Towards Recursive Mathematics Curricula:

A Complexified Hermeneutic Journey

by

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Abstract

Derived from Doll's (1993) seminal conceptualization of a post-modern curriculum with the criteria of 4Rs (i.e., richness, relations, recursion and rigor), the present research continues the effort to complexify and theorize recursion and recursive curriculum. This study re-conceptualizes mathematics curriculum as recursive through the lens of complexity thinking (Davis & Sumara, 2006), which studies complex systems that are adaptive, such as cognition and knowledge. A mathematics curriculum often seems be designed or delivered as linear: a sequence of predetermined, sometimes unrelated, topics with few chances for learners to revisit them from different perspectives. This suggests learning as accumulation with predictable outcomes. Learning, observed through a complexified world view, is neither linear nor predictable. Learning is a self-organizing process through which a learner and her environment co-evolve, and a recursive elaboration through which a learner transforms her previous understanding (Davis & Sumara, 2002; Davis, Sumara, & Luce-Kapler, 2008). Both learners and school subjects are complex systems with a biological structure (Davis & Sumara, 2002) that emerges. This view demands a recursive curriculum that centers on reviewing previously encountered ideas with an orientation towards newness and changes along its formation. What might such mathematics curriculum be like, particularly at high school level, in theory and practice is my research focus.

The research methodology follows the tradition of hermeneutics (Gadamer, 1989/2013) that attends to language and emphasizes emerging understanding through iterative loops of interpretations. The interpretations in this research are informed by three kinds of entry texts, my personal reflections about recursive curriculum, teaching documents (i.e., programs of studies and textbooks), and conversations with teachers, serving to provoke my thinking and generate

further reflection subjected to new rounds of interpretations. Several high school mathematics teaching documents from Canada and China were examined to see in what ways a planned curriculum might afford recursion. Conversations with experienced high school mathematics teachers were conducted in professional development workshops and/or individual meetings. Teachers were invited to reflect on their learning and teaching experiences and comment on several teaching and learning practices (e.g., reviewing), and work with me to revise or generate curriculum materials to promote such practices orientated towards helping students to learn something new from what they have encountered before.

This study aims to make a contribution in the field of mathematics education by addressing the gap between the perceived importance of recursive mathematics curricula and the insufficiency of research about them. I expect that this study speaks to a reinterpretation of reviewing, and potentially provokes learners (both teachers and students) to interpret mathematics and curriculum differently and inspires learners to (re)embark a complexified hermeneutic inquiry on recursive mathematics curricula for the purpose of both learning and teaching. This study has led to a metaphorical and iconic image (see the image on p. v or Figure 9.4.8) of recursive curricula that represents abundant curriculum possibilities rather than a fixed one. Such visualization can provide theoretical and practical references for mathematics educators and education researchers to draw inspirations from when designing towards recursive mathematics curricula.

Preface

This research is an original study by Lixin Luo. The research project, of which this thesis is a part, received research ethics approval from the University of Alberta Research Ethics Board, (project name "Towards a Recursive Mathematics Curriculum", No. 00054834) on March 10, 2015. The same project, under the title "Towards a Recursive Mathematics Curriculum: A Hermeneutic Inquiry", also received research ethics approval from four school districts' ethics boards in Edmonton AB, Canada: Elk Island Public Schools on April 17, 2015, Edmonton Catholic Schools on June 3, 2015, St Albert Public Schools on June 12, 2015, and Edmonton Public Schools on July 21, 2015.

Parts of chapters 1, 2 and 3 of this thesis have been published in L. Luo, 2014, "Recursion in the mathematics curriculum," *Philosophy of Mathematics Education Journal*, 28. Parts of chapter 8 have been published in L. Luo, 2015, "Repetition as a means of encouraging Tall's met-befores," *Delta-k* (Journal of the Mathematics Council of the Alberta Teachers' Association), vol. 52, issue 2, 15-17. Parts of chapters 1, 2, 3, and 4 have been published in L. Luo, 2019, "A recursive path to infinity," In M. Quinn (Ed.), *Complexifying Curriculum Studies: Reflections on the Generative and Generous Gifts of William E. Doll, Jr*. (pp. 94-101), New York, NY: Routledge. Parts of the abstract and the image on p. v or Figure 9.4.8 have been published in L. Luo, 2019, "Abundant recursive mathematics curricula possibilities" [Image participated in the Image of Research Competition 2019 at the University of Alberta]. Available from https://era.library.ualberta.ca/items/7c7edca1-b704-4c11-9467-980bda1af444

This work is dedicated to

Bill Doll

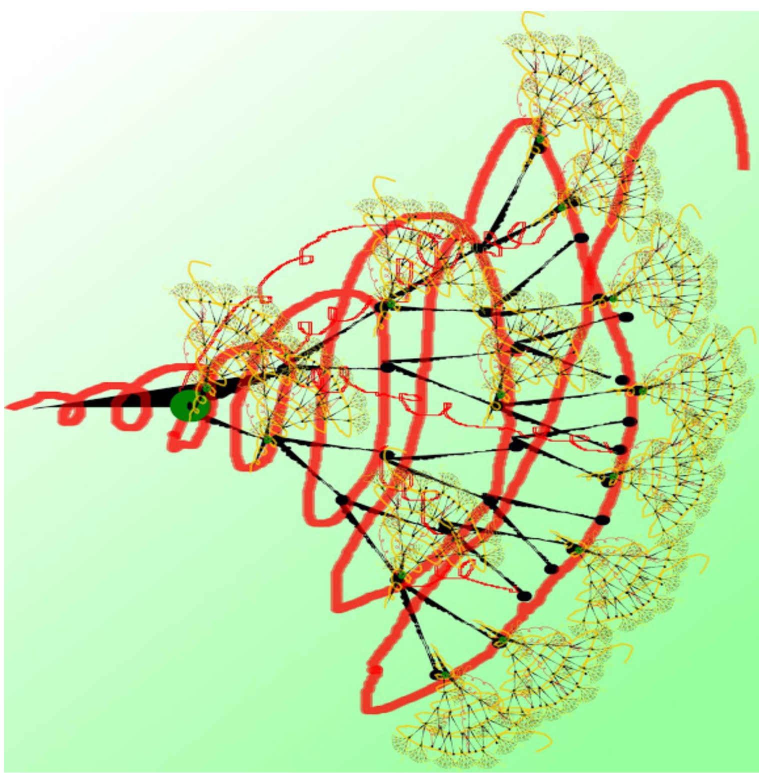
Who taught me complexity thinking through his enactment and praxis

Who cherished my recursive wrestling with recursion

&

Lixin

Who struggled and survived



Acknowledgements

Writing an acknowledgement, from a perspective of complexity thinking, is challenging.

How can I thank all the others who have contributed to my study, when I have broadened the concept of others to include things and processes beyond living beings; when I recognize the Butterfly Effect and accept learning as being occasioned instead of being caused; and when the lines between personal and academic, between physical and mental, and between individual and collective are no more rigid? In what ways, for example, can I give credit to people like a yoga instructor who taught me to listen to my body without judging, to spaces like the Rutherford library, the river valley, Internet, and society at large, to daily processes like cooking, cleaning, and walking, to random ideas like those I bumped into on the radio, and to diverse issues and questions that arose from trying to love and respect imperfect self and other beings?

Exactly as Dewey (1929/2004) says, education is "a process of living" (p. 19). The possible contributors for my research are infinite. Here, I can only try to thank a few individuals whose direct involvement in this thesis has made a difference.

This research would not have started and continued without Dr. William E. Doll, Jr.'s teaching. Bill's enactment and praxis of complexity thinking has led me into a learning process that has transformed all aspects of my life. I am also blessed to have Dr. Tom Kieren as a cheerful thinking partner in my research journey. The humbleness and excitement both Bill and Tom exhibited when talking with a novice like me have been influential for my understanding of recursion.

I am grateful for my supervisor Dr. Elaine Simmt. Elaine's pointed questions and practical guidance have kept me from wandering around forever in frustration and in awe and getting trapped in the many rabbit holes that I dug myself into. Our debates have helped me reshape many messy and fuzzy concepts that are often all-at-once. Gratitude also goes to my supervisor committee and examination committee members. I thank Dr. Florence Glanfield for her caring manner and teaching me as if we were walking alongside with each other. I appreciate Dr. Kent den Heyer's warm encouragements and sharp philosophical insights that helped me to focus on what matters. I thank Dr. Lynn McGarvey and Dr. Jayne Fleener for offering me constructive and intriguing questions and comments about my thesis. I appreciate all professors' effort to keep the final defense challenging and conversational, allowing me to learn further.

This project is not feasible without the effort and support of my teacher participants. These teachers' interest in learning reflectively together and openness towards new ideas have made thinking collectively and creatively during our conversations possible. Special thanks go to Maxine (a pseudonym). Her commitment to all six workshops has ensured the continuity of my conversation text generation with necessary repetitions.

Thanks also go to my fellow graduate students who help me engender, experience, and enact hermeneutic conversations that grow in the space between us and keep our thinking going. Particularly I thank Priscila Dias Correa, Xiong Wang, Jihye Yoon, and Jing Cui for showing me curiosity, sincerity, and care.

Finally, I thank my parents for cultivating an interest of learning and reflection in me. I thank my Chinese family, particularly, my first niece and my mom, for offering me opportunities to study strong complex learning systems recursively and to relearn about past and reimagine future from a different perspective. I thank my first Canadian friend/sister Sheila Rose Richardson, my husband Abram and his family for opening my eyes to many possibilities, engaging me with interesting ideas, and inviting me to new experiences. I thank my families in two cultures for supporting me over the years and affording me a hermeneutic inquiry about love, respect, and faith. Particularly, I would not have been here without Abram, who continues to amaze me with contagious passion for learning and living a responsible fulfilling life.

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1 A Calling to Recursion

"There is always a story that happened once upon a time", and such story is exactly where a hermeneutic work starts (Jardine, 2015, p. 238).

I was called to the idea of recursion.

The first time I encountered "recursion", I was in Bill Doll's graduate class. Along with richness, relations, and rigor, recursion was introduced to me as one of the four criteria of Doll's (1993) post-modern curriculum. It reminded me of Confucius's teaching in *Analects (论语 lun yu)*: 温故知新 *wen gu zhi xin*, translated as "gain new knowledge by reviewing old; understand the present by reviewing the past" (Chinese Academy of Social Sciences, 2002, p. 2003). The concept of recursion appeared to me as iterations of the common emphasis of reflection (反思 *fan si*), review (复习 *fu xi*), and repetition (重复 *chong fu*) throughout my schooling and living in China. Still, revisiting these ideas and connecting them to recursion in a new context (i.e., post-modernism, complexity thinking), I thought of them differently (e.g., I learned to reflect by thinking critically and aim reflection towards relations and different understandings rather than *the* truth) and came up with some practical applications of recursion in teaching (see Luo, 2004). Yet, I found the concept of recursion suspiciously straight forward.

The idea of recursion got more puzzling when I started to implement it in my mathematics classes. Interpreting recursion as recursive reflection, I tried out various methods to encourage students to reflect, including reflection journals, retests, learning method reviews and mathematics content reviews. It was my hope that through learning from what I modeled and promoted in class, students could become more reflective. Meanwhile, my professional growth benefited tremendously from constantly rereading the textbooks, reflecting on my practices and modifying them, and mostly by teaching the same course or courses related to each other in different classes over years. While my learning process seemed to be recursive, my students' learning processes appeared to remain primarily linear: Many students showed little willingness and efficiency to learn from their previously learned topics and mistakes. The activities I used to encourage reflection remained at the margin of the lived curriculum. We spent most of class time rushing through topics: There seemed so little time for so many curriculum contents for which my students showed low readiness. I sensed, somehow, there was more to do to make a curriculum recursive than simply allocating space and time for reflection and offering time between reflections.

Confused, I revisited Doll's concept of recursion during the first year of my doctoral program. This reencounter with recursion brought forth new insights about the concept (see Luo, 2014). Subtly yet saliently, my focus, on my interpretation of recursion as “recursive reflection”, shifted from “reflection” to “recursive”. This emphasis, following complexity thinking, suggests recursion more as a continuous generative looping back movement. The question in recursion, then, is not just *how to reflect*, but also *how to loop back*. With this new interpretation, I reconsidered the idea of recursive curriculum, stressing practices with a structure of looping back. Reviewing, in which students go over what they have learned before, logically came to focus. I started to wonder, what kind of reviewing can afford recursion? More generally, how can one design looping back processes that help learners build connections and see something new from what they have encountered before? Or, what might a recursive curriculum be?

These questions are worth asking. As shown in the literature review in Chapter 3, there is a gap in how little we (i.e., mathematics educators and mathematics education researchers) know about and enact a recursive curriculum and how important it is for mathematical learning. The word curriculum here refers to braid that includes (at least) planned, lived and hidden dimensions.

Although mathematics curricula may be designed with a recursive quality, my teaching experience in Canadian high schools led me to a concern that the lived mathematics curricula tend to be linear: Mathematics topics tend to be covered with little chance of being revisited from different perspectives later, resulting in a fragmental view of mathematics knowledge. As such, mathematics learning can easily become a process of linearly accumulating disconnected topics. To enhance cognitive growth, a nonlinear curriculum centered on recursion is a promising direction through the lens of complexity thinking. A recursive mathematics curriculum fits with how mathematical understanding is thought to develop: According to Pirie and Kieren's (1994) model, the growth in mathematical understanding follows a recursive path.

Motivated by the rich potential of recursion, I embarked on a hermeneutic inquiry about recursive mathematics curricula. The leading question in this research is, "*What might a high school recursive mathematics curriculum informed by complexity thinking be?*" To answer this question, I sought inspirations through reinterpreting multiple texts, including autobiographical reflections, teaching documents (i.e., programs of studies and textbooks), and conversations with experienced teachers. Three sub-questions are used to guide my text generation and interpretation:

1) How do my teaching and learning experiences inform my understanding of recursive mathematics curriculum?
2) How do teaching documents inform my understanding of recursive mathematics curriculum?
3) How do conversations with experienced high school mathematics teachers inform my understanding of recursive mathematics curriculum?

The purpose of the research is to enrich the interpretations of the concept of recursive mathematics curriculum. This research aims to contribute to mathematics education by narrowing the gap between the perceived importance of the recursive mathematics curriculum and the insufficiency of research about it. Given that I entered this study with an initial focus on the affordance of a recursive curriculum for students' mathematics learning but was charmed later (as one can see in Chapter 4 on the research process) by a recursive curriculum's affordance for the mathematics learning of teachers, I believe an inquiry about recursive curriculum is beneficial for the inquirer to develop mathematical and pedagogical understanding thus it is worthwhile for both teachers and students to take on. I use "learners" in this dissertation to refer to both students and teachers and use a learner's pedagogical understanding to refer to an understanding of learning process that can be used to guide one's and others' learning. I expect that this research speaks to a reinterpretation of reviewing, and potentially provokes learners to interpret mathematics and curriculum differently and inspires learners to (re)embark on a hermeneutic inquiry on recursive mathematics curricula for the purpose of both learning and teaching. In short, the goal of this study is to inspire, to rekindle, and to continue growing the seed(ling) that has been planted.

Before moving on to the details of the study, it is necessary to specify a few writing strategies used in this thesis. First, italics are frequently used as a way to interrupt, nudge and perturb, and to call for rereading, re-sensing and reconsideration, particularly towards any subtle differences that one's reading might bring forth. Second, all the figures I created during this study were heuristic visualizations, meaning that their generation was my process of thinking through images instead of outputting completed thoughts. Therefore visual spatial elements, such as layout, line, curve, shape, color, font, empty space etc., were not used for decorative purpose but for eliciting unconscious knowing and incurring emerging possibilities. Particularly, I often used colors and fonts in certain ways without rational reasons until their appearance spoke to me some unnoticed relations, which prompted me to visualize again. Some figures turned out having multiple elements to signify the same message. For instance, I used both font and capitalization to differentiate class from case in Figure 8.2.3. In addition, many images are like doodling instead of clear-cut diagrams, signifying their incompleteness and organic becoming. The redundancy and incompleteness of the images are kept in this thesis to preserve the role of images as both medium and unfinished product of my thinking process, and to invite readers to review the images and visualize again. Third, to emphasize part-whole relationship, sections and figures are labeled in a way indicating their respective relations to a bigger whole where they locate (e.g., section 8.2 is the second section in Chapter 8, and Figure 8.2.3 is the third figure in section 8.2).

One must note that although I am interested in nonlinearity in this study and the research (including writing) process is nothing but linear, this does not negate a "final" text of this study presented as a linear document. Given that spiral movements are well presented through multiple rounds of reinterpretations of the same idea (e.g., a form of re-viewing, re-viewing as a whole, recursion, recursive curriculum) in one chapter or across chapters, this linear form is recognized as an adequate one before a more effective form of writing is established. Given that I got to experience recursion in different ways in this study and consequently transformed my understanding of hermeneutic research process, recursion, recursive curriculum, mathematics, education, and self, it would not be surprising for a reader to hear similar stories resonating when reading different parts of the dissertation. Meanwhile, as a result of this study being influenced by hermeneutics and complexity thinking, each attempt of looking at the study as a whole or ending the writing led to a new round of reinterpretation. This is reflected particularly in the last three chapters, with each chapter trying to offer a holistic view of the study outcome yet only resulting into renewed interpretations of recursion and recursive curriculum.

Another thing worth noting is that several metaphors (e.g., loops of spiral, hermeneutic circle, story, fractal tree-spiral) are used in this thesis as mediums for thinking, expressing, and playing with ideas. My affinity with metaphors came more from my experience of growing up in Chinese language and culture, in which analogies (including metaphors) are omnipresent, and less from literature related to metaphors. While I agree with Lakoff and Johnson (1980) that metaphors influence thoughts, how the particular metaphors used in my research influence my study and how metaphors in general influence thoughts is not the focus of the research at current stage. Rather, the returning to metaphors as mediums of thoughts and contemplating on their influences on thinking belong to the next research cycle.

Here I give an overview of the rest of the thesis. In Chapter 2, I set up my study by articulating its theoretical framework (i.e., complexity thinking) and methodology (i.e., hermeneutic inquiry). In Chapter 3, I provide a working definition of recursion based on a preliminary concept study. I then establish the rationale of the study, by arguing for the importance of recursion in education through drawing support from complexity thinking and mathematics education research and by contrasting the insufficient amount of research about recursive curricula with their importance. In Chapter 4, I describe the research design and its dynamic process, reflect on how the process transformed my understanding of hermeneutic inquiry and recursion, and propose an interpretation of the hermeneutic study process as fractal-like. Chapter 5 is a review of the process of reviewing through which I complexify it as re-viewing, a form of recursion. I propose and conceptualize three forms of re-viewing (i.e., re-languaging, re-imaging, and re-inbodying) in Chapter 6 by drawing concepts and terminology from semiosis and by interpreting and reinterpreting some lived experiences of re-viewing. I then look at these three forms of re-viewing as a whole and reinterpret it as re-storying in Chapter 7. In Chapter 8, I re-theorize re-viewing as re-encountering and reinterpret recursion and recursive curriculum, before moving on to consider the implications of such interpretations in curriculum design in Chapter 9. This attempt brings forth new visualizations of recursive curricula. Chapter 10 looks into the practicality of recursive curriculum in classroom contexts where there often are overarching linear curriculum frameworks at play. This connects back to my personal struggling as a classroom teacher and my relationship with mathematics, resulting into transformations in both self-understanding and interpretations of recursive curriculum.

2 Theoretical Underpinning and Methodology

Complexity thinking is the theoretical framework for this study, and hermeneutic inquiry is chosen as a suitable methodology for it. In this chapter, I provide an overview of complexity thinking and an interpretation of education from this theoretical perspective. Then I examine the hermeneutic inquiry as a research methodology in general. Lastly, I connect complexity thinking and hermeneutics and explain why hermeneutic inquiry is an appropriate methodology for my research.

2.1 Complexity Thinking

The historical development of complexity thinking

Complexity thinking, Davis and Sumara (2006) explain, "arose in the confluence of several areas of Western research, including cybernetics, system theory, artificial intelligence, chaos theory, fractal geometry, and nonlinear dynamics" (p. 7). Although many of these lines of inquiry started to develop in the 1950s and 1960s, the origin of complexity thinking can be traced back to many earlier works, such as Giambattista Vico's book *New Science* published in 1744, Henri Poincaré's work related to the three-body problem in 1903, Ludwig Von Bertalanffy's general system theory developed in the 1940s and 1950s, and Gregory Bateson's work on cybernetics in the 1940s[1].

In the 1960s, the establishment of chaos theory supported Poincaré's idea of unpredictability, a key notion in complexity thinking. Also, Prigogine's study on thermodynamic systems furthered Bertalanffy's ideas about open systems, and brought forth some key characteristics of complex systems (i.e., "dissipative structure", "far from equilibrium", and "self-organization").

[1] My understanding of the historical development of complexity thinking is largely informed by Davis and Sumara's (2006), Doll's (1986, 1989, 2002, 2008), Capra's (1996), and Fleener's (2005) works.

Starting from the 1970s, complexity research gained great momentum through the use of computer technology. Hypotheses and conjectures related to complexity systems were tested. Through computer simulations of various systems, complexity theory started to emerge as a field of inquiry in the late 1970s and early 1980s. Particularly, Benoît Mandelbrot's (1967, 1977, 1983) establishment of fractal geometry and its wide application in various disciplines further supported a new world view: "Nature embraces not simplicity but complexity" (Doll, 2002, p. 45).

The development of complexity research sped up after 1986. Complexity thinking was popularized through the increasing use of computer technology in the 1980s and 1990s. By the 1990s, complexity research was a discernible domain. Its focus shifted to the stimulation and development of complex systems. At the end of the 1990s, complexity theory had developed such rigor that it was renamed as complexity science. Complexity thinking, commonly called complexity science, was not a distinct term until the 1990s. Based on Richardson and Cilliers (2001), complexity thinking refers to a school of thinking that "focuses on the epistemological consequences of assuming the ubiquity of complexity" (p. 7), which means an attitude that "is concerned with the philosophical and pragmatic implications of assuming a complex universe, and might thus be described as representing *a way of thinking and acting"* (Davis & Sumara, 2006, p. 18).

Key ideas in complexity thinking

Complexity thinking focuses on the study of complex systems, which are pervasive in the world. Anthills, climates, ecosystems, economies, cultures, brains and living units are all examples of complex system. A complex system is a self-organizing and adaptive system that exhibit attributes that are not possessed by any of its components (Capra, 1996; Davis & Sumara, 2006). For a complex system, the whole is greater than the sum of the parts. Thus one needs to understand a complex system both holistically and analytically.

Complex systems' self-organization is sustained through the nonlinear feedback in them (Capra, 1996). The feedback loops enable the influence of one change in a system to loop back to the system itself. Complex systems are also open systems as they consistently exchange energy and matter with their environment (Davis & Sumara, 2006). Because of their self-organization and openness towards the outside world, complex systems can maintain their current structures while being open to emerging possibilities at the same time. They are unpredictable and nondeterministic: Changing one part of a system in one way does not guarantee certain behaviors from the other parts.

A complex system is fractal-like, meaning that the system demonstrates self-similarity across various scales and its development is recursive (Davis & Sumara, 2006). Fractal was coined by Mandelbrot from the Latin adjective *fractus*, which means "irregular or fragmented" (Mandelbrot, 1977, p. 4) to refer to and depict a self-similar phenomenon. Self-similarity describes a signature property of fractals, that each portion of a fractal can be viewed as a reduced-scale image of the whole (Mandelbrot, 1967), or simply put, that you can see the whole through a part of it. While the similarities between parts and whole in a strictly self-similar fractal (e.g., the Koch snowflake in Figure 3.1.2) are exact, the ones in a fractal-like system are approximate—if you zoom in on different parts of it and you see slightly different copies of the whole. Fractals are formed by recursion through infinitely many stages (Mandelbrot, 1977, 1983). The fascinating and powerful aspect of fractals is that the complexity of fractals originates from following some simple recursive rules. Often a simple recursive formula can generate complicated fractals, such as the Mandelbrot set and the Julia sets. More details can be found in an extended interpretation of recursion in Chapter 3.

Moreover, complex systems have embedded structures: A complex system often composes and comprises other complex systems. Due to the openness of the complex systems, the boundary between complex systems is heuristic rather than fixed (Davis & Sumara, 2006). The system and its environment have a nonlinear coupling relationship (Sumara & Davis, 1997). The word "coupling" refers to Maturana and Varela's (1998) notion of structural coupling, which denotes that "there is a history of recurrent interactions leading to the structural congruence between two (or more) systems" (p. 75). A coupling relationship is a mutual adaptive relationship between two or more self-organizing systems. With such relationship, a complex system and its environment co-specify each other and evolve together simultaneously. In other words, they co-emerge. Hence, to understand a complex system, one has to think relationally and holistically by connecting it with its context.

Complexity thinking and education

Education scholars started to pay attention to complexity thinking in the 1980s. Notably, Sawada and Caley (1985) and Doll (1986, 1989) are among the first group of scholars who wrote about complexity thinking in education. Over the years, other scholars (e.g., Davis & Sumara, 2000, 2006; Davis & Simmt, 2003; Davis, Sumara, & Luce-Kapler, 2008; Davis & Renert, 2014; Fleener, 1999, 2002, 2005; Gough, 2012; Reeder, 2002; Simmt & Kieren, 1999; Thom, 2012) have advocated and studied the use of complexity thinking in education. Through the lens of complexity thinking, common terms in education such as learners, learning, knowledge, and curriculum are reinterpreted and reimaged.

For example, a learner is viewed as a complex system that is a part of many embedded living systems, such as a classroom, a school, a country, a culture, and the universe. Essentially, each human being is “a ‘complex fabric of relations,’ fundamentally and inextricably intertwined with all else - biologically and phenomenologically” (Sumara & Davis, 1997, p. 415). One’s historical conditions affect oneself through the feedback loops. Hence each learner embodies his/her history. A learner also interacts with his/her surrounding constantly, consciously or unconsciously. Each learner and everything related to him/her are in constant flux and they adapt to each other (Sumara & Davis, 1997). Since complex systems tend to fold into one another as the boundary between complex systems and their surroundings is blurry, it is hard to distinguish self and others. Thus a learner is an integral part of the context rather than a solitary object situated in the context (Sumara & Davis, 1997). Whenever a part of a context changes, be it a learner or the learner’s surrounding, the whole context changes. Then the learner’s identity, which depends on the relationships between the learner and the context, changes. This fluid identity is documented in Sumara and Davis’s (2009) paper reporting an action-research study they did. The teachers in the study initially referred to the students’ parents as “they”, which was changed to “we” after their first book discussion session with the parents. Through the discussion around the book they both read, teachers and parents’ relationships changed and their identities as expressed through the interaction appeared to change too. This example shows that learners’ identities emerge through their constant interactions with their surroundings and are in constant flux.

As a complex system, a learner is an active agent that self-organizes and is capable of adapting itself to new conditions or, in other words, capable of learning. Although learning is associated with particular experiences or situations, experiences or conditions do not cause learning to happen. Learning is a kind of transformation in the learner that is simultaneously physical and behavioral, thus structural (Davis & Sumara, 2006). Learning happens as a contingent result of the interactions between a learner's historical and biological structure and her surrounding. Neither the learner's structure nor the environmental stimulus can specify what learning will happen. Hence learning is unpredictable: It can only be occasioned rather than caused (Davis & Sumara, 2006). Here the word "occasion", as used by other researchers (e.g. Kieren, Simmt, & Mgombelo, 1997) influenced by complexity thinking, is in consistent with Maturana and Varela's (1998) idea of trigger to differentiate from a cause-effect relation: "To trigger an effect" refers to

> the fact that the changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system*. The same holds true for the environment: the living being is a source of perturbations and not of instructions. (p. 96)

Knowledge, viewed by complexity thinking, is no more a thing existing internally or externally. It is knowledge-as-(inter)-action (Sumara & Davis, 1997). Rather than as objective information of the outside world that one needs to absorb, or some understanding a learner generates solely, knowledge is created through the interactions among the complex systems that the learner comprises and is composed of. Actions, seen as products of these interactions, are themselves understandings and knowledge (Sumara & Davis, 1997). The locus of cognition, therefore, is no longer understood as within an individual and neither is there a solitary truth-determining agent for knowledge (Sumara & Davis, 1997). Rather, knowledge exists and consists in the possibility for joint or shared action (Sumara & Davis, 1997) and it is determined by the shared action.

The above interpretations of learner, learning, and knowledge, through the lens of complexity thinking, have implications on curriculum. A traditional curriculum, which centers on teaching towards pre-set learning outcomes and on assessments, becomes problematic. Such curriculum, based on Doll (1986), is a measured one and is one example of the pervasive influence of classical-scientific world view summarized and guided by Newton. Newton's world is "simple, spiritual[2], and uniform or universal" (Doll, 1986, p. 11), thus a measured curriculum focuses on reductionism, certainty, quantification, and universality, and it views change as reversible and predictable. Such curriculum reflects and perpetuates control, separation, and domination (Fleener, 1999, 2002). Complexity thinking informs us that the world is not like a mechanical clock that follows universal and fixed rules. Rather, the world constantly changes through self-organization. Its change is irreversible and unpredictable. The world is "multiple, temporal, and complex" (Doll, 1986, p. 15). The measured curriculum, modeled on Newton's closed system, cannot reflect this new world view and needs to be changed. Complexity thinking invites us to understand the world as open, adaptive, emergent, unpredictable, and recursive (Davis & Sumara, 2006; Fleener, 2002, 2005; Gough, 2012). This change of worldview calls for a curriculum that is modeled on complexity.

In summary, complexity thinking has triggered a paradigm shift in many disciplines, including education. By urging educators to see learners as adaptive self-organized complex systems and learning as an emergent and unpredictable phenomenon, complexity thinking compels us to reimagine our curriculum.

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[2] "Spiritual" refers to a quality of being static, uniform, and external controlled and it is used to describe a view of universe through God's eyes (Doll, 1986).

2.2 Hermeneutic Inquiry

Hermeneutic inquiry is a qualitative research methodology informed and guided by hermeneutics, "the tradition, philosophy, and practice of interpretation" (Moules, 2002, p. 2). Sometimes hermeneutic inquiry is used interchangeable with interpretative inquiry (e.g., in Ellis, 1998; Moules, 2002). Such conflation seems inevitable, as Guba and Lincoln (1994) point out, doing interpretive inquiry situates one in a paradigm, which accepts that "individual constructions can be elicited and refined only through interaction *between and among* investigator and respondents" (p. 111) and commits to hermeneutical and dialectical methodology. However, not all interpretive inquiries can be viewed as under the influence of hermeneutics beyond simply employing hermeneutic circles as an interpretation method. According to Prasad (2005), an inquiry that employs hermeneutics in a strong sense is "research that actively engages in the interpretation of texts, and that is informed by the epistemological insights of hermeneutic philosophy", rather than only using hermeneutics to "denote the interpretive and phenomenological dimensions of qualitative inquiry" (p. 30). To avoid confusion in terminology and ensure a strong focus on understanding hermeneutic philosophy and its implication on research methodology, I use only "hermeneutic inquiry" in my following writing, and limit it as a kind of interpretative inquiry guided by hermeneutics.

Hermeneutics' textual interest and central themes

Hermeneutics views understanding as interpretation (Gadamer, 1989/2013); the one doing hermeneutics is keenly interested in interpreting with *texts* of all kind (Prasad, 2005). A text in hermeneutic study can be any instance of "mute evidence" – "things that have endured physically and that can be separated across space and time from their authors or producers"

(Prasad, 2005, p. 38), such as written records or documents, films, literature, and technology. It can also be metaphorical text, such as conversations, interactions and events (Prasad, 2005).

Hermeneutics acknowledges that interpretation is tricky, as a text does not have a fixed and uniform meaning. A text is layered, including its appearance or obvious meaning, and subtext, which is “the text underneath the surface-text” (Prasad, 2005, p. 36) or the text’s possible/hidden meanings. The task of hermeneutic researcher is to discern this subtext. However, this subtext is not a possession of the text or simply intended by the text’s immediate author. Rather it is shaped by the text, its context (e.g., the author’s cultural milieu) and the interpreter’s presuppositions and context. A good interpretation of a text cannot be achieved without connecting these areas together. As Davis (1996) puts it, “Within a focus on the deliberate act of interpretation, the interpreter, the interpreted, and the interpretive community are simultaneously presented” (p. 23). The stress of interpretation is “not upon the subjective interests of the interpreter nor upon the objective features of the work itself, but on the act of interpreting and the significance of the interpretation that is produced” (Silverman, 1994, p. 12).

According to Smith (1991), Schleiermacher has pointed out three central themes of hermeneutics. The first theme is the inherently creative character of interpretation. Interpretation is a creative action. As Gadamer (1989/2013) writes, “understanding [same as interpretation] is not merely a reproductive but always a productive activity as well” (p. 307). Interpreting a text is not simply reproducing what the author has intended to say. The meaning of a text always goes beyond its author and it is always determined by the text, the interpreter, and their historical conditions (Gadamer, 1989/2013). Further, understanding is not understanding better, but understanding in a different way (Gadamer, 1989/2013). So an interpreter needs to creatively suggest possible meanings and show relations between things in new ways (Smith, 2006).

The second theme is the interplay of part and whole during the process of interpretation. The hermeneutic rule is that "we must understand the whole in terms of the detail and the detail in terms of the whole" (Gadamer, 1989/2013, p. 302). A good interpretation "involves a playing back and forth between the specific and the general, the micro and the macro" (Smith, 1991, p. 190). A good interpreter works with the text holistically. This means that she pays close attention to the whole-part relationship as one cannot understand the part without seeing how the part is situated in the whole and vice versa. It is important to note here that for a text, the whole includes the text and its context, which includes the interpreter as well. A good interpretation of the text is inseparable from the interpreter situating the meaning of the text in relation to the whole of her own meanings or situating herself in relation to it (Gadamer, 1989/2013). For example, to understand teachers' perspectives about reviewing, it helps if an interpreter can also understand their perspectives about teaching and learning in general and their cultural milieu. She needs to relate teachers' perspectives to her own understanding of reviewing and cultural milieu too.

The third theme is the pivotal role of language in human understanding. As Ellis (1998) says, "the language available to the interpreter both enables and limits the understanding that is possible" (p. 16). This is particularly true when the interpreter and the participant are using the same word (e.g., review). Both people might think they understand each other well by viewing the word as a shared vocabulary. However, they might understand it very differently, particularly if they come from different cultural backgrounds. Thus it is important for the interpreter to ask different questions regarding key vocabularies so that a fusion of horizons can be achieved between the interpreter and the participant. A hermeneutic interpreter pays close attention to language as "hermeneutics is about an attentiveness to language, recognizing that language has a forgetfulness to it" (Moules, 2002, p. 3). Since language is a social, cultural and historical construct, hermeneutics' attentiveness to language demands a serious consideration of the text's broad context in a hermeneutic inquiry.

The hermeneutic task

Where to start? Hermeneutics starts with "what happens to us over and above our wanting and doing" (Gadamer, 1989/2013, p. xxvi). It begins, as Jardine (2015) interprets, with a kind of experience that Gadamer considers aesthetic as it draws us out of our subjectivity and into a world of abundant relations. There is something intriguing in the experience. It might be as simple as a look, a word, and a gesture or as holistic as the impression of the whole experience. These things "*strike* us, *catch* our fancy, *address* us, *speak* to us, *call* for a response, *elicit* or *provoke* something in us, *ask* something of us, *hit* us, *bowl* us over, *stop* us in our tracks, [make] us *catch our breath*" (Jardine, 2015, p. 236). Yet, we do not fully understand them. In these moments "we are drawn out of ourselves and our constructions and our methods and our 'our'-centeredness and get caught up by something, charmed by it, drawn into *its* sway, into *its* play, into *its Spiel*" (Jardine, 2015, p. 237). This is opposite to the way of studying an object that comes to meet us in our world as if it can exist without us nor reference to any other things; "it [the experience] is much more a world into which we ourselves are drawn" (Gadamer, 1994, p. 191-192; as cited in Jardine, 2015, p. 240).

Besides being rich and memorable and having something that demands our attention, such experience has something that seems both familiar and strange "as the moment of address occurs somehow 'between' this new eruption of life and some older that is awoken" (Jardine, 2015, p. 239), and "something to teach us that we could not know by ourselves" (Gadamer, 1989/2013, p. xxxiii). Such experiences have us experience the limits of our experience while being called to go beyond the limits (Jardine, 2015, p. 237). Being addressed and noting the limitation of our experience, we venture to understand. Such venture is "more a passion than an action" (Gadamer, 1989/2013, p. 375). A hermeneutic question "'occurs' to us", meaning "that it 'arises' or 'presents itself' more than that we raise it or present it" (Gadamer, 1989/2013, p. 374). "A question presses itself on us; we can no longer avoid it and persist in our accustomed opinion" (Gadamer, 1989/2013, p. 375) and such question "places us and our being-in-the-world into question" (Jardine, 2015, p. 237).

"Understanding begins… when something addresses us" (Gadamer, 1989/2013, p. 310). This is the first condition of hermeneutics, as "it is always *something* that happens that awakens our interest in pursuing interpretations" (Jardine, 2015, p. 238).

How to proceed? This question does not lead to a fixed step-by-step instruction or method. In fact, such method is exactly what hermeneutics criticizes (Jardine, 2015). In a hermeneutic inquiry, "it is impossible to establish 'correct method' in advance of an encounter with what is being investigated. This is because *what* is being investigated holds at least part of the answer to *how* it should be investigated" (Smith, 2006, p. 110). Therefore, the methods used in a hermeneutic inquiry cannot be predetermined totally.

Rather than following a method, a hermeneutic inquiry is guided by metaphors. A hermeneutic inquiry can be visualized as "a series of loops in a spiral" (Ellis, 1998, p. 19), which is specified as:

> Each loop in the spiral represents a separate inquiry activity within the study. Each loop may represent a separate 'data collection and analysis' activity or it may represent a return to a constant set of data with, however, a different question. Often the question for

each new loop has been influenced by what was uncovered in the inquiry represented by the previous loop. (Ellis, 1998, p. 20)

This description shows hermeneutic inquiry as an organic movement: Its path emerges through its process and is shaped by what the researcher has learned from previous activities. After a loop, researchers might refocus or reframe their questions for the next loop of inquiry, or they collect or generate different kinds of texts. Since the action a hermeneutic researcher takes to approach the entry question often takes the forms that are “apparently global and unfocused” (Ellis, 1998, p. 21), it is not surprised that a major turning point often comes after the first loop due to the researchers’ changed understanding of the problem or question of interest and their relationships with research participants established in the first loop. Particularly, if one has powerful activities in the first loop, one’s study can unfold in directions that few would have predicted (Ellis, 1998).

A hermeneutic study often includes multi-loop and single-loop inquiries (Ellis, 1998). The number of loops is not important and cannot be predetermined, as the goal of hermeneutic inquiry is to reach “an interpretation as coherent, comprehensive, and comprehensible as possible” (Ellis, 1998, p. 27). To that end, it is essential for one to work holistically in the spiral process: At the end of each text generation/collection, the researcher “works again with all transcripts, field notes, research notes, and artifacts, experiencing them as a whole or single text” (Ellis, 1998, p. 26).

Hermeneutic inquiries develop as a flow: A hermeneutic inquiry unfolds itself and the researcher “makes the path by walking it” (Ellis, 1998, p. 16). Therefore a hermeneutic inquiry proceeds with tentative methods that are subject to change along with the study. Moreover, a hermeneutic attitude of *letting* (Jardine, 2006) is essential: Letting the path unfold and wait and see. Such letting, as discussed later in Chapter 4 about my research process, is rather uneasy for one who lacks embodied faith in hermeneutic study. Yet, as my study will also exemplify, such letting is both a *precondition* and *goal* for a hermeneutic inquiry.

How to interpret? Again, this question leads not to methods but metaphors. The hermeneutic circle is a metaphor of the interpretation process in a hermeneutic inquiry (Gadamer, 1989/2013; Moules, 2002). It is not a method for uncovering meaning. Rather, it provides a way to conceptualize the hermeneutic researcher's process of understanding and interpretation (Moules, 2002) and "describes an element of the ontological structure of understanding" (Gadamer, 1989/2013, p. 305). The hermeneutic circle refers to an iterative back and forth movement between the micro and macro, the particular and general, the text and its context (Gadamer 1989/2013; Moules, 2002; Prasad, 2005; Smith, 1991).

The hermeneutic circle "is integral to establishing the linkage between a text and its wider context" (Prasad, 2005, p. 34). Since hermeneutic researchers recognize that they are "simultaneously affecting and affected by both the particular and the general, thus wholly embedded in the situation" (Davis, 1996, p. 22), while trying to connect a text to a broad context in hermeneutic circles, hermeneutic researchers welcome changes in their understanding of self, text, and context.

The movement of understanding and interpretation, based on Gadamer (1989/2013), is constituted of a constant process of new projection: "Interpretation begins with fore-conceptions that are replaced by more suitable ones" (p. 280). Our fore-conceptions are the fore-structures and prejudices that we have formed before we interpret a text. When we try to understand a text, we bring ourselves with us and our fore-conceptions are constantly revised along with the interpretation process. This movement is elaborated in Ellis's (1998) interpretation of hermeneutic circle. Ellis divides a hermeneutic circle into two arcs, the forward arc centered on projection process and the backward arc centered on evaluation process:

> In the forward arc…one uses 'fore-structure' to make some initial sense of the research participant, text, data. That is, one uses one's existing preconceptions, pre-understandings or prejudices – including purposes, interests, and values – to interpret…in the backward arc, one evaluates the initial interpretation and attempts to see what went unseen before. In this evaluation process, one reconsiders the interpretation by re-examining the data for confirmation, contradictions, gaps, or inconsistencies. (p. 26)

Embracing hermeneutics requires one to address one's fore-structures or prejudices (Moules, 2002). The hermeneutic inquirer recognizes the impossibility of objectivity. Moreover, treating a text as a self-identical substance is exactly an act of severance and purification that hermeneutics critiques (Jardine, 2015, p. 240). Rather than keeping her subjectivity constantly at bay, the hermeneut keeps it on check. Gadamer (1989/2013) differentiates prejudices into two kinds - the true or productive prejudices that enhance one's interpretation and the false or unproductive ones that hinder one's understanding. Believing that "detachment [from the text] distances the interpreter from the text, and consequently fosters unproductive prejudices" (Prasad, 2005, p. 37), hermeneutic researchers work with their prejudices. In practice, they can focus on making good use of their prejudices to understand in the forward arc of a hermeneutic circle, and on reflecting and addressing how their prejudices affect the ways they interact with participants, what they hear and see, and how they interpret text in the backward arc. They can also, as Gadamer (1989/2013) suggests, "not to approach the text directly…but rather explicitly to examine the legitimacy – i.e., the origin and validity – of the fore-meanings dwelling within [them]" (p. 280), as "understanding realizes its full potential only when the fore-meanings that it begins with are not arbitrary" (p. 280). In other words, there could be things that hermeneutic researchers can learn from their own (un)conscious prejudices.

Since interpretation in hermeneutics aims not the "authentic" message of the text irrelevant to its interpreter, but an understanding shaped by interaction between the text and the interpreter, the interpretive process needs to get beyond a text's superficial meaning and the interpreter's prejudices and seek deeper or more profound understandings. Therefore, sometimes interpretation involves discerning the subtext that constitutes latent and hidden meanings, such as "subjugated voices trying to speak out and express their hidden dreams, desires, and fears" and "the deceptions [that the ideological texts] practice" (Prasad, 2005, pp. 36-37). Sometimes interpretation brings forth possible and important meanings. As Moules (2002) states, interpretation starts from reflection and it

> involves careful and detailed reading and rereading of all the text, allowing for the bringing forth of general impressions, something that catches the regard of the reader and lingers, perturbing and distinctive resonances, familiarities, differences, newness, and echoes. Each re-reading of the text is an attempt to listen for echoes of something that might expand possibilities of understanding. (p. 14)

Here, interpretation is to hear *echoes* and find *inspirations* rather than merely uncovering hidden meanings or reproducing what has been said yet unheard. Such process might say more about the interpreter than the text.

Regardless, interpretation is an endless process and it does not stop until "some satisfactory level of understanding is achieved" (Prasad, 2005, p. 37). Each circle of interpretation provides certain understanding of the text and self to enable the researcher to reinterpret the text with more meaningful questions and different pre-conceptions in the next hermeneutic circle. "Alternate interpretive frameworks are purposely searched for and 'tried on'" to facilitate an interpretive process (Ellis, 1998, p. 27). This recursive back and forth movement between text study and self-introspection in hermeneutics is viewed as a dialogue or conversation between the text and its researcher, "in which the interpreter puts questions to the text, and the text in return questions the interpreter" and through which both textual and self-understanding become achievable (Prasad, 2005, p. 37). Again, interpretation is a generative rather than merely a reproductive process (Gadamer, 1989/2013). Its significance lies in its affordance of bringing forth meaningful, important, and different understandings of the interpreted and the interpreter.

How to converse with a partner? Within research, a hermeneutic conversation with participants differs from an interview in the nature of its questioning and the relationship between participants and the researcher. In an interview, a participant is viewed as informant; the interview question "involves an effort to gather information about perceptions or practices" (Carson, 1986, p. 78). Whereas, in a conversation, a participant works with the researcher together; the conversational question "implicates a revealing of something held in common" (Carson, 1986, p. 78). This common meaning might not be present in either party of the conversation prior to the conversation.

To conduct a hermeneutic conversation means to be open, to allow oneself to be conducted by the conversation, to interrogate the way we speak about the topic under discussion, and to let the language bring forth thoughts and ideas not hitherto present (Carson, 1986). Questioning, in a conversation, is to keep possibilities open: "to question means to lay open, to place in the open. As against the fixity of opinions, questioning makes the object [under discussion] and all its possibilities fluid" (Gadamer, 1989/2013, pp. 375-376). Therefore it is necessary for the researcher to "prevent questions from being suppressed by the dominant opinion" (Gadamer, 1989/2013, p. 376). Conversing is essentially a way for the researcher to think with others together. As Gadamer (1989/2013) says: "Dialectic consists not in trying to discover the weakness of what is said, but in bringing out its real strength. It is not the art of arguing… but the art of thinking" (p. 376).

To reach an understanding in a conversation, both parties of the conversation do not merely assert their individual points of views and understand what each other is saying; they are also "transformed into a communion in which [they] do not remain what [they] were" (Gadamer, 1989/2013, p. 387). In a successful conversation, conversation partners are influenced by the object under discussion and bound to each other (Gadamer, 1989/2013, p. 387). What emerges in a successful conversation belongs to neither party of the conversation but is the common meaning formed through conversation (Gadamer, 1989/2013). In other words, in a hermeneutic conversation meanings are co-constructed by both parties of the conversation. A hermeneutic conversation allows the knowledge-as-(inter)-action (Sumara & Davis, 1997) to emerge.

Evaluating a hermeneutic inquiry

Ellis (1998) suggested two criteria to evaluate a hermeneutic account: validity and practicability. Validity refers not to the issue of true or false, but "whether the interpretative account can be clarified or made more comprehensive and comprehensible" (Ellis, 1998, p. 29). Practicability points to the questions like, "Whether or not it [an inquiry or interpretation] reveals of a solution to the difficulty that motivated the inquiry" and "Whether our concern has been advanced" (Ellis, 1998, pp. 29-30).

Drawing from multiple authors' works, Moules (2002) emphasizes two criteria for hermeneutic/interpretive inquiry: rigor and validity. Rigor, or trustworthiness, is tied to believability. A hermeneutic research demonstrates rigor in multiple ways. First, the researcher shows interest not in absolute truth but rich understanding. For example, when the researcher consults participants regarding her interpretations of the collected/generated text, she searches not for “an expert evaluation of truth, but an opportunity to open the interpretations from the narrowness of one’s vision, prejudices, and focus” (Moules, 2002, p. 16). Second, the research raises more questions and is capable of extension. Here, hermeneutic research is judged not by the criteria of transferability but by the criteria of suggestiveness and potential, which seems to echo Ellis’s criteria of practicability. Third, the researcher provides sufficient and effective data evidences to help the readers to understand how she came to the interpretations she chose. Fourth, the research is consistent with the philosophical ground of the hermeneutic inquiry. For example, the researcher demonstrates that her interpretations “are arrived at in a referential and relational, rather than absolute way” (Moules, 2002, p. 16), and that she appreciates the generative nature of interpretation through creatively constructing something new out of the research text, even when the text is a combination of contradictions or differences, rather than agreement.

Validity is not tied to absolute truth, even though a valid hermeneutic research rings true to the readers. As Moules (2002) says, “A good interpretation takes the reader to a place that is recognizable, having either been there before, or in simply believing that it is possible” (p. 17). The interpretations in a valid hermeneutic research are more readily accepted than others are. These interpretations seem, based on Madison (1988, as cited in Moules, 2002), more “fruitful and promising”, and they seem to “make more and better sense of the text… [and open] up greater horizons of meaning” (p.15). In some ways, while Moules’s criteria of rigor and Ellis’s (1998) criteria of practicability seem to point at different qualities of an interpretation, with Moule’s having less concern on solutions but more about advancing relational understanding and questioning and keeping more check on the researcher's adherence to hermeneutics tradition compared to Ellis's, Moules's criteria of validity overlaps with Ellis's version: Both emphasize the fullness, richness, and comprehensivity of the interpretation.

Other than the criteria mentioned above, it is reasonable to say that a good hermeneutic study should demonstrate a fluid identity of the researcher. As the ideal of hermeneutic inquiry is to deepen both textual and self-understanding (Prasad, 2005) and hermeneutic researchers change as a result of their study and they live differently (Moules, 2002), it is important for hermeneutic research to show how the researcher has changed along with her research project.

2.3 Connecting Complexity Thinking and Hermeneutic Inquiry

I see hermeneutic inquiry as recursive due to its consistent reexamination of the research text holistically in hermeneutic circles. I agree with Moules (2002) that "the hermeneutic circle is the generative recursion between the whole and the part" (p. 15). In this sense, doing a hermeneutic inquiry embodies the process of recursion. Moreover, the unfolding quality of a hermeneutic inquiry process and the co-construction of meaning in hermeneutic conversations can find support in complexity thinking, which values emergence and openness rather than predetermination and control. Furthermore, both complexity thinking and hermeneutic inquiry stress the interplay of part and whole and they agree that to understand a whole one needs to think holistically. Therefore, hermeneutic inquiry is a suitable methodology for my research on recursion informed by complexity thinking.

