Normal Functions and the Bloch-Beilinson Filtration

by

Héctor Damián Méndez Dávila

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Department of Mathematical and Statistical Sciences
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Abstract

Let X/k be a smooth projective geometrically irreducible variety over a field k, and $\mathrm{CH}^r(X;\mathbb{Q}) := \mathrm{CH}^r(X) \otimes \mathbb{Q}$ the Chow group of codimension r cycles, modulo rational equivalence. A long standing conjecture, due by S. Bloch and fortified by A. Beilinson, is the existence of a descending filtration on $\mathrm{CH}^r(X;\mathbb{Q})$, whose graded pieces detect the complexity of $\mathrm{CH}^r(X;\mathbb{Q})$. The question then is whether one can provide an explicit geometric interpretation of this filtration in the situation where $k \subseteq \mathbb{C}$ is a subfield. This will involve a candidate filtration introduced by Lewis, the concept of cycle induced normal functions, and fields of definition of their zero locus. Towards this goal, we present some partial results, and new lines of enquiry.

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Chapter 1

Introduction

Like many areas of mathematics, the approach to studying an "object" of interest is to introduce some invariants associated to it. In algebraic geometry, where a given space X is the zero set of a finite set of polynomials over a field k, a natural way to study it is to look at the irreducible subvarieties on X of a given codimension r, and form the group $z^r(X)$ of codimension r algebraic cycles defined on it. That being said, it is really a ring structure (via intersection) that one is seeking. For instance, how does one define the self intersection of a variety? If X has no singularities, then we can take quotient group modulo a suitable "minimal" equivalent relation, which in our situation is rational equivalence. This relation is minimal in the sense that, for any adequate relation $\sim$, we have $\xi_1 \sim_{\text{rat}} \xi_2 \Rightarrow \xi_1 \sim \xi_2$. See [20] for a proof as well as the proper definition of adequate relation. The resulting group is denoted by $\text{CH}^r(X)$, called a Chow group. This will allow for a ring structure on $\oplus_{r\geq 0}\text{CH}^r(X)$. This Chow group will have a continuous part, denoted by $\text{CH}^k_{\text{alg}}(X)$ (cycles algebraically equivalent to zero modulo rational equivalence), and a countable part $\text{CH}^r(X)/\text{CH}^r_{\text{alg}}(X)$, which contains a very important subgroup $\text{CH}^r_{\text{hom}}(X)/\text{CH}^r_{\text{alg}}(X)$, called the Griffiths group, which involves a first cycle class map construction below.

Let us assume, for simplicity, that $X/\mathbb{C}$ is a projective algebraic manifold (= smooth projective variety/$\mathbb{C}$). In the 20th century, there are two well-known maps used to study $\text{CH}^r(X)$, namely, the classical betti cycle class map, viewing X as an oriented manifold,

$$\text{cl}_r : \text{CH}^r(X) \to H^{2r}(X, \mathbb{Z}(r)),$$

with kernel $\text{CH}^r_{\text{hom}}(X)$, and for which the Tate twist $\mathbb{Z}(r) \simeq \mathbb{Z}$, will be explained later. As X is also a compact complex Kähler manifold, there is a Hodge

(p,q)-decomposition

$$H^{2r}(X,\mathbb{Z}(r))\otimes\mathbb{C}=\bigoplus_{p+q=2r}H^{p,q}(X),$$

which reflects the complex structure on $X/\mathbb{C}$. Hodge speculated that those classes in $H^{2r}(X,\mathbb{Z}(r))$, induced by the inclusion $\mathbb{Z}(r)\hookrightarrow\mathbb{C}$, that map to $H^{r,r}(X)$, are precisely the image of cl_r. This conjecture proved to be false by Atiyah and Hirzebruch: they showed that non-analytic torsion integral classes of type (p,p) exist on certain projective algebraic manifolds, and more recently by others, the existence of non-analytic, non-torsion integral classes as well. See [20] and the references cited there. This prompted a revision of the conjecture taking rational coefficients instead. Even in this case X must be projective algebraic, by a counterexample due to Mumford in [31], involving a non-algebraic compact torus (a Kähler manifold). The version of the celebrated Hodge conjecture that still survives scrutiny is:

Conjecture 1.0.1 (Hodge)**.** *For smooth projective $X/\mathbb{C}$,*

$$\mathrm{cl}_r\otimes\mathbb{Q}: CH^r(X;\mathbb{Q}):=CH^r(X)\otimes\mathbb{Q}\to H^{r,r}(X,\mathbb{Q}(r)):=H^{2r}(X,\mathbb{Q}(r))\cap H^{r,r}(X),$$

is surjective.

That $\mathrm{cl}_r(\otimes\mathbb{Q})$ does not detect all cycles is well-known, for example, the zero cycles of degree zero (those whose coefficients in a linear combination of points add up to zero) belong in the kernel of the cycle class map. This led to a secondary cycle class map, called the Abel-Jacobi map

$$AJ: CH^r_{hom}(X)\to J^r(X)$$

where $J^r(X)$ is a compact complex torus called the Griffiths' jacobian. This will capture some cycles missed by the cycle class map (for example, in the previous case, if X is a curve of genus $g>0$, the Abel Jacobi map is an isomorphism). For our interests, we are mainly interested in working with $\mathbb{Q}$-coefficients. Apart form torsion considerations, the story doesn't change much but the reason for the change has to do with the decomposition of the diagonal class $\Delta_X\in\mathrm{CH}^n(X\times X;\mathbb{Q})$ into it's Künneth components, which (assuming exists) can only be guaranteed with $\mathbb{Q}$-coefficients. This is a motivic story that will be explained later. To connect both of these maps, one introduces the notion of the category of mixed Hodge structures (MHS) over $\mathbb{Q}$. This will be explained later in the text. But for now, the situation is that both maps are described as follows:

$$\mathrm{cl}_r\otimes\mathbb{Q}:\mathrm{CH}^r(X;\mathbb{Q})\to\hom_{\mathrm{MHS}}(\mathbb{Q}(0),H^{2r}(X,\mathbb{Q}(r)):=\mathrm{Ext}^0_{\mathrm{MHS}}(\mathbb{Q}(0),H^{2r}(X,\mathbb{Q}(r))),$$

$AJ\otimes\mathbb{Q} : \ker(\mathrm{cl}_r\otimes\mathbb{Q}) : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J^r(X)\otimes\mathbb{Q} \simeq \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r))),$

where the isomorphism, even on the level of $\mathbb{Z}$-coefficients,

$$J^r(X) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r))),$$

is due to Jim Carlson [5]. The next step then would be to look at a map

$$\mathrm{CH}^r_{AJ}(X;\mathbb{Q}) := \ker(AJ\otimes\mathbb{Q}) \to \mathrm{Ext}^2_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-2}(X,\mathbb{Q}(r))).$$

However, it is a given fact that $\mathrm{Ext}^\nu_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-\nu}(X,\mathbb{Q}(r))) = 0$ for $\nu \geq 2$, whereas $\mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ can be highly nontrivial (Mumford [28]).

The category MHS is too coarse, whereas a more refined conjectural category of mixed motives $\mathcal{MM}$ is what Beilinson conjecturally proposed in this situation (more generally for smooth projective X over a field k), namely,

There is a descending filtration (referred to as a Bloch-Beilinson (BB) filtration):

$$F^0 = \mathrm{CH}^r(X;\mathbb{Q}) \supset F^1 = \mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q}) \supset F^2\mathrm{CH}^r(X;\mathbb{Q}) \supset \cdots \supset F^r\mathrm{CH}^r(X;\mathbb{Q}) \supset \{0\},$$

for which

$$Gr^\nu_F\mathrm{CH}^r(X;\mathbb{Q}) \simeq \mathrm{Ext}^\nu_{\mathcal{MM}}(\mathrm{Spec}(k), h^{2r-\nu}(X)(r))), \tag{1.1}$$

where $h^r(X)$ is motivic cohomology. Here conjecturally (e.g. assuming the Hodge conjecture) $\mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$, where $\mathrm{CH}^r_{\mathrm{num}}(X;\mathbb{Q})$ is numerical equivalence (defined later), and any candidate filtration seems to indicate that $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ when $k\subset\mathbb{C}$ is a subfield. It is natural to ask whether $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$, and the evidence in [24] supports this. So the natural question is why this expectation is a reasonable one? To answer this, let us assume given a smooth projective $X/\mathbb{C}$, with $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$. Now after adjoining the coefficients of the polynomial equations defining X over say $\overline{\mathbb{Q}}$, we can write $X = X_K \times_K \mathbb{C}$, where $K/\overline{\mathbb{Q}}$ is finitely generated. One can then spread X to $\mathscr{X} \xrightarrow{\rho} \mathcal{S}$, where $\mathscr{X}$, $\mathcal{S}$ are smooth quasi-projective varieties over $\overline{\mathbb{Q}}$, ρ is smooth and proper, and if $\eta\in\mathcal{S}_{\overline{\mathbb{Q}}}$ is the generic point, then $\overline{\mathbb{Q}}(\eta)\simeq K$ via a suitable embedding $\overline{\mathbb{Q}}(\eta)\hookrightarrow\mathbb{C}$, and $X/\mathbb{C} = \mathscr{X}_\eta \times_{\overline{\mathbb{Q}}(\eta)} \mathbb{C}$. Likewise, we may assume (possibly by enlarging $\mathcal{S}$) that ξ defines a class $\tilde{\xi} \in \mathrm{CH}^r_{\mathrm{rel\ hom}}(\mathscr{X};\mathbb{Q})$, which means that for any $t\in\mathcal{S}(\mathbb{C})$, $\tilde{\xi}_t \in \mathrm{CH}^r_{\mathrm{hom}}(\mathscr{X}_t;\mathbb{Q})$ and that $\tilde{\xi}_\eta = \xi$. The key point here is that $t\in\mathcal{S}(\mathbb{C}) \mapsto \nu_{\tilde{\xi}}(t) := AJ(\tilde{\xi}_t) \in J^r(\mathscr{X}_t(\mathbb{C}))\otimes\mathbb{Q}$ determines a variational Abel-Jacobi map called a normal function. It turns out that if in this situation, the zero locus of $\nu_{\tilde{\xi}}$ is a countable union of algebraic subvarieties of $\mathcal{S}$ over $\overline{\mathbb{Q}}$, then $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$. More precisely, we adopt the candidate BB filtration in [23], that uses a $\overline{\mathbb{Q}}$-spread idea. It turns out that in this situation

$F^2\mathrm{CH}^r(X;\mathbb{Q})$ can be described (via a spread) in terms of normal functions. Determining the field of definition of the zero locus is a deep problem; albeit easy to verify if one restricts to the situation of $F^2 \cap \mathrm{CH}^2_{\mathrm{alg}}(X;\mathbb{Q})$, where one can employ a result of S. Saito ([29]), in proving that $F^2 \cap \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{alg},AJ}(X;\mathbb{Q})$. The filtration ([23]) under consideration, only guarantees that $F^\nu\mathrm{CH}^r(X;\mathbb{Q})$ induces (via a $\overline{\mathbb{Q}}$-spread), a space of normal functions, which we refer to as "arithmetic normal functions"[1]; however we are unable to characterize $F^\nu\mathrm{CH}^r(X;\mathbb{Q})$ in terms of these normal functions for $\nu > 2$. From a conjectural standpoint, we expect this characterization to still hold. Indeed a clue is that from Beilinson's formula (1.1), one can show that

$$Gr_F^\nu \mathrm{CH}^r(X;\mathbb{Q}) \simeq \mathrm{Ext}^\nu_{\mathcal{MM}}(\mathrm{Spec}(k), h^{2r-\nu}(X)(r)/N^{r-\nu+1}h^{2r-\nu}(X)(r))),$$

an observation first exploited by S. Saito, (op. cit.), and where $N^\bullet$ refers to a coniveau filtration. We explain this in the text. This idea was exploited in [25] in the situation where $X/\mathbb{C} = X_0 \times \mathbb{C}$, X_0 being defined over $\overline{\mathbb{Q}}$, but where the cycles belong to $\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$. In this situation $F^\nu\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$ can be characterized in terms of arithmetic normal functions. The difficulty then is to analyze the zero locus of such normal functions. To handle this, we restrict further to a subspace $F^\nu \cap \underline{\mathrm{CH}}^r_{\mathrm{alg}}(X;\mathbb{Q})$ where we present two arguments showing that the zero locus of the associated arithmetic normal functions is defined over $\overline{\mathbb{Q}}$. Finally we characterize $F^\nu \cap \underline{\mathrm{CH}}^r_{\mathrm{alg}}(X;\mathbb{Q})$ in terms of the Abel-Jacobi map of specific algebraic varieties (just like the case $\nu = 2$), involving the $\overline{\mathbb{Q}}$-spread.

In summary, here are the conjectural goals, for which we aspire to prove, but provide only partial answers to:

(i) Prove that $F^\nu\mathrm{CH}^r(X;\mathbb{Q})$ can be characterized in terms of arithmetic normal functions. (What we do is find a suitable class of X for which this is true.)

(ii) Show that the zero locus of such an arithmetic normal function is defined over $\overline{\mathbb{Q}}$. (In this case, we restrict to $F^\nu \cap \underline{\mathrm{CH}}^r_{\mathrm{alg}}(X;\mathbb{Q})$.)

(iii) Using the above results in (i) & (ii), characterize $F^\nu\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$ in terms of the kernel of an Abel-Jacobi map of a variety involving the $\overline{\mathbb{Q}}$-spread.

1.1 Notation

Definition 1.1.1. Let $\mathcal{V}$ be an irreducible variety defined over a field k of finite transcendence degree over $\mathbb{Q}$. A point $p \in \mathcal{V}(\mathbb{C})$ is said to be very general if

$$\{\sigma(p) \mid \sigma \in \mathrm{Gal}(\mathbb{C}/k)\},$$

[1] First introduced in [17].

is dense in $\mathcal{V}(\mathbb{C})$ is the analytic topology.

• Throughout the rest of the thesis, $k \subset \mathbb{C}$ will denote an algebraically closed subfield.

• $\mathbb{Q}(n) = (2\pi i)^n\mathbb{Q}$ is called the Tate twist. Which is a pure Hodge structure of pure weight $-2n$ and of type $(-n, -n)$.

• We abbreviate the term $\mathbb{Q}$-mixed Hodge structure (defined later) as $\mathbb{Q}$-MHS or just MHS when the ring of definition $\mathbb{Q}$ is understood. Once we define this, for a given $\mathbb{Q}$-MHS H we put

$$\Gamma(H) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H),$$

$$J(H) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H)$$

the external classes in the category of $\mathbb{Q}$-MHS not necessarily graded polarizable.

• For a smooth variety over k, $H^i(X, \mathbb{Q}) := H^i(X(\mathbb{C}), \mathbb{Q})$ in singular cohomology. If $Y \subset X$ is a Zariski closed subset, and $n = \dim X$, we can identify $H^r_Y(X, \mathbb{Q})$ with $H_{n-r}(Y, \mathbb{Q})$, via Poincaré duality.

• $\mathrm{CH}^r(X)$ denotes the Chow group of X, that is the groups of codimension r cycles modulo rational equivalence. $\mathrm{CH}^r_{\mathrm{alg}}(X)$ then denotes the subgroup of cycles algebraically equivalent to zero. These concepts will be introduced in the appropriate sections later.

• We will also put $\mathrm{CH}^r(X; \mathbb{Q}) := \mathrm{CH}^r(X) \otimes \mathbb{Q}$ and define $\mathrm{CH}^r_{\mathrm{alg}}(X; \mathbb{Q})$ We also define $\mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q})$ to be the subgroup of cycles in the kernel of the cycle class map (or homologous to zero).

• Given the Abel Jacobi map (with a proper definition given later)

$$AJ; \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \to J(H^{2r-1}(X, \mathbb{Q}(r))).$$

We denote its kernel by $\mathrm{CH}^r_{\mathrm{AJ}}(X; \mathbb{Q})$.

• Let $K \subseteq \mathbb{C}$ be a subfield, and X smooth and projective over K. The coniveau filtration, denoted by $N^\nu_K H^i(X, \mathbb{Q})$, is given by

$$N^\nu_K H^i(X, \mathbb{Q}) := \ker\left(H^i(X, \mathbb{Q}) \to \varinjlim_{Y \subset X/K, codim_X Y \geq \nu} H^i(X \setminus Y, \mathbb{Q})\right).$$

• Given X a smooth variety over $\mathbb{C}$. Let $N^\nu_H H^i(X,\mathbb{Q})$ denote the largest sub-Hodge structure in $F^\nu H^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q})$. The general Hodge conjecture (abbreviated GHC) says that $N^\nu_K H^i(X,\mathbb{Q}) = N^\nu_H H^i(X,\mathbb{Q})$, if $K=\mathbb{C}$, and is also the case for $K \subseteq \mathbb{C}$, provided that K is algebraically closed.

Chapter 2

Preliminaries

2.1 Hodge Theory

2.1.1 Cohomology involving forms

Let X be a projective algebraic manifold of dimension n, and E^r_X be the $\mathbb{C}$-space of C^∞ r -forms on X, and associated complex $(E^\bullet_X, d)$. Since the differential $d : E^r_X \to E^{r+1}_X$ satisfies $d^2 = 0$, we have the de Rham cohomology $H^r_{DR}(X, \mathbb{C}) := \frac{\ker\ d: E^r_X \to E^{r+1}_X}{dE^{r-1}_X}$.

Recall the decomposition $E^r_X = \bigoplus_{p+q=r} E^{p,q}_X$, and we have $\overline{E^{p,q}_X} = E^{q,p}_X$. Here, $E^{p,q}_X$ are the C^∞ (p,q)-forms which in local coordinates $(z_1, \dots, z_n) \in X$ are of the form

$$\sum_{|I|=p, |J|=q} f_{IJ} dz_I \wedge d\overline{z}_J \quad f_{IJ}\ \mathbb{C} - \text{valued and } C^\infty,$$

$$I = 1 \leq i_1 < \cdots < i_p \leq n,\ J = 1 \leq i_1 < \cdots < i_q \leq n,$$

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p},\ d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

Then, under this decomposition, the differential splits into $d = \overline{\partial} + \partial$, where $\partial : E^{p,q}_X \to E^{p+1,q}_X$ and $\overline{\partial} : E^{p,q}_X \to E^{p,q+1}_X$. Here we have $\overline{\partial}^2 = 0$. Then, for a fixed p, we have the complex $(E^{p,\bullet}_X, \overline{\partial})$ which gives us the Dolbeault cohomology $H^{p,q}(X, \mathbb{C})$ or just $H^{p,q}(X)$.

Since X is projective algebraic (in fact, for all compact Kähler manifolds) we have the Hodge decomposition:

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X)$$

and $\overline{H^{q,p}(X)} = H^{p,q}(X)$, where, in fact, from Hodge theory, $H^{p,q}(X)$ can be identified with $\frac{E^{p,q}_{X,d-\text{closed}}}{\partial\bar{\partial}E_X^{p-1,q-1}}$.

Now, given w_1, a r form and w_2, a $2n-r$ form over X, we define

$$(w_1, w_2) \mapsto \int_X w_1 \wedge w_2$$

which induces the following nondegenerate pairings (Poincaré/Serre dualities):

$$H^r(X, \mathbb{C}) \times H^{2n-r}(X, \mathbb{C}) \to \mathbb{C}$$
$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}$$

and thus we can make the following identifications with duality:

$$H^r(X, \mathbb{C}) \simeq H^{2n-r}(X, \mathbb{C})^\vee, \qquad \text{Poincaré Duality}$$
$$H^{p,q}(X) \simeq H^{n-p,n-q}(X)^\vee, \qquad \text{Serre Duality}$$

2.1.2 The two cycle class maps and the Hodge conjecture

Now, given a subvariety V of X of codimension r (thus, having real codimension $2r$) we assign it the element $\text{cl}_r(V) \in H^{2n-2r}(X, \mathbb{C})^\vee$ defined in the following way: for $\{w\} \in H^{2n-2r}(X, \mathbb{C})$, $\text{cl}_r(V)(w) = \int_{V^*} w$ (where V^* is the smooth part of V, that is $V^* = V/V_{sing}$). One can see that $\int_{V^*} w$ is indeed finite by taking a desingularization $\sigma : \tilde{V} \overset{\approx}{\to} V$ so that $\sigma^*(w)$ is a C^∞ form on $\tilde{V}$ and thus $\int_{V^*} w = \int_{\tilde{V}} \sigma^*(w)$. Since $\tilde{V}$ is smooth and projective (thus compact) this has finite value.

Let $z^r(X)$ denote the free abelian group generated from the set of all irreducible subvarieties of X of codimension k. Extending by linearity, we can define this map for all elements in $z^r(X)$, thus

$$\text{cl}_r : z^r(X) \to H^{2r}(X, \mathbb{C}) \simeq H^{2n-2r}(X, \mathbb{C})^\vee$$

which is well defined. cl_r is called the fundamental class map. Working with singular homology with $\mathbb{Z}$-coefficients, the argument above and Poincaré duality, it follows that $\text{cl}_r : z^r(X) \subset H^{2r}(X, \mathbb{Z})$. We observe that, for $\{w\} \in H^{p,q}(X)$, with $p+q = 2n-2r$, if $(p, q) \neq (n-r, n-r)$ we have that either $p > n-r$ or $q > n-r$, implying we have more than $n-r$ of either $dz's$ or $d\bar{z}'s$ and because $\dim V^* = n-r$, we have $\int_{V^*} w = 0$. Since we know that $\text{cl}_r(z^r(X)) \subset H^{2r}(X, \mathbb{Z})$, from the previous argument we have that $\text{cl}_r(z^r(X)) \subset H^{r,r}(X) \cap H^{2r}(X, \mathbb{Z})$. As mentioned earlier, there are counterexamples to show that this is not an equality. However, if we change the statement to rational coefficients, we arrive at the celebrated

Hodge conjecture $\mathrm{cl}_r(z^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}) = H^{r,r}(X, \mathbb{Q}) := H^{r,r}(X) \cap H^{2r}(X, \mathbb{Q})$.

There is another famous class map due to Griffiths. Using the Hodge filtration

$$F^r H^i(X, \mathbb{C}) = \bigoplus_{p+q=i,\ p \geq r} H^{p,q}(X)$$

we can define the rth Jacobian as

$$J^r(X) := \frac{H^{2r-1}(X, \mathbb{C})}{F^r H^{2r-1}(X, \mathbb{C}) + H^{2r-1}(X, \mathbb{Z})} \simeq \frac{H^{2r-1}(X, \mathbb{R})}{H^{2r-1}(X, \mathbb{Z})}$$

where the isomorphism comes from the real isomorphism

$$\frac{H^{2r-1}(X, \mathbb{C})}{F^r H^{2r-1}(X, \mathbb{C})} \simeq H^{2r-1}(X, \mathbb{R})$$

inducing a complex structure on $H^{2r-1}(X, \mathbb{R})$. Each Jacobian defined this way is thus a compact complex torus (called Griffiths Jacobian). Now, applying Serre duality to each $H^{p,q}(X)$ present in $H^{2r-1}(X, \mathbb{C})/F^r H^{2r-1}(X, \mathbb{C})$, we see that this latter term is isomorphic to $F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})^\vee$. By this, and using the isomorphism $H^{2r-1}(X, \mathbb{Z}) \simeq H_{2n-2r+1}(X, \mathbb{Z})$ between singular cohomology and homology, we can identify this homology group with its image under:

$$H_{2n-2r+1}(X, \mathbb{Z}) \to F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})^\vee$$

that maps a given $\{\gamma\}$ in $H_{2n-2r+1}(X, \mathbb{Z})$ to the map ρ_γ such that, for $\{w\} \in F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})$ we have $\rho_\gamma(\{w\}) = \int_\gamma w$ (periods). Hence we ignore torsion in $H_{2n-2r+1}(X, \mathbb{Z})$. Therefore, we arrive at

$$J^r(X) \simeq \frac{F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})^\vee}{H_{2n-2r+1}(X, \mathbb{Z})}$$

We define $z^r_{\mathrm{hom}}(X) = \ker(\mathrm{cl}_r) \subset z^r(X)$. We see that given $\xi \in z^r_{\mathrm{hom}}(X)$, then by definition we must have that ξ is the boundary of a chain ζ of real dimension 1 greater than that of ξ (thus, of real dimension $2n - 2r + 1$). Then, we define the second cycle class map to be

$$\Phi_r : z^r_{hom}(X) \to J^r(X)$$

that, given $\{w\} \in F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})$ gives us

$$\Phi_r(\xi)(\{w\}) = \int_\zeta w$$

taking quotient over the elements of $H_{2n-2r+1}(X,\mathbb{Z})$ (identified in

$$F^{n-r+1}H^{2n-2r+1}(X,\mathbb{C})^{\vee}$$

as said before). Here ζ is a chain of real dimension $2n-2r+1$ such that $\xi=\partial\zeta$.

This map is well defined and is known as the Abel-Jacobi map. Details can be found for example in [20].

2.1.3 $\mathbb{Q}$-Mixed Hodge Structures and the implementation of twists

The following definitions will be given in terms of $\mathbb{Q}$ but they can also be established for any subring $\mathbb{A}\subset\mathbb{R}$ ($\mathbb{Z}$ is a common choice as well).

