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JACOBI THETA AND DEDEKIND ETA FUNCTION IDENTITES VIA GEOMETRIC LATTICE EQUIVALENCE

by

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Abstract

Geometrical lattice equivalences are used to generate over 100 new quadratic identities involving classical modular forms, Jacobi theta functions, $\theta_2, \theta_3, \theta_4$, and the Dedekind eta function η. Generalizations are examined and a seemingly new observation on the nature of η is noted.

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Chapter 1

Introduction

In essence, we use geometrical lattice equivalences to generate identities involving the Jacobi theta function

$$\theta_3(\tau) := \sum_{m\in\mathbb{Z}} q^{\frac{m^2}{2}}$$

and the Dedekind eta function

$$\eta(\tau) := q^{1/24} \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{3m^2+m}{2}} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i\tau}$.

There are many reasons to be interested in the Jacobi theta functions and their cousin, the Dedekind eta function, so for motivational reasons we will mention a few. And of course once we convince ourselves that these functions are interesting, then it's almost automatic to find identities amongst them meaningful and useful.

We know that Ramanujan spent a significant part of his career studying these functions and generating identities amongst them. For the most part he used "only" his insightful genius to generate them, so it is quite exciting to be able to generate identities in a conceptual framework in an algorithmic fashion.

Theta functions find applications across mathematics in elliptic functions (see [12]), the theory of modular and Jacobi forms (see [23],[25]), analytic number theory (see [24]), the study of Riemann surfaces (see [3]), and the representation of affine Lie algebras (see [26]). They arise in physics as partition functions of strings (see [4]) and two-dimensional conformal field theories (see [27]).

Perhaps I'm most fascinated by them for the fact that (as you will see below in chapter 5 conclusions) their definition do not appear to be contrived and could quite "naturally" follow from some inductive exploration and yet they have the power to connect seemingly disparate areas of mathematics.

Interestingly enough lots of identities involving modular forms are floating around in the literature and are claimed to be discovered as new, but it appears that most of them are equivalent in the sense we describe below, because of their modular symmetry properties. The authors of [GL1] found hundreds of quadratic identities involving only θ_3, scattered throughout papers and books published before 1992; all can be derived (in the sense explained below in Sec. IV) from three identities in [GL1].

The lattice method of [GL1] recovered these three identities, as well as over 20 other independent ones, and is conjectured in [GL1] to give the complete list of quadratic θ_3 identities, up to equivalence.

Here, we extend their lattice method to identities involving both θ_3 and η. We also give a far-reaching generalization. We will focus for concreteness on quadratic identities but our method works for identities of any degree.

One of the particularly interesting findings in this thesis is the observation we made on the nature of η belonging to the field of fractions of T_3. This is quite unexpected and came out purely from examining some of the newly generated identities and matching and connecting them in various ways to older known/historic identities.

Chapter 2

Modular Forms Background

2.A The intuitive idea

Since the main objects of our study are modular forms, namely the Dedekind eta and Jacobi theta functions, we provide the necessary background for the exposition to follow. We will also try to provide an overview of why modular forms in their own right are a fruitful area of mathematics to study. (Why are modular forms and functions significant as connecting blocks for the whole body of mathematics? There are multiple ways to look at this question and each has its own flavour of benefits.) We will motivate our exploration with an example, we hope to learn from and generalize. In modern mathematics, still lots of objects are defined using the "Zermelo-Frankel" (ZF) axioms and hence need to be describable in terms of some sets and set theoretical language. It is of course way beyond the scope of this thesis to explore what, if anything is being sacrificed and what is gained by this assumption, nevertheless it is an interesting question to pose, for now let's just say that in some sense categories provide a broader context. This brings me to my first point.

Before we study something, we have a context usually in mind for it, or an approach to it, which affects profoundly the questions and answers our subsequent theory will generate. These assumption are of course implicit in the way we define our objects. We should celebrate these different approaches to a given object, as mathematical objects are very intricate and multifaceted.

Take the unit circle, for example, intuitively relatable by most humans, however in mathematics we need to ask ourselves how we ought to describe it. Of course, this is a crucial step as it sets the stage for what is to come, in the sense that what follows may be restricted by this very point of view, from the point of definition of our object. Sometimes the point of view we take will prefer analysis that is more topological, sometimes more algebraic, geometrical or analytical (etc.) in nature.

We start by saying that a circle is just *the set of points in the plane equidistant from a given/chosen point, the centre of the circle.* Assuming the background and tools of coordinate geometry, we express and hence translate this idea into the following definition.

$$C_1 := \{(x, y) \in \mathbb{R}^2 \| x^2 + y^2 = 1\}$$

(the "unit" circle).

Now, someone may come along and say that to her, an algebraic geometer, the circle is identified as the field

$$\mathbb{C}[x, y]/(x^2 + y^2 - 1)$$

the (polynomial) functions living on the circle. We could even take this one step further and refer to the circle by just a "polynumber", an interpretation that allows for a very broad and in a sense very pure look at the underlying object, a view that strips away, if you will, the sometimes unnecessary extra information implied by a specific contextual setting. Still again, perhaps to an algebraist, the circle is formed when we quotient out the real numbers by the integers, i.e.:

$$S^1 := \mathbb{R}/\mathbb{Z}.$$

For me, there is something very appealing in just how concise and elegant looking this definition is. Especially when we realize the particular generalizability of this idea as we explain below.

We take the real line and intuitively say: we wish to identify two points to be "essentially the same" if they differ by some integer. In other words, the elements of our new set under consideration will be all the distinct subsets of the reals whose elements differ by some integers. As we look at the idea of this construction, it still somewhat obscures it's generalizability, until we notice the following. The integers $\mathbb{Z}$, considered as a group, acts on the real numbers $\mathbb{R}$, considered as a space, by translation. i.e.: for $z \in \mathbb{Z}$ and $x \in \mathbb{R}$ we have:

$$z \circ x := x + z \in \mathbb{R}.$$

So now, if we consider the equivalence classes (also called orbits) formed by this action, we have for $x_1, x_2 \in \mathbb{R}$ that

$$x_1 \sim x_2 \iff x_2 = z \circ x_1 = x_1 + z \text{ for some } z \in \mathbb{Z}$$

i.e. $x_2 \equiv x_1 \pmod 1$. So, essentially, we just take the real interval $[0, 1]$ (a fundamental domain for this action) and wrap it up on itself by gluing the endpoints to each other.

At this point, some will wonder why and how exactly is this point of view of the circle "better" then the one rooted more in the coordinate geometrical point of view? Of course, "better" only makes sense once we fix what we want to do with the object.

Well, at the end of the day, most of us comprehend the meaning of an object using sets with structure. And a great way to explore that structure, is through functions.

But, what does this "understanding" really mean and how is it done in practice?

We could say that to a large extent mathematicians are "sorting machines". There is an infinitude of "patterns" out there and the job is to notice the ones pervasive enough to deserve their names, and then, if other ones differ "non-essentially" from the originally observed one, we consider them equivalents. We do this in practice until we discover a new pattern, that does indeed differ in its "essential" aspect from the original pattern, hence granting the introduction of a new name and a new definition for it! Two quotes come into mind, the first by Gottfried Leibniz: "The work of thought is marvellously simplified, if we adjust our definitions to our discoveries" and the second by Henry Poincaré that "Mathematics is the art of giving the same name to different things." If I may humbly add, it's often beneficial to consider the "dual view" whereby mathematics is sometimes also about giving different "names" to the same things! Meaning that sometimes it's beneficial to try and comprehend or contextualize an "intuitive" idea, like the unit circle, in as many ways, from as many angles as possible, realizing the wonderful diversity of underlying meanings each perspective has to offer. It is often tempting for humans to want to find "the right view", the "essential nature" of things set in stone, with a well-defined meaning. Whereas I think it is beneficial to open our minds to parallel interpretations, maybe even paradoxical at times, (is light a particle or wave? is matter continuous or discrete? is everything quantum mechanical or relativistic or singing the song of a string?) and appreciate the beauty of the untameable universe, and the utility of thinking about "meaning" in a more open minded fashion! Quite possibly at times notions like discrete and continuous, for example, should not be looked upon as mutually exclusive properties of things, but rather in a complimentary fashion.

To help make things precise, within each category, there are isomorphisms amongst our objects to do the job of the "essential" sorting. The question is just how to do this, how do we "probe" mathematical objects to reveal their "essential identity"? Well, of course this is an age old question, so it usually helps to put things in some historical perspective.

If, for example, we look at the pre-20th century approach to geometry, this above process of comprehension took practice by "probing" a manifold by studying curves on it, i.e.: Let M be a manifold and $\gamma : S^1 \to M$ a closed curve on M.

A great example of this way of thinking is the notion of the fundamental group, $\pi_1(M)$ of a topological space, that distinguishes spaces by how curves can live on them and do their "dance" by continuously deforming into one another; different "dance floors" (topologies) will allow different dances of the curves. Of course this is made precise by looking at closed curves up to their homotopy classes, but I digress.

The, shall we say, more "modern" approach is the "dual view", where we comprehend manifolds by the kinds of functions that, so to speak live "on" them by using them as their domains of definition, rather than functions that map to them!

If $f : M \to \mathbb{R}$ (or $\mathbb{C}$ or any ring R), then by studying these functions on M, we can use the algebraic structure of our ring R, to induce an algebraic structure on the set of functions living on M by adding and multiplying their images in the ring R:

$$(f + g)(x) := f(x) + g(x)$$

and

$$(f * g)(x) := f(x) * g(x) \in R.$$

Whatever "extra" structure M may possess, say differentiability, conformality, smoothness, we can request our mapping to respect that extra structure and be differentiable, conformal, and smooth etc.

Ok, so taking this "modern idea", how do we apply it to the "comprehension" of our object, the circle? How do we define a function that lives on the 1-D manifold, the unit circle?

If we used the "coordinate geometrical definition" of the circle, we could end up with having to define two charts, for example.

Our second way of defining the circle now comes in very handy. A function, continuous or smooth etc., living on the circle $S^1 = \mathbb{R}/\mathbb{Z}$, is simply the one living on $\mathbb{R}$ with the extra structure, that it's going to be invariant under translations by $\mathbb{Z}$.

(Note here, of course, we are not saying this is "better", but only saying it's different and therefore exposes potentially different aspects of our object. As a matter of fact it would be equally interesting if the same aspects would turn out to be "isomorphically represented" from several distinct approaches.)

Now, this is an idea that very nicely generalizes! What we are really saying is: take what's called a universal cover for the geometrical object (a simply connected covering space of a space X, is a space C such that there is a continuous map $p : C \to X$ which is a local homeomorphism onto it's image with trivial fundamental group) we would like to define, and take some symmetry subgroup of this universal cover (note that this subgroup will be isomorphic to π_1 of the geometrical object) and use the quotient of the universal cover by the group action.

In the case of the circle we note that it is a connected, real, one dimensional topological space. However, up to homeomorphism we know that $\mathbb{R}$ and S^1 are the only possibilities and the only simply connected real one dimensional topological manifold is $\mathbb{R}$. This means that $\mathbb{R}$ is the universal cover for $S^1 = \mathbb{R}/\mathbb{Z}$. S^1 locally looks like $\mathbb{R}$, but its global structure is encapsulated by $\pi_1(S^1) = \mathbb{Z}$.

So, at this point we ask what is the role of $\mathbb{Z}$ here? Well, $\mathbb{Z}$ is a subgroup of $\text{Aut}(\mathbb{R})$, the automorphism group of $\mathbb{R}$, which we know are translations.This is, of course, once we fix what aspects of $\mathbb{R}$ we are interested in preserving, in this case I'm thinking of $\mathbb{R}$ as an oriented geometrical space.

So, going back to our idea of studying spaces via functions living on them, so far we have for connected real one dimensional manifolds:

(i) $M = \mathbb{R}$ and $F(\mathbb{R})$ =all differentiable functions $\mathbb{R} \to \mathbb{R}$

(ii) $M = S^1 = \mathbb{R}/\mathbb{Z}$ and $F(S^1)$ =all differentiable functions living on $S^1 : S^1 \to \mathbb{R}$= all differentiable functions living on $\mathbb{R}$ with symmetry $f(x + n) = f(x)$ for $\forall n \in \mathbb{Z}$.

So, we can now extend this idea further by looking at "connected, real 2 dimensional topological manifolds", better known as surfaces. We will focus on conformal structure, because of the implied complex differential structure. That is, a real surface we will regard as a complex curve, since we learn that complex numbers are more fundamental and beautiful and better behaved than the reals. Complex differential structure can be reinterpreted as conformal structure in $\mathbb{R}^2$, if one prefers. After all, a locally invertible complex-differentiable map is conformal, i.e, it preserves angles but not usually lengths. So from now on when we say surface, we think of it as a complex differentiable curve or, what is essentially the same thing, a real surface with conformal structure.

Now, using the same recipe as above, the idea is to choose the underlying geometry as simple as possible by using a simply connected universal cover and factoring off some symmetry. Up to conformal equivalence, there are uncountably many different surfaces. But they can be gathered into families by their topological structure. We know that up to homeomorphism, compact real surfaces are S^2 (genus$=0$) ; torus (genus$=1$); tori (of all genus$=g$) and non-compact real surfaces are $\mathbb{R}^2$ (sphere with one point removed); cylinder (sphere with 2 points removed); and more generally any compact surface with any finite number of points removed.

However for universal covers we need the simply connected ones only! Fortunately we know these by Poincaré's Uniformization Theorem, which we now state.

Theorem 2.1: ***(a)*** *Up to conformal equivalence, the simply connected conformal real surfaces are :*

(i) $S^2 = P^1(\mathbb{C})$

(ii) $\mathbb{C}$

(iii) $\mathbb{H}$

(b) *Any conformal real surface* Σ *is conformally equivalent to* $\tilde{\Sigma}/\Gamma$ *where* $\tilde{\Sigma}$ *is the universal cover for* Σ *and* Γ *is a discrete subgroup of* $Aut(\tilde{\Sigma})$. *Moreover,* $\tilde{\Sigma_1}/\Gamma_1$ *is conformally equivalent to* $\tilde{\Sigma_2}/\Gamma_2 \iff \tilde{\Sigma_1} = \tilde{\Sigma_2}$ *and* Γ_1 *is conjugate to* Γ_2. *(i.e. Conjugate, meaning: there exists some fixed* $g \in Aut(\tilde{\Sigma})$, *such that* $g\Gamma_1 g^{-1} = \Gamma_2$.*)*

This is very powerful, as it says that not only is our way of defining the circle very elegant to have dealt with the "parametrization" issue at hand, but it's fully extensible to all surfaces!

Now, all we need to record is the automorphism groups of the above three universal covers to cover all cases! Here they are:

(i) Aut($P^1(\mathbb{C})$) $= PSL_2(\mathbb{C})$ with action: $z \mapsto \frac{az+b}{cz+d}$

(ii) Aut($\mathbb{C}$) $= \langle$translations,rotations$\rangle$

(iii) Aut($\mathbb{H}$) $= PSL_2(\mathbb{R})$ with action $\tau \mapsto \frac{az+b}{cz+d}$

Now, if we recorded up to number of punctures $= n$ and genus $= g$ the universal covers for surfaces, it becomes suddenly very apparent why we are interested in studying $\mathbb{H}$! Only $(g, n) = (0, 0)$ (the sphere), $(1, 0)$ (the torus), and some $(0, 1)$ (the plane) and $(0, 2)$ (some cylinders) have universal cover P^1 or $\mathbb{C}$!; for all other (g, n), the universal cover is $\mathbb{H}$! The generic geometry in 2-dimensions is nonEuclidean. The moral of the story is then:

Real functions of a single real variable with infinitely many symmetries are periodic functions.

Complex analytic (or more generally meromorphic) functions of a single complex variable with infinitely many symmetries are:

-periodic functions: functions living on the cylinder (singly-periodic, symmetry in one direction via $\mathbb{Z}$ such as e^z or $cos(z)$), or on the torus (doubly-periodic, with periods $\mathbb{Z}1+\mathbb{Z}\tau$, examples of such functions are ratios of elliptic functions).

-modular functions: functions living on a surface $\mathbb{H}/\Gamma$ (with symmetry Γ).

Note that more generally we consider not just analytic, but meromorphic functions here. There are many reasons for preferring meromorphic to analytic, one is that they form a field (instead of just a ring), another is that there are more of them, the only analytic functions on the torus, for example are constants. Also, as we'll explain shortly, we may need to add points to $\mathbb{H}$—he so-called cusps—to regain compactness. Excellent, so here is the generic pattern:

Functions that live on $\Sigma = \bar{\Sigma}/\Gamma$ are functions living on $\bar{\Sigma}$ with symmetry Γ i.e. we require the function to be invariant under the group action:

$$f(\gamma \circ z) = f(z)$$

for $\forall \gamma \in \Gamma$. Let's see a few examples of this:

Example 2.A.1: "the square torus" $\equiv \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with functions:

$$f(z + (m + in)) = f(z)$$

"sphere with 3 points removed" $\equiv \mathbb{H}/\Gamma(2)$ with functions:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

So, what all this finally allows us to do is to have properly contextualized the definition of modular functions and forms:

A modular function for Γ (a discrete subgroup of $PSL_2(\mathbb{R})$) is a meromorphic function (meromorphic at the cusps as well, to be defined later) that lives on the quotient surface $\mathbb{H}/\Gamma$.

On the other hand, a modular form for Γ of weight k, is a meromorphic differential $k/2$ form on $\mathbb{H}/\Gamma$.

We, now, will record all these definitions below in a compact form.

2.B Formal definitions

Let's define now a few mathematical objects that prepare us for the definition of modular forms.

Definition: Let $\mathbb{H} = \{\tau \in \mathbb{C} | \ \mathrm{Im}(\tau) > 0\}$, be the complex upper half-plane.

It's important here to note the map $q : \tau \mapsto e^{2\pi i \tau}$. This change of variables plays a crucial role in the theory of modular forms as we use it to define an analytic structure on $\mathbb{H} \cup \{i\infty\}$ about $i\infty$ (note that as $\tau \to i\infty, q \to 0$).

We can define a space D consisting of all *meromorphic functions* in the unit disc, that is, a function, $f \in D$ having a Laurent expansion at 0. We then say that a meromorphic function, $f : \mathbb{H} \to \mathbb{C}$ is *meromorphic at* $i\infty$, if there are a finite number N of rational numbers r_i and $f_i \in D$ such that $f(\tau) = \sum_{i=1}^{N} q^{r_i} f_i(q)$.

Let $\Gamma{:=}PSL_2(\mathbb{Z}) = \{\gamma \in M_2(\mathbb{Z}) | ad - bc = 1\}/\pm I$. Now that we have these two objects, $\mathbb{H}$ and Γ, we can ask, just how do these objects interact with each other?

If we take an element $\tau \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$ then it's a classical fact and straightforward to check that via:

$$\tau \mapsto \gamma \circ \tau = \frac{a\tau + b}{c\tau + d}$$

we have a group action by Γ on the upper half-plane $\mathbb{H}$ by what's called a fractional linear transformation.

By a group action we mean: given a group G and a set S, we say that G acts on S, if we have a function $A : G \times S \to S$ with the following properties:

(i) for $\forall g_1, g_2 \in G$ and $s \in S$ we have $A(g_1 \circ g_2, s) = A(g_1, A(g_2, s))$

(ii) for $e \in G$ (identity element of G) and for $\forall s \in S$ we have: $A(e, s) = s$

Note first, that the fractional linear transformation is certainly a well defined operation on $\mathbb{H}$ as $c\tau + d = 0 \Rightarrow \tau = \frac{-d}{c} \in \mathbb{R}$ which is avoided via $Im(\tau) > 0$. Also, since: $im(g \circ \tau) = \frac{Im(\tau)}{|c\tau+d|^2}$ the group action indeed maps $\mathbb{H}$ back to itself. We observe that as this group action is not transitive, when we quotient out by it, it divides $\mathbb{H}$ up into interesting equivalence classes, these are called the orbits of the action.

It is customary to choose a simply connected subset F within $\mathbb{H}$, such that each equivalence class under Γ is represented in F, but only uniquely. Such a subset F is called a *fundamental domain* for the group action.

We will briefly show that:

Lemma 2.1: *$F := \{\tau \in \mathbb{H} \| -1/2 \leq Re(\tau) < 1/2$ and $|\tau| > 1\} \cup \{|\tau| = 1 \| \pi/4 \leq Arg(\tau) \leq \pi/3\}$ is a fundamental domain for Γ*

Proof. (sketch) First, observe that $T = \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathit{\Gamma}$. We see that

$$T \circ \tau = \tau + 1$$

and

$$S \circ \tau = \frac{-1}{\tau}.$$

So first we note that we can choose any strip of width 1 as a consequence of the T action, so here we choose the strip:

$-1/2 \leq \tau \leq 1/2$ for $\tau \in \mathbb{H}$ (for historic reason and no other).

Now observe that if we choose a point τ in the above strip, the following happens when we apply the matrix S:

let $\tau = x + iy$ for $y > 0$ and $-1/2 \leq x < 1/2$, then:

$$S \circ (x + iy) = \frac{-1}{x + iy} = \frac{-(x - iy)}{x^2 + y^2} = \frac{-x + iy}{|\tau|^2}.$$

So, action by S essentially inverts τ with respect to the unit circle. Points that start out inside the unit circle are put outside and points outside are mapped inside the circle by S.

Ok, so far so good, but we have only really taken care of just what happens by a subgroup of Γ generated by the matrices S and T. We also may need to keep repeating this process as once we get inside the width 1 strip by T, we use S, if necessary to get outside circle, but this may move us back again, to outside of the vertical strip. So we may use T to move us back in, but then again we may be in the unit circle. So we use S again. This can be repeated for a long time, but the point is that it will eventually terminate. One way to see this is to let $\hat{\Gamma} = \langle S, T \rangle < \Gamma$ and take $\gamma \in \hat{\Gamma}$ and observe that, $\text{Im}(\gamma \cdot \tau) = \frac{\text{Im}(\tau)}{|c\tau+d|^2}$. As, $c, d \in \mathbb{Z}$, $|c\tau + d| > 0$ for c, d non-zero, so there is an open set around zero that contains no non-zero lattice points, implying that there is a $\gamma \in \hat{\Gamma}$ s.t. $\text{Im}(\gamma \cdot \tau)$ is maximal.

Now, by the help of the T action, raised to sufficient powers, if necessary, we can assume without loss of generality that our $\gamma \cdot \tau$ lies in the strip $-1/2 \leq \mathrm{Im}(\gamma \cdot \tau) < 1/2$. Now, if $\gamma \cdot \tau$ were not in F, then $|\gamma \cdot \tau| < 1$, but this contradicts the maximality assumption on $\mathrm{Im}(\gamma \cdot \tau)$ since :

$$\mathrm{Im}\left(\frac{\gamma \cdot \tau}{|\gamma \cdot \tau|^2}\right) > \mathrm{Im}(\gamma \cdot \tau).$$

We have to now show that we can't have two Γ equivalent points in F, other than trivially. So assume that $\exists \gamma \in \Gamma$ with $\gamma \cdot \tau_1 = \tau_2$ for $\tau_1, \tau_2 \in F$ with $Im(\tau_1) \leq Im(\tau_2)$. But this means that $|c\tau_1 + d| \leq 1$. However we see that this fails for $|c| \geq 2$. The only other cases are (i)$c = 0, d = \pm 1$, but then $\pm\gamma$ would be a translation and the only such translation is the identity within the interior of F; (ii) $c = \pm 1, d = 0$ with $\gamma = \pm T^k S$ with either $k = 0$ and τ_1, τ_2 on the unit circle, or $k = \pm 1$ and $\tau_1 = \tau_2 = \pm 1/2 + \sqrt{-3}/2$ or case (iii) with $c = d = \pm 1$ and $\tau_1 = -1/2 + \sqrt{-3}/2$ in which case $\gamma = \pm T^k \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, and either $k = 0$ and $\tau_1 = \tau_2 = -1/2 + \sqrt{-3}/1/2$ or $a = 1$ and $\tau_2 = \tau_1 + 1 = 1/2 + \sqrt{-3}/2$; or case (iv) with $c = -d = \pm 1$ and $\tau_1 = 1/2 + \sqrt{-3}/2$ and is handled similarly to case (iii). All these cases show that the only way $\tau_2 = \gamma \cdot \tau_2$ in the interior of F, if $\gamma = 1$ with $\tau_1 = \tau_2$, proving the claim.

