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THE UNIVERSITY OF ALBERTA

BOUNDARY-INITIATED WAVE PHENOMENA IN RIGID AND ELASTIC HEAT CONDUCTORS

by

Torun Sabri Öncü

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Boundary-Initiated Wave Phenomena in Rigid and Elastic Heat Conductors submitted by Torun Sabri Öncü in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The linear theory of Gurtin and Pipkin, and Chen and Gurtin is adopted to study one dimensional thermal and thermoelastic waves generated by boundary disturbances applied at the surface of a circular hole in unbounded isotropic inhomogeneous rigid and homogeneous elastic heat conductors, respectively. A ray series approach is used to generate asymptotic wavefront expansions for the field variables and general properties of the propagation processes are obtained simply and directly. Afterwards, Padé approximants are employed to extend the range of validity of these expansions for both problems. The Padé-ray solution is then specialized to the case in which the heat conduction of the media is governed by the Maxwell-Cattaneo relation and, finally, numerical results obtained for various combinations of material parameters are displayed graphically.

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CHAPTER I

Introduction

The paradox of instantantaneous propagation of thermal disturbances intrinsic to the classical theory of heat conduction in rigid materials has motivated many researchers to seek an alternative to the classical theory. Consequently, a number of theories free from this paradox have been proposed in recent years [1-6]. Almost all of these theories have also been extended to include deformable materials [7-13]. In the present work, we shall be concerned with only one of these theories. An extensive list of many others can be found in Sawatzky and Moodie [14].

Taking a departure from the fundamental concepts developed by Coleman and Noll [15] and Coleman [16], Gurtin and Pipkin [4] established their general theory of heat conduction in rigid materials with memory in 1968. This theory associates thermal disturbances with finite propagation speeds and, hence, eliminates the problem of instantaneous propagation of thermal disturbances. In 1969, Chen and Gurtin [9] extended the theory of Gurtin and Pipkin to include deformable materials with memory. The theory of Gurtin and Pipkin, and Chen and Gurtin implies two finite speeds of propagation for thermomechanical disturbances in such materials. These speeds are usually referred to as the first and second sound speeds. The first sound speed is quasi-mechanical and lies near the acoustical speed for the material whereas the second sound speed is associated with a quasi-thermal wave. Shear waves, which do not generate volume changes, remain unaffected by thermomechanical coupling. This result is in agreement with that of the thermomechanical theories based on the classical theory of heat conduction.

In this thesis we study the propagation of thermal disturbances in rigid and thermomechanical disturbances in thermoelastic materials by invoking the linearized theory of Gurtin and Pipkin, and Chen and Gurtin. We generate asymptotic wavefront expansions using a ray series approach which has been employed by Moodie and Tait [17] for rigid, Sawatzky and Moodie [14] for elastic and McCarthy, et. al. [18] for viscoelastic heat conductors. The ray series method originated from the works of Luneburg [19] in an attempt to link geometrical optics with wave optics. Constructing an asymptotic expansion for the solution of Maxwell's equation for a time harmonic field, Luneburg demonstrated that the leading term of this expansion is the geometrical optics solution. Several authors (see [20] for an extensive list) have later shown that subsequent terms account for diffraction effects. In 1946, Friedlander [21] formalized Luneburg's method to handle general-progressive waves. Using Friedlander's formulation Karal and Keller [22] extended the method to study general progressive waves in inhomogeneous isotropic elastic media. When applied to solid media, this formal asymptotic technique is often referred to as the Karal-Keller technique. The technique consists of three basic steps. First, an asymptotic series expansion involving a phase function and an infinite sequence of amplitude functions is assumed for the solution of the problem under consideration. For complicated problems, it may be necessary to express the solution as a sum of such series. This expansion or sum of expansions is then inserted into the field equations and the governing ordinary differential equations for the phase and the amplitude functions along the rays associated with the waves under consideration are found. Lastly, if possible, the initial conditions for these ordinary differential equations are obtained from the data of the problem. Otherwise, these initial conditions can be determined from the data of related canonical problems whose solutions yield the solution to the original problem after appropriate superpositions.

Since the solutions obtained from the ray series method are asymptotic wavefront expansions, they provide accurate information only near the wavefront. However, the region of validity of these expansions can be extended to larger regions of physical interest with the aid of Padé approximants. Briefly, a Padé approximant can be defined as the ratio of two polynomials that matches with the Taylor series expansion of a function up to the term whose exponent is the sum of the degrees of these polynomials plus one. Although Padé approximants have primarily been developed to sum the Taylor series expansion of a function beyond its radius of convergence, they have been successfully applied to formal asymptotic series expansions for many years [23]. All of the numerical results presented graphically in this thesis are the extensions of the ray series solutions with the use of main diagonal Padé approximants.

In Chapter II, we study the propagation of thermal transients generated by nonuniform sources applied to the boundary of a circular hole in an unbounded inhomogeneous rigid heat conductor. Nonuniform boundary sources acting on circular cavities result in both angular and radial dependency of the field variables and, consequently, complicates the application of the ray series method. To simplify the analysis, we treat those boundary disturbances which can be expressed as a finite Fourier series. This restriction enables us to employ a particular decomposition which was used by Barclay, et al. [24] in their study of unloading waves emanating from a suddenly punched hole in an axially stretched elastic plate. With this decomposition we are able to formulate several related problems in terms of the radial variable only. Although methods presented in Chapter II are developed for the boundary disturbances which can be represented by a finite Fourier series, we note that they are also useful to obtain approximate solutions for a larger class of boundary disturbances.

The propagation of one dimensional progressive waves emanating from the boundary of a circular hole in an unbounded homogeneous isotropic elastic heat conductor is studied in Chapter III. After deriving the explicit expressions for the propagation speeds of the thermoelastic waves we find that for $C_t/C_d < 1$ where C_t is the speed of purely thermal and C_d is the speed of purely elastic dilatational waves for the material, the faster and slower, whereas for $C_t/C_d \geq 1$ the slower and faster speeds are the first and second sound speeds, respectively. A similar result has been obtained by Achenbach [25], who considered only materials whose heat conduction is governed by the Maxwell-Cattaneo relation, for the cases $C_t/C_d < (1+\gamma)^{1/2}$ and $C_t/C_d > (1+\gamma)^{1/2}$ where γ is the thermoelastic coupling constant. Our result shows that the first and second sound speeds can be determined independently from γ.

Finally, conclusions based on the results obtained in Chapters II and III are summarized in Chapter IV.

CHAPTER II

Boundary-Initiated Thermal Waves in Rigid Materials

We assume that the plane rigid isotropic body is in a thermodynamic equilibrium with a uniform absolute temperature T_0. It is further assumed that the dimensions of the body in the plane are very large compared with the radius of the hole so that the body can be taken unbounded and there are no reflected waves.

Taking the origin of the plane polar coordinates r, φ on the axis of the hole of radius a, the first law of thermodynamics in the absence of external heat sources is

$$\frac{\partial e}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{q}, \tag{2.1}$$

where $e(r,\varphi,t)$ is the internal energy, $q(r,\varphi,t)$ is the heat flux and $\boldsymbol{\nabla}$ is the gradient in the plane polar coordinates.

Inhomogeneities in the conducting medium are assumed to depend on the radial coordinate r only, that is,

$$\alpha = \alpha(r,t), \quad \beta = \beta(r,t), \quad c = c(r), \quad \rho = \rho(r), \tag{2.2}$$

where α is the thermal relaxation function, β is the energy relaxation function, c is the instantaneous specific heat and ρ is the mass density. For convenience, we shall assume that the mass density is absorbed into the instantaneous specific heat and omit the mass density in the subsequent equations.

Hereafter, the mass density absorbed instantaneous specific heat will simply be referred to as the instantaneous specific heat.

With the restrictions on the heat flux and energy relaxation functions given in [4], the constitutive equations of the linearized theory of Gurtin and Pipkin are

$$e(r,\varphi,t) = e_0 + c\theta + \int_0^\infty \beta(r,s)\theta(r,\varphi,t-s)ds, \tag{2.3}$$

$$\boldsymbol{q}(r,\varphi,t) = -\int_0^\infty \alpha(r,s)\nabla\theta(r,\varphi,t-s)ds, \tag{2.4}$$

wherein

$$\theta(r,\varphi,t) = T - T_0, \tag{2.5}$$

is the temperature difference, $T(r,\varphi,t)$ the absolute temperature and e_0 the internal energy of the initial thermodynamic equilibrium.

It should be noted that the constitutive equations of the classical theory of heat conduction in rigid materials is not a special case of the above constitutive equations. However, if we choose the thermal and energy relaxation functions to be

$$\alpha(r,t) = \frac{\kappa}{\tau}e^{-t/\tau}, \quad \beta(r,t) = 0, \quad \tau > 0, \tag{2.6}$$

where $\kappa(\tau)$ is the coefficient of thermal conductivity and τ the thermal relaxation time, then the present theory reduces to the theory of heat conduction based on the Maxwell-Cattaneo relation

$$\tau \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q} = -\kappa \boldsymbol{\nabla} \theta. \tag{2.7}$$

It is convenient for the subsequent analysis to introduce the following nondimensional quantities

$$\begin{aligned} \hat{r} &= \frac{r}{a}, \quad \hat{t} = \frac{\bar{\kappa}}{a^2 \bar{c}}\, t, \quad \hat{s} = \frac{\bar{\kappa}}{a^2 \bar{c}}\, s, \quad \hat{\tau} = \frac{\bar{\kappa}}{a^2 \bar{c}}\, \tau, \\ \hat{\alpha} &= \frac{a^2 \bar{c}}{\bar{\kappa}^2}\, \alpha, \quad \hat{\beta} = \frac{a^2}{\bar{\kappa}}\, \beta, \quad \hat{c} = \frac{c}{\bar{c}}, \quad \hat{\theta} = \frac{\theta}{T_0}\, . \end{aligned} \tag{2.8}$$

where overbar indicates the quantity is evaluated at the surface of the hole. Henceforth, we shall employ these nondimensional quantities but, for convenience, omit the carets. After inserting the equations (2.3) and (2.4) into (2.1) and introducing the nondimensional quantities, we obtain

$$c\frac{\partial \theta}{\partial t} + \int_0^\infty \beta(r,s) \frac{\partial \theta}{\partial t}(r,\varphi,t-s)ds = \int_0^\infty \boldsymbol{\nabla} \cdot [\alpha(r,s) \boldsymbol{\nabla} \theta(r,\varphi,t-s)]ds. \tag{2.9}$$

Initially the body has been assumed to be in a thermodynamic equilibrium so that

$$\theta(r,\varphi,t) = \frac{\partial \theta}{\partial t}(r,\varphi,t) = 0, \quad r > 1, \quad 0 \leq 0 < 2\pi, \quad t < 0. \tag{2.10}$$

At $t = 0$ a thermal disturbance

$$\theta(1,\varphi,t) = U(\varphi,t) \tag{2.11}$$

is imposed. With the simplifying choice of a single waveform we choose

$$U = U(\varphi)Q(t) \tag{2.12}$$

where $Q(t)$ is that waveform. In this study, we shall treat those boundary temperatures which can be expressed as a finite Fourier series, that is,

$$U(\varphi) = U_1^{(0)} + \sum_{n=1}^{N}[U_1^{(n)} \cos\, n\varphi + U_2^{(n)} \sin\, n\varphi]. \tag{2.13}$$

These temperatures may therefore vanish only at isolated points and Huygens' principle guarantees that the wavefronts are circles centered on the axis of the hole.

We shall further assume that the relaxation functions α and β have well-defined Taylor series expansions about $t = 0$ [17], that is,

$$\alpha(r,t) = H(t)\sum_{i=0}^{\infty} \alpha_i^0(r)\frac{t^i}{i!}, \tag{2.14}$$

$$\beta(r,t) = H(t)\sum_{i=0}^{\infty} \beta_i^0(r)\frac{t^i}{i!}, \tag{2.15}$$

where

$$\alpha_i^0 = \frac{\partial^i \alpha}{\partial t^i}\Big|_{t=0}, \quad \beta_i^0 = \frac{\partial^i \beta}{\partial t^i}\Big|_{t=0}, \tag{2.16}$$

are the relaxation coefficients and $H(t)$ is the Heaviside step function. It suffices to specify the relaxation functions only near the time origin as the changes at the wavefronts are completely determined by the behaviour of these functions there [17]. Chen and Nunziato [26] have shown that for the second law of thermodynamics to be satisfied, α and β should obey the restrictions

$$\alpha_0^0 \geq 0, \quad \alpha_1^0 \leq 0, \quad \beta_0^0 \geq 0, \int_0^\infty \alpha(r,s)ds \geq 0. \tag{2.17}$$

We further impose the condition

$$\alpha_0^0 > 0 \tag{2.18}$$

to assure well-defined finite discontinuity wavefronts propagating at finite speeds.

(A) A Decomposition

Owing to the form of the boundary temperature given by (2.11), (2.12) and (2.13) we can separate variables in $\theta(r, \varphi, t)$ by writing

$$\theta = \theta_1^{(0)}(r,t) + \sum_{n=1}^{N} [\theta_1^{(n)}(r,t)\cos n\varphi + \theta_2^{(n)}(r,t)\sin n\varphi]. \tag{2.19}$$

It follows immediately that the coefficients in (2.19) satisfy the equations

$$c\frac{\partial \theta_\ell^{(n)}}{\partial t} + \int_0^\infty \beta(r,s)\frac{\partial \theta_\ell^{(n)}}{\partial t}(r,t-s)ds = \int_0^\infty L\theta_\ell^{(n)}(r,t-s)ds \quad \ell = 1,2, \tag{2.20}$$

where the operator L is defined by

$$L = \alpha(r,s)\frac{\partial^2}{\partial r^2} + [\frac{\partial \alpha(r,s)}{\partial r} + \frac{\alpha(r,s)}{r}]\frac{\partial}{\partial r} - \frac{n^2}{r^2}\alpha(r,s), \tag{2.21}$$

together with the initial conditions

$$\theta_\ell^{(n)} = \frac{\partial \theta_\ell^{(n)}}{\partial t} = 0, \quad r > 1, \quad t < 0 \tag{2.22}$$

and the boundary conditions

$$\theta_\ell^{(n)}\ |_{r=1} = U_\ell^{(n)} Q(t). \tag{2.23}$$

(B) Ray Series Solution

We seek progressing wave solutions to our initial value problems. These solutions are represented in terms of their asymptotic wavefront expansions as

$$\theta_\ell^{(n)} = \sum_{j=0}^{\infty} T_{\ell j}^{(n)} F_j[t - P(r)], \quad T_{\ell j}^{(n)} \equiv 0, \quad j < 0, \quad n \geq 0, \tag{2.24}$$

where the F_j's are related by

$$F_j' = F_{j-1}, \quad j = 1, 2, \ldots, \tag{2.25}$$

with the prime denoting differentiation with respect to the entire argument. The equation (2.25) enables us to determine all of the F_j's from the waveform F_0 by successive integrations.

To evaluate the coefficients $T_{\ell j}^{(n)}$ and the phase P we substitute (2.24) into (2.20), employ the expressions for α and β from (2.14) and (2.15) in the resulting equation, evaluate the terms involving integrals by means of the formula

$$\int_0^{\infty} \frac{s^k}{k!} F_j(t - s) ds = F_{j+1+k}(t), \quad j \geq 0, \tag{2.26}$$

which is obtained via integration by parts, and equate coefficients of F_j. The result of these manipulations is

$$\begin{aligned} cT_{j+1}^{(n)} = \sum_{k=0}^{j-1} \alpha_k^0 T_{\ell,j-k-1}^{(n)''} + \sum_{k=0}^{j-1} (\alpha_k^{0'} + \frac{\alpha_k^0}{r}) T_{\ell,j-k-1}^{(n)'} - 2P' \sum_{k=0}^{j} \alpha_k^0 T_{\ell,j-k}^{(n)'} \\ - P'' \sum_{k=0}^{j} \alpha_k^0 T_{\ell,j-k}^{(n)} - P' \sum_{k=0}^{j} (\alpha_k^{0'} + \frac{\alpha_k^0}{r}) T_{j-k}^{(n)} \\ + (P')^2 \sum_{k=0}^{j+1} \alpha_k^0 T_{\ell,j-k+1}^{(n)} - \frac{n^2}{r^2} \sum_{k=0}^{j-1} T_{\ell,j-k-1}^{(n)} - \sum_{k=0}^{j-1} \beta_k^0 T_{\ell,j-k}^{(n)}. \end{aligned} \tag{2.27}$$

The first of the equations in (2.26) $(j = -1)$ is

$$\{[P'(r)]^2 - c(r)/\alpha_0^0(r)\} T_{\ell 0}^{(n)} = 0. \tag{2.28}$$

Since we may require without loss of generality that $T_{\ell 0}^{(n)} \neq 0$, (2.28) gives the eikonal equation

$$[P'(r)]^2 = c(r)/\alpha_0^0(r). \tag{2.29}$$

Integration of this ordinary differential equation along a ray associated with the thermal transients gives

$$P(r) = \bar{P} \pm \int_1^r [c(\lambda)/\alpha_0^0(\lambda)]^{1/2} d\lambda \tag{2.30}$$

where $\bar{P} = P(1)$ and the $\pm$ signs are associated with outgoing and incoming waves, respectively. Since we are interested in disturbances which leave the boundary at $r = 1$ and travel into the space $1 < r < \infty$ we drop the use of double signs and choose the $+$ sign in (2.30) corresponding to outgoing waves.

