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The High Energy Logarithms in Two Loop Electroweak Bhabha Scattering

by

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Abstract

Bhabha Scattering is the scattering of an electron off of a positron; it is everywhere in particle physics and plays a special role in particle phenomenology. Its particular use as a luminosity measure in particle colliders gives great motivation to reduce the theoretical uncertainty in the cross section. We calculate the two loop contribution to Bhabha scattering from high energy Sudakov logarithms. This contribution reduces the overall uncertainty in the cross section to one per mille, the necessary level for use at the ILC.

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1 Introduction

Bhabha scattering is one of the simplest known processes in the Standard Model of particle physics. First calculated in QED in 1935 by Homi J. Bhabha, and later named in his honour, it can be thought of as the rebounding of a positron off an incident electron, or $e^+e^- \to e^+e^-$ [1]. Despite its uncomplicated nature, or in fact because of it, Bhabha scattering is of great importance to the the experimental physics community for its use as a measure of luminosity in particle accelerator experiments. For this reason a great effort has been made in recent years to push the theoretical calculation of the Bhabha scattering cross section to far higher degrees of accuracy. In this work we continue this project and calculate the high energy electroweak cross section to two loops in the logarithmic approximation. With existing results this reduces the total uncertainty in the Bhabha cross section to less than one permille, the target for next generation experiments such as the ILC.

1.1 Motivation

The problem of luminosity determination is prevalent in every particle accelerator experiment. To obtain the cross section for some process X, an experiment must measure both the frequency of the process N_X and the intensity of the beam $\mathcal{L}$, known as the luminosity. The cross section is then simply $\sigma_X = N_X/\mathcal{L}$. Although N_X changes with process to process, the luminosity $\mathcal{L}$ is a constant for an experiment and in general a complicated function of the energy and spatial distribution of particles in the beam. Any uncertainty in $\mathcal{L}$ will lead to an uncertainty in σ_X, hence the great motivation to determine the luminosity as precisely as possible.

Since any direct measurement of the luminosity would alter the beam and the entire experiment, measurements of $\mathcal{L}$ are typically done by measuring the rate of some calibration process N_{cal}. If the theoretical value for the cross section of this process is known and trusted, the luminosity can simply be taken as $\mathcal{L} = N_{\text{cal}}/\sigma_{\text{cal}}$ and the cross section for any other process can be obtained via

$$\sigma_X = \frac{\sigma_{\text{cal}}}{N_{\text{cal}}} N_X. \tag{1}$$

Clearly the choice of calibration process must be made carefully. Any candidate should have a very clear detector signature and large cross section to reduce the uncertainty in N_{cal}. The theoretical uncertainty in σ_{cal} must be sufficiently low; typically this means the process must be calculable to many[1] orders in pertubation theory. For the next generation of particle physics experiments a precision on the order of 0.1% or 1 permille will be required in σ_{cal} to achieve the desired experimental accuracy.

Bhabha scattering is one of the best candidates for a calibration process in electron positron colliders, and has been used as such for many experiments. The electron and positron in the final state are easy to detect and distinguish since they are (oppositely)

[1] At the current state of the art, 'many' $\sim$ two.

charged leptons. The process has several channels through which it can occur[2] giving it the requisite large cross section. It is the aim of the current work to fulfil the last condition; lowering the theoretical uncertainty in the Bhabha scattering cross section. Until recently, the cross section was only known accurately to one loop in QED and the full Electroweak theory. A tremendous effort over the past several years has just recently completed the full two loop calculation in QED. The Electroweak contributions at two loops to next-to-next-to-leading logarithms are presented here for the first time.

Historically, Bhabha scattering has been used successfully as a luminosity measure in high energy experiments, LEP and the SLC in particular, and the more moderate energy B-factory experiments, such as BaBar and Belle. Both LEP and the SLC used a clever trick to get around the uncertainty in the electroweak sector of Bhabha; they measured luminosity based only on scattering as small angles [2,3]. It turns out due to the relatively large W and Z mass, at small angles the Bhabha cross section is dominated by QED, and the electroweak contributions are negligible. Having neither sensitivity nor statistics in the small angle limit, BaBar and Belle were not able to take advantage of this, and used Bhabha scattering at large angles for their luminosity [4].

The next energy frontier particle collider to be built will be the International Linear Collider (ILC). It will be a TeV scale linear electron-positron collider, designed to perform precision measurements in the energy scale we have only recently begun exploring at the Large Hadron Collider. Signals of new physics beyond the Standard Model typically occur as small pertubations of the Standard Model result. High precision experiments are necessary in order to distinguish these small effects from the expected observation. Thus a precise luminosity measure is crucial to the success of the physics program at the ILC. Unfortunately, the ILC will not be able to perform the same trick as LEP and the SLC and restrict itself to measuring the luminosity from small angle Bhabha alone. Due to interactions of particles within the beam, the energy spectrum will not be monochromatic and the luminosity will vary with the specific distribution of energies in the beam. Obtaining an accurate measurement of the luminosity will require untangling this spectrum, and Bhabha scattering at large angles has been put forward as a means of performing this [5,6]. To acquire the precise results the ILC is capable of, the large angle Bhabha cross section must be known to within 1 permille. At this precision the contribution to the cross section from electroweak effects can not be ignored; the electroweak cross section will be necessary for the successful completion of the ILC program.

1.2 Outline of the Calculation

Motivated by its utility as a luminosity measure, a remarkable effort has been made in recent years to lower the theoretical uncertainty in the Bhabha scattering cross section. The QED corrections have been calculated fully to second order [7–10] and reviewed in [11]. However, the electroweak corrections have only been calculated to one loop [12].

[2]Bhabha scattering occurs through two kinematic channels, annihilation 'S' and scattering 'T'. Both of these channels get contributions from electromagnetic and weak interactions.

Calculating cross sections in the electroweak theory is a complicated business. There are two gauge groups providing particle interactions, the symmetry is broken by the Higgs mechanism, and there are several mass scales spanning several orders of magnitude.[3] These factors put a full diagrammatic calculation of the two loop electroweak Bhabha scattering cross section beyond the reach of any known computational method, even in the high energy limit. Nevertheless, significant progress can be made by restricting the calculation to the dominant contributions at high energies, the Sudakov Logarithms [13–17].

It is well known that scattering amplitudes display a logarithmic dependence on energy past leading order in pertubation theory [18, 19]. For example, in the high energy limit of an SU(N) theory the $\mathcal{O}(\alpha^n)$ correction to the scattering amplitude will be an order $2n$ polynomial in powers of $\log(s/M^2)$, where s is the squared centre of mass energy and M is the gauge boson mass. For the scales relevant here, the centre of mass energy is about 1 TeV and M is on the order of the Z mass $M_Z \approx 90$ GeV. This gives a logarithm on the order of:

$$\log\Big(\frac{s}{M^2}\Big) \approx 5. \tag{2}$$

On its own this is not so intimidating. However the leading logarithm at two loops will be $\log^4(s/M^2) \approx 625$, enhanced by two orders of magnitude over the constant[4] contribution. We will find to achieve the desired accuracy all logarithmic contributions will be required. This is referred to as the next-to-next-to-next-to-leading logs approximation, or NNNLL.

We begin by explicitly calculating the Bhabha scattering cross section at tree level in QED. The electroweak cross section will have the exact same spinor structure, and may be extracted by simply substituting a chiral-dependent coupling for the electric charge e in the final result. We demonstrate the calculation of the loop corrections in a pure SU(2) theory, where the results are still simple enough to allow them to be written explicitly in analytic form. Afterwards we move to the full electroweak model, taking into account both gauge groups and subtracting the QED contribution which is known past logarithmic accuracy. The electroweak results are presented here numerically.

[3] Two of these scales, the masses of the Higgs and the neutrinos, have not yet even been measured (or even confirmed to exist, in the case of the Higgs).

[4] Here 'constant' refers to the contribution to the scattering amplitude which is independent of energy, ie. the terms not enhanced by a power of $\log(s/M^2)$.

2 Preliminaries

Every journey has a first step. Before we descend into pertubation theory and multiloop corrections, it is beneficial to explicitly state the kinematics of Bhabha scattering in our limit and calculate the cross section at leading order. This result, also called the "tree level" or Born approximation, is simply calculated directly from the Feynman rules. Unless otherwise stated, all conventions set in the Kinematics section will be used throughout the text.

2.1 Kinematics

Bhabha Scattering is a $2 \to 2$ interaction with an electron positron pair in both the initial and final states. The initial 4-momenta of the electron and positron are p and k respectively; their final momenta are p' and k'. All calculations are performed in the centre of mass frame, $p^i + k^i = 0$. We follow the particle physics convention in defining the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

When possible, momentum dependence will be expressed through the Lorentz invariant Mandelstam variables:

$$\begin{aligned} s &\equiv (p+k)^2, \\ t &\equiv (p'-p)^2, \\ u &\equiv (k'-p)^2. \end{aligned} \tag{4}$$

We are concerned with the high energy limit $|s|, |t|, |u| \gg m_e^2$. In particular this means we can effectively set $m_e^2 = p^2 = k^2 = p'^2 = k'^2 = 0$.

Since this is a $2 \to 2$ process, all angular dependence can be expressed through a single parameter. By convention we define this scattering angle θ as the angle between the trajectories of the final and initial electrons:

$$\theta \equiv \arccos\left(\frac{\vec{p}\cdot\vec{p'}}{|\vec{p}||\vec{p'}|}\right). \tag{5}$$

By conservation of momentum and energy $p^0 = k^0 = p'^0 = k'^0 = \frac{1}{2}\sqrt{s}$. We can calculate all necessary dot products of 4-momenta:

$$
\begin{aligned}
p \cdot k = p' \cdot k' &= \frac{1}{2} s, \\
p \cdot p' = k \cdot k' &= \frac{1}{4} s \ (1 - \cos\theta), \\
p \cdot k' = p' \cdot k &= \frac{1}{4} s \ (1 + \cos\theta).
\end{aligned} \tag{6}
$$

It will be more useful to express angular dependence in terms of the variable $x \equiv \frac{1-\cos\theta}{2}$. Realizing $p \cdot p' = -\frac{1}{2} t$ and $p \cdot k' = -\frac{1}{2} u$ we can then write:

$$
\begin{aligned}
t &= -sx, \\
u &= -s(1 - x).
\end{aligned} \tag{7}
$$

A scattering event in particle physics is really just a transition from some initial state of particles $|I\rangle$ to a final state $|F\rangle$, where each state specifies all quantum numbers (charge, spin, momentum, etc.) of the initial and final particles respectively. The amplitude for transition from $|I\rangle$ to $|F\rangle$ is called the scattering amplitude $\mathcal{A}$:

$$
\mathcal{A} = \langle F | I \rangle \ . \tag{8}
$$

Finally, we just need the standard result for cross sections in terms of matrix elements. In the centre of mass frame with identical masses for all external particles, this takes the simple form [20]:

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{A}|^2}{64\pi^2 s}. \tag{9}
$$

This is all the information we need to compute the Born cross section and higher order corrections.

2.2 The QED Born Cross Section

There are two channels through which an electron and positron may interact to produce Bhabha scattering in QED. The S or annihilation channel has the initial particles meet and annihilate at a vertex, a photon propagates forward and later spawns a new electron positron pair. The T or scattering channel has the particles interact through the exchange of a virtual photon. These processes must be summed at the amplitude level before squaring to determine the cross section. This calculation is ubiquitous in quantum field theory classes and textbooks [20]. We can write the matrix elements for each of these contributions directly from their Feynman diagrams given in Fig. 1 with time flowing to the right. The amplitudes for these processes are:

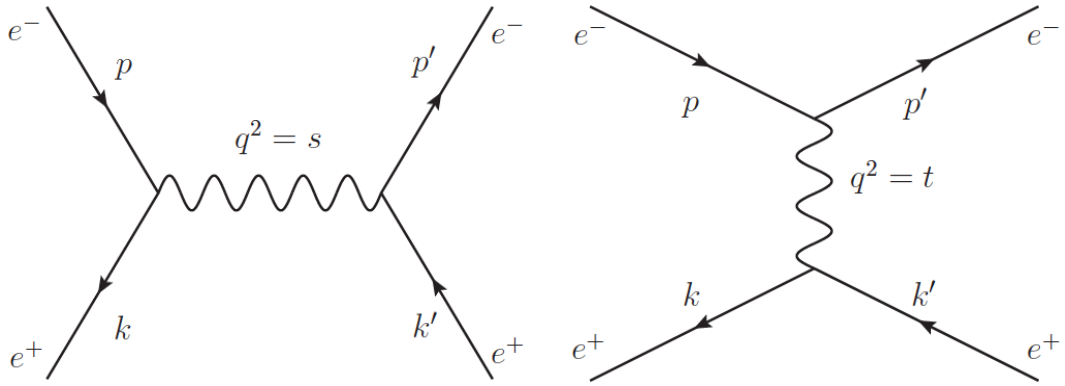


Figure 1: The Feynman diagrams contributing to Bhabha scattering in the Born Approximation

$$\mathcal{A}^S_{IJ} = \frac{e^2}{(p+k)^2}\bar{v}_s(k)\gamma^\mu P_I u_r(p)\bar{u}_{r'}(p')\gamma_\mu P_J v_{s'}(k'),$$
$$\mathcal{A}^T_{IJ} = \frac{e^2}{(p'-p)^2}\bar{u}_{r'}(p')\gamma^\mu P_I u_r(p)\bar{v}_s(k)\gamma_\mu P_J v_{s'}(k'), \tag{10}$$

where $I, J \in \{L, R\}$,

$$P_L = \frac{1-\gamma^5}{2} \text{ and } P_R = \frac{1+\gamma^5}{2}.$$

We work in the chiral basis for all γ^μ. To calculate the cross section we need the square matrix element:

$$|\mathcal{A}_{IJ}|^2 = \underbrace{|\mathcal{A}^S_{IJ}|^2}_{S} + \underbrace{|\mathcal{A}^T_{IJ}|^2}_{T} + \underbrace{2\mathrm{Re}\left((\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ}\right)}_{interference}. \tag{11}$$

Making use of $(\bar{u}_1\gamma^\mu P_I u_2)^* = \bar{u}_2\gamma^\mu P_I u_1$ we can write:

$$|\mathcal{A}^S_{IJ}|^2 = \frac{e^4}{s^2}\mathrm{tr}(\bar{v}_s(k)\gamma^\mu P_I u_r(p)\bar{u}_r(p)\gamma^\nu P_I v_s(k))\,\mathrm{tr}(\bar{u}_{r'}(p')\gamma_\mu P_J v_{s'}(k')\bar{v}_{s'}(k')\gamma_\mu P_J u_{r'}(p'))\,,$$
$$|\mathcal{A}^T_{IJ}|^2 = \frac{e^4}{t^2}\mathrm{tr}(\bar{u}_{r'}(p')\gamma^\mu P_I u_r(p)\bar{u}_r(p)\gamma^\nu P_I u_{r'}(p'))\,\mathrm{tr}(\bar{v}_s(k)\gamma_\mu P_J v_{s'}(k')\bar{v}_{s'}(k')\gamma_\mu P_J v_s(k))\,,$$
$$(\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ} = \frac{e^4}{st}\mathrm{tr}(\bar{u}_{r'}(p')\gamma^\mu P_I u_r(p)\bar{u}_r(p)\gamma^\nu P_I v_s(k)\bar{v}_s(k)\gamma_\mu P_J v_{s'}(k')\bar{v}_{s'}(k')\gamma_\mu P_J u_{r'}(p'))\,. \tag{12}$$

The cyclic property of the trace allows us to write $\mathrm{tr}\,(\bar{u}\cdots u)$ as $\mathrm{tr}\,(u\bar{u}\cdots)$, after which we can sum over spins using the completeness relations:

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{13}$$

We work here in the high energy limit, where $s, t, u \gg m_e^2$, so the electron mass in the above expressions will be neglected. Assuming an unpolarized e^+e^- beam, we average over initial spins and sum over final. After using the completeness relations an expression is obtained containing only the trace of products of gamma matrices:

$$\begin{aligned}
|\mathcal{A}^S_{IJ}|^2 &= \frac{e^4}{4s^2}\mathrm{tr}(\not{k}\gamma^\mu P_I\ \not{p}\gamma^\nu P_I)\,\mathrm{tr}(\not{p}'\gamma_\mu P_J\ \not{k}'\gamma_\mu P_J)\,, \\
|\mathcal{A}^T_{IJ}|^2 &= \frac{e^4}{4t^2}\mathrm{tr}(\not{p}'\gamma^\mu P_I\ \not{p}\gamma^\nu P_I)\,\mathrm{tr}(\not{k}\gamma_\mu P_J\ \not{k}'\gamma_\mu P_J)\,, \\
(\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ} &= \frac{e^4}{4st}\mathrm{tr}(\not{p}'\gamma^\mu P_I\ \not{p}\gamma^\nu P_I\ \not{k}\gamma_\mu P_J\ \not{k}'\gamma_\mu P_J)\,.
\end{aligned} \tag{14}$$

Using the trace identities of γ^μ, these products can be written in terms of dot products of their momenta. Simple algebra gives:

$$\begin{aligned}
|\mathcal{A}^S_{IJ}|^2 &= 4\frac{e^4}{s^2}\Big(\underbrace{\delta_{IJ}(p\cdot k')(p'\cdot k)}_{\text{Same Chirality}} + \underbrace{(1-\delta_{IJ})(p\cdot p')(k\cdot k')}_{\text{Different Chirality}}\Big), \\
|\mathcal{A}^T_{IJ}|^2 &= 4\frac{e^4}{t^2}\Big(\delta_{IJ}(p\cdot k')(p'\cdot k) + (1-\delta_{IJ})(p\cdot k)(p'\cdot k')\Big), \\
\mathrm{Re}\left((\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ}\right) &= 4\frac{e^4}{st}\Big(\delta_{IJ}(p\cdot k')(p'\cdot k)\Big).
\end{aligned} \tag{15}$$

At this point the chiral structure of the cross section in made manifest. Since QED is a parity conserving theory it does not distinguish between left and right chiral states. Cross sections depend only on whether the states are the same or different, not on their exact chirality.

