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AUTOMORPHISMS AND TWISTED FORMS OF DIFFERENTIAL LIE CONFORMAL SUPERALGEBRAS

by

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Abstract

Given a conformal superalgebra $\mathscr{A}$ over an algebraically closed field $\Bbbk$ of characteristic zero, a twisted loop conformal superalgebra $\mathcal{L}$ based on $\mathscr{A}$ has a differential conformal superalgebra structure over the differential Laurent polynomial ring $\mathcal{D} = (\Bbbk[t^{\pm 1}], \frac{d}{dt})$. In this context, $\mathcal{L}$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$ with respect to an étale extension of differential rings $\mathcal{D} \to \mathcal{D}_m = (\Bbbk[t^{\pm\frac{1}{m}}], \frac{d}{dt})$, and hence is a $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$ for $\widehat{\mathcal{D}} = \varinjlim \mathcal{D}_m$. Such a perspective reduces the problem of classifying the twisted loop conformal superalgebras based on $\mathscr{A}$ to the computation of the non-abelian cohomology set of its automorphism group functor.

The primary goal of this dissertation is to classify the twisted loop conformal superalgebras based on $\mathscr{A}$ when $\mathscr{A}$ is one of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras. To achieve this, we first explicitly determined the automorphism group of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{A} \otimes_{\Bbbk} \widehat{\mathcal{D}}$ in each case. We then computed the corresponding non-abelian continuous cohomology set, and obtained the classification of our objects up to isomorphism over $\mathcal{D}$. Finally, by applying the so-called "centroid trick", we deduced from isomorphisms over $\mathcal{D}$ to isomorphisms over $\Bbbk$, thus accomplishing the classification over $\Bbbk$.

Additionally, in order to understand the representability of the automorphism group functors of the $N = 1, 2, 3$ and small $N = 4$ conformal superalgebras, we discuss the $\mathcal{R}$–points of these automorphism group functors for an *arbitrary* differential ring $\mathcal{R} = (R, \mathsf{d})$. In particular, if R is an integral domain (with certain additional assumptions in the small $N = 4$ case), these automorphism groups have been completely determined.

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Contents

Notation

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}/m\mathbb{Z}$	the finite cyclic group of order m
$\Bbbk$	an algebraically closed field of characteristic zero
$\mathbb{Q}$	the field of rational numbers
$\mathbb{C}$	the field of complex numbers
δ_{ij}	the Kronecker symbol, which is 1 if $i = j$ and is 0 if $i \neq j$
ϵ_{ijk}	the sign of a cycle (i, j, k), where $i, j, k \in \{1, 2, 3\}$
$\mathbf{i}$	a square root of -1 in $\Bbbk$
ζ_m	the standard m-th primitive root of unity in $\Bbbk$, i.e., $\zeta_m = e^{\frac{2\pi \mathbf{i}}{m}}$
D	the Laurent polynomial ring $\Bbbk[t^{\pm 1}]$
D_m	the ring $\Bbbk[t^{\pm \frac{1}{m}}]$ which is an étale extension of D
$\widehat{D}$	the ring $\varinjlim D_m$
$\mathcal{D}$	the $\Bbbk$–differential ring $(D, \frac{d}{dt})$
$\mathcal{D}_m$	the $\Bbbk$–differential ring $(D_m, \frac{d}{dt})$
$\widehat{\mathcal{D}}$	the $\Bbbk$–differential ring $(\widehat{D}, \frac{d}{dt})$
$\mathrm{Spec}(R)$	the spectrum of R
$\mathrm{Mat}_2(R)$	the set of 2×2-matrices with coefficients in the ring R

Chapter 1

Introduction

Infinite dimensional Lie (super)algebras emerged in the study of theoretical physics in the 1960s. They turned out to be one of the most useful mathematical tools to describe supersymmetric phenomena. Presently, there are (at least) two families of infinite dimensional Lie (super)algebras of particular interest in physics: one is the family of affine Kac-Moody algebras and the other is the family of the so-called superconformal algebras.

Kac-Moody algebras appeared in mathematics as a generalization of finite dimensional simple Lie algebras over the field $\mathbb{C}$ of complex numbers. In Kac-Moody theory, the affine Kac-Moody algebras play a very special role. While a general Kac-Moody algebra is defined by Chevalley-Serre relations (a useful approach, but one that makes it impossible to see what the objects look like), an affine Kac-Moody algebra (derived modulo its center) has a beautiful explicit realization as a twisted loop Lie algebra of the form $L(\mathfrak{g}, \sigma)$ for a finite dimensional complex simple Lie algebra $\mathfrak{g}$ with respect to an automorphism σ of $\mathfrak{g}$ of finite order m (cf. Chapters 7 and 8 of [Kac90]). Concretely, one may assign a natural structure of a Lie algebra on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm\frac{1}{m}}]$ by defining

$$[a \otimes r, b \otimes s] = [a, b] \otimes rs, \quad a, b \in \mathfrak{g}, r, s \in \mathbb{C}[t^{\pm\frac{1}{m}}].$$

The automorphism σ of $\mathfrak{g}$ is extended to an automorphism $\sigma \otimes \psi$ on the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm\frac{1}{m}}]$, where

$$\psi : \mathbb{C}[t^{\pm\frac{1}{m}}] \to \mathbb{C}[t^{\pm\frac{1}{m}}], \quad t^{\frac{1}{m}} \mapsto \zeta_m^{-1} t^{\frac{1}{m}},$$

and ζ_m is an m-th primitive root of unity. The twisted loop Lie algebra $L(\mathfrak{g}, \sigma)$ is defined to be the sub-Lie algebra of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm\frac{1}{m}}]$ consisting of elements fixed by $\sigma \otimes \psi$.

These twisted loop Lie algebras have been further generalized to twisted multi-loop Lie algebras by replacing the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$ with the multiloop ring $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$. Such generalizations play an important role in the theory of extended affine Lie algebras (cf. [ABFP09]).

In recent years, inspired by the twisted loop construction of an affine Kac-Moody algebra, algebraic-geometric methods, including non-abelian Galois cohomology and descent theory, have been brought into the study of affine Kac-Moody algebras and twisted multiloop Lie algebras. The basic ideas behind these methods, which are succinctly exposed in [Pia05], are the following observations:

- Every twisted loop Lie algebra $L(\mathfrak{g}, \sigma)$ is not only a Lie algebra over the field $\mathbb{C}$, but also a Lie algebra over the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$.
- Viewed as a Lie algebra over $\mathbb{C}[t^{\pm 1}]$, every twisted loop Lie algebra $L(\mathfrak{g}, \sigma)$ is a twisted form of the untwisted loop Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ which is "trivialized" by the étale extension of rings $\mathbb{C}[t^{\pm 1}] \to \mathbb{C}[t^{\pm \frac{1}{m}}]$. Furthermore, the affine Kac-Moody Lie algebras account for all the twisted forms.

This point of view has prompted two avenues of investigations. One is the exploration of the structure and representation theory of a twisted (multi)loop Lie algebra by applying descent theory to the corresponding untwisted (multi)loop Lie algebra. This idea has been successfully used in the research occurring on the following topics:

- the central extensions of twisted forms of Lie algebras in [PPS07, Sun09],
- the derivations of twisted forms of Lie algebras in [Pia10],
- the conjugacy theorem of maximal abelian diagonalizable subalgebras (analogues of Cartan subalgebras) of twisted loop Lie algebras and affine Kac-Moody algebras in [Pia04, CGP11, CEGP12],
- the finite-dimensional irreducible representations of a twisted form of a Lie algebra in [Lau10, LP13].

The other is the investigation of the automorphism groups of Lie algebras and the torsors of an affine group scheme over $\mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$. In the classical theory of twisted forms, every isomorphism class of twisted forms of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ which are split by an étale extension of $\mathbb{C}[t^{\pm 1}]$ corresponds to an element in the non-abelian étale cohomology set $\mathrm{H}^1_{\text{ét}}(\mathbb{C}[t^{\pm 1}], \mathbf{Aut}(\mathfrak{g}))$, where $\mathbf{Aut}(\mathfrak{g})$ is the automorphism group functor of $\mathfrak{g}$. In the situation where $\mathfrak{g}$ is a finite dimensional Lie algebra, $\mathbf{Aut}(\mathfrak{g})$ is representable by an affine group scheme of finite type. In addition, the classes in $\mathrm{H}^1_{\text{ét}}(\mathbb{C}[t^{\pm 1}], \mathbf{Aut}(\mathfrak{g}))$ can also be interpreted in terms of torsors over $\mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ under $\mathbf{Aut}(\mathfrak{g})$. This lead to a detailed study of the automorphism group schemes of Lie algebras or Lie superalgebras (cf. [GP04, GP08b]), and of the theory of torsors of reductive group schemes over $\mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$, or more generally over $\mathrm{Spec}(\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}])$ (cf. [CGP12, GP05, GP07, GP08a, Pia05]).

In 2009, V. Kac, M. Lau, and A. Pianzola [KLP09] found that the above strategy for studying twisted loop Lie algebras by cohomological means can be used to understand the so-called twisted superconformal algebras in theoretical physics. This discovery is the main motivation for the work presented in this dissertation. Superconformal algebras are infinite dimensional Lie superalgebras used to describe supersymmetries in conformal field theory. Many important examples, including the Neveu-Schwarz algebra, and the Ramond algebra, have been known for decades.

The key feature of such a Lie superalgebra is that its Lie superbracket involves the operator product expansion of formal distributions. In the 1990s, V. G. Kac introduced the notion of conformal superalgebras to deal with Lie superalgebras with operator product expansions. In subsequent work of V. G. Kac and his collaborators, finite simple conformal superalgebras over $\mathbb{C}$ (the field of complex numbers) were classified (cf. [Kac98a]).

Along with the usual (untwisted) superconformal algebras, physicists have also created another family of Lie superalgebras called twisted superconformal algebras in [SS87] and [STVP88]. The readers are encouraged to look at these papers for the relevance to the physics of the objects we will study. Based on Kac's theory of conformal superalgebras, a twisted superconformal algebra is the Lie superalgebra induced by a twisted loop conformal superalgebra. Analogous to the classification of twisted loop Lie algebras in terms of torsors, one of the main observations used in [KLP09] to study twisted loop conformal algebras is that a twisted loop conformal algebra based on a given $\mathbb{C}$–conformal superalgebra $\mathscr{A}$ has a conformal superalgebra structure over the differential Laurent polynomial ring $\mathcal{D} := (\mathbb{C}[t^{\pm 1}], \mathsf{d}_t)$, that is the pair consisting of the ring $\mathbb{C}[t^{\pm 1}]$ and the derivation $\mathsf{d}_t := \frac{d}{dt}$. The derivation d_t has been introduced here because the commutative associative algebra structure on $\mathbb{C}[t^{\pm 1}]$ is not sufficient to define the affinization of a conformal superalgebra (or more generally, is not sufficient to define the change of base rings for conformal superalgebras).

As a result, conformal superalgebras over the field $\mathbb{C}$ were generalized to differential conformal superalgebras over a differential ring (that is, a pair consisting of a commutative associative algebra over the base field $\Bbbk$ together with a derivation). With this generalization, one can define the "change of base" over differential rings, and this turns out to be the correct (and crucial) ingredient needed in the conformal superalgebra situation to deal with "local triviality" in the étale sense. As expected, every twisted loop conformal superalgebra is a twisted form of the corresponding untwisted loop conformal superalgebra with respect to an extension of differential rings of the form $\mathcal{D} := (\Bbbk[t^{\pm 1}], \frac{d}{dt}) \to \mathcal{D}_m := (\Bbbk[t^{\pm \frac{1}{m}}], \frac{d}{dt})$.

Additionally, the change of base differential rings motives the definition of the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ of a given conformal superalgebra $\mathscr{A}$ over a base differential ring $\mathcal{R}$. It has been proved in [KLP09] that, for faithfully flat extension $\mathcal{S}/\mathcal{R}$ of differential rings, the $\mathcal{R}$–isomorphism classes of $\mathcal{S}/\mathcal{R}$–twisted forms of a conformal superalgebra $\mathscr{A}$ over $\mathcal{R}$ bijectively correspond to the classes in the non-abelian cohomology set $\mathrm{H}^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$.

Unlike the case for a usual finite dimensional algebra, the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ of a $\Bbbk$–conformal superalgebra $\mathscr{A}$ fails to be representable in the usual sense. Nonetheless, since our primary concern is the classification of twisted loop conformal superalgebras based on $\mathscr{A}$, we may specialize extensions of differential rings to be the extension $\mathcal{D} \to \mathcal{D}_m$ for some positive integer m. Moreover, instead of considering the extension $\mathcal{D} \to \mathcal{D}_m$ for each m individually, we may consider $\widehat{\mathcal{D}} := \varinjlim \mathcal{D}_m$ and the extension $\mathcal{D} \to \widehat{\mathcal{D}}$. The result is that all twisted loop conformal superalgebras based on $\mathscr{A}$ can be classified by $\mathrm{H}^1(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$ up to isomorphism of conformal superalgebras over $\mathcal{D}$.

When $\mathscr{A}$ satisfies a certain finiteness condition, the non-abelian cohomology set $\mathrm{H}^1(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$ can be further identified with the non-abelian continuous cohomology set $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}}))$, where $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$.

In the context of the differential conformal superalgebra theory developed in [KLP09], this dissertation will focus on the classification of twisted loop conformal superalgebras based on $\mathscr{A}$, where $\mathscr{A}$ is one of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras.

One of the key ingredients in the classification is to find the automorphism group $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$ for each of the conformal superalgebras $\mathscr{A}$ listed above. In fact, for the $N = 2$ or small $N = 4$ conformal superalgebra $\mathscr{A}$, $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$ has been determined in [KLP09]. We will explicitly compute $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$, where $\mathscr{A}$ is one of the $N = 1, 2, 3$ or large $N = 4$ conformal superalgebras. Summarizing our results and the automorphism group of the small $N = 4$ algebra obtained in [KLP09], we have the following Table 1.1:

Table 1.1: automorphism groups

$\mathscr{A}$	$N = 1, 2, 3$	small $N = 4$	large $N = 4$
$\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$	$\mathbf{O}_N(\widehat{D})$	$\frac{\mathbf{SL}_2(\widehat{D})\times\mathbf{SL}_2(\Bbbk)}{\langle(-I_2,-I_2)\rangle}$	$\left(\frac{\mathbf{SL}_2(\widehat{D})\times\mathbf{SL}_2(\widehat{D})}{\langle(-I_2,-I_2)\rangle}\times\mathbf{G}_a(\widehat{D})\right)\rtimes\mathbb{Z}/2\mathbb{Z}$

where $\widehat{D} = \varinjlim \Bbbk[t^{\pm\frac{1}{m}}]$, and $\mathbf{O}_N, \mathbf{SL}_2, \mathbf{G}_a$ are the group scheme of $N \times N$ orthogonal matrices, special linear group scheme, and additive group scheme, respectively.

These results seem to suggest that the automorphism groups $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$ are closely related to the $\widehat{D}$–points of certain affine group schemes. With this as a motivation, we further study the relationships between the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ and affine group schemes whose $\mathcal{R}$–points yield the abstract group $\mathbf{Aut}(\mathscr{A})(\mathcal{R})$ for an arbitrary $\Bbbk$–differential ring $\mathcal{R}$. For one of the $N = 1, 2, 3$ or small $N = 4$ conformal superalgebras $\mathscr{A}$, we found that the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ has a subgroup functor $\mathbf{GrAut}(\mathscr{A})$, which coincides with $\mathbf{Aut}(\mathscr{A})$ when evaluated on a $\Bbbk$–differential ring whose underlying ring is an integral domain. Moreover, $\mathbf{GrAut}(\mathscr{A})$ can be obtained as a lift of an affine group scheme (viewed as functors from the category of commutative associative $\Bbbk$-algebras to the category of groups) by composing certain functors from the category of $\Bbbk$–differential rings to the category of commutative associative $\Bbbk$-algebras.

After determining the automorphism group $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$, we move on to compute the non-abelian continuous cohomology set $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}}))$. In the case where $\mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}})$ coincides with the $\widehat{D}$-points $\mathbf{G}(\widehat{D})$ of a reductive group scheme $\mathbf{G}$, the non-abelian cohomology set $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}(\widehat{D}))$ can be further identified with the non-abelian étale cohomology set $\mathrm{H}_{\text{ét}}(D, \mathbf{G})$ (cf. [GP08a]). Such an identification allows us to connect our problem to the theory of torsors over the punched affine line $\mathrm{Spec}(D)$ under $\mathbf{G}$.

Once we determined $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathscr{A})(\widehat{\mathcal{D}}))$, we obtained the classification of all twisted loop conformal superalgebras based on $\mathscr{A}$ up to isomorphism over $\mathcal{D}$. The passage from isomorphism over $\mathcal{D}$ to isomorphism over $\Bbbk$ can be done by using the so-called "centroid trick", which was developed in [KLP09]. We will discuss them in Section 2.5 where we will provide a new explicit description of the centroid of the twisted loop conformal superalgebras under certain assumptions (cf. Propositon 2.12). These assumptions are fulfilled by any twisted loop conformal superalgebra based on each of the $\Bbbk$–conformal superalgebra $\mathscr{A}$ listed in Table 1.1.

The structure of this dissertation is as follows. In Chapter 2, we will provide a review of the general theory of differential conformal superalgebras and their twisted forms, which is part of the preliminaries required for our subsequent discussions. In Chapter 3, we will review basic terminology and facts from non-abelian Galois cohomology theory. Chapters 4,5, and 6 comprise the main body of this dissertation. In Chapter 4, we will focus on the $N = 1, 2, 3$ conformal superalgebras $\mathscr{K}_N$: we will deduce the structure of the automorphism group functor $\mathbf{Aut}(\mathscr{K}_N)$, complete the classification of twisted loop conformal superalgebras based on $\mathscr{K}_N$, and discuss the Lie superalgebras determined by these non-isomorphic twisted loop conformal superalgebras. In Chapter 5, we will concentrate on the properties of the automorphism functor $\mathbf{Aut}(\mathscr{W})$ of the small $N = 4$ conformal superalgebra $\mathscr{W}$, and provide a review of the classification of twisted loop conformal superalgebras based on $\mathscr{W}$ obtained in [KLP09]. A similar classification for the large $N = 4$ conformal superalgebra $\mathscr{M}$ will be completed in Chapter 6 based on an explicit computation of the automorphism group $\mathbf{Aut}(\mathscr{M})(\widehat{\mathcal{D}})$ and the non-abelian cohomology set $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{Aut}(\mathscr{M})(\widehat{\mathcal{D}}))$.

Chapter 2

Differential Conformal Superalgebras

This chapter is a review of the general theory of differential conformal superalgebras developed in [KLP09].

2.1 Conformal superalgebras

Before going into a discussion on differential conformal superalgebras, we first review some facts about conformal superalgebras over the base field $\Bbbk$, and identify the relations between conformal superalgebras and formal distribution Lie superalgebras. The terminologies and notions presented in this section were introduced by V. Kac in [Kac98b].

Let $\mathfrak{g}$ be a Lie superalgebra[1] (usually infinite dimensional) over $\Bbbk$. A $\mathfrak{g}$–*valued formal distribution* is a formal series $a(w)$ of the form

$$a(w) = \sum_{n\in\mathbb{Z}} a_n w^{-n-1}, \quad a_n \in \mathfrak{g}. \tag{2.1.1}$$

Two $\mathfrak{g}$-valued formal distributions $a(w)$ and $b(w)$ are called *mutually local* if there is a positive integer N such that

$$(z-w)^N[a(z), b(w)] = 0, \tag{2.1.2}$$

where

$$[a(z), b(w)] = \sum_{m,n\in\mathbb{Z}} [a_m, b_n] z^{-m-1} w^{-n-1}, \tag{2.1.3}$$

if $a(w) = \sum_{n\in\mathbb{Z}} a_n w^{-n-1}$ and $b(w) = \sum_{n\in\mathbb{Z}} b_n w^{-n-1}$.

It has been shown in Corollary 2.2 of [Kac98b] that two $\mathfrak{g}$–valued formal distributions $a(w)$ and $b(w)$ are mutually local if and only if $[a(z), b(w)]$ can be written

[1] I will not restate the definition of a Lie superalgebra here since it can be easily found in many literatures such as the long paper [Kac77] by V. G. Kac, and the book [CW12] by S. Cheng and W. Wang.

as a finite sum

$$[a(z), b(w)] = \sum_{j\in\mathbb{Z}_+} c_j(w)\partial_w^{(j)}\delta(z-w), \tag{2.1.4}$$

where ∂_w is the formal derivative with respect to w, $\partial_w^{(j)} := \partial_w^j/j!$, and

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} w^n z^{-n-1}. \tag{2.1.5}$$

The expansion (2.1.4) is called the *operator product expansion (OPE)* of $[a(z), b(w)]$, and the $c_j(w), j \in \mathbb{Z}_+$ are called the OPE coefficients.

A *formal distribution Lie superalgebra over* $\mathbb{k}$ is a pair $(\mathfrak{g}, \mathcal{F})$ consisting of a Lie superalgebra $\mathfrak{g}$ over $\mathbb{k}$ and a set $\mathcal{F}$ of mutually local $\mathfrak{g}$–valued formal distributions such that $\mathfrak{g}$ is spanned (as a $\mathbb{k}$–vector space) by coefficients of elements in $\mathcal{F}$.

Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, one strategy to investigate the properties of the Lie superalgebra $\mathfrak{g}$ is to consider the operator product expansion of formal distributions in $\mathcal{F}$.

To illustrate, we consider the centreless Virasoro algebra $\mathfrak{v} = \oplus_{n\in\mathbb{Z}}\mathbb{k}\mathrm{L}_n$ with the Lie bracket defined by

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, \quad m, n \in \mathbb{Z}. \tag{2.1.6}$$

Then $\mathfrak{v}$ (as a $\mathbb{k}$–vector space) is spanned by the coefficients of the formal distribution

$$\mathrm{L}(z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_n z^{-n-2}, \tag{2.1.7}$$

i.e., if we take $\mathcal{F} = \{\mathrm{L}(z)\}$, then $(\mathfrak{v}, \mathcal{F})$ is a formal distribution Lie algebra. By considering the operator product expansion, we have

$$[\mathrm{L}(z), \mathrm{L}(w)] = (\partial_w \mathrm{L}(w))\delta(z-w) + 2\mathrm{L}(w)\partial_w\delta(z-w). \tag{2.1.8}$$

Let $\overline{\mathcal{F}} = \mathrm{span}_{\mathbb{k}}\{\partial_w^j \mathrm{L}(w) | j \in \mathbb{Z}_+\}$. Then all OPE coefficients of $[a(z), b(w)]$ for $a(z), b(z) \in \overline{\mathcal{F}}$ are contained in $\overline{\mathcal{F}}$, i.e., $\overline{\mathcal{F}}$ is closed under taking OPE coefficients. Hence, the OPE yields an algebraic structure on $\overline{\mathcal{F}}$, which motivates the definition of a conformal superalgebra.

In a $\mathbb{Z}/2\mathbb{Z}$–graded $\mathbb{k}$–vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we always use $p(a, b)$ to denote the sign $(-1)^{p(a)p(b)}$ for two homogeneous elements a and b in V, where $p(a)$ and $p(b)$ are the parity of a and b, respectively.

Definition 2.1 (Definition 2.7 of [Kac98b]). A *(Lie) conformal superalgebra* over $\Bbbk$ is a $\mathbb{Z}/2\mathbb{Z}$–graded $\Bbbk[\partial]$–module $\mathscr{A} = \mathscr{A}_{\bar{0}} \oplus \mathscr{A}_{\bar{1}}$, on which there is a $\Bbbk$–bilinear product $-_{(n)}-$ for each $n \in \mathbb{Z}_+$ satisfying:

(C0) $a_{(n)}b = 0$ for $n \gg 0$,

(C1) $(\partial_{\mathscr{A}} a)_{(n)}b = -na_{(n-1)}b$,

(C2) $a_{(n)}b = -p(a,b)\sum_{j=0}^{\infty}(-1)^{j+n}\partial^{(j)}(b_{(n+j)}a)$,

(C3) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m}\binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + p(a,b)b_{(n)}(a_{(m)}c)$,

where $\partial^{(j)} = \partial^j/j!$, $a,b,c \in \mathscr{A}$, and $m,n \in \mathbb{Z}_+$.

One can also define an associative conformal superalgebra by modifying axioms (C2) and (C3) above (cf. Section 2.10 of [Kac98b]). Since all conformal superalgebras considered in this thesis are Lie conformal superalgebras, we simply say a conformal superalgebra instead of a Lie conformal superalgebra.

For convenience, we also use the λ–bracket notation for all n–th products of a conformal superalgebra $\mathscr{A}$:

$$[a_\lambda b] = \sum_{n=0}^{\infty}\lambda^{(n)}(a_{(n)}b), \tag{2.1.9}$$

for $a,b \in \mathscr{A}$, where λ is an indeterminate and $\lambda^{(n)} := \lambda^n/n!$. Adopting this notation, the axiom (C0) is equivalent to that $[a_\lambda b]$ is a polynomial in λ with coefficients in $\mathscr{A}$. The axioms (C1)-(C3) can be rewritten as

$(\text{C1})_\lambda$ $[(\partial_{\mathscr{A}} a)_\lambda b] = -\lambda[a_\lambda b]$.

$(\text{C2})_\lambda$ $[a_\lambda b] = -p(a,b)[b_{-\lambda-\partial_{\mathscr{A}}}a]$.

$(\text{C3})_\lambda$ $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + p(a,b)[b_\mu[a_\lambda c]]$.

As an example, the set of formal distributions $\overline{\mathcal{F}}$ associated to the centreless Virasoro algebra $\mathfrak{v}$ form a conformal superalgebra $\mathscr{V}$, on which the n-th product $a(w)_{(n)}b(w)$ is defined to be the n-th OPE coefficient in the OPE of $[a(z),b(w)]$ for

$a(w), b(w) \in \overline{\mathcal{F}}$. The conformal superalgebra $\mathscr{V}$ is called the *centreless Virasoro conformal algebra*. More precisely,

$$\mathscr{V} = \Bbbk[\partial]\mathrm{L} \tag{2.1.10}$$

is a free $\Bbbk[\partial]$–module of rank 1, on which the λ–bracket is given by

$$[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}. \tag{2.1.11}$$

Another example is a *current conformal superalgebra*. Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\Bbbk$. Then

$$\mathrm{Cur}(\mathfrak{g}) = \Bbbk[\partial] \otimes_{\Bbbk} \mathfrak{g} \tag{2.1.12}$$

is a conformal superalgebra under the n-th product defined by

$$a_{(n)}b = \delta_{n,0}[a, b], \tag{2.1.13}$$

where $a, b \in \mathfrak{g}, n \in \mathbb{Z}_+$.

The passage from the centreless Virasoro algebra $\mathfrak{v}$ to the centreless Virasoro conformal algebra $\mathscr{V}$ can be generalized to all formal distribution Lie superalgebras as follows. For a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, the closure $\overline{\mathcal{F}}$ (the minimal set of formal distributions which is closed under OPE and contains $\mathcal{F}$) is equipped with a structure of a conformal superalgebra through OPE. This conformal superalgebra is denoted by $\mathscr{A}(\mathfrak{g}, \mathcal{F})$ (cf. Section 2.7 of [Kac98b] for an explicit description of this passage).

Conversely, a conformal superalgebra $\mathscr{A}$ over $\Bbbk$ also realizes a Lie superalgebra. Such a realization requires us to consider the *affinization of a conformal superalgebra*.

Throughout this dissertation, D always denotes the Laurent polynomial ring $\Bbbk[t^{\pm 1}]$. Given a conformal superalgebra $\mathscr{A}$, there is a conformal superalgebra structure on the $\Bbbk$–vector space $\mathscr{A} \otimes_{\Bbbk} D$ given by

$$\widehat{\partial}(a \otimes r) = \partial_{\mathscr{A}}(a) \otimes r + a \otimes \mathsf{d}_t(r), \tag{2.1.14}$$

for $a \in \mathscr{A}, r \in D$, and the n-th product

$$(a \otimes r)_{(n)}(b \otimes s) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b) \otimes \mathsf{d}_t^{(j)}(r)s, \tag{2.1.15}$$

for $a, b \in \mathscr{A}, r, s \in D$, where $\mathsf{d}_t = \frac{d}{dt}$ is the derivative with respect to t and $\mathsf{d}_t^{(j)} := \mathsf{d}_t^j / j!$. The resulting conformal superalgebra is denoted by $\mathscr{A}_{\mathcal{D}} := \mathscr{A} \otimes_{\Bbbk} \mathcal{D}$, where $\mathcal{D} = (D, \mathsf{d}_t)$.

The conformal superalgebra $\mathscr{A}_{\mathcal{D}}$ determines a Lie superalgebra

$$\mathrm{Alg}(\mathscr{A}) := (\mathscr{A} \otimes_{\Bbbk} \mathcal{D}) / \widehat{\partial}(\mathscr{A} \otimes_{\Bbbk} \mathcal{D}) \tag{2.1.16}$$

with Lie superbracket induced by the 0-th product of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$. Furthermore, if $\mathscr{A} = \mathscr{A}(\mathfrak{g}, \mathcal{F})$ is the conformal superalgebra associated to a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, then $\mathrm{Alg}(\mathscr{A}) \cong \mathfrak{g}$ as Lie superalgebras over $\Bbbk$ (cf. Theorem 2.7 of [Kac98b]).

Analogous to the twisted loop construction for Lie algebras, we may define twisted loop conformal superalgebras. Starting with a conformal superalgebra $\mathscr{A}$ and an automorphism σ of $\mathscr{A}$ of order m, we define the conformal superalgebra $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m$ using (2.1.14) and (2.1.15), where $D_m = \Bbbk[t^{\pm \frac{1}{m}}]$ and $\mathcal{D}_m = (D_m, \mathsf{d}_t)$. We then extend σ to an automorphism $\sigma \otimes \psi$ of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m$, where

$$\psi : D_m \to D_m, \quad t^{\frac{1}{m}} \mapsto \zeta_m^{-1} t^{\frac{1}{m}}$$

and ζ_m is an m-th primitive root of unity. Let Γ be the group generated by $\sigma \otimes \psi$. Then the set of fixed points of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m$ under Γ

$$\mathcal{L}(\mathscr{A}, \sigma) := (\mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m)^{\Gamma} = \{\eta \in \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m | (\sigma \otimes \psi)(\eta) = \eta\} \tag{2.1.17}$$

is a sub conformal superalgebra of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m$, called the *twisted loop conformal superalgebra based on $\mathscr{A}$ with respect to σ*. In particular, when σ is the identity map, $\mathcal{L}(\mathscr{A}, \mathrm{id}) \cong \mathscr{A} \otimes_{\Bbbk} \mathcal{D}$.

More explicitly, since σ is of finite order m, $\mathscr{A}$ is decomposed into a direct sum with respect to σ, i.e., $\mathscr{A} = \oplus_{i=0}^{m-1} \mathscr{A}_i$, where $\mathscr{A}_i = \{a \in \mathscr{A} | \sigma(a) = \zeta_m^i a\}$ for $i \in \mathbb{Z}$. In particular $\mathscr{A}_i = \mathscr{A}_{i+m}$ and hence

$$\mathcal{L}(\mathscr{A}, \sigma) = \bigoplus_{i \in \mathbb{Z}} (\mathscr{A}_i \otimes \Bbbk t^{\frac{i}{m}}). \tag{2.1.18}$$

Similar to (2.1.16), the conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ also yields a Lie superalgebra

$$\mathrm{Alg}(\mathscr{A}, \sigma) := \mathcal{L}(\mathscr{A}, \sigma)/\widehat{\partial}\mathcal{L}(\mathscr{A}, \sigma), \tag{2.1.19}$$

with Lie superbracket induced by the 0-th product of $\mathcal{L}(\mathscr{A}, \sigma)$. The Lie superalgebra $\mathrm{Alg}(\mathscr{A}, \sigma)$ is not necessary a formal distribution Lie superalgebra in the usual sense, but rather is interpreted as a Γ–twisted formal distribution Lie superalgebra (cf. [KLP09]). Moreover, the central extensions of these Lie superalgebras realize the so-called twisted superconformal algebras in physics literatures (cf. [SS87, STVP88]).

2.2 Differential conformal superalgebras

Recall that a twisted loop Lie algebra is not only a Lie algebra over the base field $\mathbb{k}$ but also a Lie algebra over the ring $\mathbb{k}[t^{\pm 1}]$. Differential conformal superalgebras arose from the efforts to adapt the concept of a twisted loop conformal superalgebra to a structure over $\mathbb{k}[t^{\pm 1}]$. The critical obstacle is that the ring structure of $\mathbb{k}[t^{\pm 1}]$ is not sufficient to define a concept of conformal algebra over $\mathbb{k}[t^{\pm 1}]$: one requires a differential structure, i.e., to consider the derivation $\frac{d}{dt}$ on $\mathbb{k}[t^{\pm 1}]$. In other words, one can define differential conformal superalgebras over $(\mathbb{k}[t^{\pm 1}], \frac{d}{dt})$ and, more generally, over a differential ring $\mathcal{R} = (R, \mathsf{d})$. In this section, we will review the definition of a differential conformal superalgebra, which was originally introduced by V. Kac, M. Lau, and A. Pianzola in [KLP09].

Let $\mathbb{k}$**-rng** denote the category of unital commutative associative algebras over $\mathbb{k}$. A $\mathbb{k}$–differential ring is a pair $\mathcal{R} = (R, \mathsf{d}_R)$ consisting of an object R in $\mathbb{k}$**-rng** and a $\mathbb{k}$–linear derivation $\mathsf{d}_R : R \to R$. For example, 0 is a derivation on every object R in $\mathbb{k}$**-rng**, i.e., $(R, 0)$ is a $\mathbb{k}$–differential ring. In particular, $\mathbb{k}$ (viewed as a $\mathbb{k}$-differential ring) refers to the $\mathbb{k}$–differential ring $(\mathbb{k}, 0)$. The Laurent polynomial ring $D = \mathbb{k}[t^{\pm 1}]$ paired with the derivation $\mathsf{d}_t = \frac{d}{dt}$ (the derivative with respect to t) gives a $\mathbb{k}$–differential ring $\mathcal{D} = (D, \mathsf{d}_t)$.

A morphism $f : \mathcal{R} = (R, \mathsf{d}_R) \to \mathcal{S} = (S, \mathsf{d}_S)$ of $\mathbb{k}$–differential rings is a morphism $f : R \to S$ in $\Bbbk$**-rng** such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ {\scriptstyle \mathsf{d}_R}\downarrow & & \downarrow{\scriptstyle \mathsf{d}_S} \\ R & \xrightarrow[f]{} & S \end{array}$$

commutes. The collection of all $\Bbbk$–differential rings together with the morphisms given above form a category, which is denoted by $\Bbbk$**-drng**.

Definition 2.2 (Definition 1.3 of [KLP09])**.** Let $\mathcal{R} = (R, \mathsf{d})$ be an object in $\Bbbk$**-drng**. A *differential (Lie) conformal superalgebra* over $\mathcal{R}$ is a triple $(\mathscr{A}, \partial_{\mathscr{A}}, (-_{(n)}-)_{n\in\mathbb{Z}_+})$ consisting of

(i) a $\mathbb{Z}/2\mathbb{Z}$-graded R–module $\mathscr{A} = \mathscr{A}_{\bar{0}} \oplus \mathscr{A}_{\bar{1}}$,

(ii) a $\Bbbk$–linear map $\partial_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ preserving the $\mathbb{Z}/2\mathbb{Z}$–grading of $\mathscr{A}$,

(iii) a $\Bbbk$-bilinear product $(a, b) \mapsto a_{(n)}b, a, b \in \mathscr{A}$ for each $n \in \mathbb{Z}_+$,

satisfying the following axioms for $r \in R, a, b, c \in \mathscr{A}$, and $m, n \in \mathbb{Z}_+$:

(DC0) $a_{(n)}b = 0$ for $n \gg 0$,

(DC1) $\partial_{\mathscr{A}}(a)_{(n)}b = -na_{(n-1)}b$ and $a_{(n)}\partial_{\mathscr{A}}(b) = \partial_{\mathscr{A}}(a_{(n)}b) + na_{(n-1)}b$,

(DC2) $\partial_{\mathscr{A}}(ra) = r\partial_{\mathscr{A}}(a) + \mathsf{d}(r)a$,

(DC3) $a_{(n)}(rb) = r(a_{(n)}b)$ and $(ra)_{(n)}b = \sum_{j\in\mathbb{Z}_+} \mathsf{d}^{(j)}(r)(a_{(n+j)}b)$,

(DC4) $a_{(n)}b = -p(a, b) \sum_{j\in\mathbb{Z}_+}(-1)^{j+n}\partial_{\mathscr{A}}^{(j)}(b_{(n+j)}a)$, and

(DC5) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m} \binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + p(a, b)b_{(n)}(a_{(m)}c)$,

where $\mathsf{d}^{(j)} = \mathsf{d}^j/j!$ and $\partial_{\mathscr{A}}^{(j)} = \partial_{\mathscr{A}}^j/j!$ for $j \in \mathbb{Z}_+$.

The axioms (DC4) and (DC5) are the analogues of the supersymmetry axiom and Jacobi identity that hold for a Lie superalgebra. One can also define a differential associative conformal superalgebra over $\mathcal{R}$ by replacing (DC4) and (DC5) with an appropriate conformal associativity axiom (cf. Section 2.10 of [Kac98b]). In this thesis, all differential conformal superalgebras over $\mathcal{R}$ are assumed to be differential Lie conformal superalgebras. Hence, for the remainder of this thesis, we will say an *$\mathcal{R}$–conformal superalgebra* instead of a differential Lie conformal superalgebra over $\mathcal{R}$.

The $\mathcal{R}$–conformal superalgebras are natural generalizations of conformal superalgebras over the field $\Bbbk$, since every conformal superalgebra over $\Bbbk$ described by Definition 2.1 is a $\Bbbk$–conformal superalgebra (i.e., a differential conformal superalgebra over $(\Bbbk, 0)$).

Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra and σ an automorphism of $\mathscr{A}$ of finite order. As is also the case for a twisted loop Lie algebra, the twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ is not only a $\Bbbk$–conformal superalgebra but also a $\mathcal{D}$–conformal superalgebra, where $\mathcal{D} = (\Bbbk[t^{\pm 1}], \mathsf{d}_t)$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathcal{R}$–conformal superalgebras. A *homomorphism of $\mathcal{R}$–conformal superalgebras* is a map $\phi : \mathscr{A} \to \mathscr{B}$ satisfying:

(i) ϕ is R–linear and preserves the $\mathbb{Z}/2\mathbb{Z}$–gradings,

(ii) $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$, for all $a, b \in \mathscr{A}$ and $n \in \mathbb{Z}_+$.

(iii) $\partial_{\mathscr{B}} \circ \phi = \phi \circ \partial_{\mathscr{A}}$.

A homomorphism of $\mathcal{R}$–conformal superalgebras $\phi : \mathscr{A} \to \mathscr{B}$ is called an isomorphism if it is bijective. In particular, an isomorphism of $\mathcal{R}$–conformal algebras $\phi : \mathscr{A} \to \mathscr{A}$ is called an *$\mathcal{R}$–automorphism* of $\mathscr{A}$. The set of all $\mathcal{R}$–automorphisms of $\mathscr{A}$ is a group under composition, and is denoted by $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{A})$.

2.3 The automorphism group functor of a conformal superalgebra

Analogous to the affinization of a conformal superalgebra, one can define the change of base differential rings for differential conformal superalgebras.

Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra and $\mathcal{R} = (R, \mathsf{d}_R) \to \mathcal{S} = (S, \mathsf{d}_S)$ a morphism in $\Bbbk$-**drng**. We define $\mathscr{A}_{\mathcal{S}} := \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S}$ to be the $\mathcal{S}$–conformal superalgebra with underlying $\mathbb{Z}/2\mathbb{Z}$–graded S–module $\mathscr{A} \otimes_R S$, on which the derivation is given by

$$\partial_{\mathscr{A} \otimes_{\mathcal{R}} \mathcal{S}}(a \otimes s) := \partial_{\mathscr{A}}(a) \otimes s + a \otimes \mathsf{d}_S(s), \tag{2.3.1}$$

for $a \in \mathscr{A}$, $s \in S$, and the n-th product is defined by

$$(a \otimes r)_{(n)}(b \otimes s) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b) \otimes \mathsf{d}_S^{(j)}(r)s \tag{2.3.2}$$

for $a, b \in \mathscr{A}$, $r, s \in S$, where $\mathsf{d}_S^{(j)} = \mathsf{d}_S^j / j!$.

The above change of base differential rings defines a functor from the category of $\mathcal{R}$–conformal superalgebras to the category of $\mathcal{S}$–conformal superalgebras. Every homomorphism of $\mathcal{R}$–conformal superalgebras $\phi : \mathscr{A}_1 \to \mathscr{A}_2$ determines a homomorphism of $\mathcal{S}$–conformal superalgebras $\phi \otimes \mathrm{id} : \mathscr{A}_1 \otimes_{\mathcal{R}} \mathcal{S} \to \mathscr{A}_2 \otimes_{\mathcal{R}} \mathcal{S}$.

The change of base differential rings is associative. Concretely, let $f_i : \mathcal{R}_i = (R_i, \mathsf{d}_{R_i}) \to \mathcal{S} = (S, \mathsf{d}_S), i = 1, 2$ be morphisms in $\mathbb{k}$-**drng** such that S is an R_1–R_2–bimodule, and let $h : \mathcal{R}_2 \to \mathcal{S}' = (S', \mathsf{d}_{S'})$ be a morphism in $\mathbb{k}$-**drng**. For an $\mathcal{R}_1$–conformal superalgebra $\mathscr{A}$, we have

$$(\mathscr{A} \otimes_{\mathcal{R}_1} \mathcal{S}) \otimes_{\mathcal{R}_2} \mathcal{S}' \cong \mathscr{A} \otimes_{\mathcal{R}_1} (\mathcal{S} \otimes_{\mathcal{R}_2} \mathcal{S}'), \tag{2.3.3}$$

where $\mathcal{S} \otimes_{\mathcal{R}_2} \mathcal{S}' = (S \otimes_{R_2} S', \mathsf{d}_S \otimes \mathrm{id} + \mathrm{id} \otimes \mathsf{d}_{S'})$.

Given an object $\mathcal{R}$ in $\mathbb{k}$-**drng**, we consider the category $\mathcal{R}$-**ext**, in which an object is a morphism $f : \mathcal{R} \to \mathcal{S}$ in $\mathbb{k}$-**drng** and a morphism from $f_1 : \mathcal{R} \to \mathcal{S}_1$ to $f_2 : \mathcal{R} \to \mathcal{S}_2$ is a morphism $h : \mathcal{S}_1 \to \mathcal{S}_2$ such that the diagram

$$\begin{array}{ccc} \mathcal{S}_1 & \xrightarrow{\quad h \quad} & \mathcal{S}_2 \\ {\scriptstyle f_1} \nwarrow & & \nearrow {\scriptstyle f_2} \\ & \mathcal{R} & \end{array}$$

commutes. We simply use $\mathcal{S}$ to represent an object $f : \mathcal{R} \to \mathcal{S}$ in $\mathcal{R}$-**ext** since f is determined by the R–module structure on S.

Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra and $h : \mathcal{S}_1 \to \mathcal{S}_2$ a morphism in $\mathcal{R}$-ext. Then h induces a group homomorphism

$$\mathrm{Aut}_{\mathcal{S}_1\text{-conf}}(\mathscr{A}_{\mathcal{S}_1}) \to \mathrm{Aut}_{\mathcal{S}_2\text{-conf}}(\mathscr{A}_{\mathcal{S}_2}), \quad \phi \mapsto h_*(\phi), \tag{2.3.4}$$

where $h_*(\phi) : \mathscr{A}_{\mathcal{S}_2} \to \mathscr{A}_{\mathcal{S}_2}$ is the homomorphism of S_2–modules defined by

$$h_*(\phi)(a \otimes 1) = \sum a_i \otimes h(s_i), \tag{2.3.5}$$

if $\phi(a \otimes 1) = \sum a_i \otimes s_i$ for $a \in \mathscr{A}$. This leads to the definition of the *automorphism* *group functor* of an $\mathcal{R}$–conformal superalgebra $\mathscr{A}$:

$$\mathbf{Aut}(\mathscr{A}) : \mathcal{R}\textbf{-ext} \to \mathbf{grp}, \qquad \mathcal{S} \mapsto \mathrm{Aut}_{\mathcal{S}\text{-conf}}(\mathscr{A}_{\mathcal{S}}). \tag{2.3.6}$$

In fact, the category $\Bbbk$**-ext** is equivalent to the category $\Bbbk$**-drng**, and the automorphism group functor of a $\Bbbk$–conformal superalgebra $\mathscr{A}$ is a functor from the category $\Bbbk$**-drng** to the category of groups.

