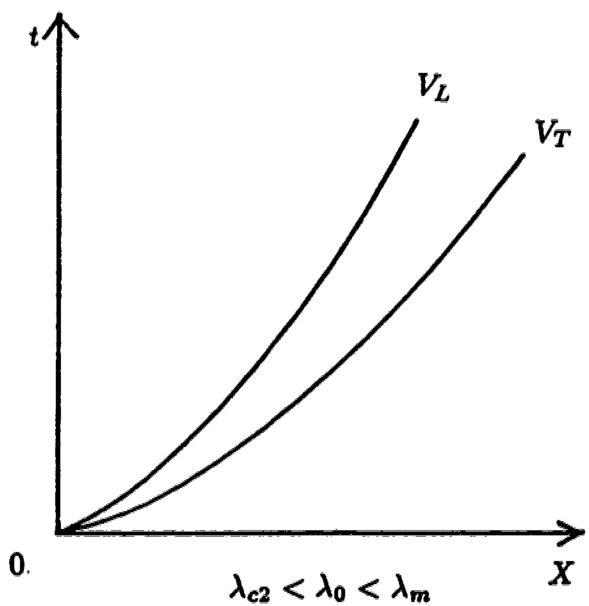


Fig5.3-11

0 to θ_m. $\theta(0,t)$ is known from equation (5.3-9) . Since $\Lambda_2 > (\lambda\Lambda_2)'$ this implies that the curve $\theta = \theta_1$, say, $0 < \theta_1 < \theta_m$, moves into a region where λ increases, as X increases from zero. Since $\Lambda_2(\lambda)$ decreases with increasing λ in this region the $\theta = \theta_1$ curve slows as X increases until it meets the λ shock. θ is, as mentioned above, continuous across the shock but the value of $\frac{dX}{dt}$ has a jump discontinuity there, jumping to the value $\Lambda_2(\lambda_0)$ ahead of the shock.

If $\lambda_0 < \lambda_{c2} < \lambda_m$ the situation is slightly different. In the $T - \lambda$ plane let the line of slope $\Lambda_2^2(\lambda_0)$ meet the curve again at $\hat{\lambda}_0$ so that $\frac{T(\hat{\lambda}_0)}{\hat{\lambda}_0} = \frac{T(\lambda_0)}{\lambda_0}$. If $\lambda_m < \hat{\lambda}_0$ the θ discontinuity described above again precedes the λ shock travelling with speed $\Lambda_2(\lambda_0)$ (see Fig.5.3-9). Since $\lambda(0,t)$ decreases from λ_m to λ_0 the line $x = \Lambda_2(\lambda_{c2})(t - t_*)$, where t_* is chosen so that $\lambda(0,t_*) = \lambda_{c2}$, divides the region behind the shock into two parts. Clearly the $\theta(x,t) =$ constant curve when $\lambda = \lambda_{c2}$ coincides with this line as does the λ characteristic. For $\lambda > \lambda_{c2}$, $\Lambda_2 < (\lambda\Lambda_2)'$ so that $\theta =$ constant moves to smaller values of λ, Λ_2 decreases and the speed $\frac{dX}{dt}$ decreases. The argument for $\lambda < \lambda_{c2}$ is similar and the speed again decreases, as X, t increase. If $\lambda_m > \hat{\lambda}_0$, the λ shock precedes the θ discontinuity initially. The λ characteristic from $X = 0$ carrying the value λ_{c2} meets the shock at some point. The curve $\theta(X,t) = \theta_m$ emanating from $X = 0$, $t = 0$ has slope $\frac{dX}{dt} = \Lambda_2(\lambda)$ so that if it meets the λ characteristic carrying λ_{c2} it is tangent to it and will continue as a straight line. Otherwise it remains below that characteristic and must meet the shock so that thereafter the θ discontinuity will precede (see Fig.5.3-10). The case $\lambda_{c2} < \lambda_0 < \lambda_m$ gives no difficulty with the shock preceding the θ discontinuity (see Fig.5.3-11). Typical graphs of λ and θ as functions of X at a given time and with $\lambda \simeq \lambda_{c2}$ are shown in Fig.6.5-5 and Fig.6.5-6 in chapter 6. In these figures, we compare the results obtained by the present method with those obtained by the direct numerical method described in chapter 6.

Chapter 6

Numerical Analysis

6.1 Finite Difference Scheme in Conservation Form

Consider the initial value problem for the conservation law

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u})}{\partial x} = \boldsymbol{0}, \qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),$$
$$-\infty < x < +\infty, \qquad t > 0, \tag{6.1-1}$$

where $\boldsymbol{u} = (u_1, u_2, ..., u_n)^T$, $\boldsymbol{H}(\boldsymbol{u}) = (H_1(\boldsymbol{u}), ..., H_n(\boldsymbol{u}))^T$. Suppose $\boldsymbol{u}(x,t)$ is a solution of (6.1-1). We define $\boldsymbol{u}_i^n$ by

$$\boldsymbol{u}_i^n = \boldsymbol{u}(x_i, t_n),$$
$$x_i = ih, \quad h = \Delta x, \quad i = 0, \pm 1, \pm 2, ...$$
$$t_n = nk, \quad k = \Delta t, \quad n = 0, 1, 2, ... \tag{6.1-2}$$

If we use a finite difference method to solve equation (6.1-1) and $\boldsymbol{v}_i^n$ is the numerical solution at the grid point $(x,t) = (i\Delta x, n\Delta t)$, then we say $\boldsymbol{v}_i^n$ is an approximation to $\boldsymbol{u}_i^n$.

It is well known that the approximate solutions obtained from a finite difference method may not converge to a weak solution of the system (6.1-1). For example [16], we consider a scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \qquad -\infty < x < \infty, \quad t > 0 \tag{6.1-3}$$

with data

$$u(x,0) = u_0(x) = \begin{cases} 1 & x < 0, \\ 0, & x \geq 0. \end{cases} \tag{6.1-4}$$

Since equation (6.1-3) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{6.1-5}$$

then the approximation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t}\Big|_{(x,t)=(i\Delta x, n\Delta t)} &\approx \frac{v_i^{n+1} - v_i^n}{\Delta t}, \\ \frac{\partial u(x,t)}{\partial x}\Big|_{(x,t)=(i\Delta x, n\Delta t)} &\approx \frac{v_i^n - v_{i-1}^n}{\Delta x}, \\ u(i\Delta x, n\Delta t) &\approx v_i^n \end{aligned} \tag{6.1-6}$$

allows equation (6.1-5) to be approximated by the finite difference scheme

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} v_i^n (v_i^n - v_{i-1}^n) \tag{6.1-7}$$

with data

$$v_i^0 = \begin{cases} 1, & i < 0, \\ 0, & i \geq 0. \end{cases} \tag{6.1-8}$$

Clearly the scheme gives $v_i^1 = v_i^0$ for all i and hence $v_i^n = v_i^0$ for all i. Thus the numerical solution converges to the function $u(x,t) = u_0(x)$ which is not a weak solution of equation (6.1-3). This example shows that a numerical solution may not converge to a weak solution of the corresponding conservation law no matter how one refines the step sizes Δt and Δx. In order to overcome this difficulty, we need to introduce a conservative numerical method.

A finite difference method for equation (6.1-1) is said to be in *conservation form* if it has the form

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x}(G_{i+1/2}^n - G_{i-1/2}^n), \tag{6.1-9}$$

where

$$G_{i+1/2}^n = G(v_{i-l+1}^n, v_{i-l+2}^n, ..., v_{i+l}^n),$$
$$G_{i-1/2}^n = G(v_{i-l}^n, v_{i-l+1}^n, ..., v_{i+l-1}^n). \tag{6.1-10}$$

The function G with $2l$ arguments is called the *numerical flux function*. The numerical flux must be consistent with the physical flux in the sense that

$$G(u, ..., u) = H(u). \tag{6.1-11}$$

If v_i^n $(i = 0, \pm1, \pm2, ...,\quad n = 0, 1, 2, ...)$ is an approximate solution of system (6.1-1) corresponding to a finite difference method in conservation form, we define a continuous function $v(x,t)$ by setting

$$v(x,t) = v_i^n, \qquad i = [x/\Delta x], \quad n = [t/\Delta t], \tag{6.1-12}$$

where $[\alpha]$ denotes the maximum integer which is not greater than the real number α. The following theorem shows why it is important that a finite difference approximation to a conservation law be in conservation form.

Theorem 6.1-1 (Lax, Wendroff [25])

Suppose that the solution $v(x, n\Delta t)$ of a finite difference method in conservation form converges boundly almost everywhere to some function $u(x,t)$ as Δx and Δt approach zero. Then $u(x,t)$ is a weak solution of the system (6.1-1).

The above theorem does not indicate whether the weak solution $u(x,t)$ is a physically relevant solution. We define the physically relevant solution as that which is the limit as $\varepsilon \to 0$ of a solution $u(\varepsilon)$ of the viscous equations

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad \varepsilon > 0. \tag{6.1-13}$$

If system (6.1-1) possesses an entropy function $U(u)$ satisfying the conditions,
1) U is a convex function of u, i.e. U_{uu} is positive definite,
2) $U_u H_u = F_u$, where F is some other function called entropy flux, it can be shown that a smooth solution of system (6.1-1) also satisfies

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0. \tag{6.1-14}$$

Limit solutions of equation (6.1-13) satisfy the following inequality in the weak sense

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0. \tag{6.1-15}$$

We say that the finite difference scheme (6.1-9) is consistent with the entropy condition (6.1-15) if

$$U_i^{n+1} \leq U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n), \tag{6.1-16}$$

where

$$U_i^{n+1} = U(v_i^{n+1}), \quad U_i^n = U(v_i^n),$$
$$F_{i+1/2}^n = F(v_{i-l+1}^n, v_{i-l+2}^n, \ldots, v_{i+l}^n),$$
$$F_{i-1/2}^n = F(v_{i-l}^n, v_{i-l+1}^n, \ldots, v_{i+l-1}^n). \tag{6.1-17}$$

Theorem (6.1-1) was extended by Harten, Lax and Van Leer as follows [14]:

Theorem (6.1-2)

Suppose the difference scheme (6.1-9) is consistent with the conservation law (6.1-1) and with the entropy condition (6.1-15). Let v_i^n be a solution of the scheme (6.1-9), with initial values $v_i^0 = u_0(i\Delta x)$. Suppose that for some sequence of grids indexed by $l = 1, 2, \cdots$, with mesh parameters $(\Delta t)_l, (\Delta x)_l \to 0$ as $l \to \infty$, where $\frac{(\Delta t)_l}{(\Delta x)_l} = constant$. If $v(x,t)$ converges boundly almost everywhere to some function $u(x,t)$ as $l \to \infty$, then the limit $u(x,t)$ is a weak solution of the system (6.1-1) and satisfies the weak form of entropy condition

$$\int_0^\infty \int_{-\infty}^{+\infty} (\frac{\partial w}{\partial t} U + \frac{\partial w}{\partial x} F) dx dt + \int_{-\infty}^{+\infty} w(x,0) U(u_0(x)) dx \leq 0, \qquad (6.1\text{-}18)$$

where $w(x,t)$ is any nonnegative smooth test function with compact support.

In the next section, we will discuss a well known scheme proposed by Godunov.

6.2 Godunov's Scheme

In this section, we describe a method known as Godunov's method which depends on the exact solution of the Riemann problem. For details of the solution of Riemann problem and further information on Godunov's method see [24] and [25].

Consider the Riemann problem defined by the system

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \qquad (6.2\text{-}1)$$

with initial condition

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \qquad (6.2\text{-}2)$$

where u_L, u_R are constant states. The similarity solution which depends on u_L, u_R and the ratio x/t, is denoted by $R(x/t, u_L, u_R)$. Suppose that v_i^n

is given. We construct a piecewise constant function by setting

$$\boldsymbol{v}_i^n(x) = \boldsymbol{v}_i^n, \qquad \text{for} \quad x \in I_i = [(i-1/2)\Delta x, (i+1/2)\Delta x], \tag{6.2-3}$$

and now consider the initial-value problem given by (6.2-1) and (6.2-3). On each interval $[i\Delta x, (i+1)\Delta x]$ the initial value problem defines a Riemann problem; thus the initial value problem (6.2-1) and (6.2-3) defines a sequence of Riemann problems. If we choose

$$\frac{\Delta t}{\Delta x}|a_{max}| < 1/2, \tag{6.2-4}$$

where $|a_{max}|$ is the largest signal velocity, then since

$$\begin{aligned} \boldsymbol{R}(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R) &= \boldsymbol{u}_L, \qquad x/t \le a_L, \\ \boldsymbol{R}(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R) &= \boldsymbol{u}_R, \qquad x/t \ge a_R, \end{aligned} \tag{6.2-5}$$

where a_L and a_R are the smallest and largest signal velocities, there is no interaction between neighboring Riemann problems under the condition (6.2-4). Therefore the exact solution of the initial-value problem (6.2-1) and (6.2-3) can be expressed as

$$\begin{aligned} &\boldsymbol{v}^e(x,t) = \boldsymbol{R}\left(\frac{x-(i+1/2)\Delta x}{t-n\Delta t}, \boldsymbol{v}_i^n, \boldsymbol{v}_{i+1}^n\right), \\ &i\Delta x \le x \le (i+1)\Delta x, \qquad n\Delta t \le t \le (n+1)\Delta t, \end{aligned} \tag{6.2-6}$$

then $\boldsymbol{v}_i^{n+1}$ can be obtained by

$$\boldsymbol{v}_i^{n+1} = \frac{1}{\Delta x}\int_{I_i} \boldsymbol{v}^e(x,(n+1)\Delta t)dx. \tag{6.2-7}$$

By the integral form of conservation law, one has

$$\int_{I_i} \boldsymbol{v}^e(x,(n+1)\Delta t)dx - \int_{I_i} \boldsymbol{v}^e(x,n\Delta t)dx$$
$$+ \int_{n\Delta t}^{(n+1)\Delta t} \boldsymbol{H}(\boldsymbol{v}^e((i+1/2)\Delta x,t))dt - \int_{n\Delta t}^{(n+1)\Delta t} \boldsymbol{H}(\boldsymbol{v}^e((i-1/2)\Delta x,t))dt = 0. \quad (6.2\text{-}8)$$

By equation (6.2-6), we have

$$\boldsymbol{v}^e((i-1/2)\Delta x,t) = \boldsymbol{R}(0,\boldsymbol{v}_{i-1}^n,\boldsymbol{v}_i^n),$$
$$\boldsymbol{v}^e((i+1/2)\Delta x,t) = \boldsymbol{R}(0,\boldsymbol{v}_i^n,\boldsymbol{v}_{i+1}^n). \quad (6.2\text{-}9)$$

Using equations (6.2-7) and (6.2-9), equation (6.2-8) can be written as

$$\boldsymbol{v}_i^{n+1} = \boldsymbol{v}_i^n - \frac{\Delta t}{\Delta x}(\boldsymbol{H}_{i+1/2}^n - \boldsymbol{H}_{i-1/2}^n), \quad (6.2\text{-}10)$$

where

$$\boldsymbol{H}_{i-1/2}^n = \boldsymbol{H}(\boldsymbol{R}(0,\boldsymbol{v}_{i-1}^n,\boldsymbol{v}_i^n)),$$
$$\boldsymbol{H}_{i+1/2}^n = \boldsymbol{H}(\boldsymbol{R}(0,\boldsymbol{v}_i^n,\boldsymbol{v}_{i+1}^n)).$$

The finite difference scheme (6.2-10) is called Godunov's scheme. Clearly, Godunov's scheme is in conservation form. Since the exact solution $\boldsymbol{v}^e(x,t)$ satisfies the entropy condition

$$\frac{\partial U(\boldsymbol{v}^e(x,t))}{\partial t} + \frac{\partial F(\boldsymbol{v}^e(x,t))}{\partial x} \le 0, \quad (6.2\text{-}11)$$

then integrating (6.2-11) over the rectangle $I_i \times [n\Delta t,(n+1)\Delta t]$, we have

$$\int_{I_i} U(\boldsymbol{v}^e(x,(n+1)\Delta t))dx - \int_{I_i} U(\boldsymbol{v}^e(x,n\Delta t))dx$$
$$+ \int_{n\Delta t}^{(n+1)\Delta t} F(\boldsymbol{v}^e((i+1/2)\Delta x,t))dt - \int_{n\Delta t}^{(n+1)\Delta t} F(\boldsymbol{v}^e((i-1/2)\Delta x,t))dt \le 0. \quad (6.2\text{-}12)$$

Notice that

$$v^e(x, n\Delta t) = v_i^n, \qquad for \quad x \in I_i,$$

$$v^e((i-1/2)\Delta x, t) = R(0, v_{i-1}^n, v_i^n),$$

$$v^e((i+1/2)\Delta x, t) = R(0, v_i^n, v_{i+1}^n),$$

so that inequality (6.2-12) can be simplified to

$$\int_{I_i} U(v^e(x,(n+1)\Delta t))dx \leq \Delta x U_i^n - \Delta t(F_{i+1/2}^n - F_{i-1/2}^n), \qquad (6.2\text{-}13)$$

where

$$U_i^n = U(v_i^n),$$

$$F_{i+1/2}^n = F(R(0, v_i^n, v_{i+1}^n)),$$

$$F_{i-1/2}^n = F(R(0, v_{i-1}^n, v_i^n)).$$

Since U is a convex function, Jensen's inequality [21] holds in the form

$$U(\frac{1}{\Delta x}\int_{I_i} u(x,t)dx) \leq \frac{1}{\Delta x}\int_{I_i} U(u(x,t))dx. \qquad (6.2\text{-}14)$$

By using inequalities (6.2-13) and (6.2-14), we can verify that the entropy inequality is satisfied, namely

$$U_i^{n+1} \leq U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n). \qquad (6.2\text{-}15)$$

By theorem (6.1-2), the approximate solution obtained by Godunov's scheme will approach a weak solution of system (6.2-1) which satisfies the weak form of entropy condition as we refine the grid sizes, provided it converges.

6.3 Approximate Riemann Solvers

Godunov's scheme depends on exact solutions of Riemann problems. The exact solution of the Riemann problem for a general system of n conservation laws with $n > 2$. is known only when the left state $\boldsymbol{u}_L$ is close to the right state $\boldsymbol{u}_R$. If $\boldsymbol{u}_L$ is not close to $\boldsymbol{u}_R$ and if $n > 2$, then the solution of the Riemann problem is not known in general except for some specific problems (for example, Michael Shearer [22]). Even for the problems for which the exact solution of the Riemann problem is known, it can be very costly to apply the exact solution to Godunov's scheme. On the other hand, since Godunov's scheme does not make use of all information in the exact solution of the Riemann problem, this implies that we may replace the exact solution of the Riemann problem $\boldsymbol{R}(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R)$ by an approximate solution $\boldsymbol{w}(x/t, \boldsymbol{u}_l, \boldsymbol{u}_R)$ as long as it does not violate conservation properties and the entropy inequality. The numerical schemes based on approximate solutions of the Riemann problem are called *Riemann solvers*.