Putting $j = 0$ in (2.27) gives the next equation in the sequence, the first of the transport equations, that is,

$$T_{\ell 0}^{\prime(n)} + \Big[\frac{P''}{2P'} + \frac{\alpha_0^{0'}}{2\alpha_0^0} + \frac{1}{2r} + \frac{\beta_0^0}{2P'\alpha_0^0} - \frac{P'\alpha_1^0}{2\alpha_0^0}\Big] T_{\ell 0}^{(n)} = 0. \qquad (2.31)$$

The solution of (2.31) is

$$T_{\ell 0}^{(n)} = \bar{T}_{\ell 0}^{(n)} \Big[\frac{\bar{c}\bar{\alpha}_0^0}{c(r)\alpha_0^0(r)}\Big]^{1/4} \Big[\frac{1}{r}\Big]^{1/2} e^{-W(r)}, \qquad (2.32)$$

where

$$W(r) = \frac{1}{2}\int_1^r \Big[\frac{\beta_0^0(\lambda)\alpha_0^0(\lambda) - c(\lambda)\alpha_1^0(\lambda)}{[c(\lambda)]^{1/2}[\alpha_0^0(\lambda)]^{3/2}}\Big] d\lambda. \qquad (2.33)$$

The higher order transport equations for determining $T_{\ell j}^{(n)}$, $j \geq 1$, may be obtained from (2.27) and solved to yield

$$\begin{aligned} T_{\ell j}^{(n)} = {} & \bar{T}_{\ell j}^{(n)} \Big[\frac{\bar{c}\bar{\alpha}_0^0}{c(r)\alpha_0^0(r)}\Big]^{1/4} \Big[\frac{1}{r}\Big]^{1/2} e^{-W(r)} \\ & + \frac{1}{2} e^{-W(r)} \int_1^r \Big\{ [c(\lambda)\alpha_0^0(\lambda)]^{1/2} \Big[\frac{\lambda}{r}\Big]^{1/2} \Big[\frac{c(\lambda)\alpha_0^0(\lambda)}{c(r)\alpha_0^0(r)}\Big]^{1/4} \\ & \times e^{W(\lambda)} \sum_{i=1}^{j} L_i T_{\ell, j-i}^{(n)}(\lambda) \Big\} d\lambda, \quad j = 1, 2, \ldots, \end{aligned} \qquad (2.34)$$

where L_i is the second order ordinary differential operator defined by

$$\begin{aligned} L^i = {} & \alpha_{i-1}^0 \frac{d^2}{dr^2} + \big[\alpha_{i-1}^{0'} + \frac{\alpha_{i-1}^0}{r} - 2P'\alpha_i^0\big] \frac{d}{dr} - P''\alpha_i^0 - P'\big(\alpha_i^{0'} + \frac{\alpha_i^0}{r}\big) \\ & + \frac{c}{\alpha_0^0}\, \alpha_{i+1}^0 - \beta_i^0 - \frac{n^2}{r^2}\, \alpha_{i-1}^0. \end{aligned} \qquad (2.35)$$

The solution of (2.20) subject to (2.22) and (2.23) is given by (2.24) with $P(r)$ and $T_{\ell j}^{(n)}$, determined from (2.33) and (2.34). The initial conditions $\bar{P}$ and $\bar{T}_j^{(n)}$ and the waveform F_0 are determined fr[illegible] boundary condition

$$\sum_{j=0}^{\infty} \bar{T}_{\ell j}^{(n)} F_j(t - \bar{P}) = U_\ell^{(n)} Q(t). \tag{2.36}$$

From (2.36) we may choose

$$T_{\ell j}^{(n)} = \begin{cases} U_\ell^{(n)}, & j = 0, \\ 0, & j > 0 \quad \text{or} \quad j < 0, \end{cases} \tag{2.37}$$

$$\bar{P} = 0, \quad F_0(t) = Q(t).$$

In general we shall choose $Q(t) = f(t)H(t)$ so that Duhamel's principle enables us to write

$$F_j(t) = \frac{H(t)}{j!} \frac{\partial}{\partial t} \int_0^t (t - \tau)^j f(\tau) d\tau. \tag{2.38}$$

We now specify the properties of the material by choosing

$$\alpha_i^0 = \overline{\alpha_i^0} r^{2\delta}, \quad \beta_i^0 = \overline{\beta_i^0} r^{2\omega}, \quad c = \bar{c} r^{2\omega} \tag{2.39}$$

with the superimposed bar indicating the quantity evaluated at the boundary of the hole. For economy in calculations, we have chosen the exponents for the energy relaxation coefficients and the instantaneous specific heat to be the same. It should be emphasized, however, that employing a different exponent for the instantaneous specific heat introduces no essential complications.

From (2.30), (2.37) and (2.39) we obtain

$$P(r) = \frac{(\bar{c}/\bar{\alpha}_0^0)^{1/2}}{p+1}\,[r^{p+1} - 1], \quad p \neq -1 \tag{2.40}$$

where

$$p = \omega - \delta. \tag{2.41}$$

In (2.40), $(\bar{c}/\bar{\alpha}_0^0)^{1/2}$ is the reciprocal of the speed of propagation of the thermal waves in a homogeneous medium whose properties coincide with the local values of these properties at the boundary of the hole. From (2.34), (2.37) and (2.40) it may be shown by induction that

$$T_{\ell j}^{(n)}(r) = r^{-(s+1)/2} e^{-\frac{M}{p+1}(r^{p+1}-1)} \sum_{m=-j}^{j} \tau_{\ell m j}^{(n)} r^{m(p+1)} \tag{2.42}$$

where

$$M = \frac{1}{2}\frac{\bar{\alpha}_0^0 \bar{\beta}_0^0 - \bar{c}\bar{\alpha}_1^0}{\bar{c}^{1/2}\bar{\alpha}_0^{0\,3/2}}, \tag{2.43}$$

$$s = \delta + \omega. \tag{2.44}$$

and

$$\tau_{\ell m j}^{(n)} = \begin{cases} \frac{1}{m(p+1)} \sum_{i=1}^{j-|m|+1} \sum_{k=0}^{2} \tau_{\ell, m+1-k, j}^{(n)} A_{i,k,m+1-k}^{(n)} & \text{if} \quad j > 0,\ m \neq 0,\ -j \leq m \leq j \\ -\sum_{k=1}^{j} \{\tau_{\ell,-k,j}^{(n)} + \tau_{\ell k j}^{(n)}\} & \text{if} \quad j > 0,\ m = 0 \\ U_\ell^{(n)} & \text{if} \quad j = m = 0 \\ 0 & \text{if} \quad j < 0 \text{ or } m < -j \text{ or } m > j \end{cases} \tag{2.45}$$

The auxiliary coefficients in (2.45) are

$$
\begin{aligned}
A^{(n)}_{i,0,k} &= \frac{\alpha_{i-1}}{2(\bar{c}\alpha^0_0)^{1/2}}\{[k(p+1)-\frac{(s+1)}{2}][k(p+1)-p-\frac{(s-1)}{2}]-n^2\},\\
A^{(n)}_{i,1,k} &= -\frac{k(p+1)}{(\bar{c}\bar{\alpha}^0_0)^{1/2}}\{\bar{\alpha}^0_i(\frac{\bar{c}}{\bar{\alpha}^0_0})^{1/2}+\bar{\alpha}^0_{i-1}M\},\\
A^{(n)}_{i,2,k} &= \frac{1}{2(\bar{c}\bar{\alpha}^0_0)^{1/2}}\{\bar{\alpha}^0_{i-1}M^2+2\bar{\alpha}^0_i M(\frac{\bar{c}}{\bar{\alpha}^0_0})^{1/2}+(\frac{\bar{c}}{\bar{\alpha}^0_0}\bar{\alpha}^0_{i+1}-\bar{\beta}^0_i)\}.
\end{aligned}
\tag{2.46}
$$

Eqs. (2.42) and (2.43) reveal that any discontinuous change in temperature is damped at the wavefront at a rate depending upon the factors in (2.43) and the restrictions (2.17), (2.18) guarantee that the temperature at the wavefront decays to zero with r unless $p < -1$ and $s < -1$.

As a check on our results, we construct the wavefront expansion of $\theta(r,\varphi,t)$ from the transform solution by specializing α to be given by the nondimensional form of (2.6), $\beta = 0$, $\delta = 0.5$, $\omega = 0.5$, and the boundary condition $\theta(1,\varphi,t) = H(t)$. The Laplace integral representation of $\theta(r,\varphi,t)$ for this problem is

$$\theta = \frac{1}{r}\{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\tilde{p}t-(r-1)\sqrt{\tau\tilde{p}^2+\tilde{p}}}}{\tilde{p}}\,d\tilde{p}\}. \tag{2.47}$$

Employing the asymptotic expansion of the integrand in (2.47) as $\tilde{p} \to \infty$ and calculating the first four integrals in the resulting sum, we recover the same wavefront expansion, up to four terms, obtained from the formal ray solution.

(C) <u>Padé Extensions and Numerical Results</u> In this section we carry out a detailed numerical examination of the problem treated previously by choosing an unbounded rigid conductor whose relaxation coefficients are given explicitly by

$$\alpha(r,t) = \frac{r^{2k}}{\tau} e^{-t/\tau}, \quad \beta(r,t) = 0. \tag{2.48}$$

Furthermore, we restrict our attention to the thermal disturbances for which the function $f(t) = 1$ so that

$$F_j(t) = H(t)\frac{t^j}{j!}. \tag{2.49}$$

If we let

$$x = t - P(r) \tag{2.50}$$

then at a particular station $r, \theta_\ell^{(n)}$ has the formal power series representation

$$\theta_\ell^{(n)} = \sum_{j=0}^{\infty} \frac{T_{\ell j}^{(n)}}{j!} x^j, \quad x > 0. \tag{2.51}$$

The recursive nature of the coefficients of the series (2.51) makes it relatively easy to handle for numerical computations. On the other hand, estimation of the truncation error is practically impossible owing to the relatively complicated nature of these recursion relations.

For each value of r, we sum the series until the magnitude of the difference of the last two consequtive partial sums is less than a tolerance value. For practical purposes we choose the tolerance value to be 10^{-10}. This procedure reveals that, given r, the series is convergent for a certain range of x (time elapsed after passage of the wavefront) and the rate of convergence slows down as x increases. The quantity $x_c = x_c(r)$ is the value of x beyond which the series is divergent.

To extend the numerical results beyond x_c, we employ the main diagonal Padé approximants. The L, M Padé approximant to the function represented by the series (2.51) is by definition [27]

$$[L/M] = P_L(x)/Q_M(x), \tag{2.52}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The coefficients of $P_L(x)$ and $Q_M(x)$ are determined from the series by the equation

$$\sum_{j=0}^{\infty} \frac{T_{\ell j}^{(n)}}{j!} x^j - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}), \tag{2.53}$$

with the normalization condition

$$Q_M(0) = 1. \tag{2.54}$$

For any r, we construct the sequence of $[N/N]$ Padé approximants up to $N = 10$ by solving the coefficients of the numerator and denominator polynomials directly from the linear system of equations obtained from (2.53) and (2.54) . Since the series (2.51) represents an entire function in x of exponential type, the diagonal sequence is not uniformly convergent and an oscillatory behaviour at $x = \infty$ is expected (see Baker [27], pp. 123-126).

However, analytic properties of the series manifest a strong improvement of the numerical convergence [28]. To choose a Padé approximant from the constructed finite sequence we use the following ad-hoc procedure: obtain numerical results for $0 \leq x \leq 20$ from each term in the sequence, eliminate those having apparently unstable behaviour if there are any, and choose the highest order Padé approximant from the remaining terms.

In order to examine the effect of material properties on the propagation of thermal transients, we have evaluated the Padé-ray-wavefront solutions and have displa[illegible] the results graphically in Figs. 2.1-2.4. The results given in Figs. 2.1 and 2.3 are for the boundar[illegible]ance $\theta(1, \varphi, t) = H(t)$, whereas in Figs. 2.2 and 2.4 $\theta(1, \varphi, t) = (1 +$ [illegible] $)$.

Figs. 2.1 and 2.2 reveal that thermal transients propagate with higher speeds if the degree of inhomogeneity of the heat flux relaxation function exceeds that of the instantaneous specific heat. In other words, the influence of a thermal distrubance spreads quicker in materials for which the instantaneous conductivity increases faster or decreases slower with r than the instantaneous specific heat. Since the discontinuous jump and the decay of the temperature at the wavefront depend not only on the relative but also on the absolute magnitudes of the degree of inhomogeneities, a general statement for the effect of inhomogeneities on these quantities is not possible.

In Figs. 2.3 and 2.4, we examine how the size of the thermal relaxation time affects the propagation pattern of thermal disturbances. These plots affirm that the behaviour of the temperature tends to the one predicted by the classical theory of heat conduction as τ decreases. Larger thermal relaxation times result in slower propagation speeds and larger discontinuous jumps together with smaller decay rates at the wavefront.

Fig. 2.5 is a snapshot of a particular thermal wave generated by the boundary disturbance of Figs. 2.2 and 2.4. Discontinuities appearing at the boundary of the hole are due to the resolution of the graph.

Although we have dealt only with thermal disturbances which can be represented by a finite Fourier series, the method presented here is applicable to a broader class of boundary temperatures of physical interest. If a function $g(\varphi)$ is continuous in $[-\pi, \pi]$, $g(-\pi) = g(\pi)$ and $g'(\varphi)$ is sectionally continuous, then the Fourier series for $g(\varphi)$ converges uniformly [29]. In this case, summing the series using a finite number of terms provides a good approximation of the function. As an illustrative example, we choose the boundary temperature to be

$$\theta(1, \varphi, t) = H(t)(\pi^2 - \varphi^2) \tag{2.55}$$

and compare the function (2.55) and its 10 term Fourier series approximation graphically in Fig. 2.6. The results obtained for the 10 term Fourier series expansion of the boundary condition (2.55) are given graphically in Fig. 2.7.

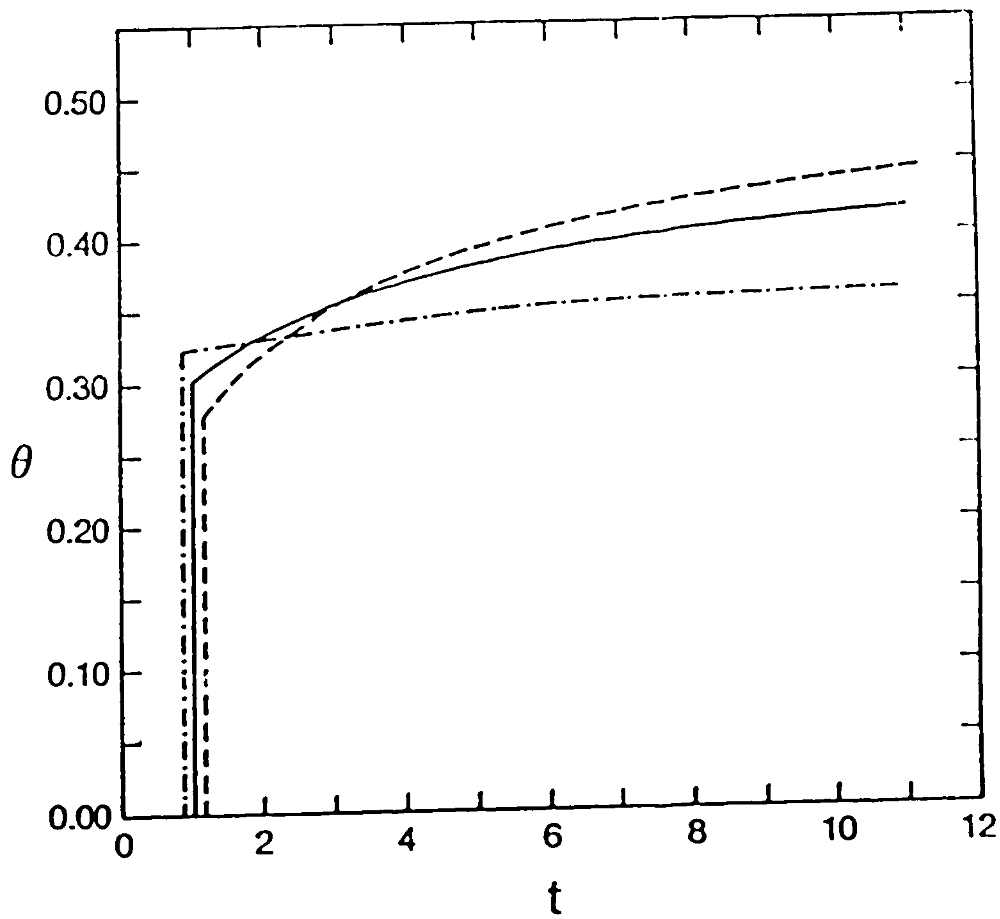

Fig. 2.1. Variation of non-dimensional temperature with non-dimensional time at $r = 2.0$ for $\tau = 1.0$, $\delta = 0.5$, $\omega = 0.5(-)$, $\delta = 0.3$, $\omega = 0.7(--)$, $\delta = 0.7$, $\omega = 0.3(-\cdot-)$

