Unbounded convergences in vector lattices

by

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Abstract

Abstract. Suppose X is a vector lattice and there is a notion of convergence $x_\alpha \xrightarrow{\sigma} x$ in X. Then we can speak of an "unbounded" version of this convergence by saying that $x_\alpha \xrightarrow{u\sigma} x$ if $|x_\alpha - x| \wedge u \xrightarrow{\sigma} 0$ for every $u \in X_+$. In the literature the unbounded versions of the norm, order and absolute weak convergence have been studied. Here we create a general theory of unbounded convergence, but with a focus on *uo*-convergence and those convergences deriving from locally solid topologies. We also give characterizations of minimal topologies in terms of unbounded topologies and *uo*-convergence. At the end we touch on the theory of bibases in Banach lattices.

Preface

The research in this thesis is an amalgamation of the papers *Unbounded topologies and uo-convergence in locally solid vector lattices*, *Metrizability of minimal and unbounded topologies*, *Completeness of unbounded convergences*, and *Extending topologies to the universal σ-completion of a vector lattice*. The second paper was done in collaboration with Marko Kandić, and appeared in the *Journal of Mathematical Analysis and Applications*. The third paper is published in the *Proceedings of the American Mathematical Society*. There are also many new results in this thesis that do not appear in the aforementioned papers. The section on bibases in Banach lattices has not yet been submitted for publication, but there are plans to do so.

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Contents

1. The basics on *uo*-convergence

We begin by recalling the fundamental results on *uo*-convergence that will be used freely throughout this thesis. The concept of *uo*-convergent nets appears intermittently in the vector lattice literature (see [Na48], [De64], [Wi77], [Ka99] as well as the notion of order*-convergence in [Frem04] and the notion of L-convergence in [Pap64]), but it was not until the recent papers [GX14], [Gao14], [GTX17], and [GLX17] that it gained significant traction. This section will be completely minimalistic, as we will just recall what will be needed for the study of unbounded topologies. There are several beautiful areas that will not be investigated, most notably, the *uo*-dual of [GLX17], and the applications to financial mathematics. For more on these directions we refer the reader to the work of N. Gao, D. Leung and F. Xanthos. The contributions to *uo*-convergence that I have made will appear in later sections.

Throughout this thesis, for convenience, all vector lattices are assumed Archimedean. This is a minor assumption. Notice, for example, that by [AB03, Theorem 2.21] every vector lattice which admits a Hausdorff locally solid topology is automatically Archimedean.

1.1. **The four pillars of *uo*-theory.** Throughout this section, X is an (Archimedean) vector lattice. We begin with the definition of order convergence:

Definition 1.1. *A net $(x_\alpha)_{\alpha\in A}$ in X is said to* ***order converge*** *to $x \in X$, written as $x_\alpha \xrightarrow{o} x$, if there exists another net $(y_\beta)_{\beta\in B}$ in X satisfying $y_\beta \downarrow 0$ and for any $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. We say that a net (x_α) is* ***order Cauchy*** *if the double net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')}$ order converges to zero.*

This leads to the definition of *uo*-convergence:

Definition 1.2. *A net (x_α) in X is said to* ***unbounded order converge*** *(or* ***uo-converge****) to $x \in X$, written as $x_\alpha \xrightarrow{uo} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$. (x_α) is said to be uo-**Cauchy** if the double net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')}$ uo-converges to zero.*

Remark 1.3. The initial motivation for *uo*-convergence was the following observation: Suppose (Ω, Σ, μ) is a semi-finite measure space and X is a regular sublattice of $L_0(\mu)$. Then for a sequence (x_n) in X, $x_n \xrightarrow{o} 0$ in X iff (x_n) is order bounded in X and $x_n \xrightarrow{a.e.} 0$. It seems natural to try to remove the order boundedness condition from this equivalence and, indeed, *uo*-convergence allows one to do this: $x_n \xrightarrow{uo} 0$ in X iff $x_n \xrightarrow{a.e.} 0$. Therefore, *uo*-convergence can be thought of as a generalization of convergence almost everywhere to vector lattices. For details see [GTX17].

The most important fact about *uo*-convergence is that it passes freely between regular sublattices. This was proved in [GTX17, Theorem 3.2], but can be traced as far back as [Pap64]:

Theorem 1.4. *Let Y be a sublattice of a vector lattice X. TFAE:*

(i) Y is regular;
(ii) For any net (y_α) in Y, $y_\alpha \xrightarrow{uo} 0$ in Y implies $y_\alpha \xrightarrow{uo} 0$ in X;
(iii) For any net (y_α) in Y, $y_\alpha \xrightarrow{uo} 0$ in Y if and only if $y_\alpha \xrightarrow{uo} 0$ in X.

Theorem 1.4 is not true for order convergence. Indeed, c_0 is a regular sublattice of ℓ_∞, the unit vector basis of c_0 converges in order to zero in ℓ_∞, but fails to order converge in c_0.

The next important fact is [GTX17, Corollary 3.6]; it states that *uo*-convergence, although strong enough to have unique limits, is quite a weak convergence:

Theorem 1.5. *Let (x_n) be a disjoint sequence in X. Then $x_n \xrightarrow{uo} 0$ in X.*

Remark 1.6. See Lemma 18.10 for a very simple observation that demonstrates the extent to which *uo* and *o*-convergence crucially differ. In Corollary 15.3 we show that *uo* and *o*-convergence disagree on nets in every infinite dimensional vector lattice.

The next theorem will be crucial for our analysis. One of my main tricks is to pass *uo*-convergence from X to the universal completion of X, prove something up there, and then pass back down. This technique works very well because universally σ-complete vector lattices have remarkable properties. Here is one of them:

Theorem 1.7. *A sequence (x_n) in a universally σ-complete vector lattice is uo-Cauchy iff it is order convergent.*

Remark 1.8. I will eventually show that *uo*-convergence is sequentially complete iff X is universally σ-complete.

The next theorem is very useful, and can be found in [CL17]. It tells us exactly which subsets of X_+ suffice as "test vectors" for *uo*-convergence:

Theorem 1.9. *Let X be a vector lattice and A a subset of X_+. TFAE:*

(i) The band generated by A is X;
(ii) For any $x \in X_+$, $x \wedge a = 0$ for all $a \in A$ implies $x = 0$;
(iii) For any net (x_α) in X, $|x_\alpha| \wedge a \xrightarrow{o} 0$ for all $a \in A$ implies $x_\alpha \xrightarrow{uo} 0$.

All other results on *uo*-convergence will be pulled from their original sources. Fortunately, the literature on *uo*-convergence is not too scattered, and makes for easy reference. For standard definitions we refer to the excellent monograph [AB03]. Our notation is completely consistent with that book except the definition of order convergence (see the appendix), the updated terminology regarding minimal topologies, and the standard synonyms such as "order complete" and "Dedekind complete", "vector lattice" and "Riesz space", etc. Note that we will use the notation $x_\alpha \xrightarrow{o_1} x$ to denote the order convergence from [AB03], and $x_\alpha \xrightarrow{uo_1} x$ to denote its unbounded counterpart.

2. Unbounded topologies

If *uo*-convergence is meant to act as convergence almost everywhere in vector lattices, we best start looking for a notion of convergence in measure. Indeed, we will find such a notion in the form of the "minimal topology", but, to begin, we set the table with many facts of unbounded topologies.

Definition 2.1. *A (not necessarily Hausdorff) topology* τ *on a vector lattice* X *is said to be* ***locally solid*** *if it is linear and has a base at zero consisting of solid sets. A pair* (X, τ) *where* X *is a vector lattice and* τ *a locally solid topology is known as a* ***locally solid vector lattice****.*

Starting from any locally solid topology on X we can define, in analogy with *uo*-convergence, a new convergence on X.

Definition 2.2. *Let* (X, τ) *be a locally solid vector lattice and* $A \subseteq X$ *an ideal. We say a net* (x_α) *in* X ***unbounded*** τ***-converges to*** $x \in X$ ***with respect to*** A *if* $|x_\alpha - x| \wedge |a| \xrightarrow{\tau} 0$ *for all* $a \in A$ *or, equivalently, if* $|x_\alpha - x| \wedge a \xrightarrow{\tau} 0$ *for all* $a \in A_+$*.*

Remark 2.3. The assumption that A is an ideal in the last definition presents no loss in generality since $|x_\alpha - x| \wedge |a| \xrightarrow{\tau} 0$ for all $a \in A$ if and only if $|x_\alpha - x| \wedge |a| \xrightarrow{\tau} 0$ for all $a \in I(A)$ (the ideal generated by A in X).

Remark 2.4. A natural thought is, why do we introduce the ideal A at all? Why don't we do this for *uo*-convergence? The next example and remark should clarify this.

Example 2.5. Compare [KLT17, Example 2.1] with [KMT17, Theorem 2.3]: In $C[0, 1]$ the unbounded norm convergence (with respect to $C[0, 1]$) agrees with the norm convergence, but there exist ideals A of $C[0, 1]$ such that the unbounded norm convergence with respect to A is Hausdorff and strictly weaker than the norm convergence.

Remark 2.6. By analogy, for any ideal $A \subseteq X$, one can declare that a net (x_α) in X unbounded order converges to $x \in X$ with respect to A if $|x_\alpha - x| \wedge a \xrightarrow{o} 0$ for each $a \in A_+$. However, it is immediately deduced from Theorem 1.9 that if A is order dense in X then this convergence agrees with *uo*-convergence, and if A is not order dense, then this convergence won't have unique limits. This is why we are only interested in *uo*-convergence. When dealing with Hausdorff locally solid topologies, however, there is a "gap" between being order dense and τ-dense (see [AB03, Exercise 2.7]), and this is what makes the theory interesting.

The next theorem shows that the unbounded τ-convergence with respect to A corresponds to the convergence of a locally solid topology on X.

Theorem 2.7. *If A is an ideal of a locally solid vector lattice (X, τ) then the unbounded τ-convergence with respect to A is a topological convergence on X. Moreover, the corresponding topology, $u_A\tau$, is locally solid.*

Proof. Since τ is locally solid it has a base $\mathcal{N}_0^\tau$ at zero consisting of solid neighbourhoods. For each $V \in \mathcal{N}_0^\tau$ and $a \in A_+$ define $V_a := \{x \in X : |x| \wedge a \in V\}$. We claim that the collection $\mathcal{N}_0^{u_A\tau} := \{V_a : V \in \mathcal{N}_0^\tau, a \in A_+\}$ is a base of neighbourhoods of zero for a locally solid topology; we will call it $u_A\tau$. Notice that (x_α) unbounded τ-converges to x with respect to A iff every set in $\mathcal{N}_0^{u_A\tau}$ contains a tail of the net $(x_\alpha - x)$. This means the unbounded τ-convergence with respect to A is exactly the convergence given by this topology. Notice also that $V \subseteq V_a$ and, since V is solid, so is V_a.

We now verify that $\mathcal{N}_0^{u_A\tau}$ is a base at zero. Trivially, every set in $\mathcal{N}_0^{u_A\tau}$ contains 0. We now show that the intersection of any two sets in $\mathcal{N}_0^{u_A\tau}$ contains another set in $\mathcal{N}_0^{u_A\tau}$. Take $V_{a_1}, W_{a_2} \in \mathcal{N}_0^{u_A\tau}$. Then $V_{a_1} \cap W_{a_2} = \{x \in X : |x| \wedge a_1 \in V \;\&\; |x| \wedge a_2 \in W\}$. Since $\mathcal{N}_0^\tau$ is a base we can find $U \in \mathcal{N}_0^\tau$ such that $U \subseteq V \cap W$. We claim that $U_{a_1 \vee a_2} \subseteq V_{a_1} \cap W_{a_2}$. Indeed, if $x \in U_{a_1 \vee a_2}$, then $|x| \wedge (a_1 \vee a_2) \in U \subseteq V \cap W$. Therefore, since $|x| \wedge a_1 \leq |x| \wedge (a_1 \vee a_2) \in V \cap W \subseteq V$ and V is solid, we have $x \in V_{a_1}$. Similarly, $x \in W_{a_2}$.

We know that for every V there exists W such that $W + W \subseteq V$. From this we deduce that for all V and all $a \in A_+$, if $x, y \in W_a$ then

$$|x + y| \wedge a \leq |x| \wedge a + |y| \wedge a \in W + W \subseteq V \tag{2.1}$$

so that $W_a + W_a \subseteq V_a$.

If $|\lambda| \leq 1$ then $\lambda V_a \subseteq V_a$ because V_a is solid. It follows from $V \subseteq V_a$ that V_a is absorbing. This completes the verification by [AlB06, Theorem 5.6]. □

Remark 2.8. Notice that the convergence defined in Definition 2.2 still makes sense if τ is only defined on the ideal A (rather than all of X). However, Theorem 2.7 can fail in this case.

Example 2.9. Let $A = \ell_\infty$ and $X = \mathbb{R}^\mathbb{N}$, the universal completion of A, and $x = (1, 2, 3, \dots)$. It is easy to see that $\frac{1}{n}x$ does not unbounded norm converge to zero with respect to A, showing that the unbounded norm topology with respect to A is not linear. Note, however, that an inspection of the proof of Theorem 2.7 shows that the linearity problem that occurs in Definition 2.2 when τ is only defined on A can be avoided if τ is assumed to be σ-Lebesgue.

It is also easy to see that the convergence in Definition 2.2 defines a locally solid topology on X iff $u\tau$ admits a (not necessarily unique) locally solid extension $(u\tau)^X$ from A to X. The convergence in Definition 2.2 corresponds to the topology $u_A((u\tau)^X)$. In particular, if τ extends from A to a locally solid topology on X, then $u\tau$ does as well. The converse is not true, and this will play a major role in the theory of minimal topologies.

Remark 2.10. The reason Theorem 2.7 fails when τ is only defined on A is because of scalar multiplication. Unbounded convergence can also be defined on lattice ordered groups, where one defines a locally solid topology as a *group* topology with a neighbourhood base at zero consisting of solid sets.

Remark 2.11. Notice the change in notation from [KLT17]. For each ideal A of X, one may think of $u_A : \tau \mapsto u_A\tau$ as a map from the set of locally solid topologies on X to itself. In particular, if B is some other ideal of X then $u_A u_B \tau$ should make sense notationally and equal $u_A(u_B\tau)$.

It is clear that $(u_A\tau)|_A = u_A(\tau|_A)$ and if $A \subseteq B$ then $u_A\tau \subseteq u_B\tau \subseteq \tau$. It can be checked that if A and B are ideals of a locally solid vector lattice (X,τ) then $u_A(u_B\tau) = u_B(u_A\tau) = u_{A\cap B}\tau$. Throughout, we will abbreviate $u\tau := u_X\tau$; notice that $u(u_A\tau) = u_A\tau$. This justifies the following definition:

Definition 2.12. *A locally solid topology τ is* ***unbounded*** *if $\tau = u\tau$ or, equivalently, (since the map $\tau \mapsto u\tau$ is idempotent) if $\tau = u\sigma$ for some locally solid topology σ.*

Remark 2.13. If τ is the norm topology on a Banach lattice X, the corresponding $u\tau$-topology is called *un*-topology; it has been studied in [KMT17], [KLT17] and [DOT17]. It is easy to see that the weak and absolute weak topologies on X generate the same unbounded convergence[1] and, since the absolute weak topology is locally solid, this convergence is locally solid. It has been denoted *uaw* and was studied in [Zab17].

Now that we have constructed topologies, we ask when they are Hausdorff.

Proposition 2.14. *Let A be an ideal of a locally solid vector lattice (X,τ). Then $u_A\tau$ is Hausdorff iff τ is Hausdorff and A is order dense in X.*

Proof. Routine modification of the proof of Proposition 1.4 in [KLT17]. □

Corollary 2.15. *For an order dense ideal A of X, the map $\tau \mapsto u_A\tau$ sends the set of Hausdorff locally solid topologies on X to itself.*

We next present a few easy corollaries of Theorem 2.7 for use later in the thesis.

Corollary 2.16. *Lattice operations are uniformly continuous with respect to $u_A\tau$, and $u_A\tau$-closures of solid sets are solid.*

Proof. The result follows immediately from Theorem 8.41 and Lemma 8.42 in [AlB06]. □

[1] Even though the weak topology is generally not locally solid we may still define $x_\alpha \xrightarrow{uaw} x$ iff $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for each $u \in X_+$. The point is that the operation of unbounding can convert non-locally solid topologies into locally solid topologies. See Section 16 for the full discussion.

In the next corollary, the Archimedean property is not assumed. Statement (iii) states that, under very mild topological assumptions, it is satisfied automatically. Statements (ii) and (iv) are efficient generalizations of Lemma 1.2 and Proposition 4.8 in [KMT17].

Corollary 2.17. *Suppose τ is both locally solid and Hausdorff and A is an order dense ideal of (X,τ), then:*

(i) The positive cone X_+ is $u_A\tau$-closed;
(ii) If $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{u_A\tau} x$, then $x_\alpha \uparrow x$;
(iii) X is Archimedean;
(iv) Every band in X is $u_A\tau$-closed.

Proof. The result follows immediately from Theorem 8.43 in [AlB06]. □

From now on, unless explicitly stated otherwise, throughout this thesis the minimum assumption is that X is an Archimedean vector lattice and τ is a locally solid topology on X. The following straightforward result should be noted. It justifies the name *unbounded* τ-convergence.

Proposition 2.18. *Let A be an ideal of (X,τ). If $x_\alpha \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{u_A\tau} 0$. The converse holds for nets that are order bounded in A.*

From this we deduce that the $u_A\tau$-topology is canonical in the following sense:

Proposition 2.19. *Let A be an ideal of a locally solid vector lattice (X,τ). Then $u_A\tau$ is the coarsest locally solid topology on X that agrees with τ on the order intervals of A.*

Proof. By Proposition 2.18, $u_A\tau$ and τ agree on the order intervals of A. Suppose σ is locally solid and agrees with τ on the order intervals of A. Then $u_A\sigma$ does as well, but $u_A\tau = u_A\sigma \subseteq \sigma$. □

We next work towards a version of Proposition 3.15 in [GTX17] that is applicable to locally solid topologies. The proposition is recalled here along with a definition.

Proposition 2.20. *Let X be a vector lattice, and Y a sublattice of X. Then Y is uo-closed in X if and only if it is o-closed in X.*

Definition 2.21. *A locally solid topology τ on a vector lattice is said to be* ***Lebesgue*** *(or* ***order continuous****) if $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\tau} 0$.*

Proposition 2.22. *Let X be a vector lattice, τ a Hausdorff Lebesgue topology on X, and Y a sublattice of X. Y is $u\tau$-closed in X if and only if it is τ-closed in X.*

Proof. If Y is $u\tau$-closed in X it is clearly τ-closed in X. Suppose now that Y is τ-closed in X and let (y_α) be a net in Y that $u\tau$-converges in X to some $x \in X$. Since lattice operations are $u\tau$-continuous we have that $y_\alpha^\pm \xrightarrow{u\tau} x^\pm$ in X. Thus, WLOG, we may assume that $(y_\alpha) \subseteq Y_+$ and $x \in X_+$. Observe that for every $z \in X_+$,

$$|y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z \xrightarrow{\tau} 0. \tag{2.2}$$

In particular, for any $y \in Y_+$, $y_\alpha \wedge y \xrightarrow{\tau} x \wedge y$. Since Y is τ-closed, $x \wedge y \in Y$ for any $y \in Y_+$. On the other hand, given any $0 \leq z \in Y^d$, we have $y_\alpha \wedge z = 0$ for all α so that $x \wedge z = 0$ by (2.2) and the assumption that τ is Hausdorff. Therefore, $x \in Y^{dd}$, which is the band generated by Y in X. It follows that there is a net (z_β) in the ideal generated by Y such that $0 \leq z_\beta \uparrow x$ in X. Furthermore, for every β there exists $w_\beta \in Y$ such that $z_\beta \leq w_\beta$. Then $x \geq w_\beta \wedge x \geq z_\beta \wedge x = z_\beta \uparrow x$ in X, and so $w_\beta \wedge x \xrightarrow{o} x$ in X. By the Lebesgue property, $w_\beta \wedge x \xrightarrow{\tau} x$ in X. Since $w_\beta \wedge x \in Y$ and Y is τ-closed, we get $x \in Y$. ☐

Remark 2.23. We will use Proposition 2.22 in the next section to prove a much deeper statement: see Theorem 3.16.

Question 2.24. Can the Lebesgue assumption be removed in Proposition 2.22? Can we extend the results to sequentially closed sublattices (while avoiding an assumption like the countable sup property)?

3. A connection between unbounded topologies and the universal completion

The connection between unbounded topologies, minimal topologies, *uo*-convergence, and the universal completion is, in my opinion, the most interesting part of this thesis. Therefore, although we do not yet have the technical machinery to fully investigate these connections, I wish to briefly introduce the concepts in the next two sections. This study will be completed in Section 10.

Definition 3.1. *A locally solid topology τ on a vector lattice is said to be uo-**Lebesgue** (or **unbounded order continuous**) if $x_\alpha \xrightarrow{uo} 0$ implies $x_\alpha \xrightarrow{\tau} 0$.*

It is clear that the *uo*-Lebesgue property implies the Lebesgue property but not conversely:

Example 3.2. The norm topology of c_0 is order continuous but not unbounded order continuous.

We begin with a simple observation; the converse will be proved shortly:

Proposition 3.3. *If τ is Lebesgue then $u\tau$ is uo-Lebesgue. In particular, $u\tau$ is Lebesgue.*

Proof. Suppose $x_\alpha \xrightarrow{uo} x$, i.e., $\forall u \in X_+$, $|x_\alpha - x| \wedge u \xrightarrow{o} 0$. The Lebesgue property implies that $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ so that $x_\alpha \xrightarrow{u\tau} x$. □

Recall that a vector lattice is ***universally complete*** if it is both laterally complete and order complete. We refer the reader to [AB03] for a study of universally complete vector lattices and a proof of the existence of a (or rather *the*) universal completion of a vector lattice. For our purposes, we must recall Theorem 7.54 in [AB03]:

Theorem 3.4. *For a vector lattice X we have the following:*

(i) X can admit at most one Hausdorff Lebesgue topology that extends to its universal completion as a locally solid topology;
(ii) X admits a Hausdorff Lebesgue topology if and only if X^u does.

We can now add an eighth and nineth equivalence to Theorem 7.51 in [AB03]. For convenience of the reader, and since we will need nearly all these properties, we recall the entire theorem. Dominable sets will play a major role later in this thesis, and their definition can be found in [AB03]. Recall that a locally solid topology is ***σ-Fatou*** if it has a base $\mathcal{N}_0$ at zero consisting of solid sets with the property that $(x_n) \subseteq V \in \mathcal{N}_0$ and $0 \leq x_n \uparrow x$ implies $x \in V$.

Theorem 3.5. *For a Hausdorff locally solid vector lattice (X,τ) with the Lebesgue property the following statements are equivalent.*

(i) τ *extends to a Lebesgue topology on* X^u*;*
(ii) τ *extends to a locally solid topology on* X^u*;*
(iii) τ *is coarser than any Hausdorff* σ*-Fatou topology on* X*;*
(iv) Every dominable subset of X_+ *is* τ*-bounded;*
(v) Every disjoint sequence of X_+ *is* τ*-convergent to zero;*
(vi) Every disjoint sequence of X_+ *is* τ*-bounded;*
(vii) The topological completion $\widehat{X}$ *of* (X,τ) *is Riesz isomorphic to* X^u*, that is,* $\widehat{X}$ *is the universal completion of* X*;*
(viii) Every disjoint net of X_+ *is* τ*-convergent to zero;*[2]
(ix) τ *is unbounded.*

Proof. First we prove (v)⇔(viii). Clearly (viii)⇒(v). Assume (v) holds and suppose there exists a disjoint net (x_α) which is not τ-null. Then there exists a solid τ-neighbourhood V of zero such that for every α there exists $\beta > \alpha$ with $x_\beta \notin V$. Inductively, we find an increasing sequence (α_k) of indices such that $x_{\alpha_k} \notin V$. Hence the sequence (x_{α_k}) is disjoint but not τ-null.

We now prove that (v)⇔(ix). Assume τ is unbounded and is a Hausdorff Lebesgue topology. It follows that $\tau = u\tau$ is *uo*-Lebesgue and, therefore, disjoint sequences are τ-null.

Now assume (v) holds so that every disjoint sequence of X_+ is τ-convergent to zero. Since τ is Hausdorff and Lebesgue, so is $u\tau$. Since τ-convergence implies $u\tau$-convergence, every disjoint positive sequence is $u\tau$-convergent to zero so that, by (ii), $u\tau$ extends to a locally solid topology on X^u. We conclude that τ and $u\tau$ are both Hausdorff Lebesgue topologies that extend to X^u as locally solid topologies. By Theorem 3.4, $\tau = u\tau$. □

Theorem 3.5(vii) yields the following:

[2] In statements such as this we require that the index set of the net have no maximal elements. See page 9 of [AB03] for a further discussion on this minor issue.

Corollary 3.6. *Let τ be an unbounded Hausdorff Lebesgue topology on a vector lattice X. τ is complete if and only if X is universally complete.*

Remark 3.7. Compare this with [KMT17] Proposition 6.2 and [AB03] Theorem 7.47. It can also be deduced that if τ is a topologically complete unbounded Hausdorff Lebesgue topology then it is the only Hausdorff Fatou topology on X. See Theorem 7.53 of [AB03].

By Exercise 5 on page 72 of [AB03], if an unbounded Hausdorff Lebesgue topology τ is extended to a Lebesgue topology τ^u on X^u, then τ^u is also Hausdorff. By Theorem 7.53 of [AB03] it is the only Hausdorff Lebesgue (even Fatou) topology X^u can admit. It must therefore be unbounded. By the uniqueness of Hausdorff Lebesgue topologies on X^u we deduce uniqueness of unbounded Hausdorff Lebesgue topologies on X since these types of topologies always extend to X^u.

We summarize in a theorem:

Theorem 3.8. *Let X be a vector lattice. We have the following:*

(i) X admits at most one unbounded Hausdorff Lebesgue topology. (It admits an unbounded Hausdorff Lebesgue topology if and only if it admits a Hausdorff Lebesgue topology);

(ii) Let τ be a Hausdorff Lebesgue topology on X. τ is unbounded if and only if τ extends to a locally solid topology on X^u. In this case, the extension of τ to X^u can be chosen to be Hausdorff, Lebesgue and unbounded.

Example 3.9. Let X be an order continuous Banach lattice. Both the norm and *un* topologies are Hausdorff and Lebesgue. Since these topologies generally differ, it is clear that a space can admit more than one Hausdorff Lebesgue topology. Notice, however, that when X is order continuous, *un* is the same as *uaw*. The deep reason for this is that *un* and *uaw* are two unbounded Hausdorff Lebesgue topologies, so, by the theory just presented, they must coincide.

Recall that every Lebesgue topology is Fatou; this is Lemma 4.2 of [AB03]. Also, if τ is a Hausdorff Fatou topology on a universally complete vector lattice X then (X, τ) is τ-complete. This is Theorem 7.50 of [AB03] and can also be deduced from previous facts presented in this thesis.

Corollary 3.10. *Suppose (X, τ) is Hausdorff and Lebesgue. Then $u\tau$ extends to an unbounded Hausdorff Lebesgue topology $(u\tau)^u$ on X^u and $(X^u, (u\tau)^u)$ is topologically complete.*

Example 3.11. Recall by Theorem 6.4 of [AB03] that if X is a vector lattice and A an ideal of $X^\sim$ then the absolute weak topology $|\sigma|(A, X)$ is a Hausdorff Lebesgue topology on A. This means that the topology $u|\sigma|(A, X)$ is the unique unbounded Hausdorff Lebesgue topology on A. In particular, if X is a Banach lattice then $u|\sigma|(X^*, X)$ is the unique unbounded Hausdorff Lebesgue topology on X^* so, if X^* is (norm) order continuous, then $un = uaw = u|\sigma|(X^*, X)$ on X^*.

We next characterize the *uo*-Lebesgue property:

Theorem 3.12. *Let (X, τ) be a Hausdorff locally solid vector lattice. TFAE:*

(i) τ is uo-Lebesgue, i.e., $x_\alpha \xrightarrow{uo} 0 \Rightarrow x_\alpha \xrightarrow{\tau} 0$;
(ii) τ is Lebesgue and unbounded.

Proof. (ii)$\Rightarrow$(i) is known since if τ is Lebesgue then $u\tau$ is *uo*-Lebesgue and, therefore, since $\tau = u\tau$, τ is *uo*-Lebesgue.

Assume now that τ is *uo*-Lebesgue. Note first that this trivially implies that τ is Lebesgue. Assume now that (x_n) is a disjoint sequence in X_+. By known results, (x_n) is *uo*-null. Since τ is *uo*-Lebesgue, (x_n) is τ-null. Therefore, τ satisfies condition (v) of Theorem 3.5. It therefore also satisfies (ix) which means τ is unbounded. □

Question 3.13. Does the above theorem remain valid if *uo* is replaced by uo_1? In other words, is the *uo*-Lebesgue property independent of the definition of order convergence?

Remark 3.14. Example 13.3 demonstrates that sequentially *uo*-Lebesgue and sequentially uo_1-Lebesgue are different concepts.

Remark 3.15. Since *un*-convergence implies *uo*-convergence in $C(K)$-spaces and atomic Banach lattices (and hence in products of such spaces), it is unlikely one can classify those topologies τ in which τ-convergence implies *uo*-converge, even in Banach lattices.

There are two natural ways to incorporate unboundedness into the literature on locally solid vector lattices. The first is to take some property relating order convergence to topology and then make the additional assumption that the topology is unbounded. The other is to take said property and replace order convergence with *uo*-convergence. For the Lebesgue property, these approaches are equivalent: τ is unbounded and Lebesgue iff it is *uo*-Lebesgue (with the overlying assumption τ is Hausdorff). Later on we will study the Fatou property and see that these approaches differ.

We can now strengthen Proposition 2.22. Compare this with Theorem 4.20 and 4.22 in [AB03]. The latter theorem says that all Hausdorff Lebesgue topologies induce the same topology on order bounded subsets. It will now be shown that, furthermore, all Hausdorff Lebesgue topologies have the same topologically closed sublattices.

Theorem 3.16. *Let τ and σ be Hausdorff Lebesgue topologies on a vector lattice X and let Y be a sublattice of X. Then Y is τ-closed in X if and only if it σ-closed in X.*

Proof. By Proposition 2.22, Y is τ-closed in X if and only if it is $u\tau$-closed in X and it is σ-closed in X if and only if it is $u\sigma$-closed in X. By Theorem 3.8, $u\tau = u\sigma$, so Y is τ-closed in X if and only if it σ-closed in X. □

Next we present a partial answer to the question of whether unbounding and passing to the order completion is the same as passing to the order completion and then unbounding. First, a proposition:

Proposition 3.17. *Let (X, τ) be a Hausdorff locally solid vector lattice with the uo-Lebesgue property. Then Y is a regular sublattice of X iff $\tau|_Y$ is a Hausdorff uo-Lebesgue topology on Y.*

Proof. The reader should convince themselves that the subspace topology defines a Hausdorff locally solid topology on Y; we will prove that $\tau|_Y$ is *uo*-Lebesgue when Y is regular. Suppose (y_α) is a net in Y and $y_\alpha \xrightarrow{uo} 0$ in Y. Since Y is regular, $y_\alpha \xrightarrow{uo} 0$ in X, so that $y_\alpha \xrightarrow{\tau} 0$ as τ is *uo*-Lebesgue. This is equivalent to $y_\alpha \xrightarrow{\tau|_Y} 0$.

For the converse, assume $\tau|_Y$ is Hausdorff and *uo*-Lebesgue, and $y_\alpha \downarrow 0$ in Y. Then $y_\alpha \xrightarrow{\tau|_Y} 0$, hence $y_\alpha \xrightarrow{\tau} 0$. Since (y_α) is decreasing in X, $y_\alpha \downarrow 0$ in X be [AB03] Theorem 2.21. This proves that Y is regular in X. ☐

Let σ be a Hausdorff Lebesgue topology on a vector lattice X. By Theorem 4.12 in [AB03] there is a unique Hausdorff Lebesgue topology σ^δ on X^δ that extends σ. We have the following:

Lemma 3.18. *For any Hausdorff Lebesgue topology τ on a vector lattice X, $u(\tau^\delta) = (u\tau)^\delta = (u\tau)^u|_{X^\delta}$.*

Proof. As stated, τ extends uniquely to a Hausdorff Lebesgue topology τ^δ on X^δ. $u(\tau^\delta)$ is thus a Hausdorff *uo*-Lebesgue topology on X^δ.

Alternatively, since τ is a Hausdorff Lebesgue topology, $u\tau$ is a Hausdorff *uo*-Lebesgue topology. This topology extends uniquely to a Hausdorff Lebesgue topology $(u\tau)^\delta$ on X^δ. It suffices to prove that $(u\tau)^\delta$ is *uo*-Lebesgue since then, by uniqueness of such topologies, it must equal $u(\tau^\delta)$.

Since $u\tau$ is Hausdorff and *uo*-Lebesgue, it also extends to a Hausdorff *uo*-Lebesgue topology $(u\tau)^u$ on X^u. By page 187 of [AB03], the universal completions of X and X^δ coincide so we can restrict $(u\tau)^u$ to X^δ. By Proposition 3.17, this gives a Hausdorff *uo*-Lebesgue topology, $(u\tau)^u|_{X^\delta}$, on X^δ that extends $u\tau$. By uniqueness of Hausdorff Lebesgue extensions to X^δ, $(u\tau)^u|_{X^\delta} = (u\tau)^\delta$ and so $(u\tau)^\delta$ is *uo*-Lebesgue. ☐

In particular, if τ is also unbounded, so that τ is a Hausdorff *uo*-Lebesgue topology, then $u(\tau^\delta) = \tau^\delta$. This means we can also include *uo*-Lebesgue topologies in [AB03] Theorem 4.12:

Corollary 3.19. *Let τ be a Hausdorff Lebesgue topology on a vector lattice X. Then τ is uo-Lebesgue iff τ^δ is uo-Lebesgue.*

In Proposition 6.26 we will show that $u(\tau^\delta) = (u\tau)^\delta$ still holds if Lebesgue is weakened to Fatou. Note that by [AB03, Exercise 8, page 73], every locally solid topology on X extends to a locally solid topology on X^δ. However, we do not have any necessary criterion for such an extension to preserve unboundedness, as our arguments make use of the uniqueness clause in [AB03, Theorem 4.12]. For the extension τ^* constructed in the solution to [AB03, Exercise 2.8] it is clear that $u(\tau^*) \subseteq (u\tau)^*$.

Remark 3.20. The process of extending a Hausdorff *uo*-Lebesgue topology to the universal completion is quite simple. First, extend τ to the *uo*-Lebesgue topology τ^δ on X^δ described in Corollary 3.19 and note that X^δ is an order dense ideal of X^u. The extension τ^u of τ to X^u can be described as follows: for a net (x_α) in X^u, $x_\alpha \xrightarrow{\tau^u} 0$ iff $|x_\alpha| \wedge u \xrightarrow{\tau^\delta} 0$ for all $u \in X^\delta_+$. In other words, (keeping in mind the small notational inconsistency mentioned in Example 2.9) $\tau^u = u_{X^\delta}\tau^\delta$.

The same logic shows that every unbounded σ-Lebesgue topology on an order complete vector lattice extends to a σ-Lebesgue topology on X^u, and this has some major consequences.

In the papers [Lux67], [Frem75] and [BL88] it is shown that certain natural vector lattice questions are logically equivalent to the statement "every cardinal is nonmeasurable" [3]. Here we reformulate [AB78, Open problem 7.1]; this question was "solved" in [BL88].

Proposition 3.21. *The following statements are logically equivalent:*

(i) Every Hausdorff locally solid topology on a universally complete vector lattice is Lebesgue;

(ii) Every Hausdorff σ-Lebesgue topology on an order complete vector lattice is Lebesgue.

Proof. (i)⇒(ii): Suppose τ is a Hausdorff σ-Lebesgue topology on an order complete vector lattice X. Then $u\tau$ has the same properties and,

[3] I am being deliberately vague here because there are certain variants of measurable cardinals; see the papers for precise statements.

as mentioned, extends to a Hausdorff locally solid topology on X^u. This extension is also σ-Lebesgue, as one can easily check. By assumption, the extension is Lebesgue, and hence $u\tau$ is Lebesgue. It follows that τ is Lebesgue. (ii)⇒(i) follows from [AB03, Theorem 7.49]. □

As we see, although it is true that one can extend every Hausdorff locally solid topology to a Hausdorff locally solid topology on the order completion, the σ-Lebesgue property is generally lost during this process. In a way this shows that [AB03, Theorem 4.12] is "sharp".

3.1. **A brief introduction to minimal topologies.** In this section we will see that *uo*-convergence "knows" exactly which topologies are minimal. Much work has been done on minimal topologies and, unfortunately, the section in [AB03] is out-of-date both in terminology and sharpness of results. First we fix our definitions; they are inconsistent with [AB03].

Definition 3.22. *A Hausdorff locally solid topology τ on a vector lattice X is said to be* ***minimal*** *if it follows from $\tau_1 \subseteq \tau$ and τ_1 a Hausdorff locally solid topology that $\tau_1 = \tau$.*

Definition 3.23. *A Hausdorff locally solid topology τ on a vector lattice X is said to be* ***least*** *or, to be consistent with* [Con05], ***smallest****, if τ is coarser than any other Hausdorff locally solid topology σ on X, i.e., $\tau \subseteq \sigma$.*

A crucial result, not present in [AB03], is Proposition 6.1 of [Con05]:

Proposition 3.24. *A minimal topology is a Lebesgue topology.*

Proof. The proof is just simple modification of the arguments in [AB03, Theorem 7.67] which proves the result for least topologies. □

This allows us to prove the following result; the equivalence of (i) and (ii) has already been established by Theorem 3.12, but is collected here for convenience.

Theorem 3.25. *Let τ be a Hausdorff locally solid topology on a vector lattice X. TFAE:*

(i) τ is uo-Lebesgue;

(ii) τ *is Lebesgue and unbounded;*
(iii) τ *is minimal.*

Proof. Suppose τ is minimal. By the last proposition, it is Lebesgue. It is also unbounded since $u\tau$ is a Hausdorff locally solid topology and $u\tau \subseteq \tau$. Minimality forces $\tau = u\tau$.

Conversely, suppose that τ is Hausdorff, Lebesgue and unbounded and that $\sigma \subseteq \tau$ is a Hausdorff locally solid topology. It is clear that σ is then Lebesgue and hence σ-Fatou. By Theorem 3.5(iii), τ is coarser than σ. Therefore, $\tau = \sigma$ and τ is minimal. $\square$

In conjunction with Theorem 3.8, we deduce the (already known) fact that minimal topologies, if they exist, are unique. They exist if and only if X admits a Hausdorff Lebesgue topology. One can also deduce this as follows: Suppose τ and σ are minimal topologies. Then

$$x_\alpha \xrightarrow{\tau} 0 \Leftrightarrow |x_\alpha|\wedge u \xrightarrow{\tau} 0 \text{ for each } u \in X_+ \overset{Amemiya}{\Leftrightarrow} |x_\alpha|\wedge u \xrightarrow{\sigma} 0 \text{ for each } u \in X_+ \Leftrightarrow x_\alpha \xrightarrow{\sigma} 0.$$

Amemiya's theorem ([AB03, Theorem 4.22]) is one of the deepest results in vector lattice theory and the above trick demonstrates the power achieved one combines it with unboundedness. Actually, many of the results in the last two sections can be proved via Amemiya's theorem and such tricks alone.

We also get many generalizations of results on unbounded topologies in Banach lattices. The following is part of [AB03] Theorem 7.65, but is really just a trivial consequence of the fact that the norm topology is the finest locally solid topology a Banach lattice can admit:

Theorem 3.26. *If* X *is an order continuous Banach lattice then* X *has a least topology.*

The least topology on an order continuous Banach lattice is, simply, *un*. This proves that *un* is "special", and also that it has been implicitly studied before. If X is any Banach lattice then, since the topology $u|\sigma|(X^*, X)$ is Hausdorff and *uo*-Lebesgue, it also has a minimality property:

Lemma 3.27. *Let X be a Banach lattice. $u|\sigma|(X^*, X)$ is the (unique) minimal topology on X^*.*

Example 3.28. $L_\infty[0,1]$ admits a minimal topology but no least topology (see [AB03, Theorem 7.75]); by [AB03, Theorem 7.70] ℓ_p admits a least topology for $1 \leq p \leq \infty$.

For more examples of spaces that admit minimal and least topologies the reader is referred to [AB03] and [Con05] (but be wary that there are a few mistakes in the latter paper). Interestingly, the process of unbounding a topology can convert the finest topology into the least topology; this happens with the norm topology on an order continuous Banach lattice: see [AB03, Theorem 5.20 and Theorem 7.65].

4. Miscellaneous facts on unbounded topologies

We next proceed to generalize many results from the aforementioned papers on *un*-convergence. For the sake of simplicity, several results in this thesis are stated only for $u\tau$-convergence; we leave it to the reader to extend these results to general $u_A\tau$. For the most part, results about $u\tau$ extend trivially to $u_A\tau$, at least when A is order dense. However, in some cases (for example regarding topological completeness) this is not true, and we will mention some of the subtleties. Note that in the case τ is Hausdorff and Lebesgue and A is order dense, $u_A\tau = u\tau$ automatically holds (by minimality), so no additional verification is needed.

First we present Lemmas 2.1 and 2.2 of [KMT17] which carry over with minor modification. The proofs are similar and, therefore, omitted.

Lemma 4.1. *Let X be a vector lattice, $u \in X_+$ and U a solid subset of X. Then $U_u := \{x \in X : |x| \wedge u \in U\}$ is either contained in $[-u, u]$ or contains a non-trivial ideal. If U is, further, absorbing, and U_u is contained in $[-u, u]$, then u is a strong unit.*

Next we present a trivialized version of Theorem 2.3 in [KMT17].

Proposition 4.2. *Let (X, τ) be a locally solid vector lattice and suppose that τ has a neighbourhood U of zero containing no non-trivial ideal. If there is a $u\tau$-neighbourhood contained in U then X has a strong unit.*

Proof. Let $\mathcal{N}_0$ be a solid base at zero for τ and suppose there exists $U \in \mathcal{N}_0$ and $u > 0$ s.t. $U_u \subseteq U$. We conclude that U_u contains no non-trivial ideal and, therefore, u is a strong unit. $\square$

Corollary 4.3. *If τ is unbounded and X does not admit a strong unit then every neighbourhood of zero for τ contains a non-trivial ideal.*

Definition 4.4. *A subset A of a locally solid vector lattice (X, τ) is* ***τ-almost order bounded*** *if for every solid τ-neighbourhood U of zero there exists $u \in X_+$ with $A \subseteq [-u, u] + U$. In the literature one may search for this concept under the name* ***(quasi) order τ-precompact****.*

It is easily seen that for solid U, $x \in [-u, u] + U$ is equivalent to $(|x| - u)^+ \in U$. The proof is the same as the norm case. This leads to a generalization of Lemma 2.9 in [DOT17]; the proof is left to the reader.

Proposition 4.5. *If $x_\alpha \xrightarrow{u\tau} x$ and (x_α) is τ-almost order bounded then $x_\alpha \xrightarrow{\tau} x$.*

In a similar vein, the following can easily be proved; just follow the proof of Proposition 3.7 in [GX14] or notice it is an immediate corollary of Proposition 4.5.

Proposition 4.6. *Let (X, τ) be a locally solid vector lattice. TFAE:*

(i) τ is Lebesgue;
(ii) If (x_α) is τ-almost order bounded and uo-converges to x, then (x_α) τ-converges to x.

Remark 4.7. Let $u \in X_+$, (x_α) a net in X, and $x \in X$. Then, by a standard lattice identity, $|x_\alpha - x| = |x_\alpha - x| \wedge u + (|x_\alpha - x| - u)^+$. One can think of the unbounded τ-convergence as controlling the first term; if the net (or at least a tail) is τ-almost order bounded, then the second term is also controlled.

One direction of [DOT17] Theorem 4.4 can also be generalized. The proof is, again, easy and left to the reader. The (necessarily partial) converse is more difficult and will be presented in Corollary 11.9.

Proposition 4.8. *Let (x_n) be a sequence in (X,τ) and assume τ is Lebesgue. If every subsequence of (x_n) has a further subsequence which is uo-null then (x_n) is $u\tau$-null.*

Although [DOT17] provides an example to show that the Lebesgue property cannot be removed in Proposition 4.8, one can reformulate the proposition to see that the Lebesgue property is pretty much necessary. In the Banach lattice case it is completely necessary.

Proposition 4.9. *Suppose (X,τ) is an unbounded locally solid vector lattice. TFAE:*

(i) $x_n \xrightarrow{uo} 0 \Rightarrow x_n \xrightarrow{\tau} 0$*;*

(ii) (x_n) *is τ-null whenever every subsequence of (x_n) has a further subsequence that is uo-null.*

Recall that a net (x_α) in a vector lattice X is *uo-**Cauchy*** if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')}$ *uo*-converges to zero. X is *uo-**complete*** if every *uo*-Cauchy net is *uo*-convergent. A study of *uo*-complete spaces was undertaken in [CL17]. A weaker property involving norm boundedness was introduced in [GTX17]. Here is a generalization of both definitions to locally solid vector lattices.

Definition 4.10. *A locally solid vector lattice (X,τ) is **boundedly** uo-**complete** (respectively, **sequentially boundedly** uo-**complete**) if every τ-bounded uo-Cauchy net (respectively, sequence) is uo-convergent.*

Proposition 4.11. *Let (X,τ) be a locally solid vector lattice. If (X,τ) is boundedly uo-complete then it is order complete. If (X,τ) is sequentially boundedly uo-complete then it is σ-order complete.*

Proof. Let (x_α) be a net in X such that $0 \leq x_\alpha \uparrow\leq x$ for some $x \in X$. By [AB03] Theorem 2.19, (x_α) is τ-bounded. By [CL17] Lemma 2.1, (x_α) is order Cauchy and hence *uo*-Cauchy. By the assumption that (X,τ) is boundedly *uo*-complete and the order boundedness of (x_α), we conclude that $x_\alpha \xrightarrow{o} y$ for some $y \in X$. Since (x_α) is increasing, $y = \sup x_\alpha$. The sequential argument is similar. □

Notice that a vector lattice X is *uo*-complete if and only if X equipped with the trivial topology (which is locally solid) is boundedly *uo*-complete. Thus, this is a more general concept than both *uo*-complete vector lattices and boundedly *uo*-complete Banach lattices. Notice also that the order completeness assumption in [CL17] Proposition 2.8 may now be dropped:

Corollary 4.12. *Let X be a vector lattice. If X is uo-complete then it is universally complete. Conversely, if X is universally complete, and, in addition, has the countable sup property, then it is uo-complete.*

Here, of course, a vector lattice has the ***countable sup property (CSP)*** if for every net (x_α) of X, $x_\alpha \downarrow 0$ implies there exists an increasing sequence of indices α_n with $x_{\alpha_n} \downarrow 0$.

Remark 4.13. Whether the countable sup property can be dropped was an important open question in [CL17]. Upon searching the literature, however, I realized that the answer can be deduced from a paper on *uo*-convergence from the 1960s. The details follow:

Theorem 4.14. *Let X be a vector lattice. TFAE:*

(i) X is uo-complete;
(ii) Positive increasing uo-Cauchy nets have supremum;
(iii) X is universally complete.

Proof. (i)⇒(ii) is trivial, and (ii)⇒(iii) since the argument for lateral completeness in [CL17] only deals with positive increasing nets, as does the argument for order completeness in Proposition 4.11. To prove (iii)⇒(i) note that the *uo*-completion constructed in [Pap64] constitutes an order dense extension of X, and universally complete vector lattices admit no proper order dense extension by [AB03, Theorem 7.15]. □

Corollary 4.15. *Let (x_α) be a uo-Cauchy net in a vector lattice X. Then (x_α) has a uo-limit in X^u. In other words, the uo-completion is just the universal completion.*

Remark 4.16. [Pap64, Lemma 8.17] is used at the end of the construction of the *uo*-completion, and is quite interesting in its own right since *uo*-convergence is intrinsically very non-monotone yet the lemma shows that in some cases one can connect *uo* with monotone objects.

A locally solid vector lattice (X, τ) is said to satisfy the ***Levi property***[4] if every increasing τ-bounded net of X_+ has a supremum in X. The Levi and Fatou properties together are enough to ensure that a space is boundedly *uo*-complete. The formal statement is Theorem 6.30. Recall that a locally solid topology is ***Fatou*** if it has a base at zero consisting of solid order closed sets. Although it is not obvious by definition, the Fatou property is independent of the definition of order convergence. This can be easily deduced as an argument similar to that of Lemma 1.15 in [AB03] shows that a solid set is o-closed if and only if it is o_1-closed. In fact, more is true: see Theorem 18.24.

Remark 4.17. It should be noted that if τ is Hausdorff and locally solid then every τ-convergent *uo*-Cauchy net *uo*-converges to its τ-limit. This follows since lattice operations are τ-continuous and the positive cone is τ-closed: see [AB03] Theorem 2.21. If the assumption that τ is locally solid is dropped, then it is possible that a net converges topologically and *uo*, but to different limits: see [Gao14, Lemma 3.3]. The following is a slight generalization of Proposition 4.2 in [GX14]. The proof is similar but is provided for convenience of the reader.

Proposition 4.18. *Suppose that τ is a complete Hausdorff Lebesgue topology on a vector lattice X. If (x_α) is a τ-almost order bounded uo-Cauchy net in X then (x_α) converges uo and τ to the same limit.*

Proof. Suppose (x_α) is τ-almost order bounded and *uo*-Cauchy. Then the net $(x_\alpha - x_{\alpha'})$ is τ-almost order bounded and is *uo*-convergent to zero. By Proposition 4.6, $(x_\alpha - x_{\alpha'})$ is τ-null. It follows that (x_α) is τ-Cauchy and thus τ-convergent to some $x \in X$ since τ is complete. By Remark 4.17, (x_α) *uo*-converges to x. □

[4] This property generalizes the concept of monotonically complete Banach lattices appearing in [MN91].

5. Products, quotients, sublattices, and completions

In this section we ask how well unboundedness is preserved under taking products, quotients, sublattices, and topological completions.

5.1. **Products.** Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \mathcal{A}}$ be a family of locally solid vector lattices and let $X = \prod X_\alpha$ be the Cartesian product, ordered componentwise, and equipped with the product topology $\prod \tau_\alpha$. It is known that X has the structure of a locally solid vector lattice. See [AB03] pages 8 and 56 for details.

Theorem 5.1. *Let $\{(X_\alpha, \tau_\alpha)\}$ be a family of locally solid vector lattices, and for each α let A_α be an ideal of X_α. Then $(\prod X_\alpha, u_{\prod A_\alpha} \prod \tau_\alpha) = (\prod X_\alpha, \prod u_{A_\alpha} \tau_\alpha)$.*

Proof. First note that $\prod A_\alpha$ is an ideal of $\prod X_\alpha$, so that the notation is well-defined. Let $(x_\beta)_\beta$ be a net in $\prod X_\alpha$. Then:

$x_\beta \xrightarrow{u_{\prod A_\alpha} \prod \tau_\alpha} 0$ means that for each $u \in (\prod A_\alpha)_+ = \prod (A_\alpha)_+$, $|x_\beta| \wedge u \xrightarrow{\prod \tau_\alpha} 0$. However, the product topology is just componentwise convergence, and the lattice operations are componentwise, so this is the same as $|x_\beta^\alpha| \wedge u^\alpha \xrightarrow{\tau_\alpha} 0$ in β for each α, where the $\cdot^\alpha$ notation denotes components. Note that $u^\alpha \in A_\alpha$ for all α.

