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by

Shirley Elizabeth Mills

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ABSTRACT

Much work has been done on outliers in normal populations and recently in exponential populations. Here we extend the study to the examination of outliers in three competing life-testing distributions. Assuming the exchangeable model of random variables, we examine the concepts of outlier-prone and outlier-resistant families as they apply to the Gamma, Lognormal, and Weibull families of density functions. We examine also the detection of outliers for these distributions and the estimation of parameters in the presence of one or more spurious observations.

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LIST OF SYMBOLS

NOTATION	DESCRIPTION	PAGE
n	sample size	6
F	distribution function	6
G	distribution function	6
m	number of outliers	6
τ	location parameter	6
θ	scale parameter	6
S_n	sample of size n	6
$f(x;\underline{\theta})$	probability density function of target population P	6
$f(x;\underline{\xi})$	probability density function of spurious population Q	7
$L(\underline{x};\underline{\theta},\underline{\xi})$	likelihood function for the exchangeable model	7
$N(\tau,\theta^2)$	normal distribution with mean τ and variance θ^2	7
$EXP(\theta)$	exponential distribution with mean θ	7
k^*	coefficient of spuriosity	
I	set of spurious observations	8
$\mathcal{I}$	set of all $\binom{n}{m}$ possible sets of spurious observations	8
$P(k,n\|F)$	$P\{x_{(n)}-x_{(n-1)}>k(x_{(n-1)}-x_{(1)})\|x_i \sim F\}$	9
$\Pi_1(k,n\|\mathcal{F})$	$\sup_{F\in\mathcal{F}} P(k,n\|F)$	10
$C(\xi,\theta)$	Cauchy distribution centered at ξ, scale θ	12

$u(r;n,k^*)$	$P\{X_{(r)}$ is spurious in a sample of size $n\}$	14
$\Psi(x)$	$dG(x)/dF(x)$	14
$T_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)} + mx_{(n-m)}}{n-m+1}$	16
$\hat{\theta}_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$	17
T_t	$\dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n}$	17
$D_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} - x_{(n-1)} < k(x_{(n-1)} - x_{(1)}) \\ T_t & \text{otherwise} \end{cases}$	17
$T_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} < C\bar{x}\ ,\ C > 0 \\ T_t & \text{otherwise} \end{cases}$	17
$\hat{\theta}_i$	$\sum_{j=1, j\neq i}^{n} x_j/n$	18
$\hat{\theta}_{CK}$	$\sum_{i=1}^{n} \omega_i \hat{\theta}_i,\ \omega_i = \dfrac{2r_i}{n(n+1)},\ r_i$ = rank of x_i	18
$GAM(\theta,\eta,\tau)$	$f(x,\theta,\eta,\tau) = \dfrac{(x-\tau)^{\eta-1} e^{-(x-\tau)/\theta}}{\theta^{\eta}\Gamma(\eta)}$, $x > \tau$, $\theta,\ \eta > 0,\ -\infty < \tau < \infty$	22
η	shape parameter	22
χ^2_{ν} df	Chi-square density with ν degrees of freedom	24
$h(x)$	HF; hazard function $= \dfrac{f(x)}{1-F(x)}$	24
E	event that $x_{(n)}$ is a (k,n)-outlier	26
$P(E\|\eta,k^*)$	$P(k,n\|L)$	26
$I_r(\eta,k^*)$	$\int_E (n-1)! \prod_{i\neq r} f(x_{(i)};\eta)(f(x_{(r)};k^*\eta)dx_{(1)}\cdots dx_{(r)}$	27
$u(r;n,\eta,k^*)$	$P(X_{(r)}$ is the spurious observation)	27

$I'_n(\eta,k^*)$		28
$\vartheta(\nu,t)$		28
$\hat{\theta}_{MLE}$	$\bar{x}/\eta$ maximum likelihood estimator of scale	38
$\tilde{x}$	$(\prod_{i=1}^{n} x_i)^{1/n}$	38
$\psi(z)$	Euler's psi function $\frac{d}{dz} \ln \Gamma(z)$	38
γ	Euler's constant .5772157	38
$\hat{\eta}_{MLE}$	Maximum likelihood estimator of shape	38
$\hat{\eta}_M$	Moment estimator of shape	39
$\hat{\theta}_M$	Moment estimator of scale	39
$\hat{\hat{\eta}}$	Lilliefors' estimator of shape η	39
$\hat{\hat{\theta}}$	Lilliefors' estimator of scale θ	39
η^*	Thom's estimator of shape η	40
θ^*	Thom's estimator of scale θ	40
M	$\ln(\bar{x}/\tilde{x}) = -\ln S_1$	40
$L(\underline{x};\theta,\eta,k^*,I)$		42
$\Lambda(\mu,\sigma)$	lognormal density $f(x;\mu,\sigma) = \frac{1}{\sigma x\sqrt{2x}} \exp\{-\frac{1}{2}\frac{(\ln x-\mu)^2}{\sigma^2}\}$	53
$L(\underline{x};\sigma,k^*)$		59
$\phi(z)$	standard normal p.d.f.	71
$\Phi(z)$	standard normal distribution function at z	71
$u(r;n,\sigma,k^*)$	$P\{X_{(r)}$ is the spurious observation$\}$	75
$\hat{\mu}$	M.L.E., B.L.I.E. of μ, ; $\hat{\mu} = \bar{w} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$	78
$\hat{\sigma}^2$	M.L.E. of σ^2	78
$\tilde{\sigma}^2$	B.L.I.E. of σ^2	78