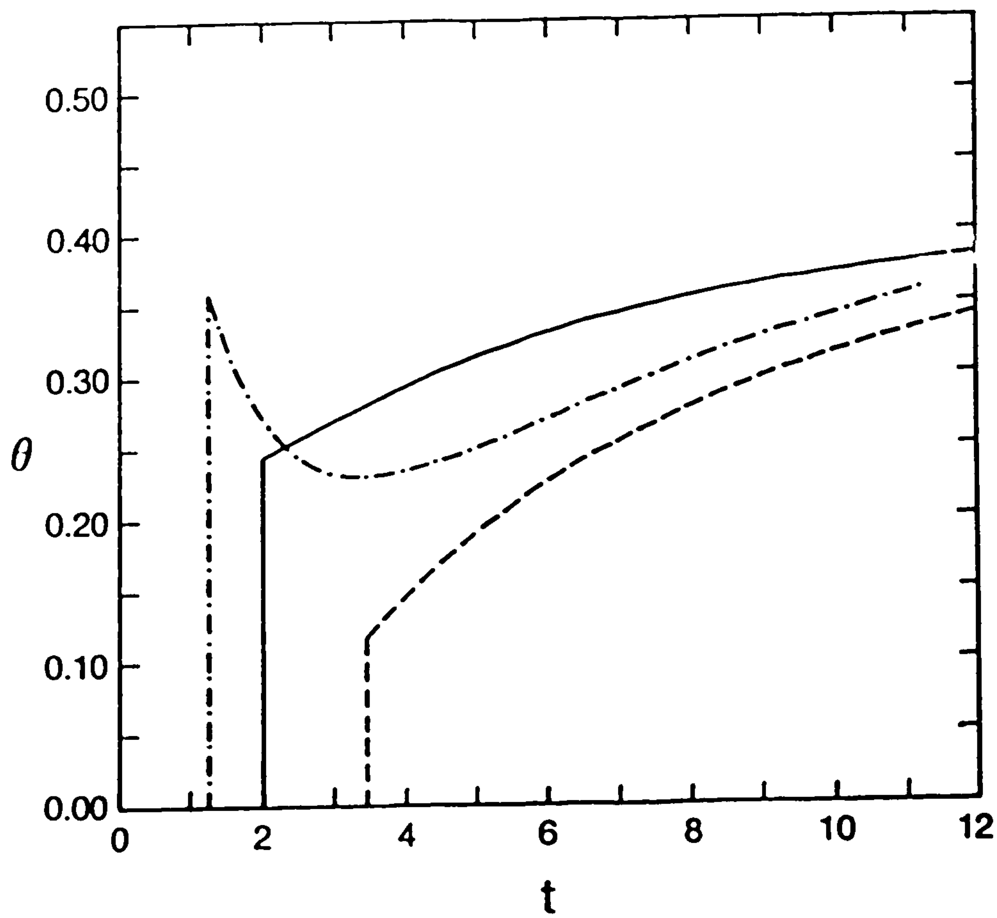

Fig. 2.2. Variation of non-dimensional temperature with non-dimensional time at $r = 3.0$, $\varphi = 0$ for $\tau = 1.0$, $\delta = 0.5$, $\omega = 0.5(-)$, $\delta = 0.3$, $\omega = 0.7(--)$, $\delta = 0.7$, $\omega = 0.3(-\cdot-)$

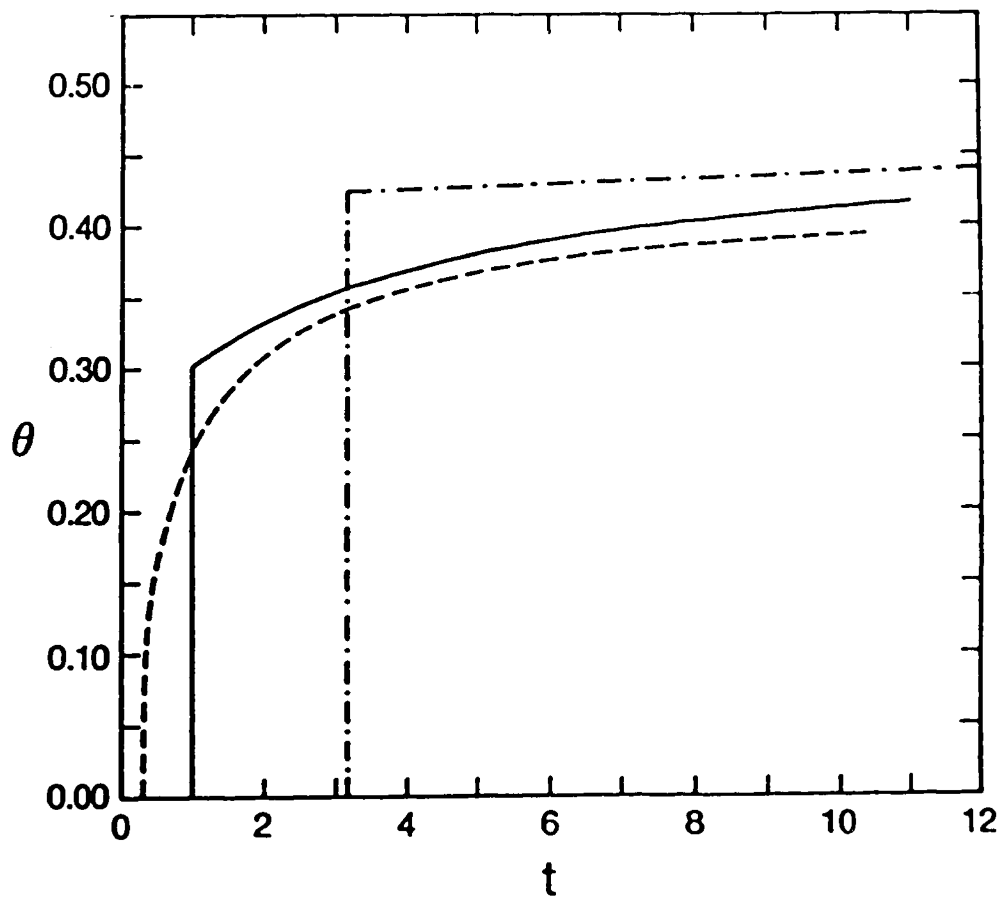

Fig. 2.3. Variation of non-dimensional temperature with non-dimensional time at $r = 2.0$ for $\delta = 0.5$, $\omega = 0.5$, $\tau = 1.0(-)$, $\tau = 0.1(---)$, $\tau = 10.0(- \cdot -)$

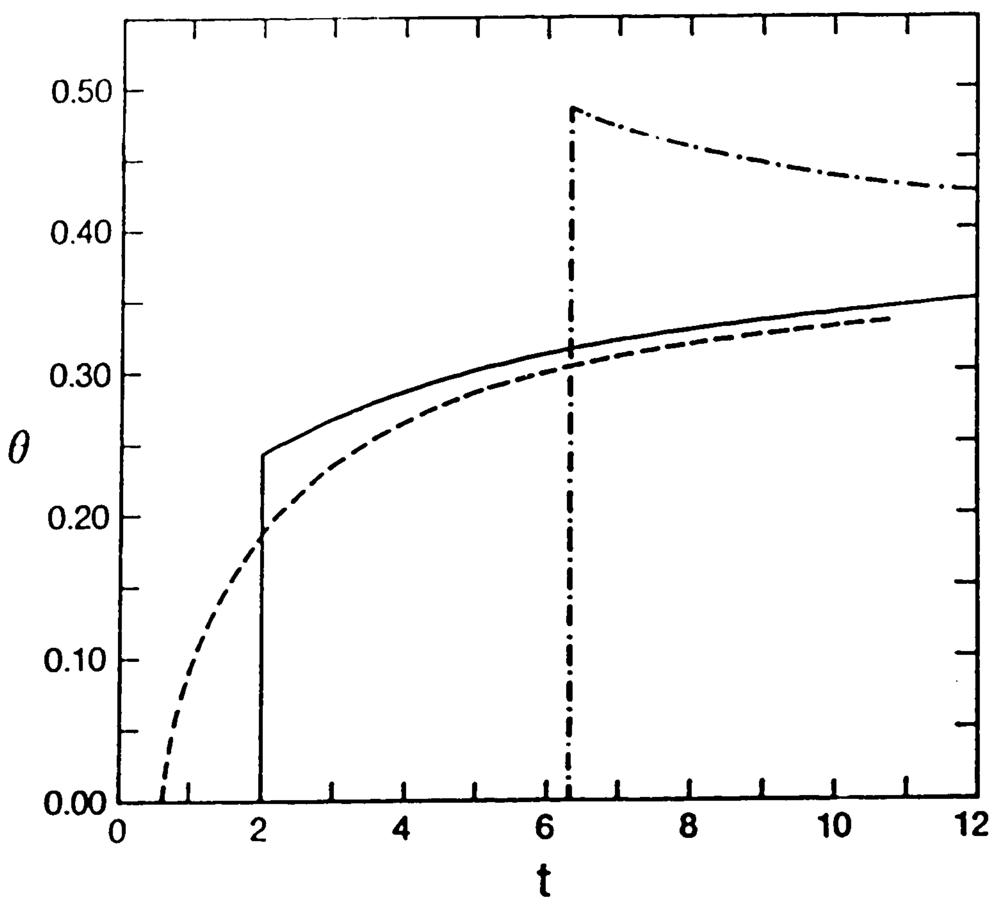

Fig. 2.4. Variation of non-dimensional temperature with non-dimensional time at $r = 3.0$, $\varphi = 0$ for $\delta = 0.5$, $\omega = 0.5$, $\tau = 1.0(-)$, $\tau = 0.1(--)$, $\tau = 10.0(- \cdot -)$

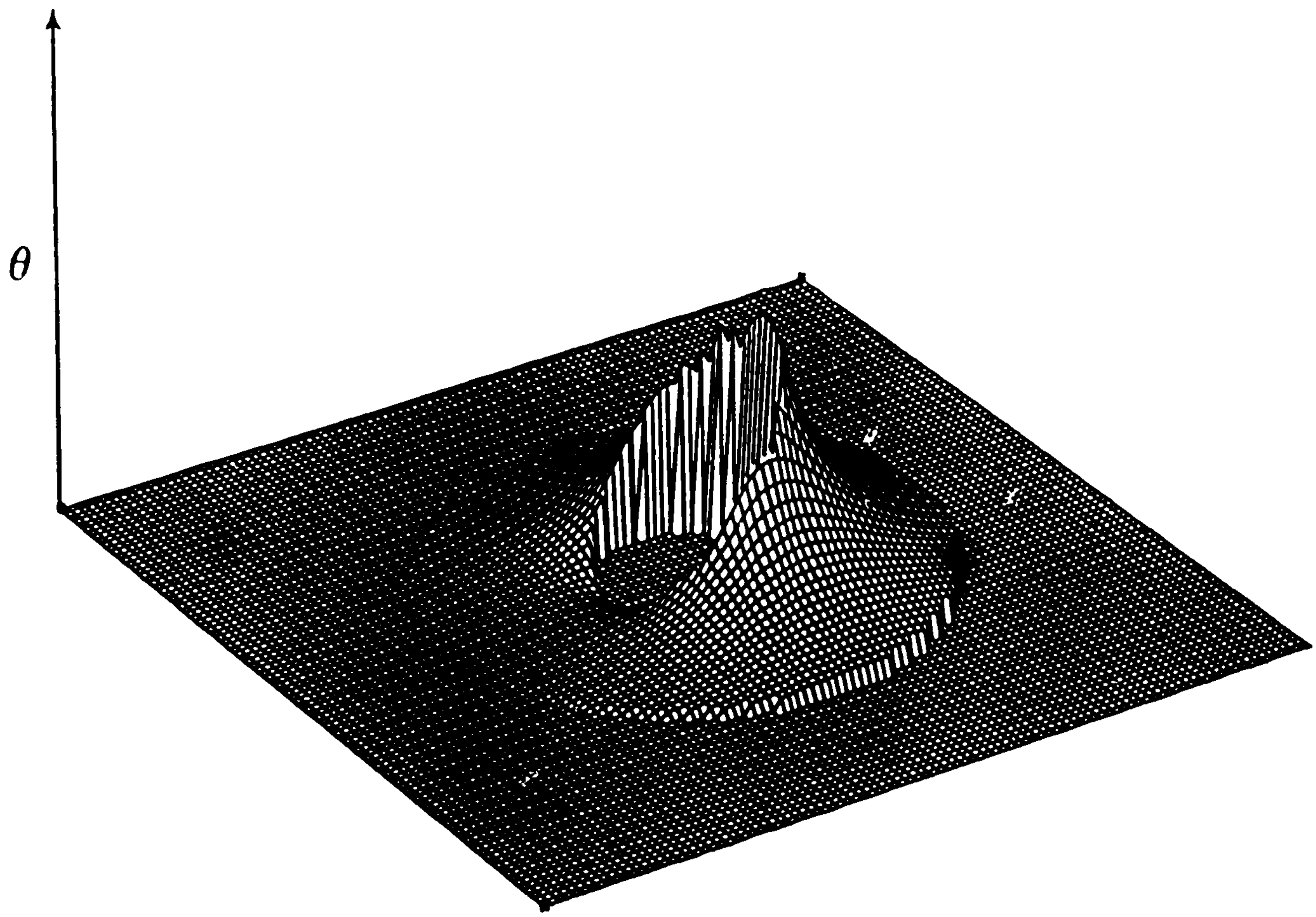

Fig. 2.5. **Variation of non-dimensional** temperature with non-dimensional

r and φ at $t = 2$ for $\delta = 0.5$, $\omega = 0.5$, $\tau = 1.0$

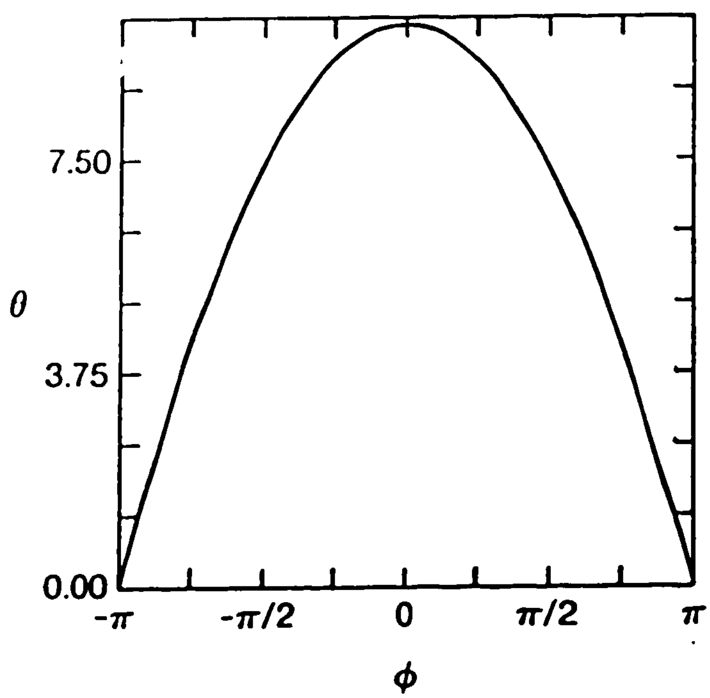

Fig. 2.6. Comparison of the boundary temperature (2.55) and its 10-term Fourier series expansion

