UNIVERSITY OF ALBERTA

The Design of Neutral Elastic Inclusions in the Case of Non-Uniform External Loading

BY

DANIEL VAN VLIET ©

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Abstract

In this paper, we consider the problem of a single elastic inhomogeneity embedded within an infinite elastic matrix in anti-plane shear. In particular, we examine the design of this inhomogeneity to achieve (stress) neutrality when a *non-uniform* stress field is prescribed in the surrounding matrix. Since it is known that neutral elastic inhomogeneities do not exist when the inhomogeneity is assumed to be perfectly bonded to the matrix, the design method presented here is based on the assumption of imperfect interface and the appropriate choice of the (single) interface parameter (characterizing the imperfect interface) to achieve the desired neutrality. Specifically, in the case of a homogeneously imperfect interface, it is shown that the circular inhomogeneity is neutral if and only if the prescribed non-uniform stress field in the surrounding matrix belongs to a certain class of polynomial functions. In the case of an inhomogeneously imperfect interface, neutrality is established for circular and elliptic inhomogeneities for specific classes of prescribed states of stress in the surrounding matrix. The results in this paper affirm the feasibility of designing a neutral elastic inhomogeneity by controlling the (imperfect) interface parameter describing the inhomogeneity-matrix interface.

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1 Introduction

It is commonly believed that a hole made in an elastic body will inevitably disturb the original stress field and often lead to a stress concentration. Mansfield [1] was one of the first who recognized the feasibility of designing a reinforced "neutral" hole which does not alter the original stress distribution in the cut elastic body. For related works see, for example, [2-11].

The analogous problem of a neutral elastic inclusion, which does not cause any stress disturbance in the surrounding elastic body, has been studied for a body subjected to constant stress [12-14]. The limited results in the area of neutral inclusions may be attributed to the fact that the design of such inclusions is impossible under the assumptions of the traditional model of an inclusion-matrix composite involving a perfectly bonded interface [14].

The design method followed in this thesis is that proposed by Ru [14] in which the interface is modelled as imperfect. In this interface model, tractions are continuous across the interface, while jumps in displacement are proportional to their respective traction components in terms of the interface parameter. This model has originally been proposed to describe the imperfectly bonded interfaces appearing in various composite materials and structures, see, for example, [15-22]. In this thesis, we will extend the current model to problems involving non-constant stress in the region exterior to the inclusion [23].

Consider a homogeneous elastic body, finite or infinite in extent and simply or multiply connected, undergoing a non-uniform stress state under the prescribed loading system. Assume that the elastic body is now cut out over a number of simply connected sub-domains and filled up with some homogeneous elastic inclusions. The problem raised in this thesis is how to design the interfaces between the inclusions and the elastic body such that the embedded inclusions are "neutral" in the sense that they do not disturb the original stress field in the uncut elastic body. In other words, the concept of a neutral inclusion defined here emphasizes the undisturbed stress state *outside* the inclusion. (As will be seen below, it also implies the state of stress inside the inclusion is identical to the state of stress that the elastic body was under before the introduction of the inclusion.) This is obviously different from the "equal-strain inclusion" in the sense of Eshelby [24], which usually destroys the uniformity of the stress field outside the inclusion and then is not "neutral" (see [25-26], and [12]). It is believed that the concept of a neutral elastic inclusion will find its applications in many practical problems where the stress concentration caused by material mismatch is of utmost concern.

Since this problem for multiple embedded inclusions reduces to the single inclusion problems for each of the embedded inclusions [14], the thesis focuses on the design of a single neutral elastic inclusion.

2 Foundations of the Problem

2.1 Kinematics

We consider the equilibrium deformation of a two-part composite deformable solid body. In its unstressed state the inclusion, D_2, occupies an open cylindrical region whose generators are parallel to the X_3 axis of a rectangular Cartesian coordinate system. The boundary of D_2 is denoted by Γ. The surrounding matrix, D_1, occupies the region $\mathbb{R}^3 \backslash (D_2 \cup \Gamma)$. The inclusion and the surrounding matrix are made of different isotropic linearly elastic materials. This is illustrated in Figure 1.

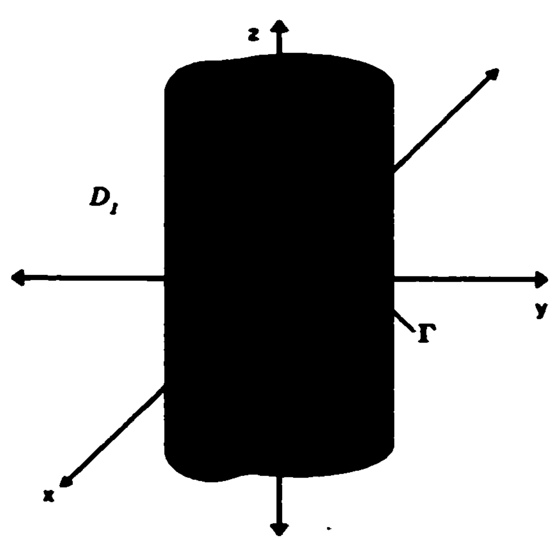


Figure 1: A general cylindrical inclusion

We consider deformations of the composite of the form

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w_3(X_1, X_2), \tag{1}$$

so that the displacement field is given by

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = w_3(X_1, X_2). \tag{2}$$

These functions are defined on $D_1 \cup D_2$, and not necessarily on the interface.

The composite is assumed to be long enough in the X_3 - direction so that the end effects are negligible.

Deformations of this type are called *antiplane shear deformations [27].*

2.2 Formulation of the Boundary Value Problem

We continue the development of the model by assuming that each of the regions D_1 and D_2 is composed of a homogeneous isotropic linearly elastic solid and consider infinitesimal deformations. Thus, we need not distinguish between material and spatial coordinates [14]. We adopt the convention that the axes are named x, y, z, rather than X_1, X_2, X_3. Due to the symmetry of the problem, we discard references to the z - coordinate.

Under the assumptions of local homogeneity, isotropy, and linearity, the problem of antiplane shear simplifies to solving Laplace's equation for w_3 on each of the regions. That is [27],

$$\frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_3}{\partial y^2} = 0, \qquad (x,y) \in D_i. \tag{3}$$

The condition of harmonic equilibrium (3) gives boundedness of w_3 and its partial derivatives on D_1 and D_2, but it does not preclude singular behavior on Γ. A further assumption is needed here. To ensure physical tractability of the model, we must assume that w_3 and its partial derivatives are *uniformly* bounded on D_2 and on any bounded subset of D_1.[1] This allows us to extend the domain of w_3 (and its partial derivatives) to include the interface Γ. Define $\frac{\partial^n}{\partial \xi^n} w_i : D_i \cup \Gamma \to R$ for $i = 1, 2$ by

$$\frac{\partial^n}{\partial \xi^n} w_i = \begin{cases} \frac{\partial^n}{\partial \xi^n} w_3\left(x,y\right), & (x,y) \in D_i, \\ \lim_{(x,y) \in D_i \to \Gamma} \frac{\partial^n}{\partial \xi^n} w_3\left(x,y\right), & (x,y) \in \Gamma, \end{cases}$$

where ξ is some unit vector. Note that for $(x,y) \in \Gamma$, $\frac{\partial^n}{\partial \xi^n} w_1\left(x,y\right)$ and

[1] In Section 1.4, it will be seen that imposing the condition of neutrality implicitly necessitates that the partial derivatives of w_3 be uniformly bounded on any bounded subset of D_1. However, it is important to observe that this boundedness is necessary for tractability of any problems dealing with inclusions of this type, not just those dealing with neutrality.

$\frac{\partial^n}{\partial \xi^n} w_2(x, y)$ need not be the same. Henceforth, the subscripts 1 and 2 will be used to refer to the regions D_1 and D_2, respectively.

It still remains to describe the interface conditions under which (3) is to be solved. A perfectly bonded interface between regions $D_1$and D_2 would be described by [12]

$$\begin{aligned} w_1(x, y) &= w_2(x, y), \qquad (x, y) \in \Gamma, \\ \mu_1 \frac{\partial}{\partial n} w_1(x, y) &= \mu_2 \frac{\partial}{\partial n} w_2(x, y), \qquad (x, y) \in \Gamma, \end{aligned}$$

where n denotes the outward normal direction to Γ, and μ_i is the shear modulus of region D_i. These interface conditions mean, respectively, that the displacements and the shear stress on an element of the boundary surface Γ are both continuous across Γ. It has been shown [14] that a perfectly bonded inclusion interface between two different isotropic linear elastic materials cannot give rise to a neutral inclusion.

In order to permit the design of a neutral inclusion, an imperfectly bonded interface is used. Physically, this condition represents an interfacial layer which is soft. That is, the thin interfacial layer can undergo large differences in displacement across its thickness. Following the model used by Ru (1996), we employ an imperfect interface across which tractions are continuous and jumps in the displacement are proportional to their respective traction components in terms of the interface parameters. This condition is written as

$$\mu_1 \frac{\partial}{\partial n} w_1(x,y) = \mu_2 \frac{\partial}{\partial n} w_2(x,y), \qquad (x,y) \in \Gamma, \tag{4}$$

$$h(x,y)\left[w_1(x,y) - w_2(x,y)\right] = \mu_1 \frac{\partial}{\partial n} w_1(x,y), \qquad (x,y) \in \Gamma, \tag{5}$$

where $h(x,y) > 0$ is an interface function. In particular, $h(x,y) = 0$ represents a traction-free interface, and $h(x,y) = \infty$ represents a perfectly bonded interface.

In practice, the interface Γ will represent an adhesive layer between the inclusion and the body [19]. As such, $h(x,y)$ should be inversely proportional to the thickness or directly proportional to the density of the adhesive layer (see, for example [15-16] and [19]). In this way, the function $h(x,y)$ can be 'chosen' by varying the properties of the adhesive layer. The only restriction is that $h(x,y)$ must be non-negative everywhere.[2]

There is still no formal boundary condition under which the problem is to be solved. This is because, for most of the problems addressed here, the displacement field in the matrix D_1 will be prescribed. Thus, this prescribed

[2]This requirement is needed only for practical design constraints, not mathematical necessity. A negative interface parameter would necessitate an adhesive material with a negative shear modulus. It is this constraint that makes certain neutral inclusion design problems unsolvable.

exterior displacement function $w_1(x, y)$, coupled with the interface conditions (4 and 5), effectively determine boundary conditions under which the interior displacement function $w_2(x, y)$ is to be solved.