$L(\underline{x};\mu,\mu_1,\sigma,I)$		81
$\hat{\mu}_{het}$	M.L.E. of μ } under exchangeable model	82
$\hat{\mu}_{1_{het}}$	M.L.E. of μ_1 } under exchangeable model	82
$\hat{\sigma}^2_{het}$	M.L.E. of σ^2 } under exchangeable model	82
$WEI(\theta,\eta,\tau)$	Weibull distribution $f(x;\theta,\eta,\tau) = \frac{\eta(x-\tau)^{\eta-1}}{\theta^\eta} \exp\{-(\frac{x-\tau}{\theta})^\eta\}$	86
$EV_I(\xi,b)$	Extreme-Value distribution $F(y) = 1 - \exp\{-\frac{\exp(y-\xi)}{b}\}$	88
$Q(k,n\mid\eta)$		91
$L(\underline{x};\eta,k^*)$		92
$u(r;n,k^*)$		99
$L(u,p)$	Laplace transform of e^{-u}, Re $u > 0$	100
$\hat{\eta}$	M.L.E. of η	109
$\hat{\theta}$	M.L.E. of θ	109
$\tilde{b}_{BLIE}$	Mann and Fertig's (1973) BLIE of b	109
$\hat{b}_k$	Mann and Fertig's adaptation of Hassanein's estimator of b	110
b^*	Unbiased form of $\hat{b}_k$	110
b^*_{BLIE}	BLIE based on b^*	110
r	size of censored sample	110
$\hat{b}_B$	Bain's estimator of b	110
$k_{r,n}$		110
$\hat{\eta}_B$	Bain's estimator of η	111
$\hat{b}_s$	Englehardt and Bain's (1973) estimator of b	111
$\tilde{b}_{MF}$	Mann and Fertig's (1975) BLIE adaptation of $\hat{b}_s$	111
$\ell_{k,n}$		112

$\hat{b}_{EB}$	Englehardt and Bain's (1977) estimator of b	112
k_n		112
b^*_{EB}	BLIE based on $\hat{b}_{EB}$	113
$\hat{b}_{MN}$	Menon's estimator of b	113
$\hat{\xi}_{MN}$	Menon's estimator of ξ	113
$\hat{\eta}_{MN}$	Menon's estimator of η	113
$\hat{\theta}_{MN}$	Menon's estimator of θ	113
$\hat{\eta}_D$	Dubey's estimator of η	113
$\hat{b}_{MS}$	Murthy-Swartz estimator of b	114
$\hat{b}_{RA}$		117
$\hat{b}_{RW}$		117
$\hat{b}_{RS}$		117
$\hat{\xi}_{RA}$		117
$\hat{\xi}_{RW}$		117
$\hat{\xi}_{RS}$		117
$\hat{\mu}_A$	A-rule estimator for mean of normal distribution	129
$\hat{\sigma}^2_A$	A-rule estimator for variance of normal distribution	129
$\hat{\mu}_W$	W-rule estimator for mean of normal distribution	130
$\hat{\mu}_S$	S-rule estimator for mean of normal distribution	130
$\hat{\sigma}^2_W$	W-rule estimator for variance of normal distribution	130
$\hat{\sigma}^2_S$	S-rule estimator for variance of normal distribution	130
$\zeta(x)$	Reimann's zeta function	185

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CHAPTER I

OUTLIERS AND OUTLIER-PRONENESS

What is an outlier and what is the outlier problem? To quote Ferguson (1961):

> In a sample of moderate size taken from a certain population it appears that one or two values are surprisingly far way from the main group. The experimenter is tempted to throw away the apparently erroneous values, and not because he is certain that the values are spurious. ... It is rather because he feels that other explanations are more plausible, and that the loss in the accuracy of the experiment caused by throwing away a couple of good values is small compared to the loss caused by keeping even one bad value.

Barnett and Lewis (1978) indicate that in the light of developments in outlier methodology over the last 15 years, Ferguson's formulation may be unduly restrictive. Outlying values need not be "bad" or "erroneous"; in fact they may be welcomed as indicating a useful treatment or a strain that was unexpectedly good.

1.1 What is an outlier?

What do we mean by an "outlier"? We shall use Barnett and Lewis' definition (1978): "an outlier in a set of data (is) ... an observation (or subset of observations) which appears to be inconsistent with the remainder of that set of data." We use the term "outlier" to characterize an observation that stands out in contrast to the rest of the data and is not consistent with what our mind feels constitutes a reasonable data set, nor with our initial view of an appropriate probability model to describe the generation of our data. An observation is "spurious" if it is statistically unreasonable on the basis of some prescribed probability model. This would include an observation known to be generated by a different probability model; however a spurious observation will not necessarily show up as an outlier.

1.2 The outlier problem

Experimental scientists and others who deal with data are forced to make decisions about outliers - whether or not to include them in analysis, whether to make allowances for them on some compromise basis, etc. What worries an experimenter is whether or not some observations are genuine members of the main population. If these observations appear in the midst of the data they may not be conspicuous and are unlikely to distort inferences seriously. However, should the outlier appear extreme, it could create problems in attempts to represent the population and it could contaminate estimates and tests of population parameters.

Once we decide that outliers exist in a data set, we must decide on how to react to them. Methods to support outright rejection or to adjust values, prior to processing the principal mass of data will be related to any postulated model for that population. We are concerned with whether the extreme values are so extreme as to be unreasonable under our original model. This may indicate the data contains a "mistake" (that should be rejected or corrected) or it may indicate an alternative model for which the complete data set appears as a homogeneous sample. The outlier may be a 'foreign' random influence in an otherwise homogeneous data set - interesting to study in itself or only serving to obstruct analysis of the main data mass.

