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TOWARD MATHESIS: A PARADIGM FOR THE DEVELOPMENT OF HUMANISTIC MATHEMATICS CURRICULA

by

© WILLIAM COOKE HIGGINSON

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "TOWARD MATHESIS: A PARADIGM FOR THE DEVELOPMENT OF HUMANISTIC MATHEMATICS CURRICULA" submitted by WILLIAM C. HIGGINSON in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics Education.

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Date 30 August 1973.

This study is dedicated with

affection and gratitude

to

M. M. H. and T. E. K.,

model mathematicians

ABSTRACT

This synthetic, conceptual study develops some of the educational aspects of the concept of mathesis, or humanistic mathematics. In it a heuristic framework, or propaedeutic paradigm, for the construction of mathesis curricula is outlined, elaborated, illustrated and rationalized. The study bridges two different approaches to the area of mathesis: the philosophical, concerning the nature of mathesis and the practical, relating to the humanizing of mathematics curricula. Two pervasive themes are those of man as 'active organism' and Popper's epistemic method of conjectures and refutations. The dissertation is seen as being a broad, impressionistic and initial statement regarding mathesis; more detailed consideration of certain aspects of the study are planned.

Central to the concept of humanism are the 'essential attributes' of man: his capacities for rational thought, symbol-making, social organization, technical production and game-playing. Mathematics is considered to be the science of structure. The humanistic aspects of mathematics, centred on persons and processes, rather than on products, is referred to as mathesis. The ten components of the mathesis paradigm are listed. Its two foundation pieces are the epistemological and psychological bases. Popper's 'Critical Fallibilism' is the theory of knowledge which underlies the

paradigm and the study as a whole. The cognitive psychological basis is provided by Piaget's work and the affective basis by the writings of the third force psychologists such as Maslow. The remaining components are the process bias, the potentially-rich situation matrix, the pedagogic mode and five characteristics, genetic, interdisciplinary, contemporary, high-information and intrinsic-interest.

Using Dewey's idea of reflective thought, the importance of problems and problem-solving to humanistic education is argued. Ten characteristics of potentially-rich situations, the component of the paradigm most closely related to problem-solving, are also given here. These characteristics are: accessibility, breadth, depth, connectivity, generalizability, pattern-latency, concrete representability, empiricality, identifiability and symbolizability. Examples of two particular potentially-rich situations, polytopes and polyominoes, are given and are examined in detail.

An analysis of the mathesis paradigm, in particular from the viewpoint of validity and utility. Does the paradigm do what it purports to do, and how useful is it at doing this? The structural aspects of the validity and utility of the paradigm are judged by using informal interpretations of the formal concepts of consistency, completeness and independence. Indications are found that the paradigm has some measure of 'completeness' and 'consistency'. It appears that the non-independence of the components may be essential to the successful implememtation of the paradigm. The non-structural aspects of the paradigm are examined using the criteria of 'existence' and 'falsifiability'. The work of Davis, Papert and the members of the Association of Teachers of Mathematics are taken as 'partial-existence' examples of mathematics curricula. The importance of the role of the individual teacher in mathesis curricula is emphasized.

PREFACE

This study is, by nature, open, and synthetic. It attempts to give an overall, impressionistic view, from an educational standpoint, of the extremely large and virtually unconsidered problem of 'humanistic mathematics'.

In considering this question it would seem that there are two distinct approaches one might take. From a purely philosophic position one could look at the 'What is humanistic mathematics?' problem in detail. From a purely practical vantage one might proceed by examining existing practice in mathematics education for its humanistic aspects. In this study a position which combines aspects of both of these approaches is taken. At the beginning of the study the nature of humanistic mathematics is briefly outlined, and at the end of it classroom practice is considered. The main purpose of the work is, however, to present a framework which will bridge these two different aspects of the problem. The bulk of the study is therefore devoted to the statement, elaboration, and analysis of a paradigm for the production of humanistic mathematics curricula. It is intended that at a later date more complete statements of the paradigm and the philosophical and practical aspects of the problem will be given.

The question of assumptions is one which is crucial to the study. Fundamental to almost all of the dissertation is the conception of man as an active organism. This assumption, which is closely connected to the long-standing philosophical dispute between determinism and free will, is by itself sufficient to bring the study into conflict with much of contemporary 'learning theory', which is ultimately based on a vision of man as a reactive organism. The study can be seen, in many ways, as an attempt to build a position from a set of fundamental assumptions. If one assumes certain things about the nature of mathematics and the nature of man, then what are the implications of these assumptions for education?

The scope of the study is such that it has not been feasible to refer in the text of the dissertation to all of the sources which have been influential in the construction of the paradigm. Because it seemed desirable to have a record of these influences available to the reader, a list of some of the major sources has been included in the dissertation as Appendix One. The sources have been classified as falling into one of five major areas: Mathematics, Philosophy, Psychology, Social Issues, or Education. These areas in turn have been subdivided into some twenty-five sub-areas. Readers wishing, for example, to follow up references on Popper should consult some of the literature in the Philosophy of Science subsection of the Philosophy area. For reasons of space, with a few exceptions, priority has been given to book titles in this appendix.

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CHAPTER ONE

INTRODUCTION

1.1 Introduction to the Study

The motivation for this work lies in a complex of social and cultural factors currently present in the western world. For a number of reasons social and educational planners have seen fit to recommend that educational institutions and their curricula become more 'humanistic'. Despite these recommendations there seems, at the moment, to be few comprehensive plans which describe how this humanization might be brought about. This study is an attempt to consider the problem of humanistic curricula in relation to the discipline of mathematics. The major part of the study is given over to the presentation and analysis of a paradigm which curriculum developers can use as a framework or generator for the construction of humanistic mathematics curricula.

In the first part of this chapter the social background to and motivation for the study is described. This is followed by a short description of humanism and mathematics. The way in which these two concepts are related is briefly examined in the next section and it is suggested that the word 'mathesis' can be used to mean 'humanistic mathematics'. The chapter concludes with two sections which define the terms 'curriculum' and 'paradigm' as thay are used in the study, and a concise overview of the study as a whole.

1.2 The Social Background to and Motivation for the Study

The roots of the demands for educational reform can be found in the unprecedented rate of social change which has occurred throughout the world in the last quarter-century. Variations in patterns of living which in the past might have evolved over a period of centuries have, in many parts of the world, taken place in decades or even years. The driving force behind this social change has been the technological application of scientific research. The general direction of this change has been toward growing urbanization, greater industrialization, and increased bureaucratization.

While there has been a considerable improvement in the standard of living, at least in the western world, during this period, social critics have pointed out that this has been purchased at great cost, both to individuals and to the environment. The concomitant hazards of this rapid social change and the forces which have brought it about have now become better known. The spectre of an over-populated, famine-threatened, environment-exhausted world, menaced by nuclear annihilation, is one which intelligent beings of the nineteen-seventies are only too familiar.

In recent years there has been increased public awareness of what may be called humanistic issues. These are situations which necessitate a reconsideration of the concept of human nature. Several of these issues have arisen from the results of bio-medical research. The reality of effective birth-control methods, the advent of 'transplant' technology and the prospect of 'xeroxed beings' brought into the world by cloning techniques has challenged many people's ideas of humanness. The question of inherent characteristics in humans has surfaced, noticeably in relation to the problem of sexual and racial discrimination. The man-nature relationship has undergone renewed scrutiny because of growing concern for the ecological balance.

A number of thinkers have attempted to extrapolate from the contemporary situation and to forecast what the future path of world development might be. Among the forecasts of this group of futurists there is, as might be expected, a wide range of possible scenarios. In spite of this there seems to be general agreement on four points: the only certainty about the future is that it will involve continuing change, that there is a strong possibility that man as a species will cease to exist, that this possibility of extinction will be decreased if the people of the western world will lessen their emphasis on materialistic values, and that at the moment few individuals are adequately prepared to cope with change.

The last conclusion leads most of the futurists to be quite critical of contemporary educational institutions. This is a characteristic they share with many other writers. A large number of social critics, academics, teachers, and students have focused their attention on the present-day system of education. While the prognoses may differ, the diagnoses sound uncannily the same. Schools as joyless, boring, archaic, destructive, impersonal and mindless places are among the most dehumanizing institutions of our culture. They do little to help their clients achieve their potential and even less to aid them in coping with the world around them.

Educational planners, obviously influenced by both the futurists and the critics of schooling, have produced reports which prescribe educational systems with a strong humanistic flavour. The Hall-Dennis Commission in Ontario (1968) with its theme of a "child-centred learning continuum (p. 179)", the Worth Commission in Alberta (1972) with its "alternative future...a person-centred society (p. 31)" and the Faure Commission (UNESCO) (1973), which sees the fundamental aim for education as being "the physical, intellectual, emotional and ethical integration of the individual into a complete man (p. 156)", can be seen as examples of this phenomenon.

The immediate response by the educational establishment to this criticism has not, in general, been particularly cogent. Many of the proposals for humanizing education have been exceedingly vague; others have concentrated on trivial aspects of the problem. Few writers seem to have dealt at all successfully with the problem in its totality. This probably reflects the fact that few of them have made it clear just what they mean when they use the term humanistic. It is this question which we consider in the next section.

1.3 On Humanism

The word 'humanistic' is directly derived from 'humanist' which in turn is directly derived from 'human'. Hence one's interpretation of 'humanistic' depends entirely on what one understands by the word 'human'. The problem thus reduces to defining what it means to be human, or equivalently, to describing 'human nature', or to describing how man differs from his fellow animals.

Unfortunately, this is one of the oldest and most disputed problems known to man. The proposed answers to it have been many and varied. At different times and in different places the case has been made for man as: the thinking reed, the time-binding animal, the animal who knows he is going to die, and the animal with a soul. From ancient times many thinkers have proposed man as animal rationale, while more recently the cases have been argued for homo faber and homo politicus. From a mathematical point of view the positions of Cassirer (1970) and Huizinga (1971) seem particularly attractive. They make strong cases for, respectively, animal symbolicum, man the symbol-making animal, and homo ludens, man the game-playing animal.

To tie oneself to a single characteristic view of human nature is, however, to put oneself in a vulnerable position. (As Plato is reputed to have found when he defined man as a 'featherless biped' only to have Diogenes produce a plucked rooster called 'Plato's man' (Laertius, 1853, p. 231).) A more reasonable position would seem to be that taken by Fromm and Xirau (1968) who consider human nature in terms of not one, but several 'essential attributes', such as the capacity for symbol-making. They state:

> of course man is not wholly definable, but what we have termed his "essential attributes" can give us an approximate, and at the same time, rather accurate approach to what we may call man's nature (p. 9).

We accordingly take the view in this study that man is an active organism whose nature is to a great extent characterized by his capacities for symbol-making, rational thought, social organization, technical production and game-playing.

In many historical periods groups of men have declared themselves to be philosophical humanists. Traditionally there are two ways of arriving at a humanist position. Panofsky (1970) has written:

> Historically the word humanitas has had two clearly distinguishable meanings, the first arising from a contrast between man and what is less than man; the second between man and what is more. In the first case humanitas means a value, in the second a limitation...It is from this ambivalent conception of humanitas that humanism was born. It is not so much a movement as an attribute which can be defined as the conviction of the dignity of man, based both on the insistence on human values (rationality and freedom) and the acceptance of human limitations (fallibility and fraility): from this two postulates result - responsibilty and tolerance (pp. 23-24).

The first of these humanist groups arose during the Renaissance in reaction to the strict doctrines of the mediaeval church. Perhaps the most recently-formed humanist school is that of the Third-Force psychologists; this group was formed largely as a reaction to the doctrines of Behaviourism and Freudianism. Although separated by a period of several hundred years, these humanist groups share a dedication to anti-authoritarianism, a high regard for the creations of man and a faith in the potential of man.

That not all humanist groups have emphasized the same aspects of humanness can be seen from the names they have used to describe themselves. At one time or another, there have been movements known as Existential humanism, Christian humanism, Evolutionary humanism, Naturalistic humanism, Academic humanism, Marxist humanism, Pragmatic humanism and Scientific humanism. While each of these movements has stressed different aspects of man's "essential attributes", there are several value positions to which all of them adhere. It is this core of value positions which constitutes a humanistic position in this study. Hence a humanist here is an individual who is concerned for the worth, welfare and dignity of the individual; sees man as active, capable and responsible; and who values man's essential attributes, the capacity for rational thought, symbol-making, technical production, social organization and game-playing.

From this basic statement many 'corollaries' about humanism follow. For example, because of their faith in man's capacity for rational thought, humanists have a high regard for the methodology of science. However, because of their concern for human welfare, they will not use this methodology indiscriminately; they would not accept a purely rationalist position which would argue for value-free science. In a similar manner one can show that while humanists respect the achievements of man and are in that sense traditionalists, they do not accept authoritarianism and thus cannot be regarded as being conservatives.

It follows that in this study, curricula, educational goals, and educational systems can be said to be humanistic if they are consistent with this humanist value position.

In so far as humanists view man as being active, responsible for his own actions and capable of influencing his destiny, they have what Reese and Overton (1970) have called "an active organism model of man (p. 133)". This holistic or organismic model differs radically from the mechanistic or "reactive organism model" which sees man's activity as a "result of external or peripheral forces (p. 131)." Reese and Overton contend that these two world views are "logically independent and cannot be assimilated to each other. They reflect different ways of looking at the world and, as such, are incompatible in their implications (p. 116)".

An immediate and important consequence of this contention is that much of contemporary 'educational theory' is simply not applicable to humanistic education. For example, most 'learning theories' are directly derived from the reactive model of man. To attempt to evaluate the results of work done within the context of one world view by the criteria developed within the framework of another world view is to invite confusion.

1.4 On Mathematics

The gap between the practitioner and the general public is probably greater in mathematics than in any other discipline. The general public has virtually no idea of either what contemporary mathematics is about or what it is that research mathematicians do (Halmos 1968; Friedman 1966). The view that mathematics is developed in an ultra-logical fashion by a select group of brilliant but cold researchers is quite mistaken but widely held nonetheless (Hudson, 1970).

For many people the subject of mathematics has quite unpleasant associations. Traditional mathematics teaching with its emphasis on calculation did much to develop these misconceptions and negative associations (Henry, 1966). It was hoped, therefore, that the 'new math' curriculum reforms of the last decade might succeed in giving students a more valid picture of the nature of mathematics and in promoting more positive attitudes toward the subject. Unfortunately, there seems to be little evidence to show that this has happened (Kline, 1973). In fact, the so-called 'new-math revolution' would appear in many ways to have been a media-inflated fraud and failure. Bell (1972) speaks of "disillusionment, puzzlement (p. 153)" and Davis (1970) writes of the "dissemination problem...somehow we can't do very often and in very many places at once what we can do in a few places now and then (p. 7)". Sarason (1971) concludes his examination of the introduction of new math programmes as follows:

> It is perhaps too charitable to conclude that 'the more things change the more they remain the same'; if only because so many people continue to be

> unaware that basically nothing has changed; in addition, and perhaps more to to the point, many of those who are aware that intended outcomes have not been achieved have no clear understanding of the factors contributing to failure (p. 46).

It is difficult to give a satisfactory definition of mathematics. Those given by professional mathematicians have often seemed unnecessarily enigmatic or cryptic. Russell wrote (1970) that mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true (p. 60)", and Benjamin Pierce (1881) called it "the science which draws necessary conclusions (p. 97)". In this study mathematics is considered to be the science of structure. This view of mathematics is the one which has been advanced by the polycephalous Nicholas Bourbaki (1950) and it has now come to be the one accepted by many influential contemporary mathematicians (Albert, Browder, Herstein, Kaplansky, and MacLane, 1965; Weissinger, 1969). This structuralist view subsumes the earlier conception of mathematics as the science of number and space. (If the science of structure view proves difficult to grasp, one can go a long way thinking of mathematics as the study, in the abstract, of patterns and relationships (Sawyer, 1963; Whitehead, 1941).) As well as having proved to be a powerful conceptual tool in pure and applied mathematics, the structuralist approach has yielded impressive results when applied to such diverse fields as psychology (Piaget, 1971a), anthropology (Levi-Strauss,

1967) and linguistics (Chomsky, 1965). Information-processing, both by machines (Cherry, 1970) and by humans (Biggs, 1971; Miller, 1970), is another area where the structuralist approach has been fruitful.

In this study making mathematics is considered to be a natural human activity (Gattegno, 1970). In this way it is seen as being similar to fields like art, literature and music in that it is an activity in which all humans can participate. Some people, the research mathematicians, will necessarily be highly skilled in this activity. However, their existence should not diminish the satisfaction the amateur mathematician gets from his work any more than the existence of professionals limits the satisfaction possible for amateurs in any other field. The view that a learner of mathematics is capable of doing authentic mathematical activity is borne out by Hadamard's (1954) statement:

> Between the work of the student who tries to solve a problem in geometry and algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature.

The sources of mathematics are seen to lie in human experience. Von Neumann (1961) has stated, "mathematical ideas originate in empirics, although the geneology is sometimes long and obscure (p. 9)". Since human experience is to a great extent culturally determined, this view seems to be consistent with Wilder's (1965) contention that:

"mathematics is what we make it (p. 299)" and that (Wilder, 1952), "the state and directions of growth of mathematics are determined by the general complex of cultural forces both within and without mathematics (p. 270)".

1.5 On Mathesis

Having described what the terms humanism and mathematics are to mean in the study, we can now turn to a consideration of possible interpretations of 'humanistic mathematics'. Does it make sense to speak of humanistic mathematics, or are the terms mutually contradictory? What, to use Ryle's (1968) phrase, is the "logical geography (p. 9)" of the relationship between humanism and mathematics?

We have stated that man's humanness is closely bound up with his capacity for rational thought, for symbol-making and for game-playing. Strong cases can be made for mathematics as being the purest product of rational thought, the most sophisticated symbol-system man has created and the highest form of game. In this study it is therefore assumed that the term humanistic mathematics is in no way inherently contradictory.

Etymologically the word mathematics is closely related to the concepts of thinking and learning for it is derived from the Greek root 'mathein' meaning 'to learn'. The assumption in the part of the Greeks that mathematics was in some way at the heart of rational activity was also reflected by the fact that their word 'reason' meant both 'to think logically' and 'to calculate arithmetically'. (See, for example, Plato's Republic, 1971, p. 292). Until fairly recently this same idea was expressed in the English language by the word 'mathesis' meaning "learning, mental discipline, especially mathematics (Webster's New International Dictionary, Second Unabridged Edition)". In this sense, and often personalized in verse, it was used by Peele in 1593, "And clothest Mathesis in rich ornaments,/ That admirable Mathematique skill, (Oxford English Dictionary)", and by Pope (1966), in 1742, in The Dunciad, "Mad Mathesis alone was unconfined,/ Too mad for mere material chains to bind (p. 551)". The polymathic Leibnitz started to formulate a foundation for reasoning in all of the sciences which he called 'mathēsis universalis'.