Godunov's scheme was extended by Harten, Lax and Van Leer [14] in the following theorem

Theorem 6.3-1

Let $\boldsymbol{w}(x/t, \boldsymbol{u}_l, \boldsymbol{u}_R)$ be an approximation to the solution of the Riemann problem that satisfies the following conditions:

$$1) \qquad \int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R)dx = \frac{\Delta x}{2}(\boldsymbol{u}_L + \boldsymbol{u}_R) - \Delta t(\boldsymbol{H}_R - \boldsymbol{H}_L), \qquad (6.3\text{-}1)$$

for $\Delta x/2 > \Delta t \max|\Lambda_k(\boldsymbol{u})|$, where $\boldsymbol{H}_L = \boldsymbol{H}(\boldsymbol{u}_L)$, $\boldsymbol{H}_R = \boldsymbol{H}(\boldsymbol{u}_R)$,

$$2) \qquad \int_{-\Delta x/2}^{\Delta x/2} U(\boldsymbol{w}(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R))dx \le \frac{\Delta x}{2}(U_L + U_R) - \Delta t(F_R - F_L), \qquad (6.3\text{-}2)$$

for $\Delta x/2 > \Delta t \max|\Lambda_k(\boldsymbol{u})|$, where $F_L = F(\boldsymbol{u}_L)$, $F_R = F(\boldsymbol{u}_R)$. Define a

Godunov-type scheme as follows

$$v_i^{n+1} = \frac{1}{\Delta x}\int_0^{\Delta x/2} w(x/t, v_{i-1}^n, v_i^n)dx + \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} w(x/t, v_i^n, v_{i+1}^n)dx. \quad (6.3\text{-}3)$$

Then if conditions 1) and 2) are satisfied, the Godunov-type scheme is in conservation form consistent with system (6.1-1) and satisfies the entropy inequality (6.1-16).

It can be shown that Godunov's scheme is of Godunov type.

The simplest approximate Riemann solver contains only one intermediate state [14]. If a_L denotes the lower bound of the smallest signal velocity and a_R denotes the upper bound of the largest signal velocity, we define the approximate Riemann solver by

$$w(x/t, u_L, u_R) = \begin{cases} u_L, & x/t < a_L, \\ u_{LR}, & a_L < x/t < a_R, \\ u_R, & a_R < x/t, \end{cases} \quad (6.3\text{-}4)$$

where the intermediate state u_{LR} is determined by the integral form of the conservation law (6.3-1) as

$$u_{LR} = \frac{a_R u_R - a_L u_L}{a_R - a_L} - \frac{H_R - H_L}{a_R - a_L}. \quad (6.3\text{-}5)$$

The above Riemann solver can be extended to a two-step Riemann solver which satisfies the integral form of the conservation law and the entropy condition.

6.4 Two-Step Riemann Solver

The simplest Riemann solver with only one intermediate state described in the last section can be extended to a *two-step Riemann solver* as follows.

Consider the Riemann problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u})}{\partial x} = \boldsymbol{0},$$
$$\boldsymbol{u}(x, n\Delta t) = \begin{cases} \boldsymbol{u}_L, & x \in [-\Delta x/2, 0), \\ \boldsymbol{u}_R, & x \in (0, \Delta x/2], \end{cases} \tag{6.4-1}$$

on the region $[-\Delta x/2, \Delta x/2] \times [n\Delta t, (n+1)\Delta t]$. If we denote the simplest Riemann solver given in last section by $\boldsymbol{w}_H(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R)$, then

$$\boldsymbol{w}_H(x/t, \boldsymbol{u}_L, \boldsymbol{u}_R) = \begin{cases} \boldsymbol{u}_L, & x/t < a_L, \\ \boldsymbol{u}_{LR}, & a_L < x/t < a_R, \\ \boldsymbol{u}_R, & a_R < x/t, \end{cases} \tag{6.4-2}$$

where

$$\boldsymbol{u}_{LR} = \frac{a_R \boldsymbol{u}_R - a_L \boldsymbol{u}_L}{a_R - a_L} - \frac{\boldsymbol{H}_R - \boldsymbol{H}_L}{a_R - a_L}. \tag{6.4-3}$$

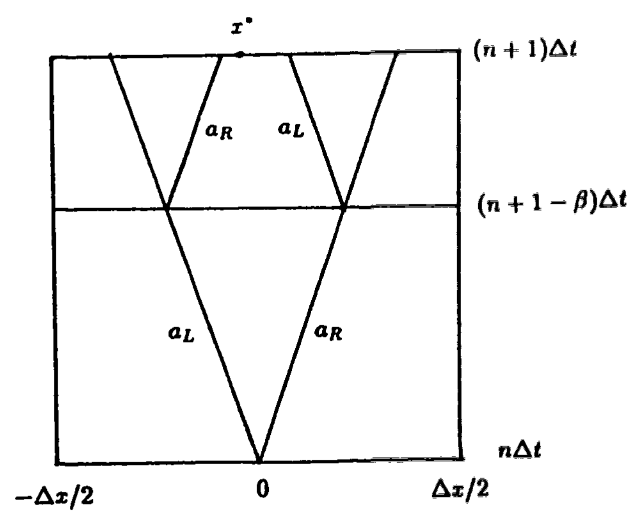


Fig.6.4-1

We apply the Riemann solver $\boldsymbol{w}_H(\frac{x}{t}, \boldsymbol{u}_L, \boldsymbol{u}_R)$ to the region $(x,t) \in [-\frac{\Delta x}{2}, \frac{\Delta x}{2}]$ $\times [n\Delta t, (n+1-\beta)\Delta t]$, where $0 \le \beta < 1$, (see Fig.6.4-1). Then, when $t = (n+1-\beta)\Delta t$, the approximate Riemann solver $\boldsymbol{w}_H$ has three constant states;

these states constitute two neighbouring Riemann problems with initial conditions

$$u(x,(n+1-\beta)\Delta t) = \begin{cases} u_L, & -\Delta x/2 < x < a_L(1-\beta)\Delta t, \\ u_{LR}, & a_L(1-\beta)\Delta t < x < a_R(1-\beta)\Delta t, \end{cases} \tag{6.4-4}$$

and

$$u(x,(n+1-\beta)\Delta t) = \begin{cases} u_{LR}, & a_L(1-\beta)\Delta t < x < a_R(1-\beta)\Delta t, \\ u_R, & a_R(1-\beta)\Delta t < x < \Delta x/2. \end{cases} \tag{6.4-5}$$

With suitable translation of the origin, we can apply the Riemann solver w_H for the Riemann problems with data given by equations (6.4-4) and (6.4-5) (see Fig.6.4-1). We shall denote the two step Riemann solver constructed above as $w_\beta(x,t)$. Then $w_\beta(x,t)$ has five constant states when $t=(n+1)\Delta t$. These states are given by

$$\begin{aligned} u_A &= u_L, \qquad u_B = u_{LR}, \qquad u_C = u_R, \\ u_{AB} &= \frac{a_R u_{LR} - a_L u_L}{a_R - a_L} - \frac{H(u_{LR}) - H_L}{a_R - a_L}, \\ u_{BC} &= \frac{a_R u_R - a_L u_{LR}}{a_R - a_L} - \frac{H(u_R) - H(u_{LR})}{a_R - a_L}. \end{aligned} \tag{6.4-6}$$

In order that the neighbouring Riemann problems have no interaction, we require

$$a_L(1-\beta)\Delta t + a_R\beta\Delta t \le a_R(1-\beta)\Delta t + a_L\beta\Delta t. \tag{6.4-7}$$

Since $a_R - a_L > 0$, we then have

$$0 \le \beta \le \frac{1}{2}. \tag{6.4-8}$$

Next, we will show that the two-step Riemann solver $w_\beta(x,t)$ satisfies the integral form of conservation law (6.3-1). Clearly, we have

$$\int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}_H \left(\frac{x}{(n+1-\beta)\Delta t}, \boldsymbol{u}_L, \boldsymbol{u}_R\right) dx = \frac{\Delta x}{2}(\boldsymbol{u}_L + \boldsymbol{u}_R) - (1-\beta)\Delta t(\boldsymbol{H}_R - \boldsymbol{H}_L). \tag{6.4-9}$$

If x^* is some point such that (see Fig.6.4-1)

$$a_L(1-\beta)\Delta t + a_R\beta\Delta t \le x^* \le a_R(1-\beta)\Delta t + a_L\beta\Delta t, \tag{6.4-10}$$

then

$$\int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}_\beta(x,(n+1)\Delta t)dx = \int_{-\Delta x/2}^{x^*} \boldsymbol{w}_\beta dx + \int_{x^*}^{\Delta x/2} \boldsymbol{w}_\beta dx. \tag{6.4-11}$$

It can be shown that

$$\int_{-\Delta x/2}^{x^*} \boldsymbol{w}_\beta(x,(n+1)\Delta t)dx = (\Delta x/2 + a_L(1-\beta)\Delta t)\boldsymbol{u}_L$$
$$+ (x^* - a_L(1-\beta)\Delta t)\boldsymbol{u}_{LR} - \beta\Delta t(\boldsymbol{H}_{LR} - \boldsymbol{H}_L), \tag{6.4-12}$$

and

$$\int_{x^*}^{\Delta x/2} \boldsymbol{w}_\beta(x,(n+1)\Delta t)dx = (\Delta x/2 - a_R(1-\beta)\Delta t)\boldsymbol{u}_R$$
$$+ (a_R(1-\beta)\Delta t - x^*)\boldsymbol{u}_{LR} - \beta\Delta t(\boldsymbol{H}_R - \boldsymbol{H}_{LR}), \tag{6.4-13}$$

where $\boldsymbol{H}_{LR} = \boldsymbol{H}(\boldsymbol{u}_{LR})$. By equations (6.4-11), (6.4-12) and (6.4-13) , we have

$$\int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}_\beta(x,(n+1)\Delta t)dx = (\Delta x/2 + a_L(1-\beta)\Delta t)\boldsymbol{u}_L +$$
$$(\Delta x/2 - a_R(1-\beta)\Delta t)\boldsymbol{u}_R + (a_R - a_L)(1-\beta)\Delta t\boldsymbol{u}_{LR} - \beta\Delta t(\boldsymbol{H}_R - \boldsymbol{H}_L), \tag{6.4-14}$$

since

$$\int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}_H(\frac{x}{(n+1-\beta)\Delta t}, \boldsymbol{u}_L, \boldsymbol{u}_R)dx = (\Delta x/2 + a_L(1-\beta)\Delta t)\boldsymbol{u}_L + (\Delta x/2 - a_R(1-\beta)\Delta t)\boldsymbol{u}_R + (a_R - a_L)(1-\beta)\Delta t \boldsymbol{u}_{LR}, \quad (6.4\text{-}15)$$

then by equations (6.4-9), (6.4-14) and (6.4-15), we have

$$\int_{-\Delta x/2}^{\Delta x/2} \boldsymbol{w}_\beta(x, (n+1)\Delta t)dx = \frac{\Delta x}{2}(\boldsymbol{u}_L + \boldsymbol{u}_R) - \Delta t(\boldsymbol{H}_R - \boldsymbol{H}_L). \quad (6.4\text{-}16)$$

Hence the two-step Riemann solver $\boldsymbol{w}_\beta$ satisfies the integral form of the conservation law (6.3-1). We will show that $\boldsymbol{w}_\beta$ also satisfies the entropy condition (6.3-2). We need to show

$$\int_{-\Delta x/2}^{\Delta x/2} U(\boldsymbol{w}_\beta(x, (n+1)\Delta t))dx \leq \frac{\Delta x}{2}(U_L + U_R) - \Delta t(F_R - F_L). \quad (6.4\text{-}17)$$

We already know that

$$\int_{-\Delta x/2}^{\Delta x/2} U(\boldsymbol{w}_H(x/(n+1-\beta)\Delta t, \boldsymbol{u}_L, \boldsymbol{u}_R))dx \leq \frac{\Delta x}{2}(U_L + U_R) - (1-\beta)\Delta t(F_R - F_L). \quad (6.4\text{-}18)$$

Since

$$\int_{-\Delta x/2}^{\Delta x/2} U(\boldsymbol{w}_H(x/(n+1-\beta)\Delta t, \boldsymbol{u}_L, \boldsymbol{u}_R))dx = (\Delta x/2 + a_L(1-\beta)\Delta t)U_L + (\Delta x/2 - a_R(1-\beta)\Delta t)U_R + (a_R - a_L)(1-\beta)\Delta t U_{LR}, \quad (6.4\text{-}19)$$

where $U_{LR} = U(\boldsymbol{u}_{LR})$, then using equations (6.4-18) and (6.4-19), we can show that

$$U_{LR} \leq \frac{a_R U_R - a_L U_L}{a_R - a_L} - \frac{F_R - F_L}{a_R - a_L}. \quad (6.4\text{-}20)$$

Now we consider a Riemann problem in the region $(x,t) \in [-\Delta x/2, x^*] \times [(n+1-\beta)\Delta t, (n+1)\Delta t]$ with initial conditions

$$u(x,(n+1-\beta)\Delta t) = \begin{cases} u_L, & -\Delta x/2 \le x < a_L(1-\beta)\Delta t, \\ u_{LR}, & a_L(1-\beta)\Delta t < x \le x^*. \end{cases} \tag{6.4-21}$$

If $u^e(x,t)$ denotes the exact solution of the above Riemann problem, then

$$\frac{\partial U(u^e)}{\partial t} + \frac{\partial F(u^e)}{\partial x} \le 0. \tag{6.4-22}$$

Integrating the inequality (6.4-22) over $[-\Delta x/2, x^*] \times [(n+1-\beta)\Delta t, (n+1)\Delta t]$, we have

$$\int_{-\Delta x/2}^{x^*} U(u^e(x,(n+1)\Delta t))dx - \int_{-\Delta x/2}^{x^*} U(u^e(x,(n+1-\beta)\Delta t))dx -$$
$$\int_{(n+1-\beta)\Delta t}^{(n+1)\Delta t} F(u^e(x^*,t))dt - \int_{(n+1-\beta)\Delta t}^{(n+1)\Delta t} F(u^e(-\Delta x/2,t))dt \le 0. \tag{6.4-23}$$

Notice that $u^e(x^*,t) = u_{LR}$ and $u^e(-\Delta x/2,t) = u_L$, so that by (6.4-23)

$$\int_{-\Delta x/2}^{x^*} U(u^e(x,(n+1)\Delta t))dx \le (\Delta x/2 + a_L(1-\beta)\Delta t)U_L -$$
$$a_L(1-\beta)\Delta t U_{LR} - \beta\Delta t(F_{LR} - F_L). \tag{6.4-24}$$

We will show

$$\int_{-\Delta x/2}^{x^*} U(w_\beta(x,(n+1)\Delta t))dx \le \int_{-\Delta x/2}^{x^*} U(u^e(x,(n+1)\Delta t))dx. \tag{6.4-25}$$

Since we have $w_\beta(x,(n+1)\Delta t) = u^e(x,(n+1)\Delta t) = u_L$ when $x \in [-\Delta x/2, a_L\Delta t)$ and $w_\beta(x,(n+1)\Delta t) = u^e(x,(n+1)\Delta t) = u_{LR}$ when $x \in (a_L\Delta t + (a_R - a_L)\beta\Delta t, x^*]$, we need only show

$$\int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} U(w_\beta(x,(n+1)\Delta t))dx \le \int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} U(u^e(x,(n+1)\Delta t))dx. \tag{6.4-26}$$

Since when $a_L\Delta t \le x \le a_L\Delta t + (a_R - a_L)\beta\Delta t$ and $t = (n+1)\Delta t$, we have

$$w_\beta = \frac{1}{(a_R - a_L)\beta\Delta t}\int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} u^e(x,(n+1)\Delta t)dx = u_{AB} \qquad (6.4\text{-}27)$$

and since U is convex, then by Jensen's inequality

$$\begin{aligned}&\int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} U(w_\beta(x,(n+1)\Delta t))dx = (a_R-a_L)\beta\Delta t U(w_\beta)\\ &= (a_R-a_L)\beta\Delta t U(\frac{1}{(a_R-a_L)\beta\Delta t}\int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} u^e(x,(n+1)\Delta t)dx)\\ &\le (a_R-a_L)\beta\Delta t\frac{1}{(a_R-a_L)\beta\Delta t}\int_{a_L\Delta t}^{a_L\Delta t+(a_R-a_L)\beta\Delta t} U(u^e(x,(n+1)\Delta t))dx.\end{aligned} \qquad (6.4\text{-}28)$$

Thus inequality (6.4-26) holds and hence inequality (6.4-25) is satisfied. We then have

$$\begin{aligned}&\int_{-\Delta x/2}^{x^*} U(w_\beta(x,(n+1)\Delta t))dx \le (\Delta x/2 + a_L(1-\beta)\Delta t)U_L -\\ &a_L(1-\beta)\Delta t U_{LR} - \beta\Delta t(F_{LR} - F_L).\end{aligned} \qquad (6.4\text{-}29)$$

Similarly, we can show that

$$\begin{aligned}&\int_{x^*}^{\Delta x/2} U(w_\beta(x,(n+1)\Delta t))dx \le (\Delta x/2 - a_R(1-\beta)\Delta t)U_R +\\ &a_R(1-\beta)\Delta t U_{LR} - \beta\Delta t(F_R - F_{LR}).\end{aligned} \qquad (6.4\text{-}30)$$

Therefore

$$\begin{aligned}&\int_{-\Delta x/2}^{\Delta x/2} U(w_\beta(x,(n+1)\Delta t))dx \le\\ &\frac{\Delta x}{2}(U_L + U_R) + (a_L U_L - a_R U_R)(1-\beta)\Delta t +\\ &(a_R - a_L)(1-\beta)\Delta t U_{LR} - \beta\Delta t(F_R - F_L).\end{aligned} \qquad (6.4\text{-}31)$$

By using inequality (6.4-20), inequality (6.4-31) can be simplified as

$$\int_{-\Delta x/2}^{\Delta x/2} U(\boldsymbol{w}_\beta(x,(n+1)\Delta t))dx \le \frac{\Delta x}{2}(U_L + U_R) - \Delta t(F_R - F_L). \qquad (6.4\text{-}32)$$

Thus inequality (6.4-17) is satisfied. Therefore the two step Riemann solver $\boldsymbol{w}_\beta(x,t)$ satisfies all conditions in theorem (6.3-1) and the Godunov-type scheme

$$\boldsymbol{v}_i^{n+1} = \frac{1}{\Delta x}\int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} \boldsymbol{w}_\beta(x,(n+1)\Delta t)dx \qquad (6.4\text{-}33)$$

will be in conservation form and satisfy the entropy inequality

$$U_i^{n+1} \le U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n). \qquad (6.4\text{-}34)$$