An assessment of the discordancy of some outliers is just the first stage of the study of outliers. Once an outlier is judged discordant we may:

1) decide to reject (or correct) it and analyse the remaining (modified) data on the original model,

2) decide to modify the model to incorporate the outliers in a non-discordant manner,

3) refine the analysis of the entire data set to accommodate outliers (i.e. render the analysis relatively impervious to their presence),

4) focus attention on the discordant outliers because they identify factors of unexpected value.

As examples of each of these situations, consider the following:

Fifteen residuals (about a simple model) of observations of the vertical semi-diameter of Venus, in seconds, made by Lt. Herndon in 1846 were:

-0.30	+0.06	-0.05
-0.44	+0.63	+0.20
+1.01	-0.13	+0.18
+0.48	-1.40	+0.39
-0.24	-0.22	+0.10

Chauvenet (1863) declared -1.40 and +1.01 outliers. We may choose to <u>reject</u> them as "gross errors"; <u>incorporate</u> them by changing our model to a non-normal one or; <u>accommodate</u> them by using a robust estimation technique that employs Winsorization or an α-trimmed sample.

Consider also the data described by Pearson and Pearson (1931) on the capacity (in c.c.) of a sample of 17 male Moriori skulls.

1230	1318	1380	1420	1630	1378
1348	1380	1470	1445	1360	1410
1540	1260	1364	1410	1545	

The observation 1630 is suspicious and as such may be rejected as a measurement/recording error; or we may incorporate it in a non-discordant way by assuming a non-normal model (since biological data often require skew distributions); or identification of the outlier may reflect the presence of a small number of another species in the population being studied.

1.3 Models for discordancy

There are several possible models for discordancy.

i) Deterministic: this covers the cases of outliers caused by obvious identifiable gross measurement errors, etc. If the data set $\{x_i\}_{i=1}^{n}$ contains one observation, x_i, clearly resulting from measurement/recording error, we reject H: all $x_i \in F$ in favor of H_1: all $x_j \in F$ $(j \neq i)$ and x_i is different (requiring rejection or correction).

ii) Inherent: here we reject H: all observations are from F in favor of H_1: all observations are from G, where G has more or different inherent variability than F.

iii) Mixture: this model allows contamination of the sample by a few members of a population other than F. We reject H: all $x_i \in F$ in favor of H_1: all $x_i \in pF+(1-p)G$. Thus outliers reflect probability 1-p that observations arise from G.

iv) Slippage: here H_1 states all observations apart from some small number m arise independently from the initial model F indexed by location and scale parameters τ and θ, while the remaining m are independent observations from a modified version of F in which τ or θ is shifted. Which observation(s) come(s) from the shifted distribution is not specified. In most published work, F represents the normal distribution.

v) Exchangeable: this is an extension of the slippage alternative. A set of n observations $S_n = \{x_1,\ldots,x_n\}$ ideally comes from a target population P with probability density function (p.d.f.) $f(x;\underline{\theta})$. However the suspicion is that one of the observations, say x_i, is not from the target population P but from a different population Q with p.d.f. $f(x;\underline{\xi})$. Prior to the experiment, there is no way to identify the possible (at most one spurious observation). Hence, a priori, each observation is equally likely to be the discordant one. Thus the random variables $X_1,\ldots,X_n$ are not independent but are exchangeable. For one outlier, the likelihood of the sample is given by

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i;\underline{\xi}) \prod_{j\neq i} f(x_j;\underline{\theta}) .$$

Definition 1.3.1: Let $\mathcal{F}$ be a family of absolutely continuous univariate distribution functions $\{F(x;\underline{\theta})\}$ indexed by a parameter $\underline{\theta} \in \Theta$. Then the random variables $X_1,\ldots,X_n$ are said to be exchangeable random variables based on $\mathcal{F}$ if their joint p.d.f. is of the form

$$\frac{1}{n}\sum_{r=1}^{n} \prod_{i\neq r} F'(x_i;\underline{\theta}_1)F'(x_r;\underline{\theta}_2), \underline{\theta}_1,\underline{\theta}_2 \in \Theta$$

where $F'(x;\underline{\theta}) = \frac{\partial}{\partial x} F(x;\underline{\theta})$.

Anscombe (1960) used this model for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a\theta,\theta^2)$. Guttman and Smith (1969,1971) and Guttman (1973a) used it for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a,\theta^2)$. Kale and Sinha (1971), Joshi (1972b), Kale (1974), Chikkagoudar and Kunchur (1980) and Rauhut (1982) have used it for $f(x;\underline{\theta})$ as $EXP(\theta)$ and $f(x;\underline{\xi})$ as $EXP(\theta/k^*)$, $0 < k^* \leq 1$.

