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NONPARAMETRIC REGRESSION ANALYSIS - MULTIVARIATE CASE

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Nonparametric Regression Analysis - Multivariate Case", submitted by George E.J. Smith, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis the multivariate multiple regression model, $Y_{ij} = \underset{\sim}{x}'_{ij}\underset{\sim}{\beta} + e_{ij}$, $i = 1,\ldots,n$; $j = 1,\ldots,p$, is considered. The $\{Y_{ij}\}$ are observations, $\{\underset{\sim}{x}_{ij}\}$ are q-vectors $(q \geq 1)$ of known regression constants, $\{e_{ij}: j=1,\ldots,p\}$, $i = 1,\ldots,n$, are independent and identically distributed error vectors and $\underset{\sim}{\beta}$ is a q-vector of unknown parameters.

Nonparametric tests and estimates for $\underset{\sim}{\beta}$, based on signed rank statistics, are proposed using both the joint and separate ranking procedures. The methods used are extensions of the ideas in Koul (1967) where only the univariate case is considered and the estimates are based on Wilcoxon scores. In the present work the multivariate case is considered with more general scores (see conditions (6.1)).

The asymptotic distribution of the test statistics is obtained under both the null hypothesis and a sequence of contiguous alternatives. Also, the large sample existence and asymptotic normality of the proposed estimates are discussed. To do this, some needed convergence theorems in stochastic processes are proved in Chapter V. Next the asymptotic efficiency of these procedures relative to the classical ones is obtained. Finally, some examples of score functions satisfying the conditions of Chapter VI are given.

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TABLE OF CONTENTS

LIST OF TABLES

SUMMARY

In Chapter I, the regression model is described along with some history of the problem, the assumptions and the notations.

In Chapter II a class of tests and estimates for β , based on the statistics $M_n(Y)$ (given by (2.2) for the separate ranking case and (2.8) for the joint ranking case), are defined for the multivariate case and the general rank scores. The proposed estimate $\hat{\beta}_n$ for β is defined as the centre of gravity of a confidence region determined by $M_n(Y)$. Koul (1967) defines this estimate for the special case where the underlying distribution is univariate and the scores are Wilcoxon. For this class of estimates, translation invariance is proved and, when the error vectors are diagonally symmetric (see definition 2.1), unbiasedness is shown. Both the joint and the separate ranking procedures are defined and discussed (see section 1.2).

In Chapter III, the asymptotic distribution of the test statistic $M_n(Y)$ is derived for both the joint and the separate ranking procedures. For the separate ranking case (see lemmas 3.3, 3.4, 3.5, and 3.6) the results are extensions of the work of Hàjek (1962) and Mehra (1969). For the joint ranking case (see lemmas 3.7, 3.8, and 3.9) some additional conditions are needed on either the underlying distribution or the regression scores. Three such sets of

conditions are discussed in (3.1), (3.4), and (3.6), and the asymptotic distribution is found.

The principal result of Chapter IV is contained in theorem 4.4. There, the asymptotic distributions of $M_n(\underset{\sim}{Y})$ is found under a sequence of contiguous alternatives. Similar results were proved by Hàjek (1962) for the univariate case, and by Mehra (1969) for the multivariate case with unsigned rank statistics. In the present work the multivariate case is considered with signed rank statistics. Finally, conditions [see (4.15)] are given which ensure the asymptotic normality of the least squares estimates. These results are used to obtain the asymptotic efficiency of the tests of Chapter II relative to the classical tests.

Chapter V gives a number of convergence theorems for stochastic processes. Theorem 5.1 is the main theorem of this chapter and enables us to deal with the multivariate case. Lemmas 5.1 to 5.7 are the required but immediate extensions of the results of Koul (1967), which, coupled with subsequent results - lemma 5.8 onward - enable us to demonstrate, in Chapter VI and VII, the large sample existence and asymptotic normality of the estimates.

In Chapter VI, the joint ranking case is considered. Conditions on the score functions are given under which the large sample existence of $\hat{\underset{\sim}{\beta}}_n$ is proved [see theorem 6.6]. Theorem 6.7 gives the asymptotic distribution of $\hat{\underset{\sim}{\beta}}_n$.

Chapter VII proves results similar to those of Chapter VI but for the separate ranking procedure. The estimate based on the sign statistic is also considered. Finally it is shown that estimates based on the joint and the separate ranking procedures are asymptotically equivalent whenever the assumed conditions for both procedures are satisfied.

In Chapter VIII, the efficiency of the proposed estimates with respect to the least squares estimates is discussed. Also, some examples of score functions satisfying the assumed conditions are given.

The Appendix consists of a few lemmas which are used in other chapters. Lemmas A4 and A7 contain results which make it possible to discuss the case of general scores by providing bounds for certain remainder terms occurring in the proof of theorem 6.1.

CHAPTER I

INTRODUCTION

1.1 *Historical Note*

A statistical model with wide application is the regression model which can be written as

$$(1.1) \qquad Y_{nij} = \sum_{k=1}^{q} \beta_{jk} x_{nijk} + e_{nij} \qquad \begin{array}{l} j = 1,\ldots,K_i \\ i = 1,\ldots,n \end{array}$$

where $\{Y_{nij}\}$ are the observations, $\{x_{nijk}\}$ are known regression constants, $\{\beta_{jk}\}$ are unknown regression parameters, and $\{e_{nij}:j=1,\ldots,K_i\}$ are independent random vectors denoting the error terms.

The problem of testing and estimation of the regression parameter has been extensively dealt with in statistical literature. The least squares (L.S.) estimates have been shown to be optimum in the sense of minimum variance in the class of linear unbiased estimates. However, under severe departures of the underlying distribution from normality, these estimates and test procedures have been shown to be very inefficient. Thus if little is known concerning the underlying distribution, tests and estimates based on L.S. are of dubious value.

In view of this it is of benefit to attack such problems from the nonparametric point of view, giving procedures that are "robust" against changes in the underlying distribution. One of the first attempts at comparing L.S. and nonparametric methods was the consideration of the one and two sample problems of shift by Hodges and Lehmann (1956). There it was shown that the sign and Wilcoxon tests were more robust than the classical t and normal tests against changes in the underlying distribution. They also showed that these procedures were more robust against gross errors.

Hàjek (1962) considered the univariate regression model with two regression parameters, i.e. (1.1) with $K_i = 1$, $q = 2$, $\beta_{1k} = \beta_k$, $x_{ni11} = 1$, and $\{e_{ni1}\}$ independent and identically distributed (i.i.d.). He discussed the problem of testing β_2 against a sequence of contiguous (which he defines) alternatives, and found asymptotically most powerful tests. The same model was discussed by Adichie (1967a) and (1967b) where tests for (β_1,β_2) versus a sequence of contiguous alternatives were obtained along with estimates for (β_1,β_2) using the Hodges and Lehmann (1963) approach.

Mehra (1969) considered (1.1) under the assumptions $q = 2$, $x_{nij1} = 1$, $\beta_{jk} = \beta_k$, and some restictions on the joint distribution of $\{e_{ij}:j=1,\ldots,K_i\}$ and $\{x_{nij2}\}$. He proved the asymptotic normality of certain rank statistics under the hypothesis $\beta_2 = 0$ and under a sequence of contiguous alternatives. This extends some results of Hàjek (1962) to the case where certain types of dependence exist.

Koul (1967) considered (1.1) with $K_i = 1$ (i.e. the univariate case). Asymptotic normality of certain test statistics and estimates based on Wilcoxon scores was proved. The estimates of the β's were formed by taking the centre of gravity of an appropriate confidence region. The present work is an extension of Koul's approach to the multivariate case and more general scores.

(Recently a paper by Jurecková (1969) has appeared in the *Annals of Mathematical Statistics* which considers (1.1) with $K_i = 1$, and $\{e_{i1}: i=1,\ldots,p\}$ i.i.d. It is shown that certain rank statistics based on fairly general scores can be uniformly approximated in probability by a linear function of the β's for alternatives that are "contiguous" to the hypothesis. The methods used there are, however, different from the ones used in this thesis.

Also, Puri and Sen (1969) have considered, for the multivariate multiple regression model, just the testing problem using statistics based on unsigned ranks and the separate ranking procedure. Their methods are also different from ours.)

1.2 *The Problem: Testing and Estimation*

The following regression model will be considered

$$Y_{nij} = \underset{\sim}{x}'_{nij}\underset{\sim}{\beta} + e_{nij} \qquad \begin{array}{l} i = 1,\ldots,n \\ j = 1,\ldots,p \end{array} \tag{1.2}$$

where $\{Y_{nij}: i=1,\ldots,n;\ j=1,\ldots,p\}$ are the observations, $\{\underset{\sim}{x}'_{nij}\} = \{(x_{nij1},\ldots,x_{nijq})\}$ are known q-vectors of regression constants, $\{e_{nij}: j=1,\ldots,p\}$ are error vectors which are i.i.d., and $\underset{\sim}{\beta}' = (\beta_1,\ldots,\beta_q)$ are unknown parameters. If the subscript "n" is suppressed, (1.2) may be written in the following equivalent forms:

$$(1.3)\quad \begin{cases} Y_{ij} = \underset{\sim}{x}'_{ij}\underset{\sim}{\beta} + e_{ij} & i = 1,\ldots,n;\quad j = 1,\ldots,p \\ \underset{\sim}{Y}_i = \underset{\sim}{X}'_{ni}\underset{\sim}{\beta} + \underset{\sim}{e}_i & i = 1,\ldots,n \\ \underset{\sim}{Y} = \underset{\sim}{X}'_n\underset{\sim}{\beta} + \underset{\sim}{e} & \end{cases}$$

where

$$\begin{array}{lll} \underset{\sim}{Y}'_i = (Y_{i1},\ldots,Y_{ip}) & (\underset{\sim}{Y}_i & \text{is } p \times 1) \\ \underset{\sim}{X}'_{ni} = (\underset{\sim}{x}_{i1},\ldots,\underset{\sim}{x}_{ip}) & (\underset{\sim}{X}_{ni} & \text{is } q \times p) \\ \underset{\sim}{X}_n = (\underset{\sim}{X}_{n1},\ldots,\underset{\sim}{X}_{nn}) & (\underset{\sim}{X}_n & \text{is } q \times (np)) \\ \underset{\sim}{Y}' = (\underset{\sim}{Y}'_1,\ldots,\underset{\sim}{Y}'_n) & (\underset{\sim}{Y} & \text{is } (np) \times 1) \\ \underset{\sim}{e}'_i = (e_{i1},\ldots,e_{ip}) & (\underset{\sim}{e}_i & \text{is } p \times 1) \\ \underset{\sim}{e} = (\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_n) & (\underset{\sim}{e} & \text{is } (np) \times 1) \end{array}$$

The approach used here for testing and estimation of $\underset{\sim}{\beta}$ involves ranking the observations $\{Y_{ij}\}$. For the multivariate case there are two ways of doing this. One consists of ranking the j^{th} components of the vector observations separately for each j. This is called the *separate ranking procedure*. The second is to rank all the np Y_{ij}'s jointly. This is known as the *joint ranking procedure*. Both methods seem justifiable if the marginal distributions of e_{ij} and $e_{ij'}$ are the same for all $j \neq j'$ – in the latter case at least under some additional assumptions on the joint distribution (see section 3.4). Joint ranking makes little sense, however, if the marginal distributions are not the same.

In the sequel, the testing and the estimation problem using both the separate and joint ranking procedures will be considered. In the testing problem, a set of statistics will be introduced, and from these a statistic, M_n, will be obtained, appropriate for testing $H_o : \underset{\sim}{\beta} = \underset{\sim}{\beta}_o$. The distribution of M_n under H_o and a sequence of contiguous alternatives will be obtained. For estimation, the centre of gravity of a confidence region involving M_n is considered. Its asymptotic normality will be proved and asymptotic efficiency discussed. The estimation procedure uses the ideas of Koul (1967) and generalizes them to the multivariate case with more general scores than Wilcoxon.

Let us now introduce the following notation.

$$(1.5) \qquad \begin{cases} F(\underset{\sim}{w}) = P(\underset{\sim}{e}_i \leq \underset{\sim}{w}) & \text{where } \underset{\sim}{w}' = (w_1, \ldots, w_p) \\ F_j(w) = P(e_{ij} \leq w) & \\ F_j^*(w) = [F_j(w) - F_j(-w)] I(w \geq 0) & \end{cases}$$

The vectors $\underset{\sim}{e}_i$ are assumed to be independent and identically distributed for different i. The following assumptions are made throughout this work concerning F and $\{x_{ijk}\}$ —

$$(1.6)\begin{cases} \text{(i)} & F(\underset{\sim}{w}) \text{ is continuous on } E_p. \\ \text{(ii)} & \lim_{n\to\infty} \dfrac{\max_{1\le i\le n} x_{ijk}^2}{\sum_{i=1}^{n} x_{ijk}^2} = 0 \text{ for all } j,\ k. \\ \text{(iii)} & \underset{\sim}{X}_n \underset{\sim}{X}_n' \text{ is a } q \times q \text{ positive definite matrix and} \\ & \lim_{n\to\infty} n^{-1} \underset{\sim}{X}_n \underset{\sim}{X}_n' \text{ exists and is positive definite.} \\ \text{(iv)} & F_j(w) \text{ is symmetric about zero, i.e. } F_j(w) = 1 - F_j(-w) \end{cases}$$

Note: Assumption (1.6)-(iv) can be dispensed with while proving normality and other asymptotic results.

We now define the rank scores. Let $a_{N,1},\ldots,a_{N,N}$ be sequences of real numbers such that $a_{N,i} \le a_{N,i+1}$ for $i = 1,\ldots,N-1$, and $a_{N,1} < a_{N,N}$. For each N, define

$$(1.7)\begin{cases} \text{(i)} & \psi_N(u) = \psi_N(\frac{k}{N+1}) = a_{N,k} \quad \text{for } \frac{k-1}{N} < u \le \frac{k}{N}, \quad k = 1,\ldots,n \\ \text{(ii)} & \psi_N(u) = -\psi_N(-u) \quad \text{for } -1 \le u \le 0 \end{cases}$$

Condition (ii) is not necessary. The extension of the domain of ψ_N to $(-1,1)$ is a convenient device used to simplify the representation of certain integrals (eg. (1.10)). Further, suppose there exists a function ψ on $[0,1)$, and extended to $(-1,1)$ by $\psi(-u) = -\psi(u)$, which satisfies

$$(1.8)\quad \begin{cases} \text{(i)} & \psi(u) \text{ is monotone nondecreasing on } [0,1) \\ \text{(ii)} & \psi(0) = 0 \\ \text{(iii)} & \int_0^1 \psi^2(u)du < \infty \\ \text{(iv)} & \lim_{n\to\infty} \int_0^1 [\psi_n(u)-\psi(u)]^2 du = 0 \\ \text{(v)} & \lim_{n\to\infty} \dfrac{\max_{1\le k\le n} [\psi_n(k/(n+1))-\overline{\psi}_n]^2}{\sum_{k=1}^{n} [\psi_n(k/(n+1))-\overline{\psi}_n]^2} = 0 \\ & \quad \text{where } \overline{\psi}_n = n^{-1} \sum_{k=1}^{n} \psi_n(k/(n+1)) \end{cases}$$

The following assumption is also made concerning both the $\{x_{ijk}\}$ and ψ .

$$(1.9)\qquad \underset{\sim}{\Sigma}_n \text{ and } \underset{\sim}{\Sigma} = \lim_{n\to\infty} \underset{\sim}{\Sigma}_n \text{ exist and are positive definite,}$$

where

$$(1.10)\quad \begin{cases} \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1q} \\ \vdots & & \vdots \\ \sigma_{q1} & \cdots & \sigma_{qq} \end{bmatrix} \\ \sigma_{kk'} = \sum_{j=1}^{p} \sum_{j'=1}^{p} \lambda_{jj'} n^{-1} \sum_{i=1}^{n} x_{ijk}x_{ij'k'} \qquad k,k' = 1,\ldots,q \\ \lambda_{jj} = \int_0^1 \psi^2(u)du \quad \text{for all } j \\ \lambda_{jj'} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi[2F_j(u)-1]\psi[2F_j(v)-1]dH_{jj'}(u,v) \\ H_{jj'}(u,v) = P(e_{ij}\le u, e_{ij'}\le v) \quad \text{for all } j \ne j'. \end{cases}$$

CHAPTER II

PROPOSED TESTS AND ESTIMATES

In this section the general testing and estimation problem will be outlined and a condition for unbiasedness of the proposed estimate given.

2.1 *Separate Ranking Procedure*

Let

$$R_{ij} = \text{rank of } |Y_{ij}| \text{ in the ranking of } |Y_{\alpha j}| \quad \alpha = 1,\ldots,n. \tag{2.1}$$

Consider the following signed rank statistics

$$\left\{ \begin{aligned} T_k &= T_k(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi_n\left(\frac{R_{ij}}{n+1}\right) \text{ sign } Y_{ij} \\ T_k^* &= T_k^*(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi_n(F_j^*(|Y_{ij}|)) \text{ sign } Y_{ij} \\ S_k &= S_k(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi(F_j^*(|Y_{ij}|)) \text{ sign } Y_{ij} \\ M_n &= M_n(\underset{\sim}{Y}) = \underset{\sim}{T}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{T} \end{aligned} \right. \tag{2.2}$$

where $\underset{\sim}{T}' = \underset{\sim}{T}'(\underset{\sim}{Y}) = (T_1,\ldots,T_q)$, $\underset{\sim}{\Sigma}_n^{-1}$ is given in (1.10), and sign $Y_{ij} = 2I(Y_{ij}\geq 0) - 1$ where I is the indicator function. Under certain further assumptions, it will be shown that M_n provides a test for $H_o : \underset{\sim}{\beta} = \underset{\sim}{\beta}_o$ where $\underset{\sim}{\beta}_o$ is some fixed q-vector.

Now define

$$(2.3) \qquad R_n(\underset{\sim}{Y}) = \{\underset{\sim}{\beta} : M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\beta}) \leq k_{n\alpha}\} \subset E_q$$

where $P[M_n(\underset{\sim}{Y})\geq k_{n\alpha}] = \alpha$ under $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, and E_q is q-dimensional Euclidean space. Let us define the "estimate"

$$(2.4) \qquad \hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) = \frac{1}{\lambda[R_n(\underset{\sim}{Y})]} \int_{R_n(\underset{\sim}{Y})} \underset{\sim}{t}\, d\lambda(\underset{\sim}{t})$$

where λ is the Lebesgue measure on E_q. $\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y})$ is the centre of gravity of the confidence region $R_n(\underset{\sim}{Y})$.

Lemma 2.1:

If $\hat{\underset{\sim}{\beta}}_n$ *exists,* $\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}+\underset{\sim}{X}_n'\underset{\sim}{b}) = \hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) + \underset{\sim}{b}$ *where* $\underset{\sim}{b}$ *is any* $q \times 1$ *vector of constants.*

Proof:

Follows as in lemma 2.1 of Koul (1967).

Definition 2.1:

The random vector $\underset{\sim}{e}' = (e_1,\ldots,e_p)$ *is* *diagonally symmetric* *if* $\underset{\sim}{e}$ *and* $-\underset{\sim}{e}$ *have the same distribution.*

It may be noted that diagonal symmetry implies that for each j, e_j and $-e_j$ have the same distribution.

The next result is proved under the condition of diagonal symmetry of $\underset{\sim}{e}_i$, i.e.

$$F(\underset{\sim}{w}) \text{ satisfies } P(\underset{\sim}{e}_i \leq \underset{\sim}{w}) = P(\underset{\sim}{e}_i \geq -\underset{\sim}{w}) \text{ for all } \underset{\sim}{w} \in E_p \tag{2.5}$$

and shows that this is sufficient to ensure unbiasedness of $\hat{\underset{\sim}{\beta}}_n$.

Lemma 2.2:

If $\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y})$ *exists and if* $F(\underset{\sim}{w})$ *satisfies* (2.5), *then* $\underset{\sim}{\beta}_n - \underset{\sim}{\beta}_o$ *is diagonally symmetrically distributed about* $\underset{\sim}{0}$.

Proof:

Let $P_{\underset{\sim}{\beta}_o}$ denote that the probability is calculated when $\underset{\sim}{\beta} = \underset{\sim}{\beta}_o$. We must show

$$P_{\underset{\sim}{\beta}_o}(\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) - \underset{\sim}{\beta}_o \leq -\underset{\sim}{b}) = P_{\underset{\sim}{\beta}_o}(\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) - \underset{\sim}{\beta}_o \geq -\underset{\sim}{b}) \qquad \forall \underset{\sim}{b} \in E_q$$

In view of lemma 2.1, it suffices to prove this for $\underset{\sim}{\beta}_o = \underset{\sim}{0}$ only, i.e. show $P_{\underset{\sim}{0}}(\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \geq \underset{\sim}{b}) = P_{\underset{\sim}{0}}(\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \leq -\underset{\sim}{b})$. From (2.2), $T_k(\underset{\sim}{Y}) = -T_k(-\underset{\sim}{Y})$ since the R_{ij} remain unchanged and sign $Y_{ij} = -$ sign $(-Y_{ij})$. It now follows that $M_n(\underset{\sim}{Y}) = M_n(-\underset{\sim}{Y})$. Therefore

$$\begin{aligned} R_n(-\underset{\sim}{Y}) &= \{\underset{\sim}{\beta}: M_n(-\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\beta}) \leq k_{n\alpha}\} \\ &= \{\underset{\sim}{\beta}: M_n(\underset{\sim}{Y}+\underset{\sim}{X}_n'\underset{\sim}{\beta}) \leq k_{n\alpha}\} \\ &= -\{\underset{\sim}{\beta}: M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\beta}) \leq k_{n\alpha}\} \\ &= -R_n(\underset{\sim}{Y}) . \end{aligned}$$

Thus,
$$\begin{aligned} \hat{\underset{\sim}{\beta}}_n(-\underset{\sim}{Y}) &= \frac{1}{\lambda[R_n(\underset{\sim}{Y})]} \int_{R_n(-\underset{\sim}{Y})} \underset{\sim}{t}\, d\lambda(\underset{\sim}{t}) \\ &= \frac{1}{\lambda[R_n(\underset{\sim}{Y})]} \int_{R_n(\underset{\sim}{Y})} (-\underset{\sim}{s})\, d\lambda(\underset{\sim}{s}) \quad \text{where } \underset{\sim}{s} = -\underset{\sim}{t} \\ &= -\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \end{aligned}$$

Thus, $P_{\underset{\sim}{0}}[\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \leq -\underset{\sim}{b}] = P_{\underset{\sim}{0}}[\hat{\underset{\sim}{\beta}}_n(-\underset{\sim}{Y}) \leq -\underset{\sim}{b}]$ since $\underset{\sim}{Y}$ and $-\underset{\sim}{Y}$ have the same distribution, and so

$$= P_{\underset{\sim}{0}}[-\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \leq -\underset{\sim}{b}] = P_{\underset{\sim}{0}}[\hat{\underset{\sim}{\beta}}_n(\underset{\sim}{Y}) \geq \underset{\sim}{b}].$$

2.2 *Joint Ranking Procedure*

In this case we let

$$(2.6) \qquad R_{ij} = \text{rank of } |Y_{ij}| \text{ in the joint ranking of } |Y_{\alpha\beta}|$$

$$\alpha = 1,\ldots,n; \quad \beta = 1,\ldots,p$$

To avoid the possibility of ties and to ensure that the observations are comparable, it is assumed that

$$(2.7) \qquad \begin{cases} F_j(w) = F_1(w) & \text{for all } w \in (-\infty,\infty) \\ P(e_{ij} = e_{ij'}) = 0 & \text{if } j \neq j' \end{cases}$$

Consider the following signed rank statistics

$$(2.8) \qquad \begin{cases} T_k = T_k(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi_{np}\left(\frac{R_{ij}}{np+1}\right) \text{ sign } Y_{ij} \\ T_k^* = T_k^*(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi_{np}(F_1^*(|Y_{ij}|)) \text{ sign } Y_{ij} \\ S_k = S_k(\underset{\sim}{Y}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi(F_1^*(|Y_{ij}|)) \text{ sign } Y_{ij} \\ M_n = M_n(\underset{\sim}{Y}) = \underset{\sim}{T}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{T} \end{cases}$$

where $\underset{\sim}{T}' = \underset{\sim}{T}'(\underset{\sim}{Y}) = (T_1,\ldots,T_q)$ and $\underset{\sim}{\Sigma}_n$ is given in (1.10). Now let us define the "estimate" based on joint ranks just as in (2.3) and (2.4) but with M_n defined by (2.8). Since the proofs of lemmas 2.1 and 2.2 do not depend on the ranking procedure, the results apply here also and we have unbiasedness of the estimate if $F(\underset{\sim}{w})$ is diagonally symmetric.

Remark:

Without loss of generality, when testing for $H_o : \underset{\sim}{\beta} = \underset{\sim}{\beta}_o$, it may be assumed that $\underset{\sim}{\beta}_o = 0$. If it is not, consider the model $\underset{\sim}{Y}^* = \underset{\sim}{Y} - \underset{\sim}{X}_n'\underset{\sim}{\beta}_o = \underset{\sim}{X}_n'(\underset{\sim}{\beta}-\underset{\sim}{\beta}_o) + \underset{\sim}{e}$. $\underset{\sim}{\beta} = \underset{\sim}{\beta}_o$ is equivalent to $\underset{\sim}{\beta} - \underset{\sim}{\beta}_o = \underset{\sim}{\beta}^* = 0$.

CHAPTER III

LIMIT THEOREMS UNDER THE HYPOTHESIS $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$.

The object of this chapter is to prove the asymptotic normality of $\underset{\sim}{T}' = (T_1,\ldots,T_q)$ for both the separate and joint ranking procedures. To do this, additional conditions will be needed in some cases.

3.1 *A Definition and a Lemma*

Definition 3.1:

A random vector $\underset{\sim}{e} = (e_1,\ldots,e_p)$ *is* *quadrant symmetric* *if, for every* p-*vector* $\underset{\sim}{\alpha} = (\alpha_1,\ldots,\alpha_p)$ *where* $\alpha_j = +1$ or -1, $j = 1,\ldots,p$, $\underset{\sim}{e}$ *and* $(\alpha_1 e_1,\ldots,\alpha_p e_p)$ *have the same distribution.*

In terms of probability, if A is any measurable set in E_p, then for any vector $(e_1,\ldots,e_p) \in E_p$,

$$P(A) = P\{(e_1,\ldots,e_p):(\alpha_1 e_1,\ldots,\alpha_p e_p)\in A\} \tag{3.1}$$

If a density $f(e_1,\ldots,e_p)$ of F exists, the condition can be more simply stated as $f(e_1,\ldots,e_p) = f(|e_1|,\ldots,|e_p|)$. In the literature, quadrant symmetric has also been referred to as *sign exchangeable*.

Lemma 3.1:

If a distribution on E_p *is*

(a) *quadrant symmetric, then the vector of ranks of* $\{|e_{ij}|:i=1,\ldots,n;j=1,\ldots,p\}$ *is independent of the vector* $\{\text{sign } e_{ij}:i=1,\ldots,n;j=1,\ldots,p\}$,

(b) *diagonally symmetric, then the vector of ranks of* $\{|e_{ij}|:i=1,\ldots,n;j=1,\ldots,p\}$ *is independent of* $\text{sign } e_{\alpha\beta}$ *for any fixed* α, β.

Proof:

(a) Firstly, the components of $\{\text{sign } e_{ij}:i=1,\ldots,n;j=1,\ldots,p\}$ are mutually independent. This follows since

$P(\text{sign } e_{ij}=\alpha_{ij}:i=1,\ldots,n) = \prod_{i=1}^{n} P(\text{sign } e_{ij}=\alpha_{ij}:j=1,\ldots,p)$ by independence. Then, because of quadrant symmetry, the probability mass in each of the 2^p quadrants is the same- $P(\text{sign } e_{ij}=\alpha_{ij}:j=1,\ldots,p) = 2^{-p}$, and the above expression is 2^{-np}. Similarly, $\prod_{i=1}^{n}\prod_{j=1}^{p} P(\text{sign } e_{ij}=\alpha_{ij}) = 2^{-np}$, and the above claimed independence is immediate.

Now consider $p^* = P(|e_{ij}|\leq x_{ij}:i=1,\ldots,n;j=1,\ldots,p|\ \text{sign } e_{ij} = \alpha_{ij}: i=1,\ldots,n;j=1,\ldots,p)$. Let us take the case $\alpha_{ij} = 1$ for all i, j. Then

$$p^* = \frac{P(0\le e_{ij}\le x_{ij}:i=1,\ldots,n;j=1,\ldots,p)}{P(\text{sign } e_{ij}=1:i=1,\ldots,n;j=1,\ldots,p)}$$

$$= 2^{np}P(0\le e_{ij}\le x_{ij}:i=1,\ldots,n;j=1,\ldots,p)$$

From the definition of quadrant symmetry, because there are 2^p quadrants and the e_{ij} are independent in i, $p^* = P(|e_{ij}|\le x_{ij}:i=1,\ldots,n;j=1,\ldots,p)$. The same is true for any choice of $\{\alpha_{ij}\}$. Hence $\{|e_{ij}|\}$ is independent of $\{\text{sign } e_{ij}\}$. Hence any measurable function of $\{|e_{ij}|\}$ is independent of $\{\text{sign } e_{ij}\}$. The ranks are such a function, hence the result follows.

(b) If we show $\{|e_{ij}|:i=1,\ldots n;j=1,\ldots,p\}$ is independent of sign $e_{\alpha\beta}$, the result follows by the concluding remarks of (a). To show the above, consider $p_1 = P\{|e_{ij}|\le x_{ij}, \text{sign } e_{\alpha\beta}=1:i=1,\ldots,n; j=1,\ldots,p) = P\{|e_{ij}|\le x_{ij}, 0\le e_{\alpha\beta}\le x_{\alpha\beta}:(i,j)\ne(\alpha,\beta)\} = P\{|e_{ij}|\le x_{ij}: i\ne\alpha, j=1,\ldots,p\} \cdot P\{0\le e_{\alpha\beta}\le x_{\alpha\beta}, |e_{\alpha j}|\le x_{\alpha j}:j\ne\beta\}$. Because of diagonal symmetry $\underset{\sim}{e}_i$ and $-\underset{\sim}{e}_i$ have the same distribution. Hence $P\{0\le e_{\alpha\beta}\le x_{\alpha\beta}, |e_{\alpha j}|\le x_{\alpha j}:j=1,\ldots,p\} = \frac{1}{2} P\{|e_{\alpha j}|\le x_{\alpha j}:j=1,\ldots,p\}$ $= P\{|e_{\alpha j}|\le x_{\alpha j}:j=1,\ldots,p\} \cdot P\{\text{sign } e_{\alpha\beta}=1\}$. The same results hold if sign $e_{\alpha\beta} = -1$. Thus the above two equalities imply the result.

3.2 *Asymptotic Distribution of* $\underset{\sim}{S}' = (S_1,\ldots,S_q)$ *under* $H_0 : \underset{\sim}{\beta} = \underset{\sim}{0}$.

$\{S_k\}$, as defined in (2.2) (separate ranking case) is more general than that defined in (2.8). Thus any result proved for the $\{S_k\}$ of (2.2) will hold for that of (2.8).

Lemma 3.2:

Under conditions (1.6), (1.8), (1.9), $\underset{\sim}{S}' = (S_1,\ldots,S_q)$ *converges in law to a joint normal distribution with mean* $\underset{\sim}{0}$ *and covariance matrix* $\underset{\sim}{\Sigma}$.

Proof:

From (1.6)-(iv), it follows that $E(S_k) = 0$ since $|Y_{ij}|$ and $\text{sign } Y_{ij}$ are independent.

(i) Calculation of covariance matrix of $\underset{\sim}{S}$.

From (2.2),

$$\text{cov}(S_k,S_{k'}) = n^{-\frac{1}{2}} \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{i'j'k'}E\{\psi[F_j^*(|Y_{ij}|)]\psi[F_{j'}^*(|Y_{i'j'}|)]$$

$$\cdot \text{ sign } Y_{ij} \text{ sign } Y_{i'j'}\}$$

The terms for which $i \neq i'$ vanish since F_j is symmetric about zero and the Y's are independent. Now for $i = i'$, the above expectation is $\lambda_{jj'}$ where $\lambda_{jj'}$ is given by (1.10). Hence the above expression becomes

$$\text{cov } (S_k,S_{k'}) = n^{-1} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{ij'k'}\lambda_{jj'}$$

Thus the required covariance matrix is $\underset{\sim}{\Sigma}_n$ given in (1.10).

(ii) Asymptotic normality of $\underset{\sim}{s}$.