Many seventeenth and eighteenth-century thinkers perceived an intimate relationship between rational thought and mathematics. John Locke (1966) wrote:

> Would you have a man reason well, you must use him to it betimes, exercise his mind in observing the connection of ideas and following them in train. Nothing does this better than mathematics, which therefore I think should be taught to all those who have the time and the opportunity, not as much as to make them mathematicians as to make them reasonable creatures (p. 4).

Condorcet expressed quite similar thoughts (quoted in Polya, 1972):

> Mathematics is the science that yields the best opportunity to observe the working of the mind...and has the advantage that by cultivating it we may acquire the habit of a method of reasoning which may be applied afterwards to the study of any subject and can guide us in the pursuit of life's object (p. 71)".

The most distinguished contemporary proponent of the Locke-Condorcet school of thought is Polya (1957, 1967, 1968) who holds that through mathematics one can teach people how to think. Papert (1972a, b, c), who talks of having children 'learn how to learn', uses mathematics as a vehicle for this purpose, and Bruner and Kenney (1965) suggest that "learning mathematics may be viewed as a microcosm of intellectual developmemt (p. 59)". The recent work of Piaget on the possible isomorphism between cognitive and mathematical structures (Beth and Piaget, 1966) and the position of the Chomskian school of linguistics (Lyons, 1970) with its neo-Kantian views on inherent mental abilities, may both bring support to the Polya view if research bears out their conjectures.

The distinction between knowledge and learning is one that has been made many times. This distinction, as Panofsky (1970) points out, has existed at least since Roman times.

> A subtle difference exists in Latin between scientia and eruditio, and in English between knowledge and learning. Scientia and knowledge, denoting a mental possession rather than a mental process, can be identified with the natural sciences; eruditio and learning, denoting a

> process rather than a possession, with the humanities. The ideal aim of science would seem to be something like mastery, that of the humanities something like wisdom (pp. 49-50).

Although most dictionaries now consider the word mathesis to be obsolete, it will be used in this study to denote humanistic mathematics. It is the eruditio and not the scientia aspect of mathematics; the learning part and not the knowledge part; the process but not the product. Mathesis is more qualitative than quantitative, more directed to wisdom than to mastery; it is mathematics as a humanity. It is the rational, thinking, reasoning part of animal symbolicum and exists in every member of the species who has mastered natural language.

Its sense was caught with precision by Carlyle (1949), himself a mathematician of no small aptitude, when he wrote,

> More specifically, I appoint that five of the 'John Welsh Bursaries' shall be given for best proficiency in Mathematics (I would rather say 'in Mathesis' if that were a thing to be judged of from competition (p. 197)).

1.6 On Curriculum

Considerable debate has been waged within educational circles in recent years concerning the nature of curriculum. A number of models for curriculum development have been proposed and several approaches to curriculum theorizing have emerged (Short and Marconnit, 1969). As of yet, however, this activity has not produced a unified approach. MacDonald (1971a) has written:

> Curriculum theory and theorizing may be characterized as being in a rather formative condition. One would suspect that theory would be focused upon a clearly identified realm of phenomena. Unfortunately this is not so in curriculum for the definitions of curriculum are as narrow as 'the subject matter to be learned' and as broad as 'all the experiences students have in school' (p. 196).

In this study the term curriculum is used to refer to a dynamic system which has as components: teachers, learners, content, setting, values, goals and inter-relationships among these components. This view of curriculum which is holistic, dynamic, person-centred and multi-dimensional, differs considerably from many of the contemporary curriculum models, such as Johnson's (1967), which tend to be reductivist, goal-centred, static and linear. Many aspects of MacDonald's (1971b, 1972) circular model for curriculum development are, however, quite consistent with the given view of curriculum, as are several of Eisner's (1969) positions, particularly his concept of expressive objectives.

When used in conjunction with humanistic and mathematics, the term curriculum indicates that the relations between humanism and mathematics are being viewed from, and will be utilized in, some educational context. This educational context need not neccessarily be institutional.

1.7 On Paradigms

The central problem of the study is concerned with humanistic mathematics curricula. The conceptual mechanism which has been chosen to approach this problem is that of a paradigm. Etymologically derived from a Greek word meaning 'to show side by side', the term paradigm has traditionally been used to mean a pattern, a model, or an exemplar. It is still widely used in this sense in fields such as grammar and philosophy (Black, 1962; Wittgenstein, 1967). In the last few years, however, the term has been used in a much wider sense by sociologists such as Merton (1967), and philosophers of science such as Kuhn (1970).

Kahn and Bruce-Briggs (1972) have modified Merton's idea of an analytical paradigm to create a propaedeutic or heuristic paradigm. It is this conception of paradigm which is used throughout the study. In this sense a paradigm is:

> an explicitly structured set of assumptions, definitions, typologies, conjectures, analyses, and questions giving both a framework and a pattern of relationships: it is halfway between an analogy and a model, more rigorous than an analogy, not a model, relevant to the subject, but not a theory. It is a set of interrelated questions, typologies, conjectures, speculations, tentative theories, intuitions, insights, lists and so on which cover a subject as about as far as you can go. It offers you a framework, at least, for thinking about the subject (p. 89).

This conception of a paradigm differs from Kuhn's (1970) highly stimulating but rather amorphous (Masterman, 1970) idea of a 'paradigm of science' in that it is explicit rather than being implicit. The community aspect of a paradigm of science is, however, also important to propaedeutic paradigms, since many of the insights and intuitions will be derived from the experiences of a group of practitioners who share the basic assumptions.

1.8 Summary and Overview

In the previous sections of this chapter we have examined some of the forces militating for the introduction of humanistic curricula in education and have indicated the criteria which these curricula must meet. The two concepts fundamental to the study, humanism and mathematics, have been briefly reviewed and some comments regarding the nature of humanistic mathematics have been made. It has been suggested that mathesis can be used to mean humanistic mathematics and it has been implied that mathesis curricula may serve as vehicles for the achievement of humanistic goals in education. The way in which the terms curriculum and paradigm are to be used throughout the study have been noted. We now turn to the consideration, in the remainder of the dissertation, of the mathesis curriculum paradigm: a framework which will facilitate the construction and implementation of humanistic mathematics curricula.

The next chapter is devoted to the presentation of the components of the paradigm; in the third chapter we examine one of these components in depth. In the last two chapters we analyse the paradigm with respect to its validity and utility. In the fourth chapter the analysis is of the structural aspects of the paradigm and in the fifth chapter its non-structural aspects are examined.

CHAPTER TWO

THE COMPONENTS OF THE MATHESIS CURRICULUM PARADIGM

2.1 Introduction

The purpose of this chapter is to outline the mathesis curriculum paradigm (hereafter referred to as the mathesis paradigm): a framework which educators can employ to construct humanistic mathematics curricula. The paradigm is made up of ten components. They are, in the order in which they will be considered: the epistemological and psychological bases; the process bias; the potentially-rich situation matrix; the five characteristics, genetic, interdisciplinary, contemporary, high-information, intrinsic-interest; and the pedagogic mode. No attempt is made, at this stage, to rationalize the choice of the components or to analyze their inter-relationships; this is consistent with the nature of propaedeutic paradigms.

It is possible that one may grasp the essential ideas of some of the components more easily if one compares and contrasts the implications of the component with current practice in mathematics education. In the examples given to illustrate aspects of some of the components, the learners in most cases are assumed, using Piagetian terminology, to have reached the stage of 'formal operations'. This should not be taken to imply that the paradigm is inappropriate for constructing curricula for younger learners.

2.2 The Epistemological Basis

The epistemological position which seems most appropriate as a basis for mathesis curricula is that of 'Critical Fallibilism'. This theory of the origin, nature, limits and methods of knowledge is due to Popper (1968, 1969). According to Popper criticism is fundamental to knowledge. He writes:

> The process of learning, of the growth of subjective knowledge, is always fundamentally the same. It is imaginative criticism. This is how we transcend our local and temporal environment by trying to think of circumstances beyond our experience: by criticizing the universality, or the structural necessity of what may, to us appear (or what philosophers may describe) as the 'given' or as 'habit'; by trying to find, construct, invent, new situations - that is test situations, critical situations; and by trying to locate, detect and challenge our prejudices and habitual assumptions.
>
> This is how we lift ourselves by our bootstraps out of the morass of our ignorance (1972, p. 148).

The rack and pinion of this critical approach is the method of conjectures and refutations. Popper's position is that:

> The way in which knowledge progresses, and especially our scientific knowledge, is by unjustified (and unjustifiable) anticipations, by guesses, by tentative solutions to our problems, by conjectures. These conjectures are controlled by criticism; that is, by attempted refutations, which include severely critical tests. They may survive these tests; but they can never be positively justified: they can neither be established as certainly true nor even as 'probable' (in the sense of the probability calculus). Criticism of our conjectures is of decisive importance: by bringing out our mistakes

> it makes us understand the difficulties of the problem we are trying to solve. This is how we become better acquainted with our problem, and able to propose more mature solutions: the very refutation of a theory - that is, of any serious tentative solution to our problem - is always a step forward that takes us nearer to the truth. And this is how we can learn from our mistakes (1969, p. ix).

Popper's theories have been submitted to quite severe tests in several disciplines; in mathematics by Lakatos (1963), in education by Dawson (1969) and in science by Medawar (1969b) and Eccles (1970). In none of these tests have his conjectures been refuted. Popper himself has shown how his general epistemic approach can be applied to history (1961) and to political science (1966). Putting Popper's theories into practice in mathesis curricula would mean placing a premium on such activities as generating problems from given mathematical situations, making conjectures about the solutions to these problems, testing and modifying these conjectures and constructing alternate methods of solving these problems.

Popper sees mistakes in a very positive light; " all our knowledge grows only through the correcting of our mistakes (1969, p. ix)". This is antithetic to the position taken in most of today's schools. Holt (1969), Henry (1966) and Bell (1972) have documented the intensity of student's commitment to the 'right answer-please the teacher' game in today's schools. That this is by no means a recent phenomenon can be deduced from the fact that forty years ago Dewey (1933) was warning teachers of the dangers of having students "satisfying the teacher instead of the problem (p. 61)". Two mathematics educators who have used 'Popperian' techniques are Davis (1966) with his idea of "torpedoing (p. 117)" and Papert (1972b) who uses a "debugging (p. 261)" approach to problem-solving with computers.

Papert's highly personalized approach to problem-solving is quite consistent with the views of Polanyi (1964, 1967), another philosopher of science whose ideas are of relevance to mathesis. Of particular interest are his remarks on scientific objectivity and intellectual passions and his concepts of personal knowledge and tacit knowing. It is Polanyi's contention that "we know more than we can tell and we can tell nothing without relying on our awareness of things we may not be able to tell (1964, p. x)".

2.3 The Psychological Basis

From an affective standpoint the work of the third-force psychologists is particularly apposite for mathesis curricula. Rogers's (1961) conception of a "helping relationship" which he defines as being one in which "at least one of the parties has the intent of promoting the growth, development, maturity, improved functioning, improved coping with life of the other (p. 40)", would seem to be a useful one for humanistic teachers. His work on methods of heightening self-awareness and improving interpersonal skills also seem appropriate.

Something of a more theoretical base for the affective aspects of humanistic education can be found in the writing of Maslow (1968, 1970, 1971). The theory of the hierarchy of basic human needs (Maslow, 1970; Goble, 1972) and his concept of self-actualization (1968) are fundamental to Maslow's position. According to Maslow (1971):

> the goal of education - the human goal, the humanistic goal, the goal so far as human beings are concerned - is ultimately the "self-actualization" of a person, the fully becoming human, the development of the fullest height that the human species can stand up to or that the particular individual can come to. In a less technical way, it is helping the person to become the best that he is able to become (pp. 168-169).

From a cognitive viewpoint mathesis curricula can be built on the foundation provided by the research of Piaget (Piaget, 1963; Piaget, 1970; Piaget, 1971b; Piaget and Inhelder, 1969; Beth and Piaget, 1966; Inhelder and Piaget, 1958). Of the many aspects of Piaget's work which mathesis curriculum developers need to consider, two are of particular importance. The first of these is the equilibration process and the second concerns the relations between cognitive structures and mathematical structures. The assimilation-accommodation model of equilibration, which is fundamental to Piaget's theory of cognitive growth, strongly reflects his academic background in biology. It also shows clearly that Piaget (1968) has an 'active' view of man; he sees man as interacting with his environment rather than reacting to it, subduing it, or submitting passively to it. He has written:

> equilibrium is not an extrinsic or added characteristic but rather an intrinsic and constitutive property of organic and mental life. A pebble may be in states of stable, unstable, or indifferent equilibrium with respect to its surroundings and this makes no difference to its nature. By contrast, an organism presents with respect to its milieu, multiple forms of equilibrium, from postures to homeostasis. These forms are necessary to its life, hence are intrinsic characteristics; durable disequilibria constitute pathological organic or mental states...all behavior is an <u>assimilation</u> of reality to prior schemata (schemata which, in varying degrees, are due to heredity) and all behavior is at the same time an <u>accommodation</u> of these schemata to the actual situation. The result is that developmental theory necessarily calls upon the concept of equilibrium, since all behavior tends toward assuring an equilibrium between internal and external factors or, speaking more generally, between assimilation and accommodation (pp. 102-103).

Hence cognitive structures change through "the assimilative activity of incorporating objects into ongoing schemes and the accommodative activity of modifying the schemes to new objects (Overton, 1972, p. 107)".

The second aspect of Piaget's work to which we wish to draw attention is his conjecture that mathematical structures and cognitive structures are closely related. Although it would seem that this conjecture should be of great interest to mathematics educators since it raises, for example, interesting questions about the nature of 'mathematical aptitude', it has attracted minimal attention. Little research seems to have been done to follow up the hypothesis. The conjecture does, however, strongly support the mathetic view that cognition and mathematics are intimately connected. The Bourbakist school, as has been noted earlier, sees mathematics as the science of structure. It is their contention (Bourbaki, 1950) that all mathematics stems from three "mother-structures"; algebraic structures, order structures and topological structures. The group is the prototype of the algebraic structures, the lattice of the order structures and the topological space of the topological structures. By combining two or three of the mother-structures in some fashion, complex mathematical entities, or "multiple-structures", such as topological algebras are formed.

Piaget (Beth and Piaget, 1966) had independently concluded that "there are three kinds of elementary structures in the child, corresponding to operators of classes (then of numbers etc.), relations and continuous transformations (p. 186)". On hearing Dieudonne, one of the leading Bourbakists, expound the mother-structures view, Piaget was understandably "astonished by the resemblance between these two types of structure (Ripple and Rockcastle, 1964, p. 36)". After investigating the two sets of structures in detail Piaget concluded that the three mathematical mother-structures "correspond to something in natural intelligence and specifically in the thought of the child (Ripple and Rockcastle, 1964, p. 35)". Piaget contends that this correspondence can be represented;

> under the heading , not of a formal isomorphism (which would be untenable from the viewpoint of generality and validity) but of a genetic relationship (Beth and Piaget, 1966, p. 188).

There are two other remarks about Piaget and his work which should be made here. The first which has not been widely noted, is that Piaget considers an 'active' approach to learning to be appropriate for adults as well as for children (Ripple and Rockcastle, 1964, p. 3). The second is to recognize that Piaget has become to a great extent an educational cult figure. Many writers, particularly those from North America, who have made educational extrapolations from his theories of genetic epistemology seem not to have fully appreciated the <u>Weltanschauung</u> from which Piaget's work emerges. Those writers who criticize Piaget's work on the basis of his 'unsound' research methodology are perhaps the most flagrant offenders here.

Much of 'Piagetian' research in North America has focused on the acquisition of conservation concepts. Piaget seems to find this emphasis, with its concomitant assumption that it is desirable to accelerate this process, somewhat bemusing. He refers to it as "the American question (E.

Hall, 1970, p. 30)". McDonald (1964) has illustrated how much of the widespread acceptance of Dewey's philosophy in the early years of this century was a result of Dewey's catching "the spirit of the times in a way that Thorndike did not (p. 14)". In a similar way Piaget seems to have caught the educational Zeitgeist of today. It would be unfortunate if some of the more wretched excesses carried out in Dewey's name were to be perpetuated in our time by educators who put a particular interpretation on Piaget's research. The recommendations made by educators on the basis of 'Piagetian research' should be entertained with considerable skepticism. It is directly on the work of Piaget and his Genevan colleagues, and not on any 'extrapolation' of this work that the mathesis paradigm is based.

2.4 The Process Bias

Mathematics teaching has traditionally been content-oriented. In mathesis curricula the bias will be toward process learning. The mathematical processing abilities which a learner brings to bear on a new problem will be considered more important than the mathematical products to which he has been exposed. Thus teachers of mathesis will be person, process and present-oriented whereas mathematics teachers traditionally have been subject, product and past-oriented. Parker and Rubin (1966), who define content as "a rhetoric of conclusions to be transferred to the student (p. 2)" and process as "all the random or ordered operations which can be associated with knowledge and with human activities (p. 2)" point out that the difference between content and process learning is essentially "the difference between passive and active approaches to learning (p. 3)". They also observe that "where the stress is upon process, the assimilation of knowledge is not derogated, but greater importance is attached to the methods of its acquisition and to its subsequent utilization (p. 2)". In fact, one cannot have process without content, for content is the grist of the process mill.

The rationale for process-based curriculum in general has been argued by Bruner (1968), and by Biggs (1971, 1973). The implications of process-based curricula in mathematics have been outlined by Dienes (1967), Kolb (1970), Scandura (1971), Morley (1973) and Sigurdson and Kieren (1973). Commonly accepted processing abilities in mathematics are, classifying, symbolizing, detecting patterns, making conjectures, testing conjectures, generalizing and proving.

The two extremes within the process-education camp are the process-purist position and the product-through-process position. For process-purists content serves solely as a vehicle for the improvement of general process abilities. Product-through-process advocates see the utilization of process skills as the most effective means of learning some given piece of content. The two different stances may well only reflect dispositions toward different types of educational goal. Mathesis curricula will be biased toward the process-purist position. Mathematical processes will be emphasized, but not to the exclusion of mathematical content.