6.5 Numerical Analysis of the Normal Impact of a Nonlinear Elastic String

The governing system of equations of the wave motion in an elastic string is in nondimensional form (see Chapter 3),

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u})}{\partial X} = \boldsymbol{0}, \qquad 0 < X < 1, \quad t > 0, \qquad (6.5\text{-}1)$$

where

$$\begin{aligned} \boldsymbol{u} &= (\lambda\cos\theta, \lambda\sin\theta, u, v)^T, \\ \boldsymbol{H} &= -(u, v, T(\lambda)\cos\theta, T(\lambda)\sin\theta)^T. \end{aligned} \qquad (6.5\text{-}2)$$

The function $T(\lambda)$ has been written in nondimensional form

$$T(\lambda) = \frac{1}{\mu}\sum_{i=1}^{3}\mu_i(\lambda^{\alpha_i-1} - \lambda^{-\frac{\alpha_i}{2}-1}), \qquad (6.5\text{-}3)$$

where Ogden's values

$$\mu_1 = 1.491\mu, \quad \mu_2 = 0.003\mu, \quad \mu_3 = -0.0237\mu,$$
$$\alpha_1 \doteq 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0 \tag{6.5-4}$$

are used. The graph of the function $T(\lambda)$ can be found in Fig.3.2-1. The eigenvalues of the Jacobian matrix $\frac{\partial H(\boldsymbol{u})}{\partial \boldsymbol{u}}$ are $\pm\Lambda_1$ and $\pm\Lambda_2$ where

$$\Lambda_1 = (T'(\lambda))^{1/2}, \qquad \Lambda_2 = (T(\lambda)/\lambda)^{1/2}. \tag{6.5-5}$$

The initial condition is

$$\boldsymbol{u}(X,0) = (\lambda_0, 0, 0, 0)^T. \tag{6.5-6}$$

The boundary conditions are

$$u(0,t) = 0, \quad u(1,t) = v(1,t) = 0,$$
$$v(0,0^+) = q,$$
$$\frac{dv(0,t)}{dt} = 2\alpha T(\lambda(0,t)) \sin\theta(0,t). \tag{6.5-7}$$

For this problem, we may define an entropy function U and entropy flux F by

$$U = \frac{1}{2}(u^2 + v^2) + \int_{\lambda_0}^{\lambda} T(s)ds,$$
$$F = -T(\lambda)(u\cos\theta + v\sin\theta). \tag{6.5-8}$$

It can be verified that

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial X} = 0,$$
$$U_{\boldsymbol{uu}} > 0, \quad \text{(positive definite)}$$
$$U_{\boldsymbol{u}} H_{\boldsymbol{u}} = F_{\boldsymbol{u}}. \tag{6.5-9}$$

Consider first the application of the simplest Riemann solver with only one intermediate state

$$w_H(X/t, u_L, u_R) = \begin{cases} u_L, & X/t < a_L, \\ u_{LR}, & a_L < X/t < a_R, \\ u_R, & a_R < X/t, \end{cases} \tag{6.5-10}$$

where a_L and a_R are the lower and upper bounds the of signal velocity respectively. Then we have

$$-a_L = a_R = a. \tag{6.5-11}$$

The intermediate state u_{LR} in equation (6.5-10) is then given by the following

$$u_{LR} = \frac{u_R + u_L}{2} - \frac{H_R - H_L}{2a}. \tag{6.5-12}$$

Recall that

$$v_i^{n+1} = \frac{1}{\Delta X} \int_0^{\Delta X/2} w(X/t, v_{i-1}^n, v_i^n) dX + \frac{1}{\Delta X} \int_{-\Delta X/2}^{0} w(X/t, v_i^n, v_{i+1}^n) dX. \tag{6.5-13}$$

We then obtain the numerical scheme for the simplest Riemann solver

$$v_i^{n+1} = (1 - ar) v_i^n + \frac{r}{2}(a(v_{i+1}^n + v_{i-1}^n) - (H(v_{i+1}^n) - H(v_{i-1}^n)), \tag{6.5-14}$$

where $r \equiv \frac{\Delta t}{\Delta x}$. The above scheme can be written in conservation form

$$v_i^{n+1} = v_i^n - r(G_{i+1/2}^n - G_{i-1/2}^n), \tag{6.5-15}$$

where

$$
\begin{aligned}
G^n_{i+1/2} &= G(v^n_i, v^n_{i+1}) \\
&= \frac{1}{2}(a(v^n_i - v^n_{i+1}) + H(v^n_{i+1}) + H(v^n_i)), \\
G^n_{i-1/2} &= G(v^n_{i-1}, v^n_i) \\
&= \frac{1}{2}(a(v^n_{i-1} - v^n_i) + H(v^n_i) + H(v^n_{i-1})),
\end{aligned}
\tag{6.5-16}
$$

and it can be verified that

$$
G(u, u) = H(u). \tag{6.5-17}
$$

The corresponding numerical scheme for the two-step Riemann solver is

$$
\begin{aligned}
v^{n+1}_i = (1 - ar)v^n_i + \frac{r(1-\beta)}{2}(a(v^n_{i+1} + v^n_{i-1}) - (H(v^n_{i+1}) - H(v^n_{i-1})) \\
- \beta r(H(\frac{v^n_{i+1} + v^n_i}{2} - \frac{H^n_{i+1} - H^n_i}{2}) - H(\frac{v^n_{i+1} + v^n_i}{2} - \frac{H^n_{i+1} - H^n_i}{2})).
\end{aligned}
\tag{6.5-18}
$$

If $\beta = 0$, then scheme (6.5-18) reduces to the scheme (6.5-14). Scheme (6.5-18) can be written in conservation form as

$$
v^{n+1}_i = v^n_i - r(F^n_{i+1/2} - F^n_{i-1/2}), \tag{6.5-19}
$$

where

$$\begin{aligned}
\boldsymbol{F}^n_{i+1/2} &= \boldsymbol{F}(\boldsymbol{v}^n_i, \boldsymbol{v}^n_{i+1}) \\
&= \frac{1}{2}(1-\beta)(a(\boldsymbol{v}^n_i - \boldsymbol{v}^n_{i+1}) + \boldsymbol{H}(\boldsymbol{v}^n_{i+1}) + \boldsymbol{H}(\boldsymbol{v}^n_i)) + \\
&\quad \beta \boldsymbol{H}(\frac{\boldsymbol{v}^n_{i+1} + \boldsymbol{v}^n_i}{2} - \frac{\boldsymbol{H}^n_{i+1} - \boldsymbol{H}^n_i}{2a}), \\
\boldsymbol{F}^n_{i-1/2} &= \boldsymbol{F}(\boldsymbol{v}^n_{i-1}, \boldsymbol{v}^n_i) \\
&= \frac{1}{2}(1-\beta)(a(\boldsymbol{v}^n_{i-1} - \boldsymbol{v}^n_i) + \boldsymbol{H}(\boldsymbol{v}^n_i) + \boldsymbol{H}(\boldsymbol{v}^n_{i-1})) + \\
&\quad \beta \boldsymbol{H}(\frac{\boldsymbol{v}^n_i + \boldsymbol{v}^n_{i-1}}{2} - \frac{\boldsymbol{H}^n_i - \boldsymbol{H}^n_{i-1}}{2a}).
\end{aligned} \tag{6.5-20}$$

It can be verified that

$$\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{H}(\boldsymbol{u}). \tag{6.5-21}$$

As before, we denote the initial stretch and the maximum stretch by λ_0 and λ_m. If $1 < \lambda_0 < \lambda_m < \lambda_{c1}$, then $\Lambda_1(\lambda_0) > \Lambda_1(\lambda) > \Lambda_2(\lambda)$ for $\lambda_0 < \lambda \le \lambda_m$ and hence we may take $a = \Lambda_1(\lambda_0)$.

In Fig.6.5-1, we plot the graph of λ as a function of X when $\lambda_0 = 1.5$, $q = 0.2$ and $\alpha = 1.0$ using the simplest Riemann solver and the results of the perturbation method. In Fig.6.5-2, we plot the graph of θ as a function of X for the same data as in Fig.6.5-1.

If $\lambda_i < \lambda_0 < \lambda_m < \lambda_{c2}$, then $\Lambda_2(\lambda_0) > \Lambda_2(\lambda) > \Lambda_1(\lambda) > \Lambda_1(\lambda_0)$ for $\lambda_0 < \lambda \le \lambda_m$ and hence we may take $a = \Lambda_2(\lambda_0)$.

In Fig.6.5-3, we plot the graph of λ as a function of X when $\lambda_0 = 2.8$, $q = 0.3$ and $\alpha = 1.0$ using the simplest Riemann solver and the results of the perturbation method. In Fig.6.5-4, we plot the graph of θ as a function of X for the same data as in Fig.6.5-3.

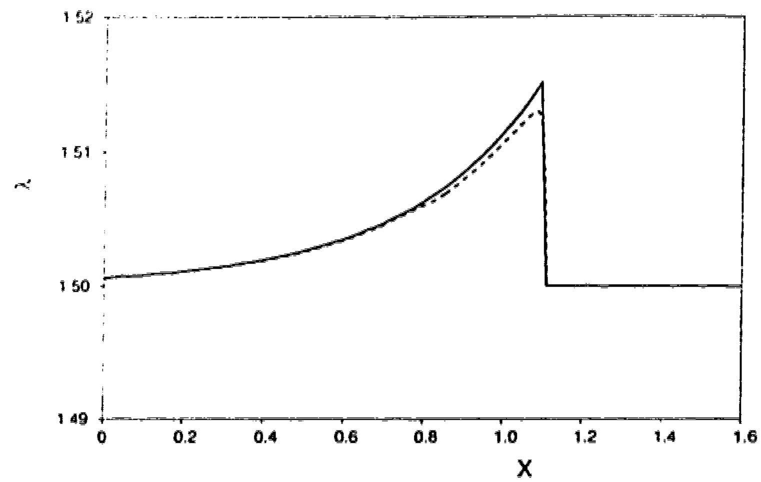


Fig.6.5-1 $\lambda_0 = 1.5$, $q = 0.2$, $\alpha = 1.0$, $t = 1.0$.

—perturbation method,

- - -direct numerical method, dt=0.0005, r=0.9.

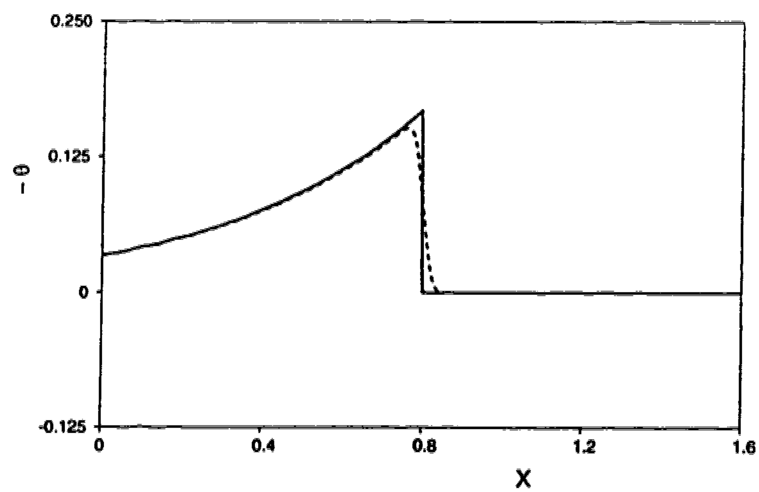


Fig.6.5-2 $\lambda_0 = 1.5$, $q = 0.2$, $\alpha = 1.0$, $t = 1.0$.

—perturbation method,

- - -direct numerical method, dt=0.0005, r=0.9.

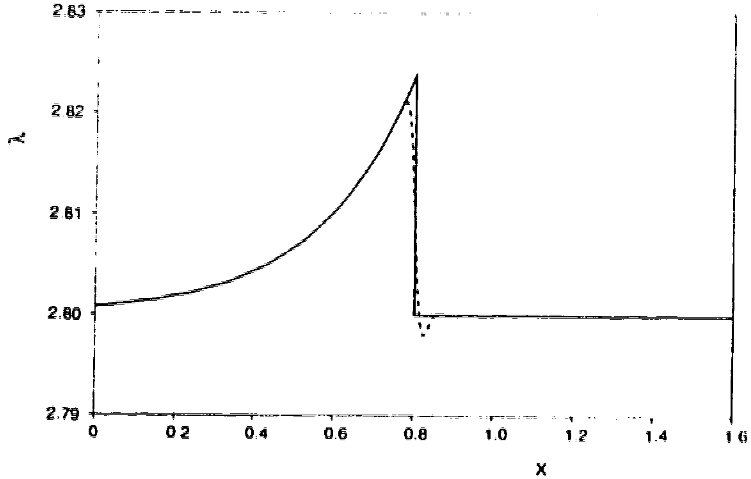


Fig.6.5-3 $\lambda_0 = 2.8$, $q = 0.3$, $\alpha = 1.0$, $t = 1.0$.

—perturbation method,

- - -direct numerical method, dt=0.0005, r=0.9.

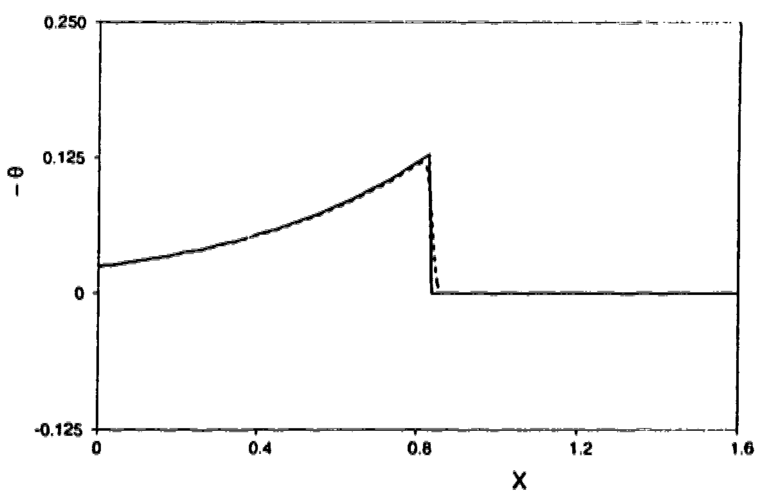


Fig.6.5-4 $\lambda_0 = 2.8$, $q = 0.3$, $\alpha = 1.0$, $t = 1.0$.

—perturbation method,

- - -direct numerical method, dt=0.0005, r=0.9.

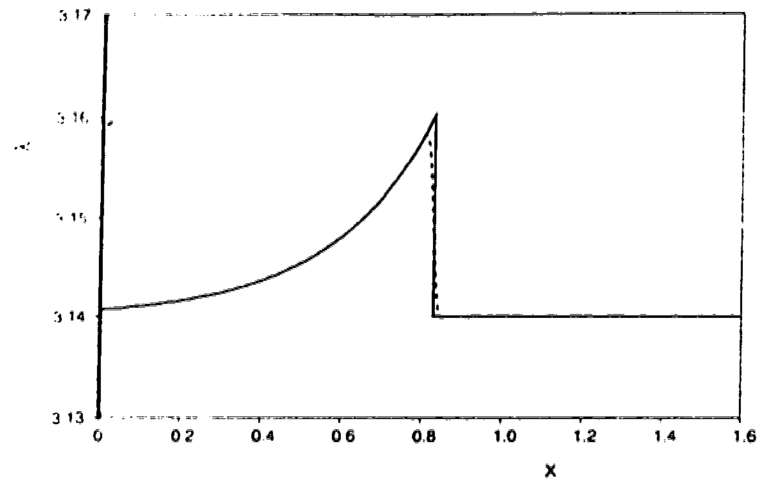


Fig.6.5-5 $\lambda_0 = 3.14$, $q = 0.3$, $\alpha = 1.0$, $t = 1.0$.

—approximation method,

- - -direct numerical method, dt=0.0005, r=0.9.

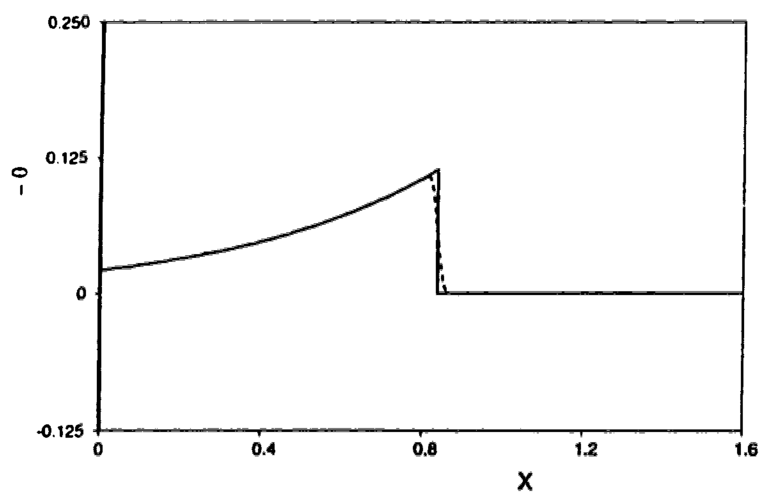


Fig.6.5-6 $\lambda_0 = 3.14$, $q = 0.3$, $\alpha = 1.0$, $t = 1.0$.

—approximation method,

- - -direct numerical method, dt=0.0005, r=0.9.

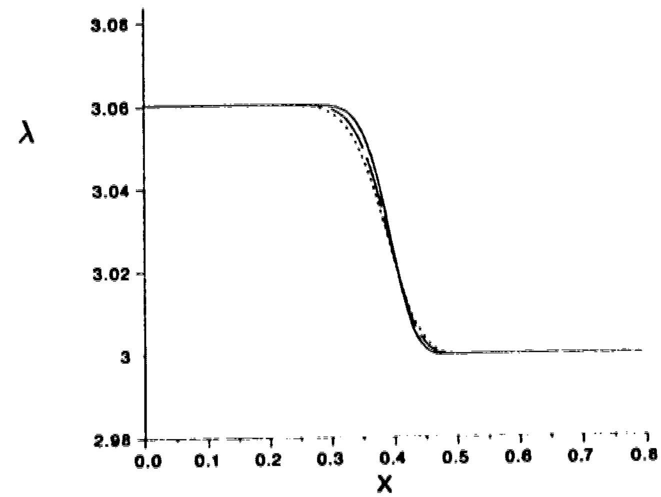


Fig.6.5-7 $\lambda_0 = 3.0$, $q = 0.5$, $t = 0.5$, $\alpha = 0.0$, $r = 0.25$, $\Delta t = 0.001$.

- - - - - -simplest Riemann solver,

— — —two step Riemann solver with $\beta = 0.25$,

———two step Riemann solver with $\beta = 0.5$.