We can use our knowledge of kinematics to cast this in a more useful form.

$$\begin{aligned}
|\mathcal{A}^S_{IJ}|^2 &= \frac{e^4}{s^2}\Big(\delta_{IJ}\ u^2 + (1-\delta_{IJ})t^2\Big), \\
|\mathcal{A}^T_{IJ}|^2 &= \frac{e^4}{t^2}\Big(\delta_{IJ}\ u^2 + (1-\delta_{IJ})s^2\Big), \\
\mathrm{Re}\left((\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ}\right) &= \frac{e^4}{st}\delta_{IJ}\ u^2.
\end{aligned} \tag{16}$$

Writing the angular dependence explicitly:

$$
\begin{aligned}
|\mathcal{A}^S_{IJ}|^2 &= e^4\Big(\delta_{IJ}(1-x)^2 + (1-\delta_{IJ})\ x^2\Big),\\
|\mathcal{A}^T_{IJ}|^2 &= e^4\Big(\delta_{IJ}\frac{(1-x)^2}{x^2} + (1-\delta_{IJ})\frac{1}{x^2}\Big),\\
\mathrm{Re}\left((\mathcal{A}^S_{IJ})^*\mathcal{A}^T_{IJ}\right) &= -e^4\delta_{IJ}\frac{(1-x)^2}{x}.
\end{aligned}
\tag{17}
$$

Finally we can sum over all polarizations to acquire the total squared matrix element in the Born approximation

$$
|\mathcal{A}_B|^2 = 4e^4\left(\frac{1-x+x^2}{x}\right)^2. \tag{18}
$$

In terms of the QED fine structure constant $\alpha = \frac{e^2}{4\pi}$ we get:

$$
|\mathcal{A}_B|^2 = 64\pi^2\alpha^2\left(\frac{1-x+x^2}{x}\right)^2. \tag{19}
$$

The differential cross section for Bhabha scattering in the Born approximation is then:

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Born}} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2. \tag{20}
$$

2.3 Electroweak Suppression at Small Angles

It is quick to demonstrate the QED dominance of the Bhabha cross section at small angles. The first thing to notice is the kinematic dependence of the S and T channel cross sections:

$$
|\mathcal{A}_S|^2 \propto \frac{t^2+u^2}{s^2} \quad \text{and} \quad |\mathcal{A}_T|^2 \propto \frac{s^2+u^2}{t^2}\ . \tag{21}
$$

In the limit of small angles $\theta \to 0$ and hence $x \to 0$. It is clear then $t \to 0$, $u \to -s$ and the T channel will dominate in the cross section. The T channel interaction at tree level involves the exchange of a virtual gauge boson between the electron and positron. In QED this is the massless photon γ and in the weak interaction it is the massive Z boson Z^0 [20]. We have seen the QED amplitude in (10), for $I = J = L$ it reads:

$$
\mathcal{A}^T_{\mathrm{QED}} = \frac{e^2}{t}\bar{u}_L(p')\gamma^\mu u_L(p)\bar{v}_L(k)\gamma_\mu v_L(k')\ . \tag{22}
$$

Since only left-chiral particles interact via the Electroweak SU(2) symmetry, this channel will have the largest contribution from Weak effects. The Feynman rules for Z^0 exchange are well known [21]. The resulting amplitude in the high energy limit $m_e \to 0$ is:

$$\mathcal{A}^T_{\text{Weak}} = \frac{e^2}{t - M_Z^2} \frac{\left(\frac{1}{2} - \sin\theta_W\right)^2}{\cos\theta_W \sin\theta_W} \bar{u}_L(p')\gamma^\mu u_L(p)\bar{v}_L(k)\gamma_\mu v_L(k') \ , \tag{23}$$

where θ_W is the Weinberg angle. and M_Z is the Z^0 mass. To determine which effect is stronger as $\theta \to 0$ we take the ratio of (23) and (22):

$$\frac{\mathcal{A}^T_{\text{Weak}}}{\mathcal{A}^T_{\text{QED}}} = \frac{\left(\frac{1}{2} - \sin\theta_W\right)^2}{\cos\theta_W \sin\theta_W} \frac{t}{t - M_Z^2} \tag{24}$$

As $\theta \to 0$ the behaviour becomes:

$$\frac{\mathcal{A}^T_{\text{Weak}}}{\mathcal{A}^T_{\text{QED}}} \approx -\frac{t}{M_Z^2} \ . \tag{25}$$

The Weak interaction is suppressed by a factor of t/M_Z^2 at small angles, hence the ability of LEP and the SLC to obtain an accurate luminosity measure using only the QED cross section. Of course, at large angles and high energy the factor $t/(t - M_Z^2)$ in (24) is of the order of unity, and both QED and Weak effects contribute with approximately equal magnitudes, and the full electroweak model should be employed in the calculation of scattering amplitudes.

2.4 Electroweak corrections

In the Standard Model of particle physics, electromagnetic and weak phenomena have been unified into the collective Electroweak interaction. The Electroweak theory postulates a universal gauge symmetry:

$$SU(2)_L \times U(1)_Y \ . \tag{26}$$

The $SU(2)_L$ symmetry couples only to left-chiral particles, and the $U(1)_Y$ symmetry couples to the hypercharge Y. Such a symmetry gives rise to four massless gauge bosons, two charged and two neutral. The charged bosons $W^\pm$ acquire masses M_W through the Higgs Mechanism. Although all four bosons independently interact with the Higgs field, a particular linear combination of the neutral bosons decouples from the field entirely [20]. This boson is the QED photon, it remains massless and invariant under a $U(1)_Q$ symmetry. The orthogonal combination (the Z^0) acquires a mass M_Z through the Higgs Mechanism. We can define the Weinberg (or Weak-Mixing) angle θ_W as the angle between the $\{\gamma, Z^0\}$ basis and the original neutral bosons. This leads to a simple connection between the Z and W masses: $M_W = M_Z \cos\theta_W$.

The inclusion of electroweak interactions leads to a host of corrections to the Bhabha cross section. The tree level contribution comes from the inclusion of diagrams with exchange of a Z^0 boson. If we let $I, J \in \{L, R\}$ denote the chirality of the fermion lines, the spinor structure to this amplitude is exactly that of the QED amplitude, and the amplitude can be

determined through the effective substitution:

$$e^2 \to g^2 \left(T_I^3 T_J^3 + \tan^2 \theta_W \frac{Y_I Y_J}{4} \right). \tag{27}$$

Where T^3 is the third component of the weak isospin, Y is the hypercharge, and g is the SU(2) coupling. Note the coupling now depends explicitly on the chirality of the fermions. For left handed electrons, $T_L = \frac{1}{2}$ and $Y_L = -1$, for right handed electrons $T_R = 0$ and $Y_R = -2$ [20].

In principle, the calculation of the electroweak loop corrections is a straightforward exercise in pertubation theory. As has been done in QED, we simply must sum up all the one and two loop diagrams, and the problem is reduced to one of time. Of course, nature is not so kind, and precision calculations in electroweak theory are complicated by a number of requirements of the theory. The most important problem here is that of masses. In QED the gauge boson is massless and the electron is light. To regularize infrared divergences a small photon mass must be added to the theory, which turns the calculation of the Bhabha cross section to a two scale problem.

In electroweak interactions, there are three distinct gauge boson masses, plus the mass of the electron. At two loops, explicit diagrams with the Higgs also begin appearing, and the final result will depend on the Higgs mass and the details of its coupling. This calculation now includes five distinct mass scales over many orders of magnitude. At one loop it has been completed, but the two loop corrections are beyond the power of current computational techniques.

The best we can do then is an approximation, calculating only the logarithmically enhanced contributions to cross section. Such terms appear with a power of $\log(s/M^2)$, where $M \approx M_W, M_Z$. Fortunately, this will be sufficient to reduce the uncertainty in the electroweak contributions below current experimental thresholds.

The plan is thus: we first calculate the cross section for a "pure" SU(2) theory of unbroken phase, ie. with identical masses for the gauge bosons (including the Higgs). This will then be extended to include the full electroweak theory with a massive photon of mass equal to the gauge bosons. Once this quantity is known, we subtract the portion due to photon so it may be included later with Bremsstralung. The difference between M_Z and M_W is small, so we can include it perturbatively near the end of the calculation. The simplification of the Higgs mechanism we employ introduces an uncertainty of the same order as the non logarithmic corrections, and can safely be neglected.

3 Bhabha Scattering in SU(2)

We begin our analysis with a calculation of the Bhabha Scattering cross section in a pure SU(2) gauge theory. The result will include the bulk of the effects from the full electroweak theory; simple corrections will allow us to transform it to the full physical Standard Model cross section. We perform the calculation in the equal mass approximation.

$$M \equiv M_W = M_Z = M_H \ . \tag{28}$$

In the Sudakov limit considered here $|s|, |t|, |u| \gg M^2$ and $m_e \to 0$. In this limit the only mass scale in the problem is M and all energy dependence of the amplitude will be confined to powers of the Sudakov logarithm $\log(Q^2/M^2)$, where Q^2 is the Euclidean momentum transfer of the interaction. Since we work in the high energy limit we neglect all powers of M^2/Q^2, that is we only calculate the leading order asymptotic contribution to the amplitude. The Sudakov logarithms arise from divergences in the Feynman integrals, and hence are very well structured. It is this structure we will take advantage of to calculate the two loop corrections, but first we require some notation.

3.1 Notation

The functions we will use to paramaterize the scattering amplitude are expressed as a power series in terms of the coupling constant $\alpha = \frac{g^2}{4\pi}$. We define the expansion as such:

$$f(\alpha) = \sum_{n=0}^{\infty} \left(\frac{\alpha}{4\pi}\right)^n f^{(n)}. \tag{29}$$

The factor of 4π arises naturally in SU(N) loop calculations and so is included with α in the expansion. Particles in SU(2) form doublets in weak isospin space; it will be convenient to write the amplitudes in a flavour basis $\{\mathcal{A}^\lambda, \mathcal{A}^d\}$ defined as:

$$\begin{aligned} \mathcal{A}^\lambda &= \bar{\psi}_2 \gamma^\mu t^a \psi_1 \bar{\psi}_4 \gamma_\mu t^a \psi_3 \ , \\ \mathcal{A}^d &= \bar{\psi}_2 \gamma^\mu \psi_1 \bar{\psi}_4 \gamma_\mu \psi_3 \ . \end{aligned} \tag{30}$$

where t^a are the SU(2) isospin generators. Throughout the calculation we will take the external particles to be eigenstates of weak isospin as well as chirality. The t^a matrices then simply act as projectors and the above can be simplified to:

$$\begin{aligned} \mathcal{A}^\lambda_{IJ} &= T^3_I T^3_J \bar{\psi}_2 \gamma^\mu P_I \psi_1 \bar{\psi}_4 \gamma_\mu P_J \bar{\psi}_3 \ , \\ \mathcal{A}^d_{IJ} &= \bar{\psi}_2 \gamma^\mu P_I \psi_1 \bar{\psi}_4 \gamma_\mu P_J \bar{\psi}_3 \ . \end{aligned} \tag{31}$$

For appropriately chosen ψ_i these are exactly the amplitudes for Bhabha scattering in

QED. We then define for convenience:

$$A_{IJ} \equiv \bar{v}_s(p_2)\gamma^\mu P_I u_r(p_1)\bar{u}_{r'}(p_4)\gamma_\mu P_J v_{s'}(p_3) \ . \tag{32}$$

Which is the spinor structure of the scattering amplitude. For brevity and to avoid an explosion of indices we will not explicitly denote whether the momenta p_i are chosen for the S or T channels, the specific choice will be made clear by the context. Absorbing the ψ's into A_{IJ} allows us to abbreviate our notation somewhat:

$$\begin{aligned} \mathcal{A}^\lambda_{IJ} &= T^3_I T^3_J A_{IJ} \ , \\ \mathcal{A}^d_{IJ} &= A_{IJ} \ . \end{aligned} \tag{33}$$

We can now express the Born amplitude in the S-channel as:

$$\mathcal{A}_B = \frac{ig^2}{s}\mathcal{A}^\lambda \ . \tag{34}$$

We remove the g^2 prefactor before expanding the amplitude to all orders in α.

$$\begin{aligned} \mathcal{A}^S &= \frac{ig^2}{s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \mathcal{A}^{S(n)} \ , \\ \mathcal{A}^T &= \frac{ig^2}{t}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \mathcal{A}^{T(n)} \ . \end{aligned} \tag{35}$$

for the S and T channels respectively. Only left handed particles interact via SU(2), so we will drop the I, J subscripts for the time being and assume we are dealing with $I = J = L$.

3.2 The Origin of Sudakov Logarithms

Sudakov logarithms arise in loop corrections whenever Feynman integrals encounter a divergence. The divergences are typically broken into three categories: hard, soft, and collinear. Hard divergences occur as the loop momentum $k \to \infty$; they are the controlled by renormalization and result in the running of the coupling constants, masses, etc. Soft divergences occur as $k^2 \to 0$, and are controlled by including gauge boson masses. Collinear divergences arise as the loop momentum becomes parallel with a fixed momenta $p^2 = 0$ external to the loop. Collinear divergences are mediated by including massive fermions, that is ensuring $p^2 \neq 0$. Soft and collinear divergences are collectively referred to as "Infrared" divergences, in a similar language hard divergences are collectively called "Ultraviolet". Amplitudes involving massless particles are typically divergent even after regularization, and small auxillary masses must be assigned to these particles to achieve finite results. The incorporation of real radiative corrections from soft and collinear particle emission cancels this mass dependence

and leaves cross sections finite.

In our pure SU(2) calculation with massless electrons all three divergences are present. In a single loop, each divergence will produce a contribution to the amplitude proportional to $\log(Q^2/M^2)$. However, the regions of the integration domain with soft and collinear divergences intersect. This region will produce a term proportional to the product of the individual soft and collinear contributions, that is $\log^2(Q^2/M^2)$. This is the Sudakov Double Logarithm, and is the highest degree of divergence encountered in these scattering amplitudes. Calculations including only this term are said to be accurate to Leading Logarithms (LL). In calculations past one loop the various single and double logarithms from each loop combine to form a polynomial in $\log(Q^2/M^2)$. The order of this polynomial at order n in pertubation theory (n loops) is $2n$. At each order then the amplitude can be expanded:

$$\mathcal{A}^{(n)} = \sum_{i=0}^{2n} \mathcal{A}_i^{(n)} \log^i\left(\frac{Q^2}{M^2}\right) \ . \tag{36}$$

Here we calculate the second order contribution to Bhabha scattering in the logarithmic approximation, so we need $\mathcal{A}_i^{(2)}$ for $i = 4, 3, 2, 1$.[5] The advantage of only calculating in the logarithmic approximation is the majority of contributions are going to come from products of one loop contributions, not explicitly two loop effects. The few two loop results required can be calculated for special cases, and the entire result follows.

3.3 Calculating the Scattering Amplitude

The scattering amplitudes for $f\bar{f} \to f'\bar{f}'$ in SU(2) and the full Electroweak model have been calculated to the required order in [17]. What follows is a sketch of that paper highlighting the key results of interest to our calculation. The amplitudes given in the paper are from the neutral current and so only relevant for the S channel. In this section we will not explicitly denote the channel of the amplitudes, but it is assumed always to be S.

3.3.1 The Structure of Sudakov Logarithms

To calculate the scattering amplitude in the desired approximation we must individually take into account the hard, collinear, and soft divergences of the loop integrals. The ultraviolet (hard) region has been well studied. Ultraviolet divergences lead to running of the coupling constants governed by the Callan-Symanzik equation. We will insert the running of α by hand into our calculation, and in this way account for all hard divergences. At the present accuracy we require the two loop running:

$$\alpha(Q^2) = \alpha(\mu^2)\left\{1 - \frac{\alpha(\mu^2)}{4\pi}\beta_0 \log\left(\frac{Q^2}{\mu^2}\right) + \frac{\alpha(\mu^2)^2}{16\pi^2}\left[\beta_0^2 \log^2\left(\frac{Q^2}{\mu^2}\right) - \beta_1 \log\left(\frac{Q^2}{\mu^2}\right)\right]\right\} \ . \tag{37}$$

[5] that is, all $\mathcal{A}_i^{(2)}$ except for $i = 0$

In the above, μ^2 is the renormalization scale of the theory, it is typically taken to be the energy of the interaction or one of the relevant mass scales. The standard value of the electromagnetic coupling $\alpha_{\text{EM}} \approx \frac{1}{137}$ is in fact the value measured at the electron mass $\alpha_{\text{EM}}(m_e^2)$. The real numbers β_0 and β_1 are the leading and subleading order values of the β-function, a parameter of the Callan-Symanzik equation. They depend on the specific theory being employed; β_0 in SU(2) is a different quantity than β_0 in U(1).