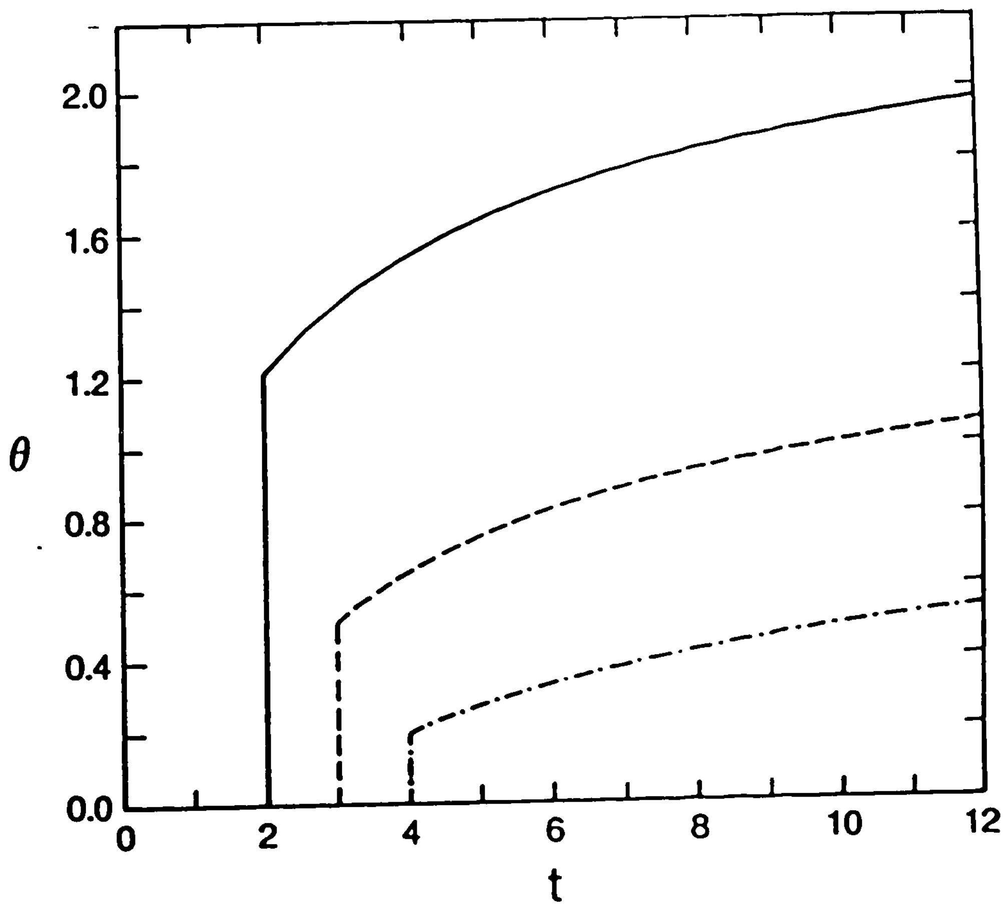

Fig. 2.7. Variation of non-dimensional temperature with non-dimensional **time at** $r = 3.0$, $\varphi = 0(-)$, $r = 4.0$, $\varphi = \pi/4(--)$, $r = 5.0$, $\varphi = \pi/2(-\cdot-)$ for $\delta = 0.5$, $\omega = 0.5, \tau = 1.0$

CHAPTER III

Boundary-Initiated Progressive Waves in Theromelastic Materials

Consider an unbounded homogeneous isotropic thermoelastic plate which occupies the region $0 < a < r < +\infty$ in a plane polar coordinate system r, φ. Initially the plate is undistorted, at rest and in thermal equilibrium with a uniform absolute temperature T_o. In any departure from this thermodynamic equilibrium the field variables are assumed to be functions of time t and radial coordinate r only. It is further assumed that departures from the equilibrium are small, that is, the displacement gradient and the relative temperature change are small at all times when compared to unity. That is,

$$\sqrt{\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\varphi}{\partial r}\right)^2} << 1 \quad \text{and} \quad |T - T_o| << T_0, \tag{3.1}$$

for all t, where $T(r,t)$ is the absolute temperature, and $u_r(r,t)$ and $u_\varphi(r,t)$ are the radial and angular displacements, respectively. As a result of the above assumptions the linearized equations of linear momentum and energy in the absence of body forces and external heat sources are

$$\rho_0 \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\varphi\varphi}), \tag{3.2}$$

$$\rho_0 \frac{\partial^2 u_\varphi}{\partial t^2} = \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{2}{r}\,\sigma_{r\varphi}, \tag{3.3}$$

$$\rho_0 \frac{\partial e}{\partial t} = tr\{\boldsymbol{\sigma}\frac{\partial \boldsymbol{\epsilon}}{\partial t}\} - \nabla \cdot \boldsymbol{q}, \tag{3.4}$$

where ρ_0 is the uniform mass density of the initial equilibrium, $\sigma_{rr}, \sigma_{r\varphi}$ and $\sigma_{\varphi\varphi}$ are the plane polar components of the symmetric stress tensor $\boldsymbol{\sigma}(r,t)$, $e(r,t)$ is the specific internal energy, $\boldsymbol{q}(r,t)$ is the heat flux and $\boldsymbol{\varepsilon}(r,t)$ is the strain tensor with the plane polar components

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{r\varphi} = \varepsilon_{\varphi r} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad \varepsilon_{\varphi\varphi} = \frac{u_r}{r}. \tag{3.5}$$

In equation (3.4), ∇ is the gradient in the appropriate coordinate system and tr denotes the trace. We also introduce the specific entropy $\eta(r,t)$, the specific free energy

$$\psi(r,t) = e - T\eta, \tag{3.6}$$

and the temperature gradient

$$\boldsymbol{g}(r,t) = \nabla T, \tag{3.7}$$

for future reference.

For the model to be completed, we assume that the present values of $\psi = \psi(r,t)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(r,t)$, $\boldsymbol{q} = \boldsymbol{q}(r,t)$, $\eta = \eta(r,t)$ and $e = e(r,t)$ are given by the response functionals $\Psi, \sum, \boldsymbol{Q}, N$ and E as

$$\begin{gathered} \psi = \Psi(\Lambda^t), \quad \boldsymbol{\sigma} = \textstyle\sum(\Lambda^t), \quad \boldsymbol{q} = \boldsymbol{Q}(\Lambda^t), \\ \eta = N(\Lambda^t) \quad \text{and} \quad e = E(\Lambda^t), \end{gathered} \tag{3.8}$$

where E is connected to Ψ and N through the relation

$$E = \Psi - TN, \tag{3.9}$$

an . . . ..al history array Λ^t for the thermoelastic plate is assumed as

$$\Lambda^t = (\varepsilon, T, T^t, g^t). \tag{3.10}$$

In identity (2.10), $\bar{T}^t$ and $\bar{g}^t$ are the summed histories of absolut tem- perature and temperature gradient at r up to time t which are defined as

$$\bar{T}^t(r,s) = \int_0^s T(r,t-\lambda)d\lambda \quad \text{and} \quad \bar{g}^t(r,s) = \int_0^s g(r,t-\lambda)d\lambda. \tag{3.11}$$

If the array (ε, T, g) and the response functionals (3.8) satisfy the hypothesis of the theorem of Chen and Gurtin [9], then the free energy functional Ψ determines $\sum, N$ and Q from the relations

$$\sum = \rho_0 \frac{\partial \Psi}{\partial \varepsilon}, \tag{3.12}$$

$$N = -\frac{\partial \Psi}{\partial T}, \tag{3.13}$$

and

$$Q \cdot h = -\rho_0 T \left[\frac{d}{dz}\psi(\varepsilon, T, \bar{T}^t, \bar{g}^t + zh)\right]_{t=0}, \tag{3.14}$$

where h is any constant nonzero vector consistent with the hypothesis of the theorem. The components of the partial derivative of Ψ with respect to the strain tensor ε are

$$\left(\frac{\partial \Psi}{\partial \varepsilon}\right)_{x_i x_j} = \frac{\partial \Psi}{\partial \varepsilon_{x_j x_i}}, \quad i,j = 1,2 \tag{3.15}$$

where $x_1 = r$ and $x_2 = \varphi$. Following Gurtin and Pipkin [4] and Chen and Gurtin [9], we then choose the free energy functional Ψ as follows:

$$\Psi(\Lambda^t) = \frac{1}{\rho_0}\hat{\psi}(\boldsymbol{\varepsilon}, T) - \int_0^\infty \beta'(s)\bar{T}^t(r,s) + \frac{1}{2\rho_0 T\alpha(0)}\Big[\int_0^\infty \alpha'(s)\boldsymbol{g}^t(r,s)ds\Big]\cdot\Big[\int_0^\infty \alpha'(s)\boldsymbol{g}^t(r,s)ds\Big], \tag{3.16}$$

where $\alpha(s)$ is the thermal relaxation function, $\beta(s)$ the energy relaxation function, $\alpha'(s)$ and $\beta'(s)$ their respective derivatives, and $\alpha(0)$ the instantaneous conductivity. We further specialize the function $\hat{\psi}(\boldsymbol{\varepsilon}, t)$ to be the free energy function of the classical theory of thermoelasticity. For isotropic materials $\hat{\psi}(\boldsymbol{\varepsilon}, t)$ takes the form [30],

$$\hat{\psi}(\boldsymbol{\varepsilon}, T) = \rho_0\hat{\psi}_0 + \mu\, tr\{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}\} + \frac{\lambda}{2}(tr\,\boldsymbol{\varepsilon})^2 - (3\lambda+2\mu)\alpha_t(T-T_0)tr\,\boldsymbol{\varepsilon} - \frac{\rho_0 c}{2T_0}(T-T_0)^2, \tag{3.17}$$

where λ and μ are the isothermal Lamé constants, α_t the coefficient of linear thermal expansion, c the specific heat at constant volume and $\hat{\psi}_0 = \Psi(\mathbf{0}, T_0, \bar{T}_0^t, \bar{\mathbf{0}}^t)$ the specific free energy in the initial equilibrium state. With the above choice of the free energy functional, the stress tensor, specific internal energy, and heat flux obtained from equations (3.9) and (3.12)-(3.14) are

$$\sigma_{rr} = (2\mu+\lambda)\varepsilon_{rr} + \lambda\varepsilon_{\varphi\varphi} - (3\lambda+2\mu)\alpha_t\theta, \tag{3.18}$$

$$\sigma_{r\varphi} = 2\mu\varepsilon_{r\varphi}, \tag{3.19}$$

$$\sigma_{\varphi\varphi} = (2\mu+\lambda)\varepsilon_{\varphi\varphi} + \lambda\varepsilon_{rr} - (3\lambda+2\mu)\alpha_t\theta, \tag{3.20}$$

$$e = e_0 + \frac{1}{\rho_0}(3\lambda + 2\mu)\alpha_t T_0(\varepsilon_{rr} + \varepsilon_{\varphi\varphi}) + c\theta - \int_0^\infty \text{[illegible]}\, s)\bar{\theta}^t(r,s)ds, \tag{3.21}$$

$$\boldsymbol{q} = \int_0^\infty \alpha'(s)\bar{\boldsymbol{g}}^t(\cdot, s)ds, \tag{3.22}$$

where

$$\theta(r,t) = T - T_0, \tag{3.23}$$

is the temperature difference and $e_0 = E(\mathbf{0}, T_0, \bar{T}_0^t, \bar{\mathbf{0}}^t)$ is the specifi energy of the initial equilibrium state. We omitted the second order terms in $\boldsymbol{\varepsilon}, \theta$ and $\boldsymbol{g}^t$ from the above constitutive equations. If we further impose the conditions

$$\lim_{s \to +\infty} s^2\alpha(s) < \infty, \quad \lim_{s \to +\infty} s^2\beta(s) < \infty, \tag{3.24}$$

the integrals in (3.21) and (3.22) may be integrated by parts [4] to yield

$$e = e_0 + \frac{1}{\rho_0}(3\lambda + 2\mu)\alpha_t T_0(\varepsilon_{rr} + \varepsilon_{\varphi\varphi}) + c\theta + \int_0^\infty \beta(s)\theta(r, t-s)ds, \tag{3.25}$$

$$\boldsymbol{q} = -\int_0^\infty \alpha(s)\nabla\theta(r, t-s)ds. \tag{3.26}$$

We again note that Fourier's law of heat conduction is not a special case of (3.26). However, if we choose the heat flux relaxation function $\alpha(s)$ as

$$\alpha(s) = \frac{\kappa}{\tau}\, e^{-s/\tau}, \quad \tau > 0, \tag{3.27}$$

where κ is the coefficient of thermal conductivity and τ is the thermal relaxation time, then (3.26) reduces to the Maxwell-Cattaneo relation

$$\tau\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q} = -\kappa\nabla\theta. \tag{3.27a}$$

In this case, if we also assume that the free energy functional is independent of the summed history of the absolute temperature or, equivalently,

$$\beta(s) = 0, \tag{3.28}$$

then the present constitutive equations reduce to the constitutive equations of the linear theory of Lord and Shulman [7].

Upon substitution of the constitutive relations (3.18)-(3.20) and (3.25),(3.26) into the field equations, together with the use of strain displacement relations (3.5), we arrive at the following thermoelastic equations

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} - \frac{1}{C_d^2}\frac{\partial^2 u}{\partial t^2} = \frac{(3\lambda + 2\mu)}{(2\mu + \lambda)}\alpha_t \frac{\partial \theta}{\partial r}, \tag{3.29}$$

$$\rho_0 c\frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\alpha_t T_0 \frac{\partial}{\partial t}[\frac{\partial u_r}{\partial r} + \frac{u_r}{r}] + \int_0^\infty \rho_0 \beta(s)\frac{\partial \theta}{\partial t}(r, t - s)ds$$
$$= \int_0^\infty \alpha(s)[\frac{\partial^2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}]\theta(r, t - s)ds, \tag{3.30}$$

and the shear equation

$$\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} - \frac{1}{C_s^2}\frac{\partial^2 u_\varphi}{\partial t^2} = 0, \tag{3.31}$$

where

$$C_d = [\frac{2\mu + \lambda}{\rho_0}]^{1/2} \tag{3.32}$$

is the isothermal velocity of dilatational waves and

$$C_s = \left[\frac{\mu}{\rho_0}\right]^{1/2} \tag{3.33}$$

is the velocity of shear waves. In (3.30) the term $tr\{\sigma \frac{\partial \epsilon}{\partial t}\}$ has been neglected since it gives rise to second order terms only. The above equations reveal that shear waves which produce no volume changes are independent from thermal effects. Thus, the theory of Gurtin and Pipkin, and Chen and Gurtin agrees with the classical theory of thermoelasticity in this regard.

In the present work we restrict our attention to the waves generated by suddenly applied uniform temperature change and uniform surface tractions at the boundary of the circular hole. Consequently, we specify the initial and the boundary conditions as

$$\theta(r,t) = \frac{\partial \theta}{\partial t}(r,t) = u_r(r,t) = \frac{\partial u_r}{\partial t}(r,t) = u_\varphi(r,t) = \frac{\partial u_\varphi}{\partial t}(r,t) = 0,$$

$$a < r < +\infty, \; -\infty < t \leq 0, \tag{3.34}$$

$$\theta(a,t) = \theta_1(t), \; \sigma_{rr}(a,t) = \sigma_1(t), \; \sigma_{r\varphi}(a,t) = \sigma_2(t) \quad 0 < t < +\infty. \tag{3.35}$$

We shall again assume that the thermal relaxation function $\alpha(t)$ and the energy relaxation function $\beta(t)$ have well-defined Taylor series expansions at $t = 0$. Therefore,

$$\alpha(t) = H(t) \sum_{i=0}^{\infty} \alpha_i^0 \frac{t^i}{i!}, \tag{3.36}$$

$$\beta(t) = H(t) \sum_{i=0}^{\infty} \beta_i^0 \frac{t^i}{i!}, \tag{3.37}$$

where

$$\alpha_i^0 = \frac{d^i\alpha}{dt^i}|_{t=0}, \quad \beta_i^0 = \frac{d^i\beta}{dt^i}|_{t=0}, \tag{3.38}$$

are the relaxation coefficients and $H(t)$ is the Heaviside step function. The relaxation coefficients still obey the restrictions (2.17) and (2.18).

Prior to further study of the problem we introduce the following nondimensional quantitities:

$$\begin{aligned}
\hat{r} &= \frac{r}{a}, & (\hat{t}, \hat{s}, \hat{\tau}) &= \frac{\kappa}{a^2\rho_0 c}(t, s, \tau) \\
\hat{\theta} &= \frac{\theta}{T_0}, & (\hat{u}_r, \hat{u}_\varphi) &= \frac{2\mu+\lambda}{(3\lambda+2\mu)\alpha_t T_0}\left(\frac{u_r}{a}, \frac{u_\varphi}{a}\right) \\
(\hat{\sigma}_{rr}, \hat{\sigma}_{r\varphi}, \hat{\sigma}_{\varphi\varphi}) &= \frac{(\sigma_{rr}, \sigma_{r\varphi}, \sigma_{\varphi\varphi})}{(3\lambda+2\mu)\alpha_t T_0}, & (\hat{C}_d, \hat{C}_s) &= \frac{a\rho_0 c}{\kappa}(C_d, C_s) \\
\hat{\alpha} &= \frac{a^2\rho_0 c}{\kappa^2}\alpha, & \hat{\beta} &= \frac{a^2\rho_0}{\kappa}\beta.
\end{aligned} \tag{3.39}$$

Henceforward, we use these nondimensional quantities but drop the carets to avoid notational complications.