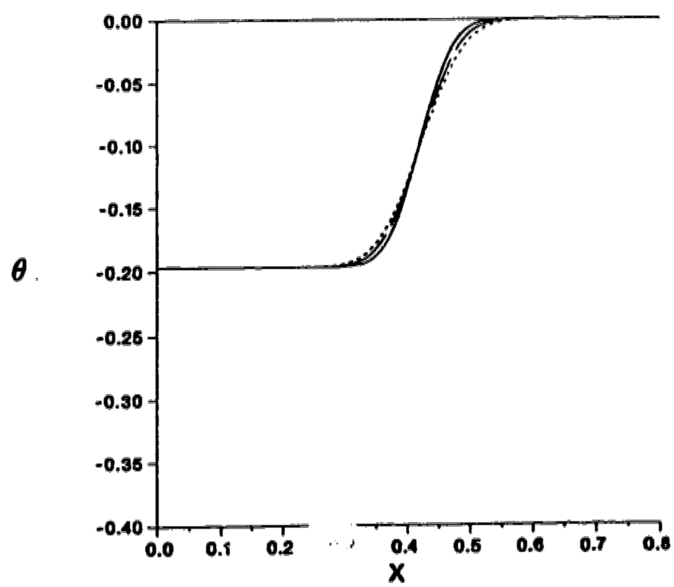


Fig.6.5-8 $\lambda_0 = 3.0$, $q = 0.5$, $t = 0.5$, $\alpha = 0.0$, $r = 0.25$, $\Delta t = 0.001$.

- - - - - -simplest Riemann solver,

— — —two step Riemann solver with $\beta = 0.25$,

———two step Riemann solver with $\beta = 0.5$.

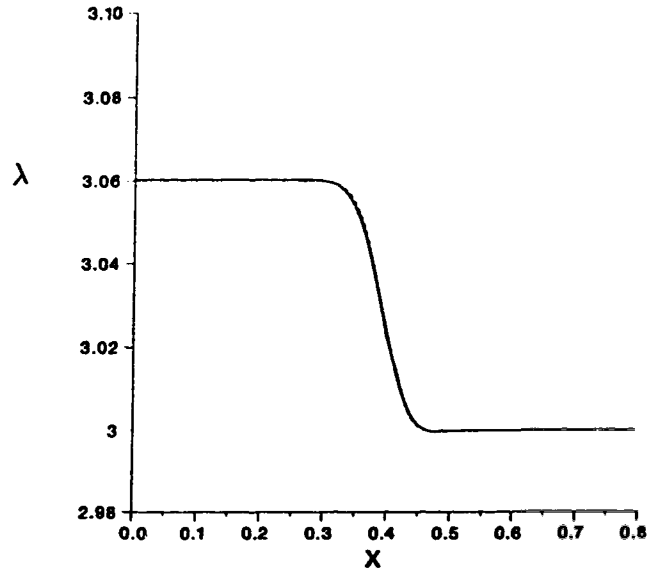


Fig.6.5-9 $\lambda_0 = 3.0$, $q = 0.5$, $t = 0.5$, $\alpha = 0.0$, $r = 0.25$.

- - - - - -simplest Riemann solver with $\Delta t = 0.001$,

———two step Riemann solver with $\beta = 0.5$, $\Delta t = 0.0005$.

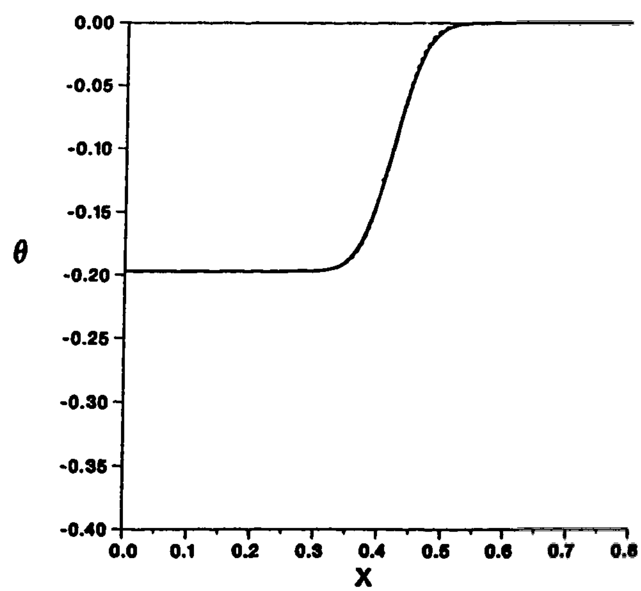


Fig.6.5-10 $\lambda_0 = 3.0$, $q = 0.5$, $t = 0.5$, $\alpha = 0.0$, $r = 0.25$.

- - - - - -simplest Riemann solver with $\Delta t = 0.001$,

———two step Riemann solver with $\beta = 0.5$, $\Delta t = 0.0005$.

In Fig.6.5-5, we plot the graph of λ as a function of X when $\lambda_0 = 3.14, q = 0.3$ and $\alpha = 1.0$ using the simplest Riemann solver and the results obtained by the approximation method described in Chapter 5. In Fig.6.5-6, we plot the graph of θ . as a function of X for the same data as in Fig.6.5-5.

In Fig.6.5-7, we plot the graph of λ as a function of X when $\lambda_0 = 3.0$, $q = 0.5$, $t = 0.5$ and $\alpha = 0.0$ by the simplest Riemann solver and by the two-step Riemann solver with $\beta = 0.25$ and $\beta = 0.5$. In Fig.6.5-8, we plot the graph of θ as a function of X for the same data as in Fig.6.5-7. We can see that for the same grid size ΔX and Δt, the results obtained by the two-step Riemann solver with $\beta > 0$ have better shock representation than those obtained by the simplest Riemann solver.

In Fig.6.5-9 and Fig.6.5-10, using the same data as those in Fig.6.5-7, we compare the λ and θ values obtained by the two-step Riemann solver with $\beta = 0.5$ and grid size $\Delta X = 0.004$, $\Delta t = 0.001$ and those obtained by the simplest Riemann solver with one half of the above grid size $\Delta X = 0.002$, $\Delta t = 0.0005$. The CPU time for the two-step Riemann solver is 78.6 Second, while the CPU time for the simplest Riemann solver is 127.4 Second. We can see that the results of the two methods are very close but the computing time of the two-step Riemann solver is le-- than that of the simplest Riemann solver.

Chapter 7

Normal Impact Problem for a Nonlinear Circular Membrane

7.1 Nonlinear Membrane Theory

A membrane theory had been developed by Green, Naghdi and Wainwright as a special case of the general theory of Cosserat surfaces [11]. They set up the theory in a way that is readily generalisable to consider thermomechanical effects, but we confine our attention here to a purely mechanical discussion and set up the membrane theory directly.

Assume that we have a fixed rectangular Cartesian coordinate system $OXYZ$ and that $\theta^\alpha(\alpha = 1, 2)$ are convected Gauss coordinates on the membrane considered as a two-dimensional surface which maintain a one-to-one correspondence with material points under deformation. If we take the undeformed state as the reference configuration , then a deformation carries the material point p with coordinates θ^α from its reference position $\boldsymbol{X}(\theta^1, \theta^2)$ to the deformed position $\boldsymbol{r}(\theta^1, \theta^2)$ in a three-dimensional Euclidean space. We define the natural basis in the deformed configuration by

$$\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha} \equiv \frac{\partial \boldsymbol{r}}{\partial \theta^\alpha}, \qquad (\alpha = 1, 2.) \tag{7.1-1}$$

If $\boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq \boldsymbol{0}$, then $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ span the tangent plane to the deformed surface at $\boldsymbol{r}$. The metric tensor has components

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \qquad a = \det(a_{\alpha\beta}) > 0, \tag{7.1-2}$$

and with $a > 0$ the reciprocal metric tensor $a^{\alpha\beta}$ and the dual basis $\boldsymbol{a}^\alpha$ are defined in the usual way with

$$a^{\alpha\mu} a_{\mu\beta} = \delta^\alpha_\beta, \quad \boldsymbol{a}^\alpha = a^{\alpha\beta} \boldsymbol{a}_\beta. \tag{7.1-3}$$

Here Greek indices take the values 1, 2, and the summation convention is used. If the bounding curve for the deformed membrane surface is denoted by l we denote the unit normal to the surface by $\hat{a}_3$, the tangent to l by $\hat{t}$, and the unit normal to l in the surface by $\hat{n}$. Then

$$\begin{aligned} \hat{a}_3 &= (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/a^{1/2}, \\ \hat{t} &= \frac{d\boldsymbol{r}}{ds} = t^\alpha \boldsymbol{a}_\alpha = \frac{d\theta^\alpha}{ds}\boldsymbol{a}_\alpha, \\ \hat{n} &= \hat{t} \times \hat{a}_3 = a^{1/2} e_{\alpha\beta}\frac{d\theta^\beta}{ds}\boldsymbol{a}^\alpha = -a^{1/2} t^\alpha e_{\alpha\beta}\boldsymbol{a}^\beta, \end{aligned} \tag{7.1-4}$$

where s is the arc length along l and $e_{12} = 1$, $e_{21} = -1$ and $e_{\alpha\beta} = 0$ otherwise. Similarly, we define basis vectors and a metric tensor on the undeformed surface as

$$\begin{aligned} \boldsymbol{A}_\alpha &= \boldsymbol{X}_{,\alpha} \equiv \frac{\partial \boldsymbol{X}}{\partial \theta^\alpha}, \\ A_{\alpha\beta} &= \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta, \quad A = \det(A_{\alpha\beta}) > 0, \\ A^{\alpha\mu} A_{\mu\beta} &= \delta^\alpha_\beta, \quad \boldsymbol{A}^\alpha = A^{\alpha\beta}\boldsymbol{A}_\beta. \end{aligned} \tag{7.1-5}$$

The deformation gradient $\mathcal{F}$ is defined by

$$d\boldsymbol{r} = \mathcal{F} d\boldsymbol{X}. \tag{7.1-6}$$

Since

$$\begin{aligned} d\boldsymbol{r} &= \boldsymbol{r}_{,\alpha} d\theta^\alpha = \boldsymbol{a}_\alpha d\theta^\alpha, \\ d\boldsymbol{X} &= \boldsymbol{X}_{,\alpha} d\theta^\alpha = \boldsymbol{A}_\alpha d\theta^\alpha, \end{aligned} \tag{7.1-7}$$

we have

$$\boldsymbol{A}_\beta \cdot d\boldsymbol{X} = \boldsymbol{A}_\beta \cdot \boldsymbol{A}_\alpha d\theta^\alpha = A_{\beta\alpha} d\theta^\alpha, \tag{7.1-8}$$

thus

$$A^{\alpha\beta}\boldsymbol{A}_\beta \cdot d\boldsymbol{X} = A^{\alpha\beta}A_{\beta\mu}d\theta^\mu = \delta^\alpha_\mu d\theta^\mu = d\theta^\alpha, \tag{7.1-9}$$

then by the first equation of (7.1-7) and equation (7.1-9), we have

$$\begin{aligned} d\boldsymbol{r} &= \boldsymbol{a}_\alpha d\theta^\alpha = \boldsymbol{a}_\alpha A^{\alpha\beta}(\boldsymbol{A}_\beta \cdot d\boldsymbol{X}) \\ &= \boldsymbol{a}_\alpha(\boldsymbol{A}^\alpha \cdot d\boldsymbol{X}) = (\boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha)d\boldsymbol{X}. \end{aligned} \tag{7.1-10}$$

Hence we have

$$\mathcal{F} = \boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha. \qquad (\alpha = 1, 2.) \tag{7.1-11}$$

If Σ denotes the undeformed membrane surface and σ the deformed membrane surface, dL an element of curve on Σ , $\hat{\boldsymbol{N}}$ the unit normal to dL in the surface Σ, dl an element of curve on σ, $\hat{\boldsymbol{n}}$ the unit normal to dl in the surface σ, and if $\boldsymbol{t}^*$ denotes the force per unit length on L, $\boldsymbol{t}$ denotes the force per unit length on l, then we have

$$\boldsymbol{t}^* dL = \boldsymbol{t} dl. \tag{7.1-12}$$

Now we define the Cauchy stress tensor $\mathcal{T}$ on the deformed surface σ by

$$\boldsymbol{t} = \mathcal{T}\hat{\boldsymbol{n}}. \tag{7.1-13}$$

With some calculation we obtain the two-dimensional Nanson's relation

$$\hat{\boldsymbol{n}} dl = J dL(\boldsymbol{a}^\alpha \otimes \boldsymbol{A}_\alpha)\hat{\boldsymbol{N}}, \tag{7.1-14}$$

where $J = (a/A)^{1/2}$. It then follows that

$$\boldsymbol{t}^* dL = \boldsymbol{t} dl = \mathcal{T}\hat{\boldsymbol{n}} dl = J\mathcal{T}(\boldsymbol{a}^\alpha \otimes \boldsymbol{A}_\alpha)\hat{\boldsymbol{N}} dL. \tag{7.1-15}$$

If we define Piola-Kirchhoff stress tensor $\mathcal{S}$ by

$$t^* = \mathcal{S}\hat{N}, \tag{7.1-16}$$

then

$$\mathcal{S} = JT\boldsymbol{a}^\alpha \otimes \boldsymbol{A}_\alpha. \tag{7.1-17}$$

If we define

$$\boldsymbol{T}^\alpha = JT\boldsymbol{a}^\alpha, \tag{7.1-18}$$

then we can write

$$\mathcal{S} = \boldsymbol{T}^\alpha \otimes \boldsymbol{A}_\alpha. \tag{7.1-19}$$

For unconstrained Green elastic materials, we assume that there exists an elastic potential energy function $W(\mathcal{F})$ per unit area of the undeformed membrane surface so that

$$\dot{W} = \mathrm{tr}(\mathcal{S}^T \dot{\mathcal{F}}). \tag{7.1-20}$$

Suppose $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is a fixed rectangular basis. We then have

$$\begin{aligned} \boldsymbol{r} &= r^i \hat{e}_i, \qquad \boldsymbol{a}_\alpha = r^i_{,\alpha} \hat{e}_i, \\ \mathcal{F} &= r^i_{,\alpha} \hat{e}_i \otimes \boldsymbol{A}^\alpha = F^i_\alpha \hat{e}_i \otimes \boldsymbol{A}^\alpha. \end{aligned} \tag{7.1-21}$$

Since

$$\begin{aligned} \dot{W} &= tr(\frac{\partial W}{\partial \mathcal{F}} \dot{\mathcal{F}}), \\ \dot{\mathcal{F}} &= \dot{r}^i_{,\alpha} \hat{e}_i \otimes \boldsymbol{A}^\alpha, \end{aligned} \tag{7.1-22}$$

then by equations (7.1-20) and (7.1-21), we have

$$\mathcal{S} = \frac{\partial W}{\partial r^i_{,\alpha}} \hat{e}_i \otimes \boldsymbol{A}_\alpha. \tag{7.1-23}$$

Comparing equation (7.1-19) and equation (7.1-23), we have

$$\boldsymbol{T}^\alpha = \frac{\partial W}{\partial r^i_{,\alpha}} \hat{e}_i. \tag{7.1-24}$$

The Cauchy stress tensor can be written in component form as

$$\mathcal{T} = T^{\alpha\beta} \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta, \qquad T^{\alpha\beta} = T^{\beta\alpha}, \tag{7.1-25}$$

and it can be shown that

$$\boldsymbol{T}^\alpha = J T^{\alpha\beta} \boldsymbol{a}_\beta. \tag{7.1-26}$$

We assume W is objective so that

$$W(\mathcal{F}) = \hat{W}(\mathcal{C}), \tag{7.1-27}$$

where

$$\begin{aligned} \mathcal{C} = \mathcal{F}^T \mathcal{F} &= (\boldsymbol{A}^\alpha \otimes \boldsymbol{a}_\alpha)(\boldsymbol{a}_\beta \otimes \boldsymbol{A}^\beta), \\ &= (\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta) \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta. \end{aligned} \tag{7.1-28}$$

Since

$$\mathcal{C} = C_{\alpha\beta} \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta, \tag{7.1-29}$$

then

$$C_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}. \tag{7.1-30}$$

It can be shown that

$$\frac{\partial W}{\partial r^i_{,\alpha}} = 2\frac{\partial \hat{W}}{\partial a_{\alpha\beta}} r^i_{,\alpha}. \tag{7.1-31}$$

Substituting equation (7.1-31) into equation (7.1-24), we have

$$\boldsymbol{T}^\alpha = 2\frac{\partial \hat{W}}{\partial a_{\alpha\beta}} \boldsymbol{a}_\beta. \tag{7.1-32}$$

By comparing equation (7.1-26) and equation (7.1-32), we get

$$T^{\beta\alpha} = T^{\alpha\beta} = \frac{2}{J}\frac{\partial \hat{W}}{\partial a_{\alpha\beta}}. \tag{7.1-33}$$

If we ignore the effect of heat transfer, then the equation of energy balance can be written as [11]:

$$\frac{D}{Dt}\iint_\sigma (\frac{1}{2}\rho \boldsymbol{v}\cdot\boldsymbol{v} + \rho U) d\sigma = \oint_l \boldsymbol{t}\cdot\boldsymbol{v} dl, \tag{7.1-34}$$

where ρ is the mass per unit area in the deformed surface σ, $\boldsymbol{v} = \frac{\partial \boldsymbol{r}}{\partial t}$ is the velocity, and U is the internal energy per unit mass. Equation (7.1-34) can be written as

$$\frac{D}{Dt}\iint_\Sigma (\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + U)\rho J d\Sigma = \oint_l \boldsymbol{t}\cdot\boldsymbol{v} dl, \tag{7.1-35}$$

where Σ refers to the undeformed surface and $J = (a/A)^{1/2}$. By conservation of mass, we have

$$\rho J = \rho_0 = constant, \tag{7.1-36}$$

where ρ_0 is the mass per unit area in the undeformed surface. By using equation (7.1-35) and equation (7.1-36), we have

$$\iint_\Sigma (\rho_0 \dot{\boldsymbol{v}}\cdot\boldsymbol{v} + \rho_0 \dot{U}) d\Sigma = \oint_l \boldsymbol{t}\cdot\boldsymbol{v} dl,$$

or

$$\iint_\sigma (\rho_0 \dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \rho_0 \dot{U}) \frac{d\sigma}{J} = \oint_l \boldsymbol{t} \cdot \boldsymbol{v} dl. \tag{7.1-37}$$

We assume that ρ, U, $\boldsymbol{t}$ remain unchanged under superposed uniform rigid body translational velocities. Then for any constant vector $\boldsymbol{b}$ we have

$$\iint_\sigma (\rho_0 \dot{\boldsymbol{v}} \cdot (\boldsymbol{v} + \boldsymbol{b}) + \rho_0 \dot{U}) \frac{d\sigma}{J} = \oint_l \boldsymbol{t} \cdot (\boldsymbol{v} + \boldsymbol{b}) dl. \tag{7.1-38}$$

By subtracting equation (7.1-37) from equation (7.1-38), we get

$$\boldsymbol{b} \cdot \left(\iint_\sigma \rho_0 \dot{\boldsymbol{v}} \frac{d\sigma}{J} - \oint_l \boldsymbol{t} dl \right) = 0.$$