Similarly, $x_\beta \xrightarrow{\prod u_{A_\alpha} \tau_\alpha} 0$ means that $x_\beta^\alpha \xrightarrow{u_{A_\alpha} \tau_\alpha} 0$ for each α. It is now easy to see that these two convergences coincide. □

Corollary 5.2. *An arbitrary product of unbounded topologies is unbounded.*

5.2. **Quotients.** For notational purposes, we now recall Theorem 2.24 in [AB03]:

Theorem 5.3. *Let (X, τ) be a locally solid vector lattice and let Q be a lattice homomorphism from X onto a vector lattice Y. Then Y equipped with the quotient topology τ_Q is a locally solid vector lattice. Moreover, if $\mathcal{N}_0$ is a solid base at zero for τ then $\{Q(U) : U \in \mathcal{N}_0\}$ is a solid base at zero for τ_Q.*

Remark 5.4. The theorem tells us that when I is an ideal of X, $(X/I, \tau/I)$ is a locally solid vector lattice. It is possible that X/I fails to be Archimedean, but we are only interested in the case when it is.

We next investigate how $u(\tau_Q)$ and $(u\tau)_Q$ relate and prove that one inclusion holds in general.

Lemma 5.5. *With the notation as in Theorem* 5.3*, for each* $U \in \mathcal{N}_0$*, and* $u \in X_+$*,* $Q(U_u) \subseteq Q(U)_{Q(u)}$*. In particular,* $u(\tau_Q) \subseteq (u\tau)_Q$ *as topologies on* Y*.*

Proof. We begin by listing the relevant bases:

- Solid base of τ at zero: $\{U_i\}_{i\in I} = \mathcal{N}_0$;
- Solid base of $u\tau$ at zero: $\{U_{i,u}\}_{i\in I, u\in X_+}$ where $U_{i,u} := \{x \in X : |x| \wedge u \in U_i\}$;
- Solid base of $(u\tau)_Q$ at zero: $\{Q(U_{i,u})\}_{i\in I,u\in X_+}$ where $Q(U_{i,u}) := \{Qx : x \in U_{i,u}\}$;
- Solid base of $u\tau_Q$ at zero: $\{Q(U_i)_z\}_{i\in I,z\in Y_+}$ where $Q(U_i)_z := \{y \in Y : |y| \wedge z \in Q(U_i)\}$.

Let $Q(U_i)_z$ with $i \in I$ and $z \in Y_+$ be a fixed solid base neighbourhood of zero in the $u\tau_Q$ topology. We must find a $j \in I$ and $u \in X_+$ such that $Q(U_{j,u}) \subseteq Q(U_i)_z$. Take $j = i$ and u any vector such that $Q(u) = z$. u exists because Q is surjective and, WLOG, u is positive since, if not, replace u with $|u|$ and notice that $Q(|u|) = |Q(u)| = |z| = z$. The claim is that $Q(U_{i,u}) \subseteq Q(U_i)_z$.

Let $x \in Q(U_{i,u})$; then $x = Qw$ where $|w| \wedge u \in U_i$. We wish to show $x \in Q(U_i)_z$ so we need only verify that $|x| \wedge z \in Q(U_i)$. This follows since

$$|x| \wedge z = |Qw| \wedge Qu = Q(|w| \wedge u) \in Q(U_i) \tag{5.1}$$

and that completes the proof. □

When the converse inclusion holds is open:

Question 5.6. Suppose τ is Hausdorff and unbounded, and A is a τ-closed ideal of X. Is τ/A unbounded? What if τ is minimal (or has some other nice property)?

uo-convergence is also not well studied in quotients and tensor products. Knowing how *uo* passes to tensor products would be useful in the study of bibases.

One could prove that the quotient of a minimal topology by a band is minimal if one could lift disjoint sequences from the quotient. Lattices with this property are studied in [M70]. Note that by [AB03, Theorem 3.10], the quotient of a Lebesgue topology is Lebesgue.

We finish with one simple result on *uo* in the quotient. It would be nice to have a more developed theory.

Proposition 5.7. *Let $\pi : X \to Y$ be an onto Riesz homomorphism. Then π has the property that $x_\alpha \xrightarrow{uo} 0$ in X implies $\pi(x_\alpha) \xrightarrow{uo} 0$ in Y iff its kernel $K_\pi = \{x \in X : \pi(x) = 0\}$ is a band of X.*

Proof. First, it is routine to check that the kernel K_π of π is an ideal of X. Suppose π preserves *uo*-convergence and that (x_α) is a net in K_π satisfying $0 \leq x_\alpha \uparrow x$ in X. Then it follows that $0 = \pi(x_\alpha) \xrightarrow{uo} \pi(x)$, so that $x \in K_\pi$. This proves that K_π is a band.

For the converse, assume that K_π is a band. By [AB03, Lemma 1.32], π is normal, from which it follows that it preserves order convergence. Suppose $x_\alpha \xrightarrow{uo} 0$ in X, and let $y \in Y_+$. Find $x \in X$ with $\pi(x) = y$; replace x with $|x|$ so that x is positive. Then $|x_\alpha| \wedge x \xrightarrow{o} 0$ in X, so that $|\pi(x_\alpha)| \wedge y \xrightarrow{o} 0$ in Y. This proves that $\pi(x_\alpha) \xrightarrow{uo} 0$ in Y. □

In general we will find that when looking for sequential analogues of our net results, we may have to choose between *uo* or uo_1 to get the proofs to go through. For example, the proof of Proposition 5.7 easily goes through if one replaces nets with sequences, bands with σ-ideals, and uses uo_1-convergence. It is not as clear whether the sequential result remains valid if one uses *uo*-convergence (but there is a good chance one can decide this after looking at the paper [AS05] in greater depth.) Although we prefer *uo* over uo_1, it is not strictly better, and in the appendix we discuss these issues in greater detail.

5.3. **Sublattices.** Let Y be a sublattice of a locally solid vector lattice (X, τ). The reader should convince themselves that Y, equipped with the subspace topology, $\tau|_Y$, is a locally solid vector lattice in its own right. It would be natural to now compare $u(\tau|_Y)$ and $(u\tau)|_Y$, but this was already implicitly done in [KMT17]. In general, $u(\tau|_Y) \subsetneq (u\tau)|_Y$, even if Y is a band. If (y_α) is a net in Y we will write $y_\alpha \xrightarrow{u\tau} 0$ in Y to mean $y_\alpha \to 0$ in $(Y, u(\tau|_Y))$. We now look for conditions that make all convergences agree.

Lemma 5.8. *Let Y be a sublattice of a locally solid vector lattice (X,τ) and (y_α) a net in Y such that $y_\alpha \xrightarrow{u\tau} 0$ in Y. Each of the following conditions implies that $y_\alpha \xrightarrow{u\tau} 0$ in X.*

(i) Y is majorizing in X;
(ii) Y is τ-dense in X;
(iii) Y is a projection band in X.

Proof. WLOG, $y_\alpha \geq 0$ for every α. (i) gives no trouble. To prove (ii), take $u \in X_+$ and fix solid τ-neighbourhoods U and V of zero (in X) with $V + V \subseteq U$. Since Y is dense in X we can find a $v \in Y$ with $v - u \in V$. WLOG, $v \in Y_+$ since V is solid and $||v| - u| = ||v| - |u|| \leq |v - u| \in V$. By assumption, $y_\alpha \wedge v \xrightarrow{\tau} 0$ so we can find α_0 such that $y_\alpha \wedge v \in V$ whenever $\alpha \geq \alpha_0$. It follows from $u \leq v + |u - v|$ that $y_\alpha \wedge u \leq y_\alpha \wedge v + |u - v|$. This implies that $y_\alpha \wedge u \in U$ for all $\alpha \geq \alpha_0$ since

$$0 \leq y_\alpha \wedge u \leq y_\alpha \wedge v + |u - v| \in V + V \subseteq U \tag{5.2}$$

where, again, we used that U and V are solid. This means that $y_\alpha \wedge u \xrightarrow{\tau} 0$. Hence, $y_\alpha \xrightarrow{u\tau} 0$ in X.

To prove (iii), let $u \in X_+$. Then $u = v + w$ for some positive $v \in Y$ and $w \in Y^d$. It follows from $y_\alpha \perp w$ that $y_\alpha \wedge u = y_\alpha \wedge v \xrightarrow{\tau} 0$. □

Even though the Hausdorff Lebesgue property only passes to regular sublattices, unboundedness of a Lebesgue topology passes to all sublattices.

Lemma 5.9. *Suppose Y is a sublattice of a vector lattice X. If τ is a Hausdorff Lebesgue topology on X then $u(\tau|_Y) = (u\tau)|_Y$.*

Proof. It is clear that $u(\tau|_Y) \subseteq (u\tau)|_Y$.

Suppose (y_α) is a net in Y and $y_\alpha \xrightarrow{u(\tau|_Y)} 0$. Since Y is majorizing in $I(Y)$, the ideal generated by Y in X, $y_\alpha \xrightarrow{u_{I(Y)}\tau} 0$. By Theorem 1.36 of [AB06], $I(Y) \oplus I(Y)^d$ is an order dense ideal in X. Let $v \in (I(Y) \oplus I(Y)^d)_+$. Then $v = a + b$ where $a \in I(Y)$ and $b \in I(Y)^d$. Notice $|y_\alpha| \wedge v \leq |y_\alpha| \wedge |a| + |y_\alpha| \wedge |b| = |y_\alpha| \wedge |a| \xrightarrow{\tau} 0$. This proves that $y_\alpha \xrightarrow{u_{I(Y)\oplus I(Y)^d}\tau} 0$. We conclude that $(u_{I(Y)\oplus I(Y)^d}\tau)|_Y \subseteq u(\tau|_Y)$. Since the other inclusion is obvious, $(u_{I(Y)\oplus I(Y)^d}\tau)|_Y = u(\tau|_Y)$.

Since $I(Y) \oplus I(Y)^d$ is order dense in X, $u_{I(Y)\oplus I(Y)^d}\tau$ is a Hausdorff locally solid topology on X. Clearly, $u_{I(Y)\oplus I(Y)^d}\tau \subseteq u\tau$ so, since $u\tau$ is a Hausdorff *uo*-Lebesgue topology and hence minimal, $u_{I(Y)\oplus I(Y)^d}\tau = u\tau$. This proves the claim. ☐

Let X be a vector lattice, τ a locally solid topology on X, and X^δ the order completion of X. It is known that one can find a locally solid topology, say, τ^*, on X^δ that extends τ. See Exercise 8 on page 73 of [AB03] for details on how to construct such an extension. Since X is majorizing in X^δ, Lemma 5.8 gives the following.

Corollary 5.10. *If (X, τ) is a locally solid vector lattice and (x_α) is a net in X then $x_\alpha \xrightarrow{u\tau} 0$ in X if and only if $x_\alpha \xrightarrow{u(\tau^*)} 0$ in X^δ (i.e., $(u\tau^*)|_X = u\tau$). Here τ^* denotes a locally solid extension of τ to X^δ.*

5.4. **Completions.** We also get the following:

Corollary 5.11. *Suppose (X, τ) is a Hausdorff locally solid vector lattice. Then $(u\widehat{\tau})|_X = u\tau$. Here $\widehat{\tau}$ denotes the topological completion of (X, τ). In particular, if $\widehat{\tau}$ is unbounded then so is τ.*

Question 5.12. Suppose τ is Hausdorff and unbounded. Is $\widehat{\tau}$ unbounded? What if τ has the Fatou property, the σ-Lebesgue property, or the pre-Lebesgue property (it is true if τ has the Lebesgue property)?

Topological completions of Hausdorff locally solid topologies are rather fascinating. The map $\tau \mapsto \widehat{\tau}$ is determined purely by the theory of topological vector spaces, but, miraculously, highly respects the vector lattice structure. Indeed, by [AB03, Theorem 2.40] the topological completion of a Hausdorff locally solid vector lattice is naturally a

Hausdorff locally solid vector lattice in its own right. Moreover, several properties of τ transfer to $\widehat{\tau}$. Many of the questions in [AB78] - which I believe are still open - ask whether sequential properties are preserved under topological completion.

For our purposes we want to know how unboundedness and topological completions interact. Note that, by [AB03, Theorem 2.46 and Exercise 2.11], a Hausdorff locally solid vector lattice (X,τ) is (sequentially) complete iff order intervals are (sequentially) complete and τ has (σ)-MCP. We first focus on completeness of order intervals. By [AB03, Theorems 2.41 and 2.42], completeness of the order intervals determines how "well" X sits in $\widehat{X}$. Now, clearly, τ has complete order intervals iff $u\tau$ does, so that X sits as well in $\widehat{(X,\tau)}$ as it does in $\widehat{(X,u\tau)}$.

If A is an order dense ideal then $u_A\tau$ possessing complete order intervals implies that $(A,\tau|_A)$ has complete order intervals (but I don't know if it implies more than this; for example, does it imply that τ has complete order intervals?[5]) However, the next example shows that τ being complete does not guarantee that $u_A\tau$ even have sequentially complete order intervals:

Example 5.13. Let $X = C[0,1]$ equipped with the norm topology and A the (norm closed) order dense ideal of functions vanishing at $\frac{1}{2}$. We claim that u_An does not have sequentially complete order intervals. This can be witnessed by the following sequence:

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2} - \frac{1}{2n}] \\ nx + \frac{1}{2} - \frac{n}{2} & x \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ 1 & x \in [\frac{1}{2} + \frac{1}{2n}, 1] \end{cases}$$

Proposition 5.14. *Let τ be a Hausdorff locally solid topology on X. If $u\tau$ satisfies MCP then so does τ. If $u\tau$ satisfies σ-MCP then so does τ.*

---

[5]There is no issue with A being τ-closed since A is an ideal and we are working on order intervals, which are always closed for Hausdorff locally solid topologies.

Proof. Suppose $0 \leq x_\alpha \uparrow$ is a τ-Cauchy net. It is then $u\tau$-Cauchy and hence $u\tau$-converges to some $x \in X$. Therefore, $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{\tau} x$. Replacing nets with sequences yields the σ-analogue. □

Remark 5.15. Therefore, since τ-convergence agrees with $u\tau$-convergence on order intervals, $u\tau$ being (sequentially) complete implies τ is (sequentially) complete.

Unbounding the norm topology in any infinite dimensional order continous Banach lattice shows that $u\tau$ need not even have σ-MCP when τ is complete. It is the inverse image of the map $\tau \mapsto u_A\tau$ that is less well-understood when it comes to completeness properties. As mentioned, it is known that $\widehat{\tau}$ being unbounded implies τ is, but it is not known whether unboundedness is preserved under the map $\tau \mapsto \widehat{\tau}$. Even less is known about how the quotient map preserves unboundedness/uo-convergence.

Question 5.16. Let A be an order dense ideal of a Hausdorff locally solid vector lattice (X, τ). If $u_A\tau$ has complete order intervals, does τ? Can Proposition 5.14 be extended to $u_A\tau$? What about Remark 5.15? In the spirit of [AB03, Theorem 2.41] one could ask to what extent X being regular in $\widehat{(X,\tau)}$ relates to X being regular in $\widehat{(X, u_A\tau)}$, or to what extent $\widehat{(X,\tau)}$ or $\widehat{(X, u_A\tau)}$ being "nice" implies the other is.

6. The map $\tau \mapsto u_A\tau$

Suppose that τ has some property. Does $u\tau$ or, more generally, $u_A\tau$ have the same property? Similarly, if $u_A\tau$ has some property does that imply τ does? This question will be investigated in the next few subsections.

6.1. **Pre-Lebesgue property and disjoint sequences.** Recall the following definition from page 75 of [AB03]:

Definition 6.1. *Let (X, τ) be a locally solid vector lattice. We say that (X, τ) satisfies the* ***pre-Lebesgue property*** *(or that τ is a* ***pre-Lebesgue topology****), if $0 \leq x_n \uparrow \leq x$ in X implies that (x_n) is a τ-Cauchy sequence.*

Recall that Theorem 3.23 of [AB03] states that in an Archimedean locally solid vector lattice the Lebesgue property implies the pre-Lebesgue property. It is also known that in a topologically complete Hausdorff locally solid vector lattice that the Lebesgue property is equivalent to the pre-Lebesgue property and these spaces are always order complete. This is Theorem 3.24 of [AB03]. The next theorem tells us exactly when disjoint sequences are $u\tau$-null. Parts (i)-(iv) are Theorem 3.22 of [AB03], (v) and (vi) are new.

Theorem 6.2. *For a locally solid vector lattice (X,τ) TFAE:*

(i) (X,τ) satisfies the pre-Lebesgue property;
(ii) If $0 \leq x_\alpha \uparrow\leq x$ holds in X, then (x_α) is a τ-Cauchy net of X;
(iii) Every order bounded disjoint sequence of X is τ-convergent to zero;
(iv) Every order bounded k-disjoint sequence of X is τ-convergent to zero;
(v) Every disjoint sequence in X is $u\tau$-convergent to zero;
(vi) Every disjoint net in X is $u\tau$-convergent to zero.

Proof. (iii)⇒(v): Suppose (x_n) is a disjoint sequence. For every $u \in X_+$, $(|x_n| \wedge u)$ is order bounded and disjoint, so is τ-convergent to zero. This proves $x_n \xrightarrow{u\tau} 0$.

(v) ⇒ (iii): Let $(x_n) \subseteq [-u,u]$ be a disjoint order bounded sequence. By (v), $x_n \xrightarrow{u\tau} 0$ so, in particular, $|x_n| = |x_n| \wedge u \xrightarrow{\tau} 0$. This proves (x_n) is τ-null.

The proof of the net version is similar to Theorem 3.5. □

Since, for a Banach lattice, the norm topology is complete, the pre-Lebesgue property agrees with the Lebesgue property. This theorem can therefore be thought of as a generalization of Proposition 3.5 in [KMT17]. Theorem 6.2 has the following corollaries:

Corollary 6.3. *τ has the pre-Lebesgue property if and only if $u\tau$ does.*

Proof. τ and $u\tau$-convergences agree on order bounded sequences. Apply (iii). □

Corollary 6.4. *If τ is pre-Lebesgue and unbounded then every disjoint sequence of X is τ-convergent to 0.*

An important question, relating to the fact that every locally solid topology on a laterally σ-complete vector lattice sends disjoint sequences to zero ([AB03, Theorem 7.49]) is the following:

Question 6.5. Suppose τ is Hausdorff and disjoint sequences are τ-convergent to zero. Is τ unbounded? Is it at least true that τ and $u\tau$ agree on sequences? The latter property is strictly weaker by [KMT17, Example 1.3]. I also conjecture that the property that uo-convergence implies τ-convergence for sequences is equivalent to τ being σ-Lebesgue and disjoint sequences being τ-null. Theorem 13.13 nearly establishes this equivalence.

We next extend our results to $u_A\tau$:

Proposition 6.6. *Let A be an ideal of a locally solid vector lattice (X,τ). TFAE:*

(i) $(A,\tau|_A)$ satisfies the pre-Lebesgue property;
(ii) Every disjoint sequence in X is $u_A\tau$-null;
(iii) Every disjoint net in X is $u_A\tau$-null;
(iv) $(X,u_A\tau)$ satisfies the pre-Lebesgue property.

Proof. To prove that (i)⇒(ii) let (x_n) be a disjoint sequence in X. Then for every $a\in A_+$, $|x_n|\wedge a$ is an order bounded disjoint sequence in A and hence τ-converges to zero by Theorem 6.2. This proves $x_n\xrightarrow{u_A\tau}0$. An argument already presented in the proof of Theorem 6.2 (or rather Theorem 3.5) proves (ii)⇔(iii). (ii)⇒(iv) is obvious.

(iv)⇒(i): Suppose $u_A\tau$ is a pre-Lebesgue topology on X. We first show that $(u_A\tau)|_A$ is a pre-Lebesgue topology on A. We again use Theorem 6.2. Let (a_n) be a disjoint order bounded sequence in A. Then (a_n) is also a disjoint order bounded sequence in X and hence $a_n\xrightarrow{u_A\tau}0$. Thus $(u_A\tau)|_A$ satisfies (iii) of Theorem 6.2 and we conclude that $(u_A\tau)|_A$ is pre-Lebesgue. Next notice that $(A,(u_A\tau)|_A)=(A,u(\tau|_A))$, so $u(\tau|_A)$ has the pre-Lebesgue property. Finally, apply Corollary 6.3. □

Example 6.7. Let $A = C[0,1]^\delta$. Since A is uniformly complete and has a strong unit, $(A, \|\cdot\|_u)$ is lattice isometric to some $C(K)$-space with K compact and Hausdorff. Let τ be the restriction of pointwise convergence from $\mathbb{R}^K$ to A. Then τ is a Hausdorff locally solid topology that satisfies the pre-Lebesgue property. Now consider the convergence $\xrightarrow{u_A\tau}$ in Definition 2.2 with $X = A^u$, and recall that this convergence is well-defined even though τ is only defined on A. Note, however, that since $C[0,1]^u$ admits no Hausdorff locally solid topology, $\xrightarrow{u_A\tau}$ does not correspond to a locally solid topology on X. From this it easily follows that there are sequences $0 \leq x_n \downarrow$ in X that are not $u_A\tau$-Cauchy, so that $u_A\tau$ fails to be "pre-Lebesgue" (in quotations since we haven't formally defined pre-Lebesgue for non-linear topologies). Compare with Proposition 6.6.

Proposition 6.8. *Let A be an ideal of a Hausdorff locally solid vector lattice (X,τ). TFAE:*

(i) $\tau|_A$ is Lebesgue;
(ii) $u(\tau|_A)$ is Lebesgue;
(iii) $u(\tau|_A)$ is uo-Lebesgue;
(iv) $u_A\tau$ is uo-Lebesgue;
(v) $u_A\tau$ is Lebesgue.

Proof. The equivalence of (i), (ii) and (iii) has already been proven.

(i)⇒(iv): Suppose $x_\alpha \xrightarrow{uo} 0$ in X where (x_α) is a net in X. Fix $a \in A_+$. Then $|x_\alpha| \wedge a \xrightarrow{uo} 0$ in X and hence in A since A is an ideal. Since the net $(|x_\alpha| \wedge a)$ is order bounded in A, this is equivalent to $|x_\alpha| \wedge a \xrightarrow{o} 0$ in A. Since $\tau|_A$ is Lebesgue this means $|x_\alpha| \wedge a \xrightarrow{\tau} 0$. We conclude that $x_\alpha \xrightarrow{u_A\tau} 0$ and, therefore, $u_A\tau$ is uo-Lebesgue. (iv)⇒(v) is trivial. (v)⇒(ii) since the restriction of a Lebesgue topology to a regular sublattice is Lebesgue, and $(u_A\tau)|_A = u(\tau|_A)$. □

Example 6.9. The *un* topology on ℓ_∞ agrees with the norm topology. One can exhibit the minimal topology on ℓ_∞ (pointwise convergence) either as $u|\sigma|(\ell_\infty, \ell_1)$ or as the unbounded norm topology induced by the ideal c_0.

We finish this section with a routine generalization of [KMT17, Corollary 3.6]:

Proposition 6.10. *Suppose τ is pre-Lebesgue and locally bounded. If every $u\tau$-null sequence in X is τ-bounded then* $\dim X < \infty$.

Proof. If $\dim X = \infty$ then there exists a sequence (e_n) of non-zero disjoint elements in X. Let V be a bounded neighbourhood of zero for τ. It is easy to see that V contains no non-trivial subspace. Scaling e_n if necessary we may assume that $e_n \notin nV$. (e_n) is $u\tau$-null but not τ-bounded. □

Remark 6.11. Clearly, one cannot replace local boundedness with metrizability in Proposition 6.10.

6.2. **σ-Lebesgue topologies.** Recall that a locally solid topology τ is ***σ-Lebesgue*** if $x_n \downarrow 0 \Rightarrow x_n \xrightarrow{\tau} 0$ or, equivalently, $x_n \xrightarrow{o_1} 0 \Rightarrow x_n \xrightarrow{\tau} 0$. Example 3.25 in [AB03] shows that the σ-Lebesgue property does not imply the pre-Lebesgue property. It should be noted that an equivalent definition is not obtained if we replace o_1-convergence with o-convergence in the latter definition of the σ-Lebesgue property. If all o-convergent sequences are τ-convergent, then, since disjoint order bounded sequences are o-null, τ is both σ-Lebesgue and pre-Lebesgue. I do not know if the σ-Lebesgue and pre-Lebesgue properties together characterize those Hausdorff locally solid topologies in which o-null sequences are τ-null (compare with Question 6.5).

Note, in particular, that Theorem 1.5 fails for uo_1-convergence. In the appendix we show that Theorem 1.4 also fails for uo_1-convergence. These two theorems are why we prefer uo over uo_1: we want our convergence to be as weak as possible (while still maintaining unique limits) and to pass easily between sublattices.

Remark 6.12. We leave it as an exercise to the reader to study how the σ-Lebesgue property and the pseudo-properties of [AB03, Definition 5.26] interact with the map $\tau \mapsto u_A\tau$. One must be careful as the definition of order convergence plays a role.

Remark 6.13. Generally, the monograph [AB03] takes much care in showing that the results that are presented are sharp. However, I can't help but notice the assymetry between [AB03, Theorem 3.8 and Theorem 3.9]. This leads to the following question:

Question 6.14. Suppose τ is Hausdorff and locally solid. Does τ satisfy the σ-Lebesgue property iff every super order dense ideal is τ-dense? If so, then σ-Lebesgue and unbounded would be equivalent to $\tau = u_A\tau$ for every super order dense ideal A of X. Recall that a sublattice Y is ***super order dense*** in X if for each $0 < x \in X$ there exists a sequence (y_n) in Y satisfying $0 \leq y_n \uparrow x$ in X.

6.3. **Entire topologies.** Following [AB03] we define the class of entire topologies:

Definition 6.15. *Let (X, τ) be a locally solid vector lattice, and let $\mathcal{N}$ denote the collection of all normal sequences consisting of solid τ-neighbourhoods of zero. We define the* ***carrier*** *C_τ of the topology τ by*

$$C_\tau = \bigcup \{N^d : \exists \{V_n\} \in \mathcal{N} \text{ such that } N = \bigcap_{n=1}^{\infty} V_n\}.$$

τ is said to be ***entire*** *if its carrier C_τ is order dense in X.*

The property of being entire is preserved completely when passing to the unbounded topology.

Proposition 6.16. *Let A be an order dense ideal of a locally solid vector lattice (X, τ). Then τ is entire iff $u_A\tau$ is entire.*

Proof. Since $C_{u_A\tau} \subseteq C_\tau$, $u_A\tau$ is entire implies τ is. Suppose τ is entire and let $0 < x \in X$. Find a $y \in C_\tau$ such that $0 < y \leq x$. Since A is order dense we may assume WLOG that $y \in A$. Now, find a normal sequence $\{V_n\}$ of solid τ-neighbourhoods of zero such that $y \in N^d$ where $N = \bigcap_{n=1}^{\infty} V_n$. Consider now the $u_A\tau$-neighbourhoods $\{(V_n)_y\}$; we claim $y \in (\bigcap (V_n)_y)^d$. Suppose $z \in \bigcap (V_n)_y$; then $|z| \wedge y \in V_n$ for each n, so that $|z| \wedge y \in N$. Hence, $0 = y \wedge |z| \wedge y = |z| \wedge y$, proving the claim. Finally, it is easy to see that $\{(V_n)_y\}$ is normal, so that $y \in C_{u_A\tau}$. $\square$

6.4. **Fatou topologies.** Using the canonical base described in Theorem 2.7, it is trivial to verify that if τ has the (σ-)Fatou property then so does $u_A\tau$. The following is open:

Question 6.17. Suppose τ is Hausdorff and locally solid. If $u\tau$ is (σ-)Fatou does that imply that τ is (σ-)Fatou?

In analogy with the Fatou property, it is natural to consider topologies that have a base at zero consisting of solid uo-closed sets. Surprisingly, this does not lead to a new concept:

Lemma 6.18. *Let $A \subseteq X$ be a solid subset of a vector lattice X. A is (sequentially) o-closed if and only if it is (sequentially) uo-closed.*

Proof. If A is uo-closed then it is clearly o-closed.

Suppose A is o-closed, $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{uo} x$. We must prove $x \in A$. By the same computation as in Lemma 3.6 of [GX14],

$$(6.1) \quad \big||x_\alpha| \wedge |x| - |x| \wedge |x|\big| \leq \big||x_\alpha| - |x|\big| \wedge |x| \leq |x_\alpha - x| \wedge |x| \xrightarrow{o} 0.$$

Thus $|x_\alpha| \wedge |x| \xrightarrow{o} |x|$. Since A is solid, $(|x_\alpha| \wedge |x|) \subseteq A$, and since A is o-closed we conclude that $|x| \in A$. Finally, using the solidity of A again, we conclude that $x \in A$.

□

A simliar proof to Lemma 6.18 gives the following. Compare with [DOT17] Lemma 2.8.

Lemma 6.19. *If $x_\alpha \xrightarrow{u\tau} x$ then $|x_\alpha| \wedge |x| \xrightarrow{\tau} |x|$. In particular, τ and $u\tau$ have the same (sequentially) closed solid sets.*

This leads to the following elegant result:

Theorem 6.20. *Let τ and σ be Hausdorff Lebesgue topologies on a vector lattice X and let A be a solid subset of X. Then A is (sequentially) τ-closed if and only if it is (sequentially) σ-closed.*

Proof. Suppose A is τ-closed. By Lemma 6.19, A is $u\tau$-closed. Since X can admit only one unbounded Hausdorff Lebesgue topology, $u\sigma = u\tau$ and, therefore, A is $u\sigma$-closed. Since $u\sigma \subseteq \sigma$, A is σ-closed. □

Remark 6.21. Actually, a stronger version of the net part of Theorem 6.20 was noticed a few months earlier by N. Gao. He proved the following:

Theorem 6.22. *Assume τ is a Hausdorff Lebesgue topology on X and $A \subseteq X$ is solid. Then A is order closed iff A is τ-closed.*

Proof. Follows immediately by combining [AB03, Lemma 4.2 and Theorem 4.20]. □

Remark 6.23. It is well known that locally convex topologies consistent with a given dual pair have the same closed convex sets. Theorem 6.20 is a similar result for locally solid topologies. It makes one wonder if a similar improvement as Theorem 6.22 can be made to Theorem 3.16, or if one can get sequential variants of these theorems. An obstruction to a sequential variant of Theorem 6.22 is that, although order closed sets do not depend on the definition of order convergence, sequentially order closed sets do (see the appendix). The latter was noticed in [Pap64], but it seems to be open whether this issue occurs for "nice" sets (like σ-ideals for example).

We can also strengthen Lemma 3.6 in [GX14]. For properties and terminology involving Riesz seminorms, the reader is referred to [AB03].

Proposition 6.24. *Let X be a vector lattice and suppose ρ is a Riesz seminorm on X satisfying the Fatou property. Then $x_\alpha \xrightarrow{uo} x \Rightarrow \rho(x) \leq \liminf \rho(x_\alpha)$.*

Proof. First we prove the statement for order convergence. Assume $x_\alpha \xrightarrow{o} x$ and pick a dominating net $y_\beta \downarrow 0$. Fix β and find α_0 such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. Since

$$(|x| - y_\beta)^+ \leq |x_\alpha|,$$

we conclude that $\rho((|x|-y_\beta)^+) \leq \rho(x_\alpha)$. Since this holds for all $\alpha \geq \alpha_0$ we can conclude that $\rho((|x| - y_\beta)^+) \leq \liminf \rho(x_\alpha)$. Since ρ is Fatou and $0 \leq (|x| - y_\beta)^+ \uparrow |x|$ we conclude that $\rho((|x| - y_\beta)^+) \uparrow \rho(x)$ and so $\rho(x) \leq \liminf \rho(x_\alpha)$.

Now assume that $x_\alpha \xrightarrow{uo} x$. Then $|x_\alpha| \wedge |x| \xrightarrow{o} |x|$. Using the above result and properties of Riesz seminorms, $\rho(x) = \rho(|x|) \leq \liminf \rho(|x_\alpha| \wedge |x|) \leq \liminf \rho(|x_\alpha|) = \liminf \rho(x_\alpha)$. $\square$

A similar result holds for sequences if one uses uo_1-convergence. Again, the issue of definition of order convergence arises due to the fact that o_1 and o-convergence are both perfectly legitimate definitions of order convergence for sequences.

Example 6.25. Here is an example where τ is Hausdorff, Fatou and unbounded but not Lebesgue. Consider the norm topology on $C[0,1]$. Since $C[0,1]$ has a strong unit, the norm topology is unbounded. It is easy to see that the norm topology is Fatou but not Lebesgue. Notice also that the norm topology on $C[0,1]$ is not pre-Lebesgue. By Theorem 4.8 in [AB03], it is easily seen that a Hausdorff locally solid topology is uo-Lebesgue if and only if it is Fatou, pre-Lebesgue and unbounded.

We next investigate how unbounded Fatou topologies lift to the order completion. Theorem 4.12 of [AB03] asserts that if σ is a Fatou topology on a vector lattice X then σ extends uniquely to a Fatou topology σ^δ on X^δ. We will use this notation in the following theorem.

Proposition 6.26. *Let X be a vector lattice and τ a Fatou topology on X. Then $u(\tau^\delta) = (u\tau)^\delta$.*

Proof. Since τ is Fatou, τ extends uniquely to a Fatou topology τ^δ on X^δ. Clearly, $u(\tau^\delta)$ is still Fatou. Suppose (x_α) is a net in X and $x \in X$. By Corollary 5.10, $x_\alpha \xrightarrow{u(\tau^\delta)} x$ in X^δ if and only if $x_\alpha \xrightarrow{u\tau} x$ in X.

Since τ is Fatou, so is $u\tau$. Therefore, $u\tau$ extends uniquely to a Fatou topology $(u\tau)^\delta$ on X^δ. Suppose (x_α) is a net in X and $x \in X$. Then $x_\alpha \xrightarrow{(u\tau)^\delta} x$ is the same as $x_\alpha \xrightarrow{u\tau} x$.

Thus, $(u\tau)^\delta$ and $u(\tau^\delta)$ are two Fatou topologies on X^δ that agree with the Fatou topology $u\tau$ when restricted to X. By uniqueness of extension $u(\tau^\delta) = (u\tau)^\delta$. $\square$

The notation in the next result is consistent with [AB03, Theorem 4.13]; the proof is similar to Proposition 6.26.

Proposition 6.27. *Let X be an almost σ-order complete vector lattice. If τ is a σ-Fatou topology on X then there exists a unique σ-Fatou topology τ^σ on the σ-order completion, X^σ, of X that induces τ on X. Moreover, $u(\tau^\sigma) = (u\tau)^\sigma$.*

Definition 6.28. *A locally solid vector lattice (X, τ) is said to be **weakly Fatou** if τ has a base $\{U_i\}$ at zero consisting of solid sets with the property that for all i there exists $k_i \geq 1$ such that whenever (x_α) is a net in U_i and $x_\alpha \xrightarrow{o} x$ we have $x \in k_i U_i$.*

Remark 6.29. It is easily seen that for solid U and $k \geq 1$, the property that $x \in kU$ whenever (x_α) is a net in U and $x_\alpha \xrightarrow{o} x$ is equivalent to the property that $x \in kU$ whenever (x_α) is a net in U and $0 \leq x_\alpha \uparrow x$. Also, note that a Banach lattice X is weakly Fatou if and only if there exists $k \geq 1$ such that $\|x\| \leq k \sup_\alpha \|x_\alpha\|$ whenever $0 \leq x_\alpha \uparrow x$ in X.

Clearly, Fatou topologies are weakly Fatou. The next theorem, and one direction of its proof, is motivated by [GLX17] Proposition 3.1.

Theorem 6.30. *Suppose (X, τ) is Hausdorff and weakly Fatou. Then τ is Levi iff (X, τ) is boundedly uo-complete.*

Proof. If τ is Levi, X is order complete by page 112 of [AB03].

Let (x_α) be a τ-bounded *uo*-Cauchy net in X. By considering the positive and negative parts, respectively, we may assume that $x_\alpha \geq 0$ for each α. For each $y \in X_+$, since $|x_\alpha \wedge y - x_{\alpha'} \wedge y| \leq |x_\alpha - x_{\alpha'}| \wedge y$, the net $(x_\alpha \wedge y)$ is order Cauchy and hence order converges to some $u_y \in X_+$. The net $(u_y)_{y \in X_+}$ is directed upwards; we show it is τ-bounded. Let U be a solid τ-neighbourhood of zero with the property that there exists $k \geq 1$ with $x \in kU$ whenever (x_α) is a net in U and $x_\alpha \xrightarrow{o} x$. Since (x_α) is τ-bounded, there exists $\lambda > 0$ such that $(x_\alpha) \subseteq \lambda U$. Since $0 \leq x_\alpha \wedge y \leq x_\alpha \in \lambda U$, $x_\alpha \wedge y \in \lambda U$ for all y and α by solidity. We conclude that $u_y \in \lambda k U$ for all y, so that (u_y) is τ-bounded.

Since τ is Levi, (u_y) increases to an element $u \in X$. Fix $y \in X_+$. For any α, α', define

$$x_{\alpha,\alpha'} = \sup_{\beta \geq \alpha, \beta' \geq \alpha'} |x_\beta - x_{\beta'}| \wedge y. \tag{6.2}$$

Since (x_α) is *uo*-Cauchy, $x_{\alpha,\alpha'} \downarrow 0$. Also, for any $z \in X_+$ and any $\beta \geq \alpha, \beta' \geq \alpha'$,

$$|x_\beta \wedge z - x_{\beta'} \wedge z| \wedge y \leq x_{\alpha,\alpha'}. \tag{6.3}$$

Taking order limit first in β' and then over $z \in X_+$, we obtain $|x_\beta - u| \wedge y \leq x_{\alpha,\alpha'}$ for any $\beta \geq \alpha$. This implies that (x_α) *uo*-converges to u.

For the converse, assume (X, τ) is boundedly *uo*-complete and let (x_α) be a positive increasing τ-bounded net in X. Following the proof of [AB03] Theorem 7.50, it is easily seen that (x_α) is dominable. By [AB03] Theorem 7.37, (x_α) has supremum in X^u, hence is *uo*-Cauchy in X^u, hence is *uo*-Cauchy in X. Since (x_α) is τ-bounded, $x_\alpha \xrightarrow{uo} x$ in X for some $x \in X$. Since (x_α) is increasing, $x = \sup x_\alpha$. This proves that τ is Levi. □

Remark 6.31. A non-trivial improvement of Theorem 6.30 is provided by Theorem 12.5 and Theorem 12.6.

In these sections we are generally looking at how order-topological properties pass from $\tau \mapsto u_A\tau$, as opposed to purely topological vector space properties (local convexity, separability, Frechet, sequential, barrelled, etc.) The latter are still of interest, and it is of interest to know when, say, minimal topologies possess them. Since a complete characterization of when convergence in measure is separable is known (see [Frem04, Page 374]), I ask the following:

Question 6.32. Characterize when minimal topologies are separable.

There are many other less important order-topological properties of locally solid topologies (the B-property, Nakano property, etc.) We leave it to the reader to investigate how these properties mesh with the map $\tau \mapsto u_A\tau$, and to form their own properties by incorporating unboundedness into the definition. We next focus our attention on some of the more important topological vector space properties.

6.5. **Submetrizability of unbounded topologies.** We next recall some notation that will be needed for the upcoming sections on metrizability. Let A be a subset of a vector lattice X. The order ideal and the band generated by A are denoted by $I(A)$ and $B(A)$, respectively. If $A = \{a\}$, we define $I_a := I(\{a\})$ and $B_a := B(\{a\})$. A positive vector $e \in X$ is said to be a ***strong unit*** if $I_e = X$. If $B_e = X$, then e is called a ***weak unit***. If A is at most countable and $B(A) = X$ then, following [LZ71], we say that X has a ***countable order basis*** (and call A a countable order basis for X). Obviously, if e is a weak unit in X, then $\{e\}$ is a countable order basis for X. A sublattice Y of X is called ***order dense*** if for each $0 \neq x \in X_+$ there exists $y \in Y$ with $0 < y \leq x$.

Let (X, τ) be a topological vector space. We say that τ is ***metrizable*** if there exists a metric on X whose metric topology equals τ. We say that τ is ***submetrizable*** if it is finer than a metrizable topology. A standard fact from topological vector spaces is that a linear topology is metrizable iff it is Hausdorff and first countable. A subset A of X is ***bounded*** if for each neighbourhood U of zero for τ there exists $\lambda > 0$ such that $\lambda A \subseteq U$. If X contains a bounded neighbourhood of zero, then X is said to be ***locally bounded***. Local boundedness is the strongest of the metrizability related notions. Indeed, if V is a bounded neighbourhood of zero, then a base at zero for τ is given by $\frac{1}{n}V$ for $n \in \mathbb{N}$. Hence, every Hausdorff locally bounded topological vector space is first countable and, therefore, metrizable. Note that a locally solid metrizable topology has a countable base at zero consisting of solid sets with trivial intersection. We say that τ is ***Riesz submetrizable*** if it is finer than a metrizable locally solid topology.

In a Hausdorff locally solid vector lattice, there is an intermediate notion between weak and strong units. Given a positive vector e in a locally solid vector lattice (X, τ), if I_e is τ-dense in X, then e is called a ***quasi-interior point*** of (X, τ). As in the case of normed lattices, it is easily checked that e is a quasi-interior point iff $x - x \wedge ne \xrightarrow{\tau} 0$ for each $x \in X_+$.

Finally, we briefly recall the basics regarding the topological completion of a Hausdorff locally solid vector lattice. Let (X,τ) be a Hausdorff locally solid vector lattice and let $(\widehat{X},\widehat{\tau})$ be the topological completion of (X,τ). Then the $\widehat{\tau}$-closure of X_+ in $\widehat{X}$ is a cone in $\widehat{X}$ and $(\widehat{X},\widehat{\tau})$ equipped with this cone is a Hausdorff locally solid vector lattice containing X as a $\widehat{\tau}$-dense vector sublattice. Moreover, $\widehat{\tau}$-closures of solid subsets of X are solid in $\widehat{X}$, and if $\mathcal{N}_0$ is a base at zero for (X,τ) consisting of solid sets, then $\{\overline{V} : V \in \mathcal{N}_0\}$ is a base at zero for $(\widehat{X},\widehat{\tau})$ consisting of solid sets. Here, $\overline{V}$ denotes the closure of V in $(\widehat{X},\widehat{\tau})$. In particular, (X,τ) is metrizable iff $(\widehat{X},\widehat{\tau})$ is metrizable.

Submetrizability of the unbounded topology was first considered in [KMT17]. It is proved in [KMT17, Proposition 3.3] that the unbounded norm topology on a Banach lattice X is submetrizable iff X has a weak unit. It is proved in [DEM1] that $u\tau$ is submetrizable if (X,τ) is a metrizable locally solid vector lattice with a weak unit. In [DEM2], the authors proved the converse statement for complete metrizable locally convex-solid vector lattices. In this section, we provide the complete answer on submetrizability of the unbounded topology.

The following example shows that the converse of [DEM1, Proposition 6], in general, does not hold.

Example 6.33. Consider the vector lattice c_{00} of all eventually null sequences, equipped with the supremum norm. Then c_{00} is a normed lattice without a weak unit, yet the unbounded norm topology is metrizable; a metric d that induces the unbounded norm topology on c_{00} is given by

$$d(x,y) = \sup_n \left\{ \frac{\min\{|x_n - y_n|, 1\}}{n} \right\}.$$

It turns out that, when considering submetrizability of the unbounded topology $u\tau$ in spaces that are not complete nor metrizable, the correct replacement for weak units is the existence of a countable order basis in X. Before showing this, we make a remark about countable order bases.

Remark 6.34. It is convenient in the definition of a countable order basis to replace the at most countable set A satisfying $B(A) = X$ with a positive increasing sequence (u_n) satisfying $B(\{u_n\}) = X$. This is easily done by enumerating $A = \{a_i\}_{i\in I}$ where $I = \mathbb{N}$ or $\{1, \dots, N\}$ and defining $u_n = |a_1| \vee \cdots \vee |a_n|$ if $n \in I$ and $u_n = u_N$ if $n \in \mathbb{N}\backslash I$. Throughout, when we say that $A = \{u_n\}$ is a countable order basis for X it is tacitly assumed that (u_n) is a positive increasing sequence.

Proposition 6.35. *Let (X, τ) be a locally solid vector lattice and A an ideal of X.*

(i) If τ is Riesz submetrizable and there is a set in A that is a countable order basis for X then $u_A\tau$ is Riesz submetrizable.
(ii) If $u_A\tau$ is submetrizable then there is a set in A that is a countable order basis for X.

Proof. (i) Suppose $\{a_n\} \subseteq A_+$ is a countable order basis for X. Let $\{U_i\}$ be a countable base at zero of solid sets for a metrizable locally solid topology σ coarser than τ. Following the proof of [Tay1, Theorem 2.3], one sees that the collection $\{U_{i,a_n}\}$ defines a solid base of neighbourhoods at zero for a locally solid topology σ_1. This topology is also Hausdorff since if $x \in U_{i,a_n}$ for all i and n then, for fixed n, $|x| \wedge a_n \in U_i$ for all i and hence $|x| \wedge a_n = 0$ since σ is metrizable and hence Hausdorff. By [CL17, Lemma 2.2], $x = 0$. Thus σ_1 is a locally solid metrizable topology that is clearly coarser than $u_A\tau$.

(ii) Suppose that $u_A\tau$ is submetrizable and let d be a metric that generates a coarser topology than $u_A\tau$. For each n, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centered at zero for the metric, that is,

$$B_{\frac{1}{n}} = \{x \in X : d(x, 0) \leq \tfrac{1}{n}\}. \tag{6.4}$$

Let $\{V_i\}$ be a basis of zero for τ consisting of solid sets. Since $u_A\tau$ is finer than the metric topology, each $B_{\frac{1}{n}}$ contains some V_{i_n,a_n} where $0 \leq a_n \in A$. Consider $\{a_n\}$. We claim that $B(\{a_n\}) = X$. Again, by [CL17, Lemma 2.2], it suffices to prove that if $x \in X_+$ satisfies $x \wedge a_n = 0$ for all n then $x = 0$. But $x \wedge a_n = 0$ implies that $x \in V_{i_n,a_n}$ and hence $x \in B_{\frac{1}{n}}$ for all n. It follows that $x = 0$. ∎

Corollary 6.36. *Let (X,τ) be a locally solid vector lattice and A an ideal of X. Then $u_A\tau$ is Riesz submetrizable if and only if τ is Riesz submetrizable and there is a set in A that is a countable order basis for X.*

Compare with the corresponding result in [KMT17]. Note that, in a Banach lattice X, a weak unit e can be constructed from a countable order basis $\{e_n\} \subseteq X_+$ via the formula $e := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{e_n}{1+\|e_n\|}$. Corollary 6.36 also answers a slightly modified version of a question posed on page 14 of [DEM1].

Remark 6.37. Note, in particular, that if τ is unbounded and Riesz (sub)metrizable then X has a countable order basis.

6.6. **Local convexity and atoms.** We now make some remarks about $u\tau$-continuous functionals and, surprisingly, generalize many results in [KMT17] whose presented proofs rely heavily on AL-representation theory and the norm.

First recall that by [AB03] Theorem 2.22, if σ is a locally solid topology on X then $(X,\sigma)^* \subseteq X^\sim$ as an ideal. $(X,\sigma)^*$ is, therefore, an order complete vector lattice in its own right. Here $(X,\sigma)^*$ stands for the topological dual and $X^\sim$ for the order dual.

Proposition 6.38. *$(X,u\tau)^* \subseteq (X,\tau)^*$ as an ideal.*

Proof. It is easy to see that the set of all $u\tau$-continuous functionals in $(X,\tau)^*$ is a linear subspace. Suppose that φ in $(X,\tau)^*$ is $u\tau$-continuous; we will show that $|\varphi|$ is also $u\tau$-continuous. Fix $\varepsilon > 0$ and let $\{U_i\}_{i\in I}$ be a solid base for τ at zero. By $u\tau$-continuity of φ, one can find an $i \in I$ and $u > 0$ such that $|\varphi(x)| < \varepsilon$ whenever $x \in U_{i,u}$. Fix $x \in U_{i,u}$. Since $U_{i,u}$ is solid, $|y| \leq |x|$ implies $y \in U_{i,u}$ and, therefore, $|\varphi(y)| < \varepsilon$. By the Riesz-Kantorovich formula, we get that

$$\big||\varphi|(x)\big| \leq |\varphi|\big(|x|\big) = \sup\Big\{\big|\varphi(y)\big| : |y| \leq |x|\Big\} \leq \varepsilon. \tag{6.5}$$

It follows that $|\varphi|$ is $u\tau$-continuous and, therefore, the set of all $u\tau$-continuous functionals in $(X,\tau)^*$ forms a sublattice. It is straightforward to see that if $\varphi \in (X,\tau)^*_+$ is $u\tau$-continuous and $0 \leq \psi \leq \varphi$ then ψ

is also $u\tau$-continuous and, thus, the set of all $u\tau$-continuous functionals in $(X,\tau)^*$ is an ideal. $\square$

Remark 6.39. An ideal is the best one can expect in Proposition 6.38 - the inclusion need not be a band or even a σ-ideal. Indeed, let $X = \ell_1$. Then $X^\sim = X^* = \ell_\infty$ but $0 \neq f \in (\ell_1, un)^*$ iff f is a finite linear combination of coordinate functionals of atoms (to be proven shortly: see Proposition 6.48). In particular, since *un* and *uo* agree in ℓ_1, the main result in the paper "Unbounded order continuous operators on Riesz spaces" by A. Bahramnezhad and K. Azar is false.

We next need some definitions. Our definition of discrete element is slightly different than [Con05] since we require them to be positive and non-zero. It is consistent with [AB03].

Definition 6.40. *Let X be a vector lattice. $x > 0$ in X is called a **discrete element** or **atom** if the ideal generated by x equals the linear span of x.*

Definition 6.41. *A vector lattice X is **discrete** or **atomic** if there is a complete disjoint system $\{x_i\}$ consisting of discrete elements in X_+, i.e., $x_i \wedge x_j = 0$ if $i \neq j$ and $x \in X$, $x \wedge x_i = 0$ for all i implies $x = 0$.*

By [AB03] Theorem 1.78, X is atomic if and only if X is lattice isomorphic to an order dense sublattice of some vector lattice of the form $\mathbb{R}^A$.

The next result is an effortless generalization of Theorem 5.2 in [KMT17]. In the upcoming results we consider the 0-vector lattice to be atomic.

Lemma 6.42. *Let τ be a Hausdorff uo-Lebesgue topology on a vector lattice X. τ is locally convex if and only if X is atomic. Moreover, if X is atomic then a Hausdorff uo-Lebesgue topology exists, it is least, and it is the topology of pointwise convergence.*

Proof. By page 291 of [Con05], a pre-L_0 space is the same as a vector lattice that admits a Hausdorff Lebesgue topology; the first part of our lemma is just a re-wording of Proposition 3.5 in [Con05]. The moreover part follows from Theorem 7.70 in [AB03] (remember, "minimal" in [AB03] means "least"). Actually, knowing that Hausdorff *uo*-Lebesgue topologies are Lebesgue and disjoint sequences are null, the entire lemma can be deduced from the statement and proof of Theorem 7.70. □

Consider Remark 4.15 in [KMT17]. It is noted that ℓ_∞ is atomic yet *un*-convergence is not the same as pointwise convergence. Lemma 6.42 tells us that ℓ_∞ does admit a least topology that coincides with the pointwise convergence. Since ℓ_∞ is a dual Banach lattice, $u|\sigma|(\ell_\infty, \ell_1)$ is defined and must be the least topology on ℓ_∞. This is an example where X^* is not an order continuous Banach lattice but $u|\sigma|(X^*, X)$ is still a least topology.