3 Recursion and Recursive Curriculum

3.1 Recursion and Cognitive Development

Recursion used here is a concept derived from Doll's (1993) post-modern perspective of curriculum and shaped by Bateson's (1979/2002) theory of mind and complexity thinking. As recursion is frequently used in mathematics and computer science and its key connotations in these two fields are also present in Doll's and Bateson's works and complexity thinking, I start this section by examining its common definitions in mathematics and computer science. Then I analyze recursion's connotations and its roles in cognitive development based on Doll's and Bateson's ideas and complexity thinking.

Recursion in mathematics and computer science

Recursion is a familiar word in mathematics. An example of recursion can be seen in this recursive formula $t_n = t_{n-1} + 10, t_1 = 1$. Basically, this formula says "To find any term (but the first term) of this sequence, add 10 to the value of the previous term", and it produces a sequence $\{1, 11, 21, 31, 41, ...\}$. From this example, we can see that recursion has connotations of continuity and repetition with variations. A recursive process reproduces earlier stages to form a later stage, but with a difference.

Recursion is also frequently used in computer science. A recursive procedure or function calls itself. Figure 3.1.1 shows an example of recursive function written as pseudo code.

```
function Product(n: integer): integer;
        begin
                if n > 1 then
                        Product := Product (n-1) * n
                else
                        Product := 1
        end;
```

Figure 3.1.1. A recursive function example.

This function calculates the nth factorial, which is the product of the first n consecutive positive integers. For instance, a call to the Product function with a parameter of $n = 3$ (i.e., Product (3)) will trigger a recursive process:

a) In function Product (3), since $n = 3 > 1$, Product (3) = Product (2) × 3, and this calls the function Product (2) before completing the multiplication.

b) In function Product (2), since $n = 2 > 1$, Product (2) = Product (1) × 2, and this calls the function Product (1) before completing the multiplication.

c) In function Product (1), since $n = 1$ is not > 1, Product (1) = 1, and this value is returned to its previous level of pending calculation.

b’[3]) In function Product (2), the pending calculation for Product (2) executes and finds Product (2) = 1 × 2 = 2. This value is returned to its previous level of pending calculation.

a’) In function Product (3), the pending calculation for Product (3) executes and finds Product (3) = 2 × 3 = 6. This value is returned to the user as the final answer for 3 factorial.

The above example shows that a recursive function or procedure has a nested structure and each layer is self-similar: A recursive function or procedure is applied within its own definition. Recursion in computer science has connotations of looping back, reflexivity, and self-referencing.

Recursion, as articulated by Doll

Drawing from post-modernism and complexity thinking, Doll (1993), a former mathematics teacher and a curriculum theorist, proposes *a* post-modern perspective of curriculum that envisions a transformative curriculum centering on richness, relations, recursion,

[3] The use of b’ and a’ here is to signify them being related to the beginning two steps (i.e., b and a) respectively. In each of the last two steps, computer returns to a previously unfinished calculation and completes it.

and rigor (the 4Rs). Doll views recursion sharing the same spirit with what T. S. Eliot (1944) writes in his poem *Little Gidding*:

> We shall not cease from exploration
> And the end of all our exploring
> Will be to arrive where we started
> And know the place for the first time. (p. 43)

Recursion is referred as “a looping back to what one has already seen/done, to see again for the first time” (Doll, 2008/2012, p. 27) and it has multiple layers of meanings.

First, recursion is *currere*-oriented. Doll (1993) connects recursion and *currere* through tracing the root of recursion: “Recursion (as well as recur) is derived from the Latin *recurrere* (to run back). In this way recursion is allied with *currere* (to run), the root word for curriculum” (p. 194). *Currere,* meaning to run the course, or the running of the course, is conceptualized by Pinar and Grumet as a way to understand curriculum from an autobiographical and biographical perspective (Pinar, 1975/2000; Pinar & Grumet, 1976). *Currere* refers to “experience in educational contexts” (Pinar, 1975/2000, p. 413). The method of *currere* attempts to disclose in what ways one’s educational experience is affected by individual and collective histories and hope (Pinar, Reynolds, Slattery, & Taubman, 2008). The study of *currere* entails a turning inward, a study of individual experience from a phenomenological and psychoanalytic perspective. Thus, the association between recursion and *currere* points to the importance of experience and process in education. A recursive curriculum, being *currere*-oriented, encourages learners to experience and to reflect upon their experience; it makes self-reflection central (Doll, 1993).

Second, recursion is recursive reflection. As Doll (2010/2012) says, "It is in this second, yet first, seeing that the richness of a situation begins to emerge; and as we become more aware of our participation in the situation, recursion turns into *recursive reflection"* (p. 181). In other words, revisiting a previous learned topic offers one a chance to reflect on both the content related to the topic, one's previous thought about the topic, and how one thinks about one's previous thought about the topic. In this sense, revisiting a topic can trigger a repetitive looping of thoughts on thoughts, or in other words, recursive reflection.

Third, recursion is hermeneutic reflection. This layer of meaning is also brought through Doll's emphasis on recursion's Latin root *recurrere* (run back). Doll (1993) says, " 'running back' means that each statement or proposition is reexamined in terms of re-looking at its original foundational assumptions" (p. 123). In other words, recursion is a reflective process during which one's interpretation of a text studied before and one's prejudices associated with the text are examined together, thus it is hermeneutic. This hermeneutic reflection allows learners to examine their process of thinking along with its products.

Fourth, recursion is neither repetition nor its synonym iteration. Doll (1993) makes it clear that repetition aims to reproduce the same action or result: In a process of repetition, "reflection plays a negative role; it breaks the process" (p. 178). Thus to secure the completion of repetition, change is not welcomed and reflection is to be avoided. The frame of repetition is closed (Doll, 1993). On the other hand, recursion, aiming to facilitate cognitive growth, welcomes changes and needs reflection: Reflection plays a positive role in recursion. The frame of recursion is open (Doll, 1993).

In short, Doll's recursion is a process of looking at one's previous thoughts critically and hermeneutically from a different perspective. It stresses on-going reflection, hermeneutic inquiry, and personal experience, affects both the process and products of thinking activities, and breeds new ideas.

Recursion lies at the heart of Doll's understanding of curriculum from a post-modern perspective. Recursion in human thoughts enables humans to make meaning and form consciousness (Doll, 1993). Through the process of recursion one can see relations among isolated topics and one's understanding becomes richer and deeper. Thus recursion brings forth the other 3Rs of Doll's post-modern curriculum - richness, relations, and rigor.

Recursion, as articulated by Bateson

Bateson's ideas, such as theory of mind and theory of levels of learning, help me to understand the process of recursion and see the connections between recursion and cognitive growth.

Bateson's (1972/1990, 1979/2002, 1987) theory of mind emphasizes recursion. For Bateson, mind is a much broader concept than brain. Any aggregate of phenomenon or any system can be viewed as mind as long as it fulfills the following six criteria:

1. *A mind is an aggregate of interacting parts or components.*

2. *The interaction between parts of mind is triggered by difference. . . .*

3. *Mental process requires collateral energy.*

4. *Mental process requires circular (or more complex) chains of determination.*

5. *In mental process, the effects of difference are to be regarded as transforms (i.e., coded versions) of events which preceded them. . . .*

6. *The description and classification of these processes of transformation disclose a hierarchy of logical types immanent in the phenomena*. (Bateson, 1979/2002, pp. 85-86)

Such category of mind covers a wide range beyond living systems, such as organisms, systems of them, and any parts of them; it includes any complex self-organizing systems consisting of living and/or nonliving parts (Bateson, 1987). Essentially Bateson's category of mind includes all complex systems. It is beyond the scope of this dissertation to explain all the six criteria of mind. Here I focus on the fourth criterion that is tied to recursion, i.e., mental process being circular. Following Bateson's conceptualization of mind, mental process is not limited to the process that happens in one's brain; rather, it can be any process (e.g., sensorimotor process) that happens in a system viewed as a mind.

Bateson (1979/2002) understands mind as a system having rich feedback loops that "carry messages *about* the behavior of the whole system" to the system itself, hence following a circular causality (p. 118). When a difference is perceived by a part of the mind, the feedback loops in the mind allow its influence to be carried through the whole system and affect every part of the system, including the beginning part. Hence, "a change in any part of the circle can be regarded as *cause* for change at a later time in any variable anywhere in the circle" (Bateson, 1979/2002, p. 56). For Bateson, this structure of mind is recursive and it allows mind to fold back to itself thus producing autonomy: "Autonomy – literally *control of the self* from the Greek *autos* (self) and *nomos* (a law) – is provided by the recursive structure of the system" (p. 118). It is clear that mental process has a looping back, circular and self-referencing nature, which is essential for a recursive process.

A recursive process can bring forth difference, which might engender a new recursive process. Difference is needed for a mental process to begin, as Bateson's (1979/2002) theory of mind states, "the interaction between parts of mind is triggered by difference" (p. 89). The difference can be either a difference between two things or a change between a thing in time 1 and the same thing in time 2 (Bateson, 1979/2002). A looking-back activity initiated by a learner, such as revisiting and reflecting on the same topic later in time or from different perspectives, can enable the learner to perceive difference between thoughts generated in different contexts. Hence, I view that recursion invites difference, which triggers one's cognitive system to operate and perceive new difference, thereby making one's recursive mental development possible.

Another role of recursion is related to Bateson's (1972/1990) theory of levels of learning. Based on this theory, there are three levels of learning in humans and animals. In Learning I, one learns to deal with a specific problem. For example, one learns to solve a particular linear equation, such as $x+1=0$. In Learning II, "the subject discovers the nature of the context itself, that is he [sic] not only solves the problems that confront him, but becomes more skilled in solving problems in general" (Berman, 1981, p. 216). Learning II is the start for a person to see the pattern within a type of problems. For instance, one learns to generalize how to solve various linear equations in the form of $mx+b=0$, or later more general equations in the form of $f(x)=0$, or even mathematical problems in general. In Learning III, "it is not a matter of one paradigm versus another, but an understanding of the nature of paradigm itself. Such changes involve a profound reorganization of personality – a change in form, not just content" (Berman, 1981, p. 217). The experience in Learning III has profound influence on one's worldview. For example, if a learner realizes that these patterns generalized in mathematics essentially are human generated rules, and that mathematics is contextual rather than universal, then the learner is in Learning III (See Luo, 2004 for a non-mathematical example of these three levels). Bateson's theory of levels of learning does not imply that learning is a process of going from a lower level to a higher level in a linear incremental manner or that a learner can only be at one level of learning at one time. Rather, it emphasizes the qualitative difference between different levels of learning. It suggests that generalization is essential for learning and it requires one to see the whole through the parts.

Seeing the whole and seeing the parts are different logically. Bateson (1979/2002) points out that the message about part and the message about whole (e.g., the message about a tree and the message about the forest to which the tree belongs, or the message about an orange and the message about fruit) belong to two different logic types: One is about an object while the other is about the context of the object. Learning about the parts only is not enough for cognitive growth. Learners who can only see parts not the whole often have difficulty to solve a problem in a context that they are not familiar with or to solve a "new" problem because they fail to recognize the family to which these seemingly different problems belong. Cognitive growth demands learning about the context of the parts (Bateson, 1979/2002). As an eminent Chinese poet, Su Shi (1037–1101), writes: "不识庐山真面目，只缘身在此山中 (You cannot see the real look of Mountain Lu because you are in the mountain)". One cannot learn about the mountain (context) without seeing the mountain (system) as a whole, and one has to leave the mountain (system) to learn about the mountain (context). This does not imply that an objective position is possible. Rather, it emphasizes the importance of holistic thinking in one's learning about context.

Helping one to learn part-whole relations is another role that I think recursion can play in one's cognitive development. As recursion "requires that the user step outside the system" (Kilpatrick, 1985, p. 5) and is a process of stepping back or distancing one's self from one's creation (Doll, 1993), recursion can help learners to learn about the contexts of their questions. Although teachers cannot cause the recursion in students' mind, they can use certain practices that have a potential to occasion students' recursion. For example, having students to revisit a set of questions one previously solved and reflect on them as a whole might trigger the process of recursion in students' mind, which helps students to see a holistic view of the questions, thus making their cognitive growth to a higher level of learning possible.

In summary, based on Bateson's theory of mind, which stresses the mind's recursive structure and recursive process, I understand that Bateson's concept of recursion has connotations of circularity, looping back, and self-referencing. An examination of Bateson's theory of mind and theory of levels of learning suggests that recursion plays important roles in one's cognitive development.

Recursion in complexity thinking

My understanding of recursion is also influenced by complexity thinking through reading the works of Bateson, Doll, and other scholars (e.g., Capra, 1996; Davis & Sumara, 2006; Mitchell, 2011). Bateson's and Doll's ideas overlap with complexity thinking in various degrees, as Bateson's ideas are influential to complexity thinking through his contributions to cybernetics and Doll's ideas are inspired by complexity thinking. So, it is not surprising to see that the idea of recursion in complexity thinking shares some similar meanings suggested by Bateson's and Doll's works. Here I focus on the meanings of recursion suggested by the parts of complexity thinking that are not covered in Doll's and Bateson's sections.

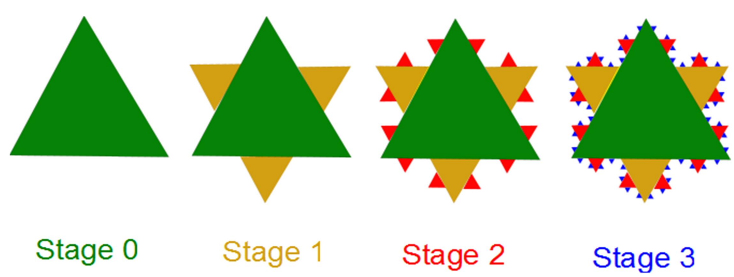

Figure 3.1.2. A Koch snowflake formation.

Recursion is the central process in the formation of complex systems and it is similar to the one in the fractal geometry. Fractals are formed through recursion (Mandelbrot, 1977, 1983). For example, to generate a Koch snowflake (see Figure 3.1.2), start with an equilateral triangle in the stage 0 and repeat the following two steps to each side of the figure in each stage of the repetition infinitely:

a) Divide each side of the figure into three equal portions;

b) Replace the middle portion with an equilateral triangle whose bottom side is removed.

The formation of a Koch snowflake exemplifies a recursive process that is “a repetitive one in which, at any particular level of computation, the new input is the output from the previous level (and the subsequent output is the input for the next round)” (Davis & Sumara, 2000, p. 827). It is through recursion that fractals gain an infinite level of self-similarity, a hallmark of fractals. This logic also applies to complex systems, which are fractal-like.

Through the lens of complexity thinking, cognition is also viewed as fractal-like. As Davis and Sumara (2000) state,

> the dynamics of cognition/knowledge are seen in much the same terms as the procedure used to generate a fractal image. It is seen as a matter of recursion, of elaborating what has come before, subjected to emergent contingencies, embedded in and part of a similarly recursive context. (p. 834)

Recursion plays an essential role in cognitive development.

In summary, complexity thinking informs us about the pervasiveness of recursion in the world. The concept of recursion emphasized in complexity thinking is a self-referencing looping back process, which consists of repetition of variations and has an open framework, and it is a central process in cognitive development.

A working definition of recursion

To conclude, recursion is a process. Recursion can be either a mental thinking process, or a process used to invoke the process of recursion in the mind. Recursion has connotations of *continuity*, *repetition with variations*, *looping back/reflexivity,* and *reflection* (see Figure 3.1.3). When used in educational contexts, recursion is hermeneutic and *currere*-oriented. It is important

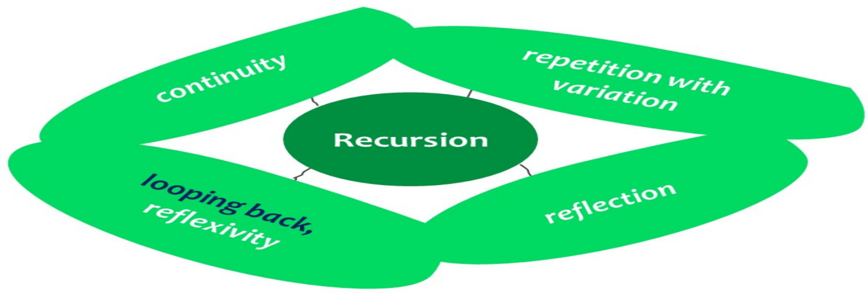

Figure 3.1.3. Four working connotations of recursion.

to note that the definition of recursion is not final and the list of connotations is not exhaustive. Also, the identified four connotations are not isolated from each other, symbolized in the model as categories overlapping with each other.

To further clarify the concept of recursion, it is necessary to differentiate recursion with some of its synonyms, such as reflection, review, and repetition. Recursion is closely tied to reflection but is more than reflection: Recursion, with its association to hermeneutics and *currere*, includes a process of continued and repeated reflecting, interpreting, and experiencing. It does not aim for the closure of one's interpretations or seek universal truths. Rather it emphasizes and promotes interpretations that are personally relevant and meaningful. It invites the learners to go through their unique learning processes rather than controlling their learning trajectories. Recursion is not the same as review. A recursive curriculum treats reviewing as a central process for learning, yet not all reviews can invoke the process of recursion for a learner. Recursion is not repetition. Repetition avoids change and reflection whereas recursion welcomes changes and needs reflection. Such difference in change, however, is the one between recollection and repetition in Kierkergaard's words: According to Caputo (1987), Kierkergaard differentiates recollection and repetition, one as reproducing a prior presence and the other as producing something and bringing forth something anew. Given the existence of contradicting interpretations of repetition, recursion is a more helpful name for a process involving novelty.

Based on the above working definition of recursion, a few guiding principles of recursive curriculum can be established: 1) a recursive curriculum aims to facilitate learners' cognitive development by trying to occasion the process of recursion in learners' mind; 2) a recursive curriculum is hermeneutic and *currere*-oriented; 3) a recursive curriculum has qualities of continuity, repetition with variations, reflexivity and reflection; and 4) a recursive curriculum emphasizes learning new through a running or looping back process, thus it centers on re-viewing, re-experiencing, and re-interpreting. These principles are heuristic rather than conclusive: They help to envision the practices in a recursive mathematics curriculum and they serve as a guideline for the text generation and interpretation in this study. Inevitably, these principles are subjected to recursive examination in this study.

3.2 Recursion and the Growth of Mathematical Understanding

In this section, I zoom in on cognitive development and focus on the growth in mathematical understanding. I examine the problems of a linear mathematics curriculum and the role recursion can play in the development of a learner's mathematical understanding.

The problems of a linear mathematics curriculum

The problem of a linear mathematics curriculum has been observed by many scholars (e.g., Davis & Sumara, 2000; Doll, 1986; Ernest, 1991; Thom, 2012). Here I focus on Ernest's arguments as they are justified from both the perspectives of mathematics and mathematics curriculum.

Ernest (1991), in his influential book, *The Philosophy of Mathematics Education*, argues forcefully that there is no a unique and fixed hierarchy in both mathematics and mathematics learning. Given that most curriculum theorists argue that "the curriculum should reflect both the knowledge and processes of inquiry of the subject discipline" (Ernest, 1991, p. 236), the mathematics curriculum should reflect the attributes of mathematics and the learners' mathematics learning process. However, as Ernest suggests, "the discipline of mathematics does not have a unique hierarchical structure, and cannot be represented as a collection of 'molecular' propositions" (p. 236).

Drawing support from cognitive science and psychology studies, Ernest (1991) rejects the view that mathematics learning is hierarchical, "meaning that there are items of knowledge and skill which are *necessary* prerequisites to the learning of subsequent items of mathematical knowledge" (p. 238). Ernest argues that the learners' understanding of one or multiple mathematical topics cannot be subsumed to a single fixed hierarchy, as there are no fixed hierarchical relations of dependence among concepts and skills and "the uniqueness of learning hierarchies is not confirmed theoretically or empirically" (p. 239). Moreover, as Ernest points out, "acquiring a concept is the process of effecting an idiosyncratic personal construction" (p. 241) that can last one's life time rather than "an all or nothing state of affairs" (p. 240). Consequently, whether one determinately forms a concept or not is an invalid claim and the proposition that one has to learn certain mathematical knowledge before the others is rejected.

Since mathematics and mathematics learning do not have a unique hierarchical structure and there is not a sequence that can best describe mathematics learning for all students, it is unjustified to impose a unique fixed hierarchical mathematics curriculum for all students. Ernest (1991) contends that the mathematics curriculum should avoid offering a collection of separate facts and skills but “allow for different ways of structuring mathematical knowledge” to reflect that “the components of mathematics are variously structured and inter-related” (p. 237). In this sense, any linear mathematics curriculum, reflecting mathematics and mathematics learning as fixed hierarchical, is ill-founded.

The recursive nature of the growth in mathematical understanding

Several mathematics education researchers view the growth of mathematical understanding as a recursive phenomenon. Here I offer three theories (i.e., Sfard, 1991; Sawada & Pothier, 1993; Pirie & Kieren, 1989, 1994) to situate my work.

Sfard (1991) applies an ontological-psychological outlook to research the formation of mathematical concepts (e.g., number or function) and develops a theoretical framework to describe the nature of mathematical conceptions and their development. A concept’s conception is defined as “the whole cluster of internal representations and associations evoked by the concept” (Sfard, 1991, p. 3). By analyzing different mathematical definitions and representations, Sfard concludes that a mathematical abstract concept can be conceived in two ways, structurally and operationally. While structural conception refers to treating mathematical concepts or notions as if they were abstract static objects, operational conception refers to treating them as processes, algorithms, and actions.

Drawn from historical examples and cognitive schema theory, Sfard (1991) shows that at both individual learning and historical development levels, mathematical concept formation tends to follow a cyclic process of transiting from operational conception to structural conception: “various processes [have] to be converted into compact static wholes to become the basic units of a new, higher level theory” (p. 16). Each recurrent transition is a long and difficult process and it follows three steps: interiorization, condensation, and reification.

At the stage of interiorization, “a learner gets acquainted with the processes which eventually give rise to a new concept” and these processes, such as counting cubes, are “operations performed on lower-level mathematical objects” (Sfard, 1991, p. 18). When a learner can carry out the processes mentally without actually performing the actions, she has interiorized the processes. At the stage of condensation, a learner becomes capable of thinking about the internalized process as a whole without noting the details of the process. Sfard (1991) says, “The phase of ‘*condensation*’ is a period of ‘squeezing’ lengthy sequences of operations into more manageable units” (p. 19). In other words, a vague entity of the concept starts to form. However, this idea remains tightly connected to a certain process so much so that the entity often serves as a shorthand for the process, e.g., viewing natural numbers as a shorthand of the process of counting. It is not until the stage of reification that this entity becomes a clear object that can be manipulated in diverse contexts (e.g., dividing natural numbers to form rational numbers). Reification, “an ontological shift – a sudden ability to see something familiar in a totally new light” (Sfard, 1991, p. 19), differs from interiorization and condensation for being a qualitative change rather than a quantitative one:

> [Reification is] a process solidifies into object, into a static structure. Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the fact of its being a member of a certain category… Processes can be performed in which the new-born object is an input. New mathematical objects may now be constructed out of the present one. (Sfard, 1991, p.20)

Thus the stage of reification of a lower-level concept (e.g., natural numbers) overlaps with the stage of interiorization of higher-level concepts (e.g., rational numbers).

It is not Sfard's (1991) interest to propose a fixed hierarchy of mathematical learning by identifying lower-level concepts and higher-level concepts. A concept's level varies for each individual and might change over time or in different contexts. Sfard's focus is the recursive movement in one's formation of mathematical conceptions: The development of mathematical understanding consists of "an intricate interplay between operational and structural conceptions of the same notions" (p. 1). One's operational and structural conceptions are different sides of the same coin. Although the operational conception tends to develop at first, any side can develop further than the other side at a particular time. The development of the two conceptions is neither linear nor a one-time event. Rather, two conceptions inform each other and they cannot be fully developed without the other. The development from one lower-level concept to a higher-level concept is not linear either. Since the phase of reification of a concept is hard to achieve, learners might develop other higher-level concepts based on mostly the operational conceptions of the concept if one can put up with a feeling of insufficient understanding and keep on drilling, and in turn further developing the concept's structural conceptions later. Overall, Sfard's mathematical concept formation framework describes a recursive development of mathematics understanding.

Sawada and Pothier (1993) recognize the recursive nature of the growth in mathematical understanding and advocate a recursive learning that benefits children's mathematical imagination. They use a working definition of recursion: "A representation (or a process in general) is recursive if at a certain point in working with it the medium becomes the message" (p. 15), which becomes the medium of further messages, and on and on. A recursive cognitive experience is a process in which one's previous thoughts (medium) becomes the message that invites further thoughts, which become the subsequent medium in the next round. Sawada and

Pothier state that the mathematical experience for mathematicians is pervasively recursive and urge educators to allow "the freedom to explore where destinations emerge from the exploration" to promote recursive learning (p. 19).

Pirie and Kieren (1989, 1994) established a theory of mathematical understanding - *transcendent recursion.* This theory characterizes mathematical understanding as a leveled but non-linear phenomenon (see Figure 3.2.1). In this model, "each level of understanding is

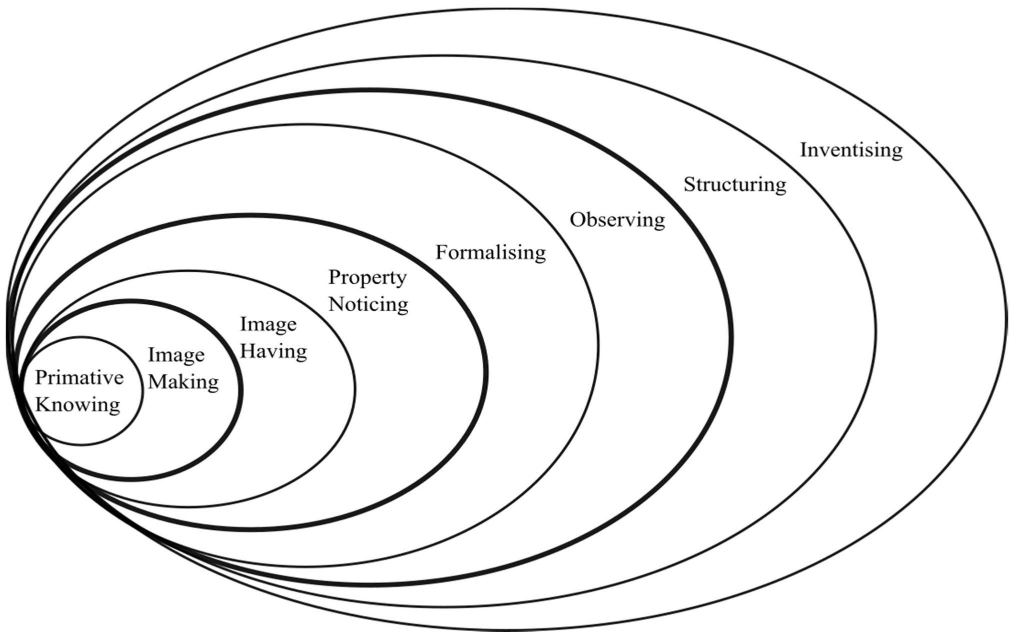

Figure 3.2.1. A recursive mathematical understanding model (adapted from Pirie & Kieren, 1994, p. 63).

contained within succeeding levels" and "any particular level is dependent on the forms and processes within and, further, is constrained by those without" (Pirie & Kieren, 1989, p. 8). This theory views mathematical understanding as fractal-like: Every level/layer is similar to its previous levels/layers yet it transcends them. Also, "inspection of any particular primitive knowing will reveal the layers of inner knowings" (Pirie & Kieren, 1994, p. 68): An example here can be that a primitive knowing of quadratic functions contains the inner layers of understanding of linear functions.

Although the growth of any particular mathematical understanding always starts from primitive knowing and has a tendency to move to an outer level, it does not follow a monodirectional linear path. Rather, as Pirie and Kieren (1994) say,

> When faced with a problem or question at any level, which is not immediately solvable, one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. Continuing with our metaphor of folding, we can say that one now has a 'thicker' understanding at the returned-to level. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. (p. 69)

The growth of mathematical understanding follows a recursive process while "recursion is seen to occur when thinking moves between levels of sophistication" (Pirie & Kieren, 1989, p. 8). Evidently, Pirie and Kieren's idea of recursion includes folding back to an inner level and returning to an outer level. And whenever one moves from one level to another, one has some newer understanding, which changes the problem one faced in the past. Thus one might repeat similar actions, but with a different focus.

In summary, the above three different groups of mathematics education researchers describe the development of mathematical understanding as recursive. Their concepts of recursion share the qualities of continuity, repetition with variations, self-referencing, and reflection in various degrees.

This section draws support from mathematics education literature to justify the nonhierarchical nature of mathematics and mathematical learning, which problematizes linear mathematics curricula and urges us to reimage and reimagine mathematics education. In line with the interpretation of the growth in mathematical understanding as a recursive phenomenon, recursive mathematics curricula become imperative.

3.3 Recursive Curricula

Some scholars, such as Doll, Bruner, Davis, and Thom, have envisioned a curriculum that has a quality of recursion with or without explicit discussion or conceptualization of recursion. These curricula are categorized as recursive here without implying that they interpret and/or emphasize recursion in the same way as I do. In this section, I examine these scholars' ideas and identify the similarities and differences between their works and my research.

Doll's post-modern curriculum

Doll's (1993) concept of recursion and its importance in his post-modern curriculum have been addressed in section 3.1. Doll problematizes the linear and deterministic nature of modern curricula, and advocates *a* transformative post-modern curriculum centering on 4Rs (richness, relations, recursion, and rigor). Recursion is viewed as an implication of complexity thinking (Doll, 2010/2012). Since my understanding of recursion is derived from Doll's concept of recursion, my vision of recursive curriculum is closely aligned with his post-modern curriculum. Despite being recursive, Doll's post-modern curriculum is directed towards education in general, with no particular focus in mathematics. It also remains largely visionary and theoretical with limited empirical study. My research is an implementation of Doll's idea of recursion in high school mathematics education. It also develops Doll's theorization further and engenders richer interpretations of recursive curriculum in general.

Bruner's spiral curriculum

Bruner (1962) advocates a spiral curriculum “that turns back on itself at higher levels” (p. 13). Bruner believes that all subjects can be taught to anybody at any age in some form because there are simple but powerful basic ideas at the heart of all subjects. He suggests a spiral curriculum that urges students to revisit these basic ideas and develop them gradually over time. Bruner writes,

> To be in command of these basic ideas, to use them effectively, requires a continual deepening of one’s understanding of them that comes from learning to use them in progressively more complex forms…The early teaching of science, mathematics, social studies, and literature should be designed to teach these subjects with scrupulous intellectual honesty, but with an emphasis upon the intuitive grasp of ideas and upon the use of these basic ideas. A curriculum as it develops should revisit these basic ideas repeatedly, building upon them until the student has grasped the full formal apparatus that goes with them. (p. 12-13)

A continual process of repetition with variations is central to Bruner’s spiral curriculum. Spiral curriculum challenges the linear metaphor of mathematics curriculum (e.g., viewing mathematics knowledge as building blocks). I see recursion in Bruner’s spiral curriculum as it invites students to revisit the profound ideas of a subject again and again to generate new understanding.

However, a spiral curriculum can become linear when executed if it is assumed that students will see the basic ideas connecting their new learning content to previous one by themselves easily. In this case, the lived spiral curriculum is like many concentric circles and when students move to different circles they are unaware of the existence of the concentric circles or they do not know where the center lies. Bruner (2006) does point out this danger:

> Many curricula are originally planned with a guiding idea much like the one set forth here [(i.e., the spiral curriculum)]. But as curricula are actually executed, as they grow and change, they often lose their original form and suffer a relapse into a certain shapelessness. It is not amiss to urge that actual curricula be reexamined with an eye to the issues of continuity and development. (p. 56)

For a curriculum to be spiral, its affordance in connecting different content is crucial. Here is where I envision a recursive curriculum can help, by making relations the central and explicit focus. Spiral curriculum is also criticized as a linear sequence of preset tasks converging to a fixed ending (Davis, Sumara, & Luce-Kapler, 2008). A recursive curriculum inspired by complexity thinking is worth exploring as a nonlinear alternative that is emergent and open to possibilities.

Since Doll's recursion is influenced by Bruner's ideas and my understanding of recursion is derived from Doll's recursion, it is not surprising to see that I view recursive curriculum close to spiral curriculum. But they are not the same. A recursive curriculum not only focuses on developing and redeveloping key ideas in a subject, but also stresses hermeneutic processes and openness towards novelty. Also, a recursive curriculum does not have to start with simplified problems associated with the basic ideas of a subject; it can start with complex fractal-like phenomena.

Davis and colleagues' fractal-informed curriculum

Informed by complexity thinking, fractal geometry and neurology studies, Davis and Sumara (2000) emphasize recursion when proposing a curriculum with a fractal-informed sensibility. They view cognition as fractal-like and recognize that recursion plays an essential role in the cognitive development. Such recursive development is incompressible, in other words, "there are no shortcuts to the eventual products" (Davis & Sumara, 2006, p. 43). Thus, the new structure of cognition can only be achieved by going through the entire learning process: "the structure emerges or the path that unfolds has to be lived through for its endpoint to be realized" (Davis & Sumara, 2000, p. 841). In this sense, Davis and Sumara's fractal-informed curriculum shares with Doll's post-modern curriculum the same focus on recursion and *currere*, thus it is compatible with my version of recursive curriculum.

Davis's later work with his colleagues offers general recursive curriculum design guidelines. Viewing a recursive curriculum as a sequence of elaborations, Davis, Sumara, and Luce-Kapler (2008) point out what matters in designing such sequence is not the linear predetermined steps, but "the manner in which each element of the sequence calls for recursive elaboration of already-established products" (p. 201). To help with thinking about how to organize tasks with a structure of recursive elaboration, they offer several lived curriculum examples. One of them is a poem writing activity. The activity started with generating a story character based on a self-selected button, it then processed through multiple rounds of alterations by adding restrictions (i.e., having two buttons meet, given a photo for the story setting), and ended with a poem writing exercise following a study of several poem examples. Based on such an example, Davis et al. suggest an image of growing a fractal tree (see an example in Figure 3.3.1) as a mnemonic device for ordering curriculum tasks and depict the tree growing process as:

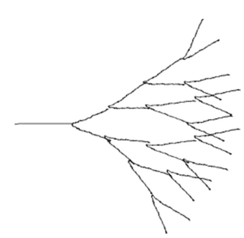

Figure 3.3.1. A fractal tree

> One begins with a seed (e.g., an enabling constraint, such as "Pick a button, and then imagine the garment it came from and what the wearer of the garment is doing"), elaborates on the product (e.g., "Have the person meet up with another

person"), elaborates on that product, and so on. (p. 201)

Guided by this growing a fractal tree metaphor, it seems that one can develop a recursive curriculum by choosing an entry task as a seed to bring forth some product, then proceed to add multiple rounds of tasks to elaborate a previously established product. There are two different developmental stages in this process: 1) initialization, during which the entry task is executed and an entity is generated as a result, or in the tree growing analogy, a seed is developed into a young plant, or a seedling; 2) elaboration, during which multiple tasks are executed in a sequence, recursively elaborating on the same entity at different stages of becoming, or a smaller tree grows into a bigger tree. Of course, the choice of seed is important for this fractal tree growing process. Davis et al. offer enabling constraints as one possible kind of seed and define them as "the sorts of questions and tasks that support both individual and collective learning" (p. 193) and specify that enabling constraints have sufficient structure to limit possibilities and also sufficient openness to allow for flexible and unanticipated responses. Put it differently, enabling constraints are "not *prescriptive* (i.e., they don't indicate what *must* be done), but *expansive* (i.e., they indicate what *might* be done, in part by indicating what must not be done)" (Davis et al., 2008, p. 193).

Other than specifying how a recursive curriculum can start (i.e., using an enabling constraint as seed) and differentiating the two stages of development, Davis et al.'s tree growing description offers little practical advice for elaboration. The use of a fractal tree seems to serve more as a conceptual metaphor to emphasize seeing the product as fractal-like and each new step in the curriculum sequence calling for a new round of elaboration, rather than suggesting what an elaboration looks like or what element in a curriculum sequence can call for elaboration, even though interesting implementations of this metaphor can be observed in Davis's later work with his colleagues, such as Davis and Renert's (2014) developing a concept study methodology in mathematics teacher education and mathematics classrooms. Similar to Doll's work, Davis and his colleagues' idea of fractal-informed curriculum is inspiring for education in general and demands further exploration for its practical implementations. I will return to this fractal-informed curriculum later in Chapter 9.

Thom's recursive curriculum (4Rs of recursion)

Recognizing the linearity of the usual elementary mathematics curriculum that interprets mathematical understanding growing through an additive action, Thom (2012) reinterprets mathematical growth, i.e., the growth of mathematical understanding, as a recursive co-emergent phenomenon through the lens of complexity thinking:

> If mathematical growth is viewed from an enactive perspective, like other forms of knowing, it is an embodied phenomenon that develops through relating and re-experiencing mathematics from an opposite side – a contrasting point of view, or seen suddenly through the eyes of an outsider. Because of this, students' learning is not achieved through repetitious acts of reproduction or sequential assembly lines of task. Doing so implies learning to be a matter of practising by redoing what one already knows or taking what one knows and adding to it in a piecemeal manner.
>
> Mathematical growth as a recursive event connotes the actual changing of one's mathematical understanding in ways that are complex and emergent. While its evolution possesses qualities of self-similarity based on primitive knowings, what it becomes and what it occasions upon each recursion is something qualitatively rather than quantitatively different. (p. 206)

Thom's concept of recursion has clear connotations of repetition with variations, looping back, reflexivity, self-referencing, and reflection.

For Thom (2012), conceptualizing mathematical growth and learning as recursive phenomena points to a need for a kind of teaching that opens ecological learning spaces for students: "It places importance on making space for students to reflect on their mathematical patterns of thinking and to revisit their mathematics inside different contexts so that they may critique what they understand from their current place of knowing" (p. 207). Thom states that opening learning spaces like this allows students to re-view and reflect on what they have seen before and relate to the mathematics in multiple ways, thus creating possibilities for students to renew their understandings in a qualitatively different manner. Thom envisions a recursive curriculum to create an ecological sense of space, which allows mathematics teaching and learning be conceived as "*holistic, organic, recursive, and co-emergent*" (p. 33) rather than "linear, hierarchical, static and deterministic" (p. 365). Thom identifies four qualities of recursion as reflecting, re-viewing, relating, and renewing (p. 206) and she views them "complement[ing] the curricular 4Rs, richness, relations, rigor, and in particular, recursion as explicated by Doll Jr. (1993)" (p. 373).

Thom's (2012) empirical study explores the ways mathematics teachers can consciously enable recursive forms of learning and create an ecological space in the classroom. Thom ran a two-year program in her mixed grade (grade 2 and 3) elementary classroom. She actively created opportunities for her 4Rs to take place – "to engage the students to reflect on, re-view, relate, and renew their mathematical understandings" (pp. 207-208). In her program, students often re-viewed what they had seen before in various contexts. Sometimes students were asked to reflect on and apply the same set of knowledge in various ways (e.g., create a riddle for others and solve a riddle created by others); sometimes students watched a mathematics film repeatedly; sometimes they revisited the same mathematics task a few times during a school year and addressed the task using their current understanding on each revisit; sometimes students worked on the same mathematics problem in various contexts (e.g., generate a statement to describe a data set, then modify this statement to fit two data sets when working with a peer, after that modify the new statement to fit four data sets when working in a group of four); and sometimes students "returned to familiar contexts but investigated them in completely different mathematical ways than before" (Thom, 2012, p. 223) (e.g., after students had investigated triangle numbers and identified square numbers, Thom invited students to study both kinds of numbers together). For Thom, this teaching structure is recursive and it opened spaces for her students to "experience recursion as an exploration of the unfamiliar within the familiar" (p. 235).

Thom (2012) summarizes some features of her teaching:

> Scripting unscripted lessons, mapping out potential as opposed to certain mathematics as part of her lesson preparation, distinguishing mathematics as both a residue and a source for learning, making familiar mathematics unfamiliar through recursive teaching structure and, attending to the class' individual and collective work. (p. 365)

She highlights that there is no predetermined path for building a recursive curriculum that has an ecological sense; one has to lay down the path while walking on the path. And rather than breaking away from their teaching methods and ways of knowing, Thom suggests teachers "re-searching [their] teaching in a systematic way" to produce prompts and means in order to lay down a different path (p. 365). In other words, similarly to students' mathematical growth, a recursive curriculum needs to be developed in a recursive path, through teachers' exploration of the unfamiliar within the familiar.

Thom's (2012) recursive curriculum is aligned with my vision of recursive curriculum. However, Thom's research differs from mine in research context and methodology: Thom's focuses on elementary mathematics education and it is a study on her own class teaching, while mine focuses on high school mathematics education and it is a hermeneutic inquiry about both lived and planned curriculum with experienced mathematics teachers involved. While Thom's recursive teaching structure is inspiring, it might have benefited from the curricular flexibility that she has as an elementary teacher who normally teaches a range of subjects to the same class. The difference in how learning is organized in high school level compared to the elementary level leaves questions about what a recursive teaching structure in a high school setting might be like.

In summary, although some scholars have explored recursive curriculum in general educational contexts and in recursive mathematics curriculum in particular, the amount of research directly addressing recursive high school mathematics curriculum is limited.

3.4 Research Necessity

Up till now, I have argued for the importance of recursion for education by drawing support from complexity thinking and mathematics education research. The importance of recursion has been exemplified in its pervasiveness in the development of many complex phenomena, including cognitive growth and mathematical understanding. Given the importance of recursion, the amount of research into recursive mathematics curriculum is insufficient. Particularly, there is no research about a recursive high school curriculum that centers on re-viewing or re-experiencing. Hence, this literature review establishes the rationale and significance of my research.

4 Path Unfolding While Walking

This chapter describes the research design and its dynamic process. The trace of the research journey is a spiral of loops, each of which is resulted from its previous loop(s) and affects what the next loop might be. Each loop can be divided into two chronological stages with different emphasis. Stage one focuses on generating an entry text, such as a personal reflection, a teaching document or a conversation with teacher participants. Such generation is affected by my interpretation of what is going on and what has happened before. Stage two focuses on interpreting the entry text and reinterpret the text existing prior to this loop. Such interpretation generates texts for further interpretation and affects the next loop. These two stages are not corresponding to data collection and data analysis executed linearly. Rather, each of them is a period of text generation and text interpretation weaving together and informing each other reciprocally. In this chapter, I specify the three categories of entry texts and then elaborate on three phase shifts, i.e., periods during which the research and I transformed. I end this chapter by offering an interpretation of the hermeneutic study process as fractal-like.

4.1 Entry Texts

The three categories of entry texts, i.e., autobiographical reflections, teaching documents, and conversations with teacher participants, correspond to different generation processes: writing, gathering, and conversing respectively. Their generation occupies different time periods in this study. The teaching documents were mostly gathered and interpreted at the beginning of the study, overlapping a 13-month period for generating autobiographical reflections and conversations with teachers. The interpretations of these three categories of texts are different, yet sharing similar methods such as contemplation, mind mapping, and freewriting. Throughout the study, these three categories of texts were woven into a whole, constantly informing my interpretations of recursive curriculum and hence affecting each other's generation and interpretation.

Autobiographical reflection

As stated in my introduction of this study, recursion is a topic that addresses me. Over the years, my personal learning, teaching, and academic experience have shaped my understanding of recursion profoundly. Certain personal experiences happened prior to the study have always obsessed me. During this study, some experiences were remembered and some incidents happened. All of these experiences and incidents were intriguing, demanding my attention. They are exactly where hermeneutics starts to work:

> Hermeneutics can only start to substantively and imaginatively appear in the face of "a case" (even though hermeneutics does not produce "case studies"). It is always *something* that happens that awakens our interest in pursuing interpretation. (Jardine, 2015, p. 238)

My work with this category of text was to hear them through doing reflective activities and attending to my bodily feelings, take note of them whenever they occurred to me, and experience them as speaking to me and as "having something to say to [me] beyond what [I] might be able, as yet, to say of it" (Jardine, 2015, p. 239). I tried general reflective tasks such as: Think back to a period (e.g., my high school years) with mathematics or learning in general. What people, event, and places do you remember? How did they make you feel? What was significant to you in shaping your teaching and learning practices? I also did the specific reflective tasks given to the teacher participants, such as "Complete the following sentences. 'Reviewing mathematics is like__________' ". These allowed me to recall some personal experiences, such as the two stories of re-naming that I present and interpret in section 6.1, as examples of lived re-viewing experiences. Some texts started with unexplained intriguing experiences and later developed into multiple levels of reflective writings, i.e., a reflection on this noticing, a reflection on this reflection, and so on. My story of re-experiencing the concept of negative numbers, for example, started with my initial unexplained resistance towards the negative tokens when they were introduced to me during the Re-experiencing Workshop. My contemplation on my initial discomfort brought forth a new understanding of negative numbers and integer tokens, and this learning experience became the entry text for the next round of reflection and interpretation. This phenomenon of one entry text emerging after an interpretation of a previous entry text exemplifies the hermeneutic quality of this study. This phenomenon is common in all three categories of texts in this study.

My reflection text was often (re)interpreted through contemplation and freewriting. The key in freewriting, in consistent with Elbow's (1998) version, is to write continuously, as long as I can, whatever comes to my mind without editing and worrying about the end-product. Here, writing is thinking and enacting hermeneutics and complexity thinking by letting things unfold. I often started with something intriguing in ways I did not fully understand. As I continued to write, trying to verbalize what seems interesting, what it might mean, what ideas are related to it, what I am learning from it, and/or what it reminds me of, ideas would flow and lead me to places that I had not thought of before.

This process of generating and interpreting personal reflection is essentially my hermeneutic conversation with myself. It is to uncover and examine the legitimacy of my assumptions and prejudices, which allows me to relate to other texts. It is also a way to learn about myself, interpret myself hermeneutically and live a hermeneutic life. A hermeneutic study needs to bring forth a better self-understanding, and the mission of hermeneutic researcher is "engaging Life hermeneutically" (Smith, 2006, p. 105). The process of personal reflections serves as a way to keep myself conscious about my use of language and my context, keep interpretation going, and occasion change in self-understanding. Moreover, generating and interpreting personal reflection is to learn something bigger than myself through learning *from* myself. It is not to reveal nothing more than my idiosyncrasies. Human experiences are always experiences of *something*: They are worldly and happen to us over and above our wanting and doing (Jardine, 2015, p. 248). My interpretation is unavoidably linked to me,

> However – and here is the paradox – *what* the interpretation is henceforth *about*, is not "me" but that topography *of which* I have had certain experiences: initiation. Initiation and its kin from a world in which I am "housed." They are not housed inside of me. (Jardine, 2015, p. 248)

Therefore, the process of interpreting my experiences and what addressed me is a movement shaping and making something of the instances, their human topographies, and myself. Through my experiences, I see the world I find myself in. Such movement is also an enactment of hermeneutic circle – considering back and forth between the part and the whole, the specific and the general. Thus through reflecting on my experiences with recursion, I begin to understand what recursive curriculum might be for other people.

Document interpretation

As teaching documents, i.e., program of studies and textbooks, largely represent the planned curriculum and play a significant role in teaching, it is important to draw inspiration from them in terms of in what ways a planned curriculum can afford recursion. Any reading of the recursive quality of a planned curriculum is a perceived potential rather than an innate quality.

Since the use of documents is for provocation rather than comprehensive overview, I focused on texts around a single topic, i.e., function, to ensure the study was feasible. This topic was chosen due to its significance across grades in the high school mathematics curriculum. However, the choice of one important topic over another should have little effect on the result of the document interpretation, because the recursive quality of a mathematics curriculum is a structural attribute rather than a content-based one. Similarly, a teaching document in any geographical or historical contexts could be informative for my inquiry. In this study, considering potential contributions for generating more relevant conversations with the teacher participants, which were recruited in Alberta due to geographical convenience, I chose the current high school (grade 9-12) mathematics program of studies and a commonly used grade 10 textbook, i.e., Davis et al. (2010) (called *Alberta Math10 textbook* in the rest of the writing), in Alberta as an entry point for document interpretation.

Other teaching documents were added to my reading and interpreting as the path unfolded itself. Following Bateson's (1979/2002) idea that mental process is triggered by difference, I pursued reading a document along with a reference to heighten my susceptibility to new experiences. I read the secondary program of studies alongside the elementary version, and the grade 10 mathematics textbook alongside other textbooks, including a grade 9 Chinese textbook, i.e., Department of Secondary Mathematics of the People's Education Press (2001) (called *Chinese Math9 textbook* in the rest of the writing), an old textbook (i.e, Jacobs, 1970, *Mathematics: A Human Endeavor*), and Saxon's (1990) Algebra 1 textbook, and a unit plan example (i.e., the function unit plan in Kalchman & Koedinger, 2005). All of these extra resources address the introduction of functions. I encountered them before and during this study, by accident or by choice, and found them relevant and intriguing for the study, so I gradually included them in my document interpretation.

Document interpretation included attuning to the resonance a document brought forth in me while reading it, pondering on the association, memory, image, emotion or sensorimotor response that it suggested or invoked, multiple rounds of rereading and re-categorizing texts, working difficulties out and generating new insights through freewriting, and following where the document interpretation led me to. I firstly read through the texts and wrote down general impressions and anything interesting. For example, both Alberta elementary and secondary programs of studies have no words about recursion or how mathematical ideas evolve over time; the summary table through which how the same theme develops across grades can be observed in the Alberta elementary program of studies does not exist in the high school version; and the tables of contents in all the texts I studied are in list rather than nonlinear forms such as diagrams or networks.