Also, it remains to be shown that actually surprisingly these are indeed the only matrices we need to care about, since Γ is generated by them. This follows from this proof and the Lemma below if we notice that the only points in F with non-trivial stabilizers are $\tau = i$ with $\Gamma_i = \{I, S\}$; $\tau = \omega = -1/2 + \sqrt{-3}/2$ with $\Gamma_\omega = \{I, ST, (ST)^2\}$; and $\tau = -\bar{\omega}$ with $\Gamma_{-\bar{\omega}} = \pm\{I, TS, (TS)^2\}$; We call these points in $\mathbb{H}$ with non-trivial stabilizers *elliptic points*. (for full details see p.100 in [20]) □

Lemma 2.2: $\Gamma = SL_2(\mathbb{Z})$ *is generated by the matrices:*

$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

Proof. Let $\hat{\Gamma} = \langle S, T \rangle < \Gamma$ be a subgroup of Γ Let $\gamma \in \Gamma$. Let $\tau \in F$ in the interior of the fundamental domain. Then we have: $\gamma \circ \tau \in \mathbb{H}$; but we have already shown above that $\exists g \in \hat{\Gamma}$ such that

$$g \circ (\gamma \circ \tau) \in \mathrm{F}.$$

However, since we started off with $\tau \in$F and we have by the definition of a group action (note the faithful action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ as seen in proof of Lemma 2.1), that: $g \circ (\gamma \circ \tau) = (g \circ \gamma) \circ \tau \in \mathrm{F} \Rightarrow g \circ \gamma = \pm I$ and hence: $g = \pm\gamma^{-1} \in \hat{\Gamma}$ showing that since $g \in \Gamma$ was an arbitrary element, that each element of Γ is indeed an element of $\hat{\Gamma}$ so

$$\Gamma = \hat{\Gamma} = \langle S, T \rangle$$

as claimed.

Alternatively, and independently of Lemma 2.1. we can see this as follows: Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Take any matrix $\gamma \in \Gamma$. We can hit it on the right by some T^k, so that $|b|$ becomes bounded above by $|a|/2$. Now, hit it on the right by S, which basically exchanges a and b (and changes a sign, but that's not important). Now hit on the right again by some power $T^{k'}$, so that the new a and b then satisfy $|b| \leq |a|/2$, as before. Keep repeating, each time b gets closer and closer to 0. Eventually b equals 0. In that case, the matrix can be written as $\pm ST^cS^{-1}$. So the net result is that we can solve for the original γ, and it will involve lots of Ss and Ts, but nothing else. So this proves Lemma

2.2, without needing Lemma 2.1.

□

It is a crucial observation in the theory that $\mathbb{H}$ is "missing" some points. The ultimate reason is compactness: there are far more holomorphic and meromorphic functions on non-compact domains, than compact domains. For example, the only holomorphic functions on the Riemann sphere are the constants, but there are lots of holomorphic functions on the plane $\mathbb{C}$ (e.g. polynomials, $\exp(z)$,...). Now, in the last section we explained that we should think of a modular function as a meromorphic function on $\mathbb{H}/\Gamma$. But it is easily seen from our fundamental domain F that $\mathbb{H}/\Gamma$ is homeomorphic to the plane, i.e. a sphere with one point missing. This missing point is where the fundamental domain touches the boundary of $\mathbb{H}$. The boundary of $\mathbb{H}$ is the circle $\mathbb{R} \cup \{i\infty\}$, which we can clearly visualize under the exponential map. The fundamental domain touches this at the point $i\infty$. So we should really add $i\infty$ to $\mathbb{H}$! We in fact should add the whole Γ-orbit of $i\infty$ to $\mathbb{H}$, or we won't have a Γ action. (Note: The Γ-orbit of $i\infty$ consists of all rational numbers, together with $i\infty$.)

So define $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup i\infty$. These extra boundary points are called *cusps*. What all this means is that we require our modular forms and functions to also be meromorphic at the cusps. Meromorphic or holomorphic at other cusps just means that $f|_k\gamma$ is meromorphic or holomorphic (respectively) at $i\infty$, for all $\gamma \in \Gamma$. Then our forms and functions will really live on the sphere. The result will be a much richer theory. If we ignore the cusps, the theory of modular forms would be essentially that of meromorphic functions on the plane. Therefore including the cusps is the nicer, prettier, more restrictive option. Fortunately "Nature" agrees: the modular forms coming from number theory, algebra, geometry, physics,... all know about the cusps!

Definition: A *modular function*, f for $\Gamma = PSL_2(\mathbb{Z})$ is a meromorphic function, everywhere including cusps $f : \bar{\mathbb{H}} \to \mathbb{C}$ satisfying:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \text{ for } \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in \mathit{\Gamma}.$$

Definition: A *modular form*, f for $\Gamma = PSL_2(\mathbb{Z})$ of weight $k \in \mathbb{Z}_{\geq 0}$ and no multiplier, is a holomorphic function, everywhere, including cusps $f : \bar{\mathbb{H}} \to \mathbb{C}$ satisfying:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for } \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in \mathit{\Gamma}.$$

Observe that a modular form is not exactly invariant under the group action of the universal cover, the extra factors seemingly appear out of nowhere, but we see they are there exactly so that the modular function's differential form does become invariant under this same action.

Example 2.B.1: for weight $k = 2$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \mathrm{d}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)(c\tau+d)^{-2} d\tau = f(\tau)\mathrm{d}\tau$$

is a differential 1-form on $\mathbb{H}/\Gamma$.

$PSL_2(\mathbb{Z})$ is only one of the possible group of symmetries. Any discrete subgroup of $PSL_2(\mathbb{R})$ would also work, but most of these never seem to arise naturally. By far the most important ones are the so-called *congruence groups*, for reasons we'll give elsewhere. These are the discrete subgroups of $PSL_2(\mathbb{R})$

containing some $\Gamma(N)$ defined shortly. Most modular forms in this thesis (or indeed in mathematics) are not modular for $PSL_2(\mathbb{Z})$, but almost all will be modular for some congruence group.

Definition: $\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathit{\Gamma} \| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N\}$,where $N \in \mathbb{N}$.

Here we mention a few basic observations. Note the following:

Lemma 2.3: $\Phi : \Gamma/\Gamma(N) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ *is an isomorphism.*

Proof. Define $\rho : \Gamma \to SL_2(\mathbb{Z}/N\mathbb{Z})$ by $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mapsto \begin{pmatrix} a \pmod N & b \pmod N \\ c \pmod N & d \pmod N \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$ is a surjective homomorphism with kernel $\Gamma(N)$ and hence induces the isomorphism Φ via the First Isomorphism Theorem, via $\Phi : \bar{\gamma} = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \Gamma/\Gamma(N) \mapsto \begin{pmatrix} a \pmod N & b \pmod N \\ c \pmod N & d \pmod N \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$. □

Lemma 2.4: $\Gamma(N)$ *is normal in* Γ.

Proof. Follows from Lemma 2.3. □

In light of this, in the next section we define the main modular forms used throughout the exposition.

2.B.1 Jacobi theta functions and the Dedekind eta

Theta functions are quite ubiquitous in mathematics. They have this mysterious power to appear in seemingly distinct areas. Just to mention a few:

- the theory of elliptic functions,
- the theory of modular and Jacobi forms,
- analytic number theory,
- the study of Riemann surfaces,

-in the representation of affine Lie algebras,

-in physics: as the partition functions of strings and two dimensional conformal field theories.

The following are the definitions of the Jacobi theta functions we need for our exposition:

$$\theta_3(\tau) := \sum_{m\in\mathbb{Z}} q^{m^2/2},$$

$$\theta_2(\tau) := \sum_{m\in\mathbb{Z}} q^{(m+1/2)^2/2},$$

$$\theta_4(\tau) := \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2/2},$$

$$\eta(\tau) := q^{1/24} \sum_{m\in\mathbb{Z}} (-1)^m q^{(3m^2+m)/2} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

$$\psi_k(\tau) := \sum_{m\in\mathbb{Z}} q^{(m+1/k)^2/2} \text{ where } k \in \mathbb{Q} \text{ is nonzero,}$$

and $\psi_\infty := \theta_3$ is convenient to define.

Also note that:

$$\psi_k = \theta_3 \iff \frac{1}{k} \in \mathbb{Z},$$

$$\psi_k = \psi_l \iff \frac{1}{k} \pm \frac{1}{l} \in \mathbb{Z}$$

(hence we will allow $k \in \mathbb{Q}$ and $k \geq 2$).

Just to calm the analyst in all of us, let's observe that these series are indeed convergent, for all $\tau \in \mathbb{H}$. As a matter of fact for example for $\theta_3(\tau)$, we get uniform convergence by the Weierstrass M-test : for $\text{Im}(\tau) > \epsilon > 0$

$$\left|q^{n^2}\right| < \exp(-\pi\epsilon)^{n^2}.$$

Similarly we can observe this for the other theta-series. As a result we will define their domains to lie in the complex upper half-plane.

These functions, $\theta_2, \theta_3, \theta_4$ and η are modular forms of weight $k = 1/2$ for $\Gamma(2)$. In fact, η is modular for $PSL_2(\mathbb{Z})$ and θ_3 for

$$\Gamma_\theta := \langle S, T^2 \rangle.$$

$\Gamma(2)$ is a subgroup of $PSL_2(\mathbb{Z})$ of index 6, and of Γ_θ of index 2. The ψ_ks, and more generally $\theta_2(r\tau), \theta_3(r\tau), \theta_4(r\tau), \eta(r\tau)$ and $\psi_k(r\tau)$ for any positive rational number r and any rational k, are modular forms for some $\Gamma(N)$.

We have the following important linear relations:

$$\theta_2(\tau) = \psi_2(\tau) \tag{2.1}$$

$$\theta_4(\tau) = 2\theta_3(4\tau) - \theta_3(\tau) \tag{2.2}$$

$$\eta(\tau) = \psi_{12}(12\tau) - \psi_{12/5}(12\tau) \tag{2.3}$$

$$\sum_{l=1}^{k} \psi_{k/l}(\tau) = \theta_3(\tau/k^2) \tag{2.4}$$

$$\sum_{l=1}^{k} \psi_{(m/kn+l/k)^{-1}}(\tau) = \psi_{n/m}(\tau/k^2). \tag{2.5}$$

These can be readily verified by brute force. For example, to verify (2.2), we need $\theta_4(\tau) + \theta_3(\tau) = 2\theta_3(4\tau)$. So as we observe from the definition of θ_3 and of θ_4 :

$$\theta_3 = 1 + 2q^{1/2} + 2q^2 + 2q^{9/2} + 2q^8 + ...$$

$$\theta_4 = 1 - 2q^{1/2} + 2q^2 - 2q^{9/2} + 2q^8 + ...$$

We notice that upon summing them, the odd terms cancel out and we are left with what looks like the definition of θ_3 except under the mapping $q \mapsto q^4$ and doubled, i.e.:

$$\theta_4(\tau) + \theta_3(\tau) = 2 + 4q^2 + 4q^8 + ... = 2\big[1 + 2q^2 + 2q^8 + ...\big] = 2\theta_3(4\tau).$$

Another way to verify these identities comes from the following powerful theorem:

Theorem 2.2: *Let $f(\tau) = \sum_{n=0}^{\infty} f_n q^n$ and $g(\tau) = \sum_{n=0}^{\infty} g_n q^n$ be two modular forms for Γ of weight k. Then $f = g \iff f_n = g_n$ for all $n < k/12$.*

This is an immediate consequence of the valence formula (see [29]), which gives the degree of the divisor of a modular form. There is an analogous theorem if we replace Γ by other (Fuchsian) groups, such as the $\Gamma(N)$ discussed last section. All we need is the valence formula for our Fuchsian group, and this is given in complete generality in [29].

Now, since for this thesis we are interested in generating identities in the θ_3 and η, we will primarily concentrate on ψ_k that can be expressed as θ_3 or

η, i.e.:

$$\psi_1(\tau) = \theta_3(\tau) = \psi_\infty(\tau) \tag{2.6}$$

$$\psi_2(\tau) = \theta_2(\tau) = \theta_3\left(\frac{\tau}{4}\right) - \theta_3(\tau) \tag{2.7}$$

$$\psi_3(\tau) = \frac{1}{2}(\theta_3\left(\frac{\tau}{9}\right) - \theta_3(\tau)) \tag{2.8}$$

$$\psi_4(\tau) = \frac{1}{2}\theta_2\left(\frac{\tau}{4}\right) \tag{2.9}$$

$$\psi_6(\tau) = \frac{1}{2}(\theta_2\left(\frac{\tau}{9}\right) - \theta_2(\tau)) \tag{2.10}$$

$$\psi_{12}(\tau) = \frac{1}{4}(\theta_3\left(\frac{\tau}{144}\right) - \theta_3\left(\frac{\tau}{9}\right) - \theta_2\left(\frac{\tau}{9}\right) - \theta_2\left(\frac{\tau}{4}\right) + 2\eta\left(\frac{\tau}{12}\right))) \tag{2.11}$$

$$\psi_{12/5}(\tau) = \frac{1}{4}(\theta_3\left(\frac{\tau}{144}\right) - \theta_3\left(\frac{\tau}{9}\right) - \theta_2\left(\frac{\tau}{9}\right) - \theta_2\left(\frac{\tau}{4}\right) - 2\eta\left(\frac{\tau}{12}\right))). \tag{2.12}$$

These are consequences of the simpler identities defined above; for example: (2.7) is derivable from (2.1) and (2.4). We will refer to these equations defined so far as *linear relations*.

At this point it is important to note the following: Why are we concentrating our efforts to these ψ_k's? What is so special about them? Why are they important? Now, of course for different people there are different answers to this, but here is a rationale for us: Most mathematicians interested in the study of modular forms are concerned in particular with the ones that have integer coefficients in their so called q-expansions. Modular forms often arise in number theory, geometry, algebra, physics, combinatorics, etc... as generating functions for a sequence of dimensions or cardinalities. For example, we can write $f_k(\tau) := \sum a_n q^n$ where a_n is the number of ways to write n as a sum of positive integers. In this case what we end up with is famously a meromorphic modular form namely $q^{1/24}\eta^{-1}$.

Anyway, whatever the motivation may be, once we fix that we are interested in modular forms with integer coefficients we need to find a source for such functions! Now, even though modular forms are known to exist for complex weights (i.e. $k \in \mathbb{C}$) and $\Gamma = PSL_2(\mathbb{Z})$ replaced with an arbitrary Fuchsian group of the first kind, still half-integer weight ($k = \frac{1}{2}z$ for $z \in \mathbb{Z}$) modular forms for congruence groups are the most important of these, as they are conjectured to be the only ones with integer coefficients.

Fortunately, we know by the Serre-Stark Theorem (see [28]) that every modular form of half-integer weight, for any congruence group is a finite linear combination of $\psi_k(n\tau)$'s. This means that, the method given in this thesis applies, in theory at least, to all weight half modular forms for congruence groups. For us, since we are primarily interested in generating identities in the θ and η it is important to know which ψ_ks are expressible purely in θ_3 and η. This and an algebraic point of view in mind inspires us to define the following.

Let $T_3^{(n)}$ denote the $\mathbb{C}$-module (i.e. vector space) of functions:

$$\sum_{j=1}^{N} \alpha_j \theta_3(a_{1j}\tau) ... \theta_3(a_{nj}\tau),$$

where each $\alpha_j \in \mathbb{C}$ and $a_{ij} > 0 \in \mathbb{Q}$ (otherwise we would no longer find convergence in $\mathbb{H}$ and modularity would be lost). We will also define $T_3 = \sum_n T_3^{(n)}$. Notice from above that T_3 contains all of $\theta_3, \theta_2, \theta_4, \psi_k$ for $k = 1, 2, 3, 4, 6$.

We will define T_3^* to be the field of fractions of T_3. We will be interested in using lattices to find functions in T_3 which are identically zero. If the function lies in T_3^n it is said to be a degree n identity.

Since our functions θ_3, η and in general the ψ_ks are modular forms, we will utilize their invariance properties with respect to the fractional linear ("modular") transformations.

We have shown above in the modular forms background section in Lemma 2.2, that the entire modular group $\Gamma = PSL_2(\mathbb{Z})$ is generated by two matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ corresponding to } \tau \mapsto \tau + 1$$

and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ corresponding to } \tau \mapsto \frac{-1}{\tau}.$$

Therefore it is sufficient to record their modular invariance properties with respect to these two transformations for further reference.

Let Λ be a lattice (defined next chapter) and f be any rapidly decreasing smooth function on $V_0(\Lambda)$. Then, we have the Poisson Summation formula:

$$\sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{\sqrt{|\Lambda|}} \sum_{\lambda^* \in \Lambda^*} \hat{f}(\lambda^*)$$

with $\hat{f}$, the Fourier transform of f, where Λ^* is the dual lattice, also defined in the next chapter. Now, choosing $\Lambda = \mathbb{Z}$ and using the Fourier transform we get: $f(x) = \exp(-\pi x^2)$ is $\hat{f}(y) = \exp(-\pi y^2) \Rightarrow \theta_3(\frac{-1}{\tau}) = (\frac{t}{i})^{1/2}\theta_3(\tau)$ (see details in [24]).

This allows us to observe that under S we have the following relations for

n and k coprime integers, with $\zeta_n := e^{2\pi i/n}$:

$$\theta_2(\frac{-1}{\tau}) = (\frac{\tau}{i})^{1/2}\theta_4(\tau), \tag{2.13}$$

$$\theta_3(\frac{-1}{\tau}) = (\frac{\tau}{i})^{1/2}\theta_3(\tau), \tag{2.14}$$

$$\theta_4(\frac{-1}{\tau}) = (\frac{\tau}{i})^{1/2}\theta_2(\tau), \tag{2.15}$$

$$\eta(\frac{-1}{\tau}) = (\frac{\tau}{i})^{1/2}\eta(\tau), \tag{2.16}$$

$$\psi_{\frac{n}{k}}(\frac{-1}{\tau}) = (\frac{\tau}{i})^{1/2}\sum_{l=0}^{n-1}{\zeta_n}^{lk}\psi_{\frac{n}{l}}(n^2\tau). \tag{2.17}$$

Under T, and much more elementary, we have the following relations:

$$\theta_2(\tau+1) = \zeta_8\theta_2(\tau), \tag{2.18}$$

$$\theta_3(\tau+1) = \theta_4(\tau), \tag{2.19}$$

$$\theta_4(\tau+1) = \theta_3(\tau), \tag{2.20}$$

$$\eta(\tau+1) = \zeta_{24}\eta(\tau), \tag{2.21}$$

$$\psi_{\frac{n}{k}}(\tau+2n) = \psi_{\frac{n}{k}}(\tau). \tag{2.22}$$

Let's derive some of these, for example for $\theta_3(\tau)$ we have the following derivation: $\theta_3(\tau+1) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2(\tau+1)} = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}e^{\pi i n^2} = (-1)^n\theta_3(\tau)$ and hence:

$$\theta_3(\tau+2) = \theta_3(\tau),$$

as desired. Let's see another one, for $\eta(\tau)$ we have the following: recall $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ and hence: $\eta(\tau+1) = e^{\frac{2\pi i(\tau+1)}{24}}\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau+1)n}) = e^{\frac{2\pi i}{24}}e^{\frac{2\pi i\tau}{24}}\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau)n}e^{2\pi i n}) = \zeta_{24}e^{\frac{2\pi i\tau}{24}}\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau)n}1) = \zeta_{24}q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = \zeta_{24}\eta(\tau)$, as desired.

In Chapter 4, we utilize these identities in the simplification and sorting of identities up to equivalence.

Actually, for θ_3, η and $\psi_{\frac{k}{l}}$ we can find formulas, more generally for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$.

We know (see [30]) that this group is generated by: the matrix S from above and by $\tau \mapsto \tau + \frac{m}{n}$, therefore we record:

$$\theta_3\left(a(\tau + m/n)\right) = \theta_3(an^2\tau) + 2\sum_{l=1}^{n-1} cos(\frac{al^2 m\pi}{n})\psi_{2n/l}(4an^2\tau),$$

$$\eta(a(\tau + m/n)) = \sum_{l=0}^{2n-1} (-1)^l \zeta_{2n}^{am(3l^2+l+1/12)} \psi_{\frac{12n}{6l+1}}(12an^2\tau),$$

for $a > 0$. Now, of course we are interested in rational translations that result in giving us back θ_3 and η. The most complicated amongst these that are still of use for us is:

$$\theta_3(a(\tau+\frac{m}{6})) = c_1\theta_3(\tau)+(c_2-c_1)\theta_3(4a\tau)+(c_3-c_1)\theta_3(9a\tau)+(1+c_1-c_2-c_3)\theta_3(36a\tau),$$

where $c_1 = \cos(ma\pi/6)$, $c_2 = \cos(2ma\pi/3)$, $c_3 = \cos(3ma\pi/2)$. As a matter of fact, $\tau \mapsto \tau + k/12$ also sends $\theta_3(a\tau)$ into a linear combination of $\theta_3(a'\tau)$, but the formula gets really complicated and we won't use it.

There are also the following useful identities for $\psi_{k/\ell}$ where k and ℓ are coprime integers.

$$\psi_{k/\ell}(a\tau+a) = \begin{cases} \zeta_{4k}^{\ell^2\,(2a/k)}\psi_{k/\ell}(a\tau) & \text{if } \frac{k}{2} \text{ is odd and } k|2a \\ \zeta_{2k}^{\ell^2\,(a/k)}\psi_{k/\ell}(a\tau) & \text{if } 4|k \text{ and } k|a \\ \zeta_{k}^{\ell^2\,(a/2k)}\psi_{k/\ell}(a\tau) & \text{if } k \text{ is odd and } 2k|a \\ \psi_{2k/\ell}(4k\tau) - \psi_{2k/(\ell+k)}(4k\tau) & \text{if } k \text{ is odd and } a = k \end{cases}, \quad (2.23)$$

where we use the notation '$k|a$' for 'k divides a', etc.

There is a long history of the study of η identities, most of which tend to look quite complicated. Two of the simplest and oldest ones are

$$\eta(2\tau)\theta_4(2\tau) = \eta(\tau)^2, \tag{2.24}$$

$$\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) = 2\eta(\tau)^3. \tag{2.25}$$

In [16] are given nine polynomial identities for η, due to Ramanujan, ranging from degree 8 to degree 24. The simplest is

$$\begin{aligned}\eta(\tau)^3\eta(3\tau)^3\eta(7\tau)\eta(21\tau) + 7\eta(\tau)\eta(3\tau)\eta(7\tau)^3\eta(21\tau)^3 = \eta(3\tau)^4\eta(7\tau)^4 - 3\eta(\tau)^2\eta(3\tau)^2 \\ \eta(7\tau)^2\eta(21\tau)^2 \\ + \eta(\tau)^4\eta(21\tau)^4.\end{aligned} \tag{2.26}$$

Four additional polynomial identities for η (from degrees 7 to 19) are in [Kr]. The simplest is

$$\eta(2\tau)^6\eta(8\tau) - \eta(\tau)^4\eta(4\tau)^2\eta(8\tau) = 4\eta(\tau)^2\eta(2\tau)\eta(4\tau)^2\eta(16\tau)^2. \tag{2.27}$$

Chapter 3

Gluing Theory and Theta Series of Lattices

In this section we introduce the necessary theory of lattices and their theta constants (theta functions) to carry out our method for generating identities in the θ and η.

We start out with some basic facts about geometrical lattices as they are one of the backbones underlying our approach to generating identities.

We furthermore introduce relevant results in the theory of gluing of lattices, as a way to construct and to decompose lattices, later used for generating identities.

Let τ be a variable that lies in the complex upper half-plane:

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid Im(\tau) > 0\}$$

and as usual write $q := e^{2\pi i\tau}$.

3.A Lattices

See [13] for an in depth introduction to geometrical lattices.

We have the following definitions:

Definition: A *lattice* Λ is a non-empty, nowhere dense set of points in some finite dimensional, real inner product space, $V = V(\Lambda)$, such that $a, b \in \Lambda$ and $k, l \in \mathbb{Z} \Rightarrow ka + lb \in \Lambda$.

Equivalently and more practically for us, a *lattice*, $\mathit{\Lambda}$, is the following $\mathbb{Z}$-span of vectors:

$$\Lambda := \{\sum_{i=1}^{n} n_i v_i \mid \forall i, n_i \in \mathbb{Z}\}$$

of a basis $\{v_1, v_2, ..., v_n\}$ in $\mathbb{R}^n$.