It is a remarkable and well known result that the collinear divergences in any amplitude depend only on the external particles, and not on the specific details of an interaction [22]. This allows us to factor out the collinear divergences from each loop integral, leaving only a reduced amplitude $\tilde{\mathcal{A}}$ which contains only soft divergences. Since the collinear divergences will be the same for all processes we may calculate them in a simple interaction, defining $\tilde{\mathcal{A}}$ to collect all remaining soft divergences. We choose to use the SU(2) form factor $\mathcal{F}$ describing scattering by an external abelian field. We need two copies of $\mathcal{F}$, one for each external fermion line. The resulting expression for the amplitude is:

$$\mathcal{A} = \frac{ig^2}{s}\mathcal{F}^2\tilde{\mathcal{A}} \, . \tag{38}$$

The reduced amplitude $\tilde{\mathcal{A}}$ will be a vector in the flavour basis. Since $\mathcal{F}$ is calculated for an external abelian field, it is independent of flavour will be a scalar in this basis. We calculate $\mathcal{F}$ and $\tilde{\mathcal{A}}$ using the infrared evolution equations, an infrared analogue to the ultraviolet Callan-Symanzik equations. The evolution equations control how an amplitude scales with the Euclidean momentum transfer Q. The theory of the infrared evolution equations was formulated and developed in [23–26]. For the form factor it takes the following form:

$$\frac{\partial}{\partial \log(Q^2)}\mathcal{F} = \left(\int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x}\gamma(\alpha(x)) + \zeta(\alpha(Q^2)) + \xi(\alpha(M^2)) \right) \mathcal{F} \, . \tag{39}$$

γ, ζ, and ξ are the parameters of anomalous dimension, and are dependent on both the theory and the specific process involved. The contribution from ζ and ξ has been split so ξ contains dependence only on $\alpha(M^2)$; it will contain only infrared divergences. Since (39) is linear, its solution may be written down immediately:

$$\mathcal{F} = \mathcal{F}_0(\alpha(M^2)) \exp\left\{ \int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x} \left(\int_{M^2}^{x} \frac{\mathrm{d}x'}{x'}\gamma(\alpha(x')) + \zeta(\alpha(x)) + \xi(\alpha(M^2)) \right) \right\} \, . \tag{40}$$

The integral is performed by expanding the anomalous dimensions in a power series in α, taking into account the running of α in (37). We illustrate the calculation for the γ term giving rise to the double logarithms. The renormalization scale μ is set as $\mu^2 = M^2$, γ is expanded to $\mathcal{O}(\alpha^2)$, and only the first order running of α is used. Any terms of higher order in α will not contribute to our result and can safely be neglected. Bare α's with no scale dependence written will be assumed to be $\alpha(M^2)$.

$$\begin{aligned}
\int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\int_{M^2}^{x}\frac{\mathrm{d}x'}{x'}\gamma(\alpha(x)) &= \int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\int_{M^2}^{x}\frac{\mathrm{d}x'}{x'}\left[\gamma^{(0)}+\frac{\alpha(x')}{4\pi}\gamma^{(1)}+\left(\frac{\alpha(x')}{4\pi}\right)^2\gamma^{(2)}\right] \\
&= \int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\int_{M^2}^{x}\frac{\mathrm{d}x'}{x'}\left[\gamma^{(0)}+\frac{\alpha}{4\pi}\gamma^{(1)}\right. \\
&\qquad \left.+\left(\frac{\alpha}{4\pi}\right)^2\left(\gamma^{(2)}-\beta_0\gamma^{(1)}\log\frac{x'}{M^2}\right)\right] \\
&= \int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\left[\left(\gamma^{(0)}+\frac{\alpha}{4\pi}\gamma^{(1)}+\left(\frac{\alpha}{4\pi}\right)^2\gamma^{(2)}\right)\log\frac{x}{M^2}\right. \\
&\qquad \left.-\left(\frac{\alpha}{4\pi}\right)^2\frac{\beta_0\gamma^{(1)}}{2}\log^2\frac{x}{M^2}\right] \\
&= \frac{1}{2}\left(\gamma^{(0)}+\frac{\alpha}{4\pi}\gamma^{(1)}+\frac{\alpha^2}{16\pi^2}\gamma^{(2)}\right)\log^2\frac{Q^2}{M^2} \\
&\qquad -\frac{\alpha^2}{16\pi^2}\frac{\beta_0\gamma^{(1)}}{6}\log^3\frac{Q^2}{M^2}
\end{aligned} \tag{41}$$

Doing the same process for ζ and ξ gives our exponentiated result for $\mathcal{F}$. From simple matching with the Born amplitude (34) we can determine the leading order values for all our parameters

$$\mathcal{F}_0^{(0)} = 1 \quad \text{and} \quad \gamma^{(0)} = \zeta^{(0)} = \xi^{(0)} = 0\ . \tag{42}$$

Simple algebra then gives:

$$\begin{aligned}
\mathcal{F} = \ &\left(1+\frac{\alpha}{4\pi}\mathcal{F}_0^{(1)}\right)\exp\left\{\frac{\alpha}{4\pi}\left(\frac{1}{2}\gamma^{(1)}\log^2\frac{Q^2}{M^2}+\left(\zeta^{(1)}+\xi^{(1)}\right)\log\frac{Q^2}{M^2}\right)\right. \\
&\left.+\frac{\alpha^2}{16\pi^2}\left(-\frac{1}{6}\beta_0\gamma^{(1)}\log^3\frac{Q^2}{M^2}+\frac{1}{2}\left(\gamma^{(2)}-\beta_0\zeta^{(1)}\right)\log^2\frac{Q^2}{M^2}+\left(\zeta^{(2)}+\xi^{(2)}\right)\log\frac{Q^2}{M^2}\right)\right\}\ .
\end{aligned} \tag{43}$$

We only expand $\mathcal{F}_0$ to $\mathcal{O}(\alpha)$ because the only contribution of $\mathcal{F}_0^{(2)}$ is to the two loop nonlogarithmic term, which is beyond the scope of the current work. The evolution equation for $\tilde{\mathcal{A}}$ takes on a much simpler form, as it only has to account for the soft divergences:

$$\frac{\partial}{\partial\log(Q^2)}\tilde{\mathcal{A}} = \chi(\alpha(Q^2))\tilde{\mathcal{A}}\ . \tag{44}$$

It is complicated however by the vectorial nature of $\tilde{\mathcal{A}}$; χ is in fact a matrix:

$$\chi = \begin{pmatrix} \chi_{\lambda\lambda} & \chi_{d\lambda} \\ \chi_{\lambda d} & \chi_{dd} \end{pmatrix} . \tag{45}$$

This complicates our expression of the solution to (44) as χ does not necessarily commute with itself for different values of α. This is taken into account by expressing the solution as the path ordered exponent:

$$\tilde{\mathcal{A}} = \mathrm{P} \exp\left\{ \int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x} \chi\left(\alpha\left(x\right)\right) \right\} \tilde{\mathcal{A}}_0(\alpha(M^2)) \ . \tag{46}$$

As with $\mathcal{F}$ we can determine χ and $\tilde{\mathcal{A}}_0$ at leading order in α by making a strict comparison with the Born amplitude (34). The result is:

$$\tilde{\mathcal{A}}_0^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{47}$$

Expanding χ in α then allows us to explicitly perform the integral in (46) just as with $\mathcal{F}$. As before, $\alpha = \alpha(M^2)$:

$$\begin{aligned} \tilde{\mathcal{A}} = \exp\left\{ \frac{\alpha}{4\pi}\chi^{(1)} \log\frac{Q^2}{M^2} + \frac{\alpha^2}{16\pi^2}\left(\chi^{(2)} \log\frac{Q^2}{M^2} - \frac{1}{2}\beta_0 \chi^{(1)} \log^2\frac{Q^2}{M^2} \right) \right\} \\ \times \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\alpha}{4\pi} \begin{bmatrix} \tilde{\mathcal{A}}_0^{\lambda(1)} \\ \tilde{\mathcal{A}}_0^{d(1)} \end{bmatrix} \right) . \end{aligned} \tag{48}$$

We now have all we need to construct the amplitude. Exponentials are expanded in a power series, and we keep only contributions up to $\mathcal{O}\left(\alpha^2 \log\frac{Q^2}{M^2}\right)$:

$$\mathcal{A} = \frac{ig^2}{s}\left\{\mathcal{A}^\lambda\left[1+\frac{\alpha}{4\pi}\left(\gamma^{(1)}\log^2\frac{Q^2}{M^2}+\left(2\zeta^{(1)}+2\xi^{(1)}+\chi^{(1)}_{\lambda\lambda}\right)\log\frac{Q^2}{M^2}+2\mathcal{F}_0^{(1)}+\tilde{\mathcal{A}}_0^{\lambda(1)}\right)\right.\right.$$
$$+\frac{\alpha^2}{16\pi^2}\left(\frac{1}{2}\left(\gamma^{(1)}\right)^2\log^4\frac{Q^2}{M^2}+\left(2\zeta^{(1)}+\chi^{(1)}_{\lambda\lambda}-\frac{1}{3}\beta_0\right)\gamma^{(1)}\log^3\frac{Q^2}{M^2}\right.$$
$$+\left(\gamma^{(2)}+\left(2\zeta^{(1)}-\beta_0\right)\zeta^{(1)}+\gamma^{(1)}\left(2\mathcal{F}_0^{(1)}+\tilde{\mathcal{A}}_0^{\lambda(1)}\right)\right.$$
$$\left.+\frac{1}{2}\left(4\zeta^{(1)}-\beta_0+\chi^{(1)}_{\lambda\lambda}\right)\chi^{(1)}_{\lambda\lambda}+\frac{1}{2}\chi^{(1)}_{\lambda d}\chi^{(1)}_{d\lambda}\right)\log^2\frac{Q^2}{M^2}$$
$$+\left(2\zeta^{(2)}+2\xi^{(2)}+2\mathcal{F}_0^{(1)}\left(2\zeta^{(1)}+\chi^{(1)}_{\lambda\lambda}\right)+\chi^{(2)}_{\lambda\lambda}\right.$$
$$\left.\left.\left.+\left(2\zeta^{(1)}+\chi^{(1)}_{\lambda\lambda}\right)\tilde{\mathcal{A}}_0^{\lambda(1)}+\chi^{(1)}_{d\lambda}\tilde{\mathcal{A}}_0^{d(1)}\right)\log\frac{Q^2}{M^2}\right)\right]$$
$$+\mathcal{A}^d\left[\frac{\alpha}{4\pi}\left(\chi^{(1)}_{\lambda d}\log\frac{Q^2}{M^2}+\tilde{\mathcal{A}}_0^{d(1)}\right)+\frac{\alpha^2}{16\pi^2}\left(\chi^{(1)}_{\lambda d}\gamma^{(1)}\log^3\frac{Q^2}{M^2}\right.\right.$$
$$+\left(\frac{1}{2}\left(4\zeta^{(1)}-\beta_0+\chi^{(1)}_{\lambda\lambda}\right)\chi^{(1)}_{\lambda d}+\gamma^{(1)}\tilde{\mathcal{A}}_0^{d(1)}\right)\log^2\frac{Q^2}{M^2}$$
$$\left.\left.+\left(2\mathcal{F}_0^{(1)}\chi^{(1)}_{\lambda d}+\chi^{(2)}_{\lambda d}+\chi^{(1)}_{\lambda d}\tilde{\mathcal{A}}_0^{\lambda(1)}+\left(2\zeta^{(1)}+\chi^{(1)}_{dd}\right)\tilde{\mathcal{A}}_0^{d(1)}\right)\log\frac{Q^2}{M^2}\right]\right\}\ . \tag{49}$$

All that remains is to evaluate the anomalous dimensions γ, ζ, ξ, χ and the initial conditions $\mathcal{F}_0$ and $\tilde{\mathcal{A}}_0$ to second and first order, respectively. The process $f\bar{f}\to f'\bar{f}'$ has been calculated fully to one loop, which allows us to effectively read off all our parameters to first order from well established existing results [17]. We sketch this procedure for the SU(N) form factor at one loop.

3.3.2 Calculating the SU(N) Form Factor

We want to calculate the one loop SU(N) form factor $\mathcal{F}$ for electron scattering in an external abelian field. This quantity will contain the collinear divergences we can factorize from the rest the full Bhabha amplitude, allowing us to treat the soft divergences independently. The tree level amplitude for this process is:

$$\mathcal{A}^\mu_B = \bar{u}(p')\gamma^\mu u(p)\ . \tag{50}$$

We define the initial momentum as p, the final as p', and the momentum transfer $q=p'-p$. The form factors are defined as the coefficients of the different Lorentz structures which appear in the amplitude at higher orders. There are two such structures, which leads to the following decomposition:

$$\mathcal{A}^\mu = \bar{u}(p')\left(F_1(q^2)\gamma^\mu+F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m_e}\right)u(p)\ . \tag{51}$$

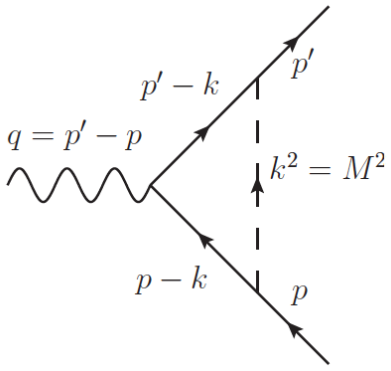


Figure 2: The Feynman diagram resulting in the one-loop form factor.

The second term F_2 is suppressed by a factor of Q^2 is negligible in the high energy limit [20]. We can then unambiguously define $\mathcal{F}$:

$$\mathcal{A}^\mu = \bar{u}(p')\ \mathcal{F}(q^2)\gamma^\mu\ u(p)\ . \tag{52}$$

This leads to the trivial tree level result:

$$\mathcal{F}^{(0)} = \mathcal{F}_B = 1\ . \tag{53}$$

The one loop value of $\mathcal{F}$ comes from a single diagram (Fig. 2) where a gauge boson of momentum k^μ is exchanged by the electrons. Its amplitude can be written:

$$\mathcal{A}^\mu = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\bar{u}(p')\,(ig\gamma^\nu t^a)\,\frac{i(\not{p}' - \not{k})}{(p-k)^2}\gamma^\mu\frac{-i}{k^2-M^2}\frac{i(\not{p} - \not{k})}{(p-k)^2}\,(igt^a\gamma_\nu)\,u(p)\ . \tag{54}$$

The gauge boson mass is M and the SU(N) coupling is g. The matrices t^a are generators of SU(N) and commute with all other elements. Using the contraction identities for gamma matrices and $p^2 = p'^2 = 0$ we can write:

$$\mathcal{A}^\mu = \bar{u}(p')\left[-2ig^2C_F\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{p} - \not{k})\gamma^\mu(\not{p}' - \not{k})}{(k^2-2p\cdot k)(k^2-2p'\cdot k)(k^2-M^2)}\right]u(p)\ . \tag{55}$$

$C_F = t^at^a$ is the quadratic Casimir operator of the fundamental representation, in SU(2):

$$C_F = \frac{N^2-1}{2N}\ . \tag{56}$$

We calculate this integral in dimensional regularization using the expansion by regions approach [27, 28]. In this method, one approximates an integral by its behaviour near poles, then extends the approximation over the entire integration domain. If all poles have been accounted for the singularities introduced by the approximation cancel, and a finite result is achieved. Although this approach is perhaps overpowered for a one loop calculation, it

has proved necessary for calculations at two loops. The advantage of using dimensional regularization is the logarithmic contributions are easily read off from the final result.

We sketch this process for the form factor integral (55). The integrand becomes singular in three obvious regions, when each of the terms in the denominator goes to zero. A fourth region is obtained in limit as $k^2 \to \infty$, as can be seen by simple power counting. If $|k| = \omega$ in the Euclidean sense, the behaviour of the integral at large ω is:

$$\int^{\infty} \mathrm{d}^4k \, \frac{k^2}{k^6} \to \int^{\infty} \mathrm{d}\omega \, \omega^3 \frac{\omega^2}{\omega^6} = \int^{\infty} \mathrm{d}\omega \frac{1}{\omega} \to \infty \, . \tag{57}$$

We then have four regions to expand in:

$$\begin{aligned} (h) \text{ hard: } & k^2 \sim q^2 \, , \\ (1c) \text{ 1-collinear: } & k \sim 2p \, , \\ (2c) \text{ 2-collinear: } & k \sim 2p' \, , \\ (s) \text{ soft: } & k^2 \sim M^2 \, . \end{aligned} \tag{58}$$

We split the domain of integration over each of these regions. Denoting the integral (55) as $\int \mathcal{I}$ we do the following decomposition:

$$\int_0^{\infty} \mathcal{I} \approx \int\limits_{k^2 \sim q^2} \mathcal{I} + \int\limits_{k \sim 2p} \mathcal{I} + \int\limits_{k \sim 2p'} \mathcal{I} + \int\limits_{k^2 \sim M^2} \mathcal{I} \, . \tag{59}$$

We then taylor expand the integrand in each of these regions in parameters considered small there and keep only leading order terms. For instance in the hard region, M^2 is far less than any other scale and makes a good expansion parameter. The leading order behaviour is obtained by simply setting $M^2 = 0$ in the integrand:

$$\int\limits_{k^2 \sim q^2} \mathcal{I} = \int\limits_{k^2 \sim q^2} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(\not{p} - \not{k})\gamma^\mu(\not{p}' - \not{k})}{(k^2 - 2p\cdot k)(k^2 - 2p'\cdot k)(k^2 - M^2)} \tag{60}$$

$$\approx \int\limits_{k^2 \sim q^2} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{(\not{p} - \not{k})\gamma^\mu(\not{p}' - \not{k})}{(k^2 - 2p\cdot k)(k^2 - 2p'\cdot k)k^2} \quad \equiv \int\limits_{k^2 \sim q^2} \mathcal{I}^{(h)} \, . \tag{61}$$

We have defined $\mathcal{I}^{(h)}$ as the leading order behaviour of $\mathcal{I}$ in the hard region. The same can be done for $\mathcal{I}^{(1c)}$, $\mathcal{I}^{(2c)}$, and $\mathcal{I}^{(s)}$. We now extend the integration limits in each term over the full domain. It should be noted in this new domain the expansion we performed is not valid. For instance, the hard integral will now include contributions from $k^2 \ll M^2$ where the our approximation becomes singular. Nevertheless, we press on and extend the limits. The integrals acquire divergences, for example $\int \mathcal{I}^{(h)}$ will have infrared divergences of both

the soft and collinear types. If dimensional regularization is used to handle these spurious[6] divergences, the final result summed over all regions will be finite. Meanwhile, we are left with integrals with one less mass scale which can be computed in a straightforward manner. The final result will be exactly equal to the initial integral, that is:

$$\int_0^\infty \mathcal{I} = \int_0^\infty \mathcal{I}^{(h)} + \int_0^\infty \mathcal{I}^{(1c)} + \int_0^\infty \mathcal{I}^{(2c)} + \int_0^\infty \mathcal{I}^{(s)} \ . \tag{62}$$

Calculating the integrals is a straightforward exercise in dimensional regularization. Integrals are Wick-rotated into the Euclidean regime and Feynman parameters are introduced. In the hard region one gets a result proportional to $(Q^2)^{-\epsilon}$, where ϵ been introduced through the dimensional regularization by taking the spacetime dimension $d = 4 - 2\epsilon$ [27]. Q is the Euclidean momentum transfer, $Q^2 = -q^2 > 0$. We can taylor expand this factor as $\epsilon \to 0$:

$$\left(Q^2\right)^{-\epsilon} \approx 1 - \epsilon \log Q^2 + \frac{1}{2}\epsilon^2 \log^2 Q^2 \ . \tag{63}$$

This factor multiplies the divergent integral $\int \mathcal{I}^{(h)}$.[7] In the limit as $\epsilon \to 0$ then, we will obtain logarithms of Q^2 only when $(Q^2)^{-\epsilon}$ multiplies the poles of $\int \mathcal{I}^{(h)}$. Hence to get an amplitude accurate only to logarithmic factors, we need calculate only the poles of the expanded integrals. Double poles $1/\epsilon^2$ will produce double logarithms $\log^2 Q^2$ and single poles will give single logs.