I then categorized the textbooks in an excel chart, using the four working qualities of recursive curricula identified earlier (i.e., continuity, repetition with variations, reflexivity, and reflection), showing how these texts embody these qualities and any possible new criteria if there was any. This process included multiple attempts and many struggles, as often when I added a text example as an embodiment of a particular quality (e.g., repetition with variations) I was confronted with an awareness that I could also add it to another category (e.g., continuity). So I had to rethink what I meant by these two categories and refine them. Often this was worked out through freewriting, followed by the next attempt for categorization. This categorization served as a hermeneutic conversation with the texts, aiming to propel thinking about recursive curriculum further rather than trying to get a comprehensive view of them. This process of repeatedly categorizing and writing helped me to redefine the four qualities of recursion and renew my interpretation of recursion and recursive curriculum (see Figure 4.1.1 for the old and new visualizations of the four working connotations of recursion).

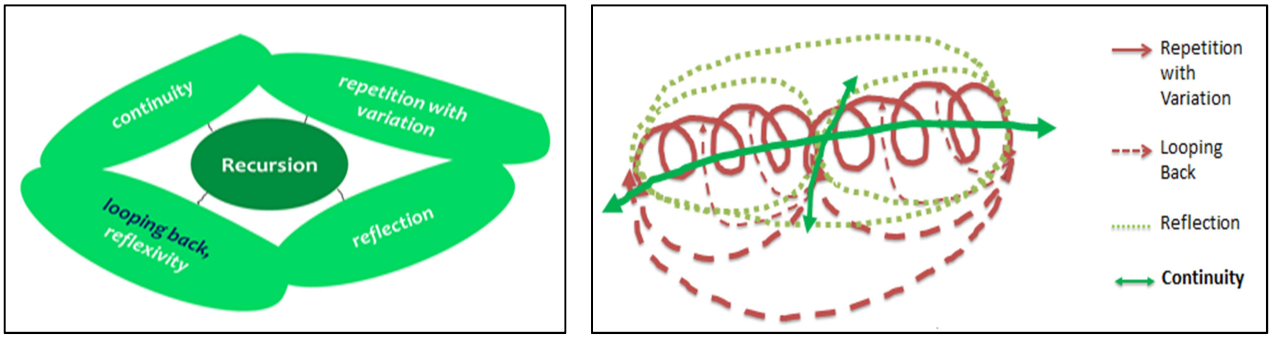

Figure 4.1.1. Two visualizations of recursion: a) the old version (on the left), and b) the new version (on the right).

These new understandings affected my later generation and interpretation of conversation and reflection, as they sensitized me differently and oriented me towards different ways to embody or enact recursion in curricula. For instance, I started to consider repetition with variations and looping back as the same kind of movement in opposite directions, reflection in relation to self-similar concepts, and continuity in two directions: development among cases of the same class (symbolized by the shorter two-way arrow) and development across classes (symbolized by the longer two-way arrow)[4]. Some intriguing examples from the document text were repeatedly reinterpreted, when I interpreted the other two layers of text or renewed my

[4] These two developments are not exactly the same as Freudenthal's (1991) horizontal and vertical mathematising. Horizontal mathematising "leads from the world of *life* to the world of *symbols*" (p. 41) and an example might be a pupil recognizing counting 5 times can be expressed as number 5. Vertical mathematising "effects the more or less sophisticated mathematical processing" (p. 41) and in the process "symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectively" (pp. 41-42). An example might be a secondary student recognizing linear and quadratic expressions are example of polynomial expressions. However, the line between life and world is relative to the learner doing the mathematising and particular situation and environment. Therefore, the same vertical mathematising example mentioned above could be an example of horizontal mathematising for a mathematician whose life mathematical objects have become parts of. So the distinction between horizontal and vertical mathematising is mostly at one focusing on objectifying and symbolizing whereas the other focusing on increasing abstraction. In my model, there is no such distinction between the world of life and the world of symbol, only the distinction between development within a class and development between classes. Whether a development is among cases of the same class or across classes is subject to individual observers. For a pupil who is trying to understanding counting 5 times as an example of number 5, this development is across classes, yet it is within the same class for her teacher while teaching natural numbers. Nevertheless, both the vertical mathematising and development across classes share a process of increasing level of abstraction.

understanding of recursion or recursive curriculum. Eventually I saw them anew: A fractal-like structure emerged alongside with a binary-tree curriculum design model (see section 8.2 for a detailed explanation).

Conversation with teachers

Much effort in this study was devoted to generate and interpret my hermeneutic conversations with experienced high school mathematics teachers. This layer of text is necessary due to the significant role that teachers play in shaping planned and lived curricula, and the need of others for one's reflection.

"Hermeneutics chooses the best players, on purpose" (Moules, 2002, p. 14). The best participants in a hermeneutic inquiry are those who can best inform understanding of the research topic (Moules, 2002; Smith, 1991). I chose to work with experienced (i.e., in-service, former, and retired) high school mathematics educators who are interested in better understanding review practice or recursive process in mathematics classrooms, as they, compared to pre-service teachers, 1) would have formed some perceptions about the recursive quality of a lived or planned mathematic curriculum through their experience of reencountering the same idea during teaching, such as teaching something students have encountered before, helping students review what they have learned before, learning something anew, and etc., and 2) would have a richer teaching experience to reflect upon, and consequently might have more to add to the discussion of recursive curriculum.

All participants could participate in this research, by 1) attending any number of monthly professional development workshop(s) in a sequence of six, and/or 2) joining an individual conversation. Such flexible involvement was allowed to attract participants while encouraging high participation and in-depth conversation. The conversation generated in both cases was used to inform my interpretations of recursive mathematics curriculum, consequently who said what is far less relevant for this study than what has been said. The participants in this study were viewed as a collective other, collaborating with me to provoke my thinking.

The number of participants in my research was not predetermined, since hermeneutic inquiry is not validated by numbers but by the fullness and depth of the interpretation of the topic under study (Moules, 2002; Smith, 1991). Participants were recruited in an on-going basis. In total, there were eleven participants in this study: Six of them only participated in the workshop activity (3 hours each, audio and video recorded), three of them only participated in the individual conversation activity (1.5- 2.5 hours each, audio recorded), and two of them participated in both activities. Most participants worked with me once or twice. One teacher participated fully in the six workshop series (see Appendix A for conversation dates, formats and topics).

Since recursion is an abstract concept and the idea of recursive curriculum is neither well-known nor well-established, it was inappropriate to assume that teachers consciously view curriculum recursive and they can talk about their interpretations of the recursive quality of the mathematics curriculum directly. Therefore, a conversation often started with individual reflective activities about a practice as a form of recursion (e.g., reviewing), and then discussion in group to identify the major themes within the group's interpretations of the practice, later continued with the group commenting on a curriculum material selected or created by the researcher or the participants. The participants were invited to examine, critique, and revise these materials with the intention to facilitate a recursive curriculum or promote the practice, and also to reflect on previous works at the end of a workshop, or at the beginning of a new one. Each conversation was designed with a tentative protocol that was informed by the work prior to it and subjected to change during its execution. The first workshop and the first individual conversation shared the same protocol. The later ones had a similar structure with different contents, or evolved into an extended conversation of the previous one(s).

In each conversation, my role as a hermeneutic conversation partner demanded me to interpret the conversation text along with its generation, and act tentatively and thoughtfully to move the conversation forward. I often wrote down my own interpretation of participants' ideas and my ideas that arose in conversation on a shared whiteboard or notebook, in order to provoke our collective thinking. After a conversation, I jotted down my impressions of the sessions, including any provocative ideas, new insights, difficulties, and possible modifications for the next session. Sometimes I continued to work on the mathematical problems I encountered in the workshop and ponder on whatever puzzled me. Before a new conversation, I went over the recording from the previous conversation, and transcribed, paraphrased, or took notes of parts that were intriguing. This resulted into a chronological text that resembled session minutes with my comments. Then I reviewed the text and I noted whatever called or beckoned by highlighting words, adding comments or coding it. When sensible or necessary, I also went over some previous sessions' digital records or notes and made new notes on them with a different color to signify revisions. Based on this work, I generated my new conversation plan. Before the final workshop session, I reviewed all previous collected texts and their associated notes, and added more notes. Upon finishing all the sessions, some sessions had been reinterpreted multiple times.

Not surprisingly, my on-going interpretation of conversation text occasioned changes in both inquiry direction and interpretation focus. Each conversation and its interpretation influenced the study in various degrees, by bringing forth new ideas or directions, (re)sensitizing me towards certain words (e.g., equivalent), consequently affecting further text (re)generation and (re)interpretation. For example, the idea of a re-imaging workshop came to me in the second workshop on re-viewing, and the idea of continuous narrative was formed during the first conversation, later became one of the foci in my interpretation of the teaching documents, and eventually turned into one of the workshop topics, i.e., re-storying. Despite constant changes in conversation plans and executions, and in my understanding of self, my study and hermeneutic conversation, a common structure remained in all conversations: We examined recursion or a form of recursion and envisioned how to design curriculum to promote this process or what might a recursive curriculum be (see Appendices B and C for original and modified plans).

4.2 Phase Shifts

I started my study with actions suitable for hermeneutic studies, actions that are "apparently global and unfocused" (Ellis, 1998, p. 21). Therefore, it was not a surprise but an affirmation of my hermeneutic work when I encountered some turning points later, during which both myself and my study were transformed to a new phase. The emergence of such phase shifts exemplifies the heuristic and generative quality of a hermeneutic study. Here I elaborate on three phase shifts, their contexts and associated significant changes in conversing, interpreting, and writing respectively, without implying any cause-effect relationship between events or their influences being limited in one dimension only.

The resolution of my struggles in conversations

The first phase shift happened after the first five conversations (i.e., three individual conversations and two workshops). While each of the sessions after the first one was the modification of its previous one(s) with changed reflective activities and curriculum example, all five sessions ended up using a protocol close to the original ones (i.e., the plans in Appendix B) with the same topic - reviewing. Although each session moved my study forward in some ways (e.g., bringing forth two conversation topics: re-imaging and re-storying), I did not seem to hear much new from my participants. I became increasingly unsatisfied with the conversations; I felt confined.

Before my work with participants, I had experienced getting lost in a conversation that seemed to have a spirit of its own, flew by itself and moved me to think of something new unexpectedly. Those first five conversations had less flow and openness than I had expected. I found myself caught in paradoxes. On the one hand, I understood that my involvement would inevitably affect what my participants would say to me. So I was cautious to avoid putting my words in others' mouths. This seemed to make me act like an observer and interviewer, contradicting with my expected role of conversation partner. I was also suspicious of my effort and found it futile to distinguish their opinions from mine and avoid influencing them. On the other hand, my effort in moving from a practical space to a theoretical space in one conversation, by including activities on the practice of reviewing at the beginning and then questions about recursion and recursive curriculum directly, seemed to limit both myself and the participants. While all teachers had much to say about reviewing, they responded little to my questions with the terms "recursive curriculum" and "recursion" (e.g., "In what ways do you see this kind of reviewing embodying a recursive curriculum?"). Such questions also forced me to battle between a need to elaborate the terms and a fear of not having a good enough explanation that would not converge participants' thinking towards mine or lead the inquiry to a wrong path. Given the high stake of each conversation, I was so concerned about my influence in it that I could move little.

My struggles led me to understand my study, hermeneutic inquiry and conversation anew. There seemed a connection between my study and hermeneutics that I overlooked before. As an inquiry into the implication (i.e., the curriculum) of an idea (i.e., recursion) that is informed by this study, my research entails circular movements: A study of an idea and a study of its implementation influence each other and themselves in return. This circular movement is one beyond the one already existing in a hermeneutic inquiry with "a moving target" (Davis, 1994, p. 21) [5]. There are *two* moving targets (i.e., the two concepts - recursion and recursive curriculum) that influence each other in this study: Not only does each concept transform itself when I interpret it, it also transforms the other. So my feeling of not knowing when and what is good enough to share with the participants was both a *cause* for and a *result* of moving in circles; my struggle was legitimate. Such understanding made exits immediately clear: 1) if I try to get a good enough version of my participants' opinions or a good enough term definition to share with my participants, I could be forever stuck in an infinite loop. Therefore I need to set off at a point and see what happens next; 2) since I am coming back to refine my understanding later, I do not have to set off at a good enough point – I can set off at any point that is *good-enough-for-now*.

My effort, then, needs not concentrate at making sure I heard enough the participants' ideas before I bring in my ideas or juggling between being a listener/interviewer and being a speaker/conversation partner. My effort needs to be at *how to keep our thinking going*. My conversation, as hermeneutic, needs to be about what the concept *might become* if we were to interpret it this way, or in Jardine's (2015) words, "What *difference* does it make if we read these words this way? What possibilities open up?" (p. 245) Therefore, the focus of conversations should be what comes out *in-between* rather than who said what, what each one of us has, or whether what we have is good enough. Let's start moving and then we will see. Thus I can feel free, after diligent preparation before a conversation, to share my thinking of recursion and

[5] As Davis (1994) points out, "For [Gadamer], an interpretation involves first an appropriation of an event and, as one comes to meaning (interprets), a transformation of that event" (p. 21). What one interprets in a hermeneutic inquiry is affected by the exact process of interpretation. Thus, "the 'object' of the hermeneutic inquiry is a moving target" (Davis, 1994, p. 21).

recursive curriculum and comment on others' ideas, but only as a way of thinking and provoking collective thinking further.

This good-enough-for-now attitude and refocusing in the space in-between released me from juggling between conflicting roles. It became easier for me to be a genuine conversation participant who thinks in the conversation and is led by the conversation. Thinking how to provoke conversation allowed me to notice task design questions, such as "how to design tasks to promote or facilitate a recursive curriculum," might be impractical for my participants. Hence I replaced them with more practical ones that have a pragmatic definition of recursive curriculum such as, "In what ways might we design/modify tasks to help students to build connections and learn something new from what they have learned/experienced/encountered before?" and "Given what a recursive curriculum does, what might a recursive curriculum look like?" I also noticed a limitation of having participants critique my curriculum examples: It did not seem to help me notice something new; rather it seemed to funnel participants' ideas towards mine. Upon consultation with my supervisor, I altered the workshop plans: I gradually stopped showing participants my examples, and invited them to generate examples and/or modify their own examples. A similar change was made on the individual conversation plan: I replaced the curriculum examples with a general question (see Appendix C for renewed protocols).

After the above changes in my questions, curriculum example tasks, attitudes, and foci, my workshops became far more unpredictable and much more interesting. By the time I got to the third workshop in the series (i.e., Workshop 3.3 on Re-experiencing), the second phase shift of the study emerged.

The emergence of an alternative interpretative framework and the unseen third dimension

There were only Maxine and I in Workshop 3.3 (Re-experiencing). Our conversation brought forth an unusual mathematics problem involving negative numbers that was new to both of us. We ended up, unexpectedly, spending an hour working on it with no final solution. This experience allowed me to re-experience some mathematics ideas. Later, I redid the mathematics problem and reflected on my experiences, with a newly heightened sensitivity towards the need for teachers to relearn mathematics as part of the demand of a recursive curriculum, which is credited to a conversation with a colleague, Tom Kieren, about recursive curricula. I was overjoyed by how much I had relearned mathematically while studying mathematics curriculum.

Connecting this experience to my earlier obsession in the idea of continuous mathematics narrative, I re-interpreted the previous two workshop texts (i.e., Workshop 3.1 on re-viewing and Workshop 3.2 on re-interpreting) with a focus on *mathematics content development*: I categorized the mathematics ideas and tasks touched in a particular workshop, traced their developments over the workshop period, and re-categorized and rearranged them towards a continuous logical development. A cognitive map and two recursive curriculum development models (see Figure 4.2.1) were established based on Workshop 3.1 (Re-viewing). A third recursive curriculum development model was also created for the mathematical tasks in Workshop 3.2 (Re-interpreting). All these models were new to me, even though the related mathematical concepts and tasks were not. The process of model establishing and visualizing helped me to renew my understanding of the related concepts and tasks, and also recursive curricula. I was thrilled for finally being able to see something new, mathematically and pedagogically, in these workshop texts. At that point my confidence in my study increased and started to learn more mathematics as my study progressed.

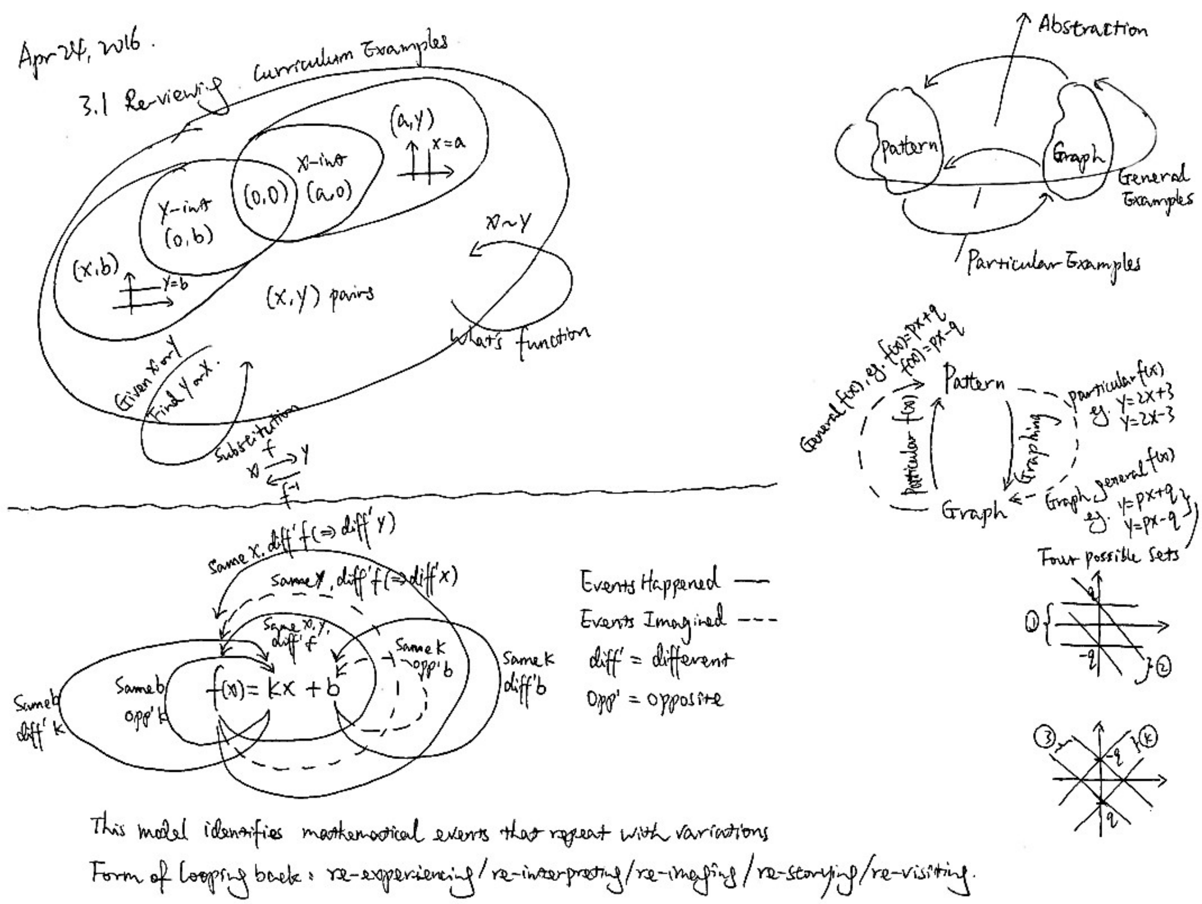

Figure 4.2.1. A cognitive map (top left) & two recursive curriculum development models (bottom left and right).

By the time I reached the second last workshop, Workshop 3.5 (Re-storying), it was clear that I had *experienced* a recursive curriculum of mathematics and mathematics curriculum in my inquiry about recursive curriculum: I was compelled to review what I have seen before repeatedly, be it a mathematical idea focused in a single workshop or multiple workshops, or the pedagogical concepts of the study, i.e., reviewing, recursion, and recursive curriculum. In each workshop (and also individual conversation), I worked with participants, switching between interpreting a form of re-viewing and designing curriculum to implement this process. Along with this back and forth movement, I re-viewed not only recursion and recursive curriculum but also mathematics. The above realization emerged and solidified through the process of visualizing all workshops/conversations. Figure 4.2.2 shows the final result of this visualization. The loops returning to the concept of recursion represent different workshops, each of which focused on a recursive process, i.e., re-viewing or a form of re-viewing. The study of all these processes informed me of the concept of recursion and recursive curriculum. Note that the first workshop loop (the left most loop) represents all the workshops and individual conversations focusing on re-viewing. The figure 8 loops at the bottom right represent the actions repeatedly conducted in each of the workshops: analyzing a form of re-viewing, designing curriculum to promote it, and re-viewing related mathematics contents.

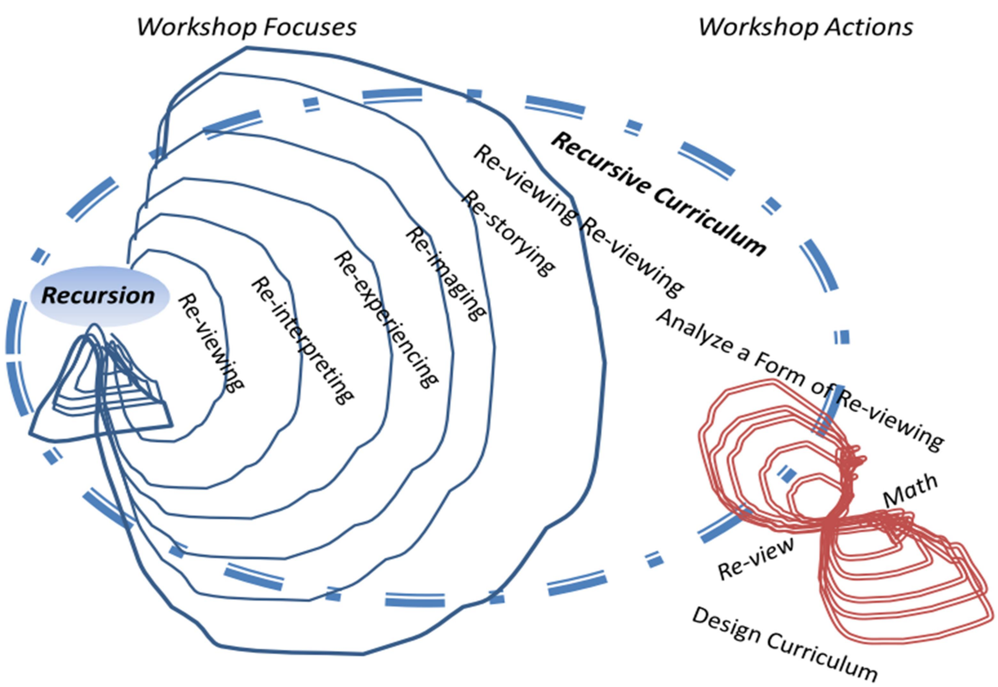

Figure 4.2.2. A holistic view of all workshops/conversations.

This learning of mathematics was an unexpected dimension in my study. It exemplified the generativity of a recursive curriculum and the boundlessness of one's growth in mathematical understanding. It also turned a study of recursive curriculum into an *experience* of it and a *learn-by-doing* process. Noting the existence of this dimension urged me to pay attention to possible mathematical content developments in conversation texts and my own experience of re-encountering mathematics afforded by this study; and to use them, as planned and lived recursive curricula, to inform my study of recursive curriculum. This turned out to be fruitful for my inquiry and in return reinforced my faith in hermeneutic inquiry and recursion.

The above experience, like my earlier struggles with conversations, taught me once again the importance and necessity of working out what came to address me over my wanting and doing in a hermeneutic inquiry and in a recursive learning journey. "Understanding begins… when something addresses us" (Gadamer, 1989/2013, p. 310). It became clear to me that this learning from oneself and from one's process of experiencing is exactly what a hermeneutic study and a curriculum as *currere* entail.

The arising of stories as attractors

The third phase shift happened quietly without much noticing during the dissertation writing process, dividing the process into two stages with different anchors. During the first stage, my writing was directed by themes. After I had come up with some themes for each form of re-viewing, I continued my interpretation through writing about them. Writing was used as thinking, rather than outputting complete thoughts. I tried to write, firstly, what kind of process each form might be, and then in what ways we might promote it through curriculum design. While writing about each theme, I intended to include at least one planned curriculum example or a lived curriculum story for provocation. However, I soon found this one to one (example or story to theme) relationship problematic. Not only were planned curriculum examples and lived curriculum stories not parallel, each of them was also not necessarily corresponding to only one theme. Some of them, like my story of re-experiencing negative numbers, were so complex that they seemed to have many things to say. Meanwhile, there seemed to be too many themes to be meaningful.

To condense the themes, I went through multiple rounds of mind mapping and outlining my writing in point form, focus on only theorizing each form of re-viewing without concerning curriculum design to promote the process at the same time. While doing so, some lived recursive curriculum stories consistently attracted my attention. To accommodate their roles of provoking my thinking rather than confirming a theme, I started to write from story to theme: I opened a section with a story and talked about how it led me to interpret a form of re-viewing in a particular way.

This reversion of order of writing was generative. My writing started to revolve around these provocative stories. Sometimes while I was writing a story and its related theme in one way, new perception and/or interpretation of the story would occur to me. These often prompted me to go back to contemplate on the story, sometimes reshape my writing of the story, and see what the story was trying to say and revising my existing themes and writing accordingly. There were often many rounds of such back and forth movements. It turned out that these stories had much to teach me than I had ever expected. A process of writing about a particular story often led me to understand the story and its related theme anew, and then this experience became a new lived recursive curriculum story for me to ponder on and write about. Through these activities, I renewed my understanding of a particular form of re-viewing, re-viewing as a whole, recursion, and recursive curricula. As such, my writing about each story became a hermeneutic inquiry into the story, propelling the overarching hermeneutic inquiry into recursive curriculum.

The arising of stories as attractors in writing marked the third phase shift in my inquiry. Once again, like what happened in the previous two phase shifts, I was educated greatly. I relearned how to trust my sense of being addressed by something, how to endure and learn from my struggles and puzzlements, and how to leave a work at a good-enough-for-now stage and come back to it later. These three phase shifts exemplified an affinity among recursion, hermeneutics, and *currere*-oriented process. Evidently, “the generativity of a hermeneutic learning process is inseparable with its recursiveness” (Luo, 2019, p. 99), and “being hermeneutic, recursion is inevitably *currere* oriented” (p. 100).

4.3 A Fractal-like Hermeneutic Inquiry

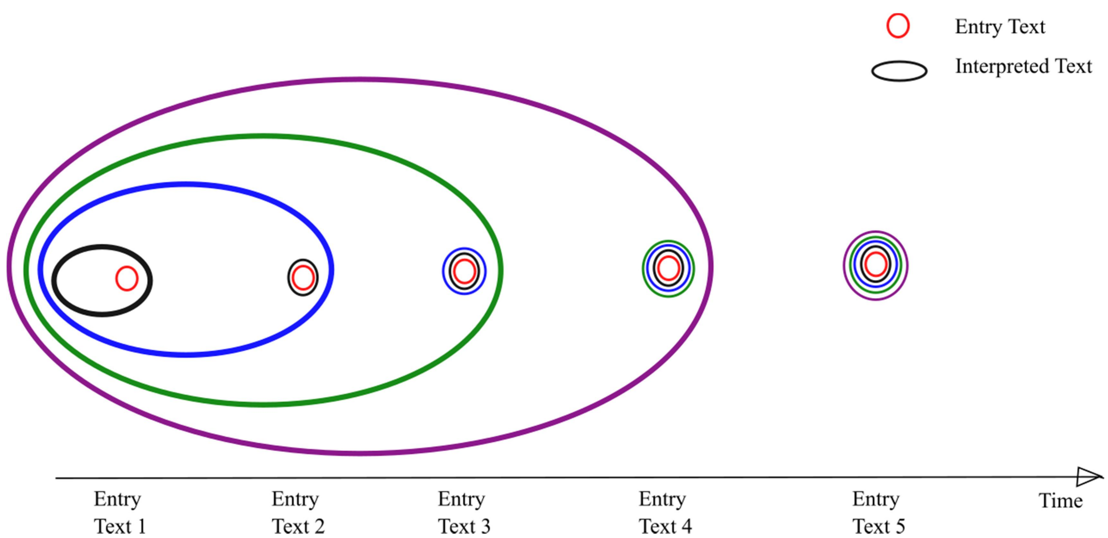

Figure 4.3.1. A visualization of the recursive structure of a hermeneutic inquiry.

When I looked back my inquiry as a whole, a fractal-like image appeared. Figure 4.3.1 shows a sequence of texts in a chronological order of generation. Each circle in red represents an entry text and each oval represents an interpreted text. Other than the first one, all circles have some colored circular boundaries, representing the influence of the previous work on the generation of an entry text. Each oval encloses all the previously generated texts, representing that each interpreted text is produced through interpreting the already present texts as a whole. For example, the first interpreted text (represented as the oval in black) was generated after the first entry text (represented as the circle in red) got interpreted. Its influence on the next text generation is represented as the black circular boundary added to the red circle in Entry Text 2. The second interpreted text (as the oval in blue) was generated through interpreting all the previous texts, and this is represented by having the blue oval enclosing all the previous texts (i.e., Entry Text 1, Interpreted Text 1, and Entry Text 2). This image shows a sequence of inquiry cycles and each of them is led by the generation of a new entry text and completed with an interpretation of the current existing text, which includes the new entry text and the old text. The result of this interpretation becomes the old text for the next loop, informing the generation of a new entry text and later the text interpretation.

This visualization is no doubt self-similar as it has a common pattern appearing across scales: Parts at different scales look like the whole. It can easily be formed through following a simple recursive formula:

Interpreted_Text(n) = Interpret (Interpreted_Text(n-1), Entry_Text(n))

This formula depicts two sub-procedures in each round of iteration:

In the *n*th round, one

1) generates an entry text, labeled as *Entry_Text(n)*, and then

2) interprets the current existing text, which include the previously interpreted text, i.e., *Interpreted_Text(n-1)*, and the newly generated entry text, i.e., *Entry_Text(n)*.

The result of this interpretation becomes the interpreted text for the next round of iteration, i.e., *Interpreted_Text(n)*.

Such a recursive structure exemplified itself in this study, more or less, at both macro and micro levels. The generation of the text as a whole and a particular layer of the text, notably the conversation text, followed this spiral process. This continuously interpreting one’s previous work and subjecting oneself to its influence also presented during the generation of each entry text and the dissertation writing process, which was used as a way to think and interpret differently rather than outputting mature completed thoughts hence made of loops of rewriting.

However, as my study has taught me, this recursive movement is not a procedure to follow but a *calling* to respond to. Hermeneutics begins with something addresses us and draws us in (Jardine, 2015). Not only does this tell me where to start and continue to ponder on, it also tells me where to go back to. During a text generation, where to go and inquiry about further was often led by what puzzled or provoked me. During a text interpretation, what to review, how much and how often was also often a response to a call to go back. For example, it puzzled me that I could not see much new in the text of Workshop 3.1 (Re-viewing), so I went back to the video text often. A breakthrough finally happened on its fourth reinterpretation. This being drawn back again and again to a particular experience or text happened even more frequently during my dissertation writing process, resulting into recursive interpretations of the same story or rounds of rewriting.

To enact a recursive inquiry as a response to a calling instead of a mechanical procedure, a hermeneutic attitude of *letting* is essential. Such "letting", as in "*Let* Eric's age be 'x'" (Jardine, 2006, p. 63), reflects a willingness to work with uncertainty, a courage to move on with a tentative good-enough to explore further, a patience to see what it brings later, a trust in learning from oneself, and a faith in the process's generativity and recursiveness (that one will be brought back to where one starts and sees it differently). Such letting, walking a path while letting the path unfold and letting one be led by the process, is rather uneasy for one who lacks embodied faith in hermeneutic study. I was led to hermeneutic inquiry by my appreciation of the theory. But I was more or less stuck in walking in circles rather than loops at the beginning of my study because I worried that my starting definition and study design might not be good enough. Once I found a way out through legitimizing leaving with a good-enough-for-now understanding in hermeneutic studies, my faith in “letting” and hermeneutic inquiry was further strengthened through seeing novel ideas emerging along my walk on the path. My hermeneutic inquiry experience has taught me that one cannot walk on a path that unfolds itself without a hermeneutic attitude of letting as it is both the *precondition* and *goal* for hermeneutic study.

5 Re(view): Re-view Reviewing

5.1 Make the Familiar Strange – Troubling Reviewing

Reviewing literally means seeing again. In my inquiry, it refers to a mental process that a learner (i.e., a student or teacher) can experience and an educational process that one can design and/or go through. The term reviewing, instead of review, is used in this study to emphasize the process of doing a review, since review can have so many meanings (e.g. the task, action, process, material, and session of a review). As a process, reviewing can be both lived and planned.

The importance of reviewing for mathematics education seems to be well recognized, easily suggested by the abundance of reviews in a planned curriculum and their frequency in a lived curriculum. For example, the Alberta Math10 textbook examined in this study has regular explicit reviews. There are review sections at the beginning of each lesson (i.e., Make Connections) and in the middle of a chapter (i.e., Checkpoint 1 and 2). Review sections also appear at the end of a teaching session, such as a lesson (i.e., Reflect), a chapter (i.e., Study Guide and Review, and Practice Test), or a unit made of two or three chapters (i.e., Cumulative Review and Project). In a lived curriculum experienced by myself and mentioned by the teacher participants, reviews are often dedicated sessions. Some appear as frequently as in Emma's grade 7- 9 mathematics classes – each of her teaching sections includes two days for new lessons and one day for review, as predictable as the ones in the high schools I taught – all grade 9-12 mathematics classes schedule a review period before tests and exams, or as grand-scaled as the ones in Dean's school – his junior high school has a six-day teaching cycle and each cycle includes a checkpoint (e.g., an assignment due day). All of these reviews offer students opportunities to see again what they have learned before. Apparently, reviewing is a routine teaching and learning process in mathematics curricula.

Yet, reviewing seems to be taken for granted both in lived and planned mathematics curricula. During my four years of teaching high school mathematics in Toronto, review sessions in class across grades took a consistent form of a period of seatwork, with students answering questions chosen from the textbook review sections or worksheets that their teacher copied from other textbooks or exercise collections. There were few discussions among teachers or between teachers and students about the process of reviewing. How to review was rarely addressed in mathematics textbooks or professional development sessions for teachers.

This taken-for-granted attitude towards reviewing seems to exemplify in teacher education programs as well. The lack of serious effort to disseminate the results of research on reviews to the educational community, observed in as early as 1991 by Dempster[6], is also noticed in the teachers' education program that I went through in Canada: Much of our effort was guided towards how to teach a new concept rather than how to review an old concept, despite that we all learned that a review of prerequisite skills at the beginning of a new lesson and a recap of the daily lesson are important for effective teaching and learning.

This lack of interest in reviewing in the mathematics education community is further demonstrated in the lack of publications about reviews in mathematics classes. An extensive literature review found between the years of 1972 and 2016[7], Suydam's (1984) brief paper on the

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[6] Dempster's (1991) literature synthesis might be the first and the latest comprehensive review on reviews in general.
[7] This result is based on a ProQuest search, conducted in the fall of 2016, on the 52 databases accessible for the libraries at the University of Alberta, including the three major education databases (CBCA Education, ERIC and ProQuest Education Journals). The search category is "su.Exact("review (reexamination)") and all("mathematics" OR "mathematical" OR "math")", aiming to locate citations that use review (reexamination) as a subject term and include any of the three specified words anywhere but the whole text.

role of review in mathematics instruction is the only discussion of mathematics reviews as a whole. Suydam's literature review is insufficient and the paper has been infrequently cited.

Does this indifference in reviewing imply that reviewing has been understood well by students and educators and effectively applied in mathematics teaching and learning? I doubt it.

Taking the frequency, timing, and content of reviewing as an example, there seems an interesting discrepancy between theory and practice: The spacing effect and the interleaving effect well recognized in educational psychology studies seems not utilized sufficiently in mathematics curricula. The spacing effect has been long documented in the memory literature (Kang, Lindey, Mozer, & Pashler, 2014), and exemplified in several mathematics education research (e.g., Gay, 1973; Rohrer & Taylor, 2006). The spacing effect describes a phenomenon whereby one learns more when one can study a material in spread out sessions rather than studying the same amount of time in one single session or a few sessions that occur close by (Dempster, 1991; Kang et al., 2014). Reviews of the same material spreading out in increasing intervals (e.g., study on day 1, 3, 9, 28) are more effective, possibly by affording *effortful* and *successful* restudy (Kang et al., 2014, p. 1549). Massed reviews, reviews that are too close together, tend to give rise to a false sense of knowing or confidence (Zechmeister & Shaughnessy, 1980). Students might think they know the materials while they do not, resulting in inactive reviewing process and lower performance in tests.

The interleaving effect recognizes that "interleaving rather than blocking practice of different skills (e.g. abcbcacab instead of aaabbbccc) usually improves subsequent test performance" (Taylor & Rohrer, 2010, p. 837). This effect has been demonstrated in mathematics learning (Rohrer, Dedrick, & Stershic, 2015; Taylor & Rohrer, 2010): The students who participated in interleaved mathematical practices (solving problems of different kinds juxtaposed in one practice session) performed better than the students who did the block practices (solving problems of the same kind). Interleaved practices demand learners to identify the problem and choose a strategy rather than repeating the same strategy. Together, the spacing effect and the interleaving effect suggest a frequent use of comprehensive reviews that make good use of forgetting.

In this line of thinking, a reading of the Alberta Math10 textbook shows me questionable implementations. The reviews in the textbook include end-of-lesson discussion questions reflecting on the key ideas in one lesson, two checkpoint reviews in each chapter summarizing what one has learned, either between a checkpoint and the beginning of the chapter or between two consecutive checkpoints, chapter reviews covering what one has learned in each chapter, and three comprehensive reviews with each covering all the previous chapters one has learned. There seems some yet inconsistent implementation of spaced reviews. As for the review content, most exercises address the freshly learned materials and the comprehensive reviews include a list of exercises categorized by learning sessions, all giving strong clues of what strategy the learner needs to use. There seems more avoidance of forgetting than taking it into good use as suggested by the interleaving effect.

Besides the frequency, timing, and content of reviewing, its form is also not well understood. In the Alberta Math10 textbook, a dominant form of reviewing is summarizing key contents and doing more exercises. Reviews in the forms of games, projects, and group works are limited. Multiple representations and embodiments of the same mathematical material are not evident.

However, it is not my interest in this study to identify effective review approaches. Rather, I am here to make the familiar strange and problematize the discrepancy between the obvious indifference in reviewing and the pervasiveness of reviewing in mathematics curricula. The above discussion about the frequency, timing, form and content of reviewing is to trouble the taken-for-granted attitude towards reviewing.

5.2 Return to the Original Difficulty – What is Reviewing?

We do not know reviewing well enough, I argue. What does review mean? What is the process of reviewing? What is reviewing for? How does reviewing help learning? These questions are largely ignored. The mathematics education literature related to reviews addresses mostly on particular approaches or forms of reviewing practice (and some of them do not specify their connection to reviewing), such as reviewing pre-requisite materials (e.g., Mokry, 2016), note taking (e.g., Hwang, Chen, Shadiev, & Li, 2011), doing assignment (e.g., Rohrer et al., 2015), using games (e.g., Goldstein, 1994), going over homework (e,g., Otten, Cirillo, & Herbel-Eisenmann, 2015), summarizing (e.g., Shimizu, 2006), and testing (e.g., Dempter, 1991). Even Dempster's (1991) comprehensive literature synthesis on reviews in general, which broadens reviews' roles in learning and forms of reviews, defines neither review nor reviewing. Rather, the word review seems to be used interchangeably with practice and repetition.

Compared to Dempster's paper, Suydam's (1984) paper about the roles and effective forms of mathematics reviews is much less comprehensive. It does start with a hermeneutic question "what does the word [review] mean?" (p. 2), but Suydam immediately closes the inquiry with an answer: "Re-view. Look again. See from a new perspective, in time if not in space" (p. 2). This short interpretation brings forth something worth noticing - Suydam continues to write: "The word carries a hidden promise of excitement or potential that we seem to miss" (p. 2). I appreciate Suydam's approach of looking for inspiration through interpreting a word. But her paper does not go further to interpret review or specify what is missed. Without answering the ontological and teleological questions such as what reviewing is as a mental learning process and what is it for, it seems impossible to answer methodological and pedagogical questions such as “What do we do in reviews? What should we review, how often and in what forms?”

So what is reviewing? What kind of process it is? What is it for? To answer these questions, one has to answer questions such as “What are mind, knowledge, and learning?” at first, for depending on what we think they are, we see reviewing differently.

Here I focus on three perspectives that have been or are currently influential in education: behaviorism, cognitivism, and enactivism. While recognizing that many alternative categorizations are possible, this particular categorization is chosen to subsume a range of educational theories in European and North American English literature while conserving their distinct views on mind, knowledge and learning. This brings a possibility that some of the theories with multiple variations, such as constructivism, might belong to two different categories in this categorization.

Behaviorism is an educational psychology theory that formed around the 1910s and established in the 1930s. For behaviorists, mind is an object situated in the brain. By viewing brain as a black box whose processes are unobservable and unmeasurable, behaviorism centers on observing, measuring and manipulating human behaviors. Knowledge is considered as an organized accumulation of connections and associations among elementary mental or behavioral units, which “may be elementary sensory impressions that combine to form percepts and concepts, or stimulus-response associations, or abstract elements of parallel, distributed networks” (Greeno, Collins, & Resnick, 1996, p. 17). Learning is viewed as a process of forming and strengthening or weakening and extinguishing associations, which one often acquires through experience (Greeno et al., 1996). Since the behaviorist view interprets a person’s behaviors as responses to stimuli in her situation, the educational models that embrace behaviorism treat learners as passive and view teaching as designing suitable stimuli to reinforce needed responses (Ertmer & Newby, 1993/2013). Since knowledge is viewed as a collection of particular stimulus-responses, the goal of education can be expressed as a list of detailed behavioral objectives, and correspondingly curriculum tasks can be analyzed and organized into a sequence of responses, ordering from simple to complex (Greeno et al., 1996). Based on such views of knowledge and learning, it is not strange for one to see reviewing as a *reinforcing* process to strengthen previously established stimulus-response associations. Thus drills and practices, during which similar stimuli are presented to students in expecting similar responses from them, seem to be ideal forms of reviewing. From a behaviorist perspective, the goal of reviewing is to acquire and reproduce the same.

Cognitivism gained acceptance during the cognitive movement in the 1950s as an alternative to behaviorism, in recognizing behaviorism's failure to explain complicated human behaviors, such as language learning. In contrast to behaviorism, traditional cognitivism views learners as active and their behaviors are not simple responses to stimuli but sophisticated outcomes of complicated mental cognitive processes, such as reasoning, solving problems, comprehending language. Mind, although still situated mainly in the brain, is like a computer from the perspective of cognitivism: It follows an input-central processing unit (CPU)-output model and it processes stimuli as input information and produces mental or behavioral changes as output. The mental action and process, or cognition, is

> the manipulation of symbols after the fashion of digital computers. In other words, cognition is *mental representation*: the mind is thought to operate by manipulating

> symbols that represent features of the world or represent the world as being a certain way. (Varela, Thompson, & Rosch, 1991/1993, p. 8)

Knowledge, then, is understood as the organized abstract structures of information (Greeno et al., 1996) and representations of the objective external worlds stored in the mind. Learning is a process of developing conceptual understanding and cognitive abilities (Greeno et al., 1996). Therefore, the educational models influenced by cognitivism emphasize teaching cognitive strategies to handle information and metacognitive skills (Ertmer & Newby, 1993/2013) and analyzing cognitive structures of subject knowledge (Greeno et al., 1996). Based on the above understanding of mind, knowledge and learning, reviewing can be interpreted as a process of enhancing one's memorization and conceptual understanding of previous learned knowledge. Hence getting students to practice and use cognitive strategies to elaborate, organize, construct their knowledge structure seems to be suitable forms for reviewing. The goal of reviewing is to *acquire and reproduce* the knowledge structures representing the objective world.

Over the years, cognitivism has developed into many different traditions, notably constructivism in the 1970s, distributed cognition, and situated cognition theories around the 1990s. Under the influence of these theories, while the focus on cognition remains unchanged in cognitivism, learning for a learner has become actively constructing their individual meaningful knowledge structure systems in a particular time, space, and environment, which include other participants, tools and artifacts. Not only do different individuals create rather than acquire different understandings, thus knowledge of the same thing, what they learn is also inseparable from where, how, and for what purposes they learned. From these perspectives, reviewing is interpreted as a process of *elaboration* and *reconstruction* and it is both individual and social at the same time. Thus the forms of reviewing can extend to include having students encounter multiple interpretations and representations of the same idea and having students study in groups or in various contexts. The acknowledgement of knowledge being contingent and individually meaningful allows reviewing to be interpreted as a process of *generating* something new rather than reproducing the old.

Although these later developed theories in cognitivism have challenged the original assumptions of cognitivism, such as learning is independent and individual, knowledge is processed by individuals and it can be transmitted, and they even started to embrace postmodernism by rejecting the idea of an objective external world and valuing individual understandings, a sense of mechanism (Bredo, 2015) emphasized in behaviorism remains. In other words, mind is still understood as an information processing machine and the mind-body dualism remains. Also knowledge is still understood as an object stored somewhere in the mind, thus can be quantified and measured. There is still a finite end of mastery for learning. Thus reviewing remains an *add-on* process for learning: Reviewing is only needed when one has forgotten or not mastered something; the ideal learning is to learn something once for all. Also, since cognition is still understood as a brain's function, physical body's engagement in learning in general is overlooked.

Enactivism is a theory of self-organized systems and enactive cognition, influenced by works in multiple disciplines, including philosophy (e.g., Merleau-Ponty's works, phenomenology), psychology (e.g., Piaget's works, radical constructivism, situated and embodied cognition theories), and biology (e.g., Maturana & Varela's works on biological autopoetic systems). Enactivism, as an illustration of complexity thinking, shares compatible views on mind, knowledge and learning with complexity thinking. Enactivism broadens the concept of mind and defines it as any autopoetic self-organized system, which includes all living systems. In other words, all living systems can learn and exemplify cognition. In enactivism, cognitive processes are no more limited in one's brain and separated from physical behaviors. Rather, "all doing is knowing and all knowing is doing" (Maturana & Varela, 1998, p. 27). Cognitive or mental processes are the ones that a living system engages to maintain structural congruence with its environment, and through which both the system and its environment evolve.[8] Therefore, different from cognitivism, physical actions are also cognitive processes. In this sense, enactivism breaks the dualism of mind-body. It echoes the emphasis of body's constitutive role in cognition as described in embodied cognition theory: "Cognition is embodied when it is deeply dependent upon features of the physical body of an agent, that is, when aspects of the agent's body beyond the brain play a significant causal or physically constitutive role in cognitive processing" (Wilson & Foglia, 2016). Enactivism is also aligned with connectionism in criticizing computational and representational models of mind centered in cognitivism. Actually, Varela et al. (1991/1993) proposed the term enactive cognition to

> emphasize the growing conviction that cognition is not the representation of a pre-given world by a pre-given mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs. (p. 9)

The enactive cognitive science is presented as "a middle path between the cognitivist version of an essentially representational mind and connectionist models of the emergence of mind from networks of neuron-like units" (Toscano, 2006, p. 169).

In enactivism, learning is defined as a kind of transformations that is simultaneously physical and behavioral thus structural for a learner (Davis & Sumara, 2006). Learning happens as a contingent result of the interactions between a learner's historical and biological structure

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[8] Cognition is exemplified through enaction, which denotes "the process whereby a world is brought forth by the interaction or structural coupling between an embodied agent and its medium or environment" (Toscano, 2006, p. 169).

and her surrounding environment. A learner's self-organization can be occasioned by environment stimuli but it is not caused by it; the learner is unpredictable and generative. Through learning, a learner and her environment co-evolve.

With the emphasis of contingence, relation, and evolution in enactivism, knowledge is no more an object stored in a learner's mind or in her external environment, or solely generated by the learner. Rather, knowledge is a contingent relationship located neither in the learner nor in the environment. Knowledge exists and consists in the possibility for joint or shared action (Sumara & Davis, 1997) and it is determined by the shared action. Thus, actions, seen as products of these interactions, are themselves understanding and knowledge (Sumara & Davis, 1997). It is knowledge-as-(inter)-action (Sumara & Davis, 1997). The locus of cognition is no more individual and there is not a solitary truth-determining agent for knowledge (Sumara & Davis, 1997). Knowledge evolves, like biological systems do.

From the perspective of enactivism, learning is not building representation to mirror the objective external world nor does it lead towards to a fixed end of universal truth. With the rejection of fixed outcome for cognitive development, comes the inevitability of reviewing for cognitive growth. Reviewing, interpreted through an enactivist perspective, is a *contingent transformative self-organization* affording emergent creativities for both a learner and its environment. Therefore reviewing, as a learning process, no matter how repetitive it might appear, can never be reproducing the same. Further, since a learner's development trajectory cannot be preset and is unpredictable, it is impossible to expect that the same reviewing task can have the same effect on different learners or even on the same learner at different reviews.

Now then, what might reviewing be like from the perspective of enactivism, or more generally, complexity thinking?

5.3 From Reviewing to Re-viewing

On the one hand, the reviewing process supported by enactivism can look much the same like the one supported by cognitivism. Albeit both theories see the end of learning differently, they both support treating learners as active agents and reviewing as a process of generating something new. Therefore, the cognitive strategies advocated in cognitivism to help one understand what one has learned before differently can still apply to the reviewing process under the theory of enactivism. So reviewing could be a process of connecting, summarizing, extending, applying, forming a bigger picture, or reconstructing. What makes a reviewing process based on enactivism different from the one based on cognitivism (particularly traditional cognitivism) is the *orientation*: Reviewing under the influence of enactivism is an endless regenerating process towards creativity rather than a finite process that leads to a pre-determined and objective truth.

Cognitive theories, as Bredo (2015) observes, have suggested that what one learns from a task is affected by how the task is perceived by the learner (e.g., Chomsky, 1959), and the task can also be appropriated in interaction (e.g., Newman, Griffin, & Cole, 1989). This is not hard to understand from an enactivist/complexity thinking perspective. So in terms of reviewing, to enact complexity thinking, it is necessary to orientate the leaner away from treating reviewing as a process to get the objective truth but towards a creative process. With a different awareness, a review task can be negotiated differently in the learner's interaction with it.