Definition: The *dual* $\mathit{\Lambda}^*$ of a lattice $\mathit{\Lambda}$ is

$$\mathit{\Lambda}^* := \{v \in \mathbb{R}^n \mid v{\cdot}\lambda \in \mathbb{Z}, \forall \lambda \in \mathit{\Lambda}\}.$$

Definition: $\mathit{\Lambda}$ is called *integral* if for all $u, v \in \mathit{\Lambda}$ the dot product $u{\cdot}v$ is an integer, i.e. $u{\cdot}v \in \mathbb{Z}$. Equivalently, $\mathit{\Lambda}$ is integral iff $\mathit{\Lambda} \subseteq \mathit{\Lambda}^*$.

Let $\mathit{\Lambda}$ and $\mathit{\Lambda}'$ be lattices with bases β and β' respectively. Recall that an orthogonal transformation $T : V \to U$ is a linear transformation between vector spaces that preserve the inner products, i.e.:

$$T(v_i){\cdot}T(v_j) = v_i{\cdot}v_j$$

for $\forall v_i, v_j \in V$. Orthogonal transformations are represented via orthogonal matrices, $A \in O_n(\mathbb{R})$, in terms of the standard basis vectors of $\mathbb{R}^n$. We recall that an orthogonal matrix, $A \in O_n(\mathbb{R})$, is a square matrix with real entries whose columns and rows are orthogonal unit vectors (i.e. orthonormal vectors).

$$A^T = A^{-1} \Rightarrow A^T A = AA^T = I,$$

where I is the identity matrix.

Definition: We call $\mathit{\Lambda}$ and $\mathit{\Lambda}'$ *integrally equivalent* if there exists an orthogonal transformation $A \in O_n(\mathbb{R})$ such that $A(\mathit{\Lambda}){=}\mathit{\Lambda}'$. We write this as $\mathit{\Lambda} \cong \mathit{\Lambda}'$.

Definition: The *determinant* $|\mathit{\Lambda}|$ of $\mathit{\Lambda}$ is the determinant of the $n \times n$ matrix $\{m_{ij}\} = v_i \cdot v_j$, where $\{v_i\}$ is a basis for Λ. The determinant is independent of the specific choice of basis (we include proof below) and is a measure of how densely packed the lattice points are.

Lemma 3.1: *The determinant of a lattice* Λ *is independent of the choice of basis,* $\{v_i\}$ *chosen for it.*

Proof. Let $\beta' = \{v_i'\}$ be another basis of Λ. Let P be the change-of-basis matrix, i.e. $v_i' = \sum_j P_{ij} v_j$. Then both P and P^{-1} must be integral matrices. Hence both $\det(P)$ and $\det(P^{-1})$ must be integers, so $\det(P) = \pm 1$. The relation between the matrices m and m' is $m' = PmP^T$, so $\det(m') = (\pm 1)^2 \det(m) = \det(m)$ and the determinant is well-defined.

□

Definition: The *orthogonal direct sum* of $\mathit{\Lambda}$ and $\mathit{\Lambda}'$ is:

$$\mathit{\Lambda} \oplus \mathit{\Lambda}' := \{(v, v') \mid v \in \mathit{\Lambda}, v' \in \mathit{\Lambda}'\}$$

where the *inner product* is defined by

$$(u, u') \cdot (v, v') = uv + u'v' \in \mathbb{Z}.$$

Definition: We call $\mathit{\Lambda}$ *orthogonal* if it has an orthogonal basis, i.e.:

$$\mathit{\Lambda} \cong (\sqrt{K_1}\mathbb{Z}) \oplus ... \oplus (\sqrt{K_n}\mathbb{Z})$$

for positive integers: $K_1, ..., K_n$. We denote this lattice via $\{K_1, ..., K_n\}$. Note that an orthogonal lattice has determinant $|\{K_1, ..., K_n\}| = \prod_i K_i$.

We have the following useful fact:

Theorem 3.1: *Any integral lattice contains an orthogonal sub-lattice of full dimension.*

Proof. Let $\mathit{\Lambda}$ be an integral lattice. Let β=$\{v_1, v_2, ..., v_n\}$ be a basis for $\mathit{\Lambda}$.

We use the Gram-Schmidt orthogonalization process to create a set of orthogonal vectors, β'=$\{b_1, ..., b_n\}$ for Λ as follows:

Let: $b_1 := v_1$;

$b_2 := v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1$;

However even though this new vector is orthogonal to b_1, it does not necessarily live in $\mathit{\Lambda}$, as $\frac{v_1 \cdot v_2}{v_1 \cdot v_1}$ is not necessarily in $\mathbb{Z}$ anymore. We can easily fix this via multiplying b_2 by $v_1 \cdot v_1$. i.e. let

$$b_2 := v_1 \cdot v_1 (v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1) = (v_1 \cdot v_1) v_2 - (v_1 \cdot v_2) v_1;$$

which, since $\mathit{\Lambda}$ is integral (i.e. for $\forall i, j$ $v_i \cdot v_j \in \mathbb{Z}$) clearly lies in the $\mathbb{Z}$-span of β. Similarly we can follow the Gram-Schmidt process for the rest of the basis vectors, multiplying each resultant orthogonal new vector b_i by $v_{i-1} \cdot v_{i-1}$. In this way we don't change the orthogonality relations, however ensure that the vectors still lie in Λ.

□

Lastly we recall a few definitions that will be useful in section 3.D. A lattice Λ is called *even* if for all vectors $v \in \Lambda$, the *norm* $\langle v, v \rangle = |v|^2$ is an even integer, and the lattice Λ of rank n is *unimodular* if its fundamental domain has volume $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$, or equivalently, if $\Lambda = \Lambda^\star$.

3.B Gluing Theory

Gluing theory is a technique for constructing and therefore decomposing lattices (see [13]). Let Λ_0 be an integral lattice. Then $\Lambda_0 \subseteq \Lambda_0{}^*$, therefore we can consider the quotient lattice, namely $\Lambda_0{}^*/\ \Lambda_0$. It will be an abelian group, since both Λ_0^* and Λ_0 are. It consists of finitely many (in fact exactly $|\Lambda_0|$) distinct cosets:

$$[g] = g + \Lambda_0, g \in \Lambda_0^*.$$

These equivalence classes are called *glue classes*.

Definition: Let $[g_1], ..., [g_h]$ be glue classes of some integral lattice Λ_0. By the *glued lattice* $\Lambda = \Lambda_0[g_1, ..., g_h]$ we mean

$$\Lambda := \Lambda_0[g_1, ..., g_h] := \Lambda_0 + \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_h\,.$$

The finite group $G := \langle [g_1], ..., [g_h] \rangle$ generated by the glue classes is called the *glue group*.

Hence a typical element of the glued lattice Λ is of the form: $v + \sum_{i=1}^{h} m_i g_i$ for $v \in \mathit{\Lambda}_0, m_i \in \mathbb{Z}$. Observe that the glue group equals $\mathit{\Lambda}/\mathit{\Lambda}_0$. We will often denote the glued lattice by $\Lambda_0[G]$.

Notice that two gluings $\Lambda_0[g_1, \dots, g_m]$ and $\Lambda_0[g'_1, \dots, g'_{m'}]$ are equal as sets iff the glue groups $\langle g_1, \dots, g_m \rangle$ and $\langle g'_1, \dots, g'_{m'} \rangle$ are equal as subsets of Λ_0^*/Λ_0.

It's important to note the following Lemma.

Lemma 3.2: *If* $\Lambda = \Lambda_0[G]$, *then* $|\Lambda_0| = |\Lambda| \|G\|^2$.

Proof. See [1]. □

Note also that provided Λ_0 is integral and $\forall i, j \;\; g_i \boldsymbol{\cdot} g_j \in \mathbb{Z}$, $\mathit{\Lambda}$ will be integral. This is apparent when we observe that vectors $v_1, v_2 \in \mathit{\Lambda}_0 + \mathbb{Z}g_1 + \mathbb{Z}g_2 + ... + \mathbb{Z}g_h$ have the form:

$$v_i = v_{0i} + z_{1i}g_1 + z_{2i}g_2 + ... + z_{hi}g_h$$

and by the integrality of the base lattice and the assumption above of $g_i \boldsymbol{\cdot} g_j \in \mathbb{Z}$, we clearly have $v_1 \cdot v_2 \in \mathbb{Z}$ as well.

Example 3.B.1: An important example is when the base lattice $\mathit{\Lambda}_0$ is the one dimensional lattice $\mathit{\Lambda}_0 = \sqrt{k}\mathbb{Z}$ for some positive $k \in \mathbb{Z}$. Then $\mathit{\Lambda}_0^* = (\sqrt{k}\mathbb{Z})^* = (1/\sqrt{k})\mathbb{Z}$ and therefore $(\sqrt{k}\mathbb{Z})^*/(\sqrt{k}\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$.

To see this, simply map: $z/(\sqrt{k}) + \sqrt{k}\mathbb{Z} \mapsto z + k\mathbb{Z}$; this is clearly surjective and injective since: $z + k\mathbb{Z} = z' + k\mathbb{Z} \Rightarrow z - z' \in k\mathbb{Z}$ and therefore $z/\sqrt{k} - z'/\sqrt{k} = (z - z')/\sqrt{k} = kz^*/\sqrt{k} = \sqrt{k}z^* \in \sqrt{k}\mathbb{Z}$ as desired.

Now, for $0 \leq a < k$, consider the equivalence class $[a] := \frac{a}{\sqrt{k}} + \sqrt{k}\mathbb{Z}$. Factoring $k = lm$, take the gluing $(\sqrt{k}\mathbb{Z})[m] = \sqrt{k}\mathbb{Z} + \mathbb{Z}\frac{m}{\sqrt{k}} = \sqrt{lm}\mathbb{Z} + \mathbb{Z}\frac{m}{\sqrt{lm}} = \frac{lm}{\sqrt{lm}}\mathbb{Z} + \mathbb{Z}\frac{m}{\sqrt{lm}} = \sqrt{\frac{m}{l}}\mathbb{Z}$. The glue group in this case is:

$$\langle [m] \rangle = \{[0/\sqrt{k}], [m/\sqrt{k}], \ldots, [(l-1)m/\sqrt{k}]\}.$$

Orthogonal lattices are direct sums of these one-dimensional lattices. Let $\Lambda = \{\sqrt{K_1}\mathbb{Z}, \sqrt{K_2}\mathbb{Z}, ..., \sqrt{K_n}\mathbb{Z}\} + \mathbb{Z}g_1 + \mathbb{Z}g_2 + ... + \mathbb{Z}g_m := \{K_1, K_2, ..., K_n\}[G]$ be the gluing of an orthogonal lattice, where the glue group is denoted by G and has generators g_i. i.e. for each i, we have: $g_i = (\frac{a_{i1}}{\sqrt{K_1}}, \frac{a_{i2}}{\sqrt{K_2}}, ..., \frac{a_{in}}{\sqrt{K_n}})$ with $a_{ij} \in \mathbb{Z}$. For notational convenience we will write this as: $[a_{i1}, a_{i2}, ..., a_{in}]$.

Theorem 3.2: *Let $\{K_1, \ldots, K_n\}[g_1, \ldots, g_m]$ be a gluing of an orthogonal lattice. Write $g_i = [a_{i1}, \ldots, a_{in}]$ for the glue vectors, where $a_{ij} \in \mathbb{Z}$.*

Then, we have:

(i) The number m of glue vectors can always be taken to be less than n (where n is the dimension of our gluing);

(ii) we may take $a_{ij} = 0$ for $i > j$;

(iii) $a_{ii}|K_i$ or $a_{ii} = 0$;

Proof. Let $a :=$gcd$(K_1, a_{11}, a_{21}, ..., a_{i1}, ..., a_{m1})$ where m denotes the number of glue vectors in the gluing. From elementary number theory we know that we can write a as the sum:

$$a = lK_1 + \sum_i l_i a_{i1}$$

where $l, l_i \in \mathbb{Z}$. Now, let $g = l(\sqrt{K_1}, 0, ..., 0) + \sum_i l_i g_i \in \Lambda$ (note $\Lambda = \sqrt{K_1}\mathbb{Z} \oplus ... \oplus \sqrt{K_n}\mathbb{Z} + \mathbb{Z}g_1 + \mathbb{Z}g_2 + ... + \mathbb{Z}g_m$) and simply replace our old first glue vector

g_1 with g, i.e.: $g'_1 := g$.

We clearly see that by subtracting a multiple of g'_1 from the other g_i we can assume that for $i > 1, a_{i1} = 0$, exactly because this new a'_{11} divides all a_{i1}s. Repeating this for the rest of the K_j's by replacing 1 with j, we see that we can assume $a_{ij} = 0$ for $i > j$ and hence part (i) and (ii) are proved. We also observe that $a_{ii}|K_i$ or $a_{ii} = 0$ proving part (iii).

To justify this, we observe the following: These new generators $[g'_i]$s give exactly the same gluing as the old ones as they are in the $\mathbb{Z}$-span of the old ones, as it's clear by the construction, and vice-versa, we can reconstruct from the new $[g'_i]$s the old $[g_i]$s using only coefficients from $\mathbb{Z}$ as the construction is fully "reversible". Therefore the new and old basis vectors are in each others' $\mathbb{Z}$-spans and hence they generate the same lattice gluing, as desired.

$\square$

3.C Two-dimensional gluings

For our exposition, we will consider two dimensional lattices as resultants of gluing a two-dimensional orthogonal lattice. Hence, the general form of the lattice gluing for our studies is the following:

$$\Lambda = \{K_1, K_2\}[a_1, a_2] = \sqrt{K_1}\mathbb{Z} \oplus \sqrt{K_2}\mathbb{Z} + \mathbb{Z}(\frac{a_1}{\sqrt{K_1}}, \frac{a_2}{\sqrt{K_2}}).$$

Let k be the order of the glue vector, $g = (\frac{a_1}{\sqrt{K_1}}, \frac{a_2}{\sqrt{K_2}})$; then provided $k > 1$, $K_1 = ka_1$ and $a'K_2 = ka_2$ with $gcd(k, a') = 1$. Hence we can write our 2-dimensional glued lattice in the following form:

Lemma 3.3: $\{ka, kb\}[a, bb'] = \{(m\sqrt{\frac{a}{k}}, n\sqrt{\frac{b}{k}}) \in \mathbb{R}^2 | m, n \in \mathbb{Z}, n \equiv mb' (\mathrm{mod}\, k)\}$

Proof. For $\lambda \in \{ka, kb\}[a, bb'] = \sqrt{ka}\mathbb{Z} \oplus \sqrt{kb}\mathbb{Z} + \mathbb{Z}\left(\frac{a}{\sqrt{ka}}, \frac{bb'}{\sqrt{kb}}\right)$ we have

$$\lambda = (\frac{z_1 ka}{\sqrt{ka}}, \frac{z_2 kb}{\sqrt{kb}}) + z_0(\frac{a}{\sqrt{ka}}, \frac{bb'}{\sqrt{kb}}) = \left((z_0 + z_1 k)\sqrt{\frac{a}{k}}, (z_0 b' + z_2 k)\sqrt{\frac{b}{k}}\right)$$

where $z_i \in \mathbb{Z}$ and hence for $m := z_0 + z_1 k$ and $n := z_0 b' + z_2 k$ we have $m, n \in \mathbb{Z}$ clearly and $n - mb' = z_0 b' + z_2 k - (z_0 + z_1 k)b' = (z_2 - z_1)k \in \mathbb{Z}k$, so $n \equiv mb'$ (mod k) as desired.

Now, on the other hand, given $v = (m\sqrt{\frac{a}{k}}, n\sqrt{\frac{b}{k}})$ with $n \equiv mb'$ (mod k), we have: $v = (m\sqrt{\frac{a}{k}}, n\sqrt{\frac{b}{k}}) = (m\sqrt{\frac{a}{k}}, (mb' + kx)\sqrt{\frac{b}{k}}) = (\frac{ma}{\sqrt{ka}}, (mb' + kx)\frac{b}{\sqrt{kb}}) = (0, \frac{xkb}{\sqrt{kb}}) + m(\frac{a}{\sqrt{ka}}, \frac{bb'}{\sqrt{kb}}) \in \sqrt{ka}\mathbb{Z} \oplus \sqrt{kb}\mathbb{Z} + \mathbb{Z}(\frac{a}{\sqrt{ka}}, \frac{bb'}{\sqrt{kb}})$ as desired.

□

Definition: The *minimum norm* (length squared) in a lattice is the smallest value of $v \cdot v$ where v runs through all non-zero lattice vectors.

Lemma 3.4: *In our case of lattices of the form:*

$$\Lambda = \{ka, kb\}[a, bb']$$

with $a < b$; The minimum norm will be either attained by: $(\sqrt{ka}, 0)$ *or by* $(ma/\sqrt{ka}, nb/\sqrt{kb})$ *where* $0 < m \leq max\{1, k/2\}$, $-b' < n \leq b'$ *with* $b'm \equiv n$ *(mod k).*

Proof. A typical vector in our lattice has the form: $v = (m\sqrt{a/k}, n\sqrt{b/k})$ with $m, n \in \mathbb{Z}$ and $n \equiv mb'$ (mod k). Then we have: $v \cdot v = m^2 a/k + n^2 b/k$, so we need to minimize $m^2 a + n^2 b$.

In case the minimum norm vector is of the form that one of its coordinates is zero, it suffices to consider: $(m\sqrt{a/k}, 0)$ with $0 \equiv mb'$ (mod k) and this implies that $k|m$ so, $m = xk$ for some $x \in \mathbb{Z}$, but then since $(xk\sqrt{a/k}, 0)$ is minimal normed, it implies that the vector has the form $(k\sqrt{a/k}, 0) = (\sqrt{ka}, 0)$ as desired.

So, if instead $v_0 = (m_0\sqrt{a/k}, n_0\sqrt{b/k})$ is of minimal norm we have the following inequality hold for all other vectors, $v \in \Lambda$ with respect to v_0 that is: $v_0 \cdot v_0 \leq v \cdot v$. So, since $((m_0 - k)\sqrt{a/k}, n_0\sqrt{b/k}) \in \Lambda$, we have: $(m_0-k)^2a+n_0^2b \geq m_0^2a+n_0^2b$ and from this it follows that: $m_0^2a-2m_0ka+k^2a \geq m_0^2a$ so, $k^2a \geq 2m_0ka$.

Hence: $k/2 \geq m_0$ and we have our first condition:

$$0 < m_0 \leq \max\{1, k/2\}.$$

Also, since $((m_0 - 1)\sqrt{a/k}, (n_0 - b')\sqrt{b/k}) \in \Lambda$, we have: $(m_0 - 1)^2a + (n_0 - b')^2b \geq m_0^2a + n_0^2b$ and from this it follows that $a + b'^2b \geq m_0^2a + n_0^2b$ so $a/b(1 - m_0^2) + b'^2 \geq n_0^2$, but since $m_0 \geq 1$ we have that $b'^2 \geq n_0^2$. That is:

$$-b' < n_0 \leq b'$$

and we have our second condition proved.

□

Given the lattice gluing $\Lambda = \sqrt{K_1}\mathbb{Z} \oplus \sqrt{K_2}\mathbb{Z} + \mathbb{Z}\,(\frac{a}{\sqrt{K_1}}, \frac{b}{\sqrt{K_2}})$, the *order* of the glue vector $(\frac{a}{\sqrt{K_1}}, \frac{b}{\sqrt{K_2}})$ is k if k is the smallest positive integer satisfying $k(\frac{a}{\sqrt{K_1}}, \frac{b}{\sqrt{K_2}}) \in \sqrt{K_1}\mathbb{Z} \oplus \sqrt{K_2}\mathbb{Z}$.

Lemma 3.5: *Consider the gluing $\Lambda = \{K_1, K_2\}[a_1, a_2]$. Then we may require:*

(a) either $a_1 = 0$ or both $0 < a_1 < K_1$ and $a_1|K_1$;

(b) if $a_1 \neq 0$, then the order k of the glue vector $[a_1, a_2]$ is $K_1/\gcd(a_1, K_1)$.

Proof. Write:

$K_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_n}^{\alpha_n}$

$a_1 = {p_1}^{\iota_1}{p_2}^{\iota_2}...{p_n}^{\iota_n}{q_1}^{\beta_1}{q_2}^{\beta_2}...{q_j}^{\beta_j}$

$k = {p_1}^{\kappa_1}{p_2}^{\kappa_2}...{p_n}^{\kappa_n}$ as their respective unique prime decompositions. Then with respect to each $i \in 1, 2, ..., n$ we have the following:

if $\alpha_i > \iota_i$, then $\kappa_i = \alpha_i - \iota_i$;

if $\alpha_i = \iota_i$, then $\kappa_i = 0$;

if $\alpha_i < \iota_i$, then $\kappa_i = 0$.

Note that dividing K_1 by the gcd(a_1,K_1) ensures exactly this prime decomposition for k, since the greatest common divisor takes the minimum power for each common prime into its account, hence we get exactly the above numbers for κ_i for each i.

□

Then for $k > 1$ we have $K_1 = ka_1$ and $a'K_2 = ka_2$, hence we can write

$$\Lambda = \{K_1, K_2\}[a_1, a_2] = \{ka, kb\}[a, bb'] = \{(m\sqrt{a/k}, n\sqrt{b/k}) \in \mathbb{R}^2 \mid m, n \in \mathbb{Z}, n \equiv mb' (\mathrm{mod}\, k)\}.$$

3.D Theta Series (Theta Functions) of Lattices

Definition: The *theta series* (or theta constant or really theta function) of a lattice is given by:

$$\Theta(\Lambda)(\tau) := \sum_{v \in \Lambda} q^{\frac{v \cdot v}{2}}.$$

More generally, the *theta series of a glue class* $[g]$ of Λ_0 is:

$$\Theta([g])(\tau) = \sum_{v\in\Lambda_0} q^{\frac{(g+v)\cdot(g+v)}{2}}.$$

Importantly, if Λ and Λ' are integrally equivalent, then they have identical theta series.

A more algebraic and equivalent definition of theta series is when it's defined as the following generating function:

$$\Theta[\Lambda(\tau) := \sum_{l} a_{2l} q^l$$

where a_m denotes the number of vectors in Λ of norm m. Note that this definition encodes the distribution of vector norms in the lattice. We might think that such information suffices to describe a lattice uniquely, however this isn't true as for example the lattices $E_8 \oplus E_8$ and D_{16}^+ have identical theta series despite being non-isomorphic lattices. (See more on this in chapter 7.)

Example 3.D.1: The 1-dimensional lattice $\sqrt{k}\mathbb{Z}$ has theta series:

$$\Theta(\sqrt{k}\mathbb{Z})(\tau) = \sum_{v\in\sqrt{k}\mathbb{Z}} q^{\frac{v\cdot v}{2}} = \sum_{v\in\sqrt{k}\mathbb{Z}} q^{\frac{\sqrt{k}z\cdot\sqrt{k}z}{2}} = \sum_{v\in\sqrt{K}\mathbb{Z}} q^{\frac{kz^2}{2}} = \theta_3(k\tau)$$

and its glue class $[a]$ has theta series:

$\Theta([a])(\tau) = \Theta(a/\sqrt{k}+\sqrt{k}\mathbb{Z})(\tau) = \sum_{v\in\sqrt{k}\mathbb{Z}} q^{\frac{(g+v)\cdot(g+v)}{2}} = \sum_{z\in\mathbb{Z}} q^{\frac{(a/\sqrt{k}+\sqrt{k}z)\cdot(a/\sqrt{k}+\sqrt{k}z)}{2}} = \sum_{z\in\mathbb{Z}} q^{\frac{a^2/k+2az+kz^2}{2}} = \sum_{z\in\mathbb{Z}} q^{\frac{k(z+a/k)^2}{2}} = \psi_{k/a}(k\tau).$

Example 3.D.2: When Λ is an *even, unimodular lattice* of *rank* n, $\Theta[\Lambda](\tau)$ will be a modular form of weight $\frac{n}{2}$. Here we only sketch the proof.

We need to check two conditions for a holomorphic modular form $f : \mathbb{H} \to \mathbb{C}$ of weight k (remember $k \in 2\mathbb{Z}^+$)

(a) $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

(b) f has a power series expansion in the variable $q = e^{2\pi i\tau}$: $\sum_{n=0}^{\infty} a_n q^n$

Proof. (a) It's suffices to show invariance under the matrices T, S.

$\Theta[\Lambda](\tau+1) = \sum_{\lambda\in\Lambda} e^{\frac{2\pi i(\tau+1)\lambda\cdot\lambda}{2}} = \sum_{\lambda\in\Lambda} e^{\frac{2\pi i(\tau)\lambda\cdot\lambda}{2}} e^{\frac{2\pi i\lambda\cdot\lambda}{2}} = e^{\frac{2\pi i(\tau)\lambda\cdot\lambda}{2}} = \Theta[\Lambda](\tau)$ as desired.