The one loop amplitude has been known for some time, and this procedure may be carried past the logarithmic terms to also produce the constant [17]. The results of this are:

$$\begin{aligned}
\mathcal{F}^{(1)}_{\text{hard}} &= C_F \left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(2\log Q^2 - 3\right) - \log^2 Q^2 + 3\log Q^2 + \frac{\pi^2}{6} - 8 \right) \ , \\
\mathcal{F}^{(1)}_{\text{collinear}} &= C_F \left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(2\log Q^2 - 4\right) + 2\log Q^2 \log M^2 - \log^2 M^2 - 4\log M^2 - \frac{5\pi^2}{6} + 4 \right) \ , \\
\mathcal{F}^{(1)}_{\text{soft}} &= C_F \left(-\frac{1}{\epsilon} + \log M^2 + \frac{1}{2} \right) \ .
\end{aligned} \tag{64}$$

Notice how the poles of the individual contributions exactly cancel in the sum. Our final result is then finite and we can read off the one loop values of γ, ζ, ξ, and $\mathcal{F}_0$. The only linear logarithmic dependence comes from the hard contribution. Since ξ was defined to depend only on infrared contributions it will be zero at this order.

[6] These divergences are 'spurious' because they are not from physical processes, merely artifacts of our appriximation.

[7] Of course there is a slight abuse of notation here, as $(Q^2)^{-\epsilon}$ comes from exactly the integral $\int \mathcal{I}^{(h)}$. Here we mean the integral with this Q^2 factored out.

$$
\begin{aligned}
\gamma^{(1)} &= -2C_F \ , \\
\zeta^{(1)} &= 3C_F \ , \\
\xi^{(1)} &= 0 \ , \\
\mathcal{F}_0^{(1)} &= -C_F \left(\frac{7}{2} + \frac{2}{3}\pi^2 \right) \ .
\end{aligned} \tag{65}
$$

The calculation of $\tilde{\mathcal{A}}$ is done in a similar fashion. This calculation has been performed up to $\mathcal{O}(\alpha^2 \log Q^2/M^2)$ in [17]. We are now ready to move on to the full amplitude.

3.3.3 Final Results for the SU(N) Amplitude

In SU(N) the reduced amplitude $\tilde{\mathcal{A}}$ is determined by [17]:

$$
\begin{aligned}
\chi_{\lambda\lambda}^{(1)} &= -2C_A \log\frac{u}{s} + 4\left(C_F - \frac{T_f}{N}\right)\log\frac{u}{t} \ , \\
\chi_{\lambda d}^{(1)} &= 4\frac{C_F T_f}{N}\log\frac{u}{t} \ , \\
\chi_{d\lambda}^{(1)} &= 4\log\frac{u}{t} \ , \\
\chi_{dd}^{(1)} &= 0 \ , \\
\tilde{\mathcal{A}}_0^{\lambda(1)} &= \left(C_F - \frac{T_f}{N}\right) f\big(\frac{u}{s},\frac{t}{s}\big) + C_A\left(\frac{85}{9} + \pi^2\right) - \frac{20}{9}T_f n_f - \frac{8}{9}T_f n_s \ , \\
\tilde{\mathcal{A}}_0^{d(1)} &= \frac{C_F T_f}{N} f\big(\frac{u}{s},\frac{t}{s}\big) \ .
\end{aligned} \tag{66}
$$

Where $C_A = N$ is the quadratic Casimir operator of the adjoint representation, n_f is the number of dirac fermions in the theory, n_s is the number of dirac scalars in the theory, T_f is the weak isospin of the fermion f, and the function $f(x,y)$ is given by:

$$
f(x,y) = -\frac{2}{x}\log y - \frac{y-x}{x^2}\left(\log^2 y + \pi^2\right) \ . \tag{67}
$$

Notice we are required to take the logarithm of a negative quantity: t/s. We resolve this through the use of analytic continuation in the next section. Also required is the SU(2) beta function:

$$
\begin{aligned}
\beta_0 &= -\frac{4}{3}N_g + \frac{43}{6} \ , \\
\beta_1 &= \frac{181}{6} \ .
\end{aligned} \tag{68}
$$

Where $N_g = 3$ is the number of generations. The second order parameters we need are not as straightforward to determine. In [17] they were found by the expansion by regions approach; expanding certain Feynman integrals around their divergences and reading off the parameters from the appropriate terms in the expansion. The results of this procedure for the form factor are:

$$\gamma^{(2)} = C_F \left(\left(\frac{-134}{9} + \frac{2}{3}\pi^2 \right) C_A + \frac{40}{9} T_f n_f + \frac{16}{9} T_f n_s \right) , \tag{69}$$

$$\begin{aligned} \zeta^{(2)} + \xi^{(2)} = C_F \Bigg(& \frac{-34}{3} N_g + \frac{749}{12} + \frac{43}{18}\pi^2 - \frac{176}{3}\zeta(3) + 5\sqrt{3}\ \pi \\ & + \frac{26}{3}\sqrt{3}\ \mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{C_F}{2}\left(3 - 4\pi^2 + 48\ \zeta(3)\right) \Bigg) . \end{aligned} \tag{70}$$

In the above $\zeta^{(2)}$ and $\xi^{(2)}$ are presented in a sum because it is independent of renormalization scheme. $\mathrm{Cl}_2(x) = -\int_0^x \log|2\sin(t/2)|\mathrm{d}t$ is the Clausen function and $\zeta(3)$ is the Reimann-Zeta function.[8] Numerically $\mathrm{Cl}_2(\pi/3) = 1.01494$ and $\zeta(3) = 1.2026$. This same procedure applied to the soft divergences gives us the required components of $\chi^{(2)}$:

$$\begin{aligned} \chi^{(2)}_{\lambda\lambda} &= \left(-\frac{20}{9} N_g + \frac{130}{9} - \frac{2}{3}\pi^2 \right) \left(-2C_A \log\frac{u}{s} + 4\left(C_F - \frac{T_f}{N}\log\frac{u}{t} \right) \right) - \beta_0 \tilde{\mathcal{A}}_0^{\lambda(1)} , \\ \chi^{(2)}_{\lambda d} &= \left(-\frac{5}{3} N_g + \frac{65}{6} - \frac{1}{2}\pi^2 \right) \log\frac{u}{t} - \beta_0 \tilde{\mathcal{A}}_0^{d(1)} . \end{aligned} \tag{71}$$

We now have all the information required to compute the SU(2) Bhabha scattering amplitude and thus obtain the desired cross section. The first step will be extracting the S and T channel amplitudes from (49) for non-Euclidean Q^2.

3.4 Crossing Symmetry and Analytic Continuation

In the previous section we followed the derivation of [17] and computed the amplitude for $f\bar{f} \to f'\bar{f}'$ scattering in SU(2). There are two complications in moving from the result to the amplitude for Bhabha scattering. The first is simply the fact that Bhabha has contributions from two channels, S and T. The result (49) was calculated exclusively for annihilation (S channel) interactions. To obtain the amplitude for T channel, we use crossing symmetry and redefine external momenta, effectively making the switch $s \leftrightarrow t$. The second complication arises from the fact that the infrared evolution equations were formulated for Euclidean momentum transfer $Q^2 < 0$ and the definition of the $f-$function in (67). These both cause the arguments of logarithms in the amplitude to occur precisely on a branch cut. We make use of the $s + i\epsilon$ prescription to resolve this in a consistent manner [17].

[8] Not to be confused with the anomalous dimension $\zeta(\alpha)$.

Our first goal is to express the amplitude obtained in (49), $\mathcal{A}(Q^2)$ as function of our kinematical invariants $s > 0$ and $t, u < 0$. Trouble may arise in two places; from the large logarithms $\log\frac{Q^2}{M^2}$ with negative argument and from the angular logarithms such as $\log\frac{u}{s}$. These latter logarithms are referred to as "angular" because they in fact only depend on the kinematic variable x. For example:

$$\log\frac{u}{s} = \log\left(\frac{-s(1-x)}{s}\right) = \log(x-1)\ . \tag{72}$$

We run into a problem above, as since $x \in [0,1]$, the argument of the above log is negative.[9] The issue with the logarithms is in consistently choosing which branch to use, effectively picking which sign to use in $\log(-x) \to \log x \pm i\pi$. This ambiguity is lifted through the use of analytic continuation by employing the conventional prescription $s \to s + i\epsilon$, where ϵ is taken to be a very small positive number. This moves the argument of the logarithm off the branch cut, making the choice of branch obvious. We illustrate the procedure for the above logarithm:

$$\log\frac{u}{s} \approx \log\frac{u}{s+i\epsilon} = \log\left(\frac{us}{s^2+\epsilon^2} + i\underbrace{\frac{-\epsilon u}{s^2+\epsilon^2}}_{\text{positive}}\right) \underset{\epsilon\to 0}{=} \log\left(\frac{-u}{s}\right) + i\pi\ . \tag{73}$$

The process is identical for all angular logs. Writing the angular dependence explicitly:

$$\begin{aligned}\log\frac{u}{s} &= \log(1-x) + i\pi\ ,\\ \log\frac{t}{s} &= \log(x) + i\pi\ .\end{aligned} \tag{74}$$

These are the only angular logs requiring analytic continuation in the SU(2) amplitude. In the S channel the momentum transfer between the fermion lines is simply the centre of mass energy $\sqrt{s}$. The S channel amplitude is then simply (49) with the identification $Q^2 = -s$. Analytic continuation with the $s + i\epsilon$ prescription gives us the correct branch for the large Sudakov logarithms:

$$\log\frac{Q^2}{M^2} = \log\left(\frac{-s}{M^2}\right) = \log\Big(\frac{s}{M^2}\Big) - i\pi\ . \tag{75}$$

These factors of $i\pi$ will cascade through the amplitude, modifying all contributions at lower orders of $\log\frac{s}{M^2}$. The T channel can be computed in the same approach as the S channel. The squared momentum transfer in T channel is t, so the identification with Q becomes $Q^2 = -t$. We can then write the Sudakov logarithms as:

$$\log\frac{Q^2}{M^2} = \log\left(\frac{-t}{M^2}\right) = \log\left(\frac{sx}{M^2}\right) = \log\Big(\frac{s}{M^2}\Big) + \log(x)\ . \tag{76}$$

[9] This of course is also trivially seen in the ratio $\frac{u}{s}$, $s > 0$ and $u < 0$.

The angular dependence of $\mathcal{A}$ also changes as one goes from S to T channels. This is accomplished through the use of crossing symmetry; the T channel amplitude is given by taking the S channel amplitude and making the substitution $s \leftrightarrow t$. It should be noted this operation does not commute with the analytic continuation above, and so must be performed on the amplitude $\mathcal{A}(Q^2)$ before the negative arguments to the logarithms are resolved. We define the crossing symmetry operator:

$$\underset{s\leftrightarrow t}{\Delta}\mathcal{A}(s,t) = \mathcal{A}(t,s) \ . \tag{77}$$

The last effect to take into account is the energy scale of the Born amplitude. To this point we have been working under the implicit assumption the scale of $g^2 = 4\pi\alpha$ in (35) is Q^2. We really should have written

$$\begin{aligned}
\mathcal{A}^S &= \frac{4\pi i\alpha(-s)}{s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \mathcal{A}^{S(n)} \ , \\
\mathcal{A}^T &= \frac{4\pi i\alpha(-t)}{t}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \mathcal{A}^{T(n)} \ .
\end{aligned} \tag{78}$$

Of course to properly sum these amplitudes they should use α measured at the same (positive) scale. To resolve this ambiguity we make the choice to set $\alpha(s)$ as the scale in the Born amplitude, using the running of α in (37). We keep the convention that all loop corrections are renormalized to $\alpha = \alpha(M^2)$, and use the $s + i\epsilon$ to resolve negative arguments to logarithms.

$$\begin{aligned}
\alpha(-s) &= \alpha(s)\left\{1 + \frac{\alpha}{4\pi}i\pi\beta_0 - \frac{\alpha^2}{16\pi^2}i\pi\beta_0^2\log\Big(\frac{s}{M^2}\Big)\right\} \\
\alpha(-t) &= \alpha(s)\left\{1 - \frac{\alpha}{4\pi}\beta_0\log(x) + \frac{\alpha^2}{16\pi^2}\beta_0^2\log(x)\log\Big(\frac{s}{M^2}\Big)\right\}
\end{aligned} \tag{79}$$

We have dropped terms at subleading orders to the present calculation. Note the coupling constant α is defined to be real, hence the need for a positive renormalization scale. The imaginary parts of the $\alpha(-s)$ expansion are formally not a part of α, they are included as a contribution to $\mathrm{Im}(\tilde{\mathcal{A}}_0)$ and are artifacts of performing the original calculation for Euclidean Q^2. With this information we can write the full expansions of $\mathcal{A}^S$ and $\mathcal{A}^T$ in terms of α and $\log\big(\frac{s}{M^2}\big)$, taking into account the running of α in the Born amplitude and the analytic continuation of $\log Q^2/M^2$. The S channel expansion is:

$$\mathcal{A}^S = \frac{ig^2(s)}{s}\sum_{n=0}^{\infty}\sum_{i=0}^{2n}\Big(\mathcal{A}_i^{(n)} + \delta_S\mathcal{A}_i^{(n)}\Big)\log^i\Big(\frac{s}{M^2}\Big) \ . \tag{80}$$

Where $\mathcal{A}_i^{(n)}$ is the $\mathcal{O}(\alpha^n \log^i \frac{Q^2}{M^2})$ term in $\mathcal{A}(Q^2)$. The correction terms $\delta_S \mathcal{A}_i^{(n)}$ are given to one loop by:

$$\begin{aligned}
\delta_S \mathcal{A}_0^{(0)} &= 0 \ , \\
\delta_S \mathcal{A}_2^{(1)} &= 0 \ , \\
\delta_S \mathcal{A}_1^{(1)} &= -2i\pi \mathcal{A}_2^{(1)} \ , \\
\delta_S \mathcal{A}_0^{(1)} &= -\pi^2 \mathcal{A}_2^{(1)} - i\pi \mathcal{A}_1^{(1)} + i\pi\beta_0 \mathcal{A}_0^{(0)} \ .
\end{aligned} \tag{81}$$

At two loops the situation is a little simpler since only the real part of the amplitude will contribute to the cross section. We define the correction terms $\delta_S \mathcal{A}_i^{(n)}$ to only include terms relevant to the cross section.

$$\begin{aligned}
\delta_S \mathcal{A}_4^{(2)} &= 0 \ , \\
\delta_S \mathcal{A}_3^{(2)} &= -4i\pi \mathrm{Im}\big(\mathcal{A}_4^{(2)}\big) \ , \\
\delta_S \mathcal{A}_2^{(2)} &= -6\pi^2 \mathrm{Re}\big(\mathcal{A}_4^{(2)}\big) - 3i\pi \mathrm{Im}\big(\mathcal{A}_3^{(2)}\big) + i\pi\beta_0 \mathrm{Im}\big(\mathcal{A}_2^{(1)}\big) \ , \\
\delta_S \mathcal{A}_1^{(2)} &= 4i\pi^3 \mathrm{Im}\big(\mathcal{A}_4^{(2)}\big) - 3\pi^2 \mathrm{Re}\big(\mathcal{A}_3^{(2)}\big) - 2i\pi \mathrm{Im}\big(\mathcal{A}_2^{(2)}\big) + i\pi\beta_0 \mathrm{Im}\big(\mathcal{A}_1^{(1)}\big) \\
&\qquad - i\pi\beta_0^2 \mathrm{Im}\big(\mathcal{A}_0^{(0)}\big) \ .
\end{aligned} \tag{82}$$

The expansion of $\mathcal{A}^T$ in α and $\log\big(\frac{s}{M^2}\big)$ has the same structure as the S channel expansion (80):

$$\mathcal{A}^T = \frac{ig^2(M^2)}{t} \sum_{n=0}^{\infty} \sum_{i=0}^{2n} \Big(\underset{s\leftrightarrow t}{\Delta} \mathcal{A}_i^{(n)} + \delta_T \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_i^{(n)} \Big) \log^i \Big(\frac{s}{M^2} \Big) \ . \tag{83}$$

The Born and one loop correction terms take the following form:

$$\begin{aligned}
\delta_T \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_0^{(0)} &= 0 \ , \\
\delta_T \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_2^{(1)} &= 0 \ , \\
\delta_T \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_1^{(1)} &= 2\log(x) \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_2^{(1)} \ , \\
\delta_T \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_0^{(1)} &= \log^2(x) \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_2^{(1)} + \log(x) \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_1^{(1)} - \beta_0 \log(x) \underset{s\leftrightarrow t}{\Delta} \mathcal{A}_0^{(0)} \ .
\end{aligned} \tag{84}$$

The two loop corrections benefit from the same simplification as in S channel; only the real part of the amplitude will be present in the cross section.

$$
\begin{aligned}
\delta_T \underset{s\leftrightarrow t}{\Delta}\mathcal{A}_4^{(2)} &= 0 \ , \\
\delta_T \underset{s\leftrightarrow t}{\Delta}\mathcal{A}_3^{(2)} &= 4\log(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_4^{(2)}\right) \ , \\
\delta_T \underset{s\leftrightarrow t}{\Delta}\mathcal{A}_2^{(2)} &= 6\log^2(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_4^{(2)}\right) + 3\log(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_3^{(2)}\right) - \beta_0\log(x)\mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_2^{(1)}\right) \\
\delta_T \underset{s\leftrightarrow t}{\Delta}\mathcal{A}_1^{(2)} &= 4\log^3(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_4^{(2)}\right) + 3\log^2(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_3^{(2)}\right) + 2\log(x)\ \mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_2^{(2)}\right) \\
&\quad - \beta_0\log(x)\mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_1^{(1)}\right) + \beta_0^2\log(x)\mathrm{Re}\left(\underset{s\leftrightarrow t}{\Delta}\mathcal{A}_0^{(0)}\right) \ . \qquad (85)
\end{aligned}
$$

We now have robust expressions for the $f\bar{f} \to f\bar{f}$ scattering amplitude in SU(2) for the S and T channels. Calculating the Bhabha cross section is a simple matter of specializing the result to $e^+e^- \to e^+e^-$.