We next use these connections between convexity and atoms to characterize when *uo*-convergence is topological.

Proposition 6.43. *uo-convergence in a vector lattice X agrees with the convergence of a locally convex-solid topology on X iff X is atomic.*

Proof. Suppose *uo*-convergence agrees with the convergence of a locally convex-solid topology τ. Since *uo*-limits are unique, τ is Hausdorff. Clearly, τ is *uo*-Lebesgue, so, by Lemma 6.42, X is atomic.

Suppose X is atomic. By [AB03] Theorem 1.78, X is lattice isomorphic to an order dense sublattice of a vector lattice of the form $\mathbb{R}^A$. Since *uo*-convergence is preserved through the onto isomorphism and the order dense embedding into $\mathbb{R}^A$, we may assume that $X \subseteq \mathbb{R}^A$. It was noted in [Gao14] that *uo*-convergence in $\mathbb{R}^A$ is just pointwise convergence. Using Theorem 3.2 of [GTX17], it is easy to see that the restriction of pointwise convergence to X agrees with *uo*-convergence on X. Hence, *uo*-convergence agrees with the convergence of a locally convex-solid topology. □

Remark 6.44. In [Pap64], *uo*-convergence was studied in the framework of commutative ℓ-groups, under the name L-convergence. In [Pap65] the author introduces α-convergence - declaring it to be more natural than *uo* in non-Archimedean spaces - and proves in [Pap65, Theorem 2.6] that α-convergence and *uo*-convergence agree in Archimedean ℓ-groups. When α-convergence is topological was studied in [Ell68, Theorem 2.1, 2.5]; it is concluded that α-convergence is topological for an ℓ-group G iff G is completely distributive. The proof, however, doesn't use standard vector lattice machinery, so Vladimir and I developed the following alternative proof:

Theorem 6.45. *uo-convergence in a vector lattice X is topological iff X is atomic.*

Proof. Suppose *uo*-convergence agrees with the convergence of a topology τ, but X is not atomic. Then there exists $0 \neq e \in X_+$ with no atom $v \in X$ satisfying $0 < v \leq e$. Consider I_e. Since I_e is an ideal of X, *uo*-convergence passes freely between I_e and X, so *uo*-convergence is topological in I_e. Note that I_e is atomless so that, replacing X with I_e, we reduce to the case that X is atomless and has a strong unit.

We identify X with an order dense and majorizing sublattice of an order complete $C(K)$-space with the strong unit e corresponding to $\mathbb{1}$.

Let $t_0 \in K$ and consider the set

$$G_{t_0} := \{t \in K : x(t) = x(t_0)\ \forall x \in X\}.$$

Then $t_0 \in G_{t_0}$. Define

$$F^{t_0} := \{x \in X : 0 \leq x \leq e \text{ and } x(t_0) = 1\}.$$

Then $e \in F^{t_0}$ and F^{t_0} is downward directed, and can be considered as a net over itself. Now, for each $t_1 \notin G_{t_0}$ find $x \in X$ such that $x(t_1) \neq x(t_0)$. The formula, $f_{t_1}(t) = \frac{|x(t)-x(t_1)|}{|x(t_0)-x(t_1)|} \wedge 1$ defines an element of F^{t_0} which vanishes at t_1. Therefore, $F^{t_0} \downarrow \chi_{G_{t_0}}$ pointwise. Suppose $\chi_{G_{t_0}} \in C(K)$. Then by order density, there exists $x \in X$ with $0 < x \leq \chi_{G_{t_0}}$. But by definition of G_{t_0}, this forces $x = \lambda \chi_{G_{t_0}}$ for some positive scalar λ. Hence, $\chi_{G_{t_0}} \in X$, and is an atom of X. This is impossible, so that $\chi_{G_{t_0}}$ is not in $C(K)$.

We next claim that $F^{t_0} \downarrow 0$ in $C(K)$, and hence in X. Indeed, since $C(K)$ is order complete there exists $g \in C(K)$ such that $F^{t_0} \downarrow g$. This forces, $0 \leq g(t) \leq \chi_{G_{t_0}}(t)$ for all $t \in K$. Suppose $g > 0$. Then, since

X is order dense in $C(K)$, there exists $x_g \in X$ with $0 < x_g \leq g$. This forces $x_g(t) \neq 0$ for some $t \in G_{t_0}$. Hence, by definition of G_{t_0}, $x_g = \lambda\chi_{G_{t_0}}$ for some $\lambda > 0$. This forces $x_g \notin C(K)$, a contradiction. Therefore, $g = 0$. Knowing that $F^{t_0} \downarrow 0$ in X, our assumptions that *uo*-convergence is topological yields that $F^{t_0} \xrightarrow{\tau} 0$.

Consider now the collection of functions $\mathcal{H}$ where $h \in \mathcal{H}$ iff $h : K \to \bigcup_{t\in K} F^t$ satisfies $h(t) \in F^t$ for each $t \in K$. $\mathcal{H}$ is non-empty because $h(t) \equiv e \in \mathcal{H}$. Ordering $\mathcal{H}$ via $h_1 \preceq h_2$ iff for each $t \in K$, $h_1(t) \geq h_2(t)$ in X, makes $\mathcal{H}$ into a directed set.

For each $t \in K$ and $h \in \mathcal{H}$, define $y_{t,h} = h(t)$. $(y_{t,h})$ is a net in X indexed by $K \times \mathcal{H}$, where we order $(t_1, h_1) \leq (t_2, h_2)$ iff $h_1 \preceq h_2$.[6]

We claim that $y_{t,h} \xrightarrow{\tau} 0$. Fix a neighbourhood V of zero, and $t \in K$. Then since $F^t \xrightarrow{\tau} 0$, there exists $h^t \in F^t$ such that for all $f \in F^t$ with $f \leq h^t$ we have $f \in V$. Define $h^* \in \mathcal{H}$ via $h^*(t) = h^t$. Then whenever $h \succeq h^*$ and $t \in K$ we have $y_{t,h} = h(t) \in V$, which proves the claim.

We next show that $y_{t,h} \not\xrightarrow{o} 0$. Suppose that $u \in X$ dominates a tail of $y_{t,h}$, say, $h \succeq h_0$. Then for each t it follows that $u(t) \geq y_{t,h_0}(t) = h_0(t)(t) = 1$. Hence, $u \geq e$, so that there is no dominating net $y_\beta \downarrow 0$ for $(y_{t,h})$. Since $(y_{t,h})$ is order bounded, we conclude that $y_{t,h} \not\xrightarrow{uo} 0$, so that *uo*-convergence cannot be topological in X. $\square$

Question 6.46. Is there a non-atomic Hausdorff locally solid vector lattice (X, τ) such that τ and *uo*-agree on sequences?

The following result is known, but is non-trivial and follows immediately:

Corollary 6.47. *A vector lattice X is atomic iff it is lattice isomorphic to a regular sublattice of some vector lattice of the form $\mathbb{R}^A$.*

Proof. The forward direction follows from Theorem 1.78 of [AB03]. For the converse, combine Theorem 6.43 with Theorem 3.2 of [GTX17]. $\square$

[6]If one insists that directed sets be partially ordered, well-order K then order the index set via $(t_1, h_1) \leq (t_2, h_2)$ iff $h_1 \prec h_2$ or $h_1 = h_2$ and $t_1 \leq t_2$. One can then proceed analogously, making use of the least element of K.

We next look to characterize the entire dual of a minimal topology. Note that Theorem 7.71 in [AB03] is a perfectly reasonable generalization of [KMT17] Corollary 5.4(ii) since the *un*-topology in order continuous Banach lattices is least. What we want, however, is to replace the least topology assumption in Theorem 7.71 with the assumption that the topology is minimal. The reason being that *uo*-convergence can "detect" if a topology is minimal, but not necessarily if it is least. To prove Corollary 5.4 in [KMT17] the authors go through the theory of dense band decompositions. A similar theory of τ-dense band decompositions can be developed, but there is an easier proof of this result utilizing the recent paper [GLX17].

Proposition 6.48. *Let τ be a uo-Lebesgue topology on a vector lattice X. If $0 \neq \varphi \in (X,\tau)^*$ then φ is a linear combination of the coordinate functionals of finitely many atoms.*

Proof. Suppose $0 \neq \varphi \in (X,\tau)^*$. Since τ is *uo*-Lebesgue and φ is τ-continuous, $\varphi(x_\alpha) \to 0$ whenever $x_\alpha \xrightarrow{uo} 0$. The conclusion now follows from Proposition 2.2 in [GLX17]. □

Remark 6.49. Notice that if X is laterally σ-complete then by [AB03, Theorem 7.8], $X^\sim = X^\sim_c$. Assuming measurable cardinals, there exists a set A and elements in $(\mathbb{R}^A)^\sim$ that are not finite linear combinations of coordinate functionals of atoms. Therefore, under the assumption of measurable cardinals, [GLX17, Proposition 2.2] cannot be improved by replacing *uo*-continuous functionals with sequentially *uo*-continuous functionals. Similarly, keeping in mind [AB03, Exercise 7.13], one cannot replace continuity with sequentially continuity in Proposition 6.48.

Question 6.50. Without appealing to measurable cardinals, can one find a vector lattice admitting a sequentially *uo*-continuous but not *uo*-continuous linear functional? Can one find a vector lattice admitting a minimal topology with a sequentially continuous but not continuous linear functional?

Loosely speaking, unbounded topologies are rarely locally convex, rarely satisfy MCP, and rarely satisfy the Levi property (yet I have no deep result for unbounded topologies that satisfy one of these properties! There should be something to say about locally convex or complete unbounded topologies...). However, sometimes this can be used to our advantage in order to distinguish τ from $u\tau$. Indeed, suppose τ is some Hausdorff locally solid topology. How do we know it is not unbounded? Well, if it satisfies one of the aforementioned properties, there is a good chance. The next result, or at least its proof, is one of the few results making heavy use of local convexity.

Proposition 6.51. *Let A be an order dense ideal of a Hausdorff locally convex-solid vector lattice (X, τ). TFAE:*

(i) Order intervals in X are $u_A\tau$-compact;
(ii) Order intervals in A are $\tau|_A$-compact and X is order complete.

Proof. (i)⇒(ii): Suppose that order intervals in X are $u_A\tau$-compact; we first show that X is order complete. Suppose $0 \leq x_\alpha \uparrow\leq x$ in X. Since $[0, x]$ is $u_A\tau$-compact, there is a subnet (z_γ) of (x_α) and $z \in [0, x]$ such that $z_\gamma \xrightarrow{\tau} z$. Since (x_α) is increasing, so is (z_γ), so that $z_\gamma \uparrow z$. It follows that $x_\alpha \uparrow z$.

Now, since A is an ideal in X, order intervals of A are order intervals of X, and, therefore, are $u(\tau|_A)$-compact since $(u_A\tau)|_A = u(\tau|_A)$. Order intervals in A are therefore $\tau|_A$-compact as $\tau|_A$ and $u(\tau|_A)$ agree on order intervals.

(ii)⇒(i): Since order intervals in A are $\tau|_A$-compact, [AB03, Corollary 6.57] yields that A is atomic, order complete, and $\tau|_A$ is Lebesgue. By [Tay1, Proposition 9.13], $u_A\tau$ is Lebesgue and, therefore, is the minimal topology on X.

Since A is order dense in X, A^u and X^u are lattice isomorphic. Since A is atomic, A^u is lattice isomorphic to $\mathbb{R}^B$ for some set B.

Since X is order complete, X is an order dense ideal in X^u. Since X^u is lattice isomorphic to $\mathbb{R}^B$, X^u is atomic. It follows that the topology, σ, of pointwise convergence is defined, and is the unique Hausdorff Lebesgue topology on X^u. Since this topology is locally convex-solid, applying [AB03, Corollary 6.57] again yields that the order intervals of X^u are σ-compact. Since X is an ideal in X^u, order intervals of X

are order intervals of X^u. Since $u_A\tau$ is minimal, $u_A\tau = \sigma|_X$. It follows that order intervals of X are $u_A\tau$-compact. □

The above proposition demonstrates once more that it is easier to work with nets than with sequences: Although we have a clean characterization of $u_A\tau$-compactness of order intervals, sequential compactness is not as clear.

For a detailed study of local convexity we refer the reader to [AB03, Chapter 6]. This chapter played only a small role in this thesis, and it is possible that some deep results could follow from a more thorough investigation. In particular, the following question arises: Given some unbounded topology τ, when can one find some order dense ideal A and some locally convex (or complete, or Levi) topology σ with $u_A\sigma = \tau$? This type of question plays some role when representing minimal topologies as convergence in measure.

The paper [DEM2] is devoted to properties of $u\tau$ when τ is locally convex-solid. However, those familiar with this thesis will have no trouble removing local convexity from many of the statements and proofs presented in that paper. In particular, local convexity can be removed from Lemma 2 and Theorem 1, and Lebesgue can be weakened to Fatou in Theorem 1. This explains the remark after [KMT17, Theorem 6.4]: Although ℓ_∞ is not Lebesgue (and hence not KB), it is Levi and Fatou. There is also no mystery surrounding [KMT17, Example 6.5]: We know that τ and $u\tau$ have the same closed solid sets, but they certainly needn't have the same closed convex sets. It is solidity of the ball, not convexity, that plays a role in [KMT17, Theorem 6.4].

7. When does $u_A\tau = u_B\sigma$?

Let A and B be ideals of a vector lattice X, and assume τ and σ are locally solid topologies on X. As we know, one can form the topologies $u_A\tau$ and $u_B\sigma$ on X. It is natural to ask how $u_A\tau$ and $u_B\sigma$ relate, and in this section we shall do that. The results are not only of intrinsic interest, but will be utilized shortly when we characterize metrizability of unbounded topologies.

Proposition 7.1. *Suppose A and B are ideals of a locally solid vector lattice (X,τ). If $\overline{A}^{\tau}=\overline{B}^{\tau}$ then the topologies $u_A\tau$ and $u_B\tau$ on X agree.*

Proof. It suffices to show that $u_A\tau = u_{\overline{A}}\tau$, where, for notational simplicity, $\overline{A}$ denotes the τ-closure of A in X. Let (x_α) be a net in X. Clearly, if $x_\alpha \xrightarrow{u_{\overline{A}}\tau} 0$ then $x_\alpha \xrightarrow{u_A\tau} 0$. To prove the converse, suppose that $x_\alpha \xrightarrow{u_A\tau} 0$. Fix $y\in \overline{A}^{\tau}_{+}$, a solid base neighbourhood V of zero for τ, and a solid base neighbourhood U of zero for τ with $U+U\subseteq V$. By definition, there exists $a\in A$ such that $a\in y+U$. WLOG $a\in A_+$ because, by solidity, $||a|-y|\le |a-y|\in U$ implies $|a|\in y+U$. By assumption, $|x_\alpha|\wedge a\xrightarrow{\tau} 0$. This implies that there exists α_0 such that $|x_\alpha|\wedge a\in U$ whenever $\alpha\ge\alpha_0$. It follows by solidity that

$$|x_\alpha|\wedge y = |x_\alpha|\wedge(y-a+a)\le |x_\alpha|\wedge|y-a|+|x_\alpha|\wedge a\in U+U\subseteq V, \tag{7.1}$$

so that $x_\alpha \xrightarrow{u_{\overline{A}}\tau} 0$. ☐

Before we state and prove Theorem 7.2 we need to recall some basic facts on $C(K)$-representations of vector lattices. Suppose X is a vector lattice with a strong unit u. For $x\in X$ we define

$$\|x\|_u := \inf\{\lambda\ge 0 : |x|\le \lambda u\}.$$

It is a standard fact that $\|\cdot\|_u$ defines a lattice norm on X, and if X is uniformly complete, then $(X,\|\cdot\|_u)$ is an AM-space with a strong unit u. By Kakutani's representation theorem [AB06, Theorem 4.29], $(X,\|\cdot\|_u)$ is lattice isometric to some $C(K)$-space for a (unique up to a homeomorphism) compact Hausdorff space K. This representation can be taken such that the vector u corresponds to the constant function $\mathbb{1}$. If X is not uniformly complete, consider its order completion X^δ. Then X is an order dense and majorizing sublattice of X^δ. Since order complete vector lattices are uniformly complete and u is also a strong unit for X^δ, by the previous case X is lattice isomorphic to an order dense and majorizing sublattice of $C(K)$ for some compact Hausdorff space K.

Theorem 7.2. *Let A and B be ideals of a vector lattice X, and suppose τ and σ are locally solid topologies on X. If $u_B\sigma\subseteq u_A\tau$ as topologies on X, then $\overline{A\cap B}^{\sigma}=\overline{B}^{\sigma}$.*

Proof. It suffices to prove that $B \subseteq \overline{A \cap B}^{\sigma}$. Let $u \in B_+$ and U a solid σ-neighbourhood of zero. Consider $U_u := \{x \in X : |x| \wedge u \in U\}$. By assumption, there exists $v \in A_+$ and V a solid τ-neighbourhood of zero such that $V_v \subseteq U_u$, where $V_v := \{x \in X : |x| \wedge v \in V\}$. This means that for all $x \in X_+$, if $x \wedge v \in V$ then $x \wedge u \in U$.

Let $x_n = (u - nv)^+$. Clearly, $x_n \downarrow$. Put $y_n = x_n \wedge v$. Then $(y_n) \subseteq A_+$, and $y_n \downarrow$.

Consider $I_{u \vee v}$, the ideal generated by $u \vee v$ in X. Since $0 \leq y_n \leq x_n \leq u \leq u \vee v$, (x_n) and (y_n) are in $I_{u \vee v}$. Also, u and v are in $I_{u \vee v}$. We identify $I_{u \vee v}$ as an order dense majorizing sublattice of $C(K)$ for some K such that $u \vee v$ corresponds to the constant function $\mathbb{1}$.

We next prove that (y_n) converges to zero point-wise in $C(K)$. Take $t \in K$. If $(u \wedge v)(t) = 0$ then $0 \leq y_n(t) = x_n(t) \wedge v(t) \leq u(t) \wedge v(t) = 0$.

If $(u \wedge v)(t) \neq 0$ then $v(t) > 0$, so that $x_n(t) = (u - u \wedge nv)(t) = u(t) - u(t) \wedge nv(t) = 0$ for sufficiently large n. Thus, for large enough n, $y_n(t) = 0$. By Dini's classical theorem (y_n) converges uniformly to zero in $C(K)$. Therefore, for any $N \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $y_n \leq \frac{1}{N}(u \vee v)$. We now go back to X. Clearly, $\frac{1}{N}(u \vee v) \xrightarrow{u_A\tau} 0$ in N in X. Since $u_A\tau$ is locally solid, $y_n \xrightarrow{u_A\tau} 0$.

Since $y_n = x_n \wedge v$ is an order bounded sequence in A, this implies that $x_n \wedge v \xrightarrow{\tau} 0$. Therefore, there exists m_0 such that for all $m \geq m_0$, $x_m \wedge v \in V$ and hence $x_m \wedge u \in U$. Since $0 \leq x_m \leq u$, we conclude that for all $m \geq m_0$ we have $x_m \in U$. In particular, $u - u \wedge m_0 v \in U$. Since $u \wedge m_0 v \in A \cap B$, we conclude $u \in \overline{A \cap B}^{\sigma}$. This proves $B \subseteq \overline{A \cap B}^{\sigma}$. ∎

Theorem 7.2 has many interesting and important consequences.

Corollary 7.3. *Let A and B be ideals of a locally solid vector lattice (X, τ). Then $u_A\tau = u_B\tau$ iff $\overline{A}^{\tau} = \overline{B}^{\tau}$. In particular, $u_A\tau = u\tau$ iff $\overline{A}^{\tau} = X$.*

It also gives the following corollary that quantitatively explains the "gap" mentioned in Remark 2.6.

Corollary 7.4. *Let A be an ideal of a Hausdorff locally solid vector lattice (X, τ). Then $u_A\tau$ is Hausdorff and not equal $u\tau$ iff A is order dense but not τ-dense in X.*

Furthermore, we deduce that minimal topologies are exactly the topologies in which there is no "choice" on how to unbound:

Corollary 7.5. *Let (X, τ) be a Hausdorff locally solid vector lattice. Then τ is minimal iff $\tau = u_A\tau$ for every order dense ideal A of X.*

Proof. The forward direction is immediate by [Tay1, Theorem 6.4] combined with [Tay1, Theorem 9.7]. Suppose $\tau = u_A\tau$ for every order dense ideal A of X, but τ is not minimal. Since τ is clearly unbounded, it follows from [Tay1, Theorem 6.4] that τ is not Lebesgue. By [AB03, Theorem 3.8], there exists an order dense ideal A of X that is not τ-dense. By Corollary 7.3 and the observation that τ is unbounded, $\tau = u\tau \neq u_A\tau$, a contradiction. □

The next result can be thought of as a topological version of [CL17, Lemma 2.2]. A result of this type was first proved in [DOT17]; in that paper, it was shown that a quasi-interior point always witnesses unbounded norm convergence. The converse was proved in [KMT17, Theorem 3.1]; a positive vector is a quasi-interior point iff it witnesses unbounded norm convergence. The [KMT17] result was extended (see [DEM1, Theorem 2]) to the setting of sequentially complete locally solid vector lattices. The following corollary improves the result of [DEM1] significantly. Not only does it drop the assumption of sequential completeness, it also characterizes general sets that witness unbounded convergence; they are precisely the sets which generate topologically dense order ideals.

Corollary 7.6. *Let (X, τ) be a locally solid vector lattice and $A \subseteq X_+$. TFAE:*

(i) $u_{I(A)}\tau = u\tau$;
(ii) $\overline{I(A)}^{\tau} = X$;
(iii) *For any net* $(x_\alpha) \subseteq X_+$, $x_\alpha \xrightarrow{u\tau} 0 \Leftrightarrow x_\alpha \wedge a \xrightarrow{\tau} 0$ *for all* $a \in A$.

Proof. (i)⇔(ii) follows from Corollary 7.3. (ii)⇒(iii) is [Tay1, Proposition 9.9]. (iii)⇒(i) is clear. □

Question 7.7. Although I expect the answer is negative, I don't know if replacing nets with sequences in Corollary 7.6(iii) will keep equivalence. A partial answer to this question is given in Proposition 7.11.

The next proposition is an analogue of [DOT17] Lemma 2.11.

Proposition 7.8. *Suppose* (X,τ) *is a locally solid vector lattice and* $E \subseteq X_+$. *Then* $x_\alpha \xrightarrow{u_{\overline{I(E)}^\tau}\tau} x$ *if and only if* $|x_\alpha - x| \wedge e \xrightarrow{\tau} 0$ *for all* $e \in E$. *In particular, if there exists* $e \in X_+$ *such that* $\overline{I_e}^\tau = X$ *then* $x_\alpha \xrightarrow{u\tau} 0$ *iff* $|x_\alpha| \wedge e \xrightarrow{\tau} 0$.

Next we present an easy generalization of Corollary 3.2 in [KLT17]; one could extend it to countable generators via diagonal arguments.

Corollary 7.9. *Suppose* A *is a* τ*-closed ideal of a metrizable locally solid vector lattice* (X,τ). *Suppose that* $e \in A_+$ *is such that* $\overline{I_e}^\tau = A$. *If* $x_\alpha \xrightarrow{u_A\tau} 0$ *in* X *then there exists* $\alpha_1 < \alpha_2 < \dots$ *such that* $x_{\alpha_n} \xrightarrow{u_A\tau} 0$.

Next we prove that every locally solid vector lattice whose unbounded topology is metrizable admits an at most countable set which generates a dense order ideal. This result will play an important role in Theorem 8.3 where we consider metrizability of unbounded topologies. Note that we do not assume τ is metrizable and, as usual, we leave it to the reader to appropriately extend the result to $u_A\tau$ (c.f. Proposition 8.7).

Proposition 7.10. *Let* (X,τ) *be a locally solid vector lattice such that* $u\tau$ *is metrizable. Then there exists* $e_n \in X_+$ $(n \in \mathbb{N})$ *such that* $\overline{I(\{e_n\})}^\tau = X$.

Proof. Assume $u\tau$ is metrizable and $\{U_i\}$ is a base at zero consisting of solid (but not even necessarily countably many) sets for τ. Let d be a metric for $u\tau$ and $B_{\frac{1}{n}}$ be the ball at zero of radius $\frac{1}{n}$ for d. Then there exists i_n and $e_n \geq 0$ such that

$$U_{i_n,e_n} \subseteq B_{\frac{1}{n}}. \tag{7.2}$$

This gives a natural choice of e_n and, indeed, it is straightforward to show that for any net $(x_\alpha) \subseteq X_+$, $x_\alpha \xrightarrow{u\tau} 0 \Leftrightarrow x_\alpha \wedge e_n \xrightarrow{\tau} 0$ for all n. This implies that $\overline{I(\{e_n\})}^\tau = X$ and that concludes the proof. □

By comparing Proposition 7.10 with Proposition 6.35 one should notice that metrizability of $u\tau$ gives the existence of a countable set which generates a topologically dense ideal while submetrizability of $u\tau$ merely gives the existence of a countable order basis.

With Proposition 7.10 in mind, we present the following sequential variant of Corollary 7.6:

Proposition 7.11. *Let (X,τ) be a Hausdorff locally solid vector lattice and (e_n) a positive increasing sequence in X. TFAE:*

(i) $\overline{I(\{e_n\})}^{\tau} = X$*;*

(ii) For every sequence (x_k) in X_+, if $x_k \wedge e_n \xrightarrow{\tau} 0$ in k for every n then $x_k \xrightarrow{u\tau} 0$.

Proof. It suffices to prove (ii)⇒(i): Fix $x \in X_+$; we will show that $x \wedge ne_n \xrightarrow{\tau} x$ or, equivalently, $(x - ne_n)^+ \xrightarrow{\tau} 0$ as a sequence of n. Fix m and put $u_m = x \vee e_m$. Now, the ideal I_{u_m} is lattice isomorphic (as a vector lattice) to some order dense and majorizing sublattice of $C(K_m)$ for some compact Hausdorff space K_m, with u_m corresponding to $\mathbb{1}$. Since $x, e_m \in I_{u_m}$, we may consider x and e_m as elements of $C(K_m)$. Note that $x \vee e_m = \mathbb{1}$ implies that x and e_m never vanish simultaneously.

For each $n \in \mathbb{N}$, we define

$$F_n^m = \{t \in K_m : x(t) \geq ne_m(t)\} \text{ and } O_n^m = \{t \in K_m : x(t) > ne_m(t)\}.$$

Clearly, $O_n^m \subseteq F_n^m$, O_n^m is open, and F_n^m is closed.

Claim 1: $F_{n+1}^m \subseteq O_n^m$. Indeed, let $t \in F_{n+1}^m$. Then $x(t) \geq (n+1)e_m(t)$. If $e_m(t) > 0$ then $x(t) > ne_m(t)$, so that $t \in O_n^m$. If $e_m(t) = 0$ then $x(t) > 0$, hence $t \in O_n^m$.

By Urysohn's Lemma, we find $z_n^{(m)} \in C(K_m)$ such that $0 \leq z_n^{(m)} \leq x$, $z_n^{(m)}$ agrees with x on F_{n+1}^m and vanishes outside of O_n^m.

Claim 2: $n(z_n^{(m)} \wedge e_m) \leq x$. Let $t \in K_m$. If $t \in O_n^m$ then $n(z_n^{(m)} \wedge e_m)(t) \leq ne_m(t) < x(t)$. If $t \notin O_n^m$ then $z_n^{(m)}(t) = 0$, so that the inequality is satisfied trivially.

Claim 3: $\big(x - (n+1)e_m\big)^+ \leq z_n^{(m)}$. Again, let $t \in K_m$. If $t \in F_{n+1}^m$ then $\big(x - (n+1)e_m\big)^+ \leq x(t) = z_n^{(m)}(t)$. If $t \notin F_{n+1}^m$ then $x(t) <$

$(n+1)e_m(t)$, so that $\big(x-(n+1)e_m\big)^+(t)=0$ and the inequality is satisfied trivially.

Denote the vector $\big(x-(n+1)e_{n+1}\big)^+$ in X by y_n. We claim that for each k, $y_n\wedge e_k\xrightarrow{\tau}0$ in X as a sequence in n. Fix k and choose $n\geq k$ arbitrarily. Then the following holds in $C(K_{n+1})$:

$$y_n\wedge e_{n+1}\leq z_n^{(n+1)}\wedge e_{n+1}\leq \tfrac{1}{n}x.$$

In particular, $y_n\wedge e_{n+1}\leq\frac{1}{n}x$ holds in $C(K_{n+1})$ and hence in X since both elements lie in X. Recalling that (e_k) is increasing in X, we conclude that $0\leq y_n\wedge e_k\leq\frac{1}{n}x$ holds in X. Since τ is locally solid, this implies that $y_n\wedge e_k\xrightarrow{\tau}0$ for each k. The assumption yields that $y_n\xrightarrow{u\tau}0$. Since $0\leq y_n\leq x$ we conclude $y_n\xrightarrow{\tau}0$. □

8. Metrizability of unbounded topologies

As Corollary 6.36 shows, the notion of Riesz submetrizability passes nicely from τ to $u\tau$ in both directions. However, the situation with metrizability is not as clean as the following example illustrates.

Example 8.1. If X is a Banach lattice then by [KMT17, Theorem 3.2] the unbounded norm topology is metrizable iff X has a quasi-interior point. Therefore, it should be clear that τ being metrizable does not guarantee that $u\tau$ is metrizable.

We now provide an example of a nonmetrizable locally solid vector lattice (X,τ) with a quasi-interior point such that $u\tau$ is metrizable.

Let $X=L_2:=L_2[0,1]$. Since X is an order continuous Banach lattice, by [Tay1, Example 5.6] the unbounded norm topology and unbounded absolute weak topology on X agree. They are metrizable because X has a quasi-interior point. Suppose that the absolute weak topology on X is metrizable. Then by [AB03, Theorem 5.6], L_2 admits a countable majorizing subset $A=\{f_n\}$. By definition, this means that $I(A)$ is majorizing in X, so that $I(A)=X$. Define $f=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|f_n|}{1+\|f_n\|}$. Then $I_f=I(A)=X$, so that L_2 has a strong unit. This is a contradiction.

In this section we consider metrizability of the unbounded topology. As was previously mentioned, if X is a Banach lattice, the unbounded norm topology is metrizable iff X has a quasi-interior point. One direction was extended in [DEM1] while the complete characterization was obtained in [DEM2, Proposition 4] only for the case of complete metrizable locally convex-solid vector lattices. In Theorem 8.3 we will provide several improvements to the latter result. We will drop the completeness and local convexity assumptions on τ, replace the existence of a countable topological orthogonal system with the weaker requirement of a sequence which generates a τ-dense ideal, and prove that metrizability of $u\tau$ is further equivalent to $\widehat{X}$ possessing a quasi-interior point.

Recall that, by Proposition 7.10, a necessary condition for $u\tau$ to be metrizable is the existence of an at most countable set $A \subseteq X_+$ with $\overline{I(A)}^{\tau} = X$. The next example shows that this condition is not sufficient.

Example 8.2. Let $X = \mathbb{R}^J$ where J is an uncountable set. Equipped with the product topology, τ, and point-wise ordering, X is a Hausdorff locally solid vector lattice with the Lebesgue property. It is a standard fact of topology that (X, τ) is not metrizable. Since the unbounded topology of a product is the product of the unbounded topologies by [Tay1, Theorem 3.1], we have $u\tau = \tau$, so that $u\tau$ is not metrizable. Notice that the function $\mathbb{1} \in \mathbb{R}^J$ is a quasi-interior point of X since τ is Lebesgue and $\mathbb{1}$ is, clearly, a weak unit.

When τ is metrizable, the following theorem provides the complete answer on metrizability of $u\tau$.

Theorem 8.3. *For a metrizable locally solid vector lattice (X, τ) the following statements are equivalent:*

- *(i) There is an at most countable set A in X such that $\overline{I(A)}^{\tau} = X$;*
- *(ii) $u\tau$ is metrizable;*
- *(iii) $u\widehat{\tau}$ is metrizable;*
- *(iv) The topological completion $\widehat{X}$ contains a quasi-interior point.*

Proof. Recall that τ is metrizable if and only if $\widehat{\tau}$ is metrizable.

(i)⇔(ii): Suppose (u_n) is a positive increasing sequence such that $A = \{u_n\} \subseteq X_+$ satisfies $\overline{I(A)}^{\tau} = X$. Let $\{U_i\}$ be a countable basis at zero for τ consisting of solid sets. Since, in particular, $B(A) = X$, as in the proof of Proposition 6.35 the collection $\{U_{i,u_n}\}$ is a base at zero for a metrizable locally solid topology $\sigma_1 \subseteq u\tau$. We claim that $\sigma_1 = u\tau$. Indeed, by [Tay1, Proposition 9.5] $u_{I(A)}\tau = u\tau$ and it is easy to see that $\sigma_1 = u_{I(A)}\tau$. We already know (ii)$\Rightarrow$(i).

(ii)$\Leftrightarrow$(iii): Suppose $u\tau$ is metrizable. It follows that there is an at most countable set A in X such that $\overline{I(A)}^{\tau} = X$. Since X is $\widehat{\tau}$-dense in $(\widehat{X}, \widehat{\tau})$, $I(A)$ is $\widehat{\tau}$-dense in $\widehat{X}$. Hence, the ideal generated by A in $\widehat{X}$ is also $\widehat{\tau}$-dense in $\widehat{X}$. This implies that $u\widehat{\tau}$ is metrizable by applying (i)$\Leftrightarrow$(ii) to $\widehat{\tau}$. Conversely, if $u\widehat{\tau}$ is metrizable then so is $u\tau$ since $(u\widehat{\tau})|_X = u\tau$ by [Tay1, Lemma 3.5].

(iii)$\Rightarrow$(iv): Since $u\widehat{\tau}$ is metrizable, there exists a sequence $(e_n) \subseteq \widehat{X}_+$ such that $\overline{I(\{e_n\})}^{\widehat{\tau}} = \widehat{X}$. Since $\widehat{\tau}$ is metrizable, there is a countable neighbourhood basis $\{V_n\}$ of zero in $\widehat{X}$ consisting of solid sets such that for each $n \in \mathbb{N}$ we have $V_{n+1} + V_{n+1} \subseteq V_n$. For each $n \in \mathbb{N}$ pick $\lambda_n > 0$ such that $\lambda_n e_n \in V_n$. We claim that the series $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in $\widehat{X}_+$. To prove this, define $s_n = \sum_{k=1}^{n} \lambda_k e_k$ and pick a solid neighbourhood V_0 of zero in $\widehat{X}$. Find $n_0 \in \mathbb{N}$ such that $V_{n_0} \subseteq V_0$. Then for $m > n \geq n_0$ we have

$$s_m - s_n = \lambda_{n+1} e_{n+1} + \cdots + \lambda_m e_m \in V_n \subseteq V_{n_0} \subseteq V_0,$$

so that the partial sums (s_n) of the series $\sum_{n=1}^{\infty} \lambda_n e_n$ form a Cauchy sequence in $\widehat{X}$. Since $(\widehat{X}, \widehat{\tau})$ is complete and Hausdorff, the series converges to an element of $\widehat{X}_+$. It is clear that $\sum_{n=1}^{\infty} \lambda_n e_n$ is a quasi-interior point of $\widehat{X}$.

(iv)$\Rightarrow$(iii): Since we have established the implication (i)$\Rightarrow$(ii) for any metrizable locally solid vector lattice, we simply apply it to $(\widehat{X}, \widehat{\tau})$. $\square$

In the case when (X, τ) is complete, Theorem 8.3 reduces to the previously obtained result for Banach lattices.

Corollary 8.4. *Let (X, τ) be a complete metrizable locally solid vector lattice. Then $u\tau$ is metrizable iff X has a quasi-interior point.*

By Example 6.33 there is no reason to believe that X has a quasi-interior point if τ and $u\tau$ are metrizable.

Remark 8.5. Example 8.1 shows that it can happen that τ is not metrizable even when $u\tau$ is metrizable and there is a countable set A such that $\overline{I(A)}^{\tau} = X$.

It so happens that (ii)⇔(iii) in Theorem 8.3 remains valid even when τ is not metrizable. We prove this now:

Proposition 8.6. *Let (X, τ) be a Hausdorff locally solid vector lattice. Then $u\tau$ is metrizable iff $u\widehat{\tau}$ is metrizable.*

Proof. If $u\widehat{\tau}$ is metrizable, then so is $u\tau$; this follows since $u\tau = (u\widehat{\tau})|_X$. Suppose $u\tau$ is metrizable and let $\{V_{U_n,u_n}\}$ be a countable basis for $u\tau$ where U_n is a solid τ-closed neighbourhood at zero in X, $u_n \in X_+$, and $V_{U_n,u_n} := \{x \in X : |x| \wedge u_n \in U_n\}$ is defined for notational convenience. Find a τ-closed solid neighbourhood U'_n of zero for τ with $U'_n + U'_n \subseteq U_n$. We claim that $\{V_{\overline{U'_n}^{\widehat{\tau}},u_n}\}$ is a basis for $u\widehat{\tau}$ where $V_{\overline{U'_n}^{\widehat{\tau}},u_n} := \{\widehat{x} \in \widehat{X} : |\widehat{x}| \wedge u_n \in \overline{U'_n}^{\widehat{\tau}}\}$.

Let $V_{\overline{Z}^{\widehat{\tau}},\widehat{x}}$ be an arbitrary base neighbourhood of zero for $u\widehat{\tau}$. Here, Z is a solid neighbourhood of zero for τ and $\widehat{x} \in \widehat{X}_+$. Find U a solid neighbourhood of zero for τ with $U + U \subseteq Z$. Since X is $\widehat{\tau}$-dense in $\widehat{X}$, there exists $x \in X_+$ with $|\widehat{x} - x| \in \overline{U}^{\widehat{\tau}}$. Find W a solid neighbourhood of zero for τ with $W + W \subseteq U$. There exists n such that $V_{U_n,u_n} \subseteq V_{W,x}$.

Let $\widehat{y} \in V_{\overline{U'_n}^{\widehat{\tau}},u_n}$; we will show that $\widehat{y} \in V_{\overline{Z}^{\widehat{\tau}},\widehat{x}}$. Find $y \in X$ with $|\widehat{y} - y| \in \overline{U'_n}^{\widehat{\tau}} \cap \overline{W}^{\widehat{\tau}}$. Then

$$|y| \wedge u_n \leq |\widehat{y} - y| \wedge u_n + |\widehat{y}| \wedge u_n \in \overline{U'_n}^{\widehat{\tau}} + \overline{U'_n}^{\widehat{\tau}} \subseteq \overline{U_n}^{\widehat{\tau}}.$$

Since U_n is τ-closed in X, $|y| \wedge u_n \in \overline{U_n}^{\widehat{\tau}} \cap X = U_n$. Therefore, $y \in V_{U_n,u_n} \subseteq V_{W,x}$. This implies that $|y| \wedge x \in W$. Hence

$$|\widehat{y}| \wedge x \leq |\widehat{y} - y| \wedge x + |y| \wedge x \in \overline{W}^{\widehat{\tau}} + W \subseteq \overline{U}^{\widehat{\tau}}.$$

Combining gives,

$$|\widehat{y}| \wedge \widehat{x} \leq |\widehat{y}| \wedge |\widehat{x} - x| + |\widehat{y}| \wedge x \in \overline{U}^{\widehat{\tau}} + \overline{U}^{\widehat{\tau}} \subseteq \overline{Z}^{\widehat{\tau}}.$$

□

We next extend our results on metrizability to $u_A\tau$. One should compare the next result with [KLT17, Theorem 3.3]:

Proposition 8.7. *Let A be a τ-closed ideal of a metrizable locally solid vector lattice (X, τ). TFAE:*

(i) $u_A\tau$ on X is metrizable.

(ii) $u(\tau|_A)$ on A is metrizable and A is order dense in X.

(iii) A contains an at most countable set B such that $\overline{I(B)}^{\tau} = A$ and B is a countable order basis for X.

Proof. (i)⇒(ii): If $u_A\tau$ is metrizable on X, then $u(\tau|_A)$ being the relative topology of $u_A\tau$ is metrizable on A. Since $u_A\tau$ is Hausdorff, A is order dense in X.

(ii)⇒(iii): Since $u(\tau|_A)$ is metrizable, by Proposition 7.10 there is an at most countable set $B \subseteq A_+$ with $\overline{I(B)}^{\tau|_A} = A$. Since A is closed, this implies that $\overline{I(B)}^{\tau} = A$. Pick $x \in X_+$ with $x \perp B$. If x is nonzero, there is $a \in A_+$ with $0 < a \leq x$. Since $a \perp B$ and $\overline{I(B)}^{\tau|_A} = A$, we have $a = 0$. This contradiction shows $x = 0$, so that by [CL17, Lemma 2.2] we conclude that B is a countable order basis for X.

(iii)⇒(i): Let $B := \{b_n\} \subseteq A_+$ be a countable order basis for X such that $\overline{I(B)}^{\tau} = A$. As always, we assume (b_n) is a positive increasing sequence. Following Proposition 6.35, the sets $U_{i,b_n} := \{x \in X : |x| \wedge b_n \in U_i\}$, where $\{U_i\}$ is a countable solid base at zero for τ, defines a metrizable locally solid topology τ_1 on X. Note that

$$x_\alpha \xrightarrow{\tau_1} 0 \Leftrightarrow \forall n \ |x_\alpha| \wedge b_n \xrightarrow{\tau} 0 \Leftrightarrow x_\alpha \xrightarrow{u_{I(B)}\tau} 0 \Leftrightarrow x_\alpha \xrightarrow{u_A\tau} 0,$$

so that $u_A\tau$ is metrizable. □

Remark 8.8. The assumption that A is τ-closed is for convenience since $u_A\tau = u_{\overline{A}^{\tau}}\tau$.

It is well known that all Hausdorff Lebesgue topologies induce the same topology on order intervals, see, for example, [AB03, Theorem 4.22]. Since minimal topologies are Hausdorff and Lebesgue, the "local" properties of minimal topologies are well studied. For example, by [AB03, Theorem 4.26], if τ is a Hausdorff Lebesgue topology then τ induces a metrizable topology on the order intervals of X if and only if X has the countable sup property. We conclude this section with a complete characterization of "global" metrizability of minimal topologies.

Let X be a vector lattice admitting a minimal topology τ. In [Con05] it was shown that, if X has a weak unit, τ is metrizable iff X has the countable sup property. Theorem 8.9 removes the weak unit assumption and characterizes metrizability of τ in terms of the vector lattice structure of X. Recall that C_τ, the carrier of the locally solid topology τ, is defined in [AB03, Definition 4.15].

Theorem 8.9. *Suppose that X is a vector lattice admitting a minimal topology τ. Then τ is metrizable if and only if X has the countable sup property and a countable order basis.*

Proof. Recall that, being minimal, τ is unbounded, Hausdorff and Lebesgue by [Tay1, Theorem 6.4].

If τ is metrizable and Lebesgue then X has the countable sup property by [AB03, Theorem 5.33] (actually, by passing through completions one can replace Lebesgue by pre-Lebesgue in this step. σ-Lebesgue is insufficient, however). Since τ is metrizable and locally solid, τ is Riesz submetrizable. Since τ is unbounded and Riesz submetrizable, X has a countable order basis by Remark 6.37.

Suppose τ is Hausdorff, unbounded and Lebesgue and that X has the countable sup property and admits a countable order basis. By [AB03, Theorem 4.17(b)], $C_\tau = X$. Let $\{u_n\}$ be a countable order basis of X. As in the proof of [AB03, Theorem 4.17(a)] there exists a normal sequence $\{U_n\}$ of solid τ-neighbourhoods of zero such that $\{u_n\} \subseteq N^d$, where $N = \bigcap_{n=1}^{\infty} U_n$. Since N^d is a band, $X = B(\{u_n\}) \subseteq N^d$, so that $X = N^d$. The sequence $\{U_n\}$ defines a metrizable locally solid topology τ' on X satisfying $\tau' \subseteq \tau$. Since τ is minimal, $\tau = \tau'$ so that τ is metrizable. □

We remark that $C[0,1]$ satisfies the countable sup property, admits a metrizable locally solid topology, and has a countable order basis, but does not admit a minimal topology. Indeed, a vector lattice admits a minimal topology iff it admits a Hausdorff Lebesgue topology, and no Hausdorff Lebesgue topology exists on $C[0,1]$ by [AB03, Example 3.2].

Recall that [AB03, Theorem 7.55] states that if a laterally σ-complete vector lattice admits a metrizable locally solid topology τ then τ is the only Hausdorff locally solid topology on X and τ is Lebesgue. Therefore, τ is the minimal topology. After reminding oneself that laterally complete vector lattices admit weak units by, say, [AB03, Theorem 7.2], the next corollary is an immediate consequence of [Tay1, Corollary 5.3]:

Corollary 8.10. *Suppose X is a vector lattice that admits a minimal topology τ. τ is complete and metrizable if and only if X is universally complete and has the countable sup property. In this case, τ is the only Hausdorff locally solid topology on X.*

One of the main purposes of unbounding topologies is to create topologies that are "characteristically weak". One way to measure how "weak" a topology is to look at disjoint sequences and, specfically, whether they are topologically null. Obviously, the coarser τ is, the more likely it is that disjoint sequences are $u\tau$-null. The next theorem looks at the worst extreme. By [AB03, Theorem 5.20], if (X,τ) is a Frechet space ([AB03, Definition 5.16]), then τ is the finest locally solid topology on X. In particular, we should not expect disjoint sequences to be $u\tau$-null. However, we will see that from every $u\tau$-null net one can extract an asymptotically disjoint sequence.

Theorem 8.11. *Let (x_α) be a net in a Frechet space (X,τ) such that $x_\alpha \xrightarrow{u\tau} 0$. Then there exists an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{\tau} 0$.*

Proof. Modify [DOT17, Theorem 3.2]. □

Remark 8.12. [KMT17, Example 1.3] shows that it is not always possible to extract a un-null sequence from a un-null net. At the cost of being moderated by a disjoint sequence, however, Theorem 8.11 says that one can extract a *norm* null sequence.

It has not been investigated to what extent Theorem 8.11 can be extended past Frechet spaces, or when one can get a version replacing τ with o. Certainly the theorem can't hold for all topologies, and may fail for o (since disjoint sequences are uo-null, and uo is not always sequential. For example, the theorem fails in any universally complete space with a minimal topology but without CSP).

By *uo* being sequential we mean that one can extract a *uo*-null sequence from a *uo*-null net. This property will be studied later when we deal with measure theoretic results. In [LT] they introduce a complementary property called OSSP. This property is needed for several results in their paper, which was motivated by recent advances in the theory of risk measures.

9. Locally bounded unbounded topologies

In this section we present a theorem and examples regarding local boundedness of unbounded topologies. If (X,τ) is locally bounded and Hausdorff, then (X,τ) is metrizable. By Theorem 8.3 we already know that metrizability of $u\tau$ is equivalent to the topological completion $\widehat{X}$ of X having a quasi-interior point. When studying Hausdorff locally bounded unbounded topologies, it is strong units of the topological completion that are of interest.

Theorem 9.1. *Let τ be a Hausdorff locally solid topology on a vector lattice X. TFAE:*

- *(i) $u\tau$ is locally bounded;*
- *(ii) $u\tau$ has an order bounded neighbourhood of zero;*
- *(iii) τ has an order bounded neighbourhood of zero;*
- *(iv) $\widehat{X}$ has a strong unit and τ is metrizable;*
- *(v) $\widehat{\tau}$ coincides with the $\|\cdot\|_u$-topology, where u is a strong unit of $\widehat{X}$, and $(\widehat{X},\|\cdot\|_u)$ is a Banach lattice which is lattice isometric to a $C(K)$-space;*
- *(vi) $u\widehat{\tau}$ is locally bounded.*

In this case, $\tau = u\tau$ and $\widehat{\tau} = u\widehat{\tau}$.

Proof. (i)⇒(ii): Suppose $u\tau$ is locally bounded. Then there exists a neighbourhood V of zero such that a base at zero for $u\tau$ is given by εV for $\varepsilon > 0$. Find U a solid neighbourhood of zero for τ and $u \in X_+$ so that, in the notation of [Tay1, Lemma 2.16], $U_u \subseteq V$. We claim that V cannot contain a non-trivial ideal, so that U_u cannot contain a non-trivial ideal, so that $U_u \subseteq [-u,u]$ by [Tay1, Lemma 2.16]. Suppose V contains a non-trivial ideal. Then there exists $x \neq 0$ such that $\lambda x \in V$

for all $\lambda > 0$. However, this implies that $x \in \varepsilon V$ for all $\varepsilon > 0$ and hence $x = 0$. This is a contradiction.

(ii)⇒(iii) because $u\tau \subseteq \tau$.

(iii)⇒(iv): Suppose τ has an order bounded neighbourhood of zero, say, $[-u, u]_X$, where the subscript denotes the space in which the order interval is taken. It follows that τ is metrizable and $\overline{[-u, u]_X}^{\widehat{\tau}}$ is a $\widehat{\tau}$-neighbourhood of zero; it is contained in $[-u, u]_{\widehat{X}}$ since the cone $\widehat{X}_+$ is $\widehat{\tau}$-closed. Therefore, $\widehat{\tau}$ has an order bounded neighbourhood of zero and thus $\widehat{X}$ has a strong unit.

(iv)⇒(v): Let u be a strong unit for $\widehat{X}$. Since τ is metrizable, $\widehat{\tau}$ is complete and metrizable. It is easy to see that $\widehat{X}$ is uniformly complete so that, by [AB03, Theorem 5.21], the $\widehat{\tau}$ and $\|\cdot\|_u$-topologies agree. That $(\widehat{X}, \|\cdot\|_u)$ is lattice isometric to a $C(K)$-space follows from Kakutani's representation theorem.

(v)⇒(vi): It follows from [KMT17, Theorem 2.3] that $\widehat{\tau} = u\widehat{\tau}$ so that $u\widehat{\tau}$ is locally bounded.

(vi)⇒(i) follows since $(u\widehat{\tau})|_X = u\tau$. For the additional clause, it has been shown that $\widehat{\tau} = u\widehat{\tau}$ from which it follows that $\tau = (u\widehat{\tau})|_X = u(\widehat{\tau}|_X) = u\tau$. □

We next show that our results cannot be extended to $u_A\tau$:

Proposition 9.2. *Let A be an order dense ideal of a Hausdorff locally solid vector lattice (X, τ). If $u_A\tau$ is locally bounded then $A = X$.*

Proof. Notice first that $u_A\tau$ is Hausdorff. Assuming $u_A\tau$ is locally bounded, there exists a neighbourhood V such that a base at zero for $u_A\tau$ is given by εV for $\varepsilon > 0$. Find U a solid neighbourhood of zero for τ and $a \in A_+$ so that, in the notation of [Tay1, Lemma 2.16], $U_a \subseteq V$. As in the proof of Theorem 9.1, $U_a \subseteq [-a, a]$. Since neighbourhoods are absorbing, a is a strong unit for X. Therefore, $X = I_a \subseteq A$. □

Example 9.3. Consider $(X_1, \tau_1) := (C[0, 1], \|\cdot\|_\infty)$. X_1 is a complete, Hausdorff, locally bounded, unbounded locally solid vector lattice that has a strong unit. On the other hand, $(X_2, \tau_2) := (C[0, 1], \|\cdot\|_2)$ is a Hausdorff, locally bounded, locally solid topology, but the topology

$u\tau_2$ is not locally bounded. This is consistent with Theorem 9.1 as $\widehat{(X_2, \tau_2)} = (L_2, \|\cdot\|_2)$ does not have a strong unit.