For invariance under S, we will have to assume two facts we wont prove: when Λ is a unimodular lattice of rank n, then $n \equiv 0 \pmod 8$ and Jacobi's identity (a corollary of the Poisson summation formula): for any lattice Λ we have $\Theta[\Lambda](\frac{-1}{\tau}) = (\frac{\tau}{i})^{\frac{n}{2}} \frac{1}{\mathrm{Vol}(\mathbb{R}^n/\Lambda)} \Theta[\Lambda^\star](\tau)$.

Since in a lattice is unimodular iff $\Lambda = \Lambda^\star$, this and the above two facts show:

$$\Theta[\Lambda](\frac{-1}{\tau}) = \tau^{\frac{n}{2}} \Theta[\Lambda](\tau)$$

as desired.

(b) It is obvious from the definition of the theta series for an even lattice that it has a power series expansion in q.

□

Lemma 3.6: *(i) The theta series of a direct sum of glue classes is the product of the theta series of the individual classes. In particular the glue vector* $[g] = [(a_1, a_2, ..., a_n)]$ *of an orthogonal lattice* $\{K_1, ..., K_n\}$ *has theta series* $\Theta([g])(\tau) = \prod_{i=1}^{n} \Theta([a_i(g)])(\tau) = \prod_{i=1}^{n} \psi_{k_j/a_j}(k_j\tau)$.

(ii) The theta series of the disjoint union of glue classes is the sum of the

theta series of the individual classes.

Proof. (i) It suffices to prove this for a direct sum of two glue classes. Let $g = (g_1, g_2)$ be a glue vector for $\Lambda_1 \oplus \Lambda_2$. Then by definition

$$\Theta([g])(\tau) = \sum_{v \in [g]} q^{v \cdot v/2} = \sum_{(v_1, v_2) \in [(g_1, g_2)]} q^{(v_1 \cdot v_1 + v_2 \cdot v_2)/2} = \sum_{v_1 \in [g_1]} \sum_{v_2 \in [g_2]} q^{v_1 \cdot v_1/2} q^{v_2 \cdot v_2/2}$$

which by distributivity equals

$$\sum_{v_1 \in [g_1]} q^{v_1 \cdot v_1/2} \sum_{v_2 \in [g_2]} q^{v_2 \cdot v_2/2} = \Theta([g_1])(\tau)\, \Theta([g_2])(\tau).$$

(ii) similarly, trivially follows. □

Example 3.D.3: : The theta series of the gluing $\Lambda = \{K_1, ..., K_n\}[G]$ is:

$$\Theta(\{K_1, ..., K_n\}[G])(\tau) = \sum_{[g] \in G} \Theta([g]) = \sum_{[g] \in G} \prod_{j=1}^{n} \Theta([a_j(g)]) = \sum_{[g] \in G} \prod_{j=1}^{n} \psi_{K_j/a_j(g)}(K_j \tau)$$

where $[g] = [(a_1(g), ..., a_n(g))]$.

Now, given this above, and the fact that any n-dimensional lattice can be obtained by gluing from an n-dimensional orthogonal sublattice, we can use this to to get the theta constant of any integral n-dimensional lattice and in general the theta constant of the glue class of any integral lattice and write it as a homogeneous degree-n polynomial in the ψ_k's. This is key to our approach to generate theta identities, as described next chapter.

Chapter 4

Identities via integral equivalences of geometrical lattices

4.A General observations on identities in the ψ_ks

Our interest is in polynomial identities in the ψ_k's. Ipsum est, we are interested in identities of the form:

$$\sum_{i=1}^{N} c_i \prod_{j=1}^{n_i} \psi_{k_{ij}}(a_{ij}\tau) = 0$$

where $c_i \in \mathbb{C}, k_{ij} \in \mathbb{Q}, a_{ij} \in \mathbb{R}, a_{ij} > 0$.

According to Theorem 4.1 in [17], it suffices to consider only the homogeneous identities, i.e. the ones with $n_i = n$, which are integral, that is $c_i, a_{ij} \in \mathbb{Z}$. All the identities given in the paper are homogeneous and integral.

It is now useful to introduce the space $\Psi^{(n)}$ of all degree-n polynomials in $\psi_k(m\tau)$ where $k, m \in \mathbb{Q}_{>0}$. An identity is any element of $\Psi^{(n)}$ which is identically 0 as a function of τ.

Once we have a set of polynomial identities $\{L_i(\tau) = R_i(\tau)\}$ of ψ's, there are simple ways to generate other ones. We can take any combinations of:

(i) Linear combinations:

$$\sum_i c_i L_i(\tau) = \sum_i c_i R_i(\tau)$$

(ii) Products:

$$\prod_i L_i(\tau) = \prod_i R_i(\tau)$$

(iii) Re-scale the arguments:

$$L_i(a\tau) = R_i(a\tau)$$

(iv) Take modular transformations by $PSL_2(\mathbb{Q})$

$$L_i(\gamma \circ \tau) = R_i(\gamma \circ \tau)$$

We say $L(\tau) = R(\tau)$ *generates* $L'(\tau) = R'(\tau)$, if the latter can be obtained from the former, by these moves. The homogeneous integral identities generate all others using these operations.

Example 4.A.1: Let's look at for example the gluing of a one dimensional lattice $\Lambda_0 = \sqrt{ml}\mathbb{Z}$, by the glue class $[m]$. We find that:

$$\Theta(\{lm\})[m](\tau) = \sum_{j=0}^{l-1} \psi_{l/j}(lm\tau) = \Theta_3(m\tau/l).$$

This is in agreement with the lattice equivalence, shown before:

$$\{lm\}[m] \equiv \{m/l\}.$$

This illustrates the point made in [GL1], which provides the backbone of our method: Integral equivalences of gluing of lattices give homogeneous identities in the ψ_ks. More precisely, given any n-dimensional lattice Λ, each pair Λ_0, Λ_1 of n-dimensional orthogonal sublattices implies a homogeneous identity in the ψ_ks. After all, $\Lambda_0[G_0] = \Lambda = \Lambda_1[G_1]$ where $G_i = \Lambda/\Lambda_i$. The theta series of each gluing $\Lambda_i[G_i]$ can be expressed polynomially in ψ_ks, as we learned last chapter. These polynomials must agree, hence we get an identity! In particular for us, the lattice equivalence:

$$\mathit{\Lambda}_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \mathit{\Lambda}_R \tag{4.1}$$

generates the following quadratic identity:

$$\sum_{i=1}^{k} \psi_{k/i}(ka\tau)\psi_{k/(b'i)}(kb\tau) = \sum_{j=1}^{l} \psi_{l/j}(lc\tau)\psi_{l/(d'j)}(ld\tau). \tag{4.2}$$

This last equation also tells us the order of the glue needed to get identities involving certain modular functions. As we can see from equations (4.1) and (4.2) above, glues of order 1,2,3,4,6 gives rise to θ_3 identities and glues of order 12 involve both θ_3 and η identities. In this thesis we are most interested in glues where k is 12 and l is $1, 2, 3, 4, 6$ or 12, as our primary interest lies in generating quadratic identities that contain not only θ_3, but η as well, hence the need for the order 12 glue.

Example 4.A.2: We observe that the glued lattice $\{2,2\}[1,1]$ is just a $\pi/4$ rotation from $\mathbb{Z}^2$ and hence, the gluing $\{2,2\}[1,1] \cong \mathbb{Z}^2$ generates the identity:

$$\theta_2(2\tau)^2 + \theta_3(2\tau)^2 = \theta_3(\tau)^2.$$

Example 4.A.3: The gluing $\{2,2\}[1,1] \cong \mathbb{Z}^2$ and $\{8,8\}[2,2] \cong \{1,4\}$ generate the *Jacobi identity*: $\theta_2^4 + \theta_4^4 = \theta_3^4$.

To see this, observe using (4.1) that our equivalence has values: $k = 2, a = 1, b = 1, b' = 1, l = 1, c = 1, d = 1, d' = 1$. Now, substituting these values in to our derived formula (4.2) yields:

$$\sum_{i=1}^{2} \psi_{2/i}(2\tau)\psi_{2/(i)}(2\tau) = \sum_{j=1}^{1} \psi_{1/j}(1\tau)\psi_{1/(j)}(1\tau)$$

which equates to:

$$\psi_2(2\tau)^2 + \psi_1(2\tau)^2 = \psi_1(\tau)^2$$

and now using eq.(2.6) and eq.(2.7) we get:

$$\theta_3(2\tau)^2 + \theta_2(2\tau)^2 = \theta_3(\tau)^2 \tag{4.3}$$

Similarly from our second equivalence, we can derive:

$$\theta_3(8\tau)^2 + \theta_2(8\tau)^2 + 1/2\theta_2(2\tau)^2 = \theta_3(\tau)\theta_3(4\tau) \tag{4.4}$$

If we consider (4.3) under the mappings: $\tau \mapsto 2 - 1/\tau$ and $-1/\tau$ respectively produces:

$$-\theta_4(\tau)^2 + \theta_3(\tau)^2 = 2\theta_2(2\tau)^2 \tag{4.5}$$

$$\theta_4(\tau)^2 + \theta_3(\tau)^2 = 2\theta_3(2\tau)^2 \tag{4.6}$$

Now using (4.3) we can rewrite (4.4) as:

$$1/2\theta_2(1/2\tau)^2 = \theta_2(\tau)^2 + \theta_3(\tau)^2 \tag{4.7}$$

And finally, using (4.5),(4.6) and (4.7) we can write:

$$\theta_3(\tau)^4 - \theta_4(\tau)^4 = (2\theta_2(2\tau)^2)(2\theta_3(2\tau)^2) = \theta_2(\tau)^4 \tag{4.8}$$

as claimed.

4.A.1 Symmetrization

A final construction permits us to extend significantly the scope of our method. Take any term P_i on the leftside of equation (4.2) and consider its image under $\tau \mapsto \tau + 1$:

$$P_i(\tau) = \psi_{k/i}(ka\tau)\,\psi_{k/b'i}(kb\tau) \mapsto P_i(\tau+1) = \psi_{k/i}(ka\tau + ka)\,\psi_{k/b'i}(kb\tau + kb)\ .$$

Assume as usual that the norm $[a, b'b] \cdot [a, b'b] = (a + b'^2 b)/k$ is an integer. Then (2.23) tells us:

(i) If k is even, then $P_i(\tau+1) = \pm P_i(\tau)$ if $(a + b'^2 b)/k$ is even/odd, respectively;

(ii) If k is odd and both a, b are even, then $P_i(\tau+1) = P_i(\tau)$;

(iii) If k is odd and at least one of a, b are odd then (2.13) does not apply and the evaluation of $P_i(\tau+1)$ will be more complicated.

By *'symmetrisation'* we mean to replace (4.2), i.e. $\sum_i P_i(\tau)$, with $\sum_i (P_i(\tau) + P_i(\tau+1))/2$. Define *'anti-symmetrisation'* similarly. Many terms can drop out, and what's left typically is a simpler identity. In particular, suppose the order k of the left glue in (4.1) is 8 or 12, while the order ℓ of the right glue lies in $\{1, 2, 3, 4, 6\}$. Then (4.2) will not be expressible as a θ_3 identity, but provided the norm $(a + b'^2 b)/k$ of the left glue is odd, then the symmetrisation will be a θ_3 identity. If both $k, \ell \in \{8, 12\}$, then the resulting symmetrised identity will be a θ_3 identity provided both norms $(a + b'^2 b)/k$ and $(c + d'^2 d)/\ell$ are odd. In these cases, the terms containing η get relegated to the anti-symmetrisation. In effect, a pure θ_3 identity (the symmetrisation) is removed from (4.2), resulting in a simpler η identity. (Anti-)symmetrisation is a way to simplify an identity.

Similarly, if $k = 24$ and $\ell \in \{1, 2, 3, 4, 6, 12\}$, and the norm $(a + b'^2 b)/k$ is odd, then the symmetrised identity will be expressible in terms of η and θ_3. If $k = 24$ but $\ell \in \{8, 24\}$, and both norms $(a + b'^2 b)/k$ and $(c + d'^2 d)/\ell$ are odd, then the resulting identity will likewise be expressible in terms of η and θ_3.

4.A.2 A note on degree one identities

We note now that the degree-one identities in $\Psi^{(1)}$ are all known, see [GL1] for details, here we state the result.

Theorem 4.1: *(a) The only degree-one identities involving any number of $\psi_k(m_k\tau)$s are linearly generated by eq.(2.4) and eq.(2.5).*

(b) The functions $\theta_3(m_k\tau)$s and $\eta(n_i\tau)$s are linearly independent:

i.e. $\sum_i c_i\theta_3(m_i\tau) + \sum_j d_j\eta(n_i\tau) = 0 \iff$ *all* $c_i = d_j = 0$.

By 'linearly generated' we mean that any linear identity $I(\tau) = 0$ can be written as a linear combination:

$$\sum_i a_i I_i(m_i\tau) = 0$$

where each $I_i(\tau) = 0$ is one of the basic identities (2.4) and (2.5). Of course (b) is an immediate corollary of (a).

Therefore next to consider are the quadratic identities. However there will almost certainly be infinitely many inequivalent identities in $\Psi^{(2)}$, so [GL1] tried something simpler: they looked at all quadratic identities involving only θ_3. We describe their results shortly. In this thesis we push the analysis to the next level: the quadratic identities involving θ_3 and η.

4.A.3 Test for equivalence

Note that not all the lattice equivalences given by (4.1) will result in independent identities. First, we expand out the identity into θ_3's and η's. This automatically takes care of all linear identities, thanks to Theorem 4.1. Note that all of the operations we are allowed to perform on one identity, in order to get another identity, is to send a term $A\theta_3(h\tau)\theta_3(j\tau)$ to a linear combination of 'comparable' terms $\theta_3(h'\tau)\theta_3(j'\tau)$, where we call two terms 'comparable' if the ratio h/j either equals $4^m 9^n h'/j'$ or $4^m 9^n j'/h'$, for some integers m, n. This means that if we have identities involving only θ_3 (as in [GL1]), two identities will necessarily be *inequivalent* unless each term in one identity is comparable to some term in the other, and vice versa. For example, we know the identities corresponding to the equivalences $\{2,2\}[1,1] \cong \{1,1\}$ and $\{3,6\}[1,2] \cong \{1,2\}$ cannot be deduced from each other, since $1/1$ is incomparable to $1/2$ or $2/1$. Although more refined tests are surely possible, in practice, when two identities are compatible in this sense, it seems they can usually be shown to be deducible from each other.

For lattice identities, this test is easy to do. Note that all terms on the left side of (4.2) involve ratios comparable to a/b, while all terms on the right side of (4.2) involve ratios comparable to c/d. [GL1] considered only $\tau \mapsto \tau + 1/2$, and not e.g. $\tau \mapsto \tau + 1/3$, and thus some identities which that paper regarded as inequivalent, may turn out to be equivalent in our sense. This analysis says that [GL1] contains at least 24 inequivalent quadratic θ_3 identities. Table 1 collects a lattice representative of each of those 24 identities, together with the 'class' (h/j) on each side. Brackets around the numbers emphasize that they are defined up to the equivalence $4^n 9^m h/j$.

Class	Lattice equivalence	Class	Lattice equivalence
(1)∼(1)	{2,2}[4,32]≅{1,1}	(2)∼(2)	{3,6}[1,2]≅{1,2}
(3)∼(3)	{4,12}[1,3]≅{1,3}	(5)∼(5)	{3,15}[1,5]≅{2,10}[1,5]
(14)∼(14)	{6,21}[2,7]≅{3,42}[1,14]	(11)∼(11)	{4,44}[1,11]≅{3,33}[1,11]
(7)∼(7)	{4,28}[1,7]≅{2,14}[1,7]	(5/3)∼(15)	{12,20}[3,5]≅{2,30}[1,15]
(7/5)∼(35)	{20,28}[5,7]≅{3,105}[1,35]	(11/5)∼(55)	{20,44}[5,11]≅{4,220}[1,55]
(13/3)∼(39)	{12,52}[3,13]≅{4,156}[1,39]	(31/5)∼(155)	{30,186}[5,31]≅{6,210}[1,155]
(29/7)∼(203)	{42,174}[7,29]≅{6,1218}[1,203]	(25/11)∼(275)	{66,150}[11,25]≅{6,1650}[1,275]
(23/13)∼(299)	{78,138}[13,23]≅{6,1794}[1,299]	(19/17)∼(323)	{102,114}[17,19]≅{6,1938}[1,323]
(17)∼(17)	{6,102}[1,17]≅{3,51}[1,17]	(13/5)∼(65)	{30,78}[5,13]≅{3,195}[1,65]
(11/7)∼(77)	{42,66}[7,11]≅{3,231}[1,77]	(23)∼(23)	{6,138}[1,23]≅{4,92}[1,23]
(19/5)∼(95)	{30,114}[5,19]≅{4,380}[1,95]	(17/7)∼(119)	{42,102}[7,17]≅{4,476}[1,119]
(15)∼(15)	{54,90}[9,15]≅{4,540}[1,135]	(13/11)∼(143)	{66,78}[11,13]≅{4,572}[1,143]

Table 1. The quadratic θ_3 identities

For the quadratic $\{\theta_3, \eta\}$-identities considered in this paper, we can say much more. Any of the operations we can perform on identities will take a term $A\,\eta(h\tau)\eta(j\tau)$ to a single term $A'\,\eta(h'\tau)\eta(j'\tau)$, where h/j *equals* either h'/j' or j'/h'. Thus almost every lattice equivalence considered here will yield an inequivalent identity.

4.B Quadratic Theta θ_3 and Eta η identities

There are infinitely many different lattice equivalences (4.1), even if we restrict to those of orders $1, 2, 3, 4, 6, 12$. But most of these won't give new identities. The hardest part of applying this method is to reduce to a finite set of lattice equivalences. This subsection accomplishes that.

Theorem 4.2 below is the key to finding all quadratic identities coming from lattice equivalences. It is the first new result in this thesis. We begin by recording a lemma that explains the constraints we need to assume on the form of allowable lattice gluings under the equivalence.

Lemma 4.1: *Without loss of generality, it suffices to consider lattice equivalences:*

$$\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R$$

subject to the following constraints:

$$ab = cd, \tag{4.9}$$

$$a \le b, c \le d, \tag{4.10}$$

$$1 \le b' \le max\{k/2, 1\} \text{ and } 1 \le d' \le max\{l/2, 1\}, \tag{4.11}$$

$$\frac{a + b'^2 b}{k} \text{ and } \frac{c + d'^2 d}{l} \in \mathbb{Z}, \tag{4.12}$$

$$gcd(a, b, \frac{a + b'^2 b}{k}) = gcd(c, d, \frac{c + d'^2 d}{l}) = 1, \tag{4.13}$$

$$(a, b', k) \neq (c, d', l), \tag{4.14}$$

$$gcd(b', k) = gcd(d', l) = 1. \tag{4.15}$$

Proof. We explain the constraints set up.

Eq.(4.9). is the determinant of the lattices in $\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R$ and since it's the determinant of the matrix comprised of the various basis vector dot-products, our lattice equivalence must preserve them. The determinant of a gluing is most easily computed from Lemma 3.2.

Eq.(4.10): By symmetry we can assume $a \leq b$ and $c \leq d$.

eq.(4.14:) If the quadruples (a, b, b', k) were (c, d, d', l) are equal, then this theta-identity will be trivial, and can be disregarded. Note that (4.9) says $a = c$ iff $b = d$, so $(a, b, b', k) \neq (c, d, d', l)$ iff eq.(4.14) holds.

eq.(4.12): We can rescale the lattices Λ_l and Λ_R so that they are integral, hence $\frac{a+b'^2 b}{k}$ and $\frac{c+d'^2 d}{l} \in \mathbb{Z}$. Now, a lattice is integral if all dot products are integral, so we can guarantee this, if we require that the dot products of the basis vectors are integral. $\beta = \{v_1 = (\sqrt{a/k}, b'\sqrt{b/k}), v_2 = (0, \sqrt{kb})\}$ is a basis and so we need:

$$v_1 \cdot v_1 = \frac{a + b'^2 b}{k} \in \mathbb{Z}$$

$$v_1 \cdot v_2 = v_2 \cdot v_1 = b'b \in \mathbb{Z}$$

$$v_2 \cdot v_2 = kb \in \mathbb{Z}$$

and so the only additional constraint we require is the $\frac{a+b'^2b}{k} \in \mathbb{Z}$ as the rest of the dot-products are integral anyway. Similarly for the right lattice Λ_R we require:

$$\frac{c+d'^2d}{l} \in \mathbb{Z}$$

but so that any smaller rescaling is not integral (rescaling the lattices in $\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R$ merely amounts to rescaling the τs in $\sum_{i=1}^{k} \psi_{k/i}(ka\tau)\psi_{k/(b'i)}(kb\tau) = \sum_{j=1}^{l} \psi_{l/j}(lc\tau)\psi_{l/(d'j)}(ld\tau)$).

To see eq.(4.13), first let p be any prime dividing a, b, and $(a + bb'^2)/k$. Then p divides all dot products on the left-side Λ_L of $\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R$. Hence the same must hold for Λ_R, and so we get that p must divide $c, d, (c + dd'^2)/\ell$. We can rescale Λ_L and Λ_R, replacing a, b, c, d with $a/p, b/p, c/p, d/p$, and all other conditions are obeyed.

□

Incidentally, note that eq.(4.13) requires that $\gcd(a, b)$ must divide k. Also, if $k = 1, 2, 3, 4, 6$, then $b' = 1$ (similarly for ℓ and d'), while if $k = 12$ then $b' = 1$ or 5.

Theorem 4.2: *The lattice equivalences*

$$\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R$$

subject to the constraints of Lemma 4.1 above fall into two classes:

(i) $k \geq 2, c|lcm(km, kn)$ *and* $c|gcd(ma, nb), klc = m^2a + n^2b \leq k^2a$ *and both* $mad' \equiv \pm nc$ *(mod* lc*) and* $\mp mc \equiv nbd'$ *(mod* lc*), for nonzero integers* m, n *satisfying* $0 < m \leq max(1, k/2), -b' < n \leq b'$, $b'm \equiv n$ *(mod* k*) and* *some choice of sign. These imply the bounds:* $a \leq b \leq (k^2 - m^2)lc/n^2$ *and* $c \leq mnk$.

(ii) $k \geq l \geq 2$, $(m^2a + n^2b)/k = ((m')^2c + (n')^2d)/l$, $m^2a + n^2b \leq k^2a$, $(m')^2c + (n')^2d \leq l^2c$, $e^2a + f^2b = klc$, $aem + bfn = km'c$, *for integers* m, n, m', n', e, f *obeying:*

-nonzero integers m, n *satisfying* $0 < m \leq$ *max*$(1, k/2)$, $-b' < n \leq b'$, $b'm \equiv n$ *(mod k);*

-nonzero integers m', n' *satisfying* $0 < m' \leq l/2$, $-d' < n' \leq d'$, $d'm' \equiv n'$ *(mod l);*

- b'e $\equiv f \not\equiv 0$*(mod k).*

These imply the bounds:

$|e| \leq l/n\sqrt{(m^2 + n^2)/((m')^2 + (n')^2)(k^2 - m^2)}$

$|e| \leq l/n\sqrt{(m^2 + n^2)/((m')^2 + (n')^2)l^2 - e^2n^2/(k^2 - m^2)}$

Proof. First, since we will be working with lattices, it's convenient to fix the bases to work with. We have shown above that a general vector in our glued lattice $\Lambda_l := \{ka, kb\}[a, bb']$ has the form:

$v = (m\sqrt{a/k}, n\sqrt{b/k})$ with $n \equiv mb'$ (mod k) and hence:

$v = (m\sqrt{a/k}, n\sqrt{b/k}) = (m\sqrt{a/k}, (mb' + xk)\sqrt{b/k}) =$

$= m(\sqrt{a/k}, b'\sqrt{b/k}) + x(0, \sqrt{bk})$ for some $x \in \mathbb{Z}$.

Hence $\beta = \{(\sqrt{a/k}, b'\sqrt{b/k}), (0, \sqrt{kb})\}$ is a basis for the glued lattice Λ_L as we clearly see that these vectors span the lattice and are linearly independent. Now, back to the proof for the sufficient and necessary conditions for the lattice equivalences.

Let $\varphi : \Lambda_L \to \Lambda_R$ be the desired lattice equivalence above. Now, lattice equivalences preserve dot products and hence a minimum norm vector must be mapped to a minimum norm vector (under a surjection), otherwise we would find the pre-image of the minimum normed vector of Λ_R, in Λ_L to be less then the assumed minimum norm in Λ_L, reaching a contradiction.