To prove asymptotic normality, we need to prove the asymptotic normality of an arbitrary linear combination of $\{S_k\}$. This follows from Wald and Wolfowitz (1944), page 371. Define

$$L_n = \sum_{j=1}^{p} \sum_{k=1}^{q} c_k S_k = \sum_{j=1}^{p} \sum_{k=1}^{q} c_k \sum_{i=1}^{n} n^{-\frac{1}{2}} x_{ijk} \psi(U_{ij}) \text{ sign } Y_{ij}$$

$$= \sum_{i=1}^{n} L_{in}$$

where $U_{ij} = F_j^*(|Y_{ij}|)$ and

$$L_{in} = \sum_{j=1}^{p} \psi(U_{ij}) \text{ sign } Y_{ij} \sum_{k=1}^{q} c_k n^{-\frac{1}{2}} x_{ijk}$$

Let $B_n^2 = \text{var } L_n = \sum_{i=1}^{n} \text{var } L_{in}$

Assume that L_n does not tend to a degenerate distribtuion, i.e. $\lim_{n\to\infty} B_n^2 = B_o^2 > 0$. The degenerate case can be treated separately. By the Lindeberg-Feller theorem (Loeve (1955), page 280), L_n is asymptotically normal with mean 0 and variance B_o^2 if for every $\varepsilon > 0$,

$$B_n^{-2} \sum_{i=1}^{n} \int_{|x|>\varepsilon B_n} x^2 dP(L_{in} \geq x) \to 0 \quad \text{as} \quad n \to \infty$$

From the above remarks, we need only to show that

$\theta_n = \sum_{i=1}^{n} \int_{|x|>\varepsilon} x^2 dP(L_{in} \leq x) \to 0$ as $n \to \infty$. Let $T_{ij} = \psi(U_{ij}) \text{sign } Y_{ij}$.

Then $\int_{|x|>\varepsilon} x^2 dP(L_{in} \leq x) = \int_{|x|>\varepsilon} x^2 dP(\sum_{j=1}^{p} T_{ij} \sum_{k=1}^{q} c_k n^{-\frac{1}{2}} x_{ijk} \leq x)$.

Thus (writing T_j for T_{ij} since the distribution of T_{ij} does not depend on i),

$$(3.2) \qquad \theta_n = \sum_{i=1}^{n} \int_{M_i} \cdots \int (\sum_{j=1}^{p} y_j \sum_{k=1}^{q} c_k n^{-\frac{1}{2}} x_{ijk})^2 dP(T_1 \leq y_1, \ldots, T_p \leq y_p)$$

where $M_i = \{(y_1, \ldots, y_p) \varepsilon E_p : |\sum_{j=1}^{p} y_j \sum_{k=1}^{q} c_k n^{-\frac{1}{2}} x_{ijk}| > \varepsilon\}$

It follows from (1.6) that $\lim_{n\to\infty} \max_{1\leq j\leq p} n^{-\frac{1}{2}} \sum_{k=1}^{q} |x_{ijk}| = 0$. Thus, given an integer M, $\exists N \ni \forall n \geq N$, $\max_{1\leq j\leq p} n^{-\frac{1}{2}} \sum_{k=1}^{q} |x_{ijk}| < \varepsilon/(pMD_1)$ where $D_1 = \max_{1\leq k\leq q} |c_k|$.

Hence if $n \geq N$, it follows that

$$M_i \subset \{(y_1, \ldots, y_p) : \sum_{j=1}^{p} |y_j| [\sum_{k=1}^{q} n^{-\frac{1}{2}} |c_k x_{ijk}|] > \varepsilon\}$$

$$\subset \{(y_1, \ldots, y_p) : (\varepsilon/pM) \sum_{j=1}^{p} |y_j| > \varepsilon\}$$

$$\subset \{(y_1, \ldots, y_p) : \max_{1\leq j\leq p} |y_j| > M\} = N \text{ , say. Thus (3.2)}$$

implies

$$\theta_n \leq \sum_{i=1}^{n} \int_{N} \cdots \int pq \sum_{j=1}^{p} y_j^2 \sum_{k=1}^{q} n^{-1} c_k^2 x_{ijk}^2 dP(T_1 \leq y_1, \ldots, T_p \leq y_p)$$

$$(3.3) \qquad \theta_n \leq pqD_2 \sum_{j=1}^{p} \int_{N} \cdots \int y_j^2 dP(T_1 \leq y_1, \ldots, T_p \leq y_p)$$

$$\text{where } D_2 = \max_{1\le k\le q} c_k^2 \cdot \max_{\substack{1\le j\le p \\ n>0}} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2$$

Now, $\int_{E_p} y_j^2 dP(T_1 \le y_1, \ldots, T_p \le y_p) = E(T_j^2)$

$$= \int_0^\infty \psi^2[F_j^*(|y|)dF_j^*(|y|) = \int_0^1 \psi^2(u)du$$

From Munroe (1959), theorem 27.1, p. 191, $\forall\varepsilon > 0$, $\exists\delta > 0 \ni$ if, for $S \subset E_p$, $P(S) < \delta$, then $\int_S y_j^2 dP(T_1 \le y_1, \ldots, T_p \le y_p) < \varepsilon/(p^2 q D_2)$ for $j = 1, \ldots, p$. But, for M sufficiently large, $P\{\max_{1\le j\le p} |T_j| > M\} < \delta$, i.e. $P(\mathcal{N}) < \delta$. Now, the use of the preceding remarks in (3.3) implies $\theta_n < \varepsilon$ if $n \ge N$, i.e. $\lim_{n\to\infty} \theta_n = 0$. Hence we conclude $B_n^{-1} \sum_{k=1}^{q} c_k S_k$ converges in law to a standard normal, and since the covariance matrix, $\underset{\sim}{\Sigma}_n$ of $\underset{\sim}{S}' = (S_1, \ldots, S_q)$ converges to $\underset{\sim}{\Sigma}$, $\underset{\sim}{S}$ converges in law to a q-variate normal $(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ distribution.

The singular case remains. If $B_n^2 \to 0$, then L_n tends to a degenerate distribution which can be thought of as a degenerate normal distribution.

3.3 *Limit Theorems for Separate Ranking Case under* $H_0 : \underset{\sim}{\beta} = \underset{\sim}{0}$.

In this section $\underset{\sim}{T}$ will be shown to converge in the mean to $\underset{\sim}{S}$. This will lead us to the asymptotic distribution of M_n (defined in (2.2)).

Lemma 3.3:

Under assumptions (1.6), (1.8), and (2.2),

$$\lim_{n\to\infty} E(T_k-T_k^*)^2 = 0 .$$

Proof:

For fixed j, $\{Y_{ij}:i=1,\ldots,n\}$ are independent and identically distributed with distribution symmetric about zero. Hence $\{|Y_{ij}|:i=1,\ldots,n\}$ is independent of $\{\text{sign } Y_{ij}:i=1,\ldots,n\}$. Thus $\{R_{ij},|Y_{ij}|:i=1,\ldots,n\}$ is independent of $\{\text{sign } Y_{ij}:i=1,\ldots,n\}$. Since $E(\text{sign } Y_{ij}) = 0$ for all i and j, $E(T_k) = E(T_k^*) = E(S_k) = 0$, and hence

$$E(T_k-T_k^*) = n^{-1}E\{\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}[\psi_n(\tfrac{R_{ij}}{n+1})-\psi_n(F_j^*(|Y_{ij}|)) \text{ sign } Y_{ij}\}^2$$

Now use the inequality $|E(AB)| \le \frac{1}{2} E(A^2+B^2)$ to obtain

$$= pn^{-1} \sum_{j=1}^{p} E\{\sum_{i=1}^{n} x_{ijk}[\psi_n(\tfrac{R_{ij}}{n+1})-\psi_n(F_j^*(|Y_{ij}|))] \text{ sign } Y_{ij}\}^2$$

and after squaring,

$$E(T_k-T_k^*) \le pn^{-1} \sum_{j=1}^{p} \sum_{i,i'=1}^{n} x_{ijk}x_{i'jk}E\{[\psi_n(\tfrac{R_{ij}}{n+1})-\psi_n(F_j^*(|Y_{ij}|))]$$

$$\cdot [\psi_n(\tfrac{R_{i'j}}{n+1})-\psi_n(F_j^*(|Y_{i'j}|))] \text{ sign } Y_{ij} \text{ sign } Y_{i'j}\}$$

From the initial remarks of the proof, it follows that terms in the above, for which $i \ne i'$, are zero. Thus

$$E(T_k-T_k^*)^2 \leq pn^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 E[\psi_n(\frac{R_{ij}}{n+1})-\psi_n(F_j^*(|Y_{ij}|))]^2$$

Because $F_j^*(|Y_{ij}|)$ has a uniform distribution, it follows from lemma 2.1 of Hàjek (1961) that

$$E(T_k-T_k^*)^2 \leq pn^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 \cdot 2 \max_{1\leq m\leq n} |\psi_n(\frac{m}{n+1})-\bar{\psi}_n|$$

$$\cdot\ n^{-1}\{2 \sum_{m=1}^{n} [\psi_n(\frac{m}{n+1})-\bar{\psi}_n]^2\}^{\frac{1}{2}}$$

Use of the Minkowski Inequality (see Loeve (1955), p. 156) and (1.8) yields $n^{-1} \sum_{m=1}^{n} [\psi_n(m/(n+1))-\bar{\psi}_n]^2 \leq n^{-1} \sum_{m=1}^{n} \psi_n(m/(n+1))$ $= \int_0^1 \psi_n^2(u)du \to \int_0^1 \psi^2(u)du$ as $n \to \infty$. Hence $\exists \alpha > 0 \ni \alpha n^{-1} \sum_{m=1}^{n} \cdot$ $[\psi_n(m/(n+1))-\bar{\psi}_n]^2 \leq 1$ for all n. In view of this, the above expression becomes

$$E(T_k-T_k^*)^2 \leq 2\sqrt{2}\ p\alpha^{-1}n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 \cdot \max_{1\leq m\leq n} |\psi(\frac{m}{n+1})- \bar{\psi}_n|$$

$$\cdot\ \{\sum_{m=1}^{n} [\psi_n(\frac{m}{n+1})-\bar{\psi}_n]^2\}^{-\frac{1}{2}}$$

Now (1.6)-(iii) implies $n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2$ tends to a finite nonzero limit, and (1.8)-(v) implies the product of the last two factors tends to zero. Hence the result is immediate.

Lemma 3.4:

Under assumptions (1.6), (1.8), and (2.2),

$$\lim_{n\to\infty} E(T_k^*-S_k)^2 = 0.$$

Proceeding as in lemma 3.3, $E(T_k^*) = E(S_k) = 0$, and

$$E(T_k^*-S_k)^2 = n^{-1}E\{\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}[\psi_n(F_j^*(|Y_{ij}|)-\psi(F_j^*(|Y_{ij}|))] \text{ sign } Y_{ij}\}^2$$

$$\leq pn^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}^2 E[\psi_n(F_j^*(|Y_{ij}|))-\psi(F_j^*(|Y_{ij}|))]^2$$

$$= p(n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}^2)\int_0^1 [\psi_n(u)-\psi(u)]^2 du$$

From (1.6)-(iii) and (1.8)-(iv), the second and third factors tend to a nonzero finite constant and to zero, respectively, as n increases. Hence result is proved.

Lemma 3.5:

Under assumptions (1.6), (1.8), (1.9), (2.2), and (2.5), $\underset{\sim}{T}' = (T_1,\ldots,T_q)$ converges in law to a normal $(\underset{\sim}{0},\underset{\sim}{\Sigma})$ distribution.

Proof:

This immediately follows from lemmas 3.2, 3.3, and 3.4 and the fact that convergence in quadratic mean implies convergence in law.

<u>*Lemma 3.6:*</u>

Under assumptions (1.6), (1.8), (1.9), (2.2), and (2.5), $M_n(\underset{\sim}{Y})$ *converges in law to a chi-square distribution with* q*-degrees of freedom.*

Proof:

Since $M_n(\underset{\sim}{Y})$ is a continuous function of $\underset{\sim}{T}$, and $\underset{\sim}{T}$ converges in law to a normal $(\underset{\sim}{0},\underset{\sim}{\Sigma})$ distribution, $M_n(\underset{\sim}{Y}) = \underset{\sim}{T}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{T}$ will converge in law to the above chi-square distribution (see Sverdrup (1952), corollary on page 5).

3.4 *Limit Theorems for Joint Ranking Case under* $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$.

In this section results corresponding to those of lemmas 3.3, 3.4, 3.5, and 3.6 will be proved for the joint ranking procedure. It turns out that these results are not valid unless certain further conditions in addition to (1.6), (1.8), (2.2) and (2.5) are made. Three sets of such conditions are listed and the above results are proved in each case.

The lemmas following are similar to theorem 3.1 of Mehra (1969). Here a signed rank rather than a rank statistic is considered, and the conditions on the underlying distribution differ. Theorem 3.1 of Hàjek (1961) proves somewhat similar results for the univariate case.

Additional Conditions

(i) *Quadrant Symmetry* - F is assumed to satisfy (3.1).

(ii) *Interchangeability* - The following conditions are assumed.

$$(3.4)\quad \begin{cases} \sum_{j=1}^{p} x_{ijk} = 0 \quad \text{for } i = 1,\ldots,n; \; j = 1,\ldots,p \\ \\ F(\underset{\sim}{e}) \text{ is the distribution function of an interchangeable} \\ \text{random vector } \underset{\sim}{e} = (e_1,\ldots,e_p). \end{cases}$$

Recall that a random vector $(e_1,\ldots,e_p)$ is interchangeable if $(e_1,\ldots,e_p)$ and $(e_{\sigma_1},\ldots,e_{\sigma_p})$ have the same distribution, where $(\sigma_1,\ldots,\sigma_p)$ is any permutation of $(1,\ldots,p)$, i.e. if $F(e_1,\ldots,e_p) = F(e_{\sigma_1},\ldots,e_{\sigma_p})$.

Although (3.4) makes an apparently unnatural assumption on the $\{x_{ijk}\}$, model (1.1), in the exchangeable case, can be made to satisfy this condition. This is done by subtracting $\overline{Y}_{i.}$ (where the "dot" and "bar" signify that the mean has been taken over all values of the missing subscript) from each observation Y_{ij}. This results in an "adjusted" model,

$$(3.5)\qquad Y_{ij} - \overline{Y}_{i.} = \sum_{k=1}^{q} (x_{ijk} - \overline{x}_{i.k})\beta_k + (e_{ij} - \overline{e}_{i.})$$

The regression coefficients now satisfy (3.4), and the joint distribution of $(e_{ij}-\bar{e}_{i.}:j=1,\ldots,p)$ is interchangeable and marginal distributions are symmetric about zero if the same is true of $(e_{ij}:j=1,\ldots,p)$. Also $P(e_{ij}-\bar{e}_{i.}=e_{ij'}-\bar{e}_{i.}) = P(e_{ij}=e_{ij'})$. Thus, in the interchangeable case, if (1.6)-(iv) and (2.7) are true for model (1.2), and (1.6)-(i), (ii), (iii), and (1.9) hold for model (3.5), then (1.6), (1.9), (2.7), and (3.4) hold for model (3.5).

(iii) Certain sums of the $\{x_{ijk}\}$ *are zero* -- In this case, the only added conditions will be ones placed on the $\{x_{ijk}\}$. It is assumed that

$$(3.6)\qquad \begin{cases} \sum_{j=1}^{p} x_{ijk} = 0 \quad \text{for} \quad i = 1,\ldots,n;\ k = 1,\ldots,q \\ \sum_{i=1}^{n} x_{ijk} = 0 \quad \text{for} \quad j = 1,\ldots,p;\ k = 1,\ldots,q \end{cases}$$

The first condition can be satisfied as in the exchangeability case. The second one could be removed by subtracting $\bar{Y}_{.j}$ from Y_{ij} for each i and j. However, doing this makes all the adjusted observations $\{Y_{ij}-\bar{Y}_{.j}\}$ dependent, and treatment of this problem will probably require more sophisticated techniques than are used here.

In some cases, however, it is possible to satisfy the second condition of (3.6). Suppose, for example, that we are able to design model (1.2) so that $x_{.jk}$ is independent of j, i.e. $\bar{x}_{.jk} = \bar{x}_{..k}$ for $j = 1,\ldots,p$. Then (1.2) can be written as

$$Y_{ij} - \bar{Y}_{i.} = \sum_{k=1}^{q} (x_{ijk}-\bar{x}_{.jk})\beta_k + (e_{ij}-\bar{e}_{i.})$$

Then $\sum_{j=1}^{p} (x_{ijk}-\bar{x}_{i\cdot k}) = p(\bar{x}_{i\cdot k}-\bar{x}_{i\cdot k}) = 0$ and $\sum_{i=1}^{n} (x_{ijk}-\bar{x}_{\cdot jk})$ $= n(\bar{x}_{\cdot jk}-\bar{x}_{\cdot\cdot k}) = 0$. Hence (3.6) is satisfied.

Lemma 3.7:

Under conditions (1.6), (1.8), (2.7), (2.8), and one of (3.1), (3.4), or (3.6), $\lim_{n\to\infty} E(T_k-T_k^*)^2 = 0$.

Proof:

(i) Suppose (3.1) is true. Then from lemma 3.1 it follows that $\{R_{ij}, |Y_{ij}| : i=1,\ldots,n\}$ is independent of $\{\text{sign } Y_{ij} : i=1,\ldots,n; j=1,\ldots,p\}$. Thus from (2.8), $E(T_k) = E(T_k^*) = E(S_k) = 0$, and hence

$$(3.7) \qquad E(T_k-T_k^*)^2 = n^{-1} \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{i'j'k} E\{[\psi_{np}(\tfrac{R_{ij}}{np+1}) - \psi_{np}(F_1^*(|Y_{ij}|))][\psi_{np}(\tfrac{R_{i'j'}}{np+1}) - \psi_{np}(F_1^*(|Y_{i'j'}|))] \text{sign } Y_{ij} \text{ sign } Y_{i'j'}\}$$

From lemma 3.1 and (1.6)-(iv), the above expectation is zero unless $i = i'$ and $j = j'$. Thus

$$(3.8) \qquad E(T_k-T_k^*)^2 = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 E[\psi_{np}(\tfrac{R_{ij}}{np+1}) - \psi_{np}(F_1^*(|Y_{ij}|))]^2$$

Because $F_1^*(|Y_{ij}|)$ has a uniform distribution, it follows from lemma A.9 and the fact that $\psi_{np}(m/(n+1)) = \psi_{np}(m/np)$ for $m = 1,\ldots,np$, (see (1.7)), that

$$E(T_k-T_k^*)^2 \le (2p)^{3/2} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 \max_{1\le m\le np} |\psi_{np}(\frac{m}{np+1}) - \bar{\psi}_{np}|$$

$$n^{-1}\{\sum_{m=1}^{np} [\psi_{np}(\frac{m}{np+1})-\bar{\psi}_{np}]^2\}^{\frac{1}{2}}$$

As in the proof of lemma 3.3, the Minkowski inequality and (1.8) imply $\exists \alpha > 0 \ni \forall n \ge 1$, $\alpha n^{-1} \sum_{m=1}^{np} [\psi_n(m/(np+1))-\bar{\psi}_n]^2 \le 1$, and so

$$E(T_k-T_k^*)^2 \le (2p)^{3/2} \alpha^{-1} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 \max_{1\le m\le np} |\psi_{np}(\frac{m}{np+1})-\bar{\psi}_{np}|$$

$$\cdot \{\sum_{m=1}^{np} [\psi_{np}(\frac{m}{np+1})-\bar{\psi}_{np}]^2\}^{-\frac{1}{2}}$$

The last two sentences of the proof of lemma 3.3 now apply and the proof is complete.

(ii) Suppose (3.4) holds. From (2.8)

$$(3.9) \qquad E(T_k) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk} E[\psi_{np}(\frac{R_{ij}}{np+1})-\psi_{np}(F_1^*(|Y_{ij}|)) \text{ sign } Y_{ij}]$$

Because the observations are independent in 'i' and interchangeable in 'j', the above expectation depends neither on 'i' nor 'j'. Thus $\sum_{j=1}^{p} x_{ijk} = 0$ implies $E(T_k) = 0$. Similarly $E(T_k) = E(T_k^*) = E(S_k) = 0$. Now, from (2.8) it is seen that (3.7) is valid here also.

Due to interchangeability, the expectation on the R.H.S. of (3.7) is independent of the pair (j,j') as well as (i,i') if

$i \neq i'$. Denote this value by a_n. When $i = i'$, it has a value, say b_n, which is independent of i and the pair (j,j') if $j \neq j'$, and takes on a third value, c_n, if $i = i'$ and $j = j'$. Consequently, (3.7) becomes

$$E(T_k-T_k^*)^2 = n^{-1}\{c_n \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 + b_n \sum_{i=1}^{n} \sum_{j\neq j'=1}^{p} x_{ijk}x_{ij'k}$$

$$+ a_n \sum_{i\neq i'=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{i'j'k}\}$$

and (3.4) implies $E(T_k-T_k^*)^2 = (c_n-b_n)n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2$. From the well known inequality $|E(AB)| \leq \frac{1}{2} E(A^2+B^2)$, $|b_n| \leq c_n$, and so

$$E(T_k-T_k^*)^2 \leq 2n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 E[\psi_{np}(R_{ij}/(np+1))-\psi_{np}(F_1^*(|Y_{ij}|))]^2.$$

Since the arguments following (3.8) employ only (1.6) and (1.8), they apply here also and the result follows.

(iii) Suppose (3.6) holds. Then (3.9) can be obtained, and by independence in i the expectation will only depend on j, say it is a_j. Then $E(T_k) = n^{-\frac{1}{2}} \sum_{j=1}^{p} a_j \sum_{i=1}^{n} x_{ijk} = 0$ by (3.6). Similarly $E(T_k) = E(T_k^*) = E(S_k) = 0$.

From (2.8), (3.7) is valid here. The expectation on the R.H.S. of (3.7) does not depend on the pair (i,i') if $i \neq i'$, say it is $a_{njj'}$, and does not depend on i if $i = i'$, say it is $b_{njj'}$. It is now evident from (3.7) that

$$E(T_k-T_k^*)^2 = n^{-1} \{ \sum_{i=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{ij'k}b_{njj'} + \sum_{i\neq i'=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{i'j'k}a_{njj'} \}$$

In view of (3.6),

$$(3.10) \qquad E(T_k-T_k^*)^2 = n^{-1} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} x_{ijk}x_{ij'k}(b_{njj'}-a_{njj'})$$

Now, from the inequality $|E(AB)| \leq \frac{1}{2} E(A^2+B^2)$, it follows that $|x_{ijk}x_{ij'k}b_{njj'}| \leq \frac{1}{2} (x_{ijk}^2 b_{njj}+x_{ij'k}^2 b_{nj'j'})$ and $|x_{ijk}x_{ij'k}a_{njj'}| \leq \frac{1}{2} (x_{ijk}^2 b_{njj}+x_{ij'k}^2 b_{nj'j'})$. Substitution of these two facts into (3.10) yields

$$E(T_k-T_k^*)^2 \leq pn^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}^2 E[\psi_{np}(\frac{R_{ij}}{np+1})-\psi_{np}(F_1^*(|Y_{ij}|))]^2$$

As in the previous two cases, the arguments following (3.8) give the result.

Lemma 3.8:

Under conditions (1.6), (1.8), and (2.8),

$$\lim_{n\to\infty} E(T_k^*-S_k)^2 = 0.$$

Proof:

Because neither T_k^* nor S_k depend on the ranking procedure, the proof follows almost verbatim that of lemma 3.4.

Lemma 3.9:

Under conditions (1.6), (1.8), (2.7), (2.8), and one of (3.1), (3.4), or (3.6),

(i) $\underset{\sim}{T}' = (T_1, \ldots, T_q)$ *converges in law to a normal* $(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ *distribution.*

(ii) $M_n(\underset{\sim}{Y})$ *converges in law to a chi-square distribution with* q- *degrees of freedom*

Proof:

(i) Follows from (3.2), (3.7), (3.8), and fact that convergence in quadratic mean implies convergence in law.

(ii) Follows by the same reasoning as in the proof of lemma 3.6.

CHAPTER IV

LIMIT THEOREMS UNDER A SEQUENCE OF CONTIGUOUS ALTERNATIVES, TESTING

In this chapter the limiting distributions of the $\underset{\sim}{T}' = (T_1,\ldots,T_q)$ defined both in (2.2) and (2.8) are found under a sequence of contiguous alternatives. To do this some additional conditions are necessary on the underlying distribtuion. From this, the limiting distributions of the two test statistics, $M_n(\underset{\sim}{Y})$, defined in (2.2) and (2.8) are obtained. The arguments resemble those of Mehra (1969). Also, Hàjek (1962) and Adichie (1967a) have used the same approach but for the univariate case.

The results following can be compared to those of theorems 6.4 and 7.4, from which the limiting distributions of $\underset{\sim}{T}$ and $M_n(\underset{\sim}{Y})$ can be found under the same sequence of alternatives but with different restrictions on the underlying distribution.

The idea of contiguity is discussed in some detail in Hàjek (1962). Let us consider the sequence of alternatives

(4.1) $$Q_n : \underset{\sim}{\beta} = n^{-\frac{1}{2}}\underset{\sim}{\xi}$$

where $\underset{\sim}{\xi} \in E_q$ is fixed.

For the remainder of this chapter, assume

(4.2)
$$\begin{cases} F(\underset{\sim}{e}) \text{ has a density, } f(\underset{\sim}{e})\text{, which is absolutely continuous} \\ \text{in each argument. } f^{(j)}(\underset{\sim}{e}) = \partial f(\underset{\sim}{e})/\partial e_j \text{ exists and is} \\ \text{finite for } j = 1,\ldots,p \text{ and almost all } \underset{\sim}{e} \in E_q\text{.} \\ \int_{E_p} \frac{[f^{(j)}(\underset{\sim}{e})]^2}{f(\underset{\sim}{e})} d\underset{\sim}{e} < \infty \quad \text{(integrand is defined to be zero if } f(\underset{\sim}{e}) = 0\text{).} \end{cases}$$

Now define

(4.3)
$$\begin{cases} \|(e_1,\ldots,e_p)\| = (e_1^2+\ldots+e_p^2)^{\frac{1}{2}} \\ s(\underset{\sim}{e}) = \sqrt{f(e)} \\ e_{ij} = Y_{ij} - n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{\zeta} \\ \underset{\sim}{e}_i = \underset{\sim}{Y}_i - n^{-\frac{1}{2}}\underset{\sim}{X}'_i\underset{\sim}{\zeta} \end{cases}$$

(4.4)
$$W_n = 2\sum_{i=1}^{n}\left\{\frac{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})}{s(\underset{\sim}{e}_i)} - 1\right\}$$

(4.5)
$$T'_n = -n^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{j=1}^{p} \underset{\sim}{x}'_{ij}\underset{\sim}{\zeta} f^{(j)}(\underset{\sim}{e}_i)/f(\underset{\sim}{e}_i)$$

Lemma 4.1:

Under conditions (4.2) and (4.3)

$$\lim_{\|\underset{\sim}{b}\|\to 0} \left|\int_{E_p}\left[\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|}\right]^2 d\underset{\sim}{e} - \int_{E_p}\left\{\sum_{j=1}^{p}\frac{b_j}{\|\underset{\sim}{b}\|} s^{(j)}(\underset{\sim}{e})\right\}^2 d\underset{\sim}{e}\right| = 0$$

Proof:

Let $b_j^{*'} = (b_1,\dots,b_j,0,\dots,0)$ and $\underset{\sim}{1}_j' = (0,\dots,0,1,0,\dots,0)$ where the "1" is the j^{th} component. Now

$$[s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})]^2 = \{\sum_{j=1}^{p} [s(\underset{\sim}{e}-\underset{\sim}{b}_j^*)-s(\underset{\sim}{e}-\underset{\sim}{b}_{j-1}^*)]\}^2$$

$$\leq p\sum_{j=1}^{p} [s(\underset{\sim}{e}-\underset{\sim}{b}_j^*)-s(\underset{\sim}{e}-\underset{\sim}{b}_{j-1}^*)]^2 \leq p\sum_{j=1}^{p} [\int_0^{b_j} s^{(j)}(\underset{\sim}{e}-\underset{\sim}{b}_{j-1}^*-x\underset{\sim}{1}_j)dx]^2$$

$$\leq p\sum_{j=1}^{p} |b_j| \int_0^{b_j} s^{(j)}(\underset{\sim}{e}-\underset{\sim}{b}_{j-1}^*-x\underset{\sim}{1}_j)]^2dx\ .$$ Thus it is evident that

$$\int_{E_p} [\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|}]^2 d\underset{\sim}{e} \leq \frac{p}{\|\underset{\sim}{b}\|^2}\sum_{j=1}^{p} |b_j| \int_{E_p}\int_0^{b_j} s^{(j)}(\underset{\sim}{e}-\underset{\sim}{b}_{j-1}^*-x\underset{\sim}{1}_j)dx d\underset{\sim}{e}$$

By changing the order of integration, it is seen that

$$(4.6)\qquad \int_{E_p} [\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|}]^2 d\underset{\sim}{e} \leq p\sum_{j=1}^{p} \int_{E_p} [s^{(j)}(\underset{\sim}{e})]^2 d\underset{\sim}{e} < \infty$$

since $\int_{E_1} [s^{(j)}(\underset{\sim}{e})]^2 d\underset{\sim}{e} = \frac{1}{4}\int_{E_p} [\frac{f^{(j)}(\underset{\sim}{e})}{f(\underset{\sim}{e})}]^2 f(\underset{\sim}{e})d\underset{\sim}{e} < \infty$ by (4.2).

Also, it is evident that

$$(4.7)\qquad \int_{E_p} \{\sum_{j=1}^{p} \frac{b_j}{\|\underset{\sim}{b}\|} s^{(j)}(\underset{\sim}{e})\}^2 d\underset{\sim}{e} \leq p\sum_{j=1}^{p} \int_{E_p} [s^{(j)}(\underset{\sim}{e})]^2 d\underset{\sim}{e} < \infty$$

Hence, from Schwartz inequality,

$$\int_{E_p} \left|\left\{\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|}\right\}^2 - \left\{\sum_{j=1}^{p} \frac{b_j}{\|\underset{\sim}{b}\|} s^{(j)}(\underset{\sim}{e})\right\}^2\right| d\underset{\sim}{e}$$

$$\leq \left\{\int_{E_p} \left[\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|} - \sum_{j=1}^{p} \frac{b_j}{\|\underset{\sim}{b}\|} s^{(j)}(\underset{\sim}{e})\right]^2 d\underset{\sim}{e}\right\}^{\frac{1}{2}}$$

$$\cdot \left\{\int_{E_p} \left[\frac{s(\underset{\sim}{e}-\underset{\sim}{b})-s(\underset{\sim}{e})}{\|\underset{\sim}{b}\|} + \sum_{j=1}^{p} \frac{b_j}{\|\underset{\sim}{b}\|} s^{(j)}(\underset{\sim}{e})\right]^2 d\underset{\sim}{e}\right\}^{\frac{1}{2}}$$

From lemma 4.1 of Mehra (1969), first factor on R.H.S. tends to zero as $\|\underset{\sim}{b}\|$ tends to zero. From (4.6) and (4.7) it follows that the second factor is bounded uniformly for $\underset{\sim}{b} \in E_p$. Thus the result follows.

Lemma 4.2:

Under conditions (1.6), (1.8), (1.9), and (4.2),

$$\lim_{n\to\infty} \{E(W_n) + \frac{1}{4} \sum_{j=1}^{p} \sum_{j'=1}^{p} [\sum_{k=1}^{q} \sum_{k'=1}^{q} \zeta_k \zeta_{k'} n^{-1} \sum_{i=1}^{n} x_{ijk} x_{ij'k'}]$$

$$\cdot \int_{E_p} [f^{(j)}(\underset{\sim}{e}) f^{(j')}(\underset{\sim}{e})/f(\underset{\sim}{e})] d\underset{\sim}{e}\} = 0$$

Proof:

$$E(W_n) = 2 \sum_{i=1}^{n} E\{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}} \underset{\sim}{X}'_{ni} \underset{\sim}{\zeta})/s(\underset{\sim}{e}_i) - 1\}$$

$$= 2 \sum_{i=1}^{n} \int_{E_p} [s(\underset{\sim}{e} - n^{-\frac{1}{2}} \underset{\sim}{X}'_{ni} \underset{\sim}{\zeta}) s(\underset{\sim}{e}) - s^2(\underset{\sim}{e})] d\underset{\sim}{e}$$

$$= - \sum_{i=1}^{n} \int_{E_p} [s^2(\underset{\sim}{e} - n^{-\frac{1}{2}} \underset{\sim}{X}'_{ni} \underset{\sim}{\zeta}) - 2s(\underset{\sim}{e} - n^{-\frac{1}{2}} \underset{\sim}{X}'_{ni} \underset{\sim}{\zeta}) + s^2(\underset{\sim}{e})] d\underset{\sim}{e}$$

by translation invariance of the integral.

$$= -\sum_{i=1}^{n} \int_{E_p} [s(\underset{\sim}{e}-n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})-s(\underset{\sim}{e})]^2 d\underset{\sim}{e}$$

$$= -\sum_{i=1}^{n} \|n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\|^2 \int_{E_p} [\frac{s(\underset{\sim}{e}-n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})-s(\underset{\sim}{e})}{\|n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\|}]^2 d\underset{\sim}{e}$$

From (1.6)-(ii) and (iii), $\lim_{n\to\infty} \max_{1\le i\le n} \|n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\| = 0$. Hence it follows from lemma 4.1 that

$$|E(W_n) + \int_{E_p} \{\sum_{j=1}^{p} n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{\zeta} s^{(j)}(\underset{\sim}{e}_i)\}^2 d\underset{\sim}{e}_i| \to 0$$

From (4.3) it is obvious that $s^{(j)}(\underset{\sim}{e}) = \frac{1}{2} f^{(j)}(\underset{\sim}{e})/\sqrt{f(\underset{\sim}{e})}$.
Substituting into the above expression and squaring yields

$$|E(W_n)+\frac{1}{4}\sum_{j=1}^{p}\sum_{j'=1}^{p}[\sum_{i=1}^{n} n^{-1}(\underset{\sim}{x}'_{ij}\underset{\sim}{\zeta})(\underset{\sim}{x}'_{ij'}\underset{\sim}{\zeta})]\int_{E_p}\frac{f^{(j)}(\underset{\sim}{e})f^{(j')}(\underset{\sim}{e})}{f(\underset{\sim}{e})}d\underset{\sim}{e}| \to 0$$

This implies the result.