On the other hand, reviewing supported by enactivism can also look quite different. Different from cognitivism, the role that environment plays in one's cognitive development has been highlighted further in enacativism: The environment shifts from being peripheral to *constitutive*. Therefore, designing suitable curricula is as important as educating students' awareness to occasion them to attend to the generative power of reviewing. Little is known what different forms a reviewing process might have from the perspective of complexity thinking and enactivism.

To specify my interest in re-interpreting reviewing practice and distinguish a new interpretation from existing ones, I use *re-viewing* to denote reviewing process interpreted through the lens of complexity thinking. This re-viewing process is similar to reviewing process in having a structure of looping back, yet differs from it with an affordance in novelty: Through re-encountering what they have encountered before, learners understand it anew. How we might loop back, i.e., the forms of re-viewing, is the focus in this study.

6 Forms of Re-viewing

The forms of re-viewing that I propose are: re-languaging, re-imaging, and re-inbodying. Each of these forms is informed by related lived re-viewing experiences, in which a learner learned something new through encountering what s/he had encountered before. In this chapter, I first situate my inquiry with some needed assumptions about signs in mathematics and prepare necessary terms to use in later sections. Then I present my inquiry into each form of re-viewing, starting with related experiences followed by interpretations of them. By looking into how these lived re-viewing experiences were triggered and what happened in them, I try to understand the re-viewing process in different forms, and seek inspirations for curricular implications to promote re-viewing. In short, to think about planned re-viewing processes I learn from the lived re-viewing processes.

Since I will be talking about re-viewing process in two dimensions, as planned and experienced, it is necessary to clarify that as a planned activity, re-viewing is a process with an orientation towards newness, and as a lived experience, re-viewing is a process with an outcome of understanding something anew. In both dimensions, re-viewing has a structure of encountering what one has encountered before again. In addition, there are many pairs of words, like re-viewing and reviewing, in this writing differing only in their prefixes. While both share a connotation of repetition, it is the one with the prefix “re-” that emphasizes the connotation of generating something new. To avoid the loss of emphasis due to an overuse of the prefix “re-“, I limit the use of “re-“ to the forms of re-viewing, connotations of recursion (e.g., re-interpreting, re-experiencing, re-encountering), and the word “re-cognize” (and its associated noun “re-cognition”), which is used for its secondary meaning as “know again” (“Recognize”, 2019, also see "Recognize", 2009 & "Cognize", 1989)[9] to refer to an experience of a learner learned something that she most likely had learned before yet forgotten. Additionally, in order to talk about some actions that might be the initial one or an iteration, I use "(re)" in front of a verb to imply these two possibilities. So a word such as "(re)enact" means "enact" or "reenact".

6.0 Assumptions – Signs in Mathematics

Agreeing with Presmeg, Radford, Roth, and Kadunz's (2016) emphasis of the significance of semiosis for mathematics, I understand that mathematics is full of signs. Aligned with complexity thinking, I follow Peirce's sign theory[10] and interpret sign as a system of three elements: *object* (the signified), *representamen*, and *interpretant.* Peirce wrote in 1908 that:

> I define a sign as anything which is so determined by something else, called its Object, and so determines an effect upon a person, which effect I call its interpretant, that the later is thereby mediately determined by the former. (Peirce Edition Project, 1998, p. 478)

When something (e.g. an object, quality, or event) or part of it indicates something else in some respects upon one's interpretation, it becomes a sign. The object, understood as *the signified* (e.g., Arkin, 2013; Presmeg et al., 2016), can be a person, thing, event, space, or idea. The representamen, understood as the *sign vehicle* (e.g., Arkin, 2013; Presmeg et al., 2016) or *the signifier* (e.g., Davis, 2012; Presmeg et al., 2016)[11], is the concrete or material form of the sign and it can be word, sound, smell, image, gesture, and any accessible or tangible element. The interpretant can be understood as "*an effect determined by the sign vehicle upon a person*"

[9] Such interpretation of "recognize" has been emphasized by other scholars (e.g., Felman, 1982; Kupfer, 1983; Thom, 2012).

[10] My reading of Peirce is based on Atkin's (2013) and Presmeg et al.'s (2016) interpretations.

[11] According to Atkin's (2013) interpretation of Peirce's sign theory, Peirce uses terms such as sign and representamen to refer to the elements that belong to something (i.e., a physical form) signifying something else and are responsible for such signification. Atkin suggests using "sign vehicle" instead to refer to such elements. This interpretation is different from Presmeg et al.'s (2016) treating the whole of the physical form as the sign vehicle, which she uses to refer to "the representamen/signifier" (p. 2). To avoid confusion, in the following writing I use these three terms (i.e., representamen, signifier, sign vehicle) interchangeably to refer to the whole physical form.

(Presmeg et al., 2016, p. 7) or “as the understanding that we have of the sign/object relation” that is determined by the sign “using certain features of the way the sign signifies its object” (Arkin, 2013). This three-fold model allows us to talk about the physical form of the sign as something accessible, sharable and distinct from its interpreted meaning, and explore the potential influence of the form on the sign user, while noting that sign vehicle co-exists with sign in interpretation. As Atkin (2013) says, “A sign signifies only in being interpreted”. I see no limitation in what can be a sign, but only the difference in how commonly one thing is socially interpreted as a sign.

Mathematical sign vehicles “are not the mathematical objects themselves but stand for them in some way” (Presmeg et al., 2016, pp. 1-2). The pervasive use of sign vehicles in mathematics is inevitable: “the objects of mathematics are ideal, general in nature, and to represent them – to others and to oneself – and to work with them, it’s necessary to employ sign vehicles” (Presmeg et al., 2016, pp. 1-2). Other than common signifiers, such as letters or characters, mathematical sign vehicles include numbers, mathematical symbols and notations, tables, lists, diagrams, graphs, photos, concrete manipulatives, gestures, and models.

According to Peirce, sign can be categorized by the *interpreted* relationship between the sign vehicle and its object. When the sign vehicle and its object are perceived to share a physical resemblance, e.g., a drawing of three apples representing 3 random apples or 3 items, the sign is *iconic*. When the sign vehicle and its object share a physical connection, e.g., a half finished mathematics equation 3+2 =, the sign is *indexical* as it invokes computation or it is a directive to perform computation. When the sign vehicle is related to its object through convention, e.g., 3 representing the concept of number 3, the sign is *symbolic*[12]. Note that a sign is essentially an

[12] I see this categorization of sign overlapping with Bruner’s (1966) typology of representations, i.e., iconic (image-based), enactive (action-based), and symbolic (language-based) representations. Although I prefer the term “enactive” than “indexical”, I keep the later term to be in consistent with Peirce’s and Presmeg’s works that I used as main references for signs and images later.

interpretation. So a sign is idiosyncratic: "different individuals may construct different interpretants from the same sign vehicle, thus effectively creating different signs for the same object" (Presmeg et al., 2016, p. 7). Therefore, categorizing sign vehicles universally is impossible. All mathematical sign vehicles can bring forth the above three types of signs as they all have spatial shapes (e.g., number 1 can be viewed as resembling one counting stick or action) thus possibly iconic, they all have element of convention (e.g., number 1 can be viewed as a conventional denotation used by a learner or a mathematical community) thus symbolic, and they can be pointers for certain actions (e.g., numbers can be viewed as an invitation of counting) thus indexical.

These multiple possibilities of the same sign vehicle allow us to consider a sign vehicle's affordances for different interpretations other than a dominate one. This opens space for us to study a sign vehicle differently, by attending to different aspects of it. For example, when viewing letters, characters, and denotations, as symbolic signs, we pay more attention to their linguistic features and historical social-cultural meanings; when viewing them as indexical, what they do or invoke us to do matters more, so we focus more on their common usages and the associated human actions; and when viewing them as iconic, their appearances or perceptual affordances matter more, thus we pay more attention to their visual-spatial and other sensible forms.

The above sign related assumptions are needed to support the following inquiry into the forms of re-viewing. Although my inquiry was not directed at signs in mathematics, it still led me to semiosis, which suggests considering a mathematics sign vehicle from multiple aspects. The re-languaging section stresses the linguistic and social-cultural meaning aspect of a sign vehicle often viewed as symbolic. The re-imaging section concerns the visual-spatial form of two dimensional (2D) sign vehicles. The re-inbodying section stresses the sensorimotor invitations of a sign vehicle.

6.1 Re-languaging

Re-languaging refers to a process of changing the language used to express or explain something with an orientation or outcome of understanding it anew. Re-languaging is closely tied to re-interpreting, yet not the same in this study. Re-languaging is a form of re-viewing with a linguistic focus, while situated in hermeneutics, re-interpreting is a more overarching category: It is often interchangeable with re-understanding, and can be achieved through both linguistic and non-linguistic means. Re-languaging also appears closely tied to paraphrasing and rewording, and verbalizing again. Yet, the term re-languaging is used to ensure that it is more than using different words to convey the same meaning: The language used can belong to different language systems (e.g., English, Chinese, sign language, computer language, and etc.) and the meaning needs not to remain the same. Re-languaging is an attempt to use linguistic means as a heuristic tool for learning.

As shifting focus

Living between two national languages (Chinese and English), I am familiar with re-languaging. Over the years I have moved from a stage of translating for the sake of translating to a stage of using translation for a better understanding. I have initiated this process more frequently after I became comfortable with using English to interpret English: I interpret a word, familiar or not, in both Chinese and English. This language changing process has done more than adding new vocabularies to me; the benefit is at the level of conceptual understanding. Sometimes, the corresponding translations of a concept (e.g., epistemology) in two different languages, neither of which made sense alone, worked together and helped me to understand it better. Sometimes, two corresponding translations of the same thing, neither of which was hard to understand, worked together and transformed my understanding. The following is one of such experiences in mathematics.

Circle. I was planning for teaching the equation of a circle after having just taught the distance formula to my grade 10 students. I have known the Chinese definition of circle ("[在同一平面内]到一个固定点等距离的点的集合") by heart since high school. While reading the same definition in English - "a set/collection[13] of points [on a plane] equal distanced away from a fixed point", I suddenly re-cognized that this definition is the same as the circle equation, had we written a circle equation using the distance formula, or more fundamentally, applying the Pythagoras theorem. The wording, "a collection of points", made much more sense to me, compared to its Chinese counterpart "点的集合". I suddenly (re)realized that this is the key feature of circle, through which a circle equation becomes a transformation of the distance formula and/or the Pythagoras theorem, and failing to appreciate this idea can lead to frustrations with algebraic representations of circles and graphs in general. There appeared a vague memory of I struggling during my high school years to understand a graph as a collection of points even though I was taught so, and I lacking flexibility to solve graph related problems, many of which can be easily solved if one realizes an implementation of the idea of "graph as a collection of points" is that one can use (x, y) to represent *any* point on a graph. This re-cognition of circle as a collection of points helped me to see that I can teach many circle related problems as implementations of the circle definition.

[13] Albeit the word "set" is used in the English definition of circle, I read it as "collection", which was more familiar and less technical for me.

Obviously, this story is not about translation between two languages, but about re-languaging: understanding something anew through language changing. The two wordings involved in the re-languaging process had identical literal meanings with similarly low demand in terminology: A translation between them can be done with little need for special mathematical vocabularies. Yet, the slight difference in the wordings seemed to make a huge difference in my mathematical understanding. Not only my conceptual understanding of circle but also that of graph in general was transformed: The change of language aided the establishment of a central idea (i.e., a graph as a set/collection of points) responsible for solving a wide range of problems compartmented in different categories before.

Literally, these two definitions of circle differ slightly in the order of their wordings. A verbatim translation of “到一个固定点等距离的点的集合” is “to a fixed point equal distance points set” in Chinese grammar order, meaning “a set of points equal distance away from a fixed point” in English grammar order, which is also the English definition of circle. So the Chinese and English definitions of circle are identical literally. Yet, “the set of points” is at the *end* of Chinese definition but *beginning* of the English definition. This difference in the wording, collaborating with other conditions at that time, contributed to my reading of this English definition in two parts with a hierarchy ordered by level of generalization (see Figure 6.1.1).

Chinese definition	到一个固定点等距离的点的集合
(verbatim translation)	(to a fixed point equal distance **points set**)
English definition	**a set of points** equal distance away from a fixed point
My reading	1) Circle is a collection of points (圆为点的集合) 2) Each point on a circle is equal distance away from a fixed point (圆上各点均到一个固定点等距)

Figure 6.1.1. Re-languaging circle’s definition.

This hierarchy shifted my focus to the first part. With the idea of "a collection of points" emphasized, my understanding of circle was reconstructed using this idea as the core, and so did my understanding of graph in general.

Such attentional shifts and cognitive growth, no doubt, are inseparable with many conditions at play during that time, such as my feeling closer to the word "collection" than "集合 (*ji he*)" [14], my being new to the circle definition in English, my English reading habit (i.e., reading word by word from left to right and concentrating on the main clause to facilitate my understanding), and my enhanced appreciation towards the distance formula and graph-point relation after finishing teaching the line equations and the distance formula prior to the circle unit. Therefore, this story is offered here neither to argue for a particular way to change language nor to attribute my conceptual growth to language change alone, but to suggest an interpretation of *re-languaging as a process of wording something in a way with a different focus and its affordance for shifting attention for a deeper understanding*. The fact that my mathematical understanding grew while I was reading a previously known Chinese definition in English does not suggest the language change from one national language to another is a decisive factor, rather it suggests us to *attend to moving a reader out of an automatic habitual reading zone through language change*. Such move might have been possible through addressing the same language differently, e.g., rereading the same wording in a sequence of parts, with certain parts highlighted, or more reflectively.

As repeatedly making language more meaningful and less arbitrary

[14] 集合(*ji he*) is a technical term in Chinese for the concept of set. I felt less connected to the term *ji he* due to my uneasiness towards technical terms in general and the concept of set in particular. Despite that *ji he* is also commonly used in daily life, meaning such as "get together" (Chinese Academy of Social Sciences, 2002, p. 908), while *ji he* is used in the Chinese definition of circle, I read it contextually as a technical term. Since *ji he* can be translated into group, collection or set, all of which are read mostly as daily used words in English for me, when I interpreted both *ji he* and set as collection, the circle definition in English became less technical and more common sensed for me.

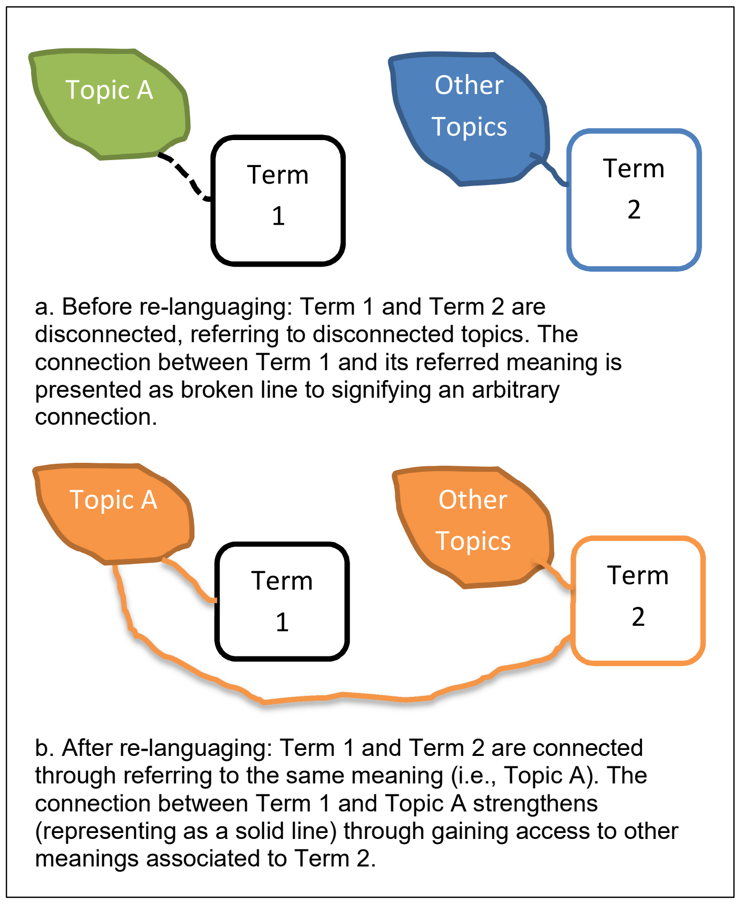
Two personal stories repeatedly came to mind when I thought about re-languaging a mathematics idea. One is about using English to interpret a previously learned mathematical concept with an obscure Chinese name. Another is an experience of returning to Chinese to interpret a previously learned mathematical concept with a hard-to-remember English name. Albeit these two stories are similar to the previous one, in that they involve language translation, I use them here as spring boards for interpreting re-languaging differently.

幂 (*mi*). When I was introduced to the exponential relations during my high school years in China, I was taught a Chinese term "幂 (*mi*)", a character that I barely knew and seemed to use rarely in non-mathematical contexts. *mi* remained as an arbitrary, thus alien, term for me for a long time. This arbitrariness of many mathematical terms made me feel disconnected with the corresponding mathematics concepts and added to my frustration towards mathematics education in general for its seemingly irrelevance to a learner's daily life. When I started to teach mathematics in English, I learned "power" [15] as the English translation for *mi* in mathematics. "Power" was a plain and familiar word for me. Suddenly exponential relations made lot more sense to me: Raising a number to its *n*th power is indeed a powerful process as the number's absolute value can get larger fast, and exponential growth is important for us to know because it is a powerful phenomenon. Since then, I started to feel more comfortable with the word of *mi*, the concept of exponential relations and mathematics in general.

[15] Since a^n is read as "*a* 的 *n* 次方/幂 (the nth *fang*/*mi* of *a*)" in Chinese, "the nth power of *a*" in English, it seems like *mi* = power. However, *mi*, by definition "表示一个数自乘若干次的形式" (Chinese Academy of Social Sciences, 2002, p. 1334) translated literally as "the form of the representation of the multiplication of a quantity by itself several times", only refers to the form and the whole of a^n, whereas power can also refer to *n*, which is the exponent (指数 *zhi shu*) or index.

Asymptote. Before I taught asymptote in high school mathematics classes in English, I rarely remembered the word “asymptote”: I had to look it up in a Chinese-English dictionary whenever I encountered it, and I thought of nothing beyond translation. When I started to teach asymptote in English and needed to remember the word, I checked its Chinese translation again. This time, I re-cognized that “渐近线 (*jian jin xian*)” is not an arbitrary name or a simple translation of asymptote. It actually tells me the key feature of asymptote. Literally translated, *jian jin xian* is “[a] gradually approaching (or getting closer) line” or paraphrased as “a line [that a graph is] gradually approaching” or “a line that is gradually approaching [a graph]” to me. It can be interpreted as an abbreviated version of the definition of asymptote, i.e., “A line which approaches nearer and nearer to a given curve, but does not meet it within a finite distance” (“Asymptote”, 1989), without losing the key mathematical feature of this concept. Along with this interpretation, an image came to me - I saw a person walking closer and closer to a line. Once I visualized it, a funny metaphor came too - an asymptote is like a dream lover whom one tries to approach but is never able to meet. With the establishment of the image and metaphor, I sensed a better understanding of asymptote and found easy to remember the word “asymptote”. I added the Chinese name of asymptote to my lessons and the dream lover metaphor always made my students laugh. Since then, I started to treat mathematical terms as a source of meaning, expecting them enriching my understanding of mathematical contents and bringing forth interesting metaphors.

In both mathematics related stories, I had difficulty to remember or understand a mathematical term. My difficulty was not about whether the term was new to me or not, but its perceived lack of meaning. I had seen neither asymptote nor *mi* outside their mathematical contexts. In fact, asymptote has only one meaning as a mathematical term ("Asymptote", 2019) while *mi*'s other two meanings as "cloth cover" and "cover" (Chinese Academy of Social Sciences, 2002, p. 1334) are rarely observed in contemporary Chinese context. Meanwhile, both asymptote and *mi* had no recognizable pictographic or semantic parts for me, at that time. Hence, they were meaningless, serving as indecomposable arbitrary signs, with a sole function of pointing away for meaning. Such signs demanded extra memorization and brought forth alienation and discontinuity in meaning (see Figure 6.1.2a for a visualization of the status before re-languaging).



a. Before re-languaging: Term 1 and Term 2 are disconnected, referring to disconnected topics. The connection between Term 1 and its referred meaning is presented as broken line to signifying an arbitrary connection.

b. After re-languaging: Term 1 and Term 2 are connected through referring to the same meaning (i.e., Topic A). The connection between Term 1 and Topic A strengthens (representing as a solid line) through gaining access to other meanings associated to Term 2.

Figure 6.1.2. A visualization of the statuses before and after re-languaging

Replacing these terms using more familiar words that have meaning beyond mathematical contexts changed the story. Both *jian jin xian* and power include words used widely in daily living contexts; there are recognizable parts (in the case of *jian jin xian*, which essentially is a compound word) or whole (in the case of power) for me. So they enlarged the meaning space from which I can draw to understand the mathematical referents. The terms that were initially strange became less alien, because of their respective associations to a more familiar word and consequently an enlarged meaning network (see Figure 6.1.2b for a visualization of the status after re-languaging).

More significantly, the literal meanings of both new terms are in consistent with their mathematical referents. Being informative for what they are referring to, these two terms serve as more than arbitrary languages pointing away for meaning (see Figure 6.1.3a) but *also self-referential* pointers that point back to themselves for a consistent meaning (see Figure 6.1.3b). By pointing towards multiple places (as discussed at the end of this section, this includes the terms and the language, knowledge, and culture systems in which they are situated) for a coherent meaning, the new terms afford more meaning making: Not only can multiple perspectives of the same referent become possible, also inconsistencies, if there is any, between these perspectives can become perceivable, offering opportunities for one to renew one's understanding. Therefore, the re-languaging process described in both stories involves *a change of language at the level of familiarity and coherence*.

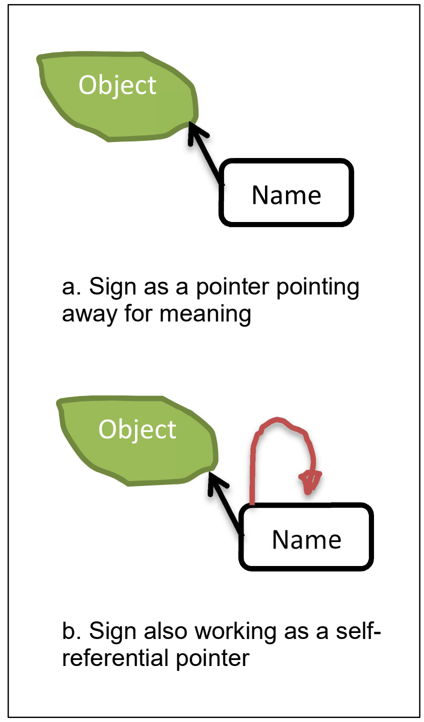


Figure 6.1.3. Sign as two different types of pointers.

Through engaging words connected better with the learner and sharing a consistent meaning with what they are referring to, the new language allows meaning making at the language level (i.e., at the sign vehicle level) possible and beneficial: The language is no more considered as a transparent bridge to some meaning waiting to be conveyed, but *an integral part* *of the meaning*. Making sense of the language is part of making sense of its referent. The increase of familiarity and coherence allows learning to resume as a continuous growth of new meaning on the old, allowing one to make sense of the referent using already familiar meaning and demanding less on rote memorization. *Re-languaging, hence, is a process of meaning making and sense growing through changing a language to less arbitrary and more meaningful one*.

A re-languaging process is essentially endless, as suggested by the later development of my re-naming story. My finding the two sensible names pointed me back to the two names that I initially found senseless and later back to the two sensible ones again:

> **Why 幂 (*mi*), asymptote, and power?** After I noted the existence of meaningful names and their affordances in sense making, the arbitrariness and meaninglessness of names such as *mi* and asymptote became questionable. There must be some good reasons why they were chosen or invented as names, and knowing these reasons could help me understand more about the related concepts (e.g., when and how these concepts emerged). This led me into etymology. I found that asymptote does have a meaningful connection with what it refers to and it has interpretable parts, as it derives from "modern Latin asymptota (linea) '(line) not meeting,' from Greek asumptōtos 'not falling together', and from a 'not' + sun 'together' + ptōtos 'apt to fall'" ("Asymptote", 2019). As for *mi*, it was used to represent exponent by Chinese mathematician Liu Hui in 263 BC. While not knowing how Liu Hui connected this Chinese character with a general meaning "cloth cover" as a noun and "cover" as a verb (Chinese Academy of Social Sciences, 2002, p. 1334) to the mathematical concept of power, I started to see *mi* having meaning from somewhere. Once *mi* became a legitimate source of meaning, my imagination took root.

> A connection between *mi* and exponent through the visual similarity between the image of the cloth covering something and the image of Pascal's Triangle, as suspected by my colleague Xiong Wang, seemed possible. In all, this inquiry into meaning drew me closer to these two names: I was more contented with them as names for believing that my further inquiry into their origins would bring forth more meaning. Later my supervisor promoted me to check the origin of the name "power". It turned out that "power" has a root of *poti*- i.e., "powerful, lord", corresponding well with "powerful" exponential results (Schwartzman, 1994, p. 170). So then I asked "Why was *jian jin xian* chosen as a name for asymptote? What other names have been used for the same mathematical concept in different times and spaces?"

The initial re-languaging processes that seemingly sent me away from the two alien terms actually paved a path for me to come back to them later. Finding more meaningful terms to replace asymptote and *mi* motivated me to proceed further in making language meaningful. Through questioning the origins of new and old terms, I saw the same language differently: Asymptote becomes a whole with meaningful parts and is consistent with its referent; *mi* and power have a story of becoming to tell. Essentially, both terms became self-referential pointers for meaning. This also opened the door for drawing meaning from the wholes (e.g., the language, knowledge, culture systems) in which the terms are situated, hence further enlarging the meaning space for their referents.

As a whole, my story with asymptote and *mi* illustrates *a back and forth meaning making process through re-languaging, in which the strange is made familiar and the familiar is made strange repeatedly, and in which coherence between language and meaning is questioned and pursued*.

As using equivalent languages to move inquiry forward and backward

Looking back at the previous re-languaging stories, I see *re-languaging as a process of meaning making through changing between equivalent languages*. Here “equivalent” is not “equal” in mathematical sense. Equivalent languages can be different wordings of the same information (as the circle definition story shows) or different languages with different information yet serving the same function (as the story of renaming shows), e.g., as a name, definition, question, answer, instruction, and so on. In both situations, re-languaging moves the learner away from the initial wording and back.

This is easy to understand when languages with the same information are involved as one basically stays in the same meaning and functional space. Different wordings of the same information serve as alternative interpretations that help to inform each other and the information. When languages with seemingly different information are involved, like asymptote and *jian jin xian*, the re-languaging process starts with a change of *anchor* – the languages are no longer anchored in the same *meaning* but the same *function*. A shift of consideration can happen, from a part to a bigger whole (e.g., a particular information about a topic to the topic itself), and from what the part means to what it does in the context of a bigger whole. This is *a step backward to see a bigger picture and to recall where one is heading.* For example, in my renaming terms story, my fundamental goal was to understand the mathematical concepts referred by the terms, but the inconsistency between the concepts and their names got into my way of sense making. Changing names allowed me not to get stuck with the meaning of some words but continue getting familiar with the concepts. The significance of this change of anchor has in keeping thinking and learning going is also illustrated in the following story of a teacher participant.

What is my question? When asked to provide an experience in which he re-interpreted a mathematical idea during Workshop 3.2 (Re-interpreting), Bill drew a diagram (see Figure 6.1.4) and explained:

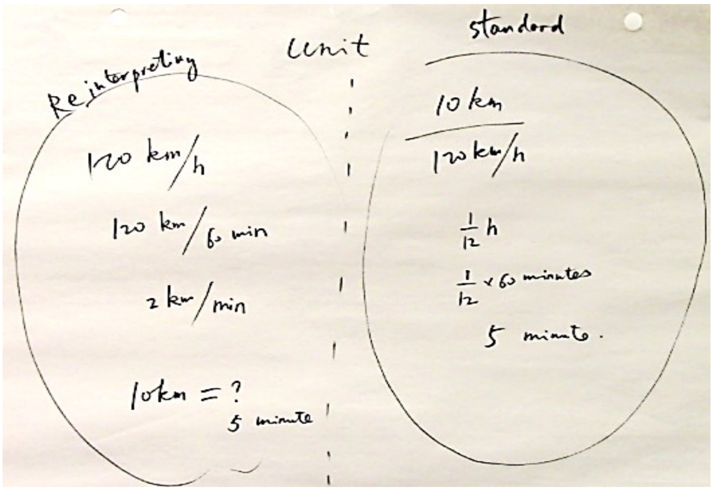


Figure 6.1.4. Bill's diagram.

> "I drive a lot… I needed to travel 10 km at 120 km/h. So I was thinking how many minutes I need. If I use the standard way, 10km divided by 120 km/h, so it's hard to have mental math, right? 1 divided by 12, it's not easy. So, usually I think, maybe I can think in another way. In my mind it's easy to think if [I] change 120 km/h to per minute. So 120 divided by 60 is 2. So 2 km/minute. If I know that, then it's easy. I know 10 km divided by 2 km/h is 5 minutes. This is much quicker. So this is based on my experience. I need to do the division, but how do I do division?"

When asked whether this is breaking the task into a smaller one, Bill responded:

> "Just change the solving method, I think. I was dividing, but I was changing, so either 10 km/ 120 or 10 km/2, which one do I divide by? So I have to rethink the question. I have to figure out "What is my question?" [upon hearing this, I wrote "re-questioning, asking a different question" on the board as an interpretation of re-interpreting] Although the question is so apparent: I need to find the time. But how do you find the time and which [method] do you use, how do you divide? So this is how I make it a little bit simpler to have mental math."

Illustrated in Bill's story, when he encountered difficulty in his task, he stepped back and reconsidered where he needed to go by asking "What is my question?" This led him to reword the problem to be solved, changing "What is the time in hours?" into "What is the time in minutes?" These two problems, albeit different, served the same function (i.e., guiding question) in Bill's task, hence they were equivalent. This change of languages moved Bill's thinking forward and kept it going, and subsequently, he solved both problems. This is similar to my renaming experience, in which my language change helped me understand and/or relate to the mathematical concepts differently and triggered me return to the names I got stuck and interpret them again.

In a sense, re-languaging is like using a different placeholder for something that one is trying to understand and employ in order to move on in the inquiry, with an openness and patience to see where one can get to later, a confidence that one can figure out what the placeholder is holding later after one goes further enough, and a commitment to come back to resolve issues later if there is still any. I see this moving on with a tentative placeholder the same as the one underlying algebra. In contrast with arithmetic, which works only with known facts towards knowing the unknown, algebra works with both known and unknown, including both of them in our logical reasoning and waiting to see where we get to then (Boole, 1909). "This method of solving problems by honest confession of one's ignorance", Boole writes, "is called Algebra" (p. 14). *The process of re-languaging, like algebra, is also motivated by an acknowledgement of temporary limitations and a faith in the generative nature of the process.*

On the other hand, re-languaging is not just changing placeholders to maintain a flow or continuity in process, nor is it moving on with something totally unknown. It is driven by meaning and essentially a meaning-making process. So coherent meaning is the primary focus.

Re-languaging demands the learner to use a *contingently* more meaningful placeholder to replace its previous one. Pragmatically, it enlists a *good-enough-for-now* placeholder for the learner to move on with a sense of coherence, continuity and confidence. The learner's mathematical understanding developed later might challenge her previous one and problematize the previously perceived coherence (i.e., the one between the more meaningful placeholder and its referent) and incoherence (i.e., the one between the less meaningful placeholder and the same referent). The same need for coherence would bring the learner back to address these issues, affording a renewed understanding of the same language and its related topic and a new sense of coherence. In my case, changing "asymptote" to "*jian jin xian*" as equivalent in function eventually allowed me to come back to see them as equivalent in meaning and consistent with what they refer to. So this moving away from one language to its equivalence is a beginning for a recursive inquiry about the same topic. *Re-languaging is a process moving forward with a determination to move backward. Using equivalent languages, it keeps inquiry going and returning.*

As attending to and enlisting equivalent languages' different affordances for thinking

My further contemplation of my renaming experience provoked me to interpret re-languaging in the light of language's affordance for thought in general.

> **The affordances of language in thinking.** When pondering on the linguistic changes involved in the two renaming stories, I noticed that the two names *jian jin xian* and power are telling me different things about their referents: The term "*jian jin xian*" describes the visual look of asymptote and paints a vivid image for me, and the term "power" captures a sense of exponential relations (i.e., quantity changing fast) as a whole. In comparison, the first name *depicts the look of the whole* and the second name *depicts the sense of the whole*. Viewed together, both names have a focus on the individual referent. This

> categorization quickly put many Chinese names in a contrast group: They inform both the referent and its relation to others. Many things belonging to the same class, mathematical or not, are named with a shared character in Chinese. For example, 2D shapes are named ending with the character of "形 *xing* (shape)" (e.g., triangle = 三角形 *san jiao xing*, square = 正方形 *zheng fang xing*), many properties of a circle are named starting with the character of "圆 yuan (circle)" (e.g., circumference = 圆周 *yuan zhou*, center of the circle = 圆心 *yuan xin*), and all two-digit numbers have "十 *shi* (ten)", located often between the names of two figures (e.g., 10 = 十 *shi*, 11 = 十一 *shi yi*, 12 = 十二 *shi er*, 20 = 二十 *er shi*, 21 = 二十一 *er shi yi*, 22 = 二十二 *er shi er*).
>
> Exactly as Pimm (1995) says, "names stress and ignore" (p. xiii), these three kinds of names, in comparison, stressed differently for me: the appearance or feature of the whole (hence parts), the sense of the whole, and the referent's relations with others respectively. Consequently, they led me to think about their individual referents differently, eliciting more visual thinking, analogical thinking, and relational thinking respectively. Obviously these three ways of thinking are helpful in my learning of the concept.

The significance of the above reflection is not just to offer some practical suggestions for creating equivalent languages, but to provoke me to think about how re-languaging might offer learners a chance to work with different kinds of languages and benefit from their different affordances for thinking, and become aware of the influence of language in thought.

Experience with equivalent languages for the same referent could bring forth conscious or unconscious differentiation and equalization. While trying to match different languages to the same referent, the different affordances of each language become noticeable: Different languages can reflect different assumptions, point to different directions, or suggest things not in harmony with each other. Such experience with equivalent languages is promising to lay a foundation for later reflections on the influence of language in thinking in general.

Re-languaging has affordance for bringing the influence of language into learners' awareness and urging them to actively use languaging as a thinking process. By inviting a learner to encounter and work with the equivalent languages of the same referent, a re-languaging process could invoke different ways of thinking and encourage reflection on the influence of language on thinking in specific or in general. In this sense, re-languaging is a process of *using equivalent languages to attend to and enlist their different affordances for thinking*.

Back to re-languaging as a whole

Many mathematics sign vehicles can be interpreted as symbolic when noting them as conventionally chosen signifiers. Rarely do etymological questions about them seem to arise. When their legitimacy is taken for granted, they become more arbitrary and less meaningful. A combination of a meaningless signifier and an abstract signified hampers the meaning making of a learner. Re-languaging, a process that problematizes arbitrary mathematics language and demands contingent meaning from it, could be helpful here. It turns language from a transparent medium for meaning delivery to a part of the meaning and a tool for meaning making. Re-languaging allows the learners to build up their understanding of the idea using language that fits their current understanding level, enabling later returns to the original wording for different interpretations. The purpose of re-languaging is not to find the perfect language, but to sustain a combination of coherence and incoherence, which is tentatively good enough for one to continue with one's inquiry yet unsatisfying enough to attract one to return and tackle later. Meaning is generated through this recursive movement, rather than at the end of this movement, and it is the drive for meaning and coherence that moves one away and back. Using equivalent languages, re-languaging ensures both coherence and continuity; it also enlists the affordances of language for thinking and opens space for the learner to think about mathematics and language differently.

6.2 Re-imaging

Re-imaging is a process of generating a different image to present something with an orientation or outcome of understanding it anew. An image can be physical (sharable and accessible for other people) or mental (accessible only for the self). Here, I use “physical” and “mental” for lack of better terms without stressing the division of external objective world and internal subjective world, and that of body and mind. A physical image contains all the information accessible for humans to generate mental images. A mental image contains the visual and spatial information perceived and interpreted by an individual viewer of a physical image, and consequently it can exclude certain details of the given image yet include information beyond what is given. Therefore, mental images are idiosyncratic: When presented the same physical image, different viewers can generate different mental images (Presmeg, 1992; Presmeg et al., 2016).

The word image is used to refer to a spatial and visual construct. In mathematical contexts, an image can be composed of elements that are commonly viewed as linguistic symbolic elements (e.g., letters, numbers, mathematical symbols and notations), elements that are commonly viewed as pictorial graphical (e.g., mathematical graphs, diagrams, drawings, pictures, and photos), spatial (e.g., three dimensional manipulatives, models), or kinesthetic (e.g., gesture, motion), and empty space (for physical images) or blurry unknown content (for mental images). My use of image here is a conflation of Presmeg et al.’s (2016) “sign vehicle” (corresponding to physical image) and “visual image” (corresponding to mental image). Yet I chose the term image in my writing 1) to include both mental and physical images; and 2) to take advantage of the common association of image and pictures in order to emphasize the picture-like aspect of a sign vehicle and destress its linguistic symbolic or kinesthetic aspect. Although the physical images discussed in this section are limited to two dimensional spatial and visual constructs, it is acknowledged that all sign vehicles have an imagery dimension, offering a sign user certain spatial-visual stimuli.

Since a re-imaging process includes image changes, I need to clarify certain image related terms and image categorizations to situate my interpretation. As I have talked about sign vehicles and their types in section 6.0, here I focus on visual images. Visual image is “a mental construct depicting visual or spatial information” (Presmeg, 1992, p. 596). This deliberately loose definition, as Presmeg (1992) conceives it, allows a broad coverage of multiple imagery possibilities without confining to the common form of imagery - “pictures-in-the-mind” (Clements, 1982, p. 36; called “concrete images” by Presmeg, 1992, p, 596). It includes imageries that “depict shape, pattern or form” (Presmeg, 1986, p. 42), which covers the “number forms” (Paivio, 1971, p. 482) generated through arranging verbal, numerical, or mathematical symbols spatially, beyond the kinds of imagery that “attains the vividness and clarity of a picture” (Presmeg, 1986, p. 42). Therefore, visual images might have a visible form: gestures, motions, three dimensional objects, photos, drawings, paintings, diagrams, graphs, tables, lists, and etc. The synonyms of visual image include visual imagery, visual representation, and mental image. Visual images are tied to visual processes or visual modes only, whereas mental model, mental representation and mental imagery (sometimes called imagery, e.g., Presmeg, 1992, p. 596) are not, as they can be constructed based on sensorimotor information in multiple modalities – visual, auditory, tactile, gustatory, and olfactory. Visual/mental images can be categorized as concrete, pattern, formula, kinesthetic, dynamic as Presmeg (1992) suggests. Yet, this categorization is also individually contextualized: A mental picture of an equation, such as $ab = 0$, can be concrete when the viewer is taking it as a special case of the pattern – a product of two quantities equals to 0, formulaic when the viewer is taking it as a formula, or pattern when it is used to represent a class of cases.

Given that the term of image in this section is a conflation of both sign vehicle and visual image and neither construct cannot be categorized without the image user's interpretation, the type of an image is relative – it is interpretive and contextualized, and an image can have multiple types at the same time. For instance, the same image of $ab = 0$ can also be viewed as symbolic (presenting an abstract and conventional mathematics equation), iconic (presenting an image made of lines and shapes), or indexical (pointing towards solving the equation) by different viewers or by the same viewer at different times.

Albeit an image's type is fluid, when comparing one image with another, we can still say that one image has more affordances to be viewed as one type than the other in certain contexts. For example, for a high school student (e.g., when reviewing solving factored equations for roots), $2(0) = 0$ is more likely be viewed as representing a specific individual case when comparing with $(\ )(\ ) = 0$, which is more likely be viewed as representing a general case. But this can change when $(\ )(\ ) = 0$ is compared with $ab = 0$: The previous equation might be more likely viewed as a concrete case than the later one[16]. It is in this kind of situation of comparison, Presmeg's (1992) two categories of visual images, i.e., concrete and pattern images, can be applied to physical images too. The change between a more concrete image and a more general pattern image will be the focus in later section.

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[16] This comment is made based on my experience with students. The presence of symbols seems to disturb students (out of their uneasiness with variables) quickly. In comparison, the presence of blank or a box seems less daunting.

The following three rounds of interpretation of re-imaging start with two related experiences of re-imaging a mathematical idea, based on which I tried to understand what changes are possible during a re-imaging process and under what conditions a re-imaging process might happen. These two experiences also led to a journey of seeing that contributed to my further interpretation of re-imaging. Although my stories are related to sighted people only, my interpretations of re-imaging are general enough to apply to images constructed based on sensorimotor information in multiple modalities.

As seeing/sensing the same image differently

Maxine's seeing a division bracket. Maxine mentioned that she used to have difficulties to teach students polynomial division using algebra tiles. Students didn't seem to understand it; they often had troubles with where to place what in their display of division work. One day, she was looking at a drawing of polynomial division with algebra tiles (e.g., $(2x^2 + 4x) \div x$ in the top section of Figure 6.2.1), she suddenly saw a division bracket (as highlighted in the bottom section of the Figure 6.2.1) standing out. She realized that she could teach polynomial division by drawing connection to the number division topic that students had learned in their primary grades. Once she started to draw a division bracket when teaching polynomial division with algebra tiles, her students had no difficulties to remember where to place what.

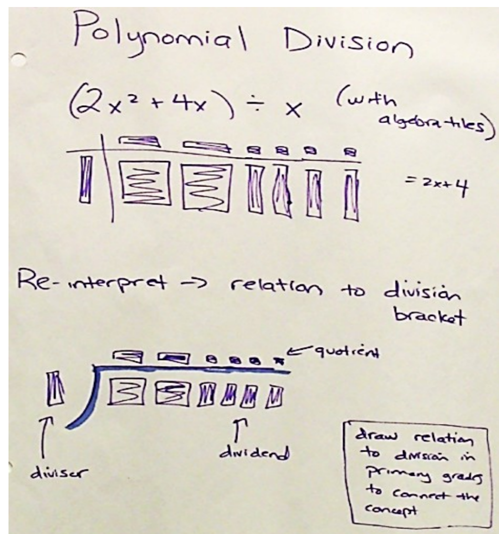

Figure 6.2.1. Maxine's seeing a division bracket.

Lixin's seeing a triangle area formula. A few months later, I was typing "Seeing proof without words, e.g., 1+2+3+…+n= n(n+1)/2" [17] on a computer while preparing for the Re-imaging Workshop 3.4. Suddenly, I *saw* the triangle area formula A=**bh/2** emerged from "**n(n+1)/2**". I could not recall that I had noticed this before. Given that I have been further sensitized towards re-imaging by Maxine's example, I knew that this seeing the same image differently might lead me to somewhere useful. I started to think how I could make a triangle to represent 1+2+3+…+n.

Both stories illustrate a kind of experience in which one gains a new insight of a mathematical idea/situation after one suddenly sees the image representing the idea/situation differently. This sudden revelation, or aha experience, is a kind of thinking and problem solving process interpreted through Gestalt psychology. Gestalt psychologists view thinking and problem solving mainly as finding the right representation of the problem, which "basically means a restructuring of the problem representation" (Schnotz et al., 2010, p. 14). A problem solver might mentally change, amplify, reorganize the given problem material and suddenly perceive new relations between elements of the material (Montgomery, 1988, cited in Schnotz et al., 2010), resulting in an altered mental representation of the problem (Schnotz et al., 2010). When the perception of the problem situation is suddenly reorganized into the "right" one, the correct solution becomes immediately obvious as it can be read off from this new perception (Schnotz et al., 2010).

Maxine's experience of revelation happened unconsciously, triggered by a noticing of "things that look alike". Maxine did not intend to see something different in the image; she was

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[17] I need to keep the expression n(n+1)/2 in this format because this is exactly what I typed. I gradually saw a triangle formula as I typed one symbol after another. The sequence of typing is important, I think, as it coincides with my habitual memorization of triangle area formula as "base times height divided by 2", or symbolically, as bh/2. To emphasize the visual similarity that I perceived in the two expressions, i.e., n(n+1)/2 and bh/2, I highlighted them in bold in the following sentence.

just looking at it while considering students' difficulties. Suddenly she noticed something new: The two intersecting lines in the representation of the polynomial division with algebra tiles became a division bracket or division sign. This "looking alike" led to a connection of polynomial division to number division, mathematically and pedagogically. With a similarity in form matched by a similarity in mathematical structure, the teaching and learning of polynomial divisions becomes a review of one's prior learning of real number divisions. Maxine found a way to make the strange and new topic (i.e., polynomial divisions) familiar and old. Maxine's teaching problem was solved.

Similar to Maxine, I also experienced a process of sudden seeing an image anew without purposefully trying: I saw n(n+1)/2 as bh/2. These two images have a level of appearance similarity relatively lower than the two in Maxine's example: At a glimpse, they do not look identical. Even both of them have a similar part "/2", but given how many specific cases can be related to this part, this similarity is not informative enough. Seeing them identical requires me to compare the two images at the level of whole and draw upon semiotic meanings: These two images became identical for them sharing the same pattern image - half the product of two quantities. When a casual notice of similarity in form was matched with mathematical structure similarity through my later semantic interpretation, this resulted into connecting two initially different topics and solving my proof problem.

The image change in Maxine's and my re-imaging experience is subtle yet salient: It only involves highlighting (and/or modifying) small parts of the original image yet it produces a different image structure that links to a different topic, consequently making this new topic's related meanings and/or associated images usable for the topic at work. Our experiences illustrate *a re-imaging process that is seeing/sensing the same image differently, and it involves a* *change of the referent (i.e., the signified) of the initial image (i.e., the sign vehicle) and might be triggered by asking what the image or parts of the image look like.*

As changing between equivalent images

Before the next round of interpretation of re-imaging, let's read the second part of my re-imaging experience which has a development different from Maxine's.

Lixin's seeing a triangle area formula (continued). After I saw a triangle area formula in n(n+1)/2, I started to think how I can make a triangle to represent 1+2+3+…+n. I went through multiple tries, representing the sequence of *n* consecutive numbers as a triangular stack of dots, and later squares, and squares with a triangle embedded (see Figure 6.2.2 for three images at different stages). The final image made the problem solution immediately obvious: The addition of *n* consecutive numbers becomes calculating the area of a quasi-triangular shape, and it is equal to the area of the triangle plus an extra area beyond the triangle, which can be found by n/2. Thus 1+2+3+…+n = n(n)/2 +n/2 = n(n+1)/2.

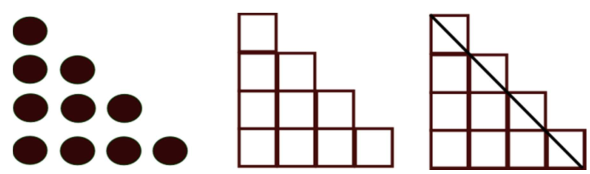


Figure 6.2.2. Representing the sum of *n* consecutive numbers.

It seemed to me that both Maxine's and my re-imaging processes started with two isolated[18] sets of image and topic associations (see Figure 6.2.3a). Upon noticing a common structure in the respective images, both Maxine and I connected images and then the two initially unconnected topics (see Figure 6.2.3b). These connections came, with one image turning into a pattern image of the other or two images becoming two concrete images for the same pattern

[18] This isolation might be the result of not having consciously connected these two topics before (e.g., grade 12 students who have never connected fraction with limit), or having forgotten that they have connected them before due to automation (e.g., experienced learners who forget that they have connected the sum of numbers to area of triangle before). This isolation is my interpretation from an observer's perspective, based on how surprised both Maxine and I were by our "newly" discovered connections.

image. Simply put, our re-imaging processes involved *a change between concrete image and pattern image*.

However, there is more to the story. In Maxine's case, since the two connected topics are mathematically equivalent, there could be an emergence of a new mental class (i.e. division) that subsumes these two topics and a new pattern image (i.e., the division sign with/out dividend, divisor, quotient parts) that captures the mathematical structure in the two related images (see Figure 6.2.4a). With a formation of a class and its associated pattern image at a higher abstract level, the two topics become two specific or concrete cases for the class and the two images become two specific or concrete cases of the pattern image. So, the re-imaging process in Maxine's example is *solely a change between concrete image and pattern image*.

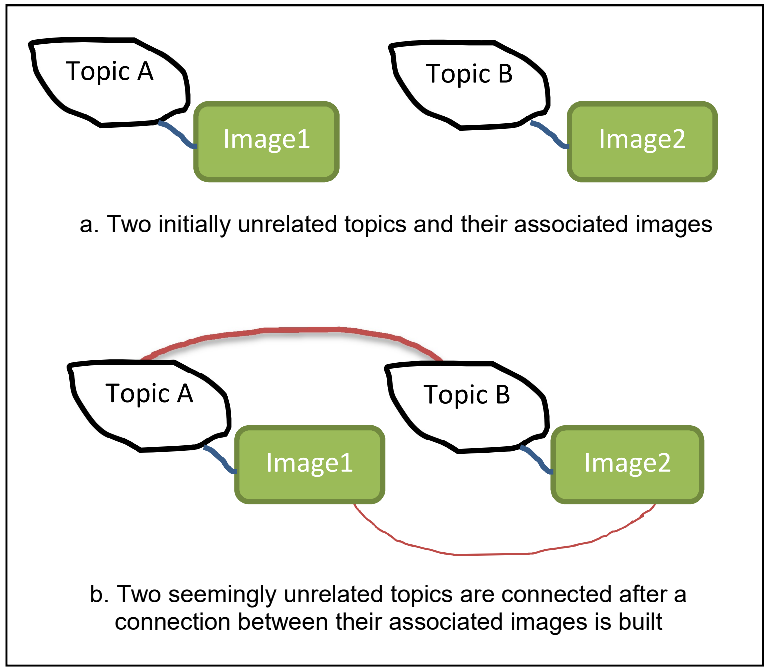


Figure 6.2.3. A visualization of a re-imaging process.