2.5 The Potentially-Rich Situation Matrix

The operating matrix in which mathesis curricula will be set is that of the potentially-rich situation. A potentially-rich situation is an environment with a high mathematical content which is conducive to the posing of problems and the making and testing of conjectures about the solutions to these problems. Potentially-rich situations generate problems in the same way that problems generate exercises. A piece of isometric graph paper is a potentially-rich situation as is a pack of playing cards, the film Dance-Squared (Jodoin, 1965) and a set of building blocks. Unlike the other components of the paradigm, the concept of a 'potentially-rich situation' per se, is unique to this study. In light of this, and of the fact that it is fundamental to mathesis curricula, most of the following chapter will be devoted to an elaboration of this concept.

2.6 The Genetic Characteristic

One reason that laymen find it so difficult to understand mathematics and mathematicians is that they have few insights into the way in which mathematicians work. They have little conception of what problems mathematicians work on, how they work on them, or why they should want to work on them in the first place (Halmos, 1968; Friedman, 1966). (To a lesser extent the same thing is true in science (Medawar, 1968a; Watson, 1968)). This is largely due to the deus ex machina style of many mathematical documents which give little evidence of the "days, weeks or even years of fumblings, doubts, certainties, minute examinations of special cases (Gattegno, 1970, p. 137)" which preceded their publication.

It is of interest that mathematicians and scientists have not always felt the compulsion to present only a brief and 'objective' version of their work. Kepler once wrote that he desired:

> not merely to impart to the reader what I have to say, but above all to convey to him the reasons, subterfuges and lucky hazards which led me to my discoveries. When Christoper Columbus, Magellan and the Portuguese relate how they went astray on their journeys, we not only forgive them, we would regret to miss their narrative (quoted in Tahta, 1969, p. 23).

In a somewhat similar vein, Leibniz is known to have felt that "Nothing is more important than to see the sources of invention which are, in my opinion, more interesting than the inventions themselves (quoted in Polya, 1957, p. 123)".

In mathesis curricula a genetic, historical or holistic approach will be taken. In seeing the way their own work develops learners may come to understand the way in which mathematics as a whole grows. Learners will be encouraged to present the whole problem, from motivation to conclusion, including wrong conjectures and mistaken assumptions. If an area has well-known historical roots then they will be revealed. 'Why was this mathematician interested in this question?', 'What were his personal skills and interests which made him want to work on this particular problem?', 'Why was the question an important one at the time?', 'What were the essential steps which led to the solution of the problem?'.

Mills (1972) has written that "the sociological imagination enables us to grasp history and biography and the relations between the two within society (p. 6)", and that "no social study that does not come back to the problems of biography, of history and of their intersections within a society has completed its intellectual journey (p. 6)". Using this terminology, the "sociological imagination" will be important in mathesis curricula since mathesis can only be fully understood within the context of the relationship between an individual and his society.

It therefore follows that the history of mathematics will be important in mathesis curricula. This is another way in which mathesis curricula will differ from contemporary mathematics curricula (Wilder, 1972). Many mathematicians, such as Abel, Babbage, Condorcet, Descartes, Galois and Wiener have led most unusual and yet highly productive lives. In mathesis curricula the historical focus will be on the mathematician as well as on his results. The fact that one of Gauss's sons became a millionaire shoe dealer in St. Louis and another helped translate the Bible into Sioux (T. Hall, 1970) is of mathetic but not of mathematical interest.

The question:

> A undertakes to throw at least one six in a single throw with six dice; B in the same way to throw at least two sixes with twelve dice; and C to throw at least three sixes with eighteen dice. Which has the best chance of succeeding (Chrystal, 1961, p. 591)?

seems a fairly typical mathematical problem. This question has a history, however, which makes it mathetically quite interesting. In 1693 the problem was being hotly debated "among men of numbers (Pepys, 1926, p. 75)" in London. One of those having a financial interest in its resolution was Samuel Pepys. Pepys appealed to his friend Newton to advise him about the problem and a long correspondence resulted. Pepys did not prove to have a natural flair for probability, "pray bee favourable to my unreadiness in keeping pace with you therein, and give mee one line of further helpe

(p. 82)". It was only after much "chawing of the question (p. 77)" that Newton was able to persuade Pepys to see the strength of A's position. This result was not particularly pleasing to Pepys who had put his money on A, B, and C having equal chances. When last heard from in the correspondence he is planning to welch on the original wager.

Mathematics educators have for some time argued the merits of a genetic approach to mathematics instruction (Ahlfors et al, 1962; Kline, 1972). Polya has written:

> having understood how the human race has acquired the knowledge of certain facts or concepts we are in a better position to judge how the human child should acquire that knowledge (1968a, p. 132).

Despite this urging there seems to have been little progress made in mathematics education toward achieving this goal.

2.7 The Interdisciplinary Characteristic

Mathesis curricula will be characterized by a strong interdisciplinary bias. The links which mathematics has with many other disciplines will be stressed. The links between mathematics and the physical sciences and engineering are well known. The Man-Made World (1971), the text produced by the Engineering Concepts Curriculum Project, illustrates many of these links very well. Many of the practical situations considered in such a text, for example, the 'Traffic-Light' problem (pp. 3-19), could be regarded as being potentially-rich situations in the mathesis sense.

The links between mathematics and art and between mathematics and music are very strong, although not widely recognized. Many social sciences have become highly 'mathematized' in recent years. The role which mathematics plays in these areas will be examined in mathesis curricula. The interdisciplinary characteristic of mathesis curricula reflects the humanistic assumption that man is an integrated being. It runs counter to the high degree of specialization and compartmentalization now current in academic fields.

As an example of the type of interdisciplinary material which might be used in a mathesis curriculum consider The Six-Sided Snowflake (1966). This booklet was written in 1611 by Kepler, "Mathematician to His Imperial Majesty", as a "New Year's Gift" for his "Master and Benefactor". The booklet provides fascinating insights into the fields of mathematics, the history and philosophy of science, physics, chemistry, crystallography, cosmology, biology, latin and late renaissance history. It is reasonable to expect that students working in a potentially-rich situation with either 'packing' or 'symmetry' aspects to start to pose problems similar in nature to the ones Kepler raises.

2.8 The Contemporary Characteristic

In contrast to most current mathematics curricula mathesis curricula will be contemporary. There are two aspects to this contemporaniety. A topic can be contemporary in the sense that it is of current interest to a significant number of research mathematicians, or it can be contemporary in the sense that it is of current use to laymen in the culture. We can refer to these two related aspects of contemporaniety as the research and relevance aspects. The basic numerical operations are examples of relevant-contemporary topics. The advent of the computer has stimulated many research-contemporary topics such as the analysis of algorithms and integer optimization. Other research-contemporary branches of mathematics include operations research, game theory, network analysis, group theory, praxeology and probability.

Von Neumann (1961) has suggested a very provocative analogy between the branches of mathematics and architectural styles. His idea is that while branches of mathematics have their roots in 'realistic' problems:

> once they are so conceived the subject begins to live a peculiar life of its own...as a mathematical discipline travels far from its empirical sources, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality" it is beset with grave dangers. It becomes more and more...l'art pour l'art... at the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up

(p. 9).

It would be absurd to contend that students can appreciate all contemporary mathematics. For example, it is extremely unlikely that a secondary school student is going to be able to comprehend homological algebra. It is possible, however, for students at this level to get some insights into those branches of mathematics which are, in von Neumann's terms, classical, for here the links with reality are still clear. For instance, the basic concerns of algebraic coding theory can be understood by adolescents. The realistic problem which gave rise to this important branch of contemporary mathematics is, 'How can messages be sent from one computer to another with maximal efficiency; that is with errors and cost both minimized?'. With more established areas of mathematics, such as number theory, the learner may only be able to see the classical trunk and not the more sophisticated outgrowths of later generations.

In any case, the important thing is that in mathesis curricula it will be realized that mathematics is a growing and vibrant discipline, peopled by active intellectuals who are often struggling with quite practical problems. The books by Stein (1969) and Beck, Bleicher and Crowe (1969) give good insights, at an elementary level, into the way in which contemporary mathematicians operate.

2.9 The High-Information Characteristic

Mathesis curricula will be high-information curricula, both in terms of being rich in physical resources and in consciously attempting to maximize 'personal-information', that is, in emphasizing inter-personal interaction and information exchange. Teachers and learners will have easy access to good library facilities, manipulative aids, stationery and equipment such as duplicating machines. Technological devices, particularly electronic computers, will be available, but will be seen as powerful tools rather than as masters. The work of Papert (1972,a,b,c) is of interest in this connection. He shows clearly how computers can be effectively used to help children 'learn how to learn'.

All learners, including the teacher, either individually or in small groups, will be encouraged to produce materials to reveal their investigations. These may take the form of written reports, displays, film-loops, newsletter articles or multi-media presentations. There will be no textbook per se; in the cases where a learner's investigation takes him into a well-known area he will be encouraged to seek out original sources. The papers of highly-creative mathematicians often have an aura of power which almost never filters through in textbooks. Euler's (1953) paper on the Koenigsberg Bridge problem, from which the areas of graph theory and topology both developed is an example of such a piece of work. Learning to find and utilize information effectively will be seen as being an integral part of mathesis curricula.

Kuhn (1970) has commented critically on the fact that in science teaching a 'unitext' approach is almost universal and that original scurces are seldom referred to. This same 'low-information' situation with similar, if perhaps not more severe, consequences can be observed in mathematics curricula. By looking only at the end product of mathetic activity, that is, the formalized mathematical structures, and not at the mathetic activity which produced these structures, the learner finds it difficult to fully come to understand the nature and methods of mathematics. Only by considering the long-term development of mathematical structures can they be completely understood. The road to formalization has almost always been a rocky one for mathematical topics. Unfortunately, at the moment, good descriptions of the evolution of mathematical topics, such as Lakatos's (1963) work on 'Euler's Relation' and Kiernan's (1972) description of the development of the Galois Theory, are not easily accessible since they can only be found in relatively esoteric journals.

2.10 The Intrinsic-Enjoyment Characteristic

Mathesis curricula will be intrinsically enjoyable. Since intrinsic enjoyment is to a great extent a function of the 'beholder's eye', this will operationally mean that students of mathesis will never have topics forced on them. This is not to say that mathesis will necessarily be the 'mathematics of joy' or that it will not involve much hard work. It is only to recognize that significant learning is unlikely to occur against the learner's will, or as Plato (1971) wrote, "knowledge which is acquired under compulsion obtains no hold on the mind (p. 306)". This intrinsic-interest aspect of mathesis curricula would seem to be in line with the Jencks Report recommendation that the "primary basis for evaluating a school should be whether the teachers and students find it a satisfying place to be (Bane and Jencks, 1972, p. 41)". It seems as well to be consistent with projections about a forthcoming 'leisure society' (Gabor, 1970).

Plato also advocated that teachers should let their "children's lessons take the form of play (p. 306)". More recently, Thom (1971) has stated, "only those topics which have a quality of 'play' have educational value (p. 696)". With the exception of Dienes (1971), however, little has been done by mathematics educators to take explicit advantage of the learning potential offered by play and the essential human attribute of the capacity for game-playing. In mathesis curricula an attempt will be made to utilize this potential.

Many mathematicians have stated that they see the aesthetic factor as being a very important one in mathematics. In mathesis curricula this aspect of mathematics will be emphasized. The mathesis teacher may well adopt here the attitude held by Mary Boole (1972) that the teacher should leave learners "alone with pure Truth (p. 14)".

2.11 <u>The Pedagogic Mode</u>

The subject of mathetic activities and the way in which they will be approached will, to a great extent, be determined by the preceding nine components of the paradigm. For example, to be consistent with the Piagetian basis of the paradigm a mathesis teacher would have to see that much of the work done by the learners was carried out in small groups, for this is part of Piaget's conception of 'activity'. He has stated:

> When I say 'active' I mean it in two senses. One is acting on material things. But the other means doing things in social collaboration, in a group effort. This leads to a critical frame of mind, where children must communicate with each other. This is an essential factor in intellectual development. Cooperation is indeed co-operation (Ripple and Rockcastle, 1964, p. 4).

Despite the constraints introduced by the nine components, however, mathesis curricula will be strongly influenced by the interests and personalities of both the teachers and learners involved. In fact, it will be impossible to fully separate 'teacher' and 'curriculum' in mathesis curricula. The single sine qua non for mathesis teachers will be that they genuinely enjoy and actively participate in mathesis. Humanistic educators should not regard the teacher-student relationship as being essentially different from any other human relationship they enter into.

At the moment, perhaps the closest model we have for an 'ideal' mathesis 'classroom' is that of an art studio where learners come voluntarily to work with a practitioner who derives satisfaction both from his work and from showing other, usually younger, people how they too might get satisfaction from the area. Contemporary practice in some British infant and junior schools might well give mathesis educators some ideas as to how to work effectively with learners. One pair of American observers (Murrow and Murrow, 1971) saw the basis of this approach as:

> an attitude which deals with children as individuals who have a right to enjoy learning and to be themselves. In order to teach in this fashion, the teacher must look beyond new projects and curriculum ideas to the basic needs and interests of each child. He must find ways in which to mesh the children's interests with his own to promote forms of learning important to them all (p. 252).

This approach is further described in Featherstone (1971) and Silberman (1970) and in particular in relation to mathematics teaching in E. Biggs (1969) and Biggs and MacLean (1969).

The experience of the Madison Project, as described by Davis (1972), is also of relevance to humanistic educators. Among the elements common to classrooms where Project classes went particularly well were:

> the teacher respected the children...there was generally an informal tone to the classroom...the teacher usually held high expectations for the quality and sophistication of the children's work...children's work was treated with respect...the study of math was undertaken as a joint exploration by teacher and children...mathematics was seen by teacher and student as something you do, something you use your ingenuity on, something you create, explore and invent...the children had bona fide participation in making a good many significant decisions ...one topic of study grew out of another...methods of solving problems were ALWAYS devised by the CHILDREN...(pp. 20-22).

Mutatis mutandis, this description would seem to be a valid one for mathesis classrooms.

CHAPTER THREE

ON POTENTIALLY-RICH SITUATIONS

3.1 Introduction

We have noted in a preceding chapter that two of the most important aspects of humanistic education are a high regard for man's capacity for rational thought and a commitment to the goal of helping the learner reach his full potential. These two aspects are, in fact, quite closely related, for from a humanistic point of view reaching one's full potential in the contemporary world depends largely on one's being able to make full use of one's capacity for rational thought. This is because the contemporary world is characterized by rapid change, and rapid change on the societal level means increased decision-making at the level of the individual. Decisions imply choice and the resolution of problems. To reach one's full potential one must make good decisions and this in turn means using one's thinking capacity effectively. Hence humanistic education in today's world must be, to a great extent, education in problem-solving, decision-making and rational thought.

The majority of this chapter is devoted to that component of the mathesis paradigm which is most closely connected to problems and problem-solving, the potentially-rich situation. A characterization of potentially-rich situations follows an examination of the role of problems and problem-solving in mathematics, and in education in general. The chapter ends with the description, in some detail, of two potentially-rich situations. The first description is a hypothetical one in the sense that it outlines some ways in which a potentially-rich situation might be developed. The second description is based on the actual response of a group of students to a potentially-rich situation which was presented to them.

3.2 On Problems and Problem-Solving

The seminal role played by problems in the field of mathematics has long been recognized. Rosenbloom (1966), for example, regards problem-solving as "the basic mathematical activity (p. 130)" and feels that "mathematical activities such as generalization, abstraction, theory building and concept formation are all based on problem-solving (p. 130)". In his famous address to the Paris Congress in 1900 Hilbert (1902) considered the importance of problems to mathematical research. He stated:

> The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as every human undertaking pursues certain objects, so also mathematical research requires its problems. It is by the solution of problems that the investigator

> tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon (p. 438).

The precociousness of many of its famous practioners has always been a characteristics of mathematics. Gauss, who reputedly (Bell, 1965, p. 242) corrected his father's pay-roll accounting before he turned three years old, and Wiener (1966) who held a Harvard doctorate at age eighteen are among the best known of these prodigies. Mathematical precocity usually is revealed in relation to problems. Polya and Kilpatrick (1973) note that:

> mathematical ability can be tested at a comparitively early age because it is manifested 'not so much by the amount of accumulated knowledge as by the originality of mind displayed in the game of grappling with difficult though elementary problems' (p. 628)".

Taton (1962) feels that it is not only in solving problems, but also in posing problems that mathematical aptitude reveals itself. He writes:

> One of the particular characteristics of mathematics is the fact that in it invention starts from the very moment that a pupil is confronted with a problem that he has to solve. Evidently this is a case of a minor effort of invention in which the subject is set in advance, and where the anticipated result should lead to no new elements. But although this is not a discovery in the proper sense of the word, the pupil must nevertheless attempt a discovery in so far as he has to produce rigorous arguments permitting him to pass from known elements to the proposition to be proved or demonstrated, or to the solution to be determined. If the pupil ignores the actual questions that are asked, but would rather make original remarks on the problem that he has to solve or if, still better, he himself poses the

> problems, then his work can no longer be distinguished from that of the creative mathematician, except by degree. The very fact of posing problems is a sign of interest in research and a curiosity of mind, which are but some of the fundamental qualities characteristic of the creative mathematician (pp. 26-27).

That problems are important to mathematics is not surprising. What is perhaps more surprising is the number of other fields where practioners consciously view themselves as problem or puzzle-solvers. Ayer (1971) writes "the philosopher's business...is rather to 'solve puzzles' than to discover truths (p. 34)" and Loeb on being asked whether he was "a neurologist, a chemist, a physicist, a psychologist, or a philosopher answered only "I solve problems (quoted in Goble, 1972, p. 21)". Kuhn (1970) views almost all scientific activity to be puzzle-solving within the framework of a given scientific paradigm or "Normal Science as Puzzle-Solving (p. 35)". (Musgrave (1971) is one of Kuhn's critics who feels that, in so far as there are no assured solutions to the questions scientists work on, " "puzzle-solving" should disappear, and "problem-solving" resume its place, as the most adequate description of scientific research (p. 293)".) Polya (1968a) and Popper (1972) go much farther; they see, respectively, that "solving problems can be regarded as the most characteristically human activity (p. ix)" and "life is problem-solving and discovery - the discovery of new facts, of new possibilities, by way of trying out new possibilities conceived on our imagination (p. 148).