3.5 SU(2) Cross Section - Analytical Results

The first step in calculating the cross section is determining exactly how our perturbative expansion affects the cross section. We structured our corrections of the amplitude in the S channel as such:

$$\mathcal{A}^S = \frac{ig^2(s)}{s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \mathcal{A}^{S(n)} \ . \qquad (86)$$

Although the g^2 factor simply gives a factor of $4\pi\alpha$, we write it separately since each factor of g comes from its own form factor. We can explicitly write out the breakdown into $\mathcal{A}^\lambda, \mathcal{A}^d$ components:

$$\mathcal{A}^S = \frac{ig^2(s)}{s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \left(\mathcal{A}^{S^{\lambda(n)}}\mathcal{A}^\lambda + \mathcal{A}^{S^{d(n)}}\mathcal{A}^d\right) \ . \qquad (87)$$

Requiring the initial and final states to be eigenstates of weak isospin turns $\mathcal{A}^\lambda$ and $\mathcal{A}^d$ into projectors:

$$\mathcal{A}^S = \frac{ig^2(s)}{s}A_{LL}^S\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \left(\left(T_e^3\right)^2\mathcal{A}^{S^{\lambda(n)}} + \mathcal{A}^{S^{d(n)}}\right) \ . \qquad (88)$$

Where A_{LL} is the spinor product defined in (32). Our perturbative corrections simply multiply the Born amplitude; all spinor and vector structure can be factored out. The same of course is true in the T channel.

$$\mathcal{A}^T = \frac{ig^2(s)}{t}A_{LL}^T\sum_{n=0}^{\infty}\left(\frac{\alpha}{4\pi}\right)^n \left(\left(T_e^3\right)^2\mathcal{A}^{T^{\lambda(n)}} + \mathcal{A}^{T^{d(n)}}\right) \ . \qquad (89)$$

We will give the corrections to the cross section individually for the S channel, T channel, and interference. The differential cross section is structured in the following manner:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\left(|\mathcal{A}^S|^2 + |\mathcal{A}^T|^2 + 2\mathrm{Re}\left(\left(\mathcal{A}^S\right)^* \mathcal{A}^T\right)\right) \ . \tag{90}$$

We present the SU(2) corrections $\mathrm{d}\sigma^{(n)}$as:

$$\begin{aligned}
|\mathcal{A}^S|^2 &= |\mathcal{A}_B^S|^2\left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_S^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_S^{(2)}\right) \ , \\
|\mathcal{A}^T|^2 &= |\mathcal{A}_B^T|^2\left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_T^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_T^{(2)}\right) \ , \\
\mathrm{Re}\left(\left(\mathcal{A}^S\right)^* \mathcal{A}^T\right) &= \mathrm{Re}\left(\left(\mathcal{A}_B^S\right)^* \mathcal{A}_B^T\right)\left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_{ST}^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_{ST}^{(2)}\right) \ .
\end{aligned} \tag{91}$$

Where $\mathcal{A}_B$ is the Born amplitude:

$$\begin{aligned}
\mathcal{A}_B^S &= \frac{ig^2(s)}{s}\left(T_e^3\right)^2 A_{LL}^S \ , \\
\mathcal{A}_B^T &= \frac{ig^2(s)}{t}\left(T_e^3\right)^2 A_{LL}^T \ .
\end{aligned} \tag{92}$$

From our calculation of the Born cross section (17) we can then write the SU(2) Bhabha cross section as:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2(s)}{4s}\left(T_e^3\right)^4 \frac{(1-x)^2}{x^2}\Bigg[x^2\left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_S^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_S^{(2)}\right) + \left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_T^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_T^{(2)}\right) \ , \\
- 2x\left(1 + \frac{\alpha}{4\pi}\mathrm{d}\sigma_{ST}^{(1)} + \frac{\alpha^2}{16\pi^2}\mathrm{d}\sigma_{ST}^{(2)}\right)\Bigg] \ .
\end{aligned} \tag{93}$$

With $\alpha = g^2/4\pi$.[10] We are now ready to present the results for the cross section. In SU(2) the Casimir operator $C_F = T_f(T_f+1) = \frac{3}{4}$ and clearly $C_A = 2$. The number of Dirac fermions $n_f = 2N_g$ and the number of scalars $n_s = 1$. Left chiral electrons have weak isospin $T_L = \frac{1}{2}$ with z-projection $T_L^3 = \frac{1}{2}$.

We expand the contributions in each channel in powers of $\log s/M^2$:

[10] We follow the convention of the previous sections; $\alpha \equiv \alpha(M^2)$

$$
\begin{aligned}
\mathrm{d}\sigma^{(1)} &= \mathrm{d}\sigma_2^{(1)} \log^2\Big(\frac{s}{M^2}\Big) + \mathrm{d}\sigma_1^{(1)} \log\Big(\frac{s}{M^2}\Big) + \mathrm{d}\sigma_0^{(1)} \ , \\
\mathrm{d}\sigma^{(2)} &= \mathrm{d}\sigma_4^{(2)} \log^4\Big(\frac{s}{M^2}\Big) + \mathrm{d}\sigma_3^{(2)} \log^3\Big(\frac{s}{M^2}\Big) + \mathrm{d}\sigma_2^{(2)} \log^2\Big(\frac{s}{M^2}\Big) + \mathrm{d}\sigma_1^{(2)} \log\Big(\frac{s}{M^2}\Big) \ . \qquad (94)
\end{aligned}
$$

The one loop corrections are well known. From the S channel:

$$
\begin{aligned}
\mathrm{d}\sigma_{S\ 2}^{(1)} &= -3 \ , \\
\mathrm{d}\sigma_{S\ 1}^{(1)} &= 9 + 2\log(1-x) - 10\log(x) \ , \\
\mathrm{d}\sigma_{S\ 0}^{(1)} &= \frac{235}{18} - 3\pi^2 + \frac{5\log(x)}{1-x} - \frac{5(1-2x)\log^2(x)}{2(1-x)^2} \, . \qquad (95)
\end{aligned}
$$

The T channel:

$$
\begin{aligned}
\mathrm{d}\sigma_{T\ 2}^{(1)} &= -3 \ , \\
\mathrm{d}\sigma_{T\ 1}^{(1)} &= 9 + 2\log(1-x) + 2\log(x) \ , \\
\mathrm{d}\sigma_{T\ 0}^{(1)} &= \frac{235}{18} + 2\pi^2 + 2\log(1-x)\log(x) \\
&\quad + \frac{(8+7x)\log(x)}{3(1-x)} + \frac{5(2-2x+x^2)\log^2(x)}{2(1-x)^2} \, . \qquad (96)
\end{aligned}
$$

The ST interference:

$$
\begin{aligned}
\mathrm{d}\sigma_{ST\ 2}^{(1)} &= -3 \ , \\
\mathrm{d}\sigma_{ST\ 1}^{(1)} &= 9 + 2\log(1-x) - 4\log(x) \ , \\
\mathrm{d}\sigma_{ST\ 0}^{(1)} &= \frac{235}{18} - \frac{1}{2}\pi^2 + \log(1-x)\log(x) \\
&\quad + \frac{(23+7x)\log(x)}{6(1-x)} + \frac{5(1+x^2)\log^2(x)}{4(1-x)^2} \, . \qquad (97)
\end{aligned}
$$

The two loop $\log s/M^2$ and $\log^2 s/M^2$ corrections to Bhabha scattering are new. We begin with the S channel:

$$
\begin{aligned}
\mathrm{d}\sigma^{(2)}_{S\,4} &= \frac{9}{2}\,,\\
\mathrm{d}\sigma^{(2)}_{S\,3} &= -\frac{143}{6} - 6\log(1-x) + 30\log(x)\,,\\
\mathrm{d}\sigma^{(2)}_{S\,2} &= -\frac{145}{4} + 11\pi^2 + \frac{89}{6}\log(1-x) + 2\log^2(1-x) - 8\log(x)\log(1-x)\\
&\quad - \frac{(535-445x)\log(x)}{6(1-x)} + \frac{(91-182x+76x^2)\log^2(x)}{2(1-x)^2}\,,\\
\mathrm{d}\sigma^{(2)}_{S\,1} &= \frac{28411}{216} - 122\zeta(3) + 26\sqrt{3}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + 15\sqrt{3}\pi - \frac{(199-127x)\pi^2}{6(1-x)}\\
&\quad + \left(\frac{125}{3} + \frac{50\pi^2}{3} + \frac{10\log(x)}{1-x} - \frac{5(1-2x)\log^2(x)}{(1-x)^2}\right)\log(1-x)\\
&\quad + \left(-\frac{1075-1250x}{6(1-x)} + \frac{(74-148x+38x^2)\pi^2}{3(1-x)^2}\right)\log(x)\\
&\quad - \frac{(631-806x)\log^2(x)}{4(1-x)^2} + \frac{19(1-2x)\log^3(x)}{(1-x)^2}\,.
\end{aligned}
\tag{98}
$$

The T channel:

$$
\begin{aligned}
\mathrm{d}\sigma^{(2)}_{T\,4} &= \frac{9}{2}\,,\\
\mathrm{d}\sigma^{(2)}_{T\,3} &= -\frac{143}{6} - 6\log(1-x) - 6\log(x)\,,\\
\mathrm{d}\sigma^{(2)}_{T\,2} &= -\frac{145}{4} + 8\pi^2 + \frac{89}{6}\log(1-x) + 2\log^2(1-x) - 14\log(x)\log(1-x)\\
&\quad + \frac{(41-131x)\log(x)}{6(1-x)} - \frac{(26-22x+11x^2)\log^2(x)}{2(1-x)^2}\,,\\
\mathrm{d}\sigma^{(2)}_{T\,1} &= \frac{28411}{216} - 122\zeta(3) + 26\sqrt{3}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + 15\sqrt{3}\pi + \frac{(8+4x)\pi^2}{1-x}\\
&\quad + \left(\frac{125}{3} + \frac{8\pi^2}{3} + \frac{(17-7x)\log(x)}{1-x} - \frac{(10-30x+15x^2)\log^2(x)}{(1-x)^2}\right)\log(1-x)\\
&\quad + 4\log(x)\log^2(1-x) + \left(\frac{1030-505x}{18(1-x)} + \frac{(80-88x+44x^2)\pi^2}{3(1-x)^2}\right)\log(x)\\
&\quad + \frac{(414-502x+263x^2)\log^2(x)}{12(1-x)^2} + \frac{(10-22x+11x^2)\log^3(x)}{(1-x)^2}\,.
\end{aligned}
\tag{99}
$$

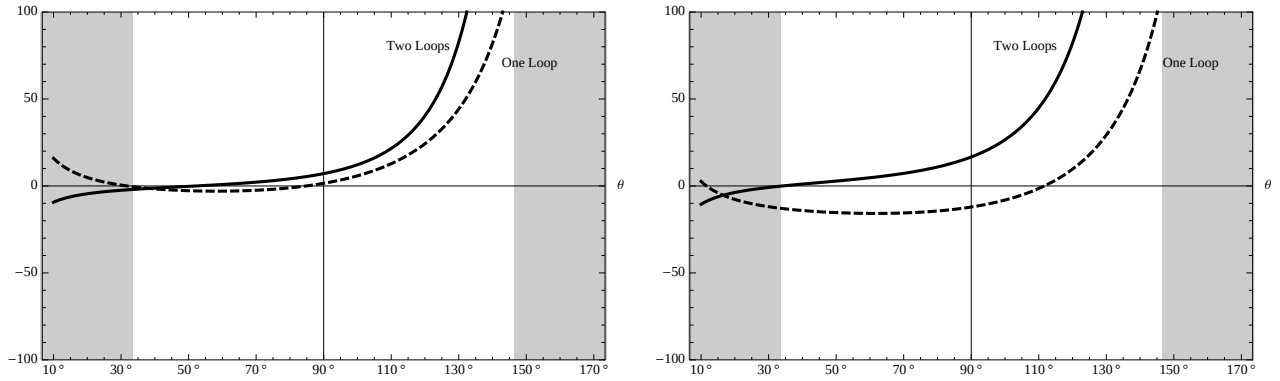
The ST interference:

$$
\begin{aligned}
\mathrm{d}\sigma^{(2)}_{ST\ 4} &= \frac{9}{2}\,, \\
\mathrm{d}\sigma^{(2)}_{ST\ 3} &= -\frac{143}{6} - 6\log(1-x) + 12\log(x)\,, \\
\mathrm{d}\sigma^{(2)}_{ST\ 2} &= -\frac{145}{4} - \frac{17}{2}\pi^2 + \frac{89}{6}\log(1-x) + 2\log^2(1-x) - 11\log(x)\log(1-x) \\
&\quad - \frac{(247-157x)\log(x)}{6(1-x)} - \frac{(7+16x+7x^2)\log^2(x)}{4(1-x)^2}\,, \\
\mathrm{d}\sigma^{(2)}_{ST\ 1} &= \frac{28411}{216} - 122\zeta(3) + 26\sqrt{3}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + 15\sqrt{3}\pi - \frac{67\pi^2}{12} \\
&\quad + \left(\frac{125}{3} + \frac{11\pi^2}{3} + \frac{(27-7x)\log(x)}{2(1-x)} - \frac{(27-64x+27x^2)\log^2(x)}{2(1-x)^2}\right)\log(1-x) \\
&\quad + 2\log(x)\log^2(1-x) - \left(\frac{2195-3245x}{36(1-x)} + \frac{(13+41x)\pi^2}{3(1-x)}\right)\log(x) \\
&\quad - \frac{(49+32x-431x^2)\log^2(x)}{24(1-x)^2} - \frac{(8+2x^2)\log^3(x)}{(1-x)^2}\,.
\end{aligned}
\tag{100}
$$

We can numerically evaluate these expressions to determine the extent to which they modify the Born cross section, and hence estimate the precision of this calculation. For the numerics, we take the argument of the logarithms M to be equal to $M_W = 80.399(23)$ GeV [29]. We are familiar with the Clausen and Reiman-Zeta functions $\mathrm{Cl}_2(\pi/3) = 1.01494$ and $\zeta(3) = 1.2026$. For the coupling, we use $\alpha = \alpha_e/s^2_W$, where α_e is the electromagnetic coupling constant and $s^2_W = \sin^2\theta_W$ is the sine of the weak mixing angle. At the Z^0 mass $\alpha_e(M_Z) = 1/(127.916 \pm 0.015)$ and $s^2_W = 0.23116(13)$ [29].

In Fig. 3 we have plotted the angular dependence of the relative correction to the Born cross section due to one loop and two loop effects for different collision energies. Even at 1 TeV the corrections are moderate, reaching the ten percent level for $\theta < 90°$. At higher energies the logarithmic dependence on s becomes evident as the two loop corrections surpass the one loop corrections for $\theta \sim 90°$, reaching almost thirty percent at 2 TeV. The shaded regions denote the kinematic area where either t or u are less than $10M^2_Z$. Our calculation worked in the Sudakov limit $s, t, u \gg M^2$. As $\theta \to 0°$ ($\theta \to 180°$) t (u) becomes small, and the limit is no longer valid. This can be seen as the corrections become singular in these regions, algebraically this is due to the presence of the angular logarithms $\log(x)$ and $\log(1-x)$ in the amplitude. As everything remains finite within these regions, these divergences may be ignored.

In Fig. 4 we have plotted the angular dependence of the relative corrections to the Born cross section for each logarithmic term; ie. $\mathrm{d}\sigma^{(2)}_i/\mathrm{d}\sigma^{(0)}$ for $i = 4, 3, 2, 1$. The most striking feature is the importance of the NNLL or $\log^2 s/M^2$ contribution at large angles, which makes up most of the two loop contribution. Being independent of θ, the leading logarithmic term simply gives a constant shift upwards, which is cancelled to the most part by the negative



(a) $\sqrt{s} = 1$ TeV (b) $\sqrt{s} = 2$ TeV

Figure 3: The percent correction to the Born cross section $\mathrm{d}\sigma^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the scattering angle θ (in degrees) from one loop and two loop effects in SU(2) Bhabha Scattering for $\sqrt{s} =$ 1 and 2 TeV.

NLL contribution. The linear log shows clear suppression relative to the higher order terms, and is virtually constant for $\theta < 120°$.

In Fig. 5 we have plotted the energy dependence of the relative corrections to the Born cross section for each logarithmic term at $\theta = 50°$. At the 1 TeV scale applicable to ILC studies the two loop corrections are less than one percent, but this relies on cancellations between the LL and NNNLL terms. As energy increases the leading logs show a clear dominance over the lower order terms.

These plots are only for our "toy" SU(2) model, so the details should not be trusted. The important information we can gain is that the subleading logarithms, especially the $\log^2 s/M^2$ contribution, are very important at large angles. From the suppression seen in the linear log term we may infer the constant contribution, N^4L or $\log^0$, will be similarly suppressed and our approach of looking only at logarithmic contributions is justified.