Definition 2.1.4. *A $\mathbb{Q}$-Hodge structure (of weight $N\in\mathbb{Z}$) consists of a finitely generated $\mathbb{Q}$-module V and a decomposition $V_{\mathbb{C}}=V\otimes_{\mathbb{Q}}\mathbb{C}=\bigoplus_{p+q=N}V^{p,q}$ such that $\overline{V^{p,q}}=V^{q,p}$ (here the bar is denoting complex conjugation).*

Remark 2.1.5. *For a $\mathbb{Q}$-Hodge structure, one can define a descending filtration*

$$V_{\mathbb{C}}\supset\cdots\supset F^{i}\supset F^{i+1}\supset\cdots\supset\{0\}$$

as $F^{r}V_{\mathbb{C}}=\bigoplus_{p+q=N,p\geq r}V^{p,q}$. We have then $V_{\mathbb{C}}=F^{r}V_{\mathbb{C}}\oplus\overline{F^{N-r+1}V_{\mathbb{C}}}$ for all r. Indeed this filtration datum is equivalent to a Hodge structure of weight N, for we can set $H^{p,q}:=F^{p}\cap\overline{F^{q}}$.

Example 2.1.6. *The main example, due to Hodge, of a Hodge structure is $H^{i}(X,\mathbb{Q})$ (of weight i) for a $X/\mathbb{C}$ smooth projective.*

Example 2.1.7. *Another example of a $\mathbb{Q}$-Hodge structure is the Tate twist $\mathbb{Q}(r)=(2\pi i)^{r}\mathbb{Q}$, which is of type $(-r,-r)$.*

Example 2.1.8. *For smooth projective $X/\mathbb{C}$, $H^{i}(X,\mathbb{Q}(r)):=H^{i}(X,\mathbb{Q})\otimes\mathbb{Q}(r)$ is a Hodge structure of weight $i-2r$.*

In his famous work on the cohomology of complex schemes, Deligne [6] formulated the notion of a generalized Hodge structure, called a mixed Hodge structure, namely,

Definition 2.1.9. *A $\mathbb{Q}$-mixed Hodge structure ($\mathbb{Q}$-MHS) is given by the following datum:*
• *A finitely generated $\mathbb{Q}$-module V.*
• *A finite descending "Hodge" filtration on $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$:*

$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\}$$

• *An increasing "weight" filtration on $V_{\mathbb{Q}}$:*

$$\{0\} \subset \cdots \subset W_{l-1} \subset W_l \subset \cdots \subset V_{\mathbb{Q}}$$

such that $\{F^r\}_{r\in\mathbb{Z}}$ induces a (pure) Hodge structure of weight l on $Gr_l^W V_{\mathbb{Q}} = W_l/W_{l-1}$.

Definition 2.1.10. *Let V_1 and V_2 be $\mathbb{Q}$-MHS. A morphism $f : V_1 \to V_2$ is a $\mathbb{Q}$-linear map such that*

$$f(W_l V_1 \otimes \mathbb{Q}) \subseteq W_l V_2 \otimes \mathbb{Q} \quad \forall l$$

$$f(F^r V_{1,\mathbb{C}}) \subset F^r V_{2,\mathbb{C}} \quad \forall r$$

According to Deligne [6], these morphisms satisfy

$$f(F^r V_{1,\mathbb{C}}) = f(V_{1,\mathbb{C}}) \cap F^r V_{2,\mathbb{C}}$$

$$f(W_l V_1 \otimes \mathbb{Q}) = f(V_1 \otimes \mathbb{Q}) \cap W_l V_2 \otimes \mathbb{Q}$$

In other words

Proposition 2.1.11. *The weight $W_\bullet$ and Hodge $F^\bullet$ functors are exact.*

We will only provide the general idea of the proof.
For a given $\mathbb{Q}$-MHS V, Deligne defines a bigrading $V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ where $F^r V_{\mathbb{C}} = \bigoplus_{p\geq r}(\bigoplus_q I^{p,q})$ and $W_l V_{\mathbb{C}} = \bigoplus_{p+q\leq l} I^{p,q}$, where $I^{p,q}$ is defined in terms of weight and Hodge filtration. Then, given a morphism $f : V_1 \to V_2$, we obtain $f(I^{p,q}(V_{1,\mathbb{C}})) \subseteq I^{p,q}(V_{2,\mathbb{C}})$ for all p, q, and the conclusion is immediate.

The following theorem by Deligne [6] will be stated without proof.

Theorem 2.1.1. *The cohomology of a complex scheme X carries a canonical and functorial MHS, which agrees with the aforementioned Hodge structure $H^i(X,\mathbb{Q})$ in the event that $X/\mathbb{C}$ is smooth projecive.*

As a blanket statement, Deligne's ideas extend to cohomology with supports, homology (singular and Borel-Moore); moreover the localization sequence of a pair (X, Z) is a LES of MHS. If $X/\mathbb{C}$ is smooth projective, then $H_i(X,\mathbb{Q})$ is a Hodge structure of weight $-i$; moreover $H_i(X,\mathbb{Q}(r)) := H_i(X,\mathbb{Q}) \otimes \mathbb{Q}(-r)$ is a Hodge structure of weight $2r - i$.

Example 2.1.12. *Let $X/\mathbb{C}$ be a smooth irreducible complex scheme of dimension n, and $Z \subset X$ a closed subvariety. Then the Poincaré duality isomorphism*

$$H^i_Z(X, \mathbb{Q}(r)) \simeq H_{2n-i}(Z, \mathbb{Q}(n-r)) := H_{2n-i}(Z, \mathbb{Q}) \otimes \mathbb{Q}(r-n),$$

is an isomorphism of MHS. (Here $H_{2n-i}(Z, \mathbb{Q}(n-r))$ is Borel-Moore homology, which agrees with singular homology in the event that Z is complete.

Remark 2.1.13. *Borel-Moore homology can be defined as follows [19]:*
We recall the construction of the simplicial homology. Given a simplicial complex N with $|N| = \bigcup_{\sigma \in N} \sigma$ and triangulation $T : |N| \to X$, we define the space of i-chains of X with respect to T, denoted by $C_i^T(X)$, to be the vector space consisting of all formal linear combinations $\xi = \sum_{\sigma \in N^i} \xi_\sigma \sigma$ where N^i denotes the set of i-simplices in N and the coefficients ξ_σ are in the field of definition and only finitely many of them are non-zero.
A refinement of T is a triangulation $T' : |N| \to X$ such that for each $\sigma' \in N$ there exists some $\sigma \in N$ such that $T'(\sigma') \subseteq T(\sigma)$. Then, let us define the space of all piecewise linear i-chains $C_i(X)$ to be the colimit of the spaces $C_i^T(X)$ under refinement. Then, the boundary maps $\partial : C_i^T(X) \to C_{i-1}^T(X)$ induce boundary maps $\partial : C_i(X) \to C_{i-1}(X)$ with $\partial^2 = 0$. Then, the simplicial homology in X is defined by

$$H_i^{\mathrm{simp}}(X) = \frac{\ker\partial : C_i(X) \to C_{i-1}(X)}{\mathrm{im}\partial : C_{i+1}(X) \to C_i(X)}$$

In the definition of $C_i^T(X)$ if we ignore the restriction that only finitely many of the coefficients of the linear combinations are non zero, we can define $C_i^T((X))$ the space of locally finite i-chains of X with respect to T. We can define $C_i((X))$ and $\partial : C_i((X)) \to C_{i-1}((X))$ in the same way as in the simplicial case. Then, the Borel-Moore homology is defined by

$$H_i^{\mathrm{BM}}(X) = \frac{\ker\partial : C_i((X)) \to C_{i-1}((X))}{\mathrm{im}\partial : C_{i+1}((X)) \to C_i((X))}$$

When X is compact, any triangulation has N finite and both the Borel-Moore and the simplicial homology coincide. Since the simplicial homology is isomorphic to the singular homology, it agrees with the Borel-Moore homology as well.

Example 2.1.14. *Let $\overline{X}$ be a compact Riemann surface (that is, a 1 dimensional smooth projective variety over $\mathbb{C}$) and $\emptyset \neq \Sigma \subsetneq \overline{X}$ a finite set of points. Let $M = \#|\Sigma|$ and $X := \overline{X} \setminus \Sigma$. Then we have the exact sequence:*

$$H^1_\Sigma(\overline{X}, \mathbb{Z}) \to H^1(\overline{X}, \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to H^2_\Sigma(\overline{X}, \mathbb{Z}) \to H^2(\overline{X}, \mathbb{Z}) \to 0$$

where we observe that

$$H^1_\Sigma(\overline{X}, \mathbb{Z}) \simeq H_1(\Sigma, \mathbb{Z}(1)) = 0,$$

$$H^2_\Sigma(\overline{X},\mathbb{Z}) \simeq H_0(\Sigma,\mathbb{Z}(1)) \simeq H^0(\Sigma,\mathbb{Z}(-1)), \text{ and } H^2(\overline{X},\mathbb{Z}) \simeq \mathbb{Z}(-1)$$

have (pure) weight 2*. Then the above exact sequence becomes*

$$0 \to H^1(\overline{X},\mathbb{Z}) \to H^1(X,\mathbb{Z}) \to \mathbb{Z}(-1)^{M-1} \to 0.$$

And, if we put $W_2 := H^1(X,\mathbb{Z})$*,* $W_1 := H^1(\overline{X},\mathbb{Z})$*, and* $W_0 = 0$*, we observe that* $W_2/W_1 \simeq \mathbb{Z}^{M-1}(-1)$ *is indeed a Hodge structure of weight 2 (and pure type* $(1,1)$*). Also* $W_1/W_0 = H^1(\overline{X},\mathbb{Z})$ *which has a Hodge structure of weight* 1 *as noted before. Thus* $H^1(X,\mathbb{Z})$ *has a MHS (with minimum weight* 1*).*

Example 2.1.15. *With the notation of the previous example, if we twisted by* $\mathbb{Z}(1)$*, then* $H^1(X,\mathbb{Z}(1))$ *would have a MHS of minimum weight of* -1 *and a maximum weight of* 0*.*

2.1.16 Alternate take on the Abel-Jacobi map

Given X smooth complex projective variety, let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$, with support $|\xi|$. Then, following the idea behind Example 2.1.14, (with the inclusion of twists that will be justified later), we have the exact sequence:

$$H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \to H^{2r-1}(X,\mathbb{Z}(r)) \to H^{2r-1}(X\setminus|\xi|,\mathbb{Z}(r))$$

$$\to H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}(X,\mathbb{Z}(r))$$

But $H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \simeq H_{2n-2r+1}(|\xi|,\mathbb{Z}(n-r)) = 0$ since $\dim_{\mathbb{R}}|\xi| = 2n-2r$. Now let

$$H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ := \ker(H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}(X,\mathbb{Z}(r)))$$

We observe that the cycle class $[\xi] \in H^{2r}_{|\xi|}(X,\mathbb{Z}(r))$, and since ξ is homologous to zero in X we have $[\xi] \in H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ$. Then, for this ξ we have the diagram

$$\begin{array}{ccccc} H^{2r-1}(X,\mathbb{Z}(r)) & \hookrightarrow & H^{2r-1}(X\setminus|\xi|,\mathbb{Z}(r)) & \twoheadrightarrow & H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ \\ \| & & \uparrow & & \uparrow \\ H^{2r-1}(X,\mathbb{Z}(r)) & \hookrightarrow & E & \hookrightarrow & \mathbb{Z}[\xi] \end{array}$$

with $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r)))$. Note that

$$H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \simeq H_{2n-2r}(|\xi|,\mathbb{Z}(n-r)),$$

has (pure) weight 0. We then put $\Phi_r(\xi) := \{\xi\}$. Carlson [5] proved that $J^r(X) \simeq \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r)))$ and that the two definitions coincide.

Explicitly, the isomorphism can be explained as follows:
given $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r-1}(X, \mathbb{Z}(r)))$ we have the exact sequence (by definition):

$$0 \to H^{2r-1}(X, \mathbb{Z}(r)) \to E \to \mathbb{Z}(0) \to 0$$

By exactness of the Hodge filtration, and shifting by $-r$ via twists, we obtain

$$0 \to F^0 H^{2r-1}(X, \mathbb{C}) \to F^0 E_{\mathbb{C}} \to \mathbb{Z} \otimes \mathbb{C} = \mathbb{C} \to 0$$

Thus, there exist an element $\mu \in F^0 E_{\mathbb{C}}$ which is mapped to 1 in $\mathbb{C}$. Likewise, over $\mathbb{Z}$, the exactness of the weight filtration implies there exists $\nu \in W_0 E$ which maps to 1 in $\mathbb{Z}(0)$. The difference $\nu - \mu$ is precisely the retraction image $r_{\mathbb{C}} : E_{\mathbb{C}} \twoheadrightarrow H^{2r-1}(X, \mathbb{C})$. Note that after "untwisting", $\mu \in F^r E_{\mathbb{C}}$, and by Hodge type, it's effect on $F^{n-r+1} H^{2n-2r+1}(X, \mathbb{C})$ is zero. On the other hand, $E \subset H^{2r-1}(X \setminus |\xi|, \mathbb{Z}(r)) \simeq H_{2n-2r+1}(X, |\xi|, \mathbb{Z}(n-r))^{\vee}$, and ν corresponds to the current $\frac{1}{(2\pi \mathrm{i})^{n-r}} \int_{\zeta} (-)$, where $\partial \zeta = \xi$. This is precisely the Griffiths Abel-Jacobi map incorporating the Tate twist.

2.1.17 Hypercohomology

Let $\mathcal{P}^{\bullet \geq 0}$ be a complex of sheaves on a 'nice' space X. Using the Cech coboundary operator δ we have the Cech double complex

$$\{C^{\bullet}(U, \mathcal{P}^{\bullet}) \mid d, \delta\}$$

where U is an open cover of X. We take the associated single complex

$$\{M^{\bullet}(U) = \bigoplus_{i+j=\bullet} C^i(U, \mathcal{P}^j), D = d \pm \delta\}$$

with $D^2 = 0$.

Definition 2.1.18. *The kth-hypercohomology of the complex $\mathcal{P}^{\bullet}$ is given by*

$$\mathbb{H}^k(\mathcal{P}^{\bullet}) := \varinjlim_{U} H^k_D(M^{\bullet})$$

Double complexes have two associated descending filtrations which are (filtered) subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences. In this case, we can denote the sequences

$$'E_2^{p,q} = H^p_{\delta}(X, H^q_d(\mathcal{P}^{\bullet}))$$

$$''E_2^{p,q} = H^p_d(X, H^q_{\delta}(\mathcal{P}^{\bullet}))$$

In the first spectral sequence $H^q_d(\mathcal{P}^{\bullet})$ denotes the cohomology of $\mathcal{P}^{\bullet}$, which gives us a way to identify when two complexes have the same associated hypercohomology.

Definition 2.1.19. *Two complexes of sheaves $\mathcal{K}_1^\bullet$, $\mathcal{K}_2^\bullet$ are quasi-isomorphic if there is a morphism $h : \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : H^q(\mathcal{K}_1^\bullet) \to H^q(\mathcal{K}_2^\bullet)$ for all q.*

Therefore, by the first spectral sequence above $'E_2^{p,q}$, two quasi-isomorphic complexes yield the same hypercohomology. Moreover, if a complex $\mathcal{P}^\bullet$ is quasi-isomorphic to a complex $(\mathcal{K}^\bullet, d)$ of acyclic sheaves (that is, $H^i(X, \mathcal{K}^j) = 0$ for $i > 0$ and for all j), the second spectral sequence $''E_2^{p,q}$ tells us that

$$\mathbb{H}^k(\mathcal{P}^\bullet) := H^k(H^0(X, \mathcal{K}^j))$$

Example 2.1.20. *The complex of sheaves of holomorphic forms $(\Omega_X^\bullet, d)$ is defined by*

$$\mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots$$

Where $\mathcal{O}_X$ denotes the sheaf of germs of holomorphic functions and Ω_X^l denotes the sheaf of germs of holomorphic differential l-forms. By the holomorphic Poincaré lemma, this complex is quasi-isomorphic to the acyclic complex

$$\mathbb{C} \to 0 \to 0 \to \cdots$$

We also have the complex of sheaves of C^∞-forms $(\mathcal{E}_X^\bullet, d)$:

$$\mathcal{E}_X^0 \to \mathcal{E}_X^1 \to \mathcal{E}_X^2 \to \cdots$$

which, by the C^∞ Poincaré lemma, is quasi-isomorphic to $(\Omega_X^\bullet, d)$. Therefore, by the previous observations we have

$$H^k(X, \mathbb{C}) \simeq \mathbb{H}^k(\mathbb{C} \to 0 \to 0 \to \cdots) \simeq \mathbb{H}^k(\Omega_X^\bullet) \simeq \mathbb{H}^k(\mathcal{E}_X^\bullet)$$

Moreover, if we denote by $F^r\Omega_X^\bullet$ and $F^r\mathcal{E}_X^\bullet$ the filtered complexes

$$0 \to 0 \to \cdots \to \Omega_X^r \xrightarrow{d} \Omega_X^{r+1} \xrightarrow{d} \cdots$$

$$0 \to 0 \to \cdots \to \mathcal{E}_X^r \xrightarrow{d} \mathcal{E}_X^{r+1} \xrightarrow{d} \cdots$$

these are quasi-isomorphic as well (see [12]). Then, since $\mathcal{E}_X^\bullet$ is acyclic, we have

$$\mathbb{H}^k(F^r\Omega_X^\bullet) \simeq \mathbb{H}^k F^r(\mathcal{E}_X^\bullet) \simeq \frac{\ker d : F^r E_X^k \to F^r E_X^{k+1}}{dF^r E_X^{k-1}} \simeq F^r H^k_{\mathrm{DR}}(X)$$

Example 2.1.21. *By GAGA, we have a quasi-isomorphism between $\Omega_X^\bullet$ and $\Omega_{X,\mathrm{alg}}^\bullet$ which is also filtered. Thus we have*

$$\mathbb{H}^i(F^r\Omega_X^\bullet) \simeq \mathbb{H}^i(F^r\Omega_{X,\mathrm{alg}}^\bullet) =: F^r H^i_{\mathrm{Zar}}(X, \mathbb{C})$$

2.2 Chow groups

2.2.1 Rational and algebraic equivalence

Let $z^r(X)$ denote the free abelian group generated from the set of all irreducible subvarieties of X of codimension r.

Example 2.2.2. *Given that X is of dimension n, the only subvariety of codimension 0, (that is, of dimension n) is X itself. Thus, $z^0(X) = \mathbb{Z}\{X\} \simeq \mathbb{Z}$.*

Example 2.2.3. *The only subvarieties of codimension n (that is, of dimension 0) are the discrete subvarieties, each consisting of a finite set of points. Thus the irreducible subvarieties are the one-point subvarieties, and we get*

$$z^n(X) = \{\sum_{j=0}^{N} n_j p_j \mid N \in \mathbb{N},\ n_j \in \mathbb{Z},\ p_j \in X\}$$

Definition 2.2.4. *We say that $\xi_1, \xi_2 \in z^r(X)$ are rationally equivalent (denoted $\xi_1 \sim_{\text{rat}} \xi_2$) if there exists a cycle $w \in z^r(\mathbb{P}^1 \times X)$ in "general position", meaning $w(t) := Pr_{2,*}((t \times X) \bullet w) \in z^r(X)$ is defined $\forall t \in \mathbb{P}^1$, such that*
$\xi_1 - \xi_2 = w(0) - w(\infty)$.

Definition 2.2.5. *We say that $\xi_1, \xi_2 \in z^r(X)$ are algebraically equivalent (denoted $\xi_1 \sim_{\text{alg}} \xi_2$) if there exists a smooth connected curve Γ, a cycle $w \in z^r(\Gamma \times X)$ in "general position", and points $p, q \in \Gamma$, such that $\xi_1 - \xi_2 = w(p) - w(q)$.*

Let $z^r_{\text{rat}}(X) = \{\xi \in z^r(X) \mid \xi \sim_{rat} 0\}$ and $z^r_{\text{alg}}(X) = \{\xi \in z^r(X) \mid \xi \sim_{\text{alg}} 0\}$. The quotient $\text{CH}^r(X) := z^r(X)/z^r_{\text{rat}}(X)$ is called the Chow group of X of codimension r and the quotient $\text{CH}^r_{\text{alg}}(X) := z^r_{\text{alg}}(X)/z^r_{\text{rat}}(X)$ is the algebraic Chow group of X of codimension r. It is also the case that $\text{CH}^r_{\text{hom}}(X) = \text{CH}^r_{\text{alg}}(X)$ for $r = 1$ and $r = n$, but is in general false for $1 < r < n$, (that being first demonstrated by Griffiths [11]).

Another notable group is the group of cycles numerically equivalent to zero, $\text{CH}^r_{\text{num}}(X)$.

Definition 2.2.6. *We say that* $\xi \in CH^r(X)$ *is numerically equivalent to zero if* $\deg(\xi \cdot \gamma)_X = 0$ *for all* $\xi \in CH_r(X)$ *(where* $(\cdot)_X$ *is the intersection pairing on* X*).*

We will see that if we assume the Hodge conjecture, a cycle is numerically equivalent to zero exactly when it is homologous to zero (it is in the kernel of the Abel Jacobi map). We first recall the hard Lefschetz theorem which will be stated without proof:

Theorem 2.2.7 (Hard Lefschetz theorem)**.** *Let* L_X *denote the operator of taking cup product with the hyperplane class of* X*. For all* k*, the map*

$$L_X^{n-i} : H^i(X; \mathbb{Q}) \to H^{2n-i}(X; \mathbb{Q})$$

is an isomorphism. The result holds true for complex coefficients as well.

And we also introduce

Conjecture 2.2.8 (hard Lefschetz conjecture)**.** *The inverse of the map* L_X^{n-i}

$$\Lambda_X^{n-i} : H^{2n-i}(X; \mathbb{Q}) \to H^i(X; \mathbb{Q})$$

is algebraic

By Hodge theory, we have that the cup product pairing

$$N_H^r H^k(X, \mathbb{Q}) \times N_H^{r+n-k} H^{2n-k}(X, \mathbb{Q}) \to H^{2n}(X, \mathbb{Q}) \simeq \mathbb{Q}$$

is nondegenerate. Here N_H^r denotes the largest sub-Hodge structure in $F^r H^i(X, \mathbb{Q}) := \{F^r H^i(X, \mathbb{C})\} \cap H^i(X, \mathbb{Q})$. If we replace N_H^r by the coniveau N^r, then nondegeneracy requires General Hodge Conjecture (GHC). Note that the GHC (stated later) $\Rightarrow$ Hodge conjecture $\Rightarrow$ hard Lefschetz conjecture, where the latter is enough to guarantee that a cycle ξ is numerically equivalent to zero exactly when it lies in the kernel of the cycle class map. As part of the construction of the Bloch Beilinson filtration below, one needs the hard Lefschetz conjecture.

2.2.9 Milnor K- theory

Given a field $\mathbb{F}$, we have the Milnor K-groups $K_n^M(\mathbb{F}), n \geq 0$, with $K_0^M(\mathbb{F}) := \mathbb{Z}$, $K_1^M(\mathbb{F}) = \mathbb{F}^\times$ and for $n \geq 2$, generated by symbols $\{a_1, \dots, a_n\}$, with $a_1, \dots, a_n \in \mathbb{F}^\times$ such that

$$\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times \to K_n^M(\mathbb{F})$$

where

$$(a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$$

is a multilinear function, and where $\{a_1, \ldots, a_n\} = 0$ if $a_i + a_j = 1$ for some $i \neq j$.

Accordingly, we define the Milnor sheaf $\mathscr{K}^M_{r,X} := (\mathcal{O}^\times_X \otimes \cdots \otimes \mathcal{O}^\times_X)/\mathscr{J}$ (r times), where $\mathscr{J}$ is the subsheaf of tensor product generated by sections of the form:

$$\{\tau_1 \otimes \cdots \otimes \tau_r \mid \tau_i + \tau_j = 1 \text{ , for some } \ i \text{ and } \ j, i \neq j\}$$

For example, $\mathscr{K}^M_{1,X} = \mathcal{O}^\times_X$. Then one has a Gersten - Milnor complex, which comes from a flasque resolution of $\mathscr{K}^M_{r,X}$ (see [8], [18]).

$$\mathscr{K}^M_{r,X} \to K^M_r(\mathbb{F}(X)) \to \bigoplus_{\mathrm{cod}_X Z=1} K^M_{r-1}(\mathbb{F}(Z)) \to \cdots$$

$$\to \bigoplus_{\mathrm{cod}_X Z=r-2} K^M_2(\mathbb{F}(Z)) \to \bigoplus_{\mathrm{cod}_X Z=r-1} K^M_1(\mathbb{F}(Z)) \to \bigoplus_{\mathrm{cod}_X Z=r} K^M_0(\mathbb{F}(Z)) \to 0$$

where $K_0(\mathbb{F}) = \mathbb{Z}$, $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_2(\mathbb{F})$ consisting of the abelian group generated by the symbols $\{a, b\}$ with $a, b \in \mathbb{F}^\times$subject to the Steinberg relations:

$$\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\}$$

$$\{a, b\} = \{b, a\}^{-1}$$

$$\{a, 1-a\} = \{a, -a\} = 1 \text{ , for } a \neq 1$$

In particular, when the field of definition is $\mathbb{C}$, we have

$$\mathscr{K}^M_{r,X} \to K^M_r(\mathbb{C}(X)) \to \bigoplus_{\mathrm{cod}_X Z=1} K^M_{r-1}(\mathbb{C}(Z)) \to \cdots$$

$$\to \bigoplus_{\mathrm{cd}_X Z=r-2} K^M_2(\mathbb{C}(Z)) \stackrel{T}{\to} \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbb{C}(Z)^\times \stackrel{\mathrm{div}}{\to} \bigoplus_{\mathrm{cd}_X Z=r-2} \mathbb{Z}$$

where div is the divisor map of zeros minus poles of a rational function and T is the Tame symbol map

$$T : \bigoplus_{\mathrm{cd}_X Z=r-2} K^M_2(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X D=r-1} \mathbb{C}(D)^\times$$

which is defined as follows: for $f, g \in \mathbb{C}(Z)^\times$, we have

$$T(\{f, g\}) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D$$

where $(\cdots)_D$ means restriction to the generic point of D, and $\nu_D(h)$ is the order of vanishing of a rational function h along D. (To see more details on this complex, one can see [14], [8], [18]). It is a Milnor K-theoretic version of the famous work of Bloch and Quillen using Quillen K-theory.