Additionally, since a homomorphism $f : \mathcal{R}_1 \to \mathcal{R}_2$ naturally induces a functor $\mathcal{R}_2$**-ext** $\to \mathcal{R}_1$**-ext**, the automorphism group functor $\mathbf{Aut}(\mathscr{A} \otimes_{\mathcal{R}_1} \mathcal{R}_2)$ is nothing but the restriction of $\mathbf{Aut}(\mathscr{A})$ to the category $\mathcal{R}_2$**-ext**, i.e., $\mathbf{Aut}(\mathscr{A}_{\mathcal{R}_2}) = \mathbf{Aut}(\mathscr{A})_{\mathcal{R}_2}$.

For the remainder of this thesis, we will be particularly concerned with the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ of a $\Bbbk$–conformal superalgebra $\mathscr{A}$. The following technical lemma will be repeatedly used in the computations of automorphism groups $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{A}_{\mathcal{R}})$ when $\mathscr{A}$ is one of the $N = 1, 2, 3$, (small or large) $N = 4$ conformal superalgebras.

Lemma 2.3. *Let $\mathscr{A} = \Bbbk[\partial] \otimes_{\Bbbk} V$ be a $\Bbbk$–conformal superalgebra which is a free $\Bbbk[\partial]$–module such that V has a $\mathbb{Z}/2\mathbb{Z}$–grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Let $\mathcal{R} = (R, \mathsf{d}_R)$ be an arbitrary object in $\Bbbk$**-drng** and $\mathscr{B}$ an arbitrary $\mathcal{R}$–conformal superalgebra. Then:*

(i) Every homomorphism of $\mathcal{R}$–conformal superalgebras $\phi : \mathscr{A} \otimes_{\Bbbk} \mathcal{R} \to \mathscr{B}$ is completely determined by its restriction to $V \cong (\Bbbk \otimes V) \otimes \Bbbk \subseteq \mathscr{A} \otimes_{\Bbbk} \mathcal{R}$.

(ii) Let $\phi : V \otimes_{\Bbbk} R \to \mathscr{B}$ be a parity-preserving R–linear map, then ϕ can be uniquely extended to $\hat{\phi} : \mathscr{A} \otimes_{\Bbbk} \mathcal{R} \to \mathscr{B}$ such that $\hat{\phi} \circ \partial_{\mathscr{A} \otimes_{\Bbbk} \mathcal{R}} = \partial_{\mathscr{B}} \circ \hat{\phi}$. In addition, if $\widehat{\phi}([(v \otimes 1)_\lambda (w \otimes 1)]) = [\phi(v \otimes 1)_\lambda \phi(w \otimes 1)]$ for all $v, w \in V$, then $\hat{\phi}$ is a homomorphism of $\mathcal{R}$–conformal superalgebras.

The above lemma is a restatement of Lemma 3.1 of [KLP09], which gives the same result in the special case that $\mathscr{B} = \mathscr{A} \otimes_{\Bbbk} \mathcal{R}$ and ϕ is an automorphism.

Automorphisms of the centreless Virasora conformal superalgebra and the current conformal algebras were studied as examples in [KLP09].

Proposition 2.4. *Let $\mathscr{V}$ be the centreless Virasoro conformal algebra over $\Bbbk$ and $\mathcal{R} = (R, \mathsf{d})$ an object in $\Bbbk$**-drng** such that R is an integral domain. Then*

$$\mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{V} \otimes_{\Bbbk} \mathcal{R}) = 1. \tag{2.3.7}$$

This was proved in Proposition 3.19 of [KLP09] for the case in which $\mathcal{R} = \widehat{\mathcal{D}}$. The general case easily follows by repeating the proof in [KLP09].

Proposition 2.5 (Corollary 3.17 of [KLP09])**.** *Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\Bbbk$ and $\mathcal{R} = (R, \mathsf{d}_R)$ an object in $\Bbbk$-**drng**. If $\mathfrak{g} \otimes_{\Bbbk} R$ is a semi-simple Lie superalgebra over R (i.e., $\mathfrak{g} \otimes_{\Bbbk} R$ has no nontrivial abelian ideal), then*

$$\mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathrm{Cur}(\mathfrak{g}) \otimes_{\Bbbk} \mathcal{R}) = \mathrm{Aut}_{R\text{-}Lie}(\mathfrak{g} \otimes_{\Bbbk} R). \tag{2.3.8}$$

Remark 2.6. If $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra over $\Bbbk$, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, where $\mathfrak{g}_i$ is a simple Lie algebra over $\Bbbk$ for $i = 1, \cdots, n$ (cf. [Bou75, I,§6.2]). Further, every ideal of the Lie algebra $\mathfrak{g} \otimes_{\Bbbk} R$ is of the form

$$\mathfrak{a} = (\mathfrak{g}_1 \otimes_{\Bbbk} I_1) \oplus \cdots \oplus (\mathfrak{g}_n \otimes_{\Bbbk} I_n), \tag{2.3.9}$$

where I_i is an ideal of R, $i = 1, \cdots, n$. Hence, $\mathfrak{a}$ is abelian if and only if $I_i^2 = 0$ for all $i = 1, \cdots, n$. In particular, if $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra over $\Bbbk$ and R is an integral domain, then (2.3.8) holds.

Similarly, if $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra over $\Bbbk$ and R is an integral domain, (2.3.8) holds as well. However, if $\mathfrak{g}$ is a finite dimensional semi-simple Lie superalgebra, (2.3.8) is not necessarily true since a finite dimensional semi-simple Lie superalgebra can not be decomposed as a direct sum of simple Lie superalgebras in general (cf. Section 5.1 of [Kac77]).

2.4 Twisted forms of conformal superalgebras

We will discuss the theory of twisted forms of differential conformal superalgebras in this section. The theory of twisted forms of a usual (associative or Lie) algebra can be found in Section II.8 of [KO74].

Definition 2.7 (Definition 2.1 of [KLP09])**.** Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra and $\mathcal{R} \to \mathcal{S}$ a morphism in $\Bbbk$-**drng**. An $\mathcal{R}$–conformal superalgebra $\mathscr{B}$ is called an $\mathcal{S}/\mathcal{R}$–form of $\mathscr{A}$ if

$$\mathscr{B} \otimes_{\mathcal{R}} \mathcal{S} \cong \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S}, \tag{2.4.1}$$

as $\mathcal{S}$–conformal superalgebras.

In particular, the twisted loop conformal superalgebras can be understood as twisted forms, i.e., we have the following:

Proposition 2.8 (Proposition 2.4 of [KLP09])**.** *Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra and σ an automorphism of $\mathscr{A}$ of order m. Then the twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathscr{A} \otimes_\Bbbk \mathcal{D}$, i.e.,*

$$\mathcal{L}(\mathscr{A}, \sigma) \otimes_\mathcal{D} \mathcal{D}_m \cong (\mathscr{A} \otimes_\Bbbk \mathcal{D}) \otimes_\mathcal{D} \mathcal{D}_m \cong \mathscr{A} \otimes_\Bbbk \mathcal{D}_m, \tag{2.4.2}$$

as $\mathcal{D}_m$–conformal superalgebras, where $\mathcal{D} = (\Bbbk[t^{\pm 1}], \mathsf{d}_t)$ and $\mathcal{D}_m = (\Bbbk[t^{\pm \frac{1}{m}}], \mathsf{d}_t)$.

Recall from (2.1.17) that the twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ is a sub conformal superalgebra of $\mathscr{A} \otimes \mathcal{D}_m$. Thus the isomorphism in Proposition 2.8 can be given as follows:

$$\begin{aligned} \varphi : \mathcal{L}(\mathscr{A}, \sigma) \otimes_\mathcal{D} \mathcal{D}_m &\to \mathscr{A} \otimes_\Bbbk \mathcal{D}_m, \\ \left(\textstyle\sum a_i \otimes r_i\right) \otimes s &\mapsto \textstyle\sum a_i \otimes r_i s. \end{aligned} \tag{2.4.3}$$

Since every twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ of a $\Bbbk$–conformal superalgebra $\mathscr{A}$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathscr{A} \otimes_\Bbbk \mathcal{D}$, the first step in the classification of the twisted loop conformal algebras of $\mathscr{A}$ is to classify the $\mathcal{D}_m/\mathcal{D}$–forms of $\mathscr{A} \otimes_\Bbbk \mathcal{D}$.

In order to interpret how the twisted forms of a differential conformal superalgebra are classified in terms of non-abelian cohomology, we recall some basic concepts from the theory of commutative rings. Let $R \to S$ be a homomorphism of unital commutative rings. The scalar extension $- \otimes_R S$ is a functor from the category of R–modules to the category of S–modules. The homomorphism $R \to S$ is called *flat (resp. faithfully flat)* if the functor $- \otimes_R S$ is exact (resp. exact and faithful)[2].

Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra and $\mathcal{R} = (R, \mathsf{d}_R) \to \mathcal{S} = (S, \mathsf{d}_S)$ a *faithfully flat morphism* in $\Bbbk$**-drng** (that is, a morphism in $\Bbbk$**-drng** such that the morphism of underlying rings is faithfully flat). We will introduce the notion of the first non-abelian cohomology set $\mathrm{H}^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$ as an analog of the Čech cohomology for a sheaf of groups over the fppf topology (cf. [Mil80, III,§3]). Since

[2]More properties of a faithfully flat ring extension can be found in [Bou72, I,§3]

$\mathbf{Aut}(\mathscr{A})$ is a functor, the following morphisms in $\mathcal{R}$**-ext**,

$$\begin{aligned}
&\rho_1 : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad s \mapsto s \otimes 1,\\
&\rho_2 : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad s \mapsto 1 \otimes s,\\
&\rho_{12} : \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad r \otimes s \mapsto r \otimes s \otimes 1,\\
&\rho_{23} : \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad r \otimes s \mapsto 1 \otimes r \otimes s,\\
&\rho_{13} : \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad r \otimes s \mapsto r \otimes 1 \otimes s,
\end{aligned}$$

induce group homomorphisms

$$\widetilde{\rho}_i : \mathbf{Aut}(\mathscr{A})(\mathcal{S}) \to \mathbf{Aut}(\mathscr{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}), \quad i = 1, 2,$$

and

$$\widetilde{\rho}_{ij} : \mathbf{Aut}(\mathscr{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}) \to \mathbf{Aut}(\mathscr{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}), \quad 1 \leqslant i < j \leqslant 3.$$

We say that an element $\mathfrak{z} \in \mathbf{Aut}(\mathscr{A})(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S})$ is a 1–*cocycle* if

$$\widetilde{\rho}_{13}(\mathfrak{z}) = \widetilde{\rho}_{23}(\mathfrak{z})\widetilde{\rho}_{12}(\mathfrak{z}). \tag{2.4.4}$$

The set of 1–cocycles is denoted by $Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$. Two 1–cocycles $\mathfrak{z}$ and $\mathfrak{z}'$ are called *cohomologous*, notation $\mathfrak{z} \sim \mathfrak{z}'$, if there is an element $\mathfrak{z}_0 \in \mathbf{Aut}(\mathscr{A})(\mathcal{S})$ such that

$$\mathfrak{z}' = \widetilde{\rho}_2(\mathfrak{z}_0) \cdot \mathfrak{z} \cdot \widetilde{\rho}_1(\mathfrak{z}_0)^{-1}. \tag{2.4.5}$$

The cohomologous relation on $Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$ is an equivalence relation. The *non-abelian cohomology set* is defined to be the set of equivalent classes

$$\mathrm{H}^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A})) = \frac{Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))}{\sim}. \tag{2.4.6}$$

Theorem 2.9 (Theorem 2.16 of [KLP09])**.** *Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra and $\mathcal{R} \to \mathcal{S}$ a faithfully flat morphism in* $\Bbbk$**-drng**. *Then the set of isomorphism classes of $\mathcal{S}/\mathcal{R}$–forms of $\mathscr{A}$ bijectively corresponds to* $\mathrm{H}^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$.

To understand the correspondence in the theorem, we briefly explain how the correspondence is established. Let $\mathscr{B}$ be an $\mathcal{S}/\mathcal{R}$–form of $\mathscr{A}$, i.e., there is an isomorphism of $\mathcal{S}$–conformal superalgebras

$$\phi : \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S} \to \mathscr{B} \otimes_{\mathcal{R}} \mathcal{S}. \tag{2.4.7}$$

Applying the change of base differential rings:

$$\rho_1 : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, s \mapsto s \otimes 1 \text{ and } \rho_2 : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, s \mapsto 1 \otimes s,$$

we obtain two isomorphisms of $\mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}$–conformal superalgebras:

$$\phi_i : \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathscr{B} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, \quad i = 1, 2.$$

Let

$$\mathfrak{z} := \phi_2^{-1} \circ \phi_1 : \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S} \to \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}. \tag{2.4.8}$$

Then it can be verified that $\mathfrak{z} \in Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A})) \subseteq \mathbf{Aut}(\mathscr{A})(\mathcal{S} \otimes \mathcal{S})$, i.e., the $\mathcal{S}/\mathcal{R}$–form $\mathscr{B}$ defines a 1–cocycle $\mathfrak{z}$.

Conversely, given a 1–cocycle $\mathfrak{z}$, one can define

$$\mathscr{B}_{\mathfrak{z}} := \left\{ \sum a_i \otimes s_i \in \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S} | \mathfrak{z}(\sum a_i \otimes s_i \otimes 1) = \sum a_i \otimes 1 \otimes s_i \right\}. \tag{2.4.9}$$

It can be verified that $\mathscr{B}_{\mathfrak{z}}$ is an $\mathcal{R}$–conformal superalgebra and an $\mathcal{S}/\mathcal{R}$–form of $\mathscr{A}$. Indeed, the map

$$\mathscr{B}_{\mathfrak{z}} \otimes_{\mathcal{R}} \mathcal{S} \to \mathscr{A} \otimes_{\mathcal{R}} \mathcal{S}, \quad (a_i \otimes s_i) \otimes s \mapsto \sum a_i \otimes s_i s$$

is an isomorphism of $\mathcal{S}$–conformal superalgebras.

In addition, for two 1-cocycles $\mathfrak{z}, \mathfrak{z}' \in Z^1(\mathcal{S}/\mathcal{R}, \mathbf{Aut}(\mathscr{A}))$, $\mathscr{B}_{\mathfrak{z}}$ is isomorphic to $\mathscr{B}_{\mathfrak{z}'}$ as $\mathcal{R}$–conformal superalgebras if and only if $\mathfrak{z}$ is cohomologous to $\mathfrak{z}'$.

Given a $\Bbbk$–conformal superalgebra $\mathscr{A}$, a key step towards classifying the twisted loop conformal superalgebras $\mathcal{L}(\mathscr{A}, \sigma)$ is to classify the $\mathcal{D}_m/\mathcal{D}$–forms of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$. As a result of Theorem 2.9, the classification problem is reduced to computing $\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$. The methods for computing this non-abelian cohomology set will be reviewed in Chapter 3.

To conclude this section, we determine a 1-cocycle $\mathfrak{z} \in Z^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$ that represents the class in $\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$ corresponding to $\mathcal{L}(\mathscr{A}, \sigma)$. Recall that $\mathscr{A}$ has the decomposition

$$\mathscr{A} = \bigoplus_{\ell=0}^{m-1} \mathscr{A}_\ell, \quad \mathscr{A}_\ell = \{a \in \mathscr{A} | \sigma(a) = \zeta_m^\ell a\}. \tag{2.4.10}$$

Let $\varphi : \mathcal{L}(\mathscr{A}, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m \to \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m$ be the isomorphism of $\mathcal{D}_m$–conformal superalgebras given by (2.4.3). We have $\varphi^{-1} : \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \to \mathcal{L}(\mathscr{A}, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m$. In fact,

$$\varphi^{-1}(a \otimes r) = (a \otimes t^{\frac{\ell}{m}}) \otimes t^{-\frac{\ell}{m}} r,$$

if $a \in \mathscr{A}_\ell$ for $\ell = 0, \cdots, m-1$ and $r \in D_m$. Applying the changes of base differential rings $\rho_i : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{S}, i = 1, 2$ to φ^{-1} and φ respectively, we obtain

$$\begin{aligned} \varphi_1^{-1} : \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m &\to \mathcal{L}(\mathscr{A}, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m, \\ a \otimes r \otimes s &\mapsto (a \otimes t^{\frac{\ell}{m}}) \otimes t^{-\frac{\ell}{m}} r \otimes s, \text{ if } a \in \mathscr{A}_\ell, \\ \varphi_2 : \mathcal{L}(\mathscr{A}, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m &\to \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m, \\ (\textstyle\sum a_i \otimes s_i) \otimes r \otimes s &\mapsto \textstyle\sum a_i \otimes r \otimes s_i s. \end{aligned}$$

Hence, we obtain a 1–cocycle associated to $\mathcal{L}(\mathscr{A}, \sigma)$:

$$\begin{aligned} \mathfrak{z} := \varphi_2 \circ \varphi_1^{-1} : \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m &\to \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m, \\ a \otimes r \otimes s &\mapsto a \otimes t^{-\frac{\ell}{m}} r \otimes t^{\frac{\ell}{m}} s, \quad \text{if } a \in \mathscr{A}_\ell. \end{aligned} \tag{2.4.11}$$

In Section 3.1, we will provide an alternative description of $\mathfrak{z}$ as a continuous 1-cocycle of $\mathbb{Z}/m\mathbb{Z}$ in $\mathbf{Aut}(\mathscr{A})(\mathcal{D}_m)$.

2.5 The centroid of conformal superalgebras

Given a $\Bbbk$–conformal superalgebra $\mathscr{A}$, the classification of the twisted forms of $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$ yields the classification of the twisted loop conformal superalgebras based on $\mathscr{A}$ up to isomorphism of $\mathcal{D}$–conformal superalgebras. To complete the classification up to isomorphism of $\Bbbk$–conformal superalgebras, one needs to deduce from isomorphisms over $\mathcal{D}$ to isomorphisms over $\Bbbk$. An essential tool to realize this passage is the centroid trick introduced in [KLP09].

The centroid trick has been used to deduce from D–linear isomorphisms to $\Bbbk$–linear isomorphisms in the case of twisted loop Lie algebras in [ABP04]. A more general discussion on the centroid of extended affine Lie algebras can be found in [BN06]. In this section, we will focus on the centroid of conformal superalgebras.

Let $\mathscr{A}$ be an $\mathcal{R}$–conformal superalgebra, the *centroid of* $\mathscr{A}$ is defined to be the set $\mathrm{Ctd}_{\mathcal{R}}(\mathscr{A})$ consisting of R–module endomorphisms $\chi : \mathscr{A} \to \mathscr{A}$ such that χ preserves the $\mathbb{Z}/2\mathbb{Z}$–grading on $\mathscr{A}$ and

$$\chi(a_{(n)}b) = a_{(n)}\chi(b), \tag{2.5.1}$$

for all $a, b \in \mathscr{A}$. By axiom (DC3), $\mathrm{Ctd}_{\mathcal{R}}(\mathscr{A})$ is an R–module. Furthermore, there is a canonical map

$$R \to \mathrm{Ctd}_{\mathcal{R}}(\mathscr{A}), \quad r \mapsto r_{\mathscr{A}}, \tag{2.5.2}$$

where $r_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}, a \mapsto ra$.

The $\mathcal{R}$–conformal superalgebra $\mathscr{A}$ can also be viewed as a $\Bbbk$–conformal superalgebra via the restriction of scalars. Thus

$$\mathrm{Ctd}_{\mathcal{R}}(\mathscr{A}) \subseteq \mathrm{Ctd}_{\Bbbk}(\mathscr{A}), \tag{2.5.3}$$

and we obtain a canonical map $R \to \mathrm{Ctd}_{\Bbbk}(\mathscr{A})$.

Proposition 2.10 (Proposition 2.35 of [KLP09])**.** *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two $\mathcal{R}$–conformal superalgebras. If*

(i) $\mathrm{Aut}_{\Bbbk}(\mathcal{R}) = 1$*, and*

(ii) the canonical maps $R \to \mathrm{Ctd}_{\Bbbk}(\mathscr{A}_i)$ are $\Bbbk$–algebra isomorphisms for $i = 1, 2$,

then $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic as $\Bbbk$–conformal superalgebras if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic as $\mathcal{R}$–conformal superalgebras.

Every twisted loop conformal superalgebra is a $\mathcal{D}$–conformal superalgebra for $\mathcal{D} = (\Bbbk[t^{\pm 1}], \frac{d}{dt})$. We immediately see that the automorphism group $\mathrm{Aut}_{\Bbbk}(\mathcal{D})$ is trivial since the only automorphism of the $\Bbbk$–algebra $\Bbbk[t^{\pm 1}]$ commuting with the derivation $\frac{d}{dt}$ is the identity map. For each of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras $\mathscr{A}$, it has been shown that the centroid of each twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ is isomorphic to $\Bbbk[t^{\pm 1}]$ (cf. [CP11], [KLP09], and [CP13], respectively). In the rest of this section, we will prove a general proposition which covers all of these results.

Lemma 2.11. *Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra and* $\mathrm{L} \in \mathscr{A}_{\bar{0}}$. *If $\mathscr{A}$ is generated by $\{a_1, \cdots, a_n\}$ as a $\Bbbk[\partial]$–module and $[\mathrm{L}_\lambda a_i] = (\partial + \Delta_i \lambda) a_i$ with $\Delta_i \neq 0$ for $i = 1, \cdots, n$, then, for an arbitrary object $\mathcal{R} = (R, \mathsf{d})$ in* $\Bbbk$**-drng**,

$$\begin{aligned}\mathscr{A}_{\mathcal{R}} &= \mathrm{span}_{\Bbbk}\{(a_i \otimes r_i)_{(j)}(\partial^{(\ell)}\mathrm{L} \otimes 1) | j = 0, 1, r_i \in R, i = 1, \cdots, n, \ell \geqslant 0\} \\ &= \mathrm{span}_{\Bbbk}\{\eta_{(j)}(\partial^{(\ell)}\mathrm{L} \otimes 1) | \eta \in \mathscr{A}_{\mathcal{R}}, j = 0, 1, \ell \geqslant 0\}.\end{aligned}$$

Proof. Since $\mathscr{A}$ is generated by $\{a_1, \cdots, a_n\}$ as a $\Bbbk[\partial]$–module, every element of $\mathscr{A}_{\mathcal{R}}$ is a $\Bbbk$–linear combination of elements of the form $(\partial^\ell a_i) \otimes s_i$ with $s_i \in R$. Hence, it suffices to show $(\partial^\ell a_i) \otimes s_i$ can be written as a $\Bbbk$–linear combination of elements of the form $(a_i \otimes r_i)_{(j)}(\partial^{(\ell')}\mathrm{L} \otimes 1)$ with $r_i \in R$ and $\ell' \geqslant 0$.

Since $[\mathrm{L}_\lambda a_i] = (\partial + \Delta_i \lambda) a_i$, from (C2), we deduce that

$$\begin{aligned}(a_i)_{(j)}\partial^{(\ell-1)}\mathrm{L} &= -\sum_{k=0}^{\infty}(-1)^{j+k}\partial^{(k)}((\partial^{(\ell-1)}\mathrm{L})_{(k+j)}a_i) \\ &= -\sum_{k=0}^{\infty}(-1)^{j+k+\ell-1}\binom{k+j}{\ell-1}\partial^{(k)}(\mathrm{L}_{(k+j-\ell+1)}a_i) \\ &= \begin{cases}(\ell(\Delta_i - 1) + j)\partial^{(\ell-j)}a_i, & \text{if } \ell \geqslant j, \\ 0, & \text{if } \ell < j,\end{cases}\end{aligned}$$

for $j \geqslant 0$ and $\ell \geqslant 1$.

If $\Delta_i \neq 1$, we have

$$\begin{aligned}&(a_i \otimes s_i)_{(1)}(\mathrm{L} \otimes 1) = \Delta_i a_i \otimes s_i, \\ &(a_i \otimes s_i)_{(0)}(\partial^{(\ell)}\mathrm{L} \otimes 1) = \sum_{j \geqslant 0}((a_i)_{(j)}\partial^{(\ell)}\mathrm{L}) \otimes \mathsf{d}^{(j)}(s_i) \\ &\qquad = (\Delta_i - 1)(\ell + 1)\partial^{(\ell+1)}a_i \otimes s_i \\ &\qquad\quad + \sum_{j=1}^{\ell+1}((\Delta_i - 1)(\ell + 1) + j)\partial^{(\ell+1-j)}a_i \otimes \mathsf{d}^{(j)}(s_i),\end{aligned}$$

for all $\ell \geqslant 0$. Since $\Delta_i \neq 0, 1$, using induction on ℓ, we obtain every $\partial^{(\ell)}a_i \otimes s_i$ is a $\Bbbk$–linear combination of elements of the form $(a_i \otimes r_i)_{(j)}(\partial^{(\ell')}\mathrm{L} \otimes 1)$ with $\ell' \geqslant 0$.

Similarly, if $\Delta_i = 1$, we deduce that

$$\begin{aligned}
&(a_i \otimes s_i)_{(1)}(\mathrm{L} \otimes 1) = a_i \otimes s_i,\\
&(a_i \otimes s_i)_{(1)}(\partial^{(\ell)}\mathrm{L} \otimes 1) = \sum_{j\geqslant 0}((a_i)_{(j+1)}\partial^{(\ell)}\mathrm{L}) \otimes \mathsf{d}^{(j)}(s_i)\\
&\qquad\qquad = \sum_{j=0}^{\ell}(j+1)\partial^{(\ell-j)}a_i \otimes \mathsf{d}^{(j)}(s_i)\\
&\qquad\qquad = \partial^{(\ell)}a_i \otimes s_i + \sum_{j=1}^{\ell}(j+1)\partial^{(\ell-j)}a \otimes \mathsf{d}^{(j)}(s_i).
\end{aligned}$$

for $\ell \geqslant 1$. Again, by induction on ℓ, every $\partial^{(\ell)}a_i \otimes s_i$ is a $\Bbbk$–linear combination of elements of the form $(a_i \otimes r_i)_{(j)}(\partial^{(\ell')}\mathrm{L} \otimes 1)$ with $\ell' \geqslant 0$. □

In a $\Bbbk$–conformal superalgebra $\mathscr{A}$, an element $\mathrm{L} \in \mathscr{A}_{\bar{0}}$ is called a *Virasoro element* if $[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}$. An element $a \in \mathscr{A}$ is called a *primary eigenvector* with respect to L of conformal weight Δ if $[\mathrm{L}_\lambda a] = (\partial + \Delta\lambda)a$.

Proposition 2.12. *Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra and σ an automorphism of $\mathscr{A}$ of order m. Suppose $\mathscr{A}$ and σ satisfy all of the following conditions:*

(i) *$\mathscr{A}$ has a Virasoro element $\mathrm{L} \in \mathscr{A}_{\bar{0}}$ fixed by σ, i.e., $\sigma(\mathrm{L}) = \mathrm{L}$.*

(ii) *$\mathscr{A}_{\bar{0}}$ is a free $\Bbbk[\partial]$–module of finite rank that has a basis $\{a_1 = \mathrm{L}, a_2, \cdots, a_{n_0}\}$ such that $[\mathrm{L}_\lambda a_i] = (\partial + \lambda)a_i$, for $i = 2, \cdots, n_0$.*

(iii) *There are $b_1, \cdots, b_{n_1} \in \mathscr{A}_{\bar{1}}$ generating $\mathscr{A}_{\bar{1}}$ as a $\Bbbk[\partial]$–module such that $[\mathrm{L}_\lambda b_i] = (\partial + \Delta'_i\lambda)b_i$ with $\Delta'_i \neq 0$ for $i = 1, \cdots, n_1$.*

Then $\mathrm{Ctd}_{\Bbbk}(\mathcal{L}(\mathscr{A}, \sigma)) = D$.

Proof. We denote $\mathcal{L}(\mathscr{A}, \sigma)$ by $\mathscr{B}$. For $r \in D$, there is an element $r_{\mathscr{B}} \in \mathrm{Ctd}_{\Bbbk}(\mathscr{B})$ given by $v \mapsto rv$, and hence $D \subseteq \mathrm{Ctd}_{\Bbbk}(\mathscr{B})$. Conversely, let $\chi \in \mathrm{Ctd}_{\Bbbk}(\mathscr{B})$. We will show that χ is of the form $r_{\mathscr{B}}$ for some $r \in D$.

We consider the $\Bbbk$–linear map[3]

$$\pi : \mathscr{A}_{\mathcal{D}_m} \to \mathscr{A}_{\mathcal{D}_m}, \quad \eta \mapsto \frac{1}{m}\sum_{i=0}^{m-1}(\sigma \otimes \psi)^i(\eta),$$

[3]The $\Bbbk$–linear map π is neither a homomorphism of $\Bbbk$–conformal superalgebras in general, nor a homomorphism of $\mathcal{D}$–conformal superalgebras.

where $\psi : D_m \to D_m, t^{\frac{1}{m}} \mapsto \zeta_m^{-1} t^{\frac{1}{m}}$. Then $\mathscr{B} = \pi(\mathscr{A}_{\mathcal{D}_m})$.

Observing that $\sigma(\mathrm{L}) = \mathrm{L}$, we obtain that $\pi(\mathrm{L} \otimes 1) = \mathrm{L} \otimes 1 \in \mathscr{B}$. We first claim that $\chi(\mathrm{L} \otimes 1) = \mathrm{L} \otimes r$ for some $r \in D$.

We write

$$\chi(\mathrm{L} \otimes 1) = \sum_{\substack{i=1,\cdots,n_0 \\ \ell \geqslant 0}} \partial^{(\ell)} a_i \otimes s_{il},$$

where $s_{il} \in D_m$ and all but finitely many $s_{il} = 0$.

Since χ is an element in the centroid, we have

$$(\mathrm{L} \otimes 1)_{(1)} \chi(\mathrm{L} \otimes 1) = 2\chi(\mathrm{L} \otimes 1), \tag{2.5.4}$$

$$(\mathrm{L} \otimes 1)_{(2)} \chi(\mathrm{L} \otimes 1) = 0. \tag{2.5.5}$$

From (2.5.4),

$$\begin{aligned} 2 \sum_{\substack{i=1,\cdots,n_0 \\ \ell \geqslant 0}} \partial^{(\ell)} a_i \otimes s_{il} &= \sum_{\substack{i=1,\cdots,n_0 \\ \ell \geqslant 0}} (\mathrm{L} \otimes 1)_{(1)} (\partial^{(\ell)} a_i \otimes s_{il}) \\ &= \sum_{\ell \geqslant 0} (\ell + 2) \partial^{(\ell)} a_1 \otimes s_{1l} + \sum_{\substack{i=2,\cdots,n_0 \\ \ell \geqslant 0}} (\ell + 1) \partial^{(\ell)} a_i \otimes s_{il}. \end{aligned}$$

We conclude that $s_{1\ell} = 0$ for $\ell \neq 0$ and $s_{i\ell} = 0$ for $i \neq 2, \cdots, n_0$ and $\ell \neq 1$. Hence,

$$\chi(\mathrm{L} \otimes 1) = \mathrm{L} \otimes s_{00} + \sum_{i=2}^{n_0} \partial a_i \otimes s_{i1}.$$

Further, we deduce from (2.5.5) that

$$0 = (\mathrm{L} \otimes 1)_{(2)} \left(\mathrm{L} \otimes s_{00} + \sum_{i=2}^{n_0} \partial a_i \otimes s_{i1} \right) = 2 \sum_{i=2}^{n_0} a_i \otimes s_{i1}.$$

This yields $s_{i1} = 0$ for $i = 2, \cdots, n_0$, and hence, $\chi(\mathrm{L} \otimes 1) = \mathrm{L} \otimes r$ where $r = s_{00}$. Note that since $\mathrm{L} \otimes 1 \in \mathscr{B}$, we obtain that $r \in D$.

Next, we will show that $\chi = r_{\mathscr{B}}$. We first observe that

$$\ell \cdot \chi(\partial^{(\ell)} \mathrm{L} \otimes 1) = \chi((\mathrm{L} \otimes 1)_{(0)} (\partial^{(\ell-1)} \mathrm{L} \otimes 1)) = (\mathrm{L} \otimes 1)_{(0)} \chi(\partial^{(\ell-1)} \mathrm{L} \otimes 1),$$

for all $\ell \geqslant 1$. By induction, we obtain

$$\chi(\partial^{(\ell)} \mathrm{L} \otimes 1) = \partial^{(\ell)} \mathrm{L} \otimes r, \quad \ell \geqslant 0.$$

Recall that $\mathscr{B} = \pi(\mathscr{A}_{\mathcal{D}_m})$. Since $\sigma(\mathrm{L}) = \mathrm{L}$, by assumption (i) we have

$$\begin{aligned}
\pi(\eta_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1)) &= \frac{1}{m}\sum_{i=0}^{m-1}(\sigma \otimes \psi)^i(\eta_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1)) \\
&= \frac{1}{m}\sum_{i=0}^{m-1}((\sigma \otimes \psi)^i(\eta))_{(k)}(\sigma \otimes \psi)^i(\partial^{(\ell)}\mathrm{L} \otimes 1) \\
&= \frac{1}{m}\sum_{i=0}^{m-1}((\sigma \otimes \psi)^i(\eta))_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1) \\
&= \pi(\eta)_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1),
\end{aligned}$$

for $\eta \in \mathscr{A}_{\mathcal{D}_m}$ and $k = 0, 1, \ell \geqslant 0$. Since $\mathscr{A}$ satisfies (ii) and (iii), by Lemma 2.11, we deduce that

$$\begin{aligned}
\mathscr{B} = \pi(\mathscr{A}_{\mathcal{D}_m}) &= \mathrm{span}_{\Bbbk}\{\pi(\eta)_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1) | \eta \in \mathscr{A}_{\mathcal{D}_m}, k = 0, 1, \ell \geqslant 0\}, \\
&= \mathrm{span}_{\Bbbk}\{\eta'_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1) | \eta' \in \mathscr{B}, k = 0, 1, \ell \geqslant 0\}.
\end{aligned}$$

We also deduce that

$$\chi(\eta_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1)) = \eta_{(k)}\chi(\partial^{(\ell)}\mathrm{L} \otimes 1) = \eta_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes r) = r(\eta_{(k)}(\partial^{(\ell)}\mathrm{L} \otimes 1)),$$

for $\eta \in \mathscr{B}$ and $k = 0, 1$ and $\ell \geqslant 0$. Hence, $\chi = r_{\mathscr{B}}$. □

Although the conditions in Proposition 2.12 seem complicated, they can easily be verified if the concrete descriptions of a $\Bbbk$–conformal superalgebra $\mathscr{A}$ and the automorphism σ are known. We will use this proposition to determine the centroid of all twisted loop conformal superalgebras based on each of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras in Propositions 4.13, 5.12, and 6.8, respectively. It is also obvious that these conditions are satisfied by the centreless Virasoro conformal superalgebra paired with any of its automorphisms of finite order. However, it can not be applied to a twisted loop conformal superalgebra based on a current conformal algebra $\mathrm{Cur}(\mathfrak{g})$ due to the absence of a Virasoro element in $\mathrm{Cur}(\mathfrak{g})$.

Chapter 3

Non-Abelian Galois Cohomology

Following the general theory of twisted forms of differential conformal superalgebras as described in Chapter 2, one of the key steps in the classification of the twisted loop conformal superalgebras based on a $\Bbbk$–conformal superalgebra $\mathscr{A}$ is to determine $\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}_{\mathcal{D}}))$, where $\mathcal{D} = (\Bbbk[t^{\pm 1}], \mathsf{d}_t), \mathcal{D}_m = (\Bbbk[t^{\pm \frac{1}{m}}], \mathsf{d}_t)$ and m is a positive integer. In this chapter, we will review the methods for computing this "differential" type of H^1, which were used in the case where $\mathscr{A}$ is the $N = 2$ or small $N = 4$ conformal superalgebras in [KLP09].

Let us briefly outline these methods. First, since $\mathcal{D}_m/\mathcal{D}$ is a Galois extension, the non-abelian cohomology set $\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}_{\mathcal{D}}))$ can be identified with the non-ableian continuous cohomology set $\mathrm{H}^1_{\text{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}))$. Further, to classify all twisted loop conformal superalgebras based on $\mathscr{A}$, we have to deal with extensions $\mathcal{D}_m/\mathcal{D}$ for all m. Instead of working on the extension $\mathcal{D}_m/\mathcal{D}$ for each m individually, we take $\widehat{\mathcal{D}} := \varinjlim \mathcal{D}_m$ and consider $\widehat{\mathcal{D}}/\mathcal{D}$. Consequently, our problem is reduced to computing $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}}))$.

In concrete examples, $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}})$ is related to the $\widehat{D}$–points of certain affine group schemes, which suggests that we should compute $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}(\widehat{D}))$ for an affine group scheme $\mathbf{G}$. When $\mathbf{G}$ is a reductive group scheme over D, this set can be further identified with the non-abelian étale cohomology set $\mathrm{H}^1_{\text{ét}}(D, \mathbf{G})$.

3.1 Non-abelian continuous cohomology

We will explain in this section how non-abelian continuous cohomology emerges into the study of conformal superalgebras. Let Γ be a *profinite group*, that is, a topological group isomorphic to the inverse limit of a projective system of finite discrete groups. A topological group G is said to be a Γ-*group* if there is a continuous action $\Gamma \times \mathrm{G} \to \mathrm{G}, (\gamma, g) \mapsto {}^{\gamma}g$ such that ${}^{\gamma}(g_1 g_2) = {}^{\gamma}g_1 {}^{\gamma}g_2$.

Given a profinite group Γ and a Γ-group G, one may define the non-abelian continuous cohomology of Γ with coefficients in G as follows: a 1-*cocycle* is a continuous map $\mathfrak{z} : \Gamma \to \mathrm{G}, \gamma \mapsto \mathfrak{z}_\gamma$ such that

$$\mathfrak{z}_{\gamma_1\gamma_2} = \mathfrak{z}_{\gamma_1} \cdot {}^{\gamma_1}(\mathfrak{z}_{\gamma_2}), \quad \forall \gamma_1, \gamma_2 \in \Gamma. \tag{3.1.1}$$

We usually write $\mathfrak{z}$ as $(\mathfrak{z}_\gamma)_{\gamma\in\Gamma}$, and the set of 1-cocycles is denoted by $Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$. Two 1-cocycles $\mathfrak{z}, \mathfrak{z}'$ are cohomologous (denoted by $\mathfrak{z} \sim \mathfrak{z}'$) if there exists an element $g \in \mathrm{G}$ such that

$$\mathfrak{z}_\gamma = g^{-1} \cdot \mathfrak{z}'_\gamma \cdot {}^{\gamma}g, \forall \gamma \in \Gamma. \tag{3.1.2}$$

It can be verified that the cohomologous relation is an equivalence relation on $Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$. The set of equivalent classes

$$\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}) = Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G})/ \sim \tag{3.1.3}$$

is called the *non-abelian cohomology set of* Γ *with coefficients in* G. For $\mathfrak{z} \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$, we use $[\mathfrak{z}]$ to denote the cohomology class in $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$ containing $\mathfrak{z}$.

Remark 3.1. $\mathrm{H}^1(\Gamma, \mathrm{G})$ does not have a group structure in general (unless G is commutative). However, we observe that $\Gamma \to \mathrm{G}, \gamma \mapsto 1$ is a 1-cocycle, where 1 is the identity element in G. This 1-cocycle is denoted by 1 and its cohomology class $[1]$ in $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$ is called the distinguished element of $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$.

Before moving on to a discussion on conformal superalgebras, we state several basic properties of $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$ below. The proofs can be found in the book [Ser02].

Proposition 3.2. *Let* Γ *be a profinite group.*

(i) If G_1 *and* G_2 *are two* Γ*–groups, then* $\mathrm{G}_1 \times \mathrm{G}_2$ *is also a* Γ*–group with the piecewise* Γ*–action. Moreover,*

$$\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1 \times \mathrm{G}_2) = \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \times \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2). \tag{3.1.4}$$

(ii) For a Γ*–group* G*,*

$$\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}) = \varinjlim \mathrm{H}^1_{\mathrm{ct}}(\Gamma/\Gamma', \mathrm{G}^{\Gamma'}), \tag{3.1.5}$$

where Γ' *runs over all open normal subgroups of* Γ *and* $\mathrm{G}^{\Gamma'}$ *is the set of points in* G *fixed by* Γ'*, i.e.* $\mathrm{G}^{\Gamma'} = \{g \in \mathrm{G} | {}^{\gamma}g = g, \forall \gamma \in \Gamma'\}$*. ([Ser02, I.5.1])* *(iii)* *Let* $1 \to \mathrm{G}_1 \to \mathrm{G}_2 \to \mathrm{G}_3 \to 1$ *be a short exact sequence of* Γ*–groups. Then the sequence of pointed sets*

$$0 \to \mathrm{G}_1^\Gamma \to \mathrm{G}_2^\Gamma \to \mathrm{G}_3^\Gamma \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3) \quad (3.1.6)$$

is exact. ([Ser02, I. Proposition 38])

(iv) *With the same assumption as in (iii), if in addition* G_1 *is central in* G_2*, then the sequence of pointed sets*

$$\begin{aligned} 0 \to &\mathrm{G}_1^\Gamma \to \mathrm{G}_2^\Gamma \to \mathrm{G}_3^\Gamma \\ &\to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3) \to \mathrm{H}^2_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \end{aligned} \quad (3.1.7)$$

is exact. ([Ser02, I. Proposition 43])

Now, we move on to the discussion on conformal superalgebras. Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra. In order to compute $\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$, we first observe that D_m/D is a Galois extension of rings with Galois group $\mathbb{Z}/m\mathbb{Z}$ in the following sense:

Definition 3.3 (Proposition 5.6 of [KO74])**.** Let R be a ring and S/R a ring extension. Let Γ be a finite group of R–automorphisms of S. Then S/R is said to be a Galois extension with Galois group Γ if S is a faithfully flat R–module and the map

$$S \otimes_R S \to \overbrace{S \times \cdots \times S}^{|\Gamma| \text{ copies}}, \quad s_1 \otimes s_2 \mapsto (\gamma(s_1)s_2)_{\gamma \in \Gamma}, \quad (3.1.8)$$

is an isomorphism of S–algebras.

For the Galois extension D_m/D, the action of $\mathbb{Z}/m\mathbb{Z} = \langle \bar{1} \rangle$ on $D_m = \Bbbk[t^{\pm \frac{1}{m}}]$ is given by ${}^{\bar{1}}t^{\frac{1}{m}} = \zeta_m^{-1} t^{\frac{1}{m}}$, and which is compatible with the derivation d_t. Hence, there is an isomorphism of $\Bbbk$–differential rings

$$\mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m \to \overbrace{\mathcal{D}_m \times \cdots \times \mathcal{D}_m}^{m \text{ copies}}. \quad (3.1.9)$$

Under this identification,

$$\begin{aligned} \mathbf{Aut}(\mathscr{A})(\mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m) &\cong \mathbf{Aut}(\mathscr{A})(\mathcal{D}_m \times \cdots \times \mathcal{D}_m) \\ &\cong \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}) \times \cdots \times \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}). \end{aligned}$$

Hence, a 1-cocycle $\mathfrak{z} \in Z^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A})) \subseteq \mathbf{Aut}(\mathscr{A})(\mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m)$, as defined in (2.4.4), yields an m-tuple $(\mathfrak{z}_\gamma)_{\gamma\in\mathbb{Z}/m\mathbb{Z}}$ under the above isomorphism. In fact, $(\mathfrak{z}_\gamma)_{\gamma\in\mathbb{Z}/m\mathbb{Z}}$ defines an element in $Z^1_{\text{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}))$. It has been proved in [KLP09] that this indeed yields an isomorphism

$$\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A})) \cong \mathrm{H}^1_{\text{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m})). \tag{3.1.10}$$

Given an automorphism σ of $\mathscr{A}$ of order m, the twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$ corresponds to the cohomology class $[\mathfrak{z}] \in \mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$, where $\mathfrak{z} \in \mathbf{Aut}(\mathscr{A})(\mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m)$ is given by

$$\begin{aligned}\mathfrak{z}: \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m &\to \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \otimes_{\mathcal{D}} \mathcal{D}_m,\\ a \otimes r \otimes s &\mapsto a \otimes t^{-\frac{\ell}{m}} r \otimes t^{\frac{\ell}{m}} s,\end{aligned}$$

if $a \in \mathscr{A}_\ell = \{a \in \mathscr{A} | \sigma(a) = \zeta_m^\ell a\}$.