(A) Ray Series Solution

Assuming the radial displacement u_r is generated by the scalar potential $\Phi(r, t)$ as

$$u_r = \frac{\partial\Phi}{\partial r}, \tag{3.40}$$

the nondimensional equations associated with thermoelastic waves obtained from (3.29), (3.30), (3.39) and (3.40) are

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} - \frac{1}{C_d^2}\frac{\partial^2\Phi}{\partial t^2} = \theta, \tag{3.41}$$

$$\frac{\partial \theta}{\partial t} + \int_0^\infty \beta(s) \frac{\partial \theta}{\partial t}(r, t-s) ds + \gamma \frac{\partial}{\partial t} [\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r}]$$
$$= \int_0^\infty \alpha(s) [\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}] \theta(r, t-s) ds, \quad (3.42)$$

where

$$\gamma = \frac{(3\lambda + 2\mu)^2 \alpha_t^2 T_0}{\rho_0 c (2\mu + \lambda)}, \quad (3.43)$$

is the thermoelastic coupling constant. Upon combining equations (3.41) and (3.42) we find that thermoelastic waves Φ, θ satisfy the integro-partial differential equation

$$L\{\Phi, \theta\} = 0, \quad (3.44)$$

where

$$Lf = \frac{\partial}{\partial t} [\frac{1}{C_d^2} \frac{\partial^2}{\partial t^2} - (1 + \gamma)(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r})] f$$
$$+ \int_0^\infty \beta(s) \frac{\partial}{\partial t} [\frac{1}{C_d^2} \frac{\partial^2}{\partial t^2} - (\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r})] f(r, t-s) ds$$
$$- \int_0^\infty \alpha(s) (\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}) [\frac{1}{C_d^2} \frac{\partial^2}{\partial t^2} - (\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r})] f(r, t-s) ds. \quad (3.45)$$

As a consequence of the assumption (3.40) the nondimensional constitutive equations of σ_{rr} and $\sigma_{\varphi\varphi}$ become

$$\sigma_{rr} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\nu}{r} \frac{\partial \Phi}{\partial r} - \theta, \quad (3.46)$$

$$\sigma_{\varphi\varphi} = \nu \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} - \theta, \quad (3.47)$$

where

$$\nu = 1 - 2\frac{C_s^2}{C_d^2}, \tag{3.48}$$

and the initial and boundary conditions which generate the thermoelastic waves considered here can be written as

$$\theta(r,t) = \frac{\partial\theta}{\partial t}(r,t) = \Phi(r,t) = \frac{\partial\Phi}{\partial t}(r,t) = 0, \quad 1 < r < +\infty, \ -\infty < t \leq 0, \tag{3.49}$$

$$\theta(1,t) = \theta_1(t), \ \frac{\partial^2\Phi}{\partial r^2} + \frac{\nu}{r}\frac{\partial\Phi}{\partial r}|_{r=1} = \sigma_1(t) + \theta_1(t), \quad 0 < t < +\infty. \tag{3.50}$$

After nondimensionalization, the shear equation remains identical to its dimensional counterpart and reads

$$\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} - \frac{1}{C_s^2}\frac{\partial^2 u_\varphi}{\partial t^2} = 0, \tag{3.51}$$

whereas the related initial and boundary conditions take the forms

$$u_\varphi(r,t) = \frac{\partial u_\varphi}{\partial t}(r,t) = 0, \quad -\infty < t \leq 0, \quad 1 \leq r < +\infty, \tag{3.52}$$

$$\sigma_{r\varphi}(1,t) = \sigma_2(t), \quad 0 < t < +\infty, \tag{3.53}$$

where the nondimensional constitutive equation for $\sigma_{r\varphi}$ is

$$\sigma_{r\varphi} = \frac{C_s^2}{C_d^2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right). \tag{3.54}$$

In this section we apply the ray series method to solve the problems given by (3.44) and (3.51) subject to conditions (3.49),(3.50) and (3.52),(3.53), respectively. Consequently, we represent the scalar potential $\Phi(r,t)$, the absolute temperature $\theta(r,t)$ and the angular displacement $u_\varphi(r,t)$ in terms of their asymptotic expansions

$$\Phi(r,t) = \sum_{n=2}^{\infty} \phi_n(r) F_n(t - P(r)), \quad \phi_n \equiv 0, \quad n < 2, \tag{3.55}$$

$$\theta(r,t) = \sum_{n=0}^{\infty} T_n(r) F_n(t - P(r)), \quad T_n \equiv 0, \quad n < 0, \tag{3.56}$$

$$u_\varphi(r,t) = \sum_{n=1}^{\infty} U_n(r) F_n(t - S(r)), \quad U_n \equiv 0, \quad n < 1, \tag{3.57}$$

where the F_n's are related by

$$F_n' = F_{n-1}, \quad n = 1, 2, \ldots, \tag{3.58}$$

with the prime denoting differentiation with respect to the entire argument. The equation (3.58) enables us to determine all of the F_n's from the waveform F_0 by successive integrations.

We first solve the thermoelastic problem given by equations (3.44), (3.49) and (3.50). The ray-series solution of the shear problem may be obtained in a similar fashion. To determine the coefficients $T_n(r)$ and the phase P, we substitute (3.56) into (3.44), employ the expressions for α and β from (2.37) and (2.38) in the resulting equation and equate the coefficients of F_{n-3}.

The terms involving integrals are evaluated by means of the formula

$$\int_0^\infty \frac{s^i}{i!} F_n(t-s)ds = F_{n+1+i}(t), \quad i \geq 0 \tag{3.59}$$

which results from integration by parts. The result of the above manipulations is

$$\frac{1}{C_d^2}T_n - (1+\gamma)\{(P')^2 T_n - P'(2T'_{n-1} + \frac{1}{r}T_{n-1}) + T''_{n-2} + \frac{1}{r}T'_{n-2}\}$$
$$= [\frac{(P')^2}{C_d^2} - (P')^4]\sum_{j=0}^{n}\alpha_j^0 T_{n-j} + [2(P')^3 - \frac{P'}{C_d^2}]\sum_{j=0}^{n-1}\alpha_j^0(2T'_{n-j-1} + \frac{1}{r}T_{n-j-1})$$
$$+ [\frac{1}{C_d^2} - 6(P')^2]\sum_{j=0}^{n-1}\alpha_j^0(T''_{n-j-1} + \frac{1}{r}T'_{n-j-2}) - (P')^2\sum_{j=0}^{n-2}\alpha_j^0 \frac{1}{r^2}T_{n-j-2}$$
$$+ P'\sum_{j=0}^{n-3}\alpha_j^0[4T'''_{n-j-3} + \frac{6}{r}T''_{n-j-3} - \frac{2}{r^2}T'_{n-j-3} + \frac{1}{r^3}T_{n-j-3}]$$
$$- \sum_{j=0}^{n-4}\alpha_j^0[T^{(iv)}_{n-j-4} + \frac{2}{r}T'''_{n-j-4} - \frac{1}{r^2}T''_{n-j-4} + \frac{1}{r^3}T'_{n-j-4}]$$
$$+ [(P')^2 - \frac{1}{C_d^2}]\sum_{j=0}^{n-1}\beta_j^0 T_{n-j-1} - P'\sum_{j=0}^{n-2}\beta_j^0(2T'_{n-j-2} + \frac{1}{r})$$
$$+ \sum_{j=0}^{n-3}\beta_j^0(T''_{n-j-3} + \frac{1}{r}T'_{n-j-3})$$
$$n = 0, 1, 2, \ldots \tag{3.60}$$

For the sake of brevity, we have omitted the terms involving P'', P''' and $P^{(iv)}$ from (3.60) since it is shown later that P' is constant.

The first equation in the sequence (3.60), that is, the equation for $n = 0$, is independent of the terms omitted and of the form

$$[\alpha_0^0(P')^4 - (1+\gamma+\frac{\alpha_0^0}{C_d^2})(P')^2 + \frac{1}{C_d^2}]T_0(r) = 0. \tag{3.61}$$

This result proves that P' is independent of r and therefore, all higher order derivatives of P vanishes. Since we may require without loss of generality that $T_0(r) \neq 0$, (3.61) reduces to the eikonal equation

$$\alpha_0^0(P')^4 - (1+\gamma+\frac{\alpha_0^0}{C_d^2})(P')^2 + \frac{1}{C_d^2} = 0, \tag{3.62}$$

whose solution can be expressed as

$$(P')^2 = \frac{1}{2\alpha_0^0}\{(1+\gamma+\frac{\alpha_0^0}{C_d^2}) \pm \Gamma^{1/2}\}, \tag{3.63}$$

where

$$\Gamma = (\frac{\alpha_0^0}{C_d^2} + \gamma - 1)^2 + 4\gamma. \tag{3.64}$$

Integrating the ordinary differential equation (3.63) along the ray associated with the thermoelastic waves we then obtain

$$\begin{aligned} P(r) &= \bar{P} \pm (r-1)P_1', \\ P(r) &= \bar{P} \pm (r-1)P_2', \end{aligned} \tag{3.65}$$

where

$$\begin{aligned} P_1' &= \{\frac{1}{2\alpha_0^0}[(1+\gamma+\frac{\alpha_0^0}{C_d^2}) + \Gamma^{1/2}]\}^{1/2}, \\ P_2' &= \{\frac{1}{2\alpha_0^0}[(1+\gamma+\frac{\alpha_0^0}{C_d^2}) - \Gamma^{1/2}]\}^{1/2}, \end{aligned} \tag{3.66}$$

$\bar{P} = P(1)$ and the $\pm$ signs designate the waves propagating in the positive and negative directions, respectively. In the subsequent analysis we drop the use of double signs and choose the $+$ sign that corresponds to waves leaving the boundary of the circular hole and propagating into the region $1 < r < +\infty$.

Equation (3.65) reveals that according to the Gurtin and Pipkin, and Chen and Gurtin theory, thermoelastic disturbances generate two wavefronts located at

$$\begin{aligned} t &= P_1(r) = \bar{P} + (r-1)P_1', \\ t &= P_2(r) = \bar{P} + (r-1)P_2' \end{aligned} \tag{3.67}$$

propagating at constant speeds

$$\begin{aligned} v_1 &= \frac{1}{P_1'} = \Big\{ \frac{1}{2\alpha_0^0}[(1+\gamma+\frac{\alpha_0}{C_d^2}) + \Gamma^{1/2}]\Big\}^{-1/2}, \\ v_2 &= \frac{1}{P_2'} = \Big\{ \frac{1}{2\alpha_0^0}[(1+\gamma+\frac{\alpha_0^0}{C_d^2}) - \Gamma^{1/2}]\Big\}^{-1/2}, \end{aligned} \tag{3.68}$$

$(v_1 < v_2)$, respectively. In their study of one dimensional progressive waves in thermoelastic half spaces Sawatzky and Moodie [14] have shown that the faster wavespeed is greater and the slower wavespeed is less than the speed of purely mechanical dilatational waves for the material. However, observing that v_1 is a decreasing and v_2 is an increasing function of the thermoelastic coupling constant, the inequalities

$$\begin{aligned} v_1 &< \Big\{ \frac{1}{2\alpha_0^0}\Big[(1+\frac{\alpha_0^0}{C_d^2}) + \Big|\frac{\alpha_0^0}{C_d^2} - 1\Big|\Big]\Big\}^{-1/2}, \\ v_2 &> \Big\{ \frac{1}{2\alpha_0^0}\Big[(1+\frac{\alpha_0^0}{C_d^2}) - \Big|\frac{\alpha_0^0}{C_d^2} - 1\Big|\Big]\Big\}^{-1/2}, \end{aligned} \tag{3.69}$$

obtained from (3.64) and (3.68) reveal that for $\alpha_0^0/C_d^2 < 1$, $v_2 > C_d$ and $v_1 < (\alpha_0^0)^{1/2}$, where $(\alpha_0^0)^{1/2}$ is the speed of purely thermal waves for the material [17], while for $\alpha_0^0/C_d^2 \geq 1$, $v_2 > (\alpha_0^0)^{1/2}$ and $v_1 < C_d$. Therefore, the speed of the fast wave is not only greater than the speed of purely mechanical dilatational waves but also greater than the speed of purely thermal waves.

Likewise, the speed of the slow wave is less than the smaller of the purely mechanical and purely thermal wavespeeds. Furthermore, setting $\gamma = 0$ in (3.68) we find that for $\alpha_0^0/C_d^2 < 1$, $v_2 = C_d$ and $v_1 = (\alpha_0^0)^{1/2}$ whereas for $\alpha_0^0/C_d^2 \geq 1$, $v_2 = (\alpha_0^0)^{1/2}$ and $v_1 = C_d$. These results lead us to the conclusion that for $\alpha_0^0/C_d^2 < 1$ the fast wave is a quasi-elastic wave and the slow wave is a quasi-thermal wave. For $\alpha_0^0/C_d^2 \geq 1$ the roles of the fast and slow waves are reversed and the fast wave is a quasi-thermal wave while the slow wave is a quasi-elastic wave.

Returning to our original analysis we observe from the above results the asymptotic wavefront expansions for Φ and θ should consist of the sum of expansions at each wavefront. Therefore, we replace the expansions (3.16) and (3.17) by

$$\Phi(r,t) = \sum_{\ell=1}^{2}\sum_{n=2}^{\infty} \phi_{\ell n}(r) F_n\big(t - P_\ell(r)\big), \tag{3.70}$$

$$\theta(r,t) = \sum_{\ell=1}^{2}\sum_{n=0}^{\infty} T_{\ell n}(r) F_n\big(t - P_\ell(r)\big), \tag{3.71}$$

where F_n satisfies (3.66) as before and the phase functions $P_1(r)$ and $P_2(r)$ are given by (3.67). When (3.71) is substituted into (3.72) the result is again (3.60) except that P', T_n are replaced by P_1', T_{1n} and P_2', T_{2n}. For $n = 0$, (3.60) gives the eikonal equation (3.62). Putting $n = 1$ in (3.60) yields the first of the so-called transport equations which is

$$T_{\ell_0}' + \Big\{\frac{1}{2r} + \frac{\big((P_\ell')^2 - 1/C_d^2\big)}{2P_\ell'}\Big[\frac{\alpha_1^0 (P_\ell')^2 - \beta_0^0}{(1 + \gamma + \frac{\alpha_0^0}{C_d^2}) - 2\alpha_0^0 (P_\ell')^2}\Big]\Big\} T_{\ell_0} = 0. \tag{3.72}$$

Solving this equation we find

$$T_{\ell_0}(r) = \bar{T}_{\ell_0} r^{-1/2} e^{-W_\ell (r-1)} \tag{3.73}$$

where

$$W_\ell = \frac{((P'_\ell)^2 - 1/C_d^2)}{2P'_\ell}\left[\frac{\alpha_1^0 (P'_\ell)^2 - \beta_0^0}{(1+\gamma+\frac{\alpha_0^0}{C_d^2}) - 2\alpha_0^0 (P'_\ell)^2}\right], \quad \ell = 1,2, \tag{3.74}$$

and $\bar{T}_{\ell_0} = T_{\ell_0}(1)$. Thus, for a thermoelastic material whose thermomechanical behaviour is characterized by the constitutive equations of the linear theory of Gurtin and Pipkin, and Chen and Gurtin, a discontinuous change in temperature is attenuated at each wavefront according to (3.73) and (3.74). The identity

$$(1+\gamma+\frac{\alpha_0^0}{C_d^2}) - 2\alpha_0^0 (P'_\ell)^2 = (-1)^\ell \Gamma^{1/2}, \tag{3.75}$$

obtained from (3.66) and the previous discussion on the wavespeeds indicate that the sign of W_ℓ is completely determined by the sign of $(\alpha_0^0 (P'_\ell)^2 - \beta_0^0)$. Therefore, in one-dimensional circular geoemetry, a discontinuous change in temperature decays with r if

$$\alpha_1^0 (P'_\ell)^2 - \beta_0^0 \leq 0, \tag{3.76}$$

and the restrictions (2.17) on α_1^0 and β_0^0 guarantee that (3.76) always holds. As will be clear later, the condition (3.76) suffices also for discontinuous changes in strain and stress to decay to zero as r increases.