The corresponding homologies of this complex define the higher Chow groups, denoted by $\mathrm{CH}^r(X,m)$ for the cases $0 \leq m \leq 2$. Specifically, one has in the given range of m, and due to flasqueness of the Gersten resolution, (proved by Stefan Müller-Stach, see [27])

$$\mathrm{CH}^r(X,m) \simeq H^{r-m}_{\mathrm{Zar}}(X, \mathscr{K}^M_{r,X}).$$

We have for example that $\mathrm{CH}^r(X,1)$ is represented by the classes of the form $\xi = \sum_j (f_j, D_j)$, where $\mathrm{codim}_X D_j = r-1$, $f_j \in \mathbb{C}(D_j)^\times$, and $\sum \mathrm{div}(f_j) = 0$ (and modulo the image of the Tame symbol).

We observe that $\mathrm{CH}^r(X) := \mathrm{CH}^r(X,0)$ is the free abelian group generated by subvarieties of codimension r in X, modulo divisors of rational functions on subvarieties of codimension $r-1$ in X.

2.2.10 Twisted version of the cycle class map

We can define the cycle class map with Milnor K-theory as follows:
Recall the d log map $\mathscr{K}^M_{r,X} \to \Omega^r_X$, defined by $\{f_1, \ldots, f_r\} \mapsto \bigwedge_j d\ \mathrm{log} f_j$. Then, by the Poincaré holomorphic lemma, this map induces the following morphism of complexes which is due to Gabber (or Müller-Stach, Elbanz-Vincent, see [8])

$$\big(\mathscr{K}^M_{r,X} \to 0 \to 0 \to \cdots\big) \to F^r\Omega^\bullet_{X,\mathrm{alg}}[r],$$

in the Zariski topology.

This, in turn, induces

$$\mathrm{CH}^r(X) = H^r_{\mathrm{Zar}}(X, \mathscr{K}^M_{r,X}) = \mathbb{H}^r\big(\mathscr{K}^M_{r,X} \to 0 \to 0 \to \cdots\big)$$

$$\to \mathbb{H}^r\big(F^r\Omega^\bullet_{X,\mathrm{alg}}[r]\big) = \mathbb{H}^{2r}\big(F^r\Omega^\bullet_{X,\mathrm{alg}}\big) \simeq F^r H^{2r}(X,\mathbb{C}).$$

Later we will see that the image lies in $H^{2r}(X,\mathbb{Q}(r))$ giving us a twisted version of the cycle class map. We observe that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on both $H^r_{\mathrm{Zar}}(X, \mathscr{K}^M_{r,X})$ and $\mathbb{H}^{2r}\big(F^r\Omega^\bullet_{X,\mathrm{alg}}\big)$, and hence compatible with this twisted version of the cycle class map. That explains the need to incorporate twists.

2.2.11 Image and kernel of the Abel-Jacobi map

It is well known that, for $r = 1$, the Abel-Jacobi map $\Phi_1 : \mathrm{CH}^1_{\mathrm{hom}}(X) \xrightarrow{\sim} J^1(X)$ is an isomorphism. It is also known that $\Phi_n : \mathrm{CH}^n_{\mathrm{hom}}(X) \twoheadrightarrow J^n(X)$ is surjective. However, surjectivity of Φ_r for $1 < r < n$ rarely holds in general.
The following result concerning the image of the Abel Jacobi map is due to Griffiths (see [11]).

Theorem 2.2.1. [***Griffiths***] *[11] If $X \subset \mathbb{P}^4$ is a quintic 3-fold, as a general hyperplane section of Fermat quintic fourfold in $\mathbb{P}^4$, then the image of the Abel Jacobi map $AJ : CH^2_{\mathrm{hom}}(X) \to J^2(X)$ contains an infinite cyclic subgroup.*

Define $J^r_{\mathrm{alg}}(X) := \Phi_r(\mathrm{CH}^r_{\mathrm{alg}}(X))$, where we recall $\mathrm{CH}^r_{\mathrm{alg}}(X) = z^r_{\mathrm{alg}}(X)/z^r_{\mathrm{rat}}(X)$ is the Chow group of cycles algebraically equivalent to zero, and Φ_r is the Abel-Jacobi map. Then, Φ_r induces a map $\mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X) \to J^r(X)/J^r_{\mathrm{alg}}(X)$. Using Hilbert schemes arguments, we know that the image of this map is countable.

Suppose that the degree d of X is large enough ($d \geq 5$) so that $H^{3,0}(X) \neq 0$. In this case, Griffith showed, by a Lefschetz pencil argument, that $J^2_{\mathrm{alg}}(X) = 0$ for general X, so that the induced map $\mathrm{CH}^2_{\mathrm{hom}}(X)/\mathrm{CH}^2_{\mathrm{alg}}(X) \to J^2(X)$ cannot be surjective. So in this example, the Abel-Jacobi map Φ_2 is nontrivial and has countable image, thus cannot be surjective.

In general the kernel of Φ_r is far from trivial, as proven by Mumford:

Theorem 2.2.2. *(Mumford) [28] Let X be a smooth complex projective surface such that $H^{2,0}(X) \neq 0$.*
Then $\ker(\Phi_2 : CH^2_{\mathrm{hom}}(X) \longrightarrow J^2(X))$ *is "enormous".*

Outline of proof Note that in this case $\mathrm{CH}^2_{\mathrm{hom}}(X) = \mathrm{CH}_0(X)_{\mathrm{deg}0}$. Looking at the N-th symmetric product $S^{(N)}(X)$, we identify it with the connected component of the Chow variety of effective 0-cycles of degree N on X. It is known to be projective algebraic, with singularities concentrated on $\{p_1 + \cdots + p_N|$ not all of the $\{p_1 + \cdots + p_N\}$ are distinct$\}$. Let $\kappa_N : S^{(N)}(X) \to \mathrm{CH}_0(X)$, $A \mapsto \{A\}$ be the natural map.
It can be proved that the fibers of this map are c-closed, that is, they are countable unions of closed subvarieties. As such, they have a unique decomposition into irreducibles (as $\mathbb{C}$ is uncountable) so we can set dimensions for them. Then we can define $\delta_N := \dim\, \kappa_N(S^{(N)}(X)) = 2N -$ min$\{$ dimensions of fibers of $\kappa_N\}$.

Using $H^{2,0}(X) \neq 0$, one can show that this is an unbounded sequence, leading us to conclude that $\ker(\Phi_2)$ is highly nontrivial.

Looking at the most simple case of the Abel-Jacobi map (Φ_1 which turns out to be an isomorphism as stated above) one would hope for either surjectivity or injectivity (or both) for the map in general.
Griffiths example shows that, for hypersurfaces of large degree, we can't expect surjectivity and Mumford's Theorem provides that we can't expect the kernel to be trivial either. Thus, the Abel-Jacobi map can be very complicated in general.

Chapter 3

Normal functions and the Bloch-Beilinson filtration

3.1 Deligne cohomology and normal functions

3.1.1 Definition of Deligne cohomology

Deligne cohomology is defined via the Deligne complex $\mathbb{Z}_{\mathcal{D}}(r)$ (or more generally, it can be defined for any subring $\mathbb{A} \subset \mathbb{R}$) for a complex manifold, which in our case will be a smooth projective variety X over $\mathbb{C}$:

$$\mathbb{Z}_{\mathcal{D}}(r): \ \mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{r-1}_X$$

Where $\mathcal{O}_X$ denotes the sheaf of germs of holomorphic functions and Ω^l_X denotes the sheaf of germs of holomorphic differential l-forms.

Definition 3.1.2. *Deligne cohomology $H^i_{\mathcal{D}}(X, \mathbb{Z}(r))$ is defined to be the ith hypercohomology of the Deligne complex, that is $H^i_{\mathcal{D}}(X, \mathbb{Z}(r)) := \mathbb{H}^i(\mathbb{Z}_{\mathcal{D}}(r))$.*

Alternatively, we can define the Deligne cohomology using the following concept.

Definition 3.1.3. *Let $h : (A^\bullet, d) \to (B^\bullet, d)$ be a morphism of complexes. We define* $\mathrm{Cone}(A^\bullet \xrightarrow{h} B^\bullet)$ *by the formula*

$$[\mathrm{Cone}(A^\bullet \xrightarrow{h} B^\bullet)]^q := A^{q+1} \oplus B^q, \quad \delta(a, b) = (-da, h(a) + db)$$

Then, $\mathrm{Cone}(\mathbb{Z}(r) \oplus F^r\Omega^\bullet_X \xrightarrow{\varepsilon - l} \Omega^\bullet_X)[-1]$ is given by:

$$\mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-2} \xrightarrow{(0,d)} (\Omega_X^r \oplus \Omega_X^{r-1})$$
$$\xrightarrow{\delta} (\Omega_X^{r+1} \oplus \Omega_X^r) \xrightarrow{\delta} \cdots \xrightarrow{\delta} (\Omega_X^n \oplus \Omega_X^{n-1}) \to \Omega_X^n$$

We observe there is a natural morphism of complexes

$$\mathbb{Z}_{\mathcal{D}}(r) \to \mathrm{Cone}(\mathbb{Z}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\varepsilon - l} \Omega_X^\bullet)[-1]$$

which is a quasi-isomorphism. Indeed, the cohomology $\mathcal{H}_d^i(\mathbb{Z}_{\mathcal{D}}(r))$ is obviously the same as that of $\mathcal{H}^{i-1}(\mathrm{Cone}(\mathbb{Z}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\varepsilon - l} \Omega_X^\bullet))$ are obviously the same for $i < r-1$. Then, any $(a,b) \in \Omega_X^r \oplus \Omega_X^{r-1}$ is mapped to $(-da, db - a) \in \Omega_X^{r+1} \oplus \Omega_X^r$. We observe then that $\delta(a,b) = (0,0)$ if and only if $da = 0$ and $a = db$, but this is the same as saying just $a = db$. Now, $\mathrm{Im}(0,d) = d\Omega_X^{r-2}$ so, by the holomorphic Poincaré lemma, $\ker\delta/\mathrm{Im}(0,d) \simeq \Omega_X^{r-1}/d\Omega_X^{r-2} = \mathcal{H}_d^{r-1}(\mathbb{Z}_{\mathcal{D}})(r)$. Then for $(a,b) \in \Omega_X^{r+j} \oplus \Omega_X^{r+j-1}$ with $j \geq 1$, we observe that the elements of $\ker\delta$ are of the form (db, b), but these can also be writen as $\delta(-b, 0)$. Therefore, the cohomology of the cone is zero in this case and the result follows.

Since the hypercohomology of two quasi-isomorphic complexes are the same, we obtain an alternate definition for the Deligne cohomology:

$$H_{\mathcal{D}}^i(X, \mathbb{Z}(r)) := \mathbb{H}^i(\mathrm{Cone}(\mathbb{Z}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\varepsilon - l} \Omega_X^\bullet)[-1])$$

Let $\mathcal{D}_X^\bullet$ be the sheaf of currents acting on C^∞ compactly supported $(2n - \bullet)$-forms, and let $\mathcal{D}_X^{p,q}$ be the sheaf of currents acting on C^∞ compactly supported $(n-p, n-q)$-forms. Then we have the decomposition $\mathcal{D}_X^\bullet = \bigoplus_{p+q=\bullet} \mathcal{D}_X^{p,q}$. Recall $\mathcal{E}_X^\bullet$, the complex of sheaves of $\mathbb{C}$-valued C^∞ forms. Then we have a morphism of complexes $\mathcal{E}_X^\bullet \hookrightarrow \mathcal{D}_X^\bullet$ induced by $\omega \mapsto (2\pi i)^{-n}\int_X \omega \wedge (-)$ and with $\mathcal{E}_X^{p,q} \hookrightarrow \mathcal{D}_X^{p,q}$ compatible with ∂ and $\overline{\partial}$.
Let us denote by $\mathcal{C}_X^\bullet = \mathcal{C}_{2n-\bullet,X}(\mathbb{Z}(r))$ the sheaf of (Borel-Moore) chains of real codimension $\bullet$. We identify the constant sheaf $\mathbb{Z}(r)$ with the complex

$$\mathbb{Z}(r) \to 0 \to \cdots \to 0$$

and we have the quasi-isomorphisms

$$\mathbb{Z}(r) \xrightarrow{\approx} \mathcal{C}_X^\bullet(\mathbb{Z}(r)), \qquad \Omega_X^\bullet \xrightarrow{\approx} \mathcal{E}_X^\bullet, \qquad \mathcal{E}_X^\bullet \xrightarrow{\approx} \mathcal{D}_X^\bullet$$

where the latter two are (Hodge) filtered. From this, we obtain

$$H_{\mathcal{D}}^i(X, \mathbb{Z}(k)) \simeq \mathbb{H}^i(\mathrm{Cone}(\mathcal{C}_X^\bullet(\mathbb{Z}(r)) \oplus F^r\mathcal{D}_X^\bullet \xrightarrow{\varepsilon - l} \mathcal{D}_X^\bullet)[-1])$$

We also observe that $\mathbb{H}^i(F^p\Omega^\bullet_X) \simeq \mathbb{H}^i(F^p\mathcal{E}^\bullet_X) \simeq F^pH^i_{\mathrm{DR}}(X)$. Thus, we have

$$\mathbb{H}^i(\Omega_X^{\bullet<p}) \simeq \frac{H^i_{\mathrm{DR}}(X)}{F^pH^i_{\mathrm{DR}}(X)}$$

And then, applying hypercohomology to the short exact sequence

$$0 \to \Omega_X^{\bullet<r}[-1] \to \mathbb{Z}_{\mathcal{D}}(r) \to \mathbb{Z}(r) \to 0$$

we obtain the short exact sequence

$$0 \to \frac{H^{i-1}(X,\mathbb{C})}{H^{i-1}(X,\mathbb{Z}(r)) + F^rH^{i-1}(X,\mathbb{C})} \to H^i_{\mathcal{D}}(X,\mathbb{Z}(r))$$

$$\to H^i(X,\mathbb{Z}(r)) \cap F^rH^i(X,\mathbb{C}) \to 0$$

3.1.4 The Milnor cycle class map revisited

Let $V \subset X$ a subvariety of codimension r in X. Let us denote $H^0(X,-)$ by Γ. Then, from the Gersten-Milnor complex, one has the following commutative diagram due to Lewis [21] where the $2\pi i$ factors enter in, due to Poincaré duality:

$$\begin{array}{ccccccc}
\Gamma K_r^M(\mathbb{C}(X)) & \to & \Gamma\bigoplus_{\mathrm{cod}_X Z=1} K^M_{r-1}(\mathbb{C}(Z)) & \to & \cdots & \to & \Gamma\bigoplus_{\mathrm{cod}_X V=r} K^M_0(\mathbb{C}(X)) \\
\int_X \frac{d\log_r}{(2\pi i)^n}\Big\downarrow & & \int_Z \frac{d\log_{r-1}}{(2\pi i)^{n-1}}\Big\downarrow & & \cdots & & \int_V \frac{d\log_0}{(2\pi i)^{n-r}}\Big\downarrow \\
\Gamma F^r\mathcal{D}^r_X & \xrightarrow{d} & \Gamma F^r\mathcal{D}^{r+1}_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Gamma F^r\mathcal{D}^{2r}_X
\end{array}$$

where $d\ \log_r(\{f_1,\dots f_r\}) = \bigwedge_{j=1}^r d\ \log f_j$, and for $\{w\} \in H^{2n-2r}_{DR}(X,\mathbb{C})$ the vertical map in the right hand is

$$\int_V \frac{d\log_0 w}{(2\pi i)^{n-r}} = \frac{1}{(2\pi i)^{n-r}}\delta_V(w)$$

where we write $\delta_V(w) := \int_{V^*} w$. Note that $\frac{1}{(2\pi i)^{n-r}}\delta_V$ is the same topologically as the corresponding homology class

$$(2\pi i)^{r-n}\{V\} \in H_{2n-2r}(X,\mathbb{Z}(n-r)) \simeq H^{2r}(X,,\mathbb{Z}(r))$$

where the latter isomorphism is Poincaré duality. This recovers the (twisted) description for the cycle class map for Chow groups through Milnor K-theory.

3.1.5 Introducing normal functions

Looking at the first and second cycle class maps, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}^k_{\mathrm{hom}}(X) & \longrightarrow & \mathrm{CH}^k(X) & \longrightarrow & \frac{\mathrm{CH}^k(X)}{\mathrm{CH}^k_{\mathrm{hom}}(X)} & \longrightarrow & 0 \\
 & & \downarrow \Phi_r & & \downarrow \psi_r & & \downarrow \mathrm{cl}_r & & \\
0 & \longrightarrow & J^r(X) & \longrightarrow & H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) & \longrightarrow & H^{r,r}(X,\mathbb{Z}(r)) & \longrightarrow & 0
\end{array}
\tag{3.1}
$$

where we take $H^{r,r}(X,\mathbb{Z}(r))$ as the inverse image of $H^{r,r}(X)$ in $H^{2r}(X,\mathbb{Z}(r))$ under the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$. Here ψ_r can be explicitly stated using the mapping cone cohomology interpretation of the Deligne cohomology ([7]):

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(r)) := \mathbb{H}^i(\mathrm{Cone}(\mathbb{Z}(r) \oplus F^r\Omega^\bullet_X \overset{\varepsilon-l}{\to} \Omega^\bullet_X)[-1])$$

through which we get the long exact sequence

$$\cdots \to H^{i-1}(X,\mathbb{Z}(r)) \oplus F^r H^{i-1}(X,\mathbb{C}) \to H^{i-1}(X,\mathbb{C})$$

$$\to H^i_{\mathcal{D}}(X,\mathbb{Z}(r)) \to H^i(X,\mathbb{Z}(r)) \oplus F^r H^i(X,\mathbb{C}) \to \cdots$$

Let $\xi \in \mathrm{CH}^r(X)$ with support $|\xi|$. Then, from the previous long exact sequence we get the long exact sequence

$$\cdots \to H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^r H^{2r-1}_{|\xi|}(X,\mathbb{C}) \to H^{2r-1}_{|\xi|}(X,\mathbb{C})$$

$$\to H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X,\mathbb{C}) \overset{\varepsilon-l}{\to} H^{2r}_{|\xi|}(X,\mathbb{C}) \to \cdots$$

We can map $\xi \mapsto \{\xi\} \in H_{2n-2r}(|\xi|,\mathbb{Z}(n-r)) \simeq H^{2r}_{|\xi|}(X,\mathbb{Z}(r))$ by Poincaré duality. Here $\{\xi\} - [\xi]$ corresponds to zero in $H^{2r}_{|\xi|}(X,\mathbb{C})$ and thus, we have the map

$$\xi \mapsto [(2\pi i)^{r-n}(\{\xi\},\delta_\xi)] \in \ker(F^r H^{2r}_{|\xi|}(X,\mathbb{C}) \to H^{2r}_{|\xi|}(X,\mathbb{C}))$$

But because the real dimension of $|\xi|$ is $2n-2r$, we have

$$H^{2r-1}_{|\xi|}(X,\mathbb{C}) \simeq H_{2n-2r+1}(|\xi|,\mathbb{C}) = 0$$

Then we can define $\mathrm{cl}_r(\xi) \in H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r))$ from the injection

$$H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \hookrightarrow H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X,\mathbb{C})$$

through the use of the forgetful map

$$H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r))$$

In terms of the cone complex, $\mathrm{cl}_r(\xi)$ can be written as $((2\pi i)^{r-n}\{\xi\},(2\pi i)^{r-n}\delta_\xi,0)$. Assume $\xi \sim_{\text{hom}} 0$, that is, $\xi = \partial\zeta$ for some ζ and $(2\pi i)^{r-n}\delta_\xi = dS$ for some $S \in F^r\mathcal{D}^{2r-1}(X)$, then we have

$$D((2\pi i)^{r-n}\zeta, S, 0) = (-(2\pi i)^{r-n}\partial\zeta, -dS, (2\pi i)^{r-n}\int_\zeta(-) - S)$$

and therefore

$$D((2\pi i)^{r-n}\zeta, S, 0) + ((2\pi i)^{r-n}\{\xi\},(2\pi i)^{r-n}\delta_\xi,0) = (0,0,(2\pi i)^{r-n}\int_\zeta(-) - S)$$

which is an element not in the image of the cycle class map as previously defined. Moreover, given $\omega \in F^{n-r+1}H^{2n-2r+1}(X,\mathbb{C})$, we see that $S(\omega)$ must be zero (since S doesn't act on this Hodge type) and thus, the third element in the previous expression is

$$\frac{1}{(2\pi i)^{n-r}}\int_\zeta \omega$$

which is the Abel-Jacobi map (twisted version) .

Combining these results, we obtain the map ψ_r depicted in the diagram 3.1 rather explicitly.

In the second row of the diagram 3.1, by Carlson's isomorphism, we can identify $J^r(X)$ with $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r))) = J(H^{2r-1}(X,\mathbb{Z}(r)))$, and $H^{r,r}(X,\mathbb{Z}(r))$ with $\hom_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2r}(X,\mathbb{Z}(r))) = \Gamma(H^{2r}(X,\mathbb{Z}(r)))$.

Now, given a smooth and proper morphism $\rho : X \to S$, with S a smooth quasiprojective variety, we set $X_t = \rho^{-1}(t)$ for $t \in S$. Using the restriction of this morphism to each fiber and the previous observations, from the previous diagram we obtain:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{CH}^r(X) & \longleftarrow & \mathrm{CH}^r_{\mathrm{relhom}}(X) & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & J(H^{2r-1}(X,\mathbb{Z}(r))) & \longrightarrow & H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) & \longrightarrow & \Gamma(H^{2r}(X,\mathbb{Z}(r))) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J(H^{2r-1}(X_t,\mathbb{Z}(r))) & \longrightarrow & H^{2r}_{\mathcal{D}}(X_t,\mathbb{Z}(r)) & \longrightarrow & \Gamma(H^{2r}(X_t,\mathbb{Z}(r))) & \longrightarrow & 0
\end{array}$$

Where $\mathrm{CH}^r_{\mathrm{relhom}}(X) := \{\xi \in \mathrm{CH}^r(X) \mid \xi \cap X_t \sim_{\mathrm{hom}} 0 \text{ in } X_t\}$.
For any $\xi \in CH^r_{\mathrm{relhom}}(X)$, using this diagram, it maps to zero in $\Gamma(H^{2r}(X_t, \mathbb{Z}(r)))$.
Thus, we can assign it a function $\nu_\xi : S(\mathbb{C}) \to \coprod_{t\in S(\mathbb{C})} J(H^{2r-1}(X_t, \mathbb{Q}(r)))$.

Definition 3.1.6. *ν_ξ is called a (cycle-induced) normal function.*

3.2 Bloch-Beilinson filtration

3.2.1 $\overline{\mathbb{Q}}$-spreads

Given a smooth projective variety X over $\mathbb{C}$, we can always write $X/\mathbb{C} = X_K \times_K \mathbb{C}$ for some finitely generated K over $\overline{\mathbb{Q}}$, where X_K denotes the underlying smooth projective variety over K. Now we can write $K = \overline{\mathbb{Q}}(\mathcal{S})$ for some smooth quasiprojective variety $\mathcal{S}$. Then, a $\overline{\mathbb{Q}}$-spread is a smooth and proper morphism ρ from quasiprojective variety $\mathscr{X}$ (over $\overline{\mathbb{Q}}$) to $\mathcal{S}$ (over $\overline{\mathbb{Q}}$), $\rho : \mathscr{X} \to \mathcal{S}$.
The generic point η of the scheme $\mathcal{S}/\overline{\mathbb{Q}}$ is the "point" of $\mathcal{S}$ such that $\overline{\mathbb{Q}}(\mathcal{S}) = \overline{\mathbb{Q}}(\eta)$ is the residue field at η, thus $K = \overline{\mathbb{Q}}(\eta)$, so the fiber $\mathscr{X}_\eta$ of the morphism ρ can be identified with X_K, via the embedding $\overline{\mathbb{Q}}(\eta) \xrightarrow{\simeq} K \subset \mathbb{C}$, and satisfies $\mathscr{X}_\eta \times_K \mathbb{C} = X/\mathbb{C}$, again with respect to the embedding $\overline{\mathbb{Q}}(\eta) \xrightarrow{\simeq} K \subset \mathbb{C}$ defining K as a subfield (hence $X/\mathbb{C}$).