Lemma 4.3:

Under conditions (1.6), (1.8), (1.9), and (4.2),

$$\lim_{n\to\infty} E[W_n - E(W_n) - T'_n]^2 = 0$$

Proof:

From (4.3) and (4.5) it is seen that

$T_n' = -2n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} \underset{\sim}{x}_{ij}'\underset{\sim}{\zeta} s^{(j)}(\underset{\sim}{e}_i)/s(\underset{\sim}{e}_i)$. Then from (4.4),

$$E[W_n - E(W_n) - T_n']^2 = 4 \sum_{i=1}^{n} E\{\frac{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta})}{s(\underset{\sim}{e}_i)} - 1$$

$$- E[\frac{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta})}{s(\underset{\sim}{e}_i)} - 1] - \sum_{j=1}^{p} n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{\zeta} s^{(j)}(\underset{\sim}{e}_i)/s(\underset{\sim}{e}_i)\}^2$$

This follows because cross terms in i are zero, a fact which follows from independence and the expectations being zero.

$$\leq 4 \sum_{i=1}^{n} \int_{E_p} \{\frac{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta}) - s(\underset{\sim}{e}_i)}{s(\underset{\sim}{e}_i)} - \sum_{j=1}^{p} n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{\zeta} \frac{s^{(j)}(\underset{\sim}{e}_i)}{s(\underset{\sim}{e}_i)}\}^2 s^2(\underset{\sim}{e}_i)d\underset{\sim}{e}_i$$

$$= 4 \sum_{i=1}^{n} ||n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta}||^2 \int_{E_p} \{\frac{s(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta}) - s(e_i)}{||n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta}||}$$

$$- \sum_{j=1}^{p} \frac{n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{\zeta}}{||n^{-\frac{1}{2}}\underset{\sim}{X}_{ni}'\underset{\sim}{\zeta}||} s^{(j)}(\underset{\sim}{e}_i)\}^2 d\underset{\sim}{e}_i$$

Lemma 4.1 of Mehra (1969) implies that the integrals tend to zero as n increases. Thus from (1.6)-(ii) and (iii) it is seen that the sum tends to zero as n increases, which completes the proof.

Theorem 4.1:

Under conditions (1.6), (1.8), (1.9), and (4.2), T_n' *converges in law to a normal distribution with mean zero and variance*

$$(4.8)\qquad \sigma^2 = \lim_{n\to\infty} \sum_{j=1}^{p} \sum_{j'=1}^{p} [\sum_{k=1}^{q} \sum_{k'=1}^{q} n^{-1} \sum_{i=1}^{n} x_{ijk}x_{ij'k'}]$$

$$\cdot \int_{E_p} [f^{(j)}(\underset{\sim}{e})f^{(j')}(\underset{\sim}{e})/f(\underset{\sim}{e})]d\underset{\sim}{e}$$

Proof:

Let $B_n^2 = \text{var } T_n'$, $B_o^2 = \lim_{n\to\infty} \text{var } T_n'$. It is seen that $E(T_n') = 0$ for all n. If $B_o^2 = 0$, then T_n' converges in law to a degenerate distribution at zero, which can be thought of as a normal distribution with both mean and variance zero.

It remains to consider the case $B_o^2 > 0$. The Lindeberg-Feller theorem will be used to show that T'/B_n converges in law to the standard normal distribution. The theorem then follows after calculating $\text{var } T_n'$.

Let $T_{ni}' = -n^{-\frac{1}{2}} \sum_{j=1}^{p} \underset{\sim}{x}_{ij}'\underset{\sim}{\xi} f^{(j)}(\underset{\sim}{e}_i)/f(\underset{\sim}{e}_i)$. Consider the expression

$$B_n^{-2} \sum_{i=1}^{n} \int_{|x|>\varepsilon B_n} x^2 dP(T_{ni}' \leq x)$$

If it is shown, for all positive ε, that this tends to zero as n increases, the proof is complete. Since $B_o^2 > 0$, this is equivalent to showing that for all positive ε, $\lim_{n\to\infty} \theta_n = 0$ where

$$\theta_n = \sum_{i=1}^{n} \int_{|x|>\varepsilon} x^2 dP(T_{ni}' \leq x).$$

Now, $\theta_n = \sum_{i=1}^{n} \int_{|x|>\varepsilon} x^2 dP\{-n^{-\frac{1}{2}} \sum_{j=1}^{p} \underset{\sim}{x}'_{ij}\underset{\sim}{\zeta} f^{(j)}(\underset{\sim}{e}_i)/f(\underset{\sim}{e}_i) \leq x\}$

$$= n^{-1} \sum_{i=1}^{n} \int_{|x|>\varepsilon n^{\frac{1}{2}}} x^2 dP\{\sum_{j=1}^{p} \underset{\sim}{x}'_{ij}\underset{\sim}{\zeta} f^{(j)}(\underset{\sim}{e}_j)/f(\underset{\sim}{e}_i) \leq x\}$$

Note that $\text{var } T'_n = \sum_{i=1}^{n} \text{var } T'_{ni}$, and since $E(T'_{ni}) = 0$,

$$= n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{j'=1}^{p} (\underset{\sim}{x}'_{ij}\underset{\sim}{\zeta})(\underset{\sim}{x}'_{ij'}\underset{\sim}{\zeta}) \int_{E_p} f^{(j)}(\underset{\sim}{e}) f^{(j')}(\underset{\sim}{e})/f(\underset{\sim}{e}) d\underset{\sim}{e}$$

This expression is the same as that given in (4.8) (except for "$\lim_{n\to\infty}$"). By arguments similar to those in (ii) of proof of lemma (3.2), it can be shown that $\lim_{n\to\infty} \theta_n = 0$.

Corollary 4.1:

W_n *converges in law to a normal distribution whose mean is given in lemma 4.1, and variance by (4.6).*

Proof:

Follows directly, in view of lemmas 4.2 and 4.3.

Lemma 4.4:

Under conditions (1.6) and (4.2), for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{1\leq i\leq n} P[|\frac{f(\underset{\sim}{e}_i - \underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})}{f(\underset{\sim}{e}_i)} - 1| > \varepsilon] = 0.$$

Proof:

Using Chebychev's inequality,

$$\max_{1\le i\le n} P[\frac{f(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})}{f(\underset{\sim}{e}_i)} - 1] \le \varepsilon^{-1} \max_{1\le i\le n} E\{|\frac{f(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})}{f(\underset{\sim}{e}_i)} - 1|\}$$

$$= \varepsilon^{-1} \max_{1\le i\le n} n^{-\frac{1}{2}} \|\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\| \int_{E_p} |\frac{f(\underset{\sim}{e} - n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}) - f(\underset{\sim}{e})}{n^{-\frac{1}{2}}\|\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\|}| d\underset{\sim}{e}$$

For simplicity, let $n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta} = (b_1,\ldots,b_p)'$, $\underset{\sim}{b}^*_j = (b_1,\ldots,b_j,0,\ldots,0)'$, $\underset{\sim}{1}_j = (0,\ldots,0,1,0,\ldots,0)'$ where the "1" is in the j^{th} position.

Consider $\|\underset{\sim}{b}^*_p\| \cdot |f(\underset{\sim}{e}-\underset{\sim}{b}^*_p) - f(\underset{\sim}{e})| \le \sum_{j=1}^{p} |b_j^{-1}| \cdot |f(\underset{\sim}{e}-\underset{\sim}{b}^*_j) - f(\underset{\sim}{e}-\underset{\sim}{b}^*_{j-1})|$

$= \sum_{j=1}^{p} |b_j^{-1}| \int_o^{b_j} f^{(j)}(\underset{\sim}{e}-\underset{\sim}{b}^*_{j-1} - x\underset{\sim}{1}_j)dx.$ Hence

$$\int_{E_p} |\frac{f(\underset{\sim}{e}-\underset{\sim}{b}^*_p) - f(\underset{\sim}{e})}{\|b^*_p\|}| d\underset{\sim}{e} \le \sum_{j=1}^{p} |b_j^{-1}| \int_{E_p} \int_o^{b_j} |f^{(j)}(\underset{\sim}{e}-\underset{\sim}{b}^*_{j-1} - x\underset{\sim}{1}_j)| dx d\underset{\sim}{e}$$

$$= \sum_{j=1}^{p} |b_j^{-1}| \int_o^{b_j} \int_{E_p} |f^{(j)}(\underset{\sim}{e})| d\underset{\sim}{e} dx \le \sum_{j=1}^{p} \int_{E_p} [f^{(j)}(\underset{\sim}{e})]^2 / f(\underset{\sim}{e}) \cdot d\underset{\sim}{e} < \infty$$

Substitution of this into the first inequalities yields

$$\max_{1\le i\le n} P[\frac{f(\underset{\sim}{e}_i - n^{-\frac{1}{2}}\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta})}{f(\underset{\sim}{e}_i)} - 1] \le \varepsilon^{-1} \max_{1\le i\le n} n^{-\frac{1}{2}} \|X'_{ni}\zeta\| \cdot \sum_{j=1}^{p} \int_{E_p} |f^{(j)}(\underset{\sim}{e})| d\underset{\sim}{e}$$

It is evident from (1.6)-(ii) and (iii) that, $\max_{1\le i\le n} n^{-\frac{1}{2}} \|\underset{\sim}{X}'_{ni}\underset{\sim}{\zeta}\| \to 0$

as n increases and thus the result follows.

Now consider the function

$$(4.9) \qquad L_n = \sum_{i=1}^{n} \log[f(e_i - n^{-\frac{1}{2}} x'_{ni}\zeta)/f(e_i)]$$

Theorem 4.2:

Under conditions (1.6), (1.8), (1.9), and (4.2),

(i) the sequence of distributions defined by $\{Q_n\}$ *in (4.1) is contiguous to the distribution under* $H_o : \beta = 0$,

(ii) for all $\varepsilon > 0$, $\lim_{n\to\infty} P(|W_n - L_n - \frac{1}{4} \text{var } W_n| \geq \varepsilon) = 0$.

Proof:

Because of lemmas 4.2 and 4.4, this follows just as in lemma 4.1 of Hàjek (1962). A similar comment is made preceding lemma 4.1 of Mehra (1969).

Corollary 4.2:

(i) Under conditions (1.6), (1.8), and (2.2), for all $\varepsilon > 0$ $\lim_{n\to\infty} P(||s-\tau||_{L_1} \geq \varepsilon) = 0$.

(ii) Under conditions (1.6), (1.8), (2.7), (2.8) and one of (3.1), (3.4), or (3.6), $\lim_{n\to\infty} P(||s-\tau||_{L_1} \geq \varepsilon) = 0$.

(iii) L_n *converges in law to a normal distribution with mean* $2 \lim_{n\to\infty} E(W_n)$ *(given by lemma 4.1), and variance given by (4.8).*

Proof:

(i) Follows from lemmas 3.3, 3.4, theorem 4.2-(i) and the fact that convergence in quadratic mean implies convergence in probability.

(ii) Similarly follows from lemmas 3.7, 3.8, and theorem 4.2-(i).

(iii) Follows from theorem 4.2-(ii) and corollary 4.1.

Theorem 4.3:

Under conditions (1.6), (1.8), (1.9), and (4.2), $\underset{\sim}{S}' = (S_1,\ldots,S_q)$ *converges, under the sequence of alternatives,* $\{Q_n\}$*, given in (4.1), to a q-variate normal distribution with mean vector* $\underset{\sim}{\rho}$ *and covariance matrix* $\underset{\sim}{\Sigma}$*, where* $\underset{\sim}{\rho}' = (\rho_1,\ldots,\rho_q)$*, and* $\rho_k = \lim_{n\to\infty} \mathrm{cov}(L_n,S_k)$ *where covariance is calculated under* $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$.

Proof:

Firstly, by arguments almost identical to those of lemma 3.2, it can be shown that, under $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, $(\underset{\sim}{S}',L_n)$ converges in law to a (q+1)-variate normal with mean vector $(\underset{\sim}{0}',-\sigma^2/2)$ and covariance matrix

$$\underset{\sim}{\Sigma}_{S,L} = \begin{bmatrix} \underset{\sim}{\Sigma} & , \underset{\sim}{\rho} \\ \underset{\sim}{\rho}' & , \sigma^2 \end{bmatrix} \tag{4.10}$$

where σ^2 is given in (4.6). Let $\underset{\sim}{b}' = (b_1,\ldots,b_q)$ and

$$(4.11)\qquad r_n = \begin{cases} \prod_{i=1}^{n} f(e_i - n^{-\frac{1}{2}} X'_{ni}\zeta)/f(e_i) & \text{if } \prod_{i=1}^{n} f(e_i) > 0 \\ 0 & \text{if } \prod_{i=1}^{n} f(e_i) = 0 \end{cases}$$

Then, using (1.3) and (4.1),

$$Q_n(s(Y)\leq b) = \int_{\{s(Y)\leq b\}} \prod_{i=1}^{n} f(e_i)de_i = \int_{\{s(e)\leq b\}} \prod_{i=1}^{n} f(e_i - n^{-\frac{1}{2}} X'_{ni}\zeta)de_i$$

$$= \int_{\substack{\{s(e)\leq b\}\\ \cap\{\pi f(e_i)>0\}}} r_n \prod_{i=1}^{n} f(e_i)de_i + \int_{\substack{\{s(e)\leq b\}\\ \cap\{\pi f(e_i)=0\}}} \prod_{i=1}^{n} f(e_i - n^{-\frac{1}{2}} X'_{ni}\zeta)de_i$$

By the contiguity result in theorem 4.2-(i), since $P\{\prod_{i=1}^{n} f(e_i)=0\} = 0$, the last term in the above equation tends to zero as n increases. From (4.10) and (4.11), $r_n = \exp(L_n)$. Hence if $F_n(v,w) = P(s(e)\leq v,\ L_n\leq w)$, and $F(v,w)$ denotes the distribution described by (4.10) and preceding, then

$$Q_n(s(Y)\leq b) \to \int_{\{v\leq b\}} e^w dF_n(v,w) \to \int_{\{v\leq b\}} e^w dF(v,w)$$

$$= \int_{\{v\leq b\}} \frac{e^w}{|\Sigma_{s,L}|^{\frac{1}{2}}(2\pi)^{(q+1)/2}} \exp\{-\tfrac{1}{2}[v', w-\sigma^2/2]\Sigma_{s,L}^{-1}[v', w-\sigma^2/2]'\, dv dw$$

Make the transformation

$$\begin{bmatrix} v \\ z \end{bmatrix} = \begin{bmatrix} I & 0 \\ a' & a_o \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}, \quad \text{i.e.} \quad \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} I & 0 \\ \rho'\Sigma^{-1} & a_o^{-1} \end{bmatrix}\begin{bmatrix} v \\ z \end{bmatrix}$$

where $a_o = [\sigma^2/(\sigma^2-\underset{\sim}{\rho}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\rho})]^{1/2}$ and $\underset{\sim}{a}' = -a_o\underset{\sim}{\rho}'\underset{\sim}{\Sigma}^{-1}$. Then

$$\begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{a}' & a_o \end{bmatrix} \underset{\sim}{\Sigma}_{s,L} \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{a}' & a_o \end{bmatrix}' = \begin{bmatrix} \underset{\sim}{\Sigma} & \underset{\sim}{\rho} \\ \underset{\sim}{\rho}' & \sigma^2 \end{bmatrix}$$

After a routine but lengthy calculation

$$Q_n(\underset{\sim}{s}\leq\underset{\sim}{b}) \to \{|\underset{\sim}{\Sigma}|^{1/2}\sigma(2\pi)^{(q+1)/2}\}^{-1}$$

$$\cdot \int_{\{\underset{\sim}{v}\leq\underset{\sim}{b}\}} \exp\{-\frac{1}{2}(\underset{\sim}{v}-\underset{\sim}{\rho})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{v}-\underset{\sim}{\rho})-(2\sigma^2)^{-1}(z-a_oz^2/2-\sigma^2/a_o)\}d\underset{\sim}{v}dw$$

$$= (2\pi)^{-q/2}|\underset{\sim}{\Sigma}|^{-1/2}\int_{\{\underset{\sim}{v}\leq\underset{\sim}{b}\}} \exp\{-\frac{1}{2}(\underset{\sim}{v}-\underset{\sim}{\rho})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{v}-\underset{\sim}{\rho})\}d\underset{\sim}{v}$$

This is the explicit form of the distribution mentioned in the statement of the theorem. Hence the theorem has been proved.

Theorem 4.4:

(i) Under conditions (1.6), (1.8), (1.9), (4.2) and the sequence of alternatives $\{Q_n\}$ *given in (4.1),* $M_n(\underset{\sim}{Y})$ *given by (2.2) converges in law to a chi-square distribution with* q*-degrees of freedom and noncentrality parameter* $\Delta = \underset{\sim}{\rho}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\rho}$. $\underset{\sim}{\rho}' = (\rho_1,\ldots,\rho_q)$ *can be calculated from*

$$\rho_k = \lim_{n\to\infty}\sum_{j=1}^{p}[n^{-1}\sum_{i=1}^{n} x_{ijk}\underset{\sim}{x}'_{ij}\underset{\sim}{\xi}]\int_{-\infty}^{\infty} f'_j(u)\psi[2F_j(u)-1]du$$

(ii) Under (2.7), one of (3.1), (3.4), or (3.6), in addition to

the above conditions, $M_n(\underset{\sim}{Y})$ *given by (2.8) converges in law to the above distribution.*

Proof:

(i) The fact that the limiting distribution is the above mentioned chi-square follows from lemmas 3.3, 3.4, corollary (4.2)-(i), and the corollary on page 5 of Sverdrup (1952).

To evaluate ρ_k, note that from lemmas 4.3 and theorem 4.2-(ii), for all $\varepsilon > 0$, $\lim_{n\to\infty} P[|T_n'-E(W_n)-L_n-\frac{1}{4}\text{ var } W_n|\geq\varepsilon) = 0$. In view of corollary 4.1, $E(W_n) \to -\frac{1}{4}\sigma^2$ and $\text{var } W_n \to \sigma^2$, where σ^2 is given in (4.8). Thus $\lim_{n\to\infty} P(|T_n'-L_n-\frac{1}{2}\sigma^2|\geq\varepsilon) = 0$, and from (4.10) and preceding, $(\underset{\sim}{s}',T_n')$ converges in law to a normal distribution with mean vector $(\underset{\sim}{\rho}',0)$ and covariance matrix $\underset{\sim\sim}{\Sigma}_{S,L}$. But if $\lim_{n\to\infty} \text{cov } (S_k,T_n')$ exists, then by theorem 21a, page 114 of Cramér (1962), $\rho_k = \lim_{n\to\infty} \text{cov } (S_k,T_n')$. Now, from (2.2) and (4.5), and fact that $E(S_k) = E(T_n') = 0$ under H_o,

$$\text{cov } (S_k,T_n') = n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{j'=1}^{p} \underset{\sim}{x}_{ij}'\underset{\sim}{\xi}x_{ij'k}\int_{E_p}\frac{f^{(j)}(\underset{\sim}{Y}_i)}{f(\underset{\sim}{Y}_i)}\psi(F_{j'}^*,(|Y_{ij'}|))$$

$$\cdot \text{ sign } Y_{ij'} \cdot f(\underset{\sim}{Y}_i)d\underset{\sim}{Y}_i$$

If $j \neq j'$, the integral in the above expression is

$$\int_{E_p} f^{(j)}(\underset{\sim}{Y}_i)\psi(F_{j'}^*,(|Y_{ij'}|)) \text{ sign } Y_{ij'}\, d\underset{\sim}{Y}_i = \int_{E_2} f_o^{(1)}(u,v)\psi(F_{j'}^*,(|v|))\text{sign } v\, dudv$$

where $f_o(u,v)$ is the joint density of Y_{ij} and $Y_{ij'}$. Thus

$$= \int_{-\infty}^{\infty} \psi(F_{j'}^{*}(|v|)) \text{ sign } v dv \int_{-\infty}^{\infty} f_o^{(1)}(u,v)du,$$ and using (4.2),

$\int_{-\infty}^{\infty} f_o^{(1)}(u,v)du = f_o(u,v)\Big|_{-\infty}^{\infty} = 0$. For $j = j'$ the integral is similarly evaluated and the result follows.

(ii) The same arguments as in (i) are used except that lemmas 3.7 and 3.8 are referred to instead of lemmas 3.3 and 3.4.

Corollary 4.4:

$$(4.12) \qquad \Delta = \lim_{n\to\infty} \underset{\sim}{\zeta}'(n^{-1}\dot{\underset{\sim}{X}}_n\dot{\underset{\sim}{X}}_n')\underset{\sim}{\Sigma}_n^{-1}(n^{-1}\dot{\underset{\sim}{X}}_n\dot{\underset{\sim}{X}}_n')\underset{\sim}{\zeta}, \quad \textit{where}$$

$$(4.13) \qquad \begin{cases} \dot{\underset{\sim}{X}}_n = (\dot{\underset{\sim}{X}}_{n1},\ldots,\dot{\underset{\sim}{X}}_{nn}) \\ \dot{\underset{\sim}{X}}_{ni} = (\underset{\sim}{x}_{i1}\kappa_1,\ldots,\underset{\sim}{x}_{ip}\kappa_p) \quad \textit{for} \quad i = 1,\ldots,n \\ \kappa_j = \{\int_{-\infty}^{\infty} f_j'(u)\psi[2F_j(u)-1]du\}^{\frac{1}{2}} \quad \textit{for} \quad j = 1,\ldots,p \end{cases}$$

Efficiency and L.S. estimation

The efficiency of the test based on $M_n(\underset{\sim}{Y})$ will be compared to that of the test based on the minimum variance least squares estimates (see Scheffé (1959), page 21). Before a comparison can be made, the least squares test statistic must be shown to converge in law to a chi-square distribution with q-degrees of freedom. To this end, the proof of asymptotic normality of the L.S. estimates will be established.

Consider model (1.3), $\underset{\sim}{Y} = \underset{\sim}{X}_n'\underset{\sim}{\beta} + \underset{\sim}{e}$. Let the $np \times np$ covariance matrix of $\underset{\sim}{Y}$ be $\underset{\sim}{B}_n$. In our case $\underset{\sim}{B}_n = \begin{bmatrix} \underset{\sim}{A} & & \underset{\sim}{O} \\ & \ddots & \\ \underset{\sim}{O} & & \underset{\sim}{A} \end{bmatrix}$ where $\underset{\sim}{A}$ is the $p \times p$ covariance matrix associated with the distribution F. The minimum variance unbiased L.S. estimate $\hat{\underset{\sim}{\beta}}^*$ of $\underset{\sim}{\beta}$ (see Scheffe (1959), page 21) is given by

$$(4.14) \quad \begin{cases} \hat{\underset{\sim}{\beta}}^* = (\underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{X}_n')^{-1} \underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{Y} \\ \\ \hat{\underset{\sim}{\beta}}^* - \underset{\sim}{\beta} = (\underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{X}_n')^{-1} \underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{e} = (\sum_{i=1}^{n} \underset{\sim}{X}_{ni} \underset{\sim}{A}^{-1} \underset{\sim}{X}_{ni}') \sum_{i=1}^{n} \underset{\sim}{X}_{ni} \underset{\sim}{A}^{-1} \underset{\sim}{e}_i \end{cases}$$

Let $\underset{\sim}{A}^{-1} = (a_{jj'})_{j,j'=1,\ldots,p}$.

Then $\underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{e} = [\sum_{i=1}^{n} \sum_{j,j'=1}^{p} x_{ij1} a_{jj'} e_{ij'}, \ldots, \sum_{i=1}^{n} \sum_{j,j'=1}^{p} x_{ijq} a_{jj'} e_{ij'}]$

Suppose

(4.15)

(i) The constants $x_{ijk}^* = \sum_{j'=1}^{p} x_{ij'k} a_{jj'}$ satisfy (1.6)-(ii).

(ii) $\underset{\sim}{X}_n \underset{\sim}{B}_n^{-1}$ satisfies (1.6)-(iii)

(iii) $\underset{\sim}{\Sigma}_n^*$ and $\underset{\sim}{\Sigma}^* = \lim_{n\to\infty} \underset{\sim}{\Sigma}_n^*$ satisfy (1.9) where

$$\underset{\sim}{\Sigma}_n^* = n^{-1} \underset{\sim}{X}_n \underset{\sim}{B}_n^{-1} \underset{\sim}{X}_n'$$

Theorem 4.5:

Under assumptions (1.6)-(i), (iv), and (4.15), *(i)* $n^{-\frac{1}{2}} X_n B_n^{-1} \mathcal{R}$ *converges in law to a normal* $(0, \Sigma^*)$ *distribution,*

(ii) $n^{\frac{1}{2}}(\hat{\beta}_n^* - \beta)$ *converges in law to a normal* $(0, \Sigma^{*-1})$ *distribution,*

(iii) *Under the sequence of alternatives* $Q_n : \beta = n^{-\frac{1}{2}} \xi$, *given in (4.1), the quadratic form* $M_n^*(Y) = nY' B_n^{-1} X_n' \Sigma_n^{*-1} X_n B_n^{-1} Y$ *converges in law to a chi-square distribution with* q*-degrees of freedom and noncentrality parameter* $\Delta^* = \lim_{n\to\infty} \xi' \Sigma^* \xi$
$= \lim_{n\to\infty} \sum_{i=1}^{n} \xi' X_{ni} A^{-1} X_{ni}' \xi$.

Proof:

(i) follows in a manner similar to that of lemma 3.2. *(ii)* and *(iii)* follow by applying the corollary on page 5 of Sverdrup (1952).

Now consider a measure of efficiency due to Pitman and generalized to the multiparameter case in definition 4.1 of Bickel (1965). Equation (5) of Hannan (1956) can be applied if both $M_n(Y)$ and $M_n^*(Y)$ converge in law to chi-square ditributions with the same number of degrees of freedom (which is true under the conditions of theorems 4.4 and 4.5). Thus the A.R.E., e_1, of the above nonparametric test with respect to the L.S. test is

$$e_1 = \frac{\Delta^*}{\Delta} = \lim_{n\to\infty} \frac{n^{-1} \xi' X_n B_n^{-1} X_n' \xi}{\xi' (n^{-1} \dot{X}_n \dot{X}_n') \Sigma_n^{-1} (n^{-1} \dot{X}_n \dot{X}_n') \xi}$$

Some properties of this expression are discussed in section 4 of Bickel(1965).

Another measure of efficiency often used, which enjoys the property of being independent of the sequence of alternatives, is the inverse ratio of the sample sizes needed to obtain the same generalized variances of the estimates on which the tests are based. This will be discussed in Chapter VIII.

CHAPTER V

CONVERGENCE THEOREMS FOR CERTAIN STOCHASTIC PROCESSES

This chapter consists of a number of results concerning weak convergence and convergence in probability. These results will be used in Chapter VI for proving large sample existence and asymptotic normality of the proposed estimates. This chapter is a generalization of the appendix of Koul (1967).

The following assumptions are made on the underlying distribution and the regression scores in addition to those of (1.6) and (2.7).

(5.1)

(i) $f_1'(x)$, and $f_1(x) = F_1'(x)$ exist, are bounded and continuous for all $x \in (-\infty,\infty)$. $f_1(x) = 0$ on at most a finite number of intervals.

(ii) $F(\underset{\sim}{e})$ is such that there exist $\eta_1 > 0$, $\eta \in (\frac{1}{2},1]$ so that for all $a,b,c,d \in (-\infty,\infty)$,

$$P(a\leq e_{ij}\leq b, c\leq e_{ij'}\leq d) \leq \eta_1[P(a\leq e_{ij}\leq b)\cdot P(c\leq e_{ij}\leq d)]^{\eta}$$

(iii) For some $a_o > 0$, $b_o > 0$,

$$(\max_{1\leq i\leq n} x_{ijk}^2)/(\sum_{i=1}^{n} x_{ijk}^2) \leq a_o/n^{b_o} \quad \text{for all } j,k$$

(iv) For some $a_1 > 0$, $b_1 > 0$, either $|x_{ijk}x_{ij'k}| = 0$ or $|x_{ijk}x_{ij'k}| \geq a_1/n^{b_1}$.

Condition (ii) is satisfied by a number of multivariate distributions. For example, it may be shown that if $(e_{ij}, e_{ij'})$ is normal with correlation coefficient ρ, then (ii) is satisfied with $\eta_1 = (1-\rho^2)^{\frac{1}{2}}$, $\eta = 1$. If $(e_{ij}, e_{ij'})$ is symmetric Cauchy (see Feller (1966), page 69) then (ii) is satisfied with $\eta_1 = \sqrt{2\pi}$, $\eta = 3/4$. Condition (iii) is a slightly stronger version of (1.6) - (ii).

For later reference, let us define, where $\underset{\sim}{t} \in E_q$,

$$(5.2) \quad \begin{cases} F^*_{np}(\underset{\sim}{t},x) = (np)^{-1} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{p} I(|Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}| \le x) \\ \mu_{nk}(\underset{\sim}{t},x) = (np)^{-1} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{p} x_{ijk} I(Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t} \le x) \text{ sign } (Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}) \\ H_{np}(\underset{\sim}{t},x) = \psi[\frac{np}{np+1} F^*_{np}(\underset{\sim}{t},x)] \\ \hat{H}_{np}(\underset{\sim}{t},x) = \psi_{np}[\frac{np}{np+1} F^*_{np}(\underset{\sim}{t},x)] \end{cases}$$

Also define (where the expectation, E, is taken for $H_o: \underset{\sim}{\beta} = \underset{\sim}{0}$)

$$(5.3) \quad \begin{cases} F^*(\underset{\sim}{t},x) = EF^*_{np}(\underset{\sim}{t},x) \\ \bar{\mu}_{nk}(\underset{\sim}{t},x) = E\mu_{nk}(\underset{\sim}{t},x) \\ \bar{H}_{np}(\underset{\sim}{t},x) = EH_{np}(\underset{\sim}{t},x) \\ L_{nk}(\underset{\sim}{t},x) = n^{\frac{1}{2}}[\mu_{nk}(\underset{\sim}{t},x)-\bar{\mu}_{nk}(\underset{\sim}{t},x)] \\ Z_n(\underset{\sim}{t},|x|) = [n^{3/2}p/(np+1)][F^*_{np}(\underset{\sim}{t},x)-F^*(\underset{\sim}{t},x)] \end{cases}$$

For any $\underset{\sim}{t}' = (t_1,\ldots,t_q) \in E_q$, $a \in (0,\infty)$, let

$$(5.4)\quad \begin{cases} \|\underset{\sim}{t}\| = \sum_{k=1}^{q} |t_k| \\ V_n(a) = \{t\varepsilon E_q : \|\underset{\sim}{t}\| \le an^{-\frac{1}{2}}\} \\ V(a) = \{t\varepsilon E_q : \|\underset{\sim}{t}\| \le a\} \end{cases}$$

Let us observe that

$$(5.5)\quad \begin{cases} \bar{\mu}_{nk}(\underset{\sim}{t},x) = (np)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}[F_1(x+\underset{\sim}{x}'_{ij}\underset{\sim}{t}) \text{ sign } x \\ \qquad\qquad - 2F(\underset{\sim}{x}'_{ij}\underset{\sim}{t})I(x\ge 0)] \\ F^*(t,x) = \begin{cases} (np)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} [F_1(x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(-x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})] & \text{if } x \ge 0 \\ 0 & \text{if } x < 0\,. \end{cases} \end{cases}$$

Next define the stochastic process

$$(5.6)\quad W_{nk}(\underset{\sim}{t},x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}[I(Y_{ij}-n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}\le x) - F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})]$$

where $\{c_{ijk}\}$ are constants.

Theorem 5.1:

Let $\{x_{ijk}\}$ *and* F *satisfy (1.6), (2.7), and (5.1)-(i), (ii). Let* $\{c_{ijk}\}$ *satisfy (5.1)-(iii) and (iv), and* $0 < \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}^2 < \infty$. *Then for each fixed* k, $\underset{\sim}{t}$, *where* $k = 1,\ldots,q$, $\underset{\sim}{t} \in E_q$, $\{W_{nk}(\underset{\sim}{t},x), -\infty\leq x\leq\infty\} \overset{D}{\to} \{W(x), -\infty\leq x\leq\infty\}$ *where* W *is a Gaussian process with continuous sample paths almost surely on* $[-\infty,\infty]$.

Also, for fixed k, $\underset{\sim}{t}$,

$$\lim_{h\to o} \lim_{n\to\infty} P[\sup_{|x-y|\leq h} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{t},y)|\geq\varepsilon] = 0 \tag{5.7}$$

Proof:

$W_{nk}(\underset{\sim}{t},x)$ is a stochastic process on $[-\infty,\infty]$. Also $W_{nk}(\underset{\sim}{t},-\infty) = W_{nk}(\underset{\sim}{t},+\infty) = 0$ for all n, with probability one. Hence $W_{nk}(\underset{\sim}{t},x) \in D[-\infty,\infty]$ where $D[-\infty,\infty]$ is defined on page 109 of Billingsley (1968).