For m outliers, we assume $x_{i_1},x_{i_2},\ldots,x_{i_{n-m}}$ come from a target population P with p.d.f. $f(x;\underline{\theta})$, while $x_{i_{n-m+1}},\ldots,x_{i_n}$ come from populations $Q_1,Q_2,\ldots,Q_m$ with p.d.f. $f(x;\underline{\xi}_1),\ldots,f(x;\underline{\xi}_m)$. Some or all of the $\underline{\xi}_i$, $i = 1,\ldots,m$ may be identical (i.e. $\underline{\xi}_i = \underline{\xi}$). The association of the different observations with the different distributions is assumed to occur at random. For the case of all $\underline{\xi}_i = \underline{\xi}$, $i = 1,\ldots,m$, the likelihood of the sample is

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{\binom{n}{m}} \sum_{I\in\mathcal{J}} \prod_{i\in I} f(x_i;\underline{\xi}) \prod_{j\notin I} f(x_j;\underline{\theta})$$

where $I = (i_1,i_2,\ldots,i_n)$ is a selection of m integers from $\{1,2,\ldots,n\}$ and $\mathcal{J}$ is the set of all $\binom{n}{m}$ possible such choices. Certain relationships must exist between $f(x;\underline{\theta})$ and $f(x;\underline{\xi})$ for it to be reasonable that the suitability of the model will be reflected in outliers. For the case of one outlier, we need the discordant observation from Q to show up at one of the sample extremes. The exchangeable model also assumes that the maximum number of possible outliers is known.

1.4 The concept of outlier-proneness

Related to model development, though not actually a model to describe the occurrence of outliers, is a concept introduced by Neyman and Scott (1971) and furthered by Green (1974,1976) and Kale (1975b, 1975c, 1976). These papers considered a method of distinguishing between families of distributions by examining the extent to which they are liable to exhibit outliers.

Let S_n be a sample of $n \geq 3$ observations and let $x_{(1)},\ldots,x_{(n)}$ be the order statistics for this sample.

Definition 1.4.1: For a positive number k, $x_{(n)} \in S_n$ is a k-outlier on the right if $x_{(n)} - x_{(n-1)} > k\{x_{(n-1)}-x_{(1)}\}$. The definition of a k-outlier on the left is analogous (i.e $x_{(1)} \in S_n$ is a k-outlier on the left if $x_{(2)} - x_{(1)} > k\{x_{(n)}-x_{(2)}\}$.

Let $P(k,n|F)$ denote the probability that a sample of n observations from a distribution F will contain a k-outlier on the right.

Then, for jointly distributed random variables

$$P(k,n|F) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)}) \, dx_{(n)}dx_{(1)}dx_{(n-1)}$$

where $g(x_{(1)},x_{(n-1)},x_{(n)})$ is the marginal joint p.d.f. of $X_{(1)}, X_{(n-1)}, X_{(n)}$ given by

$$g(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{x_{(1)}}^{x_{(n-1)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(y_1,\ldots,y_n)dy_2\ldots dy_{n-2}$$

where f is the joint p.d.f. of $X_1,\ldots,X_n$.

For the special case of i.i.d. random variables we would have

$$P(k,n|F) = n(n-1)\int_{-\infty}^{\infty} dF(x)\int_{x}^{\infty} \{F(\frac{kx+y}{k+1}) - F(x)\}^{n-2} dF(y).$$

Let $\mathcal{F}$ be the family of distributions and let $\Pi_1(k,n|\mathcal{F})$ be the least upper bound of probabilities $P(k,n|F)$ for $F \in \mathcal{F}$.

Definition 1.4.2: A family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right if $\Pi_1(k,n|\mathcal{F}) = 1$.

Definition 1.4.3: A family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right if $\Pi_1(k,n|\mathcal{F}) < 1$.

Definition 1.4.4: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right $\forall k > 0$, $\forall n > 2$ it is outlier-prone completely on the right (o.p.c.r.).

Definition 1.4.5: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right $\forall k > 0$, $\forall n > 2$ it is outlier-resistant completely on the right. (o.r.c.r.).

Theorem 1.4.6: (Green (1974))

If $\mathcal{F}$ is a family of distributions and S_n is a random sample of n i.i.d. observations from $F \in \mathcal{F}$, then $\mathcal{F}$ is outlier-prone completely on the right (o.p.c.r.) iff $\mathcal{F}$ is (k,n)-outlier-prone on the right for some $k > 0$, $n > 2$.

For i.i.d. observations, Theorem 1.4.6 shows that it is not necessary to distinguish between the concepts of "(k,n)-outlier-prone on the right" and "outlier-prone completely on the right". This leads to the following definitions and theorem for cases in which the observations are i.i.d.

Definition 1.4.7: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-prone on the right (o.p.r.) iff it is (k,n)-outlier-prone on the right for some $k > 0$ and $n > 2$, or, equivalently, iff it is outlier-prone completely on the right.

Definition 1.4.8: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-resistant on the right (o.r.r.) iff it is not o.p.r.

Theorem 1.4.9: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is o.r.r. iff $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$.

Proof: i) Assume that family $\mathcal{F}$ is o.r.r. . Then $\mathcal{F}$ is not o.p.r. . Therefore $\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \forall n > 2.$

i.e. There exists at least one $k' > 0$, $n' > 2$, $F \in \mathcal{F} \ni$.

$$\sup_{F \in \mathcal{F}} P(k',n'|F) < 1 .$$

ii) Assume $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$. Then $\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \; \forall n > 2$.

Therefore $\mathcal{F}$ is not o.p.r.

From Definition 1.4.8 it follows that $\mathcal{F}$ is o.r.r. .

We will use the same definition of a (k,n)-outlier on the right when S_n represents n observations from the exchangeable model.

The implication is that for an outlier resistant family we are justified in seeking out and eliminating outliers. On the other hand, outlier-prone families should be used to model cases where outliers are common and in these cases we should seek ways of accommodating outliers.