Example 3.2.2. *: Let*

$$X/\mathbb{C} = \mathrm{Spec}\left(\frac{\mathbb{C}[x,y,z]}{(\pi x^3 y + \sqrt{\pi} y^2 + ex + \sqrt{2})}\right)$$

where, $\mathrm{Spec}(\mathbb{A}) := \{$ *prime ideals in* $\mathbb{A}\}$ *for a ring* $\mathbb{A}$*, and put* $K = \mathbb{Q}(\pi, \sqrt{\pi}, e, \sqrt{2})$.
Observe that $X/\mathbb{C} = X_K \times_K \mathbb{C}$.
Let

$$\mathcal{R} = \frac{\mathbb{Q}[t,s,u,v]}{(t - s^2, v^2 - 2)}$$

and put $\mathcal{S} = \mathrm{Spec}(\mathcal{R})$*. Taking* $\tilde{K} = \mathrm{Quot}(\mathcal{R}) = \mathbb{Q}(\mathcal{S})$*, we can inject it in* $K \subset \mathbb{C}$ *via the "evaluation" map:* $(t,s,u,v) \mapsto (\pi, \sqrt{\pi}, e, \sqrt{2})$.
Now take

$$\mathscr{X} = \mathrm{Spec}\left(\frac{\mathbb{Q}[x,y,z,t,s,u,v]}{tx^3y + sy^2 + ux + v, t - s^2, v^2 - 2}\right)$$

so we get the map $\rho : \mathscr{X} \to \mathcal{S}$ (induced by the inclusion). Then, given a generic point $\eta \in \mathcal{S}$, the fiber $\mathscr{X}_\eta$ satisfies $\mathscr{X}_\eta \times_K \mathbb{C} = X/\mathbb{C}$, and $\tilde{K} = \mathbb{Q}(\eta)$ which is injected onto $K \subset \mathbb{C}$.

In general, we can extend this definition to k-spreads using the same arguments, where k is any algebraically closed subfield of $\mathbb{C}$.

The following result is useful:

Proposition 3.2.3. *Let X/k be a smooth projective variety defined over an algebraically closed subfield $k \subseteq \mathbb{C}$. Assume the Hodge conjecture, specifically*

$$CH^r(X/\mathbb{C};\mathbb{Q}) \twoheadrightarrow H^{r,r}(X(\mathbb{C}),\mathbb{Q}(r)).$$

Then $CH^r(X/k;\mathbb{Q}) \twoheadrightarrow H^{r,r}(X(\mathbb{C}),\mathbb{Q}(r))$ is likewise surjective.

Proof. We have $X/\mathbb{C} = X_k \times_k \mathbb{C}$. Let $\xi \in \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$. Then from the coefficients of ξ we have $\xi = \xi_K \in \mathrm{CH}^r(X_K;\mathbb{Q})$ for some finitely generated field K/k. Note that $K = k(\mathcal{S})$ for some smooth k-variety $\mathcal{S}$. Since X is already defined over k, we can define a k spread $\mathscr{X} = \mathcal{S} \times_k X$, with spread cycle $\tilde{\xi} \in \mathrm{CH}^r(\mathcal{S} \times_k X;\mathbb{Q})$. Choose a k point $t_0 \in \mathcal{S}(k)$, which is possible since k is algebraically closed. Then, applying the Künneth formula to $H^{2r}(\{\mathcal{S} \times X\}(\mathbb{C}),\mathbb{Q})$, ξ_{t_0} and ξ_K take the same image in $H^{r,r}(X(\mathbb{C}),\mathbb{Q}(r))$. By the Hodge conjecture for $X/\mathbb{C}$, we are done.

□

3.2.4 Bloch-Beilinson filtration and arithmetic normal functions

Let us return to the cycle class and Abel Jacobi maps

$$\mathrm{cl}_r : \mathrm{CH}^r(X;\mathbb{Q}) \to \Gamma H^{2r}(X,\mathbb{Q}(r)) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r}(X,\mathbb{Q}(r))$$

$$:= \mathrm{Ext}^0_{\mathrm{MHS}}(\mathbb{Q}(r), H^{2r}(X,\mathbb{Q}(r))),$$

$$AJ : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J^r(X) \simeq \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r))),$$

If we write $\mathrm{CH}^r_{AJ}(X;\mathbb{Q})$ for the kernel of AJ, we might (naively) think we can get a filtration on Chow groups following this pattern with a new element coming from the kernel of some map of the form $\mathrm{CH}^r_{AJ}(X;\mathbb{Q}) \to \mathrm{Ext}^2_{\mathrm{MHS}}(\mathbb{Q}(r), H^{2r-2}(X,\mathbb{Q}(r)))$. However this can't be the case as for any $\mathbb{Q}$-MHS H_1, H_2, $\mathrm{Ext}^i_{\mathrm{MHS}}(H_1,H_2) = 0$ for $i > 1$. This fact was proved by Beilinson [3]. For the benefit of the reader, the following argument should suffice. Carlson's formula for $\mathrm{Ext}^1_{\mathrm{MHS}}(H_1,H_2)$ show that $\mathrm{Ext}^1_{\mathrm{MHS}}(H_1,-)$ is right exact. If we assume for the moment that the category of $\mathbb{Q}$-MHS has enough injectives, then the vanishing of $\mathrm{Ext}^i_{\mathrm{MHS}}(H_1,H_2)$ for $i > 1$

is clear. In general, one still gets the vanishing via a Yoneda-Ext argument. To define a candidate BB filtration, we need the Bloch-Beilinson conjecture:

Conjecture 3.2.5. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Then the Abel-Jacobi map*

$$AJ : CH^r_{\text{hom}}(X/\overline{\mathbb{Q}}; \mathbb{Q}) \hookrightarrow J(H^{2r-1}(X(\mathbb{C}), \mathbb{Q}(r)))$$

is injective.

We need a slight variant of this conjecture. Namely,

Proposition 3.2.6. *Let us assume the Hodge conjecture. Then X in the above conjecture can be replaced by a smooth quasi-projective variety over $\overline{\mathbb{Q}}$.*

Proof. Let $X/\overline{\mathbb{Q}}$ be smooth projective, and $U/\overline{\mathbb{Q}} \subset X/\overline{\mathbb{Q}}$ an open subset, we want to show that the Abel-Jacobi map $\mathrm{CH}^r_{\text{hom}}(U; \mathbb{Q}) \to J\big(H^{2r-1}(U, \mathbb{Q}(r))\big)$ is injective. Put $Y = X \backslash U$. By a proper modification of X along Y (using blow-ups if necessary), we can assume that Y is of pure codimension one in X (which simplifies the notation). From the localization sequence on Chow groups, and after applying $\mathbb{Q}$-coefficients, one has an exact sequence

$$\mathrm{CH}^{r-1}(\tilde{Y}; \mathbb{Q}) \to \mathrm{CH}^r(X; \mathbb{Q}) \to \mathrm{CH}^r(U; \mathbb{Q}) \to 0, \tag{3.2}$$

where $\tilde{Y} \xrightarrow{\approx} Y$ is a desingularization. Correspondingly from Deligne's mixed Hodge theory, the sequence with twists

$$H^{2r-2}(\tilde{Y}, \mathbb{Q}(r-1)) \to H^{2r}(X, \mathbb{Q}(r)) \to H^{2r}(U, \mathbb{Q}(r)) \tag{3.3}$$

is exact. From the Hodge conjecture, one shows that

$$\mathrm{CH}^r_{\text{hom}}(X; \mathbb{Q}) \to \mathrm{CH}^r_{\text{hom}}(U; \mathbb{Q}),$$

is surjective. Here is how it works. We combine (3.2) and (3.3) in a commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{CH}^{r-1}(\tilde{Y}; \mathbb{Q}) & \to & \mathrm{CH}^r(X; \mathbb{Q}) & \to & \mathrm{CH}^r(U; \mathbb{Q}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{2r-2}(\tilde{Y}, \mathbb{Q}(r-1)) & \to & H^{2r}(X, \mathbb{Q}(r)) & \to & H^{2r}(U, \mathbb{Q}(r)) & &
\end{array}$$

So if $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(U;\mathbb{Q})$, then we know that $\xi' \mapsto \xi$ for some $\xi' \in \mathrm{CH}^r(X;\mathbb{Q})$. Now $[\xi'] \in H^{2r}(X,\mathbb{Q}(r)) \mapsto 0 \in H^{2r}(U,\mathbb{Q}(r))$, hence we can find an element of $H^{2r-2}(\tilde{Y},\mathbb{Q}(r-1))$ mapping to $[\xi']$ and by semi-simplicity issues, $[\xi''] \mapsto [\xi']$ for some $[\xi''] \in H^{r-1,r-1}(\tilde{Y},\mathbb{Q}(r))$. By the Hodge conjecture, we may assume that $\xi'' \in \mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})$. Thus $\xi' - \xi'' \mapsto \xi$, and $\xi' - \xi'' \in \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$. Now, one defines $\mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})^\circ \subset \mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})$ such that

$$\mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})^\circ \to \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to \mathrm{CH}^r_{\mathrm{hom}}(U;\mathbb{Q}) \to 0, \tag{3.4}$$

is exact. (Explicit:

$$\mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})^\circ = \{w \in \mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q}) \mid \sigma_*(w) \in \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})\},$$

where $\sigma : \tilde{Y} \to X$ is the natural morphism.) Next, consider the short exact sequence:

$$0 \to \frac{H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}{\ker\sigma_*} \xrightarrow{\sigma_*} H^{2r-1}(X,\mathbb{Q}(r)) \to \frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_* H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))} \to 0, \tag{3.5}$$

from which, through the long exact sequence of extension classes of MHS (and recalling that $\mathrm{Ext}^2_{\mathrm{MHS}}$ is zero) we obtain the diagram

$$\begin{array}{ccccccc}
\frac{\mathrm{CH}^{r-1}_{\mathrm{hom}}(\tilde{Y};\mathbb{Q})}{\ker\sigma_*} & \hookrightarrow & \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) & \to & \frac{\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})}{\sigma_*\mathrm{CH}^{r-1}_{\mathrm{hom}}(\tilde{Y};\mathbb{Q})} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
J\left(\frac{H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}{\ker\sigma_*}\right) & \hookrightarrow & J(H^{2r-1}(X,\mathbb{Q}(r))) & \to & J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_*H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}\right) & \to & 0
\end{array} \tag{3.6}$$

where the vertical arrows are given by the Abel Jacobi map, and by the Bloch-Beilinson conjecture for smooth projective $X/\overline{\mathbb{Q}}$, the middle arrow is injective. Next we observe that the short exact sequence 3.5 is split exact; in particular σ_* has a left inverse (call it λ_*), which by the Hodge conjecture, is cycle induced. We now argue as follows: Suppose that

$$\xi \in \ker\left[\frac{\mathrm{CH}^r_{\mathrm{hom}}(X,\mathbb{Q})}{\sigma_*\mathrm{CH}^{r-1}_{\mathrm{hom}}(\tilde{Y},\mathbb{Q})} \to J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_*H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}\right)\right].$$

We know that $\xi' \mapsto \xi$ for some $\xi' \in \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$, and so

$$AJ(\xi') \mapsto 0 \in J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_*H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}\right),$$

hence

$$AJ(\xi') \in J\left(\frac{H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}{\ker\sigma_*}\right),$$

and so

$$\lambda_* \circ \sigma_*(AJ(\xi')) = AJ(\xi').$$

By functoriality of the AJ map,

$$AJ(\xi') = AJ(\lambda_* \circ \sigma_*(\xi')).$$

Interpreted appropriately, this means that if we set $\xi'' = \lambda_* \circ \sigma_*(\xi')$, then $\xi' - \xi'' \mapsto \xi$ and $\xi' - \xi'' \in$ lies in the kernel of the middle AJ map of the diagram 3.6. Thus $\xi = 0$, and the map in the AJ map on the right hand side of the diagram 3.6 is injective. Finally, there is a short exact sequence:

$$0 \to \frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_* H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))} \to H^{2r-1}(X,\mathbb{Q}(r)) \to H^{2r}_Y(X,\mathbb{Q}(r))^\circ \to 0.$$

where $H^{2r}_Y(X,\mathbb{Q}(r))^\circ$ denotes the appropriate kernel. We observe that this is a pure Hodge structure of weight 0 and we recall that for a MHS V, $J(V)$ can be identified with (using the extension class interpretation by J.Carlson, [5])

$$\frac{W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V}$$

so the AJ maps $H^{2r}_Y(X,\mathbb{Q}(r))^\circ$ to zero. This gives us the commutative diagram

$$\begin{array}{ccccccc}
\frac{\mathrm{CH}^{r-1}(\tilde{Y};\mathbb{Q})^\circ}{\mathrm{CH}^{r-1}_{\mathrm{hom}}(\tilde{Y};\mathbb{Q})} & \to & \frac{\mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})}{\sigma_*\mathrm{CH}^{r-1}_{\mathrm{hom}}(\tilde{Y};\mathbb{Q})} & \to & \mathrm{CH}^r_{\mathrm{hom}}(U;\mathbb{Q}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma H^{2r-2}(\tilde{Y};\mathbb{Q}(r-1))^\circ & \to & J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{\sigma_* H^{2r-3}(\tilde{Y},\mathbb{Q}(r-1))}\right) & \to & J(H^{2r-1}(U,\mathbb{Q}(r))) & \to & 0
\end{array}$$

where we identify the images of the maps. By the Hodge conjecture, the left vertical map (cycle class map) is surjective and the middle arrow is injective as shown before. The proposition follows. $\square$

The concept of $\overline{\mathbb{Q}}$-spreads is used by Lewis ([24], [25]) to construct a candidate for a Bloch-Beilinson filtration for Chow groups $\mathrm{CH}^r(X;\mathbb{Q})$ (for all r).

Theorem 3.2.1. [*Lewis*] *[25] Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Then for all r, there is a filtration*

$$CH^r(X;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots$$

which satisfies the following
(i) $F^1 = CH^r_{\text{hom}}(X;\mathbb{Q})$
(ii) $F^2 \subset CH^r_{AJ}(X;\mathbb{Q})$
(iii) $F^{\nu_1}CH^{r_1}(X;\mathbb{Q}) \bullet F^{\nu_2}CH^{r_2}(X;\mathbb{Q}) \subset F^{\nu_1+\nu_2}CH^{r_1+r_2}(X;\mathbb{Q})$, where $\bullet$ is the intersection product.
(iv) F^ν is preserved under the actions of correspondences between smooth projective varieties over $\mathbb{C}$.
(v) Let $Gr^\nu_F CH^r(X;\mathbb{Q}) := F^\nu CH^r(X;\mathbb{Q})/F^{\nu+1}CH^r(X;\mathbb{Q})$ and assume that the Künneth components of the diagonal class

$$[\Delta] = \bigoplus_{p+q=2d} [\Delta_X(p,q)] \in H^{2d}(X \times X, \mathbb{Q}(d)))$$

are algebraic. Then

$$\Delta_X(2d-2r+l, 2r-l)_* \mid_{Gr^\nu_F CH^r(X;\mathbb{Q})} = \delta_{l,\nu} \ \cdot \ Identity$$

(vi) Let $D^r(X) := \cap_\nu F^\nu$, and $k = \overline{\mathbb{Q}}$. If the Bloch-Beilinson Conjecture together with the Hodge conjecture holds, then $D^r(X) = 0$.

The idea of the construction is as follows: given a spread $\rho : \mathscr{X} \to \mathcal{S}$, analogous to the situation of Deligne cohomology, there is a short exact sequence:

$$0 \to J\big(H^{2r-1}(\mathscr{X}, \mathbb{Q}(r))\big) \to H^{2r}_{\mathcal{H}}(\mathscr{X}, \mathbb{Q}(r)) \to \Gamma\big(H^{2r}(\mathscr{X}, \mathbb{Q}(r))\big) \to 0,$$

and morphism $\mathrm{CH}^r(\mathscr{X};\mathbb{Q}) \to H^{2r}_{\mathcal{H}}(\mathscr{X}, \mathbb{Q}(r))$ which is injective, under the assumption of the Bloch-Beilinson and Hodge conjectures. We use the Leray spectral sequence associated to ρ on cohomology to induce a decreasing filtration $\mathcal{F}^\nu \mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$. Here, $H^{2r}_{\mathcal{H}}(\mathscr{X}, \mathbb{Q}(r))$ denotes the Beilinson's absolute Hodge cohomology, which is an analogous to the Deligne cohomology for quasiprojective varieties, which involves "weights". The graded pieces of this filtration, denoted by $Gr^\nu_{\mathcal{F}} \mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$, map injectively to $E^{\nu,2r-\nu}_\infty(\rho)$ ([2], [17]), the ν-th graded piece of a Leray filtration associated to ρ. This term fits in a short exact sequence:

$$0 \ \to \ \underline{E}^{\nu,2r-\nu}_\infty(\rho) \ \to \ E^{\nu,2r-\nu}_\infty(\rho) \ \to \ \underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) \ \to \ 0$$

where

$$\underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) = \Gamma(H^\nu(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)))$$

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\rho) = \frac{J(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)))} \subset J(H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)))$$

Recalling here that W stands for the increasing weight filtration with corresponding $Gr_W^l := W_l/W_{l+1}$. The sheaf $R^i\rho_*\mathbb{Q}(r)$ is the direct image Leray sheaf associated to the presheaf that associates $U \subset \mathcal{S}$ open to $H^i(\rho^{-1}(U), \mathbb{Q}(r))$.

The latter inclusion comes from the short exact sequence:

$$W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r))$$

$$\twoheadrightarrow Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu}\rho_*\mathbb{Q}(r)).$$

Now, taking the direct limit over open subsets of $\mathcal{S}/\overline{\mathbb{Q}}$ we define

$$F^\nu \mathrm{CH}^r(X_K; \mathbb{Q}) = \varinjlim_{U \subset \mathcal{S}/\overline{\mathbb{Q}}} \mathcal{F}^\nu \mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}}; \mathbb{Q})$$

where $\mathscr{X}_U := \rho^{-1}(U)$.

We also define

$$E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \varinjlim_{U \subset \mathcal{S}/\overline{\mathbb{Q}}} E_{\infty}^{\nu,2r-\nu}(\rho)$$

and in the same way we define $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}})$ and $\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}})$, explicitly

$$\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \Gamma(H^\nu(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))$$

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \frac{J(W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr_W^0 H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$

giving us the short exact sequence

$$0 \to \underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to 0$$

and the injection

$$Gr_F^\nu \mathrm{CH}^r(X_K; \mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}})$$

Then, taking the direct limit over all finitely generated subfields $K \subset \mathbb{C}$ over $\overline{\mathbb{Q}}$ we arrive at

$$F^\nu \mathrm{CH}^r(X/\mathbb{C}; \mathbb{Q}) = \varinjlim_{K \subset \mathbb{C}} F^\nu \mathrm{CH}^r(X_K; \mathbb{Q})$$

which is a candidate Bloch-Beilinson filtration on $\mathrm{CH}^r(X;\mathbb{Q})$.

We will check that this filtration truncates. Let $m := \dim X_\eta$ be the dimension of the generic fiber. Then, by the hard Lefschetz theorem we have, for all i, the isomorphisms

$$L_X^{m-i} : R^i\rho_*\mathbb{Q}(r) \xrightarrow{\sim} R^{2m-i}\rho_*\mathbb{Q}(m-i+r)$$

Which induces isomorphisms

$$L_X^{m-2r+\nu} : \underline{E}_\infty^{\nu,2r-\nu} \xrightarrow{\sim} \underline{E}_\infty^{\nu,2(m-2r+\nu)-\nu}$$

$$L_X^{m-2r+\nu} : \underline{\underline{E}}_\infty^{\nu,2r-\nu} \xrightarrow{\sim} \underline{\underline{E}}_\infty^{\nu,2(m-2r+\nu)-\nu}$$

and thus inducing the isomorphism

$$L_X^{m-2r+\nu} : E_\infty^{\nu,2r-\nu} \xrightarrow{\sim} E_\infty^{\nu,2(m-2r+\nu)-\nu}$$

through the diagram

$$\begin{array}{ccccccccc} 0 & \to & \underline{E}_\infty^{\nu,2r-\nu}(\rho) & \to & E_\infty^{\nu,2r-\nu}(\rho) & \to & \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) & \to & 0 \\ & & L_X^{m-2r+\nu}\downarrow\wr & & L_X^{m-2r+\nu}\downarrow\wr & & L_X^{m-2r+\nu}\downarrow\wr & & \\ 0 & \to & \underline{E}_\infty^{\nu,2m-2r+\nu}(\rho) & \to & E_\infty^{\nu,2m-2r+\nu}(\rho) & \to & \underline{\underline{E}}_\infty^{\nu,2m-2r+\nu}(\rho) & \to & 0 \end{array}$$

Now, with the cycle class map, we define

$$\psi_0 : \mathcal{F}^0\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) := \mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) \to E_\infty^{0,2r}(\rho) = \underline{\underline{E}}_\infty^{0,2r}(\rho)$$

where the last equality comes from the fact that $\underline{E}_\infty^{0,2r}(\rho) = 0$. Then set

$$\mathcal{F}^1\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) = \ker\psi_0$$

through Lewis's construction [23] we can define an induced map

$$\psi_1 : \mathcal{F}^1\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) \to E_\infty^{1,2r-1}(\rho)$$

and set again

$$\mathcal{F}^2\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) = \ker\psi_1$$

Proceeding recurrently, we obtain maps

$$\psi_i : \mathcal{F}^i\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) \to E_\infty^{i,2r-i}(\rho)$$

where

$$\mathcal{F}^{i+1}\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) = \ker\psi_i$$

Now, with this setting, for any $j \geq 1$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{F}^{r+j}\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) & \stackrel{\psi_{r+j}}{\to} & E_\infty^{r+j,r-j}(\rho) \\ L_X^{m-2r+\nu}\downarrow & & L_X^{m-2r+\nu}\downarrow\wr \\ \mathcal{F}^{r+j}\mathrm{CH}^{m+j}(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) & \to & E_\infty^{r+j,2m-k+j}(\rho) \end{array}$$

where we observe that

$$\varinjlim_{U\subset\mathcal{S}} \mathrm{CH}^{m+r}(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) = \mathrm{CH}^{m+j}(X_\eta,\mathbb{Q}) = 0$$

And since $\mathcal{F}^{r+j+1}\mathrm{CH}^r(\mathscr{X}_U/\overline{\mathbb{Q}};\mathbb{Q}) = \ker\psi_{r+j}$, we have

$$F^{r+j}\mathrm{CH}^r(X_K;\mathbb{Q}) = F^{r+j+1}\mathrm{CH}^r(X_K;\mathbb{Q})\ \forall\ j \geq 1$$

Now, assume $\mathcal{S}$ is affine and let $V \subset \mathcal{S}(\mathbb{C})$ be a smooth, irreducible, closed subvariety of dimension $\nu - 1$. Let $\rho_V : \mathscr{X}_V \to V$ be the restriction of ρ. We have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{X}_V & \hookrightarrow & \mathscr{X}(\mathbb{C}) \\ \rho_V\downarrow & & \downarrow\rho \\ V & \hookrightarrow & \mathcal{S}(\mathbb{C}) \end{array}$$

From which we construct the commutative diagram:

$$\begin{array}{ccccccccc} & & & & Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^r(\mathscr{X};\mathbb{Q}) & \mapsto & Gr_F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q}) & & \\ & & & & \downarrow & & & & \\ 0 & \to & \underline{E}_\infty^{\nu,2r-\nu}(\rho) & \to & E_\infty^{\nu,2r-\nu}(\rho) & \to & \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \underline{E}_\infty^{\nu,2r-\nu}(\rho_V) & \to & E_\infty^{\nu,2r-\nu}(\rho_V) & \to & \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_V) & \to & 0 \end{array}$$

We observe that the sheaf $R^{2r-\nu}\rho_{V,*}\mathbb{Q}(r)$ is locally constant, and the weak Lefschetz theorem for locally constant systems over affine varieties tells us that

$H^{\nu}(V, R^{2r-\nu}\rho_{V,*}\mathbb{Q}(r)) = 0$ (since the dimension of V is below the cohomology degree). Thus $\underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V) \simeq E_{\infty}^{\nu,2r-\nu}(\rho_V)$, so for any $\xi \in Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$ we can define a "normal function" ν_ξ that, to any smooth irreducible closed $V \subset \mathcal{S}(\mathbb{C})$ of dimension $\nu - 1$, assigns an element $\nu_\xi(V) \in \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V)$. That is, ν_ξ treats the smooth irreducible closed subvarieties of $\mathcal{S}(\mathbb{C})$ as points of an open subset of a paramterizing space (open smooth subvariety of a Chow variety) that form a domain for ν_ξ.