Define

$$\left\{ \begin{array}{ll} Q_{nk}(x) = W_{nk}(\underset{\sim}{t},F_1^{-1}(x)), & x \in [0,1] \\ \\ F_1^{-1}(x) = \inf\{s:F_1(s)=x\} & \end{array} \right. \tag{5.8}$$

Since $f_1(x) = F_1'(x)$ is continuous, $F_1^{-1}(x)$ is continuous a.e., and hence $Q_{nk}(x) \in D[0,1]$.

The theorem will be proved in the following steps:

(a) $\{Q_{nk}(x):x\epsilon[0,1]\} \overset{D}{\to} \{Q(x):x\epsilon[0,1]\}$ for $Q(x) = W(F_1^{-1}(x))$.
To prove this we must prove *(i)* tightness of $\{Q_{nk}(x)\}$ and *(ii)* convergence of finite dimensional distributions (see page 35 and theorem 6.1 of Billingsley (1968)).

(b) $Q(x)$ has continuous sample functions a.s.

(c) $W(x)$ has continuous sample functions a.s.

(a)-(i) Tightness. Note that from (5.6) and (5.8),

$$Q_{nk}(x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}[I(Y_{ij}-n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}\le F_1^{-1}(x)) - F_1(F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})] \tag{5.9}$$

Consider, for $0 \le x_1 \le x \le x_2 \le 1$, the quantity

$$E^* = E\{[Q_{nk}(x)-Q_{nk}(x_1)]^2[Q_{nk}(x_2)-Q_{nk}(x)]^2\} .$$

It will be shown that theorem 15.4 of Billingsley (1968) is satisfied after obtaining suitable bounds on E^*.

Define

$$\begin{cases} \alpha_{ij} = I[F_1^{-1}(x_1)<Y_{ij}-n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}\le F_1^{-1}(x)] - [p_{ij}(x)-p_{ij}(x_1)] \\ \beta_{ij} = I[F_1^{-1}(x)<Y_{ij}-n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}\le F_1^{-1}(x_2)] - [p_{ij}(x_2)-p_{ij}(x)] \\ \qquad \text{where } p_{ij}(x) = F_1^{-1}(F_1(x)+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}) . \end{cases} \tag{5.10}$$

Hence, $Q_{nk}(x) - Q_{nk}(x_1) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}\alpha_{ij}$

and $Q_{nk}(x_2) - Q_{nk}(x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}\beta_{ij}$.

Thus expanding E^*,

$$(5.11) \qquad E^* = \sum_{\substack{i_\theta=1 \\ (\theta=1,2,3,4)}}^{n} \sum_{j_\theta=1}^{p} c_{i_1j_1k}c_{i_2j_2k}c_{i_3j_3k}c_{i_4j_4k} \cdot E(\alpha_{i_1j_1}\alpha_{i_2j_2}\beta_{i_3j_3}\beta_{i_4j_4}) .$$

Because the $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ are independent for different i, and $E(\alpha_{ij}) = E(\beta_{ij}) = 0$ for all i, j, n, it follows that

$$(5.12) \qquad E(\alpha_{i_1j_1}\alpha_{i_2j_2}\beta_{i_3j_3}\beta_{i_4j_4}) = 0 \quad \text{unless one of}$$

the following holds:

(i) $i_1 = i_2 = i_3 = i_4$

(ii) $i_1 = i_2 \neq i_3 = i_4$

(iii) $i_1 = i_3 \neq i_2 = i_4$

(iv) $i_1 = i_4 \neq i_2 = i_3$.

To find upper bounds for the remaining terms in (5.11), observe that

$$(5.13) \qquad E(\alpha_{i_1j_1}\alpha_{i_2j_2}\beta_{i_3j_3}\beta_{i_4j_4}) \leq \{E(\alpha^2_{i_1j_1}\beta^2_{i_3j_3})E(\alpha_{i_2j_2}\beta_{i_4j_4})\}^{\frac{1}{2}}$$

Set

$$(5.14)\quad \begin{cases} a_{ij} = p_{ij}(x) - p_{ij}(x_1) \\ b_{ij} = p_{ij}(x_2) - p_{ij}(x) \end{cases}$$

and square the expressions in (5.10). Then

$$(5.15)\quad \begin{cases} \alpha_{ij}^2 = I[F_1^{-1}(x_1) \le Y_{ij} - n^{-\frac{1}{2}} \underset{\sim}{x}_{ij}' \underset{\sim}{t} \le F_1^{-1}(x)][1-2a_{ij}] + a_{ij}^2 \\ \beta_{ij}^2 = I[F_1^{-1}(x) \le Y_{ij} - n^{-\frac{1}{2}} \underset{\sim}{x}_{ij}' \underset{\sim}{t} \le F_1^{-1}(x_2)][1-2b_{ij}] + b_{ij}^2 \ . \end{cases}$$

Because $0 \le a_{ij}$, $0 \le b_{ij}$, $a_{ij} + b_{ij} \le 1$, it follows from (5.10) and (5.14) that if $i \neq i'$, $E(\alpha_{ij}^2 \beta_{i'j'}^2) = E(\alpha_{ij}^2)E(\beta_{i'j'}^2)$ $= (a_{ij} - a_{ij}^2)(b_{ij} - b_{ij}^2) \le a_{ij} b_{ij}$. Similarly, $E(\alpha_{ij}^2 \beta_{ij}^2) = a_{ij} b_{ij} (b_{ij}$ $+ a_{ij} - 3a_{ij} b_{ij}) \le 2a_{ij} b_{ij}$. Finally, if $j \neq j'$, using (5.1)-(ii), $E(\alpha_{ij}^2 \beta_{ij'}^2) \le \eta_1 (a_{ij} b_{ij'})^{\eta}$. Thus

$$(5.16)\quad \begin{cases} E(\alpha_{ij}^2 \beta_{i'j'}^2) \le 2a_{ij} b_{i'j'} & \text{if } i \neq i' \\ E(\alpha_{ij}^2 \beta_{ij'}^2) \le \eta_2 (a_{ij} b_{ij'})^{\eta} & \text{where } \eta_2 = \max(\eta_1, 2) \ . \end{cases}$$

Substitution of (5.16) into (5.13), and the result, along with (5.12), into (5.11) yields

$$E^* \le n^{-2}\{\sum_{i=1}^{n}\sum_{j_\theta=1}^{p}|c_{ij_1k}c_{ij_2k}c_{ij_3k}c_{ij_4k}|\eta_2(a_{ij_1}b_{ij_2}a_{ij_3}b_{ij_4})^{\eta/2}$$

$$+\sum_{i\neq i'=1}^{n}\sum_{j_\theta=1}^{n}|c_{ij_1k}c_{ij_2k}c_{i'j_3k}c_{i'j_4k}|2(a_{ij_1}b_{ij_2}a_{i'j_3}b_{i'j_4})^{\frac{1}{2}}\}$$

Set $\gamma_{ij} = p_{ij}(x_2) - p_{ij}(x_1)$. Then, $a_{ij} \le \gamma_{ij}$ and $b_{ij} \le \gamma_{ij}$. If it is noted that for all real numbers a,b,c,d, it is true that $|abcd| \le (a^4+b^4+c^4+d^4)/4$, then

$$E^* \le \eta_2 \sum_{i=1}^{n}\sum_{j_\theta=1}^{p}|n^{-1}c_{ij_1k}c_{ij_2k}|\cdot|n^{-1}c_{ij_3k}c_{ij_4k}|\gamma_{ij}^{2\eta}$$

$$+ 2[\sum_{i=1}^{n}\sum_{j,j'=1}^{n}|n^{-1}c_{ijk}c_{ij'k}|\gamma_{ij}]^2 .$$

From assumption (5.1)-(iii), if $L = \max_{n,j} n^{-1}\sum_{i=1}^{n} c_{ijk}$, $|n^{-1}c_{ij_3k}c_{ij_4k}| \le La_o n^{-b_o} = (a_1 n^{-b_1})^{b_o/b_1}\cdot La_o a_1^{-b_o/b_1}$. From (5.1)-(iv), either $|n^{-1}c_{ij_1k}c_{ij_2k}| = 0$ or $|n^{-1}c_{ij_3k}c_{ij_4k}| \le |n^{-1}c_{ij_1k}c_{ij_2k}|^{b_o/b_1}\cdot La_o a_1^{b_o/b_1}$. Thus if $M = \eta_2^{-1}La_o a_1^{-b_o/b_1}$,

$$E^* \le M \sum_{i=1}^{n}\sum_{j,j'=1}^{p}|n^{-1}c_{ijk}c_{ij'k}|^{1+b_o/b_1}\gamma_{ij}^{2\eta}$$

$$+ 2[\sum_{i=1}^{n}\sum_{j,j'=1}^{p}|n^{-1}c_{ijk}c_{ij'k}|\gamma_{ij}]^2 .$$

Next set $\eta_o = \min(1+b_o/b_1, 2\eta)$, and $M_1 = \max_n \sum_{i=1}^{n}\sum_{j,j'=1}^{p} n^{-1}|c_{ijk}c_{ij'k}|$. Then using the well known inequality, $\sum a_i^{1+\varepsilon} \le (\sum a_i)^{1+\varepsilon}$ if $a_i > 0$ and

$\varepsilon > 0$, and the fact that there exists N such that if $n \geq N$, $\max_{i,j,j'} |n^{-1}c_{ijk}c_{ij'k}| \leq 1$ (a consequence of (1.6)-(ii) and (iii)),

$$E^* \leq M[\sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}|\gamma_{ij}]^{1+\eta_o}$$

$$+ M_1^2 [M_1^{-1} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}|\gamma_{ij}]^{1+\eta_o} .$$

Let $G_n(x) = \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}| F_1(F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})$, and $M_2 = \max (M, M_1^{1-\eta_o})$. Since $\gamma_{ij} = p_{ij}(x_2) - p_{ij}(x_1)$, it follows from (5.10) that $E^* \leq M_2[G_n(x_2)-G_n(x_1)]^{1+\eta_o}$.

Now, following the argument on pages 129-130 in Billingsley (1968), it is seen that

(5.17) $$P(w''(Q_{nk},\delta)\geq\varepsilon) \leq K\varepsilon^{-4}(\textstyle\sum_n'+\sum_n'')$$

where w'' is defined in (14.44) of Billingsley (1968), and $\sum_n'$ and $\sum_n''$ are both sums of the form $\sum_{m=1}^{r} [G_n(z_m)-G_n(z_{m-1})]^{1+\eta_o}$ where $0 \leq z_o \leq \cdots \leq z_r \leq 1$ and $\max_{1\leq m\leq r} |z_m-z_{m-1}| \leq 2\delta$. Now

(5.18) $$|\sum_{m=1}^{r} [G_n(z_m)-G_n(z_{m-1})]|^{1+\eta_o}$$

$$\leq \sup_{1\leq m\leq r} |G_n(z_m)-G_n(z_{m-1})|^{\eta_o}[G_n(1)-G_n(0)]$$

Set $G(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}|x$. Then

$$|G_n(x)-G(x)| \le \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}|\cdot|F_1(F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(F_1^{-1}(x))|$$

$$+ x\,|\sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}| - \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}||$$

$$\le 2\delta_n \sup_x F'(x) \sum_{i=1}^{n} \sum_{j,j'=1}^{p} |n^{-1}c_{ijk}c_{ij'k}| + \varepsilon$$

where $\delta_n = \max_{i,j} n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}$ and ε is arbitrary, provided $n \ge N_\varepsilon$. Clearly the R.H.S. tends to zero uniformly in x as $n \to \infty$. Thus, from (5.18)

$$\sup_{1\le m\le r} |G_n(z_k)-G_n(z_{k-1})|^{\eta_o}\cdot[G_n(1)-G_n(0)]$$

$$\to \sup_{1\le m\le r} |G(z_k)-G(z_{k-1})|^{\eta_o}[G(1)-G(0)]$$

$$\le 2\delta[G(1)-G(0)] \sup_x G'(x) = \text{constant}\cdot\delta \quad .$$

Thus, given $\varepsilon > 0$, $\varepsilon_1 > 0$, it follows from (5.17) that $P(w''(Q_{nk},\delta)\ge\varepsilon) \le \varepsilon_1$ provided n is sufficiently large and δ is sufficiently small. Hence by theorem 15.4 of Billingsley (1968), $\{Q_{nk}(x), x\varepsilon[0,1]\}$ is tight.

(a)-(ii) Asymptotic normality of finite dimensional distributions of $Q_{nk}(x)$.

Consider $\{x_s : s=1,\ldots,r; x_s\varepsilon[0,1], x_s<x_{s+1}\}$. Let

$\underset{\sim}{\gamma}' = (\gamma_1,\ldots,\gamma_r)$ be a given r-vector, and $\underset{\sim}{Q}_n' = (Q_{nk}(x_1),\ldots,Q_{nk}(x_r))$. It is sufficient to show that $\underset{\sim}{\gamma}'\underset{\sim}{Q}_n$ converges in law to a normal distribution. This is accomplished by the Lindeberg-Feller theorem (see Loeve (1955), page 280). The argument is the same as that used to prove lemma 3.2.

(b) Continuity of sample paths of Q(x).

Using (5.9), define

$$(5.19)\quad \begin{cases} \Delta_n(x,\delta) = Q_{nk}(x+\delta) - Q_{nk}(x) \\ \qquad = n^{-\frac{1}{2}} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{p} c_{ijk}\xi_{ij} \qquad \text{where} \\ \xi_{ij} = I(F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}<Y_{ij}\leq F_1^{-1}(x+\delta)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}) \\ \qquad - F_1(F_1^{-1}(x+\delta)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}) + F_1(F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}) . \end{cases}$$

Now, since $E(\xi_{ij}) = 0$ for all i and j, $\text{var}\ \xi_{ij} = E(\xi_{ij}^2) = F_1[F_1^{-1}(x+\delta)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}] - F_1[F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}] - \{F_1[F_1^{-1}(x+\delta)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}]-F_1[F_1^{-1}(x)+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}]\}^2 \to \delta - \delta^2$ uniformly in i and j as n increases, since F_1' is bounded and $\max\limits_{i,j} n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t} \to 0$. Similarly

$$\lim_{n\to\infty} \text{cov}\ (\xi_{ij},\xi_{ij'}) = \int_{E_2} I[F_1^{-1}(x)<u\leq F_1^{-1}(x+\delta)]\cdot I[F_1^{-1}(x)<v\leq F_1^{-1}(x+\delta)]\,d[H_{jj'}(u,v)-F_1(u)F_1(v)]$$

where $H_{jj'}(u,v)$ is defined in (1.10). Thus

$$\text{(5.20)} \qquad \lim_{n\to\infty} \operatorname{var} \Delta_n(x,\delta) = \sum_{j=1}^{p} \sum_{j'=1}^{p} \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} c_{ijk} c_{ij'k} E(\xi_{ij}\xi_{ij'}) .$$

It is evident from (5.1)-(ii) and the above limiting expression for $\operatorname{cov}(\xi_{ij},\xi_{ij'})$ that

$$\lim_{n\to\infty} |E(\xi_{ij}\xi_{ij'})| \le \eta_1 \{P[F_1^{-1}(x)<Y_{ij}\le F_1^{-1}(x+\delta)]P[F_1^{-1}(x)<Y_{ij'}\le F_1^{-1}(x+\delta)]\}^{\eta}+\delta^2$$

$$= \eta_1\delta^{2\eta} + \delta^2$$

This together with (5.20) implies

$$\lim_{n\to\infty} \operatorname{var} \Delta_n(x,\delta) \ge \lim_{n\to\infty} \{ \sum_{j=1}^{p} n^{-1} \sum_{i=1}^{n} c_{ijk}^2(\delta-\delta^2) - \sum_{j\ne j'=1}^{p} n^{-1} \sum_{i=1}^{n} |c_{ijk}c_{ij'k}|$$

$$\cdot (\eta_1\delta^2 + \delta^2)$$

$$\ge \sum_{j=1}^{p} \lim_{n\to\infty} (n^{-1} \sum_{i=1}^{n} c_{ijk}^2)[\delta-\delta^2-p(\eta_1\delta^{2\eta}+\delta^2)] .$$

Since $\eta > 1/2$, there exists δ_o such that $1 = (p+1)\delta_o + p\eta_1\delta_o^{2\eta-1}$, and so for $\delta < \delta_o$, $\lim_{n\to\infty} \operatorname{var} \Delta_n(x,\delta) > 0$. Also, it is evident that

$$\lim_{\delta\to o} \delta^{-1} \lim_{n\to\infty} \operatorname{var} \Delta_n(x,\delta) = \sum_{j=1}^{p} \lim_{n\to\infty} \sum_{i=1}^{n} n^{-1} c_{ijk}^2 .$$

Thus, for $\delta < \delta_o$, $\operatorname{var} \Delta_n(x,\delta) = \delta\tau_\delta < 0$, and $\lim_{\delta\to o} \tau_\delta > 0$. By the arguments of (a)-(ii) of this proof,

$\lim_{n\to\infty} [\mathrm{var}\ \Delta_n(x,\delta)]^{-\frac{1}{2}}\Delta_n(x,\delta) = (\delta\tau_\delta)^{-\frac{1}{2}}\Delta(x,\delta)$ has a standard normal distribution. To show continuity of sample paths, it will be shown that $Q(x)$ satisfies the condition of problem 3, p. 136 in Billingsley (1968), which is

$$(5.21)\qquad \lim_{\delta\to 0}\ \sup_{0\le x\le 1-\delta}\ \delta^{-1}P(|\Delta(x,\delta)|>\varepsilon) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

For a given $\varepsilon > 0$, the above remarks imply

$$P(|\Delta(x,\delta)|>\varepsilon) = P((\delta\tau_\delta)^{-\frac{1}{2}}|\Delta(x,\delta)|>(\delta\tau_\delta)^{-\frac{1}{2}}\varepsilon)$$

$$= \sqrt{2/\pi}\int_{-\infty}^{-\varepsilon/\sqrt{\delta\tau_\delta}} \exp\ (-\tfrac{1}{2}t^2)dt \quad \text{independently of } x.$$

If δ_1 is chosen so that for all $\delta < \delta_1$, $\delta < \varepsilon^2/\tau_\delta$, then $\exp\ (-\frac{1}{2}t^2) \le \exp\ (t/2)$ for $t \in (-\infty,-\varepsilon/\sqrt{\delta\tau_\delta})$ and thus

$$\lim_{\delta\to 0}\ \sup_{0\le x\le 1-\delta}\ \delta^{-1}P(|\Delta(x,\delta)|>\varepsilon) \le 2\sqrt{2/\pi}\ \lim_{\delta\to 0}\ \delta^{-1}\exp\ (-\varepsilon/\sqrt{\delta\tau_\delta})$$

$$= 0\ .$$

Hence (5.21) is satisfied and the sample paths of $Q(x)$ are continuous.

(c) Continuity of sample paths of $W(x)$.

Because of condition (5.1)-(i), $[-\infty,\infty]$ can be partitioned into a finite number of intervals (a_{i-1},a_i), $i = 1,2,\ldots,m$ such that for each given i, either $f_1(x) = 0$ or $f_1(x) > 0$ for $x \in (a_{i-1}, a_i)$.

Assume (a,b), (b,c), and (c,d) are three adjoining intervals of the partition.

Case (i): $f_1(x) = 0$ *over* (b,c).

Clearly $F_1(b) = F_1(c)$ and F_1 is constant over $[b,c]$. Thus according to (5.6), if $\delta_n = \max_{i,j} \underset{\sim}{x}'_{ij}\underset{\sim}{t}$, then $W_{nk}(\underset{\sim}{t}, b_1) = W_{nk}(\underset{\sim}{t}, c_1)$ for $b - \delta_n \leq b_1 \leq c_1 \leq c - \delta_n$. Taking limits as $n \to \infty$ (and hence $\delta_n \to 0$), and using the fact that $f_1(x) = F_1'(x)$ is bounded, along with the Chebychev inequality, it can be shown that $W(b') = W(c')$ if $b \leq b' \leq c' \leq c$. Thus $W(x)$ is continuous over (b,c) and right (left) continuous at $c(b)$.

Case (ii): $f_1(x) > 0$ *over* (b,c).

In this case $F_1^{-1}(x)$ is continuous over (b,c) and left continuous at c. Thus the same result is true for $W(x)$ with probability one, since for $x \in (b,c]$, $W(x) = Q(F_1(x))$. To show right continuity at b, note that $Q(F_1(x)) = W(F_1^{-1}(F_1(x))$ for all x. Then, since $f_1(x) > 0$ on (b,c),

$$\lim_{x \downarrow b} W(x) = \lim_{x \downarrow b} W(F_1^{-1}(F_1(x))) = \lim_{x \downarrow b} Q(F_1(x))$$

$$= Q(F_1(b)) = W(F_1^{-1}(F_1(b))) = W(a^*) \quad \text{almost surely}$$

where $a^* = \inf\{u : F_1(u) = F_1(b)\}$. Thus it is seen that $F_1(a^*) = F_1(b)$ and by case (i), $W(a^*) = W(b)$ with probability one. Hence $\lim_{x \downarrow b} W(x) = W(b)$.

The above two cases yield almost sure continuity of $W(x)$. Then (5.7) is a consequence of theorem (5.1) of Billingsley (1968).

Also, $W(x) = W(F_1^{-1}(F_1(x))$ for all x, since in case (i) it was shown that if $F_1(b) = F_1(c)$, then $W(b) = W(c)$. Thus $Q(F_1(x)) = W(x)$ and since $Q(x)$ is Gaussian, $W(x)$ is Gaussian also.

Lemma 5.1:

Under the conditions of theorem 5.1, for any fixed $t_o \in E_q$ *and* $a \in (0,\infty)$,

$$\lim_{\delta \to o} \lim_{n \to \infty} \sup_{|x-y|<\delta} \sup_{\|\underset{\sim}{t}-\underset{\sim}{t}_o\| \le a} |\bar{J}_n(\underset{\sim}{t},x)-\bar{J}_n(\underset{\sim}{t},y)-\bar{J}_n(\underset{\sim}{t}_o,x)+\bar{J}_n(\underset{\sim}{t}_o,y)| = 0$$

where $\bar{J}_n(\underset{\sim}{t},x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk} F_1(x+n^{-\frac{1}{2}} \underset{\sim}{x}_{ij}' \underset{\sim}{t})$.

Proof:

This is very similar to the proof of lemma A1 in Koul (1967) and hence the details have been omitted.

Lemma 5.2:

Under the conditions of theorem 5.1, for each fixed $\underset{\sim}{t} \in E_q$, *and for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} P[\sup_{-\infty\le x\le\infty} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)|\ge\varepsilon] = 0$$

Proof:

Theorem 8.2, page 55 in Billingsley (1968) is applied. Then the result will follow if it is shown that for all $\varepsilon > 0$

(i) $\lim_{n\to\infty} P_n[|W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)|\ge\varepsilon] = 0$

(ii) $\lim_{n\to\infty} P_n[\sup_{|x-y|\le\delta} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)-W_{nk}(\underset{\sim}{t},y)+W_{nk}(\underset{\sim}{0},y)|\ge\varepsilon] = 0$

(i) and (ii) imply that the stochastic processes $\{|W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)|, -\infty\le x\le\infty\}$ are relatively compact with degenerate process, zero, as its limit. (ii) is an immediate consequence of (5.7). Hence it remains to prove (i).

From (5.6),

$$W_{nk}(\underset{\sim}{t},x) - W_{nk}(\underset{\sim}{0},x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}U_{ij}(\underset{\sim}{t},x)$$

where $U_{ij}(\underset{\sim}{t},x) = I(Y_{ij}\le x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}) - I(Y_{ij}\le x) - F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}) + F_1(x)$

The $U_{ij}(\underset{\sim}{t},x)$ are independent for different i and all have mean 0. Thus

$$\text{var}\ [W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)]$$

$$= n^{-1} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} c_{ijk}c_{ij'k} \text{ cov } [U_{ij}(\underset{\sim}{t},x),U_{ij'}(\underset{\sim}{t},x)]$$

$$\leq n^{-1}p \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ijk}^2 \text{ var } U_{ij}(\underset{\sim}{t},x)$$

$$= n^{-1}p \sum{}^{+} c_{ijk}^2 \text{ var } U_{ij}(\underset{\sim}{t},x) + n^{-1}p \sum{}^{-} c_{ijk}^2 \text{ var } U_{ij}(\underset{\sim}{t},x)$$

where $\sum^{+}(\sum^{-})$ is the sum over the terms where $\underset{\sim}{x}_{ij}'\underset{\sim}{t} \geq 0$ $(\underset{\sim}{x}_{ij}'\underset{\sim}{t}<0)$. If $\underset{\sim}{x}_{ij}'\underset{\sim}{t} \geq 0$, then

$$\text{var } U_{ij}(\underset{\sim}{t},x) = F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t}) - F_1(x) - [F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t})-F_1(x)]^2$$

$$\leq n^{-\frac{1}{2}}\underset{\sim}{x}_{ij}'\underset{\sim}{t} \sup_x f_1(x) \to 0$$

uniformly in i, j and x as n increases. This is a consequence of (1.6)-(ii) and (iii). The same result holds if $\underset{\sim}{x}_{ij}'\underset{\sim}{t} < 0$. Therefore it follows that $\text{var}\ [W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)] \to 0$, and by the Chebychev inequality, (i) is obtained.

Lemma 5.3:

Under the conditions of theorem 5.1, for each $\varepsilon > 0$ *, there exists* $\eta > 0$ *such that for any fixed* $t_o \in E_q$,

$$\lim_{\delta\to o} \lim_{n\to\infty} P_n[\sup_{|x-y|<\delta} \sup_{\|\underset{\sim}{t}-\underset{\sim}{t}_o\|<\eta} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{t},y)-W_{nk}(\underset{\sim}{t}_o,x)$$

$$+ W_{nk}(\underset{\sim}{t}_o,y)|\geq\varepsilon] = 0$$

Proof:

This follows closely the proof of lemma A3 in Koul (1967). At one point the following result is needed.

$$\lim_{\delta\to o}\lim_{n\to\infty} P[\sup_{|x-y|\leq\delta} |W^*_{nk}(\underset{\sim}{t},x)-W^*_{nk}(\underset{\sim}{t},y)|\geq\varepsilon] = 0 \quad \text{for all} \quad \varepsilon > 0$$

where $W^*_{nk}(\underset{\sim}{t},x) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} I(Y_{ij}\leq x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}-n^{-\frac{1}{2}}\eta\|\underset{\sim}{x}_{ij}\|)$

$$- F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}-n^{-\frac{1}{2}}\eta\|\underset{\sim}{x}_{ij}\|)$$

This can be proved in exactly the same way as (5.7) since $\{n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}-n^{-\frac{1}{2}}\eta\|\underset{\sim}{x}_{ij}\| : i=1,\ldots,n;\ j=1,\ldots,p\}$ satisfy the same conditions as $\{n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}: i=1,\ldots,n;\ j=1,\ldots,p\}$.

<u>*Lemma 5.4:*</u>

Under the conditions of theorem 5.1, for all $\varepsilon > 0$ *and* $a \in (0,\infty)$

$$\lim_{\delta\to o}\lim_{n\to\infty} P_n[\sup_{|x-y|\leq\delta}\ \sup_{\underset{\sim}{t}\in V(a)} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{t},y)-W_{nk}(\underset{\sim}{0},x)$$

$$+ W_{nk}(\underset{\sim}{0},y)|\geq\varepsilon] = 0$$

where $V(a)$ *is defined in (5.4).*

Proof:

Similar to lemma A4 in Koul (1967).

Lemma 5.5:

Under the conditions of theorem 5.1, for all $\varepsilon > 0$, *there exists* $\eta > 0$ *such that for fixed* x *and* t_o,

$$\lim_{n\to\infty} P_n[\sup_{\|\underset{\sim}{t}-\underset{\sim}{t}_o\|\leq\eta} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{t}_o,x)|\geq\varepsilon] = 0$$

Proof:

Similar to lemma A5 in Koul (1967).

Lemma 5.6:

Under the conditions of theorem 5.1, for all $\varepsilon > 0$ *and* $x \in [-\infty,\infty]$,

$$\lim_{n\to\infty} P_n[\sup_{\underset{\sim}{t}\in V(a)} |W_{nk}(\underset{\sim}{t},x)-W_{nk}(\underset{\sim}{0},x)|\geq\varepsilon] = 0 .$$

Proof:

Similar to lemma A5 in Koul (1967).

Theorem 5.2:

Under the conditions of theorem 5.1, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n[\sup_{-\infty\le x\le\infty} \sup_{\underset{\sim}{t}\in V_n(a)} |L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{0},x)|\ge\varepsilon] = 0$$

Consequently, $\lim_{n\to\infty} L(\sup_{-\infty\le x\le\infty} \sup_{\underset{\sim}{t}\in V_n(a)} L_{nk}(\underset{\sim}{t},x)) = L'$ *where* L' *is a law determined by a Gaussian process with continuous sample paths almost surely, and* L_{nk} *and* $V_n(a)$ *are defined in (5.3) and (5.4) respectively.*

Proof:

In $W_{nk}(\underset{\sim}{t},x)$, defined by (5.6), let $c_{ijk} = x_{ijk}$, and use (5.2), (5.3), and (5.5) to see that with probability one,

$$L_n(n^{-\frac{1}{2}}\underset{\sim}{t},x) = \begin{cases} W_{nk}(\underset{\sim}{t},x) - 2W_{nk}(\underset{\sim}{t},0) & \text{if } x \ge 0 \\ -W_{nk}(\underset{\sim}{t},x) & \text{if } x < 0 \end{cases}$$

The result then follows as in theorem A4 of Koul (1967).

<u>*Theorem 5.3:*</u>

Under the conditions (1.6), (2.7), and (5.1)-(i) and (ii), for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n[\sup_{-\infty\le x\le\infty} \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{t},|x|)-Z_n(\underset{\sim}{0},|x|)|\ge\varepsilon] = 0$$

where $Z_n(\underset{\sim}{t},x)$ *is defined by (5.3). Consequently*

$$\lim_{n\to\infty} L(\sup_{-\infty\leq x\leq\infty} Z_n(\underset{\sim}{t},x)) = L_1$$

$$\lim_{n\to\infty} L(\sup_{\underset{\sim}{t}\varepsilon V_n(a)} Z_n(\underset{\sim}{t},x):x\varepsilon[-\infty,\infty]) = L(Z(x):x\varepsilon[-\infty,\infty])$$

where Z *is essentially a Gaussian process with continuous sample paths, and* L_1 *is determined by* Z .

Proof:

Set $c_{ijk} = 1$ for all $i,j,k,$ and n. Then from (5.6), (5.3), (5.2), and (5.5),

$$(5.22)\quad \begin{cases} Z_n(\underset{\sim}{t},|x|) = n^{\frac{1}{2}}p/(np+1) \sum_{i=1}^{n} \sum_{j=1}^{p} \{I(|Y_{ij}-n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t}|\leq|x|) \\ \qquad - F_1(x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})+F_1(-x+n^{-\frac{1}{2}}\underset{\sim}{x}'_{ij}\underset{\sim}{t})\} \\ \qquad = \begin{cases} W_{nk}(\underset{\sim}{t},x) - W_{nk}(\underset{\sim}{t},-x) & \text{if } x \geq 0 \\ W_{nk}(\underset{\sim}{t},-x) - W_{nk}(\underset{\sim}{t},x) & \text{if } x < 0 \end{cases} \\ \qquad = W_{nk}(\underset{\sim}{t},|x|) - W_{nk}(\underset{\sim}{t},-|x|) . \end{cases}$$

Then, as in theorem A5 of Koul (1967), result follows.

<u>*Corollary 5.3:*</u>

For each $\varepsilon > 0$ *and* $a > 0$, *there exists* $A > 0$ *such that*

$$\lim_{n\to\infty} P_n[\sup_{-\infty\le x\le\infty}\ \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{t},|x|)|\ge A] \le \varepsilon \ .$$

Proof:

Note that for all $\underset{\sim}{t}$ and x, $|Z_n(\underset{\sim}{t},|x|)|\le|Z_n(\underset{\sim}{t},|x|) - Z_n(\underset{\sim}{0},|x|)| + |Z_n(\underset{\sim}{0},|x|)|$. From theorem 5.3, $\forall\varepsilon>0\ \exists B \ni \lim_{n\to\infty} P_n[\sup_x \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{t},|x|)|\ge B] \le \varepsilon$. Thus

$$P_n[\sup_x \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{t},|x|)|\ge 2B] \le P_n[\sup_x \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{t},|x|)-Z_n(\underset{\sim}{0},|x|)|\ge B]$$

$$+ P_n[\sup_x \sup_{\underset{\sim}{t}\in V_n(a)} |Z_n(\underset{\sim}{0},|x|)|\ge B] \ .$$

If limits as $n \to \infty$ are taken, the first term on the R.H.S. is zero (see theorem 5.3) and the second term is bounded by ε . So, choosing $A = 2B$, the result is immediate.