2.3 Complex Variable Formulation of the Problem

To facilitate the solution of the boundary value problem given by (3), (4), and (5) it is convenient to rephrase the problem in terms of functions of one complex variable. We denote by $z = x + iy = re^{i\theta}$ the complex coordinate. The regions D_i and Γ will be used interchangeably to refer to the regions defined in Section 1.1 and also to the corresponding complex regions, depending on the context.

Define analytic functions $\chi_i : D_i \rightarrow \mathbb{C}$ such that:

$$\operatorname{Re}[\chi_i] = \mu_i w_i, \qquad (x, y) \in D_i. \tag{6}$$

The harmonic equilibrium condition (3) is then solved implicitly on each region D_i by the respective function $\chi_i(z)$, since $\chi_i(z)$ is analytic.

It still remains to translate the interface conditions into complex variable form. Rewriting (4) in complex form yields

$$
\begin{aligned}
\mu_1 \frac{\partial w_1}{\partial n} &= \mu_2 \frac{\partial w_2}{\partial n}, \qquad z \in \Gamma, \\
\mu_1 \frac{\partial \operatorname{Re}\left[\frac{\chi_1}{\mu_1}\right]}{\partial n} &= \mu_2 \frac{\partial \operatorname{Re}\left[\frac{\chi_2}{\mu_2}\right]}{\partial n}, \qquad z \in \Gamma, \\
\frac{\partial \operatorname{Re}[\chi_1]}{\partial n} &= \frac{\partial \operatorname{Re}[\chi_2]}{\partial n}, \qquad z \in \Gamma.
\end{aligned}
$$

Then, by the Cauchy-Riemann relations, we have:

$$
\frac{\partial \operatorname{Im}[\chi_1]}{\partial t} = \frac{\partial \operatorname{Im}[\chi_2]}{\partial t}, \qquad z \in \Gamma, \tag{7}
$$

where t is the clockwise tangential direction on Γ.

The only assumption made in deriving (7) is (6), an assumption about the real part of χ_i. Adding a purely imaginary constant to the function $\chi_i(z)$ will not affect its satisfaction of the condition (6). Thus, we are free to choose the value of $\operatorname{Im}[\chi_i]$ at any one point. For convenience, we choose a point $z^* \in \Gamma$ where we define $\operatorname{Im}[\chi_i(z^*)] = 0$. Then, integrating along Γ from z^* on Γ yields:

$$
\begin{aligned}
\int_{z^*}^{z} \frac{\partial \operatorname{Im}[\chi_1]}{\partial t} d\Gamma &= \int_{z^*}^{z} \frac{\partial \operatorname{Im}[\chi_2]}{\partial t} d\Gamma, \\
\operatorname{Im}[\chi_1(z)] &= \operatorname{Im}[\chi_2(z)], \qquad z \in \Gamma. \qquad (8)
\end{aligned}
$$

Translating (5) into complex form gives

$$\begin{aligned}
h(x,y)\left[w_1(x,y) - w_2(x,y)\right] &= \frac{\partial}{\partial n} w_1(x,y), \qquad (x,y) \in \Gamma, \\
h(z)\left(\frac{\mathrm{Re}[\chi_1]}{\mu_1} - \frac{\mathrm{Re}[\chi_2]}{\mu_2}\right) &= \frac{\partial\, \mathrm{Re}[\chi_1]}{\partial n}, \qquad z \in \Gamma, \\
h(z)\left(\frac{\mathrm{Re}[\chi_1]}{\mu_1} - \frac{\mathrm{Re}[\chi_2]}{\mu_2}\right) &= \mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma.
\end{aligned}$$

Here, $e^{iN(z)}$ denotes the outward unit normal, in complex form. Now we rearrange things slightly, and then add (8) to get

$$\begin{aligned}
\mathrm{Re}[\chi_1] &= \frac{\mu_1}{\mu_2}\mathrm{Re}[\chi_2] + \frac{\mu_1}{h(z)}\mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma, \\
\mathrm{Re}[\chi_1] + \mathrm{Im}[\chi_1] &= \frac{\mu_1}{\mu_2}\mathrm{Re}[\chi_2] + \mathrm{Im}[\chi_2] + \frac{\mu_1}{h(z)}\mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma, \\
\chi_1 &= \frac{\mu_1}{2\mu_2}\left[\chi_2 + \overline{\chi_2}\right] + \frac{\mu_2}{2\mu_2}\left[\chi_2 - \overline{\chi_2}\right] + \frac{\mu_1}{h(z)}\mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma, \\
\chi_1 &= \frac{\mu_1 + \mu_2}{2\mu_2}\chi_2 + \frac{\mu_1 - \mu_2}{2\mu_2}\overline{\chi_2} + \frac{\mu_1}{h(z)}\mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma, \\
\chi_1 &= \delta\chi_2 + (\delta - 1)\,\overline{\chi_2} + \frac{\mu_1}{h(z)}\mathrm{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma. \qquad (9)
\end{aligned}$$

where $\delta = \frac{\mu_1+\mu_2}{2\mu_2}$. The interface condition (9) incorporates both of the real variable interface conditions into one complex variable condition.[3]

Now the problem has been reduced to finding analytic functions of one complex variable z which satisfy the interface condition (9). Taking the

[3]To show formal equivalence of these two interface conditions, a derivation of (4) and (5) from (9) is shown in Appendix I.

imaginary part of (9) gives

$$\chi_1 - \overline{\chi_1} = -\overline{\chi_2} + \chi_2, \qquad z \in \Gamma.$$

This condition determines the value of χ_2 on Γ uniquely (in terms of χ_1) to within an arbitrary real number C_0. That is,

$$\chi_2 = \chi_1 + C_0, \qquad z \in \Gamma. \tag{10}$$

Substituting (10) into the interface condition (9) gives

$$\begin{aligned} (1-\delta)(\chi_1 + \overline{\chi_1}) &= (2\delta - 1) C_0 + \frac{\mu_1}{h(z)} \operatorname{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma, \\ 2(1-\delta)\operatorname{Re}[\chi_1] &= (2\delta - 1) C_0 + \frac{\mu_1}{h(z)} \operatorname{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma. \quad (11) \end{aligned}$$

There is a further condition relevant to the problem which allows further simplification. The problem posed here is that of finding an inclusion which, when introduced into a stressed body, results in an external loading identical to that which existed before the introduction of the inclusion. Consider that, in the prior state (with no inclusion), the elastic body was under an equilibrium antiplane shear deformation χ_{prior} which was identical to the new deformation χ_1 on the region D_1. Thus $\exists \chi_{prior} : (D_1 \cup \Gamma \cup D_2) \rightarrow \mathbb{C}$, an analytic function, such that

$$\chi_{prior} = \chi_1, \qquad z \in D_1.$$

Thus, χ_{prior} is the analytic continuation of χ_1 to $D_1 \cup \Gamma \cup D_2$. Henceforth, we refer to χ_{prior} as χ_1. Thus, χ_1 is now defined and analytic on the entire region $D_1 \cup \Gamma \cup D_2$.

We now choose the origin of our coordinates to be at the centroid of the inclusion. Choose the datum values of χ_1 and χ_2 to be zero at the origin. Observe that both the right and left-hand sides of (10) are analytic on D_2 and continuous on $D_2 \cup \Gamma$. Under these conditions, equality on Γ implies equality on the entire region $D_2 \cup \Gamma$. Then we have

$$\chi_2 = \chi_1 + C_0, \qquad z \in D_2 \cup \Gamma.$$

Evaluating this condition at the origin gives

$$\begin{aligned} \chi_2(0) &= \chi_1(0) + C_0, \\ 0 &= C_0. \end{aligned}$$

So that (10) may now be strengthened to

$$\chi_2 = \chi_1, \qquad z \in D_2 \cup \Gamma.$$

Thus, it is no longer necessary to consider χ_2 as an independent quantity.

Also notice that (11) becomes much simpler:

$$\operatorname{Re}[\chi_1] = \frac{\mu_1}{2(1-\delta)h(z)} \operatorname{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma. \tag{12}$$

This allows $h(z)$ to be isolated explicitly in terms of χ_1 and $N(z)$:

$$h(z) = \frac{\mu_1}{2(1-\delta)} \frac{\operatorname{Re}\left[e^{iN(z)}\chi_1'\right]}{\operatorname{Re}\left[\chi_1\right]}, \qquad z \in \Gamma. \tag{13}$$

3 Circular Inclusions

3.1 A Circular Inclusion with Homogeneously Imperfect Interface

We consider a single circular inclusion of radius R with a homogeneously imperfect interface characterized by the parameter $h(z) = h =$ constant. In this case, we have

$$\Gamma = \{z; |z| = R\} \text{ and } e^{iN(z)} = \frac{z}{R}, \tag{14}$$

so that (11) can be written as

$$2(1-\delta)\operatorname{Re}[\chi_1] = (2\delta - 1) C_0 + \frac{\mu_1}{h(z)} \operatorname{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma. \tag{15}$$

Consequently, the interface condition reduces to a linear ODE for χ_1:

$$\begin{aligned} \operatorname{Re}\left[\frac{\mu_1}{Rh} z\chi_1' - 2(1-\delta)\chi_1 + (2\delta-1)C_0\right] &= 0, \\ \frac{\mu_1}{Rh} z\chi_1' - 2(1-\delta)\chi_1 + Z_0 &= 0, \quad \operatorname{Re}[Z_0] = (2\delta - 1) C_0 \end{aligned} \tag{16}$$

Theorem 1 *A circular inclusion with constant interface parameter is neutral if and only if the exterior loading takes the form*

$$\begin{aligned} \chi_1 &= Az^p + \frac{Z_0}{2(1-\delta)}, \quad z \in D_1, \\ p &= \frac{2(1-\delta)Rh}{\mu_1} > 0, \quad \operatorname{Re}[Z_0] = (2\delta-1)C_0, \quad A \in \mathbb{C}, \quad C_0 \in \mathbb{R}, \end{aligned}$$

in which case the interface parameter is given by

$$h = \frac{p\mu_1}{2(1-\delta)R}. \tag{17}$$

Proof. *It is clear that since χ_1 is analytic on $D_1 \cup D_2 \cup \Gamma$, the equation (16) can be continued analytically from Γ into the domain D_1.* Sufficiency *then follows easily by direct substitution of χ_1 into (16). For necessity, note that the solution to (16) is given by*

$$\begin{aligned} \chi_1 &= Az^p + \frac{Z_0}{2(1-\delta)}, \quad p = \frac{2(1-\delta)Rh}{\mu_1}, \quad \text{Re}[Z_0] = (2\delta - 1)\,C_0, \quad A \in \mathbb{C} \\ h &= \frac{p\mu_1}{2(1-\delta)R}, \qquad z \in D_1 \cup \Gamma, \end{aligned} \tag{18}$$