Several very prominent scientists have substantiated the Kuhnian view of science by repudiating the concept of 'the scientific method' and by presenting science as problem-solving on a large scale. In Popper's view "science is merely common sense writ large (Magee, 1971, p. 77)" and Einstein (1950) wrote that "the whole of science is nothing more than a refinement of everyday thinking (p. 59)". Medawar (1969b) suggests that if one asks:

> A scientist what he conceives the scientific method to be, he will adopt an expression that is at once solemn and shifty-eyed: solemn, because he feels he ought to declare an opinion; shifty-eyed, because he is wondering how to conceal the fact that he has no opinion to declare (p. 11).

Another Nobel prize-winner, Bridgman (1950) felt that scientific method was "nothing more than doing one's damnedest with one's mind, no holds barred (p. 351)", and Conant (1952) has bluntly stated "there is no such thing as the scientific method...what the scientist does in his laboratory is simply to carry over into another frame of reference habits that go back to the caveman (p. 22)".

The reflection of these views has been evident in the trend toward inquiry-based or discovery-oriented curricula which have been developed in the last decade, particularly in the sciences. The Educational Policies Commission (Wolfle, 1966) for example, advocated that the goal of education should be "the development of persons whose approach to life as a whole is that of a person who thinks - a rational person (p. 1697)". To reach this goal the Commission recommended that all of education be "infused with the spirit of science" or "the values of rational thought". Their definition of the spirit of science was:

> In terms of seven underlying values: longing to know and to understand; questionong of all things; search for data and their meaning; demand for verification; respect for logic; consideration of premises; and consideration of consequences (p. 1697).

Exhortations to have education focus on problem-solving and the inculcation of rational methods of thought have, however, been common in educational literature for a very long time. The most explicit statement of the 'education-for-thinking' position has come from Dewey, especially in his book How We Think (1933). Because he states his position quite cogently and also because his thesis is very similar to that of Piaget and hence to the cognitive basis of the mathesis paradigm, we summarize his views here. Dewey sees reflective thought as:

> Active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends (p. 9).

The origins of reflective thought lie in 'pre-reflective' situations which are characterized by "obscurity, doubt, conflict, disturbance of some sort (p. 100)". This can be seen to be quite similar to the environment leading to 'cognitive perturbations' in a Piagetian sense. Involved in rational thinking is "an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity (Dewey, 1933, p. 12)" hence transforming the 'pre-reflective', conflict-inhabited initial situation to the "harmonious and coherent (p. 100)" 'post-reflective' situation. Dewey (1933) claims that there are five phases or aspects of reflective thought between the 'pre-reflective' and 'post-reflective' states. These "states of thinking" he summarizes as being:

> (1) suggestions, in which the mind leaps forward to a possible solution; (2) an intellectualization of the difficulty or perplexity that has been felt (directly experienced) into a problem to be solved, a question for which the answer must be sought; (3) the use of one suggestion after another as a leading idea, or hypothesis, to initiate and guide observation and other operations in collection of factual material; (4) the mental elaboration of the idea or supposition as an idea or supposition (reasoning, in the sense in which reasoning is a part, not the whole, of inference); and (5) testing the hypothesis by overt or imaginative action (p. 107).

Dewey (1933) sees reflective thought as being essential to intelligent life:

> It emancipates us from merely impulsive and merely routine activity...it enables us to direct our activities with foresight and to plan according to ends-in-view, or purposes of which we are aware. It enables us to act in deliberate and intentional fashion to attain future objects or to come into command of what is now distant and lacking...It enables us to know what we are about when we act. It converts action that is merely appetitive, blind, and impulsive into intelligent action (p. 17).

The implication of this for education is quite clear to

Dewey. He states emphatically that "upon its intellectual side education consists in the formation of wide-awake, careful, thorough habits of thinking (p. 78)."

And yet, despite this rhetoric from some of the most influential of educators, the contemporary reality is that curricula based on problem-solving have not been very effective in producing good problem-solvers. "Like the weather, problem-solving has been more talked about than predicted, controlled or understood (Kilpatrick, 1969, p. 523)". The imagination, ingenuity and tenacity shown by problem-solvers in extra-curricular situations (see, for example, De Bono, 1973), has rarely been matched within educational contexts. It is to a conjecture as to why this has been true and to a consideration of how this might be remedied to which we now turn.

Our essential contention is that what most educators have considered to be 'problems' are not problems at all, at least in any real sense of the term. In 1964 Getzels made an interesting categorization of problems from an educational viewpoint. In this categorization there are eight types of problem according to whether or not: the problem is explicitly stated; the learner knows some particular method of handling the problem; and the teacher (or someone in society) knows some method of handling the problem. Getzels makes the point that in educational situations only one of these eight possible types is typically considered: the explicitly-stated problem with 'approved' method known to both the teacher and the learners. Unfortunately, this type of problem requires little 'reflective thought' in a Deweyian sense, no real 'disequilibrium' in a Piagetian sense or "a minimum of innovation or creativeness (Getzels, 1964, p. 242)". Of the eight types of problem this type bears the least resemblence to a 'real-life problem'. In fact these 'problems' are really 'exercises' or "pseudo-problems (Getzels, 1964, p. 242)" and not true problems at all. Polya (1966) calls such 'problems' "routine (p. 126)", bemoans the "overdosage" of them in most textbooks and claims that "the routine problem has practically no chance to contribute to the mental development of the student (p. 126)". He goes on to say to teachers of mathematics:

> I shall not explain what is a nonroutine mathematical problem: if you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed some triumph and tension in one of your students, look for another job and stop teaching mathematics (p. 127).

To teachers these exercises may well have been meaningful and relevant; too often from the student's viewpoint they have represented only traps in the form of empty verbiage and totally 'unreal' situations. Dewey (1933) foresaw this difficulty arising:

> Probably the most frequent cause of failure in school to secure genuine thinking from students is

> the failure to ensure the existence of an experienced situation of such a nature as to call out thinking in the way in which...out-of-school situations do (p. 99).

Polya (1966) has also stressed that problem-solving can be an effective educational technique only if the problems appear "meaningful and relevant from the student's viewpoint (p. 127)".

The reason as to why 'exercises' have come to be the predominant form of 'problem' in educational curricula is not hard to find. The essential position held by educational psychologists in North America for the last fifty years or so has been a behaviourist one in one or another of its forms. Learning theories constructed from this authoritarian, reductivist, 'man as reactive organism' viewpoint focus on concepts like drive-reduction, sequencing and S-R (stimulus-response) interactions. One does not need to have an explicit statement of what Kaplan (1964) has called "the law of the instrument - give a small boy a hammer, and he will find that everything he encounters needs pounding (p. 28)" to realize that 'problems' with precisely controlled stimuli and equally precisely determined responses, that is, 'exercises', start to become the essence of problem-solving curricula constructed within a behaviourist framework.

To argue about the efficacy of such 'problem-solving' methods is really to argue about educational aims. To some of the more cynical educational critics, such as Reimer (1971), who sees the main purpose of schools as being custodial care, social selection and indoctrination, the 'exercise' method may well be quite appropriate. Henry (1966) who contends that through school curricula children learn "the essential nightmare. To be successful in our culture one must learn to dream of failure (p. 296)", would probably concur. There is no question, however, that this approach is antithetical to humanistic education. The authoritarian and passive bases, the framework which is so conducive to destructive competition, the frequent lack of relevance to the learner and the failure to make full use of the learner's capacity for rational thought all run contrary to a humanistic approach. Humanistic education must start with problems which the learner perceives, and not with ready-made problems. As Dewey (1933) has noted:

> Thinking is not a case of spontaneous combustion; it does not occur just on 'general principles'. There is something that occasions and evokes it. General appeals to a child (or to a grown-up) to think, irrespective of the existence in his own experience of some difficulty that troubles him and disturbs his equilibrium, are as futile as advice to lift himself by his boot-straps (p. 15)...it is artificial so far as thinking is concerned, to start with a ready-made problem...in reality such a 'problem' is simply an assigned 'task' (p. 108).

Addressing himself to a similar question Whitehead (1963) concluded:

> First-hand knowledge is the ultimate basis of intellectual life. To a large extent book-learning conveys second-hand information, and as such can

> never rise to the importance of immediate practice...the second-handedness of the learned world is the secret of its mediocrity. It is tame because it has never been scared by facts (p. 73).

Wason and Johnson-Laird (1968) indicate another reason why there is little future in educating for exercise-solving:

> The readings in this book may all be regarded as studies of how people solve problems. But a more important skill is the ability to find problems in the first place - to discover, invent, or recognize a problem. Once problems have been formulated, computers are increasingly able to solve them. But the day when computers are able to find worth-while problems is remote (p. 13).

Where the advocates of exercise-solving have failed in attempting to promote rational thinking on the part of learners is in ignoring the first two of Dewey's five stages of reflective thought. In presenting a ready-made problem they have cut out the 'suggestion' and 'intellectualization' phases of the process and in so doing they have frequently eliminated the chance of any significant learning taking place. The primary task of the humanistic educator is, therefore, to provide the problem-solving situations where the learner can fully participate in the 'suggestion' and 'intellectualization' phases. The problems which emerge then should be learner-problems and not teacher-problems. In the terminology of some Piagetian scholars (for example, Phillips, 1969, p. 110) these learner-problems should be much closer to being "optimally discrepant" relative to the learner's cognitive structures than the ready-made or teacher-produced problems have been. According to Piaget:

> The goal in education is not to increase the amount of knowledge, but to create the possibilities for a child to invent and discover...teaching means creating situations where structures can be discovered; it does not mean transmitting structures which may be assimilated at nothing other than a verbal level (Ripple and Rockcastle, 1964, p. 3).

Dewey (1933) sees the essential task of providing these situations in terms of conditions and curiosity. He writes, "the problem of method in forming habits of reflective thought is the problem of establishing conditions that will arouse and guide curiosity (p. 56)".

It is to a consideration of what would characterize such situations in the field of mathesis to which we now turn. While it is obvious that no one field can claim to be the exclusive domain for teaching learners to think effectively, it is the position of this study that mathesis is a very good area in which to do so. In fields like history and political science, values are of immediate and overwhelming importance. In mathesis human values are important, but they emerge only after mathetic activity has begun, and they tend to be individualized values like tenacity and commitment rather than social values. Hence, in a sense, mathesis is seen to be a primary area for humanistic education: an area where a learner can have not only strong cerebral, but also strong emotional experiences. For here, in a non-competitive, non-destructive situation it is possible for a learner to experience joy, frustration, hope, humility, anxiety, insight, awe, anger and confidence. To understand fully and first-hand is to have a strong emotional experience. The cognitive-affective gap is by no means as wide as some anti-rationalists would have us believe. (It is not by chance that Maslow's first 'self-actualizers' were intellectuals.) It seems reasonable to hope that as a result of mathetic activity a learner may become a more reasonable and tolerant individual. For, as Coleman (1969) has written:

> Anyone who has spent hours vainly trying to solve a well-posed problem in mathematics, if he has any sense at all, will not imagine that the vague, ill-defined and important problems of real life can be resolved by some simple-minded panacea (p. 4).

3.3 A Characterization of Potentially-Rich Situations

Those environments with a high mathematical content which are conducive to the posing of problems and to the making and testing of conjectures about the solutions to these problems, we call potentially-rich situations. They are the "conditions that will arouse and guide curiosity (Dewey, 1933, p. 56)" in mathesis. Since all mathetic activity originates in, and is formed by, such environments, potentially-rich situations can be regarded from an operational point of view as being the fundamental matrices of mathesis curricula.

The degree and quality of mathetic activity to which any given potentially-rich situation stimulates a learner will vary a great deal from individual to individual. The 'richness' of any given situation for a learner will depend largely on personal factors such as the extent to which the learner is motivated and his prior experience. A situation which is rich for one learner, and in this sense 'optimally discrepant' relative to his cognitive structures, may be much less rich for his neighbour. It is clear, however, that some situations are, on the whole, much richer than others. These situations are the ones which stimulate nearly all learners to pose significant problems and to investigate these problems by bringing their mathematical processing abilities to bear on them. These rich situations have a number of common attributes which we will call the characteristics of a potentially-rich situation. In general, the more of these characteristics which a potentially-rich situation exhibits, the greater is the probability that the potential of the situation will be actualized by the learner.

The characteristics of a potentially-rich situation are: (i) Accessibility: learners at all levels find it feasible fairly soon after encountering the situation to pose some problem related to it which they are capable of investigating;

(ii) Breadth: different learners will initially pose different types of problems related to the situation and over time any one learner will consider different types of problem;

(iii) Depth: problems generated from the situation are hierarchical, with the simpler problems being solvable by most learners with relative ease, but with the higher-order problems being more difficult to solve;

(iv) Connectivity: the links between different types of problem and between the levels of any one given type of problem are easily discernible, from both the width and height viewpoints the problems are visibly connected;

(v) Generalizability: in retrospect initial problems can be seen to be special cases, of a class of more general problems which are also solvable, the parameters involved in a given problem are distinct and the problem can be extended in a significant way by varying the parameters;

(vi) Pattern-latency: patterns of varying levels of complexity, some of which are quite visible, are inherent in the situation;

(vii) Concrete-representability: the situation lends itself easily to representation by one or more types of material, at least the lower levels of some problems generated from the situation have 'physical' manifestations;

(viii) Empiricality: the problems generated from the situation lend themselves easily to the making of conjectures and to the experimental testing of these conjectures, at least some of the conjectures are provable;

(ix) Identifiability: the problems generated from the situation can be associated with some paticular mathematician or branch of mathematics, references to similar problems and their evolution exist in mathematical literature and the learner can understand the gist of the arguments found there;

(x) Symbolizability: in working on the problems he has generated from the given situation the learner finds it convenient to invent and modify terms and symbols for the concepts he has identified.

Before turning to some examples, two general points about potentially-rich situations should be made. The first is that the effective utilization of potentially-rich situations depends very much on the mathesis teacher. It is essential that he stress the generative and openness aspects of potentially-rich situations and illustrate them in his own work. The authoritarian teacher, bent on right answers and closure, can sterilize a potentially-rich situation very quickly. Potentially-rich situations should be viewed by all learners as dynamic learning sources and not as static areas to cover. In short, there is a 'how' to potentially-rich situations as well as a 'what'.

The second point is that it is worth distinguishing between the initial, or kernel, activator presented to the learners and the potentially-rich situation to which it is related. The potentially-rich situation is the whole family of relationships which underlie the kernel-activator and to which the learner may be led by his investigations. Hence, for instance, in the following section we will speak of the potentially-rich situation of 'polytopes' although this term would certainly not be used when introducing learners to the situation. Kernel-activators, even to the same potentially-rich situation, may take many forms. Usually they will consist of a request to 'investigate' some circumstance which has been presented as a mechanism, a film-loop, a physical model, or a written statement.

The remaining two sections of this chapter are given over to the description of a pair of potentially-rich situations. In the examples the characteristics of potentially-rich situations are approached from two different vantage points. In the first we take a potentially-rich situation and illustrate how i[illegible]its each of the given characteristics by describing the mathetic activity which learners might engage in related to the situation. In the second example students' reactions to a given potentially-rich situation are described and then these reactions are related to the given characteristics.

3.4 An Example of a Potentially-Rich Situation: Polytopes

To illustrate the characteristics of a potentially-rich situation we will consider the example of polytopes. A polytope is a geometrical figure bounded by a finite number of line segments, polygons or hyperpolygons. A polygon is a two-dimensional polytope and a polyhedron is a three-dimensional polytope. To introduce learners to this particular potentially-rich situation a mathesis teacher would first make available to the learners a wide variety of materials such as: pipe-cleaners, straws, wire, card, acetate sheet, scissors, paste, tape, elastic bands, several different sizes and grids of graph paper, and supplies of various 'dimension-compatible' polygons in card and in plastic. The kernel-activator might then be a request to:

> Use these materials to construct some types of 'geometrical figure' and investigate the properties of the figures constructed.

From this starting-point we can examine some of the possible reactions of learners to this request in light of the characteristics of potentially-rich situations. We will consider the characteristics in the order in which we have listed them in the preceding section.

Accessibility: It seems fairly clear, given the materials mentioned and the foregoing rubric, that learners at almost any level can first of all construct some sort of mathematical figure, and then ask questions about that figure. Young learners, for example, will probably have had experience with 'blocks' which may influence their choice of problem. Among the most inviting problems here are the ones related to 'constructability'; the learners might will consider whether or not they can make a figure using only a certain type of polygon or a set number of polygons of various types.

Breadth: One obvious aspect of the breadth of this situation is the choice the learners must make, at least initially, between constructing two-dimensional figures, 'mosaics', or three-dimensional figures, 'solids'. Some learners may pose what are, to start with, essentially combinatorial problems, such as, 'How many different nets are there for square boxes? With tops? Without tops?'. These particular questions might soon become basically ones relating to symmetric and group-theoretic concepts.

Other learners, influenced by personal factors, may pursue problems which have their origins in gambling ("My father has these neat twelve-sided dice."), geology ("Did you ever see my quartz crystals?") or travel ("Last summer we stayed at a hotel with the funniest tiling on the bathroom floor."). What starts off as a mathematical figure for one learner may come to be much more of an artistic production by the time he has finished.

Depth: Some learners may use one type of polygon to construct mosaics and others to construct solids. They will both find their tasks easy when they use regular triangles and regular quadrilaterals. The solid-makers will probably find the pentagonal case difficult and definitely find the hexagonal case impossible. The mosaic-makers will not be able to find a pentagonal pattern, but they will be able to find a hexagonal one. Neither group is going to have any luck in looking for figures using only regular n-gons with more than six sides. They may then try to prove this fact.

Pyramid-makers and prism-builders using regular polygons for bases will encounter no such difficulties. Mosaic-makers using non-regular tiles are going to find major classification difficulties by the time they start to work with pentagons. They may well, in fact, not ever reach this point for some time if they become involved in the tiling properties of triangles and quadrialterals. Both 'truncators' and 'stellators' will find that their tasks vary in difficulty from solid to solid.

Connectedness: Learners will probably find as they progress that what appeared to be unrelated problems are, in reality, closely-related ones. One learner starting from the 'packing' properties of tetrahedra will find that his 'milk cartons' are very similar to the 'triangular dice' of the learner who started from the probability properties of polyhedra. The model produced by one learner by 'chopping the corners off a cube' will be very much like the 'house' another constructed from octagons and triangles. Nets for some solids may lead learners to look for mosaic patterns and vice-versa. The learner who works first on the symmetric aspect of solids, and then starts to look at truncation, will find that the one part of his investigation is radically affected by the other.