In my example, the connected two topics (sum of consecutive numbers and area of a triangle) are not mathematically equivalent. This connection brought forth, not a new topic class or pattern image, but a process of constructing a novel image that can represent both topics (see Figure 6.2.4b). The identification of the topic connection through the same pattern image inspired me to create a hybrid concrete image that might be otherwise unthinkable for me. Therefore, the re-imaging process in my example is *a generation of mathematically equivalent* *concrete images*. Essentially, both Maxine's and my re-imaging processes involve *a change between equivalent images*.

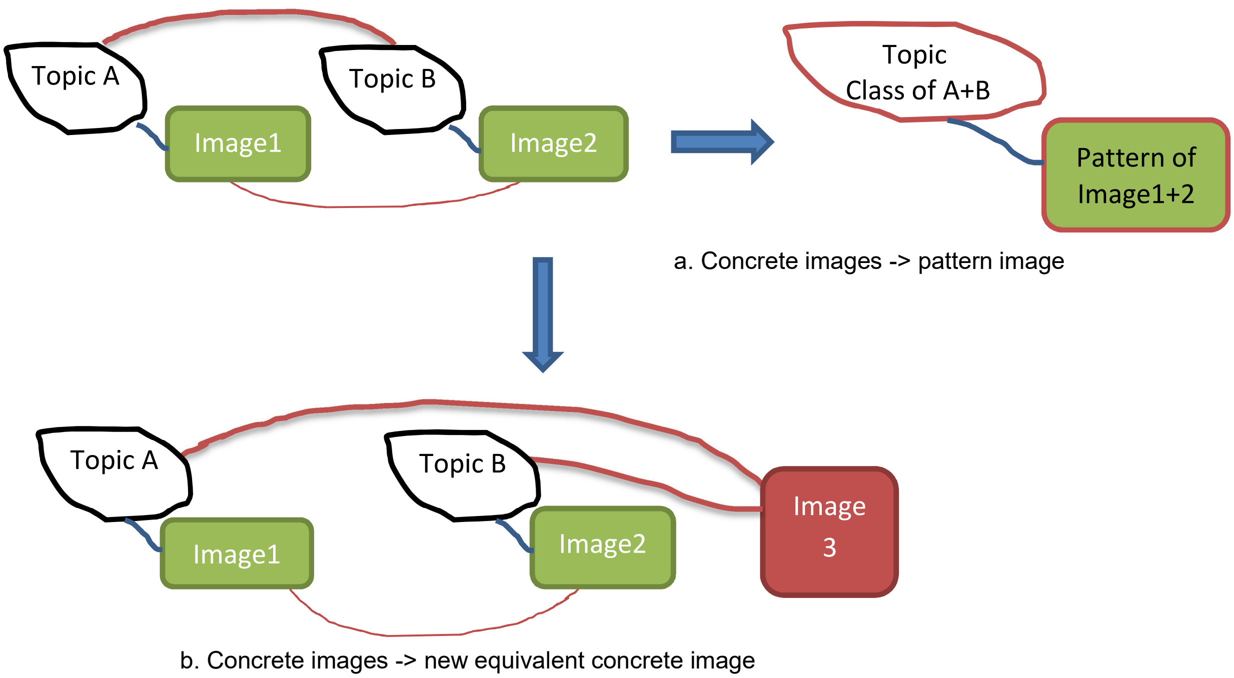


Figure 6.2.4. Two kinds of change between equivalent images.

As learning from equivalent images' affordances and limitations

My contemplation of re-imaging was further benefited from a journey of seeing that happened along with my inquiry into recursion.

A journey of seeing. I knew algebra tiles little before my PhD program and I rarely used it in my teaching. Intuitively, I felt it somewhat arbitrary thus confusing. In Workshop 3.2 (Re-interpreting), after provoked by Maxine's story of seeing a division bracket in polynomial division, I tried to connect polynomial division with area formula and multiplication table in order to help students understand division done by algebra tiles. I started to see area formula, multiplication table and division done by algebra tiles sharing the same image. The strangeness of algebra tiles decreased, yet my initial discomfort towards it remained.

Later after Workshop 3.3 (Re-experiencing), I came up with different number line representations of the same mathematics equation (i.e., $-6-2=-8$) that we worked on during the workshop. I sensed that these different images invoked me to think differently. This experience further increased my sensitivity towards the affordances of images for thinking before Workshop 3.4 (Re-imaging).

At the beginning of Workshop 3.4, Maxine mentioned that some students still go back to draw a diagram to check their multiplication of two mixed numbers, even though they had moved away from that beginning stage and could multiply by changing the numbers into improper fractions at first. I invited her to show me the diagram (see Figure 6.2.5a).

Maxine explained that the process involved in generating this diagram is the same as the one involved in the long multiplication (i.e., 27×35). From there, we went on to connect many forms of the same process (see Figure 6.2.5 presenting images drawn

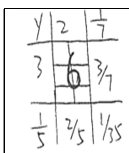
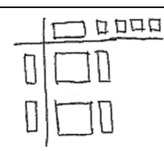
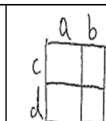
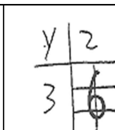
 a. Multipli-cation Chart	$(2+\frac{1}{7})(3+\frac{1}{5})$ $(2+3)(3+5)$ $(2+x)(x+5)$ b. Multiplying Binomials	 c. Multiplication / Division with Algebra Tiles	quotient $2x\overline{)2x^2+8x}$ d. Division with Polynomials	$(20+7)(30+5)$ $600+210+100+35$ e. Distributive Property	27 $\times 35$ 135 810 f. Long Multiplica-tion	 g. Area Model	 h. Multiplica-tion Chart + Area Model
G7/8	G10	G9	G10	G6[19]	G4/5	G7	G3/4

Figure 6.2.5. Equivalent images.

[19] Grade labels are credited to Maxine. Two of her labels (i.e., G6 for Figure 6.2.5e and G6 for Figure 6.2.5f) are different from the categorization in the Alberta programs of studies (i.e., G8 and G4/5 respectively). I kept the lower grade for both cases here.

on the whiteboard and arranged chronologically from left to right) and identify the grade in which a learner might start to see each of the forms.

We saw them essentially representing the FOIL method and distributive property. We agreed that learners actually encounter the same model across grades, and it would help learners to see this if teachers can bring them back to the same model image repeatedly over grades.

For such image, I recommended the area model, because it is more meaningful than a multiplication table that only shows numbers: You do not have to multiply, e.g., *a* by *c* in the top-left rectangle in Figure 6.2.5g, because your teacher tells you to do so; you multiply because that is how you can find the area of the rectangle (you can even draw many unit squares in the rectangle to show why this multiplication works to get the area), treating *a* as its width and *c* as its length. Then it is logical to add all the areas up to get the total area of the big rectangle in which all the small rectangles reside, rather than being told to do so as a rule.

While agreeing with me, Maxine mentioned something different: Her students had problems to understand how to get *x* square tile from *x* tile, even after she drew a diagram of a *x* tile besides a *x* square tile and told them to multiple *x* length and *x* width to get *x* square. So she took the algebra tiles out, and she put a *x* tile along one side of a *x* square tile and then the other side while explaining "Look, a *x* tile and a *x* square tile have the same length, [and then] the same width". Her students finally got it: "Oh, that's how you get the square." Maxine said that they finally saw the area. Maxine's story provoked me to think about my own uneasiness with the algebra tiles. Yes, the *x* tile is confusing: As a rectangle, it invoked me to view it as an area of $x \times 1$, i.e., a two dimensional construct (2D), yet to understand how it is related to the *x* square tile, it should be a one dimensional construct (1D), i.e., a length of *x*.

By the time I looked back at the video clips of the Re-interpreting Workshop, I connected my initial seeing and later seeing. Yes, an image showing multiplication can also be read as representing division. Yes, the division in algebra tile, the area model, and the multiplication table are also the same. But, wait, they are also *different*. When merging the algebra tile division and area model, seeing division in algebra tile model implies that Area / Area = Area (i.e., 2D / 2D = 2 D). When merging the multiplication table and area model, seeing division in a multiplication table implies that Length / Length = Length (i.e., 1D / 1D = 1D). Both implications are in conflict with the area model, which suggests 1D × 1D = 2 D. In addition, a multiplication table can represent grouping situations but it seems impossible in the algebra tile model. All these thinking led me to see $a \times b = c$ anew: This equation can be interpreted, before a learner encounters algebra tiles, as *Quantity* × *Quantity = Quantity, # of groups* × *Group size = Quantity, Length* × *Length = Area*, and maybe *Length* × *Area = Volume*. However none of these seems to help a learner to understand what multiplication/division with algebra tiles is about. Algebra tiles are all two-dimensional objects. Hence it can draw one away from using an algebra tile to represent a one-dimensional quantity, such as number of groups or a number. Rather it induces one to think of area model, and lead to an illogical conclusion, i.e., *Area* × *Area = Area*. I finally see what my students might see in these images that I could not see before.

In the above journey of seeing, I went through a process of seeing many different images the *same* and then a process of seeing the same images *different*. In other words, I went through a process of re-imaging in reverse directions: first changing different concrete images to the same pattern image, later changing the same pattern image back to concrete images. Of course the first change is important as it is a process of abstraction thus a key aspect of mathematising (Freudenthal, 1991). But the problem is once one sees some images the same, it can be hard to undo that and see them differently. It is harder when certain way of seeing has become a habit. The details that one experiences to reach an insight could be forgotten or simply not accessible for remembering due to automation, as Davis (2015) says, drawing support from Ericsson, Charness, Feltovich, and Hoffman's (2006) study of experts' and novices' strategies of engagement: Expert knowers tend to lose track of the difficulties encountered in their past learning and helpful parts (e.g., metaphors, exemplars) integrated in their understanding. This helps to explain why my students tend to make mistakes like $sin(a + b) = sin(a) + sin(b)$ but I cannot understand why this tendency is hard to change. For students, this equation might be logical as $sin(a + b)$ looks like $3(a + b)$, but I can no longer see that as I have seen the pattern image of $sin(a + b)$ as $f(a + b)$ for so long.

My above journey of seeing illustrated well how working with multiple equivalent images helped me to see what I cannot see anymore. Through comparing and equalizing these images, I noticed their different affordances and limitations and had to negotiate between what they induced me to think about a certain mathematics idea and what my (un)conscious understanding of the idea is. Through these activities, I learned something new mathematically and pedagogically. In all, my journey of seeing suggests a re-imaging process as *learning from equivalent images' affordances and limitations*.

Back to re-imaging as a whole

All mathematics sign vehicles can be viewed as a spatial-visual construct thus having an imagery dimension. Recognizing this possibility, a re-imaging process invites one to attend to the spatial pictographic aspect of a sign vehicle and consider what it might be imaged/imagined like, and what the same mathematics idea can be represented otherwise. Such a reflective process aims to help one to see/sense the same image differently or generate an equivalent image, and eventually understand the shared mathematics idea anew. A re-imaging process also invites one to attend to how the spatial-visual elements of a sign vehicle invokes and limits one to think differently about the same mathematics idea. It is a back and forth process of working with equivalent images of the same idea. Thus, it subjects one to the different influence of different images, and affords one opportunities to understand the idea anew by collaborating with one's automatic and unconscious understanding. Just like how the role of language changes in the process of re-languaging, the role of image in the process of re-imaging also changes from the medium of meaning to an integral part of meaning and a tool for meaning making.

6.3 Re-inbodying

Re-inbodying is a process of one attending to and/or using one's body differently to engage with an idea with an orientation or outcome of understanding it anew. Here body is referred to the biological matter that a learner has and includes both brain (i.e., the central organ of a person's nervous system) and non-brain parts. Following complexity thinking, cognitive activities are no more limited to thinking in one's head but also include physical doing (e.g., sensing, moving) and undergoing (e.g., perceiving, which includes both body responding to sensorimotor stimuli and interpreting it) that engage one's sensorimotor systems. Learning is inevitably affected by the learner's physical engagement or experience. Here, following Dewey's (1934/1980) view, experience is referred as a unity of doing and undergoing.

The inquiry about re-inbodying is an attempt to stress that having a body matters for learning and invites contemplation on the affordance of conscious and unconscious physical actions (i.e., sensation, motion, and perception) in the process of re-viewing. This opens considerations of whole body engagement, beyond the ones emphasized in re-languaging and re-imaging. Re-inbodying is closely tied to re-experiencing. Yet, they differ in the same way as how re-languaging differs from re-interpreting: Re-inbodying is a way to re-experience but re-experiencing can be achieved through means other than physical ones.

The word “re-inbody” is used here, rather than its more contemporary form “re-embody”, to activate the sense of incorporating into a body or forming a body, one of the less used meanings of “embody”, i.e., “To cause to become part of a body; to unite into one body” (“Embody | Imbody”, 1989), while still maintaining the more frequent used meaning of “embody”, which is “to give a concrete form to (what is abstract or ideal) (“Embody | Imbody”, 1989). These two meanings will be reflected in various degrees in the three rounds of interpretation of re-inbodying. Before that, let me tell an entry story that anchors the interpretations.

Re-doing as re-viewing. In Workshop 3.1 (re-viewing), both Emma and Maxine mentioned students doing the exercises they have done before as an important review process. Maxine observed that students who just flip through the notes without doing the exercises during a review do not learn well:

> They just look at it and say yah, I remember adding the opposite, yah, I remember the LCD [lowest common denominator]. But you give them harder number, different numbers, then they say “No! I don’t know how to do it”.

Emma agreed and continued to stress doing exercises again for reviews:

> … the ones [i.e. students] getting really high Bs and [those getting] As, that's the number one difference that I notice between them, is that they are not actively studying, that they are just passively flipping through the notes.

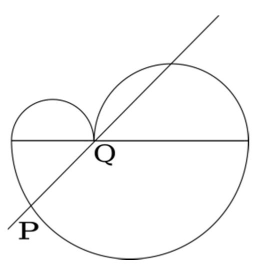
Somehow Emma and Maxine's comments stroke a chord with me. I wrote down "re-doing" and then "re-experiencing" on the group brainstorm paper as an interpretation of re-viewing.

The significance of Emma and Maxine's emphasis in students doing the same exercises again is not at reinforcing practice as a reviewing process; it is at how it seemed to have something intriguing to say about what physical reengagements of the same mathematics idea might be.

As making sense of one's bodily sense

Emma and Maxine's comments resonated with an experience of mine.

Re-sketching a heart figure. I was given a problem while attending a working group session during the CMESG (Canadian Mathematics Education Study Group) 2014 meeting. The problem says:



The figure (given at left) is made up of three semicircles. The line through *Q* divides the perimeter of this figure into two parts. What can you say about the lengths of these two parts as you move the points *P* and *Q*?

After reading the question, I automatically started to copy the figure on paper. This is a habit formed through years of geometry study in China, even when a figure has been provided. I drew the bottom semicircle (*SC1*) and then the two semicircles (*SC2* and *SC3*) above it. While drawing them, I noticed something seemingly unnoticed before. My hand movement was informing me some mathematical features of the shape: Where I started and stopped on the diameter of *SC1* to draw *SC2* and where I started and stopped on the same line segment to draw *SC3* told me the relationship between the three semicircles' diameters (*d*), i.e., $d_{\_SC1} = d_{\_SC2} + d_{\_SC3}$. I wrote down the equation; intuitively I felt its significance for solving the problem at hand. I continued to redraw this figure a few times and put *P* at different locations to examine the relationships between the two perimeters. A solution came easily through these activities.

In this experience, I perceived how my hand movements were confined by the geometry figure while sketching it, and how this limitation informed me of its properties. This unusual level of intimacy between a mathematics idea and my body sensations and movements was intriguing. I realized afterwards that I had witnessed how certain mathematics ideas, like circle and other geometric figures, can be meaningfully grounded in my body actions. Such experience and realization of embodied mathematics has been rare for me.

Sketching a given figure or sketching the same figure again is similar to the process of doing the same mathematics exercises again, mentioned earlier in the Re-doing as Re-viewing story: They both involve a physical reengagement with the same mathematics idea beyond rethinking in one's brain. It is the affordance of such physical reengagement for mathematics learning that I heard resonating in Emma's advocating and categorizing redoing mathematics exercises as learning actively.

A physical engagement with something subjects one to certain stimuli and entails certain bodily responses and hence some perceptions. Not all of these (i.e., sensations, motions, intuitive/automatic associations and interpretations, perceptions) are noticeable for the learner. As Polanyi (1966/2009) famously says, "we can know more than we can tell" (p. 4), there is a tacit dimension of knowing. In instances such as we know something yet hard to explain verbally how we know it (e.g., recognizing the mood shown on someone's face, sensing something wrong in a situation, or knowing where to start to solve a problem without being able to tell by what signs we know it), or we know something unconsciously (e.g., avoiding certain people/things/situations without knowing that we are avoiding them), tacit knowing is at display. As Davis and Renert (2014) interpret, tacit knowing is highly personal and hard to symbolize, thus it is hard to be shared, with others and even with oneself (p. 26). We are only aware of our tacit knowing to a level good enough for us to attend to something else, but not good enough for us to tell the particulars in it (Polanyi, 1966/2009). We only know these particulars in terms of their meaning, e.g., what they signify, what their effects are on the things to which we are applying them (Polanyi, 1966/2009). Using Polanyi's language, in the case that I sketch a figure, I rely on my awareness of a combination of muscular acts for attending to my performance; I attend *from* the elementary movements *to* the achievement of their joint purpose. Hence I am often unable to observe or specify these elementary acts. I know them only in terms of their effects on my task of sketching. They function as a pointer pointing *away* for meaning.

Enhancing one's sensitivity to one's tacit knowing and its particulars, or in other words, making sense of one's sense, is possible as my experience shows, and it could have mathematical significances. Besides my heightened awareness of embodied mathematics in general prior to the incident, my physical reengagement with the same figure could have helped to activate and facilitate this reflection too.

A learner's physical reengagement with the same mathematics idea might be repeating exactly what she has done previously, such as sketching or feeling about the same geometric figure again. More frequently, it would take a different form: One might follow the same mathematics procedure in a different mode of doing (e.g., enacting subtraction after working with paper and pencil), interact with the same embodiment differently (e.g., drawing a cube after touching it), or manipulate a different representation of the same mathematics idea (e.g., playing with Dienes and Golding's (1971) base three triangles after using their base three blocks, graphing a function after generating its table of values). Given such rich possibilities for one to interact with a mathematical idea, a physical reengagement of the same idea might enlist the same or a different sensorimotor system and sign vehicle. Regardless of its form, a physical reengagement with a mathematical idea can bring a *repetition* or a *contrast* of something sensed before (e.g., certain visual-spatial, audio, tangible, olfactory, tactile, kinesthetic, temporal information, or a holistic impression), inducing a general sense or feeling of something staying the same or being changed. This intuitive perception about change, a tacit knowing, could have mathematical affordance, when it is noticed and reflected upon.

Without a deliberated search for or a heightened sensitivity towards perceptual difference, one might notice this information about change easier when it is contradicting with one's expectation. For example, when one unknowingly follows the same mathematical procedure to solve different problems, certain sensible information, such as the same type of kinesthetic movements corresponding to the same procedure, or the same type of aural/visual/spatial stimuli corresponding to the same type of problem, can be reinforced, inducing an unexpected sense of rhythm or harmony. Then, one's reflection on this sense and its particulars affords one a chance to notice the common sensuous cues that resemble or hint at the shared procedure or problem. One of such experiences is offered in Samson and Schafer's (2011) vignette: A grade 9 student arrived at a general expression for a given figure pattern (made of matches) based on a counting procedure in which he repeatedly alternated between top and bottom matches. Rhythm has been recognized as part of mathematical cognition (e.g., Radford, Bardini, & Sabena, 2006) and a crucial semiotic device in the process of perceiving the general in the particular or mathematical generalization (e.g., Radford, Bardini, & Sabena, 2006, 2007). Alternatively, as my experience of (re)enacting in the next section shows, when one knowingly reengages with the same mathematical idea embodied differently, one might be induced to think of this idea in a different way due to certain automatic associations between sensible information and mathematical meaning. A reflection on the particulars of this unexpected difference and where the difference came from could lead one to notice and learn from one's unconsciousness. In both situations, triggered by a perceived harmony or conflict, one could become aware of the particulars of one's tacit knowing, what function they fulfill and what mathematical meaning they resemble or hint at. Therefore, these particulars, the elementary acts of one's body, no longer simply point *away* for meaning; they are also *self-referential* pointers that point *back* to themselves for meaning. The body, in this case, is no more a transparent medium to deliver meaning, but an *integral part of the meaning*.

Acknowledging this possibility, *a re-inbodying process invites one to attend to one's bodily sense while reengaging with a previously encountered mathematical idea, and make sense of it*. This can be a process of decomposing the same physical engagement and sensing it differently, similar to shifting focus on the same wording in a re-languaging process or seeing the same image differently in a re-imaging process.

As changing between equivalent physical engagements – a case of (re)enacting

Following the same logic used for considering re-languaging as changing between equivalent languages and re-imaging as changing between equivalent images, re-inbodying can also be interpreted as changing between equivalent physical engagements. This is well supported and informed by my experience of (re)enacting (meaning enacting and reenacting) negative number subtraction.

(Re)enacting negative number subtraction ($-6-2=$). In Workshop 3.3 (Re-experiencing), Maxine brought up integer tokens as an example of manipulatives while reflecting about re-experiencing mathematics. Being new to integer tokens, I asked her to explain. Maxine wrote $-6-2=$ on the board and showed me, through explaining, gesturing, writing, and drawing on the board, how students can use integer tokens to get the answer -8:

> You draw six negative [tokens], then you want to take away two positive, but there is no positive to take away. You make zeros. Zero is positive and negative. So by drawing zeros, my number is still -6. I can draw as many zeros as I want. It doesn't change it. Now I do have two [positives] to take away, so I take away two positives. Then I have negative 8 left over. So the answer is -8.

After that, we continued the workshop reflective activities and discussed how we can promote re-experiencing in curriculum design. At one point we talked about having students to show others mathematical ideas and procedures, and one way to show is to actively act out the mathematical ideas and processes. Maxine commented, "All the additions can be done on the number line too". This brought me a question about the equation $-6-2=-8$: "Can this be done on the number line too?" Neither Maxine nor I knew the answer. We spent about an hour trying to parallel the actions in two different representation systems to represent the same integer addition and subtraction equation. Unexpectedly, we struggled and ended the activity with confusions.

Later I watched the workshop video and reworked on the parallelizing actions task. I was overjoyed by how much I had learned about mathematics ideas (i.e., number, operation, and zero), representations, and strategies, and about re-experiencing mathematics. I had re-experienced some familiar mathematical ideas in this workshop through physically reengaging with the same mathematical task and ideas in novel ways. These reengagements brought forth several perceptual conflicts that stimulated conceptual growth.

During the workshop, I firstly redid the task “Determine $-6-2$” by following Maxine’s actions on tokens. Essentially Maxine enacted $-6-2=-6+(+\mathbf{2})+(-\mathbf{2})-2=-6+(-2)=-8$. I had to legitimize Maxine’s enactments. While doing so, I sensed some resistance towards the negative tokens but could not explain why. Later, I needed to reenact the equation on the number line. When I intuitively mapped absolute numbers to steps, “+ (plus)” and “– (minus)” to “walking right” and “walking left” respectively but could not reproduce an action to parallel with “taking away two negative tokens” that represents “$- \ -2$”, I re-cognized that number is a quantity with sign and thus negative numbers exist like tangible objects. I felt somewhat conflicted and later realized that unconsciously I did not perceive that negative numbers exist. Then when I added two representations for positive and negative numbers (i.e., Positive number *n* = a walk *n* steps to the right, Negative number *n* = a walk *n* steps to the left) and found unable to deal with tasks such as “Determine $-6--2$”, I re-cognized the dual meanings of a symbol (i.e., + and –) hence the necessity of different representations for them. I realized that I had conflated multiple meanings of a symbol without knowing. By the time I solved the paralleling task by mapping generating 0 by taking out two pairs of positive and negative tokens to generating 0 by writing two opposite walking actions as two instructions on a script, and taking out two positive tokens as crossing out a redundant instruction, I renewed my understanding of zero and negative numbers, and found that two powerful problem solving strategies used in higher grades, i.e., introducing needed symbols through making 1 or 0 (e.g., to solve grade 12 limits problems), simplifying mathematical expressions before calculations, have already appeared in lower grades through the use of negative tokens.

I finally understood that my initial discomfort towards negative tokens was to do with my previously embodied understanding of negative numbers. Unconsciously, I related negative numbers with owing something thus not existing, also with being opposite to their corresponding positive ones. Therefore, representing negative numbers using negative tokens made me feel contradictory: How can *not having something* be represented by *having something*? On the contrary, enacting negative numbers as steps opposite to their positive counterparts seemed harmonious. This explains why once I re-cognized that zero can take many forms and negativity in mathematics exists in relation to a positive counterpart making zero together (hence no need to be opposite to each other), the existence of negative tokens and their being only differ from positive ones in color was justified, and my discomfort towards them resolved. I had come to understand that any two random objects can represent positive and negative numbers as long as we *define* them making zero together.

I also understood that my initial struggles and later growth are related to the affordances and limitations of a representation system. The token system afforded differentiating two meanings of the same symbol (i.e, + and –) by representing them differently (e.g., negative number as negative token, minus as taking away), whereas the number line system helped me notice the existence of such difference through inducing me to enact both negative numbers and subtraction the same (i.e., as walking left) but only find the enactment erroneous. The token system afforded me a chance to work with various forms of zero (as any pair of positive and negative tokens) to generate equivalent expressions, whereas the number line system's inability to support addressing zeros in a linear execution made me come up with simplifying before execution. Each system brought forth a powerful problem solving strategy - making something out of nothing by using the identity property (of addition, i.e., any number plus zero is the original number, and that of multiplication, i.e., any number times one is the original number), and simplifying before computation respectively - through what is doable and not doable in the system.

My above experience illustrates that a learner's mathematical and pedagogical growth is benefited from (re)enacting a previously learned idea in novel ways. Acting out a mathematical idea or process physically demands learners to generate a consistent one-to-one correspondence between symbol/concept and object and between relation/procedure and action. This offers them chances to confront with their related unconscious embodied understanding, in turn facilitating their conceptual understanding development. In this situation, a learner actually benefits from being unaware of what she assumes, what she automatically follows, and what she used to know but now has forgotten, or from a condition of, in Davis and Renert's (2014) words, one "*not having immediate conscious access to what is known*" (p. 60). Particularly, my experience exemplifies how a learner can benefit from a re-enacting task that requires her to parallel the physical actions in two or more representation systems or models to represent the same mathematical idea/process. With their different affordances and limitations, different models can trigger certain mindsets and perceptions that are in conflict with each other and sometimes draw the learner unconsciously closer to one than the other, reflecting her unquestioned embodied assumptions and interpretations of the related mathematical ideas. Working with two systems requires repeated comparison and equalization, affording the learner many chances to confront her (un)conscious understandings and re-view them. Therefore, working with multiple models can offer learners opportunities to mathematical conceptual growth[20].

My experience has *a process of (re)enacting: One reengages with the same idea through enacting the idea, justifying the enactment and/or creating one or more kinds of physical enactments of the same idea*. (Re)enacting is a case of *re-inbodying as changing between equivalent physical engagements of the same mathematics idea*. The involvement of equivalent physical engagements could entail different objects for a learner to interact with and/or entail different kinds of interaction, thus affording a chance for one to notice and reflect on any incongruent perception of the same idea.

As invoking different ways of thinking and enlisting different modes of knowing through using the body differently or forming a different body

Emma and Maxine's comments also resonated with my contemplation on the role of "laboring" the body in mathematics learning.

[20] Kinach's (2002) study has a similar conclusion. Without knowing, my re-enacting negative number subtraction task reverses Kinach's task design in her mathematics education course – the same explanation task repeats in two contexts (firstly on number line and then later with algebra tile) – and my struggle resonates with what she observed when she assigned preservice teachers to explain integer subtraction such as 5 – (–3) to eighth graders on the number line. Supported by the preliminary success of such task design on the preservice teachers' pedagogical content understanding, Kinach proposes a teacher education model requiring teachers' explanation of mathematics topics in two instructional contexts. I see Kinach's task design an example of (re)enacting. However, while Kinach deliberately uses the first context to raise confusions and then the second context to resolve such tension and requires no paralleling actions in two contexts, I advocate using both contexts to elicit tacit knowing and raise tension, particularly through requiring paralleling actions in different contexts.

> **Laboring the body in mathematics learning.** In my experience of solving mathematics problems and helping others in mathematics, I have encountered many times that the learner (myself included) knew what to do immediately after she wrote down what she knows and is thinking. The conversation often goes like: “I know I can take this factor out, but then what?” “Write that step down. Then you will know what to do next.” “[after writing things down] Oh, I see.” I have also encountered many times that when I did the physical work myself to (re)write detailed lesson notes for a lesson that I had taught before or write solutions for a test that I just (re)generated, I noticed something I seemed to miss before and learned something new about the test topic. Supported by my positive experiences with hand writing, I insisted my students to write down test corrections rather than taking photos of my detailed solutions and to take lesson notes and graph by hand. My reflection on hand writing in mathematics learning has gone deeper over time. Of course, hand writing helped me to decrease the demand on my working memory and reflect by having my thought externalized. It provided me a sense of making something and being active[21]. It offered time needed for my reflection. But there seemed more to it, something to do with the process of writing, using hands, or “laboring” the non-brain parts of the body in general.

Emma’s categorization of doing the same exercise again as active learning and reading the previous written solution as passive, and advocating one over the other at first spoke to me the necessity of “laboring” one’s body. Later I found Emma’s words speaking less about preference or hierarchy but more about the different influences of using one’s body differently on mathematical learning.

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[21] The externalization effect and generation effect (i.e., things generated by the learner is easier to recall than reading alone) of writing have been well studied.

As shown in the previous section, using one's body differently to reengage with the same mathematics idea can enlist or bring forth different representations of the idea and invoke perceptual conflicts that can stimulate mathematics growth. It also can bring forth different modes of doing (i.e., seeing, listening, writing, typing, walking, and so on) and/or incorporate different things (i.e., pen, keyboard, handheld sensors) into the body. One can argue that such incorporation forms a different body (De Freitas & Sinclair, 2014; Polanyi, 1966/2009). Undoubtedly, these changes affect what one can do and perceive, in other words, one's sensorimotor experiences. How these changes influence ways of thinking is interesting.

A study on the influence of different note taking strategies in learning is provocative here. Mueller and Oppenheimer (2014) found students who took notes longhand performed better on conceptual questions than students who took notes on laptops. They attribute this to the extra mental processing such as summarizing, paraphrasing, and concept mapping needed for taking-note by hand contrasted with the laptop note takers' tendency of transcribing lectures verbatim. This study speaks to me that how to take notes affects how one thinks, in particular, typing notes induces detailed oriented thinking whereas writing notes encourages holistic thinking, and suggests the significance of the process of doing. It points me to consider physical limitations as enabling constraints[22] (i.e., restrictions that are empowering rather than limiting) for mathematics learning.

Here I reconsider hand writing for a thought experiment. Confined by the hand movement speed and the physical writing space, hand writing is slower and more limited by time and space compared to many other modes of doing, such as seeing, listening, speaking, and typing. This physical slowness affords time needed for the kind of thinking that is slower and

[22] This is different from Davis, Sumara, and Luce-Kapler's (2008) enabling constraint, which has been mentioned in section 3.3 and refers to curriculum tasks and activities. However, my usage of the same phrase and theirs share the same idea of considering restrictions positively instead of negatively.

requires more effort, such as thinking deliberatively, logically and reflectively[23], yet necessary for mathematics learning. Also, the time and space restriction on hand writing can encourage a learner, particularly when facing a discrepancy between how much information one is handling and what one can write down in a limited time and space (e.g., in a note taking or reading situation), to abandon details oriented techniques (e.g., transcribing words verbatim and writing long notes) and pay more attention to the key ideas and the structure of the whole. Hence, in certain situations, hand writing can help to promote holistic thinking and representing, which is part of generalization and abstraction thus essential for mathematics learning. When a learner writes while thinking, writing might help one to focus attention to a small range of information and space due to the physical need to control one's hand movement. This could be helpful when nuance (particularly the spatial-visual one) needs to be noticed. The linear execution of hand writing, producing information one after another, also seems compatible with linear processing, one thing after another, which is part of thinking logically thus mathematical thinking.

Of course, the above affordances of hand writing can be activated differently, sometimes at the expenses of dropping other kinds of affordances, depending on which mode of doing one is changing from. For example, switching from listening or speaking to hand writing might not change much one experiencing a sequence of ideas in a temporal order, but decrease affordance for auditory imageries and spatial reasoning. Changing from seeing to hand writing can establish

[23] In *Thinking, Fast and Slow*, Kahneman (2011/2013) explains two modes of thinking in the mind:

> *System 1* operates automatically and quickly, with little or no effort and no sense of voluntary control. *System 2* allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration. (pp. 20-21)

While System 1 is in charge of automatic activities, such as unconscious doing, intuitive knowing, and learned habitual actions and associations, all operations of System 2 "require attention and are disrupted when attention is drawn away" (p. 22). So System 2 only takes over when System 1encounters difficulty, such as when one tries to compute 123 x 456, and "only the slower System 2 can construct thoughts in an orderly series of steps" (p. 21). In general, System 1 is fast, automatic, intuitive, and analogical, whereas in comparison, System 2 is slower, more effortful, deliberate, and logical.

a clear temporal order and decrease the amount of information demanding for attention at once, thus enhancing affordance for effortful logical thinking, yet decreasing affordance for holistic and visual-spatial thinking.

Now, if we also consider what kinds of writing tool one is using and how familiar one is with each tool, the affordance of hand writing would change too. For example, compared to writing on paper with a pen, writing on paper with a pencil induces a more welcoming attitude for me towards errors thus decreasing my anxiety, due to the higher removability of pencil marks and my stronger association of pencil with draft. This might not be the case for learners who grew up with abundant stationary supply and higher interchangeability between these two tools: The difference between pen and pencil is too insignificant to make a difference. For such learners, writing on a computer screen with a digital pen might permit more trails and errors yet invoke less memorization due to computer's even higher removability and storage power.

Nevertheless, both the gains and losses in affordances are subjective to individuals and they both can be constructive for mathematics learning; what matters is the *change* of affordances. Different modes of doing and forms/unities of body engage sensorimotor systems differently, allowing different physical features to influence one's mathematics learning and enabling different representations of the same mathematical idea to appear. *Re-inbodying is interpreted here as redoing with the focus to afford different (e.g., pictographic, auditory, tactile, kinesthetic, olfactory) representations of mathematics ideas and alternative ways of thinking (e.g., holistic, relational, sequential-logical, visual-spatial, reflective) through diverse physical limitations of different modes of doing and forms of body*.

Back to re-inbodying as a whole

Given that a learner has the capacity to engage with an idea using multiple sensorimotor systems and any encountering with an idea involves the learner's body inevitably, mathematics learning is both mental and physical. One's physical experience can have mathematical significance as its particulars might resemble or hint at certain mathematical ideas, unveil certain bodily knowing, or invoke certain mindsets, perceptions or ways of thinking. Thus it is necessary to think about re-inbodying, an attempt to (re)ground abstract mathematical ideas to a concrete tangible physical world, particularly to one's doing and undergoing, and mobilize one's tacit knowing. Re-inbodying, as physically reengaging with the same mathematics idea, is a process of one *paying attention to one's body and listening to what it says about mathematics*. It affords learners a chance to sense and feel the properties of the mathematics idea embodied and ground the abstract idea to one's sensorimotor experiences when possible. When equivalent physical engagements of the same mathematics idea are used, e.g., in a reenactment of the idea, a process of re-inbodying can enlist different kinds of representations, interactions, and thinking and activate one's unconscious knowing. Re-inbodying opens a space for considering multiple modes of doing that engage different sensorimotor systems, afford diverse ways of thinking, and connect multiple representations for the same idea. It brings the modes and forms of experiencing and representing into focus.

6.4 Back to Re-viewing as a Whole

Up till now, I have interpreted reviewing as re-viewing and theorized three forms of re-viewing: re-languaging, re-imaging, and re-inbodying. These forms serve as different interpretations of the same process, i.e., re-viewing. They stress different dimensions (i.e., linguistic, visual-spatial, bodily) of one's reengagement with a mathematical idea. Such categorization is neither exhaustive nor absolute. There could be other forms of re-viewing and the three forms of re-viewing overlap and crisscross. For example, re-languaging and re-imaging can be subsumed under the category of re-inbodying as verbalization and visual-spatial perception belong to bodily sensorimotor capability. Re-languaging can include re-imaging and re-inbodying when both visual-spatial construction and bodily movements count as a kind of language. Re-imaging can happen along with re-languaging and re-inbodying as equivalent languages and physical engagements might appear differently or engender different mental images, or different wordings or physical engagements might reinforce the same spatial-visual stimuli to occasion the same mental image. However, in responding to the dominance of visual perception in human perceptions and the lack of emphasis in modes of doing beyond seeing in traditional learning environments, there is still value to have a dedicated interpretation of re-imaging and re-inbodying respectively. The dedicated conceptualization of these three forms of re-viewing is also an acknowledgement that all signs can be re-languaged, re-imaged, or re-inbodied, since any sign can be interpreted as symbolic, iconic and indexical.

This categorization of forms of re-viewing is also incomplete. To acknowledge the possible existence of other forms of re-viewing and the limitation of my current inquiry, the unknown forms are labeled as *re-visiting*. A heuristic visualization of the forms of re-viewing was conducted to further understand the relations among these forms (see Figure 6.4.1. for a working representation). For each of the three forms, there seems be promising trigger question: What might it be worded like? What might it be imagined like? What might it be embodied/enacted/perceived like? The word “it” refers to the mathematics idea that one re-views. Of course, these questions are general examples. More specific questions such as “What might it look/sound/smell/taste/feel/move/act like?” can also be asked to invite re-viewing that stresses a particular modality.

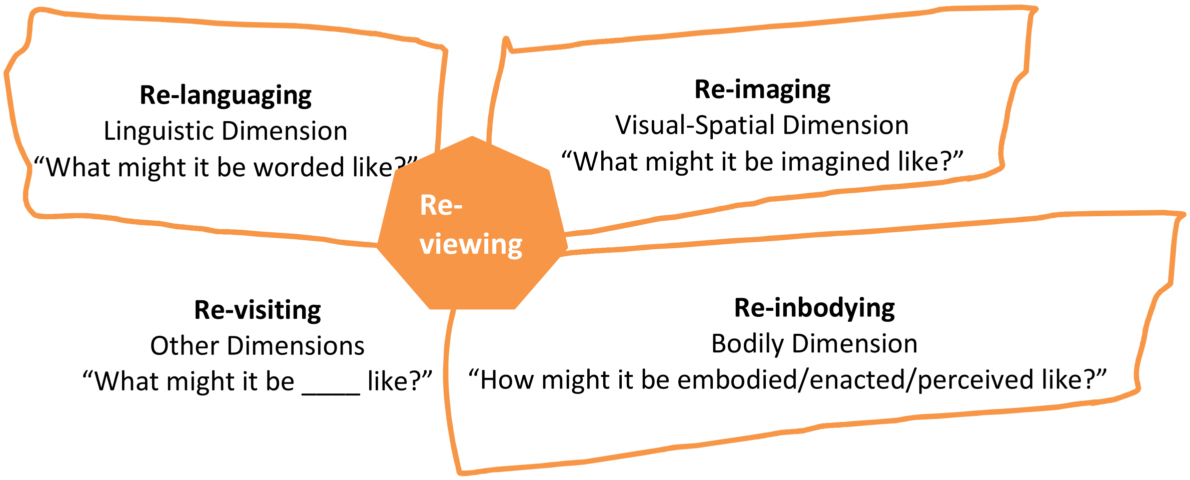

Figure 6.4.1. A tentative categorization of the forms of re-viewing ("it" refers to a mathematics idea one re-views).

The three forms of re-viewing are presented in a similar shape of different sizes, to signify that they closely resemble each other yet they are different enough to draw our attention to them separately. All three forms of re-viewing use a sign vehicle (i.e., a wording, an image and a physical engagement respectively) for meaning, and use the means and process to inform mathematical meaning making. They are a process of making a sign vehicle more mathematically meaningful, either through seeing/sensing it anew or changing between equivalent ones. The sign vehicle is no more a conventional pointer pointing away for meaning but also a self-referential one pointing back to itself for meaning. As such, the signifier becomes (part of) the signified. Besides this change, in all three forms of re-viewing, there exists a reflection triggered by some change and directed towards the re-viewing process and experience as whole, e.g., trying to make sense of what seemed to change or not change, what one does, where one is led to, how one feels, and so on. This reflection also helps one to understand the idea that one is reviewing anew. Therefore, not only does the signifier that signify a message become (part of) the message in all three forms of re-viewing, the process of working with such signifier also becomes (part of) the message. This process of a medium (a signifier or a process) that one works with becoming the message is what Sawada and Pothier (1993) define as recursion.

Given how closely these three forms are related together and how similar they are, we now turn to the next chapter to interpret them further holistically.

7 Re(Re(view))[24]: Re-view Re-viewing

When looking at the three forms of re-viewing together, a new interpretation of it emerges: They suggest re-viewing as a *re-storying* process. What this re-storying process might be is the focus of this chapter.

My interpretation, once again, was provoked by certain experiences. One experience is my sensing story emerging and changing during my reengagement with a text multiple times:

> **(Re)storying while re-teaching.** I taught a particular grade 12 mathematics course six times and the courses that feed into it multiple times in four years. Over this period, more and more mathematical problems started to look the same to me, and I sensed that they were telling me the same story. So I began to categorize examples and exercises in a lesson or multiple lessons by fewer and fewer question types and asked my students to do the same. I encouraged my students to think of these seemingly different problems (that for me were essentially of the same type) as cousins and categorize them by families. Meanwhile, a vague sense of story was forming while I re-cognized more and more topics (e.g., linear, quadratic, and polynomial relations; division, slope, tangent ratio, rational relation, and limit) across grades as variations of the same big idea. They appeared to me less as cousins but more the same person with different looks or outfits, a character developing with increasing complexity over time. Thinking and describing related mathematical objects as human characters in the same family or the same character developing over time made me feel closer to math. I also felt it was easier for me to stress mathematical relations in my teaching, when I used language such as family member and story to talk about mathematics ideas.

[24] The word "view" here is both noun and verb, representing both the content and process of viewing. "Re()" means repeating the process of viewing the scene (signified by the content in the bracket) as a whole again. Since the result and the process of this viewing become the content for the next loop of re-viewing, hence the notation of "Re(Re())".

Another experience is an intriguing example emerging during my work with three participants in the Re-storying workshop:

From a division error story to a multiplication error story. While implementing Rachel's idea of folktale to promote re-storying mathematical ideas, Valet suggested having students divided into two different groups and asking one group (say Group A) to retell a mathematical story that they heard to another group (say Group B). I asked what the Group B should do after hearing the story. We thought of it quietly. Then Maxine suggested that the group can have a story about a different question. Valet agreed and added that the new story should use the same mathematical knowledge. Pointing to the division error story mentioned by Valet earlier, in which a student's work that followed the right logic but used a "wrong" presentation was marked "wrong" by her teacher, I asked what mathematical knowledge students should learn in this story. Maxine answered, "Place value. 'Lining up' requires understanding place value". After some more discussion about Group B's story, Valet came up with an example for the division error story: "[A story in which a learner produces] the same error using the same logic reasoning, maybe not in division, but in multiplication". Later, Valet refined her idea: "Both stories are about place value. Teachers can ask group B to generate the same error, using multiplication. Or, leave it open. Students might come up with a story of division, addition, or subtraction". Here I saw an emerging idea: "Re-storying as (re)generating a similar story that has the same type of mathematical events to do with the same mathematical concept/idea". I noted that on the white board.

These two experiences addressed me – they were to tell me something about re-storying and re-viewing. My contemplation on them propelled and supported an interpretation of re-viewing as re-storying.

In this chapter, I extend Dietiker's (2013, 2015) three-layer narrative model to conceptualize mathematical story at both mathematical text interpretation and mathematical understanding levels. This modified theory is then used to define two kinds of story change in a re-viewing process and support interpreting re-viewing as re-storying. This chapter closes with the new understandings of re-viewing that are brought forth by such a reading.

7.1 Mathematical Story

Using literary story as a metaphor for mathematical planned and enacted curriculum, Dietiker (2013) conceptualizes a framework to support a narrative reading of mathematical texts. Dietiker considers mathematical ideas, a phrase that refers to "the milieu of mathematical concepts and processes" (p. 19), such as "mathematical objects, relationships, properties, and procedures" (p. 15), as story characters. She defines a mathematical event as a change of a character's mathematical state or a mathematical change for short. A mathematical narrative, Dietiker theorizes, is a system of three layers, i.e., text, story and *fabula*. Text is "the media (including its narration) in which a story is told" (p. 14)". Take a textbook as an example. This text layer includes the textbook's physical form (i.e., bound paper), the signs on paper, and their configurations (e.g., content layout design). Story "describes the chronological sequence of mathematical events encountered and experienced by a reader throughout a mathematics text" (p. 15). There is not necessarily a direct mapping between a given text and story because a story can include events beyond the text layer. The events in a mathematical story might be the ones as interpretation of a given text (e.g, when reading a mathematical solution or a sequence of topics), and/or the ones generated by the reader (e.g., when solving a task after reading it). Nevertheless, a mathematical story is the unfolding of mathematical content and developing of the mathematical ideas identified by the reader during her interaction with the text. It is a reader's sense of certain mathematical content changing across the sequence of events that the text offers the reader an experience similar to that of a literary story. Fabula is defined as "a reader's logical re-construction of the mathematical events beyond the text and story layers" at first and later as "a reader's reorganization of the logic around how certain mathematical ideas support or connect the meaning of other mathematical ideas" (p. 16). Although Dietiker's concept of fabula seems obscure and remains peripheral in her later works, it is clear that while mathematical story depends on time thus linear, mathematical fabula depends on reasoning thus not necessarily linear. A fabula is formed through, Dietiker says, not only resequencing but also "re-defining, noticing a pattern, connecting, and conjecturing" (p. 16). In short, "any point at which a reader confronts a conflicting mathematical idea that requires the logical recognition of prior mathematical understanding, the mathematical fabula is involved" (p. 16). So, essentially fabula is a non-linear construct of connected events *and* ideas and it is resulted from the logical reconstruction of events and logical (re)construction of ideas within *and* beyond the story layer. In a sense, I see fabula as a form of mathematical understanding, which is part of the reader's current mathematical knowing.

Dietiker's theory is about the role of texts in both mathematical learning and understanding, with a particular focus on texts with sequential parts. Not only can such texts afford a reader an experience of story composition or storying, affect what the story is and have a potential aesthetic effect on the readers' mathematics learning experience, but also these texts can influence the mathematical understanding that the reader might form. The sequence of parts in a text influences the fabula that is generated through changing how different aspects of the mathematical content is introduced or emphasized, thus changing its meaning (Dietiker, 2015, p. 291). Moreover, a reader's aesthetic experience while perceiving the tension between expectation and realization in a mathematical story and the reader's logical construction of fabula affect each other reciprocally (Dietiker, 2015, p. 300). In short, "the way in which the mathematical content temporally unfolds can affect both the experience for the reader and the nature of his or her mathematical conclusions" (2013, p. 19). Dietiker's theory suggests that both story and fabula are individual contingent constructs that emerge through the reader's interaction with a text and affect each other.

Seemingly, based on Dietiker's theory, mathematical story can include the stories alluded to at the beginning of this chapter: the stories I formed and changed while revisiting the same mathematical text and the two equivalent stories generated in a workshop. These two kinds of stories differ in their characters: My stories have human-like mathematical ideas as characters acting in the story, whereas the workshop story examples have fictional human characters doing something mathematically. Dietiker's theory recognizes that both mathematical characters and human characters can act in mathematical story, however the theory does not consider mathematical characters as actors: Mathematical ideas can only act when a human reader "acting *as* a mathematical character" in the story (2015, p. 295). These two kinds also differ in their formation: My stories were formed during my revisiting the same text, whereas the story examples were generated as mathematical story. There is an intriguing difference between "forming a story while reading a text and changing a story while text changed" and "forming a story while revisiting a text and changing a story while no text changed". The later pair is integral for interpreting re-viewing as re-storying yet not addressed in Dietiker's theory that focuses solely on storying.

Therefore, changes to Dietiker's theory are necessary to support re-storying. Here I propose three. The first is to include a wider range of potential factors in a text that can influence storying and understanding. Dietiker (2015) theorizes that any text can be read as a story: Despite some curriculum sequences make no sense when reading across the parts for a connected story, "all mathematical texts can be read with a narrative lens by attending to how the parts alter and constrain the meaning of the others, if they do" (p. 290). Noting that such text can be at various scales (e.g., tasks, lesson, chapter, textbook, course and beyond), Dietiker (2015) stresses,

> Fundamentally, for any sequence of "chunks" A-B-C at any grain size, it is how B builds from A and changes what we know about both A and B (and similarly how C then builds from B and changes what we know about A, B, and C) that engenders the mathematical story. (p. 291)

While these general comments suggest that the relationships among parts of a text is not limited to temporal ones, Dietiker's story layer includes only a temporal sequence thus confining the reading of a text as linearly ordering its parts. Moreover, while the text layer "focuses on the media's role for the narrative" (Dietiker, 2015, p. 288), the planned and enacted curricula discussed in Dietiker's work involve textbook and seat work only. The non-temporal (e.g., visual-spatial, semantical) relations among parts and other types of medium (e.g., oral, auditory, tactile, kinesthetic) are not the focus of Dietiker's work.

Second, although Dietiker (2015) notes that a mathematical story can be recognized in a text at various scales (e.g., tasks, lesson, chapter, textbook, course and beyond) (p. 290), all stories theoretically possible in her work seem linked to texts that can be read into a sequence of mathematical events. In a sense, all texts are read as a self-contained story. This excludes micro texts (such as a symbol or a term) and non-linear interactions with a text (e.g., perceiving a text holistically). This can change if we consider how the text parts affect and limit each other non-temporally and acknowledge that texts at any scale can be viewed as a story through being read as a part of a bigger story. For example, how the parts of a circle connected spatially can be read into a story of a character of point moving and jumping across the center and leaving a circular trace. Or a shape of circle can be read as a part of a bigger story of graphs or that of symmetry.