The parameter p must be further constrained so that the stress field remains tractable. For the displacement field to be a bounded, single-valued harmonic function, p must be constrained to be a non-negative integer (The trivial case $p = 0$ is of no interest here since it corresponds to zero stress in the region D_1). ■

Remark 2 *Isolating $(1-\delta)$ in (17) gives*

$$(1-\delta) = \frac{p\mu_1}{2Rh},$$

which, since $p > 0$, implies that $(1-\delta) > 0$. This yields the condition

$$\frac{\mu_2 - \mu_1}{2\mu_2} > 0$$

or $\mu_2 > \mu_1$. This is independent of the choice of A or p. Hence, a circular inclusion with constant interface parameter may be neutral only if the incluion is stiffer than the surrounding matrix. This is a result of the neutrality condition, and is independent of the specific form of the displacement (and stress) function. It is also interesting to note that the neutrality of the inclusion is independent of the magnitude of the stress field, and its orientation. (Both of these quantities are determined by the arbitrary complex constant A.).

This solution is consistent with that derived in [14], where the circular inhomogeneity for $p = 1$ (uniform exterior loading) was found to be given by the interface parameter $h = \frac{\mu_1}{2(1-\delta)R}$.

3.2 A Circular Inclusion Under Linearly Varying Stress

It is clear from Theorem 3.1.1 that a circular inclusion cannot be made neutral in the presence of an *arbitrary* linear stress field in the matrix when the interface is assumed to be homogeneously imperfect. However, it is of particular interest (e.g. for approximation purposes) to design a neutral inclusion for *any* prescribed linearly varying stress distribution in the matrix, if only to obtain information corresponding to the next power series approximation (after a simple constant [14]) of a general non-linear stress field in the matrix.

Here, we address this particular problem by considering a linear stress field corresponding to the analytic function

$$\chi_1 = C_2 z^2 + C_1 z, \qquad C_1, C_2 \in \mathbb{C},$$

and by trying to find an (inhomogeneous) interface parameter $h(z)$ which will make the circular inclusion neutral. For convenience, we rephrase the function χ_1 as

$$\chi_1 = A(z - z_0)^2 - A z_0^2, \qquad A = C_2, \qquad z_0 = \frac{-C_1}{2C_2}. \tag{19}$$

Remark 3 *The quantity z_0 has very special physical significance, aside from its mathematical convenience. Under a displacement of the form given by (19), there is a unique point in the elastic body at which there is zero stress. This point is z_0. Thus, there is also good physical reason for giving this point special attention.*

The orientation of the axes allows a further simplification. For example, choose the orientation of the coordinate axes such that $A \in \mathbb{R}$. Combining (19), (14), and (13) then gives

$$h(z) = \frac{\mu_1}{(1-\delta)\, R} \frac{\operatorname{Re}\left[z^2 - z_0 z\right]}{\operatorname{Re}\left[z^2 - 2 z_0 z\right]}, \quad z \in \Gamma. \tag{20}$$

As before, in order for the interface parameter given by (20) to be physically tractable, it must satisfy the condition $h(z) > 0$ everywhere on the boundary of the inclusion. This restricts the value of z_0. In fact, writing $z_0 = x_0 + iy_0$, from (20) we have:

$$
\begin{aligned}
h(z) &= \frac{\mu_1}{(1-\delta)\,R} \frac{x^2 - y^2 - x_0 x + y_0 y}{x^2 - y^2 - 2x_0 x + 2y_0 y} \\
&= \frac{\mu_1}{(1-\delta)\,R} \frac{(x - x_0/2)^2 - (y - y_0/2)^2 - ((x_0/2)^2 - (y_0/2)^2)}{(x - x_0)^2 - (y - y_0)^2 - (x_0^2 - y_0^2)}. \quad (21)
\end{aligned}
$$

To guarantee $h(z) > 0$, $\forall z \in \Gamma$, the numerator and denominator in the above expression must change sign only at the same time. In fact, from (21), the numerator and denominator's zeros are each hyperbolas passing through the origin with asymptotes at angles of $\pm\frac{\pi}{4}$ to the axes, and centered at $\frac{z_0}{2}$ and z_0, respectively. It is also important to note that the numerator and denominator do, in fact, change sign at these points.

It remains to show when (21) is strictly non-negative. This constraint implies a rather strict existence criterion.

Theorem 4 *A circular inclusion embedded in a matrix subjected to a linearly varying stress field characterized by the stress function*

$$
\chi_1 = A(z - z_0)^2 - Az_0^2, \;\; A \in \mathbb{R}, z_0 = (x_0 + iy_0)
$$

may only be made neutral if and only if

$$|x_0| = |y_0|, \quad |z_0| \notin (0, 2R),$$

in which case the interface parameter is of the form

$$h(x,y) = \frac{\mu_1}{(1-\delta)R}\frac{x \pm y - x_0}{x \pm y - 2x_0}, \tag{22}$$

where both the $\pm$ correspond to $x_0 = y_0$ and $x_0 = -y_0$, respectively.

Proof. *Sufficiency:*

(i) If $z_0 = 0$, then sufficiency is given by Theorem 1, with $p = 2$ and $Z_0 = 0$.

(ii) If $|z_0| \geq 2R$ and $|x_0| = |y_0|$, then (20) gives

$$\begin{aligned}
h(z) &= \frac{\mu_1}{2(1-\delta)R}\frac{\mathrm{Re}[z \cdot 2A(z - z_0)]}{\mathrm{Re}[A(z-z_0)^2 - Az_0^2]}, \quad z \in \Gamma, \\
&= \frac{\mu_1}{(1-\delta)R}\frac{\mathrm{Re}[z^2 - z_0 \cdot z]}{\mathrm{Re}[z^2 - 2z_0 \cdot z]}, \quad z \in \Gamma, \\
&= \frac{\mu_1}{(1-\delta)R}\frac{x^2 - y^2 - (x_0 \cdot x - y_0 \cdot y)}{x^2 - y^2 - 2(x_0 \cdot x - y_0 \cdot y)}, \quad z \in \Gamma, \\
&= \begin{cases} \frac{\mu_1}{(1-\delta)R}\frac{x^2-y^2-x_0(x-y)}{x^2-y^2-2x_0(x-y)}, & x_0 = y_0 \\ \frac{\mu_1}{(1-\delta)R}\frac{x^2-y^2-x_0(x+y)}{x^2-y^2-2x_0(x+y)}, & x_0 = -y_0 \end{cases} \\
&= \begin{cases} \frac{\mu_1}{(1-\delta)R}\frac{x+y-x_0}{x+y-2x_0}, & x_0 = y_0 \\ \frac{\mu_1}{(1-\delta)R}\frac{x-y-x_0}{x-y-2x_0}, & x_0 = -y_0 \end{cases}
\end{aligned} \tag{23}$$

It remains to show that these quantities are strictly non-negative on Γ.

In the first case ($x_0 = y_0$), the proof is as follows. The zero sets of the numerator and denominator are illustrated in Figure 2: ■

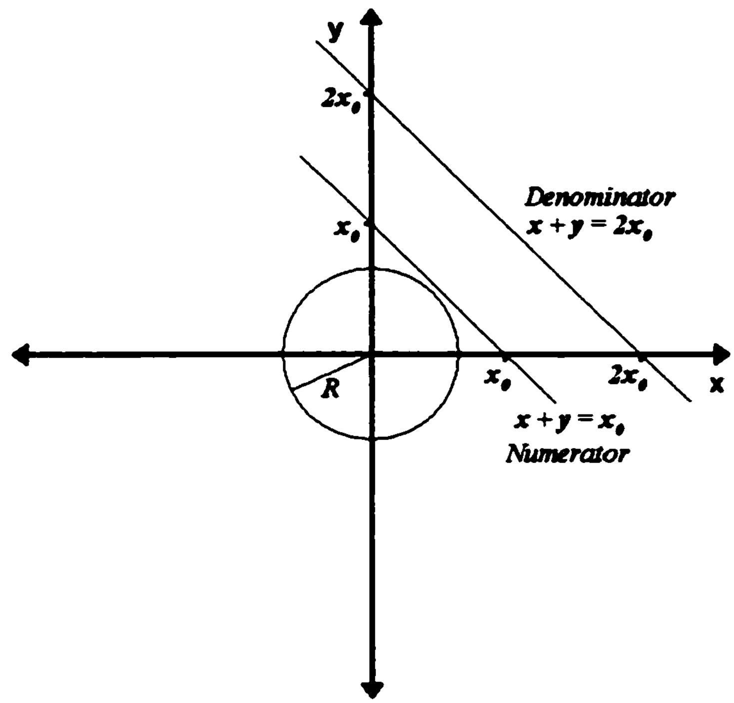


Figure 2: Zero sets of the numerator and denominator of (23)

Proof. *If $|z_0| \geq 2R$, it is clear that neither the numerator nor the denominator is ever negative on Γ. (In the case of equality, the line given by the zero set of the numerator is tangent to Γ at one point. However, it does not change sign at that point.) It is also clear that both the numerator and denominator have the same sign when evaluated on Γ. Thus, the quantity (23)* *does not change sign on* Γ. *It can thus be ensured to be strictly non-negative if* $\delta > 1$.

The argument is identical in the case when $x_0 = -y_0$, *except that the zero sets of the numerator and denominator are lines with slope of* $+1$, *instead of* -1.