Corollary 9.4. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) *τ is locally bounded;*
(ii) *X is finite dimensional.*

Proof. (i)⇒(ii): Since τ is minimal, it is Hausdorff, and $\tau = u\tau$. Theorem 9.1 implies that $\widehat{X}$ has a strong unit. By [Tay1, Theorem 5.2], X^u has a strong unit. By [AB03, Theorem 7.47], X is finite dimensional. The other direction is clear. □

The standard fact that $C(K)$ has order continuous norm iff it is finite-dimensional can be immediately deduced from the above corollary. Simply note that the norm topology on $C(K)$ is unbounded. Recall also that minimal is equivalent to Hausdorff Lebesgue and unbounded; Corollary 9.4 still holds if Lebesgue is weakened to pre-Lebesgue (use [AB03, Theorem 3.26]), but does not hold if Lebesgue is weakened to σ-Lebesgue and Fatou.

10. A thorough study of minimal topologies

First, we collect the characterizations of minimal topologies given in this thesis. We encourage the reader to retrace the thesis and collect the results on minimal topologies that have been presented under the name "uo-Lebesgue" or "unbounded and Lebesgue". Notice, in particular, that we have characterised the completions of minimal topologies, their duals, when the restriction to a sublattice is again minimal, etc.

Theorem 10.1. *Let τ be a Hausdorff locally solid topology on a vector lattice X. TFAE:*

(i) *uo-null nets are τ-null;*
(ii) *τ is Lebesgue and unbounded;*
(iii) *τ is minimal;*
(iv) *$\tau = u_A\tau$ for every order dense ideal A of X.*

Moreover, such a topology τ *exists iff* X *admits a Hausdorff Lebesgue topology* σ*. In this case,* $\tau = u\sigma$*, so minimal topologies are, in fact, unique.*

Nets cannot be replaced with sequences in (i) if equivalence is to be maintained. Indeed, [GTX17, Theorem 3.9] states that order and *uo*-convergences agree for sequences in universally σ-complete vector lattices. Combining this observation with [AB03, Theorem 7.49], we conclude that *uo*-convergent sequences are topologically convergent for any locally solid topology on a universally σ-complete space. However, [AB03, Chapter 7 Exercise 21] gives an example of a Hausdorff locally solid topology on a universally σ-complete vector lattice that fails to be Lebesgue, and thus fails to be minimal. We will study this further in the sections on σ-universal topologies.

Remark 10.2. Note that, by definition, a Hausdorff locally solid topology τ is minimal if $\sigma \subseteq \tau$ and σ Hausdorff and locally solid implies $\sigma = \tau$. Minimal topologies have also been studied in the category of Hausdorff topological vector spaces, and it is known (I learnt this from slides of W. Wnuk, who credits the discovery to N.T. Peck) that on $L_0[0,1]$ there exist Hausdorff linear topologies coarser than convergence in measure. This demonstrates the connection between order and measure theory.

Before proceeding to more properties of minimal topologies, we characterize when minimal topologies are least:

Proposition 10.3. [Con05] *Suppose* (X, σ) *is a Hausdorff locally solid vector lattice admitting a minimal topology* τ*. Then* σ *is entire iff* $\tau \subseteq \sigma$*. In particular, a minimal topology is least iff every Hausdorff locally solid topology on* X *is entire.*

Proof. Follows from [AB03, Theorem 5.48]. □

Throughout this section, X is a vector lattice and τ denotes a locally solid topology on X. We begin with a brief discussion on relations between minimal topologies and the B-property. Corollary 10.6 will be of importance as many properties of locally solid topologies are stated in terms of positive increasing nets. For minimal topologies, these properties permit a uniform and efficient treatment.

The B-property was introduced as property (B,iii) by W.A.J. Luxemburg and A.C. Zaanen in [LZ64]. It is briefly studied in [AB03] and, in particular, it is shown that the Lebesgue property does not imply the B-property. We prove, however, that if τ is unbounded then this implication does indeed hold true:

Definition 10.4. *A locally solid vector lattice (X, τ) satisfies the* ***B-property*** *if it follows from $0 \leq x_n \uparrow$ in X and (x_n) τ-bounded that (x_n) is τ-Cauchy. An equivalent definition is obtained if sequences are replaced with nets.*

Proposition 10.5. *If X is a vector lattice admitting a minimal topology τ, then τ satisfies the B-property.*

Proof. Suppose τ is minimal and (x_n) is a τ-bounded sequence satisfying $0 \leq x_n \uparrow$. By [AB03, Theorem 7.50], (x_n) is dominable. By [AB03, Theorem 7.37], (x_n) is order bounded in X^u so that $x_n \xrightarrow{uo} u$ for some $u \in X^u$. In particular, (x_n) is uo-Cauchy in X^u. It follows that (x_n) is uo-Cauchy in X. Since τ is Lebesgue, (x_n) is $u\tau$-Cauchy in X. Finally, since τ is unbounded, (x_n) is τ-Cauchy in X. □

Corollary 10.6. *Let X be a vector lattice admitting a minimal topology τ, and (x_α) an increasing net in X_+. TFAE:*

(i) (x_α) is τ-bounded;
(ii) (x_α) is τ-Cauchy.

Recall the following definition, taken from [AB03, Definition 2.43].

Definition 10.7. *A locally solid vector lattice (X, τ) is said to satisfy the* ***monotone completeness property (MCP)*** *if every increasing τ-Cauchy net of X_+ is τ-convergent in X. The σ-MCP is defined analogously with nets replaced with sequences.*

Remark 10.8. By Corollary 10.6, a minimal topology has MCP iff it is Levi.

A locally solid topology τ on a vector lattice X is ***pre-Lebesgue*** if order bounded disjoint sequences in X are τ-null. By [AB03, Theorem 3.23], the Lebesgue property implies the pre-Lebesgue property.

Lemma 10.9. *Let (X, τ) be a Hausdorff locally solid vector lattice. If τ is unbounded then TFAE:*

(i) τ has MCP and is pre-Lebesgue;
(ii) τ is Lebesgue and Levi.

Proof. It is sufficient, by [DL98, Theorem 2.5], to prove that (X, τ) contains no lattice copy of c_0. Suppose, towards contradiction, that X does contain a lattice copy of c_0, i.e., there is a homeomorphic Riesz isomorphism from c_0 onto a sublattice of X. This leads to a contradiction as the standard unit vector basis is not null in c_0, but the copy in X is by [Tay1, Theorem 4.2]. □

Lemma 10.9 is another way to prove that a minimal topology has MCP iff it is Levi. We next present the sequential analogue:

Lemma 10.10. *Let (X, τ) be a Hausdorff locally solid vector lattice. If τ is unbounded then TFAE:*

(i) τ has σ-MCP and is pre-Lebesgue;
(ii) τ is σ-Lebesgue and σ-Levi.

Proof. (i)⇒(ii) is similar to the last lemma; apply instead [DL98, Proposition 2.1 and Theorem 2.4].

(ii)⇒(i): It suffices to show that τ is pre-Lebesgue. For this, suppose that $0 \leq x_n \uparrow\leq u$; we must show that (x_n) is τ-Cauchy. Since τ is σ-Levi and order bounded sets are τ-bounded, $x_n \uparrow x$ for some $x \in X$. Since τ is σ-Lebesgue, $x_n \xrightarrow{\tau} x$. □

Putting pieces together from other papers, we next characterize sequential completeness of *uo*-convergence.

Theorem 10.11. *Let X be a vector lattice. TFAE:*

(i) X is sequentially uo-complete;
(ii) Every positive increasing uo-Cauchy sequence in X uo-converges in X;

(iii) X is universally σ-complete.

In this case, uo-Cauchy sequences are order convergent.

Proof. (i)⇒(ii) is clear. (ii)⇒(iii) by careful inspection of [CL17, Proposition 2.8], (iii)⇒(i) and the moreover clause follow from [GTX17, Theorem 3.10]. □

Remark 10.12. Recall that by [AB03, Theorem 7.49], every locally solid topology on a universally σ-complete vector lattice satisfies the pre-Lebesgue property. Using *uo*-convergence, we give a quick proof of this. Suppose τ is a locally solid topology on a universally σ-complete vector lattice X; we claim that *uo*-null sequences are τ-null. This follows since τ is σ-Lebesgue and *uo* and *o*-convergence agree for sequences by [GTX17, Theorem 3.9]. In particular, since disjoint sequences are *uo*-null, disjoint sequences are τ-null.

We next give the topological analogue of Theorem 10.11:

Lemma 10.13. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) τ is σ-Levi;
(ii) τ has σ-MCP;
(iii) X is universally σ-complete;
(iv) (X, τ) is sequentially boundedly uo-complete in the sense that τ-bounded uo-Cauchy sequences in X are uo-convergent in X.

Proof. (i)⇔(ii) follows from Lemma 10.10. We next deduce (iii). Since τ is σ-Levi, X is σ-order complete; we prove X is laterally σ-complete. Let $\{a_n\}$ be a countable collection of mutually disjoint positive vectors in X, and define $x_n = \sum_{k=1}^n a_k$. Then (x_n) is a positive increasing sequence in X, and it is *uo*-Cauchy, as an argument similar to [CL17, Proposition 2.8] easily shows. By Theorem 10.1, (x_n) is τ-Cauchy, hence $x_n \xrightarrow{\tau} x$ for some $x \in X$ since τ has σ-MCP. Since (x_n) is increasing and τ is Hausdorff, $x_n \uparrow x$. Clearly, $x = \sup\{a_n\}$.

(iii)⇒(iv) follows from Theorem 10.11; (iv)⇒(i) follows immediately after noticing that Proposition 12.3 is valid (similar proof) if "weakly Fatou Banach lattice" is replaced by "Hausdorff Fatou topology". □

The following question(s) remain open:

Question 10.14. Let X be a vector lattice admitting a minimal topology τ. Are the following equivalent?

(i) (X,τ) is sequentially complete;
(ii) X is universally σ-complete.

Question 10.15. Let (X,τ) be Hausdorff and Lebesgue. Are the following equivalent?

(i) Order intervals of X are sequentially τ-complete;
(ii) X is σ-order complete.

Remark 10.16. Question 10.14 and Question 10.15 are equivalent. Indeed, in both cases it is known that (i)⇒(ii). If Question 10.15 is true then Question 10.14 is true since we have already established that minimal topologies have σ-MCP when X is universally σ-complete. Suppose Question 10.14 is true. If X is σ-order complete, then X is an ideal in its universal σ-completion, X^s. Indeed, it is easy to establish that if Y is a σ-order complete vector lattice sitting as a super order dense sublattice of a vector lattice Z, then Y is an ideal of Z; simply modify the arguments in [AB03, Theorem 1.40]. By [AB03, Theorem 4.22] we may assume, WLOG, that τ is minimal. τ then lifts to the universal completion and can be restricted to X^s.

Question 10.15 is a special case of Aliprantis and Burkinshaw's [AB78, Open Problem 4.2], which we state as well:

Question 10.17. Suppose τ is a Hausdorff σ-Fatou topology on a σ-order complete vector lattice X. Are the order intervals of X sequentially τ-complete?

The case of complete order intervals is much easier than the sequentially complete case. The next result is undoubtedly known, but fits in nicely; we provide a simple proof that utilizes minimal topologies.

Proposition 10.18. *Suppose τ is a Hausdorff Lebesgue topology on X. Order intervals of X are complete iff X is order complete.*

Proof. If X is order complete then order intervals are complete by [AB03, Theorem 4.28].

By [AB03, Theorem 4.22] we may assume, WLOG, that τ is minimal. If order intervals are complete then X is an ideal of $\widehat{X} = X^u$ by [AB03, Theorem 2.42] and [Tay1, Theorem 5.2]. Since X^u is order complete, so is X. □

Remark 10.19. If X is an order complete and laterally σ-complete vector lattice admitting a minimal topology τ, then τ is sequentially complete. Although these conditions are strong, they do not force X to be universally complete. This can be seen by equipping the vector lattice of [AB03, Example 7.41] with the minimal topology given by restriction of pointwise convergence from the universal completion.

The key step in the proof of Theorem 12.5 is [AW97, Theorem 2.4] which states that a Banach lattice is σ-Levi if and only if it is *laterally* σ-Levi. We say that a locally solid vector lattice (X, τ) has the ***lateral σ-Levi property*** if $\sup x_n$ exists whenever (x_n) is laterally increasing and τ-bounded. For minimal topologies, the σ-Levi and lateral σ-Levi properties do not agree, as we now show:

Proposition 10.20. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) X is laterally σ-complete;
(ii) τ has the lateral σ-Levi property;
(iii) Every disjoint positive sequence, for which the set of all possible finite sums is τ-bounded, must have a supremum.

Proof. (i)⇒(iii) is clear, as is (ii)⇔(iii); we prove (ii)⇒(i). Assume (ii) and let (x_n) be a disjoint sequence in X_+. Since (x_n) is disjoint, (x_n) has a supremum in X^u. Define $y_n = x_1 \vee \dots \vee x_n$. The sequence (y_n) is laterally increasing and order bounded in X^u. By [AB03, Theorem 7.37], (y_n) forms a dominable set in X_+. By [Tay1, Theorem 5.2(iv)], (y_n) is τ-bounded, and hence has supremum in X by assumption. This implies that (x_n) has a supremum in X and, therefore, X is laterally σ-complete. □

In [Lab84] and [Lab85], many completeness-type properties of locally solid topologies were introduced. For entirety, we classify the remaining properties, which he refers to as "BOB" and "POB".

Definition 10.21. *A Hausdorff locally solid vector lattice (X,τ) is said to be* ***boundedly order-bounded (BOB)*** *if increasing τ-bounded nets in X_+ are order bounded in X. (X,τ) satisfies the* ***pseudo-order boundedness property (POB)*** *if increasing τ-Cauchy nets in X_+ are order bounded in X.*

Remark 10.22. It is clear that a Hausdorff locally solid vector lattice is Levi iff it is order complete and boundedly order-bounded. It is also clear that BOB and POB coincide for minimal topologies.

Proposition 10.23. *Let X be a vector lattice admitting a minimal topology τ. TFAE:*

(i) (X,τ) satisfies BOB;

(ii) X is majorizing in X^u.

Proof. (i)⇒(ii): Let $0 \leq u \in X^u$. Since X is order dense in X^u, there exists a net (x_α) in X such that $0 \leq x_\alpha \uparrow u$. In particular, (x_α) is order bounded in X^u, hence dominable in X by [AB03, Theorem 7.37]. By [Tay1, Theorem 5.2], (x_α) is τ-bounded. By assumption, (x_α) is order bounded in X, hence, $(x_\alpha) \subseteq [0,x]$ for some $x \in X_+$. It follows that $u \leq x$, so that X majorizes X^u.

(ii)⇒(i): Suppose (x_α) is an increasing τ-bounded net in X_+. It follows from [AB03, Theorem 7.50] that (x_α) is dominable, hence order bounded in X^u. Since X majorizes X^u, (x_α) is order bounded in X. □

Remark 10.24. By [AB03, Theorem 7.15], laterally complete vector lattices majorize their universal completions.

Remark 10.25. If τ is a Hausdorff Fatou topology on X, it is easy to see that (X,τ) satisfies BOB iff every increasing τ-bounded net in X_+ is order Cauchy in X. Compare with Problem 12.1.

We next state the σ-analogue of Proposition 10.23.

Proposition 10.26. *Let X be an almost σ-order complete vector lattice admitting a minimal topology τ. TFAE:*

(i) (X, τ) *satisfies* σ*-BOB;*

(ii) X *is majorizing in the universal* σ*-completion* X^s *of* X*.*

Proof. (i)⇒(ii) is similar to Proposition 10.23.

(ii)⇒(i): Suppose (x_n) is an increasing τ-bounded sequence in X_+. It is then dominable in X, hence in X^s by [AB03, Lemma 7.11]. It follows by [AB03, Theorem 7.38] that (x_n) is order bounded in X^s. Since X is majorizing in X^s, (x_n) is order bounded in X. □

The next definition is standard in the theory of topological vector spaces:

Definition 10.27. *Let* (E, σ) *be a Hausdorff topological vector space.* E *is* ***quasi-complete*** *if every* σ*-bounded* σ*-Cauchy net is* σ*-convergent.*

Remark 10.28. Since Cauchy sequences are bounded, there is no sequential analogue of quasi-completeness.

We finish with the full characterization of completeness of minimal topologies:

Theorem 10.29. *Let* X *be a vector lattice admitting a minimal topology* τ*. TFAE:*

(i) X *is universally complete;*

(ii) τ *is complete;*

(iii) τ *satisfies MCP;*

(iv) τ *is Levi;*

(v) τ *is quasi-complete;*

(vi) (X, τ) *is boundedly uo-complete in the sense that* τ*-bounded uo-Cauchy nets in* X *are uo-convergent in* X*.*

Proof. (i)⇔(ii) by [Tay1, Corollary 5.3] combined with [Tay1, Theorem 6.4]. Clearly, (ii)⇒(iii)⇔(iv). (iii)⇒(ii) since if τ satisfies MCP then τ is topologically complete by [AB03, Corollary 4.39]. We have thus established that (i)⇔(ii)⇔(iii)⇔(iv). It is clear that (ii)⇒(v), and (v)⇒(iii) by Corollary 10.6.

(ii)⇒(vi): Let (x_α) be a *uo*-Cauchy net in X; (x_α) is then τ-Cauchy and hence τ-convergent. The claim then follows from [Tay1, Remark 2.26].

(vi)⇒(iv): Suppose $0 \le x_\alpha \uparrow$ is τ-bounded. (x_α) is then uo-Cauchy, hence uo-convergent to some $x \in X$. Clearly, $x = \sup x_\alpha$. □

Remark 10.30. This is in good agreement with Proposition 5.14. If the minimal topology satisfies MCP then Proposition 5.14 states that every Hausdorff Lebesgue topology satisfies MCP. Universally complete spaces, however, admit at most one Hausdorff Lebesgue topology by [AB03, Theorem 7.53].

11. Measure-theoretic results

Let (Ω, Σ, μ) be a σ-finite measure space. For each $E \in \Sigma$ with $\mu(E) < \infty$ define the Riesz pseudonorm $\rho_E : L_0(\mu) \to \mathbb{R}$ via

$$\rho_E(x) = \int_E \frac{|x|}{1+|x|} d\mu.$$

The family of Riesz pseudonorms $\{\rho_E : E \in \Sigma \text{ and } \mu(E) < \infty\}$ defines a Hausdorff locally solid topology τ_μ on $L_0(\mu)$ known as the ***topology of (local) convergence in measure on*** $L_0(\mu)$. For $0 \le p \le \infty$, the topology of convergence in measure on $L_p(\mu)$ is defined, simply, as the restriction $\tau_\mu|_{L_p(\mu)}$.

The equivalence of (i) and (iii) in Theorem 10.1 has roots in classical relations between convergence almost everywhere and convergence in measure. Let (Ω, Σ, μ) be a σ-finite measure space. It is classically known that for $0 \le p < \infty$, the topology of convergence in measure is the least topology on $L_p(\mu)$, c.f., [AB03, Theorem 7.55] and [AB03, Theorem 7.74]. Theorem 10.1(i) reduces to the well-known fact that almost everywhere convergent sequences converge in measure. It can also be used, in conjunction with [GTX17, Theorem 3.2], to give a one line proof that the restriction of the topology of convergence in measure on $L_0(\mu)$ to any regular sublattice is the minimal topology on said sublattice. We note that, in L_∞, the un-topology is not minimal. The minimal topology of L_∞ is the topology $u|\sigma|(L_\infty, L_1)$; it agrees with the topology of convergence in measure in L_∞. As is shown in [AB03, Theorem 7.75], L_∞ admits no least topology.

In this section, we investigate relations between minimal topologies and *uo*-convergence and, as an application, we deduce classical results in Measure theory using our measure-free language. I would like to thank Niushan Gao, as discussions with him lead to the formulation of some of these results.

Definition 11.1. *Let X be a vector lattice. We say that uo-convergence on X is* ***sequential*** *if whenever $(x_\alpha)_{\alpha\in A}$ is a uo-null net in X there exists an increasing sequence of indices $\alpha_n \in A$ such that $x_{\alpha_n} \xrightarrow{uo} 0$.*

Theorem 11.2. *Let X be a vector lattice. TFAE:*

(i) X has a countable order basis and the countable sup property;
(ii) X^u has the countable sup property;
(iii) uo-convergence on X^u is sequential.

Moreover, in this case, uo-convergence on X is sequential.

Proof. (i)⇒(ii): Let $\{u_n\} \subseteq X_+$ be a countable order basis for X. It follows from order density of X in X^δ that $\{u_n\}$ is also a countable order basis of X^δ. It is known that X^δ inherits the countable sup property from X: see [AB03, Lemma 1.44]. Therefore, since X and X^δ have the same universal completion, we may assume, by passing to X^δ, that X is order complete. This implies that X is an ideal of X^u. Since X is order dense in X^u, $\{u_n\}$ is a countable order basis of X^u.

Let A be a non-empty disjoint subset of X^u_+. For each n, the set $A_n := A \wedge u_n$ is a non-empty order bounded disjoint subset of X_+. Since X has the countable sup property, for each n, A_n is at most countable. Hence, taking into account that A is disjoint, the set of all $a \in A$ such that $a \wedge u_n \neq 0$ for some n is at most countable. Since $\{u_n\}$ is a countable order basis for X^u, we conclude that at most countably many $a \in A$ are non-zero. [AB03, Exercise 1.15] yields that X^u has the countable sup property.

(ii)⇒(i): It is clear that X inherits the countable sup property from X^u, so we show that X has a countable order basis. Since X^u is universally complete, it has a weak unit e. Since X is order dense in X^u, there exists a net (x_α) in X_+ such that $x_\alpha \uparrow e$. By the countable sup property there is an increasing sequence (α_n) such that $x_{\alpha_n} \uparrow e$. It is easy to see that (x_{α_n}) is a countable order basis of X.

(ii)⇔(iii): It is clear that if *uo*-convergence is sequential on X^u then X^u has the countable sup property. For the converse, let e be a weak unit of X^u. It is both known and easy to check that a vector lattice X has the countable sup property iff order convergence is sequential. Using this, we conclude that for a net (x_α) in X^u,

$$x_\alpha \xrightarrow{uo} 0 \Leftrightarrow |x_\alpha| \wedge e \xrightarrow{o} 0 \Rightarrow \exists \alpha_n \ |x_{\alpha_n}| \wedge e \xrightarrow{o} 0 \Leftrightarrow x_{\alpha_n} \xrightarrow{uo} 0.$$

For the moreover clause, let (x_α) be a net in X such that $x_\alpha \xrightarrow{uo} 0$ in X. Since X is regular in X^u, $x_\alpha \xrightarrow{uo} 0$ in X^u. Since *uo*-convergence on X^u is sequential, there is an increasing sequence (α_n) of indices such that $x_{\alpha_n} \xrightarrow{uo} 0$ in X^u, hence in X. □

Remark 11.3. If *uo*-convergence on X is sequential then, of course, X has the countable sup property. However, it does not follow that X has a countable order basis. Indeed, let X be an order continuous Banach lattice. By [DOT17, Corollary 3.5], *uo*-convergence is sequential. Note that, in Banach lattices, admitting a countable order basis is the same as admitting a weak unit, and not all order continuous Banach lattices admit weak units.

Theorem 11.2 completes [CL17, Lemma 2.9]. Combining with Theorem 8.9 we get:

Corollary 11.4. *Let X be a vector lattice admitting a minimal topology τ. Then τ is metrizable if and only if X^u has the countable sup property.*

One could alternatively combine Theorem 11.2 with [AB03, Theorem 7.46] to get a simpler proof of Theorem 8.9.

Recall that a measure space (Ω, Σ, μ) is ***semi-finite*** if whenever $E \in \Sigma$ and $\mu(E) = \infty$ there is an $F \subseteq E$ such that $F \in \Sigma$ and $0 < \mu(F) < \infty$. Semi-finiteness of the measure is equivalent to the topology of convergence in measure on $L_0(\mu)$ being Hausdorff (see e.g. [Fre03, Theorem 245E]).

Proposition 11.5. *Let (Ω, Σ, μ) be a semi-finite measure space. Then $L_0(\mu)$ has the countable sup property iff μ is σ-finite.*

Proof. If μ is σ-finite, then $L_0(\mu)$ has the countable sup property by [AB03, Theorem 7.73].

For the converse, assume $L_0(\mu)$ has the countable sup property. Let $\mathcal{C}$ be the collection of all families of pairwise disjoint measurable sets of finite non-zero measure. The family $\mathcal{C}$ is partially ordered by inclusion. If $\mathcal{C}_0$ is a chain in $\mathcal{C}$, then the union of the chain is an upper bound for $\mathcal{C}_0$ in $\mathcal{C}$. Hence, by Zorn's lemma there is a maximal family $\mathcal{F}$ of pairwise disjoint measurable sets of finite non-zero measure. The set of functions $\{\chi_F : F \in \mathcal{F}\}$ is bounded above by $\mathbb{1}$ in $L_0(\mu)$. Let E be the union of all sets in $\mathcal{F}$. Since $L_0(\mu)$ has the countable sup property, [AB03, Exercise 1.15] implies $\mathcal{F}$ is at most countable, so that E is measurable. If $\mu(X \setminus E) > 0$, since μ is semi-finite, there is a measurable subset $E' \subseteq X \setminus E$ with $0 < \mu(E') < \infty$. This contradicts maximality of $\mathcal{F}$. Hence, $\mu(X \setminus E) = 0$ and μ is σ-finite. □

Remark 11.6. By [AB03, Theorem 7.73], if μ is a σ-finite measure then for $0 \leq p \leq \infty$, $L_0(\mu)$ is the universal completion of $L_p(\mu)$ and, moreover, $L_0(\mu)$ has the countable sup property. In this case, Corollary 11.4 simply states that convergence in measure in $L_p(\mu)$ is metrizable. This is consistent with classical results. Indeed, it is known that the topology of convergence in measure on $L_0(\mu)$ is metrizable iff μ is σ-finite (see e.g. [Fre03, Theorem 245E]). Proposition 11.5 suggests using the countable sup property as a replacement for σ-finiteness in general vector lattices.

It is known (see [AB03, Exercise 5.8]) that if τ is a complete metrizable locally solid topology then one can extract order convergent subsequences from τ-convergent sequences. Note that, by [AB03, Theorem 5.20], τ is the greatest locally solid topology on X in the sense that if τ' is a locally solid topology on X then $\tau' \subseteq \tau$. The next theorem shows that it is often possible to extract *uo*-convergent sequences from nets that converge in significantly weaker topologies.

Theorem 11.7. *Let (X, τ) be a Hausdorff locally solid vector lattice with the Fatou property. Assume that C_τ has a countable order basis and let $(x_\alpha)_{\alpha \in A}$ be a net in X. If $x_\alpha \xrightarrow{\tau} 0$ in X then there exists an increasing sequence of indices $\alpha_n \in A$ such that $x_{\alpha_n} \xrightarrow{uo} 0$ in X.*

Proof. Let $\{u_k\} \subseteq (C_\tau)_+$ be a countable order basis for C_τ. By [AB03, Theorem 4.12], τ extends uniquely to a Fatou topology τ^δ on X^δ. By [AB03, Exercise 4.5], $\{u_k\} \subseteq C_{\tau^\delta}$. [AB03, Theorem 4.17] tells us that $\{u_k\}$ is a countable order basis for C_{τ^δ} and C_{τ^δ} is an order dense ideal of X^δ.

Assume that a net (x_α) in X satisfies $x_\alpha \xrightarrow{\tau} 0$. As in the proof of [AB03, Theorem 4.17], choose a normal sequence $\{V_n\}$ of Fatou τ^δ-neighbourhoods of zero such that $\{u_k\} \subseteq N^d$ where $N = \bigcap_{n=1}^\infty V_n$. Since (x_α) is a τ^δ-Cauchy net of X^δ, there is an increasing sequence (α_n) of indices such that $x_{\alpha_{n+1}} - x_{\alpha_n} \in V_{n+2}$ and $x_{\alpha_n} \in V_n$ for all n. This implies that for each k and n, $|x_{\alpha_{n+1}}| \wedge u_k - |x_{\alpha_n}| \wedge u_k \in V_{n+2}$ and $|x_{\alpha_n}| \wedge u_k \in V_n$. Put $v^{*k} = \limsup_n |x_{\alpha_n}| \wedge u_k$ and $w^{*k} = \liminf_n |x_{\alpha_n}| \wedge u_k$ in X^δ. By [AB03, Lemma 4.14], $|x_{\alpha_n}| \wedge u_k - v^{*k} \in V_n$ for all n and all k. Therefore, $v^{*k} \in V_n$ for each n. We conclude that $v^{*k} \in N \cap N^d$, and, therefore, $v^{*k} = 0$. Similarly, $w^{*k} = 0$. This implies that $|x_{\alpha_n}| \wedge u_k \xrightarrow{o} 0$ in n in X^δ. Since $\{u_k\}$ is a countable order basis of C_{τ^δ}, an order dense ideal of X^δ, $\{u_k\}$ is a countable order basis of X^δ. This implies that $x_{\alpha_n} \xrightarrow{uo} 0$ in X^δ, hence in X. □

We need the following lemma in order to establish our final result:

Lemma 11.8. *Suppose X is a vector lattice admitting a Hausdorff Lebesgue topology τ. TFAE:*

(i) C_τ has a countable order basis;

(ii) X has the countable sup property and a countable order basis;

(iii) X^u has the countable sup property.

In this case, $C_\tau = X$.

Proof. Only (i)⇔(ii) requires proof.

Suppose that X has the countable sup property and a countable order basis. By [AB03, Theorem 4.17], $C_\tau = X$ and, therefore, C_τ has a countable order basis. For the converse, assume that C_τ has a countable order basis; denote it by $\{u_k\} \subseteq (C_\tau)_+$. As in the proof of [AB03, Theorem 4.17], there is a normal sequence $\{U_n\}$ of solid τ-neighbourhoods of zero such that $\{u_k\} \subseteq N^d \subseteq C_\tau$ where $N = \bigcap_{n=1}^\infty U_n$, and the disjoint complement is taken in X. Since C_τ is an ideal of X, N^d is a band of C_τ. Since $\{u_k\} \subseteq N^d \subseteq C_\tau$ and $\{u_k\}$ is a countable order basis for C_τ, $N^d = C_\tau$. This implies, by [AB03, Theorem 4.17], that C_τ is an order dense band of X and, therefore, $C_\tau = X$. Since τ is Lebesgue, $C_\tau = X$ has the countable sup property. □

The final corollary generalizes another classical relation between a.e. convergence and convergence in measure. As noted, the countable sup property assumption acts as a replacement for σ-finiteness in general vector lattices.

Corollary 11.9. *Let X be a vector lattice admitting a minimal topology τ. Assume that X^u has the countable sup property. Then a sequence (x_n) in X is τ-convergent to zero in X if and only if every subsequence of (x_n) has a further subsequence that uo-converges to zero in X.*

Proof. The result follows immediately by combining Theorem 11.7 with [Tay1, Proposition 2.22]. □

Example 11.10. The assumption that X^u has the countable sup property is crucial when trying to extract *uo*-convergent subsequences from topologically convergent sequences. Indeed, consider [KLT17, Example 9.6]. This gives an example of an order continuous Banach lattice X without a weak unit such that the minimal topology on the universal completion (which is the *un*-topology on X^u induced by X; it is locally solid by [Tay1, Theorem 9.11]) has a null sequence with no *uo*-null subsequences. Note that order continuous Banach lattices have the countable sup property and admit a countable order basis iff they admit a weak unit.

12. Some Banach lattice results

In this section we prove some results specific to Banach lattices. We note that most of these results are specific to Banach lattices because the *questions* are specific to Banach lattices. Generally speaking, if a result is true for *un* or *uaw*, and can be reasonably stated for locally solid topologies, then it will hold for a wide class (if not all) locally solid topologies. After solving the boundedly *uo*-complete problem, we make use of the ball and some properties of the norm to formulate questions specific to unbounded convergences in Banach lattices.

12.1. **Boundedly uo-complete Banach lattices.** Results equating the class of boundedly *uo*-complete Banach lattices to the class of monotonically complete Banach lattices have been acquired, under technical assumptions, by N. Gao, D. Leung, V.G. Troitsky, and F. Xanthos. The sharpest result is [GLX17, Proposition 3.1]; it states that a Banach lattice whose order continuous dual separates points is boundedly *uo*-complete iff it is monotonically complete. In this section, we remove the restriction on the order continuous dual.

The following was posed as Problem 2.4 in [CL17]:

Problem 12.1. Let (x_α) be a norm bounded positive increasing net in a Banach lattice X. Is (x_α) *uo*-Cauchy in X?

If Problem 12.1 is true, it is easily deduced that a Banach lattice is boundedly *uo*-complete iff it is monotonically complete. However, the next example answer this question in the negative, even for sequences.

Example 12.2. Let S be the set of all non-empty finite sequences of natural numbers. For $s \in S$ define $\lambda(s) = \text{length}(s)$. If $s, t \in S$, define $s \leq t$ if $\lambda(s) \leq \lambda(t)$ and $s(i) = t(i)$ for $i = 1, \dots, \lambda(s)$. For $s \in S$ with $\lambda(s) = n$ and $i \in \mathbb{N}$, define $s * i = (s(1), \dots, s(n), i)$. Put

$$X = \{x \in \ell^\infty(S) : \lim_{i\to\infty} x(s * i) = \frac{1}{2}x(s) \text{ for all } s \in S\}. \tag{12.1}$$

It can be verified that X is a closed sublattice of $(\ell^\infty(S), \|\cdot\|_\infty)$ and for $t \in S$ the element $e^t : S \to \mathbb{R}$ defined by

$$e^t(s) = \begin{cases} (\frac{1}{2})^{\lambda(s)-\lambda(t)} & \text{if } t \leq s \\ 0 & \text{otherwise} \end{cases}$$

is an element of X with norm 1. Define $f_1 = e^{(1)}, f_2 = e^{(1)} \vee e^{(2)} \vee e^{(1,1)} \vee e^{(1,2)} \vee e^{(2,1)} \vee e^{(2,2)}$, and, generally,

$$f_n = \sup\{e^t : \lambda(t) \leq n \text{ and } t(k) \leq n \ \forall k \leq \lambda(t)\}.$$

The sequence (f_n) is increasing and norm bounded by 1; it was shown in [BL88, Example 1.8] that (f_n) is not order bounded in X^u. Therefore, (f_n) cannot be *uo*-Cauchy in X for if it were then it would be *uo*-Cauchy in X^u and hence order convergent in X^u by [GTX17, Theorem 3.10]. Since it is increasing, it would have supremum in X^u; this is a contradiction as (f_n) is not order bounded in X^u.

Under some mild assumptions, however, Problem 12.1 has a positive solution. Recall that a Banach lattice is ***weakly Fatou*** if there exists $K \geq 1$ such that whenever $0 \leq x_\alpha \uparrow x$, we have $\|x\| \leq K \sup \|x_\alpha\|$.

Proposition 12.3. *Let X be a weakly Fatou Banach lattice. Then every positive increasing norm bounded net in X is uo-Cauchy.*

Proof. Let K be such that $0 \leq x_\alpha \uparrow x$ implies $\|x\| \leq K \sup \|x_\alpha\|$. Now assume that $0 \leq u_\alpha \uparrow$ and $\|u_\alpha\| \leq 1$. Let $u > 0$ and pick n such that $\|u\| > \frac{K}{n}$. If $0 \leq (\frac{1}{n}u_\alpha) \wedge u \uparrow_\alpha u$, then $\|u\| \leq \frac{K}{n}$. Therefore, there exists $0 < w \in X$ such that $(\frac{1}{n}u_\alpha) \wedge u \leq u - w$ for all α. But then $(nu - u_\alpha)^+ = n\left[u - (\frac{1}{n}u_\alpha) \wedge u\right] \geq nw > 0$ for all α, so that (u_α) is dominable. By [AB03, Theorem 7.37], (u_α) is order bounded in X^u, and hence $u_\alpha \uparrow \widehat{u}$ for some $\widehat{u} \in X^u$. This proves that (u_α) is *uo*-Cauchy in X^u, hence in X. □

The above result can be extended to Hausdorff locally solid topologies. The power of this result comes after noticing that a positive increasing net is *uo*-Cauchy iff it is dominable, and this makes dominable sets easier to work with. We next note the sequential analogue.

Proposition 12.4. *Let X be a weakly σ-Fatou Banach lattice. Then every positive increasing norm bounded sequence in X is uo-Cauchy.*

Proof. The proof is similar and, therefore, omitted. □

Even though Problem 12.1 is false, the equivalence between boundedly *uo*-complete and Levi still stands. We will show that now. First, recall that a positive sequence (x_n) in a vector lattice is said to be ***laterally increasing*** if it is increasing and $(x_m - x_n) \wedge x_n = 0$ for all $m \geq n$.

Theorem 12.5. *Let X be a Banach lattice. TFAE:*

(i) X is σ-Levi;

(ii) X is sequentially boundedly uo-complete;

(iii) Every increasing norm bounded uo-Cauchy sequence in X_+ has a supremum.

Proof. (i)⇒(ii): Let (x_n) be a norm bounded *uo*-Cauchy sequence in X. WLOG, (x_n) is positive; otherwise consider positive and negative parts. Define $e = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{x_n}{1+\|x_n\|}$ and consider B_e, the band generated by e. Then (x_n) is still norm bounded and *uo*-Cauchy in B_e. Also, B_e has the σ-Levi property for if $0 \leq y_n \uparrow$ is a norm bounded sequence in B_e, then $y_n \uparrow y$ for some $y \in X$ as X is σ-Levi. Since B_e is a band, $y \in B_e$ and $y_n \uparrow y$ in B_e. We next show that there exists $u \in B_e$ such that $x_n \xrightarrow{uo} u$ in B_e, and hence in X.

For each $m, n, n' \in \mathbb{N}$, since $|x_n \wedge me - x_{n'} \wedge me| \leq |x_n - x_{n'}| \wedge me$, the sequence $(x_n \wedge me)_n$ is order Cauchy, hence order converges to some u_m in B_e since the σ-Levi property implies σ-order completeness. The sequence (u_m) is increasing and

$$\|u_m\| \leq K \sup_n \|x_n \wedge me\| \leq K \sup_n \|x_n\| < \infty$$

where we use that σ-Levi implies weakly σ-Fatou. This can be proved by following the arguments in [MN91, Proposition 2.4.19]. Since B_e is σ-Levi, (u_m) increases to an element $u \in B_e$. Fix m. For any N, N' define $x_{N,N'} = \sup_{n \geq N, n' \geq N'} |x_n - x_{n'}| \wedge e$. Since (x_n) is *uo*-Cauchy, $x_{N,N'} \downarrow 0$. Now, for each m,

$$|x_n \wedge me - x_{n'} \wedge me| \wedge e \leq |x_n - x_{n'}| \wedge e \leq x_{N,N'} \ \forall n \geq N, n' \geq N'.$$

Taking order limit in n' yields:

$$|x_n \wedge me - u_m| \wedge e \leq x_{N,N'}$$

Since e is a weak unit in B_e, taking order limit in m now yields:

$$|x_n - u| \wedge e \leq x_{N,N'}, \ \forall n \geq N,$$

from which it follows that $|x_n - u| \wedge e \xrightarrow{o} 0$ in B_e. This yields $x_n \xrightarrow{uo} u$ in B_e by [GTX17, Corollary 3.5].

The implication (ii)⇒(iii) is clear. For the last implication it suffices, by [AW97, Theorem 2.4], to verify that every norm bounded laterally increasing sequence in X_+ has a supremum. Let (x_n) be a norm bounded laterally increasing sequence in X_+. By [AW97, Proposition 2.2], (x_n) has supremum in X^u, hence is *uo*-Cauchy in X^u. It follows that (x_n) is *uo*-Cauchy in X and, therefore, by assumption, *uo*-converges to some $x \in X$. It is then clear that $x_n \uparrow x$ in X. □

Theorem 12.6. *Let X be a Banach lattice. TFAE:*

(i) X is Levi;
(ii) X is boundedly uo-complete;
(iii) Every increasing norm bounded uo-Cauchy net in X_+ has a supremum.

Proof. If X is Levi, then X is boundedly *uo*-complete by [GLX17, Proposition 3.1]. It is clear that (ii)⇒(iii) and the proof of (iii)⇒(i) is the same as in the last theorem but with [AW97, Theorem 2.4] replaced with [AW97, Theorem 2.3]. □

The following is immediate noting that the ball is *uo*-closed iff it is order closed.

Corollary 12.7. *Let X be a Banach lattice. The ball of X is uo-complete iff X is monotonically complete and X has a Fatou norm (i.e., the* balls *are order closed).*

Remark 12.8. The previous theorems demonstrate how *uo*-convergence can be used to unify several seemingly unrelated completeness properties. Indeed, global *uo*-completeness is equivalent to universal completeness, *uo*-completeness on order bounded sets is equivalent to order completeness, and, in a Banach lattice, *uo*-completeness on norm bounded sets is equivalent to monotonic completeness. Therefore, by zooming in on the lattice we pick up all three of the classical completeness properties in the form of *uo*-completeness. Similar results are valid in the sequential case.

12.2. **Comparing convergences in Banach lattices.** In a (dual) Banach lattice we have a myriad of convergences: w, aw, w^*, aw^*, norm and order convergences. These give rise to the unbounded convergences: u_Aw, uaw^*, u_An and uo (here A is an order dense ideal). It is natural to compare these convergences globally and on the ball, for nets and for sequences.

We begin with a motivation/history. In [Wi77] the author studies interactions between *uo*-convergence and weak convergence, and in [Gao14], the author studies interactions between *uo*-convergence and w^*-convergence in dual Banach lattices. We note, however, that the weak topology is rarely locally solid and, therefore, doesn't fit into the overall scheme of this thesis. Although it is true that $0 \leq x_\alpha \leq y_\alpha$ and $y_\alpha \xrightarrow{w} 0$ implies $x_\alpha \xrightarrow{w} 0$, lattice operations are rarely w-continuous (see [AB03, Theorem 2.38]). In this case, it is easy to show that *uo*-convergence never implies w or w^*-convergence, so one has to search for "corrected" questions to ask. The answer, it turns out, has to due with a classical result in the theory of locally solid topologies. It is an analogue of the fact that the Mackey topology is the norm topology:

Theorem 12.9. *Let X be a Banach lattice. The norm topology is the finest locally solid topology on X.*

In some sense the idea when passing from a convergence to its unbounded counterpart is to "remove order boundedness". Order boundedness is a strong boundedness condition, so it makes sense to search for a substitute rather than completely remove it. The only other natural boundedness condition is topological boundedness and, since the norm topology is the finest locally solid topology on a Banach lattice, norm boundedness provides the closest approximation to order boundedness. The authors in [Wi77] and [Gao14] therefore ask when *uo*-convergence implies w or w^*-convergence when restricted to the ball, and achieve good success. This, I believe, is how Banach lattices play a role in the theory of unbounded convergences. The category of Banach lattices is too small for unbounded convergences in general, but, if one uses the ball appropriately, some stunning results can be achieved. Similar results are not expected for a general locally solid topology τ since there is no reason to believe the topology is fine enough to provide the approximation to order boundedness that norm boundedness provides. In [GLX17] the authors use this idea of norm boundedness approximating order boundedness in the study of the *uo*-dual. It is highly successful.

In the papers [DOT17], [Gao14], [GX14], [KMT17], [Wi77], and [Zab17] one can find several results characterizing the spaces in which one of the canonical convergences implies another on some subset of the Banach lattice. There are also some results establishing completeness, metrizability, compactness, etc. of these convergences, globally and on the ball.

In some sense, actually, the most important unbounded topology on a Banach lattice has not been defined. Consider the following:

Definition 12.10. *Let X be a Banach lattice. A net (f_α) in X^* is uaw^***-convergent** to $f \in X^*$ if for each $u \in X^*_+$, $|f_\alpha - f| \wedge u \xrightarrow{w^*} 0$.*

Remark 12.11. The definition of uaw^*-convergence we have given uses the Banach lattice duality, and we have not investigated to what extent the results fail if one defines this convergence relative to a Banach space predual of a Banach lattice. [Gao14] has the same (natural) assumption. For an example of a Banach space predual that is not a Banach lattice predual simply consider ℓ_2 as a predual of L_2.

As we have discussed, it is immediate that the uaw^*-topology is a minimal topology. In particular, one does not have to test against ideals and *uo*-convergence implies uaw^*-convergence. In this section we briefly establish some results on uaw^*. Of course, from the general theory of minimal topologies, we get answers to questions like, when is uaw^* complete or metrizable? The questions of interest are: when is uaw^* complete or metrizable or compact *when restricted to the ball*? We will also be interested in characterizing Banach space properties (e.g. reflexivity) in terms of uaw^* and the other convergences. In principle this could lead to a unified theory of properties of minimal topologies on the ball.

We next present several results comparing uaw^* with the other convergences. Globally, this is quite easy since uaw^* is a minimal topology (although the question of when all w or w^*-null nets/sequences are uaw^*-null is not trivial). It is easy to see that requiring uaw^*-null sequences to be either norm, aw, w, aw^*, w^* or order null implies finite dimensionality of the Banach lattice X^*. It is also easy to see that uaw^* convergence agrees with uaw (or un) on nets iff on sequences iff X^* is order continuous. All uaw^*-null nets in X^* are uo-null iff X^* is atomic. For simplicity, Table 1 only presents our results for un and uaw rather than u_An and u_Aw. We compare these convergences with uaw^* on the ball. OCN stands for order continuous norm; unknown means no complete characterization is known.

TABLE 1. When are norm bounded A-null nets/seq. B-null?

Convergence A	**Convergence B**	**Nets**	**Sequences**
uaw^*	norm	$\dim X^* < \infty$	$\dim X^* < \infty$
uaw^*	ab weak	X is reflexive	X is reflexive
uaw^*	weak	X is reflexive	X is reflexive
uaw^*	ab weak*	X has OCN	X has OCN
uaw^*	weak*	X has OCN	X has OCN
uaw^*	order	Unknown	Unknown
uaw^*	un	X^* has OCN	X^* has OCN
uaw^*	uaw	X^* has OCN	X^* has OCN
uaw^*	uo	Unknown	Unknown
uo	uaw^*	Always	Always

Let $A, B \in \{w, aw, w^*, aw^*, o, u_Cw, uaw^*, u_Cn, uo\}$ (C an ideal) and suppose at least one of A or B is unbounded. One may wonder if one can characterize when (norm bounded) A-Cauchy nets/sequences B-converge. However, it is not particularly hard to see that the ball of X^* is uaw^*-complete (since X^* is monotonically complete and τ and $u\tau$ have the same closed solid sets), so this question is not too interesting for uaw^*. For the other convergences, however, there is some interest. In the next table we present some results - most are not due to me. The theme of this section is to not be comprehensive; we leave it to the reader to search the aforementioned papers for more results, and, if they are up for it, to prove some new relations between the convergences. It is a general principle that un is the hardest to work with because it exhibits the wildest spectrum of behaviour (it can be the finest topology on X, the coarsest, or anything in between).

Table 2. When are norm bounded A-Cauchy nets/seq. B-convergent?

Convergence A	**Convergence** B	**Nets**	**Sequences**
uaw	norm	$\dim X < \infty$	$\dim X < \infty$
uaw	ab weak	X is reflexive	X is reflexive
uaw	weak	X is reflexive	X is reflexive
uaw	weak*	X has OCN	X has OCN
uaw	ab weak*	X has OCN	X has OCN
uaw	order	Unknown	Unknown
uaw	un	X is KB	X is KB
uaw	uaw^*	Always	Always
uaw	uo	Unknown	Unknown
uo	uo	X is MC	X is σ-MC
uo	norm	$\dim X < \infty$	$\dim X < \infty$
uo	ab weak	X is reflexive	X is reflexive
uo	weak	X is reflexive	X is reflexive
uo	ab weak*	X has OCN	X has OCN
uo	weak*	X has OCN	X has OCN
uo	order	Unknown	Unknown
uo	un	X is KB	X is KB
uo	uaw	X is KB	X is KB
uo	uaw^*	Always	Always

As we have mentioned, for a Banach lattice X, B_{X^*} is always uaw^*-complete, but (X^*, uaw^*) never has σ-MCP. It is not hard to show that B_{X^*} is uaw^*-compact iff X^* is atomic, and B_{X^*} is uaw^*-metrizable iff X^* has the countable sup property and a weak unit.

We next mention a subtle and oft occuring mistake in the literature, in the hopes that it clears some confusion. In [Wi77] the following theorem is proven:

Theorem 12.12. *Let X be a Banach lattice. TFAE:*

(i) Every norm bounded net in X which uo-converges to zero must converge weakly to zero;

(ii) X is order continuous and every norm bounded disjoint sequence in X converges weakly to zero;

(iii) X and X^* have order continuous norms.

It is then stated that nets cannot be replaced with sequences in (i) as in the space of all continuous real-valued functions on the one point compactification of an uncountable discrete space, norm bounded *uo*-convergent sequence must converge in norm (hence weakly). This is, however, not the case, and the reason is the definition of order convergence: o_1-convergence implying norm convergence for sequences characterizes σ-order continuity of the norm, but o-convergence implying norm convergence for sequences characterizes order continuity of the norm. The latter because disjoint order bounded sequences will be norm null. This is a very commonly occurring mistake in the literature, and it is mostly the fault of having two reasonable definitions of sequential order convergence.

12.3. **Other research that has been done in Banach lattices.** Since the modern study of unbounded convergence began in Banach lattices, we mention some other avenues - not due to me - that have been started.

The following result is stated on page 28 of [LT79]:

Theorem 12.13. *A Banach lattice* $(X, \|\cdot\|)$ *is order continuous if and only if there is an equivalent lattice norm* $\|\cdot\|_1$ *on* X *such that for every sequence* (x_n) *in* X*,*

$$x_n \xrightarrow{w} x \text{ and } \|x_n\|_1 \to \|x\|_1 \Rightarrow \|x_n - x\|_1 \to 0. \tag{12.2}$$

This motivates the question of whether one can characterize, at least up to an equivalent lattice norm, those Banach lattices in which $x_n \xrightarrow{A} x$ and $\|x_n\| \to \|x\| \Rightarrow x_n \xrightarrow{B} x$ (or similarly with sequences replaced with nets). Here A and B each stand for some "natural" convergence on a Banach lattice, at least one of which we will take to be unbounded. Note that for $1 \leq p < \infty$ it is well known that if (f_k) is a sequence in L_p and $f \in L_p$ then $f_k \xrightarrow{a.e.} f$ (i.e. $f_k \xrightarrow{uo} f$) and $\|f_k\|_p \to \|f\|_p$ implies $\|f_k - f\|_p \to 0$.

Some research has been done comparing convergences on other "nice" (e.g. relatively τ-compact or τ-almost order bounded) subsets of Banach lattices. In [GX14] the authors compare convergences on relatively weakly compact sets. Note that relatively weakly compact sets are *aw*-almost order bounded ([AB06, Theorem 4.37]). Other research (see

[Gao14]) has expanded on Remark 4.17 in an attempt to characterize the spaces in which simultaneously A and w (or w^*)-convergent (norm bounded) nets/sequences converge to the same limit.

There have also been applications of *uo*-convergence to Cesàro means, as well as to Financial Mathematics. The *uo*-dual of [GLX17] has also been quite fruitful. Note that, under appropriate assumptions, [GLX17, Theorem 2.3] can be extended to operators; compare with [AB06, Definition 5.59, and Theorem 5.60]. Some results on *un*-operators appear in [KMT17]. The idea with *uo* and operators is that *uo*-convergence turns disjointness into a convergence, in the sense that one suspects an operator to map (norm bounded) uo-null nets to null nets iff it maps order null nets to null nets and (norm bounded) disjoint sequences to zero (c.f. [GLX17, Theorem 2.3]). The idea with general unbounded convergence and operators is to extend the results from the previous section, where we view those results as characterizations of when the identity mapping is continuous with respect to the unbounded/classical convergences.