Definition 3.2.7. *ν_ξ, as defined above, is called an arithmetic normal function.*

Remark 3.2.8. *We note that, when $\nu = 1$ (and $V = \{t\}$ of dimension 0) , ν_ξ defines a traditional normal function:*

$$\underline{E}_{\infty}^{1,2r-1}(\rho_t) \subset J(H^0(\{t\}, R^{2r-1}\rho_*\mathbb{Q}(r))) = J(H^{2r-1}(X_t, \mathbb{Q}(r)))$$

This next result due to Lewis [24], points toward the main line of enquiry of my thesis:

Proposition 3.2.2. *The following statements are equivalent:*
(i) $F^2 CH^r(X;\mathbb{Q}) = CH^r_{AJ}(X;\mathbb{Q})$ for all smooth projective varieties X over $\mathbb{C}$, where $CH^r_{AJ}(X;\mathbb{Q})$ denotes the kernel of the Abel-Jacobi map.
(ii) For any smooth and proper morphism $\rho : \mathscr{X} \to S$ of smooth quasiprojective varieties over $\overline{\mathbb{Q}}$, and a normal function

$$\nu_\xi : \mathcal{S}(\mathbb{C}) \to \coprod_{t \in \mathcal{S}(\mathbb{C})} J(H^{2r-1}(X_t, \mathbb{Q}(r)))$$

defined by a cycle $\xi \in \mathcal{F}^1 CH^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$, the zero locus

$$\mathcal{Z}(\nu_\xi) = \{t \in \mathcal{S}(\mathbb{C}) \mid \nu_\xi(t) = 0\}$$

is a countable union of algebraic subvarieties over $\overline{\mathbb{Q}}$.
(iii) For any smooth and proper morphism $\rho_V : \mathscr{X}_V \to V$ of smooth quasiprojective varieties over a subfield $L \subset \mathbb{C}$ finitely generated over $\overline{\mathbb{Q}}$ and cycle induced normal function

$$\nu_\xi : V(\mathbb{C}) \to \coprod_{t \in V(\mathbb{C})} J(H^{2r-1}(X_t, \mathbb{Q}(r)))$$

defined by a cycle $\xi \in \mathcal{F}^1 CH^r(\mathscr{X}_V/L;\mathbb{Q})$, the zero locus $\mathcal{Z}(\nu_\xi)$ of ν_ξ is a countable union of algebraic subvarieties over $\overline{L}$.

Given the techniques used in this thesis, it is instructive to provide a proof.

Proof. To prove (i) $\Rightarrow$ (ii), we know that $\mathcal{Z}(\nu_\xi)$ is a countable union of analytic subvarieties. For $p \in \mathcal{Z}(\nu_\xi)$, we can use its $\overline{\mathbb{Q}}$ closure $\overline{\{p\}}$ to define $\mathscr{X}_{\overline{\{p\}}} \to \overline{\{p\}}$, which, using (i), satisfies $F^2\mathrm{CH}^r(\mathscr{X}_{\overline{\{p\}},\eta};\mathbb{Q}) = \mathrm{CH}^r_{AJ}(\mathscr{X}_{\overline{\{p\}},\eta};\mathbb{Q})$. Thus, ν_ξ is zero over all points of $\overline{\{p\}}$, so $\overline{\{p\}} \subset \mathcal{Z}(\nu_\xi)$. Since the set of all $\overline{\mathbb{Q}}$ subvarieties of $S/\overline{\mathbb{Q}}$ is countable, we conclude (ii).

For (ii) $\Rightarrow$ (i), let $\xi \in F^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$. It is obvious that if $\xi \in F^2\mathrm{CH}^r(X;\mathbb{Q})$, by the definition of normal function, ν_ξ must be zero and thus $\Phi_r(\xi) = 0$ (that is $\xi \in \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$). Therefore $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$. We now observe that given $\sigma \in Aut(\mathbb{C}/\overline{\mathbb{Q}})$, the action of σ on $\mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$ is the identity. But in the limit for a finitely generated subfield $K \subset \mathbb{C}$, we get $\sigma(\mathcal{F}^\nu\mathrm{CH}^r(X_K;\mathbb{Q})) = \mathcal{F}^\nu\mathrm{CH}^r(X_{\sigma K};\mathbb{Q})$. Thus, $\sigma : \mathcal{F}^\nu\mathrm{CH}^r(X;\mathbb{Q}) \to \mathcal{F}^\nu\mathrm{CH}^r(X_\sigma;\mathbb{Q})$ is an isomorphism. Now if we suppose $\Phi_r(\xi) = 0$, it means that the image $\nu_\xi(t) = 0$ in $J(H^{2r-1}(X_t, \mathbb{Q}(r))$ for some $t \in S(\mathbb{C})$. But applying the isomorphisms given by $Aut(\mathbb{C}/\overline{\mathbb{Q}})$, we can map t to a generic point t_0. Then if $t \in \mathcal{Z}(\nu_\xi)$, t_0 is also in $\mathcal{Z}(\nu_\xi)$, but there cannot be an algebraic subvariety over $\overline{\mathbb{Q}}$ containing t_0, so ν_ξ must be a zero normal function. Thus $\xi \in F^2\mathrm{CH}^r(X;\mathbb{Q})$. We conclude $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$.

For (ii) $\Rightarrow$ (iii) we take $\rho_V : \mathscr{X}_V \to V$ with both quasiprojective varieties over L as required. Let now $S \to \mathcal{T}$ be a $\overline{\mathbb{Q}}$-spread of V. Consider the following generic points: $\eta_{\mathcal{T}} \in \mathcal{T}/\overline{\mathbb{Q}}$ (so that $\mathcal{S}_{\eta_{\mathcal{T}}} = V/L$ and $\overline{\mathbb{Q}}(\eta_{\mathcal{T}}) = \overline{\mathbb{Q}}(\mathcal{T}) = L$), and $\eta \in \mathcal{S}/\overline{\mathbb{Q}}$ (so that $\overline{\mathbb{Q}}(\eta) = \overline{\mathbb{Q}}(\mathcal{S}) = Quot(\overline{\mathbb{Q}}(\eta_{\mathcal{T}})(S_{\eta_{\mathcal{T}}})) = L(V)$). Then we have a $\overline{\mathbb{Q}}$-spread $\mathscr{X} \to \mathcal{S}$ that fits in the following commutative diagram

$$\begin{array}{ccccc} \mathscr{X}_V & \hookrightarrow & \mathscr{X} & & \\ \downarrow & & \downarrow & & \\ \mathcal{S}_{\eta_{\mathcal{T}}} & \hookrightarrow & \mathcal{S} & \to & \mathcal{T} \end{array}$$

So that $\mathscr{X}_{\eta_{\mathcal{T}}} = \mathscr{X}_V$ (where the fiber is taken over the composite and $\mathscr{X}_V$ is identified in $\mathscr{X}$), and $\mathscr{X}_\eta = \mathscr{X}_{\eta_V}$, where η_V is a generic point of V (so that $L(\eta_V) = L(V)$).

With this construction, we see that for a cycle $\xi \in \mathcal{F}^1\mathrm{CH}^r(\mathscr{X}_V/L;\mathbb{Q})$ is the restriction of a cycle $\tilde{\xi} \in \mathcal{F}^1\mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$, and if $\Sigma \subset \mathcal{S}/\overline{\mathbb{Q}}$ is an irreducible component of $\mathcal{Z}(\nu_{\tilde{\xi}})$, then $\Sigma_{\eta_{\mathcal{T}}}$ corresponds to a component of $\mathcal{Z}(\nu_\xi)$ over $\overline{L}$ in $V/\overline{L}$.

Finally, (iii) $\Rightarrow$ (ii) is direct, taking $L = \overline{\mathbb{Q}}$ and $V = \mathcal{S}$. $\square$

One objective in this thesis is to arrive at a version of Proposition 3.2.2 for arithmetic normal functions, i.e., regarding $F^{\nu}\mathrm{CH}^{r}(X;\mathbb{Q})$ for $\nu > 2$.

Chapter 4

Working with arithmetic normal functions

4.1 Product case

We follow the ideas in Lewis [25] rather carefully. Let $K \subseteq \mathbb{C}$ be a subfield, and X smooth and projective over K. We recall the coniveau filtration, which is given by

$$N_K^{\nu} H^i(X, \mathbb{Q}) := \ker \left(H^i(X, \mathbb{Q}) \to \varinjlim_{Y \subset X/K, codim_X Y \geq \nu} H^i(X \setminus Y, \mathbb{Q}) \right)$$

Consider the coniveau subspace $N_K^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q}(r)) \subset H^{2r-\nu}(X, \mathbb{Q}(r))$. After possibly shrinking S, this subspace determines a corresponding sub VHS $N_K^{r-\nu+1} R^{2r-\nu} \rho_* \mathbb{Q}(r) \subset R^{2r-\nu} \rho_* \mathbb{Q}(r)$, giving rise to the corresponding coniveau at general points of $\mathcal{S}$, and which by semisimple considerations is a direct summand. Now let $Y \subset X/K$ be of (pure) codimension $r - \nu + 1$ such that

$$H_Y^{2r-\nu}(X, \mathbb{Q}(r)) \twoheadrightarrow N_K^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q}(r))$$

is surjective, with desingularization $\tilde{Y} \to Y$ and composite map $\sigma : \tilde{Y} \to X$. Then (ignoring twists) we have

$$H^{\nu-2}(\tilde{Y}, \mathbb{Q}) \to H_Y^{2r-\nu}(X, \mathbb{Q}) \twoheadrightarrow N_K^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q})$$

since $2r - \nu - 2 \dim Y = 2r - \nu - 2(r - \nu + 1) = \nu - 2$. Let us assume there is a K -cycle induced map

$$\tilde{P} : H^{2r-\nu}(X, \mathbb{Q}) \to H^{\nu-2}(\tilde{Y}, \mathbb{Q})$$

such that

$$P := \sigma_* \circ \tilde{P} : H^{2r-\nu}(X, \mathbb{Q}(r)) \twoheadrightarrow N_K^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q}(r)) \subset H^{2r-\nu}(X, \mathbb{Q}(r))$$

is a projector.

With this setting, we have

Proposition 4.1.1. *(i)* $P_* Gr_F^\nu CH^r(X_K, \mathbb{Q}) = 0$.

(ii) $Im(Gr_F^\nu CH^r(X_K, \mathbb{Q}) \to \underline{\underline{E}}_\infty^{\nu, 2r-\nu}(\eta_{\mathcal{S}})) \bigcap \Gamma(H^\nu(\eta_{\mathcal{S}}, N_K^{r-\nu+1} R^{2r-\nu} \rho_* \mathbb{Q}(r))) = 0$.

(iii) $\{Gr_F^\nu CH^r(X_K, \mathbb{Q})$
$\bigcap Im(J(W_{-1} H^{\nu-1}(\eta_{\mathcal{S}}, N_K^{r-\nu+1} R^{2r-\nu} \rho_* \mathbb{Q}(r))) \to \underline{\underline{E}}_\infty^{\nu, 2r-\nu}(\eta_{\mathcal{S}}))\} = 0$.

Proof. (i) Let $[\xi] \in Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$, so we have

$$P_*([\xi]) = [P_*(\xi)] = [\sigma_* \circ \tilde{P}_*(\xi)] \in Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$$

Now, $\tilde{P}_*$ lies in

$$\hom_{\mathrm{MHS}}(H^{2r-\nu}(X_K, \mathbb{Q}), H^{\nu-2}(\tilde{Y}_K, \mathbb{Q})) \simeq H^{2r-\nu}(X_K, \mathbb{Q})^\vee \oplus H^{\nu-2}(\tilde{Y}_K, \mathbb{Q})$$
$$\simeq H_{2r-\nu}(X_K, \mathbb{Q}) \oplus H^{\nu-2}(\tilde{Y}_K, \mathbb{Q}) \simeq H^{2d-2r+\nu}(X_K, \mathbb{Q}) \oplus H^{\nu-2}(\tilde{Y}_K . \mathbb{Q})$$

so is induced by a cycle in $\mathrm{CH}^{d-r+\nu-1}(X_K \times Y_K; \mathbb{Q})$.
Therefore, $\tilde{P}_* \mathrm{CH}^r(X_K; \mathbb{Q}) \subset \mathrm{CH}^{\nu-1}(\tilde{Y}_K; \mathbb{Q})$. Then, since $\tilde{P}_*$ is compatible with Gr_F^ν by Theorem 3.2.1, we have that $[\sigma_* \circ \tilde{P}_*(\xi)]$ factors through $[\tilde{P}_*(\xi)] \in Gr_F^\nu \mathrm{CH}^{\nu-1}(\tilde{Y}_K; \mathbb{Q})$ which is zero since $F^\nu \mathrm{CH}^{\nu-1}(\tilde{Y}_K; \mathbb{Q}) = F^{\nu+1} \mathrm{CH}^{\nu-1}(\tilde{Y}_K; \mathbb{Q})$ by Theorem 3.2.1 as well.
Now for (ii) we observe that, since P projects over $N_K^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q}(r))$, we have

$$Im(Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q}) \to \underline{\underline{E}}_\infty^{\nu, 2r-\nu}(\eta_{\mathcal{S}})) \bigcap \Gamma(H^\nu(\eta_{\mathcal{S}}, N_K^{r-\nu+1} R^{2r-\nu} \rho_* \mathbb{Q}(r)))$$

$$= Im(P_* Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q}) \to \underline{\underline{E}}_\infty^{\nu, 2r-\nu}(\eta_{\mathcal{S}})) \bigcap \Gamma(H^\nu(\eta_{\mathcal{S}}, N_K^{r-\nu+1} R^{2r-\nu} \rho_* \mathbb{Q}(r)))$$

which is zero by (i).
(iii) can be proved in a similar fashion. □

For the following result we recall the GHC: $N_{\mathbb{C}}^\nu H^i(X, \mathbb{Q})$ is the largest sub-Hodge structure in $F^\nu H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Q})$.

Remark 4.1.1. *$H^{r,r}(X,\mathbb{Q})$ is naturally the largest Hodge structure in $F^rH^{2r}(X,\mathbb{Q})$. Since the image of the cycle class map $cl_r CH^r(X,\mathbb{Q})$ coincides with $N^r_K H^{2r}(X,\mathbb{Q})$, GHC (with $\nu = r$, $i = 2r$) implies the classical Hodge conjecture.*

Remark 4.1.2. *As in the case for the classical Hodge conjecture, when X is defined over K algebraically closed, the GHC can be taken substituting $\mathbb{C}$ by K.*

Let X be a smooth projective variety of dimension n over an algebraically closed field K. There exists Y a possibly reducible subvariety of X of codimension r such that

$$N^r_K H^i(X,\mathbb{Q}(r)) = image(H^i_Y(X,\mathbb{Q}(r)) \longrightarrow H^i(X,\mathbb{Q}(r)))$$

with Y over $\mathbb{C}$ and X over K. If $\tilde{Y}$ denotes a desingularization of Y, since both X and $\tilde{Y}$ are pure HS, the images of $H^i_Y(X,\mathbb{Q}(r)) \simeq H_{2n-i}(Y,\mathbb{Q}(n-r))$ and $H^{i-2r}(\tilde{Y},\mathbb{Q}(0))$ (given by the composition with the desingularization map and Poincaré duality) are the same in $H^i(X,\mathbb{Q}(r))$. Thus, we can work with the smooth variety $\tilde{Y}$.

We can form a K spread $\mathcal{Y} \to \mathcal{S}$ of $\tilde{Y}$ with ρ smooth and proper, we can see this as a C^∞ fiber bundle over $\mathcal{S}$ as complex spaces. There is a point $t_1 \in \mathcal{S}(\mathbb{C})$ such that $\rho^{-1}(t_1) = \tilde{Y}$, and since K is algebraically closed we can find a point $t_0 \in \mathcal{S}(K)$. Since the fibers of the fiber bundle are diffeomorphic, the image $H^{i-2r}(\rho^{-1}(t)) \to H^i(X)$ is independent of t.

Returning to our situation, we have a cycle induced $\tilde{P}$ as stated above for $K = \mathbb{C}$. Thus, from the previous proposition we get

Corollary 4.1.2. *If we assume the GHC, we have $P_* Gr^\nu_F CH^r(X_{\mathbb{C}},\mathbb{Q}) = 0$*

Observation: If X is a variety defined over $k = \overline{k}$, we can arrange $\tilde{P}$ to be induced by a cycle over k by using a spread argument.

In the case where $\mathscr{X}$ is a product (X a variety over k) we can arrive at a characterization of $F^\nu \mathrm{CH}^r(X_K,\mathbb{Q})$ through arithmetic normal functions:

Consider then the product situation with $\mathscr{X} = \mathcal{S} \times X$ defined over k, with $K = k(\mathcal{S})$. Let $\eta_{\mathcal{S}}$ be the generic point of $\mathcal{S}/k$ and set:

$$H_0 = W_{-1}\left(H^{\nu-1}(\eta_{\mathcal{S}},\mathbb{Q}) \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)\right)$$

where $N^\nu_H H^i(X,\mathbb{Q})$ denotes the largest sub-Hodge structure in $F^\nu H^i(X,\mathbb{C}) \cap H^i(X,\mathbb{Q})$.

Lemma 4.1.3. *There is a natural map*

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to J(H_0)$$

Proof. We assume $\mathcal{S}/k$ is affine, and put

$$W_j = W_j(H^{\nu-1}(\mathcal{S},\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})(r)),$$

$$W_j^H = W_j(H^{\nu-1}(\mathcal{S},\mathbb{Q}) \otimes N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})(r))$$

We observe that, since $H^{2r-\nu}(X,\mathbb{Q})(r)$ is a pure Hodge structure of pure weight $-\nu$ (and thus $W_j(H^{2r-\nu}(X,\mathbb{Q})(r)) = \delta_{j,-\nu}$), we get

$$\begin{aligned}
Gr_W^0 &= \frac{W_0(H^{\nu-1}(\mathcal{S},\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})(r))}{W_{-1}(H^{\nu-1}(\mathcal{S},\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})(r))} \\
&= \frac{\bigoplus_{p+q=0} W_p(H^{\nu-1}(\mathcal{S},\mathbb{Q})) \otimes W_q(H^{2r-\nu}(X,\mathbb{Q})(r))}{\bigoplus_{p+q=-1} W_p(H^{\nu-1}(\mathcal{S},\mathbb{Q})) \otimes W_q(H^{2r-\nu}(X,\mathbb{Q})(r))} \\
&= \frac{W_\nu(H^{\nu-1}(\mathcal{S},\mathbb{Q})) \otimes W_{-\nu}(H^{2r-\nu}(X,\mathbb{Q})(r))}{W_{\nu-1}(H^{\nu-1}(\mathcal{S},\mathbb{Q})) \otimes W_{-\nu}(H^{2r-\nu}(X,\mathbb{Q})(r))} \\
&= \frac{W_\nu(H^{\nu-1}(\mathcal{S},\mathbb{Q}))}{W_{\nu-1}(H^{\nu-1}(\mathcal{S},\mathbb{Q}))} \otimes H^{2r-\nu}(X,\mathbb{Q})(r) \\
&= Gr_W^\nu H^{\nu-1}(S,\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})(r)
\end{aligned}$$

Let $V = Gr_W^\nu H^{\nu-1}(\mathcal{S},\mathbb{Q})$. We "untwist" things by observing that, with the previous observation, $\Gamma(Gr_W^0)$ can be identified with

$$\hom_{\mathrm{MHS}}(\mathbb{Q}(-r), V \otimes H^{2r-\nu}(X,\mathbb{Q}))$$

which in turn we identify with $\{V \otimes H^{2r-\nu}(X,\mathbb{Q})\}^{(r,r)}$.

Next, we observe that $F^\nu H^{\nu-1}(\mathcal{S}) = 0$, since there are no closed ν-forms in $H^{\nu-1}(\mathcal{S})$ together with Deligne's Hodge theory description of F^ν involving holomorphic forms with simple poles along $\overline{\mathcal{S}} \setminus \mathcal{S}$. Thus $F^\nu V_{\mathbb{C}} = 0$ as well.

We know that the minimum weight of $H^{\nu-1}(\mathcal{S},\mathbb{Q})$ is $\nu - 1$ so this holds true for V as well, and thus

$$\left\{V \otimes H^{2r-\nu}(X,\mathbb{Q})\right\}^{(r,r)} \subset V^{\nu-1,1} \otimes H^{r-\nu+1,r-1}(X) \oplus \cdots \oplus V^{1,\nu-1} \otimes H^{r-1,r-\nu+1}(X)$$

But since $F^\nu V_{\mathbb{C}} = 0$, we have

$$V_{\mathbb{C}} = V^{\nu-1,1} \oplus \cdots \oplus V^{1,\nu-1}$$

Let $V^\vee$ be the dual space of V, which is a Hodge sructure of weight $-(\nu-1)$. The dual action of $V^\vee$ on V leads to a corresponding action on $\{V \otimes H^{2r-\nu}(X,\mathbb{Q})\}^{(r,r)}$, whose image must be a Hodge structure and hence lies in $N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})$. Therefore,

$$\left\{V \otimes H^{2r-\nu}(X,\mathbb{Q})\right\}^{(r,r)} \subset V \otimes N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})$$

From which we obtain $\Gamma(Gr^0_W) \subset \Gamma(Gr^0_{W^H})$. Since the other containing is obvious, we get $\Gamma(Gr^0_W) = \Gamma(Gr^0_{W^H})$.

Using $W_j^H \hookrightarrow W_j$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & W^H_{-1} \arrow[r] \arrow[d, hook] & W^H_0 \arrow[r] \arrow[d, hook] & Gr^0_{W^H} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W_{-1} \arrow[r] & W_0 \arrow[r] & Gr^0_W \arrow[r] & 0
\end{tikzcd}$$

Then, from the long exact sequences that arise after applying the Ext operator, we obtain the commutative diagram:

$$\begin{tikzcd}
\Gamma(Gr^0_{W^H}) \arrow[r] \arrow[d, equal] & J(W^H_{-1}) \arrow[d] \\
\Gamma(Gr^0_W) \arrow[r] & J(W_{-1})
\end{tikzcd}$$

From which we get the natural map

$$\underline{E}_\infty^{\nu,2r-\nu}(\eta_S) = \frac{J(W_{-1})}{\Gamma(Gr^0_W)} \longrightarrow \frac{J(W_{-1})}{J(W^H_{-1})} = J(H_0)$$

Where the last equality comes form applying the Ext operator to the short exact sequence

$$0 \longrightarrow W^H_{-1} \longrightarrow W_{-1} \longrightarrow H_0 \longrightarrow 0$$

□

Now let $\xi \in F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$ with its corresponding class in the chow group also denoted by $\xi \in \mathrm{CH}^r(\mathcal{S} \times_k X, \mathbb{Q})$ for some affine $\mathcal{S}$. For a smooth affine closed $V \subset \mathcal{S}(\mathbb{C})$ we restrict ourselves to $\eta_V :=$ "$V \cap \eta_{\mathcal{S}}$", and let λ_ξ be the corresponding arithmetic normal function where

$$\lambda_\xi(V) \in \frac{J(W_{-1}\left[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q})(r)\right])}{\Gamma(Gr^0_W \left[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q})(r)\right])}$$

We denote by $\underline{\lambda_\xi}$ the so called reduced arithmetic normal function, which has the characteristic property that its values lies in

$$\underline{\lambda_\xi}(V) \in J\left(W_{-1}\left[H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X, \mathbb{Q})}{N^{r-\nu+1}H^{2r-\nu}(X, \mathbb{Q})}(r)\right]\right)$$

which via lemma 4.1.3 is the image of $\lambda_\xi(V)$.

Theorem 4.1.3 (Lewis)**.** *In the case where X is a smooth projective variety over k, and under the assumption of the GHC, the filtration $\{F^\nu CH^r(X_K, \mathbb{Q})\}_{\nu \geq 0}$ is characterized by the germs of reduced arithmetic normal functions.*

Proof. It is obvious that if $0 = \xi \in Gr^\nu_F \mathrm{CH}^r(X_K, \mathbb{Q})$ we have $\underline{\lambda_\xi} = 0$, so we aim to prove the converse.

Since $k = \overline{k}$, the GHC tells us that $N^{r-\nu+1}_k H^{2r-\nu}(X, \mathbb{Q}) = N^{r-\nu+1}_H H^{2r-\nu}(X, \mathbb{Q})$. If $\underline{\lambda_\xi} = 0$, ξ must lie in $N^{r-\nu+1}_H H^{2r-\nu}(X, \mathbb{Q})$ so, using the terminology from Proposition 4.1.1, $\xi = P_*\xi$. But then, by part (i) of the same Proposition 4.1.1, $\lambda_\xi = \lambda_{P_*\xi} = 0$. Therefore $\underline{\lambda_\xi} = 0 \Leftrightarrow \lambda_\xi = 0$.