If $|x_0| = |y_0|$ *and* $|z_0| \geq 2R$, *the interface parameter necessitated by the neutrality conditions is non-negative on* Γ. *Thus, the neutrality condition is satisfied by (23).*

Necessity:

(i) If $\chi_1 = A(z-z_0)^2 - Az_0^2$, $A \in \mathbb{R}$, *but* $|z_0| \in (0, 2R)$, *then at least one of the lines in Figure 2 intersect* Γ *at two distinct points. Thus, the numerator (and possibly the denominator) will change sign at two distinct points on* Γ. *If* $|z_0| \neq 0$, *then the zero sets of the numerator and denominator are clearly distinct. (They are parallel lines with intercepts equal to* x_0 *and* $2x_0$, *respectively.) Thus, the numerator and denominator change sign at different points. This implies that, for* $|z_0| \in (0, 2R)$, *the interface parameter needed to ensure neutrality of the inclusion would have to take on negative values somewhere on* Γ. *This is not admissible.*

(ii) If $\chi_1 = A(z - z_0)^2 - Az_0^2$, $A \in \mathbb{R}$, *but* $|x_0| \neq |y_0|$, *then the necessary*

interface parameter is given by (21). In order for the interface parameter to give a physically tractable solution, the interface parameter must remain non-negative. The zero sets of the numerator and denominator are each hyperbolas, centered at $(\frac{x_0}{2}, \frac{y_0}{2})$ *and* (x_0, y_0) *respectively, with asymptotes at angles of* $\pm\frac{\pi}{2}$ *to the axes. Also, each hyperbola passes through the origin. (In fact, they are tangent to each other at the origin.) Clearly then, each of the hyperbolas must intersect the inclusion boundary* Γ *at least two distinct points. Thus, the numerator and denominator each change sign at least twice.*

The only way such the inclusion can be made neutral under such a loading is if the numerator and denominator change sign at the same points. Solving for the simultaneous solution of the numerator and denominator is fairly straightforward. Isolating y *in the numerator and denominator respectively gives*

$$\begin{aligned} y &= \frac{y_0}{2} \pm \sqrt{x^2 - x \cdot x_0 + \left(\frac{y_0}{2}\right)^2}, \\ y &= y_0 \pm \sqrt{x^2 - 2x \cdot x_0 + y_0^2}. \end{aligned}$$

Equating to find the simultaneous solutions yields

$$\begin{aligned}
\frac{y_0}{2} \pm \sqrt{x^2 - x\cdot x_0 + \left(\frac{y_0}{2}\right)^2} &= y_0 \pm \sqrt{x^2 - 2x\cdot x_0 + y_0^2} \\
\pm\sqrt{x^2 - x\cdot x_0 + \left(\frac{y_0}{2}\right)^2} &= \frac{y_0}{2} \pm \sqrt{x^2 - 2x\cdot x_0 + y_0^2} \\
x^2 - x\cdot x_0 + \left(\frac{y_0}{2}\right)^2 &= \left(\frac{y_0}{2}\right)^2 \pm y_0\sqrt{x^2 - 2x\cdot x_0 + y_0^2} + x^2 - 2x\cdot x_0 + y_0^2 \\
x\cdot x_0 - y_0^2 &= \pm y_0\sqrt{x^2 - 2x\cdot x_0 + y_0^2} \\
x^2\cdot x_0^2 - 2x\cdot x_0 y_0^2 + y_0^4 &= y_0^2\left(x^2 - 2x\cdot x_0 + y_0^2\right) \\
\left(x_0^2 - y_0^2\right)x^2 &= 0
\end{aligned}$$

If $|x_0| \neq |y_0|$, *then* $\left(x_0^2 - y_0^2\right) \neq 0$ *and the only simultaneous root of the numerator and denominator is at the origin* $x = 0$ *(and thus* $y = 0$*) - the origin, which is not on* Γ. *Thus, the numerator and denominator of (21) change sign at different points on* Γ *if* $|x_0| \neq |y_0|$. *This would necessitate an interface parameter which takes negative values on* Γ. *This is not admissible, as* $h(z)$ *is non-negative by assumption.*

If $|x_0| \neq |y_0|$ *then there is no admissible interface parameter which gives rise to a neutral inclusion. This concludes the proof.* ■

The geometric implications of Theorem 4 warrant further comment. The result states that a neutral inclusion may be constructed only at specific positions in the elastic body. In an elastic body under a loading of the form

$\chi_1 = A(z - z_0)^2$, circular neutral inclusions of the type proposed here may be constructed only if they are centered on lines which are situated at angles of $\pm\frac{\pi}{4}$ to the directions of principle stress, passing through the center of the loading z_0. A composite body with such inclusions could look something like that shown in Figure 3

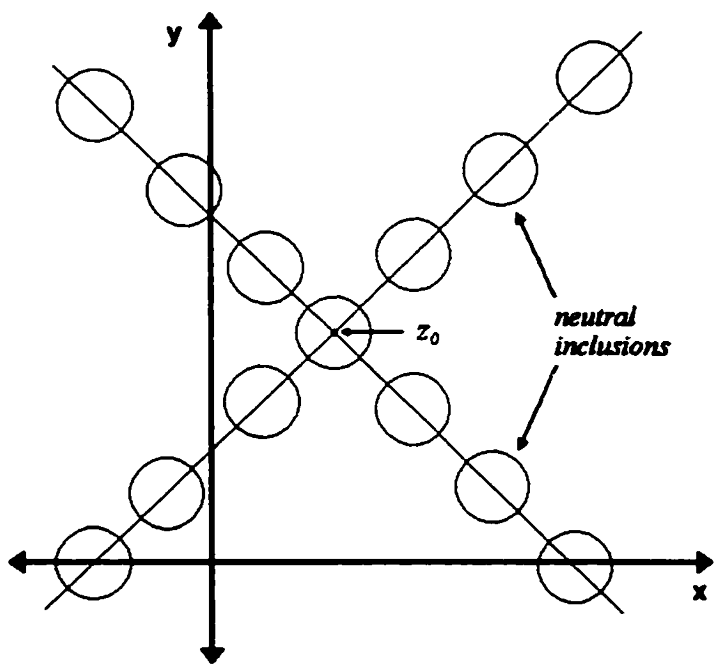


Figure 3: Possible locations of neutral circular inclusions in a composite body.

4 Elliptic Inclusions

4.1 An Elliptic Inclusion Under Simple Linearly Varying Stress

Consider an elliptic inclusion, centered at the origin, with axes of lengths $2a$ and $2b$ $(a > b)$, coincident with the x and y axes, respectively. Let the stress field in the surrounding matrix take the form characterized by the corresponding analytic function:

$$\chi_1 = Re^{i\theta_0}z^2, \qquad R \in \mathbb{R}. \tag{24}$$

Due to the lack of rotational symmetry in the ellipse, we are not free to choose $\theta_0 = 0$. We can, however, choose the axial orientation such that $0 \leq \theta_0 < \pi$.

For such an inclusion, the unit normal is given by [14]:

$$e^{iN(x,y)} = \frac{x + i\frac{a^2}{b^2}y}{a\sqrt{1 - \frac{a^2}{b^2}\left(a^{-2} - b^{-2}\right)y^2}}. \tag{25}$$

Then combining (13), (24), and (25) gives

$$
\begin{aligned}
h(z) &= \frac{\mu_1}{2(1-\delta)}\left[\frac{\mathrm{Re}[\chi_1'(z)e^{iN(z)}]}{\mathrm{Re}[\chi_1(z)]}\right], \qquad z \in \Gamma. \\
&= \frac{\mu_1}{2(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}}\left[\frac{\mathrm{Re}[2R\cdot e^{i\theta_0}z\left(x+i\frac{a^2}{b^2}y\right)]}{\mathrm{Re}[Re^{i\theta_0}z^2]}\right] \\
&= \frac{\mu_1}{(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}} \times \qquad\qquad (26) \\
&\times \left[\frac{\cos\theta_0(x^2-\frac{a^2}{b^2}y^2)-\sin\theta_0((1+\frac{a^2}{b^2})xy)}{\cos\theta_0(x^2-y^2)-\sin\theta_0(2xy))}\right], \qquad (x,y)\in\Gamma.
\end{aligned}
$$

It is clear from (26) that a solution exists if $\cos\theta_0 = 0$. In this case, the interface parameter shown above simplifies to

$$
h(z) = \frac{\mu_1\left(1+\frac{a^2}{b^2}\right)}{2\,(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\,(a^{-2}-b^{-2})\,y^2}} \qquad (27)
$$

If $\cos\theta_0 \neq 0$, then we have

$$
h\,(z) = \frac{\mu_1}{(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}}\left[\frac{x^2-\left((1+\frac{a^2}{b^2})\tan\theta_0\right)xy-\frac{a^2}{b^2}y^2}{x^2-(2\tan\theta_0)\,xy-y^2}\right]
$$

$$
\begin{aligned}
h(z) &= \frac{\mu_1}{(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}} \times \\
&\times \frac{\left(x - \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 + \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] y\right)}{(x-(\tan\theta_0+\sec\theta_0)y)\,(x-(\tan\theta_0-\sec\theta_0)y)} \qquad (28) \\
&\times \left(x - \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 - \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] y\right).
\end{aligned}
$$

The zero set of each of the factors in the numerator and denominator in (28) is a distinct line passing through the origin. Each of these lines passes through the boundary Γ of the ellipse exactly twice. Thus, both the numerator and denominator will change sign exactly four times on the boundary of the ellipse. In order for $h(z)$ to be strictly positive, the numerator and denominator must change sign at the same points. For this to be true, the two lines given by the zero sets of the factors of the numerator must be coincident with the zero sets of the factors of the denominator. This gives the following conditions:

$$
\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 \pm \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] = \tan\theta_0 \pm \sec\theta_0, \qquad (29)
$$

or

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 \pm \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] = \tan\theta_0 \mp \sec\theta_0. \tag{30}$$

In each of (29) and (30), the upper and lower instances of the $\pm$ and $\mp$ signs correspond (i.e. the $\pm$ and $\mp$ signs are either $+$ and $-$, or $-$ and $+$, respectively.). Each of these cases imposes two conditions (for each case of the $\pm$). Thus, we can solve each set of conditions explicitly for the unknown quantities $\frac{a^2}{b^2}$ and θ_0.

In either case, we can isolate, and then eliminate $\sin\theta_0$ to solve for $\frac{a^2}{b^2}$. This is done in Appendix II. The only values of $\frac{a^2}{b^2}$ which allow for the solution of (29) or (30) are found to be $\frac{a^2}{b^2} = -2, 0$, or 1. None of these values are admissible, as aspect ratios for an ellipse ($\frac{a^2}{b^2}$ is clearly positive, ruling out 0 and -2, and $\frac{a^2}{b^2} = 1$ corresponds to a circle). Thus, in the case when $\cos\theta_0 \neq 0$, we find there are no conditions under which the elliptic inclusion can be made neutral.