Neyman and Scott (1971) showed that in general families differing only in location or scale parameters are outlier-resistant. Thus we may limit studies to subfamilies with standard values for location and scale. Both the family $N(\tau,\theta^2)$ or normal distributions (with mean τ and variance θ^2) and the family $C(\xi,\theta)$ of Cauchy distributions (centered at ξ and having scale parameter θ) are outlier-resistant.

CHAPTER II

SURVEY OF THE LITERATURE

Most studies of the outlier-problem assume an initial distribution that is either exponential or normal.

2.1 Detection of outliers in exponential and normal families using the exchangeable model.

A semi-Bayesian approach was used by Kale (1969), Kale and Sinha (1971), Chikkagoudar and Kunchur (1980), and Rauhut (1982) with respect to the one-parameter exponential family. It was assumed that $X_1,\ldots,X_n$ were such that $n-1$ observations were from $f(x;\theta_1) = \frac{1}{\theta_1}\exp\{\frac{-x}{\theta_1}\}$, $x \geq 0$, $\theta_1 > 0$ while one of the X_i's was distributed as $f(x;\theta_2)$, $\theta_2 \geq \theta_1$ (i.e. $\theta_2 = \theta_1/k^*$, $0<k^*\leq 1$). *A priori*, each X_i was equally likely to be distributed as $f(x;\theta_2)$. Kale and Sinha (1971) calculated $u(r;n,k^*)$, the probability that $X_{(r)}$, the r^{th} order statistic, corresponds to the spurious observation distributed as $f(x;\theta_2)$. It was shown that

$$u(r;n,k^*) = \frac{k^*\Gamma(n)\Gamma(n-r+k^*)}{\Gamma(n+k^*)\Gamma(n-r+1)}$$

was monotone increasing in r and hence $X_{(n)}$ had maximum posterior probability of being an outlier. Mount and Kale (1973) generalized this result for the case of $n-1$ observations with distribution function F and one with distribution function G where F and G are stochastically ordered ($G<F$), where *a priori* each observation is equally likely to be the spurious one, and where $\Psi(x) = \frac{dG}{dF}$ is monotone increasing in r. Kale (1974a) generalized these results for the case of m (≥ 1) possible outliers, where *a priori* each group of m observations $(X_{i_1},X_{i_2},\ldots,X_{i_m})$ is equally likely to come from distribution function G. Then $(x_{(n-m+1)},\ldots,x_{(n)})$ has maximum

posterior probability of corresponding to the set of spurious observations, provided Ψ is monotone increasing. Kale restricted the distributions to the single-parameter exponential family. In another paper, Kale (1974b) gave a completely Bayesian approach where $n-m$ observations were distributed as $f(x;\theta)$, m_1 observations were from $f(x;\theta_j)$, $j = 1,2,\ldots,m_1$, $\theta_j \leq \theta$, and m_2 observations were from $f(x;\theta_\ell)$, $\ell = 1,2,\ldots,m_2$, $\theta_\ell \geq \theta$, $m = m_1+m_2$ and f belonged to the single-parameter exponential family. Under the exchangeable model, it was shown that $\{(x_{(1)},\ldots,x_{(m)}),(x_{(n-m+1)},\ldots,x_{(n)})\}$ had maximum posterior probability of being the set of spurious observations.

For the case of the normal family of distributions, a completely Bayesian approach has been used by Box and Tiao (1968) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau,k^*\theta^2)$, $k^* \geq 1$ (Model A) and by Dempster and Rosner (1971) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau_j,\theta^2)$, $j = 1,2,\ldots,m$, $\tau_j \geq \tau$ (Model B). For $m = 1$, Model B has been handled as a slippage test by Ferguson (1961), Kudo (1956) and Paulson (1952).

2.2 Estimation in the presence of outliers.

Anscombe (1960) suggested a premium-protection approach (see Appendix 1) to estimation that has subsequently been used by Kale and Sinha (1971), Joshi (1972b), Chikkagoudar and Kunchur (1980), and Rauhut (1982) for estimation of the mean in the single-parameter exponential distribution and by Guttman and Smith (1969, 1971, 1973a) and Veale and Huntsberger (1969) for estimation in the normal distribution.

2.2.1 Estimation for the single-parameter exponential distribution.

Under homogeneity, the optimal estimator of θ is $T_{o,n} = \frac{\sum_{i=1}^{n} x_i}{n+1}$. However, under the exchangeable model with one outlier,

$$MSE(T_{o,n}|k^*) = \theta^2\{\frac{1}{n+1} + \frac{2}{n+1}(\frac{1-k^*}{k^*})^2\} \to \infty \quad \text{as} \quad k^* \to 0.$$

Among restricted L-type estimators $T(\underline{\ell}) = \sum_{j=1}^{n-m} \ell_j x_{(j)}$ which ignore the largest m observations, Kale and Sinha (1971) and Veale and Kale (1972) advocated the one-sided Winsorized mean

$$T_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)} + m x_{(n-m)}}{n-m+1}$$

for which $MSE(T(\underline{\ell})|k^*=1)$ is minimum and $MSE(T_{m,n}|k^*<1)$ shows a gain in efficiency relative to $T_{o,n}$ for k^* sufficiently small. Joshi (1972(b)) considered choice of m and showed for k^* small, substantial gains in relative efficiency are possible. Samples up to size $n = 20$ were considered.