As proven above in Lemma 3.3, the minimum norm in the gluing Λ_L will either be ka attained by $(ka/\sqrt{ka}, 0)$ or $(m^2a + n^2b)/k$ attained by the vector $(ma/\sqrt{ka}, nb/\sqrt{kb})$. This leads to two possible cases:

Case (i): the minimum norm of Λ_R is lc. We will first show that the minimum norm of Λ_L must then be $\frac{m^2a+n^2b}{k} \leq ka$ and $\varphi(\frac{ma}{\sqrt{ka}}, \frac{nb}{kb}) = (\frac{lc}{\sqrt{lc}}, 0)$.

Note, we have only two possibilities for the form of the minimum norm vector in Λ_L that could possibly map to $(\frac{lc}{\sqrt{lc}}, 0)$. So, arguing towards contradiction, we assume that $\varphi(\frac{ka}{\sqrt{ka}}, 0) = (\frac{lc}{\sqrt{lc}}, 0)$.

We then have $ka = lc$ since the minimum norms are equal through the lattice equivalence. (I.e. $|(\frac{ka}{\sqrt{ka}}, 0)| = |(\frac{lc}{\sqrt{lc}}, 0)| \Rightarrow ka = lc$.)

The vectors in Λ_L orthogonal to $(\frac{ka}{\sqrt{ka}}, 0)$ are of the form $\mathbb{Z}(0, \frac{kb}{\sqrt{kb}})$. This is easy to see:

$\Leftarrow$: $(\frac{ka}{\sqrt{ka}}, 0) \cdot \mathbb{Z}(0, \frac{kb}{\sqrt{kb}}) = 0$

$\Rightarrow$: $(\frac{ka}{\sqrt{ka}}, 0) \cdot (\frac{ma}{\sqrt{ka}}, \frac{nb}{\sqrt{kb}}) = \frac{kma^2}{ka} + 0 = m$ i.e. $m = 0$ in their form; i.e. vectors of the form: $(\frac{ma}{\sqrt{ka}}, \frac{nb}{\sqrt{kb}}) = (0, \frac{nb}{\sqrt{kb}})$.

Similarly those in Λ_R, orthogonal to $(\frac{lc}{\sqrt{lc}}, 0)$ are of the form, $\mathbb{Z}(\frac{ld}{\sqrt{ld}}, 0)$ and we know again by definition of dot-product preservation of the isomorphism that orthogonal vectors map to orthogonal vectors.

Therefore φ sends $(0, \frac{kb}{\sqrt{kb}})$ to $\pm(\frac{ld}{\sqrt{ld}}, 0)$ and their norm kb, resp. ld must equal, $kb = ld$. We have thus far:

$$\varphi(0, \frac{kb}{\sqrt{kb}}) = (0, \pm\frac{ld}{\sqrt{ld}}),$$

$$\varphi(\frac{ka}{\sqrt{ka}},0) = (\frac{lc}{\sqrt{lc}},0).$$

This implies $\varphi(\frac{a}{\sqrt{ka}},\frac{b'b}{\sqrt{kb}}) = (\frac{c}{\sqrt{lc}},+-\frac{b'd}{\sqrt{ld}})$ and $b' = +-d'$. Hence from $ab = cd, ka = lc, kb = ld$ we then get, $(a,b',k) = (c,d',l)$ violating one of our above conditions, (4.8) in Lemma (4.1) above.

Therefore, we have to have $\varphi(\frac{ma}{\sqrt{ka}},\frac{nb}{\sqrt{kb}}) = (\frac{lc}{\sqrt{lc}},0)$ and hence

$$klc = m^2a + n^2b \text{ is derived.}$$

Now, since $\Lambda_L \equiv \Lambda_R$ there exists integers m', n' s.t. $d'm' \equiv n' \pmod{l}$ such that

$$\varphi(\frac{ka}{\sqrt{ka}},0) = (\frac{m'c}{\sqrt{lc}},\frac{n'd}{\sqrt{ld}})$$

with $ka = (m'^2c + n'^2d)/l$ as the norm is preserved under the lattice equivalence.

Also, since the dot product is preserved for any two vectors, we have that: Before isomorphism (LHS):

$$(\frac{ma}{\sqrt{ka}},\frac{nb}{kb})\cdot(\frac{ka}{\sqrt{ka}},0) = ma$$

After isomorphism (RHS) :

$$(\frac{lc}{\sqrt{lc}},0)\cdot(\frac{m'c}{\sqrt{lc}},\frac{n'd}{\sqrt{ld}}) = m'c$$

so LHS=RHS $\Rightarrow ma = m'c$ (i.e. $m' = \frac{ma}{c}$), that is $c|ma$. This means that also $d'm' \equiv n' (\bmod\, l)$ becomes: $d'\frac{ma}{c} \equiv n'(\bmod\, l) \iff d'ma \equiv cn'(\bmod\, cl)$ but $n' = \pm n \Rightarrow d'ma = \pm cn(\bmod\, cl)$.

Now, to see why $n' = \pm n$ we see that from the norm condition we had:

$$ka = \frac{m'^2 c + n'^2 d}{l};$$

but we just found above that $m' = \frac{ma}{c}$ so after substitution we have $kla = \frac{(ma)^2}{c} + n'^2 d$ but also $ab = cd$ as per (eq.4.3), so this implies:

$$n'^2 = (kla - (ma)^2/c)\frac{c}{ab} = \frac{klc}{b} - \frac{m^2 a}{b}$$

but

$$klc = m^2 a + n^2 b \Rightarrow \frac{n^2 b}{b} = n^2 = n'^2$$

and as desired we get: $n' = \pm n$.

Similarly, we also find $\varphi(0, \frac{kb}{\sqrt{kb}})$ as follows.

$$\varphi(0, \frac{kb}{\sqrt{kb}}) = (x\sqrt{c/l}, y\sqrt{d/l})$$

with $y \equiv xd'(\text{mod } l)$. Since $(\frac{ma}{\sqrt{ka}}, \frac{nb}{\sqrt{kb}})\cdot(0, \frac{kb}{\sqrt{kb}}) = (\frac{lc}{\sqrt{lc}}, 0)\cdot(x\sqrt{c/l}, y\sqrt{d/l})$ i.e. $nb = xc \Rightarrow x = nb/c$ and hence $y = xd'(l) \Rightarrow y = nbd'/c(l)$.

So far $\varphi(0, \frac{kb}{\sqrt{kb}}) = (nb/\sqrt{lc}, yd/\sqrt{ld})$, but since we had $\varphi(\frac{ka}{\sqrt{ka}}, 0) = (\frac{m'c}{\sqrt{lc}}, \frac{n'd}{\sqrt{ld}})$ also and the vectors being mapped here are orthogonal to each other, implying that the images' dot products need be zero also:

$$\varphi(\frac{ka}{\sqrt{ka}}, 0)\cdot\varphi(0, \frac{kb}{\sqrt{kb}}) = (\frac{m'c}{\sqrt{lc}}, \frac{n'd}{\sqrt{ld}})\cdot(\frac{nb}{\sqrt{lc}}, \frac{yd}{\sqrt{ld}}) = 0.$$

So $\frac{m'c}{\sqrt{lc}}\frac{nb}{\sqrt{lc}} + \frac{n'dyd}{ld} = 0 \Rightarrow \frac{m'nb + dn'y}{l} = 0 \Rightarrow y = -\frac{m'nb}{n'd}$. But, we had that $m' = ma/c$ from above, and hence: $y = -\frac{\frac{ma}{c}nb}{n'd}$. Now, we also found above that $n' = \pm n$, so $y = -\frac{\frac{ma}{c}nb}{\pm nd}$ and since $ab = cd \Rightarrow y = -\frac{\frac{ma}{c}nb}{\pm nd} = \mp m$. So as claimed, we have:

$$\varphi(0, \frac{kb}{\sqrt{kb}}) = (\frac{nb}{\sqrt{lc}}, \frac{\mp md}{\sqrt{ld}}) \in \Lambda_R.$$

Since this vector must be in Λ_R, c also must divide nb and $\mp m \equiv \frac{nbd'}{c} (\bmod\ l)$.

These conditions are sufficient for the lattice equivalence $\Lambda_L \cong \Lambda_R$ for the following reason.

A lattice equivalence is well-defined if we can determine where it send the basis vectors. We have that $\beta = \{(\sqrt{a/k}, b'\sqrt{b/k}), (0, \sqrt{kb})\}$ is a basis for the glued lattice Λ_L and since the above conditions resulting from the dot products being preserved under φ determined the images of $(k\sqrt{a/k}, 0), (0, \sqrt{kb})$ therefore by linearity we can extend these definitions to well define the images of the basis vectors as desired under the isomorphism.

Next, we choose any prime p dividing c. Let $p^\alpha, p^\beta, p^\gamma, p^\kappa, p^\mu, p^\nu$ be the exact powers of p dividing a, b, c, k, m, n respectively. Because c must divide both ma and nb, we get $\gamma \le \mu + \alpha$ and $\gamma \le \nu + \beta$.

Now, also p must divide a or b, since it must divide $ab = cd$. By considering $\frac{a+b'^2 b}{k}$, we get that $\kappa \ge \min\{\alpha, \beta\}$ since if $\alpha = 0$, then $(a+b'^2 b)/k \in \mathbb{Z}$ requires $\kappa = 0$ and if, instead say $\beta \ge \alpha > 0$, then $\gcd(a, b, (a + b'^2 b)/k) = 1$ requires $\kappa \ge \alpha$.

Case **(ii)**:

If the minimum norm of Λ_L is ka, and the minimum norm of Λ_R is $\frac{(m'^2 c+n'^2 d)}{l}$, then we just reverse Λ_L and Λ_R and we are in case (i). Therefore the only remaining possibility is:

$$\varphi(\frac{ma}{\sqrt{ka}}, \frac{nb}{\sqrt{kb}}) = (\frac{m'c}{\sqrt{lc}}, \frac{l'c}{\sqrt{ld}})$$

here the minimal norm of Λ_R is $\frac{(m'^2c+n'^2d)}{l} \leq lc$ and the minimal norm of Λ_L is $\frac{(m^2a+n^2b)}{k} \leq ka$.

Now, similarly as in case (i) above, this also gives us the quadratic norm equation $(m^2a+n^2b)/k = ((m')^2c+(n')^2d)/l$.

Also as φ is a lattice isomorphism, there must exist $e, f \in \mathbb{Z}$ s.t. $\varphi(\frac{ea}{\sqrt{ka}}, \frac{fb}{\sqrt{kb}}) = (\frac{lc}{\sqrt{lc}}, 0)$ with $eb' = f$ (mod k) with $e^2a+f^2b = klc$ again as a consequence of norm preservation.

Also, since $\varphi(\frac{ma}{\sqrt{ka}}, \frac{nb}{\sqrt{kb}}) = (\frac{m'c}{\sqrt{lc}}, \frac{l'c}{\sqrt{ld}})$ and $\varphi(\frac{ea}{\sqrt{ka}}, \frac{fb}{\sqrt{kb}}) = (\frac{lc}{\sqrt{lc}}, 0)$, dot-product preservation requires:

$$aem + bfn = km'c$$

and norm preservation requires:

$$\frac{e^2a+f^2b}{k} = lc.$$

If k were to divide both e and f, i.e. $e = ke'f = kf'$ we would obtain $k(e'^2a+f'^2b) = lc$ for integers e', f'.

Now, either $b \geq c$ or $a \geq c$: otherwise, if both $a < c$ and $b < c$, then $ab < c^2 \leq cd$, so $ab < cd$ contradicting our initial assumption that $ab = cd$.

So assume say that $b \geq c$ and since $k \geq l$, this requires that $f' = 0$ in $k(e'^2a+f'^2b) = lc$.

We have so far that:

$$\varphi(\frac{ke'a}{\sqrt{ka}}, 0) = \varphi(e'\sqrt{ak}, 0) = (\frac{lc}{\sqrt{lc}}, 0) = (\sqrt{lc}, 0)$$

and by linearity $e'\varphi(\sqrt{ak}, 0) = (\sqrt{lc}, 0)$, and so $\varphi(\sqrt{ak}, 0) = 1/e'(\sqrt{lc}, 0)$.

But since $(\sqrt{lc}, 0)$ is a primitive vector in Λ_R (i.e. $xv \in \Lambda_R$ iff $x \in \mathbb{Z}$) we must have $1/e' \in \mathbb{Z}$ so $e' = \pm 1$.

Therefore, $\varphi(\sqrt{ak}, 0) = (\sqrt{lc}, 0)$ and hence φ takes Λ_L 's x-axis to Λ_R's x-axis, and hence by orthogonality takes Λ_L's y-axis to Λ_R's y-axis, and this violates non-triviality $(a, b', k) = (c, d', l)$.

Choose any prime p dividing c. Write α, β, γ, for the exact powers of p dividing a,b,c respectively. Then $ab = cd$ implies $\alpha + \beta \geq \gamma$. We want to show that p^γ divides either k^2em or k^2fn.

Suppose p^γ fails to divide all of me, nf and k (otherwise we are done).

If p does not divide a, then p^γ must divide $b = \frac{cd}{a}$ because $ab = cd$; since now p^γ divides both $km'c = ema + fnb$ and b, it must also divide ema; i.e. p must divide a, contradicting our hypothesis.

Thus $\alpha > 0$, and similarly $\beta > 0$. So p can't divide $(a + b'^2b)/k$, otherwise we contradict $\gcd(a, b, (a + b'^2b)/k) = 1$.

So, now, if $\alpha = \beta (\geq \gamma/2)$, then p^α must divide k, so p^γ divides k^2, and we are done.

So we may assume $\alpha \neq \beta$ (without loss of generality assume $\alpha < \beta$). Then p^α divides k exactly.

Since $p^{\gamma+\alpha}$ divides $ema+fnb$, we either get that p^γ divides em (if $\gamma+\alpha \leq \beta$), or $p^{\beta-\alpha}$ divides em othewise.

So, p^γ must divide $k^2 \mathrm{lcm} em, fn$, and so must c.

Now, $(m'^2 + n'^2)c/l \leq (m'^2c + n'^2d)/l = (m^2a + n^2b)/k \leq (m^2 + n^2)b/k$ $\Rightarrow (m'^2 + n'^2)c/l \leq (m^2 + n^2)b/k \Rightarrow kc \leq \frac{m^2+n^2}{m'^2+n'^2}lb$, and $(m^2a + n^2b)/k \leq ka$ implies that $n^2b/(k^2 - m^2) \leq a$, so all together:

Since $\frac{e^2a+f^2b}{k} = lc \Rightarrow f^2b = klc - e^2a$ thus $0 \leq f^2b = lkc - e^2a \leq \frac{m^2+n^2}{m'^2+n'^2}l^2b - e^2n^2b/(k^2-m^2)$, giving us the bounds for $|e|$ and $|f|$ as follows: $0 \leq \frac{m^2+n^2}{m'^2+n'^2}l^2b-$

$e^2n^2b/(k^2-m^2) \Rightarrow$

$$|e|\leq \frac{l}{n}\sqrt{\frac{m^2+n^2}{m'^2+n'^2}(k^2-m^2)}$$

and

$$|f|\leq \sqrt{\frac{m^2+n^2}{m'^2+n'^2}l^2-\frac{e^2n^2}{k^2-m^2}}$$

as advertised.

□

Now that we have all these lattice equivalences generated by our theory a very important question remains, that is which of these lattice equivalences correspond to the same theta function identities.

First we need to consider when we call two distinct identities equivalent, i.e. we need to determine the operations we are allowed to perform on identities that constitutes the equivalence.

What we need is operations which map $\Psi^{(n)}$ to itself. Clearly we should be able to multiply identities by constants, we should be able to take linear combinations of them, rescale the arguments and take modular transformations to their arguments, as we outlined in section 4.A. above.

4.B.1 List of lattice equivalences calculated from Theorem 4.2

Here we record in the following tables the lattice equivalences generated from Theorem (4.2.). The lattice equivalences are recorder in the format of equation (4.1), i.e.:

$$\Lambda_l := \{ka, kb\}[a, bb'] \cong \{lc, ld\}[c, dd'] =: \Lambda_R.$$

Next to the equivalences we record their "class" attributes,

$$(a/b) \sim (c/d)$$

corresponding to the ratios of the arguments on left and right hand sides of the generated identities, (eq.4.2). This is an easy test to spot inequivalent identities, as identities belonging to different classes are surely independent from each other as explained above in section 4.1.A, as all of the allowable range of transformations we can do to go between identities preserves the ratios of the arguments on each side of the identities, up to some scaling. Using this test we have counted 131 inequivalent identities. For the Sage source code used to generate these identities refer to Appendix B.

Class	Lattice equivalence	Class	Lattice equivalence
8∼(2)	{48,384}[4,32]≅{3,384}[1,128]	8∼(2)	{48,384}[4,160]≅{12,384}[2,64]
3∼(3)	{36,108}[3,45]≅{9,27}[3,9]	5/4∼(5)	{192,240}[16,20]≅{3,960}[1,320]
20∼(5)	{48,960}[4,400]≅{15,192}[5,64]	7/2∼(14)	{96,336}[8,28]≅{3,672}[1,224]
20∼(5)	{48,960}[4,80]≅{60,192}[10,32]	2∼(2)	{48,96}[4,8]≅{1,32}[1,32]
14∼(14)	{48,672}[4,56]≅{84,96}[14,16]	14∼(14)	{48,672}[4,280]≅{21,96}[7,32]
1∼(1)	{72,72}[6,6]≅{1,36}[1,36]	1∼(1)	{72,72}[6,30]≅{9,36}[3,12]
5∼(5)	{24,120}[2,10]≅{1,20}[1,20]	65∼(13/5)	{24,1560}[2,130]≅{39,60}[13,20]
45/7∼315	{252,1620}[21,135]≅{36,11340}[3,945]	65∼(13/5)	{24,1560}[2,650]≅{15,156}[5,52]
13/5∼(65)	{120,312}[10,26]≅{3,780}[1,260]	77∼(11/7)	{24,1848}[2,154]≅{33,84}[11,28]
11/7∼(77)	{168,264}[14,22]≅{3,924}[1,308]	77∼(11/7)	{24,9240}[2,770]≅{21,132}[7,44]
17∼(17)	{24,408}[2,34]≅{3,204}[1,68]	17∼(17)	{24,2040}[2,170]≅{12,51}[4,17]
9/7∼(7)	{252,324}[21,27]≅{4,2268}[1,567]	63∼(7)	{36,2268}[3,945]≅{28,324}[7,81]
9/7∼63	{252,324}[21,135]≅{36,2268}[3,189]	3∼(3)	{36,108}[3,9]≅{1,27}[1,27]
55∼(11/5)	{36,1980}[3,165]≅{44,180}[11,45]	5/3∼(15)	{108,180}[9,15]≅{2,270}[1,135]
15∼(15)	{36,540}[3,225]≅{10,54}[5,27]	55∼(11/5)	{36,1980}[3,165]≅{44,180}[11,45]
15∼15	{36,540}[3,45]≅{36,540}[3,225]	15∼(15)	{36,540}[3,45]≅{54,90}[9,15]
15∼(15)	{36,540}[3,45]≅{4,540}[1,135]	5/3∼(15)	{108,180}[9,75]≅{18,270}[3,45]
5/3∼(15)	{108,180}[9,75]≅{20,108}[5,27]	195/9∼(195)	{108,2340}[9,195]≅{36,7020}[3,585]
39∼(39)	{36,1404}[3,117]≅{52,108}[13,27]	13/3∼(39)	{108,468}[9,39]≅{4,1404}[1,351]
13/3∼39	{108,468}[9,195]≅{36,1404}[3,117]		
11/5∼(55)	{180,396}[15,33]≅{4,1980}[1,495]	11/5∼(55)	{180,396}[15,165]≅{36,220}[9,55]
55∼(11/5)	{36,1980}[3,825]≅{20,396}[5,99]	11/5∼55	{180,396}[15,165]≅{36,1980}[3,825]
11/5∼55	{180,396}[15,165]≅{36,1980}[3,165]	7∼(7)	{36,252}[3,21]≅{2,126}[1,63]
7∼(7)	{36,252}[3,105]≅{18,126}[3,21]	7∼(7)	{36,252}[3,105]≅{14,18}[7,9]
7∼(7)	{36,1260}[3,105]≅{14,18}[7,9]	7∼(7)	{228,1500}[19,125]≅{12,142500}[1,11875]
175/17∼425/7	{204,2100}[17,875]≅{84,5100}[7,2125]		
119/25∼2975	{300,1428}[25,119]≅{12,35700}[1,2975]	25/11∼(275)	{132,300}[11,25]≅{3,825}[1,275]
275∼(25/11)	{12,3300}[1,275]≅{33,75}[11,25]	575∼(25/23)	{12,6900}[1,575]≅{92,100}[23,25]
25/23∼(575)	{276,300}[23,25]≅{4,2300}[1,575]	1175∼(47/25)	{12,14100}[1,1175]≅{150,282}[25,47]
47/25∼(1175)	{300,564}[25,47]≅{6,7050}[1,1175]	95/49∼4655	{588,1140}[49,95]≅{12,55860}[1,4655]
245/19∼931/5	{228,2940}[19,1225]≅{60,11172}[5,4655]	7/5∼(35)	{60,84}[5,7]≅{1,35}[1,35]
35∼(7/5)	{12,420}[1,35]≅{20,28}[5,7]	35∼(7/5)	{12,420}[1,175]≅{5,7}[5,7]
35∼(35)	{12,420}[1,35]≅{3,105}[1,35]	7/5∼(7/5)	{60,84}[5,35]≅{15,21}[5,7]
91/5∼(35/13)	{60,1092}[5,91]≅{78,210}[13,35]	455∼(35/13)	{12,5460}[1,455]≅{52,140}[13,35]