Thus, a physically tractable solution to the problem of constructing a neutral elliptic inhomogeneity under the prescribed loading exists only when $\cos\theta_0 = 0$. In this case, the interface parameter is given by

$$h(z) = \frac{\mu_1\left(1+\frac{a^2}{b^2}\right)}{2\left(1-\delta\right)a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y^2}}. \tag{31}$$

4.2 An Elliptic Inclusion Under General Linearly Varying Stress

Here, we consider the same inclusion shape and position as in the previous section, only under a slightly more general loading condition given by:

$$\chi_1 = Re^{i\theta_0}\left((z-z_0)^2 - z_0^2\right), \qquad R, \theta_0 \in \mathbb{R}, \qquad 0 \le \theta_0 < \pi, \qquad z_0 \neq 0. \quad (32)$$

The interface parameter is then given by

$$h(z) = \frac{\mu_1}{2(1-\delta)} \frac{\mathrm{Re}[\chi_1'(z)e^{iN(z)}]}{\mathrm{Re}[\chi_1(z)]}$$

$$= \frac{\mu_1}{2(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}} \times \frac{\mathrm{Re}[2R\cdot e^{i\theta_0}(z-z_0)\left(x+i\frac{a^2}{b^2}y\right)]}{\mathrm{Re}[Re^{i\theta_0}\left((z-z_0)^2-z_0^2\right)]}$$

$$= \frac{\mu_1}{(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}} \times \frac{\cos\theta_0\left((x-x_0)x-(y-y_0)\frac{a^2}{b^2}y\right) - \sin\theta_0\left((x-x_0)\frac{a^2}{b^2}y+(y-y_0)x\right)}{\cos\theta_0(x^2-y^2-2xx_0+2yy_0) - 2\sin\theta_0(xy-x_0y-xy_0)}$$

$$= \frac{\mu_1}{(1-\delta)a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})y^2}} \times \frac{x^2\cos\theta_0 - \left(1+\frac{a^2}{b^2}\right)(\sin\theta_0)xy - \left(\frac{a^2}{b^2}\right)y^2\cos\theta_0 + Ax + \frac{a^2}{b^2}By}{x^2\cos\theta_0 - 2\sin\theta_0 xy - y^2\cos\theta_0 + 2Ax + 2By} \quad (33)$$

where $A = -x_0 \cos\theta_0 + y_0 \sin\theta_0$ and $B = y_0 \cos\theta_0 + x_0 \sin\theta_0$. In order to transform the numerator and denominator of (??) into canonical form, we employ the transforation:

$$\begin{aligned} x &= \tilde{x}\cos\frac{\theta_0}{2} + \tilde{y}\sin\frac{\theta_0}{2} \\ y &= \tilde{y}\cos\frac{\theta_0}{2} - \tilde{x}\sin\frac{\theta_0}{2} \\ x_0 &= \tilde{x}_0\cos\frac{\theta_0}{2} + \tilde{y}_0\sin\frac{\theta_0}{2} \\ y_0 &= \tilde{y}_0\cos\frac{\theta_0}{2} - \tilde{x}_0\sin\frac{\theta_0}{2} \end{aligned}$$

which rotate the x, y coordinate axes through an angle of $\frac{\theta_0}{2}$ radians clockwise to give $\tilde{x}, \tilde{y}$ axes. This transformation gives[4]:

$$h(\tilde{x},\tilde{y}) = \frac{\mu_1}{(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})\,y(\tilde{x},\tilde{y})^2}} \times \frac{\left(\begin{array}{c} \tilde{x}^2\left(\cos^2\frac{\theta_0}{2}+\frac{a^2}{b^2}\sin^2\frac{\theta_0}{2}\right) - \tilde{y}^2\left(\frac{a^2}{b^2}\cos^2\frac{\theta_0}{2}+\sin^2\frac{\theta_0}{2}\right) \\ -\tilde{x}\left[\left(x_0\cos\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\sin\frac{\theta_0}{2}\right)\cos\theta_0 + \left(\frac{a^2}{b^2}\sin\frac{\theta_0}{2}x_0 - \cos\frac{\theta_0}{2}y_0\right)\sin\theta_0\right] \\ +\tilde{y}\left[\left(-x_0\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\cos\frac{\theta_0}{2}\right)\cos\theta_0 + \left(\frac{a^2}{b^2}\cos\frac{\theta_0}{2}x_0 + \sin\frac{\theta_0}{2}y_0\right)\sin\theta_0\right] \end{array}\right)}{\tilde{x}^2 - \tilde{y}^2 - 2\tilde{x}\left(x_0\cos\frac{\theta_0}{2} - y_0\sin\frac{\theta_0}{2}\right) + 2\tilde{y}\left(x_0\sin\frac{\theta_0}{2} + y_0\cos\frac{\theta_0}{2}\right)}. \tag{34}$$

The interface parameter given by (34) simplifies substantially if $\cos\theta_0 = 0$. (Recall this is one of the conditions needed for the solution given by (31).)

[4]This derivation is given in detail in Appendix III.

Recall $0 \leq \theta_0 < \pi$, so $\cos\theta_0 = 0 \Longrightarrow \theta_0 = \frac{\pi}{2}$. Thus (34) becomes:

$$\begin{aligned} h(\widetilde{x},\widetilde{y}) &= \frac{\mu_1}{(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y\left(\widetilde{x},\widetilde{y}\right)^2}} \times \\ & \frac{\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\left(\widetilde{x}^2-\widetilde{y}^2\right)-\widetilde{x}\sqrt{2}\left(\frac{a^2}{b^2}x_0-y_0\right)+\widetilde{y}\sqrt{2}\left(\frac{a^2}{b^2}x_0+y_0\right)}{\widetilde{x}^2-\widetilde{y}^2-\sqrt{2}\widetilde{x}\left(x_0-y_0\right)+\sqrt{2}\widetilde{y}\left(x_0+y_0\right)} \end{aligned}$$

$$\begin{aligned} &= \frac{\mu_1}{(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y^2}} \times \qquad (35) \\ & \frac{\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\left(\widetilde{x}^2-\widetilde{y}^2\right)-\sqrt{2}\frac{a^2}{b^2}x_0\left(\widetilde{x}-\widetilde{y}\right)+\sqrt{2}y_0\left(\widetilde{x}+\widetilde{y}\right)}{\widetilde{x}^2-\widetilde{y}^2-\sqrt{2}x_0\left(\widetilde{x}-\widetilde{y}\right)+\sqrt{2}y_0\left(\widetilde{x}+\widetilde{y}\right)}. \end{aligned}$$

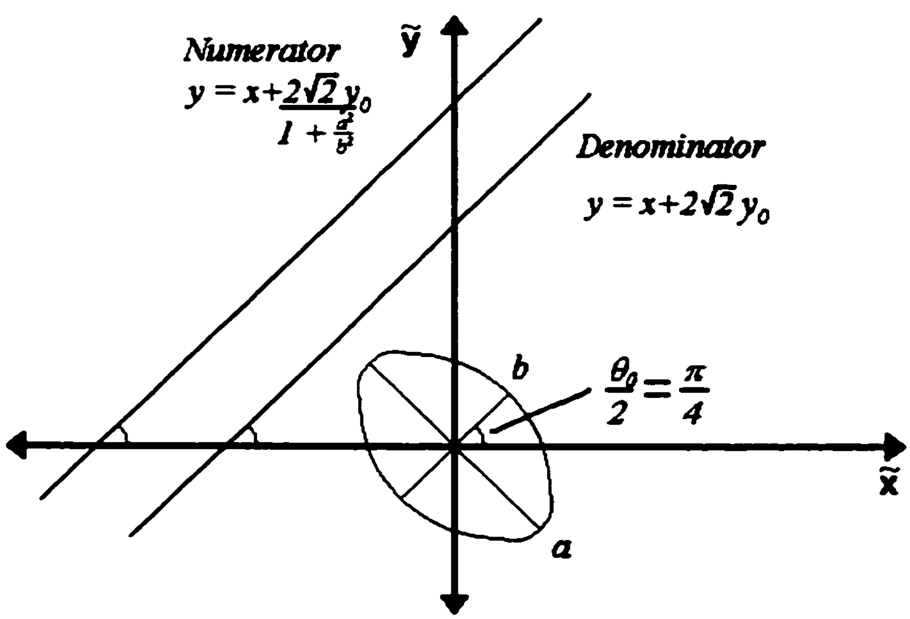
At this point, it is clear that, if either $x_0 = 0$ or $y_0 = 0$, then a linear factor can be cancelled from (35). If $x_0 = 0$, we get

$$\begin{aligned} h(\widetilde{x},\widetilde{y}) &= \frac{\mu_1}{(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y\left(\widetilde{x},\widetilde{y}\right)^2}} \frac{\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\left(\widetilde{x}^2-\widetilde{y}^2\right)+\sqrt{2}y_0\left(\widetilde{x}+\widetilde{y}\right)}{\widetilde{x}^2-\widetilde{y}^2+\sqrt{2}y_0\left(\widetilde{x}+\widetilde{y}\right)} \\ &= \frac{\mu_1}{(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y\left(\widetilde{x},\widetilde{y}\right)^2}} \frac{\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\left(\widetilde{x}-\widetilde{y}\right)+\sqrt{2}y_0}{\widetilde{x}-\widetilde{y}+\sqrt{2}y_0} \\ &= \frac{\mu_1\left(1+\frac{a^2}{b^2}\right)}{2\,(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}\left(a^{-2}-b^{-2}\right)y\left(\widetilde{x},\widetilde{y}\right)^2}} \left(\frac{\widetilde{x}-\widetilde{y}+\frac{2\sqrt{2}y_0}{\left(1+\frac{a^2}{b^2}\right)}}{\widetilde{x}-\widetilde{y}+\sqrt{2}y_0}\right) \qquad (36) \end{aligned}$$

Similarly, if $y_0 = 0$, we get:

$$h(\widetilde{x},\widetilde{y}) = \frac{\mu_1\left(1+\frac{a^2}{b^2}\right)}{2(1-\delta)\,a\sqrt{1-\frac{a^2}{b^2}(a^{-2}-b^{-2})\,y(\widetilde{x},\widetilde{y})^2}}\left(\frac{\widetilde{x}+\widetilde{y}-\frac{2\sqrt{2}\frac{a^2}{b^2}x_0}{\left(1+\frac{a^2}{b^2}\right)}}{\widetilde{x}+\widetilde{y}-\sqrt{2}x_0}\right). \tag{37}$$

There may still be conditions necessary to ensure that $h(z)$ remains non-negative. Figure 4 shows the lines given by the numerator and denominator of the second part of (36).