By applying the isomorphism in (3.1.10), the cohomology class $[\mathfrak{z}]$ can be identified with the cohomology class $[\mathfrak{z}'] \in \mathrm{H}^1_{\text{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}))$, where $\mathfrak{z}' = (\mathfrak{z}'_{\bar{i}})_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}$ is determined[1] by $\mathfrak{z}'_{\bar{1}} = \sigma \otimes \mathrm{id}_{\mathcal{D}_m} \in \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m})$.

Conversely, given a 1-cocycle $\mathfrak{z}' = (\mathfrak{z}'_{\bar{i}})_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}} \in Z^1_{\text{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m}))$, the $\mathcal{D}_m/\mathcal{D}$–form of $\mathscr{A}_{\mathcal{D}}$ associated to $[\mathfrak{z}']$ can be written as

$$\mathscr{B}_{\mathfrak{z}'} = \left\{\eta \in \mathscr{A} \otimes_{\Bbbk} \mathcal{D}_m \middle| \mathfrak{z}'_{\bar{i}}({}^{\bar{i}}\eta) = \eta, \forall \bar{i} \in \mathbb{Z}/m\mathbb{Z}\right\}.$$

To classify all twisted loop conformal superalgebras based on $\mathscr{A}$, we will compare $\mathcal{L}(\mathscr{A}, \sigma)$ and $\mathcal{L}(\mathscr{A}, \sigma')$. If σ is of order m and σ' is of order m', $\mathcal{L}(\mathscr{A}, \sigma)$ (resp. $\mathcal{L}(\mathscr{A}, \sigma')$) is a $\mathcal{D}_m/\mathcal{D}$–form (resp. a $\mathcal{D}_{m'}/\mathcal{D}$–form) of $\mathscr{A}_{\mathcal{D}}$. Considering the canonical inclusions $\mathcal{D}_m \hookrightarrow \mathcal{D}_{mm'}$ and $\mathcal{D}_{m'} \hookrightarrow \mathcal{D}_{mm'}$, both $\mathcal{L}(\mathscr{A}, \sigma)$ and $\mathcal{L}(\mathscr{A}, \sigma')$ are $\mathcal{D}_{mm'}/\mathcal{D}$–forms of $\mathscr{A}_{\mathcal{D}}$. In order to deal with all twisted loop conformal superalgebras based on $\mathscr{A}$ at once, we let $\widehat{\mathcal{D}} = \varinjlim \mathcal{D}_m$. Then every $\mathcal{L}(\mathscr{A}, \sigma)$ is a $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathscr{A}_{\mathcal{D}}$. Thus it is natural for us to compute $\mathrm{H}^1(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A}))$.

On one hand, the inclusion $\mathcal{D}_m \hookrightarrow \widehat{\mathcal{D}}$ naturally induces a map

$$\mathrm{H}^1(\mathcal{D}_m/\mathcal{D}, \mathbf{Aut}(\mathscr{A})) \to \mathrm{H}^1(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A})),$$

[1] The 1–cocycle $\mathfrak{z}'$ is determined by $\mathfrak{z}'_{\bar{1}}$ since $\mathbb{Z}/m\mathbb{Z}$ is a cyclic group.

for each m. This equation in combination with the isomorphism (3.1.10) yields a map

$$\varinjlim \mathrm{H}^1_{\mathrm{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m})) \to \mathrm{H}^1(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A})). \tag{3.1.11}$$

On the other hand, since $\mathbb{Z}/m\mathbb{Z}$ acts continuously on D_m, the profinite group $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$ acts continuously on $\widehat{D} = \varinjlim D_m$. This action is compatible with the derivation d_t, so $\widehat{\mathbb{Z}}$ acts on $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}})$ continuously. From Proposition 3.2 (ii), we have

$$\varinjlim \mathrm{H}^1_{\mathrm{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathrm{Aut}_{\mathcal{D}_m\text{-conf}}(\mathscr{A}_{\mathcal{D}_m})) \cong \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}})) \tag{3.1.12}$$

Moreover, the following proposition ensures that the map (3.1.11) is indeed a bijection when $\mathscr{A}$ satisfies a certain finiteness condition.

Proposition 3.4 (Proposition 2.29 of [KLP09])**.** *Let $\mathscr{A}$ be a $\Bbbk$–conformal superalgebra which is a finitely generated $\Bbbk[\partial]$–module. Then*

$$\mathrm{H}^1\left(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{A}_{\mathcal{D}})\right) \cong \mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-}conf}(\mathscr{A}_{\widehat{\mathcal{D}}})\right). \tag{3.1.13}$$

This proposition reduces the difficulties in classifying twisted loop conformal superalgebras based on $\mathscr{A}$ to two key computations: the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}})$ and the corresponding non-abelian continuous cohomology set. We will concretely address these issues for the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras in subsequent chapters.

3.2 Affine group schemes

According to our computations from concrete examples in the chapters to follow, the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A}_{\widehat{\mathcal{D}}})$ often turns out to be the $\widehat{D}$–points of certain affine group schemes. In this section, we will review the basic definition of an affine group scheme and state some known results on the non-abelian continuous cohomology set $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}(\widehat{D}))$ for an affine group scheme $\mathbf{G}$.

Let R be a ring. An *affine group scheme* $\mathbf{G}$ over R is a representable functor

$$\mathbf{G} : R\text{-}\mathbf{rng} \to \mathbf{grp}, \tag{3.2.1}$$

where R-**rng** is the category of commutative associative unital R–algebras and **grp** is the category of groups. The functor $\mathbf{G}$ is called *representable* if

$$\mathbf{G} = \mathrm{Hom}_{R\text{-}\mathbf{rng}}(R[\mathbf{G}], -), \tag{3.2.2}$$

for some $R[\mathbf{G}]$ in R-**rng**, which is called the coordinate ring of $\mathbf{G}$. By Yoneda's Lemma, the group structure on $\mathbf{G}$ is translated to the coassociative Hopf algebra structure on $R[\mathbf{G}]$ (cf. [Wat79]). For an object S in R-**rng**, we call the elements of $\mathbf{G}(S)$ the S–points of $\mathbf{G}$.

Example 3.5. Let R be a ring and S an object in R-**rng**. We present several affine group schemes by describing their S–points and their coordinate rings.

(i) The multiplicative group scheme $\mathbf{G}_m$:
$\mathbf{G}_m(S) = S^\times$ is the group of multiplicative units in S.
$R[\mathbf{G}_m] = R[t^{\pm 1}]$.

(ii) The additive group scheme $\mathbf{G}_a$:
$\mathbf{G}_a(S) = S$ is viewed as a group under addition.
$R[\mathbf{G}_a] = R[t]$.

(iii) The general linear group $\mathbf{GL}_n$ for $n \geqslant 1$:
$\mathbf{GL}_n(S)$ is the group of invertible $n \times n$–matrices with entries in S.
$R[\mathbf{GL}_n] = R[x_{ij}, \det(x_{ij})^{-1}]_{1 \leqslant i,j \leqslant n}$.

(iv) The special linear group $\mathbf{SL}_n$ for $n \geqslant 1$:
$\mathbf{SL}_n(S)$ is the group of $n \times n$–matrices with entries in S and determinant 1.
$R[\mathbf{SL}_n] = R[x_{ij}]_{1 \leqslant i,j \leqslant n}/\langle \det(x_{ij}) - 1\rangle$.

(v) The orthogonal group $\mathbf{O}_n$ for $n \geqslant 1$:
$\mathbf{O}_n(S) = \{A \in \mathrm{Mat}_n(S) | AA^T = I_n\}$, where A^T is the transpose of A and I_n is the $n \times n$ identity matrix.
$R[\mathbf{O}_n] = R[x_{ij}]_{1 \leqslant i,j \leqslant n}/\langle \sum_{l=1}^{n} x_{il}x_{jl} - \delta_{ij} | 1 \leqslant i,j \leqslant n\rangle$.

(vi) The special orthogonal group $\mathbf{SO}_n$ for $n \geqslant 1$:
$\mathbf{SO}_n(S) = \{A \in \mathrm{Mat}_n(S) | \det A = 1, AA^T = I_n\}$,
$R[\mathbf{SO}_n] = R[\mathbf{O}_n]/\langle \det(x_{ij}) - 1\rangle$.

(vii) The group scheme $\boldsymbol{\mu}_n$ of the n-th roots of unity for $n \geqslant 1$:
$\boldsymbol{\mu}_n(S) = \{a \in S | a^n = 1\}$.
$R[\boldsymbol{\mu}_n] = R[t]/\langle t^n - 1\rangle$.

These group schemes will be used in our description of the automorphism group functors of certain concrete conformal superalgebras. Now consider the automorphism group functor of an algebra A over R, where A is a projective R–module of finite rank. More precisely, for each S in R**-rng**,

$$\mathbf{Aut}(A)(S) := \mathrm{Aut}_{S\text{-alg}}(A \otimes_R S),$$

where $\mathrm{Aut}_{S\text{-alg}}(A \otimes_R S)$ is the group of automorphisms of the S–algebra $A \otimes_R S$. It is known that $\mathbf{Aut}(A)$ is an affine group scheme over R (cf. [DG70a, II §1.2.6]).

For a concrete example, let A be the Lie algebra $\mathfrak{sl}_2(k)$ over a field k of characteristic 0. Its automorphism group functor $\mathbf{Aut}(\mathfrak{sl}_2(k))$ is an affine group scheme. To understand $\mathbf{Aut}(\mathfrak{sl}_2(k))$, we consider the natural action of $\mathbf{GL}_2(S)$ by conjugation on $\mathfrak{sl}_2(S) := \mathfrak{sl}_2(k) \otimes_k S$ for S in k**-rng**. This action yields a morphism of group scheme

$$\mathbf{GL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(k)). \tag{3.2.3}$$

It is known that this is a quotient map[2] with respect to the étale topology, i.e., for every R in k**-rng** and every $\phi \in \mathbf{Aut}(\mathfrak{sl}_2(k))(R)$, there exists an étale cover[3] S/R such that ϕ_S is of the form

$$\phi_S(x) = AxA^{-1}, \quad \forall x \in \mathfrak{sl}_2(S), \tag{3.2.4}$$

for some $A \in \mathbf{GL}_2(S)$. Furthermore, since k is of characteristic 0 and $\det(A)$ is a unit in S, there is an étale extension S'/S and an element $s \in S'$ such that $s^2 = \det(A)$. Let $B = s^{-1}A \in \mathbf{SL}_2(S')$, then

$$\phi_{S'}(x) = AxA^{-1} = BxB^{-1}, \quad \forall x \in \mathfrak{sl}_2(S').$$

Hence, we may assume in (3.2.4) that $A \in \mathbf{SL}_2(S)$.

We now return to our discussion of non-abelian continuous cohomology. Let R be a ring and $\mathbf{G}$ an affine group scheme over R. Suppose S/R is a faithfully flat ring extension and Γ is a profinite group acting continuously on S by automorphisms of S which fix R. From the functoriality of $\mathbf{G}$, the action of Γ on S induces an action

[2] The definition of quotient requires sheafification which depends on a topology, see Expose V, of [DG70b] for more details.

[3] The definition and basic properties of étale morphisms and étale covers of schemes can be found in many books. We refer to [Mil80].

of Γ on $\mathbf{G}(S)$. This leads us to consider $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathbf{G}(S))$. In order to classify twisted loop conformal superalgebras, we restrict our attention to the special case where $R = D = \Bbbk[t^{\pm 1}]$, $S = \widehat{D} = \Bbbk[t^q | q \in \mathbb{Q}]$, and $\Gamma = \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$, which acts on $\widehat{D}$ by

$$^{\bar{1}}t^{\frac{n}{m}} = \zeta_m^{-n} t^{\frac{n}{m}}, \quad \forall m, n \in \mathbb{Z}, m \neq 0.$$

The fact that $\Bbbk$ is an algebraically closed field means the Laurent polynomial ring D has the following advantageous properties:

Proposition 3.6 (Corollary 2.10 of [GP08a])**.**

(i) Every finite connected étale cover of D is isomorphic to $D_m = \Bbbk[t^{\pm\frac{1}{m}}]$ for some positive integer m.

(ii) $\mathrm{Spec}(\widehat{D})$ is simply connected, i.e., $\widehat{D}$ has no non-trivial finite étale cover.

(iii) Let $a = \mathrm{Spec}(\overline{\Bbbk(t)})$ be the geometric point of $\mathrm{Spec}(D)$, where $\overline{\Bbbk(t)}$ is an algebraic closure of $\Bbbk(t)$. Then the algebraic fundamental group[4]

$$\pi_1(\mathrm{Spec}(D), a) \cong \varprojlim \mathbb{Z}/m\mathbb{Z} = \widehat{\mathbb{Z}}.$$

The above results have been generalized to the case of the Laurent polynomial ring in n variables $k[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ over a field k of characteristic zero, which is not necessarily algebraically closed (cf. Lemma 2.8 of [GP13]). For an affine group scheme $\mathbf{G}$ over D, the above proposition provides us with the possibility of connecting the non-abelian cohomology $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}(\widehat{D}))$ with the non-abelian étale cohomology $\mathrm{H}^1_{\text{ét}}(D, \mathbf{G})$, a set which parameterizes étale $\mathbf{G}$–torsors over D up to isomorphism. More explicitly, we have

Proposition 3.7. *Let $D = \Bbbk[t^{\pm 1}]$ and let $a = \mathrm{Spec}(\overline{\Bbbk(t)})$ be the geometric point of $\mathsf{X} := \mathrm{Spec}(D)$ as in Proposition 3.6.*

(i) If $\mathbf{G}$ is an extension of a twisted finite constant group by a reductive group[5], then

$$\mathrm{H}^1_{\mathrm{ct}}(\pi_1(\mathsf{X}, a), \mathbf{G}(\widehat{D})) \cong \mathrm{H}^1_{\text{ét}}(D, \mathbf{G}). \tag{3.2.5}$$

(cf. Corollary 2.16(3) of [GP08a])

[4]The theory of algebraic fundamental groups of a connected scheme was developed in [GR71], while a brief introduction to these objects is available in Chapter I, §5 of [Mil80].

[5]In this thesis, all reductive group schemes are assumed to be connected.

(ii) For every reductive group scheme $\mathbf{G}$ *over* D,

$$\mathrm{H}^1_{\acute{e}t}(D, \mathbf{G}) = 1. \tag{3.2.6}$$

(cf. Theorem 3.1 of [Pia05])

The above results will be repeatedly used in our concrete computations.

3.3 Twisting

In the section, we briefly review the twisting techniques from the theory of non-abelian continuous cohomology.

Let G be a Γ–group. The group $\mathrm{Aut}(\mathrm{G})$ of automorphisms of the abstract group G is also a Γ–group, where the action of Γ on $\mathrm{Aut}(\mathrm{G})$ is given by

$$({}^{\gamma}f)(g) = {}^{\gamma}(f({}^{\gamma^{-1}}g)), \quad \gamma \in \Gamma, f \in \mathrm{Aut}(\mathrm{G}), g \in \mathrm{G}.$$

Let $\mathfrak{z} = (\mathfrak{z}_\gamma)_{\gamma\in\Gamma} \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{Aut}(\mathrm{G}))$. We define a new Γ–group ${}_{\mathfrak{z}}\mathrm{G}$ as follows: the underlying group of ${}_{\mathfrak{z}}\mathrm{G}$ is G and the new Γ–action is given by

$${}^{\gamma\cdot_{\mathfrak{z}}}g = \mathfrak{z}_\gamma({}^{\gamma}g), \quad \gamma \in \Gamma, g \in \mathrm{G}.$$

We say that ${}_{\mathfrak{z}}\mathrm{G}$ is the Γ–group obtained by twisting G using $\mathfrak{z}$.

It is easy to verify that for $\mathfrak{z}, \mathfrak{z}' \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{Aut}(\mathrm{G}))$, $\mathfrak{z}$ is cohomologous to $\mathfrak{z}'$ if and only if ${}_{\mathfrak{z}}\mathrm{G}$ is isomorphic to ${}_{\mathfrak{z}'}\mathrm{G}$ as Γ–groups. However, since the isomorphism of Γ–groups is not canonical, we can not define ${}_{[\mathfrak{z}]}\mathrm{G}$ for $[\mathfrak{z}] \in \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{Aut}(\mathrm{G}))$. More precisely, the action of Γ on ${}_{\mathfrak{z}}\mathrm{G}$ does depend on the 1–cocycle $\mathfrak{z}$.

Now every $g \in \mathrm{G}$ defines an automorphism $\mathrm{Int}(g)$ of G which is given by conjugation, i.e., $\mathrm{Int}(g)(g') = gg'g^{-1}$ for $g' \in \mathrm{G}$. Hence, a 1–cocycle $\mathfrak{z} = (\mathfrak{z}_\gamma)_{\gamma\in\Gamma} \in Z^1(\Gamma, \mathrm{G})$ yields a 1–cocycle $(\mathrm{Int}(\mathfrak{z}_\gamma))_{\gamma\in\Gamma}$ in $Z^1_{\mathrm{ct}}(\Gamma, \mathrm{Aut}(\mathrm{G}))$. The Γ–group obtained by twisting G using $(\mathrm{Int}(\mathfrak{z}_\gamma))_{\gamma\in\Gamma}$ is also denoted by ${}_{\mathfrak{z}}\mathrm{G}$. In this situation,

$${}^{\gamma\cdot_{\mathfrak{z}}}g = \mathfrak{z}_\gamma \cdot {}^{\gamma}g \cdot \mathfrak{z}_\gamma^{-1}, \quad \gamma \in \Gamma, g \in \mathrm{G}.$$

Proposition 3.8 ([KMRT98, 28.8])**.** *Let* G *be a* Γ*–group and* $\mathfrak{z} \in Z^1(\Gamma, \mathrm{G})$. *Then the map*

$$\theta_{\mathfrak{z}} : \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G}) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}), \quad [\mathfrak{z}'] \mapsto [(\mathfrak{z}'_\gamma\mathfrak{z}_\gamma)_{\gamma\in\Gamma}] \tag{3.3.1}$$

is a well-defined bijection, which takes $[1]$ *in* $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G})$ *to* $[\mathfrak{z}]$ *in* $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G})$.

Moreover, the map $\theta_{\mathfrak{z}}$ is functorial in G. In fact, if we are given a homomorphism of Γ–groups $f : \mathrm{G}_1 \to \mathrm{G}_2$, it naturally induces a map

$$f_* : Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$$

and a map

$$f_* : \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2).$$

If we twist G_2 by $f_*(\mathfrak{z})$, then $f : {}_{\mathfrak{z}}\mathrm{G}_1 \to {}_{f_*(\mathfrak{z})}\mathrm{G}_2$ is also Γ–equivariant as can be easily verified. In addition, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G}_1) & \xrightarrow{f_*} & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{f_*(\mathfrak{z})}\mathrm{G}_2) \\ {\scriptstyle\theta_{\mathfrak{z}}}\downarrow & & \downarrow{\scriptstyle\theta_{f_*(\mathfrak{z})}} \\ \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) & \xrightarrow[f_*]{} & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) \end{array}$$

Since both $\theta_{\mathfrak{z}}$ and $\theta_{f_*(\mathfrak{z})}$ are bijective, the fiber of $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$ over $[f_*(\mathfrak{z})]$ bijectively corresponds to

$$\ker\left(\mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{f_*(\mathfrak{z})}\mathrm{G}_2)\right).$$

We now move on to discussing the compatibility of twisting and exact sequences of Γ–groups. Let

$$1 \to \mathrm{G}_1 \xrightarrow{i} \mathrm{G}_2 \xrightarrow{j} \mathrm{G}_3 \to 1, \tag{3.3.2}$$

be a short exact sequence of Γ–groups and let $\mathfrak{z} := (\mathfrak{z}_\gamma)_{\gamma\in\Gamma} \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$. Then $\mathfrak{z}'' = (j(\mathfrak{z}_\gamma))_{\gamma\in\Gamma} \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3)$. On the other hand, we may identify G_1 with a normal subgroup of G_2, thus every $\mathfrak{z}_\gamma$ induces an automorphism of G_1 given by conjugation, i.e.,

$$\mathfrak{z}'_\gamma : \mathrm{G}_1 \to \mathrm{G}_1, \quad g \mapsto \mathfrak{z}_\gamma g \mathfrak{z}_\gamma^{-1}.$$

It is known that $\mathfrak{z}' = (\mathfrak{z}'_\gamma)_{\gamma\in\Gamma} \in Z^1_{\mathrm{ct}}(\Gamma, \mathrm{Aut}(\mathrm{G}_1))$. The exact sequence (3.3.2) remains exact after twisting $\mathrm{G}_1, \mathrm{G}_2$ and G_3 by $\mathfrak{z}', \mathfrak{z}$ and $\mathfrak{z}''$, respectively, i.e., the following sequence of Γ–groups is exact:

$$1 \to {}_{\mathfrak{z}'}\mathrm{G}_1 \xrightarrow{i} {}_{\mathfrak{z}}\mathrm{G}_2 \xrightarrow{j} {}_{\mathfrak{z}''}\mathrm{G}_3 \to 1.$$

Applying Proposition 3.2 (iv) and Proposition 3.8 to this sequence, we obtain

Proposition 3.9 ([KMRT98, 28.11]). *Let* $1 \to \mathrm{G}_1 \to \mathrm{G}_2 \to \mathrm{G}_3 \to 1$ *be a short exact sequence of* Γ*–groups and* $\mathfrak{z} \in \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$*, then the diagram*

$$\begin{array}{ccccccc} \cdots \longrightarrow ({}_{\mathfrak{z}''}\mathrm{G}_3)^{\Gamma} & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}'}\mathrm{G}_1) & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G}_2) & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}''}\mathrm{G}_3) \\ & & & & \downarrow{\scriptstyle \theta_{\mathfrak{z}}} & & \downarrow{\scriptstyle \theta_{\mathfrak{z}''}} \\ \cdots \longrightarrow \mathrm{G}_3 & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) & \longrightarrow & \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3) \end{array}$$

is a commutative diagram with exact rows. Consequently, the fiber of $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3)$ *over* $[\mathfrak{z}'']$ *bijectively corresponds to the image of the map*

$$\mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}'}\mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, {}_{\mathfrak{z}}\mathrm{G}_2).$$

The above proposition will play important role in our computation of the non-abelian continuous cohomology sets. In particular, we consider the situation where

$$1 \to \mathrm{G}_1 \to \mathrm{G}_2 \xrightarrow{p} \mathrm{G}_3 \to 1$$

is a split exact sequence of Γ–groups, i.e., the morphism p has a section $s : \mathrm{G}_3 \to \mathrm{G}_2$. From Proposition 3.2 (iii), we have an exact sequence of pointed sets

$$1 \to \mathrm{G}_1^{\Gamma} \to \mathrm{G}_2^{\Gamma} \to \mathrm{G}_3^{\Gamma} \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2) \xrightarrow{p_*} \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3). \tag{3.3.3}$$

Since the section s of p induces a section $s_* : \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3) \to \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$, it follows that p_* is surjective. If G_2 is an abelian group, then both G_1 and G_3 are abelian groups and hence $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_i)$ is a group for $i = 1, 2, 3$. The exactness of (3.3.3) implies that the fiber $p_*^{-1}([\mathfrak{z}])$ over each $[\mathfrak{z}]$ is a coset of the subgroup $\ker(p_*)$ in $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$. Unfortunately, G_2 is not an abelian group in general and $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_2)$ is not a group. The exactness of (3.3.3) only tells us that the fiber of p_* over the trivial class is measured by $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_1)$. The above twisting technique is useful for characterizing the fibers $p_*^{-1}([\mathfrak{z}])$ over a non-trivial class $[\mathfrak{z}] \in \mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3)$.

Remark 3.10. For a short exact sequence of Γ–groups

$$1 \to \mathrm{G}_1 \to \mathrm{G}_2 \to \mathrm{G}_3 \to 1$$

with G_1 central, one can use the twisting trick to characterize the fiber of $\mathrm{H}^1_{\mathrm{ct}}(\Gamma, \mathrm{G}_3) \to \mathrm{H}^2_{\mathrm{ct}}(\Gamma, \mathrm{G}_1)$ over some $[\mathfrak{z}] \in \mathrm{H}^2_{\mathrm{ct}}(\Gamma, \mathrm{G}_1)$ (cf. Corollary 28.13 of [KMRT98]).

Next we describe how twisting looks like in the case of affine group schemes.

We focus on the following situation: let R be a ring, S/R be a faithfully flat ring extension, $\mathbf{G} = \mathrm{Hom}_{R\textbf{-rng}}(R[\mathbf{G}], -)$ be an affine group scheme over R, Γ be a profinite group, which acts continuously on S by the ring automorphisms which fix R. Then the functoriality of $\mathbf{G}$ induces an action of Γ on $\mathbf{G}(S)$. Indeed, for $g : R[\mathbf{G}] \to S \in \mathbf{G}(S)$ and $\gamma : S \to S \in \Gamma$, the action of Γ on $\mathbf{G}(S)$ is given by

$$^{\gamma}g = \gamma \circ g.$$

If we further identify $\mathbf{G}(S) = \mathrm{Hom}_{R\textbf{-rng}}(R[\mathbf{G}], S)$ with $\mathrm{Hom}_{S\textbf{-rng}}(R[\mathbf{G}] \otimes_R S, S)$, then the action of Γ on $\mathbf{G}(S)$ can be rewritten as

$$^{\gamma}g = \gamma \circ g \circ (\mathrm{id} \otimes \gamma^{-1}). \tag{3.3.4}$$

for $g : R[\mathbf{G}] \otimes_R S \to S \in \mathbf{G}(S)$ and $\gamma \in \Gamma$. In the rest of this section, an element $g \in \mathbf{G}(S)$ is understood as a homomorphism of S–algebras $g : R[\mathbf{G}] \otimes_R S \to S$.

Now, $\mathbf{G}(S)$ (viewed as an abstract group) with the Γ–action is a Γ–group. We will reinterpret the twisting trick in terms of the twisted forms of an affine group scheme. Recall that Γ acts on $\mathrm{Aut}_{\mathrm{grp}}(\mathbf{G}(S))$ (the automorphism group of the abstract group $\mathbf{G}(S)$) by

$$(^{\gamma}\sigma)(g) = {}^{\gamma}(\sigma(^{\gamma^{-1}}g)),$$

for $\gamma \in \Gamma, \sigma \in \mathrm{Aut}_{\mathrm{grp}}(\mathbf{G}(S)), g \in \mathbf{G}(S)$.

On the other hand, for the affine group scheme $\mathbf{G}$, one can define its automorphism group functor[6]

$$\mathbf{Aut}(\mathbf{G})(S') = \mathrm{Aut}_{S'\text{-Hopf}}(R[\mathbf{G}] \otimes_R S')^{\mathrm{op}},$$

for S' in R-**rng**, where $\mathrm{Aut}_{S'\text{-Hopf}}(R[\mathbf{G}] \otimes_R S')^{\mathrm{op}}$ is the opposite group of the automorphism group of the S'–Hopf algebra $R[\mathbf{G}] \otimes_R S'$. In particular, for the given faithfully flat extension S/R with Γ–action, we define an action of Γ on $\mathbf{Aut}(\mathbf{G})(S)$ by

$$^{\gamma}\varphi = (\mathrm{id} \otimes \gamma) \circ \varphi \circ (\mathrm{id} \otimes \gamma^{-1}),$$

for $\gamma \in \Gamma$ and $\varphi : R[\mathbf{G}] \otimes_R S \to R[\mathbf{G}] \otimes_R S \in \mathbf{Aut}(\mathbf{G})(S)$.

Since every $\varphi \in \mathbf{Aut}(\mathbf{G})(S)$ induces an automorphism σ_φ of the abstract group $\mathbf{G}(S)$, namely, $\sigma_\varphi(g) = g \circ \varphi$, for $g \in \mathbf{G}(S)$, we have a canonical homomorphism

[6]Note that $\mathbf{Aut}(\mathbf{G})$ is a sheaf of groups, but not necessarily a scheme.

of groups

$$\mathbf{Aut}(\mathbf{G})(S) \to \mathrm{Aut}_{\mathrm{grp}}(\mathbf{G}(S)), \quad \varphi \mapsto \sigma_\varphi. \tag{3.3.5}$$

This map is Γ–equivariant:

$$\begin{aligned}
{}^{\gamma}(\sigma_\varphi)(g) &= {}^{\gamma}(\sigma_\varphi({}^{\gamma^{-1}}g)) = {}^{\gamma}(\sigma_\varphi(\gamma^{-1}\circ g\circ(\mathrm{id}\otimes\gamma)))\\
&= {}^{\gamma}(\gamma^{-1}\circ g\circ(\mathrm{id}\otimes\gamma)\circ\varphi)\\
&= \gamma\circ\gamma^{-1}\circ g\circ(\mathrm{id}\otimes\gamma)\circ\varphi\circ(\mathrm{id}\otimes\gamma^{-1})\\
&= g\circ{}^{\gamma}\varphi\\
&= \sigma_{{}^{\gamma}\varphi}(g),
\end{aligned}$$

for $g \in \mathbf{G}(S)$, $\varphi \in \mathbf{Aut}(\mathbf{G})(S)$, and $\gamma \in \Gamma$.

Let $\mathfrak{z} = (\mathfrak{z}_\gamma)_{\gamma\in\Gamma} \in Z^1_{\mathrm{ct}}(\Gamma, \mathbf{Aut}(\mathbf{G})(S))$. Applying (3.3.5) to $\mathfrak{z}$, we obtain a 1–cocyle $(\sigma_{\mathfrak{z}_\gamma})_{\gamma\in\Gamma}$ of Γ in $\mathrm{Aut}_{\mathrm{grp}}(\mathbf{G}(S))$. Let ${}_{\mathfrak{z}}(\mathbf{G}(S))$ denote the Γ–group obtained by twisting $\mathbf{G}(S)$ by $(\sigma_{\mathfrak{z}_\gamma})_{\gamma\in\Gamma}$. Then the action of Γ on ${}_{\mathfrak{z}}(\mathbf{G}(S))$ is given by

$$^{\gamma\cdot\mathfrak{z}}g = \sigma_{\mathfrak{z}_\gamma}({}^{\gamma}g) = \gamma\circ g\circ(\mathrm{id}\otimes\gamma^{-1})\circ\mathfrak{z}_\gamma,$$

for $\gamma \in \Gamma, g \in \mathbf{G}(S)$.

Since $\mathbf{Aut}(\mathbf{G})(S) = \mathrm{Aut}_{S\text{-Hopf}}(R[\mathbf{G}]\otimes_R S)^{\mathrm{op}}$, the 1–cocycle $\mathfrak{z}$ defines a twisted form of the R–Hopf algebra $R[\mathbf{G}]$, namely,

$$A := \{x \in R[\mathbf{G}] \otimes_R S | \mathfrak{z}_\gamma({}^{\gamma}x) = x, \forall\gamma\in\Gamma\}.$$

By faithfully flat descent, A is a Hopf algebra over R and the map

$$\pi : A \otimes_R S \to R[\mathbf{G}] \otimes_R S, \quad (\textstyle\sum a_i \otimes s_i) \otimes s \mapsto \sum a_i \otimes s_i s, \tag{3.3.6}$$

is an isomorphism of S–Hopf algebras.

Now, we define a new affine group scheme

$${}_{\mathfrak{z}}\mathbf{G} = \mathrm{Hom}_{R\text{-}\mathbf{rng}}(A, -),$$

which is a functor from the category R-**rng** to the category of groups. We also conveniently identify $({}_{\mathfrak{z}}\mathbf{G})(S)$ with $\mathbf{Hom}_{S\text{-}\mathbf{rng}}(A\otimes_R S, S)$. Analogous to the action in (3.3.4), the action of Γ on S induces an action of Γ on $({}_{\mathfrak{z}}\mathbf{G})(S)$:

$${}^{\gamma}h = \gamma\circ h\circ(\mathrm{id}\otimes\gamma^{-1}),$$

for $h \in ({}_{\mathfrak{z}}\mathbf{G})(S)$, and $\gamma \in \Gamma$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{S\text{-}\mathbf{rng}}(R[\mathbf{G}] \otimes_R S, S) & \xrightarrow{\pi^*} & \mathrm{Hom}_{S\text{-}\mathbf{rng}}(A \otimes_R S, S) \\
\Big\downarrow{\scriptstyle \gamma\cdot_{\mathfrak{z}}} & & \Big\downarrow{\scriptstyle \gamma} \\
\mathrm{Hom}_{S\text{-}\mathbf{rng}}(R[\mathbf{G}] \otimes_R S, S) & \xrightarrow[\pi^*]{} & \mathrm{Hom}_{S\text{-}\mathbf{rng}}(A \otimes_R S, S)
\end{array}$$

where π^* is the group isomorphism induced by π from (3.3.6). Equivalently,

$$\begin{array}{ccc}
{}_{\mathfrak{z}}(\mathbf{G}(S)) & \xrightarrow{\pi^*} & ({}_{\mathfrak{z}}\mathbf{G})(S) \\
\Big\downarrow{\scriptstyle \gamma\cdot_{\mathfrak{z}}} & & \Big\downarrow{\scriptstyle \gamma} \\
{}_{\mathfrak{z}}(\mathbf{G}(S)) & \xrightarrow[\pi^*]{} & ({}_{\mathfrak{z}}\mathbf{G})(S)
\end{array}$$

is commutative, i.e., π^* is Γ–equivariant. Therefore, ${}_{\mathfrak{z}}(\mathbf{G}(S))$ (the abstract group $\mathbf{G}(S)$ with the twisted Γ–action defined by $\mathfrak{z}$) and $({}_{\mathfrak{z}}\mathbf{G})(S)$ (the set of S–points of the twisted form of $\mathbf{G}$ associated to $\mathfrak{z}$ with the natural Γ–action) are isomorphic as Γ–groups. We will not distinguish them and denote this Γ–group by ${}_{\mathfrak{z}}\mathbf{G}(S)$.

Chapter 4

The $N = 1, 2, 3$ conformal superalgebras[1]

In this chapter, we will focus on the automorphisms and the twisted loop conformal superalgebras based on each of the $N = 1, 2, 3$ conformal superalgebras $\mathscr{K}_N$.

A brief description of $\mathscr{K}_N$ using Grassmannian superalgebras will be reviewed in Section 4.1. We then concentrate on the descriptions of the automorphism groups of $\mathscr{K}_{N,\mathcal{R}} := \mathscr{K}_N \otimes_{\Bbbk} \mathcal{R}$ for an arbitrary object $\mathcal{R} = (R, \mathsf{d})$ in **$\Bbbk$-drng** in Section 4.2. In particular, when R is an integral domain, all automorphisms of the $\mathcal{R}$–conformal superalgebra $\mathscr{K}_{N,\mathcal{R}}$ will be explicitly constructed for $N = 1, 2, 3$.

In Section 4.3, we will deduce the group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})$, $N = 1, 2, 3$ by specializing the $\Bbbk$–differential ring $\mathcal{R}$ to $\widehat{\mathcal{D}} = (\varinjlim \Bbbk[t^{\pm\frac{1}{m}}], \frac{d}{dt})$, and complete the classification of twisted loop conformal superalgebras based on $\mathscr{K}_N$ by computing the corresponding non-abelian cohomology set and applying the centroid trick.

As a supplement, we will deal with the passage from twisted loop conformal superalgebras $\mathcal{L}(\mathscr{K}_N, \sigma)$ to the Lie superalgebra $\mathrm{Alg}(\mathscr{K}_N, \sigma)$ in Section 4.4. We will show that non-isomorphic twisted loop conformal superalgebras $\mathcal{L}(\mathscr{K}_N, \sigma)$ obtained in Section 4.3 induce non-isomorphic Lie superalgebras $\mathrm{Alg}(\mathscr{K}_N, \sigma)$.

4.1 The $N = 1, 2, 3$ conformal superalgebras

Physicists usually define a superconformal algebra by generators (also called fields by physicists) and relations. Such definitions for the $N = 1, 2, 3$ superconformal algebras are given in [ABD+76], while the twisted $N = 1, 2, 3$ superconformal algebras are described in [SS87] based on an observation of the global automorphism. As stated in Section 2.1, these Lie superalgebras are the Lie superalgebras induced by twisted loop conformal superalgebras. Realizations of the $N = 1, 2, 3$ confor-

[1] A version of this chapter has been published. Zhihua Chang and Arturo Pianzola 2011. Communications in Number Theory and Physics. 5:751-778.

mal superalgebras were obtained by V. Kac in [Kac98b], using the Grassmannian superalgebras $\Lambda(N)$ in N variables $\xi_1, \ldots, \xi_N$.

Note that $\Lambda(N)$ is generated by $\xi_1, \cdots, \xi_N$ as an associative algebra. It has both a $\mathbb{Z}/2\mathbb{Z}$-grading given by setting each $\xi_i, i = 1, \ldots, N$ to be odd and a $\mathbb{Z}$-grading in which each $\xi_i, i = 1, \ldots, N$ has degree 1. For a homogeneous element $f \in \Lambda(N)$ with respect to the $\mathbb{Z}$-grading, we use $|f|$ to denote the degree of f.

Consider the $\Bbbk$–vector space

$$\mathscr{K}_N := \Bbbk[\partial] \otimes_{\Bbbk} \Lambda(N), \tag{4.1.1}$$

where $\Bbbk[\partial]$ is the polynomial ring in one indeterminate ∂. Then $\mathscr{K}_N$ is a $\mathbb{Z}/2\mathbb{Z}$–graded vector space and a $\Bbbk[\partial]$–module. Further, there is a conformal superalgebra structure on $\mathscr{K}_N$ with the n-th product defined as follows:

$$f_{(0)}g = \left(\frac{1}{2}|f| - 1\right)\partial \otimes fg + \frac{1}{2}(-1)^{|f|}\sum_{i=1}^{N}(\partial_i f)(\partial_i g), \tag{4.1.2}$$

$$f_{(1)}g = \left(\frac{1}{2}(|f| + |g|) - 2\right) fg, \tag{4.1.3}$$

$$f_{(n)}g = 0, \quad n \geqslant 2, \tag{4.1.4}$$

where $f, g \in \Lambda(N)$ are homogenous with respect to the $\mathbb{Z}$-grading, and ∂_i is the derivative with respect to ξ_i, $i = 1, \ldots, N$. The $\Bbbk$–conformal superalgebras $\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3$ are called the $N = 1, 2, 3$ conformal superalgebras[2], respectively.

In the rest of this chapter, we will use $\mathscr{K}_{N,\mathcal{R}}$ to denote the $\mathcal{R}$–conformal superalgebra $\mathscr{K}_N \otimes_{\Bbbk} \mathcal{R}$ for an object $\mathcal{R}$ in $\Bbbk$-**drng**.

4.2 The automorphism group functor

In this section, we will compute the automorphism groups $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$ for $N = 1, 2, 3$ and $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$-**drng**, and discuss the representability of the group functor $\mathbf{Aut}(\mathscr{K}_N)$.

We observe that $\mathscr{K}_N = \Bbbk[\partial] \otimes_{\Bbbk} \Lambda(N)$ is a free $\Bbbk[\partial]$–module. The $\mathbb{Z}/2\mathbb{Z}$–graded $\Bbbk$–vector space $\Lambda(N) = \Lambda(N)_{\bar{0}} \oplus \Lambda(N)_{\bar{1}}$ can be naturally identified with the sub-

[2]These terminologies come from physics. In particular, $\mathscr{K}_4$ is also a conformal superalgebra, but it is neither isomorphic to the small $N = 4$, nor isomorphic to the large $N = 4$ conformal superalgebra considered in the following two chapters.

space $\Bbbk \otimes_{\Bbbk} \Lambda(N)$ of $\mathscr{K}_N$. Similarly, we identify $\Lambda(N) \otimes_{\Bbbk} R$ with the subspace $\Bbbk \otimes_{\Bbbk} \Lambda(N) \otimes_{\Bbbk} R$ of $\mathscr{K}_{N,\mathcal{R}}$. Instead of considering the whole group $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$, we first concentrate on the subset

$$\begin{aligned} &\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}}) \\ &\qquad := \{\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}}) | \phi\left(\Lambda(N) \otimes_{\Bbbk} R\right) \subseteq \Lambda(N) \otimes_{\Bbbk} R\}. \end{aligned} \tag{4.2.1}$$

It is obvious that $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ is closed under the composition of automorphisms. Indeed, it is a subgroup of $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$. To prove this, it suffices to show that $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ is closed under taking inverse, which follows from the lemma below.

Lemma 4.1. *Let V be a finite dimensional $\Bbbk$–vector space and $\mathcal{R} = (R, \mathsf{d})$ a $\Bbbk$–differential ring. Suppose*

$$\phi : \Bbbk[\partial] \otimes_{\Bbbk} V \otimes_{\Bbbk} R \to \Bbbk[\partial] \otimes_{\Bbbk} V \otimes_{\Bbbk} R$$

is an invertible R–linear map satisfying $\phi \circ \widehat{\partial} = \widehat{\partial} \circ \phi$, where $\widehat{\partial} = \partial \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \mathsf{d}$. If $\phi(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R$, then $\phi^{-1}(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R$.

Proof. Let $\{v_1, \cdots, v_n\}$ be a basis of V. Since ϕ is R–linear and satisfies $\phi \circ \widehat{\partial} = \widehat{\partial} \circ \phi$, ϕ is completely determined by $\phi(1 \otimes v_i \otimes 1)$. Similarly, ϕ^{-1} is also completely determined by $\phi^{-1}(1 \otimes v_i \otimes 1)$. Note that $\phi(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R$, we may write

$$\begin{aligned} \phi(1 \otimes v_i \otimes 1) &= \sum_{j=1}^{n} 1 \otimes v_j \otimes r_{ij}, \\ \phi^{-1}(1 \otimes v_i \otimes 1) &= \sum_{\substack{\ell \geqslant 0 \\ j=1,\cdots,n}} \partial^{\ell} \otimes v_j \otimes s_{ij,\ell}, \end{aligned}$$

for $r_{ij}, s_{ij,\ell} \in R$. Hence,

$$1 \otimes v_i \otimes 1 = \phi^{-1}\phi(1 \otimes v_i \otimes 1) = \sum_{\substack{\ell \geqslant 0 \\ j,k=1,\cdots,n}} \partial^{\ell} \otimes v_k \otimes r_{ij} s_{jk,\ell}.$$

Let $A = (r_{ij})_{1 \leqslant i,j \leqslant n}, B_{\ell} = (s_{ij,\ell})_{1 \leqslant i,j \leqslant n}$. Then

$$AB_0 = I_n, \text{ and } AB_{\ell} = 0, \quad \ell \geqslant 1.$$

It follows that A is invertible and $B_{\ell} = 0$ for $\ell \geqslant 1$. We conclude that $\phi^{-1}(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R$. $\square$

We have seen that $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ is a subgroup of $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$. Moreover, the construction of $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ is functorial in $\mathcal{R}$. We thus obtain a subgroup functor $\mathbf{GrAut}(\mathscr{K}_N) : \mathcal{R} \mapsto \mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$. The computation of the automorphism group $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$ can be completed in two steps: first computing $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ and second describing the relationships between $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})$ and $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{N,\mathcal{R}})$.

To simplify the notation, for a given $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$**-drng**, we set $\widehat{\partial} := \partial \otimes \mathrm{id} + \mathrm{id} \otimes \mathsf{d}$ on $\mathscr{K}_{N,\mathcal{R}}$. And for convenience we identify $f \in \Lambda(N)$ with its image $(1 \otimes f) \otimes 1$ in $\mathscr{K}_{N,\mathcal{R}}$.