Table 2. Lattice equivalences and their attributes

Class	Lattice equivalence	Class	Lattice equivalence
455∼(65/7)	{60,1092}[5,455]≅{28,260}[7,65]	35/13∼(91/5)	{156,420}[13,175]≅{30,546}[5,91]
65/7∼(91/5)	{84,780}[7,325]≅{20,364}[5,91]	65/7∼(455)	{84,780}[7,65]≅{6,2730}[1,455]
455∼(65/7)	{12,5460}[1,2275]≅{42,390}[7,65]	91/5∼(35/13)	{60,1092}[5,91]≅{78,210}[13,35]
35/13∼(455)	{156,420}[13,35]≅{4,1820}[1,455]	259/5∼(185/7)	{60,3108}[5,1295]≅{42,1110}[7,185]
1295∼(37/35)	{12,15540}[1,1295]≅{210,222}[35,37]	37/35∼(1295)	{420,444}[35,37]≅{6,7770}[1,1295]
55/17∼(935)	{204,660}[17,55]≅{6,5610}[1,935]	185/7∼(259/5)	{84,2220}[7,925]≅{30,1554}[5,259]
109/35∼3815	{420,1308}[35,109]≅{12,45780}[1,3815]	545/7∼763/5	{84,6540}[7,2725]≅{60,9156}[5,3815]
935∼(55/17)	{12,11220}[1,4675]≅{102,330}[17,55]	85/11∼(187/5)	{132,1020}[11,425]≅{30,1122}[5,187]
187/5∼(85/11)	{60,2244}[5,187]≅{66,510}[11,85]	89/55∼4895	{660,1068}[55,89]≅{12,58740}[1,4895]
445/11∼979/5	{132,5340}[11,2225]≅{60,11748}[5,979]	79/65∼5135	{780,948}[65,79]≅{12,61620}[1,25675]
395/13∼1027/5	{156,4740}[13,1975]≅{60,12324}[5,1027]	295/17∼1003/5	{204,3540}[17,1475]≅{60,12036}[5,5015]
85/59∼5015	{708,1020}[59,85]≅{12,60180}[1,5015]	19/5∼95	{60,228}[5,95]≅{12,1140}[1,95]
19/5∼(95)	{60,228}[5,19]≅{2,190}[1,95]	95∼(19/5)	{12,1140}[1,475]≅{10,38}[5,19]
115/29∼3335	{348,1380}[29,115]≅{12,40020}[1,16675]	145/23∼667/5	{276,1740}[23,725]≅{60,8004}[5,667]
155∼(31/5)	{12,1860}[1,775]≅{15,93}[5,31]	31/5∼(155)	{60,372}[5,31]≅{3,465}[1,155]
215∼(43/5)	{12,2580}[1,1075]≅{20,172}[5,43]	43/5∼(215)	{60,516}[5,43]≅{4,860}[1,215]
67/5∼(335)	{60,804}[5,67]≅{6,2010}[1,335]	335∼(67/5)	{12,4020}[1,1675]≅{30,402}[5,67]
139/5∼695	{60,1668}[5,139]≅{12,8340}[1,3475]	1127∼(49/23)	{12,13524}[1,1127]≅{138,294}[23,49]
49/23∼(1127)	{276,588}[23,49]≅{6,6762}[1,1127]	133/11∼1463	{132,1596}[11,133]≅{12,17556}[1,1463]
385/19∼1045/7	{228,4620}[19,385]≅{84,12540}[7,1045]	737/7∼469/11	{84,8844}[7,737]≅{132,5628}[11,2345]
77/67∼5159	{804,924}[67,77]≅{12,61908}[1,25795]	91/53∼4823	{636,1092}[53,91]≅{12,57876}[1,24115]
371/13∼689/7	{156,4452}[13,1855]≅{84,8268}[7,689]	17/7∼(119)	{84,204}[7,17]≅{2,238}[1,119]
119∼(17/7)	{12,1428}[1,595]≅{14,34}[7,17]	29/7∼(203)	{84,348}[7,29]≅{3,609}[1,203]
203∼(29/7)	{12,2436}[1,1015]≅{21,87}[7,29]	41/7∼(287)	{84,492}[7,41]≅{4,1148}[1,287]
287∼(41/7)	{12,3444}[1,1435]≅{28,164}[7,41]	137/7∼959	{84,1644}[7,137]≅{12,11508}[1,4795]
121/23∼2783	{276,1452}[23,121]≅{12,33396}[1,2783]	11∼(11)	{12,132}[1,11]≅{1,11}[1,11]
11∼(11)	{12,132}[1,55]≅{4,44}[1,11]		
11∼(11)	{12,132}[1,55]≅{3,33}[1,11]	13/11∼(143)	{132,156}[11,13]≅{2,286}[1,143]
143∼(143)	{12,1716}[1,143]≅{22,26}[11,13]	407∼(37/11)	{12,4884}[1,407]≅{44,148}[11,37]
37/11∼(407)	{132,444}[11,37]≅{4,1628}[1,407]	671∼(61/11)	{12,8052}[1,671]≅{66,366}[11,61]
61/11∼(671)	{132,732}[11,61]≅{6,4026}[1,671]	299∼(23/13)	{12,3588}[1,299]≅{39,69}[13,23]
23/13∼(299)	{156,276}[13,23]≅{3,897}[1,299]	767∼(59/13)	{12,9204}[1,767]≅{78,354}[13,59]
59/13∼(767)	{156,708}[13,59]≅{6,4602}[1,767]	131/13∼1703	{156,1572}[13,131]≅{12,20436}[1,1703]
323∼(19/17)	{12,3876}[1,1615]≅{51,57}[17,19]	19/17∼(323)	{204,228}[17,19]≅{3,969}[1,323]
31/17∼(527)	{204,372}[17,31]≅{4,2108}[1,527]	527∼(31/17)	{12,6324}[1,2635]≅{68,124}[17,31]
127/17∼2159	{204,1524}[17,127]≅{12,25908}[1,10795]	551∼(29/19)	{12,6612}[1,2755]≅{76,116}[19,29]
29/19∼(551)	{228,348}[19,29]≅{4,2204}[1,551]	1007∼(53/19)	{12,12084}[1,5035]≅{114,318}[19,53]
53/19∼(1007)	{228,636}[19,53]≅{6,6042}[1,1007]	23∼(23)	{12,276}[1,115]≅{6,138}[1,23]
23∼(23)	{12,276}[1,23]≅{2,46}[1,23]	23∼(23)	{12,276}[1,115]≅{4,92}[1,23]
1247∼(43/29)	{12,14964}[1,6235]≅{174,258}[29,43]	43/29∼(1247)	{348,516}[29,43]≅{6,7482}[1,1247]
41/31∼(1271)	{372,492}[31,41]≅{6,7626}[1,1271]	1271∼(41/31)	{12,15252}[1,6355]≅{186,246}[31,41]
113/31∼3503	{372,1356}[31,113]≅{12,42036}[1,17515]	107/37∼3959	{444,1284}[37,107]≅{12,47508}[1,3959]
103/41∼4223	{492,1236}[41,103]≅{12,50676}[1,21115]	101/43∼4343	{516,1212}[43,101]≅{12,52116}[1,21715]
47∼(47)	{12,564}[1,47]≅{4,188}[1,47]	47 ∼ (47)	{12,564}[1,235]≅{6,282}[1,47]
97/47∼4559	{564,1164}[47,97]≅{12,54708}[1,4559]	83/61∼5063	{732,996}[61,83]≅{12,60756}[1,5063]
71∼(71)	{12,852}[1,71]≅{6,426}[1,71]	73/71∼5183	{852,876}[71,73]≅{12,62196}[1,5183]
17∼(17)	{24,408}[2,170]≅{12,51}[4,17]		

Table 2 (cont)

It would be interesting to symmetrise these identities, and see if any new purely θ_3 identities arise since [GL1] conjecture they have all of them, so this would be a serious test of their conjecture.

Chapter 5

Generalizations

5.A Investigation of the Modular Derivative

To help us identify identities that are equivalent we investigate what happens to an identity in the θ and ηs under the action of the modular derivative, as described below. The hope is that taking modular derivatives may be another effective way to generate equivalent identities.

The modular derivative which we shall denote by capital D from now on is defined in general by:

$$D = D_k = q\frac{d}{dq} - \frac{k}{12}E_2(\tau)$$

where $q = e^{2\pi i\tau}$ and $E_2(\tau)$ is the Eisenstein series and k is the weight of our modular form.

Suppose we want to find examples of holomorphic modular forms for Γ of weight k. Eisenstein series are the "natural" examples that come to mind when we are required to construct modular forms by brute force methods.

At this point, it's convenient for notational efficiency to define the so called *slash operator* $|_k$ as follows:

$$f|_k\gamma := (c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d})$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Notice that functions that are modular are going to be invariant under the action of the slash operator and this is why we bothered with its definition. Now, for Γ, we define the following subgroup

$$G_0 := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \; \| k \in \mathbb{Z} \}.$$

The idea here is that if we can find an "initial" function, f that is invariant under G_0 we can create a new function by defining its values to be equal to the sum of all f values under translation by the group action, i.e. "average" over the group elements. A priori, there is no reason for our initial function to be invariant under G_0. The point though is that if we have to sum over the whole group Γ, there will be little or no chance for convergence. It is easy to get functions f invariant under G_0, and so for such functions we don't need to sum over the full group, but rather over G/G_0, which is easy to describe and also quite a bit smaller. Let's make this precise.

One of the simplest functions to take that is already invariant under our group "right out of the box" is just $f(\tau) = 1$ as it's trivially invariant under $|_k\gamma$ for any $\gamma \in G_0$.

Now, we will simply average over all group elements by slashing the function

1 by them and adding the results up into a series, as follows:

$$E_k(\tau) := \frac{1}{2}\sum_{\gamma\in\Gamma/G_0} 1|_k\gamma = \frac{1}{2}\sum_{c,d\in\mathbb{Z};(c.d)=1}(c\tau+d)^{-k}.$$

What we end up with then, are functions that are clearly invariant under "slashing" (except for E_2 as seen below). These series are called the Eisenstein series of weight k correspondingly.

We have the following convergence properties for these series:

$$E_k(\tau) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \text{diverges} & \text{if } k \leq 0 \\ \text{converges not absolutely} & \text{if } k = 2 \\ \text{absolutely convergent} & \text{if } k = 4, 6, 8 \end{cases}$$

So for $k \geq 2$, E_k is holomorphic throughout $\mathbb{H}$ including cusps. But the failure of absolute convergence at $k = 2$ means that E_2 is not invariant under the slash operator – in other words, it is not modular, but E_k, for $k = 4, 6, 8, ...$ is a modular form of weight k.

In particular for us, E_2 is a holomorphic and it is not quite a modular form of weight 2 as we mentioned above, it is what's called *quasi-modular* and it behaves as follows:

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2E_2(\tau) - \frac{6ic}{\pi}(c\tau+d).$$

Continuing, we need to explain the "fancy" definition of the modular derivative and why the "ordinary derivative" fails for us.

5.A.1 The Ramanujan differential operator

One of the key properties of modular forms is that they have what's called their q-expansions:

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

Let's observe what happens to f, if we just go ahead and take its ordinary derivative with respect to τ:

$$\frac{d}{d\tau} f(\tau) = 2\pi i \sum_{n=0}^{\infty} a_n n q^n.$$

This suggests to define: $' := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$. Let's see what happens if we take this *prime* operation to a modular form, f, of weight k. By definition of modularity we have for f:

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$$

then, taking *prime* to both sides, we get:

$$f'(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{k+2} f'(\tau) + \frac{kc}{2\pi i}(c\tau+d)^{k+1} f(\tau).$$

Therefore we see that f' is no longer a modular form, but it is what we call, *quasi-modular*. Recall just what we observed for E_2 above.

So, instead we define the Ramanujan differential operator to be:

$$D_k = \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2(\tau).$$

This operator takes modular forms for Γ of weight k to modular forms for Γ

of weight $k+2$.

Lemma 5.1: ***(i)*** *D takes weight-k modular forms to weight-$k+2$ modular forms.*

(ii) *D is a derivation: (i.e. if f is weight-k, and g is weight k', then $D_{k+k'}(fg) = gD_kf + fD_{k'}g$).*

5.A.2 Application of D to identities

Now, we are in position to investigate modular derivatives of our theta functions and their utility, if any, in determining identities up to equivalence. We will start with examining the modular derivative of θ_3. As we know, θ_3 is a modular form of weight $k = 1/2$. Since, $D_k = q\frac{d}{dq} - \frac{k}{12}E_2(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{24}E_2(\tau)$ we seek the derivative of θ_3 with respect to the variable τ.

As it appears, there are few ways to go about calculating the derivative. We know from theory that modular forms for $\Gamma(2)$ are polynomials in $\theta_3^4(\tau)$ and $\theta_4^4(\tau)$. In particular, a modular form of weight 2 for $\Gamma(2)$ will be a linear combination of $\theta_3^4(\tau)$ and $\theta_4^4(\tau)$. Therefore, $\theta_i^{-1}(\tau)D\theta_i(\tau)$ must be expressible as $a_i\theta_3(\tau)^4 + b_i\theta_4(\tau)^4$, for some constants a_i, b_i, for each choice of $i = 2, 3, 4$.

Note that for all modular forms f of weight k, $f^{-1}D_kf$ will be a meromorphic modular form of weight 2, for the same group as f. That ratio will have poles in $\mathbb{H}$ at any point where f has a zero. It is very rare for a modular form to have no zeros in $\mathbb{H}$, but this is true for all of $\theta_3, \theta_2, \theta_4, \eta$. So for $f(\tau) = \theta_3(m\tau)$ or $f(\tau) = \eta(m\tau)$ etc, that ratio will be a holomorphic modular form of weight 2. This is somewhat special. No zeros in $\mathbb{H}$ (only at one of the cusps)!

The simplest thing to do with modular forms is to check the first few coefficients, utilizing the power of Theorem 2.2 above. Now, $\Gamma(2)$ has index 6 in Γ, so it suffices to verify that the coefficients up to $q^{6k/12} = q^1$ vanish! This follows from the valence formula. But for $\Gamma(2)$, it's simplest just to solve for a_i, b_i directly. In particular, to get the modular derivative of $\theta_3(\tau)$, we compare the coefficients in the $q^{1/2}$-expansion of $\theta_3^{-1}(\tau)D\theta_3(\tau)$ so that they match with $a_3\theta_3(\tau)^4 + b_3\theta_4(\tau)^4$ and adjust a_3 and b_3 accordingly. We get that $a_3 = 1/24$ and $b_3 = 1/12$ producing:

$$\theta_3^{-1}(\tau)D\theta_3(\tau) = \frac{1}{24}\theta_3^4(\tau) - \frac{1}{12}\theta_4^4(\tau)$$

and hence the modular derivative of θ_3 is:

$$D\theta_3(\tau) = \theta_3(\tau)\left[\frac{1}{24}\theta_3^4(\tau) - \frac{1}{12}\theta_4^4(\tau)\right]. \tag{5.1}$$

Similarly we find the modular derivative of $\theta_4(\tau)$ to be:

$$D\theta_4(\tau) = \theta_4(\tau)\left[\frac{1}{24}\theta_4^4(\tau) - \frac{1}{12}\theta_3^4(\tau)\right]. \tag{5.2}$$

Note that another way to get the θ_4 derivative is to use the modular symmetry by $\tau \mapsto \tau + 1$ and apply it to both sides of eq.(5.1) since we know that $\theta_3(\tau + 1) = \theta_4(\tau)$ by eq.(2.19) from Section 2 above.

Now, it is also convenient to know that the modular derivative of η is zero! We can see this by noting that $\frac{1}{\eta}D(\eta)$ will be a holomorphic modular form (because η vanishes nowhere in $\mathbb{H}$) of weight 2, for $PSL_2(\mathbb{Z})$. The only such modular form is 0 (this follows from the valance formula, see p.117 of [20]). This gives rise to an alternative way to calculate the modular derivatives, since for example using Pockhammer symbols we are able to express θ_3 purely in terms of η.

Let's now calculate again (just for fun and to see this distinct approach) the modular derivative of θ_4 using the vanishing of η under the modular derivative. To do this, it would be convenient to have θ_4 expressed purely in terms of ηs. We find exactly this, using the historic identity:

$$\eta(2\tau)\theta_4(2\tau) = \eta(\tau)^2.$$

Alternatively we could also use the identity $\theta_4(\tau) = 2\theta_3(4\tau) - \theta_3(\tau)$ as we now have the modular derivative of θ_3 derived above readily available. Many other ways exist of course. We have:

$$D\theta_4(\tau) = D(\frac{\eta(\tau/2)^2}{\eta(\tau)}) = \theta_4\{1/48E_2(\tau/2) - 1/12E_2(\tau)\} \tag{5.3}$$

after using the quotient rule. Notice here again that the modular derivative of θ_3 would follow from eq.(5.3) under $\tau \mapsto \tau + 1$ as well.

Now, since the modular derivative operation raises the degree of our identities by two, (as evidenced by the appearance of E_2s), we would leave the space of quadratic identities, unless, and this was our hope all along, the same pattern repeats itself in the modular derivatives of the other θ functions; so that they all get multiplied by the same linear combination of functions that we found for the other θ-derivatives. This way we could "cancel" this common factors in the derivatives on both sides after factoring them out and our identities would remain quadratic.

What we found above, unfortunately is that even though we did get back θ_4 up to some linear combination multiple of itself, the combination is not the same as it was in the case of θ_3. This is apparent in comparing the coefficients of θ_3 to that of θ_4 in the formulas (5.1) and (5.2) respectively.

This means that modular derivatives do not appear to be an effective way to go between quadratic identities in the η and θ_3.

5.B An observation on η

There is an interesting result we have stumbled across about the Dedekind η function as a result of investigating some of the resultant identities generated by our lattice method.

We begin by investigating the identity generated by the lattice equivalence:

$$\{72,72\}[6,6] \cong \{1,36\}[1,36]$$

and here is what we get after disassembling this equivalence: $k = 12, a = 6, b = 6, b' = 1, l = 1, c = 1, d = 36, d' = 1$, implying the following identity via equation (4.2).:

$$\begin{aligned} \{\theta_3(\tau/2) - \theta_3(8\tau) - \theta_2(8\tau) - \theta_2(18\tau) + 2\eta(6\tau)\}^2 + 4\{\theta_2(8\tau) - \theta_2(72\tau)\}^2 \\ +\{\theta_3(\tau/2) - \theta_3(8\tau) - \theta_2(8\tau) - \theta_2(18\tau) - 2\eta(6\tau)\}^2 + 4\theta_2(18\tau)^2 \\ +4\{\theta_3(8\tau) - \theta_3(72\tau)\}^2 + 8\theta_2(72\tau)^2 \\ +8\theta_3(72\tau)^2 = 8\theta_3(\tau)\theta_3(36\tau) \end{aligned} \tag{5.4}$$

which after some rearrangement becomes:

$$\eta^2(\tau) = \theta_3(\tau/6)\theta_3(6\tau) - 1/2\{[\theta_2(4/3\tau) - \theta_2(12\tau)]^2 - \theta_2(3\tau)^2 - [\theta_3(4/3\tau) - \theta_3(12\tau)]^2\}$$
$$-\theta_2(12\tau)^2 - \theta_3(12\tau)^2 - 1/4[\theta_3(\tau/12) - \theta_3(4/3\tau) - \theta_2(4/3\tau) - \theta_2(3\tau)]^2. \tag{5.5}$$

So, this means that, although η is not expressible as a linear combination of θ_3's (as shown in [GL2]), its square is expressible quadratically in the θ_3's This is not that interesting, until we notice an old identity involving the square of η, namely:

$$\eta(2\tau)\theta_4(2\tau) = \eta^2(\tau)$$

and combine the two to produce:

$$\eta(2\tau) = \frac{\eta^2(\tau)}{\theta_4(2\tau)} = \frac{\text{RHS of eq.(5.5) above}}{\theta_4(2\tau)}. \tag{5.6}$$

Well, what this says then, is that the Dedekind eta belongs to the field of fractions of the ring generated by θ_3, whereas it does not belong to the ring T_3, only its square. Recall that, η doesn't belong to the ring T_3, because all linear identities are known by Theorem 4.1.

It isn't easy to find other examples of something satisfying those 3 properties. One ring that does indeed have them, is the ring

$$2\mathbb{Z}[\sqrt{2}].$$

To our knowledge and literature search, we are the first to document this peculiar fact about the "nature" of η.

5.C A story leading up to Dirichlet "twists"

And now, another philosophical interlude or fairy tale if you will, some may call "motivation" to support our following generalizations.

Mathematics throughout the ages, involved the interplay between the concrete and the abstract and in my opinion it is crucial to keep the interplay between these two active, especially nowadays when we are equipped with entire arsenals of intricately complex machinery of our "abstractions".

What do I mean here? If we trace back the evolution of thought, inevitably we will find progressions of thinking that starts in some way or form with some patterns observable in our "real" world. To not play upon any sophistication here, let's just take the example of caveman counting on their wall, for example via vertical sticks the number of prey hunted down. Sooner or later, from this "real life" scenario, as we are observing from an evolutionary perspective and generations and generations are being brought up with this way of patterning, something interesting happens. A symbolism independent of the direct life meaning (in this case number of animals hunted down), starts to take a life of its own inside the heads of people using them! No longer will a person need to tie the sticks on the wall to animals hunted down, but the sticks will become a meaningful end to themselves! And hence the process of "abstraction" begins! Suddenly, patterns are going to be observed within these symbolic abstracted entities themselves, independent of the "real-life meaning" and such notions as numbers and number systems will emerge. "Operations" get defined on these "systems". Going from the very pragmatic and practical to the abstract as the hunter understands that if he had 3 sticks on his wall and his rival had 2 yesterday and today he hunted 2 and his rival 3, then he will be able to "add" those sticks on his wall and figure out that "equality" must take place between the "number" of animals he hunted down and the ones that his rival did.

What is important to notice here, is that what was initially "concrete" eventually became "abstract" via some kind of symbolic "representation" of it, and this abstract idea starts to have a life of its own and in a sense will become again "concrete" to the next generation of thinkers to whom it will already be taught as "facts" or "real" things and the process goes on and on. Notice something very similar happening in the context of math, as things that were at one time considered abstractions and hard to comprehend and find relatable or "real", like the complex numbers, become the bread and butter of the next generation of mathematicians who are brought up using them.

Now as systems of abstract entities evolve with time and thought, like stick representations become numbers and numbers eventually morph into mathematics field theories, it becomes increasingly harder and harder in a sense to notice within the concrete structures of the time, the "new abstractions" that are going to lead to "meaningful" mathematics. This is where mathematics becomes somewhat of an art form. The more established structures we have at our disposal, the more patterns we can notice within the infinitude of them, but the harder it becomes to make up the "right" new "definitions" and notice the "right" new "axioms".

Inherently mathematics is a field of categorization. Mathematicians do a lot of categorization of our ideas. Of course it is highly non-trivial how exactly we need to go about our categorizations in accordance with the above expressed need toward meaningful mathematics, but let's examine here below some heuristic reasoning that lead us to a particular end result.

Let's assume that we are given the notion of a series to begin with. Let's follow a possible, but contrived “evolutionary” route of the progression of abstractions from this starting point on, and maybe learn something on the way. We begin by taking a typical abstracted notion of a series: $S_n = \sum_{i=0}^{n} a_i$. Then we look at the concrete examples and start to play:

(a) $S_n = \sum_{i=0}^{n} i$

(b) $S_n = \sum_{i=0}^{n} i^2$

(c) $S_n = \sum_{i=0}^{n} i^k$

Well, (a) is very “natural”, as it’s just the sum of the progression of integers. To get to (b), we could notice that within $\mathbb{Z}$ there is also the ability via another binary operation to multiply integers, so we should perhaps play with that. And once we try (b) we can always ask the ”generalized” version of the same question, hence (c).

Mathematics is also an interplay between the “constant” and the “variable” just as it’s the interplay between the “concrete” and the “abstract”, the “discrete” and the “continuous” . So knowing that these patterns are in general “principle facts” within the whole of mathematics, it is always instructive in my opinion to ask ourselves just how these things are present in our current line of investigation, and if they are not, how we could include them meaningfully, if at all.

So, with this in mind let’s again have a look at our original series and try to look at it from this perspective:

$S_n = \sum_{i=0}^{n} a_i$ what is the “constant” and what is the “variable” in this form?

We are essentially adding up using the underlying notion of the progression of integers $i = 0, ..., i = n$ things a_i that depend on them, i.e. some functions of i. Let’s look at the particular “generalization” that leads to: $S_n = \sum_{i=0}^{n} a_i x^i$.

Here, we noticed that we can add a variable x to help us with the play and get another generalization that gives us an extra "degree of freedom" to play with. Now, this is not just a mindless exchange of a freed up constant into a variable, as this define the rudiments of power series, for example. But needless to say, the point is that we can not know a-priori what will or will not be, so for this reason we can't stop playing with this kinds of "exchanges" in mathematics.

By the time we are looking at entities likes sums and such, we must already have had some kind of working "theory" or at least some kind of framework in mind that we wish to expand and prove results in and with. If this newly generalized form of the series (for examples' sake) enables us to prove previously unseen results with a possibly even larger scope of results covered, then undoubtedly our generalization or "exchange" had utility. (It's quite obvious that power series, for example, have enormous utility.)

Let's keep on going even further, so far, quite mindlessly just using substitutions and change of variables and see what we get:

$$S_n = \sum_{i=0}^{n} a_i q^i \text{ where } q = e^{2\pi i \tau}.$$

This q substitution makes sense from the point of view of Fourier series, any periodic function can be written in that way.

Let's now focus on what is the "essential feature/ structure" that is the guiding "light" of our sum: Perhaps it is this "progression":

$$i = 0, ..., n.$$

Well, one of the things about elements of $\mathbb{N}$ is that they are discrete! We now again remind ourselves that math is also born from the interplay of not just the constant and the variable, but the discrete and the continuous and there are many examples of both forms to generalize. Here we can for example extend $\mathbb{N}$ to allow for $\mathbb{Z}$ and suddenly we are having to deal with interesting phenomena like singularities theory and calculating complicated integrals using residues.

Let's now look at yet a different aspect of the "form", we call series. Let's see how we could generalize $\mathbb{Z}$ in the definition of the above series. What may be one particular essential feature of $\mathbb{Z}$ that we can potentially expand our generalization, progression into?

What do we know about $\mathbb{Z}$? What is it an example of? I.e. what structure does $\mathbb{Z}$ possess? Let's look back at our "meta organizing principles" (at least the ones we are playing with here, for the sake of exercise) for some inspiration! We started off by the comparison of concrete to that of the abstract. So let's ask now the reverse, that is, what abstraction is $\mathbb{Z}$ a concrete example of? Remember here the "dual" approach of the Poincaré quote we mentioned in the introduction.

One possible answer is that $\mathbb{Z}$ is one of the most elementary examples of what is better known as a geometrical lattice! We note here again that this question opens up another can of worms and each can leads to new and exciting directions! We just happen to chose the direction we do, as it's relevant to our end goal in mind, as of course $\mathbb{Z}$ is also an example of many other structures, besides a lattice.

So, now, can we generalize the notion of series using this particular information?