Kale (1974a) considered maximum likelihood estimation (M.L.E.) for the exchangeable model with m possible upper outliers and obtained $\hat{\theta}_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$, the trimmed mean, as MLE of θ. In comparing the trimmed mean $\hat{\theta}_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)}}{n-1}$ with the Winsorized estimator $T_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}}{n}$ for the case of a single upper outlier, it was shown that

$$MSE(\hat{\theta}_{1,n}|k^*=1) > MSE(T_{1,n}|k^*=1)$$

but $\lim_{k^*\to 0} MSE(\hat{\theta}_{1,n}|k^*) = \frac{1}{n-1} < \lim_{k^*\to 0} MSE(T_{1,n}|k^*)$. Thus $\hat{\theta}_{1,n}$ provides more protection than $T_{1,n}$ but for a higher premium (See Appendix I).

Rauhut (1982) suggested two 'testimators':

$$D_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} - x_{(n-1)} < k\{x_{(n-1)} - x_{(1)}\} \\ \frac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t & , \text{ otherwise} \end{cases}$$

$$T_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} < C\bar{x}, \quad C > 0 \\ T_t & , \text{ otherwise} \end{cases}$$

and showed $T_{k^*}(\underline{x})$ is preferred over $T_{o,n}$, T_t and D_{k^*} since the premium is small compared to the protection it provides.

Chikkagoudar and Kunchur (1980) proposed using $\hat{\theta}_{CK} = \sum_{i=1}^{n} w_i \hat{\theta}_i$

where $\hat{\theta}_i = \frac{\sum_{j=1, j \neq i}^{n} x_j}{n}$, $w_i = \frac{2r_i}{n(n+1)}$, r_i = rank of x_i in the complete sample. This estimator is more efficient than:

	TABLE 2.2.1	Alternative Estimator	
	$T_{o,n}$	$T_{m,n}$	$\hat{\theta}_{1,n}$
When to use $\hat{\theta}_{CK}$	$n \geq 4, k^* \leq .75$	$n \geq 6,\ .40 \leq k^* \leq .75$	all n, $k^* \geq .45$
	$n=3, k^* \leq .70$	$n=4,5,\ .35 \leq k^* \leq .75$	
	$n=2, k^* \leq .65$	$n=3,\ .30 \leq k^* \leq .70$	
		$n=2,\ .35 < k^* < .65$	

and it is independent of k^*.

2.2.2 Estimation for the normal distribution

Kale (1974a) has shown that the method of maximum likelihood applied to the exchangeable model involving normal distributions with change in location leads to estimators that are trimmed means.

For the case of n-1 observations from $N(\tau, \theta^2)$ and one observation from $N(\tau + k^*\theta, \theta^2)$, $k^* \geq 0$, $\bar{x}$ is UMVUE, MLE for τ if $k^* = 0$ but $\bar{x}$ is biased and $MSE(\bar{x}) = \frac{\theta^2 k^{*2}}{n}$ if $k^* > 0$. Thus an alternative estimator might be $T(\underline{x})$ where $T(\underline{x})$ is one of the following:

$$T(\underline{x}) = \begin{cases} \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t \\ \dfrac{1}{n}\left\{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}\right\} = T_{1,n} \\ A(\underline{x}) = \begin{cases} \bar{x} \quad , \text{ if } x_{(n)} - \bar{x} \leq C_\alpha S \\ \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n-1} \quad , \text{ otherwise} \end{cases} \end{cases}$$

The premium paid for using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \text{MSE}(T|k^*=0) - \frac{1}{\theta^2} \text{MSE}(\bar{x}|k^*=0).$$

The protection obtained by using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \text{MSE}(\bar{x}|k^*>0) - \frac{1}{\theta^2} \text{MSE}(T|k^*>0).$$

A detailed study of the accommodation of outliers in slippage models appears in Guttman and Smith (1969, 1971) and Guttman (1973a), using premium-protection. They consider three methods for estimating the mean $\mu = \tau$:

i) modified trimming $\hat{\mu}_A$ (A-rule)

ii) modified winsorization $\hat{\mu}_W$ (W-rule)

iii) semi-winsorization $\hat{\mu}_S$ (S-rule)

(see Appendix I).

Table 2.2.2. Comparison of $\hat{\mu}_A, \hat{\mu}_S$ and $\hat{\mu}_W$

Form of H_1 (one or two outliers)

	Location-slippage $N(\tau+k^*, \theta^2), k^*>0$	Scale-slippage $N(\tau, \theta^2 k^*), k^*>1$
$\hat{\mu}_A$	Best for large k^*	Not a contender
$\hat{\mu}_W$	Best for intermediate k^*	Best for large k^*
$\hat{\mu}_S$	Best for small k^*	Best for small k^*

The results are based on sample sizes up to $n = 20$, $\theta^2 = \sigma^2$ known, and only for $n = 3$, when θ^2 is unknown.

Guttman and Smith (1971) defined dispersion estimators $\hat{\sigma}^2_A$, $\hat{\sigma}^2_W$ and $\hat{\sigma}^2_S$ of $\hat{\sigma}^2$ analogously to $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ (see Appendix I); $\hat{\sigma}^2_W$ is not worth considering when $\mu = \tau$ is known, $\hat{\sigma}^2_W$ and $\hat{\sigma}^2_A$ are not worth considering when τ is unknown.

CHAPTER III

The Gamma Distribution

One of the most common life-testing distributions is the gamma distribution. We shall examine the exchangeable model based on the gamma distribution and show that, in the case of a shape change, it is outlier-prone completely. For changes in shape or scale parameter, we shall determine which observation is most likely to be the spurious one. We shall also consider estimation in the presence of an outlier.