Proposition 4.2. *Let $\mathcal{R} = (R, \mathsf{d})$ be an arbitrary object in $\Bbbk$**-drng**. There is an isomorphism of groups*

$$\iota_{1,\mathcal{R}} : \mathbf{O}_1(R) \xrightarrow{\sim} \mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}}), \quad a \mapsto \phi_a, \tag{4.2.2}$$

where $\phi_a \in \mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}})$ is given by:

$$\phi_a(1) = 1 \quad \textit{and} \quad \phi_a(\xi_1) = \xi_1 \otimes a. \tag{4.2.3}$$

Proof. For $a \in \mathbf{O}_1(R)$, we first show that the formulas in (4.2.3) define a homomorphism of $\mathcal{R}$–conformal superalgebras. Since $\Lambda(1) = \Bbbk 1 \oplus \Bbbk\xi_1$, formulas (4.2.3) define an R-module homomorphism $\Lambda(1)\otimes_{\Bbbk} R \to \Lambda(1)\otimes_{\Bbbk} R$. It can be uniquely extended to an R-module homomorphism $\phi_a : \mathscr{K}_{N,\mathcal{R}} \to \mathscr{K}_{N,\mathcal{R}}$, which is also denoted by ϕ_a, such that $\widehat{\partial} \circ \phi_a = \phi_a \circ \widehat{\partial}$. To show ϕ_a is a homomorphism of $\mathcal{R}$–conformal superalgebras, by Lemma 2.3 (ii), it suffices to verify that

$$\phi_a([(\eta_1 \otimes 1)_\lambda (\eta_2 \otimes 1)]) = [\phi_a(\eta_1 \otimes 1)_\lambda \phi_a(\eta_2 \otimes 1)], \tag{4.2.4}$$

where $\eta_1, \eta_2 \in \Lambda(1)$ run over a basis of $\Lambda(1)$. This follows immediately from a direct computation. Hence, ϕ_a is a homomorphism of $\mathcal{R}$–conformal superalgebras.

For $a, b \in \mathbf{O}_1(R)$, we also have

$$\phi_a \circ \phi_b(\eta \otimes 1) = \phi_{ab}(\eta \otimes 1),$$

and

$$\phi_1(\eta \otimes 1) = \eta \otimes 1,$$

for $\eta \in \Lambda(1)$. By Lemma 2.3 (i), it follows that $\phi_a \circ \phi_b = \phi_{ab}$ and $\phi_1 = \mathrm{id}$.

Hence, ϕ_a is an automorphism of the $\mathcal{R}$–conformal superalgebra $\mathscr{K}_{1,\mathcal{R}}$ since it has the inverse $\phi_{a^{-1}}$ and $\iota_{1,\mathcal{R}} : a \mapsto \phi_a$ is a group homomorphism.

Moreover, $\phi_a = \phi_b$ for $a, b \in \mathbf{O}_1(R)$ implies

$$\xi_1 \otimes a = \phi_a(\xi_1 \otimes 1) = \phi_b(\xi_1 \otimes 1) = \xi_1 \otimes b,$$

which yields $a = b$, i.e., $\iota_{1,\mathcal{R}}$ is injective.

It remains to show that $\iota_{1,\mathcal{R}}$ is surjective, namely that given an automorphism $\phi \in \mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}})$ there exists $a \in \mathbf{O}_1(R)$ such that $\phi = \phi_a$.

Since $\phi \in \mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}})$ and $(\mathscr{K}_{1,\mathcal{R}})_{\bar{1}} = \Bbbk[\partial]\xi_1 \otimes_{\Bbbk} R$, the restriction of ϕ

$$\phi| : \Bbbk\xi_1 \otimes R \to \Bbbk\xi_1 \otimes R$$

is an isomorphism of R–modules. Hence, there is a unit $a \in R$ such that

$$\phi(\xi_1 \otimes 1) = \xi_1 \otimes a.$$

We thus deduce that

$$\phi(1) = -2\phi(\xi_1)_{(0)}\phi(\xi_1) = 1 \otimes a^2.$$

Finally, $\phi(1)_{(1)}\phi(1) = -2\phi(1)$ implies that $a^4 = a^2$, which yields $a^2 = 1$ since a is a unit in R. Therefore, we obtain $a \in \mathbf{O}_1(R)$ and $\phi = \phi_a$. □

Proposition 4.3. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in $\Bbbk$-**drng** such that R is an integral domain. Then the inclusion*

$$\mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}}) \subseteq \mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{K}_{1,\mathcal{R}})$$

is an equality.

Proof. Let $\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{1,\mathcal{R}})$. It suffices to show

$$\phi(\Lambda(1) \otimes_{\Bbbk} R) \subseteq \Lambda(1) \otimes_{\Bbbk} R.$$

Since $(\mathscr{K}_{1,\mathcal{R}})_{\bar{1}} = \Bbbk[\partial]\xi_1 \otimes_{\Bbbk} R$ and ϕ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathscr{K}_{1,\mathcal{R}}$, we may assume that

$$\phi(\xi_1) = \sum_{n=0}^{M} \widehat{\partial}^n(\xi_1 \otimes s_n),$$

where $s_n \in R, n = 0, \ldots, M$ and $s_M \neq 0$. Then

$$[\phi(\xi_1)_\lambda \phi(\xi_1)] = -\frac{1}{2}\sum_{m=0}^{M}\sum_{n=0}^{M}(-\lambda)^m(\widehat{\partial}+\lambda)^n(1\otimes s_m s_n), \text{ while}$$
$$\phi([\xi_1{}_\lambda \xi_1]) = \phi\left(\left(-\frac{1}{2}\right)1\right) = -\frac{1}{2}\phi(1).$$

The leading term (i.e., the term with highest degree with respect to λ) in the right-hand sides of above two equations are $\frac{1}{2}(-1)^{M+1}\lambda^{2M}(1\otimes s_M^2)$ and $-\frac{1}{2}\phi(1)$, respectively. Since R is an integral domain, $s_M \neq 0$ implies $s_M^2 \neq 0$. Therefore, $[\phi(\xi_1)_\lambda \phi(\xi_1)] = \phi([\xi_1{}_\lambda \xi_1])$ yields $M = 0$, i.e.,

$$\phi(\xi_1) = \xi_1 \otimes s_0, \quad 0 \neq s_0 \in R,$$

and $\phi(1) = -2[\phi(\xi_1)_\lambda \phi(\xi_1)] = 1 \otimes s_0^2$. Hence,

$$\phi\left(\Lambda(1)\otimes_{\Bbbk} R\right) \subseteq \Lambda(1)\otimes_{\Bbbk} R.$$

This completes the proof. □

Remark 4.4. The integral assumption in the Proposition 4.3 is not superflous. Consider $\mathcal{R} = (R, \mathsf{d})$, where $R = \Bbbk \oplus \Bbbk\tau, \tau^2 = 0$ (the algebra of dual numbers) and $\mathsf{d} = 0$. For the $\mathcal{R}$–conformal superalgebra

$$\mathscr{K}_{1,\mathcal{R}} = \Bbbk[\partial]\otimes_{\Bbbk}(\Bbbk\oplus\Bbbk\xi_1)\otimes_{\Bbbk} R,$$

it is easy to verify that

$$\phi(\partial^\ell 1\otimes s) = \partial^\ell 1\otimes s + \partial^{\ell+1}1\otimes\tau s, \quad \ell \geqslant 0, s\in R,$$
$$\phi(\partial^\ell \xi_1\otimes s) = \partial^\ell \xi_1\otimes s + \partial^{\ell+1}\xi_1\otimes\tau s, \quad \ell \geqslant 0, s\in R,$$

define an element $\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{1,\mathcal{R}})$, which is not contained in $\mathrm{GrAut}(\mathscr{K}_{1,\mathcal{R}})$.

Proposition 4.5. *Let $\mathcal{R} = (R, \mathsf{d})$ be an arbitrary object in* $\Bbbk$-**drng**. *There is an isomorphism of groups*

$$\iota_{2,\mathcal{R}} : \mathbf{O}_2(R) \xrightarrow{\sim} \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}}), \quad A = (a_{ij})_{2\times 2} \mapsto \phi_A, \qquad (4.2.5)$$

where $\phi_A \in \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}})$ *is given by:*

$$\begin{aligned} \phi_A(1) &= 1 + \xi_1\xi_2 \otimes r, & \phi_A(\xi_1) &= \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \\ \phi_A(\xi_1\xi_2) &= \xi_1\xi_2 \otimes \det(A), & \phi_A(\xi_2) &= \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22}, \end{aligned} \tag{4.2.6}$$

and

$$\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = 2\mathsf{d}(A)A^{\mathrm{T}}. \tag{4.2.7}$$

Proof. Using similar arguments as in Proposition 4.2, the formulas (4.2.6) define an automorphism $\phi_A \in \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}})$ for $A \in \mathbf{O}_2(R)$. It is easy to show that

$$\iota_{2,\mathcal{R}} : \mathbf{O}_2(R) \to \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}}), \quad A \mapsto \phi_A$$

is an injective group homomorphism. We next show that $\iota_{2,\mathcal{R}}$ is surjective, i.e., every $\phi \in \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}})$ is of the form ϕ_A for some $A \in \mathbf{O}_2(R)$.

Since $\phi(\Lambda(2) \otimes_{\Bbbk} R) \subseteq \Lambda(2) \otimes_{\Bbbk} R$ and ϕ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathscr{K}_{2,\mathcal{R}}$, we may write

$$\phi(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \text{ and } \phi(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$$

where $a_{ij} \in R, i, j = 1, 2$.

Let $A = (a_{ij})_{2\times 2}$. Since ϕ has an inverse in $\mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}})$, the matrix A is necessarily invertible. Now

$$\phi(\xi_1\xi_2) = -\phi(\xi_1)_{(1)}\phi(\xi_2) = \xi_1\xi_2 \otimes (a_{11}a_{22} - a_{21}a_{12}) = \xi_1\xi_2 \otimes \det(A).$$

Choose $c, r \in R$ such that $\phi(1) = 1 \otimes c + \xi_1\xi_2 \otimes r$. From $\phi(1)_{(1)}\phi(\xi_1\xi_2) = -\phi(\xi_1\xi_2)$ we deduce that $c \cdot \det(A) = \det(A)$. Since A is invertible, $\det(A)$ is a unit in R and therefore $c = 1$.

Since $\phi(\xi_j)_{(0)}\phi(\xi_j) = -\frac{1}{2}\phi(1)$, we have

$$a_{1j}^2 + a_{2j}^2 = 1, \quad \text{and} \quad r = 2\big(\mathsf{d}(a_{1j})a_{2j} - a_{1j}\mathsf{d}(a_{2j})\big), \quad j = 1, 2,$$

while $\phi(\xi_1)_{(0)}\phi(\xi_2) = -\frac{1}{2}\widehat{\partial}\phi(\xi_1\xi_2)$ implies that $a_{11}a_{12} + a_{21}a_{22} = 0$. Thus $A = (a_{ij}) \in \mathbf{O}_2(R)$ and

$$r = \big(\mathsf{d}(a_{11})a_{21} - a_{11}\mathsf{d}(a_{21})\big) + \big(\mathsf{d}(a_{12})a_{22} - a_{12}\mathsf{d}(a_{22})\big)$$

$$= \big(\mathsf{d}(a_{11})a_{21} - a_{11}\mathsf{d}(a_{21})\big) + \big(\mathsf{d}(a_{12})a_{22} - a_{12}\mathsf{d}(a_{22})\big) + \mathsf{d}(a_{11}a_{21} + a_{12}a_{22})$$
$$= 2\big(\mathsf{d}(a_{11})a_{21} + \mathsf{d}(a_{12})a_{22}\big),$$

i.e.,

$$\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = 2\mathsf{d}(A)A^T.$$

It follows that $\phi(\eta \otimes 1) = \phi_A(\eta \otimes 1)$ for all $\eta \in \Lambda(2)$. Hence, $\phi = \phi_A$. □

Proposition 4.6. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in* $\Bbbk$**-drng** *such that R is an integral domain. Then the inclusion*

$$\mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}}) \subseteq \mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{K}_{2,\mathcal{R}})$$

is an equality.

Proof. Similar to the proof of Proposition 4.3, it suffices to show that

$$\phi(\Lambda(2) \otimes_{\Bbbk} R) \subseteq \Lambda(2) \otimes_{\Bbbk} R,$$

for every $\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{2,\mathcal{R}})$. We firstly write

$$\phi(\xi_1\xi_2) = \sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m) + \eta,$$

where $\eta \in \Bbbk[\partial]\xi_1\xi_2 \otimes_{\Bbbk} R$, $s_m \in R, m = 0, \ldots, M, s_M \neq 0$.

Then

$$\begin{aligned} 0 &= [\phi(\xi_1\xi_2)_\lambda \phi(\xi_1\xi_2)] \\ &= \sum_{m,n=0}^{M} (-\lambda)^m(\widehat{\partial} + \lambda)^n(-\partial 1 \otimes s_m s_n - 2 \otimes \mathsf{d}(s_m)s_n - \lambda 2 \otimes s_m s_n) \\ &\quad + \left[\sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m)_\lambda \eta\right] + \left[\eta_\lambda \sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m)\right] + [\eta_\lambda \eta]. \end{aligned}$$

Since $(\mathscr{K}_{2,\mathcal{R}})_{\bar{0}} = (\Bbbk[\partial]1 \otimes_{\Bbbk} R) \oplus (\Bbbk[\partial]\xi_1\xi_2 \otimes_{\Bbbk} R)$ and all terms in the last row of the equation above are contained in $\Bbbk[\lambda] \otimes_{\Bbbk} \Bbbk[\partial]\xi_1\xi_2 \otimes_{\Bbbk} R$, it follows that

$$0 = \sum_{m,n=0}^{M} (-\lambda)^m(\widehat{\partial} + \lambda)^n(\partial 1 \otimes s_m s_n + 2 \otimes \mathsf{d}(s_m)s_n + \lambda 2 \otimes s_m s_n).$$

By comparing the coefficients of λ, we conclude that $s_M^2 = 0$, and hence $s_M = 0$ since R is an integral domain. This contradicts our assumption that $s_M \neq 0$. Thus

$$\phi(\xi_1\xi_2) = \eta = \sum_{m=0}^{M'} \widehat{\partial}^m(\xi_1\xi_2 \otimes c_m),$$

where $c_m \in R, m = 0, \ldots, M', c_{M'} \neq 0$.

Similarly, we may assume that

$$\phi(1) = \sum_{m'=0}^{\widetilde{M}'} \widehat{\partial}^{m'}(1 \otimes r'_{m'}) + \sum_{m=0}^{\widetilde{M}} \widehat{\partial}^m(\xi_1\xi_2 \otimes r_m),$$

where $r'_{m'}, r_m \in R, m' = 0, \ldots, \widetilde{M}', m = 0, \ldots, \widetilde{M}$. Then

$$\begin{aligned}&[\phi(1)_\lambda \phi(\xi_1\xi_2)]\\ &= \sum_{m=0}^{\widetilde{M}'} \sum_{n=0}^{M'} (-\lambda)^m (\widehat{\partial} + \lambda)^n ((\partial + \lambda)\xi_1\xi_2 \otimes r'_m c_n + \xi_1\xi_2 \otimes \mathsf{d}(r'_m)c_n).\end{aligned}$$

From

$$[\phi(1)_\lambda \phi(\xi_1\xi_2)] = -(\widehat{\partial} + \lambda)\phi(\xi_1\xi_2),$$

we deduce that $r'_{\widetilde{M}'} c_{M'} = 0$ if $\widetilde{M}' + M' > 0$. Since $c_{M'} \neq 0$ and R is an integral domain, $r'_{\widetilde{M}'} = 0$ if $\widetilde{M}' + M' > 0$. Thus, $\widetilde{M}' = M' = 0$, i.e.,

$$\begin{aligned}\phi(\xi_1\xi_2) &= \xi_1\xi_2 \otimes c,\\ \phi(1) &= 1 \otimes r' + \sum_{m=0}^{\widetilde{M}} \widehat{\partial}^m(\xi_1\xi_2 \otimes r_m),\end{aligned}$$

where $0 \neq c \in R, 0 \neq r' \in R$ and $r_m \in R, m = 0, \ldots, \widetilde{M}$.

Now we consider the odd part $(\mathscr{K}_{2,\mathcal{R}})_{\bar{1}} = (\Bbbk[\partial]\xi_1 \oplus \Bbbk[\partial]\xi_2) \otimes_{\Bbbk} R$. Write

$$\phi(\xi_j) = \sum_{m=0}^{M_{1j}} \widehat{\partial}^m(\xi_1 \otimes a_{1j,m}) + \sum_{n=0}^{M_{2j}} \widehat{\partial}^n(\xi_2 \otimes a_{2j,n}),$$

where $a_{ij,m} \in R, i, j = 1, 2$, and $m = 0, \ldots, M_{ij}$. A similar consideration on

$$[\phi(\xi_1)_\lambda \phi(\xi_1\xi_2)] = -\frac{1}{2}\phi(\xi_2) \text{ and } [\phi(\xi_2)_\lambda \phi(\xi_1\xi_2)] = \frac{1}{2}\phi(\xi_1),$$

yields

$$\phi(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \text{ and } \phi(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$$

where $a_{ij} \in R, i, j = 1, 2$.

Next, we consider $\phi(1)$. We deduce from

$$[\phi(1)_\lambda \phi(\xi_i)] = -(\widehat{\partial} + \frac{3}{2}\lambda)\phi(\xi_i), i = 1, 2$$

that $\phi(1) = 1 \otimes r' + \xi_1\xi_2 \otimes r_0$, where $r', r_0 \in R$. It follows that

$$\phi\left(\Lambda(2) \otimes_{\Bbbk} R\right) \subseteq \Lambda(2) \otimes_{\Bbbk} R.$$

i.e., $\phi \in \mathrm{GrAut}(\mathscr{K}_{2,\mathcal{R}})$. □

Proposition 4.7. *Let $\mathcal{R} = (R, \mathsf{d})$ be an arbitrary object in $\Bbbk$-**drng**. There is an isomorphism of groups*

$$\iota_{3,\mathcal{R}} : \mathbf{O}_3(R) \xrightarrow{\sim} \mathrm{GrAut}(\mathscr{K}_{3,\mathcal{R}}), \quad A = (a_{ij})_{3\times 3} \mapsto \phi_A, \tag{4.2.8}$$

where $\phi_A \in \mathrm{GrAut}(\mathscr{K}_{3,\mathcal{R}})$ is given by:

$$\phi_A(1) = 1 + \sum_{l=1}^{3} \epsilon_{mnl}\xi_m\xi_n \otimes r_l, \tag{4.2.9}$$

$$\phi_A(\xi_j) = \sum_{l=1}^{3} \xi_l \otimes a_{lj} + \xi_1\xi_2\xi_3 \otimes s_j, \tag{4.2.10}$$

$$\phi_A(\xi_i\xi_j) = \epsilon_{ijl} \sum_{l'=1}^{3} \epsilon_{mnl'}\xi_m\xi_n \otimes A_{l'l}, \tag{4.2.11}$$

$$\phi_A(\xi_1\xi_2\xi_3) = \xi_1\xi_2\xi_3 \otimes \det(A), \tag{4.2.12}$$

$i, j = 1, 2, 3, i \neq j$, $A_{l'l}$ is the cofactor of $a_{l'l}$ in A and

$$\begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} = 2\mathsf{d}(A)A^{\mathrm{T}}, \quad \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix} = 2(\det A)A^{\mathrm{T}}\mathsf{d}(A). \tag{4.2.13}$$

Proof. Analogous to the proof of Proposition 4.2, the formulas (4.2.9)-(4.2.13) define an automorphism $\phi_A \in \mathrm{GrAut}(\mathscr{K}_{3,\mathcal{R}})$, and it can be shown by a direct computation that $\iota_{3,\mathcal{R}} : A \mapsto \phi_A$ is an injective group homomorphism. It remains to show that every $\phi \in \mathrm{GrAut}(\mathscr{K}_{3,\mathcal{R}})$ is of the form ϕ_A for some $A \in \mathbf{O}_3(R)$.

Observing that $(\mathscr{K}_{3,\mathcal{R}})_{\bar{1}} = \Bbbk[\partial] \otimes_{\Bbbk} \Lambda(3)_{\bar{1}} \otimes_{\Bbbk} R$, where $\Lambda(3)_{\bar{1}}$ is the $\Bbbk$–vector space spanned by $\{\xi_1, \xi_2, \xi_3, \xi_1\xi_2\xi_3\}$, and $\phi(\Lambda(3)_{\bar{1}} \otimes_{\Bbbk} R) \subseteq \Lambda(3)_{\bar{1}} \otimes_{\Bbbk} R$, we may assume that

$$\phi(\xi_j) = \xi_1 \otimes a_{1j} + \xi_2 \otimes a_{2j} + \xi_3 \otimes a_{3j} + \xi_1\xi_2\xi_3 \otimes s_j, \tag{4.2.14}$$

where $a_{ij}, s_j \in R, i, j = 1, 2, 3$. Let $A = (a_{ij})_{3\times 3}$. It follows that

$$\begin{aligned}
\phi(\xi_i\xi_j) &= -\phi(\xi_i)_{(1)}\phi(\xi_j) \\
&= \xi_1\xi_2 \otimes (a_{1i}a_{2j} - a_{2i}a_{1j}) + \xi_2\xi_3 \otimes (a_{2i}a_{3j} - a_{3i}a_{2j}) \\
&\quad + \xi_3\xi_2 \otimes (a_{3i}a_{2j} - a_{2i}a_{3j}) \\
&= \epsilon_{ijk}(\xi_1\xi_2 \otimes A_{3k} + \xi_2\xi_3 \otimes A_{1k} + \xi_3\xi_1 \otimes A_{2k}),
\end{aligned} \tag{4.2.15}$$

for $i \neq j$, where A_{ij} is the cofactor of a_{ij} in A. Similarly,

$$\begin{aligned}
\phi(\xi_1\xi_2\xi_3) &= -2\phi(\xi_1\xi_2)_{(1)}\phi(\xi_3) \\
&= \xi_1\xi_2\xi_3 \otimes (A_{13}a_{13} + A_{23}a_{23} + A_{33}a_{33}) \\
&= \xi_1\xi_2\xi_3 \otimes \det(A).
\end{aligned} \tag{4.2.16}$$

In particular, $\det(A)$ is a unit in R, and hence A is invertible.

Note that $\Lambda(3)_{\bar{0}}$ is the $\Bbbk$–vector space spanned by $\{1, \xi_1\xi_2, \xi_2\xi_3, \xi_3\xi_1\}$ and

$$\phi(\Lambda(3)_{\bar{0}} \otimes_{\Bbbk} R) \subseteq \Lambda(3)_{\bar{0}} \otimes_{\Bbbk} R.$$

We may write

$$\phi(1) = 1 \otimes c + \xi_1\xi_2 \otimes r_3 + \xi_2\xi_3 \otimes r_2 + \xi_3\xi_1 \otimes r_1, \tag{4.2.17}$$

where $c, r_j \in R, j = 1, 2, 3$.

First, we show that $c = 1$. In fact, from

$$\phi(1)_{(1)}\phi(\xi_1\xi_2\xi_3) = -\frac{1}{2}\phi(\xi_1\xi_2\xi_3),$$

we deduce that $c \cdot \det(A) = \det(A)$. Hence, $c = 1$ since $\det(A)$ is a unit in R.

Second, we show that $A \in \mathbf{O}_3(R)$, i.e.,

$$a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} = \delta_{ij},$$

for $i, j = 1, 2, 3$. These equalities follow from

$$\phi(\xi_j)_{(0)}\phi(\xi_j) = -\frac{1}{2}\phi(1)$$

when $i = j$ and from

$$\phi(\xi_i)_{(0)}\phi(\xi_j) = -\frac{1}{2}\widehat{\partial}\phi(\xi_i\xi_j)$$

when $i \neq j$.

Finally, we determine $r_j, s_j, j = 1, 2, 3$. Since

$$\phi(1)_{(1)}\phi(\xi_j) = -\frac{3}{2}\phi(\xi_j), j = 1, 2, 3,$$

we obtain

$$\frac{1}{2}\epsilon_{lmn}(a_{mj}r_n - a_{nj}r_m) = \mathsf{d}(a_{lj}), \tag{4.2.18}$$

and

$$r_1a_{1j} + r_2a_{2j} + r_3a_{3j} = s_j, \tag{4.2.19}$$

for $l, j = 1, 2, 3$. Writing the (4.2.18) in matrix form, we obtain

$$\frac{1}{2}\begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} A = \mathsf{d}(A).$$

Since $A \in \mathbf{O}_3(R)$, the first equality in (4.2.13) follows.

A direct computation shows that

$$\begin{aligned} 2A^{\mathrm{T}}\mathsf{d}(A) &= 2A^{\mathrm{T}}\mathsf{d}(A)A^{\mathrm{T}}A \\ &= A^{\mathrm{T}}\begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} A \end{aligned}$$

$$= \begin{pmatrix} 0 & \sum_{l=1}^{3} r_l A_{l3} & -\sum_{l=1}^{3} r_l A_{l2} \\ -\sum_{l=1}^{3} r_l A_{l3} & 0 & \sum_{l=1}^{3} r_l A_{l1} \\ \sum_{l=1}^{3} r_l A_{l2} & -\sum_{l=1}^{3} r_l A_{l1} & 0 \end{pmatrix}.$$

Since $A \in \mathbf{O}_3(R)$, $A_{ij} = \det(A)a_{ij}, i, j = 1, 2, 3$. So

$$\sum_{l=1}^{3} r_l A_{lj} = \det(A) \sum_{l=1}^{3} r_l a_{lj} = \det(A) s_j.$$

Hence, the second equality in (4.2.13) follows. Summarizing (4.2.14)–(4.2.17), we obtain $\phi(\eta) = \phi_A(\eta)$, for all $\eta \in \Lambda(3)$. Hence, $\phi = \phi_A$. □

Proposition 4.8. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in* $\Bbbk$**-drng** *such that R is an integral domain. Then the inclusion*

$$\operatorname{GrAut}(\mathscr{K}_{3,\mathcal{R}}) \subseteq \operatorname{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{K}_{3,\mathcal{R}})$$

is an equality.

Proof. Let $\phi \in \operatorname{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{K}_{3,\mathcal{R}})$. We have to show

$$\phi(\Lambda(3) \otimes_{\Bbbk} R) \subseteq \Lambda(3) \otimes_{\Bbbk} R.$$

We first consider the action of ϕ on the even part of $\mathscr{K}_{3,\mathcal{R}}$. Let

$$\mathscr{B} = \Bbbk[\partial] \otimes_{\Bbbk} (\Bbbk\xi_1\xi_2 \oplus \Bbbk\xi_2\xi_3 \oplus \Bbbk\xi_3\xi_1),$$

which is isomorphic to $\operatorname{Cur}(\mathfrak{so}_3(\Bbbk))$ as $\Bbbk$–conformal superalgebras and,

$$(\mathscr{K}_{3,\mathcal{R}})_{\bar{0}} = (\Bbbk[\partial]1 \otimes_{\Bbbk} R) \oplus \mathscr{B}_{\mathcal{R}}.$$

We may assume

$$\phi(\xi_i\xi_j) = \sum_{m=0}^{M} \widehat{\partial}^m (1 \otimes s_m) + \eta_{ij}, i \neq j,$$

where $s_m \in R$ and $\eta_{ij} \in \mathscr{B}_{\mathcal{R}}$. Since $\mathscr{B}_{\mathcal{R}}$ is an ideal of $(\mathscr{K}_{3,\mathcal{R}})_{\bar{0}}$ and

$$[\phi(\xi_i\xi_j)_\lambda \phi(\xi_i\xi_j)] = 0,$$

we deduce that $s_M^2 = 0$. Then $s_M = 0$ since R is an integral domain. It follows that

$$\phi(\xi_i\xi_j) = \eta_{ij} \in \mathscr{B}_{\mathcal{R}}.$$

Hence, $\phi|_{\mathscr{B}_{\mathcal{R}}}$ is an automorphism of the $\mathcal{R}$-conformal superalgebra $\mathscr{B}_{\mathcal{R}}$. Since $\mathscr{B} \simeq \mathrm{Cur}(\mathfrak{so}_3(\Bbbk))$ and $\mathfrak{so}_3(\Bbbk)$ is a finite dimensional simple Lie algebra, by Proposition 2.5, we obtain

$$\phi(\xi_i\xi_j) = \epsilon_{ijl}(\xi_1\xi_2 \otimes b_{3l} + \xi_2\xi_3 \otimes b_{1l} + \xi_3\xi_1 \otimes b_{2l}), \quad i \neq j, \tag{4.2.20}$$

where $(b_{l'l})_{3\times 3} \in \mathbf{GL}_3(R)$.

Next we consider $\phi(1)$. We claim that

$$\phi(1) = 1 + \xi_1\xi_2 \otimes r_3 + \xi_2\xi_3 \otimes r_1 + \xi_3\xi_1 \otimes r_2, \tag{4.2.21}$$

where $r_1, r_2, r_3 \in R$.

Indeed, we can write

$$\phi(1) = \sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m) + \eta$$

with $s_i \in R, i = 0, \dots, M, \eta \in \mathscr{B}_{\mathcal{R}}$. We may assume $s_M \neq 0$ since ϕ is an isomorphism. Then

$$\begin{aligned}[\phi(1)_\lambda\phi(1)] = &\sum_{m,n=0}^{M} (-\lambda)^m(\widehat{\partial} + \lambda)^n(-\partial 1 \otimes s_m s_n - 2 \otimes \mathsf{d}(s_m)s_n - \lambda 2 \otimes s_m s_n) \\ &+ \left[\sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m)_\lambda \eta\right] + \left[\eta_\lambda \sum_{n=0}^{M} \widehat{\partial}^n(1 \otimes s_n)\right] + [\eta_\lambda\eta].\end{aligned}$$

Note that all terms in the second row of the above equation are contained in $\Bbbk[\lambda] \otimes_{\Bbbk} \mathscr{B}_{\mathcal{R}}$. If $M > 0$, we deduce that $s_M^2 = 0$ by comparing the coefficients of λ^{2M+1} in $[\phi(1)_\lambda\phi(1)] = -(\widehat{\partial} + 2\lambda)\phi(1)$. Since R is an integral domain, $s_M = 0$. This contradicts $s_M \neq 0$. Hence, $M = 0$, i.e., $\phi(1) = 1 \otimes s_0 + \eta$ with $\eta \in \mathscr{B}_{\mathcal{R}}$. Simplifying the computation for $[\phi(1)_\lambda\phi(1)]$ above, we have

$$\begin{aligned}[\phi(1)_\lambda\phi(1)] = &-\partial 1 \otimes s_0^2 - 2 \otimes \mathsf{d}(s_0)s_0 - \lambda 2 \otimes s_0^2 \\ &+ [(1 \otimes s_0)_\lambda\eta] + [\eta_\lambda(1 \otimes s_0)] + [\eta_\lambda\eta],\end{aligned}$$

$$-(\widehat{\partial}+2\lambda)\phi(1) = -\partial 1\otimes s_0 - 1\otimes \mathsf{d}(s_0) - \lambda 2\otimes s_0 - (\widehat{\partial}+2\lambda)\eta.$$

It follows that $s_0^2 = s_0$, and hence $s_0 = 1$ since $s_0 \neq 0$ and R is an integral domain. Therefore, $\phi(1) = 1+\eta$ with $\eta \in \mathscr{B}_{\mathcal{R}}$.

We further write

$$\eta = \sum_{l\geqslant 0} \widehat{\partial}^l(\xi_1\xi_2\otimes r_{3l} + \xi_2\xi_3\otimes r_{1l} + \xi_3\xi_1\otimes r_{2l}),$$

where $r_{il}\in R, i=1,2,3$ and all but finitely many r_{il} are zero. Combining with (4.2.20), we obtain

$$\begin{aligned}
&[\phi(1)_\lambda \phi(\xi_i\xi_j)] = \\
&-\epsilon_{ijk}(\partial+\lambda)(\xi_1\xi_2\otimes b_{3k} + \xi_2\xi_3\otimes b_{1k} + \xi_3\xi_1\otimes b_{2k}) \\
&+\epsilon_{ijk}\sum_{l\geqslant 0}\sum_{k'=1}^{3}(-\lambda)^l \epsilon_{i'j'k'}\xi_{i'}\xi_{j'}\otimes (r_{i'l}b_{j'k} - r_{j'l}b_{i'k}).
\end{aligned}$$

Then the equalities $[\phi(1)_\lambda\phi(\xi_i\xi_j)] = -(\widehat{\partial}+\lambda)\phi(\xi_i\xi_j), i,j=1,2,3, i\neq j$ imply that

$$\epsilon_{ijk}(r_{il}b_{jk} - r_{jl}b_{ik}) = 0,$$

for all $i,j,k = 1,2,3, i\neq j, l\geqslant 1$. In the matrix form, these are equivalent to

$$\begin{pmatrix} 0 & -r_{3l} & r_{2l} \\ r_{3l} & 0 & -r_{1l} \\ -r_{2l} & r_{1l} & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33}\end{pmatrix} = 0, \quad \forall l\geqslant 1.$$

Hence, $r_{1l} = r_{2l} = r_{3l} = 0$ for all $l\geqslant 1$ since $(b_{ij})_{3\times 3}\in \mathbf{GL}_3(R)$, i.e.,

$$\phi(1) = 1 + \xi_1\xi_2\otimes r_{30} + \xi_2\xi_3\otimes r_{10} + \xi_3\xi_1\otimes r_{20},$$

where $r_{10}, r_{20}, r_{30}\in R$. This completes the proof of the claim.

Next, we consider the action of ϕ on the odd part $(\mathscr{K}_{3,\mathcal{R}})_{\bar{1}}$. First, the equalities

$$[\phi(\xi_i\xi_j)_\lambda \phi(\xi_1\xi_2\xi_3)] = 0$$

for all $i \neq j$ yield that

$$\phi(\xi_1\xi_2\xi_3) = \sum_{m=0}^{M} \widehat{\partial}^m(\xi_1\xi_2\xi_3 \otimes c_m),$$

where $c_m \in R$. Considering

$$[\phi(1)_\lambda \phi(\xi_1\xi_2\xi_3)] = -(\widehat{\partial} + \frac{1}{2}\lambda)\phi(\xi_1\xi_2\xi_3),$$

we deduce that

$$\phi(\xi_1\xi_2\xi_3) = \xi_1\xi_2\xi_3 \otimes c, \tag{4.2.22}$$

where $0 \neq c \in R$.

Finally, we apply a similar argument to $\phi(\xi_j)$. From

$$[\phi(\xi_j)_\lambda \phi(\xi_1\xi_2\xi_3)] = \epsilon_{jmn}\phi(\xi_m\xi_n)$$

and

$$[\phi(1)_\lambda \phi(\xi_j)] = -(\widehat{\partial} + \frac{3}{2}\lambda)\phi(\xi_j),$$

we obtain

$$\phi(\xi_j) = \xi_1 \otimes a_{1j} + \xi_2 \otimes a_{2j} + \xi_3 \otimes a_{3j} + \xi_1\xi_2\xi_3 \otimes s_j, \tag{4.2.23}$$

where $a_{ij}, s_j \in R$.

Summarizing (4.2.18) to (4.2.23), we obtain

$$\phi\left(\Lambda(3) \otimes_{\Bbbk} R\right) \subseteq \Lambda(3) \otimes_{\Bbbk} R.$$

This completes the proof. □

To sum up the results in this section, we have the following theorem:

Theorem 4.9. *For the $\Bbbk$–conformal superalgebra $\mathscr{K}_N$ with $N = 1, 2, 3$, the following statements hold:*

(i) For every object $\mathcal{R}$ in $\Bbbk$**-drng**, *there is an isomorphism of groups*

$$\iota_{N,\mathcal{R}} : \mathbf{O}_N(R) \to \mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}}). \tag{4.2.24}$$

(ii) The isomorphism $\iota_{N,\mathcal{R}}$ is functorial in $\mathcal{R}$, i.e.,

$$\mathbf{O}_N \circ \mathfrak{f} \cong \mathbf{GrAut}(\mathscr{K}_N), \tag{4.2.25}$$

as functors from $\Bbbk$**-drng** *to* **grp**, *where*

$$\mathfrak{f} : \Bbbk\text{-}\mathbf{drng} \to \Bbbk\text{-}\mathbf{rng}, \quad \mathcal{R} = (R, \mathsf{d}) \mapsto R. \tag{4.2.26}$$

is the forgetful functor.

(iii) For an object $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$**-drng** *such that R is an integral domain,*

$$\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}}) = \mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{K}_{N,\mathcal{R}}). \tag{4.2.27}$$

Proof. (i) and (iii) are merely summaries of Propositions 4.2-4.8.

For (ii), let $h : \mathcal{R} = (R, \mathsf{d}_R) \to \mathcal{S} = (S, \mathsf{d}_S)$ be a morphism in $\Bbbk$**-drng**. It induces a homomorphism of groups $\mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}}) \to \mathrm{GrAut}(\mathscr{K}_{N,\mathcal{S}})$. From (2.3.5) and Lemma 2.3, the image of ϕ_A under this map is $\phi_{h(A)}$ for $A \in \mathbf{O}_N(R)$, i.e., the diagram

$$\begin{array}{ccc}
\mathbf{O}_N(R) & \xrightarrow{\iota_{N,\mathcal{R}}} & \mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}}) \\
{\scriptstyle \mathbf{O}_N(h)}\downarrow & & \downarrow{\scriptstyle \mathbf{Aut}(\mathscr{K}_N)(h)} \\
\mathbf{O}_N(S) & \xrightarrow[\iota_{N,\mathcal{S}}]{} & \mathrm{GrAut}(\mathscr{K}_{N,\mathcal{R}})
\end{array}$$

is commutative. Hence, (ii) follows. ◻

Remark 4.10. From Theorem 4.9, the subgroup functor $\mathbf{GrAut}(\mathscr{K}_N)$ for $N = 1, 2, 3$ has two nice properties:

(i) $\mathbf{GrAut}(\mathscr{K}_N) = \mathbf{O}_N \circ \mathfrak{f}$ is a lift of $\mathbf{O}_N$ (viewed as a functor from $\Bbbk$**-rng** to the category of groups) by composing the forgetful functor $\mathfrak{f}$ from $\Bbbk$**-drng** to $\Bbbk$**-rng**. In particular, $\mathbf{O}_N$ is an affine group scheme of finite type.

(ii) $\mathbf{GrAut}(\mathscr{K}_N)$ gives the whole automorphism group $\mathbf{Aut}(\mathscr{K}_N)(\mathcal{R})$ when evaluating at an $\mathcal{R} = (R, \mathsf{d})$ with R an integral domain.

In general, for a $\Bbbk$–conformal superalgebra $\mathscr{A}$ whose underlying $\Bbbk[\partial]$–module is free and of finite rank, there is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$–graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ such that $\mathscr{A} = \Bbbk[\partial] \otimes_{\Bbbk} V$ as a $\Bbbk$–vector space. Choose such a

$\Bbbk$–vector space V, we can define

$$\mathrm{GrAut}_V(\mathscr{A}_{\mathcal{R}}) = \{\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{A}_{\mathcal{R}}) | \phi(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R\}, \tag{4.2.28}$$

for each $\mathcal{R}$ in $\Bbbk$**-drng**. It is a subgroup (see Lemma 4.1), and is functorial in $\mathcal{R}$. In other words, $\mathbf{GrAut}_V(\mathscr{A})$ is a subgroup functor of $\mathbf{Aut}(\mathscr{A})$, which assigns each $\mathcal{R}$ in $\Bbbk$**-drng** the group $\mathrm{GrAut}_V(\mathscr{A}_{\mathcal{R}})$.

However, the definition of $\mathbf{GrAut}_V(\mathscr{A})$ depends on the choice of V. Neither the assertion (i) nor (ii) in Remark 4.10 is necessarily true for $\mathbf{GrAut}_V(\mathscr{A})$. It is also not known, for a given $\Bbbk$–conformal superalgebra $\mathscr{A}$, whether there exists a suitable $\Bbbk$–vector space V such that $\mathbf{GrAut}_V(\mathscr{A})$ satisfies at least one of these properties. As we have seen, for $V := \Lambda(N)$, these properties are both fulfilled by $\mathbf{GrAut}_{\Lambda(N)}(\mathscr{K}_N)$ in the situation of the $N = 1, 2, 3$ conformal superalgebras. The small $N = 4$ conformal superalgebra $\mathscr{W}$ also has a subspace V, for which $\mathbf{GrAut}_V(\mathscr{W})$ has nice properties similar to (i) and (ii). This will be discussed in Section 5.2.

The following example shows that $\mathrm{GrAut}_V(\mathscr{A}_{\mathcal{R}})$ may not satisfy property (ii) in Remark 4.10 for a "bad" choice of V.

Example 4.11. Consider the $N = 2$ conformal superalgebra $\mathscr{K}_2$. Besides the realization described in Section 4.1 by using the Grassmannian superalgebra $\Lambda(2)$, $\mathscr{K}_2$ can also be realized as follows (see [Kac98b, section 5.10]):

$$\mathscr{K}_2 = \Bbbk[\partial] \otimes_{\Bbbk} (\mathrm{Der}(\Lambda(1)) \oplus \Lambda(1)),$$

where $\Lambda(1) = \Bbbk \oplus \Bbbk\xi$ and $\mathrm{Der}(\Lambda(1))$ is the superalgebra of all derivations of $\Lambda(1)$. The n-th product for $n \in \mathbb{Z}_+$ on $\mathscr{K}_2$ is given as follows:

$$\begin{aligned} &a_{(n)}b = \delta_{n0}[a, b], \quad a_{(0)}f = a(f), \quad a_{(n)}f = -\delta_{n1}p(a, f)fa, \text{ if } n \geqslant 1, \\ &f_{(0)}g = -\partial(fg), \quad f_{(n)}g = -2\delta_{n1}fg, \text{ if } n \geqslant 1, \end{aligned}$$

where $a, b \in \mathrm{Der}(\Lambda(1))$, $f, g \in \Lambda(1)$.

Let

$$V := \mathrm{Der}(\Lambda(1)) \oplus \Lambda(1) = \left(\Bbbk\frac{d}{d\xi} \oplus \Bbbk\xi\frac{d}{d\xi}\right) \oplus (\Bbbk \oplus \Bbbk\xi).$$

Then $\mathscr{K}_2 = \Bbbk[\partial] \otimes_{\Bbbk} V$. We define $\phi : \mathscr{K}_2 \to \mathscr{K}_2$ to be the unique homomorphism of $\Bbbk[\partial]$–modules satisfying

$$\phi(1) = 1 - \partial\xi\frac{d}{d\xi}, \quad \phi(\xi) = \frac{d}{d\xi}, \quad \phi\left(\frac{d}{d\xi}\right) = \xi, \quad \phi\left(\xi\frac{d}{d\xi}\right) = -\xi\frac{d}{d\xi}.$$

It is easy to verify that ϕ is a homomorphism of $\Bbbk$–conformal superalgebras and $\phi^2 = \mathrm{id}_{\mathscr{K}_2}$.

Although $\Bbbk$ is an integral domain, we have

$$\mathrm{GrAut}_V(\mathscr{K}_2) \subsetneq \mathrm{Aut}_{\Bbbk\text{-conf}}(\mathscr{K}_2),$$

since $\phi \in \mathrm{Aut}_{\Bbbk\text{-conf}}(\mathscr{K}_2)$, but $\phi \notin \mathrm{GrAut}_V(\mathscr{K}_2)$.

4.3 Twisted loop conformal superalgebras

In this section, we classify the twisted loop conformal superalgebras based on $\mathscr{K}_N$ for $N = 1, 2, 3$. Recall that every twisted loop conformal superalgebra $\mathcal{L}(\mathscr{K}_N, \sigma)$ based on $\mathscr{K}_N$ is a $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathscr{K}_{N,\mathcal{D}}$, where $\widehat{\mathcal{D}} = (\Bbbk[t^q, q \in \mathbb{Q}], \mathsf{d}_t)$ and $\mathcal{D} = (\Bbbk[t^{\pm 1}], \mathsf{d}_t)$. We first classify the $\widehat{\mathcal{D}}/\mathcal{D}$–forms of $\mathscr{K}_{N,\mathcal{D}}$ up to isomorphism of $\mathcal{D}$–conformal superalgebras.

Theorem 4.12. *Let $N = 1, 2, 3$. There are exactly two $\widehat{\mathcal{D}}/\mathcal{D}$-forms (up to isomorphism of $\mathcal{D}$-conformal superalgebras) of $\mathscr{K}_{N,\mathcal{D}} := \mathscr{K}_N \otimes_{\Bbbk} \mathcal{D}$. These are $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$, where $\omega_N : \mathscr{K}_N \to \mathscr{K}_N$ is the automorphism of the $\Bbbk$–conformal superalgebra $\mathscr{K}_N$ given by*

$\omega_1 :$	$1 \mapsto 1,$	$\xi_1 \mapsto -\xi_1,$
$\omega_2 :$	$1 \mapsto 1,$	$\xi_1 \mapsto -\xi_1,$
	$\xi_2 \mapsto \xi_2,$	$\xi_1\xi_2 \mapsto -\xi_1\xi_2,$
$\omega_3 :$	$1 \mapsto 1,$	$\xi_j \mapsto -\xi_j, j = 1, 2, 3,$
	$\xi_i\xi_j \mapsto \xi_i\xi_j, i \neq j,$	$\xi_1\xi_2\xi_3 \mapsto -\xi_1\xi_2\xi_3.$

Proof. By Theorem 2.9, the $\widehat{\mathcal{D}}/\mathcal{D}$-forms of $\mathscr{K}_{N,\mathcal{D}}$ are parametrized by the non-abelian cohomology set $\mathrm{H}^1\big(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{K}_N)\big)$. Since $\mathscr{K}_N$ is a free $\Bbbk[\partial]$–module of finite rank 2^N, Proposition 3.4 allows the following identification

$$\mathrm{H}^1\big(\widehat{\mathcal{D}}/\mathcal{D}, \mathbf{Aut}(\mathscr{K}_N)\big) \cong \mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})\big).$$

Our problem is thus reduced to compute $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})\big)$.