We finish the study of Banach lattices with a question that has applications to bibases:

Question 12.14. Suppose X is a closed sublattice of a Banach lattice Y, and (x_n) is a norm null sequence in X. Is it true that $x_n \xrightarrow{uo} 0$ in X iff $x_n \xrightarrow{uo} 0$ in Y? I actually don't know whether either implication holds, nor do I know the answer if *uo* is replaced by order convergence (although this may be in the literature).

13. σ-universal topologies

There are several ways to "complete" an Archimedean vector lattice X, depending on which properties the completion is required to satisfy, and how "large" X is to be in its completion. The most ubiquitious completion is the **order completion**, X^δ, which is characterized as follows:

Theorem 13.1. [AB03, Theorem 1.41] *Let X be an Archimedean vector lattice. Then there exists a (unique up to lattice isomorphism) order* *complete vector lattice X^δ that contains an order dense and majorizing sublattice that is lattice isomorphic to X.*

From a categorical point of view, the order completion is the largest Archimedean vector lattice that contains X as an order dense and majorizing sublattice. That is, if X is an order dense and majorizing sublattice of an Archimedean vector lattice M, then M is lattice isomorphic to a sublattice of X^δ via an isomorphism that leaves X invariant.

In a similar fashion, one can consider the ***universal completion***, X^u, of X, which is the largest Archimedean vector lattice that contains X as an order dense (but not necessarily majorizing) sublattice. In analogy with Theorem 13.1, the universal completion can be characterized as the smallest vector lattice that contains a lattice isomorphic copy of X, while at the same time possessing a certain additional property. In this case, recall that a vector lattice is ***universally complete*** if it is both order complete and laterally complete.

Theorem 13.2. [AB03, Theorem 7.21] *Let X be an Archimedean vector lattice. Then there exists a (unique up to lattice isomorphism) universally complete vector lattice X^u that contains an order dense sublattice that is lattice isomorphic to X.*

For more details on completions of vector lattices we refer the reader to [AL84].

In much of the theory of vector lattices, there are two structures at play: the order structure and a topology. Although the aforementioned completions improve the order structure, they do not concern themselves with the topological. However, by [AB03, Theorem 4.12], we know that Fatou topologies extend uniquely to the order completion, and the extension preserves the Hausdorff and Lebesgue properties. Thus, much of the topological structure is preserved when passing to X^δ.

For the universal completion, the behaviour is much more restrictive. It is known that at most one Hausdorff Lebesgue topology can extend from X to X^u, and this topology is very special. Indeed, recall that a Hausdorff locally solid topology τ is ***minimal*** if it follows from $\sigma \subseteq \tau$

and σ a Hausdorff locally solid topology that $\sigma = \tau$. It has been shown that the minimal topologies are exactly the Hausdorff Lebesgue topologies that extend to X^u.

The reason at most one Hausdorff Lebesgue topology extends to X^u is because the Hausdorff Lebesgue topologies on laterally σ-complete vector lattices are very controlled. Indeed, a laterally σ-complete vector admits at most one Hausdorff Fatou topology, which must necessarily be Lebesgue. The question of existence of Hausdorff locally solid topologies on infinite dimensional universally complete spaces that fail to be Lebesgue was considered by D.H Fremlin in [Frem75]. He concluded, stunningly, that this problem was equiconsistent with the existence of measurable cardinals.

In this section, we are interested in "countable completions" of vector lattices and the topologies such completions possess. To this end, recall that a sublattice Y of a vector lattice X is ***super order dense*** in X if for every $0 < x \in X$ there exists a sequence (y_n) in Y with $0 \leq y_n \uparrow x$ in X. The ***σ-order completion***, X^σ, of X is defined as the intersection of all sublattices between X and X^δ that are σ-order complete in their own right. If X is ***almost σ-order complete***; that is, if X is lattice isomorphic to a super order dense sublattice of a σ-order complete vector lattice, then X is super order dense in X^σ and, actually, for every $u \in X^\sigma$ there exists sequences (u_n) and (v_n) in X with $u_n \uparrow u$ and $v_n \downarrow u$ in X^σ. By [AB03, Theorem 4.13], if X is almost σ-order complete and τ is a σ-Fatou topology on X, then τ extends uniquely to a σ-Fatou topology on X^σ, and the extension preserves both the Hausdorff and σ-Lebesgue properties. Therefore, much of the topological information passes freely between X and X^σ.

A vector lattice X is said to be ***universally σ-complete*** if it is both σ-order complete and laterally σ-complete. A ***universal σ-completion*** of X is a universally σ-complete vector lattice X^s having a super order dense sublattice M that is lattice isomorphic to X. By identifying X with M we consider X as a super order dense sublattice of X^s. By [AB03, Theorem 7.42], a vector lattice has a (unique up to lattice isomorphism) universal σ-completion iff X is almost σ-order complete. The universal σ-completion of X is characterized categorically as the largest vector lattice containing X as a super order dense sublattice. Topologies that extend to X^s will be the topic of this section.

We will call a *Hausdorff* locally solid topology τ ***σ-universal*** if *uo*-null sequences are τ-null; the name will be justified shortly. It is immediate that σ-universal topologies are σ-Lebesgue and pre-Lebesgue. The next example illustrates a further subtlety involving the definition of order convergence:

Example 13.3. *σ-Lebesgue and unbounded does not imply σ-universal:* Let Ω be an uncountable set and let $X = \ell_\infty(\Omega)$, the space of all bounded real-valued functions on Ω with $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$. Clearly, X is a Banach lattice. Let Y be the closed sublattice generated by the characteristic functions of finite sets and $\mathbb{1}$. It is easy to show that Y is a σ-order continuous but not order continuous Banach lattice, the norm and un-topologies agree, and there are sequences in Y that are *uo*-null but not norm-null. Notice, however, that in this example Y is not almost σ-order complete. To verify the latter fact, apply [AB03, Exercise 3.15 and Theorem 3.23]. This example shows that, for general Hausdorff topologies, σ-universal is not equivalent to σ-Lebesgue and unbounded. Compare with [Tay1, Theorem 5.9]. If X is almost σ-order complete then it is easy to see that every unbounded Hausdorff σ-Lebesgue topology τ is σ-universal. Actually, one need only require that the τ and $u\tau$-convergences agree on sequences. This is generally weaker than unboundedness: see [KMT17, Example 1.3]. Notice also that every unbounded σ-Lebesgue topology on an almost σ-order complete vector lattice has the property that countably indexed *uo*-null nets are τ-null.

Lemma 13.4. *Let (X, τ) be a Hausdorff locally solid vector lattice. TFAE:*

(i) τ is σ-universal;

(ii) uo-Cauchy sequences are τ-Cauchy.

Proof. (i)$\Rightarrow$(ii): Suppose (x_n) is *uo*-Cauchy but not τ-Cauchy. By [AB03, Lemma 2.5], there exists a neighbourhood U of zero for τ and

$n_1 < n_2 < \dots$ with $x_{n_{k+1}} - x_{n_k} \notin U$ for each k. We get a contradiction by showing that $(x_{n_{k+1}} - x_{n_k})_k$ is *uo*-null. Fix $u \in X_+$. Since (x_n) is *uo*-Cauchy, there exists $y_\beta \downarrow 0$ such that $\forall \beta$ there exists N, for all $n, m \geq N$, $|x_n - x_m| \wedge u \leq y_\beta$. Pick k_0 such that $n_{k_0} > N$. Then for all $k \geq k_0$, $|x_{n_{k+1}} - x_{n_k}| \wedge u \leq y_\beta$, hence $|x_{n_{k+1}} - x_{n_k}| \wedge u \xrightarrow{o} 0$. This proves that $(x_{n_{k+1}} - x_{n_k})$ is *uo*-null.

(ii)$\Rightarrow$(i): Suppose $x_n \xrightarrow{uo} 0$ and define a sequence (y_n) via $y_{2n} = 0$ and $y_{2n-1} = x_n$. Then $y_n \xrightarrow{uo} 0$. It follows that (y_n) is *uo*-Cauchy, hence τ-Cauchy. Since, clearly, (y_n) has a τ-null subsequence, $y_n \xrightarrow{\tau} 0$. Therefore, $x_n \xrightarrow{\tau} 0$. ☐

Definition 13.5. *A sequence (x_n) in a vector lattice is **k-disjoint**, where $k \in \mathbb{N}$, if for every subset I of $\mathbb{N}$ with at least k elements we have $\bigwedge_{i \in I} |x_i| = 0$.*

Proposition 13.6. *Let X be a vector lattice and (x_n) a k-disjoint sequence in X. Then $x_n \xrightarrow{uo} 0$. In particular, $x_n \xrightarrow{\tau} 0$ whenever τ is σ-universal.*

Proof. Follow the proof of [AB03, Theorem 3.22]. ☐

13.1. **Extensions to the universal σ-completion.** The first result of this section characterizes those Hausdorff locally solid topologies on almost σ-order complete vector lattices that extend to the universal σ-completion X^s. First, recall the recent result characterizing those Lebesgue topologies that extend to the universal completion; it appears in [Tay1]:

Theorem 13.7. *Let (X, τ) be a Hausdorff locally solid vector lattice. TFAE:*

(i) τ is Lebesgue and extends to the universal completion as a Hausdorff Lebesgue topology;
(ii) uo-null nets are τ-null;

Moreover, the extension specified in (i) is unique.

We next present the σ-analogue of this theorem.

Theorem 13.8. *Let X be an almost σ-order complete vector lattice and τ a Hausdorff locally solid topology on X. TFAE:*

(i) τ extends to a locally solid topology on X^s;

(ii) τ is σ-universal.

Moreover, the extension specified in (i) is unique.

Proof. (i)⇒(ii): Suppose τ extends to a locally solid topology τ^s on X^s, and $x_n \xrightarrow{uo} x$ in X. Then $x_n \xrightarrow{uo} x$ in X^s hence, by [GTX17, Theorem 3.10], $x_n \xrightarrow{o} x$ in X^s. This implies that $x_n \xrightarrow{\tau^s} x$ by [AB03, Theorem 7.49]. Since (x_n) and x are in X, $x_n \xrightarrow{\tau} x$.

(ii)⇒(i): Note first that τ is σ-Lebesgue, hence σ-Fatou. Let $\mathcal{N}_0$ be a solid base at zero for τ with each $V \in \mathcal{N}_0$ satisfying $0 \leq x_n \uparrow x$ and $\{x_n\} \subseteq V$ implies $x \in V$. For each $V \in \mathcal{N}_0$, define the subset V^s of X^s by

$$V^s = \{x^* \in X^s : \exists \, \{x_n\} \subseteq V \text{ such that } 0 \leq x_n \uparrow |x^*| \text{ in } X^s\}.$$

We prove that V^s is solid and absorbing:

Solid: Clearly, $x^* \in V^s \Leftrightarrow |x^*| \in V^s$. Suppose $0 \leq y^* \leq x^* \in V^s$. Then there exists a sequence (x_n) in V with $0 \leq x_n \uparrow x^*$ in X^s. Since X is super order dense in X^s, there exists a sequence (y_n) in X with $0 \leq y_n \uparrow y^*$. Then $(x_n \wedge y_n)$ is in V, and $0 \leq x_n \wedge y_n \uparrow y^*$, which proves $y^* \in V^s$.

Absorbing: We will make use of Lemma 13.4. Let $0 < x^* \in X^s_+$ and $V \in \mathcal{N}_0$. Find (x_n) in X such that $0 \leq x_n \uparrow x^*$. Then (x_n) is uo-Cauchy in X^s, hence uo-Cauchy in X, hence τ-Cauchy in X, hence τ-bounded (since it is a sequence). Therefore, there exists $\lambda > 0$ such that $(x_n) \subseteq \lambda V$. Therefore, $V \ni \frac{x_n}{\lambda} \uparrow \frac{x^*}{\lambda}$ implies $\frac{x^*}{\lambda} \in V^s$.

It is easy to show that for $V, W \in \mathcal{N}_0$ that $(V \cap W)^s \subseteq V^s \cap W^s$ and if $W + W \subseteq V$ then $W^s + W^s \subseteq V^s$. Also, note that $V^s \cap X = V$. By the discussion on page 49 of [AB03], the collection $\mathcal{N}_0^s = \{V^s : V \in \mathcal{N}_0\}$ defines a solid base for a linear topology τ^s on X^s that extends τ.

Finally, we prove uniqueness. Let τ^* be a (necessarily σ-Lebesgue) topology on X^s that induces τ on X. Let $\mathcal{N}_0^*$ be a solid base of zero for τ^* such that whenever $0 \leq x_n^* \uparrow x^*$ and $(x_n^*) \subseteq W^* \in \mathcal{N}_0^*$, we have $x^* \in W^*$. If $W^* \in \mathcal{N}_0^*$ then $W := W^* \cap X$ is a τ-neighbourhood of zero satisfying $0 \leq x_n \uparrow x$ in X and $(x_n) \subseteq W$ implies $x \in W$. It follows that $W^s \subseteq W^*$, so that $\tau^* \subseteq \tau^s$.

Let $V \in \mathcal{N}_0$. Then there exists $W^* \in \mathcal{N}_0^*$ such that $W^* \cap X \subseteq V$. Let $w^* \in W^*$. Since X is super order dense in X^s, there exists a sequence (x_n) in X such that $0 \leq x_n \uparrow |w^*|$ in X^s. By solidity, $(x_n) \subseteq W^* \cap X$. This implies that $w^* \in V^s$, so that $W^* \subseteq V^s$. This shows that $\tau^* = \tau^s$, so that the extension is unique. □

Remark 13.9. Notice that, unlike in Theorem 13.7, there is no Lebesgue-type assumption in statement (i). This stems from the fact that every Hausdorff locally solid topology on a universally σ-complete vector lattice is σ-Lebesgue. Assuming measurable cardinals, the Lebesgue assumption in Theorem 13.7 cannot be removed; i.e., (i) cannot be replaced with "τ extends to a locally solid topology on X^u": see [AB03, Exercise 7.22].

Remark 13.10. The extension to X^s is necessarily Hausdorff by [AB03, Exercise 2.5]; it will be denoted τ^s. Notice that every *uo*-Cauchy sequence in X, when viewed as a sequence in X^s, converges in τ^s and in order in to the same limit. By the construction of τ^s we see that τ is locally convex iff τ^s is. As we have noted, τ and τ^s are necessarily pre-Lebesgue and σ-Lebesgue. In particular, if a σ-universal topology is pseudo-Lebesgue then it is minimal (see [AB03, Theorem 5.28 and Exercise 5.3] for details on pseudo-Lebesgue topologies.)

Example 13.11. Let τ be one of the topologies constructed in [AB03, Example 3.6]. Then $u\tau$ has the property that every *uo*-null sequence is $u\tau$-null but not every *uo*-null net is $u\tau$-null.

Corollary 13.12. *If an almost σ-order complete vector lattice admits a σ-universal topology then it admits a finest σ-universal topology.*

Proof. Suppose X is almost σ-order complete and admits a σ-universal topology. Then X^s admits a finest Hausdorff locally solid topology (the locally solid topology generated by the family of all Riesz pseudonorms on X). The restriction of this topology to X is as desired. □

We can say much more about the topologies which extend to X^s by studying properties in the σ-order completion X^σ. For notation, see [AB03, Theorem 4.13].

Theorem 13.13. *Let τ be a Hausdorff σ-Fatou topology on an almost σ-order complete vector lattice X. TFAE:*

(i) *τ is σ-universal;*
(ii) *Disjoint sequences in X^σ are τ^σ-null;*
(iii) *Disjoint sequences in X^σ are τ^σ-bounded;*
(iv) [Oh98] *For each positive disjoint sequence (x_k) in X^σ the set*

$$\Big\{\sum_{k\in F} x_k : F\subseteq \mathbb{N},\ F \text{ finite}\Big\},$$

of all possible finite sums, is τ^σ-bounded.

Proof. WLOG, $X = X^\sigma$. (i)⇒(ii) follows since disjoint sequences are τ-null for topologies on σ-laterally complete vector lattices. (ii)⇒(iii) is obvious.

(iii)⇒(i): Let $0 < u \in X^s$. By [DM83, Lemma 1.1], there exists a disjoint sequence (x_n) in X_+ with supremum u. Define $y_n = \sum_{k=1}^n x_k$. Then $0 \le y_n \uparrow$. We claim that (y_n) is τ-Cauchy. If not, then there exists a solid τ-neighbourhood U of zero and an increasing sequence (n_k) with $w_k := x_{n_{k+1}} + \cdots + x_{n_k+1} = y_{n_{k+1}} - y_{n_k} \notin U$ for each k. Notice that (w_k) is disjoint, hence so is (kw_k). By assumption, (kw_k) is τ-bounded so that there exists K with $\{kw_k\} \subseteq KU$. Therefore, $w_k \in U$ for $k > K$, a contradiction. It follows that $0 \le y_n \uparrow u$ and (y_n) is τ-bounded. Create an extension as in Theorem 13.8 by modifying the "absorbing" argument in the obvious way.

(i)⇒(iv): Let (x_k) be a positive disjoint sequence in X. Then (x_k) has supremum in X^s, hence the set of finite sums, which is the same as the set of finite supremums, of (x_k) is order bounded in X^s, hence τ^s-bounded, hence τ-bounded.

(iv)⇒(i): Let $0 < u \in X^s$. By [DM83, Lemma 1.1], there exists a disjoint sequence (x_n) in X_+ with supremum u. Therefore, the sequence (y_n) as above is τ-bounded and $0 \le y_n \uparrow u$. We can thus create an extension. ☐

Question 13.14. Although first passing to the σ-order completion in Theorem 13.13 is a minor inconvenience, it would be interesting to know if one *must* do this. If not, then one could justifiably say that X does not have enough disjoint sequences, compared to X^σ.

Example 13.15. By [Tay1, Proposition 5.12 and Theorem 6.4], the restriction of a minimal topology to a sublattice is minimal iff the sublattice is regular. We will show that every unbounded σ-universal topology on X arises as the restriction of a minimal topology from a vector lattice Y containing X as a σ-regular sublattice. Recall that X is a ***σ-regular*** sublattice of Y if for every sequence $x_n \downarrow 0$ in X it follows that $x_n \downarrow 0$ in Y.

Indeed, suppose τ is an unbounded and σ-universal topology on X. Then τ is Hausdorff, so the topological completion $(\widehat{X}, \widehat{\tau})$ of (X, τ) is defined as in [AB03, Theorem 2.40]. Since τ is pre-Lebesgue it follows from [AB03, Theorem 3.26] that $\widehat{\tau}$ is Lebesgue and, therefore, $u\widehat{\tau}$ is minimal. Utilizing [Tay1, Lemma 3.5] we conclude that $\tau = u\tau = u(\widehat{\tau}|_X) = (u\widehat{\tau})|_X$, so that τ is the restriction of a minimal topology to X. It is left to show that X is a σ-regular sublattice of $\widehat{X}$: If $x_n \downarrow 0$ in X then, since τ is σ-Lebesgue, $x_n \xrightarrow{\tau} 0$ in X. This yields that $x_n \xrightarrow{\widehat{\tau}} 0$ in $\widehat{X}$, which implies that $x_n \downarrow 0$ in $\widehat{X}$ by [AB03, Theorem 2.21(c)].

Question 13.16. Let X be an almost σ-order complete vector lattice. Is every σ-universal topology on X unbounded? This is equivalent to the question of whether every Hausdorff locally solid topology on a universally σ-complete vector lattice is unbounded. Is it at least true that τ and $u\tau$ agree on sequences?

We next present the sequential analogue of [AB03, Theorem 7.54].

Corollary 13.17. *Let X be an almost σ-order complete vector lattice. X admits a Hausdorff σ-Lebesgue topology iff X^s does.*

Proof. If X^s admits a Hausdorff σ-Lebesgue topology, then the restriction to X is as desired.

Suppose X admits a Hausdorff σ-Lebesgue topology τ. Then $u\tau$ is Hausdorff and σ-Lebesgue as well. By [AB03, Theorem 4.13], $u\tau$ lifts uniquely to a Hausdorff σ-Lebesgue topology $(u\tau)^\sigma$ on X^σ. It is easy to show, as in [Tay1, Proposition 8.7], that $(u\tau)^\sigma = u(\tau^\sigma)$, where τ^σ is the unique lifting of τ to a Hausdorff σ-Lebesgue topology on X^σ.

Suppose (x_n) is uo-Cauchy in X. Then (x_n) is uo-Cauchy in X^σ, and, using that the dominating net in the definition of order convergence can be taken to be a sequence, (x_n) is $u(\tau^\sigma)$-Cauchy. It follows that (x_n) is $u\tau$-Cauchy and, therefore, $u\tau$ extends to a Hausdorff locally solid σ-Lebesgue topology on X^s by Theorem 13.8. ☐

Remark 13.18. Unlike with the universal completion where at most one Hausdorff Lebesgue topology can extend, many Hausdorff σ-Lebesgue topologies can lift to X^s. See [AB03, Exercise 7.21].

Proposition 13.19. *Suppose $Y \subseteq (X, \tau)$ where Y is almost σ-order complete and τ is σ-universal. TFAE:*

(i) $\tau|_Y$ is σ-universal
(ii) Y is σ-regular in X.

Proof. Similar to [GTX17, Theorem 3.2]. ☐

Proposition 13.20 ([Oh98])**.** *Let X be an almost σ-order complete vector lattice and τ a Hausdorff σ-Fatou topology on X. TFAE:*

(i) τ extends to a locally solid topology on X^s;
(ii) Every countable dominable subset of X_+ is τ-bounded.

Proof. (i)⇒(ii): By [AB03, Lemma 7.11 and Theorem 7.38], every countable dominable subset of X_+ is order bounded in X^s, hence τ^s-bounded, hence τ-bounded.

(ii)⇒(i): Let $0 \leq u \in X^s$. Choose a sequence (x_n) in X_+ such that $0 \leq x_n \uparrow u$ in X^s. Then (x_n) is dominable, hence τ-bounded, and an extension can be created as in Theorem 13.8. ☐

This allows for a certain broadening of results in [Tay2]. One of the common techniques used to classify properties of minimal topologies is to extend the topology to the universal completion. For σ-properties, however, this is an overshoot; it suffices to extend to X^s. The proofs of the next results are nearly verbatim to those in [Tay2], and we refer the reader to [Tay2] for undefined terminology.

Theorem 13.21. *Let X be an almost σ-order complete vector lattice and τ a σ-universal topology on X. TFAE:*

(i) (X,τ) *satisfies* σ*-BOB;*

(ii) (X,τ) *satisfies* σ*-POB;*

(iii) X *is majorizing in the universal* σ*-completion* X^s *of* X*.*

Furthermore, TFAE:

(i) τ *is* σ*-Levi;*

(ii) τ *has* σ*-MCP;*

(iii) X *is universally* σ*-complete;*

(iv) (X,τ) *is sequentially boundedly uo-complete in the sense that* τ*-bounded uo-Cauchy sequences in* X *are uo-convergent in* X*.*

Furthermore, TFAE:

(i) X *is laterally* σ*-complete;*

(ii) τ *has the lateral* σ*-Levi property;*

(iii) Every disjoint positive sequence, for which the set of all possible finite sums is τ*-bounded, must have a supremum.*

Furthermore, τ *satisfies the* B*-property.*

We next characterize completeness, metrizability, and local boundedness of σ-universal topologies by reducing to the case of minimal topologies where the answer is known.

Proposition 13.22. *Let* τ *be a* σ*-universal topology on an almost* σ*-order complete vector lattice* X*. If* τ *satisfies MCP or* τ *has complete order intervals then* τ *is minimal.*

Proof. Suppose $0\le x_\alpha\uparrow x$ but (x_α) does not τ-converge to x. Since τ has MCP, (x_α) is not τ-Cauchy. This contradicts the B-property since order bounded sets are τ-bounded. It follows that τ is Lebesgue. Since τ has σ-MCP, X is universally σ-complete. By [AB03, Theorem 7.55], τ is the unique Lebesgue topology on X, hence minimal. If τ has complete order intervals then X is an ideal of $\widehat{X}$ and $\tau=\widehat{\tau}|_X$ is Lebesgue. Since τ is Lebesgue and σ-universal, it is minimal. More generally, X is regular in the completion iff minimal. See also [AB03, Theorem 2.46]. $\square$

Remark 13.23. A minimal topology is complete iff it has MCP iff X is universally complete. I do not know whether the completion of a

σ-universal topology τ is σ-universal. It is easy to see that if this is the case then τ is necessarily unbounded.

Question 13.24. Is the σ-universal property preserved under topological completion? Is the σ-universal property preserved under quotients? I am fine with assuming X is almost σ-order complete.

Proposition 13.25. *Let τ be a σ-universal topology on an almost σ-order complete vector lattice X. If τ is metrizable then τ is minimal.*

Proof. Suppose τ is metrizable and sequentially *uo*-complete. By the construction in the main theorem, τ^s has a countable base, hence is metrizable. By [AB03, Theorem 7.55], τ^s is minimal. Since X is regular in X^s, τ is minimal as well. ☐

Remark 13.26. A minimal topology is metrizable iff X has COB and CSP iff X^u has CSP.

Corollary 13.27. *Let τ be a σ-universal topology on an almost σ-order complete vector lattice X. τ is locally bounded iff X is finite-dimensional.*

Proof. Since locally bounded implies metrizable, τ is minimal. Apply the result for minimal topologies. ☐

Recall the following:

Definition 13.28. *A locally solid topology τ on a vector lattice X is* ***entire*** *if its carrier, C_τ, is order dense in X. C_τ is defined as in* [AB03, Definition 4.15].

By [AB03, Theorem 4.17(b)], minimal topologies are necessarily entire. We classify when σ-universal topologies are entire.

Theorem 13.29. *Let τ be a σ-universal topology on an almost σ-order complete vector lattice X. TFAE:*

(i) τ is entire;

(ii) X admits a minimal topology.

In particular, either every σ-universal topology on X is entire or every σ-universal topology on X fails to be entire.

Proof. (ii)⇒(i): If X admits a minimal topology τ_m then, since τ is Hausdorff and σ-Fatou, $\tau_m \subseteq \tau$. This implies that $C_{\tau_m} \subseteq C_\tau$, and since C_{τ_m} is order dense in X, so is C_τ.

(i)⇒(ii): Suppose τ is entire; we claim τ^s is entire. For this, it suffices to show that $C_{\tau^s} \supseteq C_\tau$ since C_τ being order dense in X and X being order dense in X^s yields that C_τ is order dense in X. Let $x \in C_\tau$. Then there exists a normal sequence (V_n) of solid σ-Fatou neighbourhoods of zero with $x \in N^d$ where $N = \bigcap_{n=1}^\infty V_n$. We claim that V_n^s is normal, i.e., that $V_{n+1}^s + V_{n+1}^s \subseteq V_n^s$: see [AB03, Definition 1.35, Lemma 1.36]. Take $x, y \in V_{n+1}^s$. Then there exists sequences $(x_k), (y_k) \subseteq V_{n+1}$ with $0 \leq x_k \uparrow |x|$ and $0 \leq y_k \uparrow |y|$. Therefore, $0 \leq x_k + y_k \uparrow |x| + |y|$, and, since (V_n) is normal, $x_k + y_k \in V_n$. It follows that $|x| + |y| \in V_n^s$, so that $x + y \in V_n^s$ by solidity. Define $M = \bigcap_{n=1}^\infty V_n^s$. Then $M \cap X = \bigcap_{n=1}^\infty (V_n^s \cap X) = \bigcap_{n=1}^\infty V_n = N$. Therefore, since N is a (super) order dense sublattice of M, the disjoint complement M^d of M in X^s satisfies $M^d \cap X = N^d$. Here N^d is taken in X. It follows that $x \in M^d$, so that $x \in C_{\tau^s}$ by definition.

Knowing that τ^s is entire, the proof is finished by [AB03, Exercise 7.20]. □

Corollary 13.30. *If an almost σ-order complete vector lattice X admits a minimal topology τ then there exists an order dense σ-ideal of X such that any other σ-universal topology on X induces that same topology as τ on this σ-ideal.*

Proof. Suppose τ is minimal and τ_* is any σ-universal topology on X. Then, by the proof of [AB03, Theorem 7.62], τ^s and τ_*^s agree on C_{τ^s} in X^s. By the proof of Theorem 13.29, $C_\tau \subseteq C_{\tau^s}$ so that τ and τ_* agree on C_τ. C_τ is an order dense σ-ideal of X by [AB03, Theorem 4.17]. □

Unlike with minimal topologies, we have not yet classified when σ-universal topologies are locally convex, Levi, or have BOB. We have also not classified their duals. It turns out that achieving a classification may not be so simple. Recall that a set X has a ***measurable cardinal*** if there is a probability measure μ defined on all subsets of X that vanishes on the finite subsets of X. Somewhat surprisingly, there are connections between measurable cardinals and vector lattices. See, for example, [Lux67], [Frem75, Sec. 3], and [BL88].

Example 13.31. Let X be a non-empty set and suppose there exists a finite measure μ defined on all subsets of X which vanishes on the finite subsets of X. Then the Hausdorff locally solid topology τ on $\mathbb{R}^X$ generated by the Riesz pseudonorms $\rho_x(u) = |u(x)|$ for $x \in X$ and $\rho(u) = \int \frac{|u|}{1+|u|}d\mu$, is σ-universal, Levi, but not minimal. It is Levi because pointwise convergence is Levi and coarser that τ; the rest follows from [AB03, Exercise 7.22].

Example 13.32. Suppose there exists a non-empty set X that admits a non-trivial $\{0,1\}$-valued measure defined on all subsets of X. Then, according to [BL88, Example 1.5] and [Frem75, Proposition 3.5], there exists a non-zero universally complete vector lattice X without atoms, and a Hausdorff locally convex-solid topology τ on X that does not satisfy the Levi property. In particular, τ is not minimal.

Example 13.33. Recall that a minimal topology is locally convex iff X is atomic. Although I have not worked out the details, [AB03, Exercise 7.21] gives evidence that there may exist a σ-universal topology on an atomic universally σ-complete vector lattice that is not locally convex.

Question 13.34. Can one find an explicit example, without using cardinals, of a Hausdorff locally solid topology on a universally σ-complete vector lattice that is Levi (or locally convex) but not minimal?

As in [AL84], one can define the ***lateral completion***, X^λ, of X as the intersection of all laterally complete vector lattices between X and X^u. It can then be verified that X^λ is laterally complete in its own right and, by [AB03, Exercise 7.10], $(X^\lambda)^\delta = (X^\delta)^\lambda = X^u$. The next proposition explains why we are not interested in the Hausdorff locally solid topologies that extend to X^λ.

Proposition 13.35. *Suppose X and Y are sublattices with $X \subseteq Y$ majorizing. If τ is a locally solid topology on X then τ extends to a locally solid topology τ^* on Y. If the inclusion is order dense then τ is Hausdorff iff τ^* is. In particular, if τ is a Hausdorff locally solid* *topology on a laterally complete vector lattice X, then τ extends to a Hausdorff locally solid topology on X^u.*

Proof. Let $\mathcal{N}_0$ be a base of solid τ-neighbourhoods of zero. For each $V \in \mathcal{N}_0$ put

$$V^* = \{y \in Y : \exists v \in V, |y| \leq v\}.$$

It is easy to see that each V^* is solid (and hence balanced).

Absorbing: Let $y \in Y$ and fix $V \in \mathcal{N}_0$. Since X majorizes Y, there exists $x \in X$ with $|y| \leq x$. Since $\frac{1}{n}x \xrightarrow{\tau} 0$, there exists n_0 with $\frac{1}{n_0}x \in V$. Hence, $|\frac{1}{n_0}y| \leq \frac{1}{n_0}x \in V$. This implies that $\frac{1}{n_0}y \in V^*$, so that V^* is absorbing.

Let $V, W \in \mathcal{N}_0$. Then it is easy to see that $(V \cap W)^* \subseteq V^* \cap W^*$, $V^* \cap X = V$, and $V + V \subseteq W \Rightarrow V^* + V^* \subseteq W^*$. Thus, the collection $\{V^* : V \in \mathcal{N}_0\}$ defines a locally solid topology τ^* on Y that extends τ. If the inclusion is order dense then τ is Hausdorff iff τ^* is Hausdorff by [AB03, Exercise 2.5]. If X is laterally complete then X majorizes X^u; of course, the extension to X^u is σ-Lebesgue and is Lebesgue iff τ is. ☐

Question 13.36. To what extent do the other completions defined in [AL84] commute?

For our purposes, we specialize Question 13.36 to the following:

Question 13.37. Is every laterally σ-complete vector lattice almost σ-order complete? Is lateral σ-completeness preserved under σ-order completion?

Remark 13.38. Using [DM83, Lemma 1.1] one can easily see that the property of σ-order completeness is preserved under lateral σ-completion. The last question really asks whether laterally σ-complete (and almost σ-order complete) vector lattices majorize their universal σ-completions. Note that laterally complete vector lattices majorize their universal completions. This question (and maybe a lack of nice intrinsic characterization of sets that are order bounded in X^s, i.e., dominable with respect to X^s) is the main reason why the theory of universal completions is more developed than the sequential analogue.

In this spirit, we present an improvement to [AB03, Theorem 7.38]. With our technology, the proof is quite easy.

Proposition 13.39. *Let X be an almost σ-order complete vector lattice. TFAE:*

(i) X majorizes X^s;
(ii) Every countable dominable subset of X_+ is order bounded in X.

Proof. First note that a positive increasing net (x_α) is dominable iff it is *uo*-Cauchy. Indeed, if (x_α) is dominable, then it is order bounded in X^u, hence has suprema in X^u, hence is *uo*-Cauchy. If (x_α) is *uo*-Cauchy, then since the *uo*-completion of X is X^u, it is *uo*-convergent in X^u. Since (x_α) is increasing, this means it has suprema in X^u, which implies that it is dominable.

Similarly, since X^s is sequentially *uo*-complete, a positive increasing sequence is *uo*-Cauchy in X iff it is order bounded (and hence has supremum) in X^s. We leave the actual proof of the proposition as an easy corollary to these facts.

□

Note that [AL84] shows that, even when X is not almost σ-order complete, there is a vector lattice that deserves to be called the universal σ-completion of X. However, without almost σ-order completeness we lose the universal property that the σ-order completion is the largest super order dense extension of X. I suspect, but have not shown, that most of the results in this section will fail if X is not almost σ-order complete, even though they can be stated verbatim in the non-almost σ-order complete case.

We next show that, generally, not all the completions commute. Indeed, fix some $0 < p < \infty$, and let X be the vector lattice of all Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 |f(x)|^p dx < \infty$, with the pointwise order. Let τ be the topology constructed in [AB03, Example 3.6]. Then $(u\tau)^s$ is a σ-universal topology on X^s that is not a minimal topology. We will show that $(X^s)^\delta$ is not laterally

σ-complete, i.e., lateral σ-completeness can be lost under order completion.

Suppose $(X^s)^\delta$ is laterally σ-complete, and let τ^* be a Hausdorff locally solid extension of $(u\tau)^s$ from X^s to $(X^s)^\delta$, as described in [AB03, Exercises 2.5 and 2.8]. τ^* is necessarily σ-Lebesgue by [AB03, Theorem 7.49]. Note that τ^* cannot be Fatou because if it were then by [AB03, Theorem 7.53] it would be Lebesgue, and then $\tau^*|_{X^s}$ would be minimal. Hence, the statement "Every Hausdorff locally solid topology on a universally complete vector lattice is Lebesgue" must fail by Proposition 3.21. Therefore all the statements of [Frem75, Theorem 3.3] are true. But noting the remarks in [Frem75, Section 3.6], one cannot prove that the statements of [Frem75, Theorem 3.3] are true while only working in ZFC, which is what we have just done. This gives us our contradiction.

Question 13.40. Do all Hausdorff locally solid topologies on a universally σ-complete vector lattice agree on sequences?

Remark 13.41. In [Con05, Corollary 7.4] some progress on the previous question was made, but I think there is a flaw in one of the references, or at least I can't see how to get around an issue with the lack of local convexity. Some results relating σ-universal topologies to some form of "sequential minimality" are achieved in the locally convex case in [Oh98]. It is not clear if local convexity can be dropped.

Theorem 13.21 classifies the σ-POB property for Hausdorff locally solid topologies that extend to X^s. We next present the analagous result for *uo*-convergence.

Proposition 13.42. *Let X be an almost σ-order complete vector lattice. TFAE:*

(i) Sequences in X are uo-Cauchy iff they are o-Cauchy;

(ii) Positive increasing uo-Cauchy sequences are order bounded;

(iii) X majorizes X^s;

In this case sequences are uo-null iff they are o-null.

Proof. We first show that (iii) implies that *uo*-null sequence are *o*-null. Suppose X majorizes X^s and $x_n \xrightarrow{uo} 0$ in X. Then $x_n \xrightarrow{uo} 0$ in X^s, hence, $x_n \xrightarrow{o} 0$ in X^s. It follows that (x_n) is order bounded in X^s, hence in X. It follows that *uo* and *o*-convergence agree for sequences in X. (iii)⇒(i) is similar.

(i)⇒(ii) is clear; we prove (ii)⇒(iii): Let $0 \leq x^s \in X^s$. There exists a sequence (x_n) in X such that $0 \leq x_n \uparrow x^s$. It follows that (x_n) is *uo*-Cauchy in X^s, hence in X. Therefore, (x_n) is order bounded in X, say, $0 \leq x_n \leq x$. We conclude that $x^s \leq x$. □

Question 13.43. Is the extra clause equivalent to the rest?

The following is a "complete solution" to [AB03, Exercise 8]:

Proposition 13.44. *Let X be a laterally σ-complete vector lattice and $0 \neq f \in X_n^\sim$. Then f is a linear combination of coordinate functionals of atoms.*

Proof. By the proof of [AB03, Theorem 7.49], disjoint sequences are order null (this is all that the assumption that X is laterally σ-complete is needed for). Now follow the proof of [GLX17, Proposition 2.2]. □

Notice that if X is laterally σ-complete then by [AB03, Theorem 7.8] $X^\sim = X_c^\sim$. Assuming measurable cardinals, there exists sets A and elements in $(\mathbb{R}^A)^\sim$ that are not linear combinations of coordinate functionals of atoms. Therefore, under the assumption of measurable cardinals, [GLX17, Proposition 2.2] cannot be improved by replacing *uo*-continuous functionals with sequentially *uo*-continuous functionals. Similarly, keeping in mind [AB03, Exercise 7.13], one cannot replace continuous with sequentially continuous in my characterization of the dual of a minimal topology.

We finish this section with a simple result on products: *uo*-convergence, minimal topologies, and σ-universal topologies behave very well in this regard. Note that Proposition 13.45 completely fails for order convergence, even when $I = \mathbb{N}$ and $X^i = \mathbb{R}$ for all $i \in I$.

Proposition 13.45. *Let X^i $(i \in I)$ be a family of vector lattices and form the product $\prod X^i$, with the natural Riesz space structure. Suppose $(x_\alpha) = ((x_\alpha^i))$ is a net in $\prod X^i$. Then $x_\alpha \xrightarrow{uo} 0$ in $\prod X^i$ iff $x_\alpha^i \xrightarrow{uo} 0$ in X^i for each i.*

Proof. Suppose $x_\alpha \xrightarrow{uo} 0$ in $\prod X^i$. Then for each $u \in \prod X^i$ there exists a net $y_\beta \downarrow 0$ in $\prod X_i$ such that for every β there exists α_0, for any $\alpha \geq \alpha_0$, $|x_\alpha| \wedge u \leq y_\beta$ or, equivalently, $|x^i_\alpha| \wedge u^i \leq y^i_\beta$ for each i. It should be easy to check that $y^i_\beta \downarrow 0$ in X^i for each i, which would prove the claim.

Suppose that $x^i_\alpha \xrightarrow{uo} 0$ in X^i for each i. Let $u = (u^i)$ be in $(\prod X^i)_+ = \prod (X_i)_+$. Then there exist nets $(y^i_{\beta^i})_{\beta^i \in J^i}$ decreasing to 0 in X^i such that for any $\beta^i \in J^i$, there exists α^i such that $\forall \alpha \geq \alpha^i$, $|x^i_\alpha| \wedge u^i \leq y^i_{\beta^i}$.

Let J be the collection of all finite subsets of I, directed by inclusion. Consider the index set $J \times \prod J^i$ and the net $J \times \prod J^i \to \prod X^i$ that assigns to $(i_1, \dots, i_n) \times \prod \beta^i$ the value $y^i_{\beta^i} \wedge u^i$ if $i \in \{i_1, \dots, i_n\}$ and u^i if $i \notin \{i_1, \dots, i_n\}$. It should be that this net decreases to 0.

Corresponding to the index $(i_1, \dots, i_n) \times \prod \beta^i$ we choose α_0 larger than $\alpha^{i_1}, \dots, \alpha^{i_n}$ to show that $x_\alpha \xrightarrow{uo} 0$. □

Corollary 13.46. *Arbitrary products of minimal topologies are minimal and arbitrary products of σ-universal topologies are σ-universal.*

14. Properties inherited by completions

It is interesting to know how the properties of X and its completions relate. For example, it was recently shown exactly how the countable sup property passes to X^u. Noticing that $(X^\lambda)^\delta = X^u$ and a vector lattice Y has the countable sup property iff Y^δ does, the following is immediate:

Theorem 14.1. [KT, Theorem 6.2] *Let X be a vector lattice. TFAE:*

(i) X has a countable order basis and the countable sup property;
(ii) X^u has the countable sup property;
(iii) X^λ has the countable sup property.

We next give the corresponding result for X^s:

Theorem 14.2. *Let X be an almost σ-order complete vector lattice. TFAE:*

(i) X has CSP;
(ii) X^s has CSP.

Proof. If X^s has CSP, it is clear that X also has CSP.

(i)$\Rightarrow$(ii): Suppose X has CSP. Since $(X^\sigma)^s = X^s$ and X has CSP iff X^σ has CSP, we may assume, by passing to X^σ, that X is σ-order complete. Therefore, X is an ideal of X^s: see [Tay1, Remark 3.14]. We will verify [AB03, Exercise 1.15(c)]. Let A be a non-empty bounded above disjoint subset of X^s_+. We may assume, of course, that $0 \notin A$. Since A is bounded above there exists $u \in X^s_+$ such that $A \leq u$. Find a sequence (x_n) in X with $0 \leq x_n \uparrow u$. The set $A \wedge x_n \subseteq X_+$ is non-empty order bounded and disjoint. By the countable sup property in X, $A \wedge x_n$ has at most countably many elements. Define $B := \bigcup_n A \wedge x_n$; B is countable.

Take $a \in A$; since $a \wedge x_n \uparrow a \neq 0$, there exists n_a such that $B \ni a \wedge x_{n_a} \neq 0$. If $a, b \in A$ with $a \neq b$, then $a \wedge x_{n_a} \neq b \wedge x_{n_b}$ since $a \perp b$. Therefore, the map $A \to B$, $a \mapsto a \wedge x_{n_a}$ is injective, so that A is countable. □

Example 14.3. [AB03, Example 7.41] gives an example where X is universally σ-complete, order complete and has CSP, but X^u does not have CSP.

Recall that laterally complete spaces always have weak units by [AB03, Theorem 7.2]. The next result characterizes when X^s has a weak unit.

Proposition 14.4. *Let X be an almost σ-order complete vector lattice. TFAE:*

- *(i) X has a countable order basis;*
- *(ii) X^s has a countable order basis;*
- *(iii) X^s has a weak unit.*

Proof. (i)$\Rightarrow$(ii): If (e_n) is a countable order basis for X then it is also a countable order basis for X^s since X is order dense in X^s. (ii)$\Rightarrow$(i): Let $(e_n) \subseteq X^s_+$ be a countable order basis for X^s and find sequences $0 \leq x_n^k \uparrow_k e_n$. Then $\{x_n^k\}$ should be a countable order basis for X. (iii)$\Rightarrow$(ii) is obvious, (ii)$\Rightarrow$(iii) by [Ka99, Lemma 3.1]. □

We next answer when X^s has the remaining properties in the main inclusion theorem [AB03, Theorem 1.60].

Theorem 14.5. *Let X be an almost σ-order complete vector lattice. TFAE:*

(i) *X^s has the projection property;*
(ii) *X^s is order complete;*
(iii) *$X^\sigma = X^\delta$.*

Proof. By [AB03, Theorem 1.59], X^s has the projection property iff X^s is order complete, and in this case $X^\sigma = X^\delta$.

Suppose $X^\sigma = X^\delta$. Since X and X^σ have the same universal σ-completion we may assume, by passing to X^σ, that X is order complete. The reader should recall that X is order complete iff X is an ideal in X^u.

Suppose X is order complete; we claim that X^s is order complete. As we have observed, $X \subseteq X^s \subseteq X^u$ and X is an ideal of X^u. Let $0 < x \in X^s$ and $0 < u \leq x$ with $u \in X^u$; we will show that $u \in X^s$ and, therefore, that X^s is an ideal of X^u. Since X is super order dense in X^s, there exist (x_n) with $0 \leq x_n \uparrow x$ in X^s, hence in X^u. Then $0 \leq x_n \wedge u \uparrow u$ in X^u and $(x_n \wedge u) \in X$ since X is an ideal of X^u. By construction of X^s in [AB03, Theorem 7.42], $u \in X^s$, so that X^s is an ideal of X^u, hence order complete. □

Note X is an ideal of X^u iff X is order complete and X is a ideal of X^s iff X is σ-order complete.

Remark 14.6. X is atomic iff X^u is atomic iff X^s is. If x is an atom of X^u or X^s, then there exists $0 < u \leq x$ with u an atom of X. Similarly, using that $x \in X_+$ is an atom of X iff X^δ ([LZ71, Exercise 37.23]), an element x of X is an atom of X iff X^u iff X^s.

15. An alternative to uo

In this section we go back to the root of why we study *uo*-convergence. As mentioned, *uo*-convergence was initially introduced as a generalization of convergence almost everywhere to vector lattices. However, it is not the only such convergence, and in this section we investigate another way to do it. Recall that to every vector lattice we associate two canonical completions - the order and universal completions. The order completion is the largest order dense and majorizing extension of X, and the universal completion is the largest order dense extension. By choice of definition of order convergence, the following holds for a net (x_α) in X,

$$x_\alpha \xrightarrow{o} 0 \text{ in } X \Leftrightarrow x_\alpha \xrightarrow{o} 0 \text{ in } X^\delta \Rightarrow x_\alpha \xrightarrow{o} 0 \text{ in } X^u.$$

In general, order convergence in X and X^u differ and, by the ubiquity of the universal completion, it is natural to ask how much. For this reason, we introduce the following definition:

Definition 15.1. *We say a net (x_α) in X o^u-**converges** to a vector x in X if $x_\alpha \xrightarrow{o} x$ in X^u. We will write $x_\alpha \xrightarrow{o^u} x$ to denote this convergence.*

Evidently, for a sequence (x_n) in X, $x_n \xrightarrow{uo} 0$ in X iff $x_n \xrightarrow{o^u} 0$ in X, and (x_n) is uo-Cauchy in X iff (x_n) is o^u-Cauchy in X. This means that o^u-convergence is an equally valid generalization of convergence almost everywhere to vector lattices. For general nets, order convergence implies o^u-convergence and o^u-convergence implies uo-convergence.

The key to uo-convergence is that it is stable when passing between X, X^δ, and X^u. As a general principal, one can introduce a convergence c by declaring that for a net (x_α) in X and $x \in X$,

$$x_\alpha \xrightarrow{c} x \text{ in } X \Leftrightarrow x_\alpha \xrightarrow{uo} x \text{ and a tail is order bounded in some space Y.}$$

Here if $Y = X$, then c is order convergence, but it also makes sense to let Y be X^u, the largest order dense extension of X. In this case we get o^u-convergence.

Our first result shows that uo and o^u are fundamentally different:

Proposition 15.2. *Let X be a vector lattice. Then uo and o^u-convergences agree iff X is finite-dimensional.*

Proof. If X is finite-dimensional the claim is clear. Suppose X is infinite-dimensional. By [AB03, Exercise 1.13], X admits a sequence (x_n) of non-zero positive disjoint vectors. Consider the directed set

$\mathbb{N}\times\mathbb{N}$ and the net $y_{(n,m)} := y_n^m = mx_n$. Since X is Archimedean, no tail of y_n^m is order bounded, even in X^u. Hence, y_n^m is not o^u-null. We claim it is uo-null. For this we may assume, WLOG, that X is order complete since y_n^m is uo-null in X iff in X^δ. Fix $u\in X_+$ and define $v_n = \sup_{k,l\geq n}\{y_k^l\wedge u\}$. Then for all n and m, $0\leq y_n^m\wedge u\leq v_n\downarrow$; it suffices to show that $v_n\downarrow 0$. Set $v=\inf v_n$ and define $x_n^m = y_n^m\wedge u$. Then for every m and n, $x_n^m\wedge v_{n+1} = x_n^m\wedge\sup_{k,l>n}x_k^l = \sup_{k,l>n}\{x_k^l\wedge x_n^m\} = 0$ because the x_n are pairwise disjoint. It follows from $x_n^m\wedge v\leq x_n^m\wedge v_{n+1} = 0$ that $x_n^m\wedge v = 0$ for all m and n. Hence, $v = v\wedge v_1 = v\wedge\sup_{k,l\geq 1}x_k^l = \sup_{k,l\geq 1}\{v\wedge x_k^l\} = 0$. □

One can define a laterally σ-complete vector lattice by enforcing that disjoint sequences having suprema, or arbitrary countable disjoint sets having suprema. The previous proposition shows that one must take care when working with σ-convergences since sequences and countably indexed nets in general behave very differently.

Corollary 15.3. *Order and uo-convergences agree iff X is finite-dimensional.*

Remark 15.4. [De64] showed the above result false for an analogue of *uo* in ordered vector spaces.

Remark 15.5. Although *uo* and *o*-convergences never agree in infinite dimensional vector lattices, they may or may not induce the same topology. Indeed, let $X=\mathbb{R}^{\mathbb{N}}$. If a set is *uo*-closed it is clearly *o*-closed. Conversely, suppose A is *o*-closed and $x\in\overline{A}^{uo}$. Since *uo*=pointwise and is metrizable, there exists a sequence in A with $x_n\xrightarrow{uo}x$. Since *o* and *uo*-agree on sequences (X is universally complete) and A is *o*-closed, $x\in A$. More generally, *uo* and o^u have the same closed sets whenever *uo*-convergence is sequential. Although I do not have an example where the *uo* and o^u-topologies disagree, we note that the *o* and o^u-topologies may disagree by observing that the standard unit vector basis in ℓ_1 is *o* but not o^u-closed.

Question 15.6. Do *uo* and o^u have the same closed sets? (A set A is closed if it contains the limits of all convergent nets from A).

Proposition 15.7. *Disjoint nets are o^u-null.*

Proof. Let (x_α) be a disjoint net. Then $y_\alpha = \sup_{\beta\geq\alpha}|x_\beta|$ exists in X^u and $|x_\alpha| \leq y_\alpha \downarrow$. Since X^u is order complete, $y_\alpha \downarrow y$ for some $y \in X^u$. Fix α and find $\alpha' > \alpha$. Then,

$$0 \leq y \wedge |x_\alpha| \leq \left(\sup_{\beta\geq\alpha'}|x_\beta|\right) \wedge |x_\alpha| = \sup_{\beta\geq\alpha'}\left(|x_\beta| \wedge |x_\alpha|\right) = 0,$$

so that $y \wedge |x_\alpha| = 0$. Therefore, $y = y \wedge \sup_\alpha |x_\alpha| = \sup_\alpha (y \wedge |x_\alpha|) = 0$. It follows that $x_\alpha \xrightarrow{o} 0$ in X^u and, therefore, $x_\alpha \xrightarrow{o^u} 0$ in X. □

Theorem 15.8. *Let (X, τ) be a Hausdorff locally solid vector lattice. TFAE:*

(i) o^u-convergence implies τ-convergence;
(ii) τ is minimal.