Let Λ be any geometric lattice. Let's extend our notion of "discrete" summation of things over general lattices

$$S(\Lambda)(\tau) = \sum_{v \in \Lambda} q^{v \cdot v}$$

and we arrive at the definition of theta series of lattices.

Going yet in a different direction, we could attack the integrality inherent in the definition of θ_3 and "exchange" $\mathbb{Z}$ with yet a different "aspect" of its structure. Namely, what we have in mind is what's called Dirichlet twists.

We take the structure of $\mathbb{Z}$ preserved under a particular homomorpism such that its image lets us play with elements in the upper half plane and this way can interface with our q variable compatibly in the definition of theta series.

We stop here the "philosophical progression" and expand on this direction. Of course note that we could take quite a detour here if we instead of choosing to generalize from one discrete entity $\mathbb{Z}$ to another discrete one Λ, we took the route that we generalized from discrete to a continuous entity where we would land in the field of functional analysis, in case we instead chose to play with the replacement of the discrete notion of summation with that of continuous integrals.

5.D Dirichlet "twists"

A very interesting generalization involves the "twists" of θ_3 by what's called Dirichlet characters. These characters play a big role in the theory of modular forms – for example they can twist modular forms. They are a key tool in proving that there are infinitely many primes in every arithmetic progression $i, i+n, i+2n, ...$(when $\gcd(i,n) = 1$).

Choose any positive integer $N \in \mathbb{N}$, then the numbers $\{1 \leq n \leq N\}$ coprime to N form a group $(\mathbb{Z}/N\mathbb{Z})^\times$ given by multiplication mod N. With this, we can now define:

Definition: A Dirichlet Character χ mod N is: any 1-dimensional complex representation of $(\mathbb{Z}/N\mathbb{Z})^\times$ i.e. we have this homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

We call m the period of a Dirichlet character χ if it is the smallest positive integer with this property: whenever i, j are coprime to N and $i \equiv j(\text{mod } m)$, then $\chi(i) = \chi(j)$. χ will have period dividing N. If the period is exactly N, we call χ primitive mod N. Now, in order to utilize this in our definition of θ_3 as a form of associating "something" to each $z \in \mathbb{Z}$, we need to extend this definition to all of $\mathbb{Z}$ which is possible as follows:

$$\chi(z) = \chi(z \bmod N) \text{ if } gcd(z, N) = 1;$$

$$\chi(z) = 0 \text{ if } gcd(z, N) > 1.$$

Definition: χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$.

Note that there will be the same number of Dirichlet characters mod N, as the size of $(\mathbb{Z}/N\mathbb{Z})^\times$.

Now we are in position to define how Dirichlet characters "twist" θ_3:

$$\theta(\chi, \tau) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \chi(z) q^{\frac{n^2}{2}} = \frac{1}{2} \sum_{a=1}^{N} \chi(a) \psi_{N/a}(N^2 \tau) \tag{5.7}$$

for χ nontrivial and even and

$$\theta(\chi, \tau) := \frac{1}{2}\sum_{z\in\mathbb{Z}} n\chi(z)q^{\frac{n^2}{2}} = \frac{N}{2}\sum_{a=1}^{N} \chi(a)\psi_{N/a}(N^2\tau) \tag{5.8}$$

for χ odd.

Here is a fascinating example:

Example 5.D.1: As a warmup, let's find all Dirichlet characters mod 12. First note that $(\mathbb{Z}/12\mathbb{Z})^\times = \{\pm 1, \pm 5\}$. The multiplicative group $(Z/12Z)^x$ is the direct product of the order-2 subgroup generated by -1, and the order-2 subgroup generated by 5. So any homomorphism $(\mathbb{Z}/12\mathbb{Z})^\times \to \mathbb{C}^\times$ is uniquely determined by its values on those generators. Since $7 = -5$ is the product of -1 and 5 mod 12, $\chi(7) = \chi(-1)\chi(5)$. Because -1 has order 2, it must be sent to ± 1. Similarly, 5 has order 2 (since $5^2 \equiv 1 \pmod{12}$), so it too must be sent to either ± 1. Because -1 and 5 are independent generators, any of those 4 possible choices of signs defines a Dirichlet character. If both -1 and 5 get sent to the same value, then the character will have period 6, not 12. The choice $-1 \mapsto -1$ and $5 \mapsto 1$ is also imprimitive, as it has period 4. The only primitive character at modulus 12 corresponds to the remaining choice:

$$\chi(\pm 1) = 1 \text{ and } \chi(\pm 5) = -1.$$

This is fascinating once we let $\chi, \tau/12$ do its twist to $\theta_3(\tau)$, as follows:

$$\theta(\chi, \tau/12) = \frac{1}{2}\sum_{a=1}^{12} \chi(a)\psi_{12/a}(12\tau) = \frac{1}{2}\{2\psi_{12}(12\tau) - 2\psi_{12/5}(12\tau)\} = \eta(\tau)$$

which we obtain via our linear identity (2.3) above.

Hence, interestingly the twist of θ_3 associated to the above unique primitive Dirichlet Character mod 12, reproduced $\eta(\tau)$.

Next, note that these formulae above can be inverted. For N even and n coprime to N we get:

$$\psi_{N/n}(\tau) = \frac{2}{\varphi(N)} \sum_{\chi} \chi(n)\theta(\chi, \tau/N^2) \tag{5.9}$$

and for N odd and n coprime to N we get:

$$\psi_{N/n}(\tau) = \frac{2}{N\varphi(N)} \sum_{\chi} \chi(n)\,\theta(\chi, \tau/N^2)\ ) \tag{5.10}$$

where the sums are over all characters χ (primitive and imprimitive) of $(\mathbb{Z}/N\mathbb{Z})^{\times}$. These 'twisted' theta functions are thus modular forms, and any n-dimensional Euclidean integral lattice, and any of its (finite-order) glue classes, can be expressed as a homogeneous polynomial of degree n in them.

5.E Jacobi forms

Another possible generalization of the material presented here, similarly as was done in [GL2], is to introduce another variable $z \in \mathbb{C}$, creating *Jacobi forms*, namely

$$\vartheta_3(z, \tau) := \sum_{\mathbb{Z}} e^{2m\pi i z + \pi i \tau m^2};$$

$$\psi_k(z, \tau) := \sum_{\mathbb{Z}} e^{2(m+1/k)\pi i z + \pi i \tau (m+1/k)^2}.$$

The resulting identities though are quite complicated, because the specific rotation involved in the integral equivalence of lattices will be captured in the z-dependence.

Chapter 6

Conclusions

As we now see, finding identities is a non-trivial feat. They either come to life by being "stumbled upon" as byproducts of examining certain theories, or we have to find systematic ways to generate them, as the other alternative of producing them via "brute force" methods, like Ramanujan did, are very timely and even though may be resultant of flashes of genius, they do not really shed light on the structural underpinnings that govern them.

We are therefore very fortunate to have systematics methods at our disposals to generate them, like the lattice method developed by the authors of [GL1].

What we have done here for the most part, is a generalization of their lattice method to quadratic identities that, unlike was done in [GL1,] includes not only θ_3 identities, but identities involving both θ_3 and η.

We not only utilized their lattice gluing decomposition idea to generate these identities, but we have spelled out (based on the same underlying idea) a general framework to be able to deal with identities, not only in the quadratic, but of any degree n.

We also examined ways to go in between these identities up to equivalence. We have looked at the idea of utilizing the modular symmetries of the functions we are dealing with and utilizing them in symmetrizations of the identities to reduce their complexities. This is manifest either by the disappearance of the ηs from the original identity to the symmetrized one or simply by the reduction in the number of terms involved in the identity.

We have examined another possible idea to go between identities, namely the application of the modular derivative, albeit here we arrived at a negative conclusion of their utility.

One of the particularly interesting finding in this paper is the observation we took on the nature of η belonging to the field of fractions of T_3. This is quite unexpected and came out purely from examining some of the newly generated identities and matching and connecting them in various ways to older known/historic identities.

Finally, we mention probably the most significant possible generalization of what we were doing here, namely the Dirichlet Twists, as they provide a more general framework and a better "language" within which generation of identities extend beyond the scope we examined. They provide the platform for any n-dimensional Euclidean integral lattice, and any of its (finite-order) glue classes to be expressed as a homogeneous polynomials of degree n in them. Thus they provide the natural generalization of our lattice technique to glues of arbitrary order. We would obtain long lists of polynomial identities in the $\theta(\chi, \tau)$.

Chapter 7

Appendix A: An interesting application of lattices: "Can you hear the shape of drums?"

Let's recall Mark Kac's famous question: "Can one hear the shape of a drum?" This was an exciting sounding applied rephrase of the age old math question whether the eigenvalues of the Laplacian for the Dirichlet problem determined by a planar domain determine the shape of that domain?

I chose to talk about this here as I find this a fascinating interplay between lattice theory, modular forms, geometry and physics! In no way is this material new, or original, nor is my exposition of it, as most of the material here closely follows similar expositions in [21] [31] and [32].

It turns out this questions was harder to answer than it appeared first, as it took about half a century for mathematicians to conclude with the answer in the originally posed 2D planar context. However, interestingly enough, the answer came apparent first in the context of generalizing the question to arbitrary Riemann manifolds. John Milnor provided the first counterexamples in the form of two 16-dimensional tori that “sounded” the same, even though they possessed different shapes.

I would like to present here a tentative progression of ideas one could potentially follow as a, so to speak “schedule of discovery” that could have been followed in finding the conclusive answer to the question for the generic planar domain. My reason for doing so is that via this example I will be able to point out generic methodological approaches in doing research in theoretical mathematics. In a way I will try to point out some heuristics that take place in such endeavours.

First, we set out to solve a reduced problem: Can we hear the shape of a string? We do this often in mathematics as there are invaluable insights one can gain by doing so.

The answer is a resounding yes! , as it's well known that the partial differential equation for the string gives the superposition of sine and cosine waves for the general solution, we show this now below.

A vibrating string of length L can be idealized by the interval $[O, L]$ of real numbers. We are satisfied to have a working model for this system, if we find a function $f(x,t)$ for positive t, that shall express the amplitude of the string at an given time t and location x. Since the endpoints are fixed we have the boundary conditions: $f(o,t) = f(l,t) = 0$ for all t time. We know that this system needs to satisfy the wave equation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}.$$

The standard way to seek solutions for this system is to assume via separation of variables to seek stationary solutions, $f(x,t) = g(x)g(t)$, where the general shape of the wave remains stationary, $g(x)$, and it's amplitude is adjusted by $h(t)$. We represent the fixed endpoints here via the $g(0) = g(L) = 0$ requirement.

When we substitute this above separation assumption into the wave equation, we get:

$$g(x)h''(t) = g''(x)h(t) \text{ i.e. } g''(x)/g(x) = h''(t)/h(t).$$

Since the LHS is purely a function of x and the RHS is purely a function of t, the only way to make sense of this is to set them both equal to a constant, as we can see by differentiation of the LHS by x and the RHS by t. We can show that this constant needs be negative, so it's convenient to set both sides equal $-\lambda$ Thus, we get:

$$g''(x) = -\lambda g(x) \tag{7.1}$$

$$h''(t) = -\lambda h(t). \tag{7.2}$$

One can check that the functions $sin(\sqrt{\lambda}x)$ and $cos(\sqrt{\lambda}x)$ are solutions of the spatial equation, and, in fact, the general solution turns out to be a combination of these:

$$g(x) = A * \sin(\sqrt{\lambda}x) + B * \cos(\sqrt{\lambda}x)$$

with A, B constants. From the boundary condition $g(0) = 0$, it follows (since $\cos(0) = 1$ and$\sin(0) = 0$) that $B = 0$, so $g(x) = A * \sin(\sqrt{\lambda}x)$.

The boundary condition $g(L) = 0$ implies that $\sqrt{\lambda}L$ must be an integral multiple of π, i.e.$\sqrt{\lambda}L = n\pi$, so λ must be a number of the form $\lambda = (\frac{n\pi}{L})^2$.

The general solution of equation (7.2) is:

$$h(t) = C * \sin(\sqrt{\lambda}t) + D * \cos(\sqrt{\lambda}t)$$

here the constants C, D are determined by the initial configuration and velocity of the string. $h(t)$ represents a periodic oscillation of frequency $\sqrt{\lambda}/2\pi$ But the same λ as in (7.1) appears in (7.2), so it follows that h represents an oscillation of frequency $\sqrt{\lambda}/2\pi = n/2L$ So, the basic waveform g is given by a sinusoidal function, but its frequency $n/2\pi$ must be carefully chosen in order that the function be zero at 0 and L. Thus the frequencies at which the string can vibrate are: $1/2L, 2/2L, 3/2L, ...$ and so on.

We have thus solved the one-dimensional analogue of Kac's question: The shape of a stretched string is captured completely by its length L, and we can recover L from the spectrum as half of the reciprocal of the lowest frequency. Thus one can hear the shape of a string!

Naturally, going up a dimension, we explore the 2-dimensional version of this problem, for the generic 2D-planar domain as posed in the original question we are interested in answering. We set up a PDE very similar to the 1D version, except here the spatial derivative is replaced by the Laplacian

$$\nabla = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$$

i.e. $\frac{\partial^2 f}{\partial^2 t} = \nabla f$.

We would proceed in theory similar as in the 1D case and once we get the allowable λs from the spatial equation, the temporal equation is, just as above with the allowable λs from the spatial case. However, we soon discover that mathematicians do not have an exact solution for spatial equation: $\nabla g = -\lambda g$ for a general planar domain! Numerical approximation do exist, however, the trick is that we can't expect the shape of any random planar domain to be captured by a finite number of numbers (frequencies and only finite degree of accuracy) we would gain by the numerical solutions! This is in contrast with the 1D case, where the "shape" of a string is essentially captured by just the single number, the length, L, of the string!

The lack of this solution in conjunction with the lack of obvious counterexamples leads to yet again extension/generalization of the problem to higher dimensions, that of n-dimensional manifolds. So, we ask, can we hear the shape of a Riemannian manifold? If the answer is affirmative, then clearly this must be a harder problem, since our original question is just a sub-case. On the other hand, if the answer is negative, then we may have allowed for the study of wider scope of counterexamples, from which we may draw learning onto the 2D case. In 1964, John Milnor exhibited a pair of isospectral 16-dimensional manifolds. Milnor's examples are constructed by gluing together opposite faces of a cleverly chosen 16-dimensional "parallelogram" to produce flat tori.

How does this relate to lattice theory? Well, we know a lattice is a subset in $\mathbb{R}^n$ of n linearly independent vectors, $\{v_1, ..., v_n\}$ So if we take a lattice in some Euclidean space, we can construct a quotient manifold out of the space by "rolling the space up" around the lattice points, i.e. we identify 2 points as being the same if their difference lies on a lattice point. So, similarly in $\mathbb{R}^{16}$ of Milnor's example, there exist two dissimilar lattices, $E_8 \oplus E_8$ and D_{16}^+ whose quotient tori happen to be isospectral! We can describe D_n as the sublattice of $\mathbb{Z}^n$ consisting of all points $(m_1, ..., m_n)$ with $\sum_i m_i$ even. So D_n is the checkerboard lattice. One of the glue classes of D_n is $(1/2, 1/2, ..., 1/2)+D_n$.D_n^+ is $D_n[1/2, ..., 1/2]$. This is integral iff $4|n$, in which case it is self-dual. $D_4^+ = \mathbb{Z}^4$ and $D_8^+ = E_8$.

The reason is that the spectrum of the quotient manifold happens to be determined entirely by the number of vectors of each length in the lattice we are quotienting out by, so in essence determined by its theta function!

$$\Theta_\Lambda(q) = \sum_l a_l q^l$$

where $q = e^{2\pi i z}$ and a_l is the number of vectors in Λ of length l.

So, accordingly, we call a lattice property "audible" if it's determined by the theta function.

The main problem was, then, to find in which dimensions do exist dissimilar lattices that have the same theta function! Equivalently, the same number of vectors of every length!

Now, the connection to modular forms happens as we realize that the theta functions of even unimodular lattices (lattices that have every length squared even and have just one point per unit volume) are modular forms for the full modular group $PSL_2(\mathbb{Z})$ i.e.

$$\Theta(\frac{az+b}{cz+d}) = (cz+d)^{n/2}\Theta(z)$$

for every 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$ with determinant 1 where n is the dimension of the lattice.

Now, since in 16-dimensions there happens to be only one such function up to scalars, namely:

$$\Theta(q) = 1 + 480 \sum_n \sigma_7(n) q^{2n}$$

where $\sigma_7(n)$ is the sum of the 7-th powers of the divisors of n (i.e. $\sigma_7(n) = \sum_{d|n} d^7$), therefore every 16-dimensional unimodular lattice must have this very theta function (!), since the coefficient of q^0, the number of vectors in each lattice of length zero is 1. So, just to reiterate, both of these lattice's theta series must be modular forms with their q-expansions starting with a 1, but there is only one such function, in 16-dimensions!

Now, we can look at the geodesic on a Riemannian manifold as the natural analogue of a straight line in the Euclidean plane, i.e. a curve that does not deviate from the direction in which it is travelling. The list of vibration frequencies of a vibrating manifold is closely related to the list of lengths of closed geodesics on the manifold; (this is plausible, as one might expect waves to propagate along geodesics.) Milnor's two 16-dimensional tori were chosen so that the list of lengths of closed geodesics was the same for each. It can be shown that two flat tori with the same geodesic lengths must be isospectral, so it follows that Milnor's tori both vibrate at exactly the same frequencies!

In 1984, Toshikazu Sunada of Tohoku University realized that an idea from group theory could be brought to bear on the problem of constructing isospectral manifolds. Sunada's technique involved permutation representations and linear representations of groups.

A *permutation representation* for a group G, is just a homomorphism that assigns to each group element g a permutation on a fixed set X.

A *linear representation* of a group G, is a homomorphism that assigns to each element g in the group a linear transformation T on a fixed vector space V; since each linear transformation can be represented by a matrix, we are in effect assigning to each element g in the group an $n \times n$ matrix M.

Notice that given a permutation representation for G, we can always define a corresponding linear representation, via letting the columns of our linear transformation matrix M to be determined by seeing that if the permutation,g, on the set of numbers 1 to n exchanges i to$j$, then our i-th column will have a 1 in the j-th row and rest zeros.

The interesting fact for us is that it can happen that two distinct permutation representation give rise to linear representations that are isomorphic and as it turns out, this way a key thing to notice towards a solution to the problem of constructing isospectral regions! Although Sunada's technique was very interesting, it was believed that it offered no insight on Kac's original question, as the the spaces manufactured by the method could not be regions in the Euclidean plane. In 1989, however, Pierre Berard of the Institut Fourier in Grenoble, France, discovered a new proof of Sunada's theorem that permitted wider application of the method. In 1990, Scott Wolpert et al. used Berard's discovery to construct a pair of isospectral planar regions that are not geometrically congruent, hence answering the questions:

One cannot hear the shape of a drum!

We will now outline the construction of these geometrically distinct planar domains that sound the same. Let G be the free group generated by the products of α, β, γ Let X be the set $\{1, 2, 3, 4, 5, 6, 7\}$ Let now:

α correspond to the following permutation on X: (26)(37)

β correspond to the following permutation on X: (24)(35)

γ correspond to the following permutation on X: (12)(56).

Now, we can associate to this set X and these permutations $\alpha\beta, \gamma$ a Cayley graph faithfully encoding the information of the permutations. Just draw vertices labelled with the elements from our set X and connect them with edges corresponding to our permutations α, β, γ respectively, when two numbers are permuted with either one of these permutations. Based on this diagram, we are ready now to construct one of the planar regions, as follows. We start with the "topmost" vertex 7, from the Cayley diagram. We associate to each number from our set X a triangle, with sides labelled, α, β, γ respectively. Now, we take this number 7 labelled triangle and reflect it into a copy of itself along each of it's α, β, γ labelled edges, according to which number the permutations corresponding to that side maps the number 7 into, and label the resultant triangle with the image of 7 under the permutation corresponding to the edge we are reflecting with respect to. So, 7 goes to 3 by α and is left untouched by β and γ, hence we reflect our triangle labelled 7 along it's α side to a reflected copy of itself, labelled 3 and do not create reflected copies along the sides corresponding to β, γ.

Next, we take the just drawn reflected triangle labelled 3 and see what each permutation does to it. We observe that other than α taking it back to 7, it's only β that moves it, γ leaves it untouched. So since β maps 3 to 5, we reflect our triangle labelled 3 along it's β side to a copy of itself, now labelled 5, and do not create a reflected copy along the γ side. We keep on going according to this recipe, to reach our first planar domain D1.

Now, as it turns out that G has another permutation representation, where:

α' correspond to the following permutation on X: (57)(46)

β' correspond to the following permutation on X: (24)(35)

γ' correspond to the following permutation on X: (12)(56);

This has the property that the linear representation corresponding to this permutation representation is isomorphic to the linear representation we get from the first permutation representation. It's clear, as we compare the respective Cayley graphs that the permutation representations they represent are distinct ones!

Now, just as we did before for the first representation, we can create our second planar region D2, corresponding to this representation's Cayley graph, exactly analogous to what we did above. The magic is that since these regions were created from distinct permutation representations that have isomorphic linear representations, we can use a transplantation argument, proposed by Buser to show that they are isospectral regions!

Assume, that a waveform ϕ is defined for the region D1. Since D1 is made by reflection of the same "model triangle" through it's edges multiple times, according to the recipe dictated by our permutations, we can observe few things.

For ease of argument, we denote the waveform ϕ's restrictions to each of the 7 separate model triangle regions, by A,B,C...,G

(i)First, since the outer edges of our region is fixed from vibrating, the waveform ϕ takes the value zero on those edges.

(ii) Second, since ϕ is a valid waveform on D1, it smoothly transitions through inner edges.

Notice, that any linear combinations of the separate components A,B,...,G of ϕ defined on the model triangle will create again a valid waveform for the model triangle.

We are seeking to transplant this waveform ϕ defied for D1 onto a valid form defined on D2. Note that any proposed transplanted waveform will have to obey conditions (i) and (ii) on D2 (i.e. needs be zero on the outer edges and smoothly transition across inner boarders.) Here we check these conditions for the proposed transplant solution: On the first topmost triangle the proposed waveform is: B-C+D check (i): On the β (green) edge: we see from D1 that B and C agree on the green β inner boundary as they need to transition smoothly and hence B+-C=0 ; we also note that D is zero on the green β boundary; so all together B-C+D is indeed zero on the green=β boundary edge as desired. On the γ (blue) edge: from the region D1 we observe: B is zero C=D and hence, B-C+D=0 as desired. On the α (red) edge: note this is an inner boundary edge for this triangle, so we resort to check condition (ii) here. check (ii): On the α (red) inner edge: the proposed solution is A+C+E for the adjacent triangle, so we need to check smooth transition; Note from D1 that A smoothly transition to B and D smoothly transition to E; now C on the α=red edge is zero on D1, but we can use what's called a reflection principle to extend a waveform smoothly through it's boundary by taking it's negative value on the mirror imaged reflected domain, and still get a valid waveform on the reflected extended domain. Hence, here we see that B-C+D smoothly transition to A+C+E as once again: A to B + E to D + C to -C are all smooth transitions and hence by the linear superposition principle their combination is not only a valid waveform, but a smoothly transitioning one through the boundary edge.

Similarly we can check that the proposed transplant waveform work well for the D2 domain.

So what does this show? Well, we started with a waveform with λ eigenvalue corresponding to a certain frequency of vibration on D1; we considered the restriction of this waveform onto each of the domain's 74 sub-triangle regions in order to define a valid waveform on D2 via superposition of the model regions waves. Since superposition does not change the corresponding eigenvalue/frequency of vibration we managed to be able to assign to each Laplacian of λ eigenvalue on D1 a valid Laplacian of the same eigenvalue on the sister domain D2.

By symmetry of the argument we can argue that the λ eigenspaces for the regions D1 and D2 have the same dimension, just reverse the mapping direction via the same procedure to transplant solutions on D2 to D1.

So far we have only provided a solution that we checked works, but how do we get one?

Well, we work backwards trying to constrain our solution by the conditions (i) and (ii) and in hopes of the theory propsed by Buser of distinct permutation representations having isomorphic linear representation we know that there must exist solutions obeying these constraints, with this in mind, we proceed as follows:

On D2, starting with the topmost triangle, we note that it happens to be it's red edge that is an inner transition edge, so we look for already smooth transitioning waveforms on inner red boundary edges existing within D1. We find that red=α edge inner separates triangle regions A to B and E to D. So as a first trial, we could propose that B+D could work for the topmost triangle and A+E for the below reflected one. However this only satisfies our second constrain (ii) and upon checking constrain (i) we realize that: on the green β edge, B goes to C and D goes to zero and hence the only way we will remain consistent with requirement (i) is to add a -C waveform to our mix in order to compensate for the effect of getting a C across the boundary via the extension principle. It turns out that this $\pm$ C does actually create a fully consistent waveform across all other triangles as we work our way through reflections.