Third, while Dietiker's work recognizes fabula similar to story as a contingent and individual construct whose formation is affected by a reader's experience rather than a timeless one, little aesthetic dimension is theorized in this layer. I see this is possible if we emphasize fabula as a historical construct always in the status of becoming. Dietiker's theory is in consistent with complexity thinking that acknowledges the contingence of one's mathematical understanding and its emergence through one's experience. Dietiker's emphasis in the temporal quality of mathematical texts is a critique of how little the mainstream mathematics curricula reveal "how mathematical ideas are expected to emerge through a reader's experience" (2013, p. 19). Similar to how a learner's being is affected by her becoming and always in the status of becoming, a learner's mathematical understanding is also a personal historical construct that continues to evolve over time. With such idiosyncrasy and historical experiential contingency, mathematical understanding can have an aesthetic dimension (i.e., different mathematical ideas might have different development paths associated with different experiences and sensations) and it is applicable to draw an analogy between a mathematical idea and a human character.

Even without the reader imagining herself as the mathematical character[25] as Dietiker proposes, it is possible to interpret a mathematical idea human-like.

With these three changes, it is possible for us to read any text, metaphorically, as a mathematical story. That is, such reading can happen regardless whether we can read it into a sequence of mathematical events. Also, regardless whether a mathematical idea is human-like character or not, a reader's mathematical understanding, at any particular moment, can be interpreted as a contingent logical construct made of many stories of mathematical ideas together, thus a grand story of stories. Each of these idiosyncratic stories depicts certain mathematics character's past and present and is suggestive, logically, for certain kinds of future.

7.2 Re-viewing Interpreted as Re-storying

Following the above theorization, I see two kinds of story changes possible during a re-viewing process. *One is changing text as changing story*. A reader might read a text itself as a story or as part of a story. In the first situation, a perceived text and media change is literally changing the story, while in the second situation it can be used to infer a change in the story in which the text is part of. Either way, a text change can be mapped to certain kind of story change even though the stories in the two situations might not be the same nor mathematically meaningful. What matters is the *change* perceived or interpreted. For example, I can consider the text "asymptote" as a self-contained story of letters, or I can consider it as a part in the story of asymptote. Regardless, the text change from "asymptote" to "*jian jin xian*" might be interpreted as a language change signifying a wording change for both stories, whereas the text change from "asymptote" in black to "asymptote" in yellow might be interpreted as a sentimental feature

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[25] Albeit there are multiple definitions of mathematical characters in Dietiker's writings, two characteristics, i.e., something objectified and something got fleshed out, are useful here. Dietiker (2015) writes,

> mathematical characters are "figures" brought into existence (objectified) through reference or inference in the text (e.g., naming, defining, or otherwise drawing attention to them) and which are given specifying features (a process that Bal refers to as "fleshing out"). (p. 292)

change, signifying a change in the protagonist or moral of the stories. Since a re-viewing process engages different texts to represent the same thing, such text change corresponds to a story change that happens between equivalent stories. *The other is changing mathematical understanding as changing story*. In a re-viewing situation, there exists some understanding of a particular idea that one is about to reencounter. So a story of the idea exists before the re-viewing process and gets reconstructed through the process. This story enables and influences its own re-construction. Since a re-viewing process helps one to understand an idea anew, such transformative change in understanding corresponds to a significant story change, such as a protagonist's identity shifts after character redefinition. Given how stories of different mathematical ideas can be interconnected, one story's transformation could bring forth transformation in many other stories. Putting these two kinds of story changes together, *re-viewing is a process in which a different story of the same thing is used with an orientation or outcome of generating a new story of certain mathematical idea(s)*.

Now let's turn to interpreting the lived re-viewing processes as re-storying to see what space such interpretation might open up. Although many stories can be composed based on one's interaction with a text, it is possible that a change in any of the stories triggers the same kind of change in other stories. For instance, I can read a text "asymptote" as a story of letters, part of a story of asymptote, line, graph, or terminology, part of story of my mathematical understanding or my experience with mathematics, and so on. Regardless, a change of text from asymptote to *jian jin xian* could be viewed as a change of wording in all these stories. Therefore, it is overcomplicating if I were to specify and differentiate stories. Here I use story to refer to any of these possible stories so that I can focus on *the type of story change* that might bring forth understanding something anew. To keep the following interpretation as a *contingent* possibility rather than something guaranteed, I use past tense consistently without implying that every single event happened.

In the two renaming examples discussed earlier in section 6.1, the linguistic change happened from *mi* to power and from asymptote to *jian jin xian* seemed like rewriting the same story using different words. The one written in more familiar words appeared friendlier for me at first, and then was interpreted as more informative, signifying that the story changes from an unapproachable senseless story with little content to a friendly sensible one with useful details. Such change further colored the story character(s): In the *mi*-to-power example, the characters of exponent and exponential relations seemed to change from a characterless person to a superhero due to my interpreting power as telling me that exponential relations are powerful; in the asymptote-to-*jian jin xian* example, the character of asymptote seemed to change from a boring person to a humorous/charming one due to my visualizing asymptote as a dream girl being forever unapproachable. In short, the characters became more friendly and impressive in both stories. This sentimental change in my feeling towards a few particular mathematical ideas was far from trivial: In terms of mathematical understanding, it helped me to redefine these mathematical ideas as total different characters, and it changed my understanding of mathematics terminology and my relation with mathematics more broadly.

In the circle definition example, as a byproduct of my translating from Chinese to English, the words in the circle definition were reordered: The phrase “a collection of points” got moved from the end of the sentence to the beginning of the sentence. This change seemed like reordering events in a story, helpful to bring forth a change of story center or emphasis. By the time I redefined circle and graphs as a collection of points, the change on my story of circle and story of graphs was like an identity shift for the protagonist, as significant as the one from “a Canadian Chinese” to “a Chinese Canadian”.

Not surprisingly, such change of story emphasis can also be resulted from highlighting different parts of the text. When Maxine saw a division sign standing out in the polynomial division image, the story change seemed like getting a clearer storyline (if we draw an analogy between the division sign and a storyline based on their shared function of weaving different parts of the story together) or story center (if we see the division sign as a part of the text, hence a part of the story). A polynomial division story was turned into a division story as a result; the story protagonist changed.

In the negative tokens example, the physical enactments of the negative equation $-6-2=-8$ were changed from moving tokens to walking on a number line. This seemed like casting different actors for the same character in a different setting with different actions yet keeping the story the same. This brought a need to negotiate any mismatches between actor and character (e.g., the actor of negative tokens vs. the character of negative numbers), and between actions in the old and new acting. This negotiation redefined the character of negative numbers. Other stories, with characters logically connected to negative numbers, such as numbers and subtraction, were also transformed.

Story change can happen when there is no apparent text change. How we experience and perceive the same text change over time, hence a different story can be formed. Similar to one’s experience of noticing different things while watching the same movie again, I noticed different things while drawing the circle again. My story of circle changed to a more personally related one by my noting how the character of circle controlled my body. It also became a story of embodied mathematics. The same kind of story change happened in my teaching the same course again, when there was no text change at the topics level. Over time, the same text told me a story of a different character (e.g., the story of linear functions becomes the story of polynomial functions, the story of limit becomes the story of slope and eventually the story of division) or a story about the same character rather than multiple different ones (e.g., a story of quadratic equations, quadratic trigonometry equations, and quadratic exponential equations is a story of quadratic equations, taking different forms at different times). Here the story change, sometimes, was like the change from one person's story to another person's story, and sometimes like the change from a story of an individual to a story of a collective. Such reading of one mathematical story into another helped me to feel closer to these mathematical ideas and to mathematics in general. It also transformed my mathematical understanding as it is essentially a process of pattern noticing, connection building, abstraction, and generalization.

7.3 Space Opening Up

The above interpretation of re-viewing as re-storying stresses that changing time or text (including the content, form and media) can help to invoke a re-viewing process. No doubt that changing what is being said, how it is being said, and when it is being said about the same thing can affect a learner's interpretation aesthetically and/or logically. The change in either dimension has a potential to transform the other dimension and also the learner's mathematical understanding.

Also emphasized in the above interpretation, it matters little whether a text is read as a story in a re-viewing process. It is the *perceived difference between two texts* in a re-viewing process that suggests certain story change and makes a difference in a reader's understanding. Drawing an analogy between re-viewing and re-storying is to seek inspirations from story change to help with text design.

Thinking re-viewing as re-storying enlarges the space of potential equivalent texts. Two texts might appear dramatically different yet equivalent as telling the same story. How differently two equivalent stories appear is relative to which level of abstraction they are equivalent at for a reader. For instance, the two place value stories generated in the re-storying workshop are considered the same, albeit appearing to do with different mathematical events (i.e., multiplying and dividing), for they repeating the same *kind* of mathematical errors (i.e., not lining up the units with the same place value). In a particular context and at a particular abstraction level, they differ in degree not in kind. This is the same logic why sometimes a word, a painting, a sitcom can tell the same story as a piece of music does. Similarly, any text can become a contingent equivalent pair for a mathematical one as long as they can be read as equivalent stories at a particular level of abstraction, e.g., with the same moral. Therefore, thinking re-viewing as re-storying points us to connect seemingly different stories across contexts with mathematical or non-mathematical characters together through *analogy*, breaking boundaries that alienate mathematics, such as the boundaries between mathematics on paper and embodied mathematics, between mathematics at school and mathematics in daily-life, and between mathematics and other subjects, including arts. Through stories, re-viewing stresses the importance of relational and analogical thinking for learning, and broadens the possibilities of building connections across contexts and noting the patterns that are connected[26].

Thinking re-viewing as re-storying is partially premised on a conviction of mine: That one's mathematical understanding is a story because they both are idiosyncratic contingent constructs always in the status of becoming. Considered as individual interpretation, both

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[26] In his book *Mind and Nature*, Bateson (1979/2002) identifies a pattern, which is mind, that different living creatures share. He calls this pattern as "the pattern which connects" (p. 7). His work suggests to me that there is shared commonality among different complex phenomena. Such commonality or connectedness, observed through the lens of complexity thinking, is multiple. In other words, many patterns we observe are connected, and possibly, in many different ways.

mathematical understanding and story are hermeneutic truth that is "very fragile, very mortal, very close to what happens to us" (Jardine, 2015, p. 250): They are not there to last forever. Mathematical understanding is not timeless, but instead a time bound construct. Thinking mathematical understanding as stories urges us to emphasize its aesthetic dimension and hermeneutic mortal quality, attend to what happens to us while learning mathematics, value the memories, feelings, images, and sounds evocated through our interaction with a mathematical text, and encourage us to envision mathematics learning as an aesthetic experience. Therefore, for learners who view mathematics as universal objective truth thus lifeless, re-viewing as re-storying has potential to breathe life into their mathematics learning and understanding through stories. In this case, *re-viewing is a reviving process*.

Considering re-viewing as re-storying can trigger the possibility of any of the three forms of the re-viewing process, e.g., by asking "What might a story about it be like?" ("it" refers to a certain mathematical idea at focus) or "How I can tell the same story?" when given a certain text. Each of the three forms of re-viewing can be considered as changing different dimensions of the same story. Thus, re-storying is similar to each of the three forms of re-viewing and to the whole of the three. At the same time, re-storying moves away from semiosis to narratives, in a sense, from a narrower focus on individual sign vehicles to a border view of mathematical texts. To signify such complicated relations and propel further thinking on them, I renewed the tentative visualization of re-viewing (see Figure 6.4.1) previously established. The new image (see Figure 7.3.1) presents re-storying as a form of re-viewing that transcends the other three forms and merges into the whole of re-viewing.

To deemphasize the eyes, consciousness, and logical analytical mode of knowing, to stress the whole body with kinesthetic, tacit, and analogical modes of knowing, and to

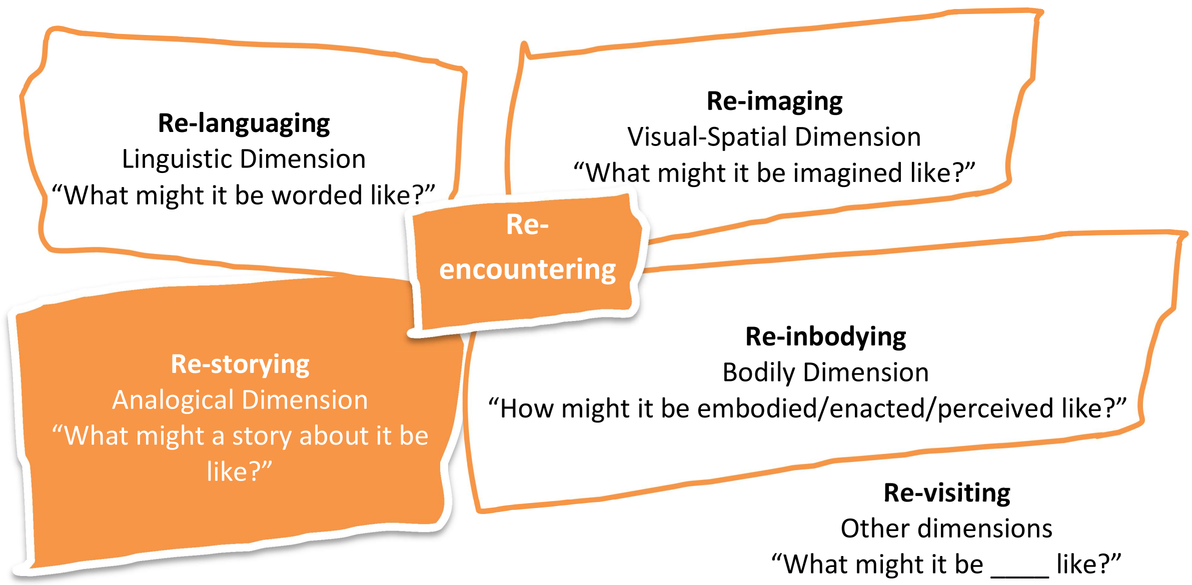


Figure 7.3.1. A renewed categorization of the forms of re-viewing.

acknowledge re-viewing as both planned and lived processes, I was compelled to change re-viewing to *re-encountering*. The word “encountering” was also chosen for its connotation of meeting something unexpectedly and informally, in order to have the theorization of re-encountering to lend support for designing a recursive curriculum that changes along its unfolding hence open for novelty and serendipity. My use of “encountering” here is somewhat in line with Badiou’s idea of encounter. An encounter with an event, as den Heyer (2009) interpreting Badiou, extends beyond a static state as the potential beginning of one’s “becoming-subject” (p. 453), a status of one being simultaneously oneself and in excess of oneself. Worded it differently, Badiou’s encounter is a prospective start of a re-engagement with familiar terms of understanding. In my theorization of recursive curriculum, it is the word “re-encountering”, which refers to the later experiences of encountering what was perceived or considered as familiar or known, that echoes Badiou’s notion of an “eventful” encounter. This is neither to exclude the possibility of first encounter in Badiou’s sense nor to deemphasize it in curriculum design. Rather this is to acknowledge the pervasiveness of the phenomenon of a learner encountering something she has previously encountered before and the lack of attention in orientating re-encountering towards transformation in theory and in practice. It is beyond the scope of this thesis to discuss Badiou's notion of encounter.

As I specified at the beginning of the methodology section, essentially, my interpretation of reviewing is an attempt to understand recursion. So now we can look back the concept of recursion to envision a recursive mathematics curriculum, with a renewed understanding of reviewing.

8 Re-encountering Recursion and Recursive Curriculum

Recursion centers in re-encountering. This does not mean that recursion is equal to re-encountering, but more so that re-encountering is a part similar to the whole of recursion. Thus, they reflect and inform each other. Re-encountering, as a renewed interpretation of the practice of re-viewing, is situated in both theory and practice. So a look at all previously discussed forms of re-encountering (including re-storying) as a whole can suggest to us some interesting qualities of both recursion and recursive curriculum. Albeit the following interpretation is divided into two parts, focusing on recursion and then recursive curriculum, these two parts are two perspectives of the same process. Therefore, it is important for one to read them as enriching and informing each other.

8.1 Recursion

At the beginning of the study, I established that recursion is a process of repetition of variations, and a process of re-interpreting and re-experiencing. Here, with the establishment of some forms of recursion, these interpretations take on new meanings and senses.

Recursion as playing with equivalency

Recursion as repetition with variations is reflected in all forms of re-encountering as learner working with equivalent ideas and texts. New understanding often emerges when a learner establishes or revises a sense of equivalency by seeing different objects as the same, seeing the same or equivalent objects differently, or generating something equivalent. Obviously, the idea of equivalency is central for re-encountering, hence recursion.

Equivalency or equivalence in general is defined as “The condition of being equivalent; equality of value, force, importance, significance, etc.” (“Equivalence”, 1989). It has some special meanings in different disciplines, for example, an equivalence relation in mathematics is "a relation between elements of a set that is reflexive, symmetric, and transitive" ("Equivalence", 1989), while equivalence means "Equality of energy or effect" in physics and the same ability to combine in chemical reactions in chemistry ("Equivalence", 1989). The idea of equivalency in re-encountering for mathematics learning, illustrated in the previous interpretation of its four forms, comprehends various kinds of equivalency; it is more *comprehensive*. For example, the different definitions of circles are equivalent in meaning; the different names of asymptote are equivalent in function; the different images of the division sign are equivalent in appearance and in both visual-spatial and mathematical structure; the paralleled actions for negative numbers subtraction are equivalent in embodiment; the story of linear functions and the story of quadratic functions are equivalent in origin (such that one can develop into the other) or logic class (such that one can be embedded in another or share the same type with the other). Putting these examples together, I see *equivalency in re-encountering is a condition of two things 1) being identical,* or *2) sharing certain quality, which can be physical (e.g., appearance, sense, impression), conceptual (e.g., meaning, interpretation), practical (e.g., function, process, value, effect), structural (e.g., pattern, structure, logic, type), historical (e.g., origin, development), and etc.*

Noting that the story of multiplication and the story of division were also viewed as equivalent in the re-storying section, there seems the third condition of equivalency in re-encountering: *Two things having a coupling relation (i.e., sharing a relational quality) can also be equivalent*. As mentioned earlier in section 2.1, the word "coupling" used in complexity thinking refers to Maturana and Varela's (1998) notion of structural coupling, which is a mutual adaptive and co-evolving relationship between two or more self-organizing systems (e.g., an organism and its environment). Two mathematics ideas can related to each other in a structural coupling relation when they are observed as complex systems through the lens of complexity thinking. This means that a change in one idea can occasion changes in the other. Here I use "coupling" in a more general way by using the literal meaning of "couple": "Two individuals (persons, animals, or things) of the same sort taken together; properly used of such as are paired or associated by some common function or relation" ("Couple", 1989). This notion of coupling subsumes structural coupling, as two things in a structural coupling relation can be considered together by an observer as the same sort of things, and it also opens other possibilities. In the context of equivalency in re-encountering, I see mathematical ideas that have a reflexive, opposite, reverse, inverse, symmetrical, or reciprocal relationship as good candidates for coupled ideas as these relations are common in mathematics and knowing one of the ideas can easily lead to knowing the other since they share the same knowledge base. For example, multiplication and division can be coupled as one can use the same grouping knowledge to understand multiplication and division, and one can get one from the other through reversion.

Fundamentally, *coupling relations are a kind of one-degree separation relation,* which I refer to the relation between two things that are one step away from each other and associated together by rule, norm or manipulation. This interpretation opens space for equivalency established through personal rules and social norms beyond mathematical conventions and through various kinds of (un)conscious manipulations (including intuitions, habits, resonances, and imaginations) beyond common mathematical operations and logics. Obviously, the more familiar the rule or norm connecting two things is and the easier the manipulation is for a learner, the stronger tie these two things have for her. Once a learner reaches to a level of automation such that seeing one from the other costs little effort in thinking, we can say that these two things are coupled for this particular learner. This seems to explain why the rational expression $\frac{y-x}{x-y}$ looks equivalent to -1 for me but not so for many of my students for a long time.

Apparently this seemingly third condition of equivalency overlaps with the second condition, as there are often meaningful, rather than arbitrary or manipulative, connections for things that one considers equivalent. Taken $\frac{y-x}{x-y}$ and -1 as example again, they are equivalent in numerical value. They can also be viewed as equivalent in appearance for a learner who has got used to -1 in the form of $\frac{-a}{a}$ or $\frac{5-3}{3-5}$. Nevertheless, it is still helpful to recognize this third interpretation of equivalency. This allows re-encountering to be triggered by questioning how any two things that one has learned to associate together might be equivalent in other dimensions (e.g., the conceptual). Bringing conditions two and three together, we can see that *some equivalency is established through partial similarity or correspondence*. In other words, drawing on the interpretation of analogy as "correspondence or resemblance between things" ("Analogy", 2010), we can say that some equivalency is established through analogy.

Obviously, *equivalency is a contingent relative observation*. Whether two mathematical ideas or texts are equivalent is subjective to the learner's current understanding and learning situation. A statement of equivalency is always made in referencing to certain conditions, e.g., a linear function can be viewed as equivalent to a quadratic function when one considers them in the context of polynomial functions or in relation to rational functions. As a learner improves mathematical understanding and broadens her view of mathematics over time, her sense of equivalency changes too. More and more mathematical ideas would be viewed as associated, coupled and eventually equivalent in meaningful dimensions. This can happen even for some mathematical ideas that are initially considered as disconnected (e.g., division and limit) or multiple steps away (e.g., $(a+b)^2$ and $a^2+2ab+b^2$). This identification of equivalency helps a learner to collapse many ideas into one, encouraging her to see a broader mathematics territory with a greater simplicity. The shifting from one kind of equivalency to another (e.g., functionally equivalent → conceptually equivalent) and the establishment of equivalency in multiple dimensions definitely help the learner to deepen her understanding.

This contingency of equivalency demands a *heuristic* use of equivalency in a re-encountering process: Learners shall be encouraged to use a tentative interpretation of equivalency to further thinking, while remaining open for other possible interpretations to emerge. There needs to be a hermeneutic and experimental attitude towards equivalency in re-encountering. Contingent equivalent sign vehicles might be multiple representations (e.g., 5, 5 objects) of the same idea (e.g., 5) or multiple representations (e.g., 5, 6 objects) corresponding to equivalent ideas respectively (e.g.,5, 6, they are equivalent when one sees them both as integers). Essentially recursion is a *play* with equivalency.

Recursion as thinking with the mediums of thought and living through the process of re-encountering

Recursion facilitates new thought about an idea looping back to previous thought, hence affording new interpretations. However, besides one's previous thought, recursion also allows one to work with the medium (i.e., a signifier/representation and process) of thought to afford re-interpreting and re-experiencing. Different forms of re-encountering engage different types of mediums: language, image, body, and story, or worded differently, linguistic, visual-spatial, physical, and narrative representation, and the process of expressing, representing, and working with the medium itself. Each type of mediums enables the learner to express and/or engage with ideas in a certain way. So mixing different types of media in a re-encountering process can help one to interpret and experience the same idea differently. However, re-interpreting and re-experiencing can happen without changing the type of medium too.

For one, re-interpreting and re-experiencing can happen through the use of equivalency in re-encountering. Such use allows the co-existence of sameness and difference. Working with equivalency requires a learner to interpret and experience the same idea differently yet negotiate the differences to maintain a sense of coherence. This demands the learner to differentiate but also synthetize and generalize, in other words, both diverge and converge. As illustrated in different forms of re-encountering, this process of diverging and converging helps to bring forth new interpretation(s) of the same idea.

Re-interpreting and re-experiencing can happen, also, when the medium is enlisted to help to shape thinking rather than being used as a transparent vehicle of thought. An expression or representation of a thought is treated different from the thought in re-encountering; rather it is a process affected by the medium of the thought. In each form of re-encountering, it is acknowledged that a medium has certain affordances and limitations for thinking: A medium affects what and how a learner perceives, feels and senses, and can induce the learner to think in certain ways. Thus a message cannot be transmitted but interpreted, and a medium can be used to activate a learner’s unconscious or tacit knowing, engage a different mode of doing, and invoke a different way of thinking. In other words, how an idea is presented and expressed affects thought. Moreover, in a re-encountering process, the medium of a certain message that one works with is turned into a message (such as a name, an image, a body action, a story, or the experience of working itself become a self-referential pointer for meaning), inviting alternative interpretations of the initial message. This change from medium to message is inseparable with the use of equivalency in a re-encountering process, which subjects a learner to the influences of two different mediums (including the same medium encountered at a different time or different mediums encountered at the same time) the same idea, affords her opportunities to perceive difference needed to trigger her to re-interpret the idea. Such difference is what Bateson (1979/2002) calls the difference that makes difference: It exists in comparison, as “it takes at least two somethings to create a difference” (p. 64), and it is needed to trigger a mind to work, as “*the interaction between parts of mind is triggered by difference*” (p. 85). The above contemplation on a medium’s role in re-viewing is echoing McLuhan’s (1964) theory “the medium is the message”, in which he argues that “it is the medium that shapes and controls the scale and form of human association and action” (p. xxx). Yet this connection is beyond the scope of this study.

Nevertheless, this acknowledgement of medium shaping thought, combined with the assumption that mathematics understanding is an infinite process, has its significant implication: It makes an expression of thought neither necessary nor possible to be final and complete. Rather, expressions (both the outcomes and processes) are tentative heuristic tools for thinking in a re-encountering process. Hence the process of thinking *with* different mediums of thought, being affected by them, and learning from one’s experience, becomes more important than getting a particular result of thinking. In other words, *currere* (as discussed in section 3.1), the running of the course and reflecting on the running, is more significant. Once again, we are reminded that there are no shortcuts for cognition development: Cognition construction is noncompressible (Davis & Sumara, 2006, p. 43); “the structure emerges or the path that unfolds has to be lived through for its endpoint to be realized” (Davis & Sumara, 2000, p. 841). A living through the re-encountering process is needed for cognition development to be realized. As to experience is “to meet with; to feel, suffer, undergo” (“Experience”, 1989), through encouraging a learner to experiment with different expressions of the same idea in order to learn from how they influence her differently, and through demanding the learner to live through the whole process, re-encountering becomes a re-experiencing process.

One must note that it is the interest in this experiencing the difference between multiple expressions and mediums that separate recursion from a process of changing forms of communication to cater to individual differences, even though it would be a byproduct of the process. As Sfard (2010) conceptualizes,

> Learning mathematics means changing forms of communication. The change may occur in any of the characteristics with the help of which one can tell one discourse from another: words and their use, verbal mediators and the ways they are operated upon, routine ways of doing things, and the narratives that are being constructed and labelled as "true" or "correct." (p. 217)

Learning throug*h change* is exactly what recursion encourages. There is no interest to label a form as good or bad absolutely or settling down with any of the forms in a recursive process. The harmony or tension that arises from the comparison of forms is the source of insights in recursion. The space for growth is *in between*.

Recursion as unclogging and playing with time

Recursion can bring forth something new for a learner. Sometimes this is not to do with something brand-new but something that the learner has known before yet somehow forgotten, or as Davis and Renert (2014) phrase, something that the learner does not have immediate access to. There are many examples in this study. In my story of re-languaging the circle definition, I re-cognized the relation between graph and points and how this relation can be implemented in solving various types of analytical geometry problems. Given how frequently I was taught to understand why a formula works in Chinese education system, this connection must have been made long ago during my high school years. In my story of renaming asymptote, I re-cognized the meaning of its Chinese name. I must have come to this realization before when I was learning asymptotes in high school. But after years of automatic use of knowledge of asymptotes, I had forgotten the name's significance and just used it as a label. In Maxine's story of seeing the bracket sign in polynomial tile divisions, it is also reasonable to believe that Maxine had seen it before due to her familiarity with the topic.

However, this seeing something one has seen before yet failed to remember is different from recollecting a lost memory. Exemplified in my re-enacting $-6-2=-8$ experience, there is more to this process. In that incident, I did not realize that I have conflated multiple meanings of the same "$-$" sign together until I had been compelled to enact $-6-2=-8$ on the number line using actions parallel with the ones executed on the negative tokens. This remembering that a "$-$" sign has multiple meanings allowed me to understand where my students' difficulty with negative numbers subtractions may lay and see what I used to see but could no longer see. Moreover, it urged me to reconstruct my concept of "$-$" sign and number as a whole at a new comprehensive level. But most significantly, it made me aware of my own recursive blindness, a phenomenon that Tom Kieren (Feb 4, 2016, personal communication) explained, while citing Maturana, as that experienced mathematics learners are not aware of what they used to know or experience. Such blindness blocked me from seeing what my students see and how I might have struggled like my students before. I had observed, as Freudenthal (1983) did,

> that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question. (p. 469)

Automation brings recursive blindness. It also makes experienced learners unable to notice what they unconsciously notice or pay attention to. I remember how strangely I felt when I suddenly *saw*, literally, quadratic trigonometry equations as quadratic equations after I had taught them for a few semesters. “Aha, this is what my students need to see too”, I thought to myself. Albeit delighted for seeing this new image, I could not believe that I had never seen something this obvious before. Now I think I was blocked by automation to see the pattern image that I can unconsciously notice.

A re-encountering process can help to undo this automation. Re-encountering something that a learner is familiar with affords her the kind of *unclogging* exercises that proposed by Arcavi and Nachmias (1989), “to direct our attention to reviewing the ‘source of our insights’ which somewhere in the past were incorporated to our mathematical background as automatisms” (p. 81). By inviting us to reencounter what we have encountered before and by engaging our senses and thoughts, a re-encountering process demands us to trouble what we have taken for granted. This can make our unconscious sensing and knowing conscious for us, thus undoing the automation process. This knowing what we know unconsciously not only enhance our content understanding but also provide new insights for pedagogical considerations. This makes re-encountering helpful for both subject learning and metacognition learning.

Thinking of recursion as unclogging reflects, once again, the underlying assumption of mathematical understanding as an infinite process. One can always understand a mathematics idea differently. Therefore, recursion is a process for learners at all mathematical fluency levels.

Thinking of recursion as unclogging, also, reminds me of the *merit* of forgetting. As I have understood, from my reading of the spacing effect (as explained in section 5.1), a certain degree of forgetting is needed to afford an *effortful* and *successful* reconstruction of previously learned knowledge. One's forgetting what one has learned or struggled before due to automation over time actually makes reconstruction of understanding inevitable and allows it to be significant. Exactly as Gadamer (1989/2013) writes,

> only by forgetting does the mind have the possibility of total renewal, the capacity to see everything with fresh eyes, so that what is long familiar fuses with the new into a many leveled unity. "Keeping in mind" is ambiguous. (p. 15)

Forgetting is necessary for one to learn things anew. *Recursion needs forgetting*. Time comes into play in recursion: It takes time to forget; it also takes time for difference between thoughts to arise thus making the old new again. *Recursion works with,* not against*, time.*

8.2 Recursive Curriculum

In this section, we move into a planned curriculum space and think about designing towards a recursive curriculum. Here, I propose two perspectives that are worth stressing in curriculum design if we were to promote recursion in mathematics curricula. One is to think curriculum as fractal-like and the other is to view it as a biological structure always in the process of becoming.

Recursive curriculum as fractal-like and as a continuous interplay of part and whole

It has been emphasized repeatedly that one works with equivalent things in recursion. Comparison is inevitably essential in a recursive curriculum. Comparison refers to "the action, or an act, of comparing, likening, or representing as similar" and also "a simile in writing or speaking" ("Comparison", 1989) as the likening of things of different sorts based on certain levels of likeness, similarity or resemblance, such as a teacher and a gardener. Thus it makes sense that comparison is alternatively interpreted as "an analogy or the quality of being similar or equivalent" ("Comparison", 2019). This implies that the things compared are somewhat considered as similar yet different, or in our words, equivalent.

Comparison, often coming in the form of parallel display of two texts side by side, seems to have been used widely in planned mathematics curricula. For example, each lesson worked example in the Alberta Math10 textbook is placed at the left side of a similar looking exercise question. In the Chinese Math9 textbook, many lesson examples and exercise questions appear in pairs: Two similar worked examples or exercise questions are often displayed side by side, and sometimes a worked example is placed on top of an exercise question on the same page. This popularity of comparison in textbooks does not suggest that there is no need for more study of comparison. Actually the study on comparison in mathematics education continues to show room for improvement in our understanding about the topic. For instance, Star et al. (2015) study the effect of an Algebra I supplemental comparison curriculum on students' mathematical knowledge. In this trial curriculum, students need to compare worked solutions of algebra problems. Such comparison appears less frequently in regular textbooks. The study shows that "greater use of the supplemental curriculum was associated with greater procedural student knowledge" (p. 41). It also suggests to me that more study on comparison is necessary. Given the significance of comparison for a recursive curriculum, the popularity of comparison in planned curricula actually demands us not to take comparison for granted but to trouble its form: In what ways might we design comparison in order to promote recursion?

To answer this question, let's follow Bateson's (1979/2002) "comparative study of the comparative method", which compares instances of comparison (p. 81). Figure 8.2.1 shows a comparison of two comparison examples. To focus on the form of comparison, only information related to the content and layout of the comparison (i.e., what the two items for comparison are and how they are displayed) is presented. For the purpose of comparison across textbooks, the

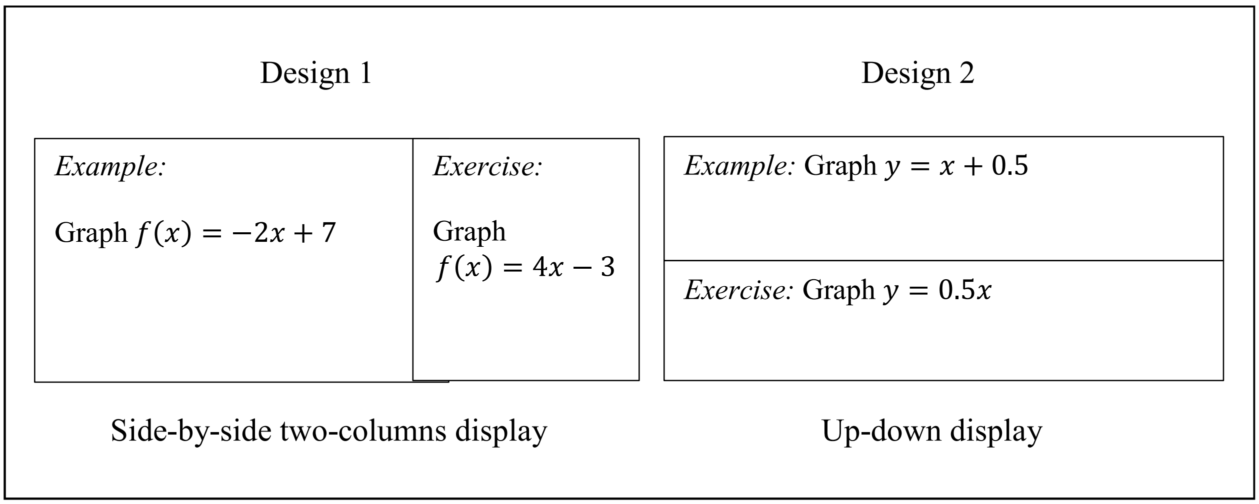


Figure 8.2.1. Two comparison design examples. Design 1 is summarized based on the example in the Alberta Math10 textbook, p. 315. Design 2 is based on the example in the Chinese Math9 textbook, p. 91

wordings of the questions cited are simplified. For example, The example in Design 1 "Graph $f(x) = -2x + 7$" was changed from "Sketch a graph of the linear function $f(x) = -2x + 7$". The example in Design 2 "Graph $y = x + 0.5$" was changed from "画出函数$y = x + 0.5$的图像 (literally translated as: sketch a graph of the function $y = x + 0.5$)" (see Appendix D for more details of the original texts).

These two examples are both the first[27] graphing a linear function example. They are also comparable at both question and solution (i.e., find at least two points on the graph then connect them) level. In both designs, a worked example and a similar-looking exercise question are

[27] The Chinese Math9 textbook has a lengthy introduction, which includes explaining how to graph$y = x$, before this first example.

displayed close by. Albeit neither design deliberately instructs the learners to compare the example and the exercise, their visual and spatial proximity is likely sufficient enough to induce comparison.

Both designs include two equivalent problems. But the pair in Design 2, compared with the pair in Design 1, looks more alike: While two numbers changed in Design 1, a number was moved and a symbol was removed in Design 2. In comparison with this close resemblance, the mathematical difference between the two in the Design 2 could become relatively more significant. In this context, this pair differs *subtly* yet *salientl*y (see Luo, 2015 for more discussions on subtle yet salient differences). This sharper contrast between the similarity in appearance and the difference in mathematical behaviors is helpful for getting a learner's attention: Seeing how a subtle change in appearance can bring forth such a dramatic behavior change can be both surprising and liberating. It draws a learner's attention to the mathematical structure of both objects and urges her to learn to discern determining factors from trivial ones. When the two objects resemble each other, seeing one can remind the other thus making learning about one is like reviewing the other.

More importantly, this addition of one thing closely resembling the previous one yet dramatically different affords one to both *differentiate* and *connect* them. This helps to bring forth abstraction and generalization. When there is a chance to interpret two things as separate categories making up a new category together, moving away from one to another is no more trivial but empowering and transformative because *learning about one is no longer about one specific case but a class of cases*. There is a change in logic type, producing embedded categories. For example, the two functions compared in Design 2 can represent two separate categories (i.e., linear functions with a non-zero y-intercept and linear functions with a zero y-intercept) and together they can make up a higher level category (i.e., linear functions). Of course, whether two things in comparison closely resemble each other is a *contingent* observation, and it is relative to what the learner has encountered before and what she encounters later. Therefore, the learner might not think much about categories until much later with either comparison design.

The issue now becomes how to trigger a learner to compare what she encounters at present *more frequently* with what she has encountered in the past. This takes us out of the small scale. Let's look at a sequence of four comparison examples extracted from four sections in the Chinese Math9 textbook in Figure 8.2.2. Again, similar to the questions in Figure 8.2.1, the

Comparison 1	Comparison 2	Comparison 3	Comparison 4
Example: Graph $y = x + 0.5$ *Exercise:* Graph $y = 0.5x$, each in a different Cartesian plane.	*Example:* Graph two linear functions ($y = 0.5x$, $y = -0.5x$) each in a different Cartesian plane.	*Example:* Graph two linear functions ($y = 2x + 1$, $y = -2x + 1$), in the same Cartesian plane.	*Exercise:* Graph four linear functions ($y = 3x$, $y = -3x$, $y = 3x + 3$, $y = -3x + 3$), in the same Cartesian plane.
from Lesson 13.3, p. 91	from Lesson 13.5, p. 99	from Lesson 13.5, p. 100	from Lesson 13.5, p. 101

Figure 8.2.2. Four equivalent comparisons.

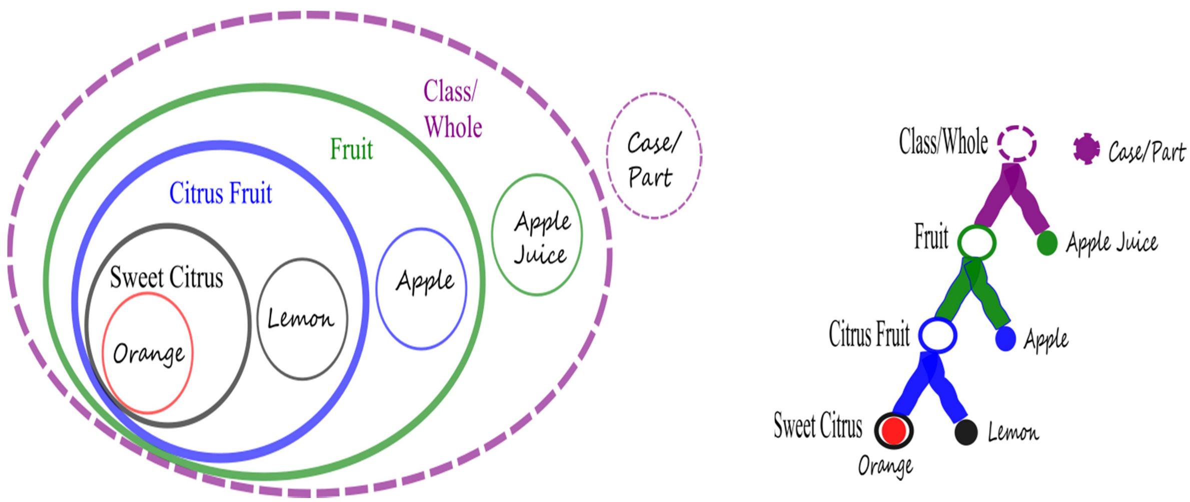
questions here (see Appendix D for more details) were translated from Chinese and worded in a simplified format to ensure comparison across samples.

These comparisons are from two lessons that are one lesson apart. Looking at them together, it is not hard to see that they consistently use the same repetition with subtle yet salient changes to design each comparison and comparisons over time. Comparison 2 starts with a previously learned function (i.e., $y = 0.5x$) and compares it with a new yet similar function (i.e., $y = -0.5x$). Comparison 3 can be viewed as a modification of Comparison 2 or it includes two modified versions of any of the two functions in Comparison 1. Comparison 4 is a modification of the previous two comparisons. This design allows a learner to be consistently reminded of her past at *multiple* levels: *Each new addition of function or comparison is a resemblance of its immediate past or the past as a whole*. It is also different enough to urge one to think about category as *each new addition can bring forth new whole(s)*. For example, when the learner encounters a comparison between $y = 0.5x$ and $y = -0.5x$, she might (un)consciously view them as different cases rather than classes. But when she moves to a comparison between $y = 2x + 1$ and $y = -2x + 1$, many classes might emerge (e.g., linear functions with zero y-intercept and non-zero y-intercept, linear functions with positive and negative slope).

An *analogy* (see Figure 8.2.3a) of this design of using multiple levels of similarity to trigger comparisons in both part and whole is helpful here. Starting from showing you an orange, if I put a lemon besides it, you can tell that they are different. By the time I add an apple, you might notice that orange and lemon are more alike than you thought before. Now if I add a bottle of apple juice, you would most likely see the three fruits the same. *Each addition brings in multiple new wholes, not just a part.* This design of *each new addition of curriculum resembling and differing from the past at both part and whole level* affords a recursive curriculum: Each new addition invites re-encountering the previously established whole. Such curriculum is self-similar

– a part at any scale represents the whole. It can be represented differently as a binary fractal tree (see Figure 8.2.3b), displayed in comparison with the analogy image.

When the same design is used to organize content at various scales, the resulted curriculum text exhibits the same pattern across scales. Such is observed in the three chapters about function in the Chinese Math9 textbook. Each new chapter introduces a new kind of



a. An analogy of the curriculum design of using multiple levels of similarity.

b. A binary tree.

Figure 8.2.3. A fractal-like curriculum design.

function using a similar plot design (i.e., each kind of function goes through a similar sequence of transformations: vertical stretch/compression/reflection → vertical shift) and has task organization in consistent with the binary tree design. Hence, each chapter is a new part that reminds one of the previous whole yet differs from it. A representation of the outline of the curriculum also shows a *fractal-like* structure (see Figure 8.2.4). Such fractal-like structure, with its abundant use of comparison and self-similar parts, supports and promotes a constant part-whole interplay.

Function –> linear function –> quadratic function –> inverse function –> Function

Linear function –> $y = 0.5x$, $y = -0.5x$ –> $y = kx$ –> $y = 2x + 1$, $y = -2x + 1$ –> $y = kx + b$

Quadratic function –> $y = \frac{1}{2}x^2$, $y = 2x^2$; $y = -x^2$, $y = x^2$ –> $y = ax^2$ –> $y = x^2 + 1$, $y = x^2 - 1$; $y = -\frac{1}{2}(x+1)^2$, $y = -\frac{1}{2}(x-1)^2$; $y = -\frac{1}{2}x^2$, $y = -\frac{1}{2}x^2 - 1$, $y = -\frac{1}{2}(x+1)^2 - 1$ –> $y = a(x-h)^2 + k$ –> Quadratic function

Inverse function –> $y = \frac{6}{x}$, $y = -\frac{6}{x}$ –> $y = \frac{k}{x}$ –> Inverse function

Figure 8.2.4. A visualization of three chapters about function.

Given that a fractal-like structure entails the same pattern observed across scales, one can learn to design a fractal-like mathematics curriculum by drawing inspirations from mathematics education literature in curriculum elements (e.g., task, example, event, process) design, such as the ones emphasizing variations (e.g., Marton & Booth, 1997; Watson & Mason, 2005, 2006), multiplicity of representations (e.g., Dienes & Golding, 1971; Radford et al., 2007), seeing the general in particular (e.g., Mason & Pimm, 1984), stressing and ignoring (e.g., Gattegno, 1970), educating attention and awareness (e.g., Gattegno, 1970, 1988; Hewitt, 1994, 1999), reflection (e.g., Freudenthal, 1991), and so on. Of course, changing one single curriculum element without changing the elements across scales is insufficient to make a recursive curriculum possible, as a recursive curriculum has holistic considerations embodied in parts. However, a thoughtful planned curriculum is insufficient either; it is even a harmful goal to strive for if it does not encourage self-transformation during its enactment. As one will read in the next section and later chapters, a more radical change in one's ways of seeing mathematics, curriculum, and one's relationship with them is critical.

Recursive curriculum as process-oriented and as a biological structure in the state of becoming

It has been established, through the concept study on recursion at the beginning of this inquiry, that a recursive curriculum is *currere*-oriented. This was interpreted, earlier in section 3.1, as that a recursive curriculum encourages learners to experience and to reflect upon their experience and it emphasizes process in education. My running a course of interpreting recursive curriculum has transformed this understanding. The following is an important experience that afforded the transformation.

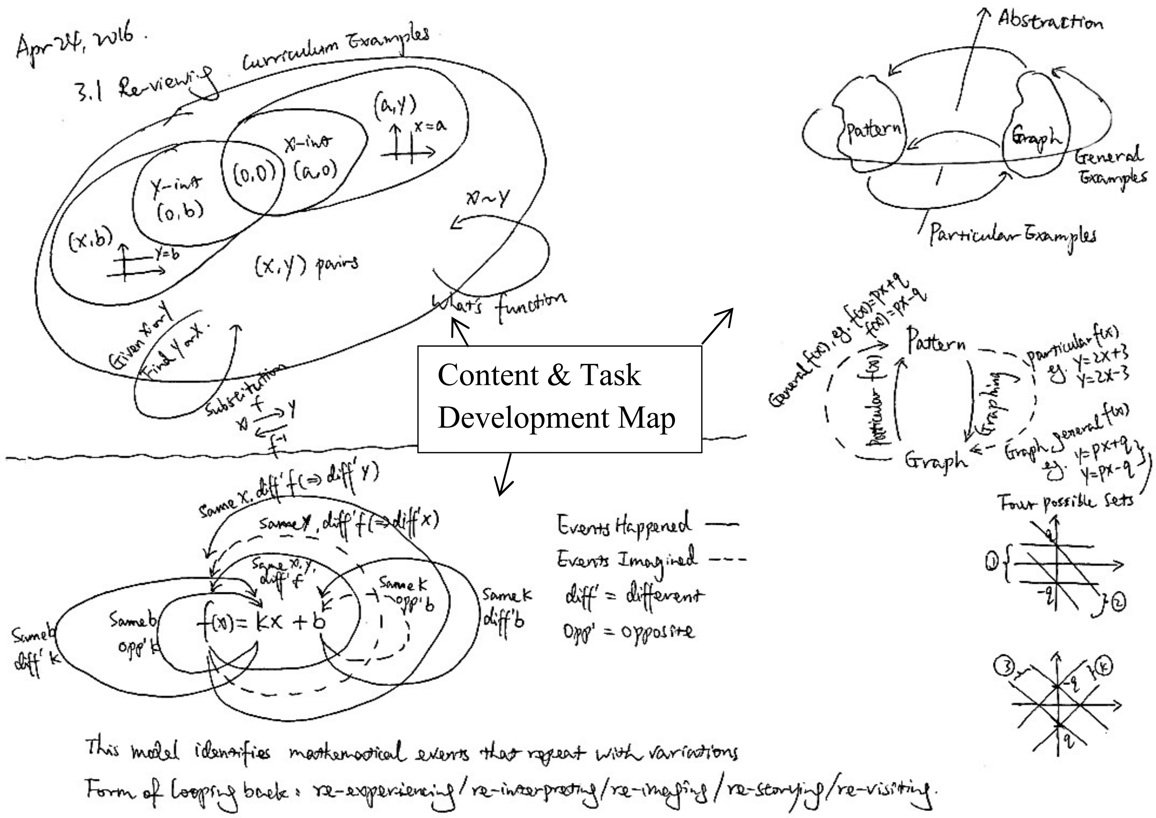


Figure 8.2.5. Curriculum development maps (set 1).

Mapping. Before my conversation with a participant about her words given in Workshop 3.1 (Re-viewing), I needed to summarize the workshop. I had interpreted the workshop before and found the curriculum tasks, which the three participants and I generated together in order to help students to learn something new from reviewing what they have encountered before, seem scattered. This time, thinking re-viewing as re-storying, I interpreted these tasks with a focus on storyline (i.e., the development of mathematical events): I categorized mathematics ideas and mathematics tasks, and analyzed how ideas are related and how one mathematics task developed into another. Unexpectedly, three content and task development maps (see Figure 8.2.5) emerged along with the process. From there, I could see that these seemingly incoherent tasks can be generated through following the two task development maps and shifting attention in the content map, along with many others that we did not mention. Not only did these three maps offer me novel views of linear relations, but also bring forth a recursive curriculum development model that participants and I had been using partially without conscious theorizing and that is helpful for future curriculum planning.