Now define

$$(5.23)\quad \begin{cases} H_{np}^{-1}(\underset{\sim}{t},y) = \inf\{x\ge 0: H_{np}(\underset{\sim}{t},x)\ge y\} \\ \overline{H}_{np}^{-1}(\underset{\sim}{t},y) = \inf\{x\ge 0: \overline{H}_{np}(\underset{\sim}{t},x)\ge y\} \\ K_{np}^{-1}(\underset{\sim}{t},y) = \inf\{x\ge 0: K_{np}(\underset{\sim}{t},x)\ge y\} \\ \overline{K}_{np}^{-1}(\underset{\sim}{t},y) = \inf\{x\ge 0: \overline{K}_{np}(\underset{\sim}{t},x)\ge y\} \end{cases}$$

where H_{np} and $\overline{H}_{np}$ are defined in (5.2) and (5.3) respectively, and

$$
(5.24)\quad \begin{cases} K_{np}(\underset{\sim}{t},|x|) = (np+1)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} I(|Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}| \le |x|) \\ \qquad = \frac{np}{np+1} F^{*}_{np}(\underset{\sim}{t},|x|) \\ \overline{K}_{np}(\underset{\sim}{t},|x|) = EK_{np}(\underset{\sim}{t},|x|) = (np+1)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} [F_1(|x|+\underset{\sim}{x}'_{ij}\underset{\sim}{t}) \\ \qquad - F_1(-|x|+\underset{\sim}{x}'_{ij}\underset{\sim}{t})] = \frac{np}{np+1} F^{*}(\underset{\sim}{t},|x|) \end{cases}
$$

Lemma 5.7:

Under the conditions of theorem 5.3, for all $\varepsilon^{*} > 0$, *there exists a set* $A \subset [0,1]$ *such that* $\lambda(A) > 1 - \varepsilon^{*}$ *and*

$$
\lim_{n\to\infty} P_n[\sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |H^{-1}_{np}(\underset{\sim}{t},y)-\overline{H}^{-1}_{np}(\underset{\sim}{t},y)| \ge \varepsilon^{*}] = 0
$$

Proof:

Because $f_1(x) = 0$ on at most a finite number of intervals $[a_i,b_i]$, $i = 1,\ldots,m$, $F_1^{-1}(x) = \inf\{u:F(u)=x\}$ is continuous on $(0,1) - \bigcup_{i=1}^{m} \{c_i\}$ where $c_i = F(a_i) = F(b_i)$.

$\forall\ \varepsilon > 0$, $F_1^{-1}(x)$ is continuous and hence uniformly continuous on the compact set $A_\varepsilon = [\varepsilon,1-\varepsilon] - \bigcup_{i=1}^{m} (c_i-\varepsilon,c_i+\varepsilon)$. Now observe that because F_1 is strictly increasing immediately to the right of b_i (left of a_i) for $i = 1,\ldots,m$, $F_1^{-1}(c_i-\varepsilon) < a_i \le b_i < F_1^{-1}(c_i+\varepsilon)$. Thus

$$\{x:f_1(x)>0\} = (-\infty,\infty) - \bigcup_{i=1}^{m} [a_i,b_i]$$

$$\supset [F_1^{-1}(\varepsilon),F_1^{-1}(1-\varepsilon)] - \bigcup_{i=1}^{m} (F_1^{-1}(c_i-\varepsilon),F_1^{-1}(c_i+\varepsilon)) = B_\varepsilon, \text{ say.}$$

Because B_ε is compact, $\exists\delta > 0 \ni f_1(x) > 2\delta$ if $x \in B_\varepsilon$. Then, from the facts that $f_1'(x)$ is bounded,

$$\lim_{n\to\infty} \max_{1\le i\le n} \max_{1\le j\le p} \sup_{\underset{\sim}{t}\in V_n(a)} |\underset{\sim}{x}_{ij}'\underset{\sim}{t}| = 0,$$

and $f_1(x)$ is symmetric, $\exists N > 0 \ni \forall n \ge N$,

$$\inf_{x\le|u|\le y} \inf_{\substack{1\le i\le n \\ 1\le j\le p}} \inf_{\underset{\sim}{t}\in V_n(a)} f(u+\underset{\sim}{x}_{ij}'\underset{\sim}{t}) > \delta$$

where $x, y \in B_\varepsilon$ and $0 < y - x < \eta$ where

$$\eta = \min_{1\le i\le m} [F_1^{-1}(c_i+\varepsilon)-F_1^{-1}(c_i-\varepsilon)]$$

Using (5.24), this implies that $\forall\underset{\sim}{t} \in V_n(a)$,

$$\overline{K}_{np}(\underset{\sim}{t},y) - \overline{K}_{np}(\underset{\sim}{t},x) \ge \delta(y-x) \quad \text{if } x, y \in B_\varepsilon \text{ and } 0 \le y - x < \eta \tag{5.25}$$

From the definition of B_ε, it is clear that $B_{2\varepsilon} \subset B_\varepsilon$ and that

$$B_{2\varepsilon} \subset \bigcup_{i=1}^{m} [F_1^{-1}(c_{i-1}+2\varepsilon), F_1^{-1}(c_i-2\varepsilon)] \cap [F_1^{-1}(2\varepsilon), F_1^{-1}(1-2\varepsilon)]$$

$$B_\varepsilon^- \subset \bigcup_{i=1}^{m} [F_1^{-1}(c_i-\varepsilon), F_1^{-1}(c_i+\varepsilon)] \cup [-\infty, F_1^{-1}(\varepsilon)] \cup [F_1^{-1}(\varepsilon), \infty]$$

Thus $B_{2\varepsilon}$ and B_ε^- are subsets of disjoint compact subsets in the extended real line. Hence for some θ,

$$\inf \{|x-y| : x \notin B_\varepsilon, y \in B_{2\varepsilon}\} = \theta > 0$$

From (5.22) and (5.24), $K_{np}(\underset{\sim}{t},x) - \overline{K}_{np}(\underset{\sim}{t},x) = n^{\frac{1}{2}} Z_n(\underset{\sim}{t},x)$. Hence corollary 5.3 implies that $\forall \varepsilon_1 > 0, \exists N_o \ni \forall n \geq N_o$

(5.26) $$P_n[\sup_x \sup_{\underset{\sim}{t} \varepsilon V_n(a)} |K_{np}(\underset{\sim}{t},x) - \overline{K}_{np}(\underset{\sim}{t},x)| < \eta_o] \geq 1 - \varepsilon_1$$

where $\eta_o = \min(\eta, \theta)$

Since $\overline{K}_{np}(\underset{\sim}{t},x)$ is continuous, the following two statements are valid with probability at least $1 - \varepsilon_1$ if $n \geq N_o$.

(i) $\exists y = y(x) \varepsilon B_\varepsilon \ni K_{np}(\underset{\sim}{t},x) = \overline{K}_{np}(\underset{\sim}{t},y) \; \forall x \varepsilon B_{2\varepsilon}, \forall \underset{\sim}{t} \varepsilon V_n(a)$.

Hence by (5.25) and (5.26),

(ii) $\exists y = y(x) \varepsilon B_\varepsilon \ni \delta|y-x| \leq |\overline{K}_{np}(\underset{\sim}{t},x) - \overline{K}_{np}(\underset{\sim}{t},y)| < \eta_o \; \forall x \varepsilon B_{2\varepsilon}$, $\forall \underset{\sim}{t} \varepsilon V_n(a)$.

Now let $r \varepsilon A_{2\varepsilon}$, $y_1 = K_{np}^{-1}(\underset{\sim}{t},r) = \overline{K}_{np}^{-1}(\underset{\sim}{t},r^*)$, and $y_2 = \overline{K}_{np}^{-1}(\underset{\sim}{t},r)$. Then $y_1 \varepsilon B_{2\varepsilon}$, and from (ii), the following string of inequalities is valid $\forall \underset{\sim}{t} \in V_n(a)$, and $\forall y_1 \in B_{2\varepsilon}$ with probability at least $1 - \varepsilon_1$ if $n \geq N_o$.

$$|K_{np}^{-1}(\underset{\sim}{t},r)-\overline{K}_{np}^{-1}(\underset{\sim}{t},r)| = |\overline{K}_{np}^{-1}(\underset{\sim}{t},r^*)-\overline{K}_{np}^{-1}(\underset{\sim}{t},r)| = |y_1-y_2|$$

$$\leq \delta^{-1}|\overline{K}_{np}(\underset{\sim}{t},y_1)-K_{np}(\underset{\sim}{t},y_2)| + \delta^{-1}|K_{np}(\underset{\sim}{t},y)-\overline{K}_{np}(\underset{\sim}{t},y_2)|$$

Using corollary (5.3), it is seen that $\exists \alpha > 0 \ni$

$$P_n[\sup_{0\leq y_1\leq 1} \sup_{\underset{\sim}{t}\in V_n(a)} |\overline{K}_{np}(\underset{\sim}{t},y_1)-K_{np}(\underset{\sim}{t},y_2)| \geq n^{-\frac{1}{2}}\alpha\} \leq \varepsilon_1$$

Also, since the jump at each discontinuity of K_{np} is $(np)^{-1}$, $|K_{np}(\underset{\sim}{t},y_1)-K_{np}(\underset{\sim}{t},y_2)| \leq (np)^{-1}$ and the previous string of inequalities yields, $\forall \varepsilon^* > 0$,

$$P_n[\sup_{r\in A_{2\varepsilon}} \sup_{\underset{\sim}{t}\in V_n(a)} |K_{np}^{-1}(\underset{\sim}{t},r)-\overline{K}_{np}^{-1}(\underset{\sim}{t},r)|>\varepsilon^*] \leq 2\varepsilon_1$$

and $\lambda(A_{2\varepsilon}) \geq 1 - (2m+2)\varepsilon$. Since ε is arbitrary and ε_1 can be made arbitrarily small as long as n is sufficiently large, the result follows.

Lemma 5.8:

Suppose $\psi(x)$ *satisfies (1.8)-(i) and (ii),* ψ' *is bounded and* $\psi^{-1}(u) = \inf\{x:\psi(x)=u\}$ *is piecewise absolutely continuous, i.e. there is a finite set of points*

$\psi(0) = a_o < a_1 < \ldots < a_{r+1} = \psi(1)$ *such that* $\psi^{-1}(u)$ *is absolutely continuous on* (a_i, a_{i+1}), $i = 0,1,\ldots,r$ *.Then, under conditions (1.6), (2.7), and (5.1)-(i) and (ii), for each* $\varepsilon > 0$ *there is a set* $A \subset [\psi(0),\psi(1)]$ *such that* $\lambda(A) > \psi(1) - \psi(0) - \varepsilon$, *where* λ *is the Lebesgue measure, and*

$$\lim_{n\to\infty} \sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}^{-1}(\underset{\sim}{t},y)-\overline{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| = 0$$

Proof:

First consider the case where ψ^{-1} si absolutely continuous on $[\psi(0),\psi(1)]$. From Taylor's expansion

$$(5.27)\quad \begin{cases} \overline{H}_{np}(\underset{\sim}{t},x) = E\psi[\frac{np}{np+1} F^*_{np}(\underset{\sim}{t},x)] \\ \qquad = \psi[\frac{np}{np+1} F^*(\underset{\sim}{t},x)] + E \frac{np}{np+1} [F^*_{np}(\underset{\sim}{t},x)-F^*(\underset{\sim}{t},x)] \\ \qquad \cdot \psi'[\frac{\theta np}{np+1} F^*_{np}(\underset{\sim}{t},x)+ \frac{(1-\theta)np}{np+1} F^*(\underset{\sim}{t},x)]\} \end{cases}$$

where $\theta = \theta(x) \varepsilon (0,1)$.

Thus, corollary A1 and the fact that ψ' is bounded imply that for each $\theta > 0$ there exists N such that for all $n \geq N_\theta$, $\underset{\sim}{t} \varepsilon V_n(a)$, the following holds

$$\overline{H}_{np}(\underset{\sim}{t},x) = \psi(\overline{K}_{np}(\underset{\sim}{t},x)) + \theta(\underset{\sim}{t},x)$$

where $\sup_x \sup_{\underset{\sim}{t} \in V_n(a)} |\theta(\underset{\sim}{t},x)| < \theta$. Hence, for $y \in [0,1]$,

$y = \overline{H}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y)) = \psi[\overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))] + \theta(\underset{\sim}{t},x)$. Because ψ' is bounded, ψ^{-1} is strictly monotone increasing and hence

$$\psi^{-1}(y-\theta(\underset{\sim}{t},x)) = \overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y)) \tag{5.28}$$

By the absolute continuity of ψ^{-1}, $\forall\delta > 0$, $\exists\eta > 0 \ni$

$$\begin{cases} \sup\limits_{\psi(0)\le y\le\psi(1)} |\psi^{-1}(y-\theta(\underset{\sim}{t},x))-\psi^{-1}(y)| < \delta \\ \\ \qquad \text{provided } \sup\limits_x \sup\limits_{\underset{\sim}{t}\in V_n(a)} |\theta(\underset{\sim}{t},x)| < \eta \end{cases} \tag{5.29}$$

Hence, from (5.26), if $n \geq N_\eta$

$$\begin{cases} \psi^{-1}(y) - \delta(\underset{\sim}{t},y) = \overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y)) \\ \\ \text{where } \sup\limits_{\psi(0)\le y\le\psi(1)} \sup\limits_{\underset{\sim}{t}\in V_n(a)} |\delta(\underset{\sim}{t},y)| < \delta \end{cases} \tag{5.30}$$

Thus

$$\overline{K}_{np}^{-1}[\underset{\sim}{t},\psi^{-1}(y)+\delta(\underset{\sim}{t},y)] = \overline{K}_{np}^{-1}[\underset{\sim}{t},\overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))] \tag{5.31}$$

Because $f_1(x) = 0$ on at most a finite number of intervals, it follows that $F_1^{-1}(x)$ is continuous on $(0,1)$ except perhaps at a finite number of points. Thus $F^{*-1}(\underset{\sim}{0},x)$ is continuous on $[0,1]$

except at a finite number of points, say $0 \leq b_1 < \dots < b_m \leq 1$.

For each $\theta_1 > 0$, $F^{*-1}(\underset{\sim}{0},x)$ is bounded on $[0,1-\theta_1]$. Hence $F^{*-1}(\underset{\sim}{0},x)$ is absolutely continuous, bounded, and strictly increasing on each of the intervals $A_i = [0,1-\theta_i] \cap (b_i,b_{i+1})$ $i = 0,1,\dots,m$, where for convience $b_o = 0$, $b_{m+1} = 1$.

From (1.6), $\exists N_1 \ni \forall n \geq N_1$, $\max_{i,j} \sup_{\underset{\sim}{t} \in V_n(a)} |\underset{\sim}{x}_{ij}' \underset{\sim}{t}| < \delta/(2 \sup_x f_1(x))$.

Then by arguments similar to those leading to (5.25), it is not hard to see that, uniformly for $\underset{\sim}{t} \in V_n(a)$, $F^{*-1}(\underset{\sim}{t},x)$, and hence $\overline{K}_{np}^{-1}(\underset{\sim}{t},x)$, is absolutely continuous and strictly increasing for $x \in \bigcup_{i=1}^{m} B_i$ where

$$B_i = [0,1-\theta_1-\delta] \cap (b_i+\delta, b_i-\delta) \tag{5.32}$$

Consider the L.H.S. of (5.31). From (5.32), $\overline{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y)+\delta(\underset{\sim}{t},y))$ may fail to be absolutely continuous for some $\underset{\sim}{t} \in V_n(a)$ only if $\psi^{-1}(y) - \delta(\underset{\sim}{t},y) \in [0,1] - \bigcup_{i=o}^{m} B_i$ $= (1-\theta_1-\delta,1] - \bigcup_{i=o}^{m} (b_i+\delta, b_{i+1}-\delta)$. Recalling (5.30), this implies $\psi^{-1}(y) \in (1-\theta_1-2\delta,1] - \bigcup_{i=o}^{m} (b_i+2\delta, b_{i+1}-2\delta)$, and since ψ^{-1} is monotone increasing and $\psi' \geq 0$, $y \in (\psi(1-\theta_1-2\delta),\psi(1)] - \bigcup_{i=o}^{m} (\psi(b_i+2\delta),\psi(b_{i+1}-2\delta)) = B$, say.

From the absolute continuity of ψ, and the fact that θ_1 and δ can be chosen arbitrarily small provided n is sufficiently large, it follows that $\forall\varepsilon > 0, \exists$ a set $B \subset [\psi(0),\psi(1)] \ni \lambda(B) > \psi(1) - \psi(0) - \varepsilon/2$ and an $N > 0 \ni \forall n \geq N$,

$$(5.33) \qquad \sup_{y\varepsilon B} \sup_{t\varepsilon V_n(a)} |\overline{K}_{np}^{-1}[\underset{\sim}{t},\psi^{-1}(y)+\delta(\underset{\sim}{t},y)]-\overline{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| < \varepsilon$$

Now consider the R.H.S. of (5.31). From (5.32), it is seen that $\overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))$ is strictly increasing for all $\underset{\sim}{t} \varepsilon V_n(a)$ if, for each such $\underset{\sim}{t}$, $\overline{H}_{np}^{-1}(\underset{\sim}{t},y) = x_t \varepsilon \bigcup_{i=1}^{m} B_i$, i.e. $y = \overline{H}_{np}(\underset{\sim}{t},x_t)$. Define

$$A^* = \{y:y = \overline{H}_{np}(\underset{\sim}{t},x) \quad \text{for some} \quad x \notin \bigcup_{i=1}^{m} B_i, \quad \underset{\sim}{t} \varepsilon V_n(a)\}$$

$$= \{y:y = \overline{H}_{np}(\underset{\sim}{t},x) \quad \text{for some} \quad x \varepsilon \bigcup_{i=o}^{m} (b_i-\delta,b_i+\delta) \cap [0,1], \quad t \varepsilon V_n(a)\}$$

From the remark above, it is seen that A^* is a superset of those y such that $\overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))$ is not strictly increasing for all $\underset{\sim}{t} \varepsilon V_n(a)$. It will be shown that $\lambda(A^*)$ can be made arbitrarily small.

Observe from (5.24), (5.2), and (5.3) that

$$\sup_{\underset{\sim}{s},\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}(\underset{\sim}{t},x_1)-\overline{H}_{np}(\underset{\sim}{s},x_2)|$$

$$\leq \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |E\{\psi(K_{np}(\underset{\sim}{t},x_2))-\psi(K_{np}(\underset{\sim}{0},x_2))\}| + |E\{\psi(K_{np}(\underset{\sim}{0},x_1))$$

$$+ \psi(K_{np}(\underset{\sim}{0},x_2))\}| + \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |E\{\psi(K_{np}(\underset{\sim}{t},x_2))-\psi(K_{np}(\underset{\sim}{0},x_2))\}|$$

$$\leq \sup_x \psi'(x)\{ \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |F^*(\underset{\sim}{t},x_1)-F^*(\underset{\sim}{0},x_1)|+|F^*(\underset{\sim}{0},x_1)-F^*(\underset{\sim}{0},x_2)|$$

$$+ \sup_{\underset{\sim}{s}\varepsilon V_n(a)} |F^*(\underset{\sim}{s},x_2)-F^*(\underset{\sim}{0},x_2)|\}$$

Using the fact that $F^*(\underset{\sim}{t},x)$ is absolutely continuous in x, and applying corollary A3, it is seen that $\forall\varepsilon > 0 \exists \eta > 0$ and $N_1 > 0 \ni \forall n \geq N_1$

$$\text{(5.34)} \qquad \sup_{\underset{\sim}{s},\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}(\underset{\sim}{t},x_1)-\overline{H}_{np}(\underset{\sim}{s},x_2)| < \varepsilon/2m$$

if $|x_1-x_2| < \eta$. Thus if $\delta < \eta/2$, then $\forall n \geq N_1$

$$\sup_{\underset{\sim}{s},\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}(\underset{\sim}{t},b_i+\delta)-\overline{H}_{np}(\underset{\sim}{s},b_i-\delta)| < \varepsilon/2m \quad i = 1,\ldots,m.$$

Let $C_{\delta,i} = \{y\varepsilon[\psi(0),\psi(1)]:y=\overline{H}_{np}(\underset{\sim}{t},x)$ for some $x\varepsilon(b_i-\delta,b_i+\delta) \cap [0,1]$, $\underset{\sim}{t} \varepsilon V_n(a)\}$.

Then $A^* = \bigcup_{i=1}^{m} C_{\delta,i}$ and $\lambda(A^*) \leq \sum_{i=1}^{m} \lambda(C_{\delta,i}) \leq \varepsilon/2$ if $n \geq N_1$.

Thus it can be concluded that

$$\overline{K}_{np}^{-1}[\underset{\sim}{t},\overline{K}_{np}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))] = \overline{H}_{np}^{-1}(\underset{\sim}{t},y)$$

except possibly for $y \varepsilon A_1$. Finally, combining this equation with (5.31) and (5.33) yields

(5.35) $$\sup_{y\varepsilon B-A^*} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}^{-1}(\underset{\sim}{t},y)-\overline{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| \leq \varepsilon$$

for n sufficiently large. The result follows since $B - A^* \subset [\psi(0),\psi(1)]$ and $\lambda(B-A^*) \geq \psi(1) - \psi(0) - \varepsilon$.

For the case where ψ^{-1} is absolutely continuous only on (a_i,a_{i+1}) $i = 0,\ldots,m$, (the $\{a_i\}$ are defined in the statement of this lemma), the above arguments can be repeated. Then (5.29) becomes

$$\sup_{y\varepsilon Q} |\psi^{-1}(y-\theta(\underset{\sim}{t},x))-\psi^{-1}(y)| < \delta \quad \text{where}$$

$$Q = [\psi(0),\psi(1)] - \bigcup_{i=1}^{r} (a_i-\eta,a_i+\eta).$$

Finally, corresponding (5.35), it will follow that

$$\sup_{y\varepsilon(B-A^*)\cap Q} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\overline{H}_{np}^{-1}(\underset{\sim}{t},y)-\overline{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| \leq \varepsilon$$

for n sufficiently large. Since η can be made arbitrarily small so long as n is large, $\lambda(Q)$ can be made arbitrarily close to $\psi(1) - \psi(0)$, from which the result follows.

Theorem 5.4:

Assume conditions (1.8)-(i) and (ii), (1.6), (2.7), and (5.1)-(i) and (ii) hold. Also assume ψ' *is bounded and* ψ^{-1} *is piecewise absolutely continuous. Then for each* $\varepsilon > 0$, *there exists a set* $A \subset [\psi(0),\psi(1)]$ *such that* $\lambda(A) > \psi(1) - \psi(0) - \varepsilon$ *and*

$$\lim_{n\to\infty} P_n[\sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |H_{np}^{-1}(\underset{\sim}{t},y)-\bar{H}_{np}^{-1}(\underset{\sim}{t},y)|\geq\varepsilon] = 0$$

Proof:

From (5.2) and (5.24), $H_{np}(\underset{\sim}{t},x) = \psi[K_{np}(\underset{\sim}{t},x)]$ for $x \geq 0$. Since ψ^{-1} is monotone increasing and one-to-one from $[\psi(0),\psi(1)]$ onto $[0,1]$,

$$\begin{aligned} H_{np}^{-1}(\underset{\sim}{t},y) &= \inf\{x\geq 0: H_{np}(\underset{\sim}{t},x)\geq y\} \\ &= \inf\{x\geq 0:\psi[K_{np}(\underset{\sim}{t},x)]\geq y\} \\ &= \inf\{x\geq 0:K_{np}(\underset{\sim}{t},x)\geq\psi^{-1}(y)\} \end{aligned}$$

Thus $H_{np}^{-1}(\underset{\sim}{t},y) = K_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))$. Now consider

$$\begin{aligned} |H_{np}^{-1}(\underset{\sim}{t},y)-\bar{H}_{np}^{-1}(\underset{\sim}{t},y)| &\leq |H_{np}^{-1}(\underset{\sim}{t},y)-K_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| \\ &+ |K_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))-\bar{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))| + |\bar{K}_{np}^{-1}(\underset{\sim}{t},\psi^{-1}(y))-\bar{H}_{np}^{-1}(\underset{\sim}{t},y)| \end{aligned}$$

If the previous equality, lemma 5.7, and lemma 5.8 are applied respectively to the terms on the right of the above inequality, it is found that the result follows.

Theorem 5.5:

Assume conditions (1.8)-(i) and (ii), (2.7), and (5.1) hold. Also assume ψ' *is bounded and* ψ^{-1} *is piecewise absolutely continuous. Then, for every* $\varepsilon > 0$, *there is a set* $A \subset [\psi(0),\psi(1)]$ *such that* $\lambda(A) > \psi(1) - \psi(0) - \varepsilon$ *and*

$$\lim_{n\to\infty} P_n[\sup_{x\in A} \sup_{\underset{\sim}{t}\in V_n(a)} |L_{nk}(\underset{\sim}{t},H_{np}^{-1}(\underset{\sim}{t},x))-L_{nk}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},x))|\geq\varepsilon] = 0$$

where $L_{nk}(\underset{\sim}{t},x)$ *is defined in (5.3).*

Proof:

Observe that $|L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{t},y)| \leq |L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{0},x)| + |L_{nk}(\underset{\sim}{0},x)-L_{nk}(\underset{\sim}{0},y)| + |L_{nk}(\underset{\sim}{0},y)-L_{nk}(\underset{\sim}{t},y)|$.

If in theorems 5.1 and 5.2, $c_{ijk} = x_{ijk}$, it follows that $\forall\varepsilon > 0$, $\forall\eta > 0$, $\exists N > 0$ and $\delta > 0 \ni$ if $n \geq N$,

$$P_n[\sup_{\underset{\sim}{t}\in V_n(a)} |L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{0},x)|\geq\varepsilon/3] \leq \eta/6$$

$$P_n[\sup_{|x-y|\leq\delta} |L_{nk}(\underset{\sim}{0},x)-L_{nk}(\underset{\sim}{0},y)|\geq\varepsilon/3] \leq \eta/6$$

$$P_n[\sup_{\underset{\sim}{t}\in V_n(a)} |L_{nk}(\underset{\sim}{0},y)-L_{nk}(\underset{\sim}{t},y)|\geq\varepsilon/3] \leq \eta/6$$

Thus, for $n \geq N$

$$(5.36) \qquad P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} \sup_{|x-y|\leq\delta} |L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{t},y)|\geq\varepsilon] \leq \eta/2$$

By theorem 5.4, N_1 can be chosen large enough so that for all $n \geq N_2 = \max(N,N_1)$

$$(5.37) \qquad P_n[\sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |H_{np}^{-1}(\underset{\sim}{t},y)-\overline{H}_{np}^{-1}(\underset{\sim}{t},y)|\geq\delta] \leq \eta/2$$

where $A \subset [\psi(0),\psi(1)]$ and $\lambda(A) \geq \psi(1) - \psi(0) - \eta$. Hence

$$(5.38) \qquad \sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |L_{nk}(\underset{\sim}{t},H_{np}^{-1}(\underset{\sim}{t},y))-L_{nk}(\underset{\sim}{t},\overline{H}_{np}^{-1}(\underset{\sim}{t},y))| \leq \varepsilon$$

unless either

(i) $\sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |H_{np}^{-1}(\underset{\sim}{t},y)-\overline{H}_{np}^{-1}(\underset{\sim}{t},y)| \leq \delta$ and (5.35) is false, or

(ii) $\sup_{y\varepsilon A} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |H_{np}^{-1}(\underset{\sim}{t},y)-\overline{H}_{np}^{-1}(\underset{\sim}{t},y)| > \delta$ and (5.38) is false.

The use of (5.36) and (5.37) shows that the union of the two events described in (i) and (ii), respectively, has probability not exceeding η if $n \geq N_2$. Hence the theorem follows.

CHAPTER VI

LIMITING DISTRIBUTION AND LARGE SAMPLE EXISTENCE OF THE ESTIMATE, $\hat{\beta}_n$, FOR THE JOINT RANKING CASE

In this chapter the existence and asymptotic normality of $\hat{\beta}_n$ are discussed. It is shown that the region $R_n(\underset{\sim}{Y})$ defined in (2.3) is bounded, and hence its centre of gravity, $\hat{\beta}_n$, exists, with probability tending to one as n increases. To show boundedness of $R_n(\underset{\sim}{Y})$, $M_n(\underset{\sim}{Y})$ is approximated by another quadratic form about which more is known.

The following assumptions, which are stronger than those of (1.8) and lemma 5.8, are made on the score function ψ. Assume that ψ is defined on $[-1,1]$, and

$$
(6.1)\quad \begin{cases}
\text{(i)} & \psi' \text{ is defined on } [-1,1] \text{ and } \psi'(x) \geq 0 \\
\text{(ii)} & \psi(x) = -\psi(-x) \text{ for } x \in [-1,1] \\
\text{(iii)} & 0 < V_{[0,1]}\psi'(x) < \infty \\
\text{(iv)} & \psi'' \text{ is defined on } (-1,0) \cup (0,1) \text{ and} \\
 & V_{(0,z)}\psi''(x) \leq (1-z)^{-1}\theta(z) \text{ for } 0 \leq z < 1 \\
\text{(v)} & \sup_{x\varepsilon(0,z)} |\psi''(x)| \leq (1-z)^{-\frac{1}{2}}\theta(z) \text{ for } 0 \leq z < 1
\end{cases}
$$

where $V_{[a,b]}\psi(x)$ is the total variation of $\psi(x)$ over $[a,b]$, and

$\theta(z)$ is a finite real valued function on $[0,1]$ such that $\lim_{z\to 1} \theta(z) = 0$

For simplicity, V will be written for $V_{[-\infty,\infty]}$.

Recalling (5.1) and (2.8), it is seen that

$$(6.2)\quad \begin{cases} \int_{-\infty}^{\infty} \hat{H}_{np}(\underset{\sim}{t},|x|)\,d\mu_{nk}(\underset{\sim}{t},x) = \sum_{i=1}^{n}\sum_{j=1}^{p} \psi_{np}(r_{ij}/(np+1)) \\ \cdot\,(x_{ijk}/np)\ \mathrm{sign}\ (Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}) = (p\sqrt{n})^{-1} T_k(\underset{\sim}{Y}-\underset{\sim}{X}'_n\underset{\sim}{t}) \\ \text{where } r_{ij} = \text{rank of } |Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}| \text{ in joint ranking} \\ \text{over } i = 1,\ldots,n;\ j = 1,\ldots,p\,. \end{cases}$$

In the following, $T_k(\underset{\sim}{t})$ will be written in place of $T_k(\underset{\sim}{Y}-\underset{\sim}{X}'_n\underset{\sim}{t})$ to emphasize the dependence on $\underset{\sim}{t}$.

Let us define, where $\overline{\mu}_{nk}$ and $\overline{H}_{np}$ are given in (5.3) and (5.5),

$$(6.3)\qquad A_{nk}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \overline{H}_{np}(\underset{\sim}{t},|x|)\,d\overline{\mu}_{nk}(\underset{\sim}{t},x)$$

Theorem 6.1:

If conditions (1.6), (2.7), (5.1), (6.1), $H_o:\underset{\sim}{\beta} = \underset{\sim}{0}$ *, and*

$$(6.4)\quad \sup_{\underset{\sim}{t}\in V_n(a)} \Big|\int_{-\infty}^{\infty} \{\psi_{np}[\tfrac{np}{np+1} F^*_{np}(\underset{\sim}{t},|x|)]-\psi[\tfrac{np}{np+1} F^*_{np}(\underset{\sim}{t},|x|)]\}\, d\mu_{nk}(\underset{\sim}{t},x) - \int_{-\infty}^{\infty} \{\psi_{np}[\tfrac{np}{np+1} F^*_{np}(\underset{\sim}{0},|x|)] - \psi[\tfrac{np}{np+1} F^*_{np}(\underset{\sim}{0},|x|)]\}\,d\mu_{nk}(\underset{\sim}{0},x)\Big| = o_p(n^{-\frac{1}{2}})$$

are satisfied, then for each $\varepsilon > 0$, *there is* $N > 0$ *so that*

$$P_n\{\sup_{t\varepsilon V_n(a)} |[(p\sqrt{n})^{-1}T_k(\underset{\sim}{t})-A_{nk}(\underset{\sim}{t})]-[(p\sqrt{n})^{-1}T_k(\underset{\sim}{0})-A_k(\underset{\sim}{0})|\geq\varepsilon\} \leq \varepsilon,$$

provided $n \geq N$.

Proof:

From (6.2) and (6.3),

$$(p\sqrt{n})^{-1}T_k(\underset{\sim}{t}) - A_{nk}(\underset{\sim}{t}) = B_{n0}(\underset{\sim}{t}) + B_{n1}(\underset{\sim}{t}) + B_{n2}(\underset{\sim}{t}) + R_n(\underset{\sim}{t})$$

where

$$(6.5)\begin{cases} B_{n0}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \{\hat{H}_{np}(\underset{\sim}{t},|x|)-H_{np}(\underset{\sim}{t},x)\}d\mu_{nk}(\underset{\sim}{t},x) \\ B_{n1}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \overline{H}_{np}(\underset{\sim}{t},|x|)d\{\mu_{nk}(\underset{\sim}{t},x)-\overline{\mu}_{nk}(\underset{\sim}{t},x)\} \\ B_{n2}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \{H_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{t},|x|)\}d\overline{\mu}_{nk}(\underset{\sim}{t},x) \\ R_n(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \{H_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{t},|x|)\}d\{\mu_{nk}(\underset{\sim}{t},x)-\overline{\mu}_{nk}(\underset{\sim}{t},x)\} \end{cases}$$

The proof will be complete if $\forall\varepsilon > 0 \ \exists \ N > 0 \ni$ if $n \geq N$, the following four inequalities hold

$$(6.6) \qquad P_n\{\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|B_{n0}(\underset{\sim}{t})-B_{n0}(\underset{\sim}{0})|\geq\varepsilon\} \leq \varepsilon$$

$$(6.7)\qquad P_n\{\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|B_{n1}(\underset{\sim}{t})-B_{n1}(\underset{\sim}{0})|\geq\varepsilon\}\leq\varepsilon$$

$$(6.8)\qquad P_n\{\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|B_{n2}(\underset{\sim}{t})-B_{n2}(\underset{\sim}{0})|\geq\varepsilon\}\leq\varepsilon$$

$$(6.9)\qquad P_n\{\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|R_n(\underset{\sim}{t})-R_n(\underset{\sim}{0})|\geq\varepsilon\}\leq\varepsilon$$

Proof of (6.6):

This follows directly from (6.4) and (5.2).