Generalizability: It is likely that most learners will at first unconsciously build in certain assumptions about the sort of figures with which they are going to work. Only later is it probable, for example, that many learners will realize that properties like convexity and uniformity place considerable restrictions on the possible types of figure and the number of possible constructible figures. At this point they should perhaps be able to see that some relationships will generalize within the limits imposed by certain definitional constraints. Euler's relation is perhaps the best example of this. Several interesting questions arise in this potentially-rich situation when learners try to generalize from the two and three-dimensional cases to the n-dimensional case. (For two particularly good investigations of this type by secondary school students see the papers by Lay (1973) and Zweig (1973).)

Pattern-latency: The visual patterns, particularly in the two-dimensional case, are quite striking in this potentially-rich situation. By choosing a good 'colour-scheme' a learner can emphasize the symmetric aspects of his mosaic or solid. Numerical patterns, such as those related to angle-sums and the vertex-face-edge relationship also permeate the situation. Other patterns, just as easy to define, may prove exceedingly difficult to unravel. Take, for instance, the sequence which represents the number of different ways a plane, convex n-gon can be divided into n-2 triangles by n-3 non-intersecting diagonals. The first three terms of this sequence, for triangles, quadrilaterals and pentagons respectively, are one, two and five. Things get more complex fairly rapidly from this point, however, and the general case yielded only to a mathematical mind of the order of Euler's (Euler, 1965; see also, Cayley, 1891).

Concrete-representability: As presented, this potentially-rich situation is particularly representable in concrete form. Polytopal figures can be constructed from a large number of diverse types of material. Frequently the type of material used influences the probability of a learner discovering a certain sort of relationship. For example, because of the 'visibility factor', learners constructing solids from acetate sheet and adhesive tape may well hit on the duality concept before those learners who are constructing the same solids from card. Almost all initial problems about polytopes will have a direct connection to some physical model. Even in 'higher-dimensional' problems, the arguments will usually proceed by analogy with the two and three-dimensional cases.

Empiricality: This situation lends itself easily to the fabrication of conjectures which can be investigated experimentally. It is likely that many of these conjectures will be capable of proof. Forms of proof which are particularly associated with this situation are: proof by example (there exists at least one convex solid having four pentagonal faces, two square faces and eight triangular faces), proof by exhaustion (there are only eight different nets for topless square boxes), and reductio ad absurdum (there exists no convex solid having ten faces, all of which are pentagons). In testing conjectures the method of disproof by counter-example can frequently be used here (there exist only seven convex solids having equilateral triangular faces).

Identifiability: The problems which the learners generate from this situation stand a good chance of being identifiable. Polytopes in their various forms have frequently been the focus of intellectual interest since very early times. The Greeks identified fire, earth, air, water and the universe with the five Platonic solids and Kepler constructed a fascinating but false model of the solar system using the same solids. In elementary mathematical journals polygons and polyhedra are the subject of a large number of articles, and books have been written on polytopes aimed at audiences as diverse as elementary school children and research mathematicians (Mold, 1967; Grunbaum, 1967). The situation has an large number of applications in different disciplines, ranging from architecture and art (Critchlow, 1971; Escher, 1970), through crystallography and chemistry (Loeb, 1968; Wells, 1968).

Symbolizability: In this potentially-rich situation learners are likely to reach a point in their investigations where their initial terminology, perhaps of 'boxes', 'houses', or 'triangle patterns', is inadequate. It may be inadequate either because it does not distinguish among several different figures of the same basic type, or because it does so in a very awkward manner. They may well then start to look for some systematic way of naming their figures. This experience would stand them in good stead when they encounter some of the many symbol systems commonly used in this area, such as the n-tuple representation of mosaics and the prefixing system used in naming the various polyhedra.

3.6 A Second Example of a Potentially-Rich Situation: Polyominoes

The potentially-rich situation utilized in this second example is that of polyominoes, shapes composed of various numbers of connected squares. In the description which follows, the reactions of a class of fourteen learners to this particular potentially-rich situation are outlined. The learners were third-year undergraduates at the University of Alberta who were preparing to become secondary school mathematics teachers. Most of them had completed the equivalent of three or four courses in university-level mathematics.

The class met for a period of two hours on each of two consecutive days. In the first session the concept of a potentially-rich situation was discussed and the students worked in small groups on the polyominoe situation for about an hour and a half. In the second session for approximately the first hour problems generated from the situation by the five different groups were described and indications were given of directions in which the group's investigations might proceed. The students then continued their investigations of their problems and the session ended with a discussion of the group's reaction to the concept and of the implications of the implementation of such a technique in schools.

In the following pages we shall outline and indicate some of the connections of the problems generated by the five groups. Following this, these reactions are examined in light of the characteristics of potentially-rich situations. For purposes of illustration we first describe in some detail the connections of the problems generated by one of the five groups. Although they are most interesting as well, the problems of the other four groups and the connections of these problems will be described much more briefly.

The learners had available to them supplies of isometric and square-grid graph paper, geoboards, elastic bands and a number of one-inch cubes. The kernel-activator with which they were presented was the following:

> Consider a unit square to be a 'one-celled animal'. Pose and investigate some problems in this 'mathematical biology'.

It should be noted here that it would be wrong to conclude from the following account that the groups found it easy to isolate and investigate their problems. None of the learners had previously participated in this sort of learning activity and at the beginning of their investigations some of the individuals in the class found the openness and lack of any explicit direction to be frustrating. This reaction did not, however, appear to last very long.

We turn now to a description of the problems generated by the groups. The outlines given of the problems and their connections summarize the activity and results of the first session and the initial half of the second session. It reflects therefore, the progress made by these groups in a total period of something like three hours.

Group A: This group, which had four members, generated a type of problem which we may identify as 'problems of mathematical ecology'. The essential question was the following: 'Given a certain polygonal enclosure, how many one-celled animals can be put in it subject to certain constraints?'. Typical of the constraints considered were those we can call the 'breathing-room' constraint and the 'electric-fence' constraint.

According to the breathing-room constraint one-celled animals must remain at least one unit apart within the given enclosure, but they may border on the sides of the enclosure. Questions that now arise are ones like: 'For a given enclosure, what is the maximum number of one-celled animals which can be put inside it, and what method will place animals optimally within any given enclosure?'. In investigating this type of problem the members of the group drew diagrams like Figure One. This diagram illustrates one way of putting four one-celled animals into a square four by four enclosure.

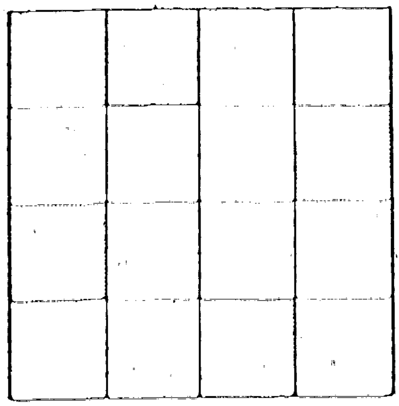

Figure One

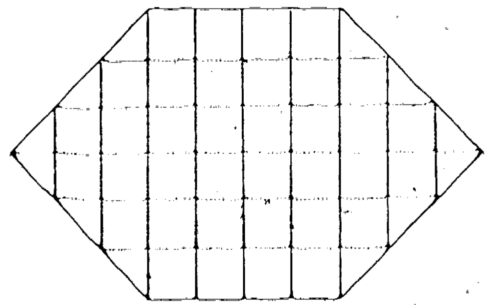

Figure Two

With the electric-fence constraint one-celled animals are allowed to touch each other, but not more than one point on any animal may come in contact with the sides of the enclosure. In investigating the relationships among types of enclosure and possible number of animals enclosed, the learners constructed diagrams like Figure Two. This diagram illustrates one type of clustering permissible under this constraint for a given hexagonal enclosure.

These problems are very similar in form to problems in two important branches of contemporary mathematics, algebraic coding theory and number theory. The problem the group has isolated by applying the breathing-room constraint, as in Figure One, is almost identical to the problem in coding theory of finding perfect error-correcting codes in the Lee metric. For the coding version of the problem instead of thinking of the animals having breathing room, one thinks in terms of code-words having a 'sphere' about them. A perfect code is one in which the 'spheres' pack closely, that is, they completely cover the enclosure without any overlapping.

In the Lee metric a 'sphere' of radius one about a code-word is the set of all 'words' with co-ordinates which differ from those of the code-word in at most one co-ordinate and then by not more than one in any direction. Hence, if in the two-dimensional case, A=(a,b) is a code-word, then the Lee metric sphere of radius one about A is the set of words: (a,b), (a,b-1), (a,b+1), (a-1,b), (a+1,b). Suppose that each co-ordinate of a code-word transmitted as (x,y) has a high probability of being correctly received, but that the most frequent errors are to have x received as x-1 or x+1 and to have y received as y-1 or y+1. Then there exists a perfect Lee metric single-error-correcting code for an alphabet of five symbols. Suppose that the alphabet is A, B, C, D, E; then encode and transmit: A as (0,0), B as (1,2), C as (2,4), D as (3,1) and E as (4,3).

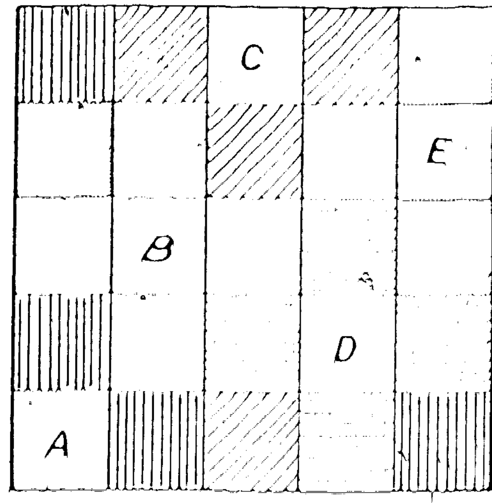

Figure Three

From Figure Three we can see that, (mod 5), we have a Lee-metric sphere of radius one about each of the code-words and that these spheres pack closely. This encoding therefore automatically corrects single occurrences of the most common types of error. For example, if the word (2,0) is received the code-word which most likely was transmitted was C=(2,4) since (2,0) is in the Lee-metric sphere of radius one about C.

There are important practical issues involved here. One can generalize the situation by varying the dimensions, the radius of the sphere or the number of symbols in the alphabet. The main results in this area can be found in Berlekamp (1968), Golomb and Posner (1964) and Golomb and Welch (1970).

The learner investigating the electric-fence constraint problem may well come to consider problems very similar in form to two well-known problems in the theory of numbers. The first and more significant of these is known as the Circle problem and the other is Pick's problem. The essence of the Circle problem is to try to establish a relationship between the number of lattice points (that is points with integral co-ordinates) inside and on a circle, and the radius of the circle. In considering this problem it becomes convenient to set up a one-to-one correspondence between lattice points and unit square and then to consider diagrams like Figure Four.

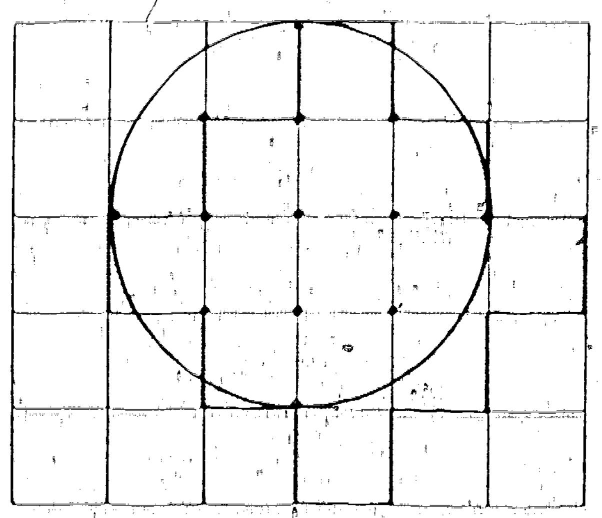

Figure Four

The similarity between Figure Four and the case of the electric-fence-constraint problem where the given enclosure is a circle is quite marked. The Circle problem is, in turn, intimately connected with another classical number-theoretic problem, that of the sum of two squares. Most secondary school students should be capable of starting to characterize integers which are expressible as the sum of two squares. That is, they should be able to prove that no integer which leaves a remainder of three upon being divided by four can be expressed as the sum of two square numbers.

The Circle problem has a long and interesting history, the last chapter of which still remains to be written. Consider the circle $u^2 + v^2 = N$ and let $R(N)$ denote the number of lattice points inside and on the circle. It is possible to prove in quite a straightforward manner (Andrews, 1971) that in the limit (that is as N increases without bound) $R(N)$ is equal to πN. In fact, it can be shown that for any value of N that $R(N)$ differs from πN by no more than some constant times a fractional power of N. That is,

$$R(N) = \pi N + O(N^{\alpha}), \text{ where } 0 < \alpha < 1.$$

The Circle problem is sometimes called Gauss's Circle problem since it was Gauss, using some of his own number-theoretic results, who first showed (1834) that:

$$R(N) = \pi N + O(N^{1/2})$$

In 1844, in relation to his work on quadratic residues, Eisenstein derived a method of attacking the problem. After Eisenstein's death Cayley (1857) translated this paper into English and noted, with respect to Eisenstein's approach to the Circle problem, that the formula:

> must, I think, have been established by geometrical considerations; ... but, as I do not perceive how this is to be done, I shall follow a reverse course, and establish the theorem from considerations based on the theory of numbers (p. 43).

Hermite (1887) also worked on the problem, but it was not until 1906 when Sierpinski showed that $\alpha < 1/3$, that any major progress was made. Working independently, Hardy (1915) and Landau proved that $\alpha > 1/4$ and in the next three decades the value of α was very slowly pushed down. Van der Corput and Nielandt took it to 27/82 in 1923, Titchmarsh (1933) to 15/46 ten years later, and Hua reduced it still further, to 13/40, in 1942. Vinogradov claimed in 1932 to have a proof which gave a value of just over 17/53 for α, but as Hua (1942) commented, this proof contained "an incurable mistake (p. 18)". Five years ago, Gendler (1968) conjectured that the value of α is, in fact, just over 1/4, but at the moment no proof of this conjecture has been forthcoming.

A second lattice problem in the theory of numbers which is closely related to the electric-fence-constraint problem is Pick's problem. In this problem one seeks to find a relationship between the area of a given polygonal figure on a lattice and the number of lattice points inside and on the boundary of the figure. The relationship is a straightforward one and the problem is particularly apposite for younger learners as it lends itself extremely well the an empirical approach (Kieren, 1968). This result appears to have been first recorded by G. Pick in 1899. The extension of this problem to the third and higher dimensions is, however, decidely non-trivial as the papers by Reeve (1957), MacDonald (1963) and Niven and Zuckerman (1967) illustrate.

Group B: This pair of learners isolated and considered two of the classical polyominoe problems (Golomb, 1966a), those of 'enumeration' and 'classification'. Having decided after some experimentation to define n-celled animals as conglomerates of one-celled animals joined by whole edges, they turned to investigating how many n-celled animals exist for small values of n. They found that for $n<5$ the problem was quite manageable and they were fairly certain of their answer of twelve for the case where $n=5$. They were, however, not able to find any distinct pattern for animal formation in general. This is not surprising in light of the fact that this is one of the unsolved problems of combinatorial theory (Harary, 1969). As late as 1962 professional mathematicians were able to find the number of ten-celled animals for the first time only by the application of considerable ingenuity and large amounts of computer time. Mathematical researchers who have addressed themselves to this problem in recent years include Read (1962; Harary and Read, 1970), Klarner (1967) and Lunnon (1971, 1972a,b).

The second problem these learners considered was that of classifying the n-celled animals they had found. They formulated conjectures regarding the existence of animals which could be described as 'straight', 'one-bent', 'two-bent', 'stair-step' and 'square' for different values of n. They found the standard 'alphabetic' method of naming polyominoes to be of interest in relation to their symbolizing work. The papers of Golomb (1966b, 1970), in which he classifies polyominoes according to their tiling properties, are also related to the investigations of this group.

Group C: This pair of learners concerned themselves almost exclusively with summation problems related to this potentially rich situation. They started out by trying to find the values of n for which 'stair-case' animals exist. They quickly realized that an s-level stair-case animal exists when n is of the form s(s+1)/2 for some positive integral value of s, that is, whenever n is a triangular number. They moved on from here to consider the problem of determining the number of 'smaller-square' animals contained in a 'large-square' n-celled animal. This question they recognized as being essentially that of finding the sum of the squares of the first n natural numbers, but they were unable either to recall or to reconstruct the 'formula'. A third problem they considered was the isometric grid analogue of the preceding problem, that is; 'How many smaller triangular animals are contained in a large triangular animal of side n?'. In this case they were not able to recognize any pattern or to construct a general formula. They did realize, however, that the cases with n odd were essentially different than the cases for which n was even. The last problem has recently been the subject of two journal articles; one by Gerrish (1970) in which a very cumbersome formula was developed, and another by Wells (1971). Wells's much more compact result was developed in conjunction with the work of a class of students.

Group D: The pair of learners in this group, after experimenting for some time with three-dimensional animals, turned to consider two 'generating' problems. The first of these was: 'If one knows all of the n-celled animals, how can one be sure of finding all of the n+1 celled animals?'. Essentially they were seeking an effective method of going from a specified set of inputs to a related set of outputs; this method should as well be capable of being precisely described and it should end after a finite number of steps. In short they were seeking an algorithm. Currently the focus of great interest because of their centrality to the field of computing science, algorithms have always been of considerable importance in mathematics. The books by Knuth (1968, 1969, 1973), give the most comprehensive contemporary treatment of algorithms and despite the fact that they contain no 'animal-generating algorithm', they would be of considerable interest to these learners.

The second problem these learners considered was that of sequence generation. They had identified one one-celled animal, one two-celled animal, two three-celled animals, four four-celled animals and eight five-celled animals. The problem they posed for investigation here was: 'How does the sequence 1, 1, 2, 4, 8 continue?'. Their conjecture was that the first two terms are 'given' and then each of the following terms is the sum of all of the terms which precede it. The errors in their empirical data notwithstanding, this sequence-generating conjecture is one which could be followed up beneficially by these learners. The properties of this particular generating rule may be relatively obvious, but certainly the fairly closely-related method of generating Fibonacci sequences leads a learner into a vast and fascinating mathematical area. Learners who have invented and investigated their own Fibonacci-like generating rules would probably find many of the articles in the <u>Fibonacci Quarterly</u> to be of interest.

<u>Group E</u>: The four members of this group considered a very vital, but quite complex problem in mathematical biology, that of 'reproduction'. Questions this group considered were ones like: 'Does an animal have an 'axis of reproduction', and if so, how might this be related to

symmetry; if n>1 does an n-celled animal grow to become an n+1-celled animal, or does it split into two n-celled animals, or does it become a 2n-celled animal?'.