Firgure 4: Zero sets of the numerator and denominator of (36)

Clearly, the numerator and denominator do not change sign if $y_0 \geq a$. Similarly, the interface parameter (37) for the case $y_0 = 0$ is non-negative whenever $x_0 \geq a$.

An elliptic inclusion embedded in a matrix subjected to antiplane shear characterized by (32)

$$\chi_1 = Re^{i\theta_0}\left((z - z_0)^2 - z_0^2\right), \qquad R, \theta_0 \in \mathbb{R}, \qquad 0 \leq \theta_0 < \pi, \qquad z_0 \neq 0$$

may be made neutral if $\theta_0 = \frac{\pi}{2}$, and z_0 is either real or purely imaginary. If these conditions hold, the interface parameter is given by

$$h(\widetilde{x}, \widetilde{y}) = \frac{\mu_1\left(1 + \frac{a^2}{b^2}\right)}{2(1-\delta)\,a\sqrt{1 - \frac{a^2}{b^2}\left(a^{-2} - b^{-2}\right)y(\widetilde{x},\widetilde{y})^2}}\left(\frac{\widetilde{x} - \widetilde{y} + \frac{2\sqrt{2}y_0}{\left(1+\frac{a^2}{b^2}\right)}}{\widetilde{x} - \widetilde{y} + \sqrt{2}y_0}\right)$$

if $z_0 = iy_0$, $y_0 \in \mathbb{R}$. If $z_0 = x_0, x_0 \in \mathbb{R}$, then the interface parameter is given by

$$h(\widetilde{x}, \widetilde{y}) = \frac{\mu_1\left(1 + \frac{a^2}{b^2}\right)}{2(1-\delta)\,a\sqrt{1 - \frac{a^2}{b^2}\left(a^{-2} - b^{-2}\right)y(\widetilde{x},\widetilde{y})^2}}\left(\frac{\widetilde{x} + \widetilde{y} - \frac{2\sqrt{2}\frac{a^2}{b^2}x_0}{\left(1+\frac{a^2}{b^2}\right)}}{\widetilde{x} + \widetilde{y} - \sqrt{2}x_0}\right)$$

Remark 5 *The conditions found here for the existence of neutral elliptic inclusions specify the arrangement of such elliptic inclusions in a manner* *nearly identical to that shown for circular inclusions in Figure 3. Recall the $\widetilde{x}$ and $\widetilde{y}$ axes are rotated from the standard axes by an angle of $\frac{\pi}{4}$. Thus, the inclusion pictured in Figure 4 is actually on one of the lines pictured in Figure 3. In fact, in a composite incorporating elliptic inclusions as specified here, the inclusions' centers would all lie on the lines pictured in Figure 3, and their axes would all be parallel with the axes in Figure 3.*

5 Conclusions and Suggestions for Future Research

The complex variable formulation of the problem was developed in a more general context than had been done previously in [14]. The problem was developed from the physical assumptions to give explicit expressions for the unknown functions without any assumptions on their specific form. The governing interface condition (12) and the corresponding interface parameter (13) were derived in a form that facilitates extension of the model to more complicated stress distributions.

The examples given indicate that, as the boundary curves and stress distributions increase in complexity, solutions become increasingly restrictive.

It may be of interest to investigate the possibility of using "smart materials" to construct interface layers which could allow for negative values of the interface parameters.

It may also be of interest to address the convergence and well-posedness of the neutrality problem. If the design parameters (i.e. the shape of the inclusion, and the magnitude of the interface parameter) are perturbed, it is important to establish the order of convergence (or if there is indeed convergence) to neutrality. This is motivated by the observation from results in this thesis that indicate error in certain design parameters gives rise to singularities in the stress distributions. It is critical to the practical application of the model that such singular behavior is properly characterized.

Appendix I. Derivation of (4) and (5) from (9)

The real variable interface conditions given by (4) and (5) are formally equivalent to the single complex variable interface condition given by (9). The derivation of the complex variable condition (9) from the two real variable conditions is shown in Section 2.2. The derivation of (4) and (5) from (9) is shown here.

Taking the imaginary part of (9) gives

$$\mathrm{Im}\left[\chi_1\right] = \mathrm{Im}\left[\chi_2\right], \qquad z \in \Gamma$$

Taking the partial derivative along Γ in the clockwise direction gives

$$\frac{\partial \, \mathrm{Im}\left[\chi_1\right]}{\partial t} = \frac{\partial \, \mathrm{Im}\left[\chi_2\right]}{\partial t}, \qquad z \in \Gamma$$

where t is a unit tangent vector to Γ. Then, applying the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial \, \mathrm{Re}\left[\varphi\right]}{\partial x} &= \frac{\partial \, \mathrm{Im}\left[\varphi\right]}{\partial y} \\ \frac{\partial \, \mathrm{Re}\left[\varphi\right]}{\partial x} &= -\frac{\partial \, \mathrm{Im}\left[\varphi\right]}{\partial y} \end{aligned}$$

(where $(x, y) \in A \cup \partial A$, and φ is an analytic function on the region A) allows one to rotate an equation of complex partial derivatives by an angle of $\frac{\pi}{2}$ so we get

$$\begin{aligned}
\frac{\partial \operatorname{Re}[\chi_1]}{\partial n} &= \frac{\partial \operatorname{Re}[\chi_2]}{\partial n}, \qquad z \in \Gamma \\
\frac{\partial w_1}{\partial n} &= \frac{\partial w_2}{\partial n}, \qquad z \in \Gamma.
\end{aligned}$$

Thus (9)$\Longrightarrow$(4).

Taking the real part of (9) gives

$$\begin{aligned}
\mu_1 w_1 &= (2\delta - 1)\,\mu_2 w_2 + \frac{\mu_1}{h(z)} \operatorname{Re}\left[e^{iN(z)}\chi_1'\right], \qquad z \in \Gamma \\
\mu_1 w_1 &= \left(\frac{\mu_1}{\mu_2}\right) \mu_2 w_2 + \frac{\mu_1}{h(z)} \frac{\partial (\mu_1 w_1)}{\partial n}, \qquad z \in \Gamma \\
h(z)\,[w_1 - w_2] &= \mu_1 \frac{\partial w_1}{\partial n}, \qquad z \in \Gamma.
\end{aligned}$$

Thus (9)$\Longrightarrow$(5).

Appendix II. Solution of (28) and (29)

Solving both instances of (28) simultaneously gives:

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 \pm \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] = \tan\theta_0 \pm \sec\theta_0$$

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)^2\left(\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}\right) = \left(\left(\frac{1-\frac{a^2}{b^2}}{2}\right)\tan\theta_0 \pm \sec\theta_0\right)^2$$

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)^2\tan^2\theta_0+\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2} = \left(\frac{1-\frac{a^2}{b^2}}{2}\right)^2\tan^2\theta_0\pm\left(1-\frac{a^2}{b^2}\right)\tan\theta_0\sec\theta_0+\sec^2\theta_0$$

$$\begin{aligned}
\left(\left(\frac{1+\frac{a^2}{b^2}}{2}\right)^2 - \left(\frac{1-\frac{a^2}{b^2}}{2}\right)^2\right)\tan^2\theta_0 \mp \left(1-\frac{a^2}{b^2}\right)\tan\theta_0\sec\theta_0 + \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2} - \sec^2\theta_0 &= 0\\
\left(\frac{a^2}{b^2}\right)\sin^2\theta_0 \mp \left(1-\frac{a^2}{b^2}\right)\sin\theta_0 + \frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\frac{a^2}{b^2}\left(1-\sin^2\theta_0\right) - 1 &= 0\\
\frac{1}{2}\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)\sin^2\theta_0 \mp \left(1-\frac{a^2}{b^2}\right)\sin\theta_0 + \frac{1}{2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\sin\theta_0 &= \frac{-\left[\mp\left(1-\frac{a^2}{b^2}\right)\right] \pm \sqrt{\left(1-\frac{a^2}{b^2}\right)^2 - \frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)\left(\left(\frac{a^2}{b^2}\right)^2+\frac{a^2}{b^2}-2\right)}}{\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)}\\
&= \frac{[\pm 1] \pm \sqrt{1-\frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2+\frac{a^2}{b^2}-2\right)\left(1-\frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}
\end{aligned}$$

In order for (28) to correspond to a physical solution, one value of θ_0 must satisfy equality in both instances of the equality - ie. when the upper of the

$\pm$ signs apply, and when the lower apply. Thus we have:

$$\text{Either of } \frac{1 \pm \sqrt{1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}$$

$$= \text{Either of } \frac{-1 \pm \sqrt{1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}$$

Clearly, the only way this can be satisfied is when

$$\frac{1 - \sqrt{1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}$$

$$= \frac{-1 + \sqrt{1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}.$$

This gives:

$$\begin{aligned}
1 &= \sqrt{1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}} \\
1 &= 1 - \frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1} \\
0 &= \frac{\frac{a^2}{b^2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)}{\left(1 - \frac{a^2}{b^2}\right)} \\
0 &= \frac{\frac{a^2}{b^2}\left(\frac{a^2}{b^2} + 2\right)\left(\frac{a^2}{b^2} - 1\right)}{\left(1 - \frac{a^2}{b^2}\right)} \\
\frac{a^2}{b^2} &= 0, 1 \text{ (possibly), or } -2.
\end{aligned}$$

Solving similarly for (29) gives:

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\left[\tan\theta_0 \pm \sqrt{\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}}\right] = \tan\theta_0 \mp \sec\theta_0$$