3.1 Characteristics of the Gamma Distribution

Consider the three-parameter gamma distribution given by

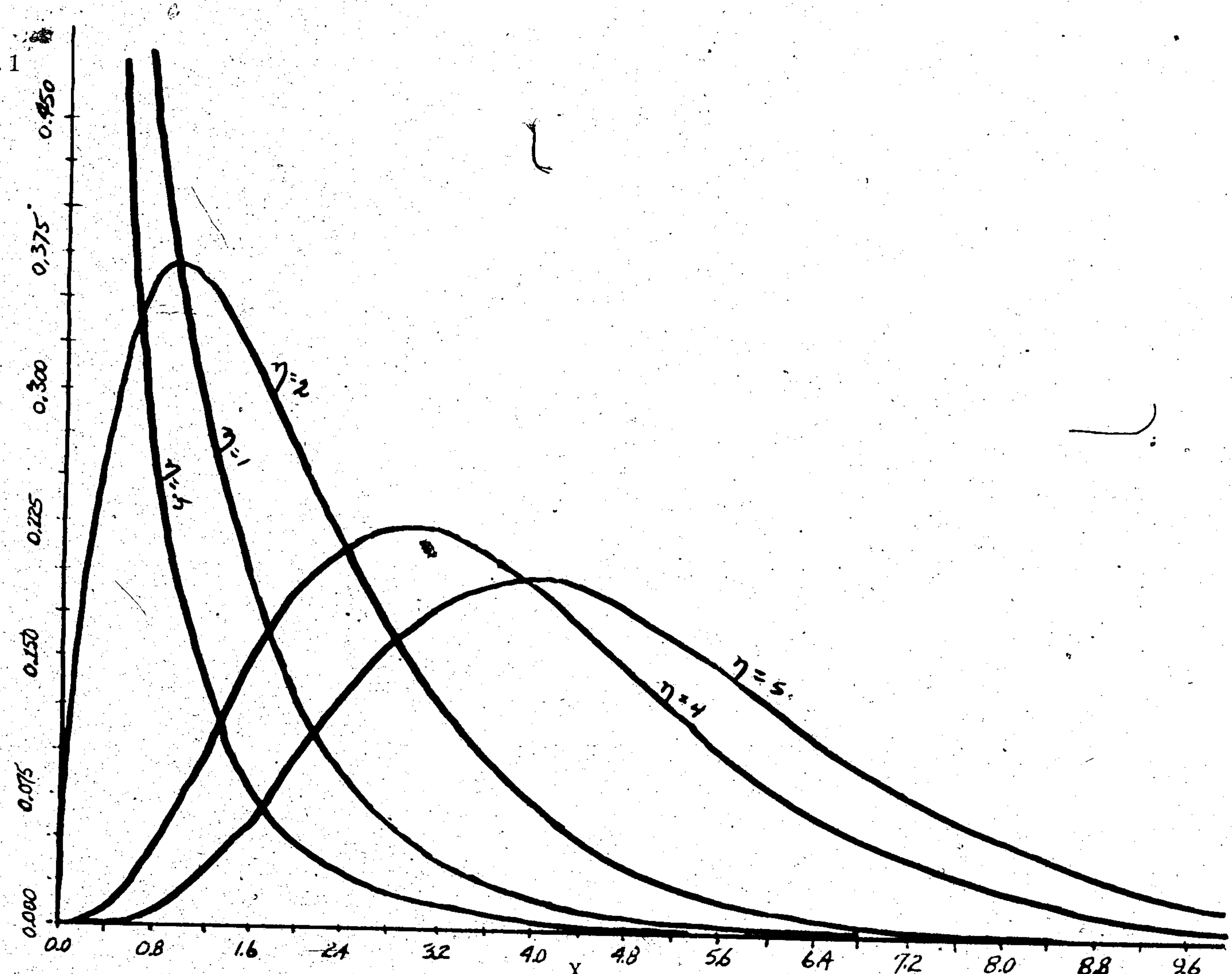
$$f(x;\theta,\eta,\tau) = \frac{(x-\tau)^{\eta-1} e^{-\frac{x-\tau}{\theta}}}{\theta^{\eta}\Gamma(\eta)}, \quad x > \tau,\ \theta,\eta > 0,\ -\infty < \tau < \infty.$$

We may denote this p.d.f. by $GAM(\theta,\eta,\tau)$. The shape parameter is η, the scale parameter θ, and the location parameter τ. For $\tau = 0$ or known, this reduces to the two-parameter gamma distribution where

$$f(x;\theta,\eta) = \frac{e^{-\frac{x}{\theta}} x^{\eta-1}}{\theta^{\eta}\Gamma(\eta)}, \quad x > 0,\ \eta,\theta > 0.$$

For [illegible]e the exponential distribution with parameter θ and for int[illegible]e have the Erlang distribution. For $\eta \leq 1$ and fixed θ, the p[illegible] decreasing in x and unbounded $(\eta<1)$ near the origin. [illegible]ean is $\theta\eta$, variance is $\theta^2\eta$, $E(X^r) = \frac{\theta^r\Gamma(\eta+r)}{\Gamma(\eta)}$ and the moment-gen[illegible]ing function is $M_X(t) = (1-\theta t)^{-\eta}$, $t < \frac{1}{\theta}$. The shape parameter [illegible] is the reciprocal of the squared coefficient of variation.