These problems are difficult ones to resolve and in the period of an hour and a half the learners in this group were only able to start to define them precisely and to investigate them. These questions are similar in many ways to those raised by scientists trying to construct self-reproducing machines. The papers of Penrose (1959), Ulam (1962) and Kemeny (1955) are of especial interest in this area. The branch of contemporary mathematics which underlies this area is that of automata theory. Sources to which these learners could be directed include Burks (1970) and Birkhoff and Bartee (1970).

We can now briefly consider the initial reactions of these learners to the polyominoe situation in light of the characteristics we have listed. It seems clear that these learners found this situation quite accessible and very broad since they posed a large number of different but mathematically quite significant problems in a relatively short period of time. While several of the problems considered were standard ones in the polyominoe area, some, particularly those generated by Group A, were quite novel from the polyominoe viewpoint. Almost all the groups generated problems that were hierarchical in nature, and some of the problems considered have resisted the efforts of even research mathematicians. The connectivity of the situation was illustrated by the fact that several groups considered more than one type of problem and also because there was considerable overlap among the problems generated, as for instance, in the animal-enumeration problem.

At least two of the groups generalized from the initial situation to consider animals whose basic unit was not the square. Both of these groups realized that while many of the questions they had raised about the square case were still meaningful, the conclusions they had reached about these questions often did not generalize to the new case. For example, while there are five different square-celled animals made up of four cells, there are only three different four-celled animals where the basic cellular unit is the equilateral triangle. By making extensive use of both graph paper and geoboards the groups found that many of their problems had their origins in attempting to foresee how some visual patterns might continue. Several numerical patterns were investigated as well, particularly from the summation viewpoint.

The learners were able to work out empirically many of the lower-level cases of their problems, but frequently found that the higher-level cases were not amenable to empirical investigation. Some of the conjectures raised, as for example, about the existence of 'stair-case' animals, were proved to be valid. One type of problem in the polyominoe situation which none of the groups considered in the time period concerned is that of tiling space with animals. There are a number of excellent places for proving conjectures in this area. Most of the groups developed their own terminology to refer to the different animals and the different classes of animal they were considering. The general problem of classification was one which most groups confronted in one way or another and this stimulated much of the symbolmaking activity.

Of the ten characteristics of a potentially-rich situation, perhaps the one which has been exhibited most strikingly here is is identifiability. Without exception, each of the five groups has raised at least one problem which is highly identifiable in some branch of mathematics. Links with some of the world's most productive mathematical minds can easily be made. The learners considering the summation problems, for example, would probably be in a good position to appreciate Jacob Bernouilli's (1967) seminal work in this area, which dates from 1713, and the posers of the reproduction problem could be referred to some of von Neumann's (1963) original papers. The learners following up references in the work of Golomb and Knuth will find themselves becoming acquainted with two of the most productive and versatile young mathematicians of the day.

3.6 Summary.

In this chapter we have made a case for the importance of problem-solving in humanistic education in which we have stressed the necessity of having the learner isolate and articulate his own problems. The primary task of the humanistic educator is to provide and maintain the environments which will allow and encourage learners to define and investigate problems. In mathesis such environments are called potentially-rich situations. In the first of these two examples some hypothetical instances of mathetic activity were considered and in the second the initial activity of a group of learners in relation to a particular potentially-rich situation was described.

CHAPTER FOUR

STRUCTURAL ANALYSIS OF THE MATHESIS PARADIGM

4.1 Introduction

Having now presented, elaborated on, and illustrated the components of the mathesis paradigm we turn in the remainder of the study to an analysis of the paradigm. The two central questions generally addressed here are those of validity and utility. To ask if the paradigm is valid is to ask if it is suited to the task for which it has been created. What reasons are there to believe that this paradigm can be used to construct curricula which are both humanistic and mathematical? To consider the utility of the paradigm is to ask how useful it is in carrying out the required task. Can the paradigm be easily utilized to construct mathesis curricula, or does it appear that the 'costs' of using it are very high? How feasible is it to think of implementing the paradigm?

The analysis of the mathesis paradigm which is described in the following pages has been carried out using two basic assumptions. The first of these relates to the primacy of two of the ten components of the paradigm. The second assumption concerns the precision with which the terms of the paradigm have been defined. The epistemological and psychological <u>bases</u>, which provide a foundation for the paradigm as a whole, are seen as being the most fundamental components of the paradigm. The five *characteristics* similarly, are seen, in this sense, to be the least fundamental. In the analysis of the paradigm, therefore, more attention has been devoted to the *bases* than to the *characteristics*.

The terms used in the mathesis paradigm have not been defined with any great degree of precision. This is partly because of the incunabular stage of the paradigm. It also reflects, however, the assumption that a lack of definitional precision may well be beneficial, particularly at this stage of the development of the paradigm. It is felt that the activity of initially defining terms is subject quite rapidly to the law of diminishing returns, and that it is better to act soon with a somewhat fuzzy set of components than to act later with a more sharply-defined set. This assumption is consistent with the view of Popper who has been very critical of the Logical Positivist school of philosophy for what he considers to be the protracted attention they have given to definitions. (In this context he has even proclaimed "the emptiness of definitions" and spoken of the "superstition that if we want to be precise we must define our terms (Magee, 1971, p. 79)".) A technique used in the following analysis is to examine the paradigm in light of criteria developed to judge the validity and utility of formal axiomatized systems. However, because the terms of a formal axiomatized system must be rigorously defined, it should be realized that this examination can only be an informal one.

In this chapter we concern ourselves mainly with the question of validity. We do this by briefly reviewing the relationship between models and paradigms and by examining the mathesis paradigm according to the criteria which characterize an idealized, formal axiom system. The focus of the chapter is therefore on the internal, structural aspects of the paradigm. Most of the issues related to the utility question and the question of existence are considered in the final chapter.

4.2 On Models and Paradigms

The conceptual tool which has been chosen to handle the problem of humanistic mathematics is that of a paradigm. In this study, as previously indicated, a paradigm is a framework which is "halfway between an analogy and a model, more rigorous than an analogy, not a model (Kahn and Bruce-Briggs, 1972, p. 89)". Unfortunately, the term 'model' has come to be used, particularly in the social sciences, in a number of different ways. Some writers use the word to mean an analogue or a replica, while for others it indicates an ideal or norm. The term is also used on occasion as a synonym for 'untested theory'. Prefixed by the word 'mathematical', a model can be anything from a quantified empirical theory to a formal axiomatized system.

In fact, what we are calling a paradigm would likely be called a model by some writers. In The Limits to Growth (Meadows and Meadows, 1972), for example, it is assumed that:

> Every person approaches his problems, wherever they occur on the space-time graph, with the help of models. A model is simply an ordered set of assumptions about a complex system. It is an attempt to understand some aspect of the infinitely varied world by selecting from perceptions and past experience a set of general observations applicable to the problem at hand (p. 26).

Of the several meanings of the term 'model', the most rigorously-defined is the idea of a model in connection with a formal axiomatized system. It is this meaning of the term which is implied when it is stated that a paradigm is more than an analogy, but less than a model. In attempting to axiomatize any theoretical system one collects a number of fundamental postulates, propositions, or components, usually called 'axioms' which form the nucleus of the system. One tries to choose components so that all 'theorems' of the theoretical system can be derived from them by some sequence of logical transformations. An ideal formal axiomatized system would satisfy three criteria; those of consistency, completeness and independence. A system is consistent if the axioms are free from contradiction. It is complete if the axioms are sufficient to generate all the theorems of the system, and it is independent if all of the axioms are necessary to generate the theorems of the system.

In the following analysis we consider the components of the mathesis paradigm to be, in a, sense, the 'axioms' of mathesis, and a mathesis curriculum to be the associated theoretical system. We then look at the mathesis paradigm from the point of view of 'consistency', 'completeness' and 'independence'. Since it can be profoundly difficult to show that even a well-defined finite system satisfies these criteria, it is quite impossible for us to think of establishing the consistency, completeness and independence of the mathesis paradigm in any precise sense. We establish informal interpretations of these criteria for our particular set of axioms, and assume that the structural validity of the mathesis paradigm varies directly with the degree to which its components satisfy these criteria.

4.3 On the Consistency of the Mathesis Paradigm

In considering the 'consistency' of the mathesis paradigm we wish to show, as far as is possible, that one cannot proceed to establish the validity of both a position and its converse by a process of logical deduction from the components. For instance, one should not be able to use the components of the paradigm to rationalize both criterion-referenced and norm-referenced modes as the central methods of learner evaluation in mathesis curricula. We assume that there are two basic necessary conditions which the components of the paradigm must satisfy in order for the paradigm as a whole to have any measure of 'consistency'. The first is that each component of the paradigm be consistent with the fundamental areas of humanism and mathematics. The second is that the components be self-consistent; that is, that any one component is not inherently in conflict with some other one. The remainder of this section is given over to an examination of the components of the paradigm from the standpoint of these two basic necessary conditions for the 'consistency' of the paradigm. We consider first the consistency of the components with the fundamental areas of mathematics and humanism.

The Popperian epistemological basis of the paradigm has strong and clear links with both humanism and mathematics. Popper, who is a member of the Advisory Council of the British Humanist Association (Ayer, 1968, p. 3), considers that it is the acceptance of the standards of rational criticism and of objective truth "which creates the dignity of the individual man; which makes him responsible morally as well as intellectually (1969, p. 384)". In light of Panofsky's (1970) contention that underlying humanism are the four fundamental concepts of rationality, freedom, fallibility and frailty, Critical Fallibilism would seem to be a particularly humanistic epistemic position. Commenting on this position Popper (1969) has stated:

> As we learn from our mistakes our knowledge grows, even though we may never know - that is know for certain. Since our knowledge can grow, there can be no reason here for despair of reason. And since we can never know for certain, there can be no authority here for any claim to authority, for conceit over knowledge, or for smugness (p. ix).

Elaborating on the importance to man of the acceptance of the standards of rational criticism and objective truth Popper (1969) writes:

> These standards may help him to discover how little he knows, and how much there is that he does not know. They may help him to grow in knowledge, and also to realize that he is growing. They may help him to become aware of the fact that he owes his growth to other people's criticisms, and that reasonableness is readiness to listen to criticism. And in this way they may even help him to transcend his animal past, and with it that subjectivism and voluntarism in which romantic and irrationalist philosophies may try to hold him captive. This is the way in which our mind grows and transcends itself. If humanism is concerned with the growth of the human mind, what then is the tradition of humanism if not a tradition of criticism and reasonableness (p. 384).

The relation between Critical Fallibilism and mathematics has been most clearly documented by Lakatos (1963), a student of Popper's, in his <u>Proofs and Refutations</u>. Lakatos's stated aim in this work is:

> to elaborate the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase in the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations (p. 6).

Polanyi, the other philosopher of science whose works contribute to the epistemolcgical basis of the paradigm, seems particularly aware of the role of emotional commitment in the development of mathematics. He writes (1966) of the "emotional colour of mathematics (p. 188)" and claims that in no other field "is intellectual beauty so deeply felt and fastidiously appreciated in its various grades and qualities (p. 188)". In another passage he reveals how similar, at least in this particular aspect of his theory, his view of the development of mathematics is to that of Popper and Lakatos. In this passage he writes (1966) of:

> The image of a living science, groping its way towards the satisfaction of the intellectual passions upholding its values. We see it originating thousands of venturesome guesses which had long obsessed their authors until they laboriously brought them to the test of completion, and often battled for them against protracted objections, until they finally gained their established places in the textbooks. And again we see the curious contrast between this image and the ideal of casting the result of this heuristic process - and by implication any further continuation of it into a strictly formalized system of axioms and symbolic operations (p. 190).

We have already noted that the basic structural approach used by Piaget is one which is essentially mathematical in nature. This is reflected, for instance, in the emphasis he places on transformations in intelligence and also in the fact that the relations between certain of these transformations are frequently of an algebraic type. The set of relations among the four fundamental transformations of Identity, Negation, Reciprocity and Correlativity, which is isomorphic to the Klein-Four group is perhaps the most obvious illustration of such relationships. While less widely discussed, it seems no more difficult to identify Piaget's research position as being humanistic. He is essentially concerned with children as they are, and with what they will become naturally, and not with what they can be made to be. On one of the few occasions when Piaget (1964) commented directly on education and its aims he contended that:

> The principal goal of education is to create men who are capable of doing new things, not simply of repeating what other generations have done - men who are creative, inventive and discoverers. The second goal of education is to form minds which can be critical, can verify, and not accept everything they are offered. The great danger today is of slogans, collective opinions, ready made trends of thoughts. We have been able to resist individually, to criticize, to distinguish between what is proven and what is not. So we need pupils who are active, who learn early to find out by themselves, partly through their own spontaneous activity and partly through material we set up for them; who learn early to tell what is verifiable and what is simply the first idea to come to them (p. 5).

With its emphasis on spontaneous activity, the utilization of man's critical faculties and the rejection of authoritarianism, this position can be seen to be quite humanistic in nature.

We take it for granted that the position of the third-force psychologists is consistent with humanism since they concern themselves explicitly with those issues which relate to human welfare and the actualization of human potential. The terms 'third-force' and 'humanistic' are, in fact, used interchangeably to describe this school. While few people would immediately identify mathematics as being one of the disciplines most closely connected with third-force psychology, the two fields are by no means oppugnant. In fact Maslow, the most perspicacious of the third-force psychologists, has on two separate occasions singled out mathematics as an area of potential for humanistic education. On one of these occasions he wrote:

> Mathematics can be just as beautiful, just as peak-producing as music; of course, there are mathematics teachers who have devoted themselves to preventing this. I had no glimpse of mathematics as a study in aesthetics until I read some books on the subject (1971, p. 178).

On the other occasion he stated:

> What I am really interested in is the new kind of education which we must develop which moves toward fostering the new kind of human being that we need, the process person, the creative person, the improvising person, the self-trusting, courageous person, the autonomous person. It just happens to be a historical accident that the art educators are the ones who went off in this direction first. It could just as easily be true of mathematical education and I hope it will be one day (1971, p. 100).

In the foregoing quotation Maslow identifies "the new kind of human being that we need" with the "process person". A process bias seems a natural one to hold as far as both humanism and mathematics are concerned. From a humanistic viewpoint an individual deserves respect for what he is as a person, for what he is able to do, for the processes he has mastered, and not for the position he holds, the products he controls, or for the the power he wields. Many mathematicians would contend that mathematics can best be seen as a process. Gauss is an interesting case in point here. In many ways he epitomized the mathematics-as-content position. His motto was pauca sed matura and he left a great deal of his work unpublished. His contention was that "after a structure was completed one should not be able to see the scaffolding (Dunnington, 1955, p. 208)", and in his papers he removed "every trace of the analysis by which he reached his results, and studies to give a proof which, while rigorous, shall be as concise and synthetical as possible (Ball, 1960, p. 454)". (His influence, in this respect, on the format of scientific papers has been profound and long-lasting.) And yet, despite this emphasis on product and the steps he took to eradicate all signs of process from his papers, it was actually from the viewpoint of mathematics-as-process and not as product that Gauss gained most of his own personal satisfaction. This is shown clearly in a letter he wrote in 1808 to his friend Bolyai.

> It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never satisfied man is so strange - if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin

> another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms again for others (Dunnington, 1955, p. 416).

While this by no means implies that Gauss considered 'process' to be more important than 'product', it does show that even for this most 'productive' mathematician, 'mathematical processing' was more enjoyable than reflecting on the results of mathematical processing.

Of contemporary mathematicians, Davis has articulated the process interpretation most clearly. He has written:

> Mathematics is in fact a process. It is not a collection of facts, definitions, algorithms, or explicit procedures, although each of these will find its place in any effort to carry out the process of actually "doing mathematics". This process is the important thing, and not its "result" or "the answer" (1967, p. 4).

In the previous chapter we have seen how the potentially-rich situation matrix of the paradigm can lead to learners becoming involved with significant mathematical problems. This component is humanistic in that it recognizes the unique abilities and insights each individual brings to a learning situation, and it permits him the freedom to investigate the problems he perceives rather than the ones the instructor sees. The contemporary, genetic, high-information, intrinsic-interest, and interdisciplinary characteristics of the paradigm are consistent with mathematics because they each reflect, in one way or another, the way in which mathematical knowledge grows. They are humanistic in that they emphasize the role of the individual in this development. (See sections 2.6 to 2.10 in Chapter Two.) The pedagogic mode we have outlined for the paradigm can be said to be humanistic because it stresses the individual needs, abilities , and interests of the learners. It is consistent with mathematics in that it is more conducive to the active 'doing' of mathematics than to the passive acceptance of mathematical results.

With regard to the internal consistency of the paradigm, we look at the two areas where it seems most critical that the components be self-consistent. We first examine the relationship between the Piagetian psychological base and the Popperian epistemological base, and second we look at the degree of compatability between the characteristics of potentially-rich situations and the other components of the paradigm.

The theories of Popper and Piaget share a number of important characteristics and seem to be quite compatible in general. Because of the extensive research he has carried out with young learners, we have categorized Piaget as a psychologist. To do this is, of course, quite arbitrary, since essentially Piaget, like Popper, is an epistemologist. In so far as Piaget has taken an 'experimental' approach to the development of his theory, we can regard him as an 'applied' epistemologist, and Popper as a 'pure' epistemologist. From an operational standpoint, however, it is probably Popper's theory which is the more 'applicable' of the two. His conjecture-refutation process seems more amenable to direct educational implementation than does Piaget's equilibration process.

The conjecture-refutation process and the accommodation-assimilation process appear, in many ways, to be variations on a single theme. In both situations the general trend is toward the construction of new schema in which pre-existing schema are embedded. The need for the construction of new schema comes when, over a period of time, anomalies arise following testing of the schema. Considering the stature of these two contemporary philosophers, it is somewhat surprising that the similarities between their epistemic positions seem not to have been directly recognized in the literature, and neither thinker appears to have commented at any length on the work of the other. Both writers have had their work compared with Kuhn's, however, and these comparisons are of interest here. Controversy over the extent to which Kuhn's theory of the growth of scientific knowledge is congruent with Popper's theories emerged in scientific and philosophical circles shortly after the publication of Kuhn's The Structure of Scientific Revolutions in 1962. The question was directly answered a few years later by Kuhn (1970b) when he stated:

> Sir Karl's view of science and my own are very nearly identical. We are both concerned with the dynamic process by which scientific knowledge is acquired rather than with the logical structure of the products of scientific research (p. 1).