The twisted loop conformal superalgebras $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$ correspond to the classes $[\mathfrak{z}]$ and $[\mathfrak{z}']$ in $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})\big)$ given by the 1–cocycles $\mathfrak{z}, \mathfrak{z}' : \widehat{\mathbb{Z}} \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})$ determined by $\bar{1} \mapsto \mathrm{id}$ and $\bar{1} \mapsto \omega_N$, respectively.

Since the isomorphism $\iota_{N,\widehat{\mathcal{D}}} : \mathbf{O}_N(\widehat{D}) \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{K}_{N,\widehat{\mathcal{D}}})$ of Theorem 4.9 is equivariant under the action of $\widehat{\mathbb{Z}}$, we are reduced to determine $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{D})\big)$. The classes in $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{D})\big)$ corresponding to $[\mathfrak{z}]$ and $[\mathfrak{z}']$ will be still denoted in the same way.

Consider the split exact sequence of $\widehat{\mathbb{Z}}$-groups

$$1 \longrightarrow \mathbf{SO}_N(\widehat{D}) \longrightarrow \mathbf{O}_N(\widehat{D}) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} \to 1, \tag{4.3.1}$$

where $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially and "$\det$" is the determinant map. It yields the exact sequence of pointed sets

$$\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SO}_N(\widehat{D})\big) \longrightarrow \mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{D})\big) \xrightarrow{\psi} \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}). \tag{4.3.2}$$

Since $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially, we have $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. Since the short exact sequence (4.3.1) is split, ψ admits a section, and hence is surjective by general considerations. This is also explicitly clear in our situation since ψ visibly maps $[\mathfrak{z}]$ and $[\mathfrak{z}']$ to the two distinct classes in $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$. It remains to show that ψ is injective.

From the exactness of sequence (4.3.2), the fiber of ψ over the trivial class of $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$ is measured by $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{SO}_N(\widehat{D})\big)$, while the fiber over the non-trivial class is measured by $\mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}'}\mathbf{SO}_N(\widehat{D})\big)$ where ${}_{\mathfrak{z}'}\mathbf{SO}_N$ is the group scheme over D obtained from $\mathbf{SO}_N$ by twisting by $\mathfrak{z}'$. By Proposition 3.7 (ii), $\mathrm{H}^1_{\text{ét}}(D, \mathbf{G})$ vanishes for every reductive group scheme $\mathbf{G}$ over R, in particular for $\mathbf{SO}_N$ and ${}_{\mathfrak{z}'}\mathbf{SO}_N$. On the other hand by the Proposition 3.7 (i), we have

$$\mathrm{H}^1_{\text{ét}}(D, \mathbf{G}) \cong \mathrm{H}^1_{\mathrm{ct}}\big(\widehat{\mathbb{Z}}, \mathbf{G}(\widehat{D})\big). \tag{4.3.3}$$

This finishes the proof of injectivity. □

Proposition 4.13. *Let $N = 1, 2, 3$ and $\mathscr{B} = \mathcal{L}(\mathscr{K}_N, \sigma)$ be a twisted loop conformal superalgebra of $\mathscr{K}_N$ with respect to an automorphism σ of finite order. Then* $\mathrm{Ctd}_{\Bbbk}(\mathscr{B}) \cong D$.

Proof. As a $\Bbbk[\partial]$–module, $\mathscr{K}_N = \Bbbk[\partial] \otimes_{\Bbbk} \Lambda(N)$ is free of rank 2^N. It has Virasoro element $\mathrm{L} = -1 \in \Lambda(N)$ and the monomials in $\Lambda(N)$ form a set of generators of $\mathscr{K}_N$ as a $\Bbbk[\partial]$–module satisfying all the assumptions of (ii) and (iii) of Proposition 2.12. From Theorem 4.9, and the explicit construction of automorphisms of $\mathscr{K}_N$ in Propositions 4.2,4.5, and 4.7 in the case of $N = 1, 2, 3$, respectively, we know $\sigma(\mathrm{L}) = \mathrm{L}$. By Proposition 2.12, $\mathrm{Ctd}_{\Bbbk}(\mathscr{B}) \cong D$. □

Theorem 4.14. *There are exactly two twisted loop conformal superalgebras (up to isomorphism of* $\Bbbk$*–conformal superalgebras) based on each* $\mathscr{K}_N, N = 1, 2, 3$*, namely,* $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ *and* $\mathcal{L}(\mathscr{K}_N, \omega_N)$*.*

Proof. Each twisted loop conformal superalgebra based on $\mathscr{K}_N$ is an $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathscr{K}_{N,\mathcal{D}}$. It follows from Theorem 4.12 that there exist exactly two of them up to isomorphism of $\mathcal{D}$–conformal superalgebras, namely $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$. By Propositions 2.12 and 4.13, we conclude that $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$ remain non-isomorphic when viewed as $\Bbbk$–conformal superalgebras. □

4.4 The corresponding twisted Lie superalgebras

In the previous section, we have shown that, for each $N = 1, 2, 3$, there are only two twisted loop conformal superalgebras based on $\mathscr{K}_N$ up to isomorphism of $\Bbbk$–conformal superalgebras, namely $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$. Recall from (2.1.19) that every twisted loop conformal superalgebra $\mathcal{L}(\mathscr{K}_N, \sigma)$ determines a Lie superalgebra

$$\mathrm{Alg}(\mathscr{K}_N, \sigma) = \mathcal{L}(\mathscr{K}_N, \sigma)/\widehat{\partial}\mathcal{L}(\mathscr{K}_N, \sigma),$$

with Lie superbracket induced by the 0-th product of $\mathcal{L}(\mathscr{K}_N, \sigma)$. Hence, the two non-isomorphic twisted loop conformal superalgebras $\mathcal{L}(\mathscr{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathscr{K}_N, \omega_N)$ determine two Lie superalgebras $\mathrm{Alg}(\mathscr{K}_N, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_N, \omega_N)$, respectively. As claimed in [CP11], the two Lie superalgebras $\mathrm{Alg}(\mathscr{K}_N, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_N, \omega_N)$ are not isomorphic for $N = 1, 2, 3$. However, the proof in [CP11] is inaccurate. We will provide amended proofs in this section.

We first prove some basic properties of the centreless Virasoro algebra $\mathfrak{v}$. Recall from (2.1.6) that $\mathfrak{v}$ has a $\Bbbk$–basis $\{\mathrm{L}_n | n \in \mathbb{Z}\}$ satisfying $[\mathrm{L}_m, \mathrm{L}_n] = (m - n)\mathrm{L}_{m+n}$ for $m, n \in \mathbb{Z}$.

Lemma 4.15. *Let* $\mathfrak{v}$ *be the centreless Virasoro algebra. Then the following hold:*

(i) Every abelian Lie subalgebra of $\mathfrak{v}$ has dimension at most one.

(ii) Let $\phi : \mathfrak{v} \to \mathfrak{v}$ be an endomorphism of $\mathfrak{v}$. Then either $\phi = 0$ or ϕ is injective. In particular, every injective endomorphism ϕ is of the form

$$\phi(\mathrm{L}_m) = \frac{1}{\ell} a^m \mathrm{L}_{\ell m}, \quad \forall m \in \mathbb{Z},$$

for some non-zero integer ℓ and some nonzero $a \in \Bbbk$.

(iii) Every automorphism $\phi : \mathfrak{v} \to \mathfrak{v}$ is of the form

$$\phi(\mathrm{L}_m) = \pm a^m \mathrm{L}_{\pm m}, \quad \forall m \in \mathbb{Z},$$

for some nonzero $a \in \Bbbk$.

Proof. (i) It suffices to show that for $0 \neq x, y \in \mathfrak{v}$, $[x, y] = 0$ implies that x and y are proportional. Write

$$x = a\mathrm{L}_M + x', \text{ and } y = b\mathrm{L}_N + y',$$

such that $a, b \neq 0$, $x' \in \mathfrak{v}_{<M} := \oplus_{m<M} \Bbbk \mathrm{L}_m$, and $y' \in \mathfrak{v}_{<N}$. Then

$$0 = [x, y] = [a\mathrm{L}_M + x', b\mathrm{L}_N + y'] = ab(M-N)\mathrm{L}_{M+N} + a[\mathrm{L}_M, y'] + b[x', \mathrm{L}_N] + [x', y'].$$

Note that $[\mathrm{L}_M, y'], [x', \mathrm{L}_N], [x', y'] \in \mathfrak{v}_{<M+N}$. Hence, $ab(M-N) = 0$, i.e., $M = N$.

Now, we know that $[x, bx - ay] = 0$ and $bx - ay \in \mathfrak{v}_{<M}$. We thus conclude that $bx - ay = 0$, i.e., x and y are proportional.

(ii) Since $\mathfrak{v}$ is a simple Lie algebra and $\ker \phi$ is an ideal of $\mathfrak{v}$, we know that either $\phi = 0$ or ϕ is injective.

Now we assume that ϕ is injective. We first claim that $\phi(\mathrm{L}_0) = \frac{1}{\ell}\mathrm{L}_0$ for some nonzero integer ℓ. It has been shown in [Su02] that every three dimensional subalgebra of $\mathfrak{v}$ is of the form $\Bbbk\mathrm{L}_{-n} \oplus \Bbbk\mathrm{L}_0 \oplus \Bbbk\mathrm{L}_n$ for some nonzero $n \in \mathbb{Z}$. Since ϕ is injective, $\phi(\Bbbk\mathrm{L}_{-n} \oplus \Bbbk\mathrm{L}_0 \oplus \Bbbk\mathrm{L}_n)$ is a three dimensional subalgebra of $\mathfrak{v}$ for every $0 \neq n \in \mathbb{Z}$. Hence, there is $0 \neq m_n \in \mathbb{Z}$ such that

$$\phi(\Bbbk\mathrm{L}_{-n} \oplus \Bbbk\mathrm{L}_0 \oplus \Bbbk\mathrm{L}_n) = \Bbbk\mathrm{L}_{-m_n} \oplus \Bbbk\mathrm{L}_0 \oplus \Bbbk\mathrm{L}_{m_n}.$$

The injectivity of ϕ also implies that there are $n \neq n' \in \mathbb{Z}$ such that $m_n \neq m_{n'}$.

Hence,

$$\phi(\mathrm{L}_0) \in \phi(\Bbbk \mathrm{L}_{-n} \oplus \Bbbk \mathrm{L}_0 \oplus \Bbbk \mathrm{L}_n) \cap \phi(\Bbbk \mathrm{L}_{-n'} \oplus \Bbbk \mathrm{L}_0 \oplus \Bbbk \mathrm{L}_{n'}) = \Bbbk \mathrm{L}_0,$$

i.e., $\phi(\mathrm{L}_0) = b\mathrm{L}_0$ for some $0 \neq b \in \Bbbk$. Furthermore, we deduce from

$$-\phi(\mathrm{L}_1) = [\phi(\mathrm{L}_0), \phi(\mathrm{L}_1)] = b[\mathrm{L}_0, \phi(\mathrm{L}_1)]$$

that $\phi(\mathrm{L}_1)$ is an eigenvector of $\mathrm{ad}(\mathrm{L}_0)$ with eigenvalue $-\frac{1}{b}$, which is an integer. Hence, $b = \frac{1}{\ell}$ for some nonzero integer ℓ. This proves the claim.

Note that $\phi(\mathrm{L}_n)$ is an eigenvector of $\mathrm{ad}(\mathrm{L}_0)$ with eigenvalue $-\frac{n}{b} = -\ell n$. It follows that $\phi(\mathrm{L}_n) = \frac{1}{\ell} a_n \mathrm{L}_{\ell n}$ for some $0 \neq a_n \in \Bbbk$. We put $a = a_1$. Then

$$2\phi(\mathrm{L}_0) = [\phi(\mathrm{L}_1), \phi(\mathrm{L}_{-1})]$$

implies that $a_{-1} = a^{-1}$. Similarly, we deduce that $a_n = a^n$ for all $n \in \mathbb{Z}$. Therefore, ϕ is of the form

$$\phi(\mathrm{L}_m) = \frac{1}{\ell} a^m \mathrm{L}_{\ell m}, \quad \forall m \in \mathbb{Z},$$

for some nonzero integer ℓ and nonzero $a \in \Bbbk$.

(iii) Let ϕ be an automorphism of $\mathfrak{v}$. By (ii), ϕ is of the form $\phi(\mathrm{L}_m) = \frac{1}{\ell} a^m \mathrm{L}_{\ell m}$ for some nonzero integer ℓ and nonzero $a \in \Bbbk$. We deduce that

$$\phi(\mathfrak{v}) = \bigoplus_{m \in \mathbb{Z}} \Bbbk \mathrm{L}_{\ell m},$$

which is equal to $\mathfrak{v}$ only if $\ell = \pm 1$. Since ϕ is an automorphism, we conclude that $\ell = \pm 1$. This completes the proof. □

Remark 4.16. The assertion (iii) in the lemma above is a special case of the automorphisms of generalized Virasoro algebras, which has been determined in [Su02]. We give the proof of this lemma here since part (ii) was not established in [Su02].

In fact, the assertion (ii) was stated in [Zha92]. However, the proof for (ii) in [Zha92] is invalid since the Lemma 10 of [Zha92] is inaccurate. For the Virasoro algebra $\hat{\mathfrak{v}} = \mathfrak{v} \oplus \Bbbk c$ (a central extension of $\mathfrak{v}$), Lemma 10 of [Zha92] states that for $x \in \hat{\mathfrak{v}}$, if $\mathrm{ad}x$ has infinitely many linearly independent eigenvectors, then $x \in \Bbbk \mathrm{L}_0 \oplus \Bbbk c$. A counterexample to this is the following: we consider $x = \mathrm{L}_0 + \mathrm{L}_{-1} \in \mathfrak{v}$, then

$$y_m = \sum_{i=0}^{m} \binom{m+1}{i+1} \mathrm{L}_i + \mathrm{L}_{-1},$$

is an eigenvector of x with eigenvalue $-m$ for each positive integer m.

Next we will describe the Lie superalgebras $\mathrm{Alg}(\mathscr{K}_N, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_N, \omega_N)$, and prove these two Lie superalgebras are not isomorphic for each $N = 1, 2, 3$. We always use $\overline{a}$ to denote the image of $a \in \mathcal{L}(\mathscr{K}_N, \sigma)$ under the canonical map $\mathcal{L}(\mathscr{K}_N, \sigma) \to \mathrm{Alg}(\mathscr{K}_N, \sigma) = \mathcal{L}(\mathscr{K}_N, \sigma)/\widehat{\partial}\mathcal{L}(\mathscr{K}_N, \sigma)$, where $\sigma = \mathrm{id}$ or ω_N.

Case $N = 1$:

In $\mathrm{Alg}(\mathscr{K}_1, \mathrm{id})$, we let $\mathrm{L}_m := \overline{-1 \otimes t^{m+1}}$ for $m \in \mathbb{Z}$ and $\mathrm{G}_{m'} := \overline{2\xi_1 \otimes t^{m'+\frac{1}{2}}}$ for $m' \in \frac{1}{2} + \mathbb{Z}$. Then $\mathrm{Alg}(\mathscr{K}_1, \mathrm{id})$ has a basis $\{\mathrm{L}_m, \mathrm{G}_{m'} | m \in \mathbb{Z}, m' \in \frac{1}{2} + \mathbb{Z}\}$, satisfying

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, \ [\mathrm{L}_m, \mathrm{G}_{n'}] = (\tfrac{1}{2}m - n')\mathrm{G}_{m+n'}, \ [\mathrm{G}_{m'}, \mathrm{G}_{n'}] = 2\mathrm{L}_{m'+n'},$$

for $m, n \in \mathbb{Z}$, and $m', n' \in \frac{1}{2} + \mathbb{Z}$.

For $\mathrm{Alg}(\mathscr{K}_1, \omega_1)$, we know that $\omega_1(1) = 1$ and $\omega_1(\xi_1) = -\xi_1$. Hence,

$$\mathcal{L}(\mathscr{K}_1, \omega_1) = (\Bbbk[\partial]1 \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]) \oplus (\Bbbk[\partial]\xi_1 \otimes_{\Bbbk} t^{\frac{1}{2}}\Bbbk[t^{\pm 1}]).$$

Set $\mathrm{L}_m = \overline{-1 \otimes t^{m+1}}$ and $\mathrm{G}_m = \overline{2\xi_1 \otimes t^{m+\frac{1}{2}}}$ for $m \in \mathbb{Z}$. Then $\{\mathrm{L}_m, \mathrm{G}_m | m \in \mathbb{Z}\}$ is a basis of $\mathrm{Alg}(\mathscr{K}_1, \omega_1)$, satisfying

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, \quad [\mathrm{L}_m, \mathrm{G}_n] = (\tfrac{1}{2}m - n)\mathrm{G}_{m+n}, \quad [\mathrm{G}_m, \mathrm{G}_n] = 2\mathrm{L}_{m+n},$$

for $m, n \in \mathbb{Z}$.

Proposition 4.17. *The two Lie superalgebras* $\mathrm{Alg}(\mathscr{K}_1, \mathrm{id})$ *and* $\mathrm{Alg}(\mathscr{K}_1, \omega_1)$ *are not isomorphic.*

Proof. Suppose that $\phi : \mathrm{Alg}(\mathscr{K}_1, \mathrm{id}) \to \mathrm{Alg}(\mathscr{K}_1, \omega_1)$ is an isomorphism. We first observe that the even parts $\mathrm{Alg}(\mathscr{K}_1, \mathrm{id})_{\bar{0}}$ and $\mathrm{Alg}(\mathscr{K}_1, \omega_1)_{\bar{0}}$ are both isomorphic to the centreless Virasoro algebra $\mathfrak{v}$. Then ϕ induces an automorphism of $\mathfrak{v}$. By Lemma 4.15 (iii), $\phi(\mathrm{L}_0) = \pm \mathrm{L}_0$.

Note that

$$\mathrm{Alg}(\mathscr{K}_1, \omega_1)_{\bar{1}} = \bigoplus_{n \in \mathbb{Z}} \Bbbk \mathrm{G}_n$$

and $[\mathrm{L}_0, \mathrm{G}_n] = -n\mathrm{G}_n$ for all $n \in \mathbb{Z}$. It follows that if $x \in \mathrm{Alg}(\mathscr{K}_1, \omega_1)_{\bar{1}}$ such that $[\mathrm{L}_0, x] = ax$ for some $a \in \Bbbk$, then a is an integer. We consider $\phi(\mathrm{G}_{\frac{1}{2}}) \in \mathrm{Alg}(\mathscr{K}_1, \omega_1)_{\bar{1}}$. It satisfies

$$[\pm\mathrm{L}_0, \phi(\mathrm{G}_{\frac{1}{2}})] = [\phi(\mathrm{L}_0), \phi(\mathrm{G}_{\frac{1}{2}})] = -\frac{1}{2}\phi(\mathrm{G}_{\frac{1}{2}}),$$

which yields a contradiction since $\frac{1}{2}$ is not an integer. Hence, $\mathrm{Alg}(\mathscr{K}_1, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_1, \omega_1)$ are not isomorphic. $\square$

Case $N = 2$:

The Lie superalgebra $\mathrm{Alg}(\mathscr{K}_2, \mathrm{id})$ has a basis consisting of

$$\mathrm{L}_m = \overline{-1 \otimes t^{m+1}}, \qquad \mathrm{U}_m = \overline{2\mathbf{i}\xi_1\xi_2 \otimes t^m}, \qquad \mathrm{G}^{\pm}_{m'} = \overline{(\xi_1 \pm \mathbf{i}\xi_2) \otimes t^{m'+\frac{1}{2}}},$$

where $m \in \mathbb{Z}$ and $m' \in \frac{1}{2} + \mathbb{Z}$. The Lie superbracket is written as follows:

$$\begin{aligned}
&[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, && [\mathrm{L}_m, \mathrm{U}_n] = -n\mathrm{U}_{m+n},\\
&[\mathrm{L}_m, \mathrm{G}^{\pm}_{n'}] = (\tfrac{1}{2}m - n')\mathrm{G}^{\pm}_{m+n'}, && [\mathrm{U}_m, \mathrm{G}^{\pm}_{n'}] = \pm\mathrm{G}^{\pm}_{m+n'}\\
&[\mathrm{U}_m, \mathrm{U}_n] = 0, && [\mathrm{G}^{+}_{m'}, \mathrm{G}^{+}_{n'}] = [\mathrm{G}^{-}_{m'}, \mathrm{G}^{-}_{n'}] = 0,\\
&[\mathrm{G}^{+}_{m'}, \mathrm{G}^{-}_{n'}] = \mathrm{L}_{m'+n'} + \tfrac{1}{2}(m'-n')\mathrm{U}_{m'+n'}
\end{aligned}$$

for $m, n \in \mathbb{Z}$ and $m', n' \in \frac{1}{2} + \mathbb{Z}$. The Lie superalgebra $\mathrm{Alg}(\mathscr{K}_2, \mathrm{id})$ is indeed isomorphic to the $N = 2$ Neveu-Schwarz algebra (modulo the center), which is also isomorphic to the $N = 2$ Ramond algebra (modulo the center) as claimed in [SS87].

In a parallel manner, we recall from Theorem 4.12 that

$$\omega_2(1) = 1, \quad \omega_2(\xi_1) = \xi_1, \quad \omega_2(\xi_2) = -\xi_2, \text{ and } \omega_2(\xi_1\xi_2) = -\xi_1\xi_2.$$

Hence,

$$\begin{aligned}
\mathcal{L}(\mathscr{K}_2, \omega_2)_{\bar{0}} &= (\Bbbk[\partial]1 \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]) \oplus (\Bbbk[\partial]\xi_1\xi_2 \otimes_{\Bbbk} t^{\frac{1}{2}}\Bbbk[t^{\pm 1}]),\\
\mathcal{L}(\mathscr{K}_2, \omega_2)_{\bar{1}} &= (\Bbbk[\partial]\xi_1 \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]) \oplus (\Bbbk[\partial]\xi_2 \otimes_{\Bbbk} t^{\frac{1}{2}}\Bbbk[t^{\pm 1}]).
\end{aligned}$$

In $\mathrm{Alg}(\mathscr{K}_2, \omega_2)$, we set

$$\mathrm{L}_m = \overline{-1 \otimes t^{m+1}}, \ \ \mathrm{U}_{m'} = 2\overline{\xi_1\xi_2 \otimes t^{m'}}, \ \ \mathrm{G}^1_{m'} = \overline{2\xi_1 \otimes t^{m'+\frac{1}{2}}}, \ \ \mathrm{G}^2_m = \overline{2\xi_2 \otimes t^{m+\frac{1}{2}}},$$

for $m \in \mathbb{Z}$ and $m' \in \frac{1}{2}+\mathbb{Z}$. Then $\{\mathrm{L}_m, \mathrm{U}_{m'}, \mathrm{G}^1_{m'}, \mathrm{G}^2_m | m \in \mathbb{Z}, m' \in \mathbb{Z}\}$ is a basis of $\mathrm{Alg}(\mathscr{K}_2, \omega_2)$, satisfying the following relations:

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, \qquad [\mathrm{L}_m, \mathrm{U}_{n'}] = -n'\mathrm{U}_{m+n'}, \qquad [\mathrm{U}_{m'}, \mathrm{U}_{n'}] = 0,$$

$$[\mathrm{L}_m, \mathrm{G}^1_{n'}] = (\tfrac{1}{2}m - n')\mathrm{G}^1_{m+n'}, \qquad [\mathrm{U}_{m'}, \mathrm{G}^1_{n'}] = \mathrm{G}^2_{m'+n'}, \qquad [\mathrm{G}^1_{m'}, \mathrm{G}^1_{n'}] = 2\mathrm{L}_{m'+n'},$$

$$[\mathrm{L}_m, \mathrm{G}^2_n] = (\tfrac{1}{2}m - n)\mathrm{G}^2_{m+n}, \qquad [\mathrm{U}_{m'}, \mathrm{G}^2_n] = -\mathrm{G}^1_{m'+n}, \qquad [\mathrm{G}^2_m, \mathrm{G}^2_n] = 2\mathrm{L}_{m+n},$$

$$[\mathrm{G}^1_{m'}, \mathrm{G}^2_n] = (n-m')\mathrm{U}_{m'+n},$$

for $m, n \in \mathbb{Z}$ and $m', n' \in \frac{1}{2}+\mathbb{Z}$.

Proposition 4.18. *The two Lie superalgebras* $\mathrm{Alg}(\mathscr{K}_2, \mathrm{id})$ *and* $\mathrm{Alg}(\mathscr{K}_2, \omega_2)$ *are not isomorphic.*

Proof. We will show that the two Lie superalgebras $\mathrm{Alg}(\mathscr{K}_2, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_2, \omega_2)$ indeed have non-isomorphic even parts.

Let σ be one of the automorphisms id and ω_2. Note that $\mathrm{Alg}(\mathscr{K}_2, \sigma)_{\bar{0}}$ is isomorphic to the semidirect product[3] $\mathfrak{s}(\sigma) \rtimes \mathfrak{v}$, where $\mathfrak{v} = \mathrm{span}_{\Bbbk}\{\mathrm{L}_m | m \in \mathbb{Z}\}$ is isomorphic to the centreless Virasoro algebra and $\mathfrak{s}(\sigma) = \mathrm{span}_{\Bbbk}\{\mathrm{U}_m | m \in \varepsilon + \mathbb{Z}\} \subseteq \mathrm{Alg}(\mathscr{K}_2, \sigma)_{\bar{0}}$ is an abelian ideal of $\mathrm{Alg}(\mathscr{K}_2, \sigma)_{\bar{0}}$, where $\varepsilon = 0$ if $\sigma = \mathrm{id}$ and $\varepsilon = \frac{1}{2}$ if $\sigma = \omega_2$.

Suppose that $\phi : \mathrm{Alg}(\mathscr{K}_2, \mathrm{id})_{\bar{0}} \to \mathrm{Alg}(\mathscr{K}_2, \omega_2)_{\bar{0}}$ is an isomorphism of Lie algebras. Then the composition

$$\tilde{\phi} : \mathrm{Alg}(\mathscr{K}_2, \mathrm{id})_{\bar{0}} \xrightarrow{\phi} \mathrm{Alg}(\mathscr{K}_2, \omega_2)_{\bar{0}} \twoheadrightarrow \mathrm{Alg}(\mathscr{K}_2, \omega_2)_{\bar{0}}/\mathfrak{s}(\omega_2) \cong \mathfrak{v},$$

is a surjective homomorphism. It follows that $\tilde{\phi}(\mathfrak{s}(\mathrm{id}))$ is an abelian ideal of $\mathfrak{v}$. Since $\mathfrak{v}$ is a simple Lie algebra, we conclude that $\tilde{\phi}(\mathfrak{s}(\mathrm{id})) = 0$, i.e., $\phi(\mathfrak{s}(\mathrm{id})) \subseteq \mathfrak{s}(\omega_2)$. Indeed, $\phi(\mathfrak{s}(\mathrm{id})) = \mathfrak{s}(\omega_2)$ since the same argument shows $\phi^{-1}(\mathfrak{s}(\omega_2)) \subseteq \mathfrak{s}(\mathrm{id})$.

Hence, the surjective homomorphism $\tilde{\phi}$ induces a surjective homomorphism

$$\bar{\phi} : \mathfrak{v} \cong \mathrm{Alg}(\mathscr{K}_2, \mathrm{id})_{\bar{0}}/\mathfrak{s}(\mathrm{id}) \to \mathrm{Alg}(\mathscr{K}_2, \omega_2)/\mathfrak{s}(\omega_2) \cong \mathfrak{v},$$

which is indeed an automorphism since $\mathfrak{v}$ is a simple Lie algebra. By Lemma 4.15 (iii), we know $\bar{\phi}(\mathrm{L}_0) = \pm\mathrm{L}_0$, i.e., $\phi(\mathrm{L}_0) = \pm\mathrm{L}_0 + x$ for some $x \in \mathfrak{s}(\omega_2)$.

[3] Given two Lie algebras $\mathfrak{g}, \mathfrak{g}'$, and a Lie algebra homomorphism $\mathfrak{g}' \to \mathrm{Der}_{\Bbbk}(\mathfrak{g})$ (the Lie algebra of derivations of $\mathfrak{g}$), one can define the semidirect product Lie algebra $\mathfrak{g} \rtimes \mathfrak{g}'$ (cf. [Bou75, I,§1.8]).

Since ϕ is an isomorphism,

$$\mathrm{Alg}(\mathscr{K}_2,\omega_2)_{\bar 0} = \phi(\mathrm{Alg}(\mathscr{K}_2,\mathrm{id})_{\bar 0}) = \bigoplus_{m\in\mathbb{Z}}(\Bbbk\phi(\mathrm{L}_m)\oplus\Bbbk\phi(\mathrm{U}_m)),$$

satisfying $[\phi(\mathrm{L}_0),\phi(\mathrm{L}_m)] = -m\phi(\mathrm{L}_m)$ and $[\phi(\mathrm{L}_0),\phi(\mathrm{U}_m)] = -m\phi(\mathrm{U}_m)$ for all $m\in\mathbb{Z}$. We deduce that if $y\in\mathrm{Alg}(\mathscr{K}_2,\omega_2)_{\bar 0}$ such that $[\phi(\mathrm{L}_0),y] = ay$ for some $a\in\Bbbk$, then a is an integer. However, $\mathrm{U}_{\frac12}\in\mathrm{Alg}(\mathscr{K}_2,\omega_2)_{\bar 0}$ satisfies

$$[\phi(\mathrm{L}_0),\mathrm{U}_{\frac12}] = [\pm\mathrm{L}_0 + x,\mathrm{U}_{\frac12}] = \mp\frac{1}{2}\mathrm{U}_{\frac12},$$

in which $\frac12$ is not an integer. This is a contradiction. □

Case $N = 3$:

The Lie superalgebra $\mathrm{Alg}(\mathscr{K}_3,\mathrm{id})$ has a basis consisting of

$$\mathrm{L}_m = -\overline{1\otimes t^{m+1}}, \qquad \mathrm{G}^i_{m'} = 2\overline{\xi_i\otimes t^{m'+\frac12}},$$
$$\mathrm{T}^i_m = 2\mathbf{i}\epsilon_{ijl}\overline{\xi_j\xi_l\otimes t^m}, \qquad \Psi_{m'} = -2\mathbf{i}\overline{\xi_1\xi_2\xi_3\otimes t^{m'-\frac12}},$$

where $i = 1,2,3, m\in\mathbb{Z}, m'\in\frac12+\mathbb{Z}$. The Lie superbracket on $\mathrm{Alg}(\mathscr{K}_3,\mathrm{id})$ is given by:

$$[\mathrm{L}_m,\mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, \qquad [\mathrm{L}_m,\mathrm{T}^i_n] = -n\mathrm{T}^i_{m+n}, \qquad [\mathrm{T}^i_m,\mathrm{T}^j_n] = \mathbf{i}\epsilon_{ijk}\mathrm{T}^k_{m+n},$$
$$[\mathrm{L}_m,\Psi_{n'}] = -(\tfrac12 m+n')\Psi_{m+n'}, \qquad [\mathrm{T}^i_m,\Psi_{n'}] = 0, \qquad [\Psi_{m'},\Psi_{n'}] = 0,$$
$$[\mathrm{L}_m,\mathrm{G}^i_{n'}] = (\tfrac12 m-n')\mathrm{G}^i_{m+n'}, \qquad [\mathrm{T}^i_m,\mathrm{G}^j_{n'}] = \mathbf{i}\epsilon_{ijk}\mathrm{G}^k_{m+n'} + \delta_{ij}m\Psi_{m+n'},$$
$$[\mathrm{G}^i_{m'},\Psi_{n'}] = \mathrm{T}^i_{m'+n'}, \qquad [\mathrm{G}^i_{m'},\mathrm{G}^j_{n'}] = 2\delta_{ij}\mathrm{L}_{m'+n'} + \mathbf{i}\epsilon_{ijl}(m'-n')\mathrm{T}^k_{m'+n'},$$

for $m,n\in\mathbb{Z}, m',n'\in\frac12+\mathbb{Z}, i,j = 1,2,3$.

For $\mathrm{Alg}(\mathscr{K}_3,\omega_3)$, we know that ω_3 acts on the even part $(\mathscr{K}_3)_{\bar 0}$ as the identity and acts on the odd part $(\mathscr{K}_3)_{\bar 1}$ as $-\mathrm{id}$. Hence

$$\mathcal{L}(\mathscr{K}_3,\omega_3) = ((\mathscr{K}_3)_{\bar 0}\otimes_\Bbbk \Bbbk[t,t^{-1}])\oplus((\mathscr{K}_3)_{\bar 1}\otimes_\Bbbk t^{\frac12}\Bbbk[t,t^{-1}]). \tag{4.4.1}$$

In $\mathrm{Alg}(\mathscr{K}_3,\omega_3)$, let

$$\mathrm{L}_m = -\overline{1\otimes t^{m+1}}, \qquad \mathrm{G}^i_m = 2\overline{\xi_i\otimes t^{m+\frac12}},$$
$$\mathrm{T}^i_m = 2\mathbf{i}\epsilon_{ijl}\overline{\xi_j\xi_l\otimes t^m}, \qquad \Psi_m = -2\mathbf{i}\overline{\xi_1\xi_2\xi_3\otimes t^{m-\frac12}},$$

for $i=1,2,3, m\in\mathbb{Z}$. Then $\{\mathrm{L}_m, \mathrm{T}^i_m, \mathrm{G}^i_m, \Psi_m | i=1,2,3, m\in\mathbb{Z}\}$ is a basis of $\mathrm{Alg}(\mathscr{K}_3,\omega_3)$, satisfying

$$[\mathrm{L}_m,\mathrm{L}_n]=(m-n)\mathrm{L}_{m+n},\qquad [\mathrm{L}_m,\mathrm{T}^i_n]=-n\mathrm{T}^i_{m+n},\quad [\mathrm{T}^i_m,\mathrm{T}^j_n]=\mathbf{i}\epsilon_{ijk}\mathrm{T}^k_{m+n},$$

$$[\mathrm{L}_m,\Psi_n]=-(\tfrac{1}{2}m+n)\Psi_{m+n},\quad [\mathrm{T}^i_m,\Psi_n]=0,\qquad [\Psi_m,\Psi_n]=0,$$

$$[\mathrm{L}_m,\mathrm{G}^i_n]=\left(\tfrac{1}{2}m-n\right)\mathrm{G}^i_{m+n},\quad [\mathrm{T}^i_m,\mathrm{G}^j_n]=\mathbf{i}\epsilon_{ijk}\mathrm{G}^k_{m+n}+\delta_{ij}m\Psi_{m+n},$$

$$[\mathrm{G}^i_m,\Psi_n]=\mathrm{T}^i_{m+n},\qquad [\mathrm{G}^i_m,\mathrm{G}^j_n]=2\delta_{ij}\mathrm{L}_{m+n}+\mathbf{i}\epsilon_{ijk}(m-n)\mathrm{T}^k_{m+n},$$

for $i,j=1,2,3$ and $m,n\in\mathbb{Z}$.

To prove the two Lie superalgebras $\mathrm{Alg}(\mathscr{K}_3,\mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_3,\omega_3)$ are not isomorphic, we first observe that both the two Lie superalgebras have the isomorphic even part $\mathfrak{g}$, which is isomorphic to the semidirect product of Lie algebras $\mathfrak{g}=\mathfrak{s}\rtimes\mathfrak{v}$, where $\mathfrak{s}:=\mathfrak{sl}_2(\Bbbk)\otimes_{\Bbbk}\Bbbk[t^{\pm1}]$ is the loop Lie algebra based on $\mathfrak{sl}_2(\Bbbk)$ and $\mathfrak{v}=\mathrm{span}_{\Bbbk}\{\mathrm{L}_m|m\in\mathbb{Z}\}$ is the centreless Virasoro algebra, which acts on $\mathfrak{s}$ via

$$[\mathrm{L}_m, x\otimes t^n]=-n\mathbf{x}\otimes t^{m+n},$$

for $m,n\in\mathbb{Z}$ and $\mathbf{x}\in\mathfrak{sl}_2(\Bbbk)$.

Lemma 4.19. *Let $\mathfrak{g}=\mathfrak{s}\rtimes\mathfrak{v}$ as above and $\phi:\mathfrak{g}\to\mathfrak{g}$ an automorphism of the Lie algebra $\mathfrak{g}$. Then*

(i) $\phi(\mathfrak{s})=\mathfrak{s}$. Hence, the restriction of ϕ to $\mathfrak{s}$ yields an automorphism of $\mathfrak{s}$;

(ii) $\phi(\mathrm{L}_0)=\pm\mathrm{L}_0+x$ for some $x\in\mathfrak{s}$.

Proof. (i) Since ϕ is an automorphism, the composition

$$\tilde{\phi}:\mathfrak{g}\xrightarrow{\phi}\mathfrak{g}\twoheadrightarrow\mathfrak{g}/\mathfrak{s}\cong\mathfrak{v},$$

is a surjective homomorphism of Lie algebras.

Note that $\mathfrak{s}=\mathfrak{s}_1\oplus\mathfrak{s}_2\oplus\mathfrak{s}_3$, where $\mathfrak{s}_i=\mathrm{span}_{\Bbbk}\{\mathrm{T}^i_m|m\in\mathbb{Z}\}$ is an abelian subalgebra of $\mathfrak{g}$ for $i=1,2,3$. From Lemma 4.15 (i), we deduce that $\tilde{\phi}(\mathfrak{s})\subseteq\tilde{\phi}(\mathfrak{s}_1)+\tilde{\phi}(\mathfrak{s}_2)+\tilde{\phi}(\mathfrak{s}_3)$ is an ideal of $\mathfrak{v}$ of dimension at most three. Hence, $\tilde{\phi}(\mathfrak{s})=0$, i.e., $\phi(\mathfrak{s})\subseteq\mathfrak{s}$. Applying the same argument to the automorphism ϕ^{-1}, we know that $\phi^{-1}(\mathfrak{s})\subseteq\mathfrak{s}$. Hence, $\phi(\mathfrak{s})=\mathfrak{s}$.

(ii) By (i), the surjective homomorphism $\tilde{\phi}$ induces a surjective homomorphism

$$\bar{\phi}:\mathfrak{v}\cong\mathfrak{g}/\mathfrak{s}\to\mathfrak{g}/\mathfrak{s}\cong\mathfrak{v},$$

which is indeed an automorphism of $\mathfrak{v}$ since $\mathfrak{v}$ is simple. By Lemma 4.15 (iii), we know that $\bar{\phi}(\mathrm{L}_0) = \pm\mathrm{L}_0$, i.e., $\phi(\mathrm{L}_0) = \pm\mathrm{L}_0 + x$ for some $x \in \mathfrak{s}$. $\square$

Next we consider the odd parts $\mathrm{Alg}(\mathscr{K}_3, \mathrm{id})_{\bar{1}}$ and $\mathrm{Alg}(\mathscr{K}_3, \omega_3)_{\bar{1}}$. We will show that for $\sigma = \mathrm{id}$ or σ, the $\mathbb{k}$–subspace of $\mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}}$ consisting of elements annihilated by all $\mathrm{ad}(y), y \in \mathfrak{s}$ is exactly equal to the vector space spanned by $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$, where $\varepsilon = \frac{1}{2}$ if $\sigma = \mathrm{id}$ and $\varepsilon = 0$ if $\sigma = \omega_3$, i.e., we have the following

Lemma 4.20. *Let σ be one of the automorphisms id and ω_3. Let $\mathfrak{s} = \mathrm{span}_{\mathbb{k}}\{\mathrm{T}^i_m | i = 1, 2, 3, m \in \mathbb{Z}\} \subseteq \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{0}}$. Then the $\mathbb{k}$–subspace*

$$\mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}}) := \{u \in \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}} | [y, u] = 0, \forall y \in \mathfrak{s}\}$$

is spanned by $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$, where $\varepsilon = \frac{1}{2}$ if $\sigma = \mathrm{id}$ and $\varepsilon = 0$ if $\sigma = \omega_3$.

Proof. It is obvious that any $\mathbb{k}$–linear combination of $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$ is contained in $\mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}})$. It suffices to show that every $u \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}})$ is a $\mathbb{k}$–linear combination of $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$.

Since $u \in \mathrm{Alg}(\mathscr{K}_3, \sigma)_{\bar{1}}$, we write

$$u = \alpha_1 \mathrm{G}^1_{m_1} + \cdots + \alpha_p \mathrm{G}^1_{m_p} + \beta_1 \mathrm{G}^2_{n_1} + \cdots + \beta_q \mathrm{G}^2_{n_q} + \gamma_1 \mathrm{G}^3_{k_1} + \cdots + \gamma_r \mathrm{G}^3_{k_r} + u',$$

where $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q, \gamma_1, \cdots, \gamma_r \in \mathbb{k}$ are all nonzero, $m_1 < \cdots < m_p$, $n_1 < \cdots < n_q, k_1 < \cdots < k_r$ and u' is a $\mathbb{k}$–linear combination of $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$.

Suppose that one of $p, q, r > 0$. Without loss of generality, we assume $p > 0$. Then

$$[\mathrm{T}^1_1, u] = \alpha_1 \Psi_{m_1+1} + \cdots + \alpha_p \Psi_{m_p+1} + u'' \neq 0,$$

where $u'' \in \mathrm{span}_{\mathbb{k}}\{\mathrm{G}^i_{m'} | i = 1, 2, 3, m' \in \frac{1}{2} + \mathbb{Z}\}$. This contradicts the assumption that $[y, u] = 0$ for all $y \in \mathfrak{s}$. Hence, $p = q = r = 0$, i.e., $u = u'$ is a $\mathbb{k}$–linear combination of $\{\Psi_n | n \in \varepsilon + \mathbb{Z}\}$. $\square$

Proposition 4.21. *The two Lie superalgebras $\mathrm{Alg}(\mathscr{K}_3, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{K}_3, \omega_3)$ are not isomorphic.*

Proof. Suppose that $\phi : \mathrm{Alg}(\mathscr{K}_3, \mathrm{id}) \to \mathrm{Alg}(\mathscr{K}_3, \omega_3)$ is an isomorphism of Lie superalgebras. Then it induces an automorphism on $\mathfrak{g} = \mathfrak{s} \rtimes \mathfrak{v}$, which is isomorphic to both of the even parts $\mathrm{Alg}(\mathscr{K}_3, \mathrm{id})_{\bar{0}}$ and $\mathrm{Alg}(\mathscr{K}_3, \omega)_{\bar{0}}$. By Lemma 4.19, $\phi(\mathfrak{s}) = \mathfrak{s}$.

Let $u \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \mathrm{id})_{\bar{1}})$. Then

$$[\phi(y), \phi(u)] = \phi([y, u]) = 0, \quad \forall y \in \mathfrak{s}.$$

It follows that $\phi(u) \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \omega_3)_{\bar{1}})$ since $\phi(\mathfrak{s}) = \mathfrak{s}$.