Proof. If τ is minimal then *uo*-convergence implies τ-convergence. Conversely, if o^u-convergence implies τ-convergence then τ is Lebesgue and disjoint sequences are τ-null. This is enough to imply that τ is minimal. □

Theorem 15.9. *o^u-convergence is topological iff X is finite-dimensional.*

Proof. Suppose o^u-convergence agrees with the convergence of a topology τ. Clearly, τ is Hausdorff, linear, and lattice operations are continuous. By [AT07, Theorem 2.23], the cone X_+ is τ-normal. Let $\mathcal{B}$ be a base of full neighbourhoods of zero. Let $U \in \mathcal{B}$ and choose $V \in \mathcal{B}$ such that $x \in V$ implies $|x| \in U$. Suppose $x \in V$ and $|y| \leq |x|$; then $y \in [-|x|, |x|]$, so that $y \in U$ since U is full. Define $W := \{y \in X : \text{there exists } x \in V \text{ with } |y| \leq |x|\}$. W is a solid neighbourhood of zero contained in U, hence τ has a base of solid sets. Knowing τ is locally solid, it follows from the previous result that τ must be minimal. This means that *uo*-convergence implies τ-convergence, a contradiction to Corollary 15.3. □

Proposition 15.10. *Let X be a vector lattice. TFAE:*

(i) Nets are o^u-null iff they are o-null;
(ii) Positive increasing o^u-Cauchy nets are order bounded;
(iii) X majorizes X^u;

Proof. (i)⇒(ii) is clear since positive increasing o-Cauchy nets are order bounded; we prove (ii)⇒(iii): Let $0 \leq x^u \in X^u$. There exists a net (x_α) in X such that $0 \leq x_\alpha \uparrow x^u$. It follows that (x_α) is o-Cauchy in X^u, hence o^u-Cauchy in X. By assumption, (x_α) is order bounded in X, say, $0 \leq x_\alpha \leq x$. We conclude that $x^u \leq x$, so that X majorizes X^u. (iii)⇒(i) by [GTX17, Theorem 2.8]. □

Proposition 15.11. *Let X be a vector lattice. TFAE:*

(i) o^u-convergence is complete;
(ii) Positive increasing o^u-Cauchy nets are o^u-convergent;
(iii) X is universally complete.

Proof. If X is universally complete, $o = o^u$, and we are done by [GTX17, Proposition 2.3]. (i)⇒(ii) is clear. For the last implication, let $0 \leq x^u \in X^u$. There exists a net (x_α) in X such that $0 \leq x_\alpha \uparrow x^u$. It follows that (x_α) is o-Cauchy in X^u, hence o^u-Cauchy in X. We conclude that $x_\alpha \xrightarrow{o^u} x$ in X for some $x \in X$. By uniqueness of order limits, $x^u = x \in X$ and, therefore, $X = X^u$. □

Lemma 15.12. *Every o^u-Cauchy, uo-convergent net is o^u-convergent. Every o-Cauchy uo-convergent net is o-convergent.*

Proof. Suppose (x_α) is o^u-Cauchy and uo-convergent to $x \in X$. Notice first that (x_α), being o-Cauchy in X^u, has an order bounded tail in X^u. Since $x_\alpha \xrightarrow{uo} x$ in X, the same is true in X^u. Since (x_α) has an order bounded tail in X^u, $x_\alpha \xrightarrow{o} x$ in X^u. By definition, then, $x_\alpha \xrightarrow{o^u} x$ in X. The second statement is easier. □

Although we could continue to show that o^u-convergence and uo-convergence are essentially interchangeable when it comes to the general theory, we finish with one final result and some remarks. The definition of boundedly o^u-complete Banach lattice is exactly what one expects:

Theorem 15.13. *Let X be a Banach lattice. TFAE:*

(i) X is boundedly uo-complete;
(ii) X is boundedly o^u-complete.

Proof. (i)⇒(ii): Let (x_α) be a norm bounded o^u-Cauchy net in X. Since o^u-convergence implies *uo*-convergence, (x_α) is *uo*-Cauchy. By assumption, $x_\alpha \xrightarrow{uo} x$ for some $x \in X$. By Lemma 15.12, $x_\alpha \xrightarrow{o^u} x$.

(ii)⇒(i) is similar to the proof of [Tay1, Theorem 2.6]. □

Remark 15.14. By the same logic presented at the start of this section, one can similarly define o^s***-convergence*** by declaring that $x_\alpha \xrightarrow{o^s} x$ in X if $x_\alpha \xrightarrow{o} x$ in X^s. Here when X is not almost σ-order complete (so that X^s does not exist by our requirement that X be super order dense in X^s) by X^s we mean the universal σ-completion constructed in [AL84], i.e., the intersection of all universally σ-complete vector lattices between X and X^u. Clearly, order convergence implies o^s-convergence and o^s-convergence implies o^u-convergence. Since o^s and o^u agree on sequences, o^s is, again, an equally valid generalization of almost everywhere convergence to vector lattices. We will not develop the theory of o^s-convergence here, but we will note that one should not expect it to be as close to *uo*-convergence as o^u is. We demonstrate that there may be possible surprises with this convergence via the following result:

Proposition 15.15. *o^s-convergence is complete iff X is laterally σ-complete and order complete.*

Proof. If X is laterally σ-complete and order complete then $o = o^s$ is complete. Suppose o^s-convergence is complete; by a similar argument as above, $X = X^s$ so that X is laterally σ-complete and $o = o^s$. Since, then, order convergence is complete, X is order complete. □

Curiously, the discrepancy between *uo*, o^u, and o^s-completeness is invisible in the classical case since $L_0(\mu)$ is universally σ-complete and is universally complete iff it is order complete (because it has a weak unit).

Remark 15.16. Since $(X^\lambda)^\delta = (X^\delta)^\lambda = X^u$, a net (x_α) in X satisfies $x_\alpha \xrightarrow{o} 0$ in X^λ iff $x_\alpha \xrightarrow{o} 0$ in X^u: We do not have to develop a theory of o^λ-convergence!

16. Producing locally solid topologies from linear topologies

In this section we elaborate on Remark 2.13.

Let X be an (Archimedean) ordered vector space. A subset A of X is said to be ***full***[7] if for each $a \in A_+$, $[-a, a] \subseteq A$. A linear topology τ on X is said to be ***normal*** if τ has a base at zero consisting of full sets. Normal topologies are characterized as follows:

Theorem 16.1. [AT07, Theorem 2.23] *Suppose τ is a linear topology on an ordered vector space X. TFAE:*

(i) τ is normal;
(ii) If two nets (x_α) and (y_α) of X satisfy $0 \leq x_\alpha \leq y_\alpha$ for each α and $y_\alpha \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{\tau} 0$.

One can justly argue that continuity of lattice operations is the natural compatibility between topology and lattices, and normality is the natural compatibility between topology and the ordered vector space structure. However, just considering these two compatibilities on their own does not yield nearly as rich of a theory as considering them together. In this section we will indicate how to remedy this situation by extending the definition of locally solid topologies to ordered vector spaces, and introducing a process of converting linear topologies into locally solid topologies.

Proposition 16.2. *Suppose τ is a linear topology on an ordered vector space X with generating cone. Let $\mathcal{N}_0$ be a balanced base for τ and for each $V \in \mathcal{N}_0$ define $V^S := \{x \in X : \text{there exists } y \in V \text{ with } \pm x \leq y\}$. Then the collection $\mathcal{N}_0^S := \{V^S : V \in \mathcal{N}_0\}$ is a base for a normal topology τ^S on X.*

Proof. Trivially, every set in $\mathcal{N}_0^S$ contains 0. We now show that the intersection of any two sets in $\mathcal{N}_0^S$ contains another set in $\mathcal{N}_0^S$. Take $V^S, W^S \in \mathcal{N}_0^S$. Since $\mathcal{N}_0$ is a base we can find $U \in \mathcal{N}_0$ such that $U \subseteq V \cap W$. We claim that $U^S \subseteq V^S \cap W^S$. Indeed, if $x \in U^S$, then

[7] [AT07] and [AB03] call a subset A full if for each pair $x, y \in A$ we have $[x, y] \subseteq A$. This doesn't affect normality by [AT07, Exercise 5, p. 80].

there exists $u \in U$ with $\pm x \le u$. Since $u \in V \cap W$, the claim follows.

We know that for every V there exists W such that $W + W \subseteq V$. Suppose $x, y \in W^S$ so that there exists $w_1 \in W$ with $\pm x \le w_1$ and $w_2 \in W$ with $\pm x \le w_2$. Then

$$\pm (x + y) \le w_1 + w_2 \in W + W \subseteq V \tag{16.1}$$

so that $W^S + W^S \subseteq V^S$.

If $|\lambda| \le 1$ then $\lambda V^S \subseteq V^S$ because V is balanced. To show that V^S is absorbing, let $x \in X$. Since the cone is generating there exists $y \in X_+$ with $\pm x \le y$. Since V is absorbing, there exists $\lambda > 0$ in $\mathbb{R}$ with $\lambda y \in V$. Then $\pm\lambda x \le \lambda y \in V$ implies $\lambda x \in V^S$. This completes the verification that τ^S is a linear topology by [AlB06, Theorem 5.6].

Suppose $0 \le x_\alpha \le y_\alpha \xrightarrow{\tau^S} 0$. Then for each $V \in \mathcal{N}_0$ there exists α_0, for any $\alpha \ge \alpha_0$, there exists $z_\alpha \in V$ with $\pm y_\alpha \le z_\alpha$. It then follows from $-z_\alpha \le 0 \le x_\alpha \le y_\alpha \le z_\alpha$ that $\pm x_\alpha \le z_\alpha$. Hence $x_\alpha \xrightarrow{\tau^S} 0$. One can also prove this by showing that V^S satisfies the (weaker) definition of a full set. □

Remark 16.3. The assumption that the cone is generating cannot be dropped. Indeed, consider $\mathbb{R}^2$ with its standard topology τ, but with positive cone $K := \{(x, 0) : x \ge 0\}$. It is clear that $\frac{1}{n} \cdot (0, 1)$ is not τ^S-null.

Remark 16.4. If V is further assumed to be full then $V^S \subseteq V$ and hence $\tau \subseteq \tau^S$ when τ is normal; [AB03, Example 2.18] demonstrates that this inclusion may fail if τ is not normal.

Remark 16.5. Notice that $V \cap X_+ \subseteq V^S \cap X_+$, and hence if V is full then $V \cap X_+ = V^S \cap X_+$. It follows that $(V^S)^S = V^S$ and hence $(\tau^S)^S = \tau^S$. It also follows that for a positive net (x_α) in X that $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{\tau^S} 0$; the converse holds when τ is normal. Notice also that local convexity is preserved when passing from τ to τ^S.

[Ng71] defines a ***solid subset*** of an ordered vector space with generating cone to be a non-empty subset A that is full and satisfies that for each $a \in A$ there exists $b \in A$ with $\pm a \leq b$. This definition agrees with the standard definition in vector lattices.

Proposition 16.6. *Let τ be a linear topology on an ordered vector space X with generating cone. TFAE:*

(i) τ has a base at zero consisting of solid sets;
(ii) $\tau = \tau^S$.

Proof. (i)⇒(ii): Suppose τ has a base $\mathcal{N}_0$ at zero of solid sets. Since solid sets are full and balanced, τ is normal, so for each $V \in \mathcal{N}_0$ $V^S \subseteq V$. For the converse, suppose $x \in V$. Then there exists $y \in V$ with $\pm x \leq y$. Hence, by definition, $y \in V^S$. It follows that $V = V^S$, so that $\tau = \tau^S$.

(ii)⇒(i): Suppose $\tau = \tau^S$ and $\mathcal{N}_0$ is a base of balanced sets for τ. Then V^S is full by construction; solidity follows from $V \cap X_+ \subseteq V^S \cap X_+$. □

Definition 16.7. *We call a linear topology τ on an ordered vector space X with generating cone **locally solid** if $\tau = \tau^S$; this definition extends the standard definition in vector lattices.*

Consider now the weak topology on a Banach lattice X; it is normal but generally not locally solid. As in Remark 2.13, we can still define the unbounded weak convergence; it agrees with the *uaw*-convergence. However, since the weak topology is not locally solid, we lose the basic fact that $u\tau \subseteq \tau$. To see this note that the Rademacher sequence is weakly null but not *uaw*-null. To remedy this situation, we think of uw as $u(w^S)$, where $w^S = |\sigma|(X, X^*)$.

In general, for a normal topology τ on a vector lattice X and an ideal A of X, we can declare that $x_\alpha \xrightarrow{u_A\tau} x$ if $|x_\alpha - x| \wedge a \xrightarrow{\tau} 0$ for each $a \in A_+$. It is easy to see that $u_A\tau = u_A(\tau^S)$, and, since τ^S is locally solid, we can use the results of this thesis on $u_A(\tau^S)$. This procedure shows that the map $\tau \mapsto u_A\tau$ sends normal topologies to locally solid topologies; if A is order dense then it sends Hausdorff normal topologies to Hausdorff locally solid topologies.

On the other hand, since a linear topology is locally solid iff it is normal and has continuous lattice operations, we can suppose instead of normality that the lattice operations are τ-continuous. In this case it is true that $u_A\tau \subseteq \tau$, but now one must worry about $u_A\tau$ being locally solid. Thanks are due to N. Juselius and P. Dudar for patiently listening to me ramble about lattices, and alerting me to [Kh82, Page 94] which gives an example of an ordering on c_0 which makes the map $x \mapsto x^+$ continuous but not uniformly continuous in the norm topology. See also [AB03, Example 2.18] for a linear topology that is not locally solid but $x \mapsto x^+$ is continuous at zero. In relation with this example, it may be worth noting that a pseudonorm ρ satisfying $\rho(x) = \rho(|x|)$ is not a Riesz pseudonorm iff there is a $u \in X_+$ such that ρ_u is not a Riesz pseudonorm. Here $\rho_u(x) = \rho(|x| \wedge u)$. This follows since if $|x| \leq |y|$ and each ρ_u is a Riesz pseudonorm then $\rho(x) = \rho(|x|) = \rho_{|y|}(|x|) \leq \rho_{|y|}(|y|) = \rho(y)$.

Remark 16.8. Having now a good understanding of locally solid topologies on laterally σ-complete spaces, it may be of interest to study a wider class of topologies in such spaces. One can do this by weakening one of the components of solidity (linearity, normality, and continuous lattice operations). As mentioned, if A is an ideal of a vector lattice X, then the $u_A\tau$-convergence can be defined on X even when τ is only defined on A; the $u_A\tau$-topology has a base of solid (but not necessarily absorbing) sets. This gives one way non-linear topologies on vector lattices appear "naturally". Other examples of non-locally solid topologies include the weak topology of a Banach lattice, which is normal but generally not locally solid, and the discrete topology on $\mathbb{R}$ which is a locally solid group topology (on a universally complete space) that is not linear. In the next section we study certain "maximal" topologies, where linearity will play a key role.

Note that global convergence in measure falls into the class of convergences that would be locally solid, if only they were linear.

Remark 16.9. At this point it is rather tempting to try to extend the theory of unbounded convergence to more general ordered algebraic structures. The question is, which ones? The observant reader may have noticed that scalar multiplication does not play a prominent role in the theory, and in Example 2.9 it even causes trouble. For this reason, it is natural to extend the results to the setting of (locally solid) lattice ordered groups. In this case, the conversion process is generally quite straightforward; to get a flavour on how to do it see [Pap64]. For a notion of convergence in general lattices, see [Frem04]'s order $*$-convergence.

In [De64] a definition of unbounded order convergence in ordered vector spaces was given. For positive nets there isn't much debate on how to define uc-convergence given c-convergence. One simply declares that a positive net $x_\alpha \xrightarrow{uc} 0$ if every order bounded net (y_α) with $0 \leq y_\alpha \leq x_\alpha$ c-converges to 0. However, the definition for general nets is not as transparent. Notice that the modulus in a locally solid vector lattice implies that if $x_\alpha \xrightarrow{\tau} 0$ then there exists a net (y_α) (choose $y_\alpha = |x_\alpha|$) such that $\pm x_\alpha \leq y_\alpha \xrightarrow{\tau} 0$. In other words, every null net is dominated in sign by a positive null net. One can then declare that $x_\alpha \xrightarrow{u\tau} 0$ if one can find a net (y_α) with $\pm x_\alpha \leq y_\alpha \xrightarrow{u\tau} 0$. From this one recovers the fact that τ-convergence implies $u\tau$-convergence.

Since it is not completely clear how to define unbounded convergence[8] in ordered vector spaces (and I am no expert in this area), we will not undertake this endeavour here.

17. The other end of the spectrum

Uniform convergence is one of the most important convergences on a vector lattice, but it has not played much of a role in this theses. I wish to give a brief explanation for this. Given the definition of unbounded order convergence, unbounded topologies, and o^u-convergence, the following is very natural.

[8]In DeMarr's Definition 4 I would have declared an arbitrary net (x_α) to be uo-null if there exists a (positive) uo-null net (y_β) such that for any β there exists α_β, for any $\alpha \geq \alpha_\beta$, $-y_\beta \leq x_\alpha \leq y_\beta$. Actually, in his setting the claim that o-convergence implies uo-convergence is false; it would be true with my definition.

Definition 17.1. *Let X be a vector lattice. A net (x_α) uru-**converges** to a vector $x \in X$ if for each $a \in X_+$*[9]*, $|x_\alpha - x| \wedge a \xrightarrow{u} 0$. (x_α) is said to u^u-**converge** to x if $x_\alpha \xrightarrow{u} x$ in X^u.*

One of the reasons I have not studied these convergences is that they do not generalize convergence almost everywhere or convergence in measure. Indeed, in L_∞, *uru* agrees with norm convergence. The following example illustates a similar finding with u^u-convergence.

Remark 17.2 ([Tuc74])**.** Let $L = \{(\lambda_k) : \lambda_k \in \mathbb{R}, 0 \leq \lambda_k \uparrow \infty\}$. Then, clearly, $\mathbb{R}^L$ is a universally complete vector lattice. However, there is a sequence that is order null (i.e. a.e. null for counting measure on the discrete σ-algebra) but not u-null. Consider $f_n \in \mathbb{R}^L$ such that $f_n((\lambda_k)) = \frac{1}{\lambda_n}$. Then (f_n) order converges to zero in X. However, whenever $0 \leq \lambda_n \uparrow \infty$ in $\mathbb{R}$, $(\lambda_n f_n)$ does not order converge to zero since $\lambda_n f_n((\lambda_n)) = 1$ for all n. Therefore, order convergence is not stable hence cannot agree with uniform convergence on sequences by [LZ71, Theorem 16.3].

Also, *uru*-convergence exhibits wild behaviour. If X is an order continuous Banach lattice, *uru* is *uo*, if X has a strong unit, *uru* is norm convergence (the finest locally solid topology on X), and if X is $\mathbb{R}^{\mathbb{N}}$, *uru* is pointwise convergence (the coarsest Hausdorff locally solid topology on $\mathbb{R}^{\mathbb{N}}$). I believe u^u-convergence has more regulated properties (disjoint sequences are u^u-null, what about nets? c.f. Lemma 18.10), but have not looked at it deeply. Note that uniform convergence implies τ-convergence for every locally solid topology τ, and the norm topology is the finest locally solid topology on a Banach lattice. This is really the reason the unbounded versions of these convergences do not appear in this thesis - we are investigating the other end of the spectrum; specifically, the minimal topologies and those that in some sense can be classified as "weak". As Table 1 demonstrates, it is easy to characterize interactions between uaw^* and the other convergences. *uaw* is also manageable - as demonstrated by [Zab17] - because disjoint sequences

[9]Or more generally, one can test unbounded uniform convergence against an ideal A; you will get a different convergence. It even matters whether you evaluate the uniform convergence in X or in A: Consider the sequence (e_k) in $\ell_\infty \subseteq \mathbb{R}^{\mathbb{N}}$.

are uaw-null. For un, some results are known, but the overall picture is not as developed, so I have not presented it here.

As will be evident after reading the next paragraph, every vector lattice admits a finest locally solid topology, and hence a finest unbounded locally solid topology. Often (for example in Banach lattices) the finest locally solid topology is just the $*$-convergence induced by uniform convergence. Given that uru-convergence implies τ-convergence for all unbounded locally solid τ, taking its $*$-convergence can (in certain spaces) identify concretely the sequential convergence of the finest unbounded locally solid topology.

As mentioned, uniform convergence implies τ-convergence for every locally solid topology τ. However, uniform convergence itself is generally not topological. Does a vector lattice admit a finest locally (convex)-solid topology? The answer is yes, and it is easy to see. Indeed, the family of all lattice pseudonorms (resp. seminorms) on X defines a locally (convex)-solid topology on X and, by [AB03, Theorems 2.25 and 2.28], it is the finest locally (convex)-solid topology on X. Note, of course, that we make no claim that these topologies are Hausdorff. By [AB03, Theorem 4.7], a locally solid topology is Fatou iff it is generated by a family of Riesz pseudonorms with the Fatou property, so one can similarly study the topology generated by all Fatou pseudonorms on X. Similarly, one can define Lebesgue pseudonorms, and look at the topology defined by all of them; replacing pseudonorms with seminorms incorporates convexity. The point is that there are many "maximal" topologies on a vector lattice. This motivates the following possible directions:

Question 17.3. Let P be some property of a locally solid topology, and order the Hausdorff locally solid topologies on X that satisfy P by inclusion. Are the minimal/least elements of this set interesting? When do they exist? Of course, if P is the Lebesgue property, one gets back minimal topologies. However, choosing P to be, say, local convexity, may give something interesting. It may also be worth looking for minimal elements in the category of normal topologies to see to whether normality better approximates linearity or local solidity in Remark 10.2.

Similarly, let A be an ideal of a locally solid vector lattice (X, τ), and suppose τ has P. Proposition 2.19 motivates the study of minimal/coarsest locally solid topologies satisfying P and agreeing with τ on the order intervals of A. Of course, if $u_A\tau$ inherits P from τ, then this is not interesting. However, important properties such as local convexity are lost in the process of unbounding.

On the other hand, Proposition 2.19 motivated the following result:

Proposition 17.4. *If A is an ideal of a locally solid vector lattice (X, τ) then there is a finest locally solid topology $F_A\tau$ on X that agrees with τ on the order intervals of A.*

Proof. Let $\mathcal{N}_0^{u_A\tau}$ be a solid base at zero for $u_A\tau$. We define $\mathcal{N}_0^{F_A\tau}$ by $U \in \mathcal{N}_0^{F_A\tau}$ iff there exists a sequence of solid and absorbing sets (U^n) with $U^1 = U$, $U^{n+1} + U^{n+1} \subseteq U^n$ for all n, and for each $a \in A_+$ and each n there exists $W \in \mathcal{N}_0^{u_A\tau}$ with $W \subseteq U_a^n$. That $\mathcal{N}_0^{F_A\tau}$ defines a locally solid topology can be verified as in Theorem 2.7. It is clear that τ and $F_A\tau$ agree on the order intervals of A, and that $\tau \subseteq F_A\tau$. Since the definition of $F_A\tau$ doesn't explicitly mention τ, only $u_A\tau$, $F_A\tau$ is the finest locally solid topology agreeing with τ on the order intervals of A.

□

Remark 17.5. By a similar proof, if A is an ideal of a locally solid vector lattice (X, τ) and τ has the Fatou property, then there is a finest locally solid topology $F_A^F\tau$ on X that satisfies the Fatou property and agrees with τ on the order intervals of A. One can similarly construct $F_A^P\tau$ for other properties P which are determined by the neighbourhood base at zero. For example, P can be the σ-Fatou property or convexity.

The maps F_A send the set of locally solid topologies on X to itself; note that taking $A = 0$, $F_A\tau$ recovers the finest locally solid topology on X, independent of τ. Similarly, F_A^F maps the set of Fatou topologies to itself, and taking $A = 0$ gives the finest Fatou topology on X.

Suppose τ is Hausdorff and Lebesgue. Since the Lebesgue property is defined by the behaviour of τ on the order bounded sets, $F_X\tau$ must also be Lebesgue. Hence, using Amemiya's theorem, $F_X\tau$ is the finest Lebesgue topology on X (in particular, such a topology exists).

Obviously, if τ is Hausdorff then $F_A\tau$ is Hausdorff for each ideal A. We next show when the converse is true. For this we use our knowledge of unbounded topologies:

Proposition 17.6. *Let X be a vector lattice admitting a Hausdorff locally solid topology. TFAE:*

(i) A is order dense;
(ii) τ is Hausdorff whenever τ is a locally solid topology on X for which $F_A\tau$ is Hausdorff.

Proof. (i)$\Rightarrow$(ii): Since A is order dense and $F_A\tau$ is Hausdorff, $u_A\tau = u_A(F_A\tau)$ is Hausdorff. Since $u_A\tau \subseteq \tau$, τ is Hausdorff.

Let τ a Hausdorff locally solid topology on X. Since τ and $u_A\tau$ agree on the order bounded subsets of A, $F_A\tau = F_A(u_A\tau)$, and this topology is Hausdorff. By assumption, this implies that $u_A\tau$ is Hausdorff, which implies that A is order dense. □

This is as far as we will go here with $F_A^P\tau$; a more thorough study is forthcoming.

We next mention one other relatively unexplored direction; we will use the language of pseudonorms but the reader can easily convert it to the language of base neighbourhoods of zero. Motivated by the $\mathfrak{S}$-topologies introduced on [AB06, Page 149], for each Riesz pseudonorm ρ and each $\emptyset \neq A \subseteq X$, define $\rho_A : X \to \mathbb{R}$ via $\rho_A(x) = \sup_{a\in A} \rho(|x| \wedge |a|)$. One easily checks that ρ_A is a Riesz pseudonorm dominated by ρ. Now, if $\mathfrak{P}$ is a collection of Riesz pseudonorms generating a locally solid topology τ on X, and $\mathfrak{S}$ is a non-empty collection of non-empty subsets of X, then the collection $\mathfrak{P}^{\mathfrak{S}} = \{\rho_A : \rho \in \mathfrak{P},\ A \in \mathfrak{S}\}$ defines a locally solid topology $u_{\mathfrak{S}}\tau$ on X that is coarser than τ. For A any ideal of X, taking $\mathfrak{S} = \{\{a\}\}_{a\in A}$ gives us back our topology $u_A\tau$; if $X \in \mathfrak{S}$ (for example) then $u_{\mathfrak{S}}\tau = \tau$. Thus, to every τ we associate a collection of topologies $u_{\mathfrak{S}}\tau$ that are coarser than τ. We leave it as an exercise to find a locally solid topology τ and a Hausdorff locally solid topology $\sigma \subseteq \tau$ such that there is no $\mathfrak{S}$ with $u_{\mathfrak{S}}\tau = \sigma$. In general, convergence in the $u_{\mathfrak{S}}\tau$-topology looks kind of like unbounded convergence but "uniformly" against the members of $\mathfrak{S}$.

If one replaces $\mathfrak{S}$ with $\mathfrak{S}' := \{\{a\} : a \in \bigcup \mathfrak{S}\}$ then $u_{\mathfrak{S}'}\tau = u_{I(\bigcup \mathfrak{S})}\tau$, and this disintegration procedure preserves the Hausdorff property. Thus, given a non-empty subset $B \subseteq X$ and a locally solid topology τ on X, one gets a map $\mathfrak{S} \mapsto u_{\mathfrak{S}}\tau$. Here $\mathfrak{S}$ is a collection of non-empty subsets of X with $\bigcup \mathfrak{S} = B$. The topology $u_{\mathfrak{S}'}\tau$ will be the coarsest element of the image.

We leave it to the interested reader to explore this direction more thoroughly. One other natural direction is to study *uo*-convergence, universal completions, and locally solid topologies in free vector/Banach lattices ([dPW15] and [ART18]) or in the universal Banach lattice of [LLOT]. The latter may have applications for *uo*-bibases.

18. Appendix on order convergence

In this section we review order convergence. Order convergence, although natural, is a very subtle subject, and our focus will be to emphasize these subtleties. This will, hopefully, clear up some confusion that still persists in the literature. With this in mind, we will not spend much time convincing the reader that order convergence is important and has many applications; rather, we will stress the distinction in definitions that appear in the literature, note that order convergence is not topological, and emphasize the features that distinguish it from most other convergences encountered in Analysis.

Throughout this section, X denotes an Archimedean vector lattice. This section is presented for completeness and (most of) the results within should not be attributed to me.

18.1. **Why our definition of order convergence?** As the theory of vector lattices evolved, so did the concept of order convergence. In early literature, the following definition was standard:

Definition 18.1. *A net $(x_\alpha)_{\alpha\in A}$ in X is said to be **order convergent** to $x \in X$ if there exists a net $(y_\alpha)_{\alpha\in A}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for all $\alpha \in A$. We will denote this convergence by $x_\alpha \xrightarrow{o_1} x$.*

However, this definition has a major flaw; the next example shows that o_1-convergence is not a "tail property".

Example 18.2. Consider the net (x_α) in $\mathbb{R}$ whose index set consists of two consecutive copies of $\mathbb{N}$ and

$$(x_\alpha) = (1, 2, 3, \dots, 0, 0, 0, \dots).$$

Since this net is identically zero on the second copy of $\mathbb{N}$, it is natural to expect that it converges to zero. However, it is very easy to show that there is no decreasing net on the same index set which dominates (x_α). In particular, o_1-convergence does not even agree with the usual notion of convergence in $\mathbb{R}$.

Although o_1-convergence is outdated, it has been engrained in the vector lattice literature and cannot be avoided. In particular, this definition is used in [AB03] and [AB06], which are two of the most important and standard references. The goal of the appendix is to justify our choice of definition of order convergence, and to show that most properties of vector lattices and locally solid topologies are independent of this choice.

In [AS05], two ways to patch the tail issue were proposed:

Definition 18.3. *A net $(x_\alpha)_{\alpha\in A}$ in X is said to be **order convergent** to $x \in X$ if there exists a net $(y_\beta)_{\beta\in B}$ such that $y_\beta \downarrow 0$ and for each $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_0$. We will denote this convergence by $x_\alpha \xrightarrow{o_2} x$ but, after some explanation, also denote it by $x_\alpha \xrightarrow{o} x$.*

Definition 18.4. *A net $(x_\alpha)_{\alpha\in A}$ in X is said to be **order convergent** to $x \in X$ if there exists a net $(y_\alpha)_{\alpha\in A}$ with $y_\alpha \downarrow 0$ and an index $\alpha_0 \in A$ such that $|x_\alpha - x| \le y_\alpha$ for all $\alpha \ge \alpha_0$. We will denote this convergence by $x_\alpha \xrightarrow{o_3} x$.*

Clearly,

$$x_\alpha \xrightarrow{o_1} x \Rightarrow x_\alpha \xrightarrow{o_3} x \Rightarrow x_\alpha \xrightarrow{o_2} x, \tag{18.1}$$

and o_1 and o_3-convergences agree on sequences. The next example - essentially due to Fremlin - shows that o_2 and o_3-convergence generally disagree on sequences.

Example 18.5. Let X be the space of functions from $\mathbb{R}$ to $\mathbb{R}$ of the form $c\mathbb{1} + f$ where $c \in \mathbb{R}$ and f is a function with finite support. It is easy to see that X is a vector lattice under point-wise operations. For each $n \in \mathbb{N}$, let x_n be the characteristic function of the singleton $\{n\}$. We will show that $x_n \xrightarrow{o_2} 0$ but $x_n \not\xrightarrow{o_3} 0$.

To see that $x_n \xrightarrow{o_2} 0$, put Λ to be the set of all finite subsets of $\mathbb{R}$ directed by inclusion. For every $\alpha \in \Lambda$, let z_α be the characteristic function of α. Clearly, $(z_\alpha)_{\alpha\in\Lambda}$ is a net, and $z_\alpha \uparrow \mathbb{1}$. Put $y_\alpha = \mathbb{1} - z_\alpha$, then $y_\alpha \downarrow 0$. We claim that (y_α) dominates (x_n). Indeed, for each $\alpha \in \Lambda$ there exists $m \in \mathbb{N}$ such that $m > \max \alpha$ and, therefore, $x_n \leq y_\alpha$ for all $n \geq m$. This proves $x_n \xrightarrow{o_2} 0$.

On the other hand, $x_n \not\xrightarrow{o_1} 0$. Indeed, suppose that (u_n) is a sequence in X such that $u_n \downarrow 0$ and $x_n \leq u_n$ for every n. It follows that $x_k \leq u_n$ for all $k \geq n$, so that u_n is greater than or equal to $\mathbb{1}$ except on a finite set, say, F_n. Take any $t \notin \bigcup_{n=1}^{\infty} F_n$, then $u_n(t) \geq 1$ for every n. This contradicts $u_n \downarrow 0$.

We next show that such an example cannot exist in an order complete vector lattice.

Proposition 18.6. *For a net (x_α) in an order complete vector lattice X and $x \in X$, TFAE:*

(i) $x_\alpha \xrightarrow{o_2} x$;
(ii) $x_\alpha \xrightarrow{o_3} x$.

Proof. Clearly only (i)⇒(ii) needs proof. WLOG, passing to a tail, we may assume that (x_α) is order bounded. Now it is a standard fact that for an order bounded net (x_α) in an order complete vector lattice,

$$x_\alpha \xrightarrow{o_2} x \quad \text{iff} \quad \inf_\alpha \sup_{\beta\geq\alpha} |x_\beta - x| = 0 \quad \text{iff} \quad x = \inf_\alpha \sup_{\beta\geq\alpha} x_\beta = \sup_\alpha \inf_{\beta\geq\alpha} x_\beta.$$

Hence we may choose $y_\alpha = \sup_{\beta\geq\alpha}|x_\beta - x|$ in the definition of o_3-convergence. ☐

The reason we use o_2-convergence is because of the following important theorem:

Theorem 18.7. *For any net (x_α) and vector x in X we have $x_\alpha \xrightarrow{o_2} x$ in X iff $x_\alpha \xrightarrow{o_2} x$ in X^δ. Here, as usual, X^δ denotes the order completion of X.*

In other words, o_2-convergence is just the restriction of the (unambigiously defined) notion of order convergence on X^δ.

With Theorem 18.7 in mind, we will denote o_2-convergence simply by o-convergence. Theorem 18.7 follows from the next result, which is of interest in its own right. It is not really any more general than Theorem 18.7, given that X^δ is the largest order dense and majorizing extension of X.

Theorem 18.8. *Suppose that Y is an order dense and majorizing sublattice of a vector lattice X, (y_α) is a net in Y, and $y \in Y$. Then $y_\alpha \xrightarrow{o} y$ in Y iff $y_\alpha \xrightarrow{o} y$ in X.*

Proof. See [GTX17, Theorem 2.8]. ☐

Theorem 18.8 highlights a subtle issue with order convergence, in that it depends on the enveloping space. The following example shows that the order completion in Theorem 18.7 cannot be replaced with the universal completion.

Example 18.9. Let E be any (infinite dimensional) Banach lattice and (x_n) any normalized disjoint sequence in E. The sequence (nx_n) is not norm bounded, and hence cannot be order bounded in E. It follows that the sequence (nx_n) is not order null in E. Although (x_n) fails to be order null in E, it is order null when viewed as a sequence in the universal completion of E. This follows from the next simple lemma:

Lemma 18.10. *Let X be a vector lattice. TFAE:*

(i) Disjoint nets in X that have countable index sets are u-null;
(ii) Disjoint sequences in X are order null;
(iii) Countable disjoint collections in X are order bounded;

We will call such vector lattices σ***-laterally bounded.***

Proof. (i)⇒(ii) is clear, as is (ii)⇒(iii). Suppose that $(x_\alpha)_{\alpha\in\Lambda}$ is a countably indexed disjoint net in X. Find a bijection $\varphi:\Lambda\to\mathbb{N}$ and define $y_\alpha=\varphi(\alpha)|x_\alpha|$; $\{y_\alpha\}$ is, by assumption, order bounded by some $e\in X_+$. Therefore, $\varphi(\alpha)|x_\alpha|\leq e$ for all α, and this can be shown to imply the uniform convergence of (x_α). □

Example 18.11. *A vector lattice that is laterally bounded but not laterally σ-complete:* Let X be the vector lattice of sequences (x_n) such that $x_{2n}=x_{2n+1}$ for large enough n. Then $X^u=\mathbb{R}^{\mathbb{N}}$ and X majorizes X^u. In particular, any disjoint collection in X is order bounded. It is trivial to see that X is not laterally σ-complete.

It may be worthwhile to remark that boundedly σ-laterally complete vector lattices have been studied in the literature. X is said to be ***boundedly*** σ***-laterally complete***[10] if each countable order bounded disjoint subset of X_+ has supremum. Evidently, a vector lattice is laterally σ-complete iff it is both boundedly σ-laterally complete and σ-laterally bounded. To my knowledge, the concept of laterally σ-bounded vector lattices has not been isolated in the literature and, in particular, it has not been studied to what extent they behave like laterally σ-complete vector lattices. It seems like a worthwhile endeavour to decide which properties (e.g. PPP or uniform completeness or almost σ-order completeness) either come for free or force a σ-laterally bounded space to be laterally σ-complete, and decide exactly which results in [AB03, Chapter 7] generalize. Is there any relation between X being σ-laterally bounded and order and *uo*-convergences agreeing on sequences? (c.f. Proposition 13.42) Are Hausdorff locally solid topologies on such spaces necessarily σ-Lebesgue?

[10]or ***conditionally laterally*** σ***-complete***, or some variant.

One could also define a space X to be ***laterally bounded*** if disjoint collections in X_+ are order bounded. Are laterally bounded spaces exactly the vector lattices that majorize X^u?

18.2. **Which properties depend on the definition of order convergence?** In this subsection we show that most properties of vector lattices and locally solid topologies are independent of the choice of order convergence. This subtlety was explored in [AS05][11], where the authors studied to what extent definitions involving operators depend on the definition of order convergence. We will skip this direction since it does not play a role in this thesis, and focus more on order-topological properties. Recall the following:

Definition 18.12. *A locally solid topology τ on a vector lattice X is said to be* ***Lebesgue*** *if $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{\tau} 0$.*

The following is easy to see:

Proposition 18.13. *Fix $i \in \{1, 2, 3\}$. Then τ is Lebesgue iff $x_\alpha \xrightarrow{o_i} 0$ implies $x_\alpha \xrightarrow{\tau} 0$.*

We next show that the σ-Lebesgue property *does* depend on the definition of order convergence:

Definition 18.14. *A locally solid topology τ is* ***σ-Lebesgue*** *if $x_n \downarrow 0$ implies $x_n \xrightarrow{\tau} 0$.*

It is easy to see that τ is σ-Lebesgue iff $x_n \xrightarrow{o_1} 0$ implies $x_n \xrightarrow{\tau} 0$. However, we cannot replace o_1-convergence with o-convergence:

Example 18.15. It is known that disjoint sequences are uo-null (see Theorem 1.5). Disjoint order bounded sequences are, therefore, o-null. If τ has the property that o-convergence implies τ-convergence for sequences, then every order bounded disjoint sequence would be τ-null. It follows by [AB03, Theorem 3.22] that τ is pre-Lebesgue. In particular, if this property is equivalent to the σ-Lebesgue property then every σ-Lebesgue topology would be pre-Lebesgue. This contradicts

[11] But it can be traced back to very old papers on lattice ordered groups and more general ordered structures. Start, for example, with [Pap64] and the references therein.

[AB03, example 3.25]. I am not sure whether o-null sequences being τ-null characterizes (for Hausdorff locally solid topologies) τ possessing both the σ-Lebesgue and pre-Lebesgue properties. As mentioned earlier, it is not known whether the uo-Lebesgue property depends on the definition of order convergence; the sequential uo-Lebesgue property does.

The next goal is to show that the property of being order closed does not depend on the definition of order convergence. This was first proved in [Imh12]; see also [HK].

Definition 18.16. *Fix $i \in \{1, 2, 3\}$. For a subset A of a vector lattice X, we define the o_i-**closure** of A, denoted $\overline{A}^{o_i}$, to be the set of all $x \in X$ such that $x_\alpha \xrightarrow{o_i} x$ in X for some net (x_α) in A. We say that A is o_i-closed in X if $A = \overline{A}^{o_i}$.*

Before showing that the property of being order closed does not depend on the definition of order convergence, we note two subtleties. The first is that order closures need not be order closed (and, in particular, the order closure of A may be a proper subset of the intersection of all order closed sets containing A). This is demonstrated by the following theorem:

Theorem 18.17. *Let X be a σ-order complete Banach lattice. TFAE:*

(i) X is order continuous;
(ii) $\overline{Y}^{o}$ is order closed for every sublattice Y of X;
(iii) $\overline{Y}^{o} = \overline{Y}^{uo}$ for every sublattice Y of X.

Proof. The proof is technically difficult and omitted: see [GL18, Theorem 2.7]. Note that a sublattice is uo-closed iff it is o-closed, and contrast this with statement (iii). ☐

Remark 18.18. It is of interest to know to what extent Theorem 18.17 differs if "sublattice" is replaced with "solid subset". Note that, just as for sublattices, a solid set is order closed iff it is uo-closed (Lemma 6.18). Using recent work of N. Gao and C. Munari, we can actually prove more:

Proposition 18.19. *Let A be a solid subset of a vector lattice X. Then $\overline{A}^o = \overline{A}^{uo}$, and this set is order closed.*

Proof. Arguing as in [AB03, Theorem 2.19(ii)], both $\overline{A}^o$ and $\overline{A}^{uo}$ are solid. Suppose $x \in \overline{A}^{uo}$; it suffices to prove that $|x| \in \overline{A}^o$. Since $|x| \in \overline{A}^{uo}$, find a net (x_α) in A with $x_\alpha \xrightarrow{uo} |x|$. Replacing (x_α) with $(|x_\alpha|)$ we may assume that $x_\alpha \geq 0$. By [GM, Lemma 6b], $|x| = \sup_\gamma z_\gamma$ for some increasing net (z_γ) in A. Therefore, $|x| \in \overline{A}^o$.

We now prove that $\overline{A}^o$ is order closed. For this it suffices to show that if $x \in \overline{\overline{A}^o}^o \cap X_+$, then $x \in \overline{A}^o$. Since $x \in \overline{\overline{A}^o}^o \cap X_+$, there exists a net $(z_\gamma) \in \overline{A}^o$ such that $0 \leq z_\gamma \uparrow x$. Define $Z_\gamma := \{a \in A : 0 \leq a \leq z_\gamma\}$, and $Z := \cup_\gamma Z_\gamma \subseteq A_+$. It is easy to see that Z is directed upward, and hence can be viewed as an increasing net indexed over itself. Now, for each γ, $z_\gamma \in \overline{A}^o$, so there exists a net in A_+ that increases to z_γ. This proves that $\sup Z = x$, and hence $x \in \overline{A}^o$.

□

Remark 18.20. For information on how order and *uo*-closures of sublattices and solid sets relate to finance, see [GM], [GL18] and, generally, the work of N. Gao and F. Xanthos.

Next we note that when taking order closures it matters where the order convergence comes from:

Example 18.21. (*Being order closed depends on the ambient space:*) Let $X = \ell_p \subseteq \mathbb{R}^{\mathbb{N}}$. The set $\{e_k\}$, where e_k is the standard unit vector basis of ℓ_p, is order closed in ℓ_p but $0 \in \overline{\{e_k\}}^o$ when we use the order convergence induced on X from $\mathbb{R}^{\mathbb{N}}$.

We now begin showing that order closed sets are independent of definition. Note that, by (18.1), it suffices to prove that o_1 and o_2-convergence lead to the same order closed sets. So, from now on, unless explicitly stated, $i \in \{1, 2\}$.

We will call the complements of o_i-closed sets o_i-open. Note that if A is o_2-closed then it is o_1-closed. The proof of the next proposition is straightforward:

Proposition 18.22. *For $i = 1, 2$ define $\tau_{o_i} := \{U \subseteq X : U$ is o_i-open$\}$. Then τ_{o_i} is a topology on X and if $A \subseteq X$ is o_i-open, $y \in X$ and $\lambda \in \mathbb{R} \setminus \{0\}$ then $A + y$ and λA are o_i-open.*

Now that we have topologies, they induce convergences, which we will denote by $x_\alpha \xrightarrow{\tau_{o_i}} x$. As we know, $x_\alpha \xrightarrow{o_1} 0 \Rightarrow x_\alpha \xrightarrow{o_2} 0$; the next step is to show that $x_\alpha \xrightarrow{o_2} 0 \Rightarrow x_\alpha \xrightarrow{\tau_{o_1}} 0$.

Lemma 18.23. *For a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X, $x_\alpha \xrightarrow{o_2} 0 \Rightarrow x_\alpha \xrightarrow{\tau_{o_1}} 0$.*

Proof. Suppose $x_\alpha \xrightarrow{o_2} 0$ is a net in X and $U \subseteq X$ is an o_1-open set with $0 \in U$. Then we can find a net $(y_\beta)_{\beta \in B} \downarrow 0$ such that $\forall \beta \in B\ \exists \alpha_0 \in A$ with $|x_\alpha| \leq y_\beta\ \forall \alpha \geq \alpha_0$. We claim that there exists β_1 with $[-y_{\beta_1}, y_{\beta_1}] \subseteq U$. Indeed, if not, then $\forall \beta \in B$, $[-y_\beta, y_\beta] \nsubseteq U$. By the axiom of choice, for each β, pick $z_\beta \in [-y_\beta, y_\beta] \setminus U$. Then $z_\beta \xrightarrow{o_1} 0$, and since this net is contained in the o_1-closed set U^c, $0 \in U^c$. This contradicts the assumption that $0 \in U$.

Corresponding to β_1, we can find α_1 such that $\forall \alpha \geq \alpha_1$, $|x_\alpha| \leq y_{\beta_1}$. This means that $\forall \alpha \geq \alpha_1$, $x_\alpha \in [-y_{\beta_1}, y_{\beta_1}] \subseteq U$. We conclude that a tail of (x_α) is contained in U, so that $x_\alpha \xrightarrow{\tau_{o_1}} 0$. □

Theorem 18.24. *If $A \subseteq X$ then A is o_1-closed $\Leftrightarrow$ A is o_2-closed. Therefore, $\tau_{o_1} = \tau_{o_2} = \tau_{o_3} =: \tau_o$.*

Proof. $\Leftarrow$: Clear since $x_\alpha \xrightarrow{o_1} x \Rightarrow x_\alpha \xrightarrow{o_2} x$.

$\Rightarrow$: Let $A \subseteq X$ be o_1-closed and $x_\alpha \xrightarrow{o_2} x$ a net in A converging to some $x \in X$. Then $(x_\alpha - x) \xrightarrow{o_2} 0$ so, by the previous lemma, $(x_\alpha - x) \xrightarrow{\tau_{o_1}} 0$. By translation invariance of the topology, $x_\alpha \xrightarrow{\tau_{o_1}} x$. Assume $x \notin A$. Then, since A^c is o_1-open and $x \in A^c$, there exists α_0 such that $\forall \alpha \geq \alpha_0$, $x_\alpha \in A^c$. This contradicts the assumption that (x_α) is a net in A, so we conclude that $x \in A$. A is, therefore, o_2-closed.

Since τ_{o_1} and τ_{o_2} have the same closed sets, they coincide as topologies. Since every o_2-closed set is o_3-closed and every o_3-closed set is o_1-closed, o_3-convergence also has the same closed sets. □

Corollary 18.25. *The Fatou property does not depend on the definition of order convergence, nor do bands.*

Definition 18.26. *An ideal A of X is said to be a σ-**ideal** if whenever $(x_n) \subseteq A$ and $0 \leq x_n \uparrow x$ imply $x \in A$. Equivalently, A is a σ-ideal iff it is sequentially o_1-closed.*

Proposition 18.27. *σ-ideals are sequentially o-closed. Moreover, if A is an ideal then the sequential order closure of A is sequentially order closed.*

Proof. Follows from the proof of [Z83, Lemma 105.4]. Compare the proposition with the remark after Proposition 5.7. □

Remark 18.28. I haven't checked whether the σ-Fatou property depends on the definition of order convergence, but, regardless, the proper way to define it is in terms of monotone sequences: see [AB03, Lemma 1.15].

Question 18.29. Does the σ-Fatou property depend on the definition of order convergence? It may be trivial, but I don't have an example of a solid set or sublattice that is sequentially o_1-closed but not sequentially o-closed.

We next return to Example 18.5:

Example 18.30. Let the notation be as in Example 18.5 and let $A = \{x_n\}_{n\in\mathbb{N}}$. As was shown in Example 18.5, $0 \in \overline{A}^{o_2}$, so we should expect $0 \in \overline{A}^{o_1}$; we explicitly constuct a net (w_α) in A such that $w_\alpha \xrightarrow{o_1} 0$. To do this, let Λ be as before. Recall that $\lceil\cdot\rceil$ is the ceiling function and define $(w_\alpha)_{\alpha\in\Lambda}$ via $w_\alpha = \chi_{\{\lceil\max_{x\in\alpha}|x|\rceil+1\}}$, so w_α is just a characteristic function of a singleton set containing a natural number and is thus contained in A. Define $y_\alpha \downarrow 0$ as before. Then $|w_\alpha| = w_\alpha \leq y_\alpha$ and so $w_\alpha \xrightarrow{o_1} 0$. This proves $0 \in \overline{A}^{o_1}$ as expected.

We end this subsection by noting the following. Since there are three definitions of order convergence, there are three potential definitions of unbounded order convergence: For each $i \in \{1, 2, 3\}$ one could say that $x_\alpha \xrightarrow{uo_i} x$ if $|x_\alpha - x| \wedge u \xrightarrow{o_i} 0$ for each $u \in X_+$. However, it is easy to see that uo_1-convergence coincides with uo_3-convergence. It is absolutely crucial that we use uo_2-convergence in the general theory of uo-convergence, because, if not, major results such as Theorem 1.4 and Theorem 1.5 fail.

18.3. **Order convergence is not topological.** Now that we can unambiguously talk about the ***order topology***, τ_o, we may ask when order convergence and τ_o-convergence agree. Some results in this direction were stated in [Imh12] but, unfortunately, there is a subtle but critical error in the proof as a certain directed set happens to be empty. Although it is clear that τ_o is always a T_1-topology, the next example shows that it needn't be Hausdorff, and hence needn't be linear.

Example 18.31. Suppose τ_o defines a Hausdorff topology on $C[0,1]$. Let U and V be open neighbourhoods of zero and $\mathbb{1}$, respectively, with $U \cap V = \emptyset$. Let (a_k) be an enumeration of the rational numbers in $(0,1)$. For each $k, n \in \mathbb{N}$, let $x_{k,n}$ be the continuous functions such that $x_{k,n}(a_k) = 1$, $x_{k,n}$ vanishes outside of $[a_k - \frac{1}{n}, a_k + \frac{1}{n}]$, and is linear on $[a_k - \frac{1}{n}, a_k]$ and on $[a_k, a_k + \frac{1}{n}]$. For each fixed k, $x_{k,n} \downarrow 0$ as $n \to \infty$. It follows that $x_{k,n} \xrightarrow{o} 0$ and, therefore, $x_{k,n} \xrightarrow{\tau_o} 0$ as $n \to \infty$. Choose n_1 such that $y_1 := x_{1,n_1} \in U$; $y_1 \vee x_{2,n} \downarrow y_1$. It follows from $y_1 \in U$ that $y_2 := y_1 \vee x_{2,n_2} \in U$ for some n_2. Iterating this process, we construct sequences (n_k) in $\mathbb{N}$ and (y_k) in U such that $y_k = y_{k-1} \vee x_{k,n_k}$ for every $k > 1$. It follows that $y_k \uparrow$ and $y_k(a_i) = 1$ as $i = 1, \dots, k$, which yields $y_k \uparrow \mathbb{1}$. Therefore, there exists k_0 such that $y_{k_0} \in V$, which contradict the disjointness of U and V.