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)^2\left(\tan^2\theta_0 + \frac{2\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}\right) = \left[\left(\frac{1-\frac{a^2}{b^2}}{2}\right)\tan\theta_0 \mp \sec\theta_0\right]^2$$

$$\left(\frac{1+\frac{a^2}{b^2}}{2}\right)^2\tan^2\theta_0 + \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2} = \left(\frac{1-\frac{a^2}{b^2}}{2}\right)^2\tan^2\theta_0 \mp \left(1-\frac{a^2}{b^2}\right)\tan\theta_0\sec\theta_0 + \sec^2\theta_0$$

$$\left(\frac{a^2}{b^2}\right)\tan^2\theta_0 \pm \left(1-\frac{a^2}{b^2}\right)\tan\theta_0\sec\theta_0 + \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2} - \sec^2\theta_0 = 0$$

$$\left(\frac{a^2}{b^2}\right)\sin^2\theta_0 \pm \left(1-\frac{a^2}{b^2}\right)\sin\theta_0 + \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2}\left(1-\sin^2\theta_0\right) - 1 = 0$$

$$\left(\frac{a^2}{b^2} - \frac{a^2}{b^2}\left(\frac{1+\frac{a^2}{b^2}}{2}\right)\right)\sin^2\theta_0 \pm \left(1-\frac{a^2}{b^2}\right)\sin\theta_0 - 1 + \left(\frac{1+\frac{a^2}{b^2}}{2}\right)\frac{a^2}{b^2} = 0$$

$$\frac{1}{2}\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)\sin^2\theta_0 \pm \left(1-\frac{a^2}{b^2}\right)\sin\theta_0 + \frac{1}{2}\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right) = 0$$

$$\begin{aligned}\sin\theta_0 &= \frac{-\left[\pm\left(1-\frac{a^2}{b^2}\right)\right] \pm \sqrt{\left(1-\frac{a^2}{b^2}\right)^2 - \left(\frac{a^2}{b^2}\right)\left(1-\frac{a^2}{b^2}\right)\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)}}{\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)} \\ &= \frac{[\mp 1] \pm \sqrt{1 - \left(\frac{a^2}{b^2}\right)\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1-\frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}\end{aligned}$$

Similar to (28), we have

$$
\begin{aligned}
& \text{Either of } \frac{1 \pm \sqrt{1 - \left(\frac{a^2}{b^2}\right)\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}} \\
= \; & \text{Either of } \frac{-1 \pm \sqrt{1 - \left(\frac{a^2}{b^2}\right)\left(\left(\frac{a^2}{b^2}\right)^2 + \frac{a^2}{b^2} - 2\right)\left(1 - \frac{a^2}{b^2}\right)^{-1}}}{\frac{a^2}{b^2}}
\end{aligned}
$$

Here, we observe that the equality conditions are identical to those found for (28).

Appendix III. Derivation of (34) from (33)

The quotient in (33) which requires simplification is

$$\frac{\begin{array}{c} x^2\cos\theta_0 - \left(1+\frac{a^2}{b^2}\right)(\sin\theta_0)\,xy - \left(\frac{a^2}{b^2}\right)y^2\cos\theta_0 \\ +\left(-x_0\cos\theta_0 + y_0\sin\theta_0\right)x + \frac{a^2}{b^2}\left(y_0\cos\theta_0 + x_0\sin\theta_0\right)y \end{array}}{\begin{array}{c} x^2\cos\theta_0 - 2\sin\theta_0 xy - y^2\cos\theta_0 \\ +2\left(-x_0\cos\theta_0 + y_0\sin\theta_0\right)x + 2\left(y_0\cos\theta_0 + x_0\sin\theta_0\right)y \end{array}} \tag{38}$$

Using the substitution:

$$\begin{aligned}
x &= \widetilde{x}\cos\frac{\theta_0}{2} + \widetilde{y}\sin\frac{\theta_0}{2} \\
y &= \widetilde{y}\cos\frac{\theta_0}{2} - \widetilde{x}\sin\frac{\theta_0}{2} \\
x^2 &= \widetilde{x}^2\cos^2\frac{\theta_0}{2} + 2\widetilde{x}\widetilde{y}\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2} + \widetilde{y}^2\sin^2\frac{\theta_0}{2} \\
xy &= -\widetilde{x}^2\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + \widetilde{x}\widetilde{y}\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right) + \widetilde{y}^2\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2} \\
y^2 &= \widetilde{x}^2\sin^2\frac{\theta_0}{2} - 2\widetilde{x}\widetilde{y}\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + \widetilde{y}^2\cos^2\frac{\theta_0}{2}
\end{aligned}$$

we get

$$\frac{\left(\begin{array}{c}\widetilde{x}^2\left[\cos^2\frac{\theta_0}{2}\cos\theta_0+\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}-\left(\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\cos\theta_0\right]\\ +\widetilde{x}\widetilde{y}\left[2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}\cos\theta_0-\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\left(\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\right)+2\left(\frac{a^2}{b^2}\right)\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\cos\theta_0\right]\\ +\widetilde{y}^2\left[\sin^2\frac{\theta_0}{2}\cos\theta_0-\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}-\left(\frac{a^2}{b^2}\right)\cos^2\frac{\theta_0}{2}\cos\theta_0\right]\\ +\widetilde{x}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\cos\frac{\theta_0}{2}-\frac{a^2}{b^2}(y_0\cos\theta_0+x_0\sin\theta_0)\sin\frac{\theta_0}{2}\right]\\ +\widetilde{y}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}(y_0\cos\theta_0+x_0\sin\theta_0)\cos\frac{\theta_0}{2}\right]\end{array}\right)}{\left(\begin{array}{c}\widetilde{x}^2\left[\cos^2\frac{\theta_0}{2}\cos\theta_0+2\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\cos\theta_0\right]\\ +\widetilde{x}\widetilde{y}\left[2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}\cos\theta_0-2\sin\theta_0\left(\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\right)+2\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\cos\theta_0\right]\\ +\widetilde{y}^2\left[\sin^2\frac{\theta_0}{2}\cos\theta_0-2\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}-\cos^2\frac{\theta_0}{2}\cos\theta_0\right]\\ +2\widetilde{x}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\cos\frac{\theta_0}{2}-(y_0\cos\theta_0+x_0\sin\theta_0)\sin\frac{\theta_0}{2}\right]\\ +2\widetilde{y}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\sin\frac{\theta_0}{2}+(y_0\cos\theta_0+x_0\sin\theta_0)\cos\frac{\theta_0}{2}\right]\end{array}\right)}$$

Collecting terms gives

$$\frac{\left(\begin{array}{c} \widetilde{x}^2\left[\left(\cos^2\frac{\theta_0}{2}-\left(\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\right)\cos\theta_0+\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right] \\ +\widetilde{x}\widetilde{y}\left[\begin{array}{c}\left(2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}+2\left(\frac{a^2}{b^2}\right)\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right)\cos\theta_0 \\ -\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\left(\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\right)\end{array}\right] \\ +\widetilde{y}^2\left[\left(\sin^2\frac{\theta_0}{2}-\left(\frac{a^2}{b^2}\right)\cos^2\frac{\theta_0}{2}\right)\cos\theta_0-\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right] \\ +\widetilde{x}\left[\left(-x_0\cos\theta_0+y_0\sin\theta_0\right)\cos\frac{\theta_0}{2}-\frac{a^2}{b^2}\left(y_0\cos\theta_0+x_0\sin\theta_0\right)\sin\frac{\theta_0}{2}\right] \\ +\widetilde{y}\left[\left(-x_0\cos\theta_0+y_0\sin\theta_0\right)\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}\left(y_0\cos\theta_0+x_0\sin\theta_0\right)\cos\frac{\theta_0}{2}\right] \end{array}\right)}{\left(\begin{array}{c} \widetilde{x}^2\left[\left(\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\right)\cos\theta_0+2\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right] \\ +\widetilde{x}\widetilde{y}\left[4\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}\cos\theta_0-2\sin\theta_0\left(\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}\right)\right] \\ +\widetilde{y}^2\left[\left(\sin^2\frac{\theta_0}{2}-\cos^2\frac{\theta_0}{2}\right)\cos\theta_0-2\sin\theta_0\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right] \\ +2\widetilde{x}\left[\left(-x_0\cos\theta_0+y_0\sin\theta_0\right)\cos\frac{\theta_0}{2}-\left(y_0\cos\theta_0+x_0\sin\theta_0\right)\sin\frac{\theta_0}{2}\right] \\ +2\widetilde{y}\left[\left(-x_0\cos\theta_0+y_0\sin\theta_0\right)\sin\frac{\theta_0}{2}+\left(y_0\cos\theta_0+x_0\sin\theta_0\right)\cos\frac{\theta_0}{2}\right] \end{array}\right)}$$

Noting that $2\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}=\sin\theta_0$, and $\cos^2\frac{\theta_0}{2}-\sin^2\frac{\theta_0}{2}=\cos\theta_0$, this becomes