Since $\boldsymbol{b}$ is arbitrary, we have

$$\iint_\sigma \rho_0 \dot{\boldsymbol{v}} \frac{d\sigma}{J} = \oint_l \boldsymbol{t} dl. \tag{7.1-39}$$

The Stokes-Green theorem is given as follows (see A.E.Green[12])

$$\iint_\sigma a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \psi^\alpha) d\sigma = \oint_l \psi^\alpha n_\alpha dl, \tag{7.1-40}$$

where n_α is defined by $\boldsymbol{n} = n_\alpha \boldsymbol{a}^\alpha$. If we take

$$\psi^\alpha = \boldsymbol{t}^\alpha \cdot \boldsymbol{c},$$

where $\boldsymbol{t}^\alpha$ is defined by $\boldsymbol{t} = \boldsymbol{t}^\alpha n_\alpha$ and $\boldsymbol{c}$ is an arbitrary constant vector, then

$$\begin{aligned} a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \psi^\alpha) &= a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \boldsymbol{t}^\alpha \cdot \boldsymbol{c}) \\ &= a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \boldsymbol{t}^\alpha) \cdot \boldsymbol{c}, \end{aligned} \tag{7.1-41}$$

hence we have

$$\oint (\boldsymbol{t}^\alpha \cdot \boldsymbol{c}) n_\alpha dl = \iint_\sigma a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \boldsymbol{t}^\alpha) \cdot \boldsymbol{c} d\sigma, \tag{7.1-42}$$

since $\boldsymbol{c}$ is arbitrary, we have

$$\oint_l \boldsymbol{t} dl = \iint_\sigma a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \boldsymbol{t}^\alpha) d\sigma. \tag{7.1-43}$$

Since σ is arbitrary, then by equation (7.1-39) and equation (7.1-43) we get

$$J a^{-1/2} \frac{\partial}{\partial \theta^\alpha} (a^{1/2} \boldsymbol{t}^\alpha) = \rho_0 \dot{\boldsymbol{v}}. \tag{7.1-44}$$

Since

$$\boldsymbol{t} = \mathcal{T} \hat{n} = \mathcal{T} \boldsymbol{a}^\alpha n_\alpha = \frac{\boldsymbol{T}^\alpha}{J} n_\alpha, \tag{7.1-45}$$

it is clear that

$$\boldsymbol{t}^\alpha = \frac{1}{J} \boldsymbol{T}^\alpha. \tag{7.1-46}$$

Notice that $J = (a/A)^{1/2}$, then equation (7.1-44) can be written as

$$A^{-1/2} \frac{\partial}{\partial \theta^\alpha} (A^{1/2} \boldsymbol{T}^\alpha) = \rho_0 \dot{\boldsymbol{v}}. \tag{7.1-47}$$

The above equation is the equation of motion of an elastic membrane.

7.2 Governing Equations for the Impact Problem

Consider a two-dimensional circular elastic membrane with radius B in the undeformed state. We assume that the membrane is perfectly flexible so that we can ignore the bending moment and the transverse shearing force. The membrane is pre-stretched such that it is subjected to an equibiaxial stretch $\lambda_0 > 1$, so that the radius of the membrane becomes $b = \lambda_0 B$. We fix the edge of the membrane so that no displacement is allowed, the membrane is then subjected to a normal impact in the region of a circle with radius $r = a < b$, and the initial impact velocity v_0 is given. We take $A = a/\lambda_0$.

We take $\theta^1 = R$, $\theta^2 = \Theta$, where R, Θ are cylindrical polar coordinates. We denote the position of a point in the reference surface by $\boldsymbol{X} = X^i \hat{e}_i$, where $\hat{e}_i$, $i = 1, 2, 3$, is the fixed retangular basis, we have

$$X^1 = R\cos\Theta, \qquad X^2 = R\sin\Theta, \qquad X^3 = 0. \tag{7.2-1}$$

The position of a point in the deformed surface is described by $\boldsymbol{r} = r^i \hat{e}_i$, where

$$r^1 = r(R,t)\cos\Theta, \qquad r^2 = r(R,t)\sin\Theta, \qquad r^3 = z(R,t). \tag{7.2-2}$$

Recall that $\boldsymbol{A}_\alpha = \boldsymbol{X}_{,\alpha}$, $\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}$, we have

$$\begin{aligned}
\boldsymbol{A}_1 &= \cos\Theta\hat{e}_1 + \sin\Theta\hat{e}_2,\\
\boldsymbol{A}_2 &= -R\sin\Theta\hat{e}_1 + R\cos\Theta\hat{e}_2,\\
\boldsymbol{a}_1 &= \frac{\partial r}{\partial R}(\cos\Theta\hat{e}_1 + \sin\Theta\hat{e}_2) + \frac{\partial z}{\partial R}\hat{e}_3,\\
\boldsymbol{a}_2 &= -r\sin\Theta\hat{e}_1 + r\cos\Theta\hat{e}_2.
\end{aligned} \tag{7.2-3}$$

It can be shown that

$$\begin{aligned}
(A_{\alpha\beta}) &= \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix},\\
(A^{\alpha\beta}) &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix},\\
(a_{\alpha\beta}) &= \begin{pmatrix} (\frac{\partial r}{\partial R})^2 + (\frac{\partial z}{\partial R})^2 & 0 \\ 0 & r^2 \end{pmatrix}.
\end{aligned} \tag{7.2-4}$$

Since $\boldsymbol{A}^\alpha = A^{\alpha\beta}\boldsymbol{A}_\beta$, we have

$$\boldsymbol{A}^1 = \boldsymbol{A}_1, \qquad \boldsymbol{A}^2 = \frac{1}{R^2}\boldsymbol{A}_2. \tag{7.2-5}$$

If we define

$$\hat{e}_R = \cos\Theta\hat{e}_1 + \sin\Theta\hat{e}_2,$$
$$\hat{e}_\Theta = \hat{e}_3 \times \hat{e}_R = -\sin\Theta\hat{e}_1 + \cos\Theta\hat{e}_2, \tag{7.2-6}$$

then

$$\boldsymbol{A}_1 = \hat{e}_R, \quad \boldsymbol{A}_2 = R\hat{e}_\Theta, \quad \boldsymbol{A}^1 = \hat{e}_R, \quad \boldsymbol{A}^2 = \frac{1}{R}\hat{e}_\Theta,$$
$$\boldsymbol{a}_1 = \frac{\partial r}{\partial R}\hat{e}_R + \frac{\partial z}{\partial R}\hat{e}_3, \qquad \boldsymbol{a}_2 = r\hat{e}_\Theta. \tag{7.2-7}$$

Hence the deformation gradient tensor $\mathcal{F}$ and the right Cauchy-Green deformation tensor $\mathcal{C} = \mathcal{F}^T\mathcal{F}$ are given by

$$\mathcal{F} = \boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha = \frac{\partial r}{\partial R}\hat{e}_R \otimes \hat{e}_R + \frac{\partial z}{\partial R}\hat{e}_3 \otimes \hat{e}_R + \frac{r}{R}\hat{e}_\Theta \otimes \hat{e}_\Theta,$$
$$\mathcal{C} = \lambda_t^2\hat{e}_R \otimes \hat{e}_R + \lambda_\theta^2\hat{e}_\Theta \otimes \hat{e}_\Theta, \tag{7.2-8}$$

where $\lambda_t^2 = (\frac{\partial r}{\partial R})^2 + (\frac{\partial z}{\partial R})^2$ and $\lambda_\theta^2 = (r/R)^2$. $\mathcal{F}$ has a polar decomposition $\mathcal{F} = \mathcal{R}\mathcal{U}$, where

$$\mathcal{U} = \lambda_t\hat{e}_R \otimes \hat{e}_R + \lambda_\theta\hat{e}_\Theta \otimes \hat{e}_\Theta,$$
$$\mathcal{R} = \cos\alpha\hat{e}_R \otimes \hat{e}_R + \sin\alpha\hat{e}_3 \otimes \hat{e}_R + \hat{e}_\Theta \otimes \hat{e}_\Theta, \tag{7.2-9}$$

where

$$\cos\alpha = \frac{\partial r}{\partial R}/\lambda_t, \qquad \sin\alpha = \frac{\partial z}{\partial R}/\lambda_t. \tag{7.2-10}$$

Using equations (7.1-19), (7.1-32) and note that $a_{11} = \lambda_t^2$, $a_{12} = a_{21} = 0$ and $a_{22} = R^2\lambda_\theta^2$, then the Biot stress tensor $\mathcal{T}^{(1)}$ (see Ogden[19]) can be calculated as

$$\mathcal{T}^{(1)} = \frac{1}{2}(\mathcal{S}^T\mathcal{R} + \mathcal{R}^T\mathcal{S}) = \frac{\partial W_0}{\partial \lambda_t}\hat{e}_R \otimes \hat{e}_R + \frac{\partial W_0}{\partial \lambda_\theta}\hat{e}_\Theta \otimes \hat{e}_\Theta, \qquad (7.2\text{-}11)$$

where a superposed T denotes the transpose. Following Haddow, Wegner and Jiang[13], the quantities

$$T_1(\lambda_t, \lambda_\theta) = \frac{\partial W_0(\lambda_t, \lambda_\theta)}{\partial \lambda_t}, \qquad T_2(\lambda_t, \lambda_\theta) = \frac{\partial W_0(\lambda_t, \lambda_\theta)}{\partial \lambda_\theta}. \qquad (7.2\text{-}12)$$

are taken as Biot principal stresses. Here, we adopt the elastic potential energy function W_0 proposed by Haddow, Wagner and Jiang [13]

$$W_0(\lambda_t, \lambda_\theta) = \frac{\mu}{2}(\gamma(\lambda_t^2 + \lambda_\theta^2 + \lambda_t^{-2}\lambda_\theta^{-2} - 3) + (1-\gamma)(\lambda_t^{-2} + \lambda_\theta^{-2} + \lambda_\theta^2\lambda_t^2 - 3)), \quad (7.2\text{-}13)$$

with μ the modified shear modulus for infinitesimal deformation from the undeformed state and $0 < \gamma \leq 1$. By equation (7.2-4), we have

$$A = det(A_{\alpha\beta}) = R^2. \qquad (7.2\text{-}14)$$

The equation of motion (7.1-47) can now be written as

$$\frac{1}{R}(R\boldsymbol{T}^1)_{,1} + \frac{1}{R}(R\boldsymbol{T}^2)_{,2} = \rho_0\dot{\boldsymbol{v}}, \qquad (7.2\text{-}15)$$

where, by equation (7.1-31)

$$\begin{aligned} \boldsymbol{T}^1 &= 2\frac{\partial W_0}{\partial a_{11}}\boldsymbol{a}_1 + 2\frac{\partial W_0}{\partial a_{12}}\boldsymbol{a}_2 = 2\frac{\partial W_0}{\partial a_{11}}\boldsymbol{a}_1, \\ \boldsymbol{T}^2 &= 2\frac{\partial W_0}{\partial a_{21}}\boldsymbol{a}_1 + 2\frac{\partial W_0}{\partial a_{22}}\boldsymbol{a}_2 = 2\frac{\partial W_0}{\partial a_{22}}\boldsymbol{a}_2. \end{aligned} \qquad (7.2\text{-}16)$$

It can be shown that

$$\boldsymbol{T}^1 = T_1 \cos\alpha\hat{e}_R + T_1 \sin\alpha\hat{e}_3, \qquad \boldsymbol{T}^2 = \frac{1}{R}T_2\hat{e}_\Theta. \qquad (7.2\text{-}17)$$

Notice that

$$\frac{\partial \hat{e}_\Theta}{\partial \Theta} = -\hat{e}_R, \qquad \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \ddot{r}\hat{e}_R + \ddot{z}\hat{e}_3, \tag{7.2-18}$$

and as a result we have the equation of motion in component form as

$$\begin{aligned} \frac{\partial}{\partial R}(T_1 \cos\alpha) + \frac{T_1 \cos\alpha - T_2}{R} &= \rho_0 \ddot{r}, \\ \frac{\partial}{\partial R}(T_1 \sin\alpha) + \frac{T_1 \sin\alpha}{R} &= \rho_0 \ddot{z}. \end{aligned} \tag{7.2-19}$$

The compatibility equations are

$$\frac{\partial(\lambda_t \cos\alpha)}{\partial t} = \frac{\partial u}{\partial R}, \qquad \frac{\partial(\lambda_t \sin\alpha)}{\partial t} = \frac{\partial v}{\partial R}, \qquad \frac{\partial \lambda_\theta}{\partial t} = \frac{u}{R}, \tag{7.2-20}$$

where $u = \frac{\partial r}{\partial t} = \dot{r}, \quad v = \frac{\partial z}{\partial t} = \dot{z}.$

Next, we will consider the boundary conditions for this problem. The boundary conditions at $R = B$ are

$$\begin{aligned} z(B,t) = 0, \qquad & w|_{R=B} = \frac{\partial z}{\partial t}|_{R=B} = 0, \\ u|_{R=B} = \frac{\partial r}{\partial t}|_{R=B} = 0, \qquad & \lambda_\theta|_{R=B} = \lambda_0. \quad (t \geq 0) \end{aligned} \tag{7.2-21}$$

The boundary conditions for $R \leq A$ are

$$\begin{aligned} (M + \Delta M)\ddot{z}(R,t) &= 2\pi a \boldsymbol{t}|_{R=A} \cdot \hat{\boldsymbol{e}}_3, \\ \lambda_\theta &= \lambda_0, \quad ,(0 \leq R \leq A, t \geq 0), \\ u = 0, \quad \lambda_t &= \lambda_0, (0 \leq R < A, \quad t \geq 0) \end{aligned} \tag{7.2-22}$$

where M is the mass of the impact object, $\Delta M = \pi A^2 \rho_0$ is the mass of the membrane inside the circle $R \leq A$ (or $r \leq a$), with ρ_0 being the density per unit area in the undeformed configuration. Since

$$t = T\hat{n}, \qquad \hat{n} = \frac{a_1}{|a_1|} = \frac{a_1}{\lambda_t},$$
$$T^{\alpha\beta} = \frac{2}{J}\frac{\partial W_0}{\partial a_{\alpha\beta}}, \qquad J = (a/A)^{1/2} = \lambda_t \lambda_\theta,$$

then

$$T\hat{n} = T^{\alpha\beta}(a_\alpha \otimes a_\beta)\frac{a_1}{\lambda_t} = T^{11}\lambda_t a_1,$$

so the first equation in (7.2-22) can be written as

$$(M + \Delta M)\ddot{z}(R,t) = \frac{2\pi a}{\lambda_\theta} T_1 \sin\alpha, \qquad (R = A). \tag{7.2-23}$$

The nonslip assumption implies that $\lambda_\theta|_{R=A} = \lambda_0$. Equation (7.2-23) can be written as

$$(M + \Delta M)\ddot{z}(R,t) = \frac{2\pi a}{\lambda_0} T_1 \sin\alpha, \qquad (R = A). \tag{7.2-24}$$

If we introduce the nondimensional variables

$$\hat{R} = R/B, \quad \hat{t} = \frac{tC_0^*}{B}, \quad C_0^* = (\frac{4\mu}{\rho_0})^{1/2}, \quad \hat{A} = A/B, \quad \hat{B} = 1, \quad \hat{r} = r/B,$$
$$\hat{z} = z/B, \quad \hat{u} = u/C_0^*, \hat{v} = v/C_0^*, \quad \hat{T}_1 = \frac{T_1}{4\mu}, \quad \hat{T}_2 = \frac{T_2}{4\mu}, \quad \hat{W} = \frac{W}{4\mu}, \tag{7.2-25}$$

then equations (7.2-19) and equations (7.2-20) remain unchanged when the hats are dropped except that ρ_0 is replaced by 1. These equations can be written in system form

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial R} + B(u, R) = 0, \qquad A < R < 1, \tag{7.2-26}$$

where

$$\begin{aligned}
\boldsymbol{u} &= (\lambda_t \cos\alpha, \lambda_t \sin\alpha, \lambda_\theta, u, v)^T, \\
\boldsymbol{H} &= -(u, v, 0, T_1 \cos\alpha, T_1 \sin\alpha)^T, \\
\boldsymbol{B} &= -(0, 0, u/R, (T_1 \cos\alpha - T_2)/R, T_1 \sin\alpha/R)^T,
\end{aligned} \tag{7.2-27}$$

where a superposed T denotes the transpose. The condition (7.2-24) can be written as

$$\dot{v} = KT_1 \sin\alpha, \qquad K = \frac{2\Delta M}{M + \Delta M}\left(\frac{1}{A}\right), \qquad 0 \le R \le A. \tag{7.2-28}$$

The complete description of nondimensional boundary conditions and initial conditions will be given in Chapter 8.

The linearised version of the above problem has been treated by Farrar[10]. If we set

$$\begin{aligned}
&r = \lambda_0 R, \quad u = 0, \quad T_2 = T_0, \quad T_1 \cos\alpha = T_0, \\
&\frac{\partial z}{\partial r} = \tan\alpha, \quad T_0 = T_1(\lambda_0, \lambda_0) = T_2(\lambda_0, \lambda_0),
\end{aligned} \tag{7.2-29}$$

in equation (7.2-19) we get

$$\rho\ddot{z} = \frac{\lambda_0 T_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right), \qquad a < r < b. \tag{7.2-30}$$

The linearised version of the first equation of (7.2-22) is

$$(M + \Delta M)\ddot{z}(a,t) = \frac{2\pi a T_0}{\lambda_0}\frac{\partial z}{\partial r}(a,t). \tag{7.2-31}$$

In Farrar's case the initial thickness of the sheet is denoted by h_F, the density of the material ρ_F is considered constant. The tension of the sheet T_F is also considered constant. Then

$$\rho_0 = \lambda_0^2 h_F \rho_F, \qquad T_0 = \lambda_0 h_F T_F, \tag{7.2-32}$$

and we recover Farrar's form

$$\frac{1}{C_F^2}\ddot{z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right), \qquad a < r < b, \quad t > 0, \quad C_F^2 = T_F/\rho_F. \tag{7.2-33}$$

Equation (7.2-31) can be written as

$$(M + \Delta M)\ddot{z} = 2\pi a h T_F \frac{\partial z}{\partial r}, \qquad r = a. \tag{7.2-34}$$

The complete description of the linearised boundary conditions and initial conditions will be given in Chapter 8.

We leave the linearised equations in dimensional form. As a result when comparison is made between the linear and nonlinear cases it is necessary to convert the results to dimensional form.