Remark 18.32. By [M68, Example 2.2] it is even possible that the topology induced by uniform convergence agrees with the topology induced by uo-convergence.

We next show, following [DEM], that order convergence is topological iff X is finite-dimensional.

Definition 18.33. *Order convergence on X is said to be* ***topological*** *if there exists a topology τ on X such that for every net (x_α) and vector x in X, $x_\alpha \xrightarrow{o} x$ iff $x_\alpha \xrightarrow{\tau} x$.*

Lemma 18.34. *If there is a linear topology τ on X such that for every net (x_α) in X, $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{o} 0$ then X has a strong unit e. In this case, for any net (x_α) in X, $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{\|\cdot\|_e} 0$, where $\|x\|_e := \inf\{\lambda > 0 : |x| \le \lambda e\}$.*

Proof. Let $\mathcal{N}_0$ be a base at zero for τ and order $\mathcal{N}_0$ via $U_1 \le U_2$ iff $U_1 \supseteq U_2$. Define $\Lambda := \{(U, y) : U \in \mathcal{N}_0 \text{ and } y \in U\}$, and order Λ lexicographically. This order makes Λ into a directed set. For each $\alpha = (U, y)$ in Λ let $x_\alpha = y$. Then $x_\alpha \xrightarrow{\tau} 0$. By assumption, this implies that $x_\alpha \xrightarrow{o} 0$. Therefore, a tail of (x_α) is order bounded; there exists $e \in X_+$ and $(U_0, y_0) \in \Lambda$ such that for all $(U, y) \ge (U_0, y_0)$, $x_{(U,y)} = y \in [-e, e]$. Find $U_1 \in \mathcal{N}_0$ such that $U_1 \subsetneq U_0$. It follows that $U_1 \subseteq [-e, e]$. Since neighbourhoods are absorbing, e is a strong unit.

Since e is a strong unit, $I_e = X$ and $(X, \|\cdot\|_e)$ is a normed lattice. Suppose $x_\alpha \xrightarrow{\tau} 0$ and let U_1 be as above. Then for any $\varepsilon > 0$, noting that εU_1 is a zero neighbourhood for τ, there exists α_ε such that for all $\alpha \ge \alpha_\varepsilon$ $x_\alpha \in \varepsilon U_1$. Hence $|x_\alpha| \le \varepsilon e$, which implies that $x_\alpha \xrightarrow{\|\cdot\|_e} 0$. □

Now, if we want to classify when uniform convergence is topological, we are done:

Proposition 18.35. *Uniform convergence is topological iff X has a strong unit.*

Proof. If uniform convergence is topological, X has a strong unit by Lemma 18.34 (noting that uniform convergence implies order convergence). Conversely, if X has a stong unit e then $x_\alpha \xrightarrow{u} 0$ iff $\|x_\alpha\|_e \to 0$, so that uniform convergence is topological and, actually, normable. □

Theorem 18.36. *Let X be a vector lattice. Order convergence is topological iff X is finite-dimensional.*

Proof. It is clear that order convergence is topological in finite-dimensional spaces. Suppose order convergence agrees with the convergence of a topology τ. Since order convergence has unique limits and respects addition and scalar multiplication, τ is a Hausdorff linear topology. By Lemma 18.34, X has a strong unit e, $(X = I_e, \|\cdot\|_e)$ is a normed lattice, and τ-convergence implies $\|\cdot\|_e$-convergence. Since $\|\cdot\|_e$-convergence implies order convergence which, by assumption, implies τ-convergence, for any net (x_α) in X, $x_\alpha \xrightarrow{\tau} 0$ iff $x_\alpha \xrightarrow{\|\cdot\|_e} 0$. Therefore, $(X, \|\cdot\|_e)$ is an order continuous normed lattice. Since the $\|\cdot\|_e$-topology is unbounded (which follows easily by identifying $X = I_e$ with an order dense and majorizing sublattice of $C(K)$ with e corresponding to $\mathbb{1}$), it follows that *uo*-convergence implies norm-convergence. This is impossible unless X is finite-dimensional. Simply take any disjoint normalized sequence (x_n) and notice that (nx_n) is *uo*-null but not norm null.

We also present an alternative argument:

Identify X as an order dense and majoring sublattice of some order complete $C(K)$ with e corresponding to $\mathbb{1}$. Let $t_0 \in K$ and define G_{t_0} and F^{t_0} as in Theorem 6.45. As before, $F^{t_0} \downarrow \chi_{G_{t_0}}$ pointwise.

Suppose $\chi_{G_{t_0}} \notin C(K)$. Arguing as in Theorem 6.45, $F^{t_0} \downarrow 0$ in $C(K)$, and hence in X. Knowing that $F^{t_0} \downarrow 0$ in X, our assumptions that order convergence is topological yields that $F^{t_0} \xrightarrow{\|\cdot\|_e} 0$, which contradicts that $\|f\|_e \geq 1$ for each $f \in F^{t_0}$.

Therefore, $\chi_{G_{t_0}} \in C(K)$, and hence G_{t_0} is open. Write $K = \cup_{t \in K} G_t$. Since K is compact, there exists $t_1 \dots t_n \in K$ with $K = G_{t_1} \cup \dots \cup G_{t_n}$. Therefore, each $x \in X$ can take only finitely many distinct values. It follows that $\dim X < \infty$.

$\square$

Remark 18.37. Although order convergence is never topological on the whole space, it can easily be shown to be topological on the order intervals of atomic vector lattices. This follows since *uo*-convergence is topological in such spaces.

Remark 18.38. The reader is referred to [LZ71, Ch. 2 Sec. 16] to begin a study of sequential variants of the results in this section. Sequential variants appear much more commonly in the literature as in older papers sequential o_1-convergence was all that was considered. Note that in c_0, norm, uniform, and order convergence agree on sequences. This illustrates the importance of using nets in Theorem 18.36, and hints that sequential results will have a different flavour. For lots of information in this direction see the thesis of Theresa K.Y. Chow Dodds. She proves, amongst other things, exactly when the sequential o_1-closure operator is actually a closure operator.

For us, it is the distinction in definition of sequential order convergence that causes issues - o_1-convergence for sequences is generally well-understood, and we would like to have as good an understanding of o-convergent sequences.

Example 18.39. As mentioned, sequentially order closed sets are studied in [LZ71], and in that book they use o_1-convergence. In Example 18.5 and Example 18.30 we considered the set $A = \{x_n\}_{n\in\mathbb{N}}$ and showed that we can find a sequence in A that o_2-converges to zero, and a net in A that o_1-converges to zero. However, one can further show that there is no *sequence* in A that o_1-converges to zero. This illustrates that the choice of whether to use o_1 or o_2-convergence to define sequential closures does crucially effect the theory. Note also that since A is an order bounded set, the same conclusions can be made for uo versus uo_1-convergence. For historical purposes, it seems the first distinction between sequentially o-closed and sequentially o_1-closed sets was made in [Pap64]. As I have stressed, sequential aspects of order convergence are not well understood. In particular, there is no known sequential variants of Theorem 18.17 or Proposition 18.19.

Remark 18.40. Although there are two sequential order topologies, $\tau^s_{o_1}$ and $\tau^s_{o_2}$, the convergence of sequences in these topologies is easy to describe. Indeed, by [Pap65, Theorem 6.1], $x_n \xrightarrow{\tau^s_{o_i}} x$ iff every subsequence of (x_n) has a further subsequence that o_i-converges to x (this is another way to see that the sequential order topologies disagree). This notion of "$*$-convergence" has been a part of the literature for many years (see for example A.L. Peressini's book on ordered topological vector spaces), but the $*$-version of uo has not yet been studied. It may be interesting, since it is even weaker than uo, yet maintains uniqueness of limits for sequences (but not for nets!) A nice consequence of [Pap65, Theorem 6.1] is that for an order bounded sequence $x_n \xrightarrow{\tau^s_{o_i}} 0$ iff $x_n \xrightarrow{\tau^s_{uo_i}} 0$. It is not immediate that there is a net version of this result.

Remark 18.41. One may wonder whether the order topology "explains" the topological terminology used in vector lattices (order bounded, order complete, order continuous, etc.). For example, one may ask whether an operator is (sequentially) o_i-continuous iff it is continuous in the (sequential) o_i-topology. The answer is yes for functionals (modify [LZ71, p. 81]), but, generally, [AS05] shows that order continuity depends on the definition of order convergence.

18.4. **When do the order convergence definitions agree?** If one only cares about nets, it is almost always beneficial to use o-convergence over o_3-convergence. However, there is something "natural" about sequences witnessing the order convergence of sequences, and so in some contexts we cannot entirely dismiss o_3-convergence. In this section, we classify the spaces in which such distinctions arise.

Lemma 18.42. *Let X be a vector lattice and $(x_\alpha)_{\alpha\in\Lambda}$ a net in X. If there exists a sequence $y_m \downarrow 0$ such that for any m there exists α_m satisfying $|x_\alpha| \leq y_m$ for all $\alpha \geq \alpha_m$, then $x_\alpha \xrightarrow{o_3} 0$. The converse holds if Λ is countable. The converse holds for every net in X iff X has CSP.*

Proof. Let (x_α) be a net in X and suppose that there exists a sequence $y_m \downarrow 0$ in X such that for any m there exists α_m, for all $\alpha \geq \alpha_m$, $|x_\alpha| \leq y_m$. WLOG, (x_α) does not contain a tail of zeros, i.e., for any $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ with $x_\beta \neq 0$. If this is not the case then it is clear that $x_\alpha \xrightarrow{o_3} 0$. WLOG, we may assume that $\alpha_1 \leq \alpha_2 \leq \dots$. Let $\alpha \in \Lambda$. If $\alpha \ngeq \alpha_1$, define $z_\alpha = y_1$. If $\alpha \geq \alpha_1$ then there exists m such that $\alpha \ngeq \alpha_m$. Indeed, if not then $|x_\alpha| \leq y_m \downarrow 0$ for all m, so that $x_\alpha = 0$, which contradicts that (x_α) has no tail of zeros. Let m be such that $\alpha \geq \alpha_m$ but $\alpha \ngeq \alpha_{m+1}$. Define $z_\alpha = y_m$. It is easy to see that the net $(z_\alpha)_{\alpha\in\Lambda}$ satisfies $z_\alpha \downarrow 0$ and for each $\alpha \geq \alpha_1$, $|x_\alpha| \leq z_\alpha$, so that $x_\alpha \xrightarrow{o_3} 0$.

Suppose Λ is countable and $x_\alpha \xrightarrow{o_3} 0$. Then there exists $y_\alpha \downarrow 0$ and $\alpha_0 \in \Lambda$ with $|x_\alpha| \leq y_\alpha$ for all $\alpha \geq \alpha_0$. Enumerate $\Lambda = \{\alpha_1, \alpha_2, \dots\}$ and define $z_m = y_{\alpha_1} \wedge \dots \wedge y_{\alpha_m}$. Then $z_m \downarrow 0$. Fix m and choose $\beta \geq \alpha_0, \alpha_1 \dots, \alpha_m$. Then for all $\alpha \geq \beta$, $|x_\alpha| \leq y_\alpha \leq y_\beta \leq y_{\alpha_1} \wedge \dots \wedge y_{\alpha_m} = z_m$.

CSP part: Suppose $x_\alpha \downarrow 0$. Then $x_\alpha \xrightarrow{o_3} 0$. By assumption there exists a sequence $y_m \downarrow 0$ such that for any m there exists α_m, for all $\alpha \geq \alpha_m$, $|x_\alpha| \leq y_m$. It follows that $x_{\alpha_m} \downarrow 0$, so that X has CSP. Suppose X has CSP and $x_\alpha \xrightarrow{o_3} 0$. Then there exists $y_\alpha \downarrow 0$ and $\alpha_0 \in \Lambda$ with $|x_\alpha| \leq y_\alpha$ for all $\alpha \geq \alpha_0$. Since X has CSP there exists $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots$ with $y_{\alpha_n} \downarrow 0$, from which the conclusion follows. □

Corollary 18.43. *If X has the countable sup property then for every net (x_α) in X, $x_\alpha \xrightarrow{o} 0$ iff $x_\alpha \xrightarrow{o_3} 0$.*

Recall that a net $(x_\alpha)_{\alpha\in\Lambda}$ is o_3-Cauchy if the double net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in\Lambda\times\Lambda}$ is o_3-null; i.e, if there exists a net $y_{\alpha,\beta} \downarrow 0$ and indices α_0, β_0 with $|x_\alpha - x_\beta| \leq y_{\alpha,\beta}$ for all $(\alpha,\beta) \geq (\alpha_0,\beta_0)$. Equivalently, by defining $z_\alpha := y_{\alpha,\alpha} \downarrow 0$, (x_α) is o_3-Cauchy iff there is a net $z_\alpha \downarrow 0$ and an index α_0 such that for any $\alpha, \beta \geq \alpha_0$, $|x_\beta - x_\alpha| \leq z_\alpha$. This is more natural, especially in the sequential case.

Proposition 18.44. *Let X be a vector lattice. TFAE:*

- *(i) X is almost σ-order complete;*
- *(ii) Monotone order bounded sequences are o_3-Cauchy iff they are o-Cauchy;*
- *(iii) o and o_3-convergences agree on countably indexed nets;*
- *(iv) uo and uo_1-convergences agree on countably indexed nets.*

Proof. (iii)⇒(ii) is immediate and (ii)⇔(i) is in [AL74] after noting that monotone order bounded sequences are o-Cauchy.

(i)⇒(iii): Suppose X is almost σ-order complete, Λ is countable, and $(x_\alpha)_{\alpha\in\Lambda} \xrightarrow{o} 0$ in X. It follows that $(x_\alpha)_{\alpha\in\Lambda} \xrightarrow{o} 0$ in X^σ so that one can find α_0 with $\{x_\alpha\}_{\alpha\geq\alpha_0}$ order bounded in X^σ. For each $\alpha \geq \alpha_0$ define $z_\alpha = \sup_{\beta\geq\alpha} |x_\beta|$, which exists in X^σ since Λ is countable. Then $z_\alpha \downarrow 0$ in X^σ and for all $\alpha \geq \alpha_0$, $|x_\alpha| \leq z_\alpha$. From the lemma there exists a sequence $y_m \downarrow 0$ in X^σ such that for every m there exists α_m, for any $\alpha \geq \alpha_m$, $|x_\alpha| \leq y_m$. Using that X is almost σ-order complete, for each m find a sequence $(x_{m,k})$ in X with $x_{m,k} \downarrow_k y_m$. Define $w_m = \bigwedge_{1\leq i,j\leq m} x_{i,j}$. Clearly, $y_m \leq w_m \downarrow 0$ and (w_m) is in X. Using the lemma again, we conclude that $x_\alpha \xrightarrow{o_3} 0$ in X. (iii)⇔(iv) is immediate. □

Question 18.45. I don't know if one can replace countably indexed nets in (iii) by sequences and keep equivalence. I also do not know if there is a version of Proposition 18.44 for general nets (of course, o and o_3-convergences agreeing on all nets in X is equivalent to uo and uo_1-convergences agreeing on all nets in X). Although not important to the general theory, I have actually not even seen a net in a σ-order complete vector lattice that is o but not o_3-null. The "standard" result is that o and o_3 agree on nets when X is order complete, and agree on sequences when X is σ-order complete. Clearly, these are not sharp.

The final objective is to highlight another difference between the definitions of order convergence. The following fact is standard and appears, for example, in [Z83]. We provide a proof for convenience.

Proposition 18.46. *Every order Cauchy net in an order complete vector lattice is order convergent. Every order Cauchy sequence in a σ-order complete vector lattice is order convergent.*

Proof. Let (x_α) be an order Cauchy net in an order complete vector lattice. It is easy to see that (x_α) has an order bounded tail; it follows that WLOG we may assume that (x_α) is order bounded. Put

$$x = \inf a_\alpha \text{ where } a_\alpha = \sup_{\beta\geq\alpha} x_\beta \text{ and } y = \sup b_\alpha \text{ where } b_\alpha = \inf_{\beta\geq\alpha} x_\beta.$$

Clearly, $b_\alpha \leq y \leq x \leq a_\alpha$ for every α; it suffices to show that $x = y$. Let $(c_{\alpha,\beta})$ be a net such that $c_{\alpha,\beta} \downarrow 0$ and $|x_\alpha - x_\beta| \leq c_{\alpha,\beta}$. Fix a pair (α_0, β_0). Let α be such that $\alpha \geq \alpha_0$ and $\alpha \geq \beta_0$. For every β with $\beta \geq \alpha$, we have $(\alpha, \beta) \geq (\alpha_0, \beta_0)$, so that $\||x_\alpha - x_\beta| \leq c_{\alpha_0,\beta_0}$. It follows that $x_\beta \in [x_\alpha - c_{\alpha_0,\beta_0}, x_\alpha + c_{\alpha_0,\beta_0}]$, which yields $a_\alpha, b_\alpha \in [x_\alpha - c_{\alpha_0,\beta_0}, x_\alpha + c_{\alpha_0,\beta_0}]$ and, therefore, $a_\alpha - b_\alpha \leq 2c_{\alpha_0,\beta_0}$, and hence $x - y \leq 2c_{\alpha_0,\beta_0}$. If follows that $x - y = 0$.

The proof for sequences is similar. □

Remark 18.47. The converse is true as well. A vector lattice is order complete iff every order Cauchy net order converges, and is σ-order complete iff every order Cauchy sequence order converges. This follows from the next lemma, which states that monotone order bounded nets are, actually, *always* o-Cauchy. Recall that X^δ denotes the order completion of X.

Corollary 18.48. *Every order bounded positive increasing net in X is order Cauchy.*

Proof. Let (x_α) be a net in X such that $0 \leq x_\alpha \uparrow \leq x$ for some $x \in X$. Then (x_α) is order convergent in X^δ, and, therefore, order Cauchy in X^δ. It follows that (x_α) is order Cauchy in X by Theorem 18.7. $\square$

Remark 18.49. In the literature one may encounter ***order Cauchy complete vector lattices***, which are defined as the vector lattices in which all o_1-Cauchy sequences are o_1-convergent. This is a *strictly* weaker property than σ-order completeness, as there are generally not enough o_1-Cauchy sequences to witness σ-order completeness. Order Cauchy completeness is complementary to almost σ-order completeness in the sense that a vector lattice is σ-order complete iff it is order Cauchy complete and almost σ-order complete. Compare also [AB03, Exercise 1.29] with [Z83, Exercise 101.8]. Analagously with Proposition 18.44 we may reduce order Cauchy completeness to the monotone case: A vector lattice is order Cauchy complete iff every monotone o_1-Cauchy sequence o_1-converges (i.e. has supremum).

I am not sure if vector lattices in which all o_3-Cauchy nets o_3-converge have been studied in the literature, and, in particular, whether it is known if this property is strictly weaker than order completeness, or, even, how it compares to σ-order completeness.

18.5. **Questions on order convergence.** As this appendix has demonstrated, the concept of order convergence is quite subtle. We warn the reader that there are many errors in the literature concerning order convergence, as it is easy to take for granted the "basic" parts of the theory, and overlook the subtleties highlighted in this chapter. We close with three more basic questions on order convergence; answering them may lead to a better understanding of the subject.

Question 18.50. Do uo-closed sets depend on the definition of order convergence?

Question 18.51. Does the map $\mathcal{P}(X) \to \mathcal{P}(X)$, $A \mapsto \overline{A}^{o_i}$ depend on the definition of order convergence? What about the map $A \mapsto \overline{A}^{uo_i}$?

Question 18.52 ([GL18])**.** Is $\overline{Y}^{uo}$ order closed for every sublattice of a vector lattice X? What if X is a Banach lattice? In the Banach lattice setting, is it always true that $\overline{\overline{Y}^{o}}^{o}$ is order closed?

Part II: Bibases in Banach lattices

In this second portion of the thesis we are primarily interested in basic sequences whose partial sums simultaneously norm and order converge. The results in this section are not as polished or complete as the rest of the thesis, as we are still actively developing the theory of bibases in Banach lattices. With that being said, there are some results that seem intrinsically interesting and I wish to include them. I have also chosen to include many open questions that we are currently thinking about; hopefully there will be solutions to some of these questions in the published version of this work.

For convenience of the reader who has not read the rest of this thesis, we recall what it means to converge in order. Also, since many of our results are applicable in the more general setting of Schauder decompositions, we recall some standard definitions.

In an Archimedean vector lattice X there are three classical notions of sequential convergence:

- $x_n \xrightarrow{u} x$ if there exists $e \in X_+$ and a sequence $\epsilon_m \downarrow 0$ in $\mathbb{R}$ satisfying:
$$\forall m\ \exists n_m\ \forall n \geq n_m,\ |x_n - x| \leq \epsilon_m e.$$
- $x_n \xrightarrow{o_1} x$ if there exists a sequence $y_m \downarrow 0$ in X satisfying:
$$\forall m\ \exists n_m\ \forall n \geq n_m,\ |x_n - x| \leq y_m.$$
- $x_n \xrightarrow{o} x$ if there exists a net $y_\beta \downarrow 0$ in X satisfying:
$$\forall \beta\ \exists n_\beta\ \forall n \geq n_\beta,\ |x_n - x| \leq y_\beta.$$

Clearly, $x_n \xrightarrow{u} x \Rightarrow x_n \xrightarrow{o_1} x \Rightarrow x_n \xrightarrow{o} x$; neither converse implication holds in general. In a Banach lattice, $x_n \xrightarrow{u} x \Rightarrow x_n \xrightarrow{\|\cdot\|} x$, and there is no general relation between norm and o-convergence. Norm convergent sequences do have uniformly convergent subsequences, however.

A sequence (M_k) of closed non-zero subspaces of a Banach space E is called a ***Schauder decomposition*** of $[M_k]$, the smallest closed subspace containing $\bigcup M_k$, if every element x of $[M_k]$ has a unique norm convergent expansion $x = \sum_{k=1}^{\infty} x_k$ where $x_k \in M_k$ for each $k \in \mathbb{N}$. It is well known that a sequence (M_k) of closed non-zero subspaces of E is a Schauder decomposition of $[M_k]$ iff there exists a constant K such that for all m and sequences (x_k) with $x_k \in M_k$ for all k we have $\bigvee_{i=1}^{m} \|\sum_{k=1}^{i} x_k\| \leq K\|\sum_{k=1}^{m} x_k\|$: See [SinII, Theorem 15.5]. For each $x = \sum_{k=1}^{\infty} x_k$ in $[M_k]$ and each i we define $P_i x = \sum_{k=1}^{i} x_k$; each P_i is a continuous linear projection on $[M_k]$.

Before moving on I would like to acknowledge V.G. Troitsky for his collaboration on this portion of the thesis. I would also like to thank W.B. Johnson for the highly beneficial conversations we had during my visit to Texas A&M University.

20. Bidecompositions

Throughout the remainder of this thesis, X denotes a Banach lattice. We begin with a theorem which identifies and gives several equivalent characterizations of the bases we are interested in:

Theorem 20.1. *Let X be a Banach lattice and (M_k) a sequence of subspaces of X that form a Schauder decomposition of $[M_k]$. TFAE:*

(i) For all $x \in [M_k]$, $P_i x \xrightarrow{u} x$;
(ii) For all $x \in [M_k]$, $P_i x \xrightarrow{o} x$;
(iii) For all $x \in [M_k]$, $(P_i x)$ is order bounded in X;
(iv) For all $x \in [M_k]$, $(\bigvee_{i=1}^{m} |P_i x|)$ is norm bounded;
(v) There exists $M \geq 1$ such that for any $m \in \mathbb{N}$ and elements $x_1, \dots, x_m$ with $x_k \in M_k$ for each k one has

$$\left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} x_k \right| \right\| \leq M \left\| \sum_{k=1}^{m} x_k \right\|. \tag{20.1}$$

Proof. (i)⇒(ii)⇒(iii)⇒(iv) is clear.

(iv)⇒(v): Let

$$X_n := \left\{ x \in [M_k] : \sup_m \left\| \bigvee_{i=1}^{m} |P_i x| \right\| \leq n \right\}.$$

Then X_n is a closed set (since the projections and lattice operations are continuous) and $\bigcup_{n=1}^{\infty} X_n = [M_k]$ by (iv). By Baire Category theorem, there exists n_0 such that X_{n_0} has non-empty interior in $[M_k]$, i.e., there exists $x_0 \in X_{n_0}$ and $\varepsilon > 0$ such that $x \in X_{n_0}$ whenever $x \in [M_k]$ and $\|x - x_0\| \leq \varepsilon$. Let $x \in [M_k]$ with $\|x\| \leq 1$. Then for each $m \in \mathbb{N}$,

$$\bigvee_{i=1}^{m} |P_i x| \leq \varepsilon^{-1} \left(\bigvee_{i=1}^{m} |P_i(x_0 + \varepsilon x)| + \bigvee_{i=1}^{m} |P_i x_0| \right)$$

so that $\left\| \bigvee_{i=1}^{m} |P_i x| \right\| \leq \frac{2n_0}{\varepsilon}$. It follows that for general $x \in [M_k]$, $\sup_m \left\| \bigvee_{i=1}^{m} |P_i x| \right\| \leq \frac{2n_0}{\varepsilon} \|x\|$. Now given m and $x_1, \ldots, x_m$ with $x_k \in M_k$, define $x = \sum_{k=1}^{m} x_k$ to get (v) with $M = \frac{2n_0}{\varepsilon}$.

(v)⇒(i): Let $x \in [M_k]$ and let (x_k) be the unique sequence of elements with $x_k \in M_k$ for each k and $P_n x = \sum_{k=1}^{n} x_k \xrightarrow{\|\cdot\|} x$. Then there exists a subsequence $(P_{n_m} x)$ such that $P_{n_m} x \xrightarrow{u} x$. WLOG, passing to a further subsequence and using that $(P_n x)$ is Cauchy, we may assume that for all $i > n_m$, $\| \sum_{k=n_m+1}^{i} x_k \| < \frac{1}{2^m}$.

Define $u_m = \bigvee_{i=n_m+1}^{n_{m+1}} \left| \sum_{k=n_m+1}^{i} x_k \right|$. Applying (v) yields $\|u_m\| \leq M \left\| \sum_{k=n_m+1}^{n_{m+1}} x_k \right\| < \frac{M}{2^m}$. Define $e := \sum_{m=1}^{\infty} m u_m$. Then for every $m \in \mathbb{N}$, $u_m \leq \frac{e}{m}$, so that $u_m \xrightarrow{u} 0$.

Since $|P_{n_m} x - x| + u_m \xrightarrow{u} 0$ there exists a vector $e_1 > 0$ with the property that for any $\varepsilon > 0$ there exists m^*, for any $m \geq m^*$, $|P_{n_m} x - x| + u_m \leq \varepsilon e_1$. Fix ε, and find the required m^*. Let $i \in \mathbb{N}$ with $i > n_{m^*}$. Then we can find $m \geq m^*$ such that $n_m < i \leq n_{m+1}$, so that

$$|P_i x - x| \leq |P_{n_m} x - x| + |P_i x - P_{n_m} x| = |P_{n_m} x - x| + \left| \sum_{k=n_m+1}^{i} x_k \right| \leq |P_{n_m} x - x| + u_m \leq \varepsilon e_1.$$

This shows that $P_i x \xrightarrow{u} x$. □

Remark 20.2. Notice, in particular, that when Theorem 20.1 is applied to basic sequences, the Fatou[12] assumption can be removed from [GKP15, Theorem 2.6].

Definition 20.3. *Let X be a Banach lattice. A sequence (M_k) of closed non-zero subspaces of X will be called a* ***bidecomposition*** *of $[M_k]$ if (M_k) forms a Schauder decomposition of $[M_k]$ and any of the equivalent conditions in Theorem 20.1 are satisfied. We will define bi-FDD's, bibasic sequences, and bibases in the obvious way. The infimum over all M satisfying Equation* (20.1) *will be called the* ***bibasis constant****.*

Remark 20.4. We have changed the definition of bibasic sequences slightly from [GKP15]. Example 23.5 shows that a bibasis in our sense need not be a bibasis in the sense of [GKP15]. We do not know if the [GKP15] definition of bibasis implies ours, in general.

Question 20.5. Suppose (x_k) is a basis and an order schauder basis (in the sense of [GKP15]). Is it a bibasis? Explicitly, if for each $x \in X$ there exist unique scalars (a_k) such that $\sum_{k=1}^m a_k x_k \xrightarrow{\|\cdot\|} x$ and there exist unique scalars (b_k) such that $\sum_{k=1}^m b_k x_k \xrightarrow{o_1} x$, does it follow that $a_k = b_k$ for all k?

Since neither order nor uniform convergence pass freely between sublattices, the following is unexpected:

Corollary 20.6. *Let X be a closed sublattice of a Banach lattice Y and $(M_k) \subseteq X$ a Schauder decomposition of $[M_k]$. Then (M_k) is a bidecomposition of $[M_k]$ with respect to X iff it is a bidecomposition of $[M_k]$ with respect to Y.*

Proof. Statement (iv) of Theorem 20.1 holds in X iff it holds in Y. □

In particular, since X is a sublattice of X^{**}, replacing X with X^{**} we may assume that the ambient Banach lattice is monotonically complete

[12]This is a different, and very much stronger, version of the Fatou property than what has been used in this thesis. From now on, we revert back to our old definition of Fatou, but tailor it to the isometric nature of the norm. Specifically, X is said to have a ***Fatou norm*** if $0 \leq x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$ for every net (x_α) in X. In our language, [GKP15] proved Theorem 20.1 under the assumption that X is KB.

and has a Fatou norm. If (x_k) is basic, one can then pass to $B(\{x_k\})$ in X^{**} to further attain a weak unit.

Remark 20.7. Let (M_k) be a bidecomposition of $[M_k]$ and for $x \in [M_k]$ define $\|x\|_{BSD} = \sup_m \left\| \bigvee_{i=1}^m |P_i x| \right\|$. Then $\| \cdot \|_{BSD}$ is a norm on the Banach *space* $[M_k]$ and, by (v), it is equivalent to the original norm. However, this new norm may be inferior to the old one in that it no longer "respects" the order. Also, there is generally no reason that one can evaluate Equation (20.1) in this norm because lattice operations may take one out of the subspace $[M_k]$. Note that the ability to take $[M_k] \neq X$ emphasizes that Theorem 20.1 is not a theorem on Banach lattices, but a theorem on embedding Banach spaces into Banach lattices. In this sense, it is not surprising that every basic sequence in $C[0,1]$ is bibasic, since every separable Banach space embeds isometrically into $C[0,1]$.

We have the following connection between bidecompositions, bi-FDD's and bibasic sequences:

Corollary 20.8. *Let X be a Banach lattice and (M_k) a sequence of closed non-zero subspaces of X. We have the following:*

(i) If (M_k) forms a bidecomposition of $[M_k]$ then each sequence (x_k) with $0 \neq x_k \in M_k$ is bibasic.

(ii) If each sequence (x_k) satisfying $0 \neq x_k \in M_k$ is bibasic, then there exists an integer N such that $(M_k)_{k \geq N}$ is a bidecomposition of $[M_k]_{k \geq N}$. If, in addition, $\dim M_k < \infty$ for all k then we can take $N = 1$.

Proof. The first part is immediate from statement (v) of Theorem 20.1.

(ii): By [SinII, Theorem 15.21b] there exists N such that $(M_k)_{k \geq N}$ is a Schauder decomposition of $[M_k]_{k \geq N}$; if $\dim M_k < \infty$ for all k we can choose $N = 1$ by [SinII, Theorem 15.22(4)]. It is easy to see from the assumptions in (ii) that any sequence $(x_k)_{k \geq N}$ with $0 \neq x_k \in M_k$ is bibasic. To finish the proof we show that $(M_k)_{k \geq N}$ satisfies condition (i) in Theorem 20.1. Let $x \in [M_k]_{k \geq N}$. Then there exist unique $x_k \in M_k$ $(k \geq N)$ such that $x = \sum_{k=N}^{\infty} x_k$. Let $k \geq N$; if $x_k \neq 0$, define $y_k = x_k$ and $a_k = 1$. If $x_k = 0$, put $a_k = 0$ and let y_k be an arbitrary non-zero element of M_k. As noted, $(y_k)_{k\geq N}$ is bibasic. It follows that the series $x=\sum_{k=N}^{\infty}x_k=\sum_{k=N}^{\infty}a_ky_k$ converges uniformly. □

In the spirit of [GKP15, Problem 1.3] we also mention the following open question. We say that a sequence (x_k) in a Banach lattice X is a ***u*-basis** if for each $x\in X$ there exist unique scalars (a_k) such that $\sum_{k=1}^{m}a_kx_k\xrightarrow{u}x$. Clearly, every bibasis is a u-basis.

Question 20.9. Is every u-basis a bibasis?

Remark 20.10. When X is order continuous, uniform and order convergences agree, and Question 20.9 reduces to [GKP15, Problem 1.3].

Remark 20.11. In this thesis we are always working in the setting of bibases in Banach lattices. However, the concepts of u-bases and order Schauder bases can be formulated in vector lattices: see [GKP15, Definition 1.1]. We note, however, that [GKP15, Definition 1.1] is highly dependent on the choice of ambient vector lattice "E". Indeed, the Haar basis is an order Schauder basis of L_2 when we take $E=F=L_2$ in [GKP15, Definition 1.1], but fails to be an order Schauder basis when we choose $F=L_2$ and $E=L_0$. The latter because order convergence for sequences in L_0 is almost everywhere convergence, and there are multiple almost everywhere expansions of zero for the Haar basis. The importance of Corollary 20.6 to the theory of bibases cannot be overemphasized.

Originally, we were looking to prove Theorem 20.1 for basic sequences. However, the proof immediately generalized to decompositions. We next show that the result does not hold for frames:

Definition 20.12. *Let E be a Banach space. A sequence (x_k,f_k) in $E\times E^*$ is called a* ***quasibasis*** *or* ***frame*** *of E if for every $x\in E$, $\sum_{k=1}^{i}f_k(x)x_k\xrightarrow{\|\cdot\|}x$. As expected, for $x\in E$ and $i\in\mathbb{N}$ we define $P_ix=\sum_{k=1}^{i}f_k(x)x_k$.*

Each statement in Theorem 20.1 has meaning for frames (c.f. [Liu10, Proposition 2.3(a)(i)] if you are unsure how to modify (v)). The arguments we have presented then show that (i)⇒(ii)⇒(iii)⇒(iv)⇔(v). We next show that (iii)⇏(ii):

Example 20.13. Let $X = L_p[0,1]$, $1 < p < \infty$, and let (t_n) be the "typewriter" sequence, $t_n = \chi_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}$ where $k \geq 0$ is such that $2^k \leq n < 2^{k+1}$. Let (h_n, h_n^*) denote the Haar basis with its coordinate functionals, and choose $f \in X^*$ with $f(\mathbb{1}) = 1$. We define our sequence (x_k, f_k) as

$$(x_k) = (h_1, t_1, -t_1, h_2, t_2, -t_2, h_3, t_3, -t_3, \dots),$$
$$(f_k) = (h_1^*, f, f, h_2^*, f, f, h_3^*, f, f, \dots).$$

It is then easy to see that (x_k, f_k) is a frame and, moreover, for every $x \in X$, $P_k(x)$ is order bounded, but $P_k(\mathbb{1}) \overset{a.e.}{\not\longrightarrow} \mathbb{1}$.

I do not know to what extent the other implications in Theorem 20.1 fail for frames.

21. Stability of bibases

Theorem 21.1. *Let (x_k) be a bibasic sequence in a Banach lattice X with basis constant K and bibasis constant M. Let (y_k) be a sequence in X with*

$$2K \sum_{k=1}^{\infty} \frac{\|x_k - y_k\|}{\|x_k\|} =: \theta < 1.$$

Then (y_k) is bibasic with bibasis constant at most $\frac{M+\theta}{1-\theta}$.

Proof. Fix scalars $a_1, \dots, a_m$, put $x = \sum_{k=1}^m a_k x_k$, $y = \sum_{k=1}^m a_k y_k$. Then

$$\|x - y\| \leq \sum_{k=1}^m |a_k| \|x_k - y_k\| \leq 2K\|x\| \sum_{k=1}^m \frac{\|x_k - y_k\|}{\|x_k\|} \leq \theta\|x\|.$$

This implies that $\|x\| \leq \|x-y\| + \|y\| \leq \theta\|x\| + \|y\|$, so that $\|x\| \leq \frac{\|y\|}{1-\theta}$. Define $u := \sum_{k=1}^\infty \frac{|x_k - y_k|}{\|x_k\|}$. Then $\|u\| \leq \theta/(2K)$. For every $n = 1, \dots, m$ we have

$$\left|\sum_{k=1}^n a_k y_k\right| \leq \left|\sum_{k=1}^n a_k x_k\right| + \sum_{k=1}^n |a_k| \cdot |x_k - y_k| \leq \left|\sum_{k=1}^n a_k x_k\right| + 2K\|x\|u.$$

Therefore,

$$\bigvee_{n=1}^m \left|\sum_{k=1}^n a_k y_k\right| \leq \bigvee_{n=1}^m \left|\sum_{k=1}^n a_k x_k\right| + 2K\|x\|u.$$

Yielding, after an application of the bibasis inequality for (x_k)

$$\Big\| \bigvee_{n=1}^{m} \Big| \sum_{k=1}^{n} a_k y_k \Big| \Big\| \leq M\|x\| + 2K\|x\|\|u\| \leq \frac{M+\theta}{1-\theta}\|y\|.$$

Therefore, (y_k) satisfies the bibasis inequality and the conclusion follows. □

The next result follows from the bibasis inequality:

Proposition 21.2. *Let (x_k) be a bibasic sequence in a Banach lattice X. Then every block basic sequence*

$$u_n = \sum_{k=m_{n-1}+1}^{m_n} a_k x_k,\ 0 = m_0 < m_1 < \dots,\ u_n \neq 0$$

is a bibasic sequence with a bibasis constant that does not exceed that of (x_k).

It is obvious that every closed infinite-dimensional sublattice of a Banach lattice admits a bibasic sequence (in the form of a disjoint sequence). The next corollary says that the property of having a bibasic sequence passes down to closed *subspaces*.

Corollary 21.3. *Let (x_k) be a bibasic sequence in a Banach lattice X. Let Y be a closed infinite dimensional subspace of $[x_k]$. Then Y contains a bibasic sequence.*

Proof. Use the argument in Proposition 1.a.11 of [LT77], i.e., pass to a block sequence and perturb. One can even replace the bibasic sequence (x_k) with a bi-FDD, and, up to an ε error, control the bibasis constant of the sequence in Y. □

Question 21.4. Let X be a Banach lattice and Y a closed infinite dimensional subspace of X. Does Y contain a (uo-)bibasic sequence?

We will provide a partial answer to Question 21.4 in Proposition 25.7. Here a sequence (x_k) is said to be *uo-**bibasic*** if it is basic and for each $x = \sum_{k=1}^{\infty} a_k x_k \in [x_k]$ we have $\sum_{k=1}^{m} a_k x_k \xrightarrow{uo} x$; we will introduce this concept formally later on in the thesis.

22. Shades of unconditionality

In this section we investigate varying degrees of unconditionality for bidecompositions.

22.1. **Unconditional bidecompositions.** A Schauder decomposition (M_k) of $[M_k]$ is said to be ***unconditional*** if for each $x \in [M_k]$ the expansion $x = \sum_{k=1}^{\infty} x_k$, $(x_k \in M_k)$ converges unconditionally. We will say that a sequence (M_k) of closed non-zero subspaces is an ***unconditional bidecomposition*** if (M_k) is simultaneously an unconditional Schauder decomposition and a bidecomposition of $[M_k]$.

Corollary 22.1. *Let X be a Banach lattice and (M_k) a sequence of closed non-zero subspaces of X. TFAE:*

(i) (M_k) is an unconditional bidecomposition of $[M_k]$;

(ii) There exists $L_1 \geq 1$ such that for each m and $x_k \in M_k$, $k = 1, \dots, m$, we have

$$\sup_{\delta_k = 0,1} \left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} \delta_k x_k \right| \right\| \leq L_1 \left\| \sum_{k=1}^{m} x_k \right\|. \tag{22.1}$$

(iii) There exists $L_2 \geq 1$ such that for each m and $x_k \in M_k$, $k = 1, \dots, m$, we have

$$\sup_{\varepsilon_k = \pm 1} \left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} \varepsilon_k x_k \right| \right\| \leq L_2 \left\| \sum_{k=1}^{m} x_k \right\|. \tag{22.2}$$

(iv) There exists $L_3 \geq 1$ such that for each m, $x_k \in M_k$, and scalars β_k with $|\beta_k| \leq 1$, $k = 1, \dots, m$, we have

$$\left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} \beta_k x_k \right| \right\| \leq L_3 \left\| \sum_{k=1}^{m} x_k \right\|. \tag{22.3}$$

Proof. (ii)⇒(i): Clearly, Equation (22.1) implies Equation (20.1) as well as inequality (4) in [Marc17, Theorem 3.1]. It follows that (x_k) is an unconditional bi-Schauder decomposition.

(i)⇒(ii) follows from:

$$\left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} \delta_k x_k \right| \right\| \leq M \left\| \sum_{i=1}^{m} \delta_k x_k \right\| \leq MC_2 \left\| \sum_{k=1}^{m} x_k \right\|. \tag{22.4}$$

with $L_1 = MC_2$ and C_2 the constant in [Marc17, Theorem]. (i)⇔(iii) and (i)⇔(iv) are similar. □

However, we have the following subtlety:

Example 22.2. *The bibasis property may be lost after permutation:* Let (x_k) denote the Haar basis of L_2. In [Gr14] it is stated that one can find a function $x \in L_2$ and a rearrangement of the Haar basis so that the sequence of partial sums becomes divergent almost everywhere on $[0, 1]$. This shows that a rearrangement of an unconditional bibasis may not be a bibasis. See also [KS89, Page 96].

22.2. **Permutable decompositions.**

Definition 22.3. *An unconditional decomposition is said to be* ***permutable*** *if every permutation is a bidecomposition.*

Let (M_k) be a permutable bidecomposition and σ a permutation of $\mathbb{N}$. By definition of permutability, $(M_{\sigma(k)})$ is a bidecomposition, so we have an associated bidecomposition constant M^σ. We next show that $\sup_\sigma M^\sigma < \infty$.

Theorem 22.4. *Suppose (M_k) is permutable. Then the supremum of the bibasis constants over all permutations remains finite.*

Proof. Suppose not and consider $(M_k)_{k\geq 2}$. This is a permutable bidecomposition and we claim that it also has the property that the supremum of the bibasis constants over all permutations is infinite. Indeed, if not then there exists M such that for any distinct $n_1, \ldots, n_m$ with $n_k \neq 1$ and any $x_{n_k} \in M_{n_k}$ we have $\| \bigvee_{i=1}^m \left|\sum_{k=1}^i x_{n_k}\right| \| \leq M \left\|\sum_{k=1}^m x_{n_k}\right\|$. So if we take any indices $n_1, \ldots, n_m$, say, with $n_{k^*} = 1$, and any $x_{n_k} \in M_{n_k}$ we have

$$\left\| \bigvee_{i=1}^m \left|\sum_{k=1}^i x_{n_k}\right| \right\| \leq \|x_1\| + M \| \sum_{k\neq k^*} x_{n_k}\| \leq (K_u + MK_u) \| \sum_{k=1}^m x_{n_k}\|,$$

where K_u is the unconditional constant. This contradicts that the supremum of the bibasis constant of $(M_k)_{k\geq 1}$ over all permutations is infinite. Clearly, we can repeat the process and deduce that for each N, $(M_k)_{k\geq N}$ is a permutable bidecomposition with supremum over permutations of the bibasis constant being infinite.

Since the supremum over permutations of the bibasis constants of $(M_k)_{k\geq 1}$ is infinite there exists distinct indices $n_1^1, \ldots, n_{m_1}^1$ and vectors $x_{n_1^1}, \ldots, x_{n_{m_1}^1}$ with $x_{n_k^1} \in M_{n_k^1}$ for $k = 1, \ldots, m_1$ and

$$\left\| \bigvee_{i=1}^{m_1} \left| \sum_{k=1}^{i} x_{n_k^1} \right| \right\| > \left\| \sum_{k=1}^{m_1} x_{n_k^1} \right\|.$$

Define $N_1 = \max\{n_1^1, \ldots, n_{m_1}^1\}$. Then $(M_k)_{k>N_1}$ is a permutable bidecomposition and, as we have noted, its bibasis constant over permutations is unbounded.

We may therefore repeat and find distinct $n_1^2, \ldots, n_{m_2}^2 > N_1$ and $x_{n_1^2}, \ldots, x_{n_{m_2}^2}$ with $x_{n_k^2} \in M_{n_k^2}$ for $k = 1, \ldots, m_2$ such that

$$\left\| \bigvee_{i=1}^{m_2} \left| \sum_{k=1}^{i} x_{n_k^2} \right| \right\| > 2\left\| \sum_{k=1}^{m_2} x_{n_k^2} \right\|.$$

We then repeat the process in the obvious way; the elements of $\mathbb{N}$ that are missed we enumerate as $k_1, \ldots,$. We then order the natural numbers as $n_1^1, \ldots, n_{m_1}^1, k_1, n_1^2, \ldots, n_{m_2}^2, k_2, \ldots,$ and notice that under this permutation, say, σ, $(M_{\sigma(k)})$ fails to be a bidecomposition. $\square$

Corollary 22.5. *Let (M_k) be a sequence of closed non-zero subspaces of X. TFAE:*

(i) (M_k) is a permutable bidecomposition of $[M_k]$.

(ii) There exists $M^P \geq 1$ such that for any distinct indices $n_1, \ldots, n_m$ and any $x_{n_k} \in M_{n_k}$, $k = 1, \ldots, m$, we have

$$\left\| \bigvee_{i=1}^{m} \left| \sum_{k=1}^{i} x_{n_k} \right| \right\| \leq M^P \left\| \sum_{k=1}^{m} x_{n_k} \right\|. \tag{22.5}$$

Before starting the next subsection recall that a Banach lattice is said to be ***sequentially monotonically bounded (σ-MB)*** if each increasing norm bounded sequence is order bounded. Clearly, every dual Banach lattice, being monotonically complete, is σ-MB.

22.3. **Absolute decompositions.** Suppose (M_k) is an unconditional decomposition of $[M_k]$ and denote by P_i^σ the ith canonical projection associated to $(M_{\sigma(k)})$. If (M_k) is permutable, then Theorem 20.1 holds for each permutation σ, so we have statements such as:

(iii$^\sigma$) For all $x \in [M_k]$ there exists $u^\sigma \in X$ with $|P_i^\sigma x| \leq u^\sigma$ for all i.

(iv$^\sigma$) For all $x \in [M_k]$ there exists C^σ such that $\|\bigvee_{i=1}^m |P_i^\sigma x|\| \leq C^\sigma$ for all i.

(v$^\sigma$) There exists $M^\sigma \geq 1$ such that for any $m \in \mathbb{N}$ and elements $x_{\sigma(1)}, \dots, x_{\sigma(m)}$ with $x_{\sigma(k)} \in M_{\sigma(k)}$ for each k one has

$$\left\| \bigvee_{i=1}^m \left| \sum_{k=1}^i x_{\sigma(k)} \right| \right\| \leq M^\sigma \left\| \sum_{k=1}^m x_{\sigma(k)} \right\|. \tag{22.6}$$

By Corollary 22.5, however, one can choose M^σ independent of σ in Equation (22.6) and, consequently, C^σ independent of σ in Item (iv$^\sigma$). Can one choose u independent of σ in Item (iii$^\sigma$)? We will investigate this question in this subsection, and how it relates to a further modification of the basis inequality.

Definition 22.6. *Let X be a Banach lattice and (M_k) a sequence of closed non-zero subspaces of X. We will refer to (M_k) as an* ***absolute decomposition*** *if there exists M^* such that for all m and all sequences (x_k) with $x_k \in M_k$ for each k we have*

$$\left\| \sum_{k=1}^m |x_k| \right\| \leq M^* \left\| \sum_{k=1}^m x_k \right\|.$$

The smallest M^* will be referred to as the ***absolute basis constant****.*

There are many ways to arrive at this definition. Essentially, we are just pulling the sup over ϵ in Equation (22.2) (or [Heil10, Theorem 6.7b]) inside the norm and using $\bigvee_{\varepsilon_i=\pm 1} \sum_{i=1}^n \varepsilon_i x_i = \sum_{i=1}^n |x_i|$. Pulling the sup over δ in Equation (22.1) gives the same condition since $\sum_{i=1}^n |x_i| \geq \bigvee_{\delta_i=0,1} |\sum_{i=1}^n \delta_i x_i| \geq \frac{1}{2} \sum_{i=1}^n |x_i|$ (use Yudin's theorem).

Remark 22.7. A Schauder decomposition (M_k) of $[M_k]$ is absolute iff for each $x = \sum_{k=1}^\infty x_k \in [M_k]$, $\sup_m \| \sum_{k=1}^m |x_k| \| < \infty$. Indeed, since $\sum_{k=1}^m |x_k| = |P_1 x| + |P_2 x - P_1 x| + |P_3 x - P_2 x| + \cdots + |P_m x - P_{m-1} x|$ we see that, for each m, the map $T_m(x) = \sum_{k=1}^m |x_k|$ is continuous. One can then argue as in Theorem 20.1 as the map T_m is sublinear and positively homogeneous. If (M_k) is absolute the formula $\|x\|_A =$

$\sup_m \|\sum_{k=1}^m |x_k|\|$ $(x = \sum_{k=1}^\infty x_k \in [M_k])$ defines an equivalent norm on $[M_k]$.

Proposition 22.8. *Let X be a σ-MB Banach lattice, and (M_k) an unconditional decomposition of $[M_k]$ in X. TFAE:*

(i) (M_k) is absolute;
(ii) For each $x \in [M_k]$ there exists $u \in X$ such that $|P_i^\sigma x| \leq u$ for all i and all permutations σ.

Proof. (i)⇒(ii) is straightforward, and (ii)⇒(i) by lattice identities. (ii)⇒(i) does not need the assumption that X is σ-MB. □

Remark 22.9. Although I do not know if the assumption that X is σ-MB can be removed from Proposition 22.8, we may always view an absolute decomposition as being contained X^{**}, and apply Proposition 22.8 there.

Question 22.10. Can the assumption that X is σ-MB be removed from Proposition 22.8? The simplest place to look for a counterexample is c_0, but this may be too simple.

It is not hard to see that every absolute decomposition is an unconditional Schauder decomposition of $[M_k]$, and in fact is a permutable bidecomposition. Unlike with bibases, the absolute condition is stable under permutation. It is also easy to see that results analagous to Corollary 20.8 (follow the proof in [SinII]), Theorem 21.1, **??**, Proposition 21.2, and Corollary 21.3 hold for absolute basic sequences. From the lattice equality mentioned above, the concepts of absolute and unconditional coincide in AM-spaces.

We next give an example to show that permutable does not imply absolute:

Example 22.11. Let $1 \leq p < \infty$; we will show that the Rademacher sequence (r_k) is permutable but not absolute. Let (x_k) be a permutation of the Rademacher sequence and fix scalars $a_1, \dots, a_m$. Let $f_n = \sum_{k=1}^n a_k x_k$ as $n = 1, \dots, m$. It is easy to see that $(f_n)_{n=1}^m$ is a martingale with difference sequence $d_k = a_k x_k$. The associated square function is $S(f) = (\sum_{k=1}^m a_k^2)^{\frac{1}{2}}\,\mathbb{1}$. By applying the Burkholder-Gundy-Davis inequality followed by Khintchine's inequality, we get

$$\left\| \bigvee_{n=1}^{m} \left| \sum_{k=1}^{n} a_k x_k \right| \right\|_p \leq C \left(\sum_{k=1}^{m} a_k^2 \right)^{\frac{1}{2}} \leq C' \left\| \sum_{k=1}^{m} a_k x_k \right\|_p$$

for some constants C, C'. This shows that (x_k) is bibasic, so that (r_k) is permutable.