The higher order transport equations for $T_{\ell_1}, T_{\ell_2}, \ldots$ determined from (3.60) are now solved to yield

$$T_{\ell n}(r) = r^{-1/2} e^{-W_\ell(r-1)} \{ T_{\ell n} + \int_1^r (r')^{1/2} e^{W_\ell(r'-1)} Q_{\ell n}(r') dr' \}, \qquad (3.77)$$

$$\ell = 1,,2, \; j = 1,2,\ldots$$

where

$$Q_{\ell n}(r) = \frac{(-1)^\ell}{2P'_\ell \Gamma^{1/2}} \{ \sum_{k=1}^{n} (L^1_{\ell k} T_{\ell, n-k} + L^2_{\ell k} T_{\ell, n-1-k} + L^3_{\ell k} T_{\ell, n-2-k}) \}, \qquad (3.78)$$

and $L^1_{\ell k}, L^2_{\ell k}$ and $L^3_{\ell k}$ are the ordinary differential operators defined by

$$\begin{aligned} L^1_{\ell k} f \equiv & \left[(1+\gamma)\delta_{1k} + \left(\frac{1}{C_d^2} - 6(P'_\ell)^2 \right) \alpha^0_{k-1} \right] (f'' + \frac{1}{r} f') \\ & + P'_\ell \left[\left(2(P'_\ell)^2 - \frac{1}{C_d^2} \right) \alpha^0_k - \beta^0_{k-1} \right] (2f' + \frac{1}{r} f) \\ & - \left((P'_\ell)^2 - \frac{1}{C_d^2} \right) \left((P'_\ell)^2 \alpha^0_{k+1} - \beta^0_k \right) f - (P'_\ell)^2 \alpha^0_{k-1} \frac{f}{r^2}, \\ L^2_{\ell k} f \equiv & P'_\ell \alpha^0_{k-1} (4f''' + \frac{6}{r} f'' - \frac{2}{r^2} f' + \frac{1}{r^3} f) + \beta^0_{k-1} (f'' + \frac{1}{r} f'), \\ L^3_{\ell k} f \equiv & -\alpha^0_{k-1} (f^{(iv)} + \frac{2}{r} f''' - \frac{1}{r^2} f'' + \frac{1}{r^3} f'). \end{aligned} \qquad (3.79)$$

In (3.79), δ_{1k} indicates the Kronecker delta. Since $T_{\ell n} \equiv 0$ for $n < 0$, some of the summations in (3.78) have been extended to $k = n$. It can be proved by induction that the amplitude functions $T_{\ell n}$ are of the form

$$T_{\ell n}(r) = r^{-1/2} e^{-W_\ell(r-1)} \sum_{j=-n}^{n} t_{\ell j n} r^j, \quad \ell = 1,2, \quad n = 0,1,2,\ldots \qquad (3.80)$$

Substituting (3.80) into (3.77) and simplifying gives the recursion relation

$$t_{\ell j n} = \begin{cases} \dfrac{(-1)^{\ell}}{2jP'_{\ell}\Gamma^{1/2}} \displaystyle\sum_{k=1}^{n-|j|+1} \Big(\sum_{m=0}^{2} A^{m}_{\ell k j} t_{\ell, j+1-m, n-k} + \sum_{m=0}^{3} B^{m}_{\ell k j} t_{\ell, j+2-m, n-1-k} & \\ \quad + \displaystyle\sum_{m=0}^{4} C^{m}_{\ell k j} t_{\ell, j+3-m, \ldots -k} \Big) & j \neq 0, \quad -n \le j \le n, \\ \tilde{T}_{\ell n} - \displaystyle\sum_{k=1}^{n} (t_{\ell,-k,n} + t_{\ell,k,n}), & j = 0, \ n \geq 0 \\ 0 & j < -n \quad \text{or} \quad j > n, \end{cases} \tag{3.81}$$

where

$$A^{0}_{\ell k j} = (j + \frac{1}{2})^2 \{ (1 + \gamma)\delta_{1k} + (\frac{1}{C_d^2} - 6(P'_{\ell})^2)\alpha^{0}_{k-1} \} - (P'_{\ell})^2 \alpha^{0}_{k-1},$$

$$A^{1}_{\ell k j} = -2j \{ W_{\ell} [(1+\gamma)\delta_{1k} + (\frac{1}{C_d^2} - 6(P'_{\ell})^2)\alpha^{0}_{k-1}]$$

$$- P'_{\ell} [(\ldots)^2 - \frac{1}{C_d^2})\alpha^{0}_{k} - \beta^{0}_{k-1}] \},$$

$$A^{2}_{\ell k j} = W^{2}_{\ell} \ldots \delta_{1k} + (\frac{1}{C_d^2} - 6(P'_{\ell})^2)\alpha^{0}_{k-1} \}$$

$$- 2W_{\ell}P'_{\ell} \{ (2(P'_{\ell})^2 - \frac{1}{C_d^2})\alpha^{0}_{k} - \beta^{0}_{k-1} \} - ((P'_{\ell})^2 - \frac{1}{C_d^2})(\alpha^{0}_{k+1}(P'_{\ell})^2 - \beta^{0}_{k}),$$

$$B^{0}_{\ell k j} = P'_{\ell}\alpha^{0}_{k-1} M_4(j+2),$$

$$B^{1}_{\ell k j} = \beta^{0}_{k-1} M_2(j+1) - 2P'_{\ell}W_{\ell}\alpha^{0}_{k-1} M_3(j+1),$$

$$B^{2}_{\ell k j} = \{ 6P'_{\ell}W^{2}_{\ell}\alpha^{0}_{k-1} - W_{\ell}\beta^{0}_{k-1} \} M_1(j),$$

$$B^{3}_{\ell k j} = W^{2}_{\ell}\beta^{0}_{k-1} - 4P'_{\ell}W^{3}_{\ell}\alpha^{0}_{k-1},$$

$$C^{0}_{\ell k j} = -\alpha^{0}_{k-1} M_5(j+3), \quad C^{1}_{\ell k j} = W_{\ell}\alpha^{0}_{k-1} M_4(j+2),$$

$$C^{2}_{\ell k j} = -W^{2}_{\ell}\alpha^{0}_{k-1} M_3(j+1), \quad C^{2}_{\ell k j} = 2W^{3}_{\ell}\alpha^{0}_{k-1} M_1(j), \quad C^{4}_{\ell k j} = -W^{4}_{\ell}\alpha^{0}_{k-1}. \tag{3.82}$$

In the above equations the auxiliary functions $M_1(j)$ to $M_5(j)$ are

$$
\begin{aligned}
M_1(j) &= 2K_1(j) + 1, \\
M_2(j) &= K_1(j) + K_2(j), \\
M_3(j) &= 6K_2(j) + 6K_1(j) - 1, \\
M_4(j) &= 4K_3(j) + 6K_2(j) - 2K_1(j) + 1, \\
M_5(j) &= K_4(j) + 2K_3(j) - K_2(j) + K_1(j).
\end{aligned}
\tag{3.83}
$$

where

$$
\begin{aligned}
&K_1(j) = j - \tfrac{1}{2}, && K_2(j) = (j - \tfrac{1}{2})(j - \tfrac{3}{2}), \\
&K_3(j) = (j - \tfrac{1}{2})(j - \tfrac{3}{2})(j - \tfrac{5}{2}), && K_4(j) = (j - \tfrac{1}{2})(j - \tfrac{3}{2})(j - \tfrac{5}{2})(j - \tfrac{7}{2}).
\end{aligned}
\tag{3.84}
$$

Repeating the above procedure for the amplitude coefficients $\phi_{\ell n}(r)$ in the expansion (3.31) for $\Phi(r,t)$, we then obtain

$$
\phi_{\ell n}(r) = r^{-\frac{1}{2}} e^{-W_\ell (r-1)} \sum_{j=-(n-2)}^{n-2} \varphi_{\ell j n} r^j, \quad \ell = 1, 2, \; n = 2, 3, \ldots \tag{3.85}
$$

where

$$
\varphi_{\ell j n} = \begin{cases}
\dfrac{(-1)^{\varphi}}{2j P'_\ell \Gamma^{1/2}} \displaystyle\sum_{k=1}^{n-|j|-1} \Big(\sum_{m=0}^{2} A^m_{\ell k j} \varphi_{\ell, j+1-m, n-k} + \sum_{m=0}^{3} B^m_{\ell k j} \varphi_{\ell, j+2-m, n-1-k} \\
\quad + \displaystyle\sum_{m=0}^{4} C^m_{\ell k j} \varphi_{\ell, j+3-m, n-2-k} \Big) & j \neq 0, \; -(n-2) \leq j \leq n-2, \\
\bar{\Phi}_{\ell n} - \displaystyle\sum_{k=1}^{n-2} (\varphi_{\ell, -k, n} + \varphi_{\ell, k, n}), & j = 0, \; n \geq 2, \\
0, & j < -(n-2) \text{ or } j > (n-2).
\end{cases}
\tag{3.86}
$$

However, the amplitude coefficients $T_{\ell n}(r)$ and $\phi_{\ell n}(r)$ are not free but connected through the field equations (3.41) and (3.42) for θ and Φ. It is easy to verify that inserting the expansions (3.70) and (3.71) into one of the field equations, say (3.41), after some algebra the relationship between $T_{\ell n}$ and $\Phi_{\ell n}$ can be found as

$$
\begin{aligned}
(j+\frac{3}{2})^2\varphi_{\ell,j+2,n} &- 2(j+1)W_\ell\varphi_{\ell,j+1,n} + W_\ell^2\varphi_{\ell,j,n} \\
&- 2(j+1)P_\ell'\varphi_{\ell,j+1,n+1} + 2P_\ell'W_\ell\varphi_{\ell,j,n+1} \\
&+ ((P_\ell')^2 - \frac{1}{C_d^2})\varphi_{\ell,j,n+2} = t_{\ell jn}, \\
&\ell = 1,2, \quad j = 0,1,\dots,n, \quad n = 0,1,2,\dots
\end{aligned}
\tag{3.87}
$$

For $j=0$, the use of (3.81) and (3.86) in (3.87) gives the relationship between $\bar{T}_{\ell n}$ and $\bar{\phi}_{\ell n}$ which is

$$
\begin{aligned}
&((P_\ell')^2 - \frac{1}{C_d^2})\bar{\phi}_{\ell,n+2} + 2P_\ell'W_\ell\bar{\phi}_{\ell,n+1} + W_\ell^2\bar{\phi}_{\ell,n} - \bar{T}_{\ell n} \\
&= ((P_\ell')^2 - \frac{1}{C_d^2})\sum_{k=1}^{n}(\varphi_{\ell,-k,n+2} + \varphi_{\ell,k,n+2}) + 2P_\ell'W_\ell\sum_{k=1}^{n-1}(\varphi_{\ell,-k,n+1} + \varphi_{\ell,k,n+1}) \\
&\quad + W_\ell^2\sum_{k=1}^{n-2}(\varphi_{\ell,-k,n} + \varphi_{\ell,k,n}) - \frac{9}{4}\varphi_{\ell,2,n} + 2W_\ell\varphi_{\ell,1,n} + 2P_\ell'\varphi_{\ell,1,n+1} \\
&\quad - \sum_{k=1}^{n}(t_{\ell,-k,n} + t_{\ell,k,n}), \qquad \ell = 1,2, \quad n = 0,1,2,\dots
\end{aligned}
\tag{3.88}
$$

On the other hand, the recursive use of (3.81) and (3.82), together with (3.88), reduces (3.87) to an identity for $j \geq 1$.

So far, we have obtained the solution of the thermoelastic problem (3.44) subject to initial and boundary conditions (3.49) and (3.50). This solution is given by (3.31) and (3.32) where $P_\ell(r), T_{\ell n}(r)$ and $\Phi_{\ell n}(r)$ are determined from (3.66), (3.67) and (3.80)-(3.86). The initial values $\bar{P}, \bar{T}_{\ell n}$ and $\bar{\Phi}_{\ell n}$ are to be found from the source conditions (3.49) and (3.50) with the use of (3.88). In order to complete the solution, the waveform F_0, which in turn fixes the wavefunctions F_n from the relations (3.58), should be determined from the source conditions as well.

Let us consider the following source conditions

$$\theta(1,t) = H(t), \quad \frac{\partial^2\Phi(r,t)}{\partial r^2} + \frac{\nu}{r}\frac{\partial\Phi(r,t)}{\partial r}|_{r=1} = 0, \tag{3.89}$$

$$\theta(1,t) = 0, \quad \frac{\partial\Phi(r,t)}{\partial r^2} + \frac{\nu}{r}\frac{\partial\Phi(r,t)}{\partial r}|_{r=1} = H(t), \tag{3.90}$$

which correspond to purely thermal and purely mechanical unit step disturbances, respectively. We shall refer to these canonical problems as Problem 1 and Problem 2 for convenience. The solution to the problem corresponding to the general source conditions (3.49), (3.50) can then be obtained with the appropriate superposition of the solutions to Problems 1 and 2 and the use of Duhamel's theorem.

Let $\theta^{(1)}, \Phi^{(1)}$ and $\theta^{(2)}, \Phi^{(2)}$ be the respective solutions to the two canonical problems. Substituting (3.70) and (3.71) into (3.89) and (3.90) and using (3.81), (3.82) we get

$$\sum_{\ell=1}^{2}\sum_{n=0}^{\infty} \bar{T}_{\ell n}^{(m)} F_n^{(m)}(t - \bar{P}_\ell^{(m)}) = \begin{cases} H(t), & m = 1, \\ 0, & m = 2, \end{cases} \tag{3.91}$$

$$\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\{(P'_\ell)^2\varphi^{(m)}_{\ell,j,n+2}+2P'_\ell W_\ell\varphi^{(m)}_{\ell,j,n+1}+W_\ell^2\varphi^{(m)}_{\ell jn}$$
$$+\left[(j+3/2)^2-2(j+3/2)\frac{C_s^2}{C_d^2}\right]\varphi^{(m)}_{\ell,j+2,n}-\left[2(j+1)-2\frac{C_s^2}{C_d^2}\right]W_\ell\varphi^{(m)}_{\ell,j+1,n}$$
$$-\left[2(j+1)-2\frac{C_s^2}{C_d^2}\right]P'_\ell\varphi_{\ell,j+1,n+1}\}F_n^{(m)}(t-\bar{P}_\ell^{(m)})=\begin{cases}0, & m=1,\\ H(t), & m=0.\end{cases} \tag{3.92}$$

From these two equations we then choose

$$\bar{P}_\ell^{(m)}=0,\quad F_0^{(m)}(t)=H(t),\quad m=1,2,\quad \ell=1,2 \tag{3.93}$$

$$\bar{T}_{10}^{(1)}+\bar{T}_{20}^{(1)}=1,\quad (P'_1)^2\bar{\phi}_{12}^{(1)}+(P'_2)^2\bar{\phi}_{12}^{(1)}=0, \tag{3.94}$$

$$\bar{T}_{10}^{(2)}+T_{20}^{(2)}=0,\quad (P'_1)^2\bar{\phi}_{12}^{(2)}+(P'_2)^2\bar{\phi}_{22}^{(2)}=1, \tag{3.95}$$

$$\sum_{\ell=1}^{2}\bar{T}_{\ell n}^{(m)}=0,\quad m=1,2,\ n=1,2\ \ldots \tag{3.96}$$

$$\sum_{\ell=1}^{2}\sum_{j=-n}^{n}\{(P'_\ell)^2\varphi^{(m)}_{\ell,j,n+2}+2P'_\ell W_\ell\varphi^{(m)}_{\ell,j,n+1}+W_\ell^2\varphi^{(m)}_{\ell jn}$$
$$+\left[(j+3/2)^2-2(j+3/2)\frac{C_s^2}{C_d^2}\right]\varphi^{(m)}_{\ell,j+2,n}-\left[2(j+1)-2\frac{C_s^2}{C_d^2}\right]W_\ell\varphi^{(m)}_{\ell,j+1,n}$$
$$-\left[2(j+1)-2\frac{C_s^2}{C_d^2}\right]P'_\ell\varphi_{\ell,j+1,n+1}\}=0,\quad m=1,2,\ n=1,2,\ldots \tag{3.97}$$

We further note that for more general source conditions where the Heaviside step function $H(t)$ is replaced by arbitrary functions $f^{(1)}(t)$ and $f^{(2)}(t)$ in Problems 1 and 2, respectively, the wavefunctions $F_n^{(m)}(t)$, $n\geq 1$ are determined from the waveform $F_0^{(m)}(t) = f^{(m)}(t)$ with the use of Duhamel's theorem [14] as

$$F_n^{(m)}(t) = \frac{H(t)}{n!}\frac{\partial}{\partial t}\int_0^t (t-\xi)^n f^{(m)}(\xi)d\xi. \tag{3.98}$$

It is easy to solve from (3.50), (3.88) and (3.89) that the coefficients $\bar{\phi}_{\ell,n+2}^{(m)}$ and $T_{\ell,n}^{(m)}$ for $n = 1, 2, \ldots$ are

$$\begin{aligned}
\varphi_{\ell,n+2}^{(m)} = &\sum_{k=1}^{n}(\varphi_{\ell,-k,n+2}^{(m)} + \varphi_{\ell,k,n+2}^{(m)}) + (-1)^{\ell}\alpha_0^0\Gamma^{-1/2}\sum_{q=1}^{2}\{2P_q'W_q\varphi_{q,0,n+1}^{(m)} \\
&+ W_q^2\varphi_{q,0,n}^{(m)} + (9/4 - 3C_s^2(\frac{1}{C_d^2} - (P_{3-\ell}')^2))\varphi_{q,2,n}^{(m)} \\
&- 2(1 - C_s^2(1/C_d^2 - (P_{3-\ell}')^2))W_q\varphi_{q,1,n}^{(m)} \\
&- 2(1 - C_s^2(1/C_d^2 - (P_{3-\ell}')^2))P_q'\varphi_{q,1,n+1}^{(m)} \\
&+ C_d^2(1/C_d^2 - (P_{3-\ell}')^2))S_{qn}^{(m)} + C_d^2(P_{3-\ell}')^2\sum_{k=1}^{n}(t_{q,-k,n} + t_{q,k,n})\},
\end{aligned} \tag{3.99}$$

$$\begin{aligned}
T_{\ell n}^{(m)} = &\sum_{k=1}^{n}(t_{\ell,-k,n}^{(m)} + t_{\ell,k,n}^{(m)}) + ((P_\ell')^2 - 1/C_d^2)\varphi_{\ell,0,n+2}^{(m)} \\
&+ 2P_\ell'W_\ell\varphi_{\ell,0,n+1}^{(m)} + W_\ell^2\varphi_{\ell,0,n}^{(m)} + \frac{9}{4}\varphi_{\ell,2,n}^{(m)} \\
&- 2W_\ell\varphi_{\ell,1,n}^{(m)} - 2P_\ell'\varphi_{\ell,1,n+1}^{(m)}, \quad m = 1,2,\ \ell = 1,2,\ \ = 1,2,\ldots,
\end{aligned}$$

where

$$\begin{aligned}
S_{\ell n}^{(m)} = &\sum_{\substack{j=-n \\ j\neq 0}}^{n}\{(P_\ell')^2\varphi_{\ell,j,n+2}^{(m)} + 2P_\ell'W_\ell\varphi_{\ell,j,n+1}^{(m)} \\
&+ W_\ell^2\varphi_{\ell jn}^{(m)} + [(j+3/2)^2 - 2(j+3/2)\frac{C_s^2}{C_d^2}]\varphi_{\ell,j+2,n}^{(m)} \\
&- [2(j+1) - 2\frac{C_s^2}{C_d^2}]W_\ell\varphi_{\ell,j+1,n}^{(m)} \\
&- [2(j+1) - 2\frac{C_s^2}{C_d^2}]P_\ell'\varphi_{\ell,j+1,n+1},\}, \quad m = 1,2,\ n = 1,2,\ldots.
\end{aligned} \tag{3.100}$$