We know from Lemma 4.20 that $\mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \omega_3)_{\bar{1}}) = \mathrm{span}_{\Bbbk}\{\Psi_n | n \in \mathbb{Z}\}$, in which $[\mathrm{L}_0, \Psi_n] = -n\Psi_n$ for all $n \in \mathbb{Z}$. It follows that if $u' \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \omega_3)_{\bar{1}})$ such that $[\mathrm{L}_0, u'] = au'$ for some $a \in \Bbbk$, then a is an integer. By Lemma 4.19 (i), $\phi(\mathrm{L}_0) = \pm\mathrm{L}_0 + x$ for some $x \in \mathfrak{s}$. Considering $\Psi_{\frac{1}{2}} \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \mathrm{id})_{\bar{1}})$, we have $\phi(\Psi_{\frac{1}{2}}) \in \mathrm{C}(\mathfrak{s}, \mathrm{Alg}(\mathscr{K}_3, \omega_3)_{\bar{1}})$ satisfying

$$-\frac{1}{2}\phi(\Psi_{\frac{1}{2}}) = [\phi(\mathrm{L}_0), \phi(\Psi_{\frac{1}{2}})] = [\pm\mathrm{L}_0 + x, \phi(\Psi_{\frac{1}{2}})] = \pm[\mathrm{L}_0, \phi(\Psi_{\frac{1}{2}})],$$

which yields a contradiction since $\frac{1}{2}$ is not an integer. □

Remark 4.22. It was inaccurately stated in Lemma 5.1 of [CP11] that, for the Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{v}$, every automorphism ϕ of $\mathfrak{g}$ satisfies $\phi(\mathrm{L}_0) = \pm\mathrm{L}_0$. A counterexample to this is the following: we consider the standard basis $\{h, e, f\}$ of $\mathfrak{sl}_2(\Bbbk)$ satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Then we can verify that for a fixed $\ell \in \mathbb{Z}$, the linear transformation $\mathrm{ad}(e \otimes t^{\ell})$ is locally nilpotent on $\mathfrak{g}$. It follows that $\sigma = \exp(\mathrm{ad}(e \otimes t^{\ell}))$ is an automorphism of $\mathfrak{g}$. More explicitly,

$$\begin{aligned} \sigma(\mathrm{L}_m) &= \mathrm{L}_m + \ell e \otimes t^{m+\ell}, \\ \sigma(h \otimes t^m) &= h \otimes t^m - 2e \otimes t^{m+\ell}, \\ \sigma(e \otimes t^m) &= e \otimes t^m, \\ \sigma(f \otimes t^m) &= f \otimes t^m + h \otimes t^{m+\ell} - e \otimes t^{m+2\ell}. \end{aligned}$$

for $m \in \mathbb{Z}$. If $\ell \neq 0$, we know $\sigma(\mathrm{L}_0) = \mathrm{L}_0 + \ell e \otimes t^{\ell} \neq \pm\mathrm{L}_0$.

We also observe that such an automorphism σ does not come from an automorphism of the conformal superalgebra $\mathscr{A}$ associated to $\mathfrak{g}$. In other words, given a conformal superalgebra $\mathscr{A}$, the automorphism group of its associated Lie superalgebra $\mathrm{Alg}(\mathscr{A}, \mathrm{id})$ is not necessarily equal to the automorphism group of the conformal superalgebra $\mathscr{A}$.

Chapter 5

The small $N = 4$ conformal superalgebra[1]

In this chapter, we will concentrate on the small $N = 4$ conformal superalgebra $\mathscr{W}$, which is simply called the $N = 4$ conformal superalgebra in general. We emphasize the small $N = 4$ conformal superalgebra here to distinguish it with the large $N = 4$ conformal superalgebra that will be discussed in the next chapter.

Indeed, the twisted loop conformal superalgebras based on $\mathscr{W}$ have been classified in [KLP09] by computing the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{W}_{\widehat{\mathcal{D}}})$, where $\mathscr{W}_{\widehat{\mathcal{D}}} := \mathscr{W} \otimes_{\Bbbk} \widehat{\mathcal{D}}$ is the $\widehat{\mathcal{D}}$–conformal superalgebra obtained by the change of base differential ring $\Bbbk \to \widehat{\mathcal{D}}$.

The key ingredient of this chapter is to characterize the automorphism group functor $\mathbf{Aut}(\mathscr{W})$. We will consider the group $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_{\mathcal{R}})$ for an arbitrary $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$**-drng** and prove that $\mathbf{Aut}(\mathscr{W})$ has a subgroup functor $\mathbf{GrAut}(\mathscr{W})$ which has nice properties similar to (i) and (ii) in Remark 4.10.

5.1 The small $N = 4$ conformal superalgebra

The (small) $N = 4$ superconformal algebra was described in [SS87]. The corresponding conformal superalgebra $\mathscr{W}$, which is called the small $N = 4$ conformal superalgebra, can be defined as follows:

As a $\mathbb{Z}/2\mathbb{Z}$–graded $\Bbbk[\partial]$–module, $\mathscr{W} = \mathscr{W}_{\bar{0}} \oplus \mathscr{W}_{\bar{1}}$, where

$$\mathscr{W}_{\bar{0}} = \Bbbk[\partial]\mathrm{L} \oplus \Bbbk[\partial]\mathrm{T}^1 \oplus \Bbbk[\partial]\mathrm{T}^2 \oplus \Bbbk[\partial]\mathrm{T}^3,$$
$$\mathscr{W}_{\bar{1}} = \Bbbk[\partial]\mathrm{G}^1 \oplus \Bbbk[\partial]\mathrm{G}^2 \oplus \Bbbk[\partial]\overline{\mathrm{G}}^1 \oplus \Bbbk[\partial]\overline{\mathrm{G}}^2.$$

The λ–bracket on $\mathscr{W}$ is given by

[1] A version of the first two sections of this chapter has been submitted for publication. A preprint version [Cha13] is available on Arxiv.

$$
\begin{aligned}
&[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}, && [\mathrm{L}_\lambda \mathrm{T}^i] = (\partial + \lambda)\mathrm{T}^i,\\
&[\mathrm{L}_\lambda \mathrm{G}^p] = \left(\partial + \tfrac{3}{2}\lambda\right)\mathrm{G}^p, && \left[\mathrm{L}_\lambda \overline{\mathrm{G}}^p\right] = \left(\partial + \tfrac{3}{2}\lambda\right)\overline{\mathrm{G}}^p,\\
&[\mathrm{T}^i{}_\lambda \mathrm{T}^j] = \mathbf{i}\epsilon_{ijl}\mathrm{T}^l, && [\mathrm{G}^p{}_\lambda \mathrm{G}^q] = \left[\overline{\mathrm{G}}^p{}_\lambda \overline{\mathrm{G}}^q\right] = 0,\\
&[\mathrm{T}^i{}_\lambda \mathrm{G}^p] = -\tfrac{1}{2}\textstyle\sum_{q=1}^2 \sigma^i_{pq}\mathrm{G}^q, && \left[\mathrm{T}^i{}_\lambda \overline{\mathrm{G}}^p\right] = \tfrac{1}{2}\textstyle\sum_{q=1}^2 \sigma^i_{qp}\overline{\mathrm{G}}^q,\\
&\left[\mathrm{G}^p{}_\lambda \overline{\mathrm{G}}^q\right] = 2\delta_{pq}\mathrm{L} - 2(\partial + 2\lambda)\textstyle\sum_{i=1}^3 \sigma^i_{pq}\mathrm{T}^i,
\end{aligned}
$$

where $i, j = 1, 2, 3$, $p, q = 1, 2$, and $\sigma^i, i = 1, 2, 3$ are the Pauli spin matrices:

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1.1}
$$

Before computing the automorphism group $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_\mathcal{R})$ for an object $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$-**drng**, we first introduce some notations to simplify the relations above. These notations will be used to explicitly write down the automorphisms of the $\mathcal{R}$–conformal superalgebra $\mathscr{W}_\mathcal{R}$ in a nice matrix form.

For $\mathbf{x} = (x_{ij})_{2\times 2} \in \mathfrak{sl}_2(\Bbbk)$ and $\mathbf{u} = (u_{ij})_{2\times 2} \in \mathrm{Mat}_2(\Bbbk)$, we set

$$
\begin{aligned}
\mathrm{T}(\mathbf{x}) &:= (x_{12} + x_{21})\mathrm{T}^1 + \mathbf{i}(x_{12} - x_{21})\mathrm{T}^2 + 2x_{11}\mathrm{T}^3,\\
\mathrm{G}(\mathbf{u}) &:= u_{22}\mathrm{G}^1 + u_{11}\overline{\mathrm{G}}^1 - u_{12}\mathrm{G}^2 + u_{21}\overline{\mathrm{G}}^2.
\end{aligned}
$$

Then $\mathscr{W}$ is a $\Bbbk[\partial]$–module generated by $\mathrm{L}, \mathrm{T}(\mathbf{x}), \mathrm{G}(\mathbf{u}), \mathbf{x} \in \mathfrak{sl}_2(\Bbbk), \mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$. The λ–bracket on $\mathscr{W}$ can be rewritten as follows:

$$
\begin{aligned}
&[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}, &&\\
&[\mathrm{L}_\lambda \mathrm{T}(\mathbf{x})] = (\partial + \lambda)\mathrm{T}(\mathbf{x}), && [\mathrm{T}(\mathbf{x})_\lambda \mathrm{T}(\mathbf{y})] = \mathrm{T}([\mathbf{x}, \mathbf{y}]),\\
&[\mathrm{L}_\lambda \mathrm{G}(\mathbf{u})] = \left(\partial + \tfrac{3}{2}\lambda\right)\mathrm{G}(\mathbf{u}), && [\mathrm{T}(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})] = \mathrm{G}(\mathbf{xu}),\\
&[\mathrm{G}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})] = 2\mathrm{tr}(\mathbf{uv}^\dagger)\mathrm{L} + (\partial + 2\lambda)\mathrm{T}(\mathbf{uv}^\dagger - \mathbf{vu}^\dagger), &&
\end{aligned}
$$

where $\mathbf{x}, \mathbf{y} \in \mathfrak{sl}_2(\Bbbk), \mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(\Bbbk)$, $\mathrm{tr} : \mathrm{Mat}_2(\Bbbk) \to \Bbbk$ is the trace map and

$$
{}^\dagger : \mathrm{Mat}_2(\Bbbk) \to \mathrm{Mat}_2(\Bbbk), \quad \mathbf{u} = (u_{ij}) \mapsto \mathbf{u}^\dagger = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} \tag{5.1.2}
$$

is the standard sympletic involution on $\mathrm{Mat}_2(\Bbbk)$ (cf. [Row80, 2.5]).

5.2 The automorphism group functor

This section encompasses a detailed discussion of the automorphism group functor $\mathbf{Aut}(\mathscr{W})$ of the small $N = 4$ conformal superalgebra $\mathscr{W}$.

As for the $N = 1, 2, 3$ conformal superalgebra $\mathscr{K}_N$, we fix the $\mathbb{k}$–vector space V spanned by $\{\mathrm{L}, \mathrm{T}(\mathbf{x}), \mathrm{G}(\mathbf{u}) | \mathbf{x} \in \mathfrak{sl}_2(\mathbb{k}), \mathbf{u} \in \mathrm{Mat}_2(\mathbb{k})\}$. Then $\mathscr{W} = \mathbb{k}[\partial] \otimes_{\mathbb{k}} V$ is a free $\mathbb{k}[\partial]$–module of rank 8. We identify V with the subspace $\mathbb{k} \otimes V$ in $\mathscr{W}$. For an object $\mathcal{R} = (R, \mathsf{d})$ in $\mathbb{k}$**-drng**, we define

$$\mathrm{GrAut}(\mathscr{W}_{\mathcal{R}}) = \{\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_{\mathcal{R}}) | \phi(V \otimes_{\mathbb{k}} R) \subseteq V \otimes_{\mathbb{k}} R\}. \tag{5.2.1}$$

Then $\mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$ is a subgroup of $\mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_{\mathcal{R}})$ and the construction is functorial in $\mathcal{R}$, i.e., we have defined a subgroup functor $\mathbf{GrAut}(\mathscr{W}) : \mathcal{R} \mapsto \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$.

We first concentrate on determining $\mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$ for an given $\mathcal{R} = (R, \mathsf{d})$ in $\mathbb{k}$**-drng**. In $\mathscr{W}_{\mathcal{R}}$, we use $\widehat{\partial}$ to denote $\partial \otimes \mathrm{id} + \mathrm{id} \otimes \mathsf{d}$. We also conveniently write $\mathrm{T}(r\mathbf{x}) := \mathrm{T}(\mathbf{x}) \otimes r$ and $\mathrm{G}(r\mathbf{u}) := \mathrm{G}(\mathbf{u}) \otimes r$ for $r \in R, \mathbf{x} \in \mathfrak{sl}_2(\mathbb{k}), \mathbf{u} \in \mathrm{Mat}_2(\mathbb{k})$.

Lemma 5.1. *For an arbitrary object $\mathcal{R} = (R, \mathsf{d})$ in $\mathbb{k}$**-drng**, there is a group homomorphism*

$$\iota_{\mathcal{R}} : \mathbf{SL}_2(R) \times \mathbf{SL}_2(R_0) \to \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}}), \quad (A, B) \mapsto \theta_{A,B}, \tag{5.2.2}$$

where $R_0 = \ker \mathsf{d}$, and $\theta_{A,B} \in \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$ is defined by

$$\theta_{A,B}(\mathrm{L}) = \mathrm{L} + \mathrm{T}(\mathsf{d}(A)A^{-1}), \tag{5.2.3}$$

$$\theta_{A,B}(\mathrm{T}(\mathbf{x})) = \mathrm{T}(A\mathbf{x}A^{-1}), \tag{5.2.4}$$

$$\theta_{A,B}(\mathrm{G}(\mathbf{u})) = \mathrm{G}(A\mathbf{u}B^{-1}), \tag{5.2.5}$$

for $\mathbf{x} \in \mathfrak{sl}_2(\mathbb{k}), \mathbf{u} \in \mathrm{Mat}_2(\mathbb{k})$. In addition, the homomorphism $\iota_{\mathcal{R}}$ is functorial in $\mathcal{R}$.

Proof. The formulas in (5.2.3)-(5.2.5) define a homomorphism $\theta_{A,B}$ of R–modules $V \otimes_{\mathbb{k}} R \to V \otimes_{\mathbb{k}} R$. It determines a homomorphism of R–modules $\mathscr{W}_{\mathcal{R}} \to \mathscr{W}_{\mathcal{R}}$ which preserves the $\mathbb{Z}/2\mathbb{Z}$–gradings and satisfies $\widehat{\partial} \circ \theta_{A,B} = \theta_{A,B} \circ \widehat{\partial}$. This map is also denoted by $\theta_{A,B}$.

To show $\theta_{A,B}$ is a homomorphism of conformal superalgebras, by Lemma 2.3, it suffices to show

$$\theta_{A,B}([(\eta_1 \otimes 1)_\lambda (\eta_2 \otimes 1)]) = [\theta_{A,B}(\eta_1 \otimes 1)_\lambda \theta_{A,B}(\eta_2 \otimes 1)] \tag{5.2.6}$$

for all $\eta_1, \eta_2 \in V$. This can be accomplished by a direct computation.

For instance, let $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(\Bbbk)$, then

$$\begin{aligned}
&\theta_{A,B}([\mathrm{G}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})]) \\
&= \theta_{A,B}\left(2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{L} + (\partial + 2\lambda)\mathrm{T}(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)\right) \\
&= 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{L} + 2\mathrm{T}(\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathsf{d}(A)A^{-1}) \\
&\quad + (\widehat{\partial} + 2\lambda)\mathrm{T}(A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1}), \\
&[\theta_{A,B}(\mathrm{G}(\mathbf{u}))_\lambda \theta_{A,B}(\mathrm{G}(\mathbf{v}))] \\
&= [\mathrm{G}(A\mathbf{u}B^{-1})_\lambda \mathrm{G}(A\mathbf{v}B^{-1})] \\
&= 2\mathrm{tr}(A\mathbf{u}B^{-1}(A\mathbf{v}B^{-1})^\dagger)\mathrm{L} \\
&\quad + (\partial + 2\lambda)\mathrm{T}(A\mathbf{u}B^{-1}(A\mathbf{v}B^{-1})^\dagger - A\mathbf{v}B^{-1}(A\mathbf{u}B^{-1})^\dagger) \\
&\quad + 2\mathrm{T}(\mathsf{d}(A\mathbf{u}B^{-1})(A\mathbf{v}B^{-1})^\dagger - A\mathbf{v}B^{-1}\mathsf{d}(A\mathbf{u}B^{-1})^\dagger).
\end{aligned}$$

A straightforward computation shows that

- $(\mathbf{u}\mathbf{v})^\dagger = \mathbf{v}^\dagger\mathbf{u}^\dagger$ for $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(R)$.
- $\mathbf{u}\mathbf{v}^\dagger + \mathbf{v}\mathbf{u}^\dagger = \mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)I$ for $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(R)$, where I is the identity matrix.
- $A^{-1} = A^\dagger$ if $A \in \mathbf{SL}_2(R)$.

Hence, $A\mathbf{u}B^{-1}(A\mathbf{v}B^{-1})^\dagger = A\mathbf{u}B^{-1}B\mathbf{v}^\dagger A^{-1} = A\mathbf{u}\mathbf{v}^\dagger A^{-1}$. Note that $\mathsf{d}(B) = 0$ and $\mathsf{d}(A^{-1}) = -A^{-1}\mathsf{d}(A)A^{-1}$, we obtain

$$\begin{aligned}
&2(\mathsf{d}(A\mathbf{u}B^{-1})(A\mathbf{v}B^{-1})^\dagger - A\mathbf{v}B^{-1}\mathsf{d}(A\mathbf{u}B^{-1})^\dagger) \\
&= 2(\mathsf{d}(A)\mathbf{u}B^{-1}B\mathbf{v}^\dagger A^{-1} - A\mathbf{v}B^{-1}B\mathbf{u}^\dagger\mathsf{d}(A^{-1})) \\
&= \mathsf{d}(A)(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1} + A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)\mathsf{d}(A^{-1}) \\
&\quad + \mathsf{d}(A)(\mathbf{u}\mathbf{v}^\dagger + \mathbf{v}\mathbf{u}^\dagger)A^{-1} - A(\mathbf{u}\mathbf{v}^\dagger + \mathbf{v}\mathbf{u}^\dagger)\mathsf{d}(A^{-1}) \\
&= \mathsf{d}(A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1}) + \mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)(\mathsf{d}(A)A^{-1} - A\mathsf{d}(A^{-1})) \\
&= \mathsf{d}(A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1}) + 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathsf{d}(A)A^{-1}.
\end{aligned}$$

It follows that

$$\theta_{A,B}([\mathrm{G}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})]) = [\theta_{A,B}(\mathrm{G}(\mathbf{u}))_\lambda \theta_{A,B}(\mathrm{G}(\mathbf{v}))].$$

Similarly, it is easy to verify that equation (5.2.6) holds for all $\eta_1, \eta_2 \in V$. Hence, $\theta_{A,B}$ is a homomorphism of $\mathcal{R}$–conformal superalgebras.

It can also be directly verified that

$$\theta_{A_1A_2,B_1B_2}(\eta \otimes 1) = \theta_{A_1,B_1} \circ \theta_{A_2,B_2}(\eta \otimes 1),$$
$$\theta_{I,I}(\eta \otimes 1) = \eta \otimes 1,$$

for all $\eta \in V$. It follows from Lemma 2.3 that $\theta_{A_1A_2,B_1B_2} = \theta_{A_1,B_1} \circ \theta_{A_2,B_2}$ and $\theta_{I,I} = \mathrm{id}$. Hence, $\theta_{A,B}$ has the inverse $\theta_{A^{-1},B^{-1}}$, which implies $\theta_{A,B} \in \mathrm{GrAut}(\mathscr{W}_\mathcal{R})$. It also follows that $\iota_\mathcal{R} : \mathbf{SL}_2(R) \times \mathbf{SL}_2(R_0) \to \mathrm{GrAut}(\mathscr{W}_\mathcal{R})$ is a group homomorphism. From the definition of $\theta_{A,B}$, we know that $\iota_\mathcal{R}$ is functorial in $\mathcal{R}$. □

Lemma 5.2. *For every object $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$-**drng**,*

$$\ker(\iota_\mathcal{R}) \cong \boldsymbol{\mu}_2(R_0), \tag{5.2.7}$$

where $R_0 = \ker \mathsf{d}$. Additionally, the isomorphism is functorial in $\mathcal{R}$.

Proof. Let $(A, B) \in \ker(\iota_\mathcal{R})$, where $A \in \mathbf{SL}_2(R)$ and $B \in \mathbf{SL}_2(R_0)$. Since $\theta_{A,B} = \mathrm{id}$,

$$\mathrm{T}(\mathbf{x}) = \theta_{A,B}(\mathrm{T}(\mathbf{x})) = \mathrm{T}(A\mathbf{x}A^{-1}),$$

for all $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk)$. Hence, $A\mathbf{x} = \mathbf{x}A$ for all $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk)$. It follows that $A = aI$ for some $a \in \boldsymbol{\mu}_2(R)$.

Then

$$\mathrm{G}(\mathbf{u}) = \theta_{A,B}(\mathrm{G}(\mathbf{u})) = \mathrm{G}(A\mathbf{u}B^{-1}),$$

for all $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$, so that $a\mathbf{u} = A\mathbf{u} = \mathbf{u}B$ for all $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$, and so $B = aI$ and $a \in R_0$ as $B \in \mathbf{SL}_2(R_0)$. Therefore, $(A, B) = (aI, aI)$ for $a \in \boldsymbol{\mu}_2(R_0)$.

Conversely, for $a \in \boldsymbol{\mu}_2(R_0)$, it is obvious that $(aI, aI) \in \ker(\iota_\mathcal{R})$. Hence, $\ker(\iota_\mathcal{R}) \cong \boldsymbol{\mu}_2(R_0)$. □

By Lemmas 5.1 and 5.2, we obtain the following theorem:

Theorem 5.3. *For every object $\mathcal{R} = (R, \mathsf{d})$ in* $\Bbbk$**-drng**, *there is an exact sequence of groups*

$$1 \to \boldsymbol{\mu}_2(R_0) \to \mathbf{SL}_2(R) \times \mathbf{SL}_2(R_0) \xrightarrow{\iota_{\mathcal{R}}} \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}}), \tag{5.2.8}$$

where $R_0 = \ker \mathsf{d}$. Furthermore, the exact sequence is functorial in $\mathcal{R}$. $\square$

In general, $\iota_{\mathcal{R}}$ fails to be surjective. However, it has properties analogous to the surjectivity of the quotient morphisms of $\Bbbk$–group schemes. More precisely, we have the following Propositions 5.4 and 5.5. We say a morphism $\mathcal{R} = (R, \mathsf{d}_R) \to \mathcal{S} = (S, \mathsf{d}_S)$ in $\Bbbk$**-drng** is *étale* if the homomorphism $R \to S$ of rings is étale.

Proposition 5.4. *Let $\mathcal{R} = (R, \mathsf{d}_R)$ be an object in* $\Bbbk$**-drng** *such that R is an integral domain, and $\phi \in \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$. Then there is an étale extension $\mathcal{S} = (S, \mathsf{d}_S)$ of $\mathcal{R}$, an element $A \in \mathbf{SL}_2(S)$, and an element $B \in \mathbf{SL}_2(S_0)$ such that*

$$\phi_{\mathcal{S}} = \theta_{A,B} = \iota_{\mathcal{S}}(A, B), \tag{5.2.9}$$

where $S_0 = \ker \mathsf{d}_S$, and $\phi_{\mathcal{S}}$ is the image of ϕ under $\mathrm{GrAut}(\mathscr{W}_{\mathcal{R}}) \to \mathrm{GrAut}(\mathscr{W}_{\mathcal{S}})$.

Proof. We first write $\phi(\mathrm{L}) = \mathrm{L} \otimes r + \mathrm{T}(\mathbf{x}_0)$, where $r \in R, \mathbf{x}_0 \in \mathfrak{sl}_2(R)$. Then

$$\begin{aligned} \phi([\mathrm{L}_\lambda \mathrm{L}]) &= (\widehat{\partial} + 2\lambda)(\mathrm{L} \otimes r + \mathrm{T}(\mathbf{x}_0)), \\ [\phi(\mathrm{L})_\lambda \phi(\mathrm{L})] &= (\widehat{\partial} + 2\lambda)(\mathrm{L} \otimes r^2 + \mathrm{T}(r\mathbf{x}_0)). \end{aligned}$$

We deduce from $\phi([\mathrm{L}_\lambda \mathrm{L}]) = [\phi(\mathrm{L})_\lambda \phi(\mathrm{L})]$ that $r^2 = r$ and $r\mathbf{x}_0 = \mathbf{x}_0$. Since R is an integral domain, $r = 0$ or 1. If $r = 0$, we obtain $\mathbf{x}_0 = r\mathbf{x}_0 = 0$, and so $\phi(\mathrm{L}) = 0$. This contradicts the injectivity of ϕ. Hence, $r = 1$, i.e.,

$$\phi(\mathrm{L}) = \mathrm{L} + \mathrm{T}(\mathbf{x}_0). \tag{5.2.10}$$

Recall that $\{\sigma^i, i = 1, 2, 3\}$ is a $\Bbbk$–basis of $\mathfrak{sl}_2(\Bbbk)$. Let $\mathscr{B}$ be the $\Bbbk[\partial]$–submodule of $\mathscr{W}$ generated by $\mathrm{T}(\sigma^i), i = 1, 2, 3$. Then $\mathscr{B}$ is isomorphic to $\mathrm{Cur}(\mathfrak{sl}_2(\Bbbk))$. We write $\phi(\mathrm{T}(\sigma^i)) = \mathrm{L} \otimes r_i + \mathrm{T}(\mathbf{x}_i)$ for $r_i \in R, \mathbf{x}_i \in \mathfrak{sl}_2(R), i = 1, 2, 3$. Thus,

$$\phi([\mathrm{L}_\lambda \mathrm{T}(\sigma^i)]) = (\widehat{\partial} + \lambda)(\mathrm{L} \otimes r_i + \mathrm{T}(\mathbf{x}_i)),$$

$$[\phi(\mathrm{L})_\lambda \phi(\mathrm{T}(\sigma^i))] = (\partial + 2\lambda)\mathrm{L}\otimes r_i + (\partial+\lambda)\mathrm{T}(\mathbf{x}_i) \\ + \lambda \mathrm{T}(r_i\mathbf{x}_0) + \mathrm{T}(r_i\mathsf{d}(\mathbf{x}_0)) + \mathrm{T}([\mathbf{x}_0, \mathbf{x}_i]).$$

We deduce from $\phi([\mathrm{L}_\lambda \mathrm{T}(\sigma^i)]) = [\phi(\mathrm{L})_\lambda \phi(\mathrm{T}(\sigma^i))], i = 1,2,3$ that $r_i = 0$ and $\mathsf{d}(\mathbf{x}_i) = [\mathbf{x}_0, \mathbf{x}_i]$, $i = 1,2,3$. It follows that $\phi(\mathrm{T}(\sigma^i)) = \mathrm{T}(\mathbf{x}_i), i = 1,2,3$. Hence, $\phi(\mathscr{B}_{\mathcal{R}}) \subseteq \mathscr{B}_{\mathcal{R}}$ and $\phi|_{\mathscr{B}_{\mathcal{R}}}$ is an automorphism of $\mathscr{B}_{\mathcal{R}}$.

By Proposition 2.5, there is an R-linear automorphism $\overline{\phi}$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)\otimes_{\Bbbk} R = \mathfrak{sl}_2(R)$ such that $\phi(\mathrm{T}(\mathbf{x})) = \mathrm{T}(\overline{\phi}(\mathbf{x}))$ for all $\mathbf{x}\in\mathfrak{sl}_2(R)$.

It is known that $\mathbf{SL}_2(R)$ acts on $\mathfrak{sl}_2(\Bbbk)\otimes_{\Bbbk} R$ functorially via conjugation, which yields a morphism $\pi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(\Bbbk))$ of k-group schemes. From (3.2.4), there exists an étale extension S of R and $A \in \mathbf{SL}_2(S)$ such that

$$\overline{\phi}_S(\mathbf{x}) = A\mathbf{x}A^{-1}, \quad \mathbf{x}\in\mathfrak{sl}_2(S). \tag{5.2.11}$$

Since $R \to S$ is an étale ring homomorphism, there is a unique $\Bbbk$–derivation d_S of S extending d_R. Hence, $\mathcal{S} = (S, \mathsf{d}_S)$ is an étale extension of $\mathcal{R}$ (cf. Chapter 0, Corollary 20.5.8 of [Gro64], or Lemma 1.16 of [Gil02]).

Now we consider the image $\phi_{\mathcal{S}}$ of ϕ in $\mathrm{GrAut}(\mathscr{W}_{\mathcal{S}})$. From (5.2.10) and (5.2.11), we obtain

$$\phi_{\mathcal{S}}(\mathrm{L}) = \mathrm{L} + \mathrm{T}(\mathbf{x}_0), \text{ and } \phi_{\mathcal{S}}(\mathrm{T}(\sigma^i)) = \mathrm{T}(A\sigma^i A^{-1}), \quad i = 1,2,3.$$

Then we deduce from $[\phi_{\mathcal{S}}(\mathrm{L})_\lambda \phi_{\mathcal{S}}(\mathrm{T}(\sigma^i))] = (\widehat{\partial}+\lambda)\phi_{\mathcal{S}}(\mathrm{T}(\sigma^i))$ that

$$\mathsf{d}_S(A\sigma^i A^{-1}) = [\mathbf{x}_0, A\sigma^i A^{-1}], \quad i = 1,2,3.$$

Hence, a direct computation shows that $\mathbf{x}_0 = \mathsf{d}_S(A)A^{-1}$.

Let $\psi = \phi_{\mathcal{S}}\circ\theta_{A,I}^{-1}$. Then $\psi \in \mathrm{GrAut}(\mathscr{W}_{\mathcal{S}})$ and $\psi|_{(\mathscr{W}_{\mathcal{S}})_{\bar{0}}} = \mathrm{id}$. Next we consider $\psi|_{(\mathscr{W}_{\mathcal{S}})_{\bar{1}}}$. Suppose

$$\psi(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\nu(\mathbf{u})), \quad \mathbf{u}\in\mathrm{Mat}_2(S),$$

where $\nu : \mathrm{Mat}_2(S) \to \mathrm{Mat}_2(S)$ is a bijective S–linear map. Now we deduce from

$$\psi([\mathrm{T}(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})]) = [\psi(\mathrm{T}(\mathbf{x}))_\lambda \psi(\mathrm{G}(\mathbf{u}))]$$

that $\mathbf{x}\cdot\nu(\mathbf{u}) = \nu(\mathbf{x}\mathbf{u})$ for all $\mathbf{x}\in\mathfrak{sl}_2(S)$ and $\mathbf{u}\in\mathrm{Mat}_2(S)$. Then a straightforward computation shows that there is an element $B \in \mathbf{GL}_2(S)$ such that $\nu(\mathbf{u}) = \mathbf{u}B^{-1}$ for all $\mathbf{u} \in \mathrm{Mat}_2(S)$.

Next we show $B \in \mathbf{SL}_2(S_0)$. Let $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(\Bbbk)$, then

$$\begin{aligned}\psi([\mathrm{G}(\mathbf{u})_{(1)}\mathrm{G}(\mathbf{v})]) &= 2\mathrm{T}(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger),\\ [\psi(\mathrm{G}(\mathbf{u}))_{(1)}\psi(\mathrm{G}(\mathbf{v}))] &= 2\mathrm{T}(\mathbf{u}B^{-1}(\mathbf{v}B^{-1})^\dagger - \mathbf{v}B^{-1}(\mathbf{u}B^{-1})^\dagger)\\ &= 2\mathrm{T}(\det(B^{-1})(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)).\end{aligned}$$

It follows that $\det(B^{-1}) = 1$, so $B \in \mathbf{SL}_2(S)$.

For $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$, we consider

$$\begin{aligned}\psi([\mathrm{G}(\mathbf{u})_\lambda \mathrm{L}]) &= \left(\frac{1}{2}\widehat{\partial} + \frac{3}{2}\lambda\right)\mathrm{G}(\mathbf{u}B^{-1}),\\ [\psi(\mathrm{G}(\mathbf{u}))_\lambda \psi(\mathrm{L})] &= \left(\frac{1}{2}\partial + \frac{3}{2}\lambda\right)\mathrm{G}(\mathbf{u}B^{-1}) + \frac{3}{2}\mathrm{G}(\mathbf{u}\mathsf{d}(B^{-1})).\end{aligned}$$

These yield that $\mathsf{d}(B^{-1}) = 0$, and hence $B \in \mathbf{SL}_2(S_0)$. Therefore, $\psi = \phi_{\mathcal{S}} \circ \theta_{A,I}^{-1} = \theta_{I,B}$, i.e., $\phi_{\mathcal{S}} = \theta_{I,B} \circ \theta_{A,I} = \theta_{A,B}$. □

Proposition 5.5. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in $\Bbbk$-**drng**. If R is an integral domain and the étale cohomology set $\mathrm{H}^1_{\acute{e}t}(R, \boldsymbol{\mu}_2)$ is trivial, then $\iota_{\mathcal{R}}$ is surjective.*

Proof. Given $\phi \in \mathrm{GrAut}(\mathscr{W}_{\mathcal{R}})$, as in Proposition 5.4, its restriction to $\mathscr{B}_{\mathcal{R}}$ yields an R–linear automorphism $\overline{\phi}$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk) \otimes_{\Bbbk} R$, where $\mathscr{B} = \Bbbk[\partial]\mathrm{T}^1 \oplus \Bbbk[\partial]\mathrm{T}^2 \oplus \Bbbk[\partial]\mathrm{T}^3 \cong \mathrm{Cur}(\mathfrak{sl}_2(\Bbbk))$. It is known that there is an short exact sequence of $\Bbbk$–group schemes

$$1 \to \boldsymbol{\mu_2} \to \mathbf{SL}_2 \to \mathbf{Aut}(\mathfrak{sl}_2(\Bbbk)) \to 1,$$

which yields a long exact sequences:

$$1 \to \boldsymbol{\mu_2}(R) \to \mathbf{SL}_2(R) \to \mathbf{Aut}(\mathfrak{sl}_2(\Bbbk))(R) \to \mathrm{H}^1_{\text{ét}}(R, \boldsymbol{\mu}_2) \to \cdots$$

Hence, the triviality of $\mathrm{H}^1_{\text{ét}}(R, \boldsymbol{\mu}_2)$ yields that $\overline{\phi}$ is the image of an element $A \in \mathbf{SL}_2(R)$. i.e.,

$$\phi(\mathrm{T}(\mathbf{x})) = A\mathbf{x}A^{-1}, \mathbf{x} \in \mathfrak{sl}_2(R).$$

Then the proof of this proposition can be completed by similar arguments as in Proposition 5.4. □

Proposition 5.6. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in* $\Bbbk$**-drng**. *Then*

$$\boldsymbol{\mu}_2(R) = \boldsymbol{\mu}_2(R_0), \tag{5.2.12}$$

where $R_0 = \ker \mathsf{d}$.

Proof. It is obvious that $\boldsymbol{\mu}_2(R_0) \subseteq \boldsymbol{\mu}_2(R)$. Conversely, let $r \in \boldsymbol{\mu}_2(R)$, then $r^2 = 1$. Thus $2r\mathsf{d}(r) = 0$, and so $r\mathsf{d}(r) = 0$, from which we obtain $\mathsf{d}(r) = rr\mathsf{d}(r) = 0$, i.e., $r \in R_0$. This yields that $r \in \boldsymbol{\mu}_2(R_0)$. Therefore, $\boldsymbol{\mu}_2(R) = \boldsymbol{\mu}_2(R_0)$. □

Theorem 5.7. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in* $\Bbbk$**-drng** *with R an integral domain. Then*

$$\mathrm{GrAut}(\mathscr{W}_{\mathcal{R}}) = \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_{\mathcal{R}}). \tag{5.2.13}$$

Proof. Let $\phi \in \mathrm{Aut}_{\mathcal{R}\text{-conf}}(\mathscr{W}_{\mathcal{R}})$. It suffices to show $\phi(V \otimes_{\Bbbk} R) \subseteq V \otimes_{\Bbbk} R$.

Recall that $\mathscr{B} = \Bbbk[\partial]\mathrm{T}(\sigma^1) \oplus \Bbbk[\partial]\mathrm{T}(\sigma^2) \oplus \Bbbk[\partial]\mathrm{T}(\sigma^3)$. If we write

$$\phi(\mathrm{T}(\sigma^i)) = \sum_{m=0}^{M_i} \widehat{\partial}^m(\mathrm{L} \otimes r_{im}) + \eta_i,$$

$r_{im} \in R, \eta_i \in \mathscr{B}_{\mathcal{R}}$, $i = 1, 2, 3$, then

$$\begin{aligned}
&\phi([\mathrm{T}(\sigma^i)_\lambda \mathrm{T}(\sigma^i)]) = 0,\\
&[\phi(\mathrm{T}(\sigma^i))_\lambda \phi(\mathrm{T}(\sigma^i))]\\
&\quad= \sum_{m,n=0}^{M_i} (-\lambda)^m(\widehat{\partial} + \lambda)^n((\partial + 2\lambda)(\mathrm{L} \otimes r_{im}r_{in}) + \mathrm{L} \otimes \mathsf{d}(r_{im})r_{in}) + \eta_i',
\end{aligned}$$

where $\eta_i' \in \Bbbk[\lambda] \otimes_{\Bbbk} \mathscr{B}_{\mathcal{R}}$. By comparing the degree and coefficients of λ in

$$\phi([\mathrm{T}(\sigma^i)_\lambda \mathrm{T}(\sigma^i)]) = [\phi(\mathrm{T}(\sigma^i))_\lambda \phi(\mathrm{T}(\sigma^i))], i = 1, 2, 3,$$

we obtain $M_i = 0$ and $r_{iM_i}^2 = 0$, $i = 1, 2, 3$. Thus $r_{iM_i} = 0$, $i = 1, 2, 3$, since R is an integral domain, i.e., $\phi(\mathrm{T}(\sigma^i)) \in \mathscr{B}_{\mathcal{R}}$. Since $\mathscr{B} \cong \mathrm{Cur}(\mathfrak{sl}_2(\Bbbk))$, by Proposition 2.5,

$$\phi(\mathrm{T}(\sigma^i)) \subseteq (\Bbbk\mathrm{T}(\sigma^1) \oplus \Bbbk\mathrm{T}(\sigma^2) \oplus \Bbbk\mathrm{T}(\sigma^3)) \otimes_{\Bbbk} R \subseteq V \otimes_{\Bbbk} R.$$

More precisely, there is an R–linear automorphism of the Lie algebra $\mathfrak{sl}_2(R)$

$$\varphi : \mathfrak{sl}_2(R) \to \mathfrak{sl}_2(R), \quad \sigma^i \mapsto \mathbf{x}_i, i = 1, 2, 3 \tag{5.2.14}$$

such that $\phi(\mathrm{T}(\sigma^i)) = \mathrm{T}(\mathbf{x}_i), i = 1, 2, 3.$

A similar argument using $\phi([\mathrm{L}_\lambda \mathrm{L}]) = [\phi(\mathrm{L})_\lambda \phi(\mathrm{L})]$ yields that

$$\phi(\mathrm{L}) = \mathrm{L} + \sum_{m=0}^{M} \widehat{\partial}^m \mathrm{T}(\mathbf{y}_m),$$

for $\mathbf{y}_m \in \mathfrak{sl}_2(R)$. Then $\phi([\mathrm{L}_\lambda \mathrm{T}(\sigma^i)]) = [\phi(\mathrm{L})_\lambda \phi(\mathrm{T}(\sigma^i))]$ implies

$$[\mathbf{y}_m, \mathbf{x}_i] = 0, \text{ for } i = 1, 2, 3, m > 0.$$

Applying the inverse φ^{-1} of the automorphism φ given by (5.2.14), we obtain

$$[\varphi^{-1}(\mathbf{y}_m), \sigma^i] = 0, \text{ for } i = 1, 2, 3, m > 0.$$

Since $\{\sigma^i, i = 1, 2, 3\}$ is a basis of $\mathfrak{sl}_2(\Bbbk)$ and $\varphi^{-1}(\mathbf{y}_m) \in \mathfrak{sl}_2(R)$, we deduce that $\varphi^{-1}(\mathbf{y}_m) = 0$ for $m > 0$, and so $\mathbf{y}_m = 0, m > 0$. Hence,

$$\phi(\mathrm{L}) = \mathrm{L} + \mathrm{T}(\mathbf{y}_0),$$

i.e., $\phi(\mathrm{L}) \in V \otimes_\Bbbk R$.

Next we consider the odd part. By considering

$$\phi([\mathrm{L}_\lambda \mathrm{G}(\mathbf{u})]) = [\phi(\mathrm{L})_\lambda \phi(\mathrm{G}(\mathbf{u}))],$$

for $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$, we obtain $\phi(\mathrm{G}(\mathbf{u})) \in V \otimes_\Bbbk R$. Hence, $\phi \in \mathrm{GrAut}(\mathscr{W}_\mathcal{R})$. □

Corollary 5.8. *Let $\mathcal{R} = (R, \mathsf{d})$ be an object in $\Bbbk$-**drng**. If R is an integral domain and $\mathrm{H}^1_{ét}(R, \boldsymbol{\mu_2})$ is trivial, then*

$$\mathrm{Aut}_{\mathcal{R}\text{-}conf}(\mathscr{W}_\mathcal{R}) \cong \frac{\mathbf{SL}_2(R) \times \mathbf{SL}_2(R_0)}{\boldsymbol{\mu}_2(R_0)}, \tag{5.2.15}$$

where $R_0 = \ker \mathsf{d}$.

Remark 5.9.

- Since $\Bbbk$ is an algebraically closed field of characteristic zero, $\mathrm{H}^1_{\text{ét}}(\Bbbk, \boldsymbol{\mu_2})$ is trivial. Hence,

$$\mathrm{Aut}_{\Bbbk\text{-conf}}(\mathscr{W}) \cong \frac{\mathbf{SL}_2(\Bbbk) \times \mathbf{SL}_2(\Bbbk)}{\langle(-I_2, -I_2)\rangle}. \tag{5.2.16}$$

- For $\widehat{\mathcal{D}} = (\Bbbk[t^q | q \in \mathbb{Q}], \frac{d}{dt})$, the étale cohomology set $\mathrm{H}^1_{\text{ét}}(\widehat{D}, \boldsymbol{\mu_2})$ is trivial ([GP08a, Theorem 2.9]) and $\widehat{D}_0 = \ker(\frac{d}{dt}) = \Bbbk$. Hence,

$$\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{W}_{\widehat{\mathcal{D}}}) \cong \frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\Bbbk)}{\langle(-I_2, -I_2)\rangle}, \tag{5.2.17}$$

which is the result of Proposition 3.69 in [KLP09].

Recall that, for one of the $N = 1, 2, 3$ conformal superalgebras $\mathscr{K}_N$, the automorphism group functor $\mathbf{Aut}(\mathscr{K}_N)$ has a subgroup functor $\mathbf{GrAut}(\mathscr{K}_N)$ satisfying nice properties (i) and (ii) of Remark 4.10. For the small $N = 4$ conformal superalgebra $\mathscr{W}$, $\mathbf{Aut}(\mathscr{W})$ also has a subgroup functor $\mathbf{GrAut}(\mathscr{W})$ defined by (5.2.1). Analogous to property (ii) of Remark 4.10 in the $N = 1, 2, 3$ cases, Theorem 5.7 shows that $\mathbf{GrAut}(\mathscr{W})$ coincides with $\mathbf{Aut}(\mathscr{W})$ when evaluating at $\mathcal{R} = (R, \mathsf{d})$ with R an integral domain.

Next we will discuss the properties of $\mathbf{Aut}(\mathscr{W})$ analogous to (i) of Remark 4.10, which states that $\mathbf{GrAut}(\mathscr{K}_N) \cong \mathbf{O}_N \circ \mathfrak{f}$, where

$$\mathfrak{f} : \Bbbk\textbf{-drng} \to \Bbbk\textbf{-rng}, \quad \mathcal{R} = (R, \mathsf{d}) \mapsto R \tag{5.2.18}$$

is the forgetful functor, and $\mathbf{O}_N$ is the group functor of $N \times N$–orthogonal matrices. In particular, $\mathbf{O}_N$ is representable by an affine group scheme of finite type. Base on the result:

$$\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{W}_{\widehat{\mathcal{D}}}) \cong (\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\Bbbk))/\langle(-I_2, -I_2)\rangle$$

obtained in [KLP09], it was conjectured in [CP11] that $\mathbf{GrAut}(\mathscr{W}) = \mathbf{G} \circ \mathfrak{f}$, where $\mathbf{G} = (\mathbf{SL}_2 \times \mathbf{SL}_2(\Bbbk))/\boldsymbol{\mu}_2$, and $\mathbf{SL}_2(\Bbbk)$ is understood as the affine constant group scheme defined by the abstract group $\mathbf{SL}_2(\Bbbk)$. Theorem 5.3 shows this conjecture fails to be true.