Proof of (6.7):

From (5.3) it is seen that

$$n^{\frac{1}{2}}[B_{n1}(\underset{\sim}{t})-B_{n1}(\underset{\sim}{0})] = \int_{-\infty}^{\infty} \overline{H}_{np}(\underset{\sim}{t},|x|)dL_{nk}(\underset{\sim}{t},x) - \int_{-\infty}^{\infty} \overline{H}_{np}(\underset{\sim}{0},|x|)dL_{nk}(\underset{\sim}{0},x)$$

and integrating by parts

$$= \overline{H}_{np}(\underset{\sim}{t},|x|)L_{nk}(\underset{\sim}{t},x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} L_{nk}(\underset{\sim}{t},x)d\overline{H}_{np}(\underset{\sim}{t},|x|)$$

$$- \overline{H}_{np}(\underset{\sim}{0},|x|)L_{nk}(\underset{\sim}{0},x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} L_{nk}(\underset{\sim}{0},x)d\overline{H}_{np}(\underset{\sim}{0},|x|)$$

From (5.3) and (5.2) it follows that $L_{nk}(\underset{\sim}{t},\pm\infty) = 0$, and $\overline{H}_{np}(\underset{\sim}{t},|x|)$ is monotone increasing in $|x|$. Thus it readily follows that

$$\sup_{\underset{\sim}{t}\varepsilon V_n(a)} |n^{\frac{1}{2}}[B_{n1}(\underset{\sim}{t})-B_{n1}(\underset{\sim}{0})]| \leq \sup_{\underset{\sim}{t}\varepsilon V_n(a)} \{|\int_{-\infty}^{\infty} [L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{0},x)]$$

$$d\bar{H}_{np}(\underset{\sim}{t},|x|)| + |\int_{-\infty}^{\infty} L_{nk}(\underset{\sim}{0},x)\,d[\bar{H}_{np}(\underset{\sim}{t},|x|)-\bar{H}_{np}(\underset{\sim}{0},|x|)]|\}$$

$$\leq \sup_x \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |L_{nk}(\underset{\sim}{t},x)-L_{nk}(\underset{\sim}{0},x)|\cdot\psi(1)$$

$$+ \sup_x |L_{nk}(\underset{\sim}{0},x)| \sup_{\underset{\sim}{t}\varepsilon V_n(a)} V[\bar{H}_{np}(\underset{\sim}{t},|x|)-\bar{H}_{np}(\underset{\sim}{0},|x|)]$$

Applying theorem 5.2 to the first term, and theorem 5.2 and lemma A8 to the second term it is seen that their sum tends to zero in probability as n increases, and hence (6.6) is proved.

Proof of (6.8):

From (6.5), observe that

$$(6.10)\quad \begin{cases} n^{\frac{1}{2}}[B_{n2}(\underset{\sim}{t})-B_{n2}(\underset{\sim}{0})] = n^{\frac{1}{2}}\{\int_{-\infty}^{\infty} [H_{np}(\underset{\sim}{t},|x|)-\bar{H}_{np}(\underset{\sim}{t},|x|)] \\ \cdot d\bar{\mu}_{nk}(\underset{\sim}{t},x) - \int_{-\infty}^{\infty} [H_{np}(\underset{\sim}{0},|x|)-\bar{H}_{np}(\underset{\sim}{0},|x|)]d\bar{\mu}_{np}(\underset{\sim}{0},x) \end{cases}$$

To find bounds on the above expression, first consider the following Taylor series expansion. From (5.2), (5.3), and (5.24),

$$(6.11)\quad \begin{cases} H_{np}(\underset{\sim}{t},|x|) = \psi[K_{np}(\underset{\sim}{t},|x|)] \\ \qquad = \psi[\overline{K}_{np}(\underset{\sim}{t},|x|)] + \psi'[\overline{K}_{np}(\underset{\sim}{t},|x|)][K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)] \\ \qquad + \frac{1}{2}\psi''(y_1)[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)] \end{cases}$$

where y_1 is some real number depending on $\underset{\sim}{t}$ and x and lying between $K_{np}(\underset{\sim}{t},|x|)$ and $\overline{K}_{np}(\underset{\sim}{t},|x|)$. After taking expectations on each side,

$$(6.12)\qquad \overline{H}_{np}(\underset{\sim}{t},|x|) = \psi[\overline{K}_{np}(\underset{\sim}{t},|x|)] + \frac{1}{2} E\{\psi''(y_1)[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)\}$$

In view of (6.11) and (6.12), the first term on the right of (6.10) becomes

$$(6.13)\quad \begin{cases} n^{\frac{1}{2}}\int_{-\infty}^{\infty}\{\psi'[\overline{K}_{np}(\underset{\sim}{t},|x|)]\cdot[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},x)] \\ \qquad + \frac{1}{2}[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)]^2\psi''(y_1) \\ \qquad + \frac{1}{2}E\{[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)]^2\psi''(y_1)\}\}d\overline{\mu}_{nk}(\underset{\sim}{t},x) \end{cases}$$

Now, it follows from (5.5) and (1.6)-(iii) that $\overline{\mu}_{nk}(\underset{\sim}{t},x)$ is of bounded variation uniformly in $\underset{\sim}{t}$ and n. Also, $n^{\frac{1}{2}}[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)] = Z_n(\underset{\sim}{t},|x|)$, and corollary 5.3 shows the supremum over $\underset{\sim}{t}$ and x of this quantity is bounded in probability. From (5.24) it is clear that $y_1 \leq np/(np+1)$.

Thus, from the foregoing remarks, the absolute value of the second term in (6.13) is bounded by $\frac{1}{2} n^{-\frac{1}{2}} A^2 \theta(np/(np+1))$ $\cdot \sup_{\underset{\sim}{t} \varepsilon V_n(a)} \sup_n V[\bar{\mu}_{nk}(\underset{\sim}{t},x)]$ with probability at least $1 - \varepsilon$, where A is defined in corollary 5.3, and θ is defined following (6.1). Hence this term is $o_p(1)$ since $\lim_{n\to\infty} \theta(np/(np+1)) = 0$.

Now consider the third term on the right of (6.13). According to corollary A1, (5.24), and the above remarks, the supremum over $\underset{\sim}{t} \varepsilon V_n(a)$ of this is $o_p(1)$. Thus it follows that (6.13) becomes

$$n^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi'[\bar{K}_{np}(\underset{\sim}{t},|x|)][K_{np}(\underset{\sim}{t},|x|)-\bar{K}_{np}(\underset{\sim}{t},|x|)]d\bar{\mu}_{nk}(\underset{\sim}{t},x) + o_p(1)$$

and substitution into (6.10) yields

$$n^{\frac{1}{2}}[B_{n2}(\underset{\sim}{t})-B_{n2}(\underset{\sim}{0})] = n^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi'[\bar{K}_{np}(\underset{\sim}{t},|x|)]\cdot[K_{np}(\underset{\sim}{t},|x|)- \bar{K}_{np}(\underset{\sim}{t},|x|)]d\bar{\mu}_{nk}(\underset{\sim}{t},x) - n^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi'[\bar{K}_{np}(\underset{\sim}{0},|x|)]\cdot[K_{np}(\underset{\sim}{t},|x|)- \bar{K}_{np}(\underset{\sim}{0},|x|)]d\bar{\mu}_{nk}(\underset{\sim}{0},x) + o_p(1),$$ where $o_p(1)$ is

independent of $\underset{\sim}{t}$ for $\underset{\sim}{t} \varepsilon V_n(a)$. Further, this may be written as

$$(6.14) \qquad n^{\frac{1}{2}}[B_{n2}(\underset{\sim}{t})-B_{n2}(\underset{\sim}{0})] = D_1(\underset{\sim}{t}) + D_2(\underset{\sim}{t}) + D_3(\underset{\sim}{t})$$

where

$$(6.15)\quad \begin{cases} D_1(\underset{\sim}{t}) = n^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi'[\overline{K}_{np}(\underset{\sim}{t},|x|)]\cdot[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|)] \\ \qquad d[\overline{\mu}_{nk}(\underset{\sim}{t},x)-\overline{\mu}_{nk}(\underset{\sim}{0},x)] \\ D_2(\underset{\sim}{t}) = n^{\frac{1}{2}} \int_{-\infty}^{\infty} \psi'[\overline{K}_{np}(\underset{\sim}{t},|x|)]\cdot[K_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{t},|x|) \\ \qquad - K_{np}(\underset{\sim}{0},|x|)+\overline{K}_{np}(\underset{\sim}{0},|x|)]d\overline{\mu}_{nk}(\underset{\sim}{0},x) \\ D_3(\underset{\sim}{t}) = n^{\frac{1}{2}} \int_{-\infty}^{\infty} \{\psi'[\overline{K}_{np}(\underset{\sim}{t},|x|)] - \psi'[\overline{K}_{np}(\underset{\sim}{0},x)]\}\cdot \\ \qquad [K_{np}(\underset{\sim}{0},|x|)-\overline{K}_{np}(\underset{\sim}{0},|x|)]d\overline{\mu}_{nk}(\underset{\sim}{0},x) \end{cases}$$

First observe that

$$\sup_{\underset{\sim}{t}\varepsilon V_n(a)} |D_1(t)| \leq \sup_{0\leq x\leq 1} \psi'(x)\cdot \sup_{x} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|K_{np}(\underset{\sim}{t},|x|)-$$

$$\overline{K}_{np}(\underset{\sim}{t},|x|)| \cdot \sup_{\underset{\sim}{t}\varepsilon V_n(a)} V[\overline{\mu}_{nk}(\underset{\sim}{t},x)-\overline{\mu}_{nk}(\underset{\sim}{0},x)].$$

(6.1), theorem 5.3, (5.3), and (5.24) imply that the product of the first two factors on the right are bounded in probability. Finally, the third factor is bounded by

$$\sup_{\underset{\sim}{t}\varepsilon V_n(a)} (np)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} |x_{ijk}|\{V[F_1(x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(x)]+V[F_1(-x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(-x)]\}.$$

Because $\lim_{n\to\infty} \max_{i,j} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\underset{\sim}{x}'_{ij}\underset{\sim}{t}| = 0$ it follows from lemma A.2 that this tends to zero as n increases and hence $\sup_{\underset{\sim}{t}\varepsilon V_n(a)} D_1(\underset{\sim}{t}) = o_p(1)$.

Now consider $D_2(\underset{\sim}{t})$. From (5.3) and (5.24), its absolute value is bounded by

$$\sup_x \psi'(x) \cdot \sup_x \sup_{\underset{\sim}{t} \varepsilon V_n(a)} |Z_n(\underset{\sim}{t},|x|)-Z_n(\underset{\sim}{0},|x|)| \cdot V[\overline{\mu}_{nk}(\underset{\sim}{0},x)]$$

(6.1)-(iii) and (5.5) imply the first and third factors respectively are bounded, and theorem 5.3 implies the second factor tends to zero in probability. Thus $\sup_{\underset{\sim}{t} \varepsilon V_n(a)} |D_2(\underset{\sim}{t})| = o_p(1)$.

Next, let us observe that $|D_3(\underset{\sim}{t})|$ is bounded by

$$\sup_{0 \le x \le 1} \psi''(npx/(np+1)) \cdot \sup_x \sup_{\underset{\sim}{t} \varepsilon V_n(a)} |\overline{K}_{np}(\underset{\sim}{t},|x|)-\overline{K}_{np}(\underset{\sim}{0},|x|)|$$

$$\cdot \sup_x |Z_n(\underset{\sim}{0},|x|)| \cdot V[\overline{\mu}_{nk}(\underset{\sim}{0},x)].$$

Scrutiny of each factor, with references to (6.1)-(iv), theorem 5.3, and some of the remarks immediately above, it follows that $\sup_{\underset{\sim}{t} \varepsilon V_n(a)} |D_3(\underset{\sim}{t})| = o_p(1)$.

Thus we have shown $\sup_{\underset{\sim}{t} \varepsilon V_n(a)} |D_m(\underset{\sim}{t})| = o_p(1)$ for $m = 1,2,3$. Thus (6.15) and (6.14) imply (6.8).

Proof of (6.9):

It follows from (6.5) and integration by parts that

$$n^{\frac{1}{2}}|R_n(t)| = |n^{\frac{1}{2}}[H_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{t},|x|)][\mu_{nk}(\underset{\sim}{t},x)-\overline{\mu}_{nk}(\underset{\sim}{t},x)]|_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} L_{nk}(\underset{\sim}{t},x)\,d[H_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{t},|x|)].$$

From (5.2) it follows that the first term on the right is zero, thus $n^{\frac{1}{2}}|R_n(\underset{\sim}{t})| \leq R_{n1}(\underset{\sim}{t}) + R_{n2}(\underset{\sim}{t})$ where

$$R_{n1}(\underset{\sim}{t}) = \left|\int_{-\infty}^{0} L_{nk}(\underset{\sim}{t},x)\,d[H_{np}(\underset{\sim}{t},x)-\bar{H}_{np}(\underset{\sim}{t},x)]\right|$$

$$R_{n2}(\underset{\sim}{t}) = \left|\int_{0}^{\infty} L_{nk}(\underset{\sim}{t},x)\,d[H_{np}(\underset{\sim}{t},-x)-\bar{H}_{np}(\underset{\sim}{t},-x)]\right|$$

By changing the variable of integration, and letting $D = [\psi(0),\psi(1)]$, it is seen that

$$R_{n1}(\underset{\sim}{t}) \leq \int_D |L_{nk}(\underset{\sim}{t},H_{np}^{-1}(\underset{\sim}{t},y))-L_{nk}(\underset{\sim}{t},\bar{H}_{np}^{-1}(\underset{\sim}{t},y))|\,dy$$

From theorem 5.2 it follows that $\forall\delta > 0, \exists\alpha \ni P[\sup_x \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |L_{nk}(\underset{\sim}{t},x)|>\alpha]<\delta/2$.

From theorem 4.5, $\forall\varepsilon > 0$, $\exists$ a set $A \subset [0,1]$ and $N > 0 \ni$ $\lambda(A) > \psi(1) - \psi(0) - \varepsilon/2\alpha$, and if $n \geq N$, $P_n\{\sup_x \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |L_{nk}(\underset{\sim}{t},H_{np}^{-1}(\underset{\sim}{t},x)) - L_{nk}(\underset{\sim}{t},\bar{H}_{np}^{-1}(\underset{\sim}{t},x))| > \varepsilon/[2(\psi(1)-\psi(0))]\} < \delta/2$. Applying these remarks to the above integral, $\forall\underset{\sim}{t} \varepsilon V_n(a)$,

$$R_{n1}(\underset{\sim}{t}) = \{\int_A + \int_{D-A}\}|L_{nk}(\underset{\sim}{t},H_{np}^{-1}(\underset{\sim}{t},y))-L_{nk}(\underset{\sim}{t},\bar{H}_{np}^{-1}(\underset{\sim}{t},y))|\,dy$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ with probability at least } 1 - \delta,$$

i.e. $\sup_{\underset{\sim}{t}\varepsilon V_n(a)} R_{n1}(\underset{\sim}{t}) = o_p(1)$.

A very similar argument shows $\sup_{\underset{\sim}{t} \in V_n(a)} R_{n2}(\underset{\sim}{t}) = o_p(1)$.
Hence $\sup_{\underset{\sim}{t} \in V_n(a)} n^{\frac{1}{2}}|R_n(\underset{\sim}{t})| = o_p(1)$, which implies $n^{\frac{1}{2}}|R_n(\underset{\sim}{0})| = o_p(1)$, and these imply (6.9).

Hence (6.6) through (6.9) have been proved and the theorem is true.

Theorem 6.2:

Under conditions (1.6), (2.7), (5.1), (6.1), and $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, $\lim_{n\to\infty} \sup_{\underset{\sim}{t} \in V_n(a)} n^{\frac{1}{2}}|A_{nk}(\underset{\sim}{t}) - A_{nk}(\underset{\sim}{0}) - B_{nk}(\underset{\sim}{t}) + B_{nk}(\underset{\sim}{0})| = 0$,

where

$$(6.16) \qquad B_{nk}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \psi[\overline{K}_{np}(\underset{\sim}{t},|x|)]\,d\overline{\mu}_{nk}(\underset{\sim}{t},x)$$

Proof:

From (6.3) and (6.16), it is clear that

$$n^{\frac{1}{2}}[A_{nk}(\underset{\sim}{t}) - B_{nk}(\underset{\sim}{t})] = n^{\frac{1}{2}} \int_{-\infty}^{\infty} \{\overline{H}_{np}(\underset{\sim}{t},|x|) - \psi[\overline{K}_{np}(\underset{\sim}{t},|x|)]\}\,d\overline{\mu}_{nk}(\underset{\sim}{t},x),$$

and substitution from (6.12) yields

$$= \frac{1}{2} n^{\frac{1}{2}} \int_{-\infty}^{\infty} E\{[K_{np}(\underset{\sim}{t},|x|) - \overline{K}_{np}(\underset{\sim}{t},|x|)]^2 \psi''(y_1)\}\,d\overline{\mu}_{nk}(\underset{\sim}{t},x)$$

From corollary A.1 and (6.1)-(iv), this is bounded by $n^{-\frac{1}{2}}(np+1)^{\frac{1}{2}}\cdot\theta(np/(np+1))V[\bar{\mu}_{nk}(\underset{\sim}{t},x)]$, and since $\lim_{z\to 1}\theta(z) = 0$ and $V[\bar{\mu}_{nk}(\underset{\sim}{t},x)]$ is bounded uniformly in n and $\underset{\sim}{t} \in V_n(a)$, the result follows.

Theorem 6.3:

Under conditions (1.6), (2.7), (5.1), (6.1), and $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, $\lim_{n\to\infty} \sup_{\underset{\sim}{t}\in V_n(a)} n^{\frac{1}{2}}|B_{nk}(\underset{\sim}{t})-B_{nk}(\underset{\sim}{0})+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}_k| = 0,$

where

$$(6.17)\qquad \dot{\underset{\sim}{A}}_k = (np)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}\underset{\sim}{x}_{ij}\int_{-\infty}^{\infty}\psi'[2F_1(x)-1]f_1^2(x)dx$$

Proof:

In view of (6.16), (5.24), (5.5), and (5.3), it follows that, if for simplicity $\underset{\sim}{x}_{ij}'\underset{\sim}{t}$ is denoted by δ_{ij}, $d\mu_{nk}(\underset{\sim}{t},x) = ((\text{sign } x)/np)\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}dF_1(x+\delta_{ij})$, and for $x < 0$, $\psi[\frac{np}{np+1}F^*(\underset{\sim}{t},|x|)] = -\psi[\frac{np}{np+1}\sum_{i=1}^{n}\sum_{j=1}^{p}\{F_1(x+\delta_{ij})-F_1(-x+\delta_{ij})\}]$. Hence it follows that

$$(6.18)\qquad B_{nk}(\underset{\sim}{t}) = (np)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ijk}\int_{-\infty}^{\infty}\psi[\frac{1}{np+1}\sum_{i'=1}^{n}\sum_{j'=1}^{p}\{F_1(x-\delta_{ij}+\delta_{i'j'}) - F_1(-x+\delta_{ij}+\delta_{i'j'})\}]dF_1(x)$$

Taylor series expansion of the argument of ψ yields, since $f_1(x) = f_1(-x)$,

$$(6.19)\begin{cases} (np+1)^{-1} \sum_{i'=1}^{n} \sum_{j'=1}^{p} \{F_1(x-\delta_{ij}+\delta_{i'j'})-F_1(x+\delta_{ij}+\delta_{i'j'})\} \\ = \frac{np}{np+1} [F_1(x)-F_1(-x)-2\delta_{ij}f_1(x)+\rho_{ij}(x)] \\ \text{where } \rho_{ij}(x) = (2np)^{-1} \sum_{i'=1}^{n} \sum_{j'=1}^{p} \{(\delta_{i'j'}-\delta_{ij})^2 f_1'(x_2)- \\ (\delta_{i'j'}+\delta_{ij})^2 f_1'(x_1)\}, \text{ with } x_1 \text{ and } x_2 \text{ depending on} \\ i,j,i' \text{ and } j', \text{ and } |x_2| \le |\delta_{i'j'}-\delta_{ij}|, \ |x_1| \le |\delta_{i'j'}+\delta_{ij}|. \end{cases}$$

Next let us substitute (6.19) into (6.18), and expand by Taylor series. Then the integral in (6.18) becomes

$$\int_{-\infty}^{\infty} \psi[\frac{np}{np+1} \{F_1(x)-F_1(-x)\}]dF_1(x)-2\delta_{ij} \int_{-\infty}^{\infty} \psi'[\frac{np}{np+1} \{F_1(x)-F_1(-x)\}\cdot$$

$$f_1(x)dF_1(x) + \int_{-\infty}^{\infty} \psi'[\frac{np}{np+1} \{F_1(x)-F_1(-x)\}\rho_{ij}(x)dF_1(x)$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \psi''(y)[-2\delta_{ij}f_1(x)+\rho_{ij}(x)]^2 dF_1(x)$$

where y is some value satisfying Taylor's formula. The first term is zero since $\psi(-x) = -\psi(x)$ and $f_1(x) = f_1(-x)$, hence substitution into (6.18) results in

$$B_{nk}(\underset{\sim}{t}) = (np)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\{-2\delta_{ij} \int_{-\infty}^{\infty} \psi'[\frac{np}{np+1} (F_1(x)-F_1(-x))]f_1(x)dF_1(x)$$

$$+ \int_{-\infty}^{\infty} \psi'[\frac{np}{np+1} (F_1(x)-F_1(-x))]\rho_{ij}(x)dF_1(x)$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \psi''(y)[-2\delta_{ij}f_1(x)+\rho_{ij}(x)]^2 dF_1(x)\}$$

Using the facts that f_1, f_1', and ψ' are bounded, ψ'' satisfies (6.1)-(v), $\lim_{n\to\infty} \max_{i,j} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\underset{\sim}{x}_{ij}'\underset{\sim}{t}| = 0$, along with (1.6)-(iii) and (iv), it can be shown by a tedious but routine calculation that the sum of the last two terms in the above expression is $o(n^{-\frac{1}{2}})$ uniformly for $\underset{\sim}{t} \varepsilon V_n(a)$. Hence it follows that

$$(6.20) \qquad \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |B_{nk}(\underset{\sim}{t})+(2/np) \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\underset{\sim}{x}_{ij}'\underset{\sim}{t} \int_{-\infty}^{\infty} \psi'[\frac{np}{np+1} \cdot (F_1(x)-F_1(-x))]f_1(x)dF_1(x)| = o_p(n^{-\frac{1}{2}}).$$

To conclude the proof, note the following --

(i) $B_{nk}(\underset{\sim}{0}) = 0$. This follows directly from (3.18).

(ii) $$|\int_{-\infty}^{\infty} \{\psi'[\frac{np}{np+1} (F_1(x)-F_1(-x))]-\psi'[F_1(x)-F_1(-x)]\}f_1(x)dF_1(x)$$

$$\leq \alpha \int_{0}^{1-1/n} |\psi'[\frac{np}{np+1} (2y-1)]-\psi'(2y-1)|dy$$

$$+ \alpha \int_{1-1/n}^{1} |\psi'[\frac{np}{np+1} (2y-1)-\psi'(2y-1)|dy$$

where $\alpha = \sup_x f_1(x)$

The use of the mean value theorem and (6.1)-(v) yields

$$\leq \alpha \sup_{0<x<1} |\psi''(\tfrac{nx}{n+1})|(1-\tfrac{1}{n})(np+1)^{-1} + 2\alpha \sup_{0\leq x\leq 1} \psi'(x)(1-\tfrac{1}{n})$$

$$= o_p(1).$$

Putting (i) and (ii) together with (6.20) gives the result.

Theorem 6.4:

Under conditions (1.6), (1.9), (2.7), (2.8), (5.1), (6.1), (6.4), one of (3.1), (3.4), or (3.6), and $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, *for each* $\varepsilon > 0$ *there exists* N *so that for all* $n \geq N$,

$$P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|(1/pn^{\frac{1}{2}})[T_k(\underset{\sim}{t})-S_k]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}_k|\geq\varepsilon] \leq \varepsilon$$

where S_k *is given in (2.8).*

Proof:

If the results of theorems 6.1, 6.2, and 6.3 are combined, it follows that

$$P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|(1/pn^{\frac{1}{2}})[T_k(\underset{\sim}{t})-T_k(\underset{\sim}{0})]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}_k|\geq\varepsilon] \leq \varepsilon \quad \text{for} \quad n \geq N.$$

If lemmas 3.7 and 3.8 are now used, the desired result follows.

Theorem 6.5:

Under the conditions of theorem 6.4, for each $\varepsilon > 0$ *there exists* $N > 0$ *so that for all* $n \geq N$

$$P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} |M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})-Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})| \geq \varepsilon] \leq \varepsilon$$

where

$$(6.21)\quad \begin{cases} Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t}) = \underset{\sim}{w}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{w} \\ \underset{\sim}{w}' = \underset{\sim}{w}'(\underset{\sim}{t}) = (S_1-2pn^{\frac{1}{2}}\underset{\sim}{t}'\dot{\underset{\sim}{A}}_1,\ldots,S_q-2pn^{\frac{1}{2}}\underset{\sim}{t}'\dot{\underset{\sim}{A}}_q) \\ \qquad = (w_1,\ldots,w_q) \end{cases}$$

Proof:

From (2.8),

$$|M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})-Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})| \leq |[\underset{\sim}{\tau}(\underset{\sim}{t})-\underset{\sim}{w}(\underset{\sim}{t})]'(\underset{\sim}{\Sigma}_n^{-1}-\underset{\sim}{\Sigma}^{-1})[\underset{\sim}{\tau}(\underset{\sim}{t})+\underset{\sim}{w}(\underset{\sim}{t})]|$$

$$+ |[\underset{\sim}{\tau}(\underset{\sim}{t})-\underset{\sim}{w}(\underset{\sim}{t})]'\underset{\sim}{\Sigma}^{-1}[\underset{\sim}{\tau}(\underset{\sim}{t})+\underset{\sim}{w}(\underset{\sim}{t})]|$$

where $\underset{\sim}{\tau}(\underset{\sim}{t})$ is the $\underset{\sim}{\tau}(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})$ defined following (2.8). Now from (1.9), $\lim_{n\to\infty} \underset{\sim}{\Sigma}_n^{-1} = \underset{\sim}{\Sigma}$ and from (6.17) and (1.6) $\sup_{1\leq n<\infty} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |2pn^{\frac{1}{2}}\underset{\sim}{t}'\dot{\underset{\sim}{A}}_k|$ is bounded. Thus from lemma 3.2 and theorem 6.4, both S_k and $T_k(\underset{\sim}{t})$ are bounded in probability uniformly in n and $\underset{\sim}{t} \in V_n(a)$. Hence w_k is bounded in probability. Also, theorem 6.4 implies $\sup_{\underset{\sim}{t}\in V_n(a)} \|\underset{\sim}{T}(\underset{\sim}{t})-\underset{\sim}{w}(\underset{\sim}{t})\| \leq \sum_{k=1}^{q} \sup_{\underset{\sim}{t}\in V_n(a)} |T_k(\underset{\sim}{t})-w_k(\underset{\sim}{t})|$ tends to zero in probability. If the foregoing remarks are applied to the above inequality, the result is obtained.

A direct consequence of (6.17) and (6.21) is

$$(6.22)\quad \begin{cases} \underset{\sim}{w}(\underset{\sim}{t}) = \underset{\sim}{S} - 2n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}(\underset{\sim}{x}'_{ij}\underset{\sim}{t}) \int_{-\infty}^{\infty} \psi'[2F_1(x)-1]f_1^2(x)dx \\ \qquad = \underset{\sim}{S} - 2n^{-\frac{1}{2}}\underset{\sim}{X}_n\underset{\sim}{X}'_n\underset{\sim}{t} \int_{-\infty}^{\infty} \psi'[2F_1(x)-1]f_1^2(x)dx \\ \text{where } \underset{\sim}{S}' = (S_1,\ldots,S_q) \end{cases}$$

Let us define the following subset of E_q.

$$(6.23)\qquad R_n^* = \{\underset{\sim}{t}:Q_n(\underset{\sim}{Y}-\underset{\sim}{X}'_n\underset{\sim}{t})\leq k_{n\alpha}\} = \{\underset{\sim}{t}:\underset{\sim}{w}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{w}\leq k_{n\alpha}\}$$

where $k_{n\alpha}$ is defined following (2.3). $Q_n(\underset{\sim}{Y}-\underset{\sim}{X}'_n\underset{\sim}{t})$ is a quadratic form in $\underset{\sim}{t}$, and R_n^* is an ellipsoid in E_q whose centre of gravity is given by $\underset{\sim}{\tilde{\beta}}$ in the equation

$$(6.24)\qquad \underset{\sim}{S} = n^{\frac{1}{2}}\underset{\sim}{\Omega}_n\underset{\sim}{\tilde{\beta}}\ , \quad \text{where}$$

$$(6.25)\qquad \underset{\sim}{\Omega}_n = 2n^{-1}\underset{\sim}{X}_n\underset{\sim}{X}'_n \int_{-\infty}^{\infty} \psi'[2F_1(x)-1]f_1^2(x)dx$$

The purpose of the following results, culminating in theorem 6.6, is to show that M_n becomes large for values of $\underset{\sim}{t}$ which differ greatly from the true parameter point (assumed to be $\underset{\sim}{0}$). From this it will follow that $R_n(\underset{\sim}{Y})$ is bounded (and hence $\hat{\underset{\sim}{\beta}}$ defined by (2.4) exists) with probability approaching one as n increases.

Let us define

$$(6.26)\qquad \begin{cases} g_n(\gamma,\underset{\sim}{\theta}) = (\underset{\sim}{\theta}'\underset{\sim}{\Sigma}_n\underset{\sim}{\theta})^{-\frac{1}{2}}(\underset{\sim}{\theta}'\underset{\sim}{s}-n^{\frac{1}{2}}\gamma\underset{\sim}{\theta}'\underset{\sim}{Q}_n\underset{\sim}{\theta}) \\ h_n(\gamma,\underset{\sim}{\theta}) = (\underset{\sim}{\theta}'\underset{\sim}{\Sigma}_n\underset{\sim}{\theta})^{-\frac{1}{2}}\underset{\sim}{\theta}'\underset{\sim}{r}(\gamma\underset{\sim}{\theta}) \\ ||\underset{\sim}{\theta}|| = \sum_{k=1}^{q} |\theta_k| \quad \text{where } \underset{\sim}{\theta}' = (\theta_1,\ldots,\theta_q) \end{cases}$$

Lemma 6.1:

Under the conditions of theorem 6.4, for all $\varepsilon > 0$ *and* $b > 0$, *there exists* $a > 0$ *so that if* $n \geq N(\varepsilon,b)$, *then*

$$(6.27)\qquad P_n[\inf_{||\underset{\sim}{\theta}||=1}\ \inf_{|\gamma|=an^{-\frac{1}{2}}} |g_n(\gamma,\underset{\sim}{\theta})|\geq b] \geq 1-\varepsilon$$

$$(6.28)\qquad P_n[\inf_{||\underset{\sim}{\theta}||=1}\ \inf_{|\gamma|=an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})|\geq b] \geq 1-\varepsilon$$

$$(6.29)\qquad P_n[\inf_{|\gamma|<an^{-\frac{1}{2}}} |g_n(\gamma,\underset{\sim}{\theta})| = 0\ \forall\ ||\underset{\sim}{\theta}||=1] \geq 1-\varepsilon$$

$$(6.30)\qquad P_n[\inf_{|\gamma|<an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})|\leq\varepsilon\ \forall\ ||\underset{\sim}{\theta}||=1] \geq 1-\varepsilon$$

Proof:

The proofs of the above four statements are similar to the proofs of similar statements in lemma 3.2 of Koul (1967). Hence the details have been omitted.

Lemma 6.2:

Under the conditions of theorem 6.4, for all $\varepsilon > 0$ *and* $b > 0$, *there exists* $a > 0$ *and* $N > 0$ *such that*

(6.31) $$P_n[\inf_{\|\underset{\sim}{\theta}\|=1} \inf_{|\gamma|\geq an^{-\frac{1}{2}}} |g_n(\gamma,\underset{\sim}{\theta})|\geq b] \geq 1 - \varepsilon$$

(6.32) $$P_n[\inf_{\|\underset{\sim}{\theta}\|=1} \inf_{|\gamma|\geq an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})|\geq b] \geq 1 - \varepsilon$$

Proof:

Proof of (6.31):

From (6.26) it follows that $g_n(\gamma,\underset{\sim}{\theta})$ is monotone decreasing in γ for each fixed $\underset{\sim}{\theta}$. From (6.27) and (6.29), $\forall\underset{\sim}{\theta} \in \{\underset{\sim}{\theta}\in E_q : \|\underset{\sim}{\theta}\| = 1\}$, $\exists\gamma_\theta \ni |\gamma_\theta| < an^{-\frac{1}{2}}$, so that the following two inequalities hold simultaneously.

$$P_n[\sup_{\|\underset{\sim}{\theta}\|=1} |g_n(\gamma_\theta,\underset{\sim}{\theta})|=0] \geq 1 - \varepsilon$$

$$P_n[\inf_{\|\underset{\sim}{\theta}\|=1} \inf_{|\gamma|=an^{-\frac{1}{2}}} |g_n(\gamma,\underset{\sim}{\theta})|>b] \geq 1 - \varepsilon$$

Therefore $g_n(\gamma,\underset{\sim}{\theta})$ is monotone increasing for $|\gamma| \geq an^{-\frac{1}{2}}$ and (6.31) follows.