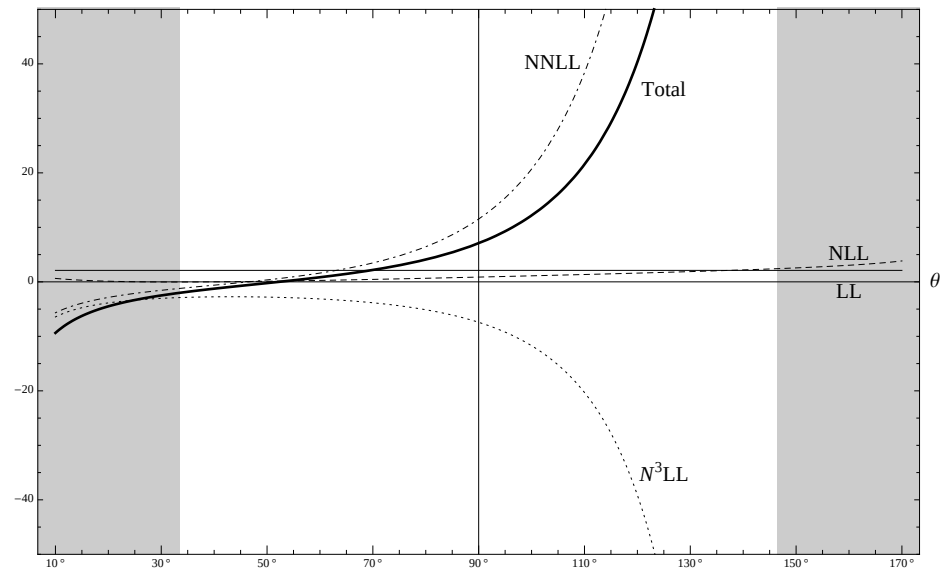


Figure 4: The percent correction to the Born cross section $\mathrm{d}\sigma_i^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the scattering angle θ (in degrees) from individual logarithmic two loop effects in SU(2) Bhabha Scattering for $\sqrt{s} = 1$ TeV.

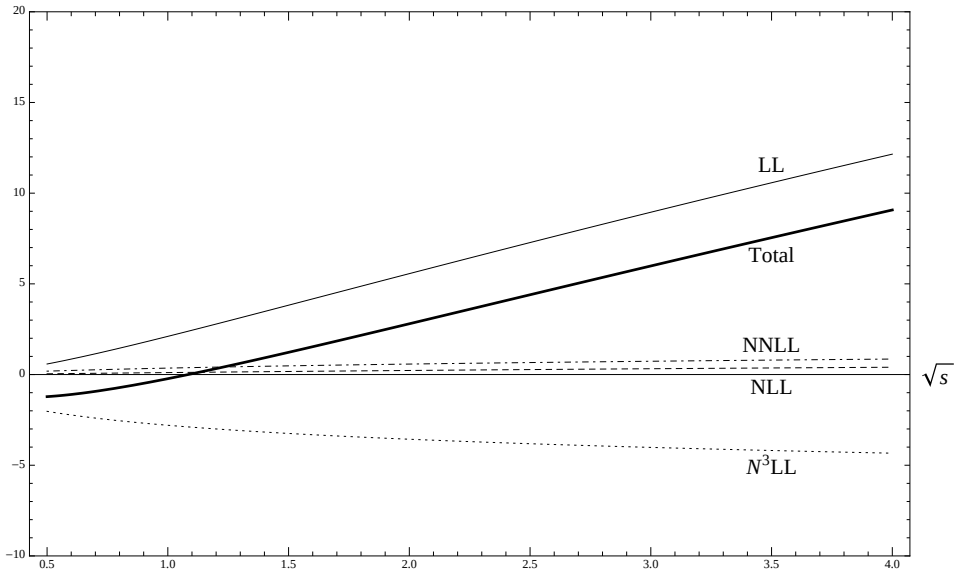


Figure 5: The percent correction to the Born cross section $\mathrm{d}\sigma_i^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the centre of mass energy $\sqrt{s}$ (in GeV) from individual logarithmic two loop effects in SU(2) Bhabha Scattering for $\theta = 50°$.

4 Bhabha Scattering in the Standard Model

The Standard Model of particle physics accurately describes the phenomenology of all elementary particles by postulating just three fundamental interactions; the strong force, the weak force, and electromagnetism [20]. Strong interactions are modelled by Quantum Chromodynamics (QCD), an SU(3) gauge theory which couples to colour charge. Weak and electromagnetic interactions have been unified in the Glashow-Weinberg-Salam (GWS) model [30,31]. This model contains the U(1) interaction of Quantum Electrodynamics (QED, the quantum theory of electromagnetism) as a subgroup; the remainder gives the the $W^{\pm}$ and Z^0 interactions associated with the weak force.

In the study of Bhabha scattering our primary interest is the interactions between electrons and positrons. Having no colour charge they are unable to interact directly via QCD and we may consider the Bhabha cross section to be completely electroweak in nature.[11] The most complicated part of the electroweak interaction is the contribution from the SU(2) gauge group we have calculated in the previous section. We now extend this result to the full electroweak correction by including coupling to the hypercharge U(1) gauge group. Since the photon is massless divergences resulting from photon exchange must be regulated in a different manner than those from the W and Z bosons. We subtract off the QED contribution to the cross section so it can be included later with its own regularization scheme. Finally we account for the mass splitting between the Z and W perturbatively, and present the electroweak Bhabha cross section numerically to next-to-next-to-leading logarithmic accuracy.

4.1 The Electroweak Theory

The fundamental principle of the GWS Electroweak theory is the gauge symmetry

$$SU(2)_L \times U(1)_Y \ . \tag{101}$$

Particles of left-handed chirality 'see' the SU(2) interaction and form doublets, whereas right handed particles are oblivious to this interaction and are SU(2) singlets. Each generator t^a of SU(2) in the fundamental representation gives rise to a gauge boson. There are three such generators:

$$t^a = \frac{\sigma^a}{2}, \quad a = 1, 2, 3. \tag{102}$$

Where σ^a are the Pauli sigma matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{103}$$

[11] This is not to say QCD does not contribute to the Bhabha cross section. At the two loop level, we must include effects such as gluon exchange by quarks in closed fermion loops. However, these processes only affect the running of α and have been accounted for in the calculation of $\alpha(M_Z^2)$.

The Pauli matrices show themselves in elementary quantum mechanics when considering the spin of particles. In this tradition, the quantum number of this SU(2) interaction is referred to as the weak isospin T. Identifying the t^a with spin operators, the quadratic Casimir operator in the fundamental representation C_F is simply the spin-squared operator:

$$C_F \equiv t^a t^a = T(T+1) \ . \tag{104}$$

Summation is implied over repeated indices. Left handed leptons form doublets in the t^a space:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \ . \tag{105}$$

The vertex for SU(2) coupling of strength g to fermions is of the following form:

$$ig\gamma^\mu t^a \ . \tag{106}$$

In Bhabha scattering always have electrons (or positrons) scattering into electrons (or positrons). At tree level, each fermion line has a single interaction vertex, which carries with it a t^a. Since electrons and positrons are vectors $(0 \ \ 1)$ in SU(2) space, t^1 and t^2 will give zero contribution to the amplitude as they are off diagonal. The only contribution at tree level will then be from t^3. Defining the third component of weak isospin T_f^3 for some fermion f in the obvious way:

$$t^3 f = T_f^3 f \ . \tag{107}$$

We can write the $f\bar{f} \to f'\bar{f}'$ amplitude in SU(2) as:

$$\mathcal{A} = \frac{ig^2}{s} T_f^3 T_{f'}^3 \ \bar{v}_f \gamma^\mu u_f \ \bar{u}_{f'} \gamma_\mu v_{f'} \ . \tag{108}$$

The U(1) gauge group generates as interaction with hypercharge Y having the exact same structure as QED.[12] The only difference between the two is the charge they couple to, $Y \neq Q$, and that the $U(1)_Y$ gauge boson is massive. Let us define the SU(2) gauge bosons as A^1, A^2, and A^3 and the $U(1)_Y$ boson as B. If the masses of these bosons are generated by the Higgs mechanism, as they are in the Standard Model, then there will be a unique combination of A^3 and B which will be exactly massless [20]:

$$A = \sin\theta_W A^3 + \cos\theta_W B \ . \tag{109}$$

The angle θ_W is the weak-mixing (or Weinberg) angle. This state is identified as the photon. It couples to with electric charge Q defined as:

$$Q = T^3 + \frac{Y}{2} \ . \tag{110}$$

[12] This is obvious, since QED is also a U(1) gauge theory.

Taking $Q = -1$ for electrons, this tells us $Y_e = -1, -2$ for left and right handed electrons respectively. The magnitude of this interaction will be given by the coupling constant e:

$$e = g \sin\theta_W \ . \tag{111}$$

We see that the complete QED is contained within the electroweak theory. The Z^0 boson is defined as the combination of A^3 and B orthogonal to the photon:

$$Z^0 = \cos\theta_A A^3 - \sin\theta_W B \ . \tag{112}$$

The $W^\pm$ are defined as appropriate linear combinations of A^1 and A^2. Their masses will be related by $M_W = \cos\theta_W M_Z$. For now we will work in the full electroweak theory, ie. the A^a, B basis, and will not break into the mass eigenstates $W^\pm$, Z^0, A. This has the advantage that the SU(2) and U(1) interactions decouple completely, and may be treated independently. This would be impossible if we were dealing with QED explicitly, as it is a combination of the couplings from both groups.

In this scheme the U(1) coupling is:

$$g' = g \tan\theta_W \ . \tag{113}$$

The $f\bar{f} \to f'\bar{f}'$ amplitude can then be written in U(1) as:

$$\mathcal{A}_{IJ} = \frac{ig^2}{s} \tan^2\theta_W \frac{Y_f Y_{f'}}{4} \ \bar{v}_f \gamma^\mu u_f \ \bar{u}_{f'} \gamma_\mu v_{f'} \ . \tag{114}$$

Where I, J is the chirality of f, f' respectively. We can now write the amplitude for the entire electroweak interaction. For brevity we adopt the following shorthand:

$$\begin{aligned} s_W \equiv \sin\theta_W \ , \qquad t_W \equiv \tan\theta_W \ , \\ A_{IJ} = \bar{v}_f \gamma^\mu u_f \ \bar{u}_{f'} \gamma_\mu v_{f'} \ . \end{aligned} \tag{115}$$

The last line is simply a restatement of (32) for the S channel. In this notation we may write the polarized $f\bar{f} \to f'\bar{f}'$ amplitude as:

$$\mathcal{A}_{IJ} = \frac{ig^2}{s} \left(T_f^3 T_{f'}^3 + t_W^2 \frac{Y_f Y_{f'}}{4} \right) A_{IJ} \ . \tag{116}$$

Where $T_f^3 = 0$ for right-handed fermions. We see here the exact substitution rule quoted in (27). This is enough information to calculate the required Born cross sections for Bhabha scattering. We can now move to loop corrections.

4.2 Loop Corrections in the Electroweak Theory

The study of loop corrections in the electroweak model is very similar to its study in the pure SU(2) theory. Working in the equal mass approximation $M \equiv M_Z = M_W$ with fields

of unbroken phase; that is, the exact fields A^a and B. In this scheme the only change in moving from SU(2) to electroweak interactions is the addition of the U(1) gauge interaction. This can be directly accounted for within the infrared evolution equation methods of the SU(2) calculation.

The infrared evolution equations quantify the dependence of a scattering amplitude on the Euclidean momentum transfer Q^2. This is accomplished by 'running' the amplitude from the known value at some initial scale[13] to the desired momentum Q^2. The advantage of using fields of unbroken phase is that the SU(2) A^a fields with coupling g up to three loops run completely independently of the U(1) field B with coupling g'. The running of each interaction then factorizes, and multiplies the total initial amplitude [15]. For the form factor $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}_{SU(2)}\mathcal{F}_{U(1)}\mathcal{F}_0 \ . \tag{117}$$

Where each factor obeys its own evolution equation with the trivial initial condition $\mathcal{F}_0 = 1$:

$$\begin{aligned}
\frac{\partial}{\partial \log(Q^2)}\mathcal{F}_{SU(2)} &= \left(\int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\gamma_{SU(2)}(\alpha(x)) + \zeta_{SU(2)}(\alpha(Q^2)) + \xi_{SU(2)}(\alpha(M^2))\right)\mathcal{F}_{SU(2)} \ , \\
\frac{\partial}{\partial \log(Q^2)}\mathcal{F}_{U(1)} &= \left(\int_{M^2}^{Q^2}\frac{\mathrm{d}x}{x}\gamma_{U(1)}(\alpha'(x)) + \zeta_{U(1)}(\alpha'(Q^2)) + \xi_{U(1)}(\alpha'(M^2))\right)\mathcal{F}_{U(1)} \ .
\end{aligned} \tag{118}$$

Where $\alpha' = g'^2/4\pi = t_W^2\alpha$. The running of α (or α') in each equation will also be different, as the beta function differs between U(1) and SU(2).

$$\begin{aligned}
\beta_{0SU(2)} &= -\frac{4}{3}N_g + \frac{43}{6} \ , \\
\beta_{0U(1)} &= \left(-\frac{20}{9}N_g - \frac{1}{6}\right)t_W^2 \ .
\end{aligned} \tag{119}$$

We do not include the U(1) β_1 here, as the full electroweak calculation may only be taken to $\mathcal{O}(\alpha^2\log^2(s/M^2))$. At this order only β_0 is required, as can be seen in (37). We have written the t_W^2 factor explicitly in the U(1) beta function so the expansion of the running of the coupling (37) is still in terms of α and not α'. $N_g = 3$ is still the number of generations.

The initial condition $\mathcal{F}_0$ is taken as the sum of the contributions from each interaction:

$$\mathcal{F}_0 = 1 + \frac{\alpha}{4\pi}\left(\mathcal{F}_{0\ SU(2)}^{(1)} + \mathcal{F}_{0\ U(1)}^{(1)}\right) \ . \tag{120}$$

[13]The initial scale of the infrared evolution equations in our analysis is always set to be M^2.

The anomalous dimensions γ, ζ, and ξ as well as the initial condition $\mathcal{F}_0(M^2)$ may be calculated in U(1) in the expansion-by-regions approach as with SU(2), but it is easier to simply use our SU(2) results directly. This is a simple procedure since we calculated $\mathcal{F}$ in SU(2) for scattering off of an external Abelian field; there is no untoward matrix structure to deal with in the translation to U(1). In fact, the only SU(2) parameter entering the final results for these quantities at one loop is the Casimir operator C_F. This quantity enters the amplitude through the two interaction vertices from the SU(N) boson exchange; each vertex carries a copy of t^a, and their product is summed over. We can compare the interaction vertices for U(1) and SU(2):

$$\text{U(1): } ig'\frac{Y}{2}\gamma^\mu \ , \qquad \text{SU(2): } ig\gamma^\mu t^a \ . \tag{121}$$

C_F in SU(2) is just the square of the coefficient of $ig\gamma^\mu$. We can then reproduce the U(1) form factor from our SU(2) result by making the substitution:

$$C_F \to t_W^2 \frac{Y^2}{4} \ . \tag{122}$$

At the current time $\xi^{(2)}$ is unknown at two loops in the full electroweak model. We then present the electroweak contribution to Bhabha scattering at two loops only to the $\log^2 s/M^2$ term, or in the next-to-next-to-leading logarithmic approximation. At this order we still need $\gamma^{(2)}$ in SU(2) and U(1), and it depends directly on the weak isospin T_f of the theory, the Casimir operator C_A, and the number of dirac fermions n_f. U(1) has no matrix structure; its only generator is unity. Therefore $T_f \to 1$ in U(1). The adjoint representation of a Lie Group is the representation formed by matrices of the structure constants f^{abc} which define the commutation relation for the group generators. Being an Abelian group, the structure constants for U(1) are all identically zero and hence the quadratic Casimir operator in the adjoint representation $C_A = 0$. The correct values for n_f including Higgs coupling are given in [15]. For U(1): $T_f n_f \to 5N_g/3$ and $T_f n_s \to 1/2$. The form factor in U(1) is then completely determined in our precision by:

$$\begin{aligned}
\gamma^{(1)} &= -2t_W^2 \frac{Y^2}{4} \ , \\
\gamma^{(2)} &= t_W^4 \frac{Y^2}{4}\left(\frac{200}{27}N_g + \frac{8}{9}\right) \ , \\
\zeta^{(1)} &= 3t_W^2 \frac{Y^2}{4} \ , \\
\xi^{(1)} &= 0 \ , \\
\mathcal{F}_0^{(1)} &= -t_W^2 \frac{Y^2}{4}\left(\frac{7}{2} + \frac{2\pi^2}{3}\right) \ .
\end{aligned} \tag{123}$$

$\gamma^{(2)}$ gains an extra power of t_W^2 because it is one order higher in the expansion around

$\alpha/4\pi$. It is important to remember both left and right chiral particles may interact through U(1), but they have different hypercharges Y_I and must be distinguished in the final amplitude.