$$
\frac{\left(\begin{array}{c}
\widetilde{x}^2\left[\left(\cos\theta_0+\left(1-\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\right)\cos\theta_0+\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\sin^2\theta_0\right] \\
+\widetilde{x}\widetilde{y}\left[\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\cos\theta_0-\left(1+\frac{a^2}{b^2}\right)\sin\theta_0\cos\theta_0\right] \\
+\widetilde{y}^2\left[\left(-\cos\theta_0+\left(1-\frac{a^2}{b^2}\right)\cos^2\frac{\theta_0}{2}\right)\cos\theta_0-\frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\sin^2\theta_0\right] \\
+\widetilde{x}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\cos\frac{\theta_0}{2}-\frac{a^2}{b^2}(y_0\cos\theta_0+x_0\sin\theta_0)\sin\frac{\theta_0}{2}\right] \\
+\widetilde{y}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}(y_0\cos\theta_0+x_0\sin\theta_0)\cos\frac{\theta_0}{2}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2\left[\cos^2\theta_0+\sin^2\theta_0\right] \\
+2\widetilde{x}\widetilde{y}\left[\sin\theta_0\cos\theta_0-\sin\theta_0\cos\theta_0\right] \\
+\widetilde{y}^2\left[-\cos^2\theta_0-\sin^2\theta_0\right] \\
+2\widetilde{x}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\cos\frac{\theta_0}{2}-(y_0\cos\theta_0+x_0\sin\theta_0)\sin\frac{\theta_0}{2}\right] \\
+2\widetilde{y}\left[(-x_0\cos\theta_0+y_0\sin\theta_0)\sin\frac{\theta_0}{2}+(y_0\cos\theta_0+x_0\sin\theta_0)\cos\frac{\theta_0}{2}\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\widetilde{x}^2\left[\left(\cos\theta_0 + \left(1-\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\right)\cos\theta_0 + \frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\sin^2\theta_0\right] \\
+\widetilde{y}^2\left[\left(-\cos\theta_0 + \left(1-\frac{a^2}{b^2}\right)\cos^2\frac{\theta_0}{2}\right)\cos\theta_0 - \frac{1}{2}\left(1+\frac{a^2}{b^2}\right)\sin^2\theta_0\right] \\
+\widetilde{x}\left[\begin{array}{c} -x_0\cos^3\frac{\theta_0}{2} + x_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + 2y_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} \\ -\frac{a^2}{b^2}y_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + \frac{a^2}{b^2}y_0\sin^3\frac{\theta_0}{2} - 2x_0\frac{a^2}{b^2}\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} \end{array}\right] \\
+\widetilde{y}\left[\begin{array}{c} -x_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + x_0\sin^3\frac{\theta_0}{2} + 2y_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} \\ +\frac{a^2}{b^2}y_0\cos^3\frac{\theta_0}{2} - \frac{a^2}{b^2}y_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + 2\frac{a^2}{b^2}x_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} \end{array}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2 - \widetilde{y}^2 \\
+2\widetilde{x}\left[\begin{array}{c} -x_0\cos^3\frac{\theta_0}{2} + x_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + 2y_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} \\ -y_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + y_0\sin^3\frac{\theta_0}{2} - 2x_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} \end{array}\right] \\
+2\widetilde{y}\left[\begin{array}{c} -x_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + x_0\sin^3\frac{\theta_0}{2} + 2y_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} \\ +y_0\cos^3\frac{\theta_0}{2} - y_0\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + 2x_0\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} \end{array}\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\widetilde{x}^2 \left[\begin{array}{c}
\left(\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right) + \left(1 - \frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\right)\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right) \\
+ \left(1 + \frac{a^2}{b^2}\right) 2\sin^2\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2}
\end{array}\right] \\
+\widetilde{y}^2 \left[\begin{array}{c}
\left(-\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right) + \left(1 - \frac{a^2}{b^2}\right)\cos^2\frac{\theta_0}{2}\right)\left(\cos^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\right) \\
- \left(1 + \frac{a^2}{b^2}\right) 2\sin^2\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2}
\end{array}\right] \\
+\widetilde{x} \left[\begin{array}{c}
\left(-\cos^3\frac{\theta_0}{2} + \left(1 - 2\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right) x_0 \\
+ \left(\left(2 - \frac{a^2}{b^2}\right)\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + \frac{a^2}{b^2}\sin^3\frac{\theta_0}{2}\right) y_0
\end{array}\right] \\
+\widetilde{y} \left[\begin{array}{c}
\left(\sin^3\frac{\theta_0}{2} + \left(2\frac{a^2}{b^2} - 1\right)\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2}\right) x_0 \\
+ \left(\left(2 - \frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + \frac{a^2}{b^2}\cos^3\frac{\theta_0}{2}\right) y_0
\end{array}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2 - \widetilde{y}^2 \\
+2\widetilde{x}\left[\left(-\cos^3\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2}\right) x_0 + \left(\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + \sin^3\frac{\theta_0}{2}\right) y_0\right] \\
+2\widetilde{y}\left[\left(\sin\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2} + \sin^3\frac{\theta_0}{2}\right) x_0 + \left(\sin^2\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + \cos^3\frac{\theta_0}{2}\right) y_0\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\widetilde{x}^2 \left[\begin{array}{c} \cos^4 \frac{\theta_0}{2} - \left(1 + \frac{a^2}{b^2}\right) \sin^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_0}{2} + \frac{a^2}{b^2} \sin^4 \frac{\theta_0}{2} \\ + \left(1 + \frac{a^2}{b^2}\right) 2 \sin^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_0}{2} \end{array}\right] \\
-\widetilde{y}^2 \left[\begin{array}{c} \frac{a^2}{b^2} \cos^4 \frac{\theta_0}{2} - \left(1 + \frac{a^2}{b^2}\right) \sin^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_0}{2} + \sin^4 \frac{\theta_0}{2} \\ + \left(1 + \frac{a^2}{b^2}\right) 2 \sin^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_0}{2} \end{array}\right] \\
-\widetilde{x} \left[\begin{array}{c} \left(\cos^2 \frac{\theta_0}{2} + \left(2\frac{a^2}{b^2} - 1\right) \sin^2 \frac{\theta_0}{2}\right) x_0 \cos \frac{\theta_0}{2} \\ - \left(\left(2 - \frac{a^2}{b^2}\right) \cos^2 \frac{\theta_0}{2} + \frac{a^2}{b^2} \sin^2 \frac{\theta_0}{2}\right) y_0 \sin \frac{\theta_0}{2} \end{array}\right] \\
+\widetilde{y} \left[\begin{array}{c} \left(\sin^2 \frac{\theta_0}{2} + \left(2\frac{a^2}{b^2} - 1\right) \cos^2 \frac{\theta_0}{2}\right) x_0 \sin \frac{\theta_0}{2} \\ + \left(\left(2 - \frac{a^2}{b^2}\right) \sin^2 \frac{\theta_0}{2} + \frac{a^2}{b^2} \cos^2 \frac{\theta_0}{2}\right) y_0 \cos \frac{\theta_0}{2} \end{array}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2 - \widetilde{y}^2 \\
-2\widetilde{x} \left[\left(\cos^2 \frac{\theta_0}{2} + \sin^2 \frac{\theta_0}{2}\right) x_0 \cos \frac{\theta_0}{2} - \left(\cos^2 \frac{\theta_0}{2} + \sin^2 \frac{\theta_0}{2}\right) y_0 \sin \frac{\theta_0}{2}\right] \\
+2\widetilde{y} \left[\left(\cos^2 \frac{\theta_0}{2} + \sin^2 \frac{\theta_0}{2}\right) x_0 \sin \frac{\theta_0}{2} + \left(\sin^2 \frac{\theta_0}{2} + \cos^2 \frac{\theta_0}{2}\right) y_0 \cos \frac{\theta_0}{2}\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\widetilde{x}^2\left[\cos^4\frac{\theta_0}{2}+\left(1+\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2}+\frac{a^2}{b^2}\sin^4\frac{\theta_0}{2}\right]\\
-\widetilde{y}^2\left[\frac{a^2}{b^2}\cos^4\frac{\theta_0}{2}+\left(1+\frac{a^2}{b^2}\right)\sin^2\frac{\theta_0}{2}\cos^2\frac{\theta_0}{2}+\sin^4\frac{\theta_0}{2}\right]\\
-\widetilde{x}\left[\left(\cos\theta_0+2\frac{a^2}{b^2}\sin^2\frac{\theta_0}{2}\right)x_0\cos\frac{\theta_0}{2}-\left(2\cos^2\frac{\theta_0}{2}-\frac{a^2}{b^2}\cos\theta_0\right)y_0\sin\frac{\theta_0}{2}\right]\\
+\widetilde{y}\left[\left(2\frac{a^2}{b^2}\cos^2\frac{\theta_0}{2}-\cos\theta_0\right)x_0\sin\frac{\theta_0}{2}+\left(2\sin^2\frac{\theta_0}{2}+\frac{a^2}{b^2}\cos\theta_0\right)y_0\cos\frac{\theta_0}{2}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2-\widetilde{y}^2\\
-2\widetilde{x}\left[x_0\cos\frac{\theta_0}{2}-y_0\sin\frac{\theta_0}{2}\right]\\
+2\widetilde{y}\left[x_0\sin\frac{\theta_0}{2}+y_0\cos\frac{\theta_0}{2}\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\widetilde{x}^2\left[\cos^2\frac{\theta_0}{2}+\frac{a^2}{b^2}\sin^2\frac{\theta_0}{2}\right]\\
-\widetilde{y}^2\left[\frac{a^2}{b^2}\cos^2\frac{\theta_0}{2}+\sin^2\frac{\theta_0}{2}\right]\\
-\widetilde{x}\left[\left(x_0\cos\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\sin\frac{\theta_0}{2}\right)\cos\theta_0+\left(\frac{a^2}{b^2}\sin\frac{\theta_0}{2}x_0-\cos\frac{\theta_0}{2}y_0\right)2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}\right]\\
+\widetilde{y}\left[\left(-x_0\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\cos\frac{\theta_0}{2}\right)\cos\theta_0+\left(\frac{a^2}{b^2}\cos\frac{\theta_0}{2}x_0+\sin\frac{\theta_0}{2}y_0\right)2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{2}\right]
\end{array}\right)}{\left(\begin{array}{c}
\widetilde{x}^2-\widetilde{y}^2\\
-2\widetilde{x}\left[x_0\cos\frac{\theta_0}{2}-y_0\sin\frac{\theta_0}{2}\right]\\
+2\widetilde{y}\left[x_0\sin\frac{\theta_0}{2}+y_0\cos\frac{\theta_0}{2}\right]
\end{array}\right)}
$$

$$
= \frac{\left(\begin{array}{c}
\tilde{x}^2\left[\cos^2\frac{\theta_0}{2}+\frac{a^2}{b^2}\sin^2\frac{\theta_0}{2}\right] \\
-\tilde{y}^2\left[\frac{a^2}{b^2}\cos^2\frac{\theta_0}{2}+\sin^2\frac{\theta_0}{2}\right] \\
-\tilde{x}\left[\left(x_0\cos\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\sin\frac{\theta_0}{2}\right)\cos\theta_0+\left(\frac{a^2}{b^2}\sin\frac{\theta_0}{2}x_0-\cos\frac{\theta_0}{2}y_0\right)\sin\theta_0\right] \\
+\tilde{y}\left[\left(-x_0\sin\frac{\theta_0}{2}+\frac{a^2}{b^2}y_0\cos\frac{\theta_0}{2}\right)\cos\theta_0+\left(\frac{a^2}{b^2}\cos\frac{\theta_0}{2}x_0+\sin\frac{\theta_0}{2}y_0\right)\sin\theta_0\right]
\end{array}\right)}{\left(\begin{array}{c}
\tilde{x}^2-\tilde{y}^2 \\
-2\tilde{x}\left[x_0\cos\frac{\theta_0}{2}-y_0\sin\frac{\theta_0}{2}\right] \\
+2\tilde{y}\left[x_0\sin\frac{\theta_0}{2}+y_0\cos\frac{\theta_0}{2}\right]
\end{array}\right)}
$$

Which is the final form given in (34).

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