Chapter 8

Numerical Methods for the Impact Problem of a Nonlinear Membrane

8.1 Numerical Procedures

The nondimensional governing equations for the dynamical impact problem for a nonlinear elastic membrane were derived in Chapter 7 and are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{H}(\boldsymbol{u})}{\partial R} + \boldsymbol{B}(\boldsymbol{u}, R) = \boldsymbol{0}, \qquad A < R < 1, \tag{8.1-1}$$

where

$$\begin{aligned} \boldsymbol{u} &= (\lambda_t \cos\alpha, \lambda_t \sin\alpha, \lambda_\theta, u, v)^T, \\ \boldsymbol{H} &= -(u, v, 0, T_1 \cos\alpha, T_1 \sin\alpha)^T, \\ \boldsymbol{B} &= -(0, 0, u/R, (T_1 \cos\alpha - T_2)/R, T_1 \sin\alpha/R)^T, \end{aligned} \tag{8.1-2}$$

where a superposed T denotes the transpose. The notation is explained in Chapter 7 and we use the same nondimensionalisation. System (8.1-1) can be written in the matrix form

$$\frac{\partial \boldsymbol{u}}{\partial t} + A(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial R} + \boldsymbol{B}(\boldsymbol{u}, R) = \boldsymbol{0}, \qquad A < R < 1, \tag{8.1-3}$$

where $A(\boldsymbol{u}) = \frac{\partial H(\boldsymbol{u})}{\partial \boldsymbol{u}}$, its nonzero components are given by

$$
\begin{aligned}
A_{14} &= -1, \qquad A_{25} = -1, \\
A_{41} &= -\frac{\partial T_1}{\partial \lambda_t}(\cos\alpha)^2 - \frac{T_1}{\lambda_t}(\sin\alpha)^2, \\
A_{42} = A_{51} &= (\frac{T_1}{\lambda_t} - \frac{\partial T_1}{\partial \lambda_t}) \sin\alpha\cos\alpha, \\
A_{43} &= -\frac{\partial T_1}{\partial \lambda_\theta}\cos\alpha, \\
A_{52} &= -\frac{\partial T_1}{\partial \lambda_t}(\sin\alpha)^2 - \frac{T_1}{\lambda_t}(\cos\alpha)^2, \\
A_{53} &= -\frac{\partial T_1}{\partial \lambda_\theta}\sin\alpha.
\end{aligned}
\tag{8.1-4}
$$

The eigenvalues of $A(\boldsymbol{u})$ are

$$\Lambda_0 = 0, \qquad \Lambda_{\pm 1} = \pm C_L, \qquad \Lambda_{\pm 2} = \pm C_T, \tag{8.1-5}$$

where

$$C_L = \left(\frac{\partial T_1}{\partial \lambda_t}\right)^{1/2}, \qquad C_T = \left(\frac{T_1}{\lambda_t}\right)^{1/2}. \tag{8.1-6}$$

The corresponding left eigenvectors are

$$
\begin{aligned}
\boldsymbol{L}_0 &= (0,0,1,0,0), \\
\boldsymbol{L}_{\pm 1} &= (C_L\cos\alpha, C_L\sin\alpha, C_L^{-1}\frac{\partial T_1}{\partial \lambda_\theta}, \mp\cos\alpha, \mp\sin\alpha), \\
\boldsymbol{L}_{\pm 2} &= (C_T\sin\alpha, -C_T\cos\alpha, 0, \mp\sin\alpha, \pm\cos\alpha).
\end{aligned}
\tag{8.1-7}
$$

The initial conditions for the impact problem of an elastic membrane are

$$\lambda_t(R,0) = \lambda_\theta(R,0) = \lambda_0,$$
$$u(R,0) = \alpha(R,0) = 0, \qquad 0 \le R \le 1,$$
$$v(R,0) = \begin{cases} v_0, & 0 \le R \le A, \\ 0, & A < R \le 1. \end{cases}$$

(8.1-8)

If we do not consider the restoring force, then the boundary conditions for this problem are

$$\lambda_\theta = \lambda_0, \quad v = v_0, \quad u = 0, \qquad 0 \le R \le A, \quad t > 0,$$
$$\lambda_t = \lambda_0, \quad \alpha = 0, \qquad 0 \le R < A, \quad t > 0,$$
$$v|_{R=1} = u|_{R=1} = z|_{R=1} = 0, \quad \lambda_\theta|_{R=1} = \lambda_0, \qquad t > 0. \tag{8.1-9}$$

If we consider the restoring force, then the boundary conditions are

$$\lambda_\theta = \lambda_0, \quad u = 0, \qquad 0 \le R \le A, \quad t > 0,$$
$$\dot{v} = KT_1 \sin\alpha, \qquad K = \frac{2\Delta M}{M + \Delta M}\left(\frac{1}{A}\right), \qquad 0 \le R \le A,$$
$$\lambda_t = \lambda_0, \quad \alpha = 0, \qquad 0 \le R < A, \quad t > 0,$$
$$v|_{R=1} = u|_{R=1} = z|_{R=1} = 0, \quad \lambda_\theta|_{R=1} = \lambda_0, \qquad t > 0. \tag{8.1-10}$$

Following the procedure in [13], we multiply system (8.1-1) on the left by $\boldsymbol{L} = \boldsymbol{L}_0, \boldsymbol{L}_{\pm1}, \boldsymbol{L}_{\pm2}$, we have

$$\boldsymbol{L} \cdot \frac{d\boldsymbol{u}}{dt} + \boldsymbol{L} \cdot \boldsymbol{B} = 0, \quad on \quad \frac{dR}{dt} = C, \tag{8.1-11}$$

where $C = 0, \pm C_L, \pm C_T$. When $C = 0$, then

$$\frac{d\lambda_\theta}{dt} - \frac{u}{R} = 0, \quad \text{on} \quad \frac{dR}{dt} = 0. \tag{8.1-12}$$

When $C = \pm C_L$, we have

$$C_L \frac{d\lambda_t}{dt} \mp \cos\alpha \frac{du}{dt} \mp \sin\alpha \frac{dv}{dt} + C_L^{-1} \frac{\partial T_1}{\partial \lambda_\theta} (\frac{d\lambda_\theta}{dt} - \frac{u}{R}) \mp \frac{T_2 \cos\alpha - T_1}{R} = 0,$$

$$\text{on} \qquad \frac{dR}{dt} = \pm C_L, \tag{8.1-13}$$

and when $C = \pm C_T$, we have

$$-C_T \lambda_t \frac{d\alpha}{dt} \mp \sin\alpha \frac{du}{dt} \pm \cos\alpha \frac{dv}{dt} \mp \sin\alpha \frac{T_2}{R} = 0,$$

$$\text{on} \qquad \frac{dR}{dt} = \pm C_T. \tag{8.1-14}$$

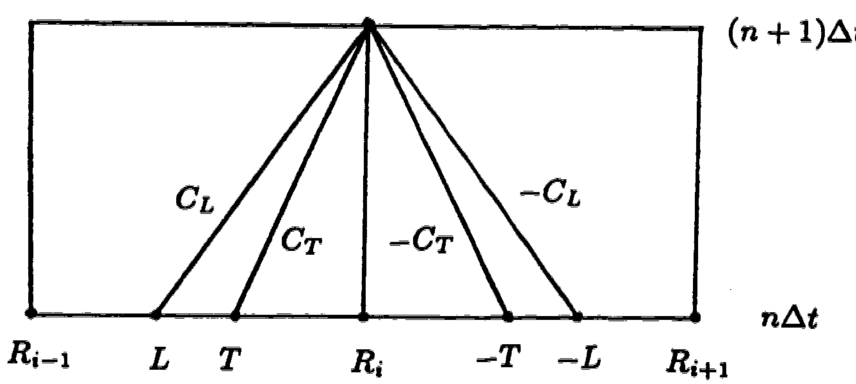


Fig.8.1-1

If we forward difference equations (8.1-12),(8.1-13) and (8.1-14), we obtain, on $\frac{dR}{dt} = 0$

$$(\lambda_\theta)_i^{n+1} = (\lambda_\theta)_i^n + \Delta t \frac{u_i^n}{R_i}, \tag{8.1-15}$$

and on $\frac{dR}{dt} = \pm C_L$

$$(C_L)_{\pm L}\frac{(\lambda_t)_i^{n+1} - (\lambda_t)_{\pm L}}{\Delta t} \mp (\cos\alpha)_{\pm L}\frac{u_i^{n+1} - u_{\pm L}}{\Delta t} \mp (\sin\alpha)_{\pm L}\frac{v_i^{n+1} - v_{\pm L}}{\Delta t}$$
$$+(C_L)_{\pm L}^{-1}(\frac{\partial T_1}{\partial \lambda_\theta})_{\pm L}(\frac{(\lambda_\theta)_i^{n+1} - (\lambda_\theta)_{\pm L}}{\Delta t} - \frac{u_{\pm L}}{R_{\pm L}}) \pm \frac{(T_1)_{\pm L} - (T_2)_{\pm L}(\cos\alpha)_{\pm L}}{R_{\pm L}} = 0. \quad (8.1\text{-}16)$$

Similarly, on $\frac{dR}{dt} = \pm C_T$ we have

$$-(\lambda_t)_{\pm T}(C_T)_{\pm T}\frac{\alpha_i^{n+1} - \alpha_{\pm T}}{\Delta t} \mp (\sin\alpha)_{\pm T}\frac{u_i^{n+1} - u_{\pm T}}{\Delta t}$$
$$\pm(\cos\alpha)_{\pm T}\frac{v_i^{n+1} - v_{\pm T}}{\Delta t} \mp \frac{(T_2\sin\alpha)_{\pm T}}{R_{\pm T}} = 0, \quad (8.1\text{-}17)$$

where the points $\pm L$, $\pm T$ are shown in Fig.8.1-1. $R_{\pm L}$, $R_{\pm T}$ are given by

$$R_{-\mu} = R_i^n + \frac{\Delta t(C_\mu)_i^n}{1 - \frac{\Delta t}{\Delta R}((C_\mu)_{i+1}^n - (C_\mu)_i^n)}, \qquad (i \geq 0),$$
$$R_\mu = R_i^n - \frac{\Delta t(C_\mu)_i^n}{1 - \frac{\Delta t}{\Delta R}((C_\mu)_{i+1}^n - (C_\mu)_i^n)}, \qquad (i \geq 1), \quad (8.1\text{-}18)$$

where $\mu = L, T$. We also have

$$e_\mu = e_i^n + \frac{e_{i-1}^n - e_i^n}{\Delta R}(R_i^n - R_\mu),$$
$$e_{-\mu} = e_i^n + \frac{e_{i+1}^n - e_i^n}{\Delta R}(R_{-\mu} - R_i^n), \quad (8.1\text{-}19)$$

where e represents one of λ_t, α, λ_θ, u, v and $\mu = L, T$. The corresponding finite difference equation for the second equation in (8.1-10) is

$$v_0^{n+1} = v_0^n + \Delta t K T_1((\lambda_t)_0^n, (\lambda_\theta)_0^n)\sin(\alpha_0^n). \quad (8.1\text{-}20)$$

If $R \neq A, 1$, we can solve $(\lambda_\theta)_i^{n+1}$ by equation (8.1-15). Then the unknowns $(\lambda_t)_i^{n+1}$, α_i^{n+1}, u_i^{n+1}, v_i^{n+1} can be found from equations (8.1-16) and (8.1-17). If $R = A$, there are three unknowns v_0^{n+1}, $(\lambda_t)_0^{n+1}$ and α_0^{n+1}. The unknown v_0^{n+1} can be obtained from equation (8.1-20), and then $(\lambda_t)_0^{n+1}, \alpha_0^{n+1}$ can be found from equations $(8.1-16)_2$ and $(8.1-17)_2$. If $R=1$, there are two unknowns $(\lambda_t)_{R=B}^{n+1}$ and $(\alpha)_{R=B}^{n+1}$. they can be found from equations $(8.1-16)_1$ and $(8.1-17)_1$.

The linear governing equation for the membrane is

$$\frac{1}{C_F^2}\ddot{z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right), \qquad a < r < b, \quad t > 0, \quad C_F^2 = T_F/\rho_F. \tag{8.1-21}$$

The corresponding initial conditions are

$$\begin{aligned} z|_{t=0} &= 0, \qquad 0 \le r \le b, \\ \frac{\partial z}{\partial t}|_{t=0} &= \begin{cases} v_0, & 0 \le r \le a, \\ 0, & a < r \le b, \end{cases} \end{aligned} \tag{8.1-22}$$

where a is the radius of the impact projectile and b is the radius of the stretched circular membrane.

If we do not consider the restoring force, then the boundary conditions for the linear problem are

$$\frac{\partial z}{\partial t}|_{r=a} = v_0, \qquad z|_{r=b} = \frac{\partial z}{\partial t}|_{r=b} = 0, \quad t > 0. \tag{8.1-23}$$

If we consider the restoring force, then the boundary conditions are

$$\begin{aligned} (M+\Delta M)\ddot{z} &= 2\pi a h_F T_F \frac{\partial z}{\partial r}, \qquad r = a, \\ z|_{r=b} &= \frac{\partial z}{\partial t}|_{r=b} = 0, \quad t > 0. \end{aligned} \tag{8.1-24}$$

where h_F is the initial thickness of the elastic sheet. The linearised equation may also be treated as a system by setting

$$v_1 = \frac{r}{C_F^2}\frac{\partial z}{\partial t}, \qquad v_2 = \frac{\partial z}{\partial r} \tag{8.1-25}$$

so that

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{v}, r)}{\partial r} = \boldsymbol{0}, \qquad a < r < b, \tag{8.1-26}$$

where

$$\boldsymbol{v} = (v_1, v_2)^T, \qquad \boldsymbol{F}(\boldsymbol{v}, r) = -\left(r v_2, \frac{C_F^2 v_1}{r}\right)^T. \tag{8.1-27}$$

The corresponding initial conditions are

$$v_1(r, 0) = v_2(r, 0) = 0, \qquad a < r < b. \tag{8.1-28}$$

The corresponding boundary conditions are either

$$v_1(a, t) = \frac{a}{C_F^2} v_0, \qquad v_1(b, t) = 0, \qquad t > 0, \tag{8.1-29}$$

or

$$\dot{v}_1(a, t) = \hat{K} v_2(a, t), \qquad v_1(b, t) = 0, \qquad t > 0, \tag{8.1-30}$$

where $\hat{K} = 2\Delta M/(M + \Delta M), \quad \Delta M = \pi a^2 h_F \rho_F$.

The approximate Riemann solver with one intermediate state proposed by Harten, Lax and Van Leer can be extended to systems of the form (8.1-26). For example, Antman and Szymczak treat their problem using the extended Riemann solver [1]. The extended Riemann solver for system (8.1-26) is

$$\boldsymbol{w}(r, t) = \begin{cases} \boldsymbol{v}_L, & r/t < -C_F, \\ \boldsymbol{v}_{LR}, & -C_F < r/t < C_F, \\ \boldsymbol{v}_R, & r/t > C_F, \end{cases} \tag{8.1-31}$$

where

$$\boldsymbol{v}_{LR} = \frac{\boldsymbol{v}_L + \boldsymbol{v}_R}{2} - \frac{\boldsymbol{F}(\boldsymbol{v}_R, r_R) - \boldsymbol{F}(\boldsymbol{v}_L, r_L)}{2C_F}. \tag{8.1-32}$$

Then we approximate $\boldsymbol{v}_i^{n+1}$ by

$$\boldsymbol{v}_i^{n+1} = \frac{1}{\Delta r}\int_{a+(i-1/2)\Delta r}^{a+(i+1/2)\Delta r} \boldsymbol{w}(r, t_{n+1})dr \qquad (8.1\text{-}33)$$

so that

$$\boldsymbol{v}_i^{n+1} = (1 - C_F\Delta t/\Delta r)\boldsymbol{v}_i^n + \frac{\Delta t}{2\Delta r}\left(C_F(\boldsymbol{v}_{i-1}^n + \boldsymbol{v}_{i+1}^n) + \boldsymbol{F}_{i-1}^n - \boldsymbol{F}_{i+1}^n\right), \qquad (8.1\text{-}34)$$

where

$$\boldsymbol{F}_{i-1}^n = \boldsymbol{F}(\boldsymbol{v}_{i-1}^n, r_{i-1}), \qquad \boldsymbol{F}_{i+1}^n = \boldsymbol{F}(\boldsymbol{v}_{i+1}^n, r_{i+1}). \qquad (8.1\text{-}35)$$

On the other hand, we may apply the characteristic method to the system (8.1-26). This system can be written in matrix form

$$\frac{\partial \boldsymbol{v}}{\partial t} + A_F(\boldsymbol{v})\frac{\partial \boldsymbol{v}}{\partial r} + \boldsymbol{B}_F(\boldsymbol{v}, r) = \boldsymbol{0}, \qquad a < r < b, \qquad (8.1\text{-}36)$$

where

$$A_F(\boldsymbol{v}) = \frac{\partial \boldsymbol{F}(\boldsymbol{v})}{\partial \boldsymbol{v}} = \begin{pmatrix} 0 & -r \\ -\frac{C_F^2}{r} & 0 \end{pmatrix},$$
$$\boldsymbol{B}_F = (-v_2, \frac{C_F^2 v_1}{r^2})^T. \qquad (8.1\text{-}37)$$

The eigenvalues of A_F are $\pm C_F$, the corresponding left eigenvectors are

$$\boldsymbol{L}_{\pm C_F} = (1, \quad \mp r/C_F). \qquad (8.1\text{-}38)$$

Multiplying $\boldsymbol{L}_{\pm C_F}$ on the left of system (8.1-36) we have

$$\boldsymbol{L}_{\pm C_F}\frac{d\boldsymbol{v}}{dt} + \boldsymbol{L}_{\pm C_F}\boldsymbol{B}_F = 0, \qquad \text{on} \quad \frac{dr}{dt} = \pm C_F. \qquad (8.1\text{-}39)$$

We then have

$$
\begin{aligned}
&\frac{dv_1}{dt} + \frac{r}{C_F}\frac{dv_2}{dt} - v_2 + \frac{C_F v_1}{r} = 0, \\
&\qquad\qquad \text{on} \quad \frac{dr}{dt} = -C_F, \\
&\frac{dv_1}{dt} - \frac{r}{C_F}\frac{dv_2}{dt} - v_2 - \frac{C_F v_1}{r} = 0, \\
&\qquad\qquad \text{on} \quad \frac{dr}{dt} = C_F.
\end{aligned}
\tag{8.1-40}
$$