The factorizing of the reduced amplitude $\tilde{\mathcal{A}}$ is made more complicated by the vector structure of the SU(2) contribution. We remind ourselves $\tilde{\mathcal{A}}$ is expressed in a flavour basis $\{\mathcal{A}^\lambda, \mathcal{A}^d\}$ where to restate (30):

$$\begin{aligned}\mathcal{A}^\lambda &= \bar{\psi}_2\gamma^\mu t^a \psi_1 \bar{\psi}_4 \gamma_\mu t^a \psi_3 \ , \\ \mathcal{A}^d &= \bar{\psi}_2\gamma^\mu \psi_1 \bar{\psi}_4 \gamma_\mu \psi_3 \ .\end{aligned} \tag{124}$$

Solving the evolution equation gave us (46):

$$\tilde{\mathcal{A}} = \mathrm{P}\exp\left\{\int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x}\chi\left(\alpha\left(x\right)\right)\right\}\tilde{\mathcal{A}}_0(\alpha(M^2)) \ . \tag{125}$$

Where the soft anomalous dimension χ is a matrix in SU(2). We can then write the reduced amplitude as:

$$\tilde{\mathcal{A}} = \exp\left\{\int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x} \underset{U(1)}{\chi}(\alpha\left(x\right))\right\}\mathrm{P}\exp\left\{\int_{M^2}^{Q^2} \frac{\mathrm{d}x}{x} \underset{SU(2)}{\chi}\left(\alpha\left(x\right)\right)\right\}\tilde{\mathcal{A}}_0 \ . \tag{126}$$

With the initial condition $\tilde{\mathcal{A}}_0$ given by:

$$\tilde{\mathcal{A}}_0 = \underset{U(1)}{\tilde{\mathcal{A}}_0}(\alpha(M^2)) + \underset{SU(2)'}{\tilde{\mathcal{A}}_0}(\alpha(M^2)) \ . \tag{127}$$

The U(1) interaction is blind to flavour, and so does not have the matrix structure of the SU(2) interaction. The U(1) soft anomalous dimension χ is a scalar, and may be determined from the $\chi_{\lambda d}$ component of the SU(2) χ matrix. At next-to-next-to-leading logarithms only the one loop value is needed. In the S channel where the chirality of the initial and final fermion lines are I and J respectively:

$$\chi^{(1)} = t_W^2 Y_I Y_J \log\left(\frac{1-x}{x}\right) \ . \tag{128}$$

The most drastic change in the switch from pure SU(2) to the electroweak model is in the initial condition $\tilde{\mathcal{A}}_0$. For mathematical consistency the U(1) contribution must be expressed as a vector in our flavour basis. The $\mathcal{A}^\lambda$ must be zero since the U(1) interaction does not contain any t^a matrices. We can then write the contribution as:

$$\underset{U(1)}{\tilde{\mathcal{A}}_{0\ IJ}} = t_W^2 \frac{Y_I Y_J}{4}\left(1 + \frac{\alpha}{4\pi}\underset{U(1)}{\tilde{\mathcal{A}}^{(1)}_{0\ IJ}}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{129}$$

The one loop value depends on chirality:

$$\underset{U(1)}{\tilde{\mathcal{A}}^{(1)}_{0\ IJ}} = t_W^2 \frac{Y_I Y_J}{4} \Big(\underbrace{\delta_{IJ} f(\frac{u}{s}, \frac{t}{s})}_{\text{Same chirality}} - \underbrace{(1-\delta_{IJ}) f(\frac{t}{s}, \frac{u}{s})}_{\text{Different chirality}} \Big) - t_W^2 \left(\frac{100}{27} N_g + \frac{4}{9} \right) \ . \tag{130}$$

Diagramatically, this term gives us the contribution from box diagrams with two B bosons. The SU(2) value for $\tilde{\mathcal{A}}_0^{(1)}$ already in our possession gives also the contribution from box diagrams with two A^a bosons. Contributions to $\mathcal{A}^d$ come when the two bosons in the loop are identical; the product of two identical t^a matrices is proportional to the identity matrix, when summed over all a we get a contribution proportional to C_F with no flavour structure. However, there is a third box diagram not accounted for by either term; the mixed box diagram with one each of the A^a and B bosons. Since this diagram includes the exchange of an A^a, both fermion lines must be left-handed for it to occur. Also, each fermion line will have a single copy of t^3, so this will contribute to the $\mathcal{A}^\lambda$ term. It is natural to include this diagram with the SU(2) contribution to $\tilde{\mathcal{A}}_0$. The magnitude of the term is $t_W^2 \frac{Y_I Y_J}{4} f(u/s, t/s)$, but this gets doubled because of the two permutations of the A^3 and B in the loop. We then define:

$$\underset{SU(2)'}{\tilde{\mathcal{A}}_0} = \underset{SU(2)}{\tilde{\mathcal{A}}_0} + \frac{\alpha}{4\pi} t_W^2 \frac{Y_I Y_J}{2} f\Big(\frac{u}{s}, \frac{t}{s}\Big) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{131}$$

This constitutes all the information we need to assemble the electroweak Bhabha amplitude in the S channel. Since each fermion line may be left or right handed we split the $\mathcal{F}^2$ from (38) into the contribution from each line separately. Since the SU(2) interaction only couples to left handed particles, we know the SU(2) contributions $\mathcal{F}_R = 1$ and $\tilde{\mathcal{A}}_{LR} = \tilde{\mathcal{A}}_{RL} = \tilde{\mathcal{A}}_{RR} = 1$. The final result will be:

$$\mathcal{A}^S_{IJ} = \frac{ig^2}{s} \mathcal{F}_I \mathcal{F}_J \tilde{\mathcal{A}}_{IJ} \ . \tag{132}$$

This was first calculated in [15]. To finish the calculation we follow the same procedure of employing crossing symmetry and analytic continuation to derive the amplitude for scattering via the T channel, then simply sum the amplitudes and take the modulus squared to determine the cross section. Due to the presence of two couplings in the electroweak model, g and gt_W, we can not perform the simple factorization as in the SU(2) calculation which allowed us to write the coupling in the Born cross section as $\alpha(s)$. Instead the full electroweak cross section will be expressed with $\alpha = \alpha(M^2)$ everywhere, even at the Born level. This will introduce new logarithmic corrections from the running of α, which are included simply by the straight application of (37). There is only one subtle problem left to address; how to differentiate the massless photon from the massive $W^\pm$ and Z^0 bosons.

4.3 QED Subtraction: Making Room for Bremsstrahlung

An important quality of the electroweak amplitude we have calculated is that it is finite. The divergences we ran into in calculating the anomalous dimension parameters only occurred because of the expansion by regions procedure; extending the limits of integration beyond the range where our approximation of the integrand was valid. For instance the 'hard' part of the form factor integral suffered from infrared divergences. The advantage of the expansion-by-regions method is these divergences cancel with those from other regions to give in the end a finite result.

However, we know physically the amplitude we have calculated should be infinite, because the photon is massless. At one loop in QED, the form factor resulting from photon exchange is infrared divergent. This divergence is traditionally regularized by assigning the photon a small mass λ [20]. This successfully makes the loop integral finite, but makes the scattering amplitude logarithmically dependent on λ. In the limit $m_e \to 0$ this dependence shows itself through terms proportional to $\log Q^2/\lambda^2$ and $\log^2 Q^2/\lambda^2$, with Q the Euclidean momentum transfer.

This exclusive scattering cross section will still be infinite as $\lambda \to 0$. The solution to this problem is to instead look at the inclusive cross section; that is the scattering cross section with real radiation. When the radiated particle is a photon, these processes are referred to as Bremsstralung radiation. Any physical detector of a finite size will only be sensitive to photons above a certain frequency ω_{res}. Photons below this frequency will not be detected by the experiment, and should be included in the total cross section. That is, a complete study of Bhabha scattering is really looking at the process $e^+e^- \to e^+e^- + n\gamma_{soft}$. When the cross section for Bremsstralung radiation is calculated, the result includes terms proportional to $\log \omega_{res}^2/\lambda^2$ and $\log^2 \omega_{res}^2\lambda^2$. The coefficients of these terms are identical to those of the logarithms of λ in the exclusive result, except they differ by a sign. In the total inclusive result then, the logarithmic dependence will be of the form $\log Q^2/\omega_{res}^2$, a fully physical quantity [32, 33].

This procedure is not necessary for the $W^\pm$ and Z^0 bosons, their mass acts as a built in infrared regulator, so their amplitudes are finite. We are now faced with the most drastic breakdown of the equal mass limit assumed to this point. We have been operating under the assumption all four electroweak bosons A^a and B are mass eigenstates with the same mass M. This has allowed us to express all Q^2 dependence in terms of a single mass scale, which is a combination of the dependence from each individual boson. However, for practical reasons we want to separate the contribution from the photon. For our result to be useful to experimental work, the dependence on the photon mass λ must be made explicit so the inclusive result with radiative corrections may be properly computed.

The procedure to acquire the full Bhabha cross section in a useful form then requires the separation of the QED contributions to the electroweak cross section. Since generally $\lambda \ll m_e$, calculations in QED can not universally set $m_e = 0$ as we have in the electroweak calculation. In fact, inclusion of the electron mass is necessary as it acts as a regulator of collinear divergences in QED. Thus when separating the QED contribution we must in principle also include terms to account for the inclusion of this new regulator. The procedure

for performing this operation is known as infrared matching, and is beyond the scope of the current work. For us, it will be enough to simply subtract the QED contribution to the amplitude so it may be properly added back in at a later date.

The process of the QED subtraction is simply performed. We introduce a factor $\mathcal{U}^{-1}$ into the scattering amplitude $\mathcal{A}$. The QED factor $\mathcal{U}$ is simply the amplitude for Bhabha scattering calculated via the infrared evolution equations in QED. Multiplying the electroweak amplitude by $\mathcal{U}^{-1}$ then simply 'devolves' the QED contribution to the amplitude from Q^2 back to M^2, eliminating the contributions of logarithms of the photon mass.

$\mathcal{U}$ is calculated in a straightforward manner from the evolution equations for the electromagnetic U(1) interaction with mass λ. To perform the subtraction we simply set $\lambda = M$. Since the goal of this procedure is only to eliminate the logarithmic dependence on λ, we have a freedom in the overall normalization of $\mathcal{U}$. For convenience we choose it so that $\mathcal{U}(Q^2 = \lambda^2) = 1$, thus causing $\mathcal{U}^{(1)} = 0$ and eliminating the need to perform the subtraction on $\mathcal{F}_0$ and $\tilde{\mathcal{A}}_0$. In terms of the photon mass λ and the electromagnetic coupling $\alpha_e = e^2/4\pi$ the QED factor $\mathcal{U}$ for the $f\bar{f} \to f'\bar{f}'$ process is given by [15]:

$$\begin{aligned}\mathcal{U}_{IJ} = \mathcal{U}_0(\alpha_e) \exp\Bigg\{ -\frac{\alpha_e(\lambda^2)}{4\pi} \Bigg[&\left(Q_f^2 + Q_{f'}^2\right) \log^2 \frac{Q^2}{\lambda^2} \\ &- \left(3\left(Q_f^2 + Q_{f'}^2\right) + 4Q_f Q_{f'} \log \frac{1-x}{x}) \right) \log \frac{Q^2}{\lambda^2} \\ + \frac{\alpha_e}{\pi} \Bigg(&\frac{8}{27}\left(Q_f^2 + Q_{f'}^2\right) N_g \log^3 \frac{Q^2}{\lambda^2} \\ &- \left(\frac{76}{27}\left(Q_f^2 + Q_{f'}^2\right) + \frac{16}{9} Q_f Q_{f'} \right) \log^2 \frac{Q^2}{\lambda^2} \Bigg) \Bigg] \Bigg\} \ . \end{aligned} \tag{133}$$

From this we can determine the anomalous dimensions for QED. Combinations such as $\left(Q_f^2 + Q_{f'}^2\right)$ come from the $\mathcal{F}_f \mathcal{F}_{f'}$ part of the amplitude, whereas mixed terms like $4Q_f Q_{f'} \log \frac{1-x}{x}$ originate in the reduced amplitude $\tilde{\mathcal{A}}$. In the case of Bhabha scattering, both fermions are electrons so $Q_f = Q_{f'} = -1$. For consistency we still expand functions in a power series in $\alpha = g^2/4\pi$. From the electroweak relation (111) we can make the relation:

$$\alpha_e = s_W^2 \alpha \ . \tag{134}$$

Simple matching with (49) then gives:

$$
\begin{aligned}
\gamma_e^{(1)} &= -s_W^2 Q_f^2 \ , \\
\gamma_e^{(2)} &= -\frac{320}{27} N_g s_W^4 Q_f^2 \ , \\
\zeta_e^{(1)} &= 3 s_W^2 Q_f^2 \ , \\
\xi_e^{(1)} &= 0 \ , \\
\chi_e^{(1)} &= -4 s_W^2 Q_f Q_{f'} \log \frac{1-x}{x} \ , \\
\beta_{0_e} &= -\frac{32}{9} N_g s_W^2 \ .
\end{aligned} \tag{135}
$$

At face value this process seems to ignore the difference between massive and massless particles. The propagator for a spin 1 particle of mass M and momentum q, the propagator used in the calculations of $\mathcal{F}$ and $\tilde{\mathcal{A}}$, has the form [21]

$$
\frac{-i(g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2})}{q^2 - M^2} \ , \tag{136}
$$

whereas the photon propagator is simply

$$
\frac{-i g_{\mu\nu}}{q^2} \ . \tag{137}
$$

Massive particles admit an additional degree of freedom (compared to the photon) because of their ability to be polarized both transversely (like the photon) and longitudinally. This would seem to indicate that the factor $\mathcal{U}$ calculated from the QED amplitude with a fictitious photon mass and not accounting for this degree of freedom, does not fully subtract the QED portion of the Electroweak amplitude. However, all terms in the electroweak amplitude corresponding to longitudinal degrees of freedom are suppressed by a factor of M^2/s, and are neglected in this calculation. At the current order of accuracy, only transverse degrees of freedom contribute to the amplitude and the QED subtraction is valid. The only other discrepancy, the extra term $q_\mu q_\nu / M^2$ in the numerator of (136), does not contribute at all due to gauge invariance [34]. Our treatment of the QED portion of the electroweak amplitude then is justified, and we may write our final amplitude for Bhabha scattering in the S channel:

$$
\mathcal{A}_{IJ} = \frac{i g^2}{s} \mathcal{F}_I \mathcal{F}_J \tilde{\mathcal{A}}_{IJ} \, \mathcal{U}_{IJ}^{-1} \ . \tag{138}
$$

4.4 The Effect of Mass Splitting

The last step before presenting the final electroweak cross section is to take into account the difference between M_W and M_Z. The corrections to the amplitude are introduced perturbatively in the small parameter δ_M.

$$\delta_M \equiv 1 - \frac{M_W^2}{M_Z^2} \approx 0.2 \ . \tag{139}$$

We only keep corrections linear in δ_M. From the SU(2) results we see that the two loop corrections enter the cross section at $\sim 1\%$. Including these mass splitting effects to first order will increase the precision of the results to about 1 permille. The second order corrections will enter another order of magnitude beyond this, and are negligible at this precision.

We define the mass appearing the in the Sudakov logarithms to be M_W. Thus the mass splitting is entirely due to effects from the Z^0 [17]. These effects can not effect either γ or ζ, since these were defined to contain only contributions from the hard region, which are independent of the mass details. It turns out the only parameters affected are the SU(2) ξ which acquires a non zero value, and the initial conditions $\mathcal{F}_0$ and $\tilde{\mathcal{A}}_0$. The modifications were presented in [17]. They are:

$$\begin{aligned}
\left.\underset{SU(2)}{\xi_I^{(1)}}\right|_{\delta_M} &= \frac{2}{3} T_I(T_I+1)\delta_M \ , \\
\left.\mathcal{F}_{0\ I}^{(1)}\right|_{\delta_M} &= -T_I(T_I+1)\delta_M \ , \\
\left.\tilde{\mathcal{A}}_{0\ IJ}^{(1)}\right|_{\delta_M} &= -4\log\frac{1-x}{x}\left(T_I^3 T_J^3 + t_W^2 \frac{Y_I Y_J}{4}\right)\delta_M \mathcal{A}^\lambda \ .
\end{aligned} \tag{140}$$

With these we finally have all the ingredients required to calculate the electroweak Bhabha scattering cross section.

4.5 Electroweak Cross Section - Numerical Results

With the electroweak amplitude for Bhabha scattering in hand, determining the cross section simply requires determining the T channel amplitude then calculating the complex modulus squared of the sum $\mathcal{A}^S + \mathcal{A}^T$ and plugging the result into the equation (9). We also have different chiral amplitudes which must be summed over, so our cross section will look like:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s} \sum_{I,J\in\{L,R\}} |\mathcal{A}_{IJ}^S + \mathcal{A}_{IJ}^T|^2 \ . \tag{141}$$

This additional chiral structure plus the additional interaction makes the electroweak cross section much longer than the SU(2) cross section, just in shear page count. Therefore we will present the results numerically, as nothing useful can be gained by staring at the analytic expression which would likely span several pages. We define a contribution to the cross section $\mathrm{d}\sigma_{IJ}$ in the following way:

$$
\begin{aligned}
|\mathcal{A}^S_{IJ}|^2 &= g^4 \mathrm{d}\sigma^S_{IJ} \ , \\
|\mathcal{A}^T_{IJ}|^2 &= g^4 \mathrm{d}\sigma^T_{IJ} \ , \\
\mathrm{Re}\left((\mathcal{A}^S_{IJ})^* \mathcal{A}^T_{IJ}\right) &= g^4 \sigma^{ST}_{IJ} \ .
\end{aligned} \tag{142}
$$

Thus we may write the cross section as:

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s} \sum_{I,J \in \{L,R\}} \left(\mathrm{d}\sigma^S_{IJ} + \mathrm{d}\sigma^T_{IJ} - 2\ \mathrm{d}\sigma^{ST}_{IJ} \right) \ . \tag{143}
$$

Note the renormalization scale of the Born cross section is M^2, not s as in the SU(2) calculation (93). This introduces logarithmic corrections from (37) which are included in the numerical results. Remembering the magnitude of the electroweak Born amplitude is $T^3_I T^3_J + t_W Y_I Y_J/4$ and our result (17) we can immediately determine the Born coefficients:

$$
\begin{aligned}
\mathrm{d}\sigma^{S\ (0)}_{LL} &= \left(\tfrac{1}{4} + \tfrac{t_W^2}{4}\right)^2 (1-x)^2, & \mathrm{d}\sigma^{S\ (0)}_{RR} &= t_W^4 (1-x)^2, & \mathrm{d}\sigma^{S\ (0)}_{LR} &= \mathrm{d}\sigma^{S\ (0)}_{RL} = \tfrac{t_W^4}{4} x^2, \\
\mathrm{d}\sigma^{T\ (0)}_{LL} &= \left(\tfrac{1}{4} + \tfrac{t_W^2}{4}\right)^2 \tfrac{(1-x)^2}{x^2}, & \mathrm{d}\sigma^{T\ (0)}_{RR} &= t_W^4 \tfrac{(1-x)^2}{x^2}, & \mathrm{d}\sigma^{T\ (0)}_{LR} &= \mathrm{d}\sigma^{T\ (0)}_{RL} = \tfrac{t_W^4}{4} \tfrac{1}{x^2}, \\
\mathrm{d}\sigma^{ST\ (0)}_{LL} &= -\left(\tfrac{1}{4} + \tfrac{t_W^2}{4}\right)^2 \tfrac{(1-x)^2}{x}, & \mathrm{d}\sigma^{ST\ (0)}_{RR} &= -t_W^4 \tfrac{(1-x)^2}{x}, & \mathrm{d}\sigma^{ST}_{LR} &= \mathrm{d}\sigma^{ST}_{RL} = 0.
\end{aligned} \tag{144}
$$