Proof of (6.32):

From (6.26) and (2.8),

$$(\underset{\sim}{\theta}'\underset{\sim}{\Sigma}_n\underset{\sim}{\theta})^{\frac{1}{2}}h_n(\gamma,\underset{\sim}{\theta}) = \sum_{k=1}^{q} \theta_k T_k(\gamma\underset{\sim}{\theta})$$

$$= \sum_{k=1}^{q} \theta_k n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\psi_{np}\left(\frac{r_{ij}}{np+1}\right) \text{ sign } (Y_{ij}-\gamma\underset{\sim}{x}'_{ij}\underset{\sim}{\theta})$$

where r_{ij} is the rank of $|Y_{ij}-\gamma\underset{\sim}{x}'_{ij}\underset{\sim}{\theta}|$ in the joint ranking over $i = 1,\ldots,n$; $j = 1,\ldots,p$.

In lemma 2.2 of Koul (1967), it is shown that the right side of the above equalities is monotone. To see this, set Koul's $x_i = \sum_{k=1}^{q} x_{ijk}\theta_k = \underset{\sim}{x}'_{ij}\underset{\sim}{\theta}$. Hence (6.28) and (6.30) imply that with probability at least $1 - 2\varepsilon$, $\inf_{\|\underset{\sim}{\theta}\|=1} \cdot \inf_{|\gamma|=an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})| \leq$ $\inf_{\|\underset{\sim}{\theta}\|=1} \inf_{|\gamma|>an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})|$. This implies (6.32).

Theorem 6.6:

Under the conditions of theorem 6.4, for all $\varepsilon > 0$ *and* $b > 0$, *there exists* $N > 0$ *and* $a > 0$ *such that*

$$(6.33) \qquad P_n[\inf_{\|\underset{\sim}{\theta}\| \geq an^{-\frac{1}{2}}} M_n(\underset{\sim}{Y}-\underset{\sim}{X}'_n\underset{\sim}{\theta}) \geq b] \geq 1 - \varepsilon$$

(6.34) $$P_n[\inf_{\|\underset{\sim}{\theta}\| \geq an^{-\frac{1}{2}}} Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\theta}) \geq b] \geq 1 - \varepsilon$$

Proof:

Using the results at the bottom of page 48 of Rao (1965), it follows from (2.8), (6.22), (6.24), (6.25), and (6.26) that

$$M_n(\underset{\sim}{Y}-\gamma\underset{\sim}{X}_n'\underset{\sim}{\theta}) = \underset{\sim}{\tau}'(\gamma\underset{\sim}{\theta})\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{\tau}(\gamma\underset{\sim}{\theta})$$

$$\geq (\underset{\sim}{\theta}'\underset{\sim}{\Sigma}_n\underset{\sim}{\theta})^{-1}[\underset{\sim}{\theta}'\underset{\sim}{\tau}(\gamma\underset{\sim}{\theta})]^2 = h_n^2(\gamma,\underset{\sim}{\theta})$$

$$Q_n(\underset{\sim}{Y}-\gamma\underset{\sim}{X}_n'\underset{\sim}{\theta}) = \underset{\sim}{w}'(\gamma\underset{\sim}{\theta})\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{w}(\gamma\underset{\sim}{\theta})$$

$$\geq (\underset{\sim}{\theta}'\underset{\sim}{\Sigma}_n\underset{\sim}{\theta})^{-1} = g_n^2(\lambda,\underset{\sim}{\theta})$$

Proof of (6.33):

If $b > 0$, then

$$P_n[\inf_{\|\underset{\sim}{\theta}\| \geq an^{-\frac{1}{2}}} M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\theta}) \geq b^2] = P_n[\inf_{\|\underset{\sim}{\theta}\| = 1}\ \inf_{|\gamma| \geq an^{-\frac{1}{2}}} M_n(\underset{\sim}{Y}-\gamma\underset{\sim}{X}_n'\underset{\sim}{\theta}) \geq b^2]$$

$$\geq P_n[\inf_{\|\underset{\sim}{\theta}\| = 1}\ \inf_{|\gamma| \geq an^{-\frac{1}{2}}} |h_n(\gamma,\underset{\sim}{\theta})| \geq b] \geq 1 - \varepsilon$$

if a is chosen as in lemma 6.2.

Proof of (6.34):

Follows in the same way as (6.33).

The next two results give the asymptotic distribution of $\hat{\beta}_n$.

Lemma 6.3:

Under the conditions of theorem 6.4, for all $\varepsilon > 0$ *there exists* $N > 0$ *such that for all* $n \geq N$,

$$P_n[n^{\frac{1}{2}} \| \hat{\beta}_n(Y) - \tilde{\beta}_n(Y) \| \geq \varepsilon] \leq \varepsilon$$

where, from (6.24) and (6.25),

$$\tilde{\beta}_n = n^{-\frac{1}{2}} \Omega_n^{-1} S = \{2 \int_{-\infty}^{\infty} \psi'[2F_1(x)-1] f_1^2(x) dx\}^{-1} n^{\frac{1}{2}} (X_n X_n')^{-1} S \quad (6.35)$$

Proof:

Although lengthy, the proof closely follows that of lemma 4.1 of Koul (1967) and hence the details have been omitted.

Theorem 6.7:

Let conditions (1.6), (1.9), (2.7), (2.8), (5.1), (6.1), (6.4), and one of (3.1), (3.4), or (3.6) be satisfied. Let L_β *denote the probability law of* $\hat{\beta}_n(Y)$ *if* β *is the true parameter point. Then, if* N *denotes the normal probability law,*

$$\lim_{n\to\infty} L_\beta[n^{\frac{1}{2}}(\hat{\beta}_n - \beta)] = N(0, \{2 \int_{-\infty}^{\infty} \psi'[2F_1(x)-1] f_1^2(x) dx\}^{-2} \Lambda^{-1} \Sigma \Lambda^{-1})$$

where $\underset{\sim}{\Lambda} = \lim_{n\to\infty} n^{-1}(\underset{\sim}{X}_n \underset{\sim}{X}_n')$ and $\underset{\sim}{\Sigma}$ is defined in (1.9).

Proof:

In view of lemma 2.1 it is sufficient to prove the theorem for $\underset{\sim}{\beta} = \underset{\sim}{0}$. From lemma 3.2 it is evident that $\underset{\sim}{s}$ converges in law to $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ as $n \to \infty$. Now (6.35) implies $n^{\frac{1}{2}}\tilde{\underset{\sim}{\beta}}_n$ converges in law to the normal distribution given in the statement of this theorem. Finally, lemma 6.3 implies that $n^{\frac{1}{2}}\hat{\underset{\sim}{\beta}}_n$ and $n^{\frac{1}{2}}\tilde{\underset{\sim}{\beta}}_n$ converge in law to the same distribution, and hence the theorem is true.

CHAPTER VII

LIMITING DISTRIBUTION AND LARGE SAMPLE EXISTENCE OF THE ESTIMATE, $\hat{\beta}_n$, FOR THE SEPARATE RANKING CASE, SIGN SCORES

The proofs for existence and asymptotic normality for the separate ranking case will follow the same lines as in Chapter VI. First of all, let us observe that if $p = 1$, theorem 5.1 remains valid if the $\{c_{ilk}\}$ merely satisfy (1.6)-(ii) instead of (5.1)-(iii) and (iv), since the stronger assumptions are used only to bound terms in which $j \neq j'$. Notice also that (2.7) and (5.1)-(ii) will not apply, since now the distribution is univariate. Theorem 5.1 now corresponds to theorem A3 of Koul (1967).

Thus, for $p = 1$, all the results of Chapter V remain valid in the absense of the above mentioned conditions. (see also the appendix of Koul (1967)).

In this chapter the following assumption is made on the underlying distribution in addition to those of (1.6).

(7.1)
(i) $f_j(x) = F_j'(x)$, and $f_j'(x)$ exist and are bounded for all $x \in (-\infty,\infty)$, $j = 1,\ldots,p$.

(ii) $f_j(x) = 0$ on at most a finite number of intervals, $j = 1,\ldots,p$.

Corresponding respectively to the quantities defined in (5.2) and (5.3), let us define, for $j = 1,\ldots,p$,

$$(7.2)\quad \begin{cases} G^{*}_{nj}(\underset{\sim}{t},x) = n^{-1}\sum_{i=1}^{n} I(|Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}|\le x) \\ \nu_{nkj}(\underset{\sim}{t},x) = n^{-1}\sum_{i=1}^{n} x_{ijk} I(Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}\le x)\ \text{sign}\ (Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}) \\ H^{\circ}_{nj}(\underset{\sim}{t},x) = \psi[\frac{n}{n+1}\, G^{*}_{nj}(\underset{\sim}{t},x)] \\ \hat{H}^{\circ}_{nj}(\underset{\sim}{t},x) = \psi_n[\frac{n}{n+1}\, G^{*}_{nj}(\underset{\sim}{t},x)] \end{cases}$$

$$(7.3)\quad \begin{cases} G^{*}_{j}(\underset{\sim}{t},x) = EG^{*}_{nj}(\underset{\sim}{t},x) \\ \overline{\nu}_{nkj}(\underset{\sim}{t},x) = E\nu_{nkj}(\underset{\sim}{t},x) \\ \overline{H}^{\circ}_{nj}(\underset{\sim}{t},x) = EH^{\circ}_{nj}\ (\underset{\sim}{t},x) \end{cases}$$

Define $V_n(a)$, $V(a)$, and $||\underset{\sim}{t}||$ as in (5.4). Then, similarly to (6.2), for $j = 1,\ldots,p$

$$(7.4)\quad \begin{cases} \int_{-\infty}^{\infty} \hat{H}^{\circ}_{nj}(\underset{\sim}{t},|x|)d\nu_{nkj}(\underset{\sim}{t},x) = \sum_{i=1}^{n} x_{ijk}\psi_n(r_{ij}/(n+1)) \\ \qquad\cdot\ \text{sign}\ (Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}) = n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{t}),\ \text{say}, \end{cases}$$

where r_{ij} = rank of $|Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}|$ in the separate ranking over $i = 1,\ldots,n$. It is clear from (2.2) that

(7.5) $$T_k(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t}) = T_k(\underset{\sim}{t}) = \sum_{j=1}^{p} T_{kj}(\underset{\sim}{t})$$

where $T_k(\underset{\sim}{t})$ is written in place of $T_k(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})$ to emphasize dependence on $\underset{\sim}{t}$.

The following four theorems will be stated without proofs. The proofs are almost the same as theorems 6.1 through 6.4, respectively, for the case $p = 1$.

Theorem 7.1:

: *If conditions (1.6), (6.1), (7.1),* $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, *and*

(7.6) $$\begin{cases} \sup\limits_{\underset{\sim}{t}\varepsilon V_n(a)} \Big| \int_{-\infty}^{\infty} \{\psi_n[\frac{n}{n+1} G^*_{nj}(\underset{\sim}{t},|x|)] - \psi[\frac{n}{n+1} G^*_{nj}(\underset{\sim}{t},|x|)\} \\ \cdot d\nu_{nkj}(\underset{\sim}{t},x) - \int_{-\infty}^{\infty} \{\psi_n[\frac{n}{n+1} G^*_{nj}(\underset{\sim}{0},|x|)] - \psi[\frac{n}{n+1} G^*_{nj}(\underset{\sim}{0},|x|)]\} \\ \cdot d\nu_{nkj}(\underset{\sim}{0},x) \Big| = o_p(n^{-\frac{1}{2}}) \end{cases}$$

are satisfied for $j = 1,\ldots,p$, *then for each* $\varepsilon > 0$ *there exists* $N > 0$ *so that for all* $n \geq N$,

$$P_n\{\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{t})-A^\circ_{nkj}(\underset{\sim}{t})]-[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{0})-A^\circ_{nkj}(\underset{\sim}{0})]|\geq\varepsilon\} \leq \varepsilon$$

where

(7.7) $$A^\circ_{nkj}(\underset{\sim}{t}) = \int_{-\infty}^{\infty} \overline{H}^\circ_{nj}(\underset{\sim}{t},|x|)d\overline{\nu}_{nkj}(\underset{\sim}{t},|x|)$$

Theorem 7.2:

Under conditions (1.6), (7.1), (6.1), *and* $H_o : \beta = 0$,

$$\lim_{n\to\infty} \sup_{t\in V_n(a)} n^{1/2}|A^\circ_{nkj}(t)-A^\circ_{nkj}(0)-B^\circ_{nkj}(t)+B^\circ_{nkj}(0)| = 0 \text{ , where}$$

$$B^\circ_{nkj}(t) = \int_{-\infty}^{\infty} \psi[\tfrac{n}{n+1}\, G^*_j(t,|x|)]\, d\bar{\nu}_{nkj}(t,x) \tag{7.8}$$

Theorem 7.3:

Under conditions (1.6), (7.1), (6.1), *and* $H_o : \beta = 0$,

$$\lim_{n\to\infty} \sup_{t\in V_n(a)} n^{1/2}|B^\circ_{nkj}(t)-B^\circ_{nkj}(0)+2t'\dot{A}^\circ_{kj}| = 0, \text{ where}$$

$$\dot{A}^\circ_{kj} = n^{-1}\sum_{i=1}^{n} x_{ijk}x_{ij} \int_{-\infty}^{\infty} \psi'[2F_j(x)-1]f_j^2(x)\,dx \tag{7.9}$$

Theorem 7.4:

Under conditions (1.6), (1.9), (2.2), (6.1), (7.1), (7.6) *and* $H_o : \beta = 0$, *for each* $\varepsilon > 0$, *there is* $N > 0$ *so that for all* $n \geq N$,

$$P_n[\sup_{t\in V_n(a)} n^{1/2}|n^{-1/2}[T_{kj}(t)-S_{kj}]+2t'\dot{A}^\circ_{kj}|\geq\varepsilon] \leq \varepsilon, \quad j = 1,\ldots,p,$$

where $S_{kj} = n^{-1/2}\sum_{i=1}^{n} x_{ijk}\psi[F^*_j(|Y_{ij}|)]\ \text{sign}\ Y_{ij}$

Corollary 7.4:

For each $\varepsilon > 0$ *there is* $N > 0$ *so that for all* $n \geq N$,

$$P_n[\sup_{\underset{\sim}{t}\in V_n(a)} n^{\frac{1}{2}}|n^{-\frac{1}{2}}[T_k(\underset{\sim}{t})-S_k]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_k^{\circ}|\geq\varepsilon] \leq \varepsilon,$$ *where*

$$\dot{\underset{\sim}{A}}{}_k^{\circ} = \sum_{j=1}^{p} \dot{\underset{\sim}{A}}{}_{kj}^{\circ} \tag{7.10}$$

Proof:

$$P_n[\sup_{\underset{\sim}{t}\in V_n(a)} n^{\frac{1}{2}}|n^{-\frac{1}{2}}[T_k(\underset{\sim}{t})-S_k]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_k^{\circ}|\geq\varepsilon]$$

$$\leq P_n[\sum_{j=1}^{p} \sup_{\underset{\sim}{t}\in V_n(a)} n^{\frac{1}{2}}|n^{-\frac{1}{2}}[T_{kj}(\underset{\sim}{t})-S_{kj}]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_{kj}^{\circ}|\geq\varepsilon]$$

$$\leq \sum_{j=1}^{p} P_n[\sup_{\underset{\sim}{t}\in V_n(a)} n^{\frac{1}{2}}|n^{-\frac{1}{2}}[T_{kj}(\underset{\sim}{t})-S_{kj}]+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_{kj}^{\circ}|\geq\varepsilon] \leq p\varepsilon$$

From this, the result is immediate.

Let us now redefine Q_n and $\underset{\sim}{w}$ as follows

$$(7.11)\quad \begin{cases} Q_n(\underset{\sim}{Y}-\underset{\sim}{X}{}_n'\underset{\sim}{t}) = \underset{\sim}{w}'\underset{\sim}{\Sigma}{}_n^{-1}\underset{\sim}{w} \quad \text{where} \\ \underset{\sim}{w}' = \underset{\sim}{w}'(\underset{\sim}{t}) = (S_1-2n^{\frac{1}{2}}\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_1^{\circ},\ldots,S_q-2n^{\frac{1}{2}}\underset{\sim}{t}'\dot{\underset{\sim}{A}}{}_q^{\circ}) \\ \qquad\qquad = (w_1,\ldots,w_q) \end{cases}$$

It can now be shown similarly to theorem 6.5 that

__Theorem 7.5:__

Under the conditions of theorem 7.4, for each $\varepsilon > 0$

there exists $N > 0$ *such that for all* $n \geq N$,

$$P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} |M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})-Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})| \geq \varepsilon] \leq \varepsilon.$$

Let us now define $\dot{\underset{\sim}{X}}_n$ as in (4.13). Thus, equations (7.9), (7.10), (7.11) and (4.13) imply

$$(7.12)\quad \begin{cases} w_k = S_k - 2n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk}\underset{\sim}{x}_{ij}'\underset{\sim}{t} \int_{-\infty}^{\infty} \psi'[2F_j(x)-1]f_j^2(x)dx \\ \underset{\sim}{w}(\underset{\sim}{t}) = \underset{\sim}{s} - 2n^{-\frac{1}{2}}\dot{\underset{\sim}{X}}_n\dot{\underset{\sim}{X}}_n'\underset{\sim}{t} \end{cases}$$

Similarly as in (6.23), define

$$(7.13)\qquad R_n^* = \{\underset{\sim}{t}:Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})\leq k_{n\alpha}\}$$
$$= \{\underset{\sim}{t}:\underset{\sim}{w}'\underset{\sim}{\Sigma}_n^{-1}\underset{\sim}{w}\leq k_{n\alpha}\}$$

where $k_{n\alpha}$ is defined following (2.3). The quantity $Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{t})$ is a quadratic form in $\underset{\sim}{t}$, and R_n^* is an ellipsoid in E_q whose centre of gravity, $\tilde{\underset{\sim}{\beta}}$, is given by

$$(7.14)\qquad \begin{cases} \underset{\sim}{s} = n^{\frac{1}{2}}\underset{\sim}{\Omega}_n\tilde{\underset{\sim}{\beta}} \qquad \text{where} \\ \underset{\sim}{\Omega}_n = 2n^{-1}\dot{\underset{\sim}{X}}_n\dot{\underset{\sim}{X}}_n' \end{cases}$$

The following additional assumption will be needed in proving the forthcoming results.

$$(7.15)\quad \begin{cases} \dot{\underset{\sim}{X}}_n \text{ is nonsingular for all } n \\ \underset{\sim}{\Lambda}_o = \lim_{n\to\infty} n^{-1}\dot{\underset{\sim}{X}}_n\dot{\underset{\sim}{X}}_n' \text{ exists and is nonsingular} \end{cases}$$

Now suppose $g_n(\lambda,\underset{\sim}{\theta})$ and $h_n(\lambda,\underset{\sim}{\theta})$ are defined as in (6.26) but with $\underset{\sim}{\Omega}_n$ given by (7.14) instead of (6.25), and $\underset{\sim}{\tau}$ by (2.2) instead of (2.8). The results of lemmas 6.1 and 6.2 are valid under (7.15) and the conditions of theorem 7.4. The proofs closely follow those of lemmas 6.1 and 6.2.

Then, corresponding to theorem 6.6,

Theorem 7.6:

If (7.15) and the conditions of theorem 7.4 are satisfied, then for all $\varepsilon > 0$ *and* $b > 0$, *there exist* $N > 0$ *and* $a > 0$ *such that for all* $n \geq N$

$$P_n[\inf_{||\underset{\sim}{\theta}|| > an^{-\frac{1}{2}}} M_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\theta}) > b] \geq 1 - \varepsilon$$

$$P_n[\inf_{||\underset{\sim}{\theta}|| \geq an^{-\frac{1}{2}}} Q_n(\underset{\sim}{Y}-\underset{\sim}{X}_n'\underset{\sim}{\theta}) > b] \geq 1 - \varepsilon$$

This implies that the region $R_n(\underset{\sim}{Y})$ given in (2.3) is bounded with large probability if n is large. Thus large sample existence of $\hat{\beta}_n$ is assured.

Lemma 7.1:

If (7.15) and the conditions of theorem 7.4 are satisfied, then for all $\varepsilon > 0$*, there is* N *so that for all* $n \geq N$,

$P_n[n^{\frac{1}{2}}||\hat{\beta}_n(Y)-\tilde{\beta}_n(Y)|| \geq \varepsilon] \leq \varepsilon$

Proof:

The details will be omitted since it is similar to that of lemma 4.1 of Koul (1967).

Theorem 7.7:

Let conditions (1.6), (1.9), (2.2), (6.1), (7.1), (7.6) and (7.15) be satisfied. Let L_β *denote the probability law of* $\hat{\beta}_n(Y)$ *if* β *is the true parameter point. Then, if* N *denotes the normal probability law,*

$\lim_{n\to\infty} L_\beta[n^{\frac{1}{2}}(\hat{\beta}_n-\beta)] = N(0,(\frac{1}{4})\Lambda_0^{-1}\Sigma\Lambda_0^{-1})$ *where* Λ_0 *and* Σ *are given in (7.15) and (1.9) respectively.*

Proof:

From lemma 2.1, it suffices to prove the theorem for $\beta = 0$. Lemma 3.2 implies S converges in law to $N(0,\Sigma)$ as $n \to \infty$. Then (7.14) and (7.15) imply $n^{\frac{1}{2}}\tilde{\beta}$ converges in law to the normal distribution given in the theorem's statement. Thus, the result is immediate after applying lemma 7.1.

The Sign Score: $\psi_n(u) = \psi(u) = 1$ *for* $u \in [0,1]$

Results similar to those proved for tests and estimates involving scores satisfying (1.8) and (6.1) will be proved. As previously, the definitions of ψ_n and ψ are extended to $[-1,1]$ by $\psi(u) = -\psi(-u)$ for $u \in (0,1]$. Hence $\psi_n(u) = \psi(u) = \text{sign } u$ for $u \in [-1,1]$.

It is readily seen that the sign score does not satisfy (1.8)-(ii) and (v) nor (6.1)-(i) and (ii) at $x = 0$.

From (2.2) it is obvious that

$$(7.16) \qquad T_k = T_k^* = S_k = n^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk} \text{ sign } Y_{ij}$$

To find the limiting distribution of $\hat{\beta}_n$, the same quantities as in (7.2) and (7.3) are defined. In this case, however, $\hat{H}^\circ_{nj}(t,x) = H^\circ_{nj}(t,x) = \overline{H}_{nj}(t,x) = 1$. Thus, from (7.4) and (7.7) respectively, it follows that

$$(7.17) \qquad \begin{cases} n^{-\frac{1}{2}} T_{jk}(t) = \nu_{nkj}(t,\infty) = n^{-1} \sum_{i=1}^{n} x_{ijk} \text{ sign } (Y_{ij} - x'_{ij}t) \\ \\ A^\circ_{nkj}(t) = \overline{\nu}_{nkj}(t,\infty) \end{cases}$$

Theorem 7.8:

Under conditions (1.6), (7.1), (6.1), *and* $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, *for each* $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n\{ \sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{t})-A^{\circ}_{nkj}(\underset{\sim}{t})]-[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{0})-A^{\circ}_{nkj}(\underset{\sim}{0})]|\geq\varepsilon\} = 0$$

Proof:

From (7.17) it follows that

$$\sup_{\underset{\sim}{t}\varepsilon V_n(a)} |[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{t})-\overset{\circ}{A}_{nkj}(\underset{\sim}{t})]-[n^{-\frac{1}{2}}T_{kj}(\underset{\sim}{0})-A^{\circ}_{nkj}(\underset{\sim}{0})]|$$

$$= \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |[\nu_{nkj}(\underset{\sim}{t},\infty)-\bar{\nu}_{nkj}(\underset{\sim}{t},\infty)]-[\nu_{nkj}(\underset{\sim}{0},\infty)-\bar{\nu}_{nkj}(\underset{\sim}{0},\infty)]|$$

$= o_p(1)$. The last equality follows from theorem 5.2 with $p = 1$. Thus the result is proved.

Theorem 7.9:

Under conditions (1.6), (6.1), (7.1), *and* $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, *for each* $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|A^{\circ}_{nkj}(\underset{\sim}{t})-A^{\circ}_{nkj}(\underset{\sim}{0})+2\underset{\sim}{t}'\dot{\underset{\sim}{A}}^{\circ}_{kj}| = 0, \quad \textit{where}$$

$$\dot{\underset{\sim}{A}}^{\circ}_{kj} = -2n^{-1}f(0) \sum_{i=1}^{n} x_{ijk}\underset{\sim}{x}_{ij}$$

Proof:

From (7.17), (7.3), and (7.2) it is evident that $A^{\circ}_{nkj}(\underset{\sim}{t}) = 0$, and

$$A^{\circ}_{nkj}(\underset{\sim}{t}) = n^{-1} \sum_{i=1}^{n} x_{ijk}[1-2F(\underset{\sim}{x}'_{ij}\underset{\sim}{t})]$$

$$= n^{-1} \sum_{i=1}^{n} x_{ijk}[1-2F(0)-2\underset{\sim}{x}'_{ij}\underset{\sim}{t}f(0)-2(\underset{\sim}{x}'_{ij}\underset{\sim}{t})^2 f'(\theta_{ij}\underset{\sim}{x}'_{ij}\underset{\sim}{t})]$$

where $0 < \theta_{ij} < 1$. Similarly to the arguments leading to (6.20), the sum of the last terms in the above expression is $o(n^{-\frac{1}{2}})$ uniformly for $\underset{\sim}{t} \varepsilon V_n(a)$. Then, since $2F(0) = 1$, $A^{\circ}_{nkj}(\underset{\sim}{t}) = -2n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ijk} \cdot (\underset{\sim}{x}'_{ij}\underset{\sim}{t})f(0) + o(n^{-\frac{1}{2}}) = -2\underset{\sim}{t}'\dot{\underset{\sim}{A}}^{\circ}_{kj} + o(n^{-\frac{1}{2}})$, from which the result follows.

<u>*Corollary 7.9:*</u>

Under the conditions of theorem 7.9, for each $\varepsilon > 0$

$$\lim_{n\to\infty} P_n[\sup_{\underset{\sim}{t}\varepsilon V_n(a)} n^{\frac{1}{2}}|n^{-\frac{1}{2}}[T_k(\underset{\sim}{t})-T_k(\underset{\sim}{0})]-2\underset{\sim}{t}'\dot{\underset{\sim}{A}}^{\circ}_{k}| \geq \varepsilon] = 0$$

where $\dot{\underset{\sim}{A}}^{\circ}_{k} = \sum_{j=1}^{p} \dot{\underset{\sim}{A}}^{\circ}_{kj}$.

Proof:

This is a direct consequence of (2.2), (7.17), and theorems 7.8 and 7.9.

Results similar to those of theorems 7.5 and 7.6, and lemma 7.1 may be obtained for the sign score case. This leads to the following result which corresponds to theorem 7.7.

Theorem 7.10:

Let conditions (1.6), (1.9), (2.2), (7.1), *and* $\psi_n(u) = \psi(u) = \text{sign } u$ *be satisfied. Let* L_{β} *denote the probability law of* $\hat{\beta}_n(Y)$ *if* β *is the true parameter point. Then*

$$\lim_{n\to\infty} L_{\beta}[n^{\frac{1}{2}}(\hat{\beta}_n-\beta)] = N(0,(\tfrac{1}{4})\Lambda_o^{-1}\Sigma\Lambda_o^{-1})$$

where Σ *is defined in* (1.10) *and* $\Lambda_o = \lim_{n\to\infty} \dot{X}_n\dot{X}_n'$ *where* $\dot{X}_n$ *is defined in* (4.13) *but with* $\kappa_j = [f_j(0)]^{\frac{1}{2}}$.

Asymptotic Equivalence of the Joint and the Separate Ranking Procedures When Both Methods are Valid.

It is of interest to note that if, in the separate ranking case, the marginal distributions are identical, i.e. $F_j(x) = F_{j'}(x)$ for all j and j', then the estimates based on both the joint and the separate ranking procedures are asymptotically equivalent. By this it is meant that the asymptotic distribution of the estimate is the same in each case. To see this, notice that the covariance matrix of $\hat{\beta}_n$ given in theorem 7.7 is, in view of (1.9), (4.13), and (7.15)

$$(\tfrac{1}{4})\Lambda_o^{-1}\Sigma\Lambda_o^{-1} = (\tfrac{1}{4}) \lim_{n\to\infty} \{\int_{-\infty}^{\infty} f_1'(u)\psi[2F_1(u)-1]du\}^{-2}(X_nX_n')^{-1}\Sigma(X_nX_n')^{-1}$$

Now, integrating by parts and using the notation of theorem 6.7, this becomes

$$(\tfrac{1}{4})\Lambda_o^{-1}\underset{\sim}{\Sigma}\Lambda_o^{-1} = \{2\int_{-\infty}^{\infty}\psi'[2F_1(x)-1]f_1^2(x)dx\}^{-2}\Lambda^{-1}\underset{\sim}{\Sigma}\Lambda^{-1} .$$

Thus it is evident that when joint ranking is valid, i.e. one of (3.1), (3.4), or (3.6) hold, then estimates based on joint and separate ranking procedures are asymptotically equivalent.

CHAPTER VIII

EFFICIENCY OF PROPOSED ESTIMATES - CONCLUDING REMARKS

8.1 *Efficiency*

Under the assumptions of both theorems 4.5 and 7.7, both the least squares estimate $\hat{\boldsymbol{\beta}}_n^*$ proposed in (4.19) and the estimate $\hat{\boldsymbol{\beta}}_n$ based on the separate ranking procedure are asymptotically normal. The efficiency of $\hat{\boldsymbol{\beta}}_n$ with respect to $\hat{\boldsymbol{\beta}}_n^*$ will now be considered in the sense of the inverse ratio of sample sizes needed to obtain the same generalized variances. From theorem 4.5 - (ii), (4.15), and theorem 7.7, the efficiency of $\hat{\boldsymbol{\beta}}$ with respect to $\hat{\boldsymbol{\beta}}^*$ is

$$e_1 = \left\{ \frac{\lim_{n\to\infty} |n^{-1}\boldsymbol{X}_n \boldsymbol{B}_n^{-1} \boldsymbol{X}_n'|}{|(\tfrac{1}{4})\boldsymbol{\Lambda}_o^{-1} \boldsymbol{\Sigma} \boldsymbol{\Lambda}_o^{-1}|} \right\}^{1/p}, \text{ hence}$$

(8.1)
$$e_1 = 4\left\{ \frac{|\boldsymbol{\Lambda}_o|^2}{|\boldsymbol{\Sigma}| \lim_{n\to\infty} |n^{-1} X_n B_n^{-1} X_n'|} \right\}^{1/p}$$

It can easily be verified that if $\hat{\boldsymbol{\beta}}$ is the nonparametric estimate based on joint ranking, and if the conditions of theorem 6.7 instead of theorem 7.7 hold, then (8.1) is valid in this case also.

8.2 *Special Cases*

(i) Univariate Case (p = 1)

In this case $\underset{\sim}{B}_n = \sigma^2 \underset{\sim}{I}$ where $\sigma^2 = \text{var } Y_{i1}$. Then from (1.9), $\underset{\sim}{\Sigma} = \int_0^1 \psi^2(u)du \cdot \lim_{n\to\infty} |n^{-1}\underset{\sim}{X}_n\underset{\sim}{X}_n'|$. Also, from theorem 7.7, $\underset{\sim}{\Lambda}_o = \int_{-\infty}^{\infty} \psi'[2F_1(x)-1]f_1^2(x)dx \cdot \lim_{n\to\infty} |n^{-1}\underset{\sim}{X}_n\underset{\sim}{X}_n'|$. Hence, for this case (8.1) reduces to

$$e_1 = 4\sigma^2\{\int_{-\infty}^{\infty} \psi'[2F_1(x)-1]f_1^2(x)dx\}^2/\int_0^1 \psi^2(u)du \tag{8.2}$$

For Wilcoxon scores, this further reduces to $e_1 = 12\sigma^2\{\int_{-\infty}^{\infty} f_1^2(x)dx\}^2$, and for sign scores, it follows from theorem 7.10 that $e_1 = 4\sigma^2 f_1^2(0)$.

(ii) Quadrant Symmetry and Identity of Marginal Distributions

From the definition of quadrant symmetry the correlation coefficients of the underlying distribution are zero. Also, from (1.10), $\lambda_{jj'} = 0$ if $j \neq j'$. Thus the expressions for $\underset{\sim}{B}_n$, $\underset{\sim}{\Sigma}$, and $\underset{\sim}{\Lambda}_o$ are similar to those for the univariate case. It is easily shown that (8.2) is valid in this case also.