Days later, I used the same method to interpret Workshop 3.2 (Re-interpreting) again. I made another set of recursive task development maps (see Figure 8.2.6), by following which one could regenerate the diverse word problems proposed in the workshop to

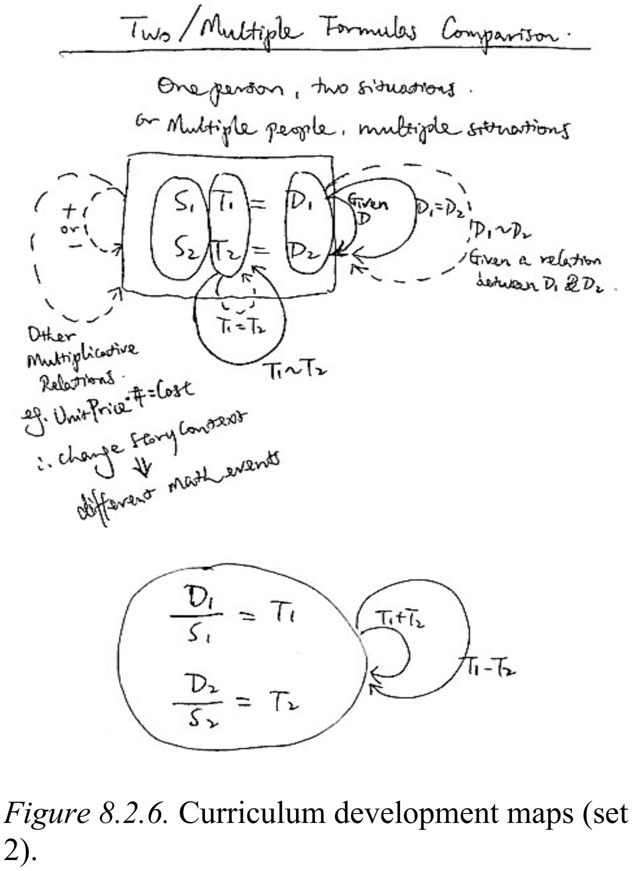


Figure 8.2.6. Curriculum development maps (set 2).

> encourage students to reinterpret what they have interpreted before and learn something new and also generate more than we did. This map allowed me to see diverse speed-time-distance problems as a whole in a novel way. I further noticed that it can be extended to subsume any system of three elements relations that fulfill $a \sim b = c$ (where ~ represents an operator). Once again, I realized that I had benefited mathematically and pedagogically from a process of mapping the territory that I have gone through, and a back-and-forth movement between map and territory seems important for a recursive curriculum.

The above story elaborates a process of re-viewing during which new mathematical and curricular understanding emerged. This process, as both re-imaging and re-storying, is also a mapping process. As Bateson (1979/2002) says, the map is not the territory. Map and territory differ in the level of abstraction. With only the structure of the territory in focus while ignoring many details, a map offers a holistic view of the territory and emphasizes how the items on the map relate to each other. Mapping, as zooming out on a territory, is essentially an abstraction process of a system of (spatial, temporal, procedural, conceptual) relations, thus significant for any mathematics curricula that intend to promote relation and mathematisation. Interestingly, given how mathematics is full of embedded ideas – so many mathematics ideas are related in a self-similar way, one can always remap a map at a more or less abstract level and in relation to a broader or narrower territory. So mapping in mathematics can be both zooming *out* and zooming *in*. While zooming out is abstraction and seeing bigger patterns in a wider territory, zooming in is a return to concrete examples or regenerate the finer details of a narrower territory. Both zooming in and zooming out are necessary for mathematical development. Although my story seems to focus mostly on the zooming out process, I ended up generating a kind of map that also depicts the details one can see when one zooms in.

Such a map is actually a set of simple rules for generating fractal-like territory. All the content and task development maps show how a set of knowledge contents and curriculum tasks are variations of the same thing. The maps are made of a common core (i.e., the shared mathematical object) with self-reflexive lines labeled with a particular change, showing the development of content and task as reviewing what one has encountered before. Compared to the forms used frequently in mathematics textbooks, e.g., the list in the table of content or the scattered idea bubbles in a review summary, such visualization of a curriculum is *nonlinear* and *dynamic* and it focuses on mathematical events rather than static mathematical objects. It signifies that both mathematics knowledge content and curriculum develop in a recursive path and they are fractal-like, and suggests us to view a recursive curriculum as a *source* of spring water and as *process-oriented*.

The above interpretation of my mapping story, like those of many other experiences that I encountered during this study, helps me to *re-inbody* the idea of recursive curriculum as *currere*-oriented. Yes, this idea points to the importance of experience and process. But it refers more than personal experience and learning process; it also refers to *mathematical* process. Both mathematical learners (again, leaners refer to teachers and students) and mathematical knowledge are complex systems - which Bateson (1979/2002) calls minds - with a biological structure that Davis and Sumara (2002) refer as "emergence" and "is always in process" (p. 412). The system of mathematical knowledge emerges and evolves like any human culture, and all human mathematical activities contribute to its evolution. Albeit the mathematical contents included in public education can remain unchanged for decades, a learner's mathematics understanding of the same content can change over time and, more or less, affect the whole body of mathematical knowledge in return. My study has shown no matter how familiar one is with a particular mathematics idea one can always interpret it differently and this change is not simply additive but often transformative. This possibility of transformation makes a learner's or a learning community's mathematical activities at the public school level no different from a mathematician's[28]: Observed from the perspective of complexity thinking, mathematical ideas that are novel and transformative for the discipline can emerge in a mathematical community at any level. This potential for new possibilities for both mathematics learners' understanding and the field of mathematics is what Doll (1999) means by saying that *a state of becoming exists in ourselves, also in school subjects*. Doll writes,

> Becoming is being that moves beyond being, away from the centered state of equilibrium to the exciting, dynamic, and perilous state of far-from-equilibrium. This state opens each of us up to the potential that exists: within life, ourselves, and the creative spirit which infuses the universe. Such a state also exists within the primordial nature of the school subjects we teach. There is an aliveness to both these subjects and ourselves (as creative creatures) if we are but willing to explore this realm far-from-equilibrium and near the edge of chaos. (p. 43)

A recursive curriculum, as a curriculum interpreted through the perspective of complexity thinking, acknowledges and supports this state of becoming in both the learner and the subject matter. It, *inevitably*, also has a biological structure always *emerging*. A curriculum, observed through the lens of complexity thinking, is not a thing but a process: It is "a learning organization"

[28] This is different from Bruner's (1962) seeing a mathematics learner as a mathematician. Bruner (1962) says "intellectual activity anywhere is the same, whether at the frontier of knowledge or in a third-grade classroom….The difference [between these activities in different contexts] is in degree, not in kind" (p. 14). He advocates having learners of a subject to act like an intellect working in that particular field, and rather than centering upon mastering a middle language, which means "classroom discussions and textbooks that talk about the conclusions in a field of intellectual inquiry", their activities should center upon the inquiry itself (p. 14). I agree with Bruner's opinions and I go one step further by invoking complexity thinking: While stressing that the intellectual activities at different academic levels are the same for that they are the same *kind* of *inquiry*, I emphasize that they are the same for they both affording transformations of a particular field of knowledge and even knowledge as a whole.

that adapts and evolves (Fleener, 2002, pp. 174-175); it is “an ongoing conversation” emerging through the interactions of teachers, students and their developing mathematical understanding (Reeder, 2002, pp. 251-253). Such curriculum cannot have a predetermined fixed ending point; what a learner and mathematics can evolve into can only be known through living through their development process. In this case, *how the path that a learner took to engage with mathematics unfolded actually matters for what the future will become*. Therefore, a learner’s idiosyncratic learning and mathematical processes are no more luxury decorations or add-ons for mathematics classrooms, ones that only get promoted when there is extra time beyond a planned curriculum. Instead, they are the essential processes that *sustain the state of becoming* in the learner and in mathematics; they generate an uncharted personal territory and a novel mathematical territory as well.

Understanding recursive curriculum as being fractal-like, process-oriented and always in the process of becoming compels all learners to focus on both educational and mathematical process and to give *time* for self, others, mathematics, and also mathematics curriculum to *unfold* a recursive path. It requires learners to only use any established fractal-like curricula, such as the ones interpreted in this study, to think *with* rather than think towards. *A planned curriculum, no matter how thoroughly designed, is always a map, not the territory*. It is significant for learners to continuously learn *both* mathematics and recursive curriculum through designing towards recursive curricula.

9 Design towards Recursive Curricula

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour
—William Blake, *Auguries of Innocence*, 1863

What could be possible implications of this study? This study has renewed the interpretations and forms of reviewing and resituated it in learning as central instead of peripheral or remedial. Meanwhile, it re-conceptualized recursion and recursive curricula. These could have implications for curriculum design, teaching, and learning. Although my study is situated in mathematics curricula, its focus on recursion makes it a good reference for curricula of other disciplines wherever the subject knowledge is also viewed as fractal-like phenomena. Therefore, the contemplation of possible implications in this chapter can be read for inspirations across subjects.

The following discussion is built as recursive elaborations on Davis, Sumara, and Luce-Kapler's (2008) work about recursive curriculum, with a particular interest in generating a more *iconic* fractal-like image that can resemble a recursive curriculum both conceptually and physically (i.e. representing both its structure and finer details) for its theoretical and practical affordances. To keep a hermeneutic and complexified sensitivity alive, it is better for one to read on while considering the images discussed and proposed in this chapter as *playful emerging analogies* to provoke and inspire, instead of rigid completed models to prescribe and instruct. Given that recursive curricula can only be prepared for rather than prescribed, curriculum design is used as thought experiments rather than predetermining a fixed curriculum. It is intended to emphasize certain ways to *cultivate* and *enact one's sensibilities* towards recursive *curricula* in order to make them possible while one is learning or teaching.

9.1 Re-encountering Davis et al.'s Recursive Curriculum Design Process

A recursive curriculum, like any other curricula, is made of a sequence of elements, such as topics, events, tasks, activities, questions, exercises, and so on. Yet one important difference is that the elements in a recursive curriculum are related to each other as recursive elaborations. To design such a sequence, three considerations are inevitable: where to start, what to elaborate on and how to elaborate or activate elaboration.

As discussed in section 3.3, Davis et al. (2008) propose a fractal-tree growing (see an example in Figure 9.1.1) as a mnemonic device to help one to think about recursive curriculum design: start from a seed, grow it into a product, then keep on growing by rounds of elaborating on the previously established product (p. 201). While Davis et al. suggest using an enabling constraint as a seed and define the enabling constraint as a question or task that is equipped with sufficient structure and openness 1) to limit possibilities and allow novelties and 2) to support both individual and collective learning, they offer little advice for elaborations, other than two intriguing curriculum examples and some general considerations for elaboration in a collective learning context.

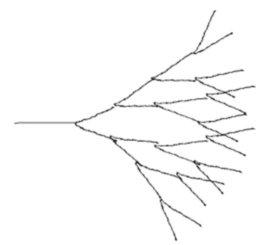


Figure 9.1.1. A fractal tree

The two examples are in English and mathematics. The poem writing activity starts with generating a story character based on a button chosen (the seed can be interpreted as "Pick a button, and then imagine the garment it came from, who the wearer of the garment is and what she or he has done"), processes through multiple rounds of changing the story with added restrictions (i.e., having two buttons meet, given a photo for the story setting), and ends with a poem writing exercise following a study of several poem examples. The multiplication of integers example starts with a prompt of "What's $^{+}3 \times {}^{-}4$? Show how you know" (p. 197) then activates repeated comparisons of equivalent products across nested learning systems: Learners first work in a small group to create a poster to show their interpretation, and then they group their posters into clusters based on common interpretations, following with a whole-class discussion to look for themes and variations.

These examples are to show elaboration as a means to represent and knit the collective knowledge that is "needed to deal with a new topic" and "usually present in the classroom collective, although not necessarily within a single person" (Davis et al., 2008, p. 202). The activities enable multiple levels of learners, i.e., nested learning systems, such as "individuals, dyads, small groups, clusters of groups, and the whole class", which are "mutually supportive and intelligent, unfolding from and enfolded in one another" (Davis et al., 2008, p. 202), to have access to each other's interpretation and elaborate knowledge at multiple systems' levels together. Given that the learning products of nested learners are similar to each other, when we visualize Davis et al.'s growing a fractal tree process as Figure 9.1.2, the fractal tree represents the

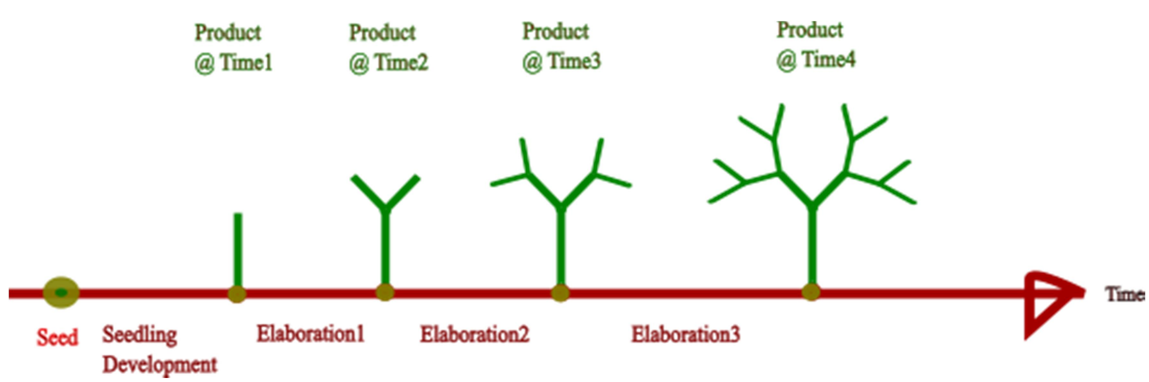


Figure 9.1.2. A visualization of growing a fractal tree.

collective learning product at different times (or phases) and also the learning product of any of the nested learners or learning systems through having parts that are similar to the whole.

Situating my study in relation to Davis et al.'s, it is clear that their work offers provocative recursive curriculum design advice for collective learning settings, whereas my work offers finer details of recursive elaborations and broadens the possible starting points of a recursive curriculum for both individual and collective learning settings. My study has theorized elaborative processes as re-encountering and proposed its four possible forms, i.e., re-languaging, re-imaging, re-inbodying, and re-storying, that can be used during individual or collective learning, at micro and macro content levels. With these forms of elaborations, what can be elaborated is a contingent and relative judgement as making an equivalent form of something is always possible. Worded differently, a*nything that can be viewed by a learner as a whole can be elaborated, or equivalently yet more provocatively, we make a whole by invoking an elaboration*. Therefore, we can start to design a recursive curriculum, by using a curriculum text or content that is of any scale (e.g., a notation, an image, a gesture, a task, a problem, a topic, a lesson, a unit, and so on) and is accessible for a learner's present level, and by using prompts to trigger forms of the re-encountering process. In other words, a recursive curriculum can *start anywhere*. At the same time, given that an elaboration as re-encountering involves changing something to its equivalence that is contingent, there are endless possibilities for one product to develop into. Therefore, a recursive curriculum can proceed in many different paths with different tentative endings. In other words, a recursive curriculum can *end anywhere* too.

However, not all curricula with a recursive structure matter. Just like drilling on statistics computation matters less than contemplating on the effects of potential data bias, what is chosen for elaborations is worthy of more considerations. As Bruner (1962) says, "a curriculum ought to be built around the great issues, principles, and values that a society deems worthy of the continual concerns of its members" (p. 52). Thus, the core around which multiple elaborations revolve in a recursive mathematics curriculum can be anything that is profound, general, and valuable for *emergent*[29] mathematics and mathematical thinking. In addition to the basic underlying and general mathematical structures and ideas, any self-similar ideas (such as recursive curricula, philosophical ideas, ways of thinking, sensitivities, and attitudes) that mathematical ideas or thinking are a part of could be a possible core.

9.2 Re-imaging Recursive Curriculum: Growing two Fractal Trees

Davis et. al.'s (2008) fractal-tree image (see Figure 9.1.1) and the formation image above (see Figure 9.1.2) tell us what a recursive curriculum does and when, more than what it looks like. Since many complex phenomena, such as cognition, individual understanding, recursive curriculum, artifact, and human knowledge (e.g., school subject like math), are fractal-like observed from a complexity thinking perspective, a fractal-tree can represent many things. When a curriculum is designed as following the development of a self-similar idea with nested parts similar to the whole or with many equivalent forms, both curriculum formation and idea formation follow a recursive path and both can be represented as growing a fractal-tree. In this case, the development of the idea and that of the curriculum *mirror* each other. A representation for this is growing *two* fractal trees as a reflection of each other at the same time (see Figure 9.2.1), suggesting that curriculum development and idea development can draw inspiration from each other, and a study of one can inform the study of another. While this close affinity[30] between idea development and curriculum development seems common for all subjects and all kinds of curriculum development, it is not often that curricula are organized as a recursive development of a *self-similar* idea. Two potential practical implications of this growing two fractal trees interpretation are: 1) one can design and study recursive curricula by following a

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[29] Here, I invoke Davis and Renert's (2014) notion of "profound understanding of *emergent* mathematics" and join them to emphasize a complexfied conception of mathematics knowledge being "ever-evolving" thus not possible thorough for any individual or community (p. 118).

[30] This close affinity is questioned by Davis, Drefs, and Francis (2015): "this conflation of curriculum structure with logical arguments may actually be an error" (p. 61).

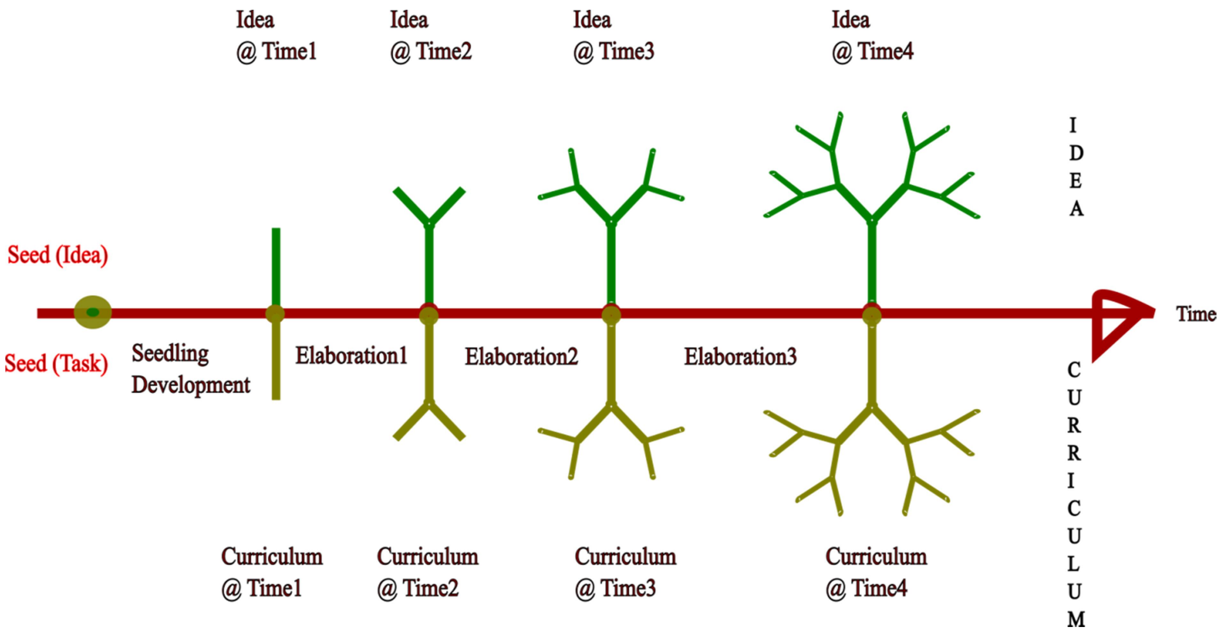


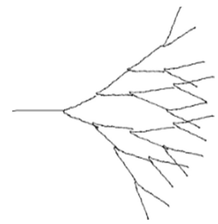
Figure 9.2.1. Idea development & curriculum development as two fractal trees growing in reflection of each other.

self-similar idea's development; and 2) one can learn subject knowledge through mapping or creating a recursive curriculum of it. So designing a recursive curriculum is not a task solely for teaching; it is also for learning. The line between teaching and learning is blurred. This blurring is not to eradicate the difference between teachers' and students' works in an educational setting, but to emphasize that both teachers and students are learners of the subject matter in a recursive curriculum and all learners can better their subject learning through *(re)designing curriculum for self or others*. Such curriculum designing process is essentially a process of mapping knowledge territory mentioned in section 8.2.

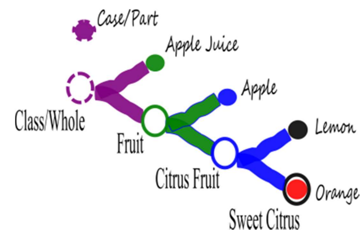
9.3 Re-imaging Recursive Curricula: Growing a Fractal Tree-Spiral

Albeit the above two tree image shows us what a recursive curriculum looks like and what it does, its resemblance to a recursive curriculum still more conceptual than iconic as it is hard to map it to a sequence or sequences of curriculum elements. To that end, a blended image of a spiral, a fractal tree, and a binary tree is more promising (see Figure 9.3.1). The image was

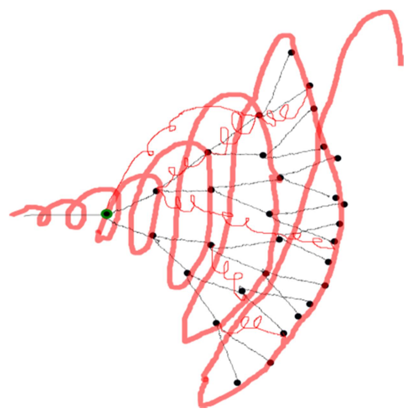
a. A spiral curriculum model displayed horizontally.



b. A fractal tree displayed horizontally.



c. A binary tree (a recursive curriculum analogy discussed in section 8.2) displayed horizontally.



d. A blended-image: A fractal tree-spiral.

Figure 9.3.1. Blending three images (i.e., the spiral, the fractal-tree and the binary tree).

resulted from a re-imaging process similar to the one used to generate the hybrid triangular pile image in section 6.2. That is, the process was one to generate an equivalent image for the two trees and the spiral and it was triggered by my noticing a visual similarity among three images. In this blended image, the fractal tree is the three previously constructed fractal trees (representing learning product, idea and curriculum respectively, as shown in Figure 9.1.2 and 9.2.1) merging together, and it is essentially a balanced version of the binary tree. The blended image represents abundant *possibilities* in learning products, contents, content developments and curricula applicable for both individual and collective learning settings.

Each node (see the highlighted dots in Figure 9.3.2) represents a possible learning product (or contents as each learning product is associated with certain subject content) at a

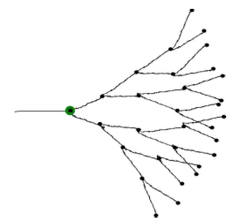


Figure 9.3.2. A fractal tree with all nodes highlighted.

particular development stage. Here, the seed (the leftmost node in Figure 9.3.2) is also viewed as a learning product: Given that the content it has needs to come from somewhere, thus it also has a historical becoming. Different from that in the vertical tree representation, the seed in this horizontal version is shown at the *branch* end of the tree truck instead of the root end, so that the model can show multiple learning products emerging after the first elaboration. More significance in this move will be explained later. The use of two branches connecting to the seed is *symbolic* rather than iconic, implying multiple possibilities existing rather than only two possibilities are possible. Multiple learning products are possible particularly when the seed is an enabling restraint that encourages novel responses or when a seed is used in a classroom learning setting with multiple learners. They are also possible when one learning product can have multiple forms or versions to be morphed into logically (e.g., $y = ax + b$ can morph into $y = 3x + b$, or $y = 3x + 6$).

Each branch represents a *developmental/inheritance* relation between the nodes at its two ends, meaning that one learning product can develop or morph into the other through *elaboration*. This means each learning product is a completed whole and makes one a different version of the other, and the change can go *both* ways. Note that this relation is *atemporal* (e.g., in a mathematics context, it might be mathematical, logical, visual spatial, and etc.) and *directional*. Put it differently, each branch that connects two nodes (say A and B) corresponds to two curriculum elements that can activate such morphing process from one node to another. The element that activates a change from A to B might be the same or different from the one activates a change from B to A. For example, a task of generating examples can enable one to change

$y = ax + b$ into $y = 3x + b$, and a task of generalization can activate a change from $y = 3x + b$ to $y = ax + b$.

Recall that each elaboration on a fractal tree is represented through adding two branches for each existing branch (see Figure 9.1.2), there are certain number of branches growing together in the same direction at each developmental stage/phase. These branches in the blended image represent the same curriculum element at work and thus the resulted nodes represent equivalent learning products or different forms of the same learning product, given that they all derived from the same seed through the same sequence of curriculum elements. The nodes formed at different developmental stages represent the different versions of the same seed. *Given that all products are derived from the same seed, they are essentially variations of the same thing*. For instance, after the question "What's $^{+}3 \times {}^{-}4$? Show how you know" is asked, using the mathematics content $^{+}3 \times {}^{-}4$ in the question, eight students come up with eight individual interpretations. These interpretations are eight possible forms of the same Stage 1 product, which is the meaning of a particular mathematical equation. They are also eight possible new versions of the content seed (i.e., $^{+}3 \times {}^{-}4$).

A spiral (noted in red in Figure 9.3.3) connects nodes across developmental stages. Each loop of the spiral represents a particular developmental stage. A sequence of loops, then, represents a sequence of stages. This spiral represents the *space of possible products* that a seed can gradually grow into, or in other words, *the space of possible variations of the same content*. Meanwhile, the spiral provides alternative curriculum path for different products to morph into each other, as there exists suitable curriculum element(s) to activate the process needed for any two products

Figure 9.3.3. An infinite fractal tree-spiral.

connected on the spiral to morph into each other. To include the possibility of different products on the same loop emerging at the same time or one after another (e.g., when multiple results are possible in a group learning setting or simply provided), a walk between nodes on the same loop can also signify a connection at first made merely through time.

The loops or developmental stages are ordered by *the level of abstraction and generalization* rather than in time. So the products at a smaller loop have content that is more abstract or general than those at a bigger loop. Given that there are infinite levels of abstraction and generalization as any category can have a subcategory and a meta-category, *this spiral and fractal-tree are infinite at both ends*. This, again, justifies the beginning move of the seed from one end to the other: The content in the seed is also developed from certain content in a previously established learning product. For example, a cycle of inquiry into "What is 3 x -4?" might be developed after a cycle of inquiry into "What is 3 x 4?" or after an invitation to generate a multiplication example of two integers for studying the meaning of multiplication. This move therefore opens a quest for the origin of the seed, and suggests that a seed for a recursive growth can be at anywhere in this tree. Here it is better to replace seed with *entry point* to avoid habitual associations between seed and upright growing direction and to broaden the direction of growth. Again, *a recursive curriculum can start anywhere*. With its openness towards emerging possibilities, it can end anywhere too. This is exactly what Doll (1993) envisions: "In a curriculum that honors, values, uses recursion, there is no fixed beginning or ending" (p. 178).

The multiple spirals (noted in lighter red in Figure 9.3.4) between two random loops are to acknowledge the possibility that any loop might be bypassed or shorten as long as a suitable curriculum element can be constructed. In the same logic, jumping across nodes on the same loop is also possible. Therefore, in this blended image, many possibilities of different paths exist for one to move from one node to another. Through branches, loops and mini spirals, a recursive curriculum starting from *any* node in this image can move in multiple directions: 1) to the node(s) on the *same* loop, i.e., those at the same level of abstraction and generalization; or 2) to the node(s) at a *different* loop, i.e., those at a *higher* or *lower* level of abstraction and generalization. Since each path is a curricular one made of a sequence of curriculum elements, this blended image shows *the abundant possibilities of recursive curricula and content developed from one entry point*. In a sense, we can say that this image shows *growing many fractal trees together*.

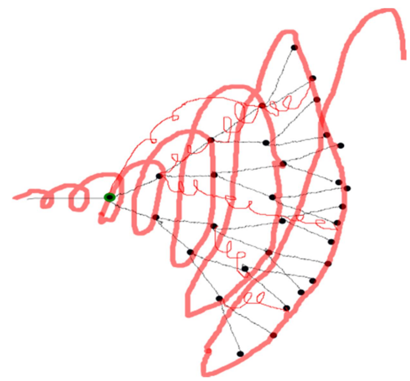


Figure 9.3.4. A blended representation of recursive curricula: A fractal tree-spiral.

9.4 Curricula Design as Recursive Learning (as thought experiments to enact, engender and experience recursive curricula)

How might this blended image/analogy help with recursive curriculum design then? While this blended image by no mean exhausts the possible visualizations of recursive curriculum, it is a complexified interpretation of recursive curriculum that affords more interpretations. In other words, this representation is recursive, exactly in the way as Sawada and Pothier (1993) define: “A representation (or a process in general) is recursive if at a certain point in working with it the medium becomes the message [, which becomes the medium of further messages, and on and on]” (p. 15). Here I consider two areas for further interpretations: curriculum visualization and curricula formation. Before that, it is helpful to recall that essentially nodes represent contents and branches represent mathematical or curricular event(s) that allow one node to change into or connect to another. Since a math task and a learning product include some content (thus ideas), a node can represent an idea, a task or a learning product. When the nodes represent learning products, the link between nodes on the same loop can be temporal relation, meaning that no manipulation needs to be done for one learning product to change into another. This way, the tree can be used to visualize situations including multiple learning products appearing at once in a group learning setting.

Recursive curriculum visualization

The blended image can support mapping and envisioning a recursive mathematics teaching and learning journey. Yet, again, map is not a territory; *visualizing a curriculum is for provocations rather than precisions and prescriptions*. Curriculum visualization is a way of thinking and learning hermeneutically about mathematics and recursive curriculum. Here is an example of such study that involves the following two visualizations.

Figure 9.4.1 shows the development of Davis et al.'s (2008) mathematics curriculum

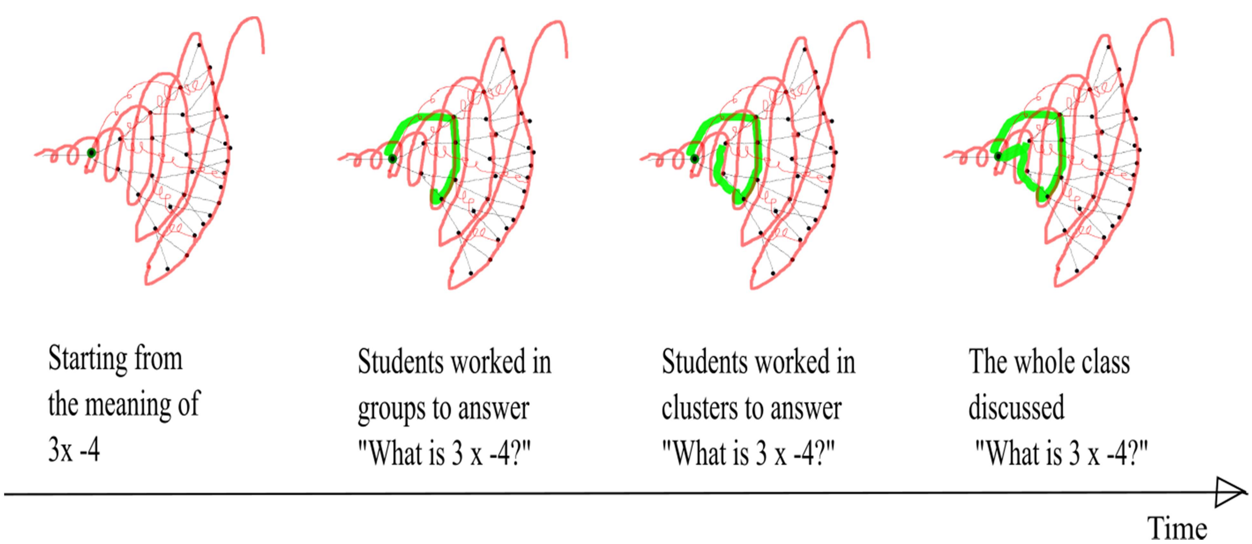


Figure 9.4.1. A recursive mathematics curriculum path (Seed: What is 3 x -4?).

example, discussed earlier, with the seed of "What's $^{+}3 \times {}^{-}4$? Show how you know". The number of students, unknown in their example, is set to eight here to permit a feasible visualization. The visualization of the curriculum formation shows a recursive curriculum path that led back to where it started[31]. The content or learning product developed in this curriculum is the meaning of the mathematical expression of 3 x -4. A quest for meaning brought forth multiple equivalent group interpretations (represented by the four nodes connected by the same loop), which then got converged to a few meaning clusters (represented by two nodes connected by the loop that has a higher abstract level) and later a synthesis of a class's interpretations (represented by merging back to the seed), hence renewing the meaning of the expression as a coherent whole again at the original abstraction and generalization level. This process of re-interpreting the same expression happened multiple times at both the individual learner level and the collective understanding level.

Figure 9.4.2 shows the development of the poem curriculum example, used by Davis et al. (2008), with the seed of "Pick a button, and then imagine the garment it came from and what the

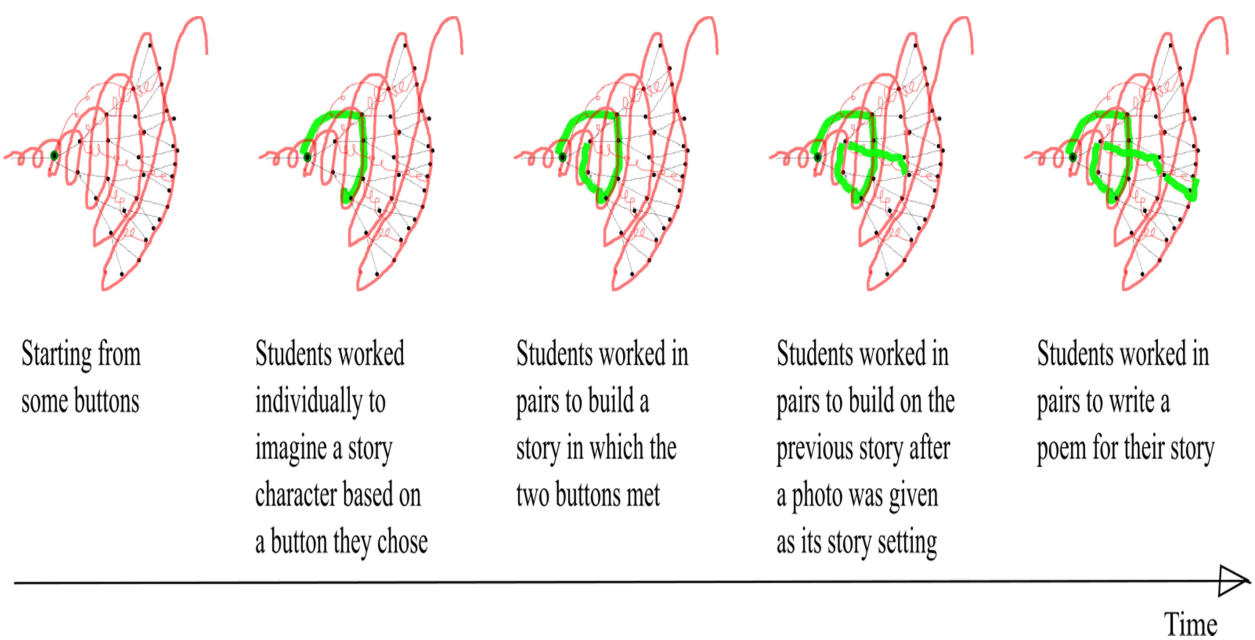


Figure 9.4.2. A recursive poem writing curriculum path (Seed: some buttons).

[31] Note that a reading of this curriculum development as a mathematical story could be a story of the mathematical character of 3 x -4. Or we can read this path as the development of any learner's, including the teacher's, understanding of this particular expression.

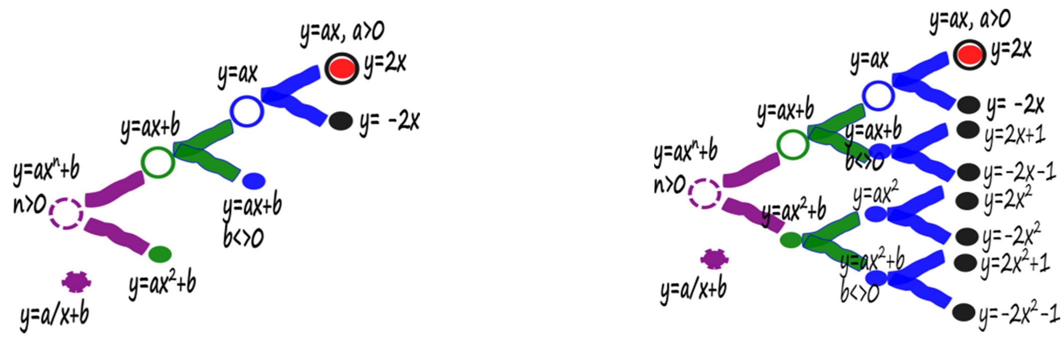
wearer of the garment is doing". The learning product developed in this curriculum is the story related to buttons. It shows a recursive curriculum path that diverged at first (producing many equivalent individual stories related to a button), converged in the middle (dropping the number of stories by half), and went on developing products with more refined details (firstly getting a restricted story setting and later a form of poetry).

The above two visualizations seem to point to two types of prompts for elaboration: While both the mathematics and poem examples employ comparing equivalent learning products at the same development stage, the poem example also includes direct invitations for changing parts of one's previously established learning product. These two approaches, using the fractal tree growing analogy, are like, respectively, changing the tree through changing its environment (i.e., what other trees it is situated with) at first and changing the tree through changing its parts at first. We can say these two approaches as changing *references* and changing *parts* for a learning product. However, these two approaches are essentially two names of the *same* thing for fractal-like products, depending on what counts as a whole at a particular time. For instance, if we view $y = x$ as a whole, then moving from it to $y = -x$ is changing its reference; the same move would be changing its parts (i.e., changing from $+$ to $-$, as like changing one person's eyes from one type to another), if we view $y = \pm x$ as a whole that is similar to both $y = x$ and $y = -x$ (meaning that three equations are equivalent rather than that $y = \pm x$ has two parts). Similarly, if we view an individual learner as a whole, then having two learners to access each other's interpretation is a prompt for changing references; the same invitation would be aiming for changing its parts if we view a dyad as a whole that is the similar to any of the two learners. This understanding suggests that we can use contents or learning products, which (seemingly) belong to different systems (e.g., learning systems, knowledge categories, or contexts) yet can be viewed as equivalent in a bigger whole, to invoke each other's elaboration and invite consideration in what ways these contents or products might be related to a whole as its fractal parts. This way, we encourage creation of novel categories through pairing two or more seemingly unrelated ideas up. Such playing with ideas and throwing ideas into every combination possible is what keeps knowledge alive (Doll, 2005).

Recursive curricula formation

The blended image, as a representation of recursive curricula possibilities, can help a learner to cultivate the sensitivity and flexibility needed to generate a recursive teaching and learning journey in responding to her own or others' emerging mathematics and mathematics thinking. To that end, growing a fractal tree-spiral is a key exercise. Again, this task is not for precisions or prescriptions but for provocations.

The binary tree analogy discussed in section 8.1 can help here: One starts with a content/task (e.g., an orange), then repeatedly add another (e.g., lemon, then apple, apple juice and so on) that resembles its immediate past yet different enough to invoke one to consider it separated from everything in the past as a whole. Essentially, each new addition is a recursive elaboration of the previous established ideas yet bringing a new whole. This way we can have a sequence of nested contents (ideas/tasks) and the sequence's being is inseparable with its becoming. To illustrate, imagine we add apple instead of lemon to compare with orange, this new addition brings forth a whole that is not "citrus fruit" any more. Once a sequence is visualized in the form of an unbalanced binary tree, one can easily use symmetry to flesh out other parts to balance this tree. This means using the existing nodes on a particular branch at a particular developmental stage as references to generate their corresponding nodes at the same stage yet located on different branches. Figure 9.4.3 shows an example (returning to function



a. A binary tree (before getting balanced, b<>0 means b≠0)

b. A balanced binary tree (the purple nodes represent possible future hence their related branches are not fully developed)

Figure 9.4.3. A recursive curriculum formation (developing from a to b).

again) for these two stages of the tree formation process.

While the above tree growing process can produce many different trees including similar or different content developments, each tree, once turned into a blended fractal tree-spiral after the spirals are added to it, can be walked on in many different ways, generating different curricula. Figure 9.4.4 shows a path allowing the concept of polynomial functions (in the form of $y = ax^n + b$) to develop from one version (i.e., linear function, in the form of $y = ax + b$) to another (i.e., quadratic function, in the form of $y = ax^2 + b$)) with each version changing from one form to another (i.e., with b=0 ➔ with b≠0). This path is a loop of loops of loops, or a loop of *nested loops*. The ideas and tasks involved in

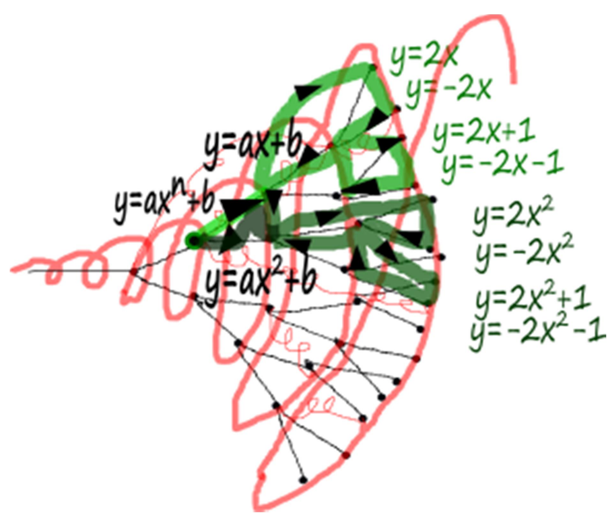


Figure 9.4.4. A recursive curriculum path on the polynomial function binary tree.

parallel loops belong to the same developmental stage hence these loops are equivalent. This path might represent a learner going through all the ideas and tasks in order. It can also represent a group learning path. For example, a class can be divided into two groups, each of which study a different set of functions (e.g., $y = ax, a > 0$; or $y = -ax, a > 0$), before having the two groups meet and re-interpret $y = ax$ together. After all, the interpretive affordance of a fractal tree-spiral image is unlimited.

The formation of such an image, though, does not have to be linear, as like growing a binary tree from a relatively smaller category to a bigger one. A fractal tree-spiral is a *logical* structure before one reads it into some temporal sequences by moving on it. So the tree can grow from root to branch, branch to root, branch to branch, or a mix of different possible directions, as long as one pays attention to *the relationship between nodes: Essentially they are related to each other as repetition with variations and one node can become another through a mathematical change or curricular arrangement*, which includes nonmathematical manipulations such as comparing/contrasting, re-imaging, and etc. Also at each developmental stage there can be more than two branches (i.e., possibilities) coming out from each node, making this tree not binary any more. Here we take a look at two more experiments.

In Experiment 1, I tried to form a fractal tree through re-imaging and asking two different sets of questions to help me form nodes at different developmental stages. Situating at the beginning of function and graph unit (learners have learned graphing using table of values), I started with $y = ax$. I thought of its variations that look alike and came up with $y = a/x$ and $y = x/a$. Then I use them to build two fractal trees by asking two sets of questions that can be interpreted as implementations of re-storying:

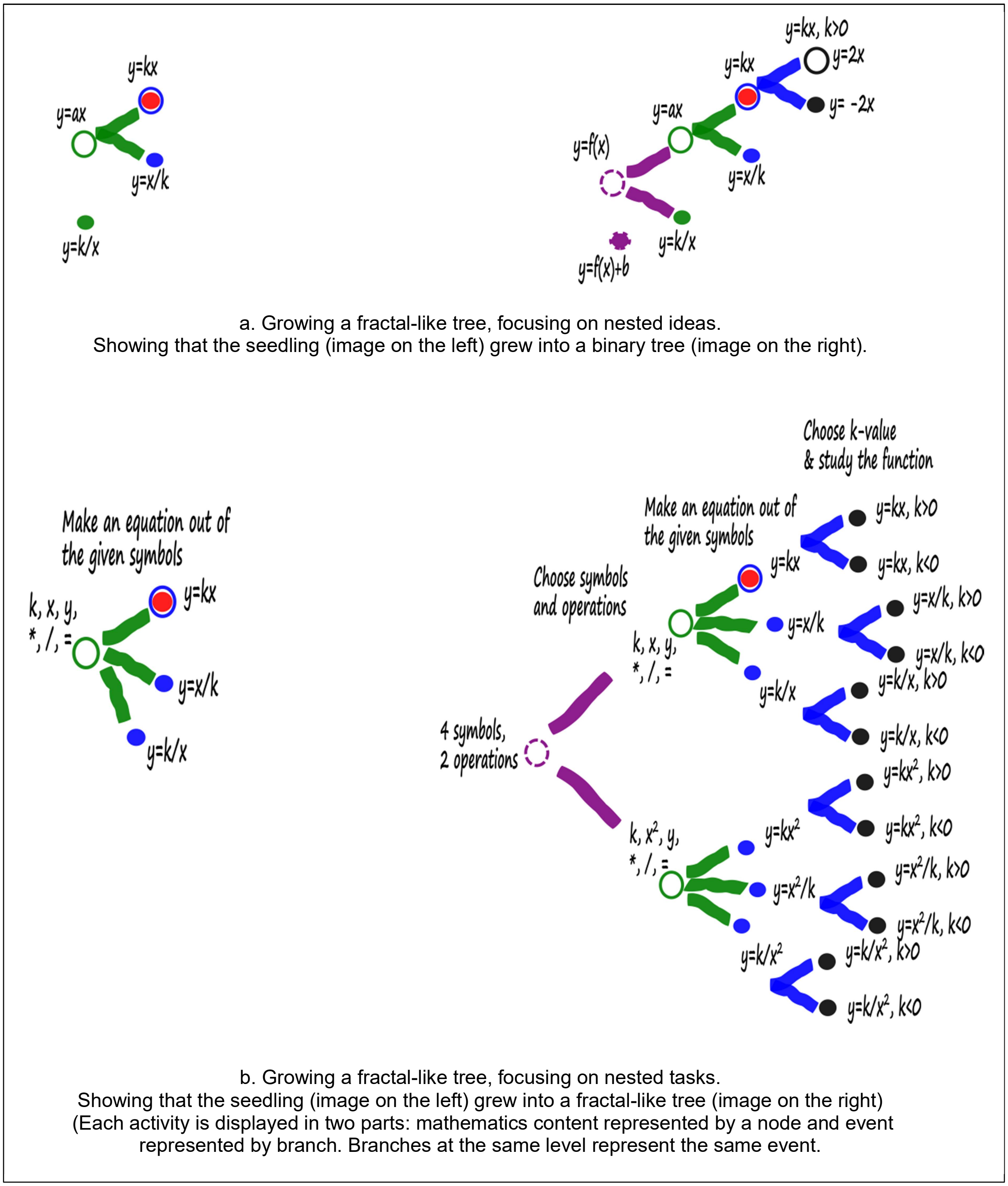
1) "From what sources might these three equations be derived? What content can these lead to?"

2) "What question might these three equations be the answers for? What answers might a question about any of these equations lead to?"

The first set stresses mathematics ideas development. I constructed a seedling by organizing the three equations in a binary tree format (see the left image in Figure 9.4.5a). The second set focuses on curriculum tasks development. A task (generated through implementing re-imaging again) that might bring forth the three equations could be: "Use four symbols (- k, x, y, =) and any of the two operations (/, *), create as many different equations as you can. Reorganize them into y =___ format[32]". So I constructed a seedling with one node and three branches (see the left image in Figure 9.4.5b). Once a seedling has been generated, it can be easily extended in two ends as long as one makes sure that parallel nodes have equivalent contents and parallel branches can be actualized through the same curriculum element.

This experiment brought forth an unbalanced binary tree that stresses a fractal-like *idea* (i.e., $y = f(x)$) with nested self-similar ideas (see the right tree in Figure 9.4.5a) and a fractal-like tree that stresses a fractal-like *task* (i.e., use some given symbols and operations to make functions for function study) with nested self-similar tasks (see the right tree in Figure 9.4.5b). Regardless how they started to grow, these two trees inevitably include both self-similar ideas and tasks. Of course, these two trees are contingent results; there could be many possibilities for these two experiments, even starting with the same seedlings. The focus of this experiment, therefore, is not at the end of the process, but the generativity of the process itself. Such experiment is helpful not merely for exploring, but more importantly for *generating self-similar ideas and tasks*. It shows what contents could be made as a repetition with variations of each other and be studied together in the same way in one learning unit.

[32] The tasks here are for planning purpose, thus the wording might be different for the same task given in class to suit particular learners' background.



a. Growing a fractal-like tree, focusing on nested ideas.
Showing that the seedling (image on the left) grew into a binary tree (image on the right).

b. Growing a fractal-like tree, focusing on nested tasks.
Showing that the seedling (image on the left) grew into a fractal-like tree (image on the right)
(Each activity is displayed in two parts: mathematics content represented by a node and event represented by branch. Branches at the same level represent the same event.

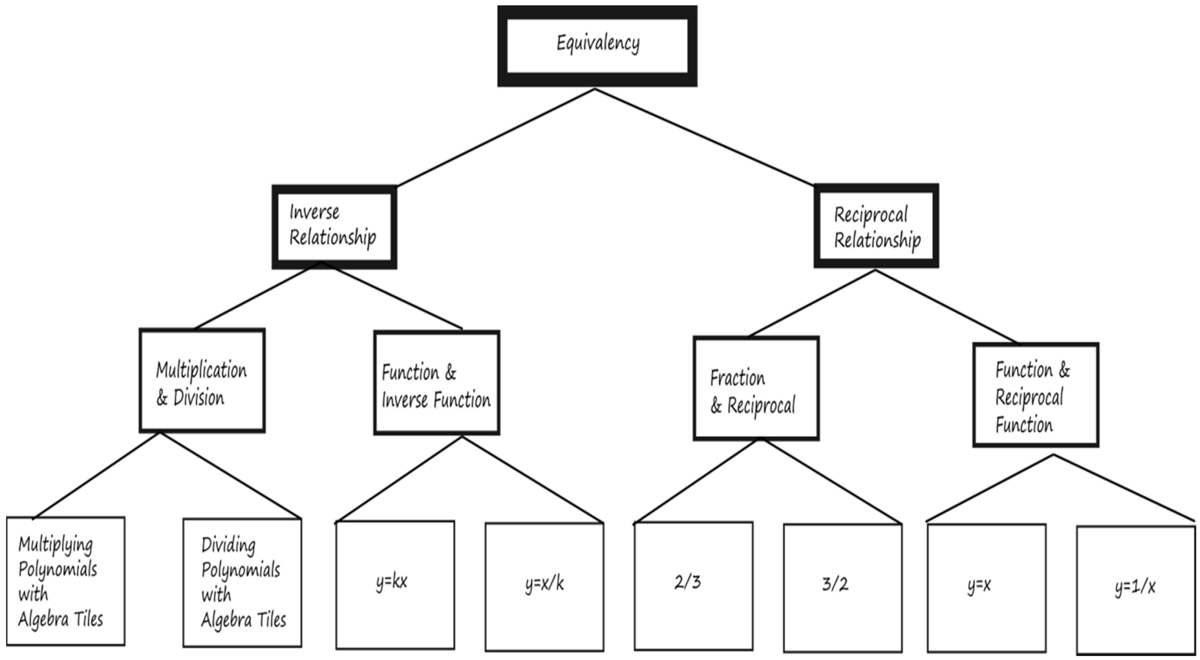
Figure 9.4.5. Two recursive curriculum formations.