Where we have used $T^3_L = -1/2$, $T^3_R = 0$, $Y_L = -1$, and $Y_R = -2$ for electrons. The electroweak Born cross section is then:

$$
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Born}} = \frac{\alpha^2}{64\ s\ x^2} \Big((1-x)^4 (1+t_W^2)^2 + 8 t_W^4 (3 - 8x + 12x^2 - 8x^3 + 3x^4) \Big) \ . \tag{145}
$$

We now present the loop corrections to the electroweak Bhabha scattering cross section up to $\mathcal{O}(\alpha^2 \log^2 s/M^2)$. The result is presented summed over chiralities and channels. That is we present the result for $\mathrm{d}\sigma^{(n)}_i$ defined as:

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s} \Bigg[& \mathrm{d}\sigma^{(0)} + \frac{\alpha}{4\pi} \left(\mathrm{d}\sigma^{(1)}_2 \log^2\left(\frac{s}{M_W^2}\right) + \mathrm{d}\sigma^{(1)}_1 \log\left(\frac{s}{M_W^2}\right) + \mathrm{d}\sigma^{(1)}_0 \right) \\
& + \left(\frac{\alpha}{4\pi}\right)^2 \left(\mathrm{d}\sigma^{(2)}_4 \log^4\left(\frac{s}{M_W^2}\right) + \mathrm{d}\sigma^{(2)}_3 \log^3\left(\frac{s}{M_W^2}\right) + \mathrm{d}\sigma^{(2)}_2 \log^2\left(\frac{s}{M_W^2}\right) \right) \Bigg] \ .
\end{aligned} \tag{146}
$$

We present numerical results for the $\mathrm{d}\sigma^{(n)}_i$ as functions of $x = (1 - \cos\theta)/2$. We use (111) to express our weak coupling constant $\alpha = g^2/4\pi$ in terms of the electromagnetic coupling $\alpha_e = e^2/4\pi$.

$$\alpha(M_Z) = \frac{\alpha_e(M_Z)}{\sin^2\theta_W} \ . \tag{147}$$

The tangent of the weak mixing angle is determined through elementary trigonometric relations:

$$\tan^2\theta_W = \frac{\sin\theta_W}{1-\sin^2\theta_W} \ . \tag{148}$$

We are then left with only four parameters to input: $\alpha_e(M_Z)$, $\sin^2\theta_W$, M_W, and M_Z. Numerical constants are taken from the Particle Data Group [29]. Since both α_e and θ_W run with interaction energy, the particular choice of renormalization scheme is important. We use the Minimal Subtraction or $\overline{MS}$ scheme [35].

$$\begin{aligned} \alpha_e(M_Z) &= \frac{1}{127.916 \pm 0.015} \ , \\ \sin^2\theta_W(M_z)_{\overline{MS}} &= 0.23116(13) \ , \\ M_Z &= 91.1876(21) \text{ GeV} \ , \\ M_W &= 80.399 \text{ GeV} \ . \end{aligned} \tag{149}$$

With these values our perturbative parameter $\alpha/4\pi$ takes the value:

$$\frac{\alpha}{4\pi} \approx 0.002691 \ . \tag{150}$$

Without further ado we present the loop corrections to electroweak Bhabha scattering. For brevity we round all numbers in the results to two decimal places. The exact one loop electroweak corrections have been known for some time [12]. The one loop results presented here are in the high energy limit with mass splitting only treated pertubatively and are presented for only comparative purposes. The one loop leading logarithm reads:

$$\mathrm{d}\sigma_2^{(1)} = -\frac{0.34 - 1.11x + 1.66x^2 - 1.11x^3 + 0.34x^4}{x^2} \, . \tag{151}$$

The one loop next-to-leading logarithm coefficient is:

$$\begin{aligned} \mathrm{d}\sigma_1^{(1)} = &\frac{1.21 - 3.33x + 5.00x^2 - 3.33x^3 + 1.21x^4}{x^2} \\ &+ \frac{0.10 - 0.52x + 0.78x^2 - 0.52x^3 + 0.10x^4}{x^2} \log(1-x) \\ &- \frac{0.02 - 0.49x + 2.05x^2 - 2.24x^3 + 0.75x^4}{x^2} \log(x) \, . \end{aligned} \tag{152}$$

The one loop constant:

$$
\begin{aligned}
\mathrm{d}\sigma_0^{(1)} =& \frac{0.49 - 1.77x - 2.62x^2 + 5.35x^3 - 2.61x^4}{x^2} + 0.03\frac{1 - x + x^2}{x}\log^2(1-x) \\
&- \frac{0.05 - 0.16x + 0.28x^2 - 0.16x^3 + 0.05x^4}{x^2}\log(1-x) \\
&+ \frac{0.31 - 0.24x - 0.61x^2 + 0.48x^3}{x^2}\log^2(x) \\
&+ \frac{1.21 - 2.52x + 2.64x^2 - 0.97x^3 + 0.05x^4}{x^2}\log(x) \\
&+ \frac{0.10 - 0.44x + 0.42x^2 - 0.13x^3}{x^2}\log(1-x)\log(x)\,. \qquad (153)
\end{aligned}
$$

Now for the two loop results. The leading logarithmic factor is:

$$
\mathrm{d}\sigma_4^{(2)} = \frac{0.34 - 1.21x + 1.81x^2 - 1.21x^3 + 0.34x^4}{x^2}\,. \qquad (154)
$$

The next-to-leading logarithm is given by:

$$
\begin{aligned}
\mathrm{d}\sigma_3^{(2)} =& -\frac{1.43 - 3.51x + 4.82x^2 - 2.92x^3 + 1.07x^4}{x^2} \\
&- \frac{0.16 - 0.81x + 1.22x^2 - 0.81x^3 + 0.16x^4}{x^2}\log(1-x) \\
&- \frac{0.18 + 0.74x - 4.23x^2 + 4.91x^3 - 1.71x^4}{x^2}\log(x)\,, \qquad (155)
\end{aligned}
$$

The next-to-next-to-leading logarithm is given by:

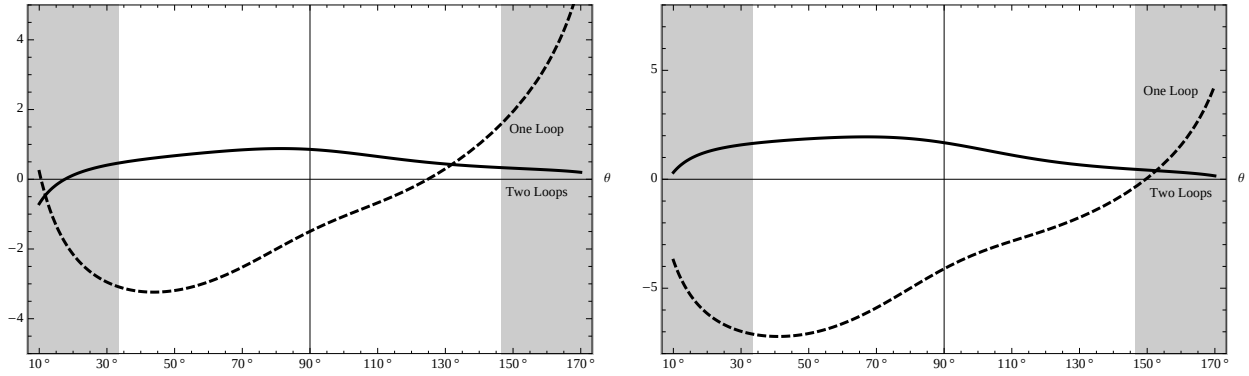
$$
\begin{aligned}
\mathrm{d}\sigma_2^{(2)} =& \frac{5.79 + 6.38x - 19.98x^2 + 8.16x^3 + 3.99x^4}{x^2} \\
&+ \frac{0.05 - 0.21x + 0.30x^2 - 0.21x^3 + 0.05x^4}{x^2}\log^2(1-x) \\
&+ \frac{0.97 - 3.28x + 3.63x^2 - 1.49x^3 - 0.13x^4}{x^2}\log(1-x) \\
&- \frac{0.28 + 0.29x - 4.06x^2 + 5.22x^3 - 1.97x^4}{x^2}\log^2(x) \\
&- \frac{3.75 - 7.57x + 9.05x^2 - 4.45x^3 + 0.43x^4}{x^2}\log(x) \\
&- \frac{0.88 - 2.76x + 2.13x^2 - 0.09x^3 - 0.28x^4}{x^2}\log(1-x)\log(x)\,. \qquad (156)
\end{aligned}
$$

This is the extent of our knowledge of the full electroweak cross section. Unfortunately, it is not enough to get the uncertainty to 1 permille, as even the NNLL contribution can contribute as much as one percent to the cross section. To achieve the accuracy we require we include the pure SU(2) contribution at the linear log (NNNLL) level. This gives us a first order approximation of the full electroweak contribution, which is sufficient for our purposes. As can be seen from Fig. 4 the SU(2) linear log term already enters at less than one percent, or a few permille. Corrections to this due to the U(1) interaction, QED subtraction, or mass splitting will enter suppressed by either a factor of t_W^2 or $s_W^2 = \delta_M$, both of which have the same small magnitude $t_W^2 \approx s_W^2 \approx 0.2$. Thus these corrections to the linear logarithm will give it an uncertainty of perhaps twenty percent. However, since the two loop linear logarithm already contributes only at the permille level, these corrections will be less than a permille and may be ignored at our precision. We have the exact expression for this term, but we reproduce it here numerically for completeness:

$$\begin{aligned}
\mathrm{d}\sigma_1^{(2)} = & -\frac{0.30 - 0.87x - 0.41x^2 + 2.21x^3 - 1.24x^4}{x^2} \\
& + \frac{4.25 - 18.23x + 36.60x^2 - 35.50x^3 + 12.89x^4}{x^2} \log(1-x) \\
& + \frac{0.63 - 0.38x + 1.88x^2 - 2.13x^3}{x^2} \log^3(x) \\
& + \frac{0.18 - 0.38x - 1.75x^2 + 0.03x^3}{x^2} \log^2(x) \\
& + \frac{18.97 - 10.32x + 6.39x^2 - 9.83x^3 - 5.21x^4}{x^2} \log(x) \\
& + \frac{0.25 - 0.75x + 0.75x^2 - 0.25x^3}{x^2} \log^2(1-x) \log(x) \\
& - \frac{0.63 - 3.56x + 5.25x^2 - 2.31x^3}{x^2} \log(1-x) \log^2(x) \\
& + \frac{0.27 - 1.60x + 2.40x^2 - 1.06x^3}{x^2} \log(1-x) \log(x) \,. \qquad (157)
\end{aligned}$$

We plot the the loop corrections relative to the Born cross section in Fig. 6. As before the shaded regions denote the kinematic domain where either t or u are less than $10M_Z^2$ and the Sudakov limit used in this calculation begins to falter. The one loop corrections are almost universally negative, about an order of magnitude larger than the two loop corrections. The two loop corrections themselves (up to $\mathcal{O}(\alpha^2 \log^2 s/M^2)$) contribute at the one percent level at 1 TeV and about twice that at 2 TeV. The breakdown of the two loop corrections into the individual logarithmic contributions is given in Fig. 7. As in SU(2), it can be seen the next-to-next-to-leading logarithms are essential. A cancellation between the leading and next-to-leading logarithms causes the bulk of the two loop corrections to be due to the $\log^2 s/M^2$ contribution. The contribution from our linear logarithmic term is highly suppressed in the

large angle area, justifying our approximation. Overall the two loop corrections enter at the 1 percent level everywhere in the angular distribution, with the largest corrections coming from the $\theta \sim 90^\circ$ region.



(a) $\sqrt{s} = 1$ TeV (b) $\sqrt{s} = 2$ TeV

Figure 6: The percent correction to the Born cross section $\mathrm{d}\sigma^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the scattering angle θ (in degrees) from one loop and two loop effects in Electroweak Bhabha Scattering for $\sqrt{s} = 1$ and 2 TeV.

In Fig. 8 we plot the running of the Bhabha scattering corrections $\mathrm{d}\sigma_i^{(1,2)}/\mathrm{d}\sigma^{(0)}$ with energy scale at $\theta = 50^\circ$. Again it can be seen that although as the energy increases the leading log contribution becomes dominant, at smaller energies in the TeV range it is the subleading next-to-next-to-leading logarithms which govern the cross section. Below 1 TeV the two loop corrections contribute far less than one percent to the Bhabha cross section.

We directly evaluate the corrections at $\sqrt{s} = 1$ TeV and $\theta = 50^\circ$ to quantitatively estimate the uncertainty.

$$\text{One loop: } -3.19\% \begin{cases} \text{LL:} & -9.45\% \\ \text{NLL:} & 7.36\% \\ \text{N}^2\text{LL:} & -1.10\% \end{cases} \qquad \text{Two loops: } 0.68\% \begin{cases} \text{LL:} & 0.62\% \\ \text{NLL:} & -0.41\% \\ \text{N}^2\text{LL:} & 1.37\% \\ \text{N}^3\text{LL:} & -0.90\% \end{cases} . \tag{158}$$

An uncertainty in the two loop linear logarithm of approximately twenty percent will only create an uncertainty on the order of 0.2% in the cross section. We have achieved the desired accuracy in the Bhabha cross section.

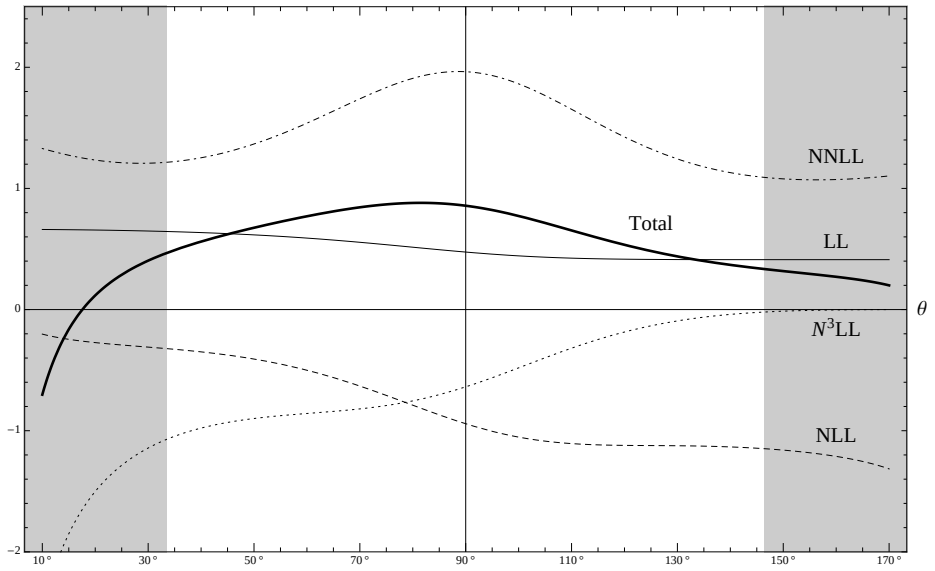


Figure 7: The percent correction to the Born cross section $\mathrm{d}\sigma_i^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the scattering angle θ (in degrees) from individual logarithmic two loop effects in Electroweak Bhabha Scattering for $\sqrt{s} = 1$ TeV.

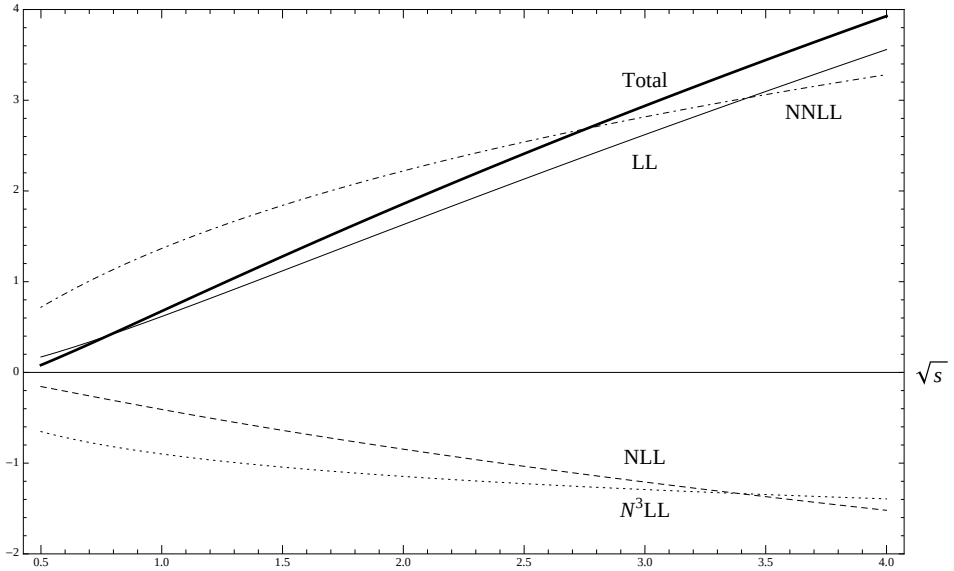


Figure 8: The percent correction to the Born cross section $\mathrm{d}\sigma_i^{(1,2)}/\mathrm{d}\sigma^{(0)}$ versus the centre of mass energy $\sqrt{s}$ (in GeV) from individual logarithmic two loop effects in Electroweak Bhabha Scattering for $\theta = 50°$.

5 Summary

We have successfully calculated the electroweak Bhabha scattering cross section up to next-to-next-to-next-to-leading logarithms at two loops. The logarithmic dependence of the scattering amplitude was determined through the use of infrared evolution equations, which reduces the problem to calculating the divergences of simple diagrams in the expansion-by-regions approach. In order to calculate the linear logarithm we neglected the electroweak mixing between $SU(2)_L$ and $U(1)_Y$ and took it to be of a pure SU(2) nature. This was sufficient to achieve a total accuracy of approximately one per mille.

This work constitutes the last theoretical piece required to attain the accuracy required to implement Bhabha scattering as a luminosity measure at the ILC. Signals of new physics tend to appear as small effects over the large Standard Model background, an experiment's ability to discover new phenomena rests heavily on its ability to make observations as precise as possible. Uncertainty in the luminosity effects every absolute measurement a particle detector makes. The inclusion of the effects calculated here will lower this uncertainty at the ILC below that of the experiments themselves, giving them the best possible chance to discover new physics and expand our knowledge of the universe.

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