Besides the forgetful functor $\mathfrak{f}$ defined by (5.2.18), there is another funtor

$$\mathfrak{t} : \Bbbk\textbf{-drng} \to \Bbbk\textbf{-rng}, \quad \mathcal{R} = (R, \mathsf{d}) \mapsto R_0 := \ker(\mathsf{d}), \tag{5.2.19}$$

involved in our discussion on $\mathbf{GrAut}(\mathscr{W})$. For $\mathcal{R} = (R, \mathsf{d})$ in $\Bbbk$**-drng**, it is easy to show that $R_0 = \ker(\mathsf{d})$ is a commutative associative ring over $\Bbbk$. R_0 is usually called the ring of constants of $\mathcal{R}$ ([Gil02]).

Now the functor $\mathbf{GrAut}(\mathscr{W})$ can be described by using the language of group functors as follows:

Proposition 5.10. *The group functor* $\mathbf{GrAut}(\mathscr{W})$ *fits into the exact sequence*[2] *of group functors*

$$1 \to \boldsymbol{\mu}_2 \circ \mathfrak{f} \to (\mathbf{SL}_2 \circ \mathfrak{f}) \times (\mathbf{SL}_2 \circ \mathfrak{t}) \xrightarrow{\iota} \mathbf{GrAut}(\mathscr{W}). \tag{5.2.20}$$

Moreover, ι *satisfies properties in Proposition 5.4.* □

Proposition 5.10 is an analogy of the property (i) of Remark 4.10. Combining with Theorem 5.7, we observe that the automorphism group functor $\mathbf{Aut}(\mathscr{W})$ is closely related to affine group scheme of finite type. These relations can be understood as the "weak representability" of $\mathbf{Aut}(\mathscr{W})$.

5.3 Twisted loop conformal superalgebras

Analogous to the $N = 1, 2, 3$ conformal superalgebras, once the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{W}_{\widehat{\mathcal{D}}})$ is known, we can classify the twisted loop conformal superalgebras based on $\mathscr{W}$. This classification has been completed by V. G. Kac, M. Lau, and A. Pianzola in [KLP09]. For the sake of completeness, we state their results in this section without repeating the proof.

Theorem 5.11 (Theorem 3.71 of [KLP09])**.** *The* $\Bbbk$*–isomorphism classes of twisted loop conformal superalgebras based on* $\mathscr{W}$ *bijectively correspond to conjugacy classes of elements of finite order in* $\mathbf{PGL}_2(\Bbbk)$*. In particular, there are infinitely many non-isomorphic twisted loop conformal superalgebras based on* $\mathscr{W}$*.*

Recall that $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{W}_{\widehat{\mathcal{D}}}) \cong \frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\Bbbk)}{\langle(-I_2, -I_2)\rangle}$ and the group $\widehat{\mathbb{Z}}$ acts on $\mathbf{SL}_2(\Bbbk)$ trivially, it has been proved in [KLP09] that the pointed set $\mathrm{H}^1_{\text{ct}}\left(\widehat{\mathbb{Z}}, \frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\Bbbk)}{\langle(-I_2, -I_2)\rangle}\right)$ is identified with the set of conjugacy classes of elements of finite order in $\mathbf{PGL}_2(\Bbbk)$. This gives the classification of twisted loop conformal superalgebras based on $\mathscr{W}$ up to isomorphism of $\mathcal{D}$–conformal superalgebras.

To deduce from isomorphisms of $\mathcal{D}$–conformal superalgebras to isomorphisms of $\Bbbk$–conformal superalgebras, the centroid of each twisted loop conformal superalgebra $\mathcal{L}(\mathscr{W}, \sigma)$ has been explicitly computed in [KLP09]. This result can also be obtained by applying our Proposition 2.12.

[2] A sequence of functors from the category $\Bbbk$-**drng** to the category of groups is called exact if it is an exact sequence of abstract groups when evaluating at each $\mathcal{R}$ in $\Bbbk$-**drng**.
For two functors $\mathbf{F}$ and $\mathbf{H}$ from $\Bbbk$-**drng** to the category of groups, their direct product $\mathbf{F} \times \mathbf{H}$ is defined to be the functor assigning each $\mathcal{R}$ to the direct product of abstract groups $\mathbf{F}(\mathcal{R}) \times \mathbf{H}(\mathcal{R})$.

Proposition 5.12. *For every $\sigma \in \mathrm{Aut}_{\Bbbk\text{-}conf}(\mathscr{W})$ of finite order, the canonical homomorphism $D \to \mathrm{Ctd}_{\Bbbk}(\mathcal{L}(\mathscr{W},\sigma))$ is an isomorphism.*

Proof. From Remark 5.9 and the explicit construction in Lemma 5.1, every $\Bbbk$–automorphism σ fixes L, i.e., $\sigma(\mathrm{L}) = \mathrm{L}$. In addition, $\mathscr{W}_{\bar{0}}$ is a free $\Bbbk[\partial]$-module generated by L and $\mathrm{T}^i, i = 1, 2, 3$, where each T^i satisfies $[\mathrm{L}_\lambda \mathrm{T}^i] = (\partial+\lambda)\mathrm{T}^i, i = 1, 2, 3$. On the other hand, $\mathscr{W}_{\bar{1}}$ is a free $\Bbbk[\partial]$–module generated by $\mathrm{G}^1, \mathrm{G}^2, \overline{\mathrm{G}}^1, \overline{\mathrm{G}}^2$, and they have the same conformal weight $\frac{3}{2}$ with respect to L. Applying Proposition 2.12, the assertion follows. $\square$

Chapter 6

The large $N = 4$ conformal superalgebra[1]

In this chapter, we will focus on the large $N = 4$ conformal superalgebra $\mathscr{M}$. The central extensions of the Lie superalgebra induced by the untwisted loop conformal algebra $\mathcal{L}(\mathscr{M}, \mathrm{id})$ are called the large (or big, or maximal) $N = 4$ superconformal algebras in physics literatures. They were discovered in [STVP88] and have inspired subsequent work such as [DST88], [Van91], and [Ras02]. In particular, the global and local automorphisms of the large $N = 4$ superconformal algebras have been studied in [DST88], based on which a twisted large $N = 4$ superconformal algebra has been created in [Van91].

The purpose of this chapter is to complete the classification of twisted loop conformal superalgebras based on $\mathscr{M}$. Since $\mathscr{M}$ is a free $\Bbbk[\partial]$–module of rank 16. The computation of the automorphisms of the large $N = 4$ conformal superalgebra is more complicated than the computation for the $N = 1, 2, 3$ and small $N = 4$ conformal superalgebras. Nonetheless, the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})$ will also be explicitly determined, where $\mathscr{M}_{\widehat{\mathcal{D}}}$ will always denote $\mathscr{M} \otimes_{\Bbbk} \widehat{\mathcal{D}}$ in this chapter. It is relevant to point out that our work shows that the automorphism group $\mathrm{Aut}_{\Bbbk\text{-conf}}(\mathscr{M})$ is in fact larger then the one described in the physics literature.

The main results of this chapter are Theorem 6.5, which characterizes the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})$, and Theorem 6.9, in which the classification of the twisted loop conformal superalgebras based on $\mathscr{M}$ have been completed. We will also pass to the two Lie superalgebras induced by the two non-isomorphic twisted loop conformal superalgebras based on $\mathscr{M}$ and show that there is no collapse occurring to the isomorphism classes of Lie superalgebras.

[1] A version of this chapter has been submitted for publication. A preprint version [CP13] is available on Arxiv

6.1 The large $N = 4$ conformal superalgebras

In this section, we will review the definition of the large $N = 4$ conformal superalgebra $\mathscr{M}$, and re-formulate the generators and relations of $\mathscr{M}$. With new generators, all automorphisms of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$ will be explicitly written down in a nice matrix form.

We start with the definition of the large $N = 4$ superconformal algebras, which are a family of Lie superalgebras $\mathfrak{g}(\gamma)$ parameterized by one parameter $\gamma \neq 0, 1$. More precisely, $\mathfrak{g}(\gamma) = \mathfrak{g}(\gamma)_{\bar{0}} \oplus \mathfrak{g}(\gamma)_{\bar{1}}$, where

$$\mathfrak{g}(\gamma)_{\bar{0}} = \mathrm{span}_{\Bbbk} \left\{ \widetilde{c}, \widetilde{\mathrm{L}}_m, \widetilde{\mathrm{T}}^{\pm i}_m, \widetilde{\mathrm{U}}_m \middle| i = 1, 2, 3, m \in \mathbb{Z} \right\},$$
$$\mathfrak{g}(\gamma)_{\bar{1}} = \mathrm{span}_{\Bbbk} \left\{ \widetilde{\mathrm{G}}^p_{m'}, \widetilde{\mathrm{Q}}^p_{m'} \middle| p = 1, 2, 3, 4, m' \in \tfrac{1}{2} + \mathbb{Z} \right\},$$

$\widetilde{c}$ is a central element of $\mathfrak{g}(\gamma)$ and the superbracket on $\mathfrak{g}(\gamma)$ is defined in [STVP88] as follows:

$$[\widetilde{\mathrm{L}}_m, \widetilde{\mathrm{L}}_n] = (m-n)\widetilde{\mathrm{L}}_{m+n} + \tfrac{m^3-m}{12}\delta_{m,-n}\widetilde{c}, \qquad [\widetilde{\mathrm{T}}^{+i}_m, \widetilde{\mathrm{T}}^{-j}_n] = 0$$

$$[\widetilde{\mathrm{L}}_m, \widetilde{\mathrm{U}}_n] = -n\widetilde{\mathrm{U}}_{m+n}, \qquad [\widetilde{\mathrm{L}}_m, \widetilde{\mathrm{T}}^{\pm i}_n] = -n\widetilde{\mathrm{T}}^{\pm i}_{m+n},$$

$$[\widetilde{\mathrm{T}}^{+i}_m, \widetilde{\mathrm{T}}^{+j}_n] = \epsilon_{ijk}\widetilde{\mathrm{T}}^{+k}_{m+n} - \tfrac{m}{12\gamma}\delta_{ij}\delta_{m,-n}\widetilde{c}, \qquad [\widetilde{\mathrm{T}}^{\pm i}_m, \widetilde{\mathrm{U}}_n] = 0,$$

$$[\widetilde{\mathrm{T}}^{-i}_m, \widetilde{\mathrm{T}}^{-j}_n] = \epsilon_{ijk}\widetilde{\mathrm{T}}^{-k}_{m+n} - \tfrac{m}{12(1-\gamma)}\delta_{ij}\delta_{m,-n}\widetilde{c}, \qquad [\widetilde{\mathrm{U}}_m, \widetilde{\mathrm{U}}_n] = -\tfrac{m\delta_{m,-n}}{12\gamma(1-\gamma)}\widetilde{c},$$

$$[\widetilde{\mathrm{L}}_m, \widetilde{\mathrm{Q}}^p_{n'}] = -\left(\tfrac{1}{2}m + n'\right)\widetilde{\mathrm{Q}}^p_{m+n'}, \qquad [\widetilde{\mathrm{U}}_m, \widetilde{\mathrm{Q}}^p_{n'}] = 0,$$

$$[\widetilde{\mathrm{L}}_m, \widetilde{\mathrm{G}}^p_{n'}] = \left(\tfrac{1}{2}m - n'\right)\widetilde{\mathrm{G}}^p_{m+n'}, \qquad [\widetilde{\mathrm{U}}_m, \widetilde{\mathrm{G}}^p_{n'}] = m\widetilde{\mathrm{Q}}^p_{m+n'},$$

$$[\widetilde{\mathrm{Q}}^p_{m'}, \widetilde{\mathrm{Q}}^q_{n'}] = -\tfrac{\delta_{pq}\delta_{m',-n'}}{12\gamma(1-\gamma)}\widetilde{c}, \qquad [\widetilde{\mathrm{T}}^{\pm i}_m, \widetilde{\mathrm{Q}}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{\pm i}_{pq}\widetilde{\mathrm{Q}}^q_{m+n'},$$

$$[\widetilde{\mathrm{T}}^{+i}_m, \widetilde{\mathrm{G}}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{+i}_{pq}(\widetilde{\mathrm{G}}^q_{m+n'} - 2(1-\gamma)m\widetilde{\mathrm{Q}}^q_{m+n'}),$$

$$[\widetilde{\mathrm{T}}^{-i}_m, \widetilde{\mathrm{G}}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{-i}_{pq}(\widetilde{\mathrm{G}}^q_{m+n'} + 2\gamma m\widetilde{\mathrm{Q}}^q_{m+n'}),$$

$$[\widetilde{\mathrm{Q}}^p_{m'}, \widetilde{\mathrm{G}}^q_{n'}] = \delta_{pq}\widetilde{\mathrm{U}}_{m'+n'} + 2\textstyle\sum_{i=1}^3(\alpha^{+i}_{pq}\widetilde{\mathrm{T}}^{+i}_{m'+n'} - \alpha^{-i}_{pq}\widetilde{\mathrm{T}}^{-i}_{m'+n'}),$$

$$[\widetilde{\mathrm{G}}^p_{m'}, \widetilde{\mathrm{G}}^q_{n'}] = 2\delta_{pq}\widetilde{\mathrm{L}}_{m'+n'} + \tfrac{1}{3}\delta_{pq}\delta_{m',-n'}(m'^2 - 1/4)\widetilde{c}$$
$$+4(n'-m')\textstyle\sum_{i=1}^3(\gamma\alpha^{+i}_{pq}\widetilde{\mathrm{T}}^{+i}_{m'+n'} + (1-\gamma)\alpha^{-i}_{pq}\widetilde{\mathrm{T}}^{-i}_{m'+n'}),$$

for $i, j = 1, 2, 3$, $p, q = 1, 2, 3, 4$, $m, n \in \mathbb{Z}$, $m', n' \in \frac{1}{2} + \mathbb{Z}$, and $\alpha^{\pm i}$ are 4×4–

matrices given by

$$\alpha^{\pm i}_{pq} = \pm\frac{1}{2}(\delta_{ip}\delta_{4q} - \delta_{iq}\delta_{4p}) + \frac{1}{2}\epsilon_{ipq}.$$

By setting

$$\mathrm{L}_n = \widetilde{\mathrm{L}}_n + (\gamma - \tfrac{1}{2})(n+1)\widetilde{\mathrm{U}}_n, \qquad \mathrm{T}^{\pm i}_n = \widetilde{\mathrm{T}}^{\pm i}_n, \quad \mathrm{U}_n = \widetilde{\mathrm{U}}_n,$$

$$\mathrm{G}^p_{n'} = \widetilde{\mathrm{G}}^p_{n'} + 2(\gamma - \tfrac{1}{2})(n' + \tfrac{1}{2})\widetilde{\mathrm{Q}}^p_{n'}, \quad \mathrm{Q}^p_{n'} = \widetilde{\mathrm{Q}}^p_{n'}, \quad c = \tfrac{1}{4\gamma(1-\gamma)}\widetilde{c},$$

for $n \in \mathbb{Z}, n' \in \frac{1}{2} + \mathbb{Z}$, the Lie superbracket on $\mathfrak{g}(\gamma)$ is written as

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n} + \tfrac{m^3-m}{12}\delta_{m,-n}c, \qquad [\mathrm{T}^{+i}_m, \mathrm{T}^{-j}_n] = 0,$$

$$[\mathrm{L}_m, \mathrm{U}_n] = -n\mathrm{U}_{m+n} - \tfrac{m^2+m}{6}\delta_{m,-n}(2\gamma-1)c, \quad [\mathrm{L}_m, \mathrm{T}^{\pm i}_n] = -n\mathrm{T}^{\pm i}_{m+n},$$

$$[\mathrm{T}^{+i}_m, \mathrm{T}^{+j}_n] = \epsilon_{ijk}\mathrm{T}^{+k}_{m+n} - \tfrac{m}{3}\delta_{m,-n}\delta_{ij}(1-\gamma)c, \quad [\mathrm{T}^{\pm i}_m, \mathrm{U}_n] = 0,$$

$$[\mathrm{T}^{-i}_m, \mathrm{T}^{-j}_n] = \epsilon_{ijk}\mathrm{T}^{-k}_{m+n} - \tfrac{m}{3}\delta_{m,-n}\delta_{ij}\gamma c, \qquad [\mathrm{U}_m, \mathrm{U}_n] = -\tfrac{m}{3}\delta_{m,-n}c,$$

$$[\mathrm{L}_m, \mathrm{Q}^p_{n'}] = -\left(\tfrac{1}{2}m + n'\right)\mathrm{Q}^p_{m+n'}, \qquad [\mathrm{U}_m, \mathrm{Q}^p_{n'}] = 0,$$

$$[\mathrm{L}_m, \mathrm{G}^p_{n'}] = \left(\tfrac{1}{2}m - n'\right)\mathrm{G}^p_{m+n'}, \qquad [\mathrm{U}_m, \mathrm{G}^p_{n'}] = m\mathrm{Q}^p_{m+n'},$$

$$[\mathrm{Q}^p_{m'}, \mathrm{Q}^q_{n'}] = -\tfrac{1}{3}\delta_{pq}\delta_{m',-n'}c, \qquad [\mathrm{T}^{\pm i}_m, \mathrm{Q}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{\pm i}_{pq}\mathrm{Q}^q_{m+n'},$$

$$[\mathrm{T}^{+i}_m, \mathrm{G}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{+i}_{pq}(\mathrm{G}^q_{m+n'} - m\mathrm{Q}^q_{m+n'}),$$

$$[\mathrm{T}^{-i}_m, \mathrm{G}^p_{n'}] = \textstyle\sum_{q=1}^4 \alpha^{-i}_{pq}(\mathrm{G}^q_{m+n'} + m\mathrm{Q}^q_{m+n'}),$$

$$[\mathrm{Q}^p_{m'}, \mathrm{G}^q_{n'}] = \delta_{pq}\mathrm{U}_{m'+n'} + 2\textstyle\sum_{i=1}^3(\alpha^{+i}_{pq}\mathrm{T}^{+i}_{m'+n'} - \alpha^{-i}_{pq}\mathrm{T}^{-i}_{m'+n'})$$
$$-\tfrac{2n'+1}{3}\delta_{m',-n'}\delta_{pq}(\gamma - \tfrac{1}{2})c,$$

$$[\mathrm{G}^p_{m'}, \mathrm{G}^q_{n'}] = 2\delta_{pq}\mathrm{L}_{m'+n'} + \tfrac{(2m')^2-1}{12}\delta_{m',-n'}\delta_{pq}c$$
$$+2(n'-m')\textstyle\sum_{i=1}^3(\alpha^{+i}_{pq}\mathrm{T}^{+i}_{m'+n'} + \alpha^{-i}_{pq}\mathrm{T}^{-i}_{m'+n'}),$$

for $i, j = 1, 2, 3$, $p, q = 1, 2, 3, 4$, $m, n \in \mathbb{Z}$, and $m', n' \in \frac{1}{2} + \mathbb{Z}$.

The Lie superalgebra $\mathfrak{g}(\gamma)/\Bbbk c$ is called the *centreless core* of $\mathfrak{g}(\gamma)$. We observe that all the $\mathfrak{g}(\gamma)/\Bbbk c$'s for $\gamma \neq 0, 1$ are isomorphic. We denote this common Lie superalgebra by $\mathfrak{g}$. In other words, every $\mathfrak{g}(\gamma)$ is a central extension of $\mathfrak{g}$.

To the Lie superalgebra $\mathfrak{g}$, one associates the conformal superalgebra $\mathscr{M}$, whose underlying $\mathbb{Z}/2\mathbb{Z}$–graded $\Bbbk[\partial]$–module is

$$\mathscr{M} = (\Bbbk[\partial] \otimes V_{\bar{0}}) \oplus (\Bbbk[\partial] \otimes V_{\bar{1}}),$$

where $V_{\bar{0}} = \Bbbk \mathrm{L} \oplus \bigoplus_{i=1}^{3}(\Bbbk \mathrm{T}^{i} \oplus \Bbbk \mathrm{T}^{-i}) \oplus \Bbbk \mathrm{U}$ and $V_{\bar{1}} = \bigoplus_{p=1}^{4}(\Bbbk \mathrm{G}^{p} \oplus \Bbbk \mathrm{Q}^{p})$. The λ–bracket on $\mathscr{M}$ is given by:

$$
\begin{aligned}
&[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}, && [\mathrm{T}^{+i}{}_\lambda \mathrm{T}^{-j}] = 0,\\
&[\mathrm{L}_\lambda \mathrm{U}] = (\partial + \lambda)\mathrm{U}, && [\mathrm{L}_\lambda \mathrm{T}^{\pm i}] = (\partial + \lambda)\mathrm{T}^{\pm i},\\
&[\mathrm{T}^{\pm i}{}_\lambda \mathrm{T}^{\pm j}] = \epsilon_{ijk}\mathrm{T}^{\pm k}, && [\mathrm{T}^{\pm i}{}_\lambda \mathrm{U}] = [\mathrm{U}_\lambda \mathrm{U}] = 0,\\
&[\mathrm{L}_\lambda \mathrm{Q}^p] = \left(\partial + \tfrac{1}{2}\lambda\right)\mathrm{Q}^p, && [\mathrm{U}_\lambda \mathrm{Q}^p] = 0,\\
&[\mathrm{L}_\lambda \mathrm{G}^p] = \left(\partial + \tfrac{3}{2}\lambda\right)\mathrm{G}^p, && [\mathrm{U}_\lambda \mathrm{G}^p] = \lambda \mathrm{Q}^p,\\
&[\mathrm{Q}^p{}_\lambda \mathrm{Q}^q] = 0, && [\mathrm{T}^{\pm i}{}_\lambda \mathrm{Q}^p] = \textstyle\sum_{q=1}^{4} \alpha_{pq}^{\pm i}\mathrm{Q}^q,\\
&[\mathrm{T}^{+i}{}_\lambda \mathrm{G}^p] = \textstyle\sum_{q=1}^{4} \alpha_{pq}^{+i}(\mathrm{G}^q - \lambda \mathrm{Q}^q), && [\mathrm{T}^{-i}{}_\lambda \mathrm{G}^p] = \textstyle\sum_{q=1}^{4} \alpha_{pq}^{-i}(\mathrm{G}^q + \lambda \mathrm{Q}^q),\\
&[\mathrm{Q}^p{}_\lambda \mathrm{G}^q] = \delta_{pq}\mathrm{U} + 2\textstyle\sum_{i=1}^{3}(\alpha_{pq}^{+i}\mathrm{T}^{+i} - \alpha_{pq}^{-i}\mathrm{T}^{-i}).\\
&[\mathrm{G}^p{}_\lambda \mathrm{G}^q] = 2\delta_{pq}\mathrm{L} - 2(\partial + 2\lambda)\textstyle\sum_{i=1}^{3}(\alpha_{pq}^{+i}\mathrm{T}^{+i} + \alpha_{pq}^{-i}\mathrm{T}^{-i}),
\end{aligned}
$$

for $i, j = 1, 2, 3$, and $p, q = 1, 2, 3, 4$.

To simplify computations, we introduce the following notation:

$$
\begin{aligned}
\mathrm{T}^{+}(\mathbf{x}) &:= -\mathbf{i}(x_{12} + x_{21})\mathrm{T}^{+1} + (x_{12} - x_{21})\mathrm{T}^{+2} + 2\mathbf{i}x_{11}\mathrm{T}^{+3}\\
\mathrm{T}^{-}(\mathbf{x}) &:= -\mathbf{i}(x_{12} + x_{21})\mathrm{T}^{-1} + (x_{12} - x_{21})\mathrm{T}^{-2} + 2\mathbf{i}x_{11}\mathrm{T}^{-3}\\
\mathrm{G}(\mathbf{u}) &:= \mathbf{i}(u_{12} + u_{21})\mathrm{G}^1 - (u_{12} - u_{21})\mathrm{G}^2 - \mathbf{i}(u_{11} - u_{22})\mathrm{G}^3 + (u_{11} + u_{22})\mathrm{G}^4\\
\mathrm{Q}(\mathbf{u}) &:= \mathbf{i}(u_{12} + u_{21})\mathrm{Q}^1 - (u_{12} - u_{21})\mathrm{Q}^2 - \mathbf{i}(u_{11} - u_{22})\mathrm{Q}^3 + (u_{11} + u_{22})\mathrm{Q}^4
\end{aligned}
$$

for $\mathbf{x} = (x_{ij}) \in \mathfrak{sl}_2(\Bbbk)$, $\mathbf{u} = (u_{ij}) \in \mathrm{Mat}_2(\Bbbk)$. With this new notation, the λ-bracket on $\mathscr{M}$ is rewritten as:

$$
\begin{aligned}
&[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L}, && [\mathrm{T}^{+}(\mathbf{x})_\lambda \mathrm{T}^{-}(\mathbf{y})] = 0,\\
&[\mathrm{L}_\lambda \mathrm{U}] = (\partial + \lambda)\mathrm{U}, && [\mathrm{L}_\lambda \mathrm{T}^{\pm}(\mathbf{x})] = (\partial + \lambda)\mathrm{T}^{\pm}(\mathbf{x}),\\
&[\mathrm{T}^{\pm}(\mathbf{x})_\lambda \mathrm{T}^{\pm}(\mathbf{y})] = \mathrm{T}^{\pm}([\mathbf{x}, \mathbf{y}]), && [\mathrm{T}^{\pm}(\mathbf{x})_\lambda \mathrm{U}] = [\mathrm{U}_\lambda \mathrm{U}] = 0,\\
&[\mathrm{L}_\lambda \mathrm{Q}(\mathbf{u})] = \left(\partial + \tfrac{1}{2}\lambda\right)\mathrm{Q}(\mathbf{u}), && [\mathrm{U}_\lambda \mathrm{Q}(\mathbf{u})] = [\mathrm{Q}(\mathbf{u})_\lambda \mathrm{Q}(\mathbf{v})] = 0,\\
&[\mathrm{L}_\lambda \mathrm{G}(\mathbf{u})] = \left(\partial + \tfrac{3}{2}\lambda\right)\mathrm{G}(\mathbf{u}), && [\mathrm{U}_\lambda \mathrm{G}(\mathbf{u})] = \lambda \mathrm{Q}(\mathbf{u}),\\
&[\mathrm{T}^{+}(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})] = \mathrm{G}(\mathbf{xu}) - \lambda \mathrm{Q}(\mathbf{xu}), && [\mathrm{T}^{+}(\mathbf{x})_\lambda \mathrm{Q}(\mathbf{u})] = \mathrm{Q}(\mathbf{xu}),\\
&[\mathrm{T}^{-}(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})] = -\mathrm{G}(\mathbf{ux}) - \lambda \mathrm{Q}(\mathbf{ux}), && [\mathrm{T}^{-}(\mathbf{x})_\lambda \mathrm{Q}(\mathbf{u})] = -\mathrm{Q}(\mathbf{ux}),
\end{aligned}
$$

$$[\mathrm{Q}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})] = 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{U} - \mathrm{T}^+(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger) + \mathrm{T}^-(\mathbf{u}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{u}),$$

$$[\mathrm{G}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})] = 4\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{L} + (\partial + 2\lambda)\left(\mathrm{T}^+(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger) + \mathrm{T}^-(\mathbf{u}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{u})\right),$$

where $\mathbf{x}, \mathbf{y} \in \mathfrak{sl}_2(\Bbbk)$, $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(\Bbbk)$, and $\dagger : \mathrm{Mat}_2(\Bbbk) \to \mathrm{Mat}_2(\Bbbk)$ is the standard sympletic involution defined by (5.1.2) in Section 5.1.

6.2 The automorphism group

We will determine the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})$ in this section by explicitly constructing all automorphisms of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$.

To simplify notations, we write $\widehat{\partial} := \partial \otimes \mathrm{id} + \mathrm{id} \otimes \mathsf{d}_t$ for short. We always use V to denote the $\Bbbk$–vector space spanned by $\{\mathrm{L}, \mathrm{T}^{\pm i}, \mathrm{U}, \mathrm{G}^p, \mathrm{Q}^p | i = 1,2,3, p = 1,2,3,4\}$. Note that $\mathscr{M}_{\widehat{\mathcal{D}}} = \Bbbk[\partial] \otimes_\Bbbk V \otimes_\Bbbk \widehat{D}$ as $\Bbbk$–vector spaces. V can be identified with the subspace $\Bbbk \otimes V \otimes \Bbbk$ in $\mathscr{M}_{\widehat{\mathcal{D}}}$. Hence, we also identify an element $\eta \in V$ with its image $1 \otimes \eta \otimes 1$ in $\mathscr{M}_{\widehat{\mathcal{D}}}$.

Lemma 6.1. *There is a group homomorphism*

$$\iota_1 : \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}) \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}), \quad (A, B) \mapsto \theta_{A,B}, \tag{6.2.1}$$

where $\theta_{A,B}$ is the automorphism of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$ given by

$$\theta_{A,B}(\mathrm{L}) = \mathrm{L} + \mathrm{T}^+(\mathsf{d}_t(A)A^{-1}) + \mathrm{T}^-(\mathsf{d}_t(B)B^{-1}),$$

$$\theta_{A,B}(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^+(A\mathbf{x}A^{-1}), \quad \theta_{A,B}(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^-(B\mathbf{x}B^{-1}),$$

$$\theta_{A,B}(\mathrm{U}) = \mathrm{U}, \qquad \theta_{A,B}(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(A\mathbf{u}B^{-1}),$$

$$\theta_{A,B}(\mathrm{G}(\mathbf{u})) = \mathrm{G}(A\mathbf{u}B^{-1}) - \mathrm{Q}(\mathsf{d}_t(A)\mathbf{u}B^{-1} - A\mathbf{u}\mathsf{d}_t(B^{-1})),$$

for $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk)$, $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$.

Proof. Recall that the underlying $\widehat{D}$–module of $\mathscr{M}_{\widehat{\mathcal{D}}}$ is $\Bbbk[\partial] \otimes_\Bbbk V \otimes_\Bbbk \widehat{D}$. The formulas define a $\widehat{D}$–module homomorphism $V \otimes_\Bbbk \widehat{D} \to V \otimes_\Bbbk \widehat{D}$, which is uniquely extended to a $\widehat{D}$–module homomorphism $\theta_{A,B} : \mathscr{M}_{\widehat{\mathcal{D}}} \to \mathscr{M}_{\widehat{\mathcal{D}}}$ satisfying $\widehat{\partial} \circ \theta_{A,B} = \theta_{A,B} \circ \widehat{\partial}$.

Based on Lemma 2.3, it can be proved that $\theta_{A,B}$ is a homomorphism of $\widehat{\mathcal{D}}$–conformal superalgebras by verifying

$$\theta_{A,B}([(\eta_1 \otimes 1)_\lambda (\eta_2 \otimes 1)]) = [\theta_{A,B}(\eta_1 \otimes 1)_\lambda \theta_{A,B}(\eta_2 \otimes 1)]$$

for all $\eta_1, \eta_2 \in V$. This can be accomplished through a direct computation. As an example, we show the proof for $\eta_1 = \mathrm{Q}(\mathbf{u})$ and $\eta_2 = \mathrm{G}(\mathbf{v})$ with $\mathbf{u}, \mathbf{v} \in \mathrm{Mat}_2(\Bbbk)$.

$$\begin{aligned}
&\theta_{A,B}([\mathrm{Q}(\mathbf{u})_\lambda \mathrm{G}(\mathbf{v})]) \\
&= 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\theta_{A,B}(\mathrm{U}) - \theta_{A,B}(\mathrm{T}^+(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)) + \theta_{A,B}(\mathrm{T}^-(\mathbf{u}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{u})) \\
&= 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{U} - \mathrm{T}^+(A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1}) + \mathrm{T}^-(B(\mathbf{u}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{u})B^{-1}), \\
&[\theta_{A,B}(\mathrm{Q}(\mathbf{u}))_\lambda \theta_{A,B}(\mathrm{G}(\mathbf{v}))] \\
&= [\mathrm{Q}(A\mathbf{u}B^{-1})_\lambda(\mathrm{G}(A\mathbf{v}B^{-1}) - \mathrm{Q}(\mathsf{d}_t(A)\mathbf{u}B^{-1} - A\mathbf{v}\mathsf{d}_t(B^{-1})))] \\
&= 2\mathrm{tr}(A\mathbf{u}B^{-1}(A\mathbf{v}B^{-1})^\dagger)\mathrm{U} \\
&\quad - \mathrm{T}^+(A\mathbf{u}B^{-1}(A\mathbf{v}B^{-1})^\dagger - A\mathbf{v}B^{-1}(A\mathbf{u}B^{-1})^\dagger) \\
&\quad + \mathrm{T}^-((A\mathbf{u}B^{-1})^\dagger A\mathbf{v}B^{-1} - (A\mathbf{v}B^{-1})^\dagger A\mathbf{u}B^{-1}) \\
&= 2\mathrm{tr}(\mathbf{u}\mathbf{v}^\dagger)\mathrm{U} - \mathrm{T}^+(A(\mathbf{u}\mathbf{v}^\dagger - \mathbf{v}\mathbf{u}^\dagger)A^{-1}) + \mathrm{T}^-(B(\mathbf{u}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{u})B^{-1}),
\end{aligned}$$

A similar computation also shows that

$$\theta_{A_1,B_1} \circ \theta_{A_2,B_2}(\eta \otimes 1) = \theta_{A_1A_2,B_1B_2}(\eta \otimes 1),$$

for $A_1, A_2, B_1, B_2 \in \mathbf{SL}_2(\widehat{D})$ and all $\eta \in V$. We thus deduce by Lemma 2.3 that

$$\theta_{A_1,B_1} \circ \theta_{A_2,B_2} = \theta_{A_1A_2,B_1B_2}.$$

We also observe that $\theta_{I_2,I_2} = \mathrm{id}$, where I_2 is the identity matrix. Hence, the above equality implies that $\theta_{A,B}$ is invertible and

$$\iota_1 : \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}) \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}), (A, B) \mapsto \theta_{A,B}$$

is a group homomorphism. ☐

Lemma 6.2. *There is a group homomorphism*

$$\iota_2 : \mathbf{G}_a(\widehat{D}) \to \mathrm{Aut}_{\widehat{D}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}), \quad s \mapsto \tau_s, \tag{6.2.2}$$

where τ_s is the automorphism of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$ defined by

$$\begin{aligned}
&\tau_s(\mathrm{L}) = \mathrm{L} + \mathrm{U} \otimes s, && \tau_s(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^+(\mathbf{x}), && \tau_s(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^-(\mathbf{x}), \\
&\tau_s(\mathrm{U}) = \mathrm{U}, && \tau_s(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\mathbf{u}) + \mathrm{Q}(s\mathbf{u}), && \tau_s(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(\mathbf{u}),
\end{aligned}$$

for $\mathbf{x} \in \mathfrak{sl}_2(\mathbb{k})$ *and* $\mathbf{u} \in \mathrm{Mat}_2(\mathbb{k})$.

Proof. An analogous argument as in Lemma 6.1 shows that the formulas define a homomorphism of $\widehat{\mathcal{D}}$–conformal superalgebras $\tau_s : \mathscr{M}_{\widehat{\mathcal{D}}} \to \mathscr{M}_{\widehat{\mathcal{D}}}$ for every $s \in \widehat{D}$ and $\tau_{s_1} \circ \tau_{s_2} = \tau_{s_1+s_2}$ for $s_1, s_2 \in \widehat{D}$. Observing that $\tau_0 = \mathrm{id}$, we obtain that τ_s has an inverse τ_{-s} and ι_2 is a group homomorphism. □

Lemma 6.3. *There is an automorphism ω of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$ such that*

$$\omega(\mathrm{L}) = \mathrm{L}, \qquad \omega(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^-(\mathbf{x}), \qquad \omega(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^+(\mathbf{x}),$$
$$\omega(\mathrm{U}) = -\mathrm{U}, \qquad \omega(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\mathbf{u}^\dagger), \qquad \omega(\mathrm{Q}(\mathbf{u})) = -\mathrm{Q}(\mathbf{u}^\dagger),$$

for $\mathbf{x} \in \mathfrak{sl}_2(\mathbb{k})$ *and* $\mathbf{u} \in \mathrm{Mat}_2(\mathbb{k})$. *In addition,* $\omega^2 = \mathrm{id}$.

Proof. The proof is similar to that of Lemma 6.1. □

Lemma 6.4.

(i) For $A, B \in \mathbf{SL}_2(\widehat{D})$ and $s \in \mathbf{G}_a(\widehat{D})$, $\tau_s \circ \theta_{A,B} = \theta_{A,B} \circ \tau_s$.

(ii) For $A, B \in \mathbf{SL}_2(\widehat{D})$, $\omega \circ \theta_{A,B} \circ \omega = \theta_{B,A}$.

(iii) For $s \in \mathbf{G}_a(\widehat{D})$, $\omega \circ \tau_s \circ \omega = \tau_{-s}$.

Proof. (i) From Lemma 2.3, it suffices to show

$$\tau_s \circ \theta_{A,B}(\eta \otimes 1) = \theta_{A,B} \circ \tau_s(\eta \otimes 1),$$

for all $\eta \in V$. This can be verified by a direct computation. The proofs for (ii) and (iii) are similar. □

Theorem 6.5. *There is a group isomorphism:*

$$\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}) \cong \left(\frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})}{\langle(-I_2, -I_2)\rangle} \times \mathbf{G}_a(\widehat{D}) \right) \rtimes \mathbb{Z}/2\mathbb{Z}. \tag{6.2.3}$$

Proof. From Lemmas 6.1-6.4, there is a group homomorphism:

$$\iota : \left(\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}) \times \mathbf{G}_a(\widehat{D}) \right) \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}),$$
$$(A, B, s, \varepsilon) \mapsto \theta_{A,B} \circ \tau_s \circ \omega^{\varepsilon}.$$

We claim that ι is surjective, i.e., every $\phi \in \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})$ is of the form $\theta_{A,B} \circ \tau_s \circ \omega^{\varepsilon}$ for some $A, B \in \mathbf{SL}_2(\widehat{D})$, $s \in \widehat{D}$, and $\varepsilon \in \{0,1\}$.

To prove the claim, we first consider the action of ϕ on the even part $(\mathscr{M}_{\widehat{\mathcal{D}}})_{\bar{0}}$. We observe

$$\mathscr{M}_{\bar{0}} = \Bbbk[\partial]\mathrm{L} \oplus \mathscr{C} \oplus \Bbbk[\partial]\mathrm{U},$$

where $\mathscr{C} = \oplus_{i=1}^{3}(\Bbbk[\partial]\mathrm{T}^{+i} \oplus \Bbbk[\partial]\mathrm{T}^{-i})$. Moreover, $\mathscr{B} := \mathscr{C} \oplus \Bbbk[\partial]\mathrm{U}$ is an ideal of $\mathscr{M}_{\bar{0}}$ and $\mathscr{B}$ is isomorphic to the current Lie conformal algebra $\mathrm{Cur}(\mathfrak{sl}_2(\Bbbk) \oplus \mathfrak{sl}_2(\Bbbk) \oplus \Bbbk)$.

Next we show that $\phi(\mathscr{B}_{\widehat{\mathcal{D}}}) \subseteq \mathscr{B}_{\widehat{\mathcal{D}}}$. Let $\eta = \mathrm{T}^{\pm i}$, $i = 1, 2, 3$ or U. Write

$$\phi(\eta) = \sum_{m=0}^{M} \widehat{\partial}^m(\mathrm{L} \otimes r_m) + \eta',$$

where $r_m \in \widehat{D}$ and $\eta' \in \mathscr{B}_{\widehat{\mathcal{D}}}$. Then

$$\begin{aligned}
0 &= \phi([\eta_\lambda \eta]) = [\phi(\eta)_\lambda \phi(\eta)] \\
&= \sum_{m,n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n ((\partial + 2\lambda)\mathrm{L} \otimes r_m r_n + 2\mathrm{L} \otimes \mathsf{d}_t(r_m) r_n) \\
&\quad + \sum_{m=0}^{M} (-\lambda)^m [(\mathrm{L} \otimes r_m)_\lambda \eta'] + \sum_{n=0}^{M} (\widehat{\partial} + \lambda)^n [\eta'_\lambda (\mathrm{L} \otimes r_n)] + [\eta'_\lambda \eta']
\end{aligned}$$

Since $[(\mathrm{L} \otimes r_m)_\lambda \eta'], [\eta'_\lambda (\mathrm{L} \otimes r_n)], [\eta'_\lambda \eta'] \in \Bbbk[\lambda] \otimes_{\Bbbk} \mathscr{B}_{\widehat{\mathcal{D}}}$, we deduce that

$$0 = \sum_{m,n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n ((\partial + 2\lambda)\mathrm{L} \otimes r_m r_n + 2\mathrm{L} \otimes \mathsf{d}_t(r_m) r_n).$$

Comparing the coefficients of λ and noting that $\widehat{D}$ is an integral domain, we obtain $M = 0$ and $r_0 = 0$, i.e., $\phi(\eta) = \eta' \in \mathscr{B}_{\widehat{\mathcal{D}}}$. Since $\mathrm{T}^{\pm i}$, $i = 1, 2, 3$ and U generate $\mathscr{B}$ as a $\Bbbk[\partial]$–module and ϕ is a $\widehat{D}$–module homomorphism satisfying $\phi \circ \widehat{\partial} = \widehat{\partial} \circ \phi$, we conclude that $\phi(\mathscr{B}_{\widehat{\mathcal{D}}}) \subseteq \mathscr{B}_{\widehat{\mathcal{D}}}$.

Furthermore, we deduce from ${\mathrm{T}^{\pm i}}_{(0)}\mathrm{T}^{\pm j} = \epsilon_{ijk}\mathrm{T}^{\pm k}$ that

$$\epsilon_{ijk}\phi(\mathrm{T}^{\pm k}) = \phi(\mathrm{T}^{\pm i})_{(0)}\phi(\mathrm{T}^{\pm j}) \in (\mathscr{B}_{\widehat{\mathcal{D}}})_{(0)}(\mathscr{B}_{\widehat{\mathcal{D}}}) \subseteq \mathscr{C}_{\widehat{\mathcal{D}}},$$

for $k = 1, 2, 3$. It yields that $\phi(\mathscr{C}_{\widehat{\mathcal{D}}}) \subseteq \mathscr{C}_{\widehat{\mathcal{D}}}$.

Therefore, the restriction $\phi|_{\mathscr{C}_{\widehat{\mathcal{D}}}}$ is an automorphism of $\mathscr{C}_{\widehat{\mathcal{D}}}$. It is known that

$$\mathscr{C}_{\widehat{\mathcal{D}}} \cong \mathrm{Cur}(\mathfrak{sl}_2(\mathbb{k}) \oplus \mathfrak{sl}_2(\mathbb{k}))_{\widehat{\mathcal{D}}}.$$

Given that $\mathfrak{sl}_2(\mathbb{k}) \oplus \mathfrak{sl}_2(\mathbb{k})$ is a semisimple Lie algebra over $\mathbb{k}$, by Proposition 2.5, there are two elements $A, B \in \mathbf{GL}_2(\widehat{D})$ such that either of the two following conditions is satisfied

$$\phi(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^+(A\mathbf{x}A^{-1}), \text{ and } \phi(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^-(B\mathbf{x}B^{-1}), \tag{6.2.4}$$

$$\phi(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^-(A\mathbf{x}A^{-1}), \text{ and } \phi(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^+(B\mathbf{x}B^{-1}). \tag{6.2.5}$$

Since any unit of $\widehat{D}$ is a square, there is no loss of generality in assuming that $A, B \in \mathbf{SL}_2(\widehat{D})$. We take $\psi := \phi \circ \theta_{A,B}^{-1}$ if ϕ satisfies (6.2.4), or $\psi := \phi \circ \omega \circ \theta_{A,B}^{-1}$ if ϕ satisfies (6.2.5). Then ψ is also an automorphism of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$ and always satisfies

$$\psi(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^+(\mathbf{x}), \text{ and } \psi(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^-(\mathbf{x}), \tag{6.2.6}$$

for all $\mathbf{x} \in \mathfrak{sl}_2(\mathbb{k})$.

To determine $\psi(\mathrm{U})$, we observe that $\mathbb{k}[\partial]\mathrm{U} \otimes_{\mathbb{k}} \widehat{\mathcal{D}}$ is the center of $\mathscr{B}_{\widehat{\mathcal{D}}}$, which is preserved under ψ. Hence, $\psi(\mathrm{U}) = P(\partial)\mathrm{U}$, where $P(\partial)$ is a polynomial in the indeterminate ∂ with coefficients in $\widehat{D}$. Then the bijectivity of ψ yields that $P(\partial) = r$ is a unit element in $\widehat{D}$, i.e., $\psi(\mathrm{U}) = \mathrm{U} \otimes r$ for a unit element $r \in \widehat{D}$.