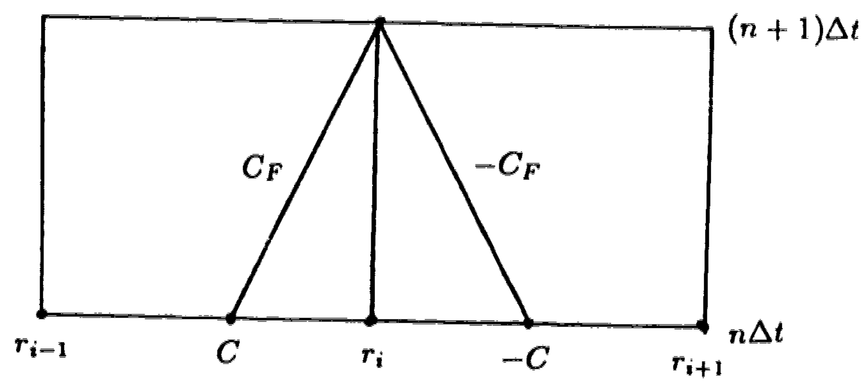


Fig.8.1-2

The corresponding finite difference equations are

$$
\begin{aligned}
&\frac{(v_1)_i^{n+1} - (v_1)_{-C}}{\Delta t} + \frac{r_{-C}}{C_F}\frac{(v_2)_i^{n+1} - (v_2)_{-C}}{\Delta t} - (v_2)_{-C} + \frac{C_F (v_1)_{-C}}{r_{-C}} = 0, \quad i \le 0, \\
&\frac{(v_1)_i^{n+1} - (v_1)_C}{\Delta t} - \frac{r_C}{C_F}\frac{(v_2)_i^{n+1} - (v_2)_C}{\Delta t} - (v_2)_C - \frac{C_F (v_1)_C}{r_C} = 0, \quad i \le 1,
\end{aligned}
\tag{8.1-41}
$$

where the points $\pm C$ are shown in Fig.8.1-2 and

$$
\begin{aligned}
r_{-C} &= r_i + C_F \Delta t, \\
(v_1)_{-C} &= (v_1)_i^n + C_F \frac{\Delta t}{\Delta r}((v_1)_{i+1}^n - (v_1)_i^n), \\
(v_2)_{-C} &= (v_2)_i^n + C_F \frac{\Delta t}{\Delta r}((v_2)_{i+1}^n - (v_2)_i^n), \\
r_C &= r_i - C_F \Delta t, \\
(v_1)_C &= (v_1)_i^n + C_F \frac{\Delta t}{\Delta r}((v_1)_{i-1}^n - (v_1)_i^n), \\
(v_2)_C &= (v_2)_i^n + C_F \frac{\Delta t}{\Delta r}((v_2)_{i-1}^n - (v_2)_i^n).
\end{aligned}
\tag{8.1-42}
$$

The unknowns $(v_1)_i^{n+1}$, $(v_2)_i^{n+1}$ can be found from equations (8.1-41)

$$
\begin{aligned}
(v_1)_i^{n+1} &= -\frac{r_{-C} P_C + r_C P_{-C}}{r_C + r_{-C}}, \\
(v_2)_i^{n+1} &= -\frac{C_F (P_C - P_{-C})}{r_C + r_{-C}},
\end{aligned}
\tag{8.1-43}
$$

where

$$
\begin{aligned}
P_C &= -(v_1)_C + \frac{r_C}{C_F}(v_2)_C - \Delta t (v_2)_C - \frac{\Delta t C_F (v_1)_C}{r_C}, \\
P_{-C} &= -(v_1)_{-C} - \frac{r_{-C}}{C_F}(v_2)_{-C} - \Delta t (v_2)_{-C} + \frac{\Delta t C_F (v_1)_{-C}}{r_{-C}}.
\end{aligned}
\tag{8.1-44}
$$

The corresponding finite difference equation for the boundary condition (8.1-30) is

$$
(v_1)_0^{n+1} = (v_1)_0^n + \Delta t \hat{K} (v_2)_0^n. \tag{8.1-45}
$$

Numerical results will be given in the next section.

8.2 Numerical Results

For the nonlinear case we need to compare our numerical results with those given by Haddow, Wegner and Jiang under the assumption of constant impact velocity [13]. Fig.8.2-1 shows the graph of the nondimensional displacement $\hat{z}$ as a function of $\hat{R}$ at different moments of nondimensional time $\hat{t}$, where the variables $\hat{z}$, $\hat{R}$ and $\hat{t}$ are defined in Chapter 7. Figure 8.2-2 shows the graph of λ_t as a function of $\hat{R}$ at different instants of nondimensional time $\hat{t}$, these results are in good agreement with those in [13].

For the linear case, we need to compare numerical results with those given by Farrar[10]. There is a minor discrepancy in some of the parameters in [10] and we rearrange these parameters so that they are consistent. We will denote these parameters by a suffix F. We have taken

$$\begin{aligned}
&\lambda_F = 1.1 = \lambda_0, \qquad h_F = 0.3 \times 10^{-2} m, \qquad a_F = 1.9 \times 10^{-2} m,\\
&b_F = 0.1275 m, \qquad V_F = 12.5 m/s, \qquad C_F = 20 m/s,\\
&E_F = 3559 \times 10^3 Kg/ms^2, \qquad \rho_F = 889.75 Kg/m^3,\\
&\Delta M = \pi a_F^2 h_F \rho_F = 3.0276 \times 10^{-3} Kg, \qquad M_F = 71 \times 10^{-3} Kg,
\end{aligned} \tag{8.2-1}$$

where h_F is the initial thickness of the elastic sheet, a_F is the radius of the projectile, b_F is the outer stretched radius of the sheet, V_F is the initial impact velocity, $C_F^2 = T_F/\rho_F$, E_F is the Young's modulus of the material and M_F is the mass of the projectile.

Figure 8.2-3 shows the graph of the displacement z as a function of r at different moments time t obtained by the linear theory with the characteristic method and the method of the extended Riemann solver.

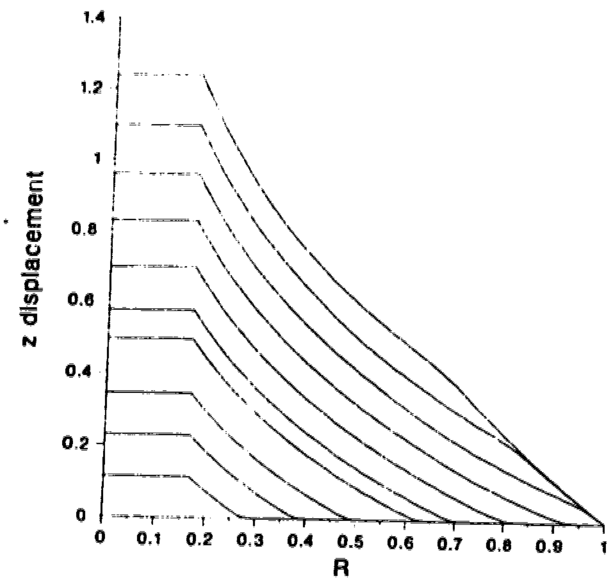


Fig.8.2-1

$\lambda_0 = 1.2$, $\hat{v}_0 = 0.5$, $\gamma = 0.9$, $B/A = 6, \hat{A} = A/B = 0.1667$, $\Delta\hat{t} = \Delta\hat{R} = 0.001, K = 0$,
$\hat{t} = 0.230, 0.461, 0.691, 0.992, 1.152, 1.398, 1.656, 1.920, 2.192, 2.474$.

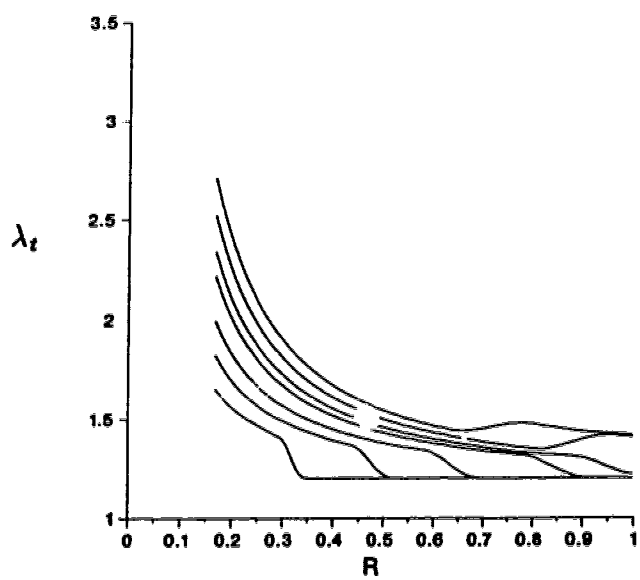


Fig.8.2-2

$\lambda_0 = 1.2$, $\hat{v}_0 = 0.5$, $\gamma = 0.9$, $B/A = 6, \hat{A} = A/B = 0.1667$, $\Delta\hat{t} = \Delta\hat{R} = 0.001, K = 0$,
$\hat{t} = 0.230, 0.461, 0.691, 0.992, 1.152, 1.398, 1.656$.

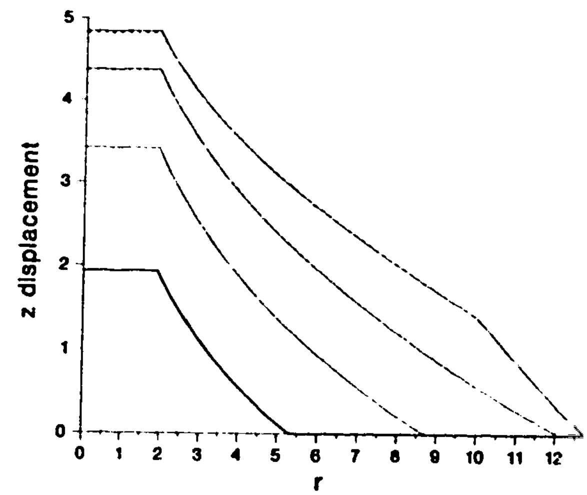


Fig.8.2-3

$\lambda_0 = 1.1$, $v_0 = 1.25cm/ms$, $a = 1.9cm$, $b = 12.75cm$, $\hat{K} = 0.0818$,

$t = 1.7ms, 3.4ms, 5.1ms, 6.8ms$,

————characteristics method with $\Delta t = 0.002ms$, $\Delta r = 0.004cm$,

- - - - - -extended Riemann solver with $\Delta t = 0.002ms$, $\Delta r = 0.008cm$.

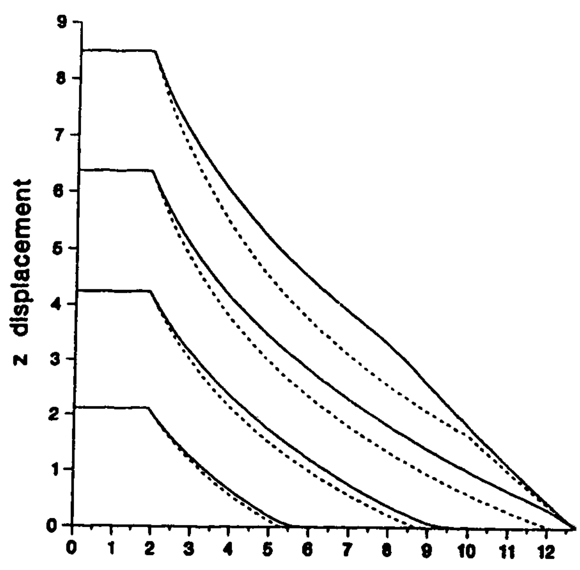


Fig.8.2-4

Graph of z as a function of r for $t = 1.7ms, 3.4ms, 5.1ms, 6.8ms$, with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

————nonlinear theory, $K = 0$. - - - - - -linear theory, $\hat{K} = 0$.

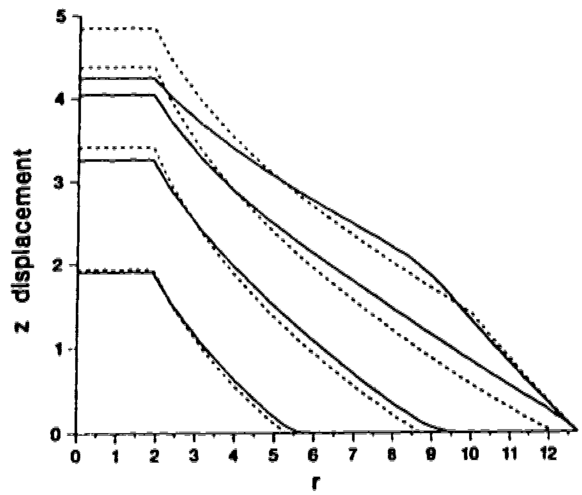


Fig.8.2-5

Graph of z as a function of r for $t = 1.7ms, 3.4ms, 5.1ms, 6.8ms$,

with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0.5489$. - - - - - -linear theory, $\hat{K} = 0.0818$.

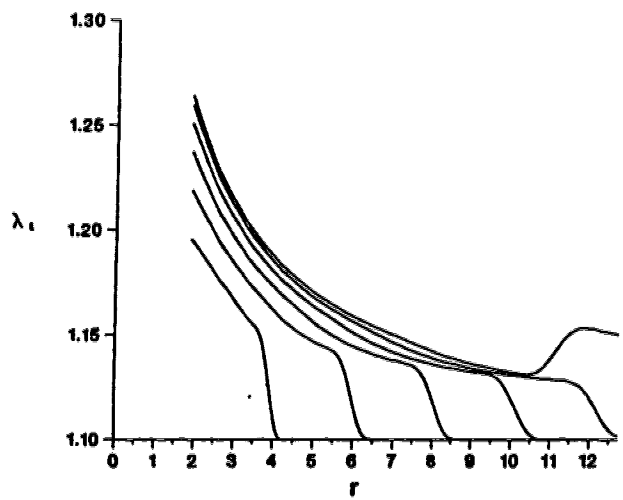


Fig.8.2-6

Graph of λ_t as a function of r for $t = 0.425ms, 0.850ms, 1.275ms, 1.700ms,$

$2.125ms, 2.550ms$ with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0.5489$.

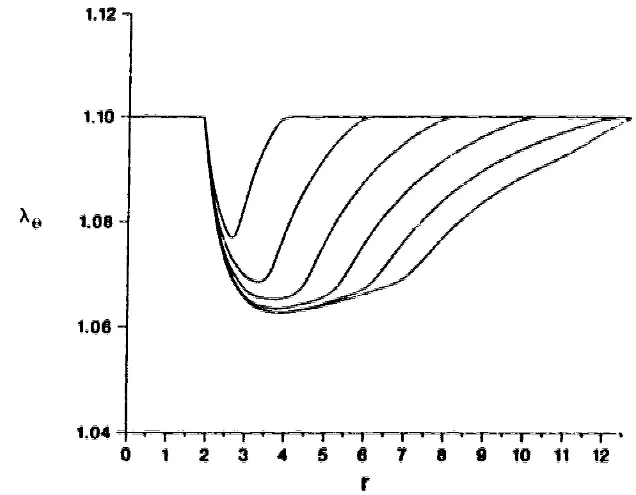


Fig.8.2-7

Graph of λ_θ as a function of r for $t = 0.425ms, 0.850ms, 1.275ms, 1.700ms, 2.125ms, 2.550ms$ with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0.5489$.

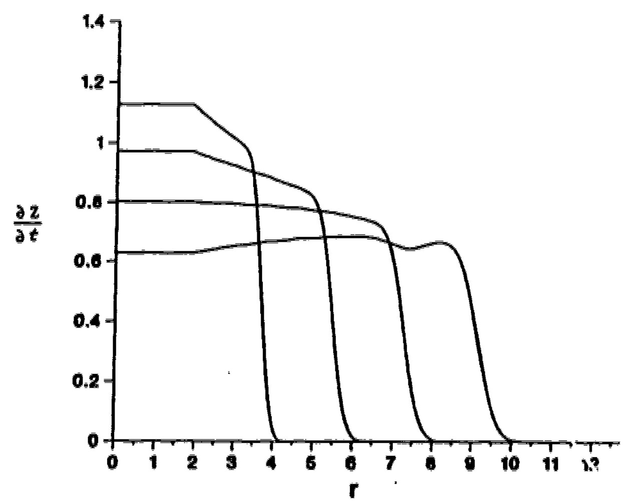


Fig.8.2-8

Graph of $v = \frac{\partial z}{\partial t}$ as a function of r for $t = 0.85ms, 1.70ms, 2.55ms, 3.40ms$ with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0.5489$.

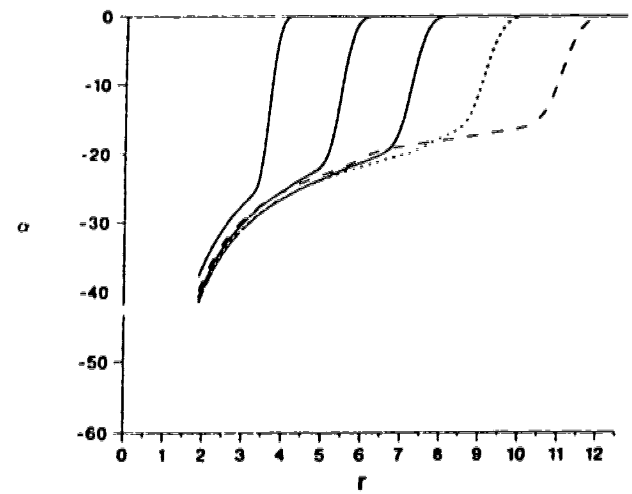


Fig.8.2-9

Graph of α as a function of r for $t = 0.85ms, 1.70ms, 2.55ms, 3.40ms, 4.25ms$ with $\lambda_0 = 1.1, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0.5489$.

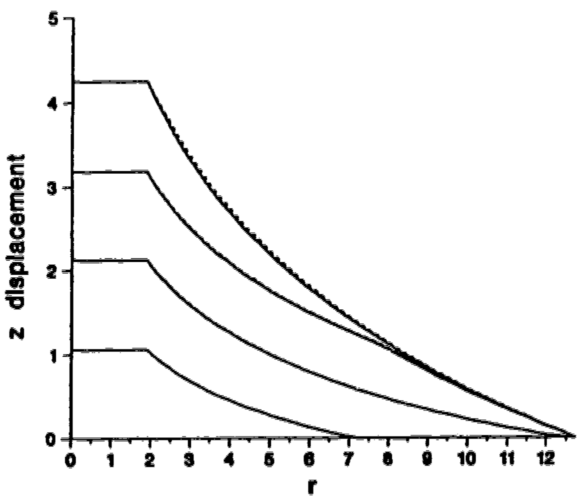


Fig.8.2-10

Graph of z as a function of r for $t = 0.850ms, 1.700ms, 2.550ms, 2.908ms$, with $\lambda_0 = 2.0, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

———nonlinear theory, $K = 0$.

- - - - - -linear theory, $\hat{K} = 0$.

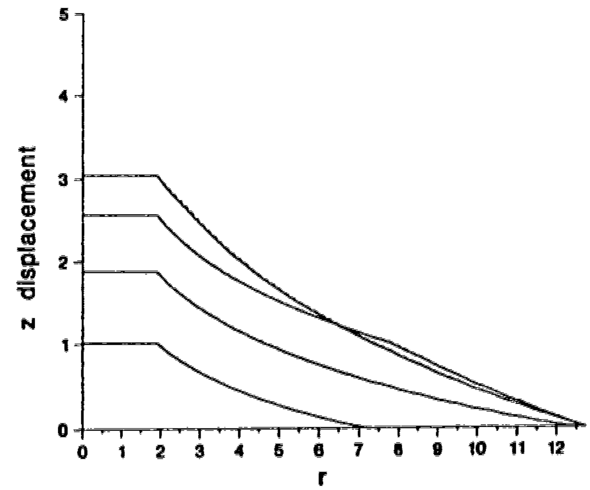


Fig.8.2-11

Graph of z as a function of r for $t = 0.850ms, 1.700ms, 2.550ms, 2.908ms$, with $\lambda_0 = 2.0, v_0 = 1.25cm/ms, a = 1.9cm, b = 12.75cm$.