For $n = 0$, the initial values $\bar{T}_{\ell_0}^{(m)}, \bar{\phi}_{\ell_2}^{(m)}$ also obtained from the same equations are

$$\bar{T}_{10}^{(1)} = \frac{C_d^2(P_2')^2((P_1')^2 - 1/C_d^2)}{(P_1')^2 - (P_2')^2}, \quad \bar{T}_{20}^{(1)} = \frac{C_d^2(P_1')^2(1/C_d^2 - (P_2')^2)}{(P_1')^2 - (P_2')^2}$$
$$\bar{\phi}_{12}^{(1)} = \frac{C_d^2(P_2')^2}{(P_1')^2 - (P_2')^2}, \quad \bar{\phi}_{22}^{(1)} = -\frac{C_d^2(P_1')^2}{(P_1')^2 - (P_2')^2}, \tag{3.101}$$

and

$$\bar{T}_{10}^{(2)} = -\bar{T}_{20}^{(2)} = \frac{C_d^2((P_1')^2 - 1/C_d^2)(1/C_d^2 - (P_2')^2)}{(P_1')^2 - (P_2')^2}$$
$$\bar{\phi}_{12}^{(2)} = \frac{C_d^2(1/C_d^2 - (P_2')^2)}{(P_1')^2 - (P_2')^2}, \quad \bar{\phi}_{22}^{(2)} = \frac{C_d^2((P_1')^2 - 1/C_d^2)}{(P_1')^2 - (P_2')^2}. \tag{3.102}$$

Thus the solutions of the canonical problems (3.89), (3.90) are given by

$$\theta^{(m)}(r,t) = \sum_{\ell=1}^{2} r^{-1/2} e^{-W_\ell(r-1)} \Big\{ \sum_{n=0}^{\infty} \frac{1}{n!}\big(t - (r-1)P_\ell'\big)^n \sum_{j=-n}^{n} t_{\ell jn}^{(m)} r^j \Big\}$$
$$\times H\big(t - (r-1)P_\ell'\big), \quad m = 1,2, \tag{3.103}$$

$$\Phi^{(m)}(r,t) = \sum_{\ell=1}^{2} r^{-1/2} e^{-W_\ell(r-1)} \Big\{ \sum_{n=2}^{\infty} \frac{1}{n!}\big(t - (r-1)P_\ell'\big)^n \sum_{j=-n}^{n} \varphi_{\ell jn}^{(m)} r^j \Big\}$$
$$\times H\big(t - (r-1)P_\ell'\big), \quad m = 1,2, \tag{3.104}$$

where the wavefunctions F_n are obtained from (3.93) and (3.98) with $f^{(m)} = 1$, $t_{\ell jn}^{(m)}$, $\varphi_{\ell jn}^{(m)}$ from (3.81), (3.86) augmented by (3.99) for $\bar{T}_{\ell n}^{(m)}$, $\bar{\phi}_{\ell n}^{(m)}$. The strains $\varepsilon_{rr}^{(m)}, \varepsilon_{\varphi\varphi}^{(m)}$ can now be determined from the relevant strain-displacement relations whereas the stresses $\sigma_{rr}^{(m)}$ $\sigma_{\varphi\varphi}^{(m)}$ can be found from (3.46), (3.47) together with (3.103) and (3.104).

We now consider the following source condition for shear waves;

$$\frac{C_s^2}{C_d^2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right)\Big|_{r=1} = H(t), \tag{3.105}$$

which we shall label as Problem 3. Proceeding as for the thermoelastic waves we find that the rays associated with shear waves are

$$S(r) = \bar{S} \pm \frac{r-1}{C_s}, \quad \bar{S} = S(1), \tag{3.106}$$

where the $\pm$ signs are associated with outgoing and incoming waves, respectively. We again choose the $+$ sign which corresponds to the waves leaving the boundary of the circular hole in the positive radial direction. The amplitude functions in (3.57) for $u_\varphi(r,t)$ are, on the o' 'r hand,

$$U_n(r) = r^{-1/2}\sum_{j=0}^{n-1} u_{jn} r^{-j}, \quad n \geq 1 \tag{3.107}$$

where

$$u_{jn} = \begin{cases} K(j)u_{j-1,n-1}, & \text{if } 1 \leq j \leq n-1, \\ -\dfrac{C_d^2}{C_s}, & \text{if } j = 0,\ n = 1, \\ -\sum_{j=1}^{n}[1 + C_s(j-1/2) + K(j)]u_{j-1,n-1}, & \text{if } j = 0,\ n \geq 2 \\ 0, & \text{if } j < 0 \text{ or } j > n-1 \end{cases} \tag{3.108}$$

and

$$K(j) = (C_s/2j)[1 - (j-1/2)^2]. \tag{3.109}$$

We have chosen

$$\tilde{S} = 0, \quad F_0(t) = H(t) \tag{3.110}$$

from the source condition (3.105). We observe that if $F_0(t) = f(t)$, for arbitrary $f(t)$, the wavefunctions $F_n(t)$ are again

$$F_n(t) = \frac{H(t)}{n!}\frac{\partial}{\partial t}\int_0^{\infty}(t-\xi)^n f(\xi)d\xi. \tag{3.111}$$

The complete expansion of the solution to Problem 3 is then given by

$$u_\varphi(r,t) = \{r^{-1/2}\sum_{n=1}^{\infty}\frac{1}{n!}(t-(r-1)/C_s)^n\sum_{j=0}^{n-1}u_{jn}r^{-j}\}H(t-(r-1)/C_s) \tag{3.112}$$

The strain $\varepsilon_{r\varphi}$ and the stress $\sigma_{r\varphi}$ corresponding to the displacement (3.112) may be composed as before.

In order to study the propagation of discontinuities in temperature, strains and stresses generated by the source conditions (3.89), (3.90) and (3.105), we now introduce the usual bracket notation

$$[f]_{t=A(r)} = f(r,t)|_{t=A^+(r)} - f(r,t)|_{t=A^-(r)}, \tag{3.113}$$

which represents the discontinuity of a function $f(r,t)$ across a wavefront $t = A(r)$. We first recall that the disturbance of Problem 3 generates shear waves only. The discontinuities in $\varepsilon_{r\varphi}$ and $\sigma_{r\varphi}$ introduced by (3.105) evolve into discontinuities in these variables across the wavefront $t = (r-1)/C_s$. These discontinuities are

$$[\varepsilon_{r\varphi}]_{t=(r-1)/C_s} = \frac{C_d^2}{C_s^2} r^{-1/2}, \quad [\sigma_{r\varphi}]_{t=(r-1)/C_s} = r^{-1/2}. \tag{3.114}$$

The disturbances (3.89) and (3.90), on the other hand, generate thermoelastic waves with the two wavefronts $t = (r-1)P_1'$ and $t = (r-1)P_2'$. In particular, we find that $\varepsilon_{\varphi\varphi}$ is continuous whereas $\theta, \varepsilon_{rr}, \sigma_{rr}$ and $\sigma_{\varphi\varphi}$ exhibit finite jump discontinuities across each wavefront. For Problem 1 where the disturbance is purely thermal

$$\begin{aligned}
[\theta^{(1)}]_{t=(r-1)P_1'} &= \frac{C_d^2(P_2')^2((P_1')^2 - 1/C_d^2)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\varepsilon_{rr}^{(1)}]_{t=(r-1)P_1'} &= \frac{C_d^2(P_1')^2(P_2')^2}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{rr}^{(1)}]_{t=(r-1)P_1'} &= \frac{(P_2')^2}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{\varphi\varphi}^{(1)}]_{t=(r-1)P_1'} &= \frac{2C_s^2(P_2')^2(1/2C_s^2 - (P_1')^2)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)},
\end{aligned} \tag{3.115}$$

and

$$\begin{aligned}
[\theta^{(1)}]_{t=(r-1)P_2'} &= \frac{C_d^2(P_1')^2(1/C_d^2 - (P_2')^2)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\varepsilon_{rr}^{(1)}]_{t=(r-1)P_2'} &= -\frac{C_d^2(P_1')^2(P_2')^2}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{rr}^{(1)}]_{t=(r-1)P_2'} &= -\frac{(P_1')^2}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{\varphi\varphi}^{(1)}]_{t=(r-1)P_2'} &= \frac{2C_s^2(P_1')^2((P_2')^2 - 1/2C_s^2)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}.
\end{aligned} \tag{3.116}$$

For Problem 2, that is, for a purely mechanical unit step disturbance

$$
\begin{aligned}
[\theta^{(2)}]_{t=(r-1)P_1'} &= \frac{C_d^2\big((P_1')^2 - 1/C_d^2\big)\big(1/C_d^2 - (P_2')^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\varepsilon_{rr}^{(2)}]_{t=(r-1)P_1'} &= \frac{C_d^2(P_1')^2\big(1/C_d^2 - (P_2')^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{rr}^{(2)}]_{t=(r-1)P_1'} &= \frac{\big(1/C_d^2 - (P_2')^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{\varphi\varphi}^{(2)}]_{t=(r-1)P_1'} &= \frac{2C_s^2\big(1/2C_s^2 - (P_1')^2\big)\big(1/C_d^2 - (P_2')^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)},
\end{aligned}
\tag{3.117}
$$

and

$$
\begin{aligned}
[\theta^{(2)}]_{t=(r-1)P_2'} &= -\frac{C_d^2\big((P_1')^2 - 1/C_d^2\big)\big(1/C_d^2 - (P_2')^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\varepsilon_{rr}^{(2)}]_{t=(r-1)P_2'} &= \frac{C_d^2(P_2')^2\big((P_1')^2 - 1/C_d^2\big)}{P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{rr}^{(2)}]_{t=(r-1)P_2'} &= \frac{\big((P_1')^2 - 1/C_d^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}, \\
[\sigma_{\varphi\varphi}^{(2)}]_{t=(r-1)P_2'} &= \frac{2C_s^2\big(1/2C_s^2 - (P_2')^2\big)\big((P_1')^2 - 1/C_d^2\big)}{(P_1')^2 - (P_2')^2}\, r^{-1/2} e^{-W_t(r-1)}.
\end{aligned}
\tag{3.118}
$$

Recalling $(P_1')^2 > 1/C_d^2$ and $1/2C_s^2 > 1/C_d^2 > (P_2')^2$ we see from (3.115), (3.116), (3.117) and (3.118) that for both canonical problems the jumps in θ, ε_{rr} and σ_{rr} are all of the same sign at the slower wavefront, whereas at the faster wavefront the jumps for θ are of opposite sign to those for ε_{rr}, σ_{rr} and $\sigma_{\varphi\varphi}$. For both canonical problems the sign of $[\sigma_{\varphi\varphi}]_{t=(r-1)P_1'}$ depends on the relative magnitudes of $(P_1')^2$ and $1/2C_s^2$. Therefore, for $(P_1')^2 < 1/2C_s^2$, that is, for $v_1^2 > 2C_s^2$ the jumps in θ and $\sigma_{\varphi\varphi}$ are of the same sign whereas for $(P_1')^2 > 1/2C_s^2$ or, equivalently, $v_1^2 < 2C_s^2$ they are of opposite signs at the slower wavefront. These results provide further verification to the related results of Sawatzky and Moodie [14] as well as offering new information pertinent to one dimensional circular geometry.

The ray series method employed in this thesis is suitable not only for the study of the propagation characteristics of thermoelastic disturbances but also for numerical evaluation of the solution behind the wavefronts. In the next section we present numerical results for the case in which the present theory reduces to the linear theory of Lord and Shulman [7].

(B) Numerial Results

In this section we specify the relaxation functions of the therm elastic plate as

$$\alpha(t) = \frac{e^{-t/\tau}}{\tau}, \quad \beta(t) = 0. \tag{3.119}$$

The relaxation functions given by (3.119) are made nondimensional according to the scheme (3.39). The propagation characteristics of thermomechanical disturbances in thermoelastic half-spaces for which the relaxation functions are defined by (3.119) have been studied in [14] in detail. We note only that with this choice of relaxation functions the relaxation coefficients in the expansions (3.36), (3.37) become

$$\alpha_i^0 = (-1)^i \tau^{-(i+1)}, \quad \beta_i^0 = 0, \tag{3.120}$$

where τ is the nondimensional thermal relaxation time.

An analogous of our Problem 1 in thermoela.tic half-spaces whose heat conduction obey the Maxwell-Cattaneo relation has been studied by several authors [31-33] by using Laplace transforms. Assuming γ is small, they inverted the transforms analytically and illustrated the numerical results graphically. Related numerical results obtained from a ray series solution have been displayed graphically in [14], as well. For comparison purposes we choose the order of magnitude of thermal parameters to be the same as in the above mentioned references and display the numerical results for Problem I in Figs. 3.1-3.6. The numerical results are obtained with the techniques introduced in Chapter II. In all of the figures, the ratio of C_d^2 to C_s^2 is chosen as three. This is because of the fact that the isothermal Lame' constants λ and μ are of the same magnitude for most of the materials [34].

In Figs. 3.1 and 3.2, we plot the nondimensional temperature and nondimensional radial stress against nondimensional time for the values of thermoelastic parameters used in [14]. These plots indicate that the discontinuities at the quasi thermal wavefront decay faster with radial distance than the discontinuities at the quasi-elastic wavefront. In Figs. 3.3 and 3.4, we depict the influence of the thermoelastic coupling constant on the evolution of discontinuities. It is clear that changes in the thermoelastic coupling constant at small values of γ have no significant influence on the response of the medium to the purely thermal disturbances. These results are in agreement with the results of the above mentioned references. The character change of the fast and slow wavefronts is examined in Figs. 3.5 and 3.6. It is seen that whether the quasi-thermal or the quasi-elastic wavefront is faster, the magnitude of the temperature immediately after the arrival of the quasi-thermal wavefront is almost identical in both cases. For both situations the temperature response of the material is the same after the arrival of the second wavefront. We also see that if the quasi-thermal wavefront is slower then the sign of the jump for the circumferential stress at this wavefront is negative whereas it is positive in the other case. This result is because of the fact that one of the thermoelastic wavespeeds is very close to C_d while the other lies near $(\alpha_0^0)^{1/2}$. Consequently, if v_1 is near C_d then the condition $v_1^2 > 2C_s^2$ holds and the jump for the circumferential stress at the slower wavefront is positive. If v_1 lies near $(\alpha_0^0)^{1/2}$ then both $v_1^2 > 2C_s^2$ and $v_1^2 < 2C_s^2$ may happen although the second condition is expected for most physical situations.