It is interesting to consider questions analogous to whether one can hear the shape of a drum, especially from what we learned from history of the exploration of the above question with the drum. There lies a wide spectrum of questions to explore, once we consider the following scheme:

Consider “objects” X (manifolds or some such, say from geometry) to which we can sensibly attach the notion of a spectrum. Consider now, in addition to our object $x \in X$ some additional ”linked” property p, that our object possess. Ask the questions:

How is the spectrum of the object x reflected in the linking of x to$p$? So, specifically for example: Once, p is known, what can we tell about the object x, holding the spectrum constant? Or:

Once, the object x is known with certain spectrum, what can we tell about property p?

For example, we could have our object x, some nonlinear classical mechanical system and the property, p dynamics of the system.

So, we know that if one makes a seemingly insignificant modification to the initial conditions, the system's long-time behaviour is affected drastically. It is of interest to comprehend if and how chaos shows up in the spectrum of the associated quantum-mechanical system.

Chapter 8

Appendix B: Sage source code

Here are the lattice equivalences generated from Theorem 4.2 above, in the following format:

$$[k, l, a, b, c, d, b', d', m, n]$$

and underneath:

$$[ka, kb][a, bb']$$

$$[lc, ld][c, dd']$$

corresponding to the variables identified in the theorem. Refer to equation (4.1.). These were calculated in Sage. We attach here the sage output and a sample algorithm for each part (i) and (ii) of the Theorem as the other ones are similar.

Satisfying part (i) constraints for k,l,a,b,c,d,b',d' pozitive integers in Theorem 4.2:

```
sage:
for k in [12]:
```

```
..for l in [12]:
... for B in [1,5]:
...     for D in [1,5]:
...         for m in [1,..,floor(k/2)]:
...             for n in [1,..,B]:
...                 for c in [1,..,k*m*n]:
...                     for b in
                            [1,..,floor((k^2-m^2)*l*c/(n^2))]:
...                         a=1
...                         while (a+B^2*b)%k!=0:
...                             a=a+1
...                         while a<=b:
...                             if (lcm(k*m,k*n)%c==0 and gcd(m*a,n*b)%c==0 and
(B*m-n)%k==0 and k*l*c==m^2*a+n^2*b and
m^2*a+n^2*b<=k^2*a and ((m*a*D+n*c)%(l*c)==0 or
(m*a*D-n*c)%(l*c)==0) and ((-m*c-n*b*D)%(l*c)==0 or
(m*c-n*b*D)%(l*c)==0) and
gcd(a,gcd(b,int((a+B^2*b)/k)))==gcd(c,gcd(D,int((c+D^2*(a*b/c))/l)))==1
and gcd(B,k)==gcd(D,l)==1 and c*c<=a*b and
(vector([a,B,k])==vector([c,D,l]))==false
and (a+B^2*b)%k==0 and (c+D^2*(a*b/c))%l==0  ):
...                         print [k,l,a,b,c,int(a*b/c),B,D,m,n]
...                         print [k*a,k*b,a,b*B];[l*c,l*int(a*b/c),c,int(a*b/c)*D]
...                             a=a+k
```

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[12, 5460, 1, 455]

[4, 12, 11, 37, 1, 407, 1, 1, 1, 1]

[44, 148, 11, 37]

[12, 4884, 1, 407]

[4, 12, 19, 29, 1, 551, 1, 5, 1, 1]

[76, 116, 19, 29]

[12, 6612, 1, 2755]

[4, 12, 17, 31, 1, 527, 1, 5, 1, 1]

[68, 124, 17, 31]

[12, 6324, 1, 2635]

[4, 12, 7, 41, 1, 287, 1, 5, 1, 1]

[28, 164, 7, 41]

[12, 3444, 1, 1435]

[4, 12, 5, 43, 1, 215, 1, 5, 1, 1]

[20, 172, 5, 43]

[12, 2580, 1, 1075]

k=12,l=3

[12, 3, 17, 19, 1, 323, 1, 1, 1, 1]

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[204, 228, 17, 19]
[3, 969, 1, 323]
[12, 3, 16, 20, 1, 320, 1, 1, 1, 1]
[192, 240, 16, 20]
[3, 960, 1, 320]
[12, 3, 14, 22, 1, 308, 1, 1, 1, 1]
[168, 264, 14, 22]
[3, 924, 1, 308]
[12, 3, 13, 23, 1, 299, 1, 1, 1, 1]
[156, 276, 13, 23]
[3, 897, 1, 299]
[12, 3, 11, 25, 1, 275, 1, 1, 1, 1]
[132, 300, 11, 25]
[3, 825, 1, 275]
[12, 3, 10, 26, 1, 260, 1, 1, 1, 1]
[120, 312, 10, 26]
[3, 780, 1, 260]
[12, 3, 8, 28, 1, 224, 1, 1, 1, 1]
[96, 336, 8, 28]
[3, 672, 1, 224]
[12, 3, 7, 29, 1, 203, 1, 1, 1, 1]
[84, 348, 7, 29]
[3, 609, 1, 203]
[12, 3, 5, 31, 1, 155, 1, 1, 1, 1]
[60, 372, 5, 31]
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[3, 465, 1, 155]
[12, 3, 4, 32, 1, 128, 1, 1, 1, 1]
[48, 384, 4, 32]
[3, 384, 1, 128]
[12, 3, 2, 34, 1, 68, 1, 1, 1, 1]
[24, 408, 2, 34]
[3, 204, 1, 68]
[12, 3, 1, 35, 1, 35, 1, 1, 1, 1]
[12, 420, 1, 35]
[3, 105, 1, 35]
[12, 3, 5, 7, 5, 7, 5, 1, 1, 5]
[60, 84, 5, 35]
[15, 21, 5, 7]
[12, 3, 6, 6, 3, 12, 5, 1, 3, 3]
[72, 72, 6, 30]
[9, 36, 3, 12]
[12, 3, 3, 9, 3, 9, 5, 1, 3, 3]
[36, 108, 3, 45]
[9, 27, 3, 9]
[12, 3, 1, 11, 1, 11, 5, 1, 5, 1]
[12, 132, 1, 55]
[3, 33, 1, 11]
[12, 3, 4, 80, 5, 64, 5, 1, 5, 1]
[48, 960, 4, 400]
[15, 192, 5, 64]
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[12, 3, 2, 130, 5, 52, 5, 1, 5, 1]

[24, 1560, 2, 650]

[15, 156, 5, 52]

k=3,l=12

[3, 12, 13, 23, 1, 299, 1, 1, 1, 1]

[39, 69, 13, 23]

[12, 3588, 1, 299]

[3, 12, 11, 25, 1, 275, 1, 1, 1, 1]

[33, 75, 11, 25]

[12, 3300, 1, 275]

[3, 12, 17, 19, 1, 323, 1, 5, 1, 1]

[51, 57, 17, 19]

[12, 3876, 1, 1615]

[3, 12, 7, 29, 1, 203, 1, 5, 1, 1]

[21, 87, 7, 29]

[12, 2436, 1, 1015]

[3, 12, 5, 31, 1, 155, 1, 5, 1, 1]

[15, 93, 5, 31]

[12, 1860, 1, 775]

k=12, l=2

[12, 2, 11, 13, 1, 143, 1, 1, 1, 1]

[132, 156, 11, 13]

[2, 286, 1, 143]

[12, 2, 9, 15, 1, 135, 1, 1, 1, 1]

[108, 180, 9, 15]

[2, 270, 1, 135]

[12, 2, 7, 17, 1, 119, 1, 1, 1, 1]

[84, 204, 7, 17]

[2, 238, 1, 119]

[12, 2, 5, 19, 1, 95, 1, 1, 1, 1]

[60, 228, 5, 19]

[2, 190, 1, 95]

[12, 2, 3, 21, 1, 63, 1, 1, 1, 1]

[36, 252, 3, 21]

[2, 126, 1, 63]

[12, 2, 1, 23, 1, 23, 1, 1, 1, 1]

[12, 276, 1, 23]

[2, 46, 1, 23]

k=2,l=12

[2, 12, 11, 13, 1, 143, 1, 1, 1, 1]

[22, 26, 11, 13]

```
[12, 1716, 1, 143]
[2, 12, 7, 17, 1, 119, 1, 5, 1, 1]
[14, 34, 7, 17]
[12, 1428, 1, 595]
[2, 12, 11, 13, 1, 143, 5, 1, 1, 1]
[22, 26, 11, 65]
[12, 1716, 1, 143]


k=12, l=1


[12, 1, 6, 6, 1, 36, 1, 1, 1, 1]
[72, 72, 6, 6]
[1, 36, 1, 36]
[12, 1, 5, 7, 1, 35, 1, 1, 1, 1]
[60, 84, 5, 7]
[1, 35, 1, 35]
[12, 1, 4, 8, 1, 32, 1, 1, 1, 1]
[48, 96, 4, 8]
[1, 32, 1, 32]
[12, 1, 3, 9, 1, 27, 1, 1, 1, 1]
[36, 108, 3, 9]
[1, 27, 1, 27]
[12, 1, 2, 10, 1, 20, 1, 1, 1, 1]
[24, 120, 2, 10]
[1, 20, 1, 20]
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[12, 1, 1, 11, 1, 11, 1, 1, 1, 1]

[12, 132, 1, 11]

[1, 11, 1, 11]

Satisfying part (ii) constraints for k,l,a,b,c,d,b',d',m,n,m',n' pozitive integers in the theorem:

```
sage:
for k in [12]:
...      for l in [12]:
...          for B in [1,5]:
...              for D in [1,5]:
...                  for m in [1,..,floor(k/2)]:
...                      for n in [-B+1,..,-1,1,..,B]:
...                          for M in [1,..,floor(l/2)]:
...                              for N in [-D+1,...,-1,1,..,D]:
...                                  for e in
 [-floor((l/n*sqrt(((m^2+n^2)/(M^2+N^2))*(k^2-m^2)))),..,
-1,1,..,floor(l/n*sqrt(((m^2+n^2)/(M^2+N^2))*(k^2-m^2)))]:
...                                       for f in
[-floor(sqrt(((m^2+n^2)/(M^2+N^2))*l^2-e^2*n^2/(k^2-m^2))),
..,-1,1,..,floor(sqrt(((m^2+n^2)/(M^2+N^2))*l^2-(e^2*n^2)/(k^2-m^2)))]:
...                                       for c in [7,..,25]:
...                                            if (e*m-(e^2)*n/f)!=0:
...                                            a=c*(k*M-k*l*n/f)/(e*m-e^2*n/f)
...                                            b=(k*l*c-(e^2)*a)/f^2
...                                            d=int(a*b/c)
```

```
...                                  if (a<=b and c<=d and a*b==c*d
 and (m^2*a+n^2*b)/k==(M^2*c+N^2*d)/l and m^2*a+n^2*b<=k^2*a
  and M^2*c+N^2*d<=l^2*c and k^2*lcm(e*m,f*n)%c==0 and
   (D*M-N)%l==0 and (B*e-f)%k==0 and f%k!=0 and
    gcd(a,gcd(b,int((a+B^2*b)/k)))==gcd(c,gcd(D,int((c+D^2*(a*b/c))/l)))==1
    and gcd(B,k)==gcd(D,l)==1
    and c*c<=a*b and (vector([a,B,k])==vector([c,D,l]))==false
    and (a+B^2*b)%k==0 and (c+D^2*(a*b/c))%l==0 ):
...                           print [k,a,b,B,l,c,d,D,m,n,M,N,e,f]
...                           print [k*a,k*b,a,b*B];[l*c,l*(a*b/c),c,(a*b/c)*D]
```

k=12, l=12

[12, 7, 737, 1, 12, 11, 469, 5, 1, 1, 5, 1, 11, 1]

[84, 8844, 7, 737]

[132, 5628, 11, 2345]

[12, 7, 689, 1, 12, 13, 371, 5, 1, 1, 5, 1, 13, 1]

[84, 8268, 7, 689]

[156, 4452, 13, 1855]

[12, 8, 28, 1, 12, 4, 56, 5, 4, 1, 5, 1, 4, 4]

[96, 336, 8, 28]

[48, 672, 4, 280]

[12, 3, 45, 1, 12, 3, 45, 5, 5, 1, 5, 1, 3, 3]

[36, 540, 3, 45]

[36, 540, 3, 225]

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[12, 3, 21, 5, 12, 3, 21, 1, 1, 1, 1, 1, 9, 3]
[36, 252, 3, 105]
[36, 252, 3, 21]
[12, 3, 9, 5, 12, 3, 9, 1, 1, 1, 1, 1, 6, 6]
[36, 108, 3, 45]
[36, 108, 3, 9]
[12, 9, 15, 5, 12, 9, 15, 1, 1, 1, 1, 1, 3, 9]
[108, 180, 9, 75]
[108, 180, 9, 15]
[12, 8, 28, 5, 12, 4, 56, 1, 2, 1, 1, 1, 4, 4]
[96, 336, 8, 140]
[48, 672, 4, 56]
[12, 16, 20, 5, 12, 4, 80, 1, 2, 1, 1, 1, 4, 4]
[192, 240, 16, 100]
[48, 960, 4, 80]
[12, 11, 469, 5, 12, 7, 737, 1, 5, 1, 1, 1, 7, 1]
[132, 5628, 11, 2345]
[84, 8844, 7, 737]
[12, 13, 371, 5, 12, 7, 689, 1, 5, 1, 1, 1, 7, 1]
[156, 4452, 13, 1855]
[84, 8268, 7, 689]

[12, 5, 667, 1, 12, 23, 145, 5, 1, 1, 2, 2, 23, 1]
[60, 8004, 5, 667]
[276, 1740, 23, 725]
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[12, 3, 189, 1, 12, 21, 27, 5, 1, 1, 2, 2, 21, 3]

[36, 2268, 3, 189]

[252, 324, 21, 135]

[12, 7, 737, 1, 12, 11, 469, 5, 1, 1, 5, 1, 11, 1]

[84, 8844, 7, 737]

[132, 5628, 11, 2345]

[12, 7, 689, 1, 12, 13, 371, 5, 1, 1, 5, 1, 13, 1]

[84, 8268, 7, 689]

[156, 4452, 13, 1855]

[12, 4, 128, 1, 12, 16, 32, 5, 4, 1, 2, 2, 8, 4]

[48, 1536, 4, 128]

[192, 384, 16, 160]

[12, 20, 64, 1, 12, 16, 80, 5, 4, 1, 2, 2, 8, 4]

[240, 768, 20, 64]

[192, 960, 16, 400]

[12, 3, 117, 1, 12, 9, 39, 5, 5, 1, 2, 2, 9, 3]

[36, 1404, 3, 117]

[108, 468, 9, 195]

[12, 9, 15, 5, 12, 9, 15, 1, 1, 1, 1, 1, 3, 9]

[108, 180, 9, 75]

[108, 180, 9, 15]

[12, 11, 469, 5, 12, 7, 737, 1, 5, 1, 1, 1, 7, 1]

[132, 5628, 11, 2345]

[84, 8844, 7, 737]

[12, 13, 371, 5, 12, 7, 689, 1, 5, 1, 1, 1, 7, 1]

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[156, 4452, 13, 1855]
[84, 8268, 7, 689]
[12, 3, 189, 5, 12, 21, 27, 5, 1, 1, 2, 2, 21, 3]
[36, 2268, 3, 945]
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[12, 4, 80, 5, 12, 16, 20, 5, 1, 2, 3, 3, 16, 4]
[48, 960, 4, 400]
[192, 240, 16, 100]
[12, 4, 56, 5, 12, 8, 28, 5, 1, 2, 5, 1, 8, 4]
[48, 672, 4, 280]
[96, 336, 8, 140]

[12, 3, 21, 1, 12, 3, 21, 5, 5, 1, 2, 2, 9, 3]
[36, 252, 3, 21]
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[36, 540, 3, 45]
[36, 540, 3, 225]
[12, 3, 21, 5, 12, 3, 21, 1, 1, 1, 1, 1, 9, 3]
[36, 252, 3, 105]
[36, 252, 3, 21]
[12, 3, 9, 5, 12, 3, 9, 1, 1, 1, 1, 1, 6, 6]
[36, 108, 3, 45]
[36, 108, 3, 9]
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k=12,l=6

[12, 5, 187, 1, 6, 11, 85, 1, 1, 1, 1, 1, 11, 1]

[60, 2244, 5, 187]

[66, 510, 11, 85]

[12, 4, 80, 1, 6, 10, 32, 1, 1, 1, 1, 1, 10, 2]

[48, 960, 4, 80]

[60, 192, 10, 32]

[12, 3, 45, 1, 6, 9, 15, 1, 1, 1, 1, 1, 9, 3]

[36, 540, 3, 45]

[54, 90, 9, 15]

[12, 5, 91, 1, 6, 13, 35, 1, 1, 1, 1, 1, 13, 1]

[60, 1092, 5, 91]

[78, 210, 13, 35]

[12, 3, 45, 5, 6, 9, 15, 1, 1, 1, 1, 1, 9, 3]

[36, 540, 3, 225]

[54, 90, 9, 15]

[12, 5, 259, 5, 6, 7, 185, 1, 5, 1, 1, 1, 7, 1]

[60, 3108, 5, 1295]

[42, 1110, 7, 185]

[12, 4, 32, 5, 6, 2, 64, 1, 5, 1, 1, 1, 2, 2]

[48, 384, 4, 160]

[12, 384, 2, 64]

[12, 1, 23, 5, 6, 1, 23, 1, 5, 1, 1, 1, 7, 1]

[12, 276, 1, 115]

[6, 138, 1, 23]

[12, 4, 56, 1, 6, 14, 16, 1, 1, 1, 1, 1, 14, 2]

[48, 672, 4, 56]

[84, 96, 14, 16]

[12, 6, 5, 187, 11, 85, 1, 1, 1, 1]

[60, 2244, 5, 187]

[66, 510, 11, 85]

[12, 6, 4, 80, 10, 32, 1, 1, 1, 1]

[48, 960, 4, 80]

[60, 192, 10, 32]

[12, 6, 3, 45, 9, 15, 1, 1, 1, 1]

[36, 540, 3, 45]

[54, 90, 9, 15]

[12, 6, 5, 91, 13, 35, 1, 1, 1, 1]

[60, 1092, 5, 91]

[78, 210, 13, 35]

[12, 6, 4, 56, 14, 16, 1, 1, 1, 1]

[48, 672, 4, 56]

[84, 96, 14, 16]

k=12,l=4

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[12, 420, 1, 35]

[20, 28, 5, 7]

[12, 3, 189, 5, 4, 7, 81, 1, 5, 1, 1, 1, 7, 1]

[36, 2268, 3, 945]

[28, 324, 7, 81]

[12, 1, 11, 5, 4, 1, 11, 1, 5, 1, 1, 1, 2, 2]

[12, 132, 1, 55]

[4, 44, 1, 11]

[12, 5, 91, 5, 4, 7, 65, 1, 5, 1, 1, 1, 7, 1]

[60, 1092, 5, 455]

[28, 260, 7, 65]

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[36, 1980, 3, 165]

[44, 180, 11, 45]

[12, 1, 35, 1, 4, 5, 7, 1, 1, 1, 1, 1, 10, 2]

[12, 420, 1, 35]

[20, 28, 5, 7]

[12, 3, 117, 1, 4, 13, 27, 1, 1, 1, 1, 1, 13, 1]

[36, 1404, 3, 117]

[52, 108, 13, 27]

k=12,l=3

[12, 3, 2, 154, 11, 28, 1, 1, 1, 1, 1, 1, 11, 1]

[24, 1848, 2, 154]

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[33, 84, 11, 28]
[12, 3, 2, 130, 13, 20, 1, 1, 1, 1, 1, 1, 13, 1]
[24, 1560, 2, 130]
[39, 60, 13, 20]
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k=12,l=2

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[12, 1, 143, 1, 2, 11, 13, 1, 1, 1, 1, 1, 11, 1]
[12, 1716, 1, 143]
[22, 26, 11, 13]
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Bibliography

[1] [GL1] T. Gannon and C. S. Lam, "Lattices and Θ-function identities I: Theta constants", *J. Math. Phys.* **33**, 854–870 (1992).

[2] [RF] H. E. Rauch and H. M. Farkas, *Theta Functions with Applications to Riemann surfaces*, (Williams & Wilkins, Baltimore 1974).

[3] [FK] H. M. Farkas and I. Kra, *Theta Constants, Riemann surfaces and the modular group*, (American Mathematical Society, Providence 2001).

[4] [GSW] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vol. 1 (Cambridge University Press, Cambridge 1987).

[5] [KLS] A. Khare, A. Lakshminarayan, and U. Sukhatme, 'Cyclic identities involving ratios of Jacobi Theta functions', math-ph/0403051.

[6] [Kr] G. Köhler, "Some eta-identities arising from theta series", *Math. Scand.* **66**, 147–154 (1990).

[7] [Mc] I. G. MacDonald, "Affine root systems and Dedekind's η-function", *Invent. Math.* **15**, 91–143 (1972).

[8] [Ma] Y. Martin, "Multiplicative η-quotients", *Trans. Amer. Math. Soc.* **348**, 4825–4856 (1996).

[9] [BG] J. M. Borwein and F. G. Garvan, “Approximations to π via the Dedekind eta function”, in: *Organic Mathematics*, ed. by J. Borwein et al (American Mathematical Society, Providence 1997), pp.89–115.

[10] [At] M. Atiyah, “The logarithm of the Dedekind η-function”, *Math. Ann.* **278**, 335–380 (1987).

[11] [PS] V. Prasolov and Y. Solovyev, *Elliptic Functions and Elliptic Integrals* (American Mathematical Society, Providence 1997).

[12] [C] K. Chandrasekharen, *Elliptic Functions* (Springer, Berlin 1985).

[13] [CS] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. (Springer, New York 1999).

[14] [RG] H. Rademacher and E. Grosswald, *Dedekind sums* (Mathematical Association of America, Washington 1972).

[15] [G] T. Gannon, *Moonshine Beyond the Monster*, (Cambridge University Press, Cambridge 2006).

[16] [BZ] B. C. Berndt and L.-C. Zhang, *“Ramanujan’s identities for eta-functions”*, *Math. Ann.* **292**, 561–573 (1992).

[17] [GL2] T. Gannon and C. S. Lam, *“Lattices and Θ-function identities II: Theta series”*, *J. Math. Phys.* **33**, 871–887 (1992).

[18] [M] D. Mumford, *Tata Lectures on Theta I* and *III*, (Birkhauser, Boston 1983 and 1991).

[19] [IR] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., (Springer, New York 1992).

[20] [K] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, 2nd ed. (Springer, New York 1993).

[21] [C2] J.H. Conway, *The sensual (quadratic) form*, (The Mathematical Association of America, USA 1997).

[22] [G1] T. Gannon, *Lattices and Theta functions (Phd. Thesis)*, (McGill University, Montreal 1991).

[23] [L] S. Lang,*Introduction to Modular Forms*, (Springer- Verlag, Berlin, 1976).

[24] [JPS] J.P. Serre,*A Course in Arithmetic*, (Springer-Verlag, Berlin, 1973).

[25] [RZ] M. Richler and D. Zagier,*The Theory of Jacobi Forms*, (Birkhauser, Boston, 1985).

[26] [VK] V.G. Kac, *Infinite Dimensional Lie Algebras*, (Cambridge U. P., Cambridge, England, 1985).

[27] [BZ-J] E. Brezin and J. Zinn-Justin, *Fields, Strings and Critical phenomena*, Les Houches Lecture notes,(North-Holland, Amsterdam, 1988).

[28] [Serre-Stark] J.P. Serre and H. M. Stark,*Modular Forms of weight 1/2*, Modular functions of one variable VI. Lecture Notes in Mathematics,vol. 627, pp. 27-68.(Berlin Heidelberg New York: Springer 1977).

[29] [TM] T. Miyaki, *Modular Forms*, (Springer, Berlin Heidelberg, 2006).

[30] [SL] S. Lang, *Algebra*, (Springer, USA, 2005).

[31] [GW] C. Gordon and D. Webb, *You can't hear the shape of a drum!*, (American Scientist,vol.84).

[32] [MK] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly 73 (1966), no. 4, 123. MR 0201237 .