Mischel (1971) has commented on the similarities between Kuhn's theories and those of Piaget. He writes:

> In both cases the "normal" tendency to assimilate anomalies to accepted paradigms through minor accommodations provides for the continuity of thought; and the development of novel systems of thought, which differ in important ways from those which preceded (Kuhn's "scientific revolutions" and Piaget's major "stages"), is explained in terms of a need for reconstructing the accepted systems of concepts so as to coordinate new insights with the intellectual gains of the past (p. 326).

We have outlined ten characteristics of potentially-rich situations. We now wish to indicate some of the ways in which the characteristics of this component relate to the other components of the paradigm. Since we have shown, in the first part of this section, how the other components of the paradigm can be seen as being consistent with mathematics and humanism, we will be showing here, by 'transitivity', that the characteristics of potentially-rich situations are consistent with mathematics and humanism. Concrete-representability relates directly to the Piagetian psychological basis of the paradigm, while empiricality and pattern-latency are particularly significant from the point of view of the Popperian basis. Potentially-rich situations which are broad and deep are likely to be those which will contribute to the intrinsic-interest component of the paradigm, while those situations which are connected are likely to contribute to the interdisciplinary characteristic. The accessibility of a potentially-rich situation is linked to the contemporary characteristic of the paradigm while those situations which are particularly generalizable and symbolizable accentuate the process bias of mathesis curricula. The high-information component of the paradigm is related to the identifiability and symbolizability of potentially-rich situations. The identifiability of potentially-rich situations is linked to the genetic characteristic of the paradigm.

4.4 On the Completeness of the Mathesis Paradigm

We consider the 'completeness' of the mathesis paradigm by observing the relationship between the paradigm and humanistic educational goals. It is assumed that the 'completeness' of the paradigm is directly related to the number of humanistic educational goals which mathetic activity helps learners to achieve. Hence, if there appear to be humanistic educational goals for which all components of the paradigm are irrelevant or antithetic we can say that the paradigm is 'incomplete'. Similarly, if mathetic activity lends itself well to meeting any humanistic educational goal we can suggest, then we can take this as an indication that the paradigm is relatively 'complete'. In this section some general humanistic goals are outlined and ways in which experience in mathesis curricula might help a learner to achieve these goals are suggested.

It is not possible to give an exhaustive list of humanistic educational goals which would meet with universal acceptance. It would seem reasonable to expect, however, that the five following goals should be considered fundamental. Any humanistic educational system should provide its learners with the opportunity to become:

(a) coping persons, individuals who can make enlightened autonomous decisions;

(b) socially-aware persons, individuals who are conscious of the factors which influence the society in which they live, understanding persons;

(c) self-aware persons, individuals who appreciate the factors which make them what they are, tolerant persons;

(d) humanistic persons, individuals who utilize and value the essential attributes of humanness, persons who are compassionate and curious;

(e) self-fulfilled persons, individuals who have actualized their potential in some area or areas, persons who obtain emotional and intellectual satisfaction from the way they spend their time.

Let us now consider how curricula generated from the mathesis paradigm might help a learner achieve these five fundamental humanistic goals. The coping person is essentially an individual who is a good problem solver. In general, when he finds himself in a situation where he is faced with some problem, a cognitive perturbation, a situation where he must accommodate rather than assimilate, the coping person is able to effectively resolve the difficulty. The experience a learner gains in mathesis curricula should help him considerably in becoming a coping person. Because of their generality, the problem-solving techniques of mathesis are applicable to problem solving in other areas. The learner's experience in perceiving, and articulating problems, in isolating the constituent variables and in applying the hypothetico-deductive process of conjectures and refutations will, mutatis mutandis, help make him a better problem solver in real life, and hence more of a coping person.

By far the single most important influence in determining the nature of contemporary western society is that of science and technology. Science and technology in turn are almost totally dependent on mathematics. The learner who gains insights into the nature of mathematics, particularly by noting the historical origin of many branches of mathematics in physical problems, is in a good position to understand science and its impact on his society. He is also in a better position to understand the dynamics of change, both social and technological. Because mathematics is perhaps the most 'transcultural' of all disciplines, the mathesis learner will have the opportunity to have a 'window' into cultures which differ extensively from his own. (The numerous proofs of the 'Pythagorean Proposition' (Loomis, 1968), can be taken as a case in point here.) Mathesis curricula, with their genetic and interdisciplinary characteristics, can therefore contribute substantially toward making a learner socially-aware and understanding.

In mathesis curricula learners will spend considerable time working in close conjunction with other individuals. They cannot help observing that their reactions to, and interpretations of, the kernel-activators of various potentially-rich situations are often significantly different from those of their colleagues. In reflecting on why this is so, they may come to realize many things about themselves as individuals. They may well also come to know that "right answers" are only "right" relative to initial assumptions and constraints, and that some conflicts are only 'word deep'. The learners may also find that the names one gives objects and the symbols one uses for concepts can greatly affect the development of theories involving these objects and concepts. Students of mathesis will get practice at evaluating their own progress and at deciding what paths are worth pursuing and which are not. Inherent in mathesis curricula are many opportunities for the learner to become a more self-aware and tolerant person.

Mathesis curricula are specifically structured so as to give the learner a chance to utilize actively his capacities for rational thought, symbol-making, technical production, social organization and game-playing. In this sense mathesis curricula offer explicit and extensive opportunities for learners to utilize and to value the essential attributes and therefore to become humanistic persons. Mathesis curricula also offer a learner the chance to identify with mathematicians of former times. A sense of personal continuity with what has gone before is an important aspect of humanism. Mathesis curricula should encourage learners to be curious, to want to know, to fully understand relationships, but they should also instil an awareness of when it is best to temper reason with compassion.

Mathesis curricula will provide, for some learners, a direct means to complete self-fulfillment. Others will not, for various reasons, find self-fulfillment in mathesis, but they should at least become aware that mathematics, like music and art, is an area where any individual can participate, and where some will experience intense emotional and intellectual satisfaction. This satisfaction will be linked to things like the great aesthetic depth of the discipline and the opportunities it offers for challenges, caring, and commitment. Mathesis curricula, it therefore seems, address themselves well to the problems of having learners achieve self-fulfillment.

The foregoing does not consider the question of completeness from the point of view of the subject matter of mathematics. This is not to deny the importance of mathematical content. In fact, potentially-rich situations can be used to generate any important area of mathematics such as calculus (Toeplitz, 1963), geometry (Stewart, 1970, and sections four and five of the previous chapter) and the mathematics of daily life (Paling, Banwell and Saunders, 1968).

4.5 <u>On the Independence of the Mathesis Paradigm</u>

The independence criterion for the axiomatization of a system is by far the least important of the three we have mentioned. In particular, when one is concerned in the initial stages of development with the operational aspects of the system, it is not essential that this criterion be met. The motivation for consideration of this criterion is aesthetic in nature; redundancy is not compatible in the long run with the mathematician's highly developed sense of logical elegance. While we have been able to suggest methods for assessing the 'consistency' and 'completeness' of the paradigm there seems to be no obvious equivalent technique for the independence criterion.

It is clear, however, that the components of the paradigm are not totally independent; there is overlapping from one component to another. One place where this happens, as we have noted, is with the Piagetian cognitive psychological basis and with the Popperian epistemological basis. The pedagogic mode, which as we have indicated, depends largely on the other nine components of the paradigm, may well be non-independent. The successful implementation of the mathesis paradigm will depend almost entirely on the individual mathesis teacher. To a certain extent, it is likely that he will be successful to the degree that he is personally able to integrate and illustrate the components of the paradigm. In a sense, therefore, non-independence is an essential criterion for the successful implementation of the mathesis paradigm. We have noted previously how several of the characteristics of potentially-rich situations are very closely related to some components of the paradigm. It might be possible to argue, for instance, that the inclusion of the contemporary characteristic of the paradigm immediately implies the accessibility of potentially-rich situations.

It should be noted here as well, however, that it is felt that any redundancy that exists in the paradigm has a definite upper bound. There is no question that the whole paradigm is dependent on the humanistic foundation of the system. Many 'axiom systems' with components quite different from those of the mathesis paradigm could be assembled and still produce humanistic curricula.

4.6 Summary of the Structural Analysis of the Mathesis Paradigm

We have considered in this chapter, from a structural viewpoint, the question of the validity of the mathesis paradigm. Taking a quasi-formal approach we have examined the paradigm for evidence of 'consistency', 'completeness', and 'independence'. We have uncovered some fairly substantial reasons for believing that the paradigm and its components have a measure of 'consistency' and 'completeness'. It would not appear to be clear that 'independence' is even a desirable property for the paradigm to have.

CHAPTER FIVE

EXISTENTIAL ANALYSIS OF THE MATHESIS PARADIGM

5.1 Introduction

The analysis of the mathesis paradigm in the preceding chapter focused on its structural aspects in relation to the question of validity. In this chapter we consider some of the non-structural aspects of the utility and validity of the paradigm, in particular from the point of view of the existence of mathematics curricula which have mathetic aspects. In addition we also consider some of the broad educational connections of the mathesis paradigm.

5.2 On Existence, Testing and Falsifiability

In considering the structural validity of the mathesis paradigm use was made of three standard criteria in this area. In this chapter , in an analogous way, use is made of two further criteria, one of which is again a standard one in the field, and the second of which is specifically due to Popper. A fourth standard criterion which an axiom system must meet, at least to be considered any sort of useful system, that is to have any utility, is that of existence. A set of axioms may be perfectly valid, in that it is consistent, complete and independent, but if it is not possible to produce some system which actually satisfies the axioms, then the set of axioms has no utility. Since our structure is explicitly defined as being less than a fully axiomatized system, it is not necessary for us to show the existence of any system which satisfies all the components of the mathesis paradigm. Once again, however, the accepted criterion can serve as the standard which we will strive to achieve in a quasi-formal way. To consider the question of existence we will see to what extent we can find evidence of mathematics curricula which satisfy the components of the mathesis paradigm.

The concept of Popper's of which we wish to make use in considering the non-structural validity and utility of the paradigm is that of falsifiability. According to Popper a theory is scientific if and only if it can be falsified; if it can be tested and refuted. If it is impossible to conceive of circumstances which would imply the refutation of the theory, the theory is not scientific. In Popper's view no number of 'confirming instances' can show the 'truth' of a theory and the scientific knowledge of any given period is simply the aggregate of those theories which have withstood the most serious tests. The theories which scientists have tried hardest to falsify without being able to succeed are said to have them most verisimilitude. We can use the concept of falsifiability as a means of assessing the non-structural validity and utility of the mathesis paradigm. Thinking in terms of falsifiability may be useful in helping to answer some of the following questions. What reasons do we have to think that this paradigm can be useful to curriculum developers hoping to meet humanistic aims? Why do we think that it is worthwhile to try to falsify this 'theory'? How severe are the tests to which it has been subjected thus far? What sort of evidence is there that the paradigm has a degree of verisimilitude? Is it realistic to think that the mathesis paradigm is clearly enough articulated to have curriculum developed from it?

5.3 On the Question of Existence

In practice the question of exhibiting the existence of a theoretical system which satisfies a given axiom set is not one which troubles mathematicians unduly. The reason for this is that customarily the process of axiomatizing follows a fairly long period of time during which mathematicians have become familiarized with the properties of the system. In a sense therefore, the formal picture of a system developing from a set of axioms is precisely the converse of the historical picture in which the axioms are the end result of the process and not the beginning. The case of group theory can be taken as an example here.

Since we are basically starting from the axiom end with the mathesis paradigm, the 'existence' question is not trivial for us. There are no ready-made mathesis curricula which we can give as examples of systems which fully satisfy the components of the paradigm. There are, however, many mathematics curricula which have aspects which are consistent with some of the components of the mathesis paradigm. In this section some of these 'partial-existence' examples are noted. In particular, four different types of 'partial-existence' example are considered. First, some individual recommendations which are highly congruent with a particular component of the paradigm are noted. Second, specific mathematics curricula which are mathetically oriented and which can be identified with one particular mathematics educator are examined. Third, the existence of a large community of mathetically inclined educators is observed, and finally, some specific tests of the mathesis paradigm, per se, are mentioned.

Mathematics educators have made recommendations for curricular change which would make mathematics curricula more like mathesis curricula if they were to be implemented. Pollock (1966) has suggested that in addition to the well-charted exercise-solving and theorem-proving aspects of mathematics courses, learners should also do some 'cross-country' mathematics starting from the position of "Here is a situation think about it (p. 117)". He elaborates:

> If one wishes to give an honest picture of what mathematics is really like, if one wishes to prepare the students for applications of the mathematics in the rich variety that is characteristic of the current work in engineering

> and in the social and physical sciences, and if one wishes to attach the best available pedagogic device to each classroom situation, then one must give the student the opportunity to explore for himself new situations, both within mathematics and outside mathematics (p. 117).

The similarities between the intent of Pollock's idea and the concept of a potentially-rich situation are quite marked.

Another mathematician who has considered the difficulties of getting learners involved in authentic mathematical activity and who has proposed generating situations of the potentially-rich situation type is May (1972a). He calls his situations "dangling problems" and gives as an example the highly identifiable problem of "Galileo sequences". According to May this is a typical dangling problem in that:

> It can be presented with little symbolism, is easily understood, has intuitive appeal, and is wide open to student initiative in experimenting, formulating questions, conjecturing and proving. Dangling such a question before a class may lead to general participation in class discussion, group projects, or on individual efforts. At the very least it provides the students with a participatory glimpse of mathematics in the making. At best it may "turn on" a potential mathematician (p. 68).

Even the <u>Report of the Cambridge Conference on School Mathematics</u> (Educational Services Inc., 1963), a generally quite content-oriented document, included a recommendation that "every opportunity be taken to let the students explore mathematical situations on their own (p. 80)". The necessity we have mentioned of having a teacher of mathesis being vitally interested in the subject of mathematics and the consequent benefits to the students of mathesis resulting from this contact with an authentic practitioner, are similar to suggestions made by Davis (1967b) regarding the possibilities of "resident mathematicians" and an "academy of mathematics (p. 69)".

While the foregoing recommendations are of undoubted merit there are few indications that they have been adopted on a broad scale. There are, however, mathematics educators whose thinking is to a considerable degree consistent with the mathesis paradigm and whose ideas are currently being implemented. We have already referred to the two North American educators who fall most naturally into this category, Davis and Papert. Davis's ideas have been made available to the mathematics education community through his extensive writing and the publications of the Madison Project. Among the cognitive or mathematical objectives of the Madison Project were:

> (i) the ability to discover pattern in abstract situations;
> (ii) the ability (or propensity) to use independent creative explorations to extend "open-ended" mathematical situations;
> (iii) the possession of a suitable set of mental symbols that serve to picture mathematical situations in a pseudo-geometrical, pseudo-isomorphic fashion... (p. 158)".

While these objectives are in themselves quite compatible with a mathetic approach, it is in the statement of the "more general objectives" that the humanistic side of the goals of the Madison Project become more apparent. These "more general objectives" include:

> (i) a belief that mathematics is discoverable;
> (ii) a realistic assessment of one's own ability to discover mathematics;
> (iii) an "emotional" recognition (or "acceptance") of the open-endedness of mathematics;
> (iv) honest personal self-critical ability;
> (v) a personal commitment to the value of abstract rational analysis;
> (vi) recognition of the valuable role of "educated intuition";
> (vii) a feeling that mathematics is "fun" or "exciting" or "worthwhile".
> Actually, there is another important objective. We want the child to know who he is in relation to the human cultural past (pp. 158-159).

Davis's monograph, Mathematics Teaching - With Special Reference to Epistemological Problems (1967c), in which he considers some of the wider philosophical problems related to mathematics instruction, is most relevant for mathesis curriculum developers. Dawson (1969), in analyzing the Madison Project from a Popperian viewpoint, concluded that it exhibited "strong Fallibilistic tendencies (p. iv)".

The ideas of Papert are less well known in the mathematics education world since he has turned his attention only fairly recently to the area. His recent writing in the field has evolved from his research in the field of artificial intelligence. His approach is characterized by the highly imaginative use of computers. Papert (1972a) writes of a:

> grander vision of an educational system in which

> technology is used not in the form of machines for processing children but as something the child himself will learn to manipulate, to extend, to apply to projects, thereby gaining a greater and more articulate mastery of the world, a sense of power of applied knowledge and a self-confidently realistic image of himself as an intellectual agent (p. 2).

In looking for vehicles to have children acquire a "mathematical way of thinking (1972b, p. 250)", Papert has been led to invent "Turtle Geometry: A Piece of Learnable and Loveable Mathematics (1972b, p. 252)". He and his colleagues at the M.I.T. Artificial Intelligence Laboratory (Papert and Solomon, 1972; Bamberger, 1972) have also used computer-linked "music boxes" as a sort of potentially-rich situation. As befits a former colleague of Piaget's, Papert strongly advocates an active, learner-centred approach to mathematics instruction. He writes that "children learn by doing and by thinking about what they do (1972a, p. 2)", and that "being a mathematician, again like being a poet, or a composer or an engineer, means <u>doing</u> rather than knowing or understanding (1972b, p. 249)".

In considering the large-scale 'community' aspect of the paradigm, that is, in looking for evidence of mathematics curricula which manifest a high degree of mathetic content and which also involve a relatively large percentage of the total teaching force, particularly at the elementary school level, it is necessary to look to the United Kingdom. As we have noted previously, reports of the general atmosphere of many British primary and junior schools indicate that they are intended to be quite humanistic. The work in mathematics in these schools, particularly those influenced by the Nuffield Project (1967) and by Biggs (1968, 1969), is very mathetic. From the Kuhnian 'community' view of paradigm, however, it is the group of mathematics educators who are active in the Association of Teachers of Mathematics who provide the strongest indication that mathesis curricula can be implemented. Although a large percentage of its 'inner core' and Executive comes from the faculties of Teacher's Colleges and University Departments of Education, the Association has members from almost all levels and areas of education in Britain. The general mathetic bias of the Association, while never explicitly stated in these terms, can be seen in its publications. The major publications of the Association are the quarterly journal, Mathematics Teaching, and the collections, Some Lessons in Mathematics (1965), Notes on Mathematics in Primary Schools (1969) and Mathematical Reflections (1970). A few members of the Association have published independently; the series by Paling, Banwell and Saunders, Making Mathematics (1968), and the book Starting Points, by Banwell, Saunders and Tahta (1972), are perhaps the best known of these; they probably come as near to being 'mathetic textbooks' as is possible.

The introductions to Notes on Mathematics in Primary Schools (1969) and Some Lessons in Mathematics (1965), are as close as the Association has come to making a complete and detailed statement of its philosophy. By noting some statements from these introductions we can see how the orientation of the Association to mathematics instruction is essentially a mathetic one.