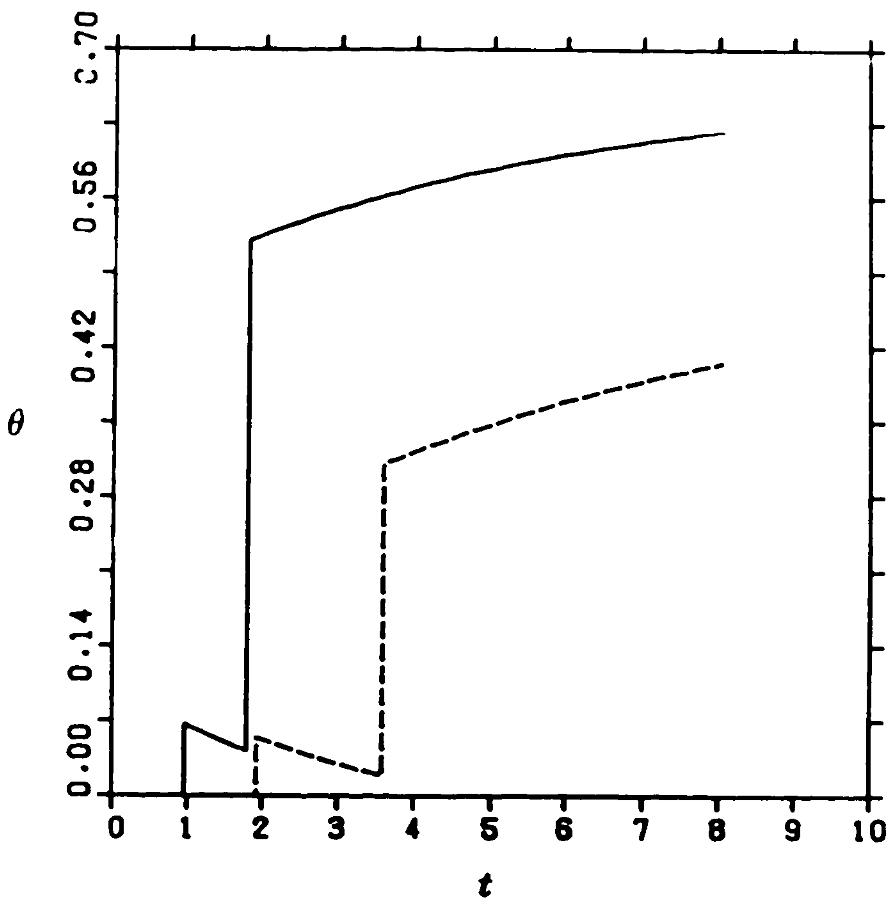

Fig. 3.1. **Variation of non-dimensional temperature** with non-dimensional time at $r = 2.0(-)$, $r = 3.0(--)$ for $C_d^2 = 1.0$, $\tau = 3.0$, $\gamma = 0.05$

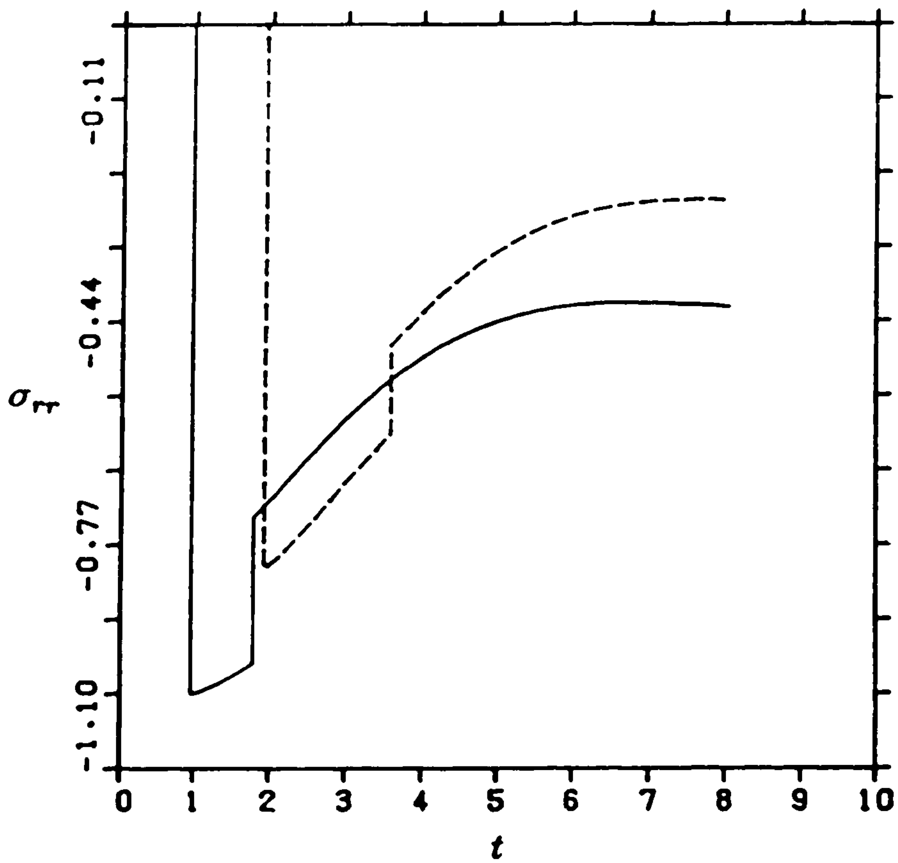

Fig. 3.2. Variation of non-dimensional radial stress with non-dimensional time at $r = 2.0(-)$, $r = 3.0(--)$ for $C_d^2 = 1.0$, $\tau = 3.0$, $\gamma = 0.05$

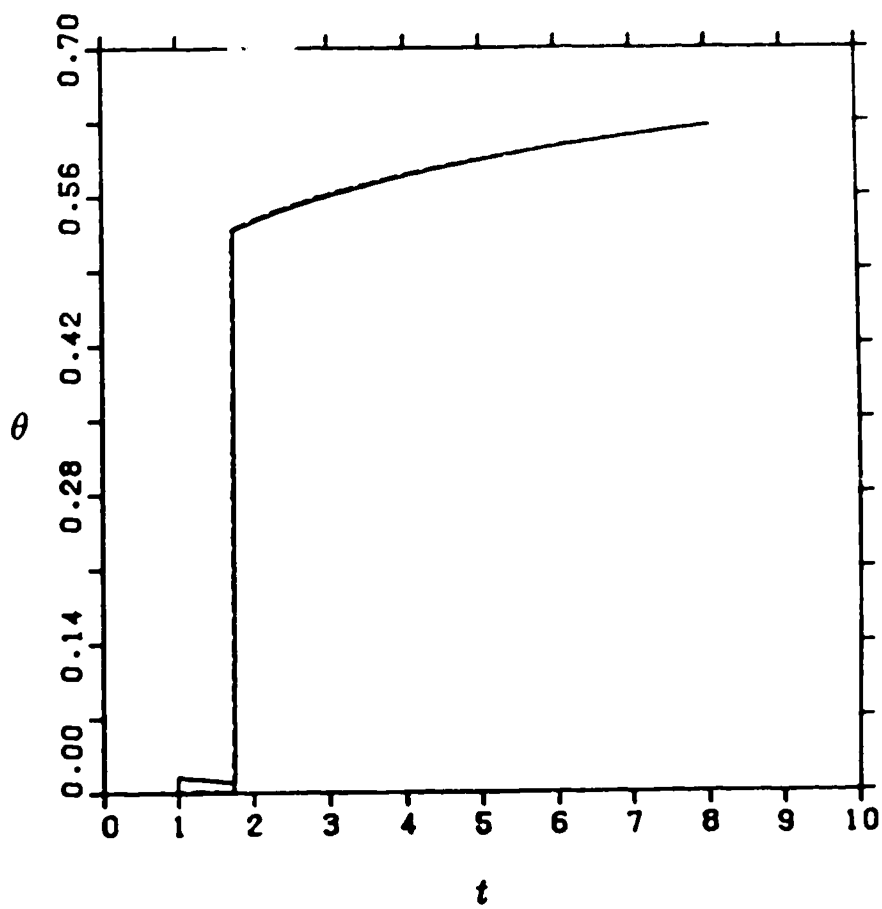

Fig. 3.3. Variation of non-dimensional temperature with non-dimensional time at $r = 2.0$ for $C_d^2 = 1.0$, $\tau = 3.0$, $\gamma = 0.01(-)$, $\gamma = 0.001(\quad)$

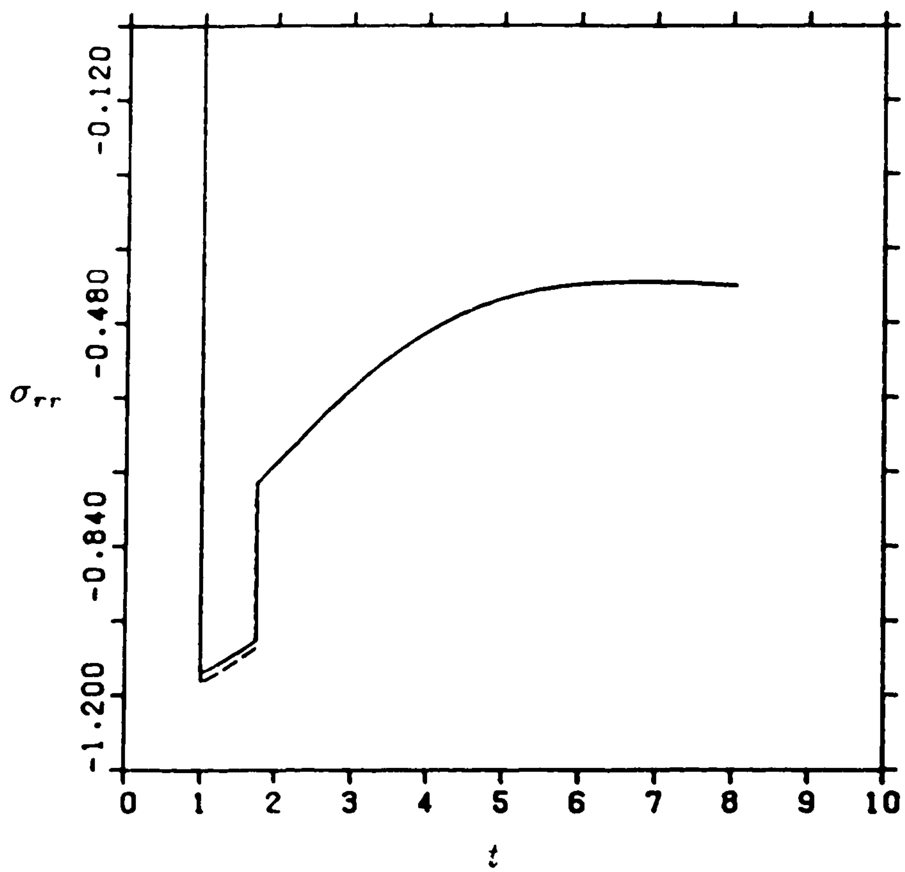

Fig. 3.4. Variation of non-dimensional radial stress with non-dimensional time at $r = 2.0$ for $C_d^2 = 1.0$, $\tau = 3.0$, $\gamma = 0.01(-)$, $\gamma = 0.001(--)$

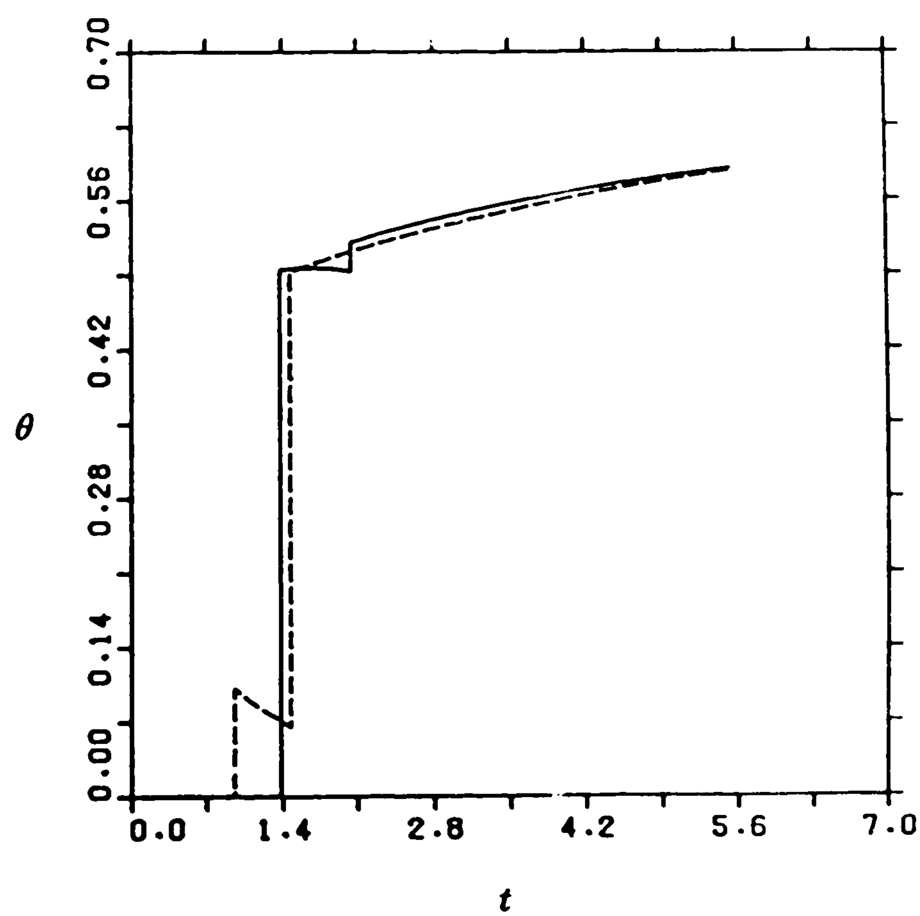

Fig. 3.5. Variation of non-dimensional temperature with non-dimensional time at $r = 2.0$ for $C_d^2 = 1.0(-)$, $C_d^2 = 0.25(\;\;)$, $r = 2.0$, $\gamma = 0.05$

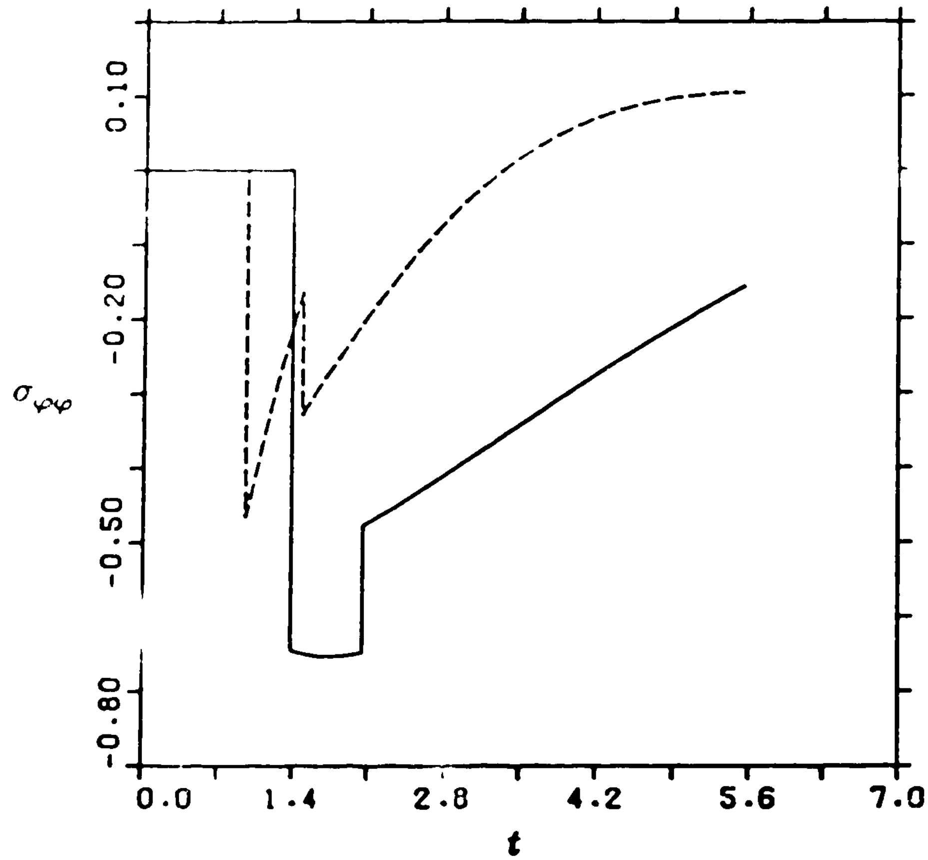

Fig. 3.6. Variation of non-dimensional circumferential stress with non-dimensional time at $r = 2.0$ for $C_d^2 = 1.0(-)$, $C_d^2 = 0.25(--)$.

$\tau = 2.0$, $\gamma = 0.05$

CHAPTER IV

Conclusions

The theory of Gurtin and Pipkin, and Chen and Gurtin introduces finite speeds for the propagation of thermal transients in rigid materials and eliminates the problem of instantaneous propagation of thermal disturbances. This theory implies two finite speeds of propagation for thermomechanical disturbances in deformable materials. According to this theory, shear waves which generate no volume changes are not affected by thermomechanical coupling. This result is in agreement with that of the theories based on the classical theory of heat conduction. In the case of thermoelastic materials, we have shown that when $\alpha_0^0/C_d^2 < 1$ quasi-elastic waves whereas $\alpha_0^0/C_d^2 \geq 1$ quasi-thermal waves propagate faster through the medium.

In this thesis we have employed the ray series methods to solve the integro-partial differential equations of the linearized theory of Gurtin and Pipkin, and Chen and Gurtin for inhomogeneous rigid and homogeneous elastic materials. The methods are formal, directly yield complete asymptotic wavefront expansions and ideally suited to numerical computation. The methods provide not only a formal procedure to derive the magnitude, decay and velocity of propagation of discontinuities at the wavefronts but also a description of the behaviour of the field variables for some distance behind the wavefronts. The Padé extended ray series solutions, on the other hand, offer a description of the behaviour of these variables for larger domains of physical interest. All of the numerical results presented in this thesis are those obtained from the ray series methods in conjunction with the Padé approximants.

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