In experiment 2, I tried out a different start to form a tree like structure that is beyond fractal-like by letting the number of branches coming out from each node differ when possible. Instead of beginning with a more concrete form of a self-similar idea/task, I started with a more abstract form of the idea/task and proceeds to generate many possible fractal parts of it (such that I can generate more loops/elaborations to explore the same idea or execute the same task), with an openness to redefine the idea/task when necessary. Figure 9.4.6a presents a tree for a self-similar idea: equivalency, a concept that a recursive curriculum can play with (see section 8.1 for its definition). Figure 9.4.6b presents a tree of a self-similar task, i.e., “In how many ways can you divide ___ in half?”, which has sufficient structure and openness to be used as an entry task for studies or preambles of many different types of quantity, such as even and odd integers, real numbers, length, area, volume, fraction, and etc.

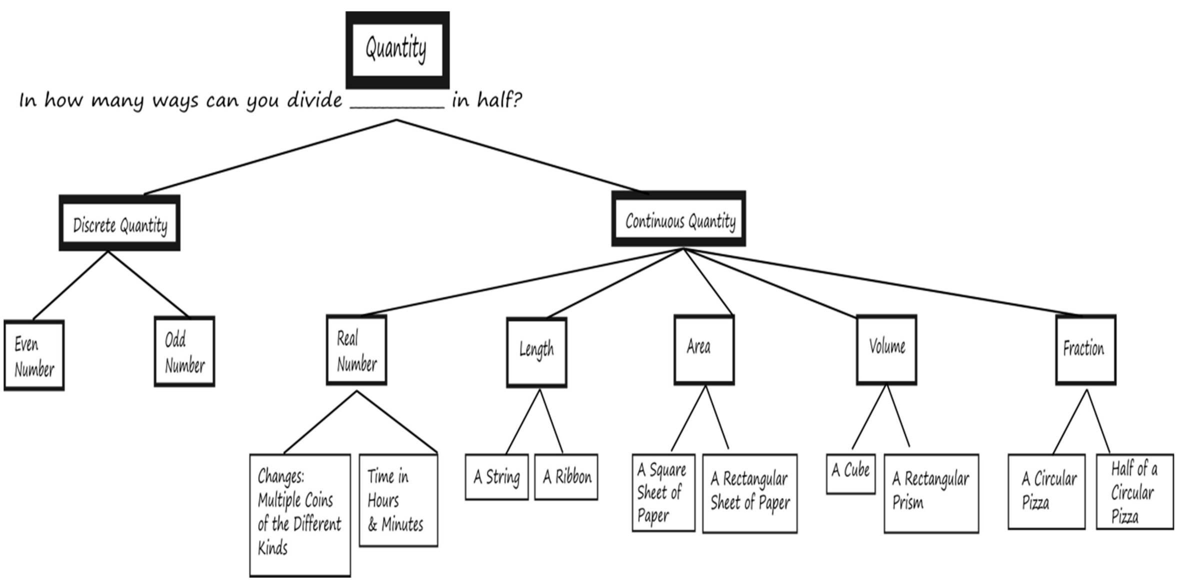
Analogically speaking, in Experiment 1, I took look at a grain (referring to a fractal part) and tried to see or *define a* universe (referring to a whole) that the grain can embody, and in Experiment 2, I took a look at a universe and tried to see or *define* a grain that embodies the universe. In return, I understood both the universe and the grain differently. Both experiments are helpful for one to identify/generate self-similar constructs (e.g., ideas or tasks) and practice discerning the same whole across scales and contexts. These constructs can be mathematical, curricular, and theoretically speaking, personal too.

Now, taking one step back and looking at self-similar ideas/tasks as a whole and connecting with the forms of re-encountering, a different kind of fractal view of self-similar ideas and tasks can be formed. Figure 9.4.7 presents a fractal view of a self-similar idea and a fractal view of a self-similar task. Both fractals are formed by collapsing a fractal tree into one cell.

Recall that each branch represents certain mathematical/curricular event(s) that allow the equivalent contents (represented by the nodes connected through the branch) change into or connect to each other, and the branches at the same development level represent the same event.



a. A fractal tree of a self-similar idea (i.e., Equivalency)



b. A tree of a self-similar task (i.e., "In how many ways can you divide ___ in half?")

Figure 9.4.6. Forming a self-similar a) idea and b) task.

All the above tree formations then help to bring forth self-similar teaching and learning cycles with the same pedagogical focus. An experience in these cycles, interpreted as an implementation of re-inbodying, is helpful to induce a sense of harmony or rhythm that hints at the underlying conceptual connections among contents. It can also be interpreted as an implementation of re-storying, focusing on or resolving around the same story, which becomes the focal or anchored story. With continuous practices and creative categorizations, it is possible to use a tree formation activity to identify or create fewer focal or anchored stories significant for mathematical learning and education over a longer time span. This opens space for increasing frequency for re-encountering the same story through exploring and generating multiple fractal parts of the story while keeping the emergence of novel fractal-like stories possible[33]. Such formation of recursive curricula is an enactment of Whitehead's (1929/1959) teaching aphorisms, i.e., keeping the main ideas one teaches few and important, teaching them thoroughly, and throwing them into all possible combinations, which, as Doll (2005) emphasizes, are important to keep knowledge alive. It is also in line with what Davis et al. (2008) advocate, to have the educational intentions be "embedded and embodied in *every* aspect of the learning experiences, as opposed to being identified as goals to be met by the end of a sequence of instruction" (p. 211). It actualizes a fractal-view of mathematics, seeing it as a complex system understood not through discovering some underlying truth or "secret in the middle", but through observing it as "holograph" in which "every piece contains the information of the whole" (Fleener, 2002, p. 138).

There is no end for recursive curriculum design and there shall not be one. All the above presented curricula representations are contingent outcomes of a thinking process that unfolds

[33] Two emergent curricula led by a few themes are worth noting here. Fowler's (1996) fractal curriculum exemplifies an emergent fractal-like mathematics curriculum, which has a few planned general themes (i.e., dimension, shape, change, quantity, uncertainty, symmetry, and discourse), uses certain fractal objects as attractors of elaborative activities, and leaves the details of the curriculum for students to construct. Reeder's (2002) case study of a middle school mathematics teacher also depicts a teacher who keeps a global perspective of curriculum and selects tasks as parts of a large and ongoing conversation emerged through the whole class community engagement with mathematics related to linear relations, variables, and functions throughout the year.

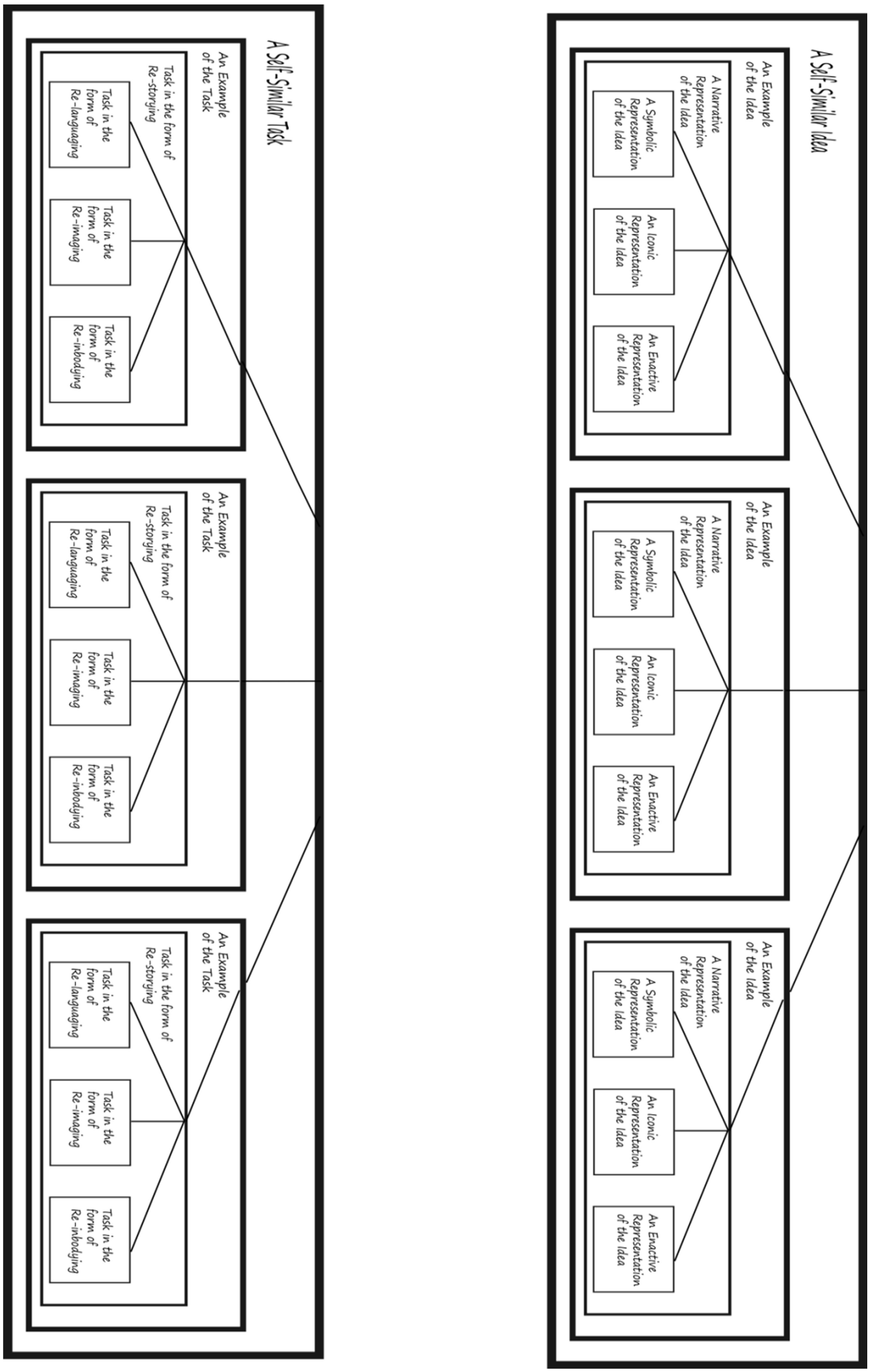


Figure 9.4.7. A fractal view of a self-similar idea (top) and a fractal view of a self-similar task (bottom).

itself and cannot be compressed. What matters is the affordances of the process rather than what it produces in the end. Designing towards a recursive curriculum is not really about generating a specific sequence of tasks to be delivered or executed. Rather, it is a reflective and creative thought experiment to *inquire* and *generate* self-similar or fractal-like contents, tasks and to *enact*, *experience*, and *engender* recursive curricula. It aims for the process's generativity for rich possibilities in both teaching and learning and it affords recursive learning experience. Teaching in class, from a complexity thinking perspective, is mindful participation in the unfolding of possibilities (Davis et al., 2008). The complexity sensibilities and hermeneutic attitude needed for such kind of teaching are exactly what designing towards recursive curricula is meant to exercise and enact. Meanwhile, the "*mathematical awareness*" (Mason & Davis, 2013, p. 183) entailed by such teaching could also be nurtured through engaging in designing recursive curricula. Educating awareness implies "*noticing more*" possibilities and "knowing more deeply and richly in the sense of having possible actions – mathematical, pedagogic, and didactic – come to mind when they are needed" (Mason & Davis, 2013, p. 192) rather than knowing more facts or theories. This is in line with designing recursive curricula towards potentials instead of prescriptions.

Here, to make my position clearer, a change from my original leading research question "*What might a high school recursive mathematics curriculum informed by complexity thinking be?*" to "*What might high school recursive mathematics curricula informed by complexity thinking be?"* is necessary, even though I had no intention to find a singular answer to start with. Consequently, the previously used research title is changed from "*Towards a Recursive Mathematics Curriculum*" to "*Towards Recursive Mathematics Curricula*". Moreover, the blended representation of recursive curricula (i.e., Figure 9.3.4) is renewed to render a fractal-like image (see Figure 9.4.8) to better represent abundant curriculum possibilities. The possibility of some nodes lying on different spirals signifies that any self-similar content can be an entry point for many recursive curricula resolving around different cores.

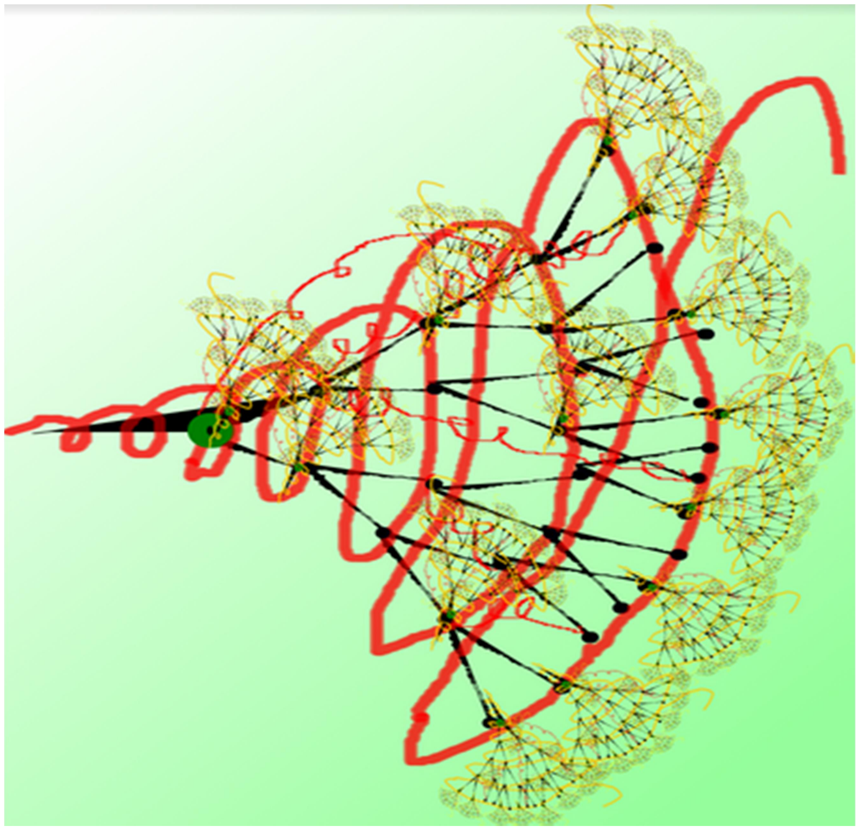


Figure 9.4.8. A fractal-like representation of recursive curricula.

10 The End is Also the Beginning

A hermeneutic inquiry has no end, yet a closure of the process can be imposed when meaningful interpretations have reached (Prasad, 2005). When I tried to do so, two well-remembered yet seemingly unrelated stories started to speak louder to me. One story is my first experience of emerging curriculum and the other is about the readiness test in my classes.

An Emerging Curriculum of *Pi*. I got a fresh taste of emerging curriculum during my first teaching practicum. In a lesson of circle for grade 9 students (with grade 5 or 6 level mathematics knowledge level), I started with a plan of telling students the circumference formula directly after we identified circumference on circular objects using a string. The object of the lesson was to apply formulas, not to discover *pi*. I did not know how to introduce *pi* meaningfully in a short time either. When I posed “How can we find the circumference of a circle?” to the class, I was looking for a reply related to the circumference formula, such as radius or diameter. A student, who was weak in mathematics and seemed engaged when I showed them where circumference is found on the circular lids that I brought to the class, volunteered an answer: “Just measure the outside of the circle”. I was not sure how I can go from his answer to my original lesson plan, yet I wanted to encourage him. So I decided to follow his lead and then see what I can do. I invited him to demonstrate what he meant. He repeated what we had done: He used a string to go around the edge of a circular lid. He went on to measure how long the string is and got a number. I suddenly sensed that his action nicely connected us back to the formula. I wrote down his number on the board, put an equal sign besides it, and asked the class for a possible circumference formula. One student said, “2 times radius”. “Oh, that's getting better, although he might have said this because he confused circumference with diameter," I thought. "Yes, very good," I said, "it is indeed related to radius. But it's not just $2 \times R$, there are other things there. It might be $2 \times R \times 3 + 7$, or $2 \times R \times 8 - 1$, something like that. What do you think? What should be there?" One student, who was the strongest mathematics student in the class, said "times 3". "Wow, that's even better, although he might have said that because he knows the formula," I thought to myself and said, "Okay, let's test it." So we found the radius of the circle measured at the beginning and calculated its circumference using the proposed formula $2 \times R \times 3$. We compared it with the measurement of the circumference. They were not the same. I said, "they are very close, but they are not the same. So the number in the formula is not 3, let's find out what number it is." So we divided the circumference by $2R$, we got 3.10. I stopped the investigation there due to time consideration, and told the students a logic conclusion I drew while seeing how the lesson unfolded: that ancient people must have done exactly what they did, i.e., measuring a circle's circumference and then dividing it by $2R$, on many circular objects, and ended up finding a constant value *pi*, which is about 3.14. I was inspired by how this lesson developed its own path, leading both my students and I to an unplanned investigation. Since I had not questioned how *pi* was discovered before, this method was also new to me. So this investigation was engaging for both my students and I.

Readiness? After I encountered a readiness test used in an undergraduate course at the University of Waterloo, I designed similar ones for my students to take at the beginning of a course. It was used as a formative assessment of students' existing levels and a device to motivate and guide them to review. For students who scored rather low in this test, I would check their student records and suggest those with a record of low mathematics marks in previous grades to take a different mathematics course instead. Given the one-time usage of the readiness test, it was unclear how much the test motivated and helped students to review, other than giving me more works and marks correlated with students' previous grade's marks. So I faded the test out eventually. When I mentioned the readiness test to my mentor Bill Doll, he suggested me to use the test to teach from. Instead of getting students to a certain level of readiness so that I can start teaching a new topic, Bill Doll urged me to start right from what they know. I wondered how I can do that in a class with diverse mathematics backgrounds.

These two seemingly unrelated accounts are memorable for me, I now understand, because they both remind me of the paradoxes that I lived in as a classroom teacher. For one, although I appreciated emergence and improvisation, I was confined by curriculum objectives and the limitation of my knowledge. There were always concerns about how to meet the curriculum goals and how to face questions that I do not have an answer for if I let the curriculum emerge. Although the emergence that I encountered during my practicum was surprisingly educational for both students and I, there was also a constant pull towards the planned lesson expectations thus its openness seemed not genuine, as if limited by a glass ceiling. For another, although it was important for me to teach from students' current knowledge level, the diversity in students' levels that I could handle in a group learning setting was limited by a modularized curriculum and my understanding of deficits. In a linear curriculum made of different learning units built on each other as if modules in an assembly line, if one had not learned the prerequisite topics or learned them well enough before a new course, one has a deficit that requires remedy. Such work is often an additional burden for both students and class teacher. Here readiness is like a concrete floor. Deep down, these two stories reflect the confinement a classroom teacher is often subjected to in a linear curriculum through speaking about restrictions at two opposite ends: where a curriculum ends and where it starts.

Now, these stories reemerged again, asking me to consider the enactment of recursive curricula in a linear curricular framework (e.g., program of studies) from which a classroom teacher is still hard to escape. Looking back the personal transformations I experienced through this inquiry, I see hope in transcending both ends through designing towards recursive curricula.

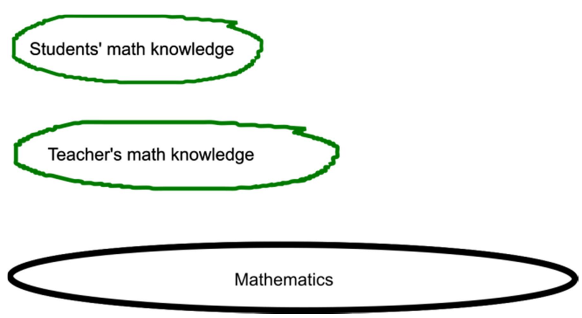
I have come to understand “deficits” differently. I started my PhD program with confidences along with many self-doubts and rejections. In terms of learning, I was eager to learn everything, but I was slow. Reading was like feeding a black hole: I was distracted and intrigued by many things yet found them hard to remember or recite as a coherent whole without time consuming reviews. Writing took forever: My ideas were often too fragmentary and fuzzy to be verbalized and too noisy and messy to flow. All these struggles made me feel inadequate and hopeless to become knowledgeable. My theoretical study of hermeneutics, while reasserting the inevitability of human limitations and recursive movements in learning, it legitimized them as essential. This started a positive spin on my understanding of my own deficits and needs for repetitions and recursive movements. During my research journey, I learned mathematics anew from my own recursive blindness and different mathematic representations’ limitations. This helped me to realize that my deficits, exactly like the prejudices and the limitations of a mathematic representation, can enable rather disable learning. Also, I was repeatedly led to fruitful reinterpretations as I being called by some stories or puzzlements without knowing why. This taught me to listen better to myself and take my confusions as the beginning and spring water of my learning. Similarly in writing, many writings came together through freewriting, wrestling with the difference among what I wrote, what I thought, and what I knew, and a process of letting go, moving on with unsatisfied incomplete pieces and coming back to them later. This allowed me to practice enduring struggles and following their lead. I finally learned to, not avoid, not merely accept or accommodate, but *appreciate* my deficits as enabling relative limitations. I became more patient with myself.

I have also come to understand “newness” in mathematics learning differently. Over the years of teaching high school mathematics, as I became more familiar with the course materials and students’ works, I saw less new in my students’ works. Albeit I still learned something new in mathematics through teaching the same course again, I thought my newness is like my students’: They are not new for the field of mathematics. My study showed me otherwise. Having experienced (re)learning something profound and beyond mathematics domain (i.e., zero is not nothing, mathematic objects are defined by humans) through re-encountering ordinary mathematic objects such as $-6 - 2 = -8$, learned to appreciate some mistakes of students such as treating $f(a + b)$ like $2(a + b)$ as creative use of pattern rather than mindless imitation after I considered re-imaging, and heard Tom Kieren, with rich knowledge in both mathematics and mathematics education, reciting excitingly stories of observing something profound in school students’ works, I was deeply moved.

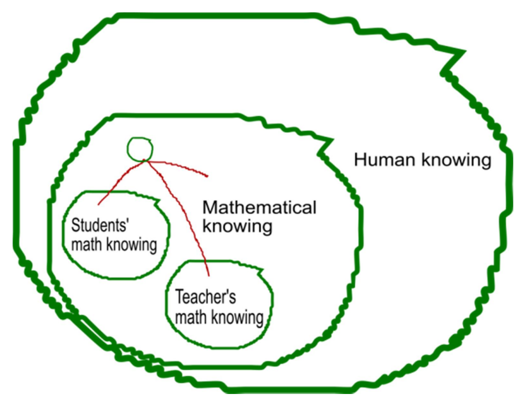
I realized that it is actually limitless what I can learn about the familiar school mathematical concepts through teaching infused with complexfied sensitivity, because my encountering with the subject and the students is a process of growing into the space in-between, through which all parties (i.e., I, subject, and students) involved evolve. Essentially the process is an encountering of three open complex systems. Such encountering has an “*I-Thou*” or subject-subject relationship (a dialogue, no objectification of the other being) instead of an “*I-It*” or subject-object relationship (a monologue, treating the other being as an object used to serve one's interest), fitting Buber's (1937) definition. Such encountering is a process of becoming, in the sense of evolution and also the sense of "bringing forth" as Tom Kieren specifies:

> [Becoming for me] is closely related to "bringing forth a world of Significance with others that can be observed to "contain?" or be mathematics or mathematical. This Becoming or Bringing Forth allows for the arising of mathematical inter-objects. (Jan 10, 2018, personal communication)

Therefore, the mathematics content at public education level is no more an established, relative static land, compared to the frontier of mathematics. It is alive. It is *ontologically*, not epistemologically, abundant (Jardine, Friesen, & Clifford, 2006, p. 88). This means that different understandings of the same mathematics topic are not multiple ways of knowing the topic that has a fixed meaning. They are ways to understand a topic that is "abundant, nonfoundationally fluid, and inherently complex" adequately, as "interpretation doesn't simply provide multiple ways of complicating a topic that is simple. Rather, interpretation transforms what it meant to *be* a topic – to be is to be-in-abundance" (Jardine et al., 2006, p. 88). Meanwhile, the connection between school mathematics and mathematics and human knowledge at large is not one-way such that it is only changed when the bigger field changes. Rather, it is connected to them as a fractal part in relation to a dynamic evolving whole. So, one can understand the whole through reengaging with this part, and whatever generated in this part can have both instant and long term effects on the whole. In other words, whatever happens in public education level is not simply reproducing well-known human knowledge; it is creating and transforming human knowledge as a whole. Such viewing of mathematics as a fractal-like dynamic complex system allowed me to see that the newness generated through learning mathematics in public education is not bound by the scale or categorization of the subject. Such view also made knowing mathematics as developing a sense of personal intimacy or relationship with mathematics as an *other*, as Handa (2011) proposes by following Buber's (1937) I-thou theory, more sensible. To iterate differently, Figure 10.1 depicts the change of my perspective. Instead of seeing myself and my students as independent unsynchronized learners of mathematics (i.e., learning about different mathematics contents at different spaces and times) and each one's learning as to reach the level of a more knowledgeable knower's knowledge (i.e., I reach the mathematical community's knowledge level, my students' knowledge reaches my level), I now see mathematics subject content as a multi-dimensional fractal-like whole that is embedded in a bigger fractal-like whole (i.e., human knowing) and within which my knowing and my students' mathematical knowing, as fractal-like parts, are embedded. A recursive curriculum can enable my encountering with my students in a mathematics classroom to bring forth new mathematical objects in a new dimension, transforming my students' and my mathematical understanding related to the same mathematics contents, and also mathematics and human knowing as a whole.



a. A static view of mathematics



b. A fractal-like view of mathematics.

Figure 10.1. A visualization of perspective change.

All these parties' growth is synchronized and none of the parties is insignificant for each other's growth. Such change of perspective, as an implication of changing language in describing learners and cognitions, turned around and made further language change necessary. As shown in Figure 10.1b, “knowledge” is replaced by “knowing” to emphasize each knowledge system’s complexity dynamics. As Davis and Renert (2014) explain: “Knowing signals an inseparability of knowledge and knower” and “reminds us of the dynamic characters of both knowers and knowledge”, which are “ever-changing, evolving forms” inseparable with their similarly dynamic contexts (p. 23). Knowing also connects knowledge to doing and highlights what a knower knows bodily and tacitly.

Looking deeper, my initial self-rejection and seeing less new from my students’ work reflect a diminished susceptibility to the self and others. I was gradually becoming, not experienced, but an “expert” who feels able to end a venture early for having seen, heard, or done enough and who become less susceptible “to the difference that the next case might bring” and “to being addressed” (Jardine, 2015, p. 252). My being out of tune with these potentials, however, might have more to do with a sense of lack than contentment. There was impatience with my own ignorance and slowness and a longing to know more and learn faster. Being busy combating with my ego or my urge to move on, I was insensitive to be addressed by self and others (including other beings, processes, and anything that can be experienced as a Thou). Deep down, I lacked the kind of learning experience that my inquiry brought forth: experiencing self and others as abundant, experiencing “trivial” and familiar as profound, and experiencing learning as an aesthetic process in which I was enchanted by something bigger than myself calling to be born.

From the onset of responding to a compelling call to a satisfactory culmination, my inquiry followed an organic path that unfolded while I walked on it, and took me to an end that is not external to but within my activity as it could not be achieved by compressing the inquiry process. Such experience was so therapeutic and aesthetic that I could not agree with Dewey

(1929/2004) more that the starting point of an educational process locates in the learner. My experience taught me that personal questions, confusions and struggles should be *central* not *peripheral* for learning. Rather than limitations or barriers, they are the driving forces for learning. Struggles are difficult, but they do not have to be painful. Each of us comes to meet each other in the midst of life, already full of stories, relations, and puzzlements that are ready to be called for and puzzled about. We are ontologically abundant (Jardine et al., 2006); each of us is "pregnant with thought, however inchoate or obscure" (Kupfer, 1983, p. 18). It is not the issue how we can get self and other people ready for a learning, rather to lead out what is already there, as the Latin origin of educate, *educere* "lead out" ("Educate", 2012), suggests. To educate ourselves and others is to conduct ourselves in a way to help one (self or others) hear one's personal story resonating.

My experience also convinced me that learning is always a creative process, maybe even more so in mathematics classes. Learning is neither reproducing known truth nor for preparation for future living (Dewey, 1929/2004). Everything a learner creates is original, no matter how it seems like repeating ordinary common sense. There is value "that intelligent search could reveal and mature among the things of ordinary experience" awaited to be realized (Dewey, 1929/1958, p. 38). Mathematics, a study of patterns made of ideas and what is possible in an imaginary land rather than what is practical, is arguably the most creative art form of mankind (Lockhart, 2009). Creating such an art work requires problems naturally arose in situations, time and space to contemplate and struggle, and openness towards unforeseen changes; teaching mathematics requires "an honest intellectual relationship with our students and our subject" (Lockhart, 2009, p. 43).

With all the above changes in myself, I see designing towards recursive curricula not just an ontological necessity, but also an *ethical* necessity. It might offer a way for educators who disagree with linear predetermined curricula to live subversively in an overarching linear curricula framework. Recursive curricula have a focus in self-similar entities (ideas or tasks). This makes the beginning and end of a planned learning process easily modifiable through changing into its equivalent. Wherever we choose to enter an educational process, there are always stories that the current beginning reminds us of and equivalent entities that the current beginning is derived from. This is particular the case when using enabling constraints (Davis et al., 2008) as entry points. Wherever we end a process, there are always equivalent entities that we can generate from the existing one. Given that the equivalence is contingent, we can change the direction of the curriculum by inviting something that resembles the past yet invites a different future. As such, we can play with different ideas' combinations and create something new, at the same time, having the curriculum keep an order by resolving around some big ideas (e.g., equivalence, symmetry, substitution/representation). Recognizing many topics are parts of the same fractal-like idea, we can have recursive elaborations on one topic rather than rushing through many topics (e.g., an introduction to trigonometry lesson might take a form of re-encountering similar triangles or equivalent ratios). In this way, we can transcend the start and the end of a linear curriculum by considering the bigger idea of which the start and the end is a fractal part respectively.

Teaching and learning, I believe, are two sides of the same coin: Educating others and educating self inform and propel each other. A curriculum that only invites students but not teachers to re-view the subject matter is not recursive. A curriculum that only changes students but neither teacher nor curriculum and subject matter is not transformative, because a transformative curriculum “continually regenerates itself and those involved within it” (Doll, 1993, p. 87). Education for students alone perpetuates domination and control and disregards teachers’ growth thus unethical. Designing towards recursive curricula is essentially an invitation for learners (i.e., teachers and students) to enter a recursive educational process in order to cultivate a capacity to *learn recursively from self and others*. It aims to bring forth transformation in all parties involved (the learners, the subject matter, and the curriculum) and cultivate an I-Thou relationship with self and others. Therefore it is not just to ask “What relationship do we want our students to have with mathematics?” as Fleener (1999) proposes for building a transformative process-oriented mathematics curriculum around experience and relationship (p. 100), but also to ask “What relationship do we as mathematics teachers want to have with mathematics?” Teachers and students are essentially co-learners of the same subject matter in a recursive curriculum, although they still have different roles to play in a recursive curriculum. Curriculum, as the running of the course, means not “students running towards an end where the teacher stands” but “students and the teacher running side by side together towards an emerging ending that is both familiar and strange for all runners”. The line between the acts of teaching and acts of learning should not be rigid. Just like the act of producing examples is not just an act of teaching but an act of learning due to its affordance in developing mathematical understanding (Watson & Mason, 2002), designing towards recursive curricula is not a task for the teacher alone. I have come to believe, through my mathematics learning experience afforded in this study, that both teachers and students can learn mathematics and self through designing towards recursive curricula together.

It has been difficult to conclude this study, as every single time I did so, new ideas emerged, invoking a renewal of my understanding of recursive curricula. So an ending here is, like what Dewey (1934/1980) says about one having an experience “when the material experienced runs its course to fulfillment”: “Its close is a consummation and not a cessation” (p. 35). Currently, my experience with recursive curricula has connected itself to curriculum in abundance (Jardine et al., 2006), aesthetic educational experience (Dewey, 1934/1980; Kupfer, 1983), and ethical encountering (related to Badiou, 2001 and Buber, 1937). The future is wide open.

> What we call the beginning is often the end
> And to make an end is to make a beginning.
> The end is where we start from. …
> —T.S. Eliot, *Little Gidding*, 1944

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Appendix A: Conversation Dates, Formats, and Topics

Date	**Format (order)**	**Topic**	**Participants (in pseudonyms)**
Part I*			
Apr 8, 2015	Conversation 1	Re-viewing	Violet
Aug 6, 2015	Workshop 1	Re-viewing	Violet, Lisa
Aug 10, 2015	Workshop 2	Re-viewing	Bill, Nancy, Emily
Sep 9, 2015	Conversation 2	Re-viewing	Dean
Nov 3, 2015	Conversation 3	Re-viewing	Tania
Part II			
Nov 22, 2015	Workshop 3.1	Re-viewing	Maxine, Bill, Emma
Dec 12, 2015	Workshop 3.2	Re-interpreting	Maxine, Bill
Jan 16, 2016	Workshop 3.3	Re-experiencing	Maxine
Feb 20, 2016	Workshop 3.4	Re-imaging	Maxine,
Apr 23, 2016	Workshop 3.5	Re-storying	Maxine, Violet, Rachel
Apr 25, 2016	Conversation 4	Re-view re-viewing, Workshop follow up	Emma
May 15, 2016	Workshop 3.6	Re-view re-viewing	Maxine, Violet

*The two workshops in Part I were conducted with different groups of participants who were former school teachers. Most of them chose to participate in one workshop only.

Appendix B: Original Workshop and Individual Conversation Plans

Original Workshop Plan

Topic: Reviewing in the Mathematics Curriculum

Part I: Reflect on the practice of reviewing – individual work, group discussion

A list of possible facilitation activities or questions:

1) Make two drawings showing what things were like for you and your students before and after a mathematics review is completed. Feel free to use thought bubbles or speech bubbles.
2) Make a list of 20 important words that come to mind for you when you think about reviewing mathematics. Then divide the list into two groups and copy them into two separate lists.
3) Complete the following sentences. “Reviewing mathematics is like___________.”
4) Close your eyes for 10 seconds and think about the practice of reviewing…is there any particular memory that comes to your mind or any particular bodily sense that you notice? … for example, a happy student, a tighten body?
5) If you were going to give advice to new mathematics teachers about the practice of having students review something they have learned before and its role in students’ learning, what would you say?

Each participant will complete some activities individually before sharing their responses with the group.

Part II: Connecting participants’ interpretations and mine

1) Identify the common themes in participants’ interpretations of reviewing – brain storm;

2) Connect them to a recursive mathematics curriculum by

- Linking the themes to the concept of recursion if possible – led by the researcher
- The researcher introducing a recursive curriculum as one that emphasizes learning new through a running or looping back process

Part III: Re-view reviewing

Examine, critique and revise the review examples provided by the researcher – group work & discussion

For example, an exercise that focuses on connections might give students an answer and ask them to find possible questions for it.

e.g.1, What can the question be to which x=5 is an answer?

e.g.2, What can the question be to which "*solving a linear equation"* is a part of the algorithm or solution?

Facilitation questions or activities:

1) In what ways do you see having students to engage with this kind of exercise as a form of reviewing?
2) In what ways do you see this kind of reviewing embodying a recursive curriculum?
3) Modify the exercise or propose a different form of reviewing to facilitate a recursive curriculum that fits your interpretation and imagination

Part IV: Back to the beginning

Re-visit Part I activities and questions and discuss new insights if there is any - Group discussion

Original Individual Conversation Plan

For the participants who chose to only participate in the conversation activity:

The conversation will have the same basic structure used in the workshops (see the sample workshop plan). In other words, a conversation will also include

1) both parties of the conversation sharing their interpretations of the recursive quality of the mathematics curriculum,
2) reflecting on a common teaching and learning practice or shared curriculum material,
3) discussing how the practice or material might be different in a recursive curriculum.

For the participants who chose to join both workshop and conversation activities:

The focus of the conversation will be around students' experience of re-encountering what they have learned before and how educators can help students learn something new from this experience. Two possible questions to start a conversation can be

1) If you were going to give advice to new mathematics teachers about the practice of having students review, revisit, or reencounter something they have learned before and its role, what would you say?
2) While looking back your experience in the past workshop(s), do you see any changes in your understanding regarding mathematics teaching and learning? In what ways did it change or not change?

The rest of the conversation can have the same basic structure used in the workshops (see the sample workshop plan). In other words, a conversation will also include

1) both parties of the conversation sharing their interpretations of the recursive quality of the mathematics curriculum,
2) reflecting on a common teaching and learning practice or shared curriculum material,

3) discussing how the practice or material might be different in a recursive curriculum.

Appendix C: Examples of Revised Workshop and Individual Conversation Plans

Workshop 3.1 (Re-viewing) Plan

Part I: Reflect on the practice of reviewing – individual work, group discussion

Complete one or two tasks selected from the following list individually before sharing with the group:

1) Complete the following sentences: "Reviewing mathematics is like___________."
2) If you were going to give advice to your younger self about <u>the practice of reviewing and its role in students' learning</u>, what would you say?
3) Make two or three drawings with different colours that symbolize how your understanding of the practice of reviewing has changed for you over time
4) If you are to study the practice of reviewing, what questions would you ask to lead your study?
5) Have your students surprised you in any way ...with what they did, said, or wanted to do.....anything at all…… related to reviewing?

Part II: Connecting participants' interpretations and mine

1) Identify the common themes in participants' interpretations of reviewing – brain storm;
2) Connect them to a recursive mathematics curriculum

   Recursive curriculum: stresses students' common experience of *re-encountering* what they have learned before and aims to help students learn something new from this experience.

   A recursive curriculum is defined as one that emphasizes <u>building connections</u> and <u>learning something new</u> through a <u>running or looping back</u> process.

Part III: Re-view reviewing

Examine, critique and revise the review curriculum examples provided by the participants and researcher – group work & discussion

Leading question: ***In what ways might we design/modify tasks to help students to build connections and learn something new from what they have learned/experienced/encountered before?***

Possible activities:

1) Review and modify a review task brought by participants,
    a. Give an example of a review task that you like. Specify what you like and wish to improve.

       Each participant writes down the task on a large piece of paper before sharing in the group
    b. Choose one to work on for the group
2) Generate review tasks that fit the participants' current teaching needs
    a. What are you teaching currently? Or pick a topic that you want to develop better tasks for
3) Review and modify a task brought by the researcher

Part IV: Back to the beginning

Re-visit Part I and the leading question in Part III, discuss new insights if there is any - Group discussion

Curriculum Examples for Workshop 3.1

1. Given the following questions:

   1) $? + 3 = 27$

   2) Solve $x + 3 = 27$

   3) Solve $2x + 3 = 27$

   4) Solve $-2x + 3 = 27$

   5) Solve $\frac{x}{2} + 3 = 27$

   6) Solve $\frac{2}{x} + 3 = 27$

   7) Solve $sin\theta + 3 = 27$

   8) ET added 3 to a number, he got 27. What is the number?

   9) ET is 3 years older than his sister. ET is 27 this year. How old is his sister?

   10) Where do the lines y=27 and y=x+3 meet?

   11) Solving a linear system:

       y=27

       y=x+3

   12) Given a sequence 4, 5, 6, 7,…, which term of the sequence will be 27?

Consider:

a) How are these questions similar?

b) If they are all related to the same problem, what other forms might this problem have?

Workshop 3.5 (Re-storying) Plan

Pre-workshop Invitations:

1) Bring an example of good math story or a curriculum example that promotes generating math stories
2) Think about one or a set of topics, lessons, assessments, units or courses in which you want to promote (re)generating math stories

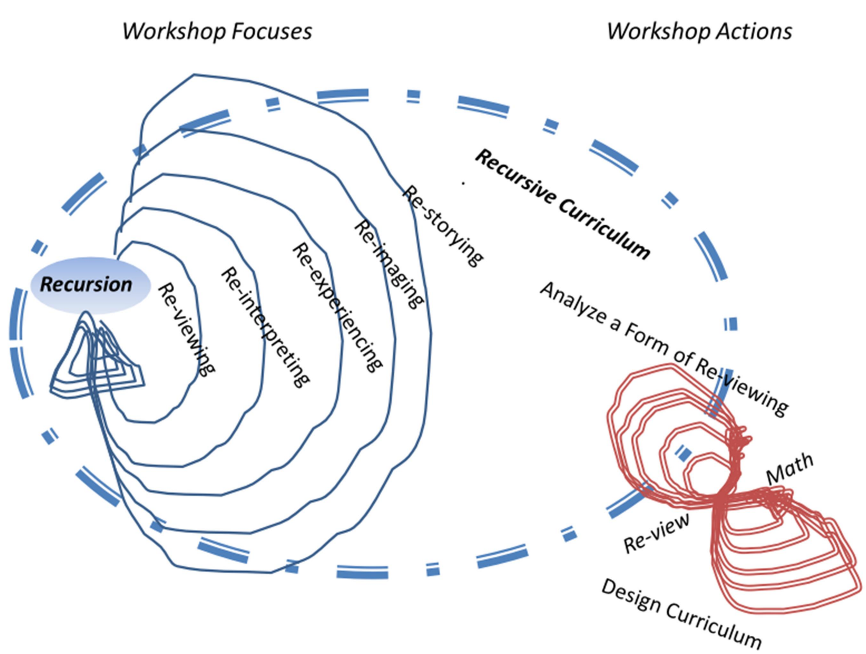
Part 0: Look back and look forward (10 minutes)

A recursive curriculum aims to help students

build connections and learn something new through re-encountering what they have learned before

Given what a recursive curriculum does, what might a recursive secondary mathematics curriculum look like?

Where have we been? Where are we now? Where are we going?



In this section, we ask: what if **re-storying** is a form of re-viewing, then

In what ways might we design curriculum to help students build connections and learn something new from what they have learned/ experienced/encountered before?

Part 1: Reflect on the process of re-storying (30 minutes)

Complete Tasks 1–4 individually, and organize the answers for Tasks 2–4 by following the format of Table 1.

Write down or draw your answers on the flip chart paper provided.

1) What comes to your mind when you hear the phrase "math story"? Write down or draw as many answers as possible.
2) Write down, respectively, the words or phrases that come to your mind when you think about good stories in general and think about good math stories. Write as many as possible.
3) Make two drawings or writings, each on a separate piece of paper, to show an experience in which you re-story a story and an experience in which you re-story (or generate a story related to) a math idea. Feel free to use icons, symbols, thought bubbles or point forms.
4) Complete the following sentences:

   "Re-storying in general is like___________."

   "Re-storying in mathematics is like ___________."

Table 1: Re-storying in General & in Math

	In General	In Math
Good story is related to…		
An experience		
Is like…		

Part 2: Connect participants' interpretations (40 minutes)

Identify the common themes in participants' interpretations of re-storying – brain storm

******10 minutes break******

Part 3: Re-story together – What might the story be? (40 minutes)

The group works together to

1) Generate stories related to one or a family of mathematical ideas (e.g., concept, process)

    a) What story might we generate to facilitate our understanding of the idea(s)?

    b) In what ways might we present the story?

2) Identify what mathematics ideas might be related to a given story

Part 4: Design curriculum to promote re-storying process (40 minutes)

Design or revise curriculum examples – group work & discussion. Possible activities:

1) Review and modify a curriculum example brought by the participants or the researcher

2) Generate a curriculum example that fits the participants' current teaching needs

Part 5: Back to the beginning (10 minutes)

Re-visit the leading workshop question in Part 0 and the results from Part 1–2, discuss:

What could a process of re-storying mathematics be?

How could a process of re-storying mathematics be prompted in curriculum design?

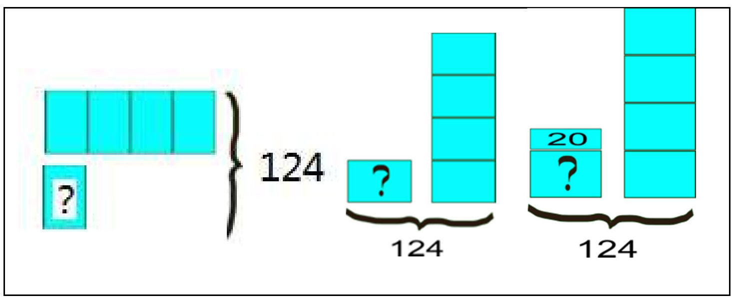
Curriculum Examples for Workshop 3.5

What math ideas might be related to the following story? What could this math story be?

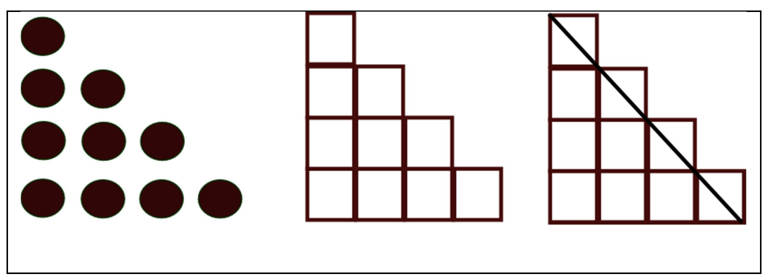
1) Given a math story in picture book format with drawings of concrete objects

2) Given a math story in abstract pictorial form, e.g.,

a.



b.



Revised Individual Conversation Plan

Questions & Activities

Study Title: Towards a Recursive Mathematics Curriculum

Note:

1) You can answer question c about recursive curriculum only.
2) If you find question c too general, you can try part I about reviewing.
3) You are welcome to go through all the questions and activities in both Part I and Part II.

Part I: Reviewing

a) Please do any number of the following activities:

1) Complete the following sentences. "Reviewing mathematics is like___________."
2) Close your eyes for 10 seconds and think about the practice of reviewing…is there any particular memory that comes to your mind or any particular bodily sense that you notice? … for example, a happy student, a tighten body?
3) Make two or three drawings with different colours that symbolize how your understanding of the practice of reviewing has changed for you over time. You can also describe your changes in writing.
4) Have your students surprised you in any way ...with what they did, said, or wanted to do.....anything at all…… related to reviewing?
5) If you were going to give advice to new mathematics teachers about the practice of having students review something they have learned before and its role in students' learning, what would you say?

b) What can reviewing in a secondary mathematics curriculum be like if we want to help students build connections and learn something new through a re-viewing process, which is a process to invite students to re-encounter what they have learned before?

Part II: Recursive Secondary Mathematics Curriculum

c) If we say that a recursive curriculum helps students to build connections and learn something new through re-encountering what they have encountered before, given what it does, what might a recursive secondary mathematics curriculum look like?

Note: You can mention general qualities of curriculum and/or give particular examples of curriculum. A curriculum example can include task, assignment, lesson plan, unit plan, course plan, assessment, and even k–12 curriculum as whole.

Appendix D: Textbook Excerpts

One comparison example from the Alberta Math10 textbook (adapted from Davis et al., 2010, p. 315)

<table>
<tr><td colspan="2">Example 2: Sketching a graph of a linear function in function notation</td></tr>
<tr><td>Sketch a graph of the linear function $f(x) = -2x + 7$.

[Solution provided below]</td><td>CHECK YOUR UNDERSTANDING

2. Sketch a graph of the linear function $f(x) = 4x - 3$.

[Answer provided below]</td></tr>
</table>

Four comparison examples from the Chinese Math9 textbook (adapted from People's Education Press, 2001, respectively p. 91, p. 99, p. 100, p.101)

Chinese Text	English Translation
例：画出函数$y = x + 0.5$的图象。 【解】 练习 1. 画出函数$y = 0.5x$的图象（先填下表，再在所给直角坐标系内描点、连线）：	Example: Sketch a graph of the function$y = x + 0.5$. [Solution provided below] Exercise: 1. Sketch a graph of the function $y = 0.5x$ (fill out the following table at first, then plot the points in the given Cartesian plane and connect the points). [A table of value, with *x* values filled and *y* values empty, provided below]
下面画正比例函数$y = 0.5x$与$y = -0.5x$的图象。先各选取两点： 再描点连线：	Next, sketch the graphs of two directly proportional functions $y = 0.5x$ and $y = -0.5x$. Firstly choose two points for each function: [two tables of values are displayed side by side] Then plot the points and connect the points: [two graphs are displayed side by side]
例 1：在同一直角坐标系内画出下列函数图象：$y = 2x + 1$，$y = -2x + 1$. 【解】	Example 1: Sketch the graphs of the following functions on the same set of Cartesian plane：$y = 2x + 1$，$y = -2x + 1$. [Solution provided below, with two tables of values displayed side by side and two graphs on the same Cartesian plane down below]
练习 1．在同一直角坐标系内画出下列函数图象：$y = 3x, y = -3x, y = 3x + 3$，$y = -3x + 3$.	Exercise: 1. Sketch the graphs of the following functions on the same set of Cartesian plane：$y = 3x, y = -3x, y = 3x + 3$，$y = -3x + 3$. [A Cartesian plane provided down below]