Now, recall that $\lambda_\xi = 0$ implies $0 = [\xi] \in \underline{\underline{E}}^{\nu, 2r-\nu}_\infty(\eta_{\mathcal{S}})$. Thus, $[\xi] \in \underline{E}^{\nu,2r-\nu}_\infty(\eta_{\mathcal{S}})$, and when its value is zero, we have $0 = \xi \in Gr^\nu_F \mathrm{CH}^r(X_K, \mathbb{Q})$. But from the proof of lemma 4.1.3 and reusing its terminology, we have $\frac{J(W_{-1})}{J(W^H_{-1})} = J(H_0)$, which together with part (iii) from Proposition 4.1.1 tells us that $[\xi] = 0 \in J(H_0)$ implies $0 = [\xi] \in \underline{E}^{\nu,2r-\nu}_\infty(\eta_{\mathcal{S}})$. Since the converse is obvious, we get $[\xi] = 0 \in J(H_0) \Leftrightarrow 0 = [\xi] \in \underline{E}^{\nu,2r-\nu}_\infty(\eta_{\mathcal{S}})$.

From the affine Lefschetz theorem, we see that

$$V_1 := H^{\nu-1}(\eta_{\mathcal{S}}, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X, \mathbb{Q})}{N^{r-\nu+1}H^{2r-\nu}(X, \mathbb{Q})}(r)$$

$$\hookrightarrow H^{\nu-1}(\eta_V, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X, \mathbb{Q})}{N^{r-\nu+1}H^{2r-\nu}(X, \mathbb{Q})}(r) =: V_2$$

is injective. From this we have the short exact sequence

$$0 \to W_{-1}V_1 \to W_{-1}V_2 \to W_{-1}(V_1/V_2) \to 0$$

Then, applying the Ext operator, together with $\Gamma(W_{-1}(V_1/V_2)) = 0$ (since Γ maps to zero weight) we obtain an injection

$$J(H_0) = J(W_{-1}V_1) \hookrightarrow J(W_{-1}V_2)$$

Thus, if $\underline{\lambda_\xi} = 0$, we have that $[\xi]$ is mapped to zero in $J(H_0)$.

Therefore, for $\xi \in F^\nu \mathrm{CH}^r(X_K;\mathbb{Q})$, $\underline{\lambda_\xi} = 0$ if and only if the image of ξ in $E_\infty^{\nu,2r-\nu}(\eta_\mathcal{S})$ is zero. But the latter implies that ξ is zero in $Gr_F^\nu \mathrm{CH}^r(X_K,\mathbb{Q})$, which is the same as saying $\xi \in F^{\nu+1}\mathrm{CH}^r(X_K;\mathbb{Q})$

□

4.2 An assumption for a more general case

Let us now consider a more general case with $\rho : \mathscr{X} \to \mathcal{S}$ over $k = \overline{k}$, $K = k(\mathcal{S})$ and generic fiber X_K. Let $\eta_\mathcal{S}$ be the generic point of $\mathcal{S}/k$.

Proposition 4.1.1 still works in this case, but for Lemma 4.1.3 I will add an additional restriction:

Let

$$H_0 = W_{-1}\left(\frac{H^{\nu-1}(\eta_\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))}{H^{\nu-1}(\eta_\mathcal{S}, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))}\right),$$

$$W_j = W_j H^{\nu-1}(\eta_\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)),$$

$$W_j^H = W_j H^{\nu-1}(\eta_\mathcal{S}, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)).$$

During the rest of this work, we will often use the following

Assumption 4.2.1. *With the above notation, let us assume that*

$$\Gamma Gr_W^0 W_0^H = \Gamma Gr_W^0 W_0, \text{ in } J(W_{-1}) \quad \forall\, \nu$$

Remark 4.2.2. *This assumption holds if $\mathscr{X} = \mathcal{S} \times X$ as shown in the proof of lemma 4.1.3, but this might not hold true in general since for the general assumption involving higher Chow groups Lewis provided a counterexample. Yet, there are still cases outside the product case in which this assumption holds true like the following:*

Let $\mathbb{P}^N$ parameterize all hypersurfaces of degree d and dimension n in $\mathbb{P}^{n+1}$, and $\overline{\mathscr{X}} := \{(t,x) \in \mathbb{P}^N \times \mathbb{P}^{n+1} \mid x \in X_t\}$, the universal family of hypersurfaces of degree d and dimension n. Put $\overline{\rho} = Pr_1 : \overline{\mathscr{X}} \to \mathbb{P}^N =: \overline{\mathcal{S}}$. Note that $Pr_2 : \overline{\mathscr{X}} \to \mathbb{P}^{n+1}$ is a $\mathbb{P}^{N-1}$-fibered projective bundle. Therefore $H^{2r-1}(\overline{\mathscr{X}}, \mathbb{Q}(r))$ is zero (since 2r-1 is odd). Note that $H^{2r-1}(\overline{\mathscr{X}}, \mathbb{Q}(r))$ maps surjectively to $W_{-1}H^{2r-1}(\mathscr{X}, \mathbb{Q}(r))$, and that non-canonically as MHS ([1]), $H^{2r-1}(\mathscr{X}, \mathbb{Q}(r)) \simeq \oplus_{p+q=2r-1} H^p(\mathcal{S}, R^q\rho_*\mathbb{Q}(r))$. Thus $W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) = 0$.

Obviously $\Gamma Gr_0^W H^{\nu-1}(\mathcal{S}, N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))$ and $\Gamma Gr_0^W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))$ are zero in $J(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))$, so assumption 4.2.1 holds in this case.

Lemma 4.2.3. *With the above notation, if we assume assumption 4.2.1, then there is a natural map*

$$\underline{E}_\infty^{\nu,2r-\nu}(\eta_\mathcal{S}) \to J(H_0)$$

Proof. The proof is basically the same as the second part of the proof for Lemma 4.1.3

Since $W_j^H \hookrightarrow W_j$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{-1}^H & \longrightarrow & W_0^H & \longrightarrow & Gr^0_{W^H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_{-1} & \longrightarrow & W_0 & \longrightarrow & Gr^0_W & \longrightarrow & 0 \end{array}$$

Then, from the long exact sequences that arise after applying the Ext operator, we obtain the commutative diagram:

$$\begin{array}{ccc} \Gamma(Gr^0_{W^H}) & \longrightarrow & J(W_{-1}^H) \\ \| & & \downarrow \\ \Gamma(Gr^0_W) & \longrightarrow & J(W_{-1}) \end{array}$$

From which we get the natural map

$$\underline{E}_\infty^{\nu,2r-\nu}(\eta_\mathcal{S}) = \frac{J(W_{-1})}{\Gamma(Gr^0_W)} \longrightarrow \frac{J(W_{-1})}{J(W_{-1}^H)} = J(H_0)$$

Where the last equality comes form applying the Ext operator to the short exact sequence

$$0 \longrightarrow W_{-1}^H \longrightarrow W_{-1} \longrightarrow H_0 \longrightarrow 0$$

□

Now let $\xi \in F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$ with its corresponding class in the chow group also denoted by $\xi \in \mathrm{CH}^r(\mathscr{X}, \mathbb{Q})$ for some affine $\mathcal{S}$. For a smooth affine closed $V \subset \mathcal{S}(\mathbb{C})$ we take η_V to be the generic point of V, with the diagram

$$\begin{array}{ccc} \mathscr{X}_V & \hookrightarrow & \mathscr{X}(\mathbb{C}) \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S}(\mathbb{C}) \end{array}$$

and let λ_ξ be the corresponding arithmetic normal function where

$$\lambda_\xi(V) \in \frac{J(W_{-1}\left[H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right])}{\Gamma(Gr_W^0\left[H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right])}$$

We again denote by $\underline{\lambda_\xi}$ the so called reduced arithmetic normal function, which has the characteristic property that its values lies in

$$\underline{\lambda_\xi}(V) \in J\left(W_{-1}\left[\frac{H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))}{H^{\nu-1}(\eta_V, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))}\right]\right)$$

which via lemma 4.2.3 is the image of $\lambda_\xi(V)$.

Theorem 4.2.1. *In the case where X is a smooth projective variety over k, and given Assumption 4.2.1, together with the GHC, the filtration $\{F^\nu CH^r(X_K, \mathbb{Q})\}_{\nu \geq 0}$ is characterized by the germs of reduced arithmetic normal functions.*

Proof. It is obvious that if $0 = \xi \in Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$ we have $\underline{\lambda_\xi} = 0$, so we aim to prove the converse.

Since $k = \overline{k}$, the GHC tells us that $N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)) = N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))$. If $\underline{\lambda_\xi} = 0$, ξ must lie in $N_H^{r-\nu+1}H^{2r-\nu}(X, \mathbb{Q})$ so, using the terminology from Proposition 4.1.1, $\xi = P_*\xi$. But then, by part (i) of the same Proposition 4.1.1, $\lambda_\xi = \lambda_{P_*\xi} = 0$. Therefore $\underline{\lambda_\xi} = 0 \Leftrightarrow \lambda_\xi = 0$.

Now, recall that $\lambda_\xi = 0$ implies $0 = [\xi] \in \underline{\underline{E}}_\infty^{\nu, 2r-\nu}(\eta_S)$. Thus, $[\xi] \in \underline{E}_\infty^{\nu, 2r-\nu}(\eta_S)$, and when its value is zero, we have $0 = \xi \in Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$. But from the proof of lemma 4.1.3 and reusing its terminology, we have $\frac{J(W_{-1})}{J(W_{-1}^H)} = J(H_0)$, which together with part (iii) from Proposition 4.1.1 tells us that $[\xi] = 0$ in $J(H_0)$ implies $[\xi] = 0$ in $\underline{E}_\infty^{\nu, 2r-\nu}(\eta_S)$. Since the converse is obvious, we get $[\xi] = 0 \in J(H_0)$ if and only if $0 = [\xi] \in \underline{E}_\infty^{\nu, 2r-\nu}(\eta_S)$.

Now, by semisimplicity considerations, we can write $R^{2r-\nu}\rho_*\mathbb{Q}(r)$ as $[N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)] \oplus [N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp}$ (where $[N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp}$ denotes the orthogonal complement), so we write:

$$V_1 := \frac{H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{H^{\nu-1}(\eta_{\mathcal{S}}, N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))} = H^{\nu-1}(\eta_{\mathcal{S}}, [N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp})$$

and similarly, for very general V,

$$V_2 := \frac{H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{H^{\nu-1}(\eta_V, N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r))} = H^{\nu-1}(\eta_V, [N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp})$$

Thus, from the affine Lefschetz theorem, we see that

$$V_1 = H^{\nu-1}(\eta_{\mathcal{S}}, [N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp}) \hookrightarrow H^{\nu-1}(\eta_V, [N^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)]^{\perp}) = V_2$$

is injective. From this we have the short exact sequence

$$0 \to W_{-1}V_1 \to W_{-1}V_2 \to W_{-1}(V_1/V_2) \to 0$$

Then, applying the Ext operator together with $\Gamma(W_{-1}(V_1/V_2)) = 0$ (since Γ maps to zero weight) we obtain an injection $J(H_0) = J(W_{-1}V_1) \hookrightarrow J(W_{-1}V_2)$. Thus, if $\underline{\lambda}_\xi = 0$, we have that $[\xi]$ is mapped to zero in $J(H_0)$.

Therefore, for $\xi \in F^\nu \mathrm{CH}^r(X_K;\mathbb{Q})$, $\underline{\lambda}_\xi = 0$ if and only if the image of ξ in $E_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}})$ is zero. But the latter implies that ξ is zero in $Gr_F^\nu \mathrm{CH}^r(X_K, \mathbb{Q})$, which is the same as saying $\xi \in F^{\nu+1}\mathrm{CH}^r(X_K;\mathbb{Q})$

□

4.3 Working with $F^2\mathrm{CH}^r_{alg}(X;\mathbb{Q})$

Let, $F^2\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) := F^2\mathrm{CH}^r(X;\mathbb{Q}) \cap \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$. The following is due to S. Saito, [29].

Proposition 4.3.1. *$F^2 CH^r_{\mathrm{alg}}(X;\mathbb{Q}) = \ker(AJ : CH^r_{\mathrm{alg}}(X;\mathbb{Q}) \to J(H^{2r-1}(X,\mathbb{Q}(r))))$*

Proof. We know $F^2\mathrm{CH}^r(X;\mathbb{Q}) \subseteq \ker(AJ : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J(H^{2r-1}(X,\mathbb{Q}(r)))$ by Theorem 3.2.1 part (ii). Thus, we have

$$F^2\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) \subseteq \ker(AJ : \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) \to J(H^{2r-1}(X,\mathbb{Q}(r)))$$

thus we just need to prove the "$\supseteq$" inclusion.

For this, we first observe that, by Theorem 3.2.1 part (i),

$$F^1\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$$

Let $\xi \in \ker(AJ : \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q}) \to J(H^{2r-1}(X,\mathbb{Q}(r))))$. By definition of algebraic equivalence, there exists a smooth projective curve C and $w_0 \in \mathrm{CH}^r(C \times X)$ such that $\xi = w_{0,*}(P') - w_{0,*}(Q')$ for some $P', Q' \in C$, where we remember that $w_{0,*}(t) = Pr_{2,*}((t \times X) \bullet w_0)$. Thus, $\xi \in w_{0,*}(\mathrm{CH}^{\mathrm{alg}}_0(C))$. We observe that

$$\mathrm{CH}^{\mathrm{alg}}_0(C) \simeq \mathrm{CH}^1_{\mathrm{alg}}(C) \simeq \mathrm{CH}^1_{\mathrm{hom}}(C)$$

and that the Abel Jacobi map is an isomorphism in this case.

(Indeed, if g is the genus of C, $\mathrm{CH}^{\mathrm{alg}}_0(C)$ can be "identified" with $\mathcal{S}^{(g)} := C^g/Sym(g)$, where Sym denotes the action of the symmetric group on g letters. Then, fixing p_0 in C, we can map each element $p_1 + \cdots + p_g \in \mathcal{S}^{(g)}$ to $p_1 + \cdots + p_g - g \bullet p_0 \in J(C)$ which can easily be checked to be a birational morphism, via the Riemann-Roch theorem).

Using this, $(P' \times X) \bullet w_0$ and $(Q' \times X) \bullet w_0$ correspond to $(P \times X) \bullet w$ and $(Q \times X) \bullet w$ for some $P, Q \in J(C)$ and some $w \in \mathrm{CH}^r(J(C) \times X)$. Thus $\xi \in w_{0,*}(J(C))$ and $\xi = w_*(P) - w_*(Q)$. But since $J(C)$ is an abelian variety, we have $\xi = w_*(P - Q)$.

Let B be the connected component of the identity in the kernel of

$$[w]_* : J(C) \to J(H^{2r-1}(X,\mathbb{Q}(r)))$$

which is given by the composition of w_* with the Abel Jacobi map. If $\dim B = 0$, then $[w]_*$ has finite kernel. Since we supposed that $[w]_*(P - Q) = AJ(\xi) = 0$, taking $P - Q \in B$ we conclude that $P - Q = 0$ and thus $\xi = 0 \in F^2\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$. Assume $b := \dim B \geq 1$, and let $w_B := w|_{B \times X}$, so we have

$$\xi \in w_{B,*}(\mathrm{CH}^{\mathrm{alg}}_0(B;\mathbb{Q})) \subset \mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$$

This is preserved on the graded level with

$$\mathrm{CH}^{\mathrm{alg}}_0(B;\mathbb{Q}) = \mathrm{CH}^{\mathrm{hom}}_0(B;\mathbb{Q}) = F^1\mathrm{CH}_0(B;\mathbb{Q})$$

that is

$$\{\xi\} \in \mathrm{Image}([w_B]_* : Gr^1_F\mathrm{CH}_0(B;\mathbb{Q}) \to Gr^1_F\mathrm{CH}^r(X;\mathbb{Q})).$$

Since w has codimension r, we see that $[w_B] \in \mathrm{CH}^r(B \times X)$ and we can decompose it into it's Künneth components: $[w_B] = \bigoplus_{l=0}^{2r}[w_B](l, 2r-l)$ where

$$\begin{aligned}
&[w_B](l, 2r-l) \in H^l(B) \otimes H^{2r-l}(X) \\
&\quad \simeq \hom_{\mathrm{MHS}}(H^l(B)^\vee, H^{2r-l}(X)) \simeq H^{2b-l}(B) \otimes H^{2r-l}(X)
\end{aligned}$$

We see that by Theorem 3.2.1 part (v) the Künneth components depend only on cohomology and when restricting to Gr_F^1 we have

$$\Delta_X(2d-2r+l, 2r-l)_* \mid_{Gr_F^1 \mathrm{CH}^r(X;\mathbb{Q})} = \delta_{l,1} \cdot \text{ Identity}$$

and similarly for $Gr_F^1\mathrm{CH}_0(B;\mathbb{Q}) \simeq Gr_F^1\mathrm{CH}^b(B;\mathbb{Q})$:

$$\Delta_B(2b-2b+l, 2b-l)_* \mid_{Gr_F^1 \mathrm{CH}_0(B;\mathbb{Q})} = \Delta_B(l, 2b-l)_* \mid_{Gr_F^1 \mathrm{CH}_0(B;\mathbb{Q})} = \delta_{l,1} \cdot \text{ Identity}$$

Thus

$$\begin{aligned}
&[w_B]_* = [w_B]_* \circ (\Delta_B(1, 2b-1)_* \mid_{Gr_F^1 \mathrm{CH}_0(B;\mathbb{Q})})_* \\
&= (\Delta_X(2d-2r+1, 2r-1) \mid_{Gr_F^1 \mathrm{CH}^r(X;\mathbb{Q})})_* \circ [w_B]_* = [w_B](1, 2r-1)_*
\end{aligned}$$

But since B is in the kernel of $[w]_*$, at least the Künneth component from the induced $[w_B]_*$ that maps over $J(H^{2r-1}(X, \mathbb{Q}(r)))$ must be zero. This Künneth component is

$$\begin{aligned}
&[w_B](1, 2r-1)_* \in H^1(B) \otimes H^{2r-1}(X) \\
&\simeq \hom_{\mathrm{MHS}}(H^1(B)^\vee, H^{2r-1}(X)) \simeq H^{2b-1}(B) \otimes H^{2r-1}(X)
\end{aligned}$$

as we see that

$$\begin{aligned}
&H^{2b-1}(B) \xrightarrow{Pr_1^*} H^{2b-1}(B \times X) \xrightarrow{\cap [w_B]} H^{2b-1+2r}(B \times X) \\
&\simeq H_{2d+2b-(2b-1+2r)}(B \times X) \xrightarrow{Pr_{2,*}} H_{2d-2r+1}(X) \simeq H^{2r-1}(X)
\end{aligned}$$

so $[w_B]_* = 0$ and thus $\{\xi\} = 0$. Therefore ξ necessarily lies in $F^2\mathrm{CH}^r_{\mathrm{alg}}(X;\mathbb{Q})$. $\square$

For any $V \subset \mathcal{S}(\mathbb{C})$ smooth, irreducible, closed subvariety of dimension $\nu - 1$, we have that the minimum weight of $H^{\nu-1}(V, \mathbb{Q})$ is $\nu - 1$ and we have the sequence:

$$H^{\nu-1}_{\bar{V} \setminus V}(\bar{V}, \mathbb{Q}) \to H^{\nu-1}(\bar{V}, \mathbb{Q}) \twoheadrightarrow W_{\nu-1}H^{\nu-1}(V, \mathbb{Q})$$

which taking direct limit gives us

$$0 \to N^1 H^{\nu-1}(\bar{V}, \mathbb{Q}) \to H^{\nu-1}(\bar{V}, \mathbb{Q}) \to W_{\nu-1}H^{\nu-1}(\eta_V, \mathbb{Q})$$

so we obtain

$$W_{\nu-1}H^{\nu-1}(\eta_V,\mathbb{Q}) \simeq \frac{H^{\nu-1}(\bar{V},\mathbb{Q})}{N^1H^{\nu-1}(\bar{V},\mathbb{Q})}$$

Let a smooth variety M/k parametrize a family $\{V_t\}_{t\in M} \subset S(\mathbb{C})$ of smooth, irreducible, closed subvarieties of dimension $\nu-1$.

Such an M arises naturally. For example, the universal family of smooth complete intersections in any $\mathbb{P}^m$ is defined over $\mathbb{Q}$: recall that any homogeneous hypersurface F contained in $\mathbb{P}^m$ of degree d in the coordinates $[z]=[z_0,...,z_m]$ is of the form

$$\sum_{\alpha\in\mathbb{Z}^d_+,[\alpha]=d} a_\alpha z^\alpha$$

where $[\alpha]$ denotes the sum of the coordinates of α and a_α denotes the α indexed coordinate of $a \in \mathbb{C}^{N(d)}$, with $N(d) = \binom{m+d-1}{d}$. In the variables a, z, it is defined over $\mathbb{Z}$. Now, let $\mathcal{S}/k \subset \mathbb{P}^m$ be a complete intersection that doesn't lie inside some $\mathbb{P}^{m-1}$. We can then write $X = V(F_1,F_2,...,F_r)$, where the F_i are homogeneous hypersurfaces of degree d_i in the coordinates $[z]=[z_0,...,z_m]$. Then, if we write $F_i = \sum_{\alpha\in\mathbb{Z}^d_+,[\alpha]=d} a_{i,\alpha}z^\alpha$ with $a_i \in \mathbb{C}^{N(d_i)}$ we can consider the product variety $W' := \mathbb{P}^{N(d_1)-1}\times ... \times \mathbb{P}^{N(d_r)-1}$ which is defined over $\mathbb{Z}$ as well. We identify $\mathbb{P}^m \times \mathbb{P}^{N(d_1)-1}\times\cdots\times\mathbb{P}^{N(d_r)-1}$ with $\mathbb{P}^m\times W'$ and consider

$$W := \{([z],[a_1],...,[a_r]) \in \mathbb{P}^m \times W| \sum_{\alpha\in\mathbb{Z}^m,[\alpha]=m} a_{i,\alpha}z^\alpha = 0 \quad \forall i\}$$

which is defined over $\mathbb{Z}$. Then, intersecting W with $\mathcal{S}\times W'$ defines the universal family of degree m complete intersections of $\mathcal{S}$, with $W\cap\{\mathcal{S}\times W'\}\to W'$ all defined over k, and it can shown it is smooth and proper over an open subset $U\subset W'$.

We can then define an arithmetic normal function ν_ξ for any cycle $\xi \in \mathcal{F}^\nu\mathrm{CH}^r(\mathcal{S}\times X;\mathbb{Q})$:

$$\nu_\xi : M \longrightarrow \coprod J(H^{2r-1}(V_t\times X,\mathbb{Q}(r)))$$

and each Jacobian by lemma 4.1.3 can be written as

$$J(W_{-1}\left(H^{\nu-1}(\eta_{V_t},\mathbb{Q})\otimes\frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)\right))$$

Now, consider the kernel of

$$\mathrm{CH}^r_{\mathrm{alg}}(\bar{V}_t \times X, \mathbb{Q}) \to J(W_{-1}\left(H^{\nu-1}(\eta_{V_t}, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r))\right) \quad (4.1)$$

where $\overline{V}_t$ can be assumed to be a smooth projective closure of V_t. The map in (4.1) factors through $\mathrm{CH}^r_{\mathrm{alg}}(V_t \times X, \mathbb{Q})$; moreover the restriction $\mathrm{CH}^r_{\mathrm{alg}}(\bar{V}_t \times X, \mathbb{Q}) \to \mathrm{CH}^r_{\mathrm{alg}}(V_t \times X, \mathbb{Q})$ is surjective. In general, we have:

Lemma 4.3.1. *Let $U \subset W$ be an inclusion of smooth varieties of the same dimension. Then the restriction $CH^r_{\mathrm{alg}}(W) \to CH^r_{\mathrm{alg}}(U)$ is surjective.*

Proof. Let $j : U \hookrightarrow W$ denote the inclusion map. We observe that if $\xi \in \mathrm{CH}^r(U)$ is algebraically equivalent to zero, then there exists a smooth connected curve Γ and a cycle $w \in z^r(\Gamma \times U)$ such that $\xi = w(p) - w(q)$ for some $p, q \in \Gamma$. Now take the closure $\overline{w} \in z^r(\Gamma \times W)$, and put $\overline{\xi} = \overline{w}(p) - \overline{w}(q)$. Then $j^*(\overline{\xi}) = \xi$ and the result is immediate. Therefore, $\mathrm{CH}^r_{\mathrm{alg}}(W) \twoheadrightarrow \mathrm{CH}^r_{\mathrm{alg}}(U)$.

□

From this, we conclude

Corollary 4.3.2. *A class $\overline{\xi}_t \in CH^r_{\mathrm{alg}}(\bar{V}_t \times X, \mathbb{Q})$ is in the kernel of the Abel-Jacobi map in (4.1) iff it's restriction $\xi_t \in CH^r_{\mathrm{alg}}(V_t \times X, \mathbb{Q})$ is.*

Now consider $\xi \in \mathcal{F}^\nu\mathrm{CH}^r(\mathcal{S} \times X; \mathbb{Q})$, with corresponding ξ_t. Let us assume for a given such fixed $t \in M$, $\xi_t \in \mathrm{CH}^r_{\mathrm{alg}}(V_t \times X; \mathbb{Q})$. Assume its closure $\overline{\xi}_t \in \mathrm{CH}^r_{\mathrm{alg}}(\bar{V}_t \times X, \mathbb{Q})$ is in the kernel of the Abel-Jacobi map in (4.1). We want to prove that it belongs to $F^2\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}_t \times X, \mathbb{Q})$. This is based on semi-simplicity considerations.