In the above two cases, e_1 was found to be independent of the regression constants, $\underset{\sim}{X}_n$. In more general cases this is not necessarily true and the choice of $\underset{\sim}{X}_n$ may be crucial in determining the validity of the estimates. This gives rise to a further problem -

that of designing the experiment so as to make $\hat{\beta}_n$ as "good" as possible, i.e. to make e_1 as large as possible.

8.3 *Examples of score functions satisfying (6.1).*

The score function for a signed rank statistic is derived from a symmetric distribution function, G, by the relation

$$\psi(u) = -g'[G^{-1}(\tfrac{u+1}{2})]/g[G^{-1}(\tfrac{u+1}{2})] \tag{8.3}$$

where $G'(x) = g(x)$, provided G is strongly unimodal, i.e. $-g'(x)/g(x)$ is monotone nondecreasing.

Hàjek (1962) showed that certain one sided tests based on such rank scores are asymptotically uniformly most powerful when the underlying distribution is G.

Some examples are given in table I where $\psi(u)$ satisfies conditions (6.1). Note that in examples 2,4, and 5, $\psi(u)$, $\psi'(u)$, and $\psi''(u)$ are bounded on $[0,1]$. In example 3, the same is true if $1/2 < a \leq 3/4$ or if $a = 1$; in fact, if $a = 1$, the Wilcoxon scores are generated. If $3/4 < a < 5/6$, $\psi(u)$ and $\psi'(u)$ are bounded but $\psi''(u)$ is not, however (6.1) is still satisfied.

It is of interest to note that if $G(x)$ has compact support, is four times differentiable on its support, and its density, $g(x)$ is bounded away from zero on its support, then $\psi(u)$, $\psi'(u)$, and

$\psi''(u)$ are all bounded. The distributions in examples 2 and 4 satisfy this property. Example 1 gives the sign score if $a = 1$.

TABLE I

Name	Distribution Function $G(x)$	Density $g(x)$	$-g'(x)/g(x)$, $x \geq 0$	$\psi(u)$
1. Double Exponential	$e^{ax}/2a$, $x < 0$ $1-e^{ax}/2a$, $x \geq 0$ where $a > 0$	$\frac{1}{2} e^{-a\|x\|}$, $-\infty < x < \infty$	a	a
2. Truncated Normal	$\frac{\Phi(x)-a}{1-2a}$, $x \varepsilon [\Phi^{-1}(a),-\Phi^{-1}(a)]$ where $0 < a < \frac{1}{2}$	$\frac{e^{-x^2/2}}{(1-2a)\sqrt{2\pi}}$, $x \varepsilon [\Phi(a),-\Phi(a)]$ 0 otherwise	x	$\Phi^{-1}(\frac{1}{2} + \frac{u-2a}{2})$
3. Generalized Logistic	$2^{2(a-1)}(1-e^{ax})^{1-2a}$, $x < 0$ $1-2^{2(a-1)}(1-e^{-ax})^{1-2a}$, $x \geq 0$ where $a > \frac{1}{2}$	$\frac{a(2a-1)2^{2(a-1)}e^{-a\|x\|}}{1-e^{-a\|x\|}}$, $-\infty < x < \infty$	$a(1-e^{-ax})$	$a[1-(1-u)^{1/(2a-1)}]$
4. Truncated Parabolic	$\frac{x^3+3a^2x+3a+1}{6a^2-a^3-1}$, $\|x\| \leq 1$ where $a > 1$	$\frac{3(a^2-x^2)}{6a^2-3a^3-1}$, $\|x\| \leq 1$ 0 otherwise	$\frac{2x}{a^2-x^2}$	$\frac{2G^{-1}(\frac{u+1}{2})}{a^2-[G^{-1}(\frac{u+1}{2})]^2}$
5. No Name	$\frac{1}{2} - \frac{\exp(1-e^{-ax})-1}{2(e-1)}$, $x < 0$ $\frac{1}{2} + \frac{\exp(1-e^{-ax})-1}{2(e-1)}$, $x \geq 0$ where $a > 0$	$\frac{a}{2(e-1)}\exp(1-a\,x - e^{-a\|x\|})$ $-\infty < x < \infty$	$a(1-e^{-ax})$	$a \log [u(e-1)+1]$

APPENDIX

Let $V_A(f)$ denote the total variation of f over the set $A \subset [-\infty,\infty]$. For simplicity write $V(f)$ to denote $V_{[-\infty,\infty]}(f)$. This agrees with the notation at the beginning of Chapter VI.

Lemma A1:

Let F^*_{np} *and* F^* *be defined as in (5.2), and (5.3) where* F_1 *is continuous. Then*

$$\sup_{\underset{\sim}{t} \in E_q} V\{E[F^*_{np}(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{t},|x|)]^2 \leq 2/n$$

Proof:

It follows from (5.2) and (5.5) that

$$V\{E[F^*_{np}(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{t},|x|)]^2\}$$

$$\leq 2V_{[0,\infty]}\{(np)^{-2} \sum_{i=1}^{n} \mathrm{var}[\sum_{j=1}^{p} \{I(|Y_{ij}-\underset{\sim}{x}'_{ij}\underset{\sim}{t}|\leq x)-F(x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})$$

$$+ F(-x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})\}]\}$$

$\leq 4/n$ independently of $\underset{\sim}{t}$. Hence result follows.

Corollary A1:

$$\sup_{-\infty\leq x\leq\infty}\ \sup_{\underset{\sim}{t}\in E_q} E[F^*_{np}(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{t},|x|)]^2 \leq 2/n$$

Proof:

Since $E[F^*_{np}(\underset{\sim}{t},\infty)-F^*(\underset{\sim}{t},\infty)]^2 = 0$, the result is a direct consequence of the above lemma.

Lemma A.2:

If $g(x)$ *is an absolutely continuous function of bounded variation, then* $\lim_{\varepsilon\to 0} V[g(x+\varepsilon)-g(x)] = 0$.

Proof:

Since $g(x)$ is of bounded variation, $\forall\delta > 0 \ \exists\ M = m - 1 \ni V_{[-M,M]}[g(x)] > V[g(x)] - \delta/12$. Thus, if $\varepsilon < 1$, $V[g(x+\varepsilon)-g(x)] \leq \delta/3 + V_{[-m,m]}[g(x+\varepsilon)-g(x)]$. Now, there is a set of real numbers $\{a_i\}$ where $-m = a_o < a_1 < \ldots < a_n = m$ so that

$$V_{[-m,m]}[g(x+\varepsilon)-g(x)] \leq \delta/3 + \sum_{k=1}^{n} |g(a_k+\varepsilon)-g(a_k)-g(a_{k-1}+\varepsilon)+g(a_{k-1})|$$

Since g is absolutely continuous, ε may be chosen $\ni\forall$ real a, $V_{[a,a+\eta]}g(x) < \delta/4n$ if $0 < \eta \leq \varepsilon$. Using this fact together with the above inequalities, it is seen that $V[g(x+\varepsilon)-g(x)] < \delta$ provided ε is sufficiently small. This implies the result.

Lemma A3:

If $F_1(x)$ *is absolutely continuous, and (1.6)-(ii) and (iii) are satisfied, then,*

$$\lim_{n\to\infty} \sup_{\underset{\sim}{t}\in V_n(a)} V[F^*(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{0},|x|)] = 0.$$

Proof:

From (5.5), $V[F^*(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{0},|x|)] \leq (2/np) \sum_{i=1}^{n} \sum_{j=1}^{p} \cdot$ $\{V[F_1(x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(x)]+V[F_1(-x+\underset{\sim}{x}'_{ij}\underset{\sim}{t})-F_1(-x)]$. It follows from (1.6) that $\lim_{n\to\infty} \max_{i,j} \sup_{\underset{\sim}{t}\in V_n(a)} |\underset{\sim}{x}'_{ij}\underset{\sim}{t}| = 0$. Thus it follows from lemma A2 that the R.H.S. of the above inequality tends to zero uniformly for $\underset{\sim}{t} \in V_n(a)$ as n increases.

Corollary A3:

$$\lim_{n\to\infty} \sup_{-\infty\leq x\leq\infty} \sup_{\underset{\sim}{t}\in V_n(a)} |F^*(\underset{\sim}{t},|x|)-F^*(\underset{\sim}{0},|x|)| = 0 .$$

Lemma A4:

Let conditions (1.6)-(ii) and (iii), (5.1)-(i), and (6.1) be satisfied. Then

$$\lim_{n\to\infty} \sup_{\underset{\sim}{t}\in V_n(a)} V\{\psi[\frac{np}{np+1} F^*(\underset{\sim}{t},|x|)]-\psi[\frac{np}{np+1} F^*(\underset{\sim}{0},|x|)]\} = 0$$

Proof:

Given $\varepsilon > 0$, let $N_o > 0$ be $\ni |\psi(\frac{N_o p}{N_o p+1}) - \psi(1)| < \varepsilon/4$. Now $F^*(\underset{\sim}{t}, |x|)$ is absolutely continuous in x since $F_1(x)$ is absolutely continuous. Then, from the continuity of ψ and corollary A3, $\exists$ $N_1 > N_o$, $\eta > 0$, and $A > 0 \ni$ if $n > N_1$

$$\text{(A.1)} \qquad \begin{cases} \sup\limits_{\underset{\sim}{t} \varepsilon V_n(a)} F(\underset{\sim}{t}, A) \leq 1 - \eta \\ \sup\limits_{\underset{\sim}{t} \varepsilon V_n(a)} |\psi[\frac{np}{np+1} F^*(\underset{\sim}{t}, A) - \psi(1)| < \varepsilon/2 \end{cases}$$

Using the notation of (5.24), i.e. $\overline{K}_{np}(\underset{\sim}{t}, |x|) = \frac{np}{np+1} F^*(\underset{\sim}{t}, |x|)$, it follows, in view of (A.1) and the monotonicity of ψ, that

$$\text{(A.2)} \qquad V\{\psi[\overline{K}_{np}(\underset{\sim}{t}, |x|)] - \psi[\overline{K}_{np}(\underset{\sim}{0}, |x|)]\}$$

$$\leq 2V_{[0,A]}\{\psi[\overline{K}_{np}(\underset{\sim}{t}, |x|)] - \psi[\overline{K}_{np}(\underset{\sim}{0}, |x|)]\} + \varepsilon$$

Hence, it is sufficient to prove that the first term on the R.H.S. of (A.2) tends to zero as n increases. For simplicity this term will be denoted by V_A.

Let $\pi(\Delta)$ denote the set of all finite partitions $\{x_i\}$ of $[0,A]$, where $0 = x_o < x_1 < \ldots < x_m = A$ and $\max\limits_{1 \leq r \leq m} |x_r - x_{r-1}| \leq \Delta$. From the definition of total variation, one sees that

$$V_A = \sup_{\pi(\Delta)} \sum_{r=1}^{m} |\psi[\overline{K}_{np}(\underset{\sim}{t},x_r)-\psi[\overline{K}_{np}(\underset{\sim}{0},x_r)]-\psi[\overline{K}_{np}(\underset{\sim}{t},x_{r-1})]-\psi[\overline{K}_{np}(\underset{\sim}{0},x_{r-1})]|$$

$$= \sup_{\pi(\Delta)} \sum_{r=1}^{m} \left| \frac{\psi[\overline{K}_{np}(\underset{\sim}{t},x_r)]-\psi[\overline{K}_{np}(\underset{\sim}{t},x_{r-1})]}{\overline{K}_{np}(\underset{\sim}{t},x_r)-\overline{K}_{np}(\underset{\sim}{t},x_{r-1})} \cdot \frac{\overline{K}_{np}(\underset{\sim}{t},x_r)-\overline{K}_{np}(\underset{\sim}{t},x_{r-1})}{x_r-x_{r-1}} \right.$$

$$\left. - \frac{\psi[\overline{K}_{np}(\underset{\sim}{0},x_r)]-\psi[\overline{K}_{np}(\underset{\sim}{0},x_{r-1})]}{\overline{K}_{np}(\underset{\sim}{0},x_r)-\overline{K}_{np}(\underset{\sim}{0},x_{r-1})} \cdot \frac{\overline{K}_{np}(\underset{\sim}{0},x_r)-\overline{K}_{np}(\underset{\sim}{0},x_{r-1})}{x_r-x_{r-1}} \right| |x_r-x_{r-1}|$$

where the quotients are defined to be zero if both numerator and denominator are zero. Using the mean value theorem, there are real numbers $\{\alpha_r,\beta_r,\gamma_r,\delta_r : r=1,\ldots,m\}$, all in $(0,1)$, so that

$$V_A = \sup_{\pi(\Delta)} \frac{np}{np+1} \sum_{r=1}^{m} |\psi'(\theta_{1r})F^{*'}(\underset{\sim}{t},\lambda_{1r})-\psi'(\theta_{2r})F^{*'}(\underset{\sim}{t},\lambda_{2r})| \cdot |x_r-x_{r-1}|$$

where

$$\text{(A.3)} \quad \begin{cases} \theta_{1r} = \frac{np}{np+1}[\alpha_r F^*(\underset{\sim}{t},x_r)+(1-\alpha_r)F^*(\underset{\sim}{t},x_{r-1})] \\ \lambda_{2r} = \frac{np}{np+1}[\beta_r x_r+(1-\beta_r)x_{r-1}] \\ \theta_{2r} = \frac{np}{np+1}[\gamma_r F^*(\underset{\sim}{0},x_r)+(1-\gamma_r)F^*(\underset{\sim}{0},x_{r-1})] \\ \lambda_{2r} = \frac{np}{np+1}[\delta_r x_r+(1-\delta_r)x_{r-1}] \end{cases}$$

Thus

$$\text{(A.4)} \quad V_A \le \sup_{\pi(\Delta)} \sum_{r=1}^{m} \{|\psi'(\theta_{1r})| \cdot |F^{*'}(\underset{\sim}{t},\lambda_{1r})-F^{*'}(\underset{\sim}{0},\lambda_{2r})|$$

$$+ |\psi'(\theta_{1r})-\psi'(\theta_{2r})| \cdot |F^{*'}(\underset{\sim}{0},\lambda_{2r})|+|\psi'(\theta_{2r})| \cdot |F^{*'}(\underset{\sim}{0},\lambda_{2r})$$

$$-F^{*}(\underset{\sim}{t},\lambda_{1r})|\}|x_r - x_{r-1}|.$$

To prove the desired result, i.e. that $V_A \to 0$ as $n \to \infty$, it will be sufficient to prove the following: $\forall \varepsilon > 0$, $\exists \Delta > 0$ and $N > 0 \ni \forall n \geq N$ and all partitions $\pi(\Delta) = \{x_r : 0 \leq r \leq m\}$

(A.5) $$\sup_{\underset{\sim}{t} \varepsilon V_n(a)} \sup_{\pi(\Delta)} |F^{*'}(\underset{\sim}{t},\lambda_{1r}) - F^{*'}(\underset{\sim}{0},\lambda_{2r})| < \varepsilon$$

(A.6) $$\sup_{\underset{\sim}{t} \varepsilon V_n(a)} \sup_{\pi(\Delta)} |\psi'(\theta_{1r}) - \psi'(\theta_{2r})| < \varepsilon$$

(A.7) $$\sup_{\underset{\sim}{t} \varepsilon V_n(a)} \sup_{\pi(\Delta)} |F^{*'}(\underset{\sim}{0},\lambda_{2r}) - F^{*'}(\underset{\sim}{t},\lambda_{1r})| < \varepsilon$$

That the above three inequalities are sufficient follows from (A.4) and the facts that both $\sup_{x \varepsilon [0,1]} \psi'(x)$ and $\sup_{x \varepsilon (-\infty,\infty)} F^{*'}(\underset{\sim}{0},|x|)$ are finite, which follow from (6.1), (5.1)-(i), and (5.5).

Proof of (A.5):

Using (5.5), the mean value theorem, and the fact that a partition in $\pi(\Delta)$ has norm at most Δ, it follows that

$$|F^{*'}(\underset{\sim}{t},\lambda_{1r}) - F^{*'}(\underset{\sim}{0},\lambda_{2r})| \leq |F^{*'}(\underset{\sim}{t},\lambda_{1r}) - F^{*'}(\underset{\sim}{t},\lambda_{2r})| + |F^{*'}(\underset{\sim}{t},\lambda_{2r}) - F^{*'}(\underset{\sim}{0},\lambda_{2r})| \leq$$

$$2 \sup_{-\infty<x<\infty} |F''(x)| [\Delta + \max_{i,j} \sup_{\underset{\sim}{t} \varepsilon V_n(a)} |\underset{\sim}{x}_{ij}' \underset{\sim}{t}|].$$ Since Δ can be chosen arbitrarily close to zero, and by virtue of (1.6)-(ii) and (iii), $\max_{i,j} \sup_{\underset{\sim}{t} \varepsilon V_n(a)} |\underset{\sim}{x}_{ij}' \underset{\sim}{t}| \to 0$ as $n \to \infty$, the above inequalities imply (A.5).

Proof of (A.7):

Similar to the proof of (A.5).

Proof of (A.6):

Application of the mean value theorem, the fact that $\theta_{1r} < F^*(\underset{\sim}{t},A)$, $\theta_{2r} < F^*(\underset{\sim}{t},A)$, and (A.1), if $n > N_1$ (defined before (A.1)), then for some w_r between θ_{1r} and θ_{2r}, $|\psi'(\theta_{1r})-\psi'(\theta_{2r})| = |\psi''(w_r)\cdot(\theta_{1r}-\theta_{2r})| \leq \sup_{0\leq x\leq 1-\eta} |\psi''(x)|\cdot \sup_{\underset{\sim}{t}\in V_n(a)} \max_{1\leq r\leq m} |\theta_{1r}-\theta_{2r}|$. Now, scrutiny of θ_{1r} and θ_{2r} defined in (A.3) reveal that $\sup_{\underset{\sim}{t}\in V_n(a)} \max_{1\leq r\leq m} |\theta_{1r}-\theta_{2r}|$ can be made arbitrarily small as long as n is sufficiently large, and Δ is sufficiently small. This completes the proof.

<u>*Lemma A.5:*</u>

If conditions (1.6)-(ii) and (iii), (5.1)-(i) and (6.1) are satisfied, then

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} \sup_{\underset{\sim}{t}\in V_n(a)} |\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)| = 0$$

where $\overline{H}_{np}$ *is defined in (5.3) and (5.2).*

Proof:

Using (5.3) and (5.2), it follows that

$|\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)| \leq \sup_{0\leq u\leq 1} \psi'(u)\cdot E|F^*_{np}(\underset{\sim}{t},|x|)-F^*_{np}(\underset{\sim}{0},|x|)|$. Now

$\forall\ \varepsilon > 0,\ \exists N > 0 \ni \forall n > N,\ \max_{i,j} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |\underset{\sim}{x}'_{ij}\underset{\sim}{t}| < \varepsilon/[4 \sup_{0\leq u\leq 1} \psi'(u)\cdot$

$\sup_{-\infty<x<\infty} F_1'(x)] = \varepsilon_1$, say. Then (5.2) implies that

$F^*_{np}(\underset{\sim}{0},|x|-\varepsilon_1) \leq F^*_{np}(\underset{\sim}{t},|x|) \leq F^*_{np}(\underset{\sim}{0},|x|+\varepsilon_1)\ \forall\ \underset{\sim}{t}\ \varepsilon\ V_n(a)$ and

$x\ \varepsilon\ (-\infty,\infty)$, provided $n > N$. Hence,

$$|\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)| \leq \sup_{0\leq u\leq 1} \psi'(u)\cdot E\{F^*_{np}(\underset{\sim}{0},|x|+\varepsilon_1)-F^*_{np}(\underset{\sim}{0},|x|-\varepsilon_1)\}$$

$$\leq 4 \sup_{0\leq u\leq 1} \psi'(u)\cdot \sup_{-\infty<x<\infty} F_1'(x)\cdot \varepsilon_1 = \varepsilon \quad \text{if} \quad n > N.$$

Hence the result follows.

Lemma A.6:

If conditions (1.6)-(ii) and (iii), (5.1)-(i), and (6.1) are satisfied, then

(i) $\forall\varepsilon > 0,\ x\ \varepsilon\ (-\infty,\infty)$,

$$0 \leq \overline{H}_{np}(\underset{\sim}{t},|x|+\varepsilon) - \overline{H}_{np}(\underset{\sim}{t},|x|) \leq 2\varepsilon \sup_{0\leq u\leq 1} \psi'(u)\cdot \sup_{-\infty<v<\infty} F_1'(v)$$

(ii) $0 \leq \psi(\frac{np}{np+1}) - \overline{H}_{np}(\underset{\sim}{t},|x|) \leq [1-F^*(\underset{\sim}{t},|x|)] \sup_{0\leq u\leq 1} \psi'(u)$

Proof:

Arguments similar to those of lemma A.5 are used and thus the details are omitted.

<u>*Lemma A.7:*</u>

If conditions (1.6)-(ii) and (iii), (5.1)-(i), and (6.1) are satisfied, then

$$\lim_{n\to\infty} \sup_{\underset{\sim}{t}\in V_n(a)} V[\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)] = 0$$

Proof:

From lemma A.6 - (i), $\overline{H}_{np}(\underset{\sim}{t},|x|)$ is monotone nondecreasing and continuous in $|x|$. Further, it is a consequence of lemma A.6 - (ii) that $\forall\varepsilon > 0, \exists A > 0, \eta > 0$ and $N > 0 \ni \forall n \geq N$,

$$\text{(A.8)}\qquad \begin{cases} F^{*}(\underset{\sim}{0},A) \leq 1 - \eta \\ \overline{H}_{np}(\underset{\sim}{0},A) \geq \psi(1) - \varepsilon/12 \end{cases}$$

Hence $V[\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)] = 2V_{[0,A]}[\overline{H}_{np}(\underset{\sim}{t},x)-\overline{H}_{np}(\underset{\sim}{0},x)]$ $+ 2V_{[A,\infty]}[\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)]$. By the monotonicity of $\overline{H}_{np}(\underset{\sim}{t},|x|)$, the choice of A, and lemma A.5, it is evident that $2V_{[A,\infty]}[\overline{H}_{np}(\underset{\sim}{t},|x|)-\overline{H}_{np}(\underset{\sim}{0},|x|)] < \varepsilon$ provided n is sufficiently large. Thus it is sufficient to show that

$$\text{(A.9)}\qquad \lim_{n\to\infty} \sup_{\underset{\sim}{t}\in V_n(a)} V_{[0,A]}[\overline{H}_{np}(\underset{\sim}{t},x)-\overline{H}_{np}(\underset{\sim}{0},x)] = 0$$

where A satisfies (A.8).

Let $\pi(\Delta_n)$ be defined as in the proof of lemma A.4. Then, in view of (5.2) and (5.24), if $\Delta_n > 0$,

$$\text{(A.10)} \quad \begin{cases} V_{[0,A]}[\bar{H}_{np}(\underset{\sim}{t},x)-\bar{H}_{np}(\underset{\sim}{0},x)] = \sup_{\pi(\Delta_n)} \sum_{r=1}^{m} |E(D_{nr}(\underset{\sim}{t}))|, \text{ where} \\ D_{nr}(\underset{\sim}{t}) = \psi[K_{np}(\underset{\sim}{t},x_r)] - \psi[K_{np}(\underset{\sim}{t},x_{r-1})] - \psi[K_{np}(\underset{\sim}{0},x_r)] \\ \qquad + \psi[K_{np}(\underset{\sim}{0},x_{r-1})] \end{cases}$$

and $\{\Delta_n\}$ is any sequence of positive real numbers. A routine calculation leads to

$$\text{(A.11)} \quad \begin{cases} \sum_{r=1}^{m} |E(D_{nr}(\underset{\sim}{t}))| = \sum_{r=1}^{m} |E\{u_r[K_{np}(\underset{\sim}{t},x_r)-K_{np}(\underset{\sim}{t},x_{r-1})] \\ \qquad + v_r[K_{np}(\underset{\sim}{t},x_r)-K_{np}(\underset{\sim}{t},x_{r-1})-K_{np}(\underset{\sim}{0},x_r)+K_{np}(\underset{\sim}{0},x_{r-1})]\}| \end{cases}$$

where

$$u_r = \frac{\psi[K_{np}(\underset{\sim}{t},x_r)]-\psi[K_{np}(\underset{\sim}{t},x_{r-1})]}{K_{np}(\underset{\sim}{t},x_r)-K_{np}(\underset{\sim}{t},x_{r-1})} - \frac{\psi[K_{np}(\underset{\sim}{0},x_r)]-\psi[K_{np}(\underset{\sim}{0},x_{r-1})]}{K_{np}(\underset{\sim}{0},x_r)-K_{np}(\underset{\sim}{0},x_{r-1})}$$

$$v_r = \frac{\psi[K_{np}(\underset{\sim}{0},x_r)]-\psi[K_{np}(\underset{\sim}{0},x_{r-1})]}{K_{np}(\underset{\sim}{0},x_r)-K_{np}(\underset{\sim}{0},x_{r-1})}$$

By the mean value theorem there is $\{\alpha_r,\beta_r : r=1,\ldots,m\}$, all in $(0,1)$ so that

$$|u_r| = |\psi'[\alpha_r K_{np}(\underset{\sim}{t},x_r)+(1-\alpha_r)K_{np}(\underset{\sim}{t},x_{r-1})]$$
$$- \psi'[\beta_r K_{np}(\underset{\sim}{0},x_r)+(1-\beta_r)K_{np}(\underset{\sim}{0},x_{r-1})]|$$

and in view of (5.24), and mean value theorem

$$|u_r| \le \min\{2 \sup_{0\le x\le 1} \psi'(x), \sup_{0\le x\le d} |\psi''(x)|\{|\alpha_r[F^*_{np}(\underset{\sim}{t},x_r)$$
$$- F^*(\underset{\sim}{t},x_r)]+(1-\alpha_r)F^*_{np}(\underset{\sim}{t},x_{r-1})-F^*(\underset{\sim}{t},x_{r-1})]$$
$$+ \beta_r[F^*_{np}(\underset{\sim}{0},x_r)-F^*(\underset{\sim}{0},x_r)]| + |\alpha_r F^*(\underset{\sim}{t},x_r)+(1-\alpha_r)F^*(\underset{\sim}{t},x_{r-1})$$
$$- \beta_r F^*(\underset{\sim}{0},x_r)-(1-\beta_r)F^*(\underset{\sim}{0},x_{r-1})|\}\}$$

where $d = \sup_{\underset{\sim}{t}\in V_n(a)} F^*_{np}(\underset{\sim}{t},A)$.

It follows from (A.8), corollary 5.3, and the absolute continuity of F^* that $\forall\varepsilon_1 > 0, \exists N_1 > 0 \ni P_n[d\le 1-\eta/2] \ge 1 - \varepsilon_1$ provided $n \ge N_1$, and hence that $\forall\varepsilon > 0, \exists N > 0 \ni \forall n \ge N$, $P_n[\max_{1\le r\le m} \sup_{\underset{\sim}{t}\in V_n(a)} |u_r|>\varepsilon] < \varepsilon$ provided $n \ge N$, and $\{x_r:r=1,\ldots,m\} \in \pi(\Delta_n)$ where the $\{\Delta_n\}$ are sufficiently small. Therefore

$$\text{(A.12)} \quad \sum_{r=1}^{m} |E\{u_r[K_{np}(\underset{\sim}{t},x_r)-K_{np}(\underset{\sim}{t},x_{r-1})]\}|$$
$$\le E\{\sum_{r=1}^{m} |u_r|[F^*_{np}(\underset{\sim}{t},x_r)-F^*_{np}(\underset{\sim}{t},x_{r-1})]\}$$
$$\le E\{\sum_{r=1}^{m} |u_r|[F^*_{np}(\underset{\sim}{t},x_r)-F^*_{np}(\underset{\sim}{t},x_{r-1})]\,|\max_{1\le r\le m} \sup_{\underset{\sim}{t}\in V_n(a)} |u_r|<\varepsilon\}$$
$$\cdot P_n[\max_{1\le r\le m} \sup_{\underset{\sim}{t}\in V_n(a)} |u_r|<\varepsilon\}$$

$$+ E\{\sum_{r=1}^{m} |u_r| [F^*_{np}(\underset{\sim}{t},x_r) - F^*_{np}(\underset{\sim}{t},x_{r-1})] \Big| \max_{1\leq r\leq m} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |u_r| \geq \varepsilon\}$$

$$\cdot P_n[\max_{1\leq r\leq m} \sup_{\underset{\sim}{t}\varepsilon V_n(a)} |u_r| \geq \varepsilon\}$$

$\leq 2\varepsilon$ if $\{x_r : r=1,\ldots,m\} \varepsilon \pi(\Delta_n)$ and $n \geq N$.

Now let us consider the second term on the R.H.S. of (A.11). After some computation requiring corollary 5.3, it follows that $\forall\varepsilon > 0, \exists N > 0 \ni P_n[\max_{1\leq r\leq m} |v_r - \psi'[F^*(\underset{\sim}{0},x_r)]| \geq \varepsilon] \leq \varepsilon$ provided $n \geq N$ and $\{x_r : r=1,\ldots,m\} \varepsilon \pi(\Delta_n)$ where $\{\Delta_n\}$ are sufficiently small.

After some computation which involves corollary 5.3 and (5.24), it follows (similarly to (A.12)) that

$$\text{(A.13)} \quad \sum_{r=1}^{m} |E\{v_r[K_{np}(\underset{\sim}{t},x_r) - K_{np}(\underset{\sim}{t},x_{r-1}) - K_{np}(\underset{\sim}{0},x_r) + K_{np}(\underset{\sim}{0},x_{r-1})]\}|$$

$$\leq \sup_{0\leq x\leq 1} \psi'(x)\{V_{[0,\infty]}[F^*(\underset{\sim}{t},x) - F^*(\underset{\sim}{0},x)]$$

$$+ 2\varepsilon E[V[F^*_{np}(\underset{\sim}{t},x) - F^*_{np}(\underset{\sim}{0},x)]]\}$$

In view of lemma A.3, the above expression can be made arbitrarily small if n is sufficiently large.

Substitution of (A.13) and (A.12) into (A.11) yields $\lim_{n\to\infty} \sup_{\pi(\Delta_n)} \sum_{r=1}^{m} |E(D_{nr}(\underset{\sim}{t}))| = 0$ provided $\{\Delta_n\}$ are sufficiently small. Thus it follows from (A.10) that (A.9) is proved and the proof is complete.

The following lemma is similar to lemma 3.1. It is an extension of lemma 2.1 of Hàjek (1961). Let $\{U_{ij}:i=1,\ldots,n;j=1,\ldots,p\}$ be random variables, independent for different i. Let

(A.14) $$P(U_{i1}\leq x_1,\ldots,U_{ip}\leq x_p) = F(x_1,\ldots,x_p)$$

where the marginal distributions of F are uniform on $[0,1]$. Also assume $P[U_{ij}=U_{ij'}] = 0$ to avoid ties with probability one.

Let R_{ij} = rank of U_{ij} in the joint ranking of the $\{U_{ij}:i=1,\ldots,n;j=1,\ldots,p\}$. Let $Z_1,\ldots,Z_{np}$ be the ordered U_{ij}'s : $Z_1 < Z_2 < \ldots < Z_{np}$. Hence $Z_\alpha = U_{ij}$ if the rank of U_{ij} is α.

Let $a_1,\ldots,a_{np}$ be a nondecreasing sequence of real numbers, and define

$$a(\lambda) = a_i \quad \text{for} \quad (i-1)/np < \lambda \leq i/np \qquad (1\leq i\leq np)$$

Note that $a_i = a(i/np) = a(i/(np+1))$. Then

Lemma A.9:

Under the above mentioned conditions, if $\bar{a} = (np)^{-1}\sum_{\alpha=1}^{np} a_\alpha$, *then*

$$E[a(U_{11})-a(R_{11}/np)]^2 \leq (2p)^{3/2} \max_{1\leq i\leq np} |a_i-\bar{a}| \; [\sum_{\alpha=1}^{np} (a_\alpha-\bar{a})^2]^{\frac{1}{2}}$$

Proof:

Using conditional expectation,

$$E[a(U_{11})-a(R_{11}/np)]^2 = \sum_{\alpha=1}^{np} E\{[a(U_{11})-a(R_{11}/np)]^2|R_{11}=\alpha\} \cdot$$

$$\{P(R_{11}=\alpha|R_{1j}=\alpha \text{ for some } j = 1,\ldots,p)\cdot P(R_{1j}=\alpha \text{ for some } j=1,\ldots,p)$$

Now $P(R_{1j}=\alpha$ for some $j=1,\ldots,p) = n^{-1}$ since the random vectors $(U_{i1},\ldots,U_{ip})$, $i = 1,\ldots,n$ are i.i.d. Now let $P(R_{1\gamma}=\alpha|R_{1j}=\alpha$ for some $j=1,\ldots,p) = r_\gamma$. Then,

$$E[a(U_{11})-a(R_{11}/np)]^2 = (r_1/n) \sum_{\alpha=1}^{np} E\{[a(Z_\alpha)-a(\alpha/np)]^2\}$$

The remainder of the proof closely follows that of lemma 2.1 of Hàjek (1961) and hence the details have been omitted. At one point, the inequality $E(K-k)^2 \leq pk(1-k/np)$, where $K = \sum_{i=1}^{n} \sum_{j=1}^{p} I[U_{ij}\leq k/np]$, is needed. This can be verified by direct calculation.

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