> Because mathematics is made by men and exists only in their minds, it must be made or re-made in the mind of each person who learns it. In this sense mathematics can only be learnt by being created. We do not believe that a clear distinction can be drawn between the activities of the mathematician inventing new mathematics and the child learning mathematics which is new to him. The child has different resources and different experiences, but both are involved in creative acts. We want to stress that the mathematics a child knows is, in a real sense, his possession, because by a personal act he has created it ... We believe that the learning of mathematics has to be seen in this way as individual creative (or re-creative) acts taking place in a social context ... We are concerned with the creative side of the child's learning and with minimizing the teacher's interference with this. Every time a teacher insists on his way of doing a piece of mathematics, rejecting any responses which do not seem to fit, he nibbles away at his pupil's ability to act mathematically. We believe in the value of the child's mathematics; that he should have freedom to make it and use it and talk about it (1969, pp. 2-5).

> Mathematics does not start with the finished theorem in the textbook; it starts from situations (1965, p. 2).

The experience, size, growth and influence of the Association of Teachers of Mathematics can serve as a strong indication that mathesis curricula can be constructed and carried out (in a way 'lived') at many educational levels. In a sense we can take their experience as a Popperian test which the mathesis paradigm has survived. Two things should be noted here, however. The first is that the experience of the Association of Teachers of Mathematics is that the process of developing a group of humanistic teachers is quite difficult and can take place only over a fairly long period of time. It seems in some ways that it is only after twenty years of existence that the bulk of British mathematics teachers have even started to realize some of the implications of the Association's basic position on mathematics instruction. The editor of Mathematics in Schools, a publication of the more traditional Mathematical Association, started his review of Starting Points as follows:

> This book will shock, perhaps even startle some older members of the Association, for it does not fit in with the commonly accepted philosophy (held implicitly by most mathematics teachers) which, to put it in its simplest terms, is mainly concerned with the acquisition of knowledge and skills appropriate for an examination or other agreed syllabus. There is no explicit statement of philosophy, but Starting Points is a manifesto for a new approach to teaching mathematics (Reynolds, 1972, p. 33).

The second point to note is that although many of the ideas for the mathesis paradigm have come from the Association this does not imply that the members of the Association would accept the paradigm as a whole. The Piagetian basis would probably be questioned by some members, and the

Popperian basis by others. Nor is it likely that the members of the Association would universally accept that it is necessarily a good thing to formalize one's assumptions; that is, to construct an explicit paradigm.

The mathesis paradigm as such has been submitted to Popperian-type tests of moderate severity and has survived. This would seem to indicate that the paradigm should now be submitted to some more-severe tests to see if it, or some parts of it in particular, can be falsified. Two of these short-term tests have been carried out with prospective teachers. The one, with prospective mathematics teachers, has been described at some length in the third chapter of this dissertation. As we observed at that point, this group was able to start from a potentially-rich situation and pose and investigate significant mathematical problems. The members of this group were generally quite enthusiastic about the concept of potentially-rich situations relative to their own mathematical development.

The second session was in some ways an even-more severe test of the viability of the paradigm as a curriculum-generating device. The paradigm was used to construct, in outline form, some units of a joint Mathematics-Art curriculum. These units were discussed and investigated during two one-and-one-half hour periods with a class of Art Education students. In a sense this session tested the interdisciplinary component in particular while the first session tested especially the potentially-rich situation component. In spite of the fact that the Art students were generally not positively disposed toward the subject of mathematics at the beginning of their three-hour experience, at the end of it the large majority of them said that they would like to be involved, as teachers, in a Mathematics-Art curriculum of the type outlined. (It was assumed that there would be two teachers, an artist and a mathematician, working in the curriculum.)

In light of the questions raised about ways of getting prospective mathematics teachers actively involved in significant mathematics (May, 1972b; Walter and Brown, 1971), it would seem that one possible place for the immediate implementation of the mathesis paradigm might be at teacher education institutions. In fact, since the individual teacher is by far the most important single factor in determining the success or failure of a mathesis curriculum, it may well be essential that the implementation of the mathesis paradigm start in teacher-education programmes. Miss B. Blackall, a doctoral student in the Department of Elementary Education at the University of Alberta, has used the mathesis paradigm as an organizing framework for a course in Early Childhood Mathematics Education and has reported that for her it proved to be a useful device for planning the course. The paradigm has been discussed with mathematics educators at several different levels of institution. Several short sessions on certain potentially-rich situations have been spent by the writer with junior-high school and secondary-school mathematics students. These sessions have helped to produce the paradigm in the form in which it now exists.

5.4 On Approaching Mathesis

The fact that some specific mathematics educators and a group of mathematics educators have been singled out for identification in the last section does not imply by any means that these are the only mathematics educators who can be considered as having a mathetic approach. In fact, all mathematics educators exhibit, to some degree or another, mathetic tendencies. To make this statement somewhat more meaningful, we can look at what might be considered stages on the approach to mathesis. For this purpose we identify some 'highly-visible' features of mathesis curricula, and then consider mathematics curricula which share these features or characteristics to a greater or lesser degree.

Among the features which all mathesis curricula will exhibit we find the following nine:

(a) process orientation: a tendency to have a process-orientation rather than a content-orientation;

(b) textbook-independency: in general no single text will be used; a wide range of sources will be utilized;

(c) interdisciplinary bias: the line between disciplines will not be finely drawn;

(d) materials orientation: a large number of diverse, inexpensive materials will be used by the learners in almost all parts of the curriculum;

(e) self-evaluation bias: learners will be expected and encouraged to evaluate their own work, formal and external evaluation taking place over a fixed time period will be minimal;

(f) learner responsibility: learners will play the major part in determining what they spend their time on and how they spend their time;

(g) learner interaction: there will be considerable interaction between learners and between groups of learners;

(h) within-group deviation: within any given group or 'class' learners will be involved in quite diverse activities at different times.

(i) person-centredness: the interests and abilities of the teacher and the students strongly influence the curriculum.

The foregoing are things which observers would perceive as being quite prominent features of a mathesis curriculum. From here it is not difficult to suggest what an 'anti-mathetic' curriculum would look like. For if mathesis curricula exhibit these characteristics to a very great extent, let us imagine a curriculum, let us say 'Curriculum A', which is characterized by the opposite of these nine features. 'Curriculum A' would then be a mathematics curriculum with a strong content bias in which the teacher followed a single textbook very closely, and in which the line between what is mathematics and what is not mathematics is finely drawn. In this curriculum the students would use no materials of a manipulative sort and formal, external evaluation would be stressed. The learners would have no say in determining what topics they studied; they would work completely independently and all learners would always be doing the same things at the same time. The curriculum would not be influenced in any significant way by the interests and abilities of different teachers and learners associated with it.

While aware of the dangers of creating a 'straw-man', it would seem that the anti-mathetic, non-humanistic mathematics 'Curriculum A' bears a very strong resemblence to a 'traditional' mathematics curriculum. In fact, to say merely that these curricula are non-humanistic because they are non-interdisciplinary and that they stress external evaluation techniques is to understate the case considerably. These curricula with their comments of the 'Marks will be deducted for the use of trigonometric methods on the algebra part of the paper' genre, even fragmented the subject of mathematics. The anti-humanistic tendencies of a system where years of invested time are 'wasted' if one performs badly in a given three hour period, cannot be overstressed.

There are of course many positions which lie between the mathetic and 'traditional' ones. If, relative to these nine features, mathesis curricula represent one extreme position on a continuum and 'traditional' curricula the other, we can consider what some of the intermediate positions might be. On a five-unit scale, for instance, running from very-low for 'traditional' to very-high for mathesis, it would be easy to consider the curricula represented by the low, moderate and high positions. We suggest that in the same way that we designated the one position 'traditional', we can suggest that these three other types of mathematics curriculum could be described as 'modified-traditional', 'structured-laboratory' and 'unstructured-laboratory'. Because it is obviously easier to change some of these features, such as the materials orientation, than it is to change others, it seems unlikely that we have any sort of 'linear' situation here. Therefore only quite general trends can be safely extracted from this particular 'model'. It would seem reasonable, however, to expect a mathematics teacher to be able to approximately locate his own practice on this continuum. He could also perhaps make use of the scale to to make his mathematics curricula more 'humanistic' if he wished to do so.

5.5 General Comments on the Mathesis Paridigm

In this, the penultimate section of the dissertation, we wish to make some general comments on the mathesis paradigm. In particular, we wish to note some general reasons for thinking that the paradigm has utility, and to mention three paradigm-related areas which seem especially promising for further research.

Examining current educational trends, there are a number of reasons for thinking that mathesis will become an increasingly attractive subject area for learners to work in, and for educational institutions to offer. One of these relates to the cost of education; the consideration of which has recently become the focus of public attention, and which promises to remain there for some time. Mathesis curricula are viable from an economic viewpoint because they are relatively inexpensive. Using computer terminology, the introduction of mathesis curricula is more of a 'software' change than of a 'hardware' change. A mathesis teacher functions as a mathematician and an exemplar, not as a classroom policeman. To build up a collection of reference books costs little more than buying a 'class-set' of textbooks, and the materials needed for mathesis are quite inexpensive.

Another contemporary educational trend is the one toward 'life-long education'. The potential of mathesis for opsimathy is considerable. Mature learners will be able to bring their diverse background experiences to bear to accentuate the interdisciplinary aspect of mathesis. To learners looking to education as a means to self-knowledge and self-fulfillment, rather than as a collection of credits, mathesis will be especially attractive. Because of their administrative flexibility , mathesis curricula could fit well into the sort of educational frameworks Illich (1971) calls "Learning Webs (p. 72)".

Two other reasons for thinking that mathesis curricula are not only feasible, but perhaps even likely, for educational institutions of the future are that they help to answer two long-standing educational problems. In mathesis curricula the person-centredness does not apply only to the students but to the teacher as well. Hence the mathesis teacher (or teacher-learner) can see the opportunity here for personal growth, both intellectual and emotional. This opportunity is not always obvious when teaching traditional mathematics curricula. Mathesis curricula also do much to bridge the gap between child-centred and subject-centred programmes, for they are essentially curricula which are simultaneously child-centred and subject-centred. The Aristotelian assumption, apparently accepted by most educators writing about 'humanizing' education, that curricula must be either subject-centred or child-centred is not one which seems to stand up under examination. It would appear quite inappropriate, in fact, to label as 'humanistic', curricula which ignore or underestimate the learner's human capacity for reason.

The concept of a mathesis curriculum raises a number of questions about the preparation of teachers who would be capable of implementing the mathesis paradigm. To successfully operate as a teacher in a mathesis curriculum an individual must be quite competent in working both with people and with mathematics. The abilities, knowledge and attitudes which the mathesis teacher will need are not the ones which are needed to teach most contemporary mathematics curricula. Specific recommendations as to how these abilities and attitudes might be developed are beyond the scope of this study. In general, however, it would seem that a mathetic 'Math Ed' programme would be required to develop teachers capable of doing justice to a mathesis curriculum.

Of the many areas touched on this study, we wish to mention here three which would seem to be particularly attractive for further investigation. The first of these has to do with the nature of mathesis. We have chosen to give fairly cursory attention to this question here, but it would appear to be one which needs considerable thought given to it. A number of the world's greatest mathematicians have held quite humanistic attitudes, in particular toward education. Whitehead (1963), for example, wrote, in a rather 'third-force' vein, of the education of the individual as the achievement of the "potentialities of that living creature (p. 48)". His colleague, Russell, wrote that:

> the humanistic conception regards a child as a gardener regards a young tree, i.e., as something with a certain intrinsic nature which will develop

> into an admirable form given proper soil and air and light (Chomsky, 1972, p. 46)".

(Russell developed this image further on another occasion when he observed that "the soil and freedom required for man's growth are immeasurably more difficult to discover and to obtain (Chomsky, 1972, p. 47)".)

And yet, despite this consciousness, very few mathematicians seem to have directly addressed the question of the humanistic aspects of mathematics. One exception to this general rule is Wilder (1968) who considers the "humanistic aspects of mathematics (p. 6)" to be related to the aesthetic pursuit of mathematics, "mathematics for its own sake (p. 6)". Another mathematician who considers some of the humanistic connections of mathematics is Sarton. He writes (1954):

> The main reason for studying this history of mathematics, or the history of science, is purely humanistic. Being men we are interested in other men, and especially in such men as helped us to fulfil our highest destiny. As soon as we realize the great part played by individual men in mathematical discoveries - for, however these may be determined, they cannot be brought about except by means of human brains - , we are anxious to know all their circumstances (p. 22).

It would appear, however, that the only mathematician to fully consider mathesis, humanistic mathematics, although it seems that he did not use the term this way, is Keyser (1922, 1933, 1971). The similarity of Keyser's outlook to the one taken in this study is shown in the following statements about education and mathematics.

> Education may be characterized by its aim, and its aim is nothing less than that of qualifying men and women to realize in fullest measure, and to represent it worthily, in their lives, their personalities, and their work, the potential dignity of man (1933, p. 99).
>
> Nothing is better entitled to rank as one of the great Humanities than Mathematics itself.... For what is the test? I hold it to be this: those subjects are best entitled to membership in the assembly of the Humanities which (1) best disclose the essential nature, the defining characteristic, of man and (2) best serve to guide our human life (1933, p. 101)."

It would seem that Keyser's work would present a good stepping off point for any researcher interested in exploring the nature of mathesis.

A second area of the paradigm which would seem to be a fruitful research area is that of the potentially-rich situation component. Worthwhile activities here would seem to be the collecting of a wide range of potentially-rich situations and investigating the reactions of different groups of learners to the same situation. Other questions here concern the completeness of the system of characteristics of potentially-rich situations and the nature of the form of kernel-activators for optimal response from learners.

A third immediate-interest research area concerns the efficacy of paradigms as general curriculum-generating devices. Propaedeutic paradigms would seem to have the merit of revealing assumptions which underlie curricula. It would seem that many of the debates regarding curriculum reform in the last decade would not have been necessary had participants realized that underneath the disputes lay fundamental disagreements about educational goals. It would seem that the use of paradigms as curriculum-generating devices might lead to a rather higher-level of curriculum debate. The question as to what aspects of the mathesis paradigm can be generalized to serve as components for humanistic paradigms in other subject areas is an interesting one. To attempt to construct a 'Kuhnian paradigm' for curriculum development, as Heimer (1971) has done, seems, however, to be unnecessarily ambitious.

5.6 Summary

In this chapter we have attempted to outline some of the reasons for thinking that the mathesis paradigm, from a non-structural viewpoint, can be thought of as having validity and utility. The relation of parts of the paradigm to contemporary practice in mathematics education has been noted. The Association of Teachers of Mathematics has been singled out as a group of mathematics educators whose approach is quite mathetic. Some indications of where mathesis curricula lies in relation to contemporary mathematics curricula have been given. Observations of how mathesis curricula are consistent with contemporary educational trends have been made and some particularly inviting areas for further research have been indicated.

At this point it is perhaps advisable to stress the fact that this study is seen as being logical rather than hortatory in nature. It is an attempt to outline a position consistent with a certain set of initial assumptions. It is hoped that it will provide a foil in which mathematics educators at all levels can examine their own basic assumptions about the nature and aims of the activity which occupies much of their time.

> Mathematics exists, not as a defined entity in some logician's or philosopher's textbook, but first and foremost as a living reality, as a fact of life. We strive to understand it as we strive to understand all the other manifold aspects of our experience, from physical nature to the nature of poetry. And we find ourselves faced with the same mixture of answered and unanswered questions of insight and puzzlement, that is everywhere characteristic of the human situation (Wittenberg, 1963, p. 1097).

In closing, we would suggest that Wittenberg's words provide an accurate picture of the world of mathesis for those of us who have chosen to pursue humanistic goals through mathematics.

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APPENDIX ONE

A LIST OF BOOKS AND ARTICLES OF RELEVANCE TO THE MATHESIS PARADIGM

A large number of sources from a wide range of disciplines have influenced the formation of the mathesis paradigm. A partial listing of these sources is given in this appendix. The sources vary considerably in the direction and in the degree to which they have contributed to the paradigm. For ease of reference the sources have been subdivided into five main classes and twenty-five sub-classes.

The sub-classes are as follows:

A: MATHEMATICS

I: The Nature of Mathematics
- (a) The Foundations and Philosophy of Mathematics
- (b) Surveys

II. The History of Mathematics
- (a) Surveys
- (b) Biographical

III. Aspects of Contemporary Mathematics
- (a) Research-Contemporary Mathematics
- (b) Some Interdisciplinary Connections

IV. Problems in Mathematics
- (a) Academic Problems
- (b) Recreational Problems

V. Mathematical Literature
- (a) Bibliographies
- (b) Journals

VI. Mathematics Education

B: PSYCHOLOGY

I. Piaget

II. Humanistic Psychology

III. Thinking, Problem-solving and Creativity

IV. General

C: PHILOSOPHY

I. The Philosophy of Humanism

II. Popper, Polanyi and Paradigms

III. Philosophy of Science

IV. General

D: EDUCATION

I. Criticism

II. Curriculum

III. Humanistic Education

IV. General

E: SOCIAL ISSUES

I. The Future

II. General

The sub-classes correspond quite closely in some cases to particular components of the paradigm. For the sake of brevity, titles have been listed only once even though many could be considered to be relevant to at least two sub-classes: book titles have been given priority over journal-article titles.

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A: MATHEMATICS

I: The Nature of Mathematics

(a) The Foundations and Philosophy of Mathematics

Benacerraf, P., & Putnam, H. (Eds.) Philosophy of mathematics: Selected readings. Oxford: Basil Blackwell, 1964.

Beth, E. W. Mathematical thought: An introduction to the philosophy of mathematics. Dordrecht, Holland: Reidel, 1965.

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Wilder, R. L. Introduction to the foundations of mathematics. (2nd ed.) New York: Wiley, 1965.

Wittgenstein, L. Remarks on the foundations of mathematics. Cambridge, Mass.: M.I.T. Press, 1967.

(b) Surveys

Aleksandrov, A. D., Kolmogorov, A. N., & Lavrent'ev, M. A. Mathematics: Its content, methods, and meaning. (2nd ed.) Cambridge, Mass.: M.I.T. Press, 1969. 3 vols.

Albert, A. A., Browder, F. E., Herstein, I. N., Kaplansky, I., & MacLane, S. Mathematics vs. Numerical analysis. Science, 1965 (16 July), 149, 243-244.

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Coleman, A. J. Mathematics in the second half of the twentieth century. Ontario Mathematics Gazette, 1970, 8, 75-88.

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