We observe that

$$\begin{aligned}
&W_{-1}\left(H^{\nu-1}(\eta_{V_t}, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)\right) \\
&\simeq W_{\nu-1}H^{\nu-1}(\eta_{V_t}, \mathbb{Q}) \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r) \\
&\simeq \frac{H^{\nu-1}(\overline{V}_t,\mathbb{Q})}{N^1H^{\nu-1}(\bar{V}_t,\mathbb{Q})} \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)
\end{aligned}$$

where

$$\frac{H^{\nu-1}(\overline{V}_t,\mathbb{Q})}{N^1H^{\nu-1}(\overline{V}_t,\mathbb{Q})} \subset H^{\nu-1}(\overline{V}_t,\mathbb{Q}), \qquad \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r) \subset H^{2r-\nu}(X,\mathbb{Q})(r)$$

Thus, by semi-simplicity considerations we have

$$J(W_{-1}\left(H^{\nu-1}(\eta_{V_t},\mathbb{Q})\otimes\frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)\right))\hookrightarrow J(H^{2r-1}(\overline{V}_t\times X,\mathbb{Q}(r))),$$

which by the Hodge conjecture, is cycle induced.

Therefore, we can consider the Abel Jacobi mapping to $J(H^{2r-1}(\overline{V}_t\times X,\mathbb{Q}(r)))$ since this doesn't alter the kernel. Thus $\nu_\xi(t)=0\in J(H^{2r-1}(\overline{V}_t\times X,\mathbb{Q}(r)))$, hence by Proposition 4.3.1, $\xi_t\in F^2\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}_t\times X,\mathbb{Q})$. Consider the case $k=\overline{\mathbb{Q}}$. We observe that, from Proposition 3.2.2, together with Corollary 4.3.2, this result tells us that for any arithmetic normal function

$$\nu_\xi: M\to\coprod J(H^{2r-1}(V_t\times X,\mathbb{Q}(r)))$$

defined by the aforementioned cycle ξ, $\nu_\xi(t)$ is zero as in (4.1) iff $\nu_\xi(\sigma(t))$ is zero as in (4.1), for all $\sigma\in Gal(\mathbb{C}/\overline{\mathbb{Q}})$.

Proof. Observe that, since $F^2\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}_t\times X,\mathbb{Q})$ is compatible with $Gal(\mathbb{C}/\overline{\mathbb{Q}})$, the kernel of the Abel-Jacobi map on $\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}_t\times X,\mathbb{Q})$ is Galois invariant. This also applies to the projectors defining the inclusions above.

□

In summary, we arrive at

Theorem 4.3.2. *Let X be a smooth projective variety with $\overline{\mathbb{Q}}$-spread $\rho:\mathcal{S}\times X=\mathscr{X}\to\mathcal{S}$. For any arithmetic normal function*

$$\nu_\xi: M\to\coprod J(H^{2r-1}(V_t\times X,\mathbb{Q}(r)))$$

defined by a cycle $\xi\in\mathcal{F}^\nu CH^r(\mathcal{S}\times X/\overline{\mathbb{Q}};\mathbb{Q})$ satisfying $\xi_t\in CH^r_{\mathrm{alg}}(V_t\times X;\mathbb{Q})$ for all $t\in M$, the zero locus of ν_ξ is a countable union of algebraic subvarieties over $\overline{\mathbb{Q}}$.

4.4 Studying the zero locus of arithmetic normal functions

The nature of the zero locus of the normal functions have been studied in several papers by Brosnan, Pearlstein and Schnell, for example in their work together [4]. The following is essentially taken from [15].

Theorem 4.4.1. *Let $\xi \in F^1 CH^r(X_{\mathcal{S}}, \mathbb{Q})$, with the zero locus of the associated normal function $\mathcal{Z}(\nu_\xi)$ defined over K/k (finitely generated). Then $\mathcal{Z}(\nu_\xi)$ is defined over a finite extension of k.*

Remark 4.4.1. *Note if $V = t \in S(\mathbb{C})$ is a point, then $H^0(V, R^{2r-1}\rho_{V,*}\mathbb{C}) = 0$ means that $H^{2r-1}(X_t, \mathbb{C}) = 0$, which would imply that all normal functions over $\mathcal{S}$ are zero. Thus this theorem is most useful when* $\dim V \geq 1$.

Proof. By [4], we know the zero locus is an algebraic subset of $\mathcal{S}$. Let $V \in \mathcal{Z}(\nu_\xi)$ be an irreducible component.

We recall the Gauss-Manin connection $\bigtriangledown = \partial \otimes 1$ which gives rise to the following flask resolution of sheaves:

$$R^{2r-1}\rho_*\mathbb{C} \hookrightarrow \mathcal{O}_S \otimes R^{2r-1}\rho_*\mathbb{C} \xrightarrow{\bigtriangledown} \Omega^1_S \otimes R^{2r-1}\rho_*\mathbb{C} \xrightarrow{\bigtriangledown} \cdots$$

From which we obtain the Gauss-Manin cohomology (For V, $H^i(V, R^{2r-i}\rho_*\mathbb{C})$), which can be described algebraically as follows:

Let $\{\Omega^\bullet_{X/\mathcal{S}}, d\}$ be the complex defined inductively by the sequence

$$0 \to \rho^*\Omega^1_{\mathcal{S}} \otimes \Omega^{\bullet-1}_{X/\mathcal{S}} \to \Omega^\bullet_X \to \Omega^\bullet_{X/\mathcal{S}} \to 0$$

where we can write $\Omega^p_X = \bigwedge^p \Omega^1_X$ and similarly $\Omega^p_{X/\mathcal{S}} = \bigwedge^p \Omega^1_{X/\mathcal{S}}$.

We can define the de Rham cohomology groups as

$$H^i_{DR}(X) := \mathbb{H}^i(X_{\text{Zar}}\Omega^\bullet_X) = H^i(\mathcal{S}_{\text{Zar}}, \mathbb{R}\rho_*\Omega^\bullet_X),$$

$$H^i_{DR}(X/S) := \mathbb{H}^i(X_{\text{Zar}}\Omega^\bullet_{X/S}) = H^i(\mathcal{S}_{\text{Zar}}, \mathbb{R}\rho_*\Omega^\bullet_{X/S})$$

where X_{Zar} and $\mathcal{S}_{\text{Zar}}$ denote the respective spaces in the Zariski topology. Now, if we define $F^m\Omega^p_X := \text{Im}(\Omega^m_{\mathcal{S}} \otimes \Omega^{p-m}_X \to \Omega^p_X)$, by the short exact sequence previously presented, we deduce that

$$\frac{F^m\Omega^p_X}{F^{m+1}\Omega^p_X} \simeq \Omega^m_{\mathcal{S}} \otimes \frac{\Omega^{p-m}_X}{\Omega^1_{\mathcal{S}} \otimes \Omega^{p-m-1}_X} \simeq \Omega^m_{\mathcal{S}} \otimes \Omega^{p-m}_{X/\mathcal{S}}$$

And we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{F^1\Omega^\bullet_X}{F^2\Omega^\bullet_X} & \longrightarrow & \frac{\Omega^\bullet_X}{F^2\Omega^\bullet_X} & \longrightarrow & \frac{\Omega^\bullet_X}{F^1\Omega^\bullet_X} & \longrightarrow & 0 \\
 & & \wr\| & & \| & & \wr\| & & \\
0 & \longrightarrow & \Omega^1_S \otimes \Omega^{\bullet-1}_{X/\mathcal{S}} & \longrightarrow & \frac{\Omega^\bullet_X}{F^2\Omega^\bullet_X} & \longrightarrow & \Omega^\bullet_{X/\mathcal{S}} & \longrightarrow & 0
\end{array}$$

From which, taking hypercohomology, we obtain the connecting homomorphism

$$\bigtriangledown : H^i_{DR}(X/\mathcal{S}) \to \Omega^1_S \otimes H^i_{DR}(X/\mathcal{S})$$

which is the Gauss-Manin connection. It can be extended to get

$$\bigtriangledown : \Omega^m_S \otimes H^i_{DR}(X/\mathcal{S}) \to \Omega^{m+1}_S \otimes H^i_{DR}(X/\mathcal{S})$$

Note that $\Omega^\bullet_{X/S}$ is a filtered complex with F^m. The corresponding spectral sequence is $E_1^{p,q} = \Omega^p_{\mathcal{S}} \otimes H^q_{DR}(X/\mathcal{S}) \Rightarrow H^{p+q}_{DR}(X_{\mathcal{S}}/k)$. This is really the Leray spectral sequence. We have analogously $E_1^{p,q} = \Omega^p_{\mathcal{S}} \otimes F^{r-p} H^q_{DR}(X/\mathcal{S}) \Rightarrow \mathbb{H}^{2r}(X_{\mathrm{Zar}} \Omega^{\bullet \geq r}_{X_{\mathcal{S}}/k})$.

Since the Gauss-Manin connection is algebraic, it commutes with the elements of $Gal(\mathbb{C}/k)$, that is, for any $\sigma \in Gal(\mathbb{C}/k)$ we have $\bigtriangledown \circ \sigma = \sigma \circ \bigtriangledown$. Now given a cycle ξ we denote by ξ_V its restriction to V, and by $[\xi_V]$ the corresponding fundamental class. Then, if we denote the action of σ over V by V^σ. By the compatibility of the Gauss-Manin connection with $Gal(\mathbb{C}/k)$ we see that, since the invariant part of $[\xi_V]$ is zero (because V is in the zero locus), the invariant part of $[\xi_{V^\sigma}]$ is also zero. We must show that V^σ is in the zero locus as well. But before doing so, see remark in passing that another possible way to prove that $[\xi_{V^\sigma}]$ is zero is by using the fact that the Leray spectral sequence is motivic, which was proved by Arapura [1]. With this we have that $Gal(\mathbb{C}/k)$ acts naturally on the Leray filtration with $\mathbb{C}$-coefficients as described above and we proceed similarly from here.

Now, for a given $\xi \in F^1\mathrm{CH}^r(X_{\mathcal{S}}, \mathbb{Q})$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & F^1\mathrm{CH}^r(X_{\mathcal{S}}, \mathbb{Q}) & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \underline{E}_\infty^{1,2r-1}(\rho) & \to & E_\infty^{1,2r-1}(\rho) & \to & \underline{\underline{E}}_\infty^{1,2r-1}(\rho) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{E}_\infty^{1,2r-1}(\rho_V) & \to & E_\infty^{1,2r-1}(\rho_V) & \to & \underline{\underline{E}}_\infty^{1,2r-1}(\rho_V) & \to & 0
\end{array}$$

where $\rho : X_{\mathcal{S}} \to \mathcal{S}$ is the spread with which the normal function associated to ξ is defined, and ρ_V denotes the restriction to V.

By hypothesis, there are no global sections over V, and thus $\underline{E}_\infty^{1,2r-1}(\rho_V) \simeq J(H^0(V, R^{2r-1}\rho_{V*}\mathbb{Q}(r)))$ is zero.

We also have $\underline{\underline{E}}_{\infty}^{1,2r-1}(\rho_V) = \Gamma(H^1(V, R^{2r-1}\rho_{V*}\mathbb{Q}(r)))$ and thus the lower part of the diagram gives us the short exact sequence

$$0 \to J(H^0(V, R^{2r-1}\rho_{V*}\mathbb{Q}(r))) \to E_{\infty}^{1,2r-1}(\rho_V) \to \Gamma(H^1(V, R^{2r-1}\rho_{V*}\mathbb{Q}(r))) \to 0$$

with the left term being zero, from which we see that (by our construction of normal functions) the restriction of the normal function ν_ξ to V takes values on $\Gamma(H^1(V, R^{2r-1}\rho_{V*}\mathbb{Q}(r)))$.

By our previous observations, and since the actions of the elements of $Gal(\mathbb{C}/k)$ take flat sections to flat sections, we get the exact sequence:

$$H^0(V^\sigma, R^{2r-1}\rho_{V^\sigma_*}\mathbb{C}) \to E_{\infty}^{1,2r-1}(\rho_{V^\sigma}) \to \Gamma(H^1(V^\sigma, R^{2r-1}\rho_{V^\sigma_*}\mathbb{Q}(r))) \to 0$$

where the left term is zero again. Then, the action of σ takes V to another component $V^\sigma \in \mathcal{Z}(\nu_\xi)$. But since we are working with $\overline{\mathbb{Q}}$ (or a finitely generated k), $\mathcal{Z}(\nu_\xi)$ has only a finite number of components, and since $Gal(\mathbb{C}/k)$ is uncountable, $\mathcal{Z}(\nu_\xi)$ must necessary be defined over a finite extension of k.

$\square$

For the general case involving arithmetic normal functions, if we substitute the hypothesis that $H^0(V, R^{2r-1}\rho_{V*}\mathbb{C}) = 0$ for V in the zero locus (in other words, that there are no global sections over the zero locus) with a similar condition, then we can carry the proof in a similar way. Let us assume given a smooth morphism $\pi : \mathcal{S} \to M$ over k, and where $\pi^{-1}(t)$ is smooth affine for $t \in M(\mathbb{C})$. Then technically speaking, $V \subset M(\mathbb{C})$. We put $\mathbf{V} := \pi^{-1}(V) \subset \mathcal{S}(\mathbb{C})$, and set $\rho_{\mathbf{V}} = \rho\big|_{\mathbf{V}} : X_{\mathbf{V}} \to \mathbf{V} \subset \mathcal{S}(\mathbb{C})$. Our assumption then is $H^{\nu-1}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{C}) = 0$.

For a given $\xi \in F^\nu \mathrm{CH}^r(X_{\mathcal{S}}, \mathbb{Q})$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & F^\nu \mathrm{CH}^r(X_{\mathcal{S}}, \mathbb{Q}) & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \underline{E}_{\infty}^{\nu,2r-\nu}(\rho) & \to & E_{\infty}^{\nu,2r-\nu}(\rho) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) & \to & E_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) & \to & 0
\end{array}$$

where, again, $\rho : X_{\mathcal{S}} \to \mathcal{S}$ is the spread with which the normal function associated to ξ is defined, and $\rho_{\mathbf{V}}$ denotes the restriction to $\mathbf{V}$.

Since $\underline{E}_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) \subset J(H^{\nu-1}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{Q}(r)))$ where, by hypothesis, the latter part is zero, and recalling that $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) = \Gamma(H^{\nu}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{Q}(r)))$, the bottom part of the previous diagram becomes

$$H^{\nu-1}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{C}) \to E_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}}) \to \Gamma(H^{\nu}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{Q}(r))) \to 0$$

with the left term being zero, from which we see that (by our construction of arithmetic normal functions) the restriction of the arithmetic normal function ν_{ξ} to $\mathbf{V}$ takes its value in $\Gamma(H^{\nu}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{Q}(r)))$, which must be zero as $\mathbf{V}$ is a component of the zero locus of ν_{ξ}.

By our previous observations, $H^{\nu-1}(\mathbf{V}^{\sigma}, R^{2r-\nu}\rho_{\mathbf{V}^{\sigma}_{*}}\mathbb{C}) = 0$ and since the actions of the elements of $Gal(\mathbb{C}/k)$ take flat sections to flat sections we have $E_{\infty}^{\nu,2r-\nu}(\rho_{\mathbf{V}^{\sigma}}) \simeq \Gamma(H^{\nu}(\mathbf{V}^{\sigma}, R^{2r-\nu}\rho_{\mathbf{V}^{\sigma}_{*}}\mathbb{Q}(r))$. As with the earlier case, another approach to this uses the fact that the Leray spectral sequence is motivic ([1]), with the Galois action being natural on the Leray filtration. Then, on $M(\mathbb{C})$, the action of σ takes V to another component V^{σ} of $\mathcal{Z}(\nu_{\xi})$. But since we are working with $\overline{\mathbb{Q}}$ (or a finitely generated $k/\mathbb{Q}$), $\mathcal{Z}(\nu_{\xi})$ has only a finite number of components, and since $Gal(\mathbb{C}/k)$ is uncountable, $\mathcal{Z}(\nu_{\xi})$ must necessary be defined over a finite extension of k. Thus, we conclude the proof as in Theorem 4.4.1.

Proposition 4.4.2. *For $\xi \in F^{\nu}CH^{r}(X_K, \mathbb{Q})$, with zero locus of the associated normal function $\mathcal{Z}(\nu_{\xi})$ defined over K/k (finitely generated) and assume that $H^{\nu-1}(\mathbf{V}, R^{2r-\nu}\rho_{\mathbf{V}*}\mathbb{C}) = 0$ for any $V \in \mathcal{Z}(\nu_{\xi})$. Then $\mathcal{Z}(\nu_{\xi})$ is defined over a finite extension of k.*

4.5 The situation restricting to a particular subspace

We recall our setting of an arithmetic normal function ν_{ξ} for any cycle $\xi \in Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^{r}(X_K;\mathbb{Q})$:

$$\nu_{\xi} : M(\mathbb{C}) \longrightarrow \coprod \frac{J(W_{-1}\left[H^{\nu-1}(\eta_{V_t}, R^{2r-\nu}\rho_{V_t,*}\mathbb{Q}(r))\right])}{\Gamma(Gr_W^0\left[H^{\nu-1}(\eta_{V_t}, R^{2r-\nu}\rho_{V_t,*}\mathbb{Q}(r))\right])}$$

where M is the aforementioned family of $\{V_t\}_{t\in M(\mathbb{C})} \subset S(\mathbb{C})$ of smooth, irreducible, closed subvarieties of dimension $\nu - 1$.

Let

$$\underline{\mathrm{CH}}^r_{alg}(X_K;\mathbb{Q}) := \mathrm{image}\big(\varinjlim_{U\cap V,\ U\subset S} \mathrm{CH}^r_{\mathrm{alg}}(\mathscr{X}_U;\mathbb{Q}) \to \mathrm{CH}^r(X_K;\mathbb{Q})\big)$$

Lemma 4.5.1.

$$\underline{CH}^r_{\mathrm{alg}}(X_K;\mathbb{Q}) = \mathrm{image}\big(CH^r_{\mathrm{alg}}(\overline{\mathscr{X}};\mathbb{Q}) \to CH^r(X_K;\mathbb{Q})\big),$$

where $\overline{\mathscr{X}}$ is any smooth compactification of $\mathscr{X}$.

We first use the well-known:

Lemma 4.5.2. *Let $W_1,\ W_2$ be smooth projective varieties of the same dimension over $k = \overline{k}$, and $f : W_1 \to W_2$ a generically finite morphism of degree N. Then $f_* : CH^r_{\mathrm{alg}}(W_1) \to CH^r_{\mathrm{alg}}(W_2)$ is surjective.*

Proof. We have $f_*f^*\mathrm{CH}^r_{\mathrm{alg}}(W_1) = N \cdot \mathrm{CH}^r_{\mathrm{alg}}(W_2)$. Indeed, we can guarantee that there is a nonempty Zariski open subset U of W_2 for which $F^{-1}(y)$ consist of N points for every $y \in U$, in particular, for U small enough we have $f_*(1) = N$. Then, by the projection formula, $f_*f^*(\xi) = \xi f_*(1) = N\xi$. Using divisibility of $\mathrm{CH}^r_{\mathrm{alg}}(W)$ for any smooth projective W/k, we have $\mathrm{CH}^r_{\mathrm{alg}}(W_2) = N \cdot \mathrm{CH}^r_{\mathrm{alg}}(W_2)$. We conclude

$$\mathrm{CH}^r_{\mathrm{alg}}(W_2) = N \cdot \mathrm{CH}^r_{\mathrm{alg}}(W_2) = f_*f^*\mathrm{CH}^r_{\mathrm{alg}}(W_2) = f_*\mathrm{CH}^r_{\mathrm{alg}}(W_1).$$

□

Proof of lemma 4.5.1 . Now let $\overline{\mathscr{X}}_1$ and $\overline{\mathscr{X}_2}$ be two smooth compactifications of $\mathscr{X}$. Then we can find a smooth compactification $\overline{\mathscr{X}}'$ that dominates them. The push-forward maps then induce $\phi_{*,1} : \mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}}') \to \mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}}_1)$ and $\phi_{*,2} : \mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}}') \to \mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}}_2)$. By the above lemma, these maps are surjective. Then for $i = 1, 2$ we have

$$\mathrm{image}\big(\mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}}';\mathbb{Q}) \to \mathrm{CH}^r(X_K;\mathbb{Q})\big) = \mathrm{image}\big(\mathrm{CH}^r_{\mathrm{alg}}(\overline{\mathscr{X}_i};\mathbb{Q}) \to \mathrm{CH}^r(X_K;\mathbb{Q})\big)$$

And the result follows. □

Now we want to prove

Theorem 4.5.1.

$$F^\nu \underline{CH}^r_{alg}(X_K;\mathbb{Q}) =$$
$$\ker\left(AJ : F^{\nu-1}\underline{CH}^r_{alg}(X_K;\mathbb{Q}) \to \frac{J(W_{-1}\left[H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right])}{\Gamma(Gr^0_W\left[H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right])}\right)$$

for a very general V of dimension $\nu - 1$ corresponding to a very general point $t \in M(\mathbb{C})$. Moreover this is independent of any Galois conjugate of V.

Proof. Since $W_\bullet$ is an exact functor, we observe that

$$W_{-1}\left(\frac{H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q})}{H^{\nu-1}(\eta_V, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q})}(r)\right) = \frac{W_{-1}(H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{W_{-1}(H^{\nu-1}(\eta_V, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$

and thus, we see that

$$W_{-1}\left(\frac{H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q})}{H^{\nu-1}(\eta_V, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q})}(r)\right)) \hookrightarrow W_{-1}(H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r)))$$

by weight and semi-simplicity reasons. Now, since the minimum weight of $H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))$ is -1, we can deal with this in a similar way: that is, for any $V \subset S(\mathbb{C})$ smooth, irreducible, closed subvariety of dimension $\nu - 1$, we have the sequence:

$$H^{\nu-1}_{\overline{V}\backslash V}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to H^{\nu-1}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow W_{-1}H^{\nu-1}(V, R^{2r-\nu}\rho_*\mathbb{Q}(r))$$

which taking direct limit gives us

$$0 \to N^1H^{\nu-1}(\bar{V}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to H^{\nu-1}(\bar{V}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to W_{-1}H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r))$$

so we obtain

$$W_{-1}H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \simeq \frac{H^{\nu-1}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q})}{N^1_{\overline{V}}H^{\nu-1}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q})}(r)$$

where we observe, by semi-simplicity reasons, that

$$\frac{H^{\nu-1}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q})}{N^1_{\overline{V}}H^{\nu-1}(\overline{V}, R^{2r-\nu}\rho_*\mathbb{Q})}(r) \hookrightarrow H^{2r-1}(\overline{\mathscr{X}}_{\overline{V}}, \mathbb{Q}(r))$$

and so we have

$$J(W_{-1}\left(\frac{H^{\nu-1}(\eta_V, R^{2r-\nu}\rho_*\mathbb{Q})}{H^{\nu-1}(\eta_V, N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q})}(r)\right)) \hookrightarrow J(H^{2r-1}(\overline{\mathscr{X}}_{\overline{V}}, \mathbb{Q}(r))).$$

The rest of the proof follows from the ideas behind Theorem 4.3.2. □

4.6 Incidence equivalence

Given a cycle $\xi \in \mathrm{CH}^r_{\mathrm{alg}}(X)$, we say that it is incident equivalent to zero if for all smooth projective varieties $\mathcal{S}$ and all cycles $w \in \mathrm{CH}^{n-r+1}(\mathcal{S} \times X)$ we have $Pr_1((\mathcal{S} \times \xi) \bullet w) = 0$. We observe that this induces the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^r_{\mathrm{alg}}(X) & \xrightarrow{w^*} & \mathrm{CH}^1_{\mathrm{alg}}(\mathcal{S}) \\ AJ \downarrow & & \downarrow \simeq \\ J^r(X) & \xrightarrow{[w]^*} & J(\mathcal{S}) \end{array}$$

From which we see that $\mathrm{CH}^r_{\mathrm{alg}}(X)_{AJ}$ is contained in $\mathrm{CH}^r_{\mathrm{alg}}(X)_{inc}$, where $\mathrm{CH}^r_{\mathrm{alg}}(X)_{\mathrm{inc}}$ and $\mathrm{CH}^r_{\mathrm{alg}}(X)_{AJ}$ denote the elements of $\mathrm{CH}^r_{\mathrm{alg}}(X)$ that are incident equivalent to zero and Abel-Jacobi equivalent to zero (that is, contained in the kernel of AJ) respectively.

We can find a complete curve C and a cycle $z \in \mathrm{CH}^r(C \times X, \mathbb{Q})$ such that its composition with the Abel-Jacobi map

$$AJ \circ [z]_* : \mathrm{CH}^1_{\mathrm{alg}}(C, \mathbb{Q}) \to J_{\mathrm{alg}}(H^{2r-1}(X, \mathbb{Q}(r)))$$

is surjective. This produces the following commutative diagram.

$$\begin{array}{ccc} \mathrm{CH}^1_{\mathrm{alg}}(C, \mathbb{Q}) & \xrightarrow{z_*} & \mathrm{CH}^r_{\mathrm{alg}}(X, \mathbb{Q}) \\ AJ \downarrow & & AJ \downarrow \\ J^1(C) & \xrightarrow{[z]_*} & J_{\mathrm{alg}}(H^{2r-1}(X, \mathbb{Q}(r))) \end{array}$$

where the horizontal map in the bottom is induced by $H^1(C, \mathbb{Q}) \to H^{2r-1}(X, \mathbb{Q})$.