Next we consider the action of ψ on the odd part $(\mathscr{M}_{\widehat{\mathcal{D}}})_{\bar{1}}$. Suppose

$$\psi(\mathrm{G}(\mathbf{u})) = \sum_{m=0}^{M_1} \widehat{\partial}^m \mathrm{G}(\nu_m(\mathbf{u})) + \sum_{n=0}^{M_2} \widehat{\partial}^n \mathrm{Q}(\nu'_n(\mathbf{u})),$$

where $\nu_m, \nu'_n : \mathrm{Mat}_2(\mathbb{k}) \to \mathrm{Mat}_2(\widehat{D})$ are $\mathbb{k}$–linear maps. Then

$$\psi([\mathrm{U}_\lambda \mathrm{G}(\mathbf{u})]) = [\psi(\mathrm{U})_\lambda \psi(\mathrm{G}(\mathbf{u}))]$$

yields

$$\lambda\psi(\mathrm{Q}(\mathbf{u})) = \sum_{m=0}^{M_1} (\widehat{\partial} + \lambda)^m (\lambda \mathrm{Q}(r\nu_m(\mathbf{u})) + \mathrm{Q}(\mathsf{d}_t(r)\nu_m(\mathbf{u}))).$$

Comparing the coefficients of λ, we obtain $M_1 = 0$, i.e.,

$$\psi(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\nu_0(\mathbf{u})) + \sum_{n=0}^{M_2} \widehat{\partial}^n \mathrm{Q}(\nu'_n(\mathbf{u})),$$
$$\psi(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(r\nu_0(\mathbf{u})).$$

Similarly, we deduce from

$$\psi([\mathrm{T}^+(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})]) = [\psi(\mathrm{T}^+(\mathbf{x}))_\lambda \psi(\mathrm{G}(\mathbf{u}))]$$

that $M_2 = 0$ and

$$\nu_0(\mathbf{xu}) = \mathbf{x}\nu_0(\mathbf{u}), \quad \nu'_0(\mathbf{xu}) = \mathbf{x}\nu'_0(\mathbf{u}), \quad r\nu_0(\mathbf{xu}) = \mathbf{x}\nu_0(\mathbf{u}), \tag{6.2.7}$$

for $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk), \mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$. Furthermore,

$$\psi([\mathrm{T}^-(\mathbf{x})_\lambda \mathrm{G}(\mathbf{u})]) = [\psi(\mathrm{T}^-(\mathbf{x}))_\lambda \psi(\mathrm{G}(\mathbf{u}))]$$

yields that

$$\nu_0(\mathbf{ux}) = \nu_0(\mathbf{u})\mathbf{x}, \quad \nu'_0(\mathbf{ux}) = \nu'_0(\mathbf{u})\mathbf{x}, \quad r\nu_0(\mathbf{ux}) = \nu_0(\mathbf{u})\mathbf{x}, \tag{6.2.8}$$

for $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk), \mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$.

From (6.2.7) and (6.2.8), we conclude that $r = 1$, and there are $s_1, s_2 \in \widehat{D}$ such that $\nu_0(\mathbf{u}) = s_1\mathbf{u}$ and $\nu'_0(\mathbf{u}) = s_2\mathbf{u}$, i.e.,

$$\psi(\mathrm{G}(\mathbf{u})) = \mathrm{G}(s_1\mathbf{u}) + \mathrm{Q}(s_2\mathbf{u}), \quad \psi(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(s_1\mathbf{u}), \quad \psi(\mathrm{U}) = \mathrm{U}.$$

Finally, the equality

$$\psi(\mathrm{Q}(\mathbf{u}))_{(0)}\psi(\mathrm{G}(\mathbf{v})) = \psi(\mathrm{Q}(\mathbf{u})_{(0)}\mathrm{G}(\mathbf{v}))$$

implies that $s_1 = \pm 1$; while the equality

$$\psi(\mathrm{G}(\mathbf{u}))_{(0)}\psi(\mathrm{G}(\mathbf{v})) = \psi(\mathrm{G}(\mathbf{u})_{(0)}\mathrm{G}(\mathbf{v}))$$

yields $\psi(\mathrm{L}) = \mathrm{L} + \mathrm{U} \otimes s_1 s_2$. Let $s = s_1 s_2$, we obtain

$$\psi(\mathrm{L}) = \mathrm{L} + \mathrm{U} \otimes s, \qquad \psi(\mathrm{G}(\mathbf{u})) = \mathrm{G}(s_1\mathbf{u}) + \mathrm{Q}(s_1 s\mathbf{u}), \tag{6.2.9}$$

$$\psi(\mathrm{U}) = \mathrm{U}, \qquad \psi(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(s_1\mathbf{u}). \tag{6.2.10}$$

Summarizing (6.2.6) and (6.2.9)-(6.2.10), we conclude that $\psi = \tau_s \circ \theta_{I,s_1 I}$. Hence,

$$\phi = \tau_s \circ \theta_{A,sB} = \theta_{A,s_1B} \circ \tau_s, \text{ or } \phi = \tau_s \circ \theta_{A,s_1B} \circ \omega = \theta_{A,s_1B} \circ \tau_s \circ \omega.$$

We complete the proof of the surjectivity of ι.

Next we will determine the kernel of ι. On one hand, it is obvious that

$$(-I_2, -I_2, 0, 0) \in \ker \iota.$$

On the other hand, we will show that $(A, B, s, \varepsilon) \in \ker \iota$ will lead to $A = B = \pm I_2$, $s = 0$ and $\varepsilon = 0$. In fact, $(A, B, s, \varepsilon) \in \ker \iota$ is equivalent to $\theta_{A,B} \circ \tau_s \circ \omega^\varepsilon = \mathrm{id}$. Hence,

$$\mathrm{U} = \theta_{A,B} \circ \tau_s \circ \omega^\varepsilon(\mathrm{U}) = (-1)^\varepsilon \mathrm{U},$$

where $\varepsilon = 0$ or 1. It follows that $\varepsilon = 0$.

Similarly, $\theta_{A,B} \circ \tau_s(\mathrm{L}) = \mathrm{L}$ yields that $s = 0$, and hence

$$\theta_{A,B}(\mathrm{T}^+(\mathbf{x})) = \mathrm{T}^+(A\mathbf{x}A^{-1}) = \mathrm{T}^+(\mathbf{x}),$$
$$\theta_{A,B}(\mathrm{T}^-(\mathbf{x})) = \mathrm{T}^-(B\mathbf{x}B^{-1}) = \mathrm{T}^-(\mathbf{x}),$$
$$\theta_{A,B}(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(A\mathbf{u}B^{-1}) = \mathrm{Q}(\mathbf{u}),$$

for all $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk)$ and all $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$, i.e.,

$$A\mathbf{x} = \mathbf{x}A, B\mathbf{x} = \mathbf{x}B, \text{ and } A\mathbf{u} = \mathbf{u}B,$$

for all $\mathbf{x} \in \mathfrak{sl}_2(\Bbbk)$ and all $\mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$. This yields that $A = B = \pm I_2$. Hence, $\ker \iota = \langle(-I_2, -I_2, 0, 0)\rangle$. Therefore, ι induces a group isomorphism

$$\begin{aligned}\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}) &\cong \frac{\left(\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}) \times \mathbf{G}_a(\widehat{D})\right) \rtimes \mathbb{Z}/2\mathbb{Z}}{\langle(-I_2, -I_2, 0, 0)\rangle} \\ &\cong \left(\frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})}{\langle(-I_2, -I_2)\rangle} \times \mathbf{G}_a(\widehat{D})\right) \rtimes \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

□

Remark 6.6. The above theorem gives an explicit description of the automorphism group of the $\widehat{\mathcal{D}}$–conformal superalgebra $\mathscr{M}_{\widehat{\mathcal{D}}}$. Using the same arguments, we also can obtain the automorphism group of the $\Bbbk$–conformal superalgebra $\mathscr{M}$. In fact,

$$\mathrm{Aut}_{\Bbbk\text{-conf}}(\mathscr{M}) \cong \left(\frac{\mathbf{SL}_2(\Bbbk) \times \mathbf{SL}_2(\Bbbk)}{\langle(-I_2, -I_2)\rangle} \times \mathbf{G}_a(\Bbbk) \right) \rtimes \mathbb{Z}/2\mathbb{Z}. \tag{6.2.11}$$

6.3 Twisted loop conformal superalgebras

The classification of the twisted loop conformal superalgebras based on $\mathscr{M}$ will be completed in this section. We will first compute the non-abelian cohomology set $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})\right)$, which yields the classification of the twisted loop conformal superalgebras based on $\mathscr{M}$ up to isomorphism of $\mathcal{D}$–conformal superalgebras. Then we will derive the classification up to isomorphism of $\Bbbk$–conformal superalgebras using the centroid trick (see Section 2.5).

Proposition 6.7. *Every $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathscr{M}_{\mathcal{D}}$ is isomorphic to either $\mathcal{L}(\mathscr{M}, \mathrm{id})$ or $\mathcal{L}(\mathscr{M}, \omega)$ as $\mathcal{D}$–conformal superalgebras, where ω is the automorphism of the $\Bbbk$–conformal superalgebra $\mathscr{M}$ defined in Lemma 6.3.*

Proof. Based on Theorem 2.9 and Proposition 3.4, $\widehat{\mathcal{D}}/\mathcal{D}$–forms of $\mathscr{M}_{\mathcal{D}}$ are parameterized by the continuous non-abelian cohomology set $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})\right)$, where $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/m\mathbb{Z}$ and the continuous action of $\widehat{\mathbb{Z}}$ on $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})$ is induced by the continuous action of $\widehat{\mathbb{Z}}$ on $\widehat{D}$ given by ${}^{\bar{1}}t^{p/q} = \zeta_q^{-p} t^{p/q}$. Hence, the crucial point of the proof is to compute the cohomology set $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})\right)$.

By Theorem 6.5, there is a split short exact sequence of groups

$$1 \to \mathrm{G} \to \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}) \to \mathbb{Z}/2\mathbb{Z} \to 1, \tag{6.3.1}$$

where

$$\mathrm{G} := \mathrm{G}_1 \times \mathbf{G}_a(\widehat{D}), \text{ and } \mathrm{G}_1 := \frac{\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})}{\langle(-I_2, -I_2)\rangle}.$$

We observe that $\widehat{\mathbb{Z}}$ continuously acts on G through the action on $\widehat{D}$ and $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially. With these $\widehat{\mathbb{Z}}$–actions, the homomorphisms in (6.3.1) are all

$\widehat{\mathbb{Z}}$–equivariant. Hence, the exact sequence (6.3.1) induces an exact sequence of non-abelian continuous cohomology sets

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}) \longrightarrow \mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})\right) \xrightarrow{\rho} \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}). \tag{6.3.2}$$

Since the exact sequence (6.3.1) is split, ρ has a section, and hence ρ is surjective. Recall that $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially, we have $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$. Since (6.3.2) is exact, the fiber of ρ over $[0]$ is measured by $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G})$. To compute $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G})$, we observe that $\widehat{\mathbb{Z}}$ piecewise acts on $\mathrm{G} = \mathrm{G}_1 \times \mathbf{G}_a(\widehat{D})$. It follows that

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}) = \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}_1) \times \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}_a(\widehat{D})). \tag{6.3.3}$$

The group G_1 fits into an exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}) \to \mathrm{G}_1 \to 1.$$

Since $\mathbb{Z}/2\mathbb{Z} = \langle(-I_2, -I_2)\rangle$ is central in $\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})$, it yields an exact sequence of pointed sets

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})) \to \mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}_1) \to \mathrm{H}^2_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}).$$

Since $\mathbf{SL}_2 \times \mathbf{SL}_2$ is a semi-simple group scheme, by Proposition 3.7 (i), the non-abelian continuous cohomology set $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})\right)$ can be identified with the non-abelian étale cohomology $\mathrm{H}^1_{\text{ét}}(D, \mathbf{SL}_2 \times \mathbf{SL}_2)$, which vanishes according to Proposition 3.7 (ii). Hence,

$$\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})\right) = 0.$$

$\mathrm{H}^2_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$ also vanishes since it can be identified with $\mathrm{H}^2_{\text{ét}}(D, \boldsymbol{\mu}_2)$, which is the 2-torsion of the Brauer group $\mathrm{H}^2_{\text{ét}}(D, \mathbf{G}_m) = 0$. Therefore,

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}_1) = 0. \tag{6.3.4}$$

From [Ser02, I.2.2, Proposition 8], we deduce that

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}_a(\widehat{D})) = \varinjlim \mathrm{H}^1_{\mathrm{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathbf{G}_a(D_m)).$$

Since D_m/D is a Galois extension with Galois group $\mathbb{Z}/m\mathbb{Z}$, $\mathrm{H}^1_{\mathrm{ct}}(\mathbb{Z}/m\mathbb{Z}, \mathbf{G}_a(D_m)) =$

$\mathrm{H}^1_{\text{ét}}(D_m/D, \mathbf{G}_{a,D})$ (see [KLP09, Remark 2.27] for details). Now, $\mathrm{H}^1_{\text{ét}}(D_m/D, \mathbf{G}_{a,D})$ can be viewed as a subset of $\mathrm{H}^1_{\text{ét}}(D, \mathbf{G}_a)$, which vanishes because our base scheme, namely $\mathrm{Spec}(D)$, is affine (see [DG70a] III.4.6.6). Hence,

$$\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbf{G}_a(\widehat{D})) = 0. \tag{6.3.5}$$

Summarizing (6.3.3), (6.3.4), and (6.3.5), we obtain $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathrm{G}) = 0$, i.e., the fiber of ρ over $[0]$ contains exactly one element.

Next we consider the fiber of ρ over $[1]$. Twisting the $\widehat{\mathbb{Z}}$–groups in (6.3.1) with respect to the cocycle $\mathfrak{z} : \widehat{\mathbb{Z}} \mapsto \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}}), \bar{1} \mapsto \omega$, we deduce that the fiber of ρ over $[1]$ is measured by $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G})$. As $\widehat{\mathbb{Z}}$–groups, ${}_{\mathfrak{z}}\mathrm{G} = {}_{\mathfrak{z}}\mathrm{G}_1 \times {}_{\mathfrak{z}}\mathbf{G}_a(\widehat{D})$. Hence, we also have

$$\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}) = \mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}_1) \times \mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathbf{G}_a(\widehat{D})). \tag{6.3.6}$$

To compute $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}_1)$, we also have an exact sequence

$$1 \to {}_{\mathfrak{z}}\mathbb{Z}/2\mathbb{Z} \to {}_{\mathfrak{z}}(\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})) \to {}_{\mathfrak{z}}\mathrm{G}_1 \to 1.$$

Since ω trivially acts on the subgroup $\langle(-I_2, -I_2)\rangle$ of $\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})$, it follows that ${}_{\mathfrak{z}}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence, there is a long exact sequence

$$\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}(\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D}))) \to \mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}_1) \to \mathrm{H}^2_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}).$$

We have seen that $\mathrm{H}^2_{\text{ct}}(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) = 0$. Further, by the same reasons given above $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}(\mathbf{SL}_2(\widehat{D}) \times \mathbf{SL}_2(\widehat{D})))$ can be identified with the non-abelian étale cohomology $\mathrm{H}^1_{\text{ét}}(D, {}_{\mathfrak{z}}(\mathbf{SL}_2 \times \mathbf{SL}_2))$, which vanishes since ${}_{\mathfrak{z}}(\mathbf{SL}_2 \times \mathbf{SL}_2)$ is also a reductive group scheme over D. Hence,

$$\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}_1) = 0. \tag{6.3.7}$$

To understand $\mathrm{H}^1_{\text{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathbf{G}_a(\widehat{D}))$, we first observe that ${}_{\mathfrak{z}}\mathbf{G}_a$ is a twisted form of $\mathbf{G}_a$ (more precisely of the D-group $\mathbf{G}_{a,D}$) associated to the cocycle $\mathfrak{z}' : \widehat{\mathbb{Z}} \to \mathrm{Aut}(\mathbf{G}_a(\widehat{D})), \bar{1} \mapsto -\mathrm{id}$, viewed in a natural way as an element of $\mathrm{H}^1_{\text{ét}}\big(D, \mathbf{Aut}(\mathbf{G}_a)\big)$. The natural D-group homomorphism $\mathbf{G}_{\mathrm{m}} \to \mathbf{Aut}(\mathbf{G}_a)$ yields a map

$$\phi : \mathrm{H}^1_{\text{ét}}\big(D, \mathbf{G}_m\big) \to H^1_{\text{ét}}\big(D, \mathbf{Aut}(\mathbf{G}_a)\big).$$

Since the class $[\mathfrak{z}']$ of $\mathfrak{z}'$ is visible in the image of ϕ and $\mathrm{H}^1_{\text{ét}}(D, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(D) = 0$, we deduce that ${}_{\mathfrak{z}}\mathbf{G}_a$ is isomorphic to $\mathbf{G}_a$ (or rather $\mathbf{G}_{a,D}$ to be precise). This yields that

$$\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathbf{G}_a(\widehat{D})) \subset \mathrm{H}^1_{\text{ét}}(D, {}_{\mathfrak{z}}\mathbf{G}_a) = \mathrm{H}^1_{\text{ét}}(D, \mathbf{G}_a) = 0. \tag{6.3.8}$$

From (6.3.6), (6.3.7), and (6.3.8), we deduce that $\mathrm{H}^1_{\mathrm{ct}}(\widehat{\mathbb{Z}}, {}_{\mathfrak{z}}\mathrm{G}) = 0$, i.e., the fiber of ρ over $[1]$ also contains exactly one element.

Consequently, $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M}_{\widehat{\mathcal{D}}})\right)$ contains exactly two elements, which correspond to $\mathcal{L}(\mathscr{M}, \mathrm{id})$ and $\mathcal{L}(\mathscr{M}, \omega)$. □

Proposition 6.8. *Let $\mathscr{B} := \mathcal{L}(\mathscr{M}, \sigma)$ be the twisted loop conformal superalgebra based on $\mathscr{M}$ with respect to an automorphism σ of order m. Then $\mathrm{Ctd}_{\Bbbk}(\mathscr{B}) \cong D$.*

Proof. From Remark 6.6, and the explicit constructions of automorphisms of $\mathscr{M}$ in Lemmas 6.1-6.3, we observe that σ satisfies either one of the following two conditions:

Condition I: $\sigma(\mathrm{L}) = \mathrm{L}$ and $\sigma(\mathrm{U}) = \mathrm{U}$.

Condition II: $\sigma(\mathrm{L}) = \mathrm{L} + \alpha\mathrm{U}$ for some $\alpha \in \Bbbk$ and $\sigma(\mathrm{U}) = -\mathrm{U}$.

If σ satisfies Condition I, then L is a Virasoro element in $\mathscr{M}$ which is fixed by σ. Moreover, $\mathscr{M}_{\bar{0}}$ is a free $\Bbbk[\partial]$–module on the basis $\{\mathrm{L}, \mathrm{T}^{\pm i}, \mathrm{U} | i = 1, 2, 3\}$, and each of $\mathrm{T}^{\pm i}$ or U has primary weight 1 with respect to L. $\mathscr{M}_{\bar{1}}$ is also a free $\Bbbk[\partial]$–module with the basis $\{\mathrm{G}^p, \mathrm{Q}^p | p = 1, 2, 3, 4\}$. Each G^p has primary weight $\frac{3}{2} \neq 0$, and each Q^p has primary weight $\frac{1}{2} \neq 0$. By Proposition 2.12, $\mathrm{Ctd}_{\Bbbk}(\mathscr{B}) \cong D$.

If σ satisfies Condition II, we replace L by $\mathrm{L}_\sigma := \mathrm{L} + \frac{\alpha}{2}\mathrm{U}$ and replace G^p by $\mathrm{G}^p_\sigma := \mathrm{G}^p + \frac{\alpha}{2}\mathrm{Q}^p$ for $p = 1, 2, 3, 4$. Then $\mathscr{M}$ is also a $\Bbbk[\partial]$–module generated by $\mathrm{L}_\sigma, \mathrm{T}^{\pm i}, \mathrm{U}, \mathrm{G}^p_\sigma, \mathrm{Q}^p, i = 1, 2, 3, p = 1, 2, 3, 4$, and in $\mathscr{M}$, we have

$$[\mathrm{L}_{\sigma\lambda}\mathrm{L}_\sigma] = (\partial + 2\lambda)\mathrm{L}_\sigma, \quad [\mathrm{L}_{\sigma\lambda}\mathrm{T}^{\pm i}] = (\partial + \lambda)\mathrm{T}^{\pm i}, \quad [\mathrm{L}_{\sigma\lambda}\mathrm{U}] = (\partial + \lambda)\mathrm{U},$$
$$[\mathrm{L}_{\sigma\lambda}\mathrm{G}^p_\sigma] = (\partial + \frac{3}{2}\lambda)\mathrm{G}^p_\sigma, \quad [\mathrm{L}_{\sigma\lambda}\mathrm{Q}^p] = (\partial + \frac{1}{2}\lambda)\mathrm{Q}^p.$$

Then $\mathrm{L}_\sigma, \mathrm{T}^{\pm i}, i = 1, 2, 3, \mathrm{U}, \mathrm{G}^p_\sigma, \mathrm{Q}^p, p = 1, 2, 3, 4$ also give a set of generators of $\mathscr{M}$ as $\Bbbk[\partial]$–module and they satisfy all assumptions in Proposition 2.12. Hence, $\mathrm{Ctd}_{\Bbbk}(\mathscr{B}) = D$. □

Theorem 6.9. *Every twisted loop conformal superalgebra based on the large $N = 4$ conformal superalgebra $\mathscr{M}$ is isomorphic to either $\mathcal{L}(\mathscr{M}, \mathrm{id})$ or $\mathcal{L}(\mathscr{M}, \omega)$ as conformal superalgebras over $\Bbbk$.*

Proof. As pointed out in Section 2.2 that every twisted loop conformal superalgebra $\mathcal{L}(\mathscr{M},\sigma)$ based on $\mathscr{M}$ is a $\widehat{\mathcal{D}}/\mathcal{D}$–form of $\mathcal{L}(\mathscr{M},\mathrm{id}) = \mathscr{M}_{\mathcal{D}}$ (Proposition 2.8). By Proposition 6.7, there are only two $\widehat{\mathcal{D}}/\mathcal{D}$–forms of $\mathscr{M}_{\mathcal{D}}$ up to isomorphisms of differential conformal superalgebras over $\mathcal{D}$. They are $\mathcal{L}(\mathscr{M},\mathrm{id})$ and $\mathcal{L}(\mathscr{M},\omega)$.

From Proposition 2.10 and 6.8, we deduce that every twisted loop conformal superalgebra based on $\mathscr{M}$ is isomorphic to either $\mathcal{L}(\mathscr{M},\mathrm{id})$ or $\mathcal{L}(\mathscr{M},\omega)$ as conformal superalgebras over $\Bbbk$. □

6.4 The corresponding twisted Lie superalgebras

As we have seen in Section 2.1, every twisted loop conformal superalgebra $\mathcal{L}(\mathscr{M},\sigma)$ based on $\mathscr{M}$ determines an infinite dimensional Lie superalgebra $\mathrm{Alg}(\mathscr{M},\sigma)$. In particular, the untwisted loop conformal superalgebra $\mathcal{L}(\mathscr{M},\mathrm{id})$ yields the Lie superalgebra $\mathfrak{g}$ described in Section 6.1, which is the centreless core of the large $N = 4$ superconformal algebras created in [STVP88].

There is another twisted loop conformal superalgebra $\mathcal{L}(\mathscr{M},\omega)$ not isomorphic to $\mathcal{L}(\mathscr{M},\mathrm{id})$. $\mathcal{L}(\mathscr{M},\omega)$ gives rise to another Lie superalgebra $\mathrm{Alg}(\mathscr{M},\omega)$. In this section, we will explicitly state the generators and relations of $\mathrm{Alg}(\mathscr{M},\omega)$, and prove that it is not isomorphic to $\mathrm{Alg}(\mathscr{M},\mathrm{id})$ as Lie superalgebras over $\Bbbk$.

Recall that

$$\mathcal{L}(\mathscr{M},\omega) = \bigoplus_{\ell\in\mathbb{Z}} \mathscr{M}_\ell \otimes \Bbbk t^{\frac{\ell}{2}},$$

where $\mathscr{M}_\ell = \{a \in \mathscr{M} | \omega(a) = (-1)^\ell a\}$. It can be directly computed that

$$\mathscr{M}_\ell = \begin{cases} \Bbbk[\partial] \otimes_\Bbbk \mathrm{span}_\Bbbk\{\mathrm{L}, \mathrm{T}^{+i} + \mathrm{T}^{-i}, \mathrm{G}^i, \mathrm{Q}^4 | i = 1,2,3\}, & \text{if } \ell \text{ is even,} \\ \Bbbk[\partial] \otimes_\Bbbk \mathrm{span}_\Bbbk\{\mathrm{U}, \mathrm{T}^{+i} - \mathrm{T}^{-i}, \mathrm{G}^4, \mathrm{Q}^i | i = 1,2,3\}, & \text{if } \ell \text{ is odd.} \end{cases}$$

As a vector space, $\mathrm{Alg}(\mathscr{M},\omega) = \mathcal{L}(\mathscr{M},\omega)/\widehat{\partial}\mathcal{L}(\mathscr{M},\omega)$. We use $\bar{\eta}$ to denote the image of an element $\eta \in \mathcal{L}(\mathscr{M},\omega)$ in $\mathrm{Alg}(\mathscr{M},\omega)$. Then the following elements,

$$\begin{aligned} \mathrm{L}_m &= \overline{\mathrm{L} \otimes t^{m+1}}, & \mathrm{U}_{m'} &= \overline{\mathrm{U} \otimes t^{m'}}, \\ \mathrm{T}^i_m &= \overline{(\mathrm{T}^{+i} + \mathrm{T}^{-i}) \otimes t^m}, & \mathrm{J}^i_{m'} &= \overline{(\mathrm{T}^{+i} - \mathrm{T}^{-i}) \otimes t^{m'}}, \\ \mathrm{G}^i_{m'} &= \overline{\mathrm{G}^i \otimes t^{m'+\frac{1}{2}}}, & \Phi_m &= \overline{\mathrm{G}^4 \otimes t^{m+\frac{1}{2}}}, \\ \mathrm{Q}^i_m &= \overline{\mathrm{Q}^i \otimes t^{m-\frac{1}{2}}}, & \Psi_{m'} &= \overline{\mathrm{Q}^4 \otimes t^{m'-\frac{1}{2}}}, \end{aligned}$$

for $i = 1, 2, 3$, $m \in \mathbb{Z}$ and $m' \in \frac{1}{2} + \mathbb{Z}$, form a basis of $\mathrm{Alg}(\mathscr{M}, \sigma)$. The superbracket on $\mathrm{Alg}(\mathscr{M}, \omega)$ can be written as:

$$
\begin{aligned}
&[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n}, && [\mathrm{L}_m, \mathrm{U}_{n'}] = -n'\mathrm{U}_{m+n'},\\
&[\mathrm{L}_m, \mathrm{T}^i_n] = -n\mathrm{T}^i_n, && [\mathrm{L}_m, \mathrm{J}^i_{n'}] = -n'\mathrm{J}^i_{m+n'},\\
&[\mathrm{L}_m, \mathrm{G}^i_{n'}] = (\tfrac{1}{2}m - n')\mathrm{G}^i_{m+n'}, && [\mathrm{L}_m, \Phi_n] = (\tfrac{1}{2}m - n)\Phi_{m+n},\\
&[\mathrm{L}_m, \mathrm{Q}^i_n] = -(\tfrac{1}{2}m + n)\mathrm{G}^i_{m+n}, && [\mathrm{L}_m, \Psi_{n'}] = -(\tfrac{1}{2}m + n')\Psi_{m+n'},\\
&[\mathrm{U}_{m'}, \mathrm{U}_{n'}] = 0, && [\mathrm{U}_{m'}, \mathrm{T}^i_n] = [\mathrm{U}_{m'}, \mathrm{J}^i_{n'}] = 0,\\
&[\mathrm{U}_{m'}, \mathrm{G}^i_{n'}] = m'\mathrm{Q}^i_{m'+n'}, && [\mathrm{U}_{m'}, \Phi_n] = m'\Psi_{m'+n},\\
&[\mathrm{U}_{m'}, \mathrm{Q}^i_n] = 0, && [\mathrm{U}_{m'}, \Psi_{n'}] = 0,\\
&[\mathrm{T}^i_m, \mathrm{T}^j_n] = \epsilon_{ijk}\mathrm{T}^k_{m+n}, && [\mathrm{T}^i_m, \mathrm{J}^j_{n'}] = \epsilon_{ijk}\mathrm{J}^k_{m+n'},\\
&[\mathrm{T}^i_m, \mathrm{G}^j_{n'}] = \epsilon_{ijk}\mathrm{G}^k_{m+n'} - m\delta_{ij}\Psi_{m+n'}, && [\mathrm{T}^i_m, \Phi_n] = m\mathrm{Q}^i_{m+n},\\
&[\mathrm{T}^i_m, \mathrm{Q}^j_n] = \epsilon_{ijk}\mathrm{Q}^k_{m+n}, && [\mathrm{T}^i_m, \Psi_{n'}] = 0,\\
&[\mathrm{J}^i_{m'}, \mathrm{J}^j_{n'}] = \epsilon_{ijk}\mathrm{T}^k_{m'+n'}, && [\mathrm{J}^i_{m'}, \mathrm{Q}^j_n] = \delta_{ij}\Psi_{m'+n},\\
&[\mathrm{J}^i_{m'}, \Psi_{n'}] = -\mathrm{Q}^i_{m'+n'}, && [\mathrm{J}^i_{m'}, \Phi_n] = -\mathrm{G}^i_{m'+n},\\
&[\mathrm{J}^i_{m'}, \mathrm{G}^j_{n'}] = \delta_{ij}\Phi_{m'+n'} - m'\epsilon_{ijk}\mathrm{Q}^k_{m'+n'}, && [\mathrm{Q}^i_m, \mathrm{Q}^j_n] = 0\\
&[\mathrm{Q}^i_m, \Psi_{n'}] = 0 && [\Psi_{m'}, \Psi_{n'}] = 0,\\
&[\mathrm{Q}^i_m, \mathrm{G}^j_{n'}] = \delta_{ij}\mathrm{U}_{m+n'} + \epsilon_{ijk}\mathrm{J}^k_{m+n'}, && [\mathrm{Q}^i_m, \Phi_n] = \mathrm{T}^i_{m+n},\\
&[\Psi_{m'}, \mathrm{G}^i_{n'}] = -\mathrm{T}^i_{m'+n'}, && [\Psi_{m'}, \Phi_n] = \mathrm{U}_{m'+n},\\
&[\mathrm{G}^i_{m'}, \Phi_n] = (n - m')\mathrm{J}^i_{m'+n}, && [\Phi_m, \Phi_n] = 2\mathrm{L}_{m+n},\\
&[\mathrm{G}^i_{m'}, \mathrm{G}^j_{n'}] = 2\delta_{ij}\mathrm{L}_{m'+n'} - \epsilon_{ijk}(m' - n')\mathrm{T}^k_{m'+n'},
\end{aligned}
$$

for $i, j = 1, 2, 3$, $m, n \in \mathbb{Z}$ and $m', n' \in \frac{1}{2} + \mathbb{Z}$.

In fact, the twisted large $N = 4$ superconformal algebra described in [Van91] is isomorphic to a central extension of the Lie superalgebra $\mathrm{Alg}(\mathscr{M}, \omega)$.

Proposition 6.10. *The two Lie superalgebras* $\mathrm{Alg}(\mathscr{M}, \mathrm{id})$ *and* $\mathrm{Alg}(\mathscr{M}, \omega)$ *are not isomorphic.*

Proof. We will show that the two Lie superalgebras $\mathrm{Alg}(\mathscr{M}, \mathrm{id})$ and $\mathrm{Alg}(\mathscr{M}, \omega)$ indeed have non-isomorphic even parts $\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}}$ and $\mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}$.

Recall from Section 6.1 that

$$\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}} = \mathrm{span}_{\Bbbk}\{\mathrm{L}_m, \mathrm{T}_m^{\pm i}, \mathrm{U}_m | i = 1, 2, 3, m \in \mathbb{Z}\},$$

in which $\mathfrak{v} := \mathrm{span}_{\Bbbk}\{\mathrm{L}_m | m \in \mathbb{Z}\}$ is isomorphic to the centreless Virasoro algebra,

$$\mathfrak{s}_i := \mathrm{span}_{\Bbbk}\{\mathrm{T}_m^{+i}, \mathrm{T}_m^{-i} | m \in \mathbb{Z}\}, \quad i = 1, 2, 3, \text{ and } \mathfrak{s}_0 := \mathrm{span}_{\Bbbk}\{\mathrm{U}_m | m \in \mathbb{Z}\},$$

are all abelian Lie subalgebras. They satisfy

$$\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}} = \mathfrak{v} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \mathfrak{s}_3.$$

Similarly, we know that

$$\mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}} = \mathrm{span}_{\Bbbk}\{\mathrm{L}_m, \mathrm{T}_m^i, \mathrm{J}_{m'}^i, \mathrm{U}_{m'} | i = 1, 2, 3, m \in \mathbb{Z}, m' \in \tfrac{1}{2} + \mathbb{Z}\},$$

in which $\mathfrak{b} := \mathrm{span}_{\Bbbk}\{\mathrm{T}_m^i, \mathrm{J}_{m'}^i, \mathrm{U}_{m'} | i = 1, 2, 3, m \in \mathbb{Z}, m' \in \frac{1}{2} + \mathbb{Z}\}$ is an ideal.

Suppose $\phi : \mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}} \to \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}$ is an isomorphism of Lie algebras. We consider the composition

$$\bar{\phi} : \mathfrak{v} \hookrightarrow \mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}} \xrightarrow{\phi} \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}} \twoheadrightarrow \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}/\mathfrak{b} \cong \mathfrak{v},$$

which is an endomorphism of the centreless Virasoro algebra. By Lemma 4.15 (ii), either $\bar{\phi} = 0$ or $\bar{\phi}$ is injective.

If $\bar{\phi} = 0$, $\phi(\mathfrak{v}) \subseteq \mathfrak{b}$. For each $i = 0, 1, 2, 3$, we consider

$$\phi_i : \mathfrak{s}_i \hookrightarrow \mathrm{Alg}(\mathscr{M}, \mathrm{id}) \xrightarrow{\phi} \mathrm{Alg}(\mathscr{M}, \omega) \twoheadrightarrow \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}/\mathfrak{b} \cong \mathfrak{v},$$

which is a homomorphism from an abelian Lie algebra into the centreless Virasoro algebra. By Lemma 4.15 (i), the image of ϕ_i has dimension at most one, i.e.,

$$\phi(\mathfrak{s}_i) \subseteq \Bbbk x_i + \mathfrak{b}$$

for some $x_i \in \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}$. It follows that

$$\phi(\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}}) \subseteq \phi(\mathfrak{v}) + \phi(\mathfrak{s}_0) + \phi(\mathfrak{s}_1) + \phi(\mathfrak{s}_2) + \phi(\mathfrak{s}_3) \subseteq \Bbbk x_0 + \Bbbk x_1 + \Bbbk x_2 + \Bbbk x_3 + \mathfrak{b},$$

which contradicts the fact that ϕ is an isomorphism.

Hence, we conclude that $\bar{\phi}$ is injective. It follows from Lemma 4.15 that $\bar{\phi}(\mathrm{L}_0) = \frac{1}{\ell}\mathrm{L}_0$ for some nonzero integer ℓ, i.e., $\phi(\mathrm{L}_0) = \frac{1}{\ell}\mathrm{L}_0 + x$ for some $x \in \mathfrak{b}$. Note that $\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}}$ has a $\Bbbk$–basis $\{\mathrm{L}_m, \mathrm{T}^{\pm i}_m, \mathrm{U}_m | i = 1, 2, 3, m \in \mathbb{Z}\}$, and

$$[\mathrm{L}_0, \mathrm{L}_m] = -m\mathrm{L}_m, \quad [\mathrm{L}_0, \mathrm{T}^{\pm i}_m] = -m\mathrm{T}^{\pm i}_m, \quad [\mathrm{L}_0, \mathrm{U}_m] = -m\mathrm{U}_m.$$

We deduce that if $z \in \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}} = \phi(\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}})$ such that $[\phi(\mathrm{L}_0), z] = az$ for some $a \in \Bbbk$, then a is an integer. However, for $\mathrm{U}_{\frac{1}{2}} \in \mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}$, we have

$$[\phi(\mathrm{L}_0), \mathrm{U}_{\frac{1}{2}}] = [\tfrac{1}{\ell}\mathrm{L}_0 + x, \mathrm{U}_{\frac{1}{2}}] = -\tfrac{1}{2\ell}\mathrm{U}_{\frac{1}{2}},$$

which yields a contradiction. Hence, $\mathrm{Alg}(\mathscr{M}, \mathrm{id})_{\bar{0}}$ is not isomorphic to $\mathrm{Alg}(\mathscr{M}, \omega)_{\bar{0}}$. □

Remark 6.11. Let $\mathscr{M}(\gamma)$ be the conformal superalgebra associated to $\mathfrak{g}(\gamma)$. From the relations

$$[\mathrm{L}_\lambda \mathrm{L}] = (\partial + 2\lambda)\mathrm{L} + \frac{1}{12}\lambda^3 c, \text{ and } [\mathrm{L}_\lambda \mathrm{U}] = (\partial + \lambda)\mathrm{U} - \frac{1}{3}\left(\gamma - \frac{1}{2}\right)\lambda^2 c,$$

in $\mathscr{M}(\gamma)$, we observe that the automorphisms τ_s with $s \in \Bbbk$ as defined in Lemma 6.2 and ω of $\mathscr{M}$ created in Lemma 6.3 can not be lifted to an automorphism of $\mathscr{M}(\gamma)$ if $\gamma \neq \frac{1}{2}$. This would seem to justify the absence of one-dimensional central extensions of $\mathrm{Alg}(\mathscr{M}, \omega)$ in [STVP88] when $\gamma \neq \frac{1}{2}$.

In contrast, both the automorphism τ_s and ω of $\mathscr{M}$ can be lifted to automorphisms $\hat{\tau}_s$ and $\hat{\omega}$ of $\mathscr{M}(\frac{1}{2})$. The action of $\hat{\tau}_s$ and $\hat{\omega}$ on $\mathscr{M}(\frac{1}{2})$ is explicitly given by:

$\hat{\tau}_s(\mathrm{L}) = \mathrm{L} + s\mathrm{U} - \frac{s^2}{6}c,$	$\hat{\tau}_f(\mathrm{U}) = \mathrm{U} - \frac{s}{3}c,$	$\hat{\tau}_s(\mathrm{T}^{\pm i}) = \mathrm{T}^{\pm i},$
$\hat{\tau}_f(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\mathbf{u}) + \mathrm{Q}(s\mathbf{u}),$	$\hat{\tau}_s(\mathrm{Q}(\mathbf{u})) = \mathrm{Q}(\mathbf{u}),$	$\hat{\tau}_s(c) = c$
$\hat{\omega}(\mathrm{L}) = \mathrm{L},$	$\hat{\omega}(\mathrm{U}) = -\mathrm{U},$	$\hat{\omega}(\mathrm{T}^{\pm i}) = \mathrm{T}^{\mp i},$
$\hat{\omega}(\mathrm{G}(\mathbf{u})) = \mathrm{G}(\mathbf{u}^\dagger),$	$\hat{\omega}(\mathrm{Q}) = -\mathrm{Q}(\mathbf{u}^\dagger),$	$\hat{\omega}(c) = c,$

for $i = 1, 2, 3, \mathbf{u} \in \mathrm{Mat}_2(\Bbbk)$.

There is a natural (injective) group homomorphism from $\mathrm{Aut}_{\Bbbk\text{-conf}}\big(\mathscr{M}(\gamma)\big) \to \mathrm{Aut}\big(\mathfrak{g}(\gamma)\big)$. The above considerations applied to the case $\gamma = \frac{1}{2}$ show that the group of automorphisms of $\mathfrak{g}(\frac{1}{2})$ is indeed larger than the one described in the physics literature.

Chapter 7

Conclusion

The main ingredients of this dissertation are the automorphism group functors and the classification of the twisted loop conformal superalgebras based on each of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras over $\Bbbk$, which are the conformal superalgebras of particular interest in theoretical physics.

For the $N = 1, 2, 3$ conformal superalgebra $\mathscr{K}_N$ over $\Bbbk$, we have completely determined the automorphism group of the $\mathcal{R}$–conformal superalgebra $\mathscr{K}_N \otimes_{\Bbbk} \mathcal{R}$ for an arbitrary $\Bbbk$–differential ring $\mathcal{R} = (R, \mathsf{d})$ with R an integral domain (see Theorem 4.9). We did the same for the small $N = 4$ conformal superalgebra $\mathscr{W}$ with the additional assumption that $\mathrm{H}^2_{\text{ét}}(R, \boldsymbol{\mu}_2)$ is trivial (see Corollary 5.8). On the one hand, these results allow us to observe relationships between the automorphism group functor $\mathbf{Aut}(\mathscr{A})$ and certain affine group schemes (see Theorem 4.9, 5.7, and Proposition 5.10) for $\mathscr{A} = \mathscr{K}_N$ with $N = 1, 2, 3$ or $\mathscr{W}$. Such relations are analogies of the representability of the automorphism group functors of usual finite dimensional algebras and motivate us to further investigate such relations in a general setting, such as for an arbitrary finite simple conformal superalgebra. On the other hand, specializing these results to the situation where $\mathcal{R} = \widehat{\mathcal{D}} = (\Bbbk[t^q | q \in \mathbb{Q}], \frac{d}{dt})$, we obtain the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A} \otimes_{\Bbbk} \widehat{\mathcal{D}})$, which is used in the classification of twisted loop conformal superalgebras. With the intent of classifying twisted loop conformal superalgebras, we also derived the automorphism group $\mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{M} \otimes_{\Bbbk} \widehat{\mathcal{D}})$ for the large $N = 4$ conformal superalgebra $\mathscr{M}$ (see Theorem 6.5).

All of the above results concerning automorphism groups were obtained by explicitly constructing all of the automorphisms. In order to write down these automorphisms, we made an appropriate choice of generators which simplified the defining relations for the small $N = 4$ and large $N = 4$ conformal superalgebras. In the small $N = 4$ case, such a choice simplifies the description of automorphisms of $\mathscr{W} \otimes_{\Bbbk} \widehat{\mathcal{D}}$ given in [KLP09] and enables us to consider the automorphisms of $\mathscr{W} \otimes_{\Bbbk} \mathcal{R}$ for an arbitrary $\Bbbk$–differential ring $\mathcal{R}$ in Chapter 5.

Using the results on automorphism groups, we completed the classification of the twisted loop conformal superalgebras based on $\mathscr{A}$ up to $\Bbbk$–linear isomorphism for each of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras $\mathscr{A}$ in Theorems 4.14, 5.11, and 6.9, respectively. The main idea behind the classification comes from the general theory of twisted forms of differential conformal superalgebras developed in [KLP09]: each twisted loop conformal superalgebra based on $\mathscr{A}$ is viewed as a $\widehat{\mathcal{D}}/\mathcal{D}$–twisted form based on $\mathscr{A} \otimes_{\Bbbk} \mathcal{D}$ and then they are classified (up to isomorphism of $\mathcal{D}$–conformal superalgebras) in terms of the non-abelian cohomology sets $\mathrm{H}^1_{\mathrm{ct}}\left(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{\mathcal{D}}\text{-conf}}(\mathscr{A} \otimes_{\Bbbk} \widehat{\mathcal{D}})\right)$.

Finally, we deduce the classification up to isomorphism over $\mathcal{D}$ to the classification up to isomorphism over $\Bbbk$ using the so-called "centroid trick", which involves justifying that the canonical map $D \to \mathrm{Ctd}_{\Bbbk}(\mathcal{L}(\mathscr{A}, \sigma))$ is a bijection for each twisted loop conformal superalgebra $\mathcal{L}(\mathscr{A}, \sigma)$. In order to prove this, we obtained a more general result about the centroid of a twisted loop conformal superalgebra satisfying certain properties (cf. Proposition 2.12).

The research presented in this dissertation is part of the increasingly active investigation of infinite dimensional Lie theory which makes use of non-abelian Galois cohomology and descent theory. Such methods are based on the viewpoint that a twisted affine Kac-Moody algebra (derived modulo its center) is a $\widehat{D}/D$–twisted form of $\mathfrak{g} \otimes_{\Bbbk} D$, for a finite dimensional split simple Lie algebra $\mathfrak{g}$ over $\Bbbk$. This viewpoint allows one to investigate twisted affine Kac-Moody algebras using descent theory. In the conformal setting, we also hope to use descent theory for the investigation of twisted loop conformal superalgebras in the near future.

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