——— nonlinear theory, $K = 0.1709$.

- - - - - linear theory, $\hat{K} = 0.02547$.

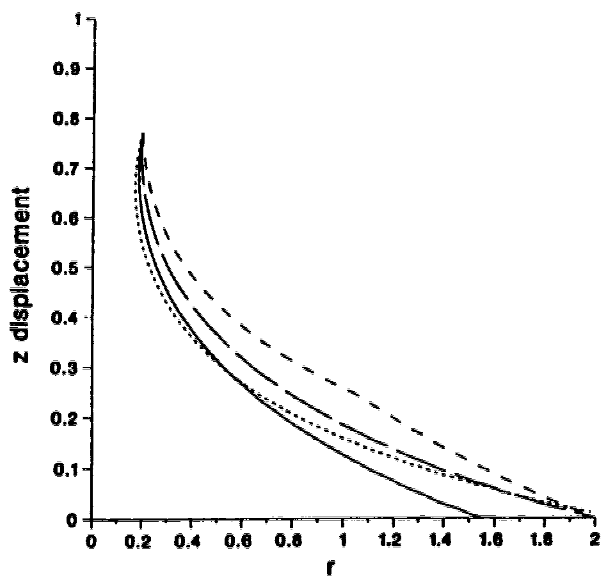


Fig.8.2-12

Graph of z as a function of r with $\gamma = 0.3, \lambda_0 = 2.0$, $\hat{A} = 0.1, \hat{B} = 1.0, B = 1.0$.

For dynamic problem, $C_0^* = 1.0$, $K = 0.0, \Delta\hat{t} = \Delta\hat{R} = 0.001$,

——— $\hat{v}_0 = 1.0$, $\hat{t} = 0.7725$,

— — — $\hat{v}_0 = 0.75$, $\hat{t} = 1.03$,

- - - - $\hat{v}_0 = 0.5$, $\hat{t} = 1.545$.

For static problem · · · · · · $\Delta R = 0.001$.

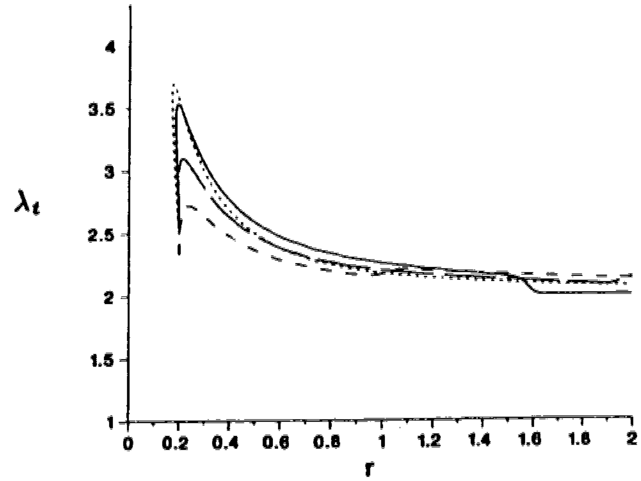


Fig.8.2-13 Graph of λ_t as a function of r with $\gamma = 0.3, \lambda_0 = 2.0$, $\hat{A} = 0.1, \hat{B} = 1.0, B = 1.0$.

For dynamic problem, $C_0^* = 1.0$, $K = 0.0, \Delta\hat{t} = \Delta\hat{R} = 0.001$,

——— $\hat{v}_0 = 1.0$, $\hat{t} = 0.7725$,

— — — $\hat{v}_0 = 0.75$, $\hat{t} = 1.03$,

- - - - $\hat{v}_0 = 0.5$, $\hat{t} = 1.545$.

For static problem · · · · · · $\Delta R = 0.001$.

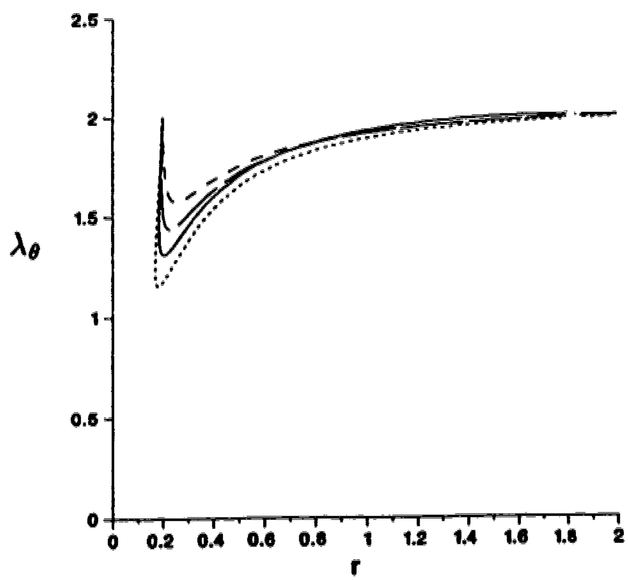


Fig.8.2-14 Graph of λ_θ as a function of r with $\gamma = 0.3, \lambda_0 = 2.0$, $\hat{A} = 0.1, \hat{B} = 1.0, B = 1.0$.

For dynamic problem, $C_0^* = 1.0$, $K = 0.0, \Delta\hat{t} = \Delta\hat{R} = 0.001$.

——— $\hat{v}_0 = 1.0$, $\hat{t} = 0.7725$,

— — — $\hat{v}_0 = 0.75$, $\hat{t} = 1.03$,

- - - - $\hat{v}_0 = 0.5$, $\hat{t} = 1.545$.

For static problem · · · · · · $\Delta R = 0.001$.

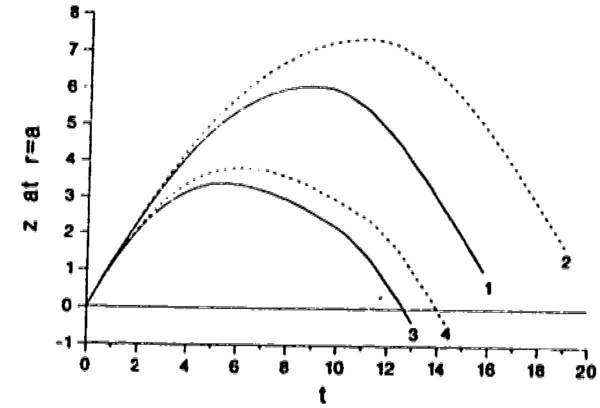


Fig.8.2-15

Graph of $z(a,t)$ as a function of t.

$\lambda_0 = 1.1$, $v_0 = 1.25cm/ms$, $a = 1.9cm$, $b = 12.75cm$.

1.——nonlinear theory, $K = 0.3$. 2.- - - -linear theory, $\hat{K} = 0.04471$.

3.——nonlinear theory, $K = 0.8$. 4.- - - -linear theory, $\hat{K} = 0.1192$.

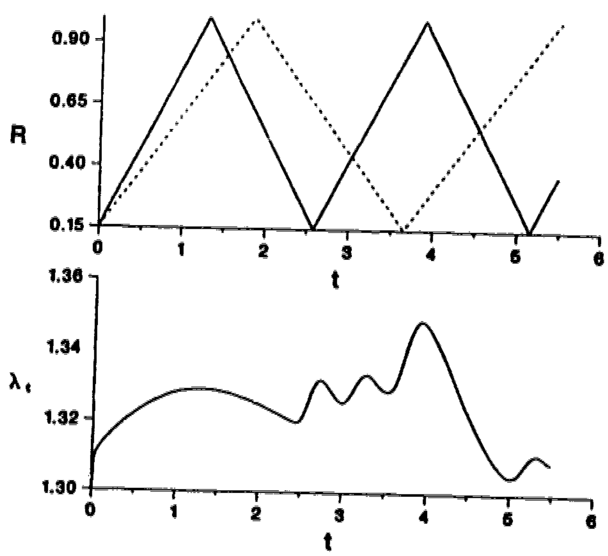


Fig.8.2-16

Top part: graphs of C_L, C_T characteristics starting at $\hat{R} = \hat{A} = 0.149, \hat{t} = 0.0$.

——— C_L characteristics,- - - - - - C_T characteristics.

bottom part: graph of $\lambda_t(\hat{A}, \hat{t})$ as a function of $\hat{t}$. Parameters are: $\lambda_0 = 1.3$, $K = 0.5, \hat{v}_0 = 0.1$, where $\hat{R}, \hat{A}, \hat{t}, \hat{v}_0$ are nondimensional variables.

To compare the linear and nonlinear cases we need to find the parameters for the nonlinear case corresponding to that given in (8.2-1), we have

$$\rho_0 = \lambda_F^2 h_F \rho_F = 3.2298 Kg/m^2, \qquad \gamma = 0.9,$$
$$\mu = 2401.1 Kg/s^2, \qquad C_0^* = (4\mu/\rho_0)^{1/2} = 54.5315 m/s. \tag{8.2-2}$$

Other scaling factors can be easily found. Note that we need to convert the results from the nonlinear case into dimensional form in order to compare the results with the linear case. Figure 8.2-4 provides a comparison of the linear and the nonlinear displacement responses for the parameters given above when the constant boundary condition is used. Figure 8.2-5 provides a comparison of the linear and the nonlinear displacement responses when the variable boundary condition is used. For the variable boundary condition case we plot the values of $\lambda_t, \lambda_\theta, v, \alpha$ as functions of r at various times in Figs. 8.2-6, 8.2-7, 8.2-8, 8.2-9 respectively.

For the elastic potential energy function given by equation (7.2-13), we have

$$C_L^2 = \frac{\partial T_1}{\partial \lambda_t} = \frac{1}{4}(\gamma + (1-\gamma)\lambda_\theta^2)(1 + 3\lambda_\theta^{-2}\lambda_t^{-4}),$$
$$C_T^2 = \frac{T_1}{\lambda_t} = \frac{1}{4}(\gamma + (1-\gamma)\lambda_\theta^2)(1 - \lambda_\theta^{-2}\lambda_t^{-4}).$$

(8.2-3)

It follows that $C_L^2 > 0$ and we require $\lambda_t^2\lambda_\theta > 1$ so that $C_T^2 > 0$. By using equations (8.2-3), we have

$$C_L^2 - C_T^2 = \lambda_t^{-4}(1 - \gamma + \gamma\lambda_\theta^{-2}). \tag{8.2-4}$$

Hence one has $C_L^2 > C_T^2 > 0$ whenever $\lambda_t^2\lambda_\theta > 1$. If λ_t is large, we have $C_L \approx C_T$. If we assume $C_L = C_T$ and that λ_θ is constant, a short calculation shows that the governing equations reduce to the linear case. Thus as the initial stretch increases we would expect the linear approximation to be close to the non-linear results at least for small times. This conclusion can be verified by the results shown in Fig.8.2-10 and Fig.8.2-11 where we plot the displacement z as a function of r for the case of constant boundary conditions and the case of variable boundary conditions respectively. For this case, the initial stretch is increased so that $\lambda_0 = 2$. Parameters $a, b, M, \mu, \rho_0, C_0^*$ are fixed as before. $\lambda_{F2} = \lambda_0 = 2$, and the new values of A and B are easily found. The other parameters can be found from

$$\Delta M = \pi a^2 h_{F2} \rho_{F2} = \pi a^2 (\rho_0 / \lambda_0^2), \qquad C_{F2} = \left(\frac{T_{F2}}{\rho_{F2}} \right)^{1/2} = \left(\frac{T_0 \lambda_0}{\rho_0} \right)^{1/2}. \quad (8.2\text{-}5)$$

Interestingly, if we choose the parameters suitably for the nonlinear case the graph of z as a function of r could be double valued. As a result, the other variables such as λ_t and λ_θ are also double valued functions of r. Similar double valued results are obtained by Roxburgh, Steigmann and Tait for the corresponding nonlinear static problem [20].

In Fig.8.2-12, we plot the graph of z as a function of r and try to compare displacement responses of the dynamic problem and the static problem. We have chosen parameters so that the dynamic problem and the static problem have the same displacement at $r = a$. Of course there are infinitely many way for the dynamic case to reach the given displacement at $r = a$. We have taken $\gamma = 0.3$, $\lambda_0 = 2, \hat{A} = 0.1$, $\hat{B} = 1.0, B = 1.0$. The other parameters are given in the figure captions.

In Fig.8.2-13, we plot the graph of λ_t as a function of r with the same parameters given in Fig.8.2-12. In Fig.8.2-14, we plot the graph of λ_θ as a function of r with the same parameters given in Fig.8.2-12.

We have continued the numerical analysis beyond the time at which reflections occur. Our assumption here is that shocks do not occur and to that extent the results obtained are subject to further examination. First we investigate the time at which the mass again attains free flight. The displacement at $r = a$ is shown in Fig.8.2-15 as a function of time in the linear and nonlinear cases for two different impacting masses. The computation is stopped when $T_1 \sin\alpha = 0$. The time and displacement at which free flight occurs is clearly dependent on the impacting mass.

Finally in Fig.8.2-16 we try to find the connection between the reflection of characteristics and the variation of λ_t at $r = a$.

Bibliography

[1] Antmann, S.S., Szymczak, W.G. *Nonlinear Elastoplastic Waves.* Contemporary Mathematics **100** , 27-54 AMS, 1989.

[2] Beatty,M.F., Haddow, J.B., *Transverse impact of a hyperelastic string,* J. Appl. Mech., **52**, 137-143, 1985.

[3] Ciarlet, Philippe G., *Mathematical elasticity, volume 1: Three-Dimensional Elasticity.* Studies in Mathematics and its applications, Eds. J.L. Lions, G. Papanicolaou, H. Fujita, H.B. Keller, **20**, North-Holland, Amsterdam, 1988.

[4] Collins, W.D. *One-dimensional non-linear wave propagation in incompressible elastic materials,* Q.J. Mech. and Appl. Math. XIX, 259-328, 1966.

[5] Collins, W.D. *The propagation and interaction of one-dimensional nonlinear waves in an incompressible isotropic elastic half -space,* Q. J. Appl. Math., **20**, 429-452, 1967.

[6] Courant, R., Freidrichs, K.O., *Supersonic Flow and Shock Waves,* Springer-Verlag, 1975.

[7] Davison, L., *Perturbation theory of nonlinear elastic wave propagation,* Int. J. Solids Structures, **4**, 301-322, 1968.

[8] Dohrenwend, C.O., Drucker, D.C., Moore, P., *Transverse impact transients.* Exp. Stress Analysis , **2**, 1-10, 1943.

[9] Engelbrecht, J., *Nonlinear wave processes of deformation in solids,* Monographs and Studies in Mathematics **16**, Main Editors, Jeffrey, J. and Douglas, R. G., Pitman Advanced Publishing Program, London, 1983.

[10] Farrar, C.L., *Impact response of a circular membrane.* Experimental Mechanics, 144-149, June 1984.

[11] Green, A.E., Naghdi, P.M., Wainwright, W.L., *A general theory of a Cosserat surface,* Archive for Rat. Mech. and Analysis **2**, 287-308, 1965.

[12] Green, A.E., Zerna, W., *Theoretical elasticity (second edition),* Oxford at the Clarendon Press, 1968.

[13] Haddow,J.B., Wegner, J.L., Jiang,L., *The dynamic response of a stretched circular hyperelastic membrane subjected to normal impact.* Wave Motion, **16**, 137-150, 1992.

[14] Harten, A., Lax, P.D., Van Leer, B., *On upstream differencing and Godunov-type schemes for hyperbolic conservation laws.* SIAM Review, **25**(1), 35-61, Jan. 1983.

[15] Jeffrey, A., *Quasilinear Hyperbolic Systems and Waves,* Research Notes in Mathematics, Pitman Publishing Co., 1976.

[16] LeVeque, R. J., *Numerical methods for conservation laws* , Lectures in Mathematics, Birkhauser Verlag, 1990.

[17] Liu, T.P., *The Riemann problem for general* 2×2 *conservation laws*, Trans. Amer. Math. Soc. **199**, 89-112, 1974.

[18] Lax, P. D., *Hyperbolic systems of conservation laws and the mathematical theory of shock waves,* CBMS Regional Conference Series in Applied Mathematics **11**, Society for Industrial and Applied Mathematics, Philadelphia, 1972.

[19] Ogden, R.W.,*Nonlinear Elastic Deformations*, Ellis Horwood Limited, Chichester, England, 1984.

[20] Roxburgh, D.G., Steigmann, D.J., Tait, R.J., *Azimuthal shearing and transverse deflection of a prestreched annular elastic membrane,* submitted, 1993.

[21] Royden, H.L., *Real analysis (third edition),* Macmillian Publishing Company, New York, Collier Macmillan Publishing, London, 1988.

[22] Shearer, M., *The Riemann problem for the planar motion of an elastic string,* J. Differential Equations, **61**, 149-163, 1986.

[23] Smoller, J., *On the solution of the Riemann problem with general step data for an extended : of hyperbolic systems,* Mich. Math. J. **16**, 201-210, 1969.

[24] Smoller, J., *Shock Waves and Reaction-Diffusion Equations* , Springer-Verlag, New York, 1983.

[25] Sod, G.A., *Numerical Methods in Fluid Dynamics,* Cambridge University Press, 1985.

[26] Tait, R.J., Duncan, D.B.,*Motion of a mass on a non-linear elastic string.* Int.J. Nonlinear Mechanics, **27**(2), 139-148, 1992.

[27] Tait,R.J., Zhong, J.L., *Perturbation methods for the impact problem of a non-linear elastic string.* Int. J. Non-Linear Mechanics. In press.

[28] Wendroff, B., *The Riemann problem for materials with noncon' ex equations of state I: Isentropic flow,* J. Math. Anal. Appl. **38**, 454-466, 1972.

[29] Wegner, J.L., Haddow, J.B., Tait, R.J. *Finite amplitude wave propagation in a stretched elastic string.* Elastic Wave Propagation, eds. M.F. McCarthy and M.A. Hayes, Proc. of second I.U.T.A.M.-I.U.P.A.P. Symposium on Elastic Wave Propagation, North Holland, Amsterdam, 161-166, 1989.

[30] Wegner, J.L., Haddow, J.B., Tait, R.J. *Unloading waves in a plucked hyperelastic string,* J. Appl. Mech. **56**, 459-465, 1989.

[31] Whitham, G.B., *Linear and Nonlinear Waves,* Pure and Applied Mathematics: A Wiley Interscience Series of Texts, Monographs and Tracts, John Wiley and Sons Ltd., 1974.