Let L_X be the operator of taking cup product with the hyperplane class on X. The strong Lefschetz theorem tells us that $L_X^{n-i} : H^i(X, \mathbb{Q}) \xrightarrow{\sim} H^{2n-i}(X, \mathbb{Q})$, for $i \leq n$. Now, assuming that the inverse $\Lambda_X^{n-i} : H^{2n-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X, \mathbb{Q})$ is algebraic, we have the isomorphism $L_X^{n-i} : N^p H^i(X, \mathbb{Q}) \xrightarrow{\sim} N^{p+n-i} H^{2n-i}(X, \mathbb{Q})$ with inverse Λ_X^{n-i}.
It follows from this (by Hodge-Riemann bilinear relations, using the fact that it is closed under the Lefschetz decomposition) that the cup product pairing

$N^pH^i(X,\mathbb{Q}) \times N^{p+n-i}H^{2n-i}(X,\mathbb{Q}) \to H^{2n}(X,\mathbb{Q}) \simeq \mathbb{Q}$ is nondegenerate and we can write $(N^pH^i(X,\mathbb{Q}))^\vee \simeq N^{p+n-i}H^{2n-i}(X,\mathbb{Q})$.

Now, focusing on the $(1, 2r-1)$ Künneth component of z we have the map

$$[z]_* : H^1(C,\mathbb{Q}(1)) \twoheadrightarrow N^{r-1}H^{2r-1}(X,\mathbb{Q}(r))$$

that when dualized gives us

$$[z]^* : N^{n-r}H^{2n-2r+1}(X,\mathbb{Q}(n-r)) \hookrightarrow H^1(C,\mathbb{Q})$$

Tensoring with $\mathbb{Q}(1)$ this becomes

$$[z]^* : N^{n-r}H^{2(n-r+2)-1}(X,\mathbb{Q}(n-r+1)) \hookrightarrow H^1(C,\mathbb{Q}(1))$$

Then we observe that

$$J_{\text{alg}}(H^1(C,\mathbb{Q}(1))) = J(H^1(C,\mathbb{Q}(1))) = J(N^0H^1(C,\mathbb{Q}(1)))$$

and $J_{\text{alg}}(H^{2r-1}(X,\mathbb{Q}(r))) = J(N^{r-1}H^{2r-1}(X,\mathbb{Q}(r)))$, so we construct the following commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^{n-r+1}_{\text{alg}}(X,\mathbb{Q}) & \xrightarrow{z^*} & \mathrm{CH}^1_{\text{alg}}(C,\mathbb{Q}) \\ \downarrow AJ & & AJ \downarrow\simeq \\ J(N^{n-r}H^{2(n-r+1)-1}(X,\mathbb{Q}(n-r+1))) & \overset{[z]^*}{\hookrightarrow} & J(H^1(C,\mathbb{Q}(1))) \end{array}$$

And since $n-r+1$ depends only on r, we can construct this diagram replacing $n-r+1$ with r:

$$\begin{array}{ccc} \mathrm{CH}^{r}_{\text{alg}}(X,\mathbb{Q}) & \xrightarrow{z^*} & \mathrm{CH}^1_{\text{alg}}(C,\mathbb{Q}) \\ \downarrow AJ & & AJ \downarrow\simeq \\ J(N^{r-1}H^{2r-1}(X,\mathbb{Q}(r))) & \overset{[z]^*}{\hookrightarrow} & J(H^1(C,\mathbb{Q}(1))) \end{array}$$

Then, any $\xi \in \mathrm{CH}^r_{\text{alg}}(X,\mathbb{Q})_{\text{inc}}$ will be mapped algebraically to zero in $\mathrm{CH}^1_{\text{alg}}(C,\mathbb{Q})$. Thus, ξ is mapped to zero in $J_{\text{alg}}(H^{2r-1}(X,\mathbb{Q}))$ meaning that $\xi \in \mathrm{CH}^r_{\text{alg}}(X)_{AJ}$.

We conclude that $\mathrm{CH}^r_{\mathrm{alg}}(X)_{\mathrm{inc}} = CH^r_{\mathrm{alg}}(X)_{AJ}$.

Now, like in the previous sections, let $\rho : X_{\mathcal{S}} \to \mathcal{S}$ be the spread with which the normal function associated to $\xi \in F^1\mathrm{CH}^r(X_{\mathcal{S}}, \mathbb{Q})$ is defined. Then, if $t \in \mathcal{S}(\mathbb{C})$ we can write $\xi_t \in \mathrm{CH}^r_{\mathrm{alg}}(X_t; \mathbb{Q})$ for the corresponding intersection.

Suppose that $\nu_\xi(t) = 0$ for a general t, in other words, that the Abel-Jacobi map is zero for the general fiber. Then ξ_t is mapped to zero in $J(H^0(\{t\}, R^{2r-1}\rho_*\mathbb{Q}(r)))$ and thus $\xi_t \in \mathrm{CH}^r_{\mathrm{alg}}(X_t)_{AJ}$.

It is known that $Gal(\mathbb{C}/k)$ acts on $\mathrm{CH}^r_{\mathrm{alg}}(X)_{\mathrm{inc}}$ so, for $\sigma \in Gal(\mathbb{C}/k)$ we have $\mathrm{CH}^r_{\mathrm{alg}}(X_t)_{\mathrm{inc}} \xrightarrow{\sim} \mathrm{CH}^r_{\mathrm{alg}}(X_{\sigma(t)})_{\mathrm{inc}}$ with the correspondence $\xi_t \mapsto \xi_{\sigma(t)}$. If V denotes the k-Zariski closure of t in $\mathcal{S}$, we see that $\nu_\xi(t) = 0$ for all general $t \in V$ and therefore ν_ξ will be the zero normal function.

We conclude that if the general fiber of ξ is in the kernel of the Abel-Jacobi map, then the associated normal function is zero. Since the converse is obvious (recalling that the image of the normal function lands on a variation of jacobians), we have that the general fiber is in the kernel of AJ iff the normal function is zero. We recall that ξ comes from a class in $\mathrm{CH}^r_{\mathrm{alg}}(X; \mathbb{Q})$ so this gives an alternate proof that $F^2\mathrm{CH}^r_{\mathrm{alg}}(X; \mathbb{Q}) = \mathrm{CH}^r_{\mathrm{alg}}(X; \mathbb{Q})_{AJ}$, based on the hard Lefschetz conjecture assumption. Since the BB filtration already requires the hard Lefschetz conjecture assumption, this becomes a moot issue.

4.7 Incidence equivalence and product case (revisited)

Let X be a projective algebraic variety of dimension n. To simplify notation, for general $t \in M$ let us write $V = V_t$. We can find a complete curve C and a cycle $z \in \mathrm{CH}^{n+\nu-r}(C \times \overline{V} \times X; \mathbb{Q})$ such that its composition with the Abel-Jacobi map $AJ \circ [z]_* : \mathrm{CH}^1(C; \mathbb{Q}) \to J_{\mathrm{alg}}(H^{2(n-r+\nu)-1}(\overline{V} \times X, \mathbb{Q}(n+\nu-r)))$ is surjective. This produces the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{CH}^1_{\mathrm{alg}}(C; \mathbb{Q}) & \xrightarrow{z_*} & \mathrm{CH}^{n-r+\nu}_{\mathrm{alg}}(\overline{V} \times X, \mathbb{Q}) \\
AJ \downarrow & & \downarrow AJ \\
J(H^1(C, \mathbb{Q}(1))) & \overset{[z]_*}{\twoheadrightarrow} & J_{\mathrm{alg}}(H^{\nu-1}(\overline{V}, \mathbb{Q}) \otimes H^{2(n-r)+\nu}(X, \mathbb{Q})(n-r+\nu))
\end{array}$$

where the bottom map is induced by $H^1(C,\mathbb{Q}) \to H^{2(n-r+\nu)-1}(\overline{V}\times X,\mathbb{Q})$.

We see that the map

$$[z]_* : H^1(C,\mathbb{Q}(1)) \twoheadrightarrow N^{n+\nu-r-1}H^{2(n-r+\nu)-1}(\overline{V}\times X,\mathbb{Q}(n-r+\nu))$$

when dualized gives us

$$[z]^* : N^{r-1}H^{2r-1}(\overline{V}\times X,\mathbb{Q}(r-1)) \hookrightarrow H^1(C,\mathbb{Q})$$

Tensoring with $\mathbb{Q}(1)$ this becomes

$$[z]^* : N^{r-1}H^{2r-1}(\overline{V}\times X,\mathbb{Q}(r)) \hookrightarrow H^1(C,\mathbb{Q}(1))$$

By the Künneth formula, we arrive at:

$$N^{r-1}\big(H^{\nu-1}(\overline{V},\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})(r)\big) \hookrightarrow H^1(C,\mathbb{Q}(1))$$

and observing that

$$J_{\mathrm{alg}}(H^{\nu-1}(\overline{V},\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})(r)) = J\big(N^{r-1}\big(H^{\nu-1}(\overline{V},\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})(r)\big)\big),$$

we construct the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}\times X,\mathbb{Q}) & \xrightarrow{z^*} & \mathrm{CH}^1_{\mathrm{alg}}(C,\mathbb{Q}) \\
\downarrow AJ & & AJ \ \downarrow\simeq \\
J(N^{r-1}H^{2r-1}(\overline{V}\times X,\mathbb{Q}(r)) & \overset{[z]^*}{\hookrightarrow} & J(H^1(C,\mathbb{Q}(1)))
\end{array}$$

We observe that, by Lemma 4.3.1, the map $V\times X \xrightarrow{\text{closure}} \overline{V}\times X$ induces $\mathrm{CH}^r_{\mathrm{alg}}(\overline{V}\times X,\mathbb{Q}) \twoheadrightarrow \mathrm{CH}^r_{\mathrm{alg}}(V\times X,\mathbb{Q})$. Thus, we can extend the diagram as follows:

$$\begin{array}{ccccc}
\mathrm{CH}^r_{\mathrm{alg}}(V\cap\eta_{\mathcal{S}}\times X,\mathbb{Q}) & \twoheadleftarrow & \mathrm{CH}^r_{\mathrm{alg}}(\overline{V}\times X,\mathbb{Q}) & \longrightarrow & CH^1_{alg}(C,\mathbb{Q}) \\
\underline{AJ}\ \downarrow & & \downarrow AJ & & AJ\ \downarrow\simeq \\
J_{\mathrm{alg}}(\Lambda) & \overset{\hookrightarrow}{\twoheadleftarrow} & J\big(N^{r-1}\big(H^{2r-1}(\overline{V}\times X,\mathbb{Q})(r)\big)\big) & \hookrightarrow & J(H^1(C,\mathbb{Q}))
\end{array}$$

where $\underline{AJ}$ is the composition of the Abel-Jacobi map with the $(2n-2r+\nu, 2r-\nu)$ Künneth projector,

$$\Lambda := N^{r-1}\left(\frac{H^{\nu-1}(\overline{V},\mathbb{Q})}{N^1_{S/\overline{\mathbb{Q}}}H^{\nu-1}(\bar{V},\mathbb{Q})}\otimes\frac{H^{2r-\nu}(X,\mathbb{Q})}{N^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)\right),$$

and where we view

$$\frac{H^{\nu-1}(\overline{V},\mathbb{Q})}{N^1_{\mathcal{S}/\overline{\mathbb{Q}}}H^{\nu-1}(\overline{V},\mathbb{Q})}\otimes\frac{H^{2r-\nu}(X,\mathbb{Q})}{N^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}(r)$$

imbedded in

$$H^{2r-1}(\overline{V}\times X,\mathbb{Q}(r)),$$

via semi-simplicity considerations.

Now consider $\xi \in \mathrm{CH}^r_{\mathrm{alg}}(V\cap\eta_{\mathcal{S}}\times X;\mathbb{Q})$ as arising from a class in $F^\nu\underline{\mathrm{CH}}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$. It naturally maps to $J_{\mathrm{alg}}(\Lambda)$ via $\underline{AJ}$. ξ is also in the image of some $\overline{\xi} \in \mathrm{CH}^r_{\mathrm{alg}}(\overline{V}\times X;\mathbb{Q})$, and we observe that $\underline{AJ}(\xi) = AJ(\overline{\xi}) \in J_{\mathrm{alg}}(\Lambda)$. Indeed, using the same construction as the projector in Proposition 4.1.1 we get the map

$$H^{\nu-1}(\overline{V},\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})(r)\to N^{r-1}\big(H^{\nu-1}(\overline{V},\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})(r)\big)$$

which is cycle induced if we assume the Hodge conjecture. Then applying the Abel-Jacobi maps takes $AJ(\overline{\xi})$ to the algebraic Jacobian

$$J\big(N^{r-1}\big(H^{2r-1}(\overline{V}\times X,\mathbb{Q})(r)\big)\big)$$

Similarly, $AJ(\xi)$ lies inside $J_{\mathrm{alg}}(\Lambda)$. Then, using the $(2n-2r+\nu, 2r-\nu)$ Künneth projector, we can identify $J_{\mathrm{alg}}(\Lambda)$ embedded in $J\big(N^{r-1}\big(H^{2r-1}(\overline{V}\times X,\mathbb{Q})(r)\big)\big)$ so by cycle induced projectors and left inverses, we can assume the image of both $\underline{AJ}(\xi)$ and $AJ(\overline{\xi})$ is the same in $J(H^{2r-1}(\overline{V}\times X,\mathbb{Q}(r)))$ and thus in $J_{\mathrm{alg}}(\Lambda)$. With this arrangement, we see that $\underline{AJ}(\xi)=0$ if and only if $AJ(\overline{\xi})=0$, but the right side of the diagram implies that this is equivalent as $\overline{\xi}\in\mathrm{CH}^r_{\mathrm{inc}}(\overline{V}\times X;\mathbb{Q})$. Since ξ comes from a class in $F^\nu\underline{\mathrm{CH}}_{\mathrm{alg}}(X_K;\mathbb{Q})$, $\mathrm{Gal}(\mathbb{C}/k)$ acts on all these objects.

Now let $t\in M$ with $V_t\subset\mathcal{S}(\mathbb{C})$. Let us write $\xi_t\in\mathrm{CH}^r_{\mathrm{alg}}(V_t\times X;\mathbb{Q})$ for the corresponding intersection. Suppose that $\nu_\xi(V_t)=0$ for a general t, in other words, that the Abel-Jacobi map is zero for the general fiber. Then ξ_t is mapped to zero in $J(H^{\nu-1}(V_t\times X,\mathbb{Q}(r)))$ and thus $\xi_t\in\mathrm{CH}^r_{\mathrm{alg}}(V_t\times X,\mathbb{Q})_{AJ}$. But then we have $\xi_t\in\mathrm{CH}^r_{\mathrm{alg}}(V_t\times X,\mathbb{Q})_{\mathrm{inc}}\simeq\mathrm{CH}^r_{\mathrm{alg}}(V_{\sigma(t)}\times X,\mathbb{Q})_{\mathrm{inc}}\ni\xi_{\sigma(t)}$. Let Z denote the k-Zariski closure of V_t in the zero locus of the arithmetic normal function. We see that $\nu_\xi(V_t)=0$ for all general $V_t\in Z$ and therefore ν_ξ will be the zero normal function.

In the same way that in the previous section, we have that a general fiber is in the kernel of AJ iff the normal function is zero. Since ξ comes form a class in $F^{\nu}\underline{\mathrm{CH}}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$, we conclude that a general fiber of ξ is in the kernel of AJ if and only if $\xi \in F^{\nu+1}\underline{\mathrm{CH}}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$.

Proposition 4.7.1. *Given $\xi \in F^{\nu}\underline{CH}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$, we have that $\xi \in F^{\nu+1}\underline{CH}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$ if and only if any given general fiber of ξ is in the kernel of AJ.*

Chapter 5

Summary results and the bigger picture

We focus in the following diagram:

$$\begin{array}{ccc} \mathscr{X} & \hookrightarrow & \overline{\mathscr{X}} \\ \rho \Big\downarrow & & \Big\downarrow \overline{\rho} \\ \mathcal{S} & \hookrightarrow & \overline{\mathcal{S}}, \end{array} \tag{5.1}$$

where all varieties are defined over $\overline{\mathbb{Q}}$, Fix a polarization $\mathcal{O}_{\overline{\mathcal{S}}}(1)$ of $\overline{\mathcal{S}}$, and for integers $\{d_1 \leq d_2 \leq \cdots \leq d_{N-1} \leq d_N\}$, where $N := \dim \overline{\mathcal{S}}$, consider the variety of full flags:

$$\mathcal{F}(\overline{\mathcal{S}}) := \big\{V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_j \subsetneq V_{N-1} \subsetneq \overline{\mathcal{S}}(\mathbb{C})\big\},$$

where $V_N := \overline{\mathcal{S}}(\mathbb{C})$ and for $j < N$, $V_j \subset \mathcal{S}(\mathbb{C})$ is a j-dimensional complete intersection of multi-degree $(d_1, ..., d_{N-j})$. With this setting, $\mathcal{F}(\overline{\mathcal{S}})$ is defined over $\overline{\mathbb{Q}}$. We can restrict $\mathcal{F}(\overline{\mathcal{S}})$ to define $\mathcal{F}(\mathcal{S})$, as well as at the generic point $\mathcal{F}(\eta_{\mathcal{S}})$, where $\eta_{\mathcal{S}}$ denotes the generic point. Observe that very general points of $\mathcal{F}(\mathcal{S})$, survive under restriction to $\mathcal{F}(\eta_{\mathcal{S}})$. For such a very general point, each V_j is irreducible and smooth (Bertini's theorem states this for any general hyperplane section not equal to $\mathcal{S}$), except possibly for V_0. By Bezout theorem, the number of intersection points of the hypersurfaces is equal to the product of their degrees (counting multiplicities), so V_0 will consist of $\deg \overline{\mathcal{S}} \cdot \prod_{j=1}^{N} d_j$ distinct and very general points.

Now let $X/\mathbb{C}$ be a smooth projective variety and $\xi \in \mathrm{CH}^r(X/\mathbb{C}; \mathbb{Q})$. Recall that there is a finitely generated field extension K with $\mathbb{C} \supset K \supset \overline{\mathbb{Q}}$, such that $X/\mathbb{C} = X_K \times_K \mathbb{C}$ and $\xi \in \mathrm{CH}^r(X_K; \mathbb{Q})$.

Also note that $\mathrm{CH}^r(X_K;\mathbb{Q}) \hookrightarrow \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$ with

$$F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) = \left\{F^\nu \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})\right\} \cap \mathrm{CH}^r(X_K;\mathbb{Q})$$

Now spread X_K out as in diagram (5.1), and accordingly let $\tilde{\xi} \in \mathrm{CH}^r(\mathscr{X}/\overline{\mathbb{Q}};\mathbb{Q})$ be the spread of ξ such that $\tilde{\xi} \mapsto \xi \in \mathrm{CH}^r(X_K;\mathbb{Q})$. Then K is given by an embedding $\overline{\mathbb{Q}}(\mathcal{S}) \hookrightarrow \mathbb{C}$, defined by evaluation at a very general point p, and hence $X_K = \mathscr{X}_p$.

Now choose a very general point in $\mathcal{F}(\mathcal{S})$ corresponding to a sequence:

$$p \in V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{N-1} \subsetneq V_N := \mathcal{S}(\mathbb{C}).$$

For simplicity, let us take $p = V_0$, and put $V_{\bullet<0} = \emptyset$. Then, by definition, $\mathscr{X}_{\eta_\mathcal{S} \cap V_0} = \mathscr{X}_p = X_K$. Next, it is clear that by restriction,

$$\gamma = 0 \in H^{2r}(\mathscr{X}_{\eta_\mathcal{S} \cap V_j}, \mathbb{Q}(r)) \Rightarrow \gamma = 0 \in H^{2r}(\mathscr{X}_{\eta_\mathcal{S} \cap V_i}, \mathbb{Q}(r)) \text{ for } i < j.$$

and using the functoriality of the Abel-Jacobi map, we have

$$[\gamma] = 0 \in J(H^{2r-1}(\mathscr{X}_{\eta_\mathcal{S} \cap V_j}, \mathbb{Q}(r))) \Rightarrow [\gamma] = 0 \in J(H^{2r-1}(\mathscr{X}_{\eta_\mathcal{S} \cap V_i}, \mathbb{Q}(r))) \text{ for } i < j.$$

And regarding the converse,

Conjecture 5.0.2 (Lewis)**.** Let $\xi \in \mathrm{CH}^r(X_K;\mathbb{Q})$.
Then $\xi \in F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) \Leftrightarrow$

(i) $\big[\tilde{\xi}\big|_{\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-1}}}\big] = 0 \in H^{2r}(\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-1}}, \mathbb{Q}(r))$, and

(ii) $AJ\big(\tilde{\xi}\big|_{\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-2}}}\big) = 0 \in J\big(H^{2r-1}(\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-2}}, \mathbb{Q}(r))\big)$.

But, using the fact that a topological invariant is defined by its normal function, and

Assumption 5.0.3. *Assume the zero locus of a cycle induced normal function respects the field of definition of the cycle.*

we can write instead:

Conjecture 5.0.4 (Lewis, Version II)**.** Put $F^0\mathrm{CH}^r(X_K;\mathbb{Q}) = \mathrm{CH}^r(X_K;\mathbb{Q})$ and $F^1\mathrm{CH}^r(X_K;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X_K;\mathbb{Q})$. Let $\xi \in \mathrm{CH}^r(X_K;\mathbb{Q})$. Then for $\nu \geq 2$,
$\xi \in F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) \Leftrightarrow$

(i) $\xi \in F^{\nu-1}\mathrm{CH}^r(X_K;\mathbb{Q})$, and

(ii) $AJ\big(\tilde{\xi}\big|_{\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-2}}}\big) = 0 \in J\big(H^{2r-1}(\mathscr{X}_{\eta_\mathcal{S} \cap V_{\nu-2}}, \mathbb{Q}(r))\big)$.

Proposition 5.0.5. *Assuming the above assumption, then both conjectures are equivalent.*

Proof. For $\nu = 0$, and $\nu = 1$, the result is immediate by Theorem 3.2.1, so we proceed by induction. Assume the conjectures are equivalent for $\nu - 1$, and we have $\nu \geq 2$. If we have $\big[\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-1}}}\big] = 0$ and $AJ\big(\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-2}}}\big) = 0$ then, as noted before, we have $\big[\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-2}}}\big] = 0$ and $AJ\big(\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-3}}}\big) = 0$, which by induction hypothesis implies $\xi \in F^{\nu-1}\mathrm{CH}^r(X_K;\mathbb{Q})$.

Now, let $\xi \in F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q})$. Theorem 4.2.1 implies that the corresponding reduced normal function (and thus the normal function itself) associated with $\tilde{\xi}$ is zero over $\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-1}}$. Under Assumption 5.0.3, the image under the normal function is defined over the same field of definition as $\tilde{\xi}$. Therefore, $\big[\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V_{\nu-1}}}\big] = 0$.

Thus, Conjecture 5.0.2 and 5.0.4 are equivalent.

□

We prove the following theorem:

Theorem 5.0.6. *Assume given the product situation, and the GHC. Then the conjectures hold for $CH^r(X_K;\mathbb{Q})$ replaced by $\underline{CH}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$.*

Proof. Let $\xi \in \underline{\mathrm{CH}}^r_{\mathrm{alg}}(X_K;\mathbb{Q})$ and suppose we have (i) and (ii) of Conjecture 5.0.4 in terms of $\underline{\mathrm{CH}}^r_{\mathrm{alg}}$. Observe that the fibers of the corresponding arithmetic normal function of ξ can be identified in $\mathscr{X}$ with inverse images of smooth subvarieties of $\mathcal{S}$ of dimension $\nu - 2$. Then, by Proposition 4.7.1 the result is immediate.

□

Example 5.0.7. *For $\nu = 2$, we have $\eta_{\mathcal{S} \cap V_{\nu-2}} = p$ and since $\mathscr{X}_p = X_K$, this conjecture states that $\xi \in F^2 CH^r(X_K;\mathbb{Q})$ if and only if $\xi \in F^1 CH^r(X_K;\mathbb{Q})$ and $AJ\big(\tilde{\xi}_{\mathscr{X}_p}\big) = 0 \in J\big(H^{2r-1}(X_K,\mathbb{Q}(r))\big)$, that is, the general fiber is in the kernel of the Abel Jacobi map. This is Proposition 4.7.1.*

Example 5.0.8. *For $\nu = 3$, the conjecture involve varieties V of dimension 1. We say that $\xi \in F^3 CH^r(X_K;\mathbb{Q})$ if and only if $\xi \in F^2 CH^r(X_K;\mathbb{Q})$, and $AJ\big(\tilde{\xi}|_{\mathscr{X}_{\eta_{\mathcal{S}} \cap V}}\big) = 0 \in J\big(H^{2r-1}(\mathscr{X}_{\eta_{\mathcal{S}} \cap V},\mathbb{Q}(r))\big)$ for a choice of general $V \ni p$ of dimension 1.*

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