Since $|r_k| = \mathbb{1}$, $\|\sum_{k=1}^m |a_k r_k|\|_p = \sum_{k=1}^m |a_k|$, while Khintchine's inequality yields that $\|\sum_{k=1}^m a_k r_k\|_p \sim (\sum_{k=1}^m a_k^2)^{\frac{1}{2}}$. As these two quantities are not equivalent, (r_k) is not absolute.

Remark 22.12. An interesting source of absolute basic sequences arise via the free Banach lattice. Indeed, by [APV, Proposition 1], any unconditional basic sequence (e_n) in a Banach space E gives rise to an absolute basic sequence $(|\delta_{e_n}|)$ in $FBL[E]$.

Proposition 22.13. *Every basic sequence (x_k) in a Banach lattice X that is equivalent to the unit vector basis of ℓ_1 is absolute.*

Proof. Being equivalent to the canonical basis (e_k), (x_k) is semi-normalized. Therefore, there exists C with $\|x_k\| \leq C$ for all k. Let M be the equivalence constant. We have for each $a_1, \dots, a_n$,

$$\left\| \sum_{k=1}^{n} |a_k x_k| \right\| \leq C \sum_{k=1}^{n} |a_k| = C \| \sum_{k=1}^{n} a_k e_k \|_{\ell_1} \leq CM \| \sum_{k=1}^{n} a_k x_k \|.$$

□

Corollary 22.14. *Every unconditional basis of ℓ_1 is absolute.*

Proof. Up to equivalence, ℓ_1 has only one normalized unconditional basis. Hence, given any unconditional basis (x_k) of ℓ_1, $(\frac{x_k}{\|x_k\|})$ is equivalent to the uvb of ℓ_1 and is absolute iff (x_k) is. □

Remark 22.15. The next example shows that similar results fail if ℓ_1 is replaced with ℓ_2:

Example 22.16. Consider the Krengel operator $T = \oplus_{k=1}^{\infty} T_k : \ell_2 \to \ell_2$ where T_k is the operator on the 2^{k-1}-dimensional Hilbert space $\ell_2^{2^{k-1}}$

determined by the recursive formulas

$$T_1 = [1],$$
$$T_{k+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} T_k & T_k \\ -T_k & T_k \end{bmatrix}.$$

Note that T is an isometry but is not order bounded. Hence, by [Kh82, p. 92] it is not even sequentially u-to-u-continuous.

Let (e_k) be the standard basis of ℓ_2 and define $f_k = Te_k$. Since T is an isometry, the sequence (f_k) is an orthonormal basis of ℓ_2; it is easy to see that (f_k) fails to be absolute.

23. Examples

As is evident from the rest of this thesis, a fourth vector lattice convergence; namely, *uo*-convergence, has become important. A Schauder decomposition $(M_k) \subseteq X$ of $[M_k]$ will be called a ***uo-decomposition*** if for all $x \in [M_k]$, $P_i x \xrightarrow{uo} x$. *uo*-bibases and *uo*-bibasic sequences will be defined analagously. Example 22.2 shows that an orthonormal basis in L_2 need not be a *uo*-bibasis. However, it is essentially obvious that every basic sequence in ℓ_2 is *uo*-bibasic. We will define ***permutable uo-decompositions*** as unconditional decompositions in which every permutation is a *uo*-decomposition. ***Permutable uo-bibasic sequences*** are defined in the obvious way.

Remark 23.1. Obviously, every bi-Schauder decomposition is a *uo*-Schauder decomposition. It is easy to see that a Schauder decomposition $(M_k) \subseteq X$ of $[M_k]$ is a *uo*-Schauder decomposition iff for all $x \in [M_k]$, $(P_i x)$ is *uo*-Cauchy. This follows by using norm continuity of lattice operations. Also, since *uo*-convergence passes freely between regular sublattices, (M_k) is *uo*-bibasic in X iff in X^δ. If we are working with basic sequences (x_k), we can then pass down to the band generated by (x_k) in X^δ to get a weak unit. The proofs in [GKP15] are easily modifiable to show that the *uo*-bibasic property is preserved under taking block basic sequences and small perturbations, and it is easy to formulate a *uo*-analogue of Corollary 20.8 and Corollary 21.3.

Possessing a *uo*-bibasis does not seem to give much structural information about the space. In particular, the following questions - which ask how "weak" the *uo*-bibasis property is - are open:

Question 23.2. Is there a Banach lattice with a basis but no *uo*-bibasis? Can one always permute an unconditional basis to make it a *uo*-bibasis (in say order continuous spaces)?

Is there a basic sequence that has no (*uo*)-bibasic block sequences, or no (*uo*-)bibasic subsequences, or at least no *uo*-FDD blocking?

Even more fundamental is the following question:

Question 23.3. Let X be a closed sublattice of a Banach lattice Y and (x_k) a basic sequence in X. Is (x_k) *uo*-bibasic wrt X iff it is *uo*-bibasic wrt Y?

At this point we pause and provide some examples. First, a remark:

Remark 23.4. Let (M_k) be a Schauder decomposition of $[M_k] \subseteq X$, where X is a Banach lattice. Since uniform convergence implies norm convergence, for each $x \in [M_k]$ there is at most one sequence (x_k) with $x_k \in M_k$ for all k satisfying $\sum_{k=1}^{i} x_k \xrightarrow{u} x$. However, the situation is very different if uniform convergence is replaced with order or *uo*-convergence. In some sense, a bibasis is simultaneously a basis with respect to norm convergence, and a "frame" with respect to order convergence.

Example 23.5. *Basis with unique o-expansion:* If X is an order continuous Banach lattice then uniform and order convergence agree, so if a basic sequence (x_k) satisfies the conditions of Theorem 20.1 then for each $x \in [x_k]$ there is a unique sequence (a_k) of scalars such that $\sum_{k=1}^{i} a_k x_k \xrightarrow{o} x$.

Basis with multiple o-expansions: Consider c with basis $e_0 = (1, 1, 1, \dots)$ and e_k as usual for $k \geq 1$. Then $e_1 + \cdots + e_n \uparrow e_0$. Actually, for every $x = (a_k) \in \ell_\infty$, $\sum_{k=1}^{m} a_k e_k \xrightarrow{\sigma o} x$; as an order series, e_0 is redundant.

Basis with unique uo-expansion but no o-expansion: Take $X = \ell_1$. Let

$$x_1 = (1, 0, 0, 0, 0, 0, 0, 0, \dots)$$
$$x_2 = (1, -1, 0, 0, 0, 0, 0, 0, \dots)$$
$$x_3 = (0, 0, 1, 0, 0, 0, 0, 0, 0, \dots)$$
$$x_4 = (0, 0, 1, -1, 0, 0, 0, 0, 0, \dots)$$
$$x_5 = (0, 0, 0, 1, -1, 0, 0, 0, 0, \dots)$$

and so forth in blocks. This will have unique *uo*-expansions, be a basis by Albiac-Kalton Lemma 9.5.3, but is not a bibasis.

Basis with unique o-expansion but multiple uo-expansions: The Haar basis of L_p $(1 < p < \infty)$ is a bibasis and the the o-expansion is unique. The *uo*-expansion is not unique, however, because we can find a "peak" that converges to zero a.e. but has norm 1.

Basis with unique o and uo-expansions: The standard bases of c_0 and ℓ_p $(1 \leq p < \infty)$ satisfy (i) in Theorem 20.1 and all expansions are unique since uo-convergence is just pointwise convergence.

Basis with multiple uo-expansions but no o-expansion: The Haar basis of L_1 is not a bibasis, and the *uo*-expansion of zero is not unique. According to [Gr14] or [KS89, Page 68], $P_k x \xrightarrow{a.e.} x$ for each $x \in L_1$, so that it is a uo-bibasis. One could also use the dual to the summing basis.

The next example shows that duality of bibases has some issues. Duality theory has not yet played a role in our results, but one may be able to play with, for example, the natural weakening of the notion of boundedly complete bases obtained by pulling the sup through the norm.

Example 23.6. It is known that if (x_k) is a basis of X then the biorthogonal functionals (f_k) are basic in X^*. We give an example of a bibasis in an order continuous Banach lattice such that the dual basic sequence is not bibasic. Let $X = c_0$ and $x_k = (1, \dots, 1, 0, \dots)$.

Then $f_k = e_k - e_{k+1}$ does not satisfy the bibasis inequality. Consider $f_1 + \cdots + f_m = e_1 - e_{m+1}$. Then $\bigvee_{i=1}^m \left|\sum_{k=1}^i f_k\right| = e_1 + \cdots + e_{m+1}$ has arbitrarily large norm.

Example 23.7. Let $1 < p < \infty$. If X is a separable L_p-space (equipped with its canonical order) then X admits a bibasis. If X is a separable L_1-space then X admits a uo-bibasis.

Proof. The Haar basis is a bibasis for L_p, $1 < p < \infty$, and a *uo*-bibasis of L_1. Since the standard basis of ℓ_p is a bibasis, the result follows immediately from the lattice isometric classification of separable L_p-spaces. See, for example, [LW76]. □

Remark 23.8. Example 23.7 can be extended to certain separable r.i. spaces based on $[0,1]$ using known properties of the Haar basis. We leave the details to the interested reader, who may find some useful information in [NS, Theorem 4.a.3, Corollary 4.a.6, Theorem 8.6].

Remark 23.9. It is known that if (x_n) is basic and $x_n \xrightarrow{w} x$ then $x = 0$: see [Heil10, Exercise 4.17]. Weak convergence cannot be replaced with order convergence as the basic sequence $x_n = e_1 + \cdots + e_n$, viewed as a sequence in c, shows.

Proposition 23.10. *Let X be a Banach lattice. Every lattice decomposition (in the sense of* [BB09]*) of X is an absolute decomposition.*

Proof. Let $x \in X$; then $x = \sum_{k=1}^\infty Q_k(x)$ where $Q_k : X \to X$ is defined via $x = \sum_{k=1}^\infty x_k \mapsto x_k$. By assumption Q_k is a lattice homomorphism. Therefore, $\sum_{k=1}^i |x_k| = \sum_{k=1}^i |Q_k(x)| \leq \sum_{k=1}^i Q_k(|x|) \leq |x|$, and this is enough to conclude that decomposition is absolute. We actually only needed the Q_k to be positive. □

24. Embedding into spaces with bi-FDDs

Suppose that X and Y are Banach lattices, and (x_k) is a basic sequence in X. If $T : [x_k] \to Y$ is an isomorphic embedding, then (Tx_k) is a basic sequence in Y. Suppose now that (x_k) is bibasic. According to Theorem 20.1, if $(\vee_{n=1}^m |Tz_n|)$ is norm bounded whenever (z_n) is a sequence in $[x_k]$ that is uniformly null in X, then (Tx_k) is bibasic in Y.

These assumptions on T are surprisingly weak, as we will soon explain. In particular, notice that it is not obvious upon first inspection that such operators respect Equation (20.1) in any reasonable way.

Our next preliminary result characterizes order bounded operators between vector lattices in terms of uniform convergence. Although elementary, there seems to be no result of this type in the literature. We will use it to illustrate how weak our assumptions in Theorem 24.9 are.

Now, typically, a linear map $T : X \to Y$ between vector lattices is said to be ***order bounded*** if it maps order bounded subsets of X to order bounded subsets of Y. Since we are interested in operators defined on subspaces of vector lattices, we will expand the definition of order bounded operators to those defined on subspaces $E \subseteq X$. In other words, we say that a linear map $T : E \subseteq X \to Y$ is ***order bounded*** if it maps subsets of E that are order bounded in X to order bounded subsets of Y.

Proposition 24.1. *Let $T : E \subseteq X \to Y$ be a linear map where X and Y are vector lattices and E is a subspace of X. TFAE:*

(i) T is order bounded;
(ii) $x_\alpha \xrightarrow{u} 0$ implies $Tx_\alpha \xrightarrow{u} 0$ for all nets (x_α) in E;
(iii) $x_\alpha \xrightarrow{u} 0$ implies $Tx_\alpha \xrightarrow{o} 0$ for all nets (x_α) in E;

In these statements the uniform convergence of (x_α) is evaluated in X.

Proof. (i)⇒(ii): Suppose (x_α) is a net in E and $x_\alpha \xrightarrow{u} 0$. Then there exists $e \in X_+$ such that for every $\varepsilon > 0$ there exists α_0, for all $\alpha \geq \alpha_0$, $|x_\alpha| \leq \varepsilon e$. Find $0 < a \in Y_+$ with $T[-e, e] \subseteq [-a, a]$. Then for ϵ and α_0 as above we have $|Tx_\alpha| \leq \varepsilon a$ for all $\alpha \geq \alpha_0$. This shows that $Tx_\alpha \xrightarrow{u} 0$. (ii)⇒(iii) is trivial.

(iii)⇒(i): Let $0 < e \in X$; it suffices to show that $T([0, e] \cap E)$ is order bounded in Y. Let $\Delta := \{(n, x) : n \in \mathbb{N},\ x \in [0, e] \cap E\}$ with the lexicographic order and consider the net $v_{n,x} = \frac{1}{n}x$. Then $v_{n,x} \xrightarrow{u} 0$, so that $Tv_{n,x} \xrightarrow{o} 0$. It follows that $(Tv_{n,x})$ has an order bounded tail, so that there exists n_0 and $u \in Y$ such that for every $x \in [0, e] \cap E$, $\frac{1}{n_0+1}x = Tv_{n_0+1,x} \in [-u, u]$. This shows that $T([0, e] \cap E)$ is order bounded in Y. ∎

Remark 24.2. Suppose X and Y are Banach lattices with X σ-MB, E is a subspace of X, and $T : E \subseteq X \to Y$ is an isomorphic embedding. If T satisfies Proposition 24.1 then, by Proposition 22.8, (Tx_k) is absolute in Y whenever $(x_k) \subseteq E$ is absolute in X.

Example 24.3. The identity map $I : c_0 \subseteq \ell_\infty \to c_0$ is not order bounded. This illustrates that the relation between E and X in Proposition 24.1 is important, and gives an example of an operator that maps absolute sequences to absolute sequences but fails the conditions in Remark 24.2.

Question 24.4. Can σ-MB be removed in Remark 24.2? Do isomorphisms satisfying Proposition 24.5 preserve absolute sequences? Is there a "larger than the obvious" class of maps that preserve uo-bibases (i.e. isomorphisms that map simultaneously uo and norm null sequences to uo-null sequences.)?

We next present the sequential analogue of Proposition 24.1 as well as an example to illustrate that these concepts are distinct, even in the classical setting when $E = X$:

Proposition 24.5. *Let $T : E \subseteq X \to Y$ be a linear map where X and Y are vector lattices and E is a subspace of X. TFAE:*

(i) $x_n \xrightarrow{u} 0$ implies $Tx_n \xrightarrow{u} 0$ for all sequences (x_n) in E;
(ii) $x_n \xrightarrow{u} 0$ implies $Tx_n \xrightarrow{o} 0$ for all sequences (x_n) in E;
(iii) $x_n \xrightarrow{u} 0$ implies (Tx_n) is order bounded for all sequences (x_n) in E.

Moreover, if X and Y are Banach lattices then T is continuous. In these statements the uniform convergence of (x_n) is evaluated in X.

Proof. Clearly, (i)⇒(ii)⇒(iii); we show (iii)⇒(i). Suppose (x_n) is a sequence in E and $x_n \xrightarrow{u} 0$ in X. Since uniform convergence is stable (see [LZ71]) there exists a sequence $0 \leq \lambda_n \uparrow \infty$ in $\mathbb{R}$ such that $\lambda_n x_n \xrightarrow{u} 0$ in X. Since $(\lambda_n x_n) \subseteq E$, $T(\lambda_n x_n)$ is order bounded in Y, so there exists $0 < e \in Y$ with $|T(\lambda_n x_n)| \leq e$ for all n. It follows that $|Tx_n| \leq \frac{1}{\lambda_n} e$, so that $Tx_n \xrightarrow{u} 0$ in Y.

For the moreover clause, suppose (x_n) is a sequence in E such that $x_n \xrightarrow{\|\cdot\|} 0$. Let (x_{n_k}) be a subsequence of (x_n). There is a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \xrightarrow{u} 0$ in X. Hence, $Tx_{n_{k_i}} \xrightarrow{u} 0$ in Y, which implies norm convergence of $(Tx_{n_{k_i}})$ to zero. Therefore, we have shown that every subsequence of (Tx_n) has a further subsequence that norm converges to zero, which implies that $Tx_n \xrightarrow{\|\cdot\|} 0$. □

Remark 24.6. From Theorem 20.1, there is a larger class of mappings than the sequentially u-u continuous isomorphisms that preserve the bibasis property; namely, those isomorphisms that map uniformly convergent sequences to sequences who's sups of moduli are norm bounded. It is natural to ask if this is actually a different class than the sequentially u-to-u continuous isomorphisms. The answer is no, of course, when the space is σ-MB. Since X^{**} is always σ-MB, the σ-MB property arises as a technicality in some of the results: The bibasis and absolute property passes freely from X to X^{**}[13] but this is not always obvious for the other concepts associated with bibases (order boundedness of partial sums under permutation, classes of operators, etc.)

This motivates the following basic question:

Question 24.7. Suppose X is a closed sublattice of Y and (x_k) is a norm null sequence in X. Does $x_k \xrightarrow{a} 0$ in X iff in Y? Here $a = u, o, uo$. For u the motivation is for sequentially u-u maps, for o to justify why the bibasis theorem doesn't depend on ambient space, (specifically statement (ii)) and for uo to prove that the uo-bibasic property doesn't depend on ambient space.

Example 24.8. *An operator which maps uniformly null sequences to uniformly null sequences, yet fails to be order bounded.* Let c_0 be equipped with the standard basis $(e_k)_{k\geq 1}$ and c equipped with the basis $(e_k)_{k\geq 0}$ where $e_0 = (1, 1, 1, \dots,)$. The surjective isomorphism $T : c \to c_0$, $T(a_n) = (a^*, a_1 - a^*, a_2 - a^*, \dots)$ $(a^* = \lim a_n)$ shows equivalence between the standard bibases of c and c_0 but is not order bounded since $(e_k)_{k\geq 1}$ is order bounded in c but not in c_0 and $Te_k = e_{k+1}$. However, since c has a strong unit, order and norm convergence agree in c, and since c_0 is order continuous, uniform and order convergence agree in c_0. Although uniform and norm convergence do

[13] Or more generally between closed sublattices

not agree on nets in c_0, it is well known that uniform convergence and norm convergence agree on *sequences* in M spaces. Therefore, T satisfies Proposition 24.5 with $E = X = c$ and $Y = c_0$.

Combing our new knowledge of bibases with a careful inspection of the proof of [GKP15, Theorem 5.1], we can now state [GKP15, Theorem 5.1] in much more generality:

Theorem 24.9. *Let (M_k) be a sequence of subspaces of a Banach lattice X that form a bi-FDD of $[M_k]$. Then there is no isomorphic embedding $T : L_1 \to [M_k]$ with $T^{-1} : T(L_1) \subseteq X \to L_1$ satisfying the equivalent conditions in Proposition 24.5.*

Note that L_1 embeds isometrically into $C[0,1]$ (which has a bibasis). The condition in the theorem says that although the inverse map of this embedding maps norm convergent sequences in its domain to norm convergent sequences in L_1, it cannot map norm convergent sequences in its domain to order (which are the same as uniformly) convergent sequences in L_1. We can also not replace "bi-FDD" with "bidecomposition" in Theorem 24.9 as disjoint positive normalized sequences in L_1 are complemented.

In the landmark paper [LLOT] the authors show that $C(\Delta, L_1)$ contains lattice isometric copies of every separable Banach lattice. This has a few consequences:

Corollary 24.10. *Let (M_k) be a sequence of subspaces of a Banach lattice X that form a bi-FDD of $[M_k]$. Then there is no isomorphic embedding $T : C(\Delta, L_1) \to [M_k]$ with $T^{-1} : T(C(\Delta, L_1)) \subseteq X \to C(\Delta, L_1)$ satisfying the equivalent conditions in Proposition 24.5.*

Notice that every basic sequence in an M-space is bibasic. In particular, given any Banach space E with a basis (x_n), we can embed E isometrically into $C[0,1]$ and the image of (x_n) will be a bibasic sequence in $C[0,1]$. However, we want to know whether a basic sequence is bibasic with respect to a fixed ambient Banach lattice. Consider the next remark:

Remark 24.11. Let X be a Banach lattice and (x_n) a basic sequence in X. Then (x_n) is bibasic in X iff it is bibasic in the separable Banach lattice $\overline{S(\{x_n\})}$. However, (x_n) is bibasic in $\overline{S(\{x_n\})}$ iff it is bibasic in $C(\Delta, L_1)$ when we view $\overline{S(\{x_n\})}$ as a closed sublattice of $C(\Delta, L_1)$. In the sense of these identifications, the ambient Banach lattice in the definition of a bibasic sequence can always be chosen to be $C(\Delta, L_1)$. The same argument works for FDDs.

Question 24.12. Is there a universal separable Banach lattice with a *uo*-bibasis? If so, Proposition 25.7 can be proven by standard block and perturb arguments. My idea would be to use tensor products, so it would be good to first decide whether the tensor product of (uo)-bibases is a (uo)-bibasis.

The next result states that being permutable is quite a lot stronger than being both unconditional and bibasic. As usual, in the statement of the theorem the uniform convergence in $T(L_p)$ is evaluated in X.

Theorem 24.13. *Suppose* $1 \leq p < \infty$ *and* (M_k) *is a sequence of subspaces of a Banach lattice* X *that form a permutable bi-FDD of* $[M_k]$. *Then there is no isomorphic embedding* $T : L_p \to [M_k]$ *with the property that* $T^{-1} : T(L_p) \subseteq X \to L_p$ *maps uniformly null sequences in* $T(L_p)$ *to uo-null sequences in* L_p.

Proof. By [KS89, Corollary 9 and Remark 10, p. 102] no Haar type system in L_p $(1 \leq p < \infty)$ is a permutable *uo*-bibasic sequence in L_p. Now proceed as in [GKP15, Theorem 5.1]. □

Remark 24.14. This result can likely be extended to certain classes of r.i. spaces, but I am no expert in this area.

As a temporary definition, let X be a vector lattice and call a sequence (x_k, f_k) in $X \times X^\sim$ a ***bi-frame*** for X if for each $x \in X$, $\sum_{k=1}^n f_k(x)x_k \xrightarrow{u} x$. If X is a Banach lattice, then every bi-frame is a frame. Does L_1 have a bi-frame? One may also wonder about permutations of bi-frames and existence of permutable bi-frames in L_p.

25. Bibasic sequences in subspaces

We begin with a technical result regarding blocks of the Haar:

Lemma 25.1. *Let (x_k) be a block sequence of the Haar basis (h_k). If (x_k) is unconditional then it is bibasic in L_1.*

Proof. Let (x_k) be an unconditional block sequence of (h_k). Fix scalars $a_1, \dots, a_m$ and let $f_n = \sum_{k=1}^n a_k x_k$ as $n = 1, \dots, m$. Since (h_k) is a martingale difference sequence, so is (x_k), hence (f_n) is a martingale. Applying the Burkholder-Gundy-Davis inequality, we get

$$\left\| \bigvee_{n=1}^{m} \left| \sum_{k=1}^{n} a_k x_k \right| \right\| \sim \left\| \left(\sum_{k=1}^{m} |a_k x_k|^2 \right)^{\frac{1}{2}} \right\|.$$

By [LT79, Theorem 1.d.6], the latter quantity is equivalent to $\int_0^1 \| \sum_{k=1}^m r_k(t) a_k x_k \| dt$, which may be viewed as the average of $\|\sum_{k=1}^m \varepsilon_k a_k x_k\|$ over all choices of signs $\varepsilon_k = \pm 1$. Since (x_k) is unconditional, the latter is equivalent to $\|\sum_{k=1}^m a_k x_k\|$. Therefore, (x_k) is bibasic. □

Corollary 25.2. *Every closed infinite-dimensional subspace of an L_1-space contains an unconditional bibasic sequence.*

Proof. Let X be a closed infinite-dimensional subspace of an L_1-space; we may assume, WLOG, that X is separable. Note that $\overline{S(X)}$ is a separable L_1-space and the bibasis property passes freely between $\overline{S(X)}$ and the original L_1-space. We may assume, therefore, that our L_1-space is separable. Up to a lattice isometry, then, it is one of the following: ℓ_1, $L_1[0,1]$, $\ell_1 \oplus_1 L_1[0,1]$, or $\ell_1^m \oplus_1 L_1[0,1]$. All these spaces can be lattice isometrically embedded into $L_1[0,1]$ - and this preserves the bibasis property in both directions - so we may assume, WLOG, that our L_1-space is $L_1 := L_1[0,1]$.

Now that we have reduced the problem to L_1, there are two cases:

Case 1: X is non-reflexive. Since L_1 is a KB-space, X contains no isomorphic copy of c_0. By [LT79, Theorem 1.c.5], X contains an isomorphic copy of ℓ_1, and, therefore, X contains a basic sequence which is equivalent to the unit vector basis of ℓ_1. By Proposition 22.13, X contains an absolute basic sequence.

Case 2: X is reflexive. Fix a normalized unconditional basic sequence (x_k) in X (see [LT79, Theorem 1.c.9] for existence of such a sequence). Since X is reflexive, (x_k) is weakly null. Passing to a subsequence and using Bessaga-Pełczyński's selection principle, we find a block sequence (u_k) of the Haar basis (h_k) such that $\|x_k - u_k\| \to 0$. Passing to further subsequences, we may assume that $\|x_k - u_k\| \to 0$ sufficiently fast so that (u_k) is a small perturbation of (x_k). It follows that (u_k) is unconditional and, therefore, bibasic by Lemma 25.1. By Theorem 21.1 we have found our unconditional bibasic sequence in X. □

Question 25.3. Can one replace "unconditional bibasic sequence" with "permutable (uo-)bibasic sequence" in Corollary 25.2? Note that [KS89, Lemma 1 p.93] states that permutable and absolute are not that different for the Haar. It is really that when you block you can't get all possible permutations.

One cannot replace "unconditional bibasic sequence" with "absolute basic sequence", as the next example shows:

Example 25.4. *A subspace of L_1 containing no absolute basic sequence:*

We claim that the closed linear span of the Rademacher's does not contain an absolute basic sequence. Since this subspace is reflexive and the absolute property is preserved under blocks and small perturbations, it suffices to show that no block of the Rademacher's is absolute.

Suppose $u_n = \sum_{k=m_{n-1}+1}^{m_n} b_k r_k$ is a block sequence.

Then

$$\|\sum_{k=1}^{l} a_k u_k\| \sim \Big(a_1^2(b_1^2 + \cdots + b_{m_1}^2) + a_l^2(b_{m_{l-1}+1}^2 + \cdots + b_{m_l}^2)\Big)^{\frac{1}{2}}$$

whereas a Khintchine estimate gives,

$$\|\sum_{k=1}^{l} |a_k u_k|\| = \sum_{k=1}^{l} |a_k| \|u_k\| \geq C\left(|a_1|(b_1^2 + \cdots + b_{m_1}^2)^{\frac{1}{2}} + \cdots + |a_l|(b_{m_{l-1}+1}^2 + \cdots + b_{m_l}^2)^{\frac{1}{2}}\right).$$

Assuming (u_n) is absolute, this yields the existence of some constant C' such that for all $a_1, \ldots, a_l$,

$$|a_1|(b_1^2+\cdots+b_{m_1}^2)^{\frac{1}{2}}+\cdots+|a_l|(b_{m_{l-1}+1}^2+\cdots+b_{m_l}^2)^{\frac{1}{2}}\leq$$
$$C'\left(a_1^2(b_1^2+\cdots+b_{m_1}^2)+\cdots+a_l^2(b_{m_{l-1}+1}^2+\cdots+b_{m_l}^2)\right)^{\frac{1}{2}}.$$

We now choose $a_k=\frac{1}{k(b_{m_{k-1}+1}^2+\cdots+b_{m_k}^2)^{\frac{1}{2}}}$, which is well-defined by definition of a block basic sequence, to get

$$\sum_{k=1}^{l}\frac{1}{k}\leq C'\left(\sum_{k=1}^{l}\frac{1}{k^2}\right)^{\frac{1}{2}}.$$

This is a contradiction.

For our next result we will use the following representation theorem:

Theorem 25.5. *Let X be an order continuous Banach lattice with a weak unit e. Then, up to a lattice isomorphism, X is a dense ideal in $L_1(\mu)$ for some finite measure μ, such that e corresponds to $\mathbb{1}$, $L_\infty(\mu)$ is a dense ideal in X corresponding to $(I_e,\|\cdot\|_e)$, and both inclusions $L_\infty(\mu)\subseteq X\subseteq L_1(\mu)$ are continuous.*

The following will also be used; it is a result from Vladimir Troitsky's book (Corollary 9.2.3 together with Remark 9.2.4).

Theorem 25.6. *Let X be an order continuous Banach lattice with a weak unit, and let $L_1(\mu)$ be an L_1-representation for X. If (x_n) is a bounded sequence in X, then either (x_n) is seminormalized in $L_1(\mu)$ or (x_n) has an asymptotically disjoint subsequence.*

Proposition 25.7. *Let Y be a closed infinite dimensional subspace of an order continuous Banach lattice X. Then Y contains an unconditional uo-bibasic sequence.*

Proof. WLOG, we may assume that Y is separable. Note that $\overline{S(Y)}$ is separable and is a regular sublattice of X. Therefore, replacing X with $\overline{S(Y)}$, we may assume that X is separable. X, therefore, has a weak unit and we can continuously embed X as an ideal in $L_1(\mu)$ as in Theorem 25.5. By continuity of the embedding, there exists $C>0$ such that $\|x\|_{L_1}\leq C\|x\|_X$ for all $x\in X$. This implies that $L_1(\mu)$ is separable as well.

Consider two cases.

Case 1: $\|\cdot\|_X$ and $\|\cdot\|_{L_1}$ are not equivalent on Y. Then there exists (y_n) in Y such that $\|y_n\|_X = 1$ for every n while $\|y_n\|_{L_1} \to 0$. Using order continuity of X, Kadeč-Pełczyński states that, passing to a subsequence, we may assume that (y_n) is almost disjoint in X, i.e., there exists a disjoint sequence (d_n) in X such that $\|y_n - d_n\|_X \to 0$. Being disjoint, (d_k) is bibasic in X. Passing to a further subsequence and using a Principle of Small Perturbations, we conclude that (y_n) is bibasic and, therefore, uo-bibasic in X. (y_n) is also, clearly, unconditional.

Case 2: $\|\cdot\|_X$ and $\|\cdot\|_{L_1}$ are equivalent on Y. Hence, Y may be viewed as a closed subspace of $L_1(\mu)$. By Corollary 25.2 there is an unconditional basic sequence (y_n) in Y *that is bibasic with respect to* $L_1(\mu)$. It is left to show that (y_n) is a uo-bibasic in X. Let $y = \sum_{n=1}^{\infty} \alpha_n y_n$, where the series converges in norm (it does not matter in which norm because $\|\cdot\|_X$ and $\|\cdot\|_{L_1}$ are equivalent on Y). Since (y_n) is a uo-bibasic sequence in $L_1(\mu)$, we have $\sum_{n=1}^{m} \alpha_n y_n \xrightarrow{uo} y$ in $L_1(\mu)$. Since X is an ideal in $L_1(\mu)$ and is, therefore, regular, we conclude that $\sum_{n=1}^{m} \alpha_n y_n \xrightarrow{uo} y$ in X.

□

Question 25.8. Can one replace "*uo*-bibasic sequence" with either "bibasic sequence" or "permutable *uo*-bibasic sequence" in Proposition 25.7?

Remark 25.9. In the proof of Proposition 25.7, X is not a closed sublattice of $L_1(\mu)$, so Corollary 20.6 doesn't apply. The issue with the proof - and why we have to downgrade to a *uo*-bibasic sequence at the end - is that we cannot pull the order convergence from $L_1(\mu)$ to X. Another way to see the issue is to note that when evaluating the bibasis inequality the lattice operations could, potentially, take one out of the subspace, where the norms are no longer equivalent.

We finish this section with some evidence in favour of Question 23.2;

Proposition 25.10. *Let (x_k) be a sequence in an order continuous Banach lattice X such that the series $x := \sum_{k=1}^{\infty} x_k$ converges unconditionally. Then there exists a permutation σ of $\mathbb{N}$ such that $\sum_{k=1}^{m} x_{\sigma(k)} \xrightarrow{uo} x$.*

Proof. We may assume that X is separable and, therefore, has a weak unit. By Theorem 25.5, X can be continuously embedded as a dense ideal into $L_1(\mu)$ for some finite measure μ. By density and continuity of the embedding, $L_1(\mu)$ is separable. By classification of separable L_1-spaces, $L_1(\mu)$ is lattice isometric to one of the following: ℓ_1, $L_1[0,1]$, $\ell_1 \oplus_1 L_1[0,1]$, or $\ell_1^m \oplus_1 L_1[0,1]$.

Let us do the case of $\ell_1 \oplus_1 L_1$. Write $x = (f, g)$ and $x_k = (f_k, g_k)$ for some $f, f_1, \dots \in \ell_1$ and $g, g_1, \dots \in L_1$. Since $\sum_{k=1}^m x_{\sigma(k)} \xrightarrow{\|\cdot\|} x$ in X, the same is true in $\ell_1 \oplus_1 L_1$ so that $\sum_{k=1}^m f_{\sigma(k)} \xrightarrow{\|\cdot\|_1} f$ and $\sum_{k=1}^m g_{\sigma(k)} \xrightarrow{\|\cdot\|_1} g$ for each σ. By [B76, Theorem 7.1] there exists a permutation σ^* such that $\sum_{k=1}^m g_{\sigma^*(k)} \xrightarrow{uo} g$ in L_1; for the same σ^* we also have $\sum_{k=1}^m f_{\sigma^*(k)} \xrightarrow{uo} f$ in ℓ_1. It follows that $\sum_{k=1}^m x_{\sigma^*(k)} \xrightarrow{uo} x$ in $\ell_1 \oplus_1 L_1$, hence in X since the inclusion is an ideal. □

Remark 25.11. In Proposition 25.10 one can instead assume X is a Banach lattice admitting a strictly positive order continuous functional. In this case, X can be represented as a regular sublattice of $L_1(\mu)$ for some measure μ by [GTX17, Theorem 4.1]. We therefore get that $\sum_{k=1}^m x_{\sigma(k)} \xrightarrow{\|\cdot\|_1} x$ in $L_1(\mu)$, hence in $\overline{S(\{x_k\})}^{\|\cdot\|_1}$, which is a separable L_1-space. Notice that *uo*-convergence passes freely between X and $L_1(\mu)$ and between $L_1(\mu)$ and $\overline{S(\{x_k\})}^{\|\cdot\|_1}$.

Remark 25.12. According to Garcia's book "Topics in a.e. convergence" p.80 Marcinkiewicz showed that any orthonormal system in L_2 can be blocked to be a bi-FDD. This motivated the latter part of Question 23.2 regarding the ability to block as a uo-FDD. See also [KS89, Theorem 7 and Corollary 6 p. 273-277].

26. Strongly bibasic sequences

Generally, Theorem 20.1 motivates our definition of a bibasic sequence over that given in [GKP15]. However, our definition of bibasic degenerates in AM-spaces. Indeed, in AM-spaces basic and bibasic are equivalent and absolute and unconditional are equivalent. Even worse, the uo-bibasic property degenerates in atomic spaces[14]. This motivates the following definition:

Definition 26.1. *A uo-bibasic sequence (x_k) in a Banach lattice X is said to be **strongly** uo-**bibasic** if $\sum_{k=1}^m a_k x_k \xrightarrow{uo} 0$ implies $a_k = 0$ for all k. Similarly,a bibasic sequence (x_k) in X is said to be **strongly bibasic** if $\sum_{k=1}^m a_k x_k \xrightarrow{o} 0$ implies $a_k = 0$ for all k.*

Example 26.2. In c, the bibasis (x_k) defined as $x_k = \sum_{n=k}^{\infty} e_n$ has unique *uo*-expansions, while the bibasis $(e_k)_{k\geq 0}$ where $e_0 = \sum_{k=1}^{\infty} e_k$ has non-unique order expansions.

The strong *uo*-bibasic property is not very stable; the next example shows that it can be lost under perturbations, blocks, and permutations.

Example 26.3. Consider $X = c_0$ and let (e_k) be the standard unit vector basis. (e_k) is absolute and has unique *uo*-expansions.

Perturbations: Fix $1 > \varepsilon > 0$ and consider the sequence $x_1 = e_1$ and for $k \geq 2$, $x_k = \frac{\varepsilon}{2^{3k}} e_{k-1} + e_k$. This is a perturbation of the absolute sequence (e_k), and it is easy to see that there is a non-trivial *uo*-expansion of zero.

Blocks: Let $X = c_0$ and consider the basis (x_k) where $x_1 = e_1$, $x_2 = \frac{1}{2^{3\cdot 3}} e_1 + e_2$ and, for $k \geq 3$, $x_k = a_k e_1 + \frac{1}{2^{3\cdot k}} e_{k-1} + e_k$. Here $a_k = \frac{1}{2^{3\cdot[k(k+1)/2-3]}}$. This is a small perturbation of (e_k), hence an absolute basis of c_0. It is easily checked that (x_k) has unique *uo*-expansions. However, if one blocks in pairs by defining $b_1 = \frac{-1}{2^{3\cdot 3}} x_1 + x_2$, $b_2 = -2^{3\cdot 3} x_3 + 2^{3(4+3)} x_4$, $b_3 = -2^{3(5+4+3)} x_5 + 2^{3(6+5+4+3)} x_6, \dots$, one can kill the alternating sums in the first coordinate and lose uniqueness of *uo*-expansion. This shows that blocks of a strong *uo*-bibasis need not be strongly *uo*-bibasic.

Permutations: One can also choose a_k in such a way that (x_k) remains a small perturbation of (e_k) (hence an absolute basis of c_0) and

[14]Note that the norm topology is the finest locally solid topology on X, uniform convergence is topological iff X has a strong unit, and *uo* is topological iff X is atomic. In this sense it is good that the *u*, *o* and *uo* are not topological, else the whole concept of bibases wouldn't exist.

the series $\frac{1}{2^{3\cdot 3}} - 2^{3\cdot 3}a_3 + 2^{3(4+3)}a_4 - 2^{3(5+4+3)}a_5 + 2^{3(6+5+4+3)}a_6 + \cdots$ converges conditionally (e.g. pick a_k so that this goes like the alternating harmonic series). One can then order the series in two ways. The first being so that it converges, so we can let the coefficent of x_1 be negative the value of the limit. This gives a non-trivial *uo*-expansion of zero. The second is so that the sum diverges to infinity, which would imply unique *uo*-expansions. That shows that the strong *uo*-bibasis property is not preserved under permutation.

Problem 26.4. Answer analagous questions for bibases. Explicitly, can the strong bibasic property be lost under blocks? If (x_k) is permutable can the strong bibasis property be lost under permutation? Is there a strong bibasis admitting arbitrarily small perturbations that are not strong (with a view towards showing that σ-order continuity cannot be removed from [GKP15, Theorem 3.2]).

The strong properties also have major issues involving the ambient space:

Example 26.5. We will show that the Schauder basis of $C[0,1]$ fails to be strong, but gains uniqueness of *uo*-expansions after $C[0,1]$ is embedded as a closed sublattice of some larger AM-space.

Let $X = C[0,1]$ and (x_n) the standard Schauder basis of $C[0,1]$ as described in [LT77, Page 3]. It can be easily verified that there is a sequence of coefficients (a_k) such that the sequence (f_n) defined via $f_n(0) = 1$, f_n is linear on $[0, \frac{1}{2^n}]$, and f_n vanishes on $[\frac{1}{2^n}, 1]$, is the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k x_k$. It follows that the series converges in order to zero, so that zero has non-unique order expansions.

For a compact Hausdorff space K, we put $c_0(K)$ to be the space of real-valued functions f on K such that the set $\{|f| > \varepsilon\}$ is finite for every $\varepsilon > 0$. One defines $CD_0(K)$ as the space of functions of the form $f + g$ where $f \in C(K)$ and $g \in c_0(K)$. It is known that $CD_0(K)$ is a Banach lattice; it is, actually, lattice isometric to $C(\tilde{K})$ for some compact Hausdorff space $\tilde{K}$. Clearly, $C(K)$ is a norm closed sublattice of $CD_0(K)$. We refer the reader to [T04] and the references therein for basic properties of $CD_0(K)$-spaces.

Put $Y = CD_0[0,1]$ and suppose that $\sum_{k=1}^m a_k x_k \xrightarrow{uo} 0$ in Y. Define $s_m := \sum_{k=1}^m a_k x_k$ and note that $s_m(0) = a_1$ for all m. It follows from $0 \le |a_1 \mathbb{1}_{\{0\}}| \le |s_m| \xrightarrow{uo} 0$ in Y that $a_1 = 0$. Therefore, $s_m(1) = a_2$ and, consequently, $|a_2 \mathbb{1}_{\{1\}}| \le |s_m|$ holds for all $m \ge 2$. It follows from $s_m \xrightarrow{uo} 0$ in Y that $a_2 = 0$. From this we deduce that $s_m(\frac{1}{2}) = a_3$ and, therefore, $|a_3 \mathbb{1}_{\{\frac{1}{2}\}}| \le |s_m|$ for all $m \ge 3$. This yields $a_3 = 0$. Proceeding in this manner $a_k = 0$ for all k.

$CD_0(K)$-spaces are somewhat exotic, and the $c_0 \subseteq \ell_\infty$ inclusion seems like a more natural place to demonstrate that the strong property depends on ambient space (in c_0 order convergence implies norm convergence but in ℓ_∞ the opposite is true). The next proposition shows that, indeed, one must venture away from the classical spaces:

Proposition 26.6. *Every basic sequence in c_0 is strongly bibasic in ℓ_∞.*

Proof. Suppose not. Then there exists a basic sequence (x_k) in c_0 and a non-trivial sequence of scalars (a_k) such that $\sum_{k=1}^n a_k x_k \xrightarrow{o} 0$ in ℓ_∞. WLOG, $a_k = 1$ for all k; otherwise, pass to the subsequence for which $a_k \neq 0$ and replace x_k with $a_k x_k$.

Put $s_n = \sum_{k=1}^n x_k$. Then $s_n \xrightarrow{o} 0$ in ℓ_∞, so (s_n) converges coordinate-wise to zero and (s_n) is order bounded in ℓ_∞. Since (x_k) is basic, the zero vector has no non-trivial norm expansions, so that (s_n) does not converge to zero in norm. It follows that there exists $\delta > 0$ such that $\|s_n\| > \delta$ for infinitely many values of n.

Fix a sequence (ε_m) in $(0, \delta/2)$ with $\varepsilon_m \to 0$. Define $P_n : \ell_\infty \to \ell_\infty$ to be the projection onto the first n coordinates and let $Q_n := I - P_n$.

Choose n_1 so that $\|s_{n_1}\| > \delta$. Since $s_{n_1} \in c_0$, there exists k_1 such that $\|Q_{k_1} s_{n_1}\| \le \varepsilon_1$. Put $v_1 = P_{k_1} s_{n_1}$. Then v_1 is supported on $[1, k_1]$ and $\|s_{n_1} - v_1\| \le \varepsilon_1$.

Since (s_n) converges to zero coordinate-wise, we can find $n_2 > n_1$ such that $\|P_{k_1} s_{n_2}\| \le \varepsilon_2$ and $\|s_{n_2}\| > \delta$. Since $s_{n_2} \in c_0$, there exists $k_2 > k_1$ such that $\|Q_{k_2} s_{n_2}\| \le \varepsilon_2$. Put $v_2 = P_{k_2} Q_{k_1} s_{n_2}$. Then v_2 is supported on $[k_1 + 1, k_2]$ and $\|s_{n_2} - v_2\| \le \varepsilon_2$.

Proceeding inductively, we produce a subsequence (s_{n_m}) of (s_n) and a disjoint sequence (v_m) such that $\|s_{n_m}\| > \delta$ and $\|s_{n_m} - v_m\| \le \varepsilon_m$. It follows from $\varepsilon_m < \delta/2$ that $\|v_m\| > \delta/2$. Since (s_n) is norm bounded, (v_m) is seminormalized. Being a disjoint seminormalized sequence in c_0, (v_m) is basic and is equivalent to (e_m). Passing to a further subsequence, if necessary, we may assume that (s_{n_m}) is also basic and is equivalent to (v_m) and, therefore, to (e_m). Hence, there is an isomorphic embedding $T : c_0 \to c_0$ with $Te_m = s_{n_m}$.

Define $n_0 = 0$ and for $m \geq 1$ put $y_m = s_{n_m} - s_{n_{m-1}}$. Then $y_m = \sum_{k=n_{m-1}+1}^{n_m} x_k$ is a block sequence of (x_k), hence is basic. Notice that $T^{-1}y_1 = e_1$ and for $m \geq 2$, $T^{-1}y_m = e_m - e_{m-1}$, so that the sequence $(e_1, e_2 - e_1, e_3 - e_2, \dots)$ is basic. This is a contradiction. □

Question 26.7. Does every Banach lattice with a (uo-)bibasis admit a strong (uo-)bibasis? Do closed infinite-dimensional subspaces of Banach lattices contain strong (uo-)bibasic sequences? These question are of interest even for AM-spaces. It is even somewhat interesting for $C[0, 1]$.

Does every Banach lattice with a uo-bibasis admit a uo-bibasis that is not strong? It is not even completely clear if every strong uo-bibasis can be perturbed to be not strong.

Example 26.8. [KS89, Theorem 2' p. 356] constructs a strong *uo*-bibasis of L_2. From [KS89, Theorem 4 p. 361 and Corollary 3] the strong *uo*-bibasis they construct is not a bibasis. This motivates the question of whether every strong *uo*-bibasis of L_2 fail to be a bibasis and, less specifically, whether there is a Banach lattice with a bibasis and a strong *uo*-bibasis but no basis possessing both these properties. With the obvious practical appeal of knowing whether the series expansions of classical bases in L_p converge almost everywhere, it is possible that counterexamples to some our questions on *uo*-bibases have already been given in the literature. This example also shows that one can rearrange a strong uo-bibasis to be not even a uo-bibasis.

27. Additional open questions

(i) Give good examples to kill duality theory. For example, a (permutable or unconditional) bibasis in a reflexive space whose dual sequence is not a (uo)-bibasis. Are there any good sufficient conditions for a duality theory?

(ii) Let (e_k) be a basic sequence in a Banach space E. Then both (δ_{e_k}) and $(|\delta_{e_k}|)$ are basic sequences in $FBL[E]$. Moreover, if (e_k) is unconditional, then $(|\delta_{e_k}|)$ is absolute. Is it always the case that (δ_{e_k}) and $(|\delta_{e_k}|)$ are (uo)-bibasic in $FBL[E]$? Can $(|\delta_{e_k}|)$ be absolute without (e_k) being unconditional? I haven't thought about it at all, so this may be trivial. It seems like a good way to find a basic sequence with no (uo-)bibasic subsequences.

(iii) Does every space with a bibasis have a conditional (or at least a non-permutable) bibasis? One could also ask the same question for *uo*-bibases. I guess one would start by trying to get a bibasis version of [AK06, Lemma 9.5.3].

(iv) It is known that a block basic sequence can be extended back up to a basis ([Sin1, Theorem 7.2]). Analogue for bibases?

(v) Let X be a Banach lattice (with a basis). If every basic sequence (every basis) in X is bibasic does that imply that X is isomorphic to an AM-space? It is a classic result that a Banach lattice is isomorphic (as a topological vector lattice) to an AM-space iff every norm null sequence is order bounded. See also [BW80] for more evidence concerning why this may be true.

(vi) There is a related open question of whether L_p $(p > 1)$ has a basis of positive vectors. It is also, I believe, of interest to consider the dual question, which would be to construct a basis of L_1 (or $C[0,1]$?) with positive coordinate functionals. In some sense we are just looking for different ways to add lattice structure to bases. This time through properties of (x_k) or the biorthogonal functions (f_k). One could also wonder if one can combine positivity and the uo-bibasis property to get even nicer bases of certain spaces.

(vii) Is there a universal separable Banach lattice with a basis of positive vectors?

Bases that are both positive and disjoint appear throughout the literature on atomic Banach lattices. Such bases are always unconditional and, moreover, given a Banach space E with an unconditional basis, one can always find a renorming and an order such that E becomes an atomic Banach lattice, and the specified basis is positive and disjoint.

Definition 27.1. *Let X be a Banach lattice, and (x_k) a basic sequence in X with (f_k) the associated biorthogonal functionals. If $x_k \geq 0$ for all k we say that (x_k) is a* ***positive basic sequence****. If each f_k extends to an element of X_+^* we say that (x_k) has* ***positive coordinate functionals.*** *We make similar definitions for w^*-Schauder bases.*

Remark 27.2. Note (x_k, f_k) is a positive basis of X iff $(f_k, \widehat{x_k})$ is a w^*-Schauder basis of X^* with positive coordinate functionals. Conversely, (x_k, f_k) is a basis of X with positive coordinate functionals iff $(f_k, \widehat{x_k})$ is a positive w^*-Schauder basis of X^*. Hence, having a positive basis is dual to having positive coordinate functionals. The final simple proposition shows that these concepts are also complementary:

Proposition 27.3. *Let (x_k) be a basis of a Banach lattice X and denote by (f_k) the associated coefficient functionals. TFAE:*

(i) Each f_k is a lattice homomorphism;
(ii) $f_k \geq 0$ and $x_k \geq 0$ for all k;
(iii) (x_k) is a sequence of positive pairwise disjoint vectors.

Proof. (i)⇒(ii): Obviously, lattice homomorphisms are positive. Let $x \in X$ so that $x = \sum_{k=1}^{\infty} f_k(x)x_k$. By uniqueness of expansion, $f_k(x_l) = \delta_{kl}$ and therefore $f_k(x_l^+) = f_k(x_l)^+ = (\delta_{kl})^+ = \delta_{kl}$. This proves that $x_l^+ = \sum_{k=1}^{\infty} f_k(x_l^+)x_k = x_l$ and hence the (x_k) are all positive vectors.

(ii)⇒(iii): Suppose $m \neq n$. Then for each k,

$$0 \leq f_k(x_m \wedge x_n) \leq f_k(x_m) \wedge f_k(x_n) = \delta_{km} \wedge \delta_{kn} = 0.$$

Therefore, $x_n \wedge x_m = \sum_{k=1}^{\infty} f_k(x_n \wedge x_m)x_k = 0$.

(iii)⇒(i): Let (x_k) be a basis with the x_k positive and pairwise disjoint. We claim that the associated coefficient functionals are lattice homomorphisms. This follows since if $x = \sum_{k=1}^{\infty} f_k(x)x_k$ then

$$\sum_{k=1}^{\infty} f_k(|x|)x_k = |x| = \left|\sum_{k=1}^{\infty} f_k(x)x_k\right| = \lim_{n\to\infty}\left|\sum_{k=1}^{n} f_k(x)x_k\right|$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} |f_k(x)x_k| = \sum_{k=1}^{\infty} |f_k(x)|x_k.$$

By uniqueness of the expansion we conclude that $f_k(|x|) = |f_k(x)|$.
□

Note that Proposition 27.3 trivially fails for frames.

We do not currently have an example of a Banach lattice with a basis but no positive basis (or basis with positive functionals). For frames this is less of an issue, as it is easy to see that admitting a positive frame gives no additional information about the Banach lattice then the BAP does (although this fails for subspaces of Banach lattices).

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