Fig. 3.1.1



Gamma p.d.f. for various values of η (θ =1, τ =0)

If T denotes the waiting time until the η^{th} occurrence in a Poisson process with intensity λ, then $T \sim GAM(\theta = \frac{1}{\lambda}, \eta, 0)$. The waiting time until the first occurrence may be denoted as $GAM(\theta = \frac{1}{\lambda}, 1, 0)$ which is $EXP(\theta = \frac{1}{\lambda})$. Thus the gamma distribution is a natural extension of the exponential distribution. The two-parameter exponential may be denoted as $GAM(\theta = \frac{1}{\lambda}, \eta = 1, \tau)$ where τ is a location parameter.

If $X_1, \ldots, X_n$ i.i.d. $GAM(\theta, m_i, 0)$ then $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta, \sum_{i=1}^{n} m_i, 0)$. Consider the case of a component with $n-1$ spare parts. If X_i, $i = 1,2,\ldots,n$ denote the lifetimes of the component and the spares and if each is distributed $GAM(\theta,1,0) = EXP(\theta = \frac{1}{\lambda})$, then the lifetime of the system, assuming use of the $n-1$ spares, is $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta, n, 0)$ or $\frac{2Y}{\theta} \sim \chi^2_{2n\,df}$. Note also that if $Y \sim GAM(2, \eta, 0)$ then $Y \sim \chi^2_{2\eta\ df}$. The hazard function (HF) $h(x) = \lambda = \frac{1}{\theta}$ for $\eta = 1$; $h(x) \to \lambda^-$ as $x \to \infty$ for $\eta < 1$ and $h(0) = 0$; $h(x) \to \lambda^+$ as $x \to \infty$ and $h(0) = \infty$ for $\eta > 1$. Consequently the gamma distribution can model systems in a regular maintenance program where the failure rate is likely to increase initially but then stabilize.

Outliers in gamma samples arise in any context where Poisson processes are appropriate basic models, e.g. traffic flow, biological aggregation, failure of electronic equipment. They also occur in the context of a shifted exponential or gamma distribution. Outliers in χ^2 samples arise in ANOVA; outliers in gamma samples of arbitrary shape parameter arise with skew-distributed data, for which the gamma distribution is often

3.2 Outlier-proneness of the exchangeable model with the Gamma distribution.

Neyman and Scott (1971) showed that for i.i.d.r.v.'s the family of gamma distributions indexed by shape parameter η is outlier-prone completely on the right. Let us consider now the exchangeable model.

CASE I: Scale change

Kale (1975b) showed that the exchangeable model with (at most) one spurious observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely. This would apply to the exchangeable model involving the gamma distribution with possible change in scale parameter θ.

CASE II: Shape change

Now consider the exchangeable model with (at most) one spurious observation based on the gamma distribution with possible change in the shape parameter η. Without loss of generality (w.l.o.g) we may take $\theta = 1$, $\tau = 0$.

Then $n-1$ observations have p.d.f. $f(x;\underline{\theta})$ which is GAM$(1,\eta,0)$ and one observation has p.d.f. $f(x;\underline{\xi})$ which is GAM$(1,k^*\eta,0)$ where $k^* \geq 1$. We shall show that this model is outlier-prone completely on the right.

Now we may write the likelihood as

$$L(x_1,\ldots,x_n;\eta,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\eta)f(x_r;k^*\eta), \quad x_i \geq 0,\ k^* \geq 1,\ \eta > 0$$

($k^* \geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Letting

E denote the event that $x_{(n)}$ is a (k,n)-outlier on the right and $P(k,n|L) = P(E|\eta,k^*)$, we must show that $\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$ in order for this family to be (k,n)-outlier-prone.

The joint p.d.f. of the order statistics $x_{(1)},\ldots,x_{(n)}$ is given by

$$g(x_{(1)},\ldots,x_{(n)};\eta,k^*) = n!L(x_{(1)},\ldots,x_{(n)};\eta,k^*)$$

$$= (n-1)! \sum_{r=1}^{n} \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta),$$

$$0 < x_{(1)} < \ldots < x_{(n)} < \infty .$$

Thus

$$P(E|\eta,k^*) = \int_E g(x_{(1)},\ldots,x_{(n)};\eta,k^*)dx_{(1)}\ldots dx_{(n)}$$

$$= \sum_{r=1}^{n} \int_E (n-1)! \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)} \cdots dx_{(n)}$$

$$= \sum_{r=1}^{n} I_r(\eta,k^*)$$

and E is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1}$ $< x_{(n)} < \infty$.

Lemma 3.2.1: $0 \le I_r(\eta,k^*) \le u(r;n,\eta,k^*)$ where $u(r;n,\eta,k^*) = P\{X_{(r)}$ is the "spurious" observation whose p.d.f. is $f(x,k^*\eta)\}$.

Proof:

$$u(r;n,\eta,k^*) = (n-1)!\int_S \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\ldots,dx_{(n)} \text{ where}$$

S is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < x_{(n)} < \infty$.

Now $I_r(\eta,k^*) = (n-1)!\int_E \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\ldots dx_{(n)}$ where

E is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Since $E \subset S$, $0 \le I_r(\eta,k^*) \le u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n$.

Theorem 3.2.2: $$\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 1, & r = n \\ 0, & r = 1,2,\ldots,n-1 \end{cases}.$$

Proof: Now $u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\eta)\}^{r-1}\{1-F(y;\eta)\}^{n-r}f(y;k^*\eta)dy$

where $F(y;\eta) = \int_{-\infty}^{y} f(x;\eta)dx$.

Consider now

$$u(n;n,\eta,k^*) = \int_{y=0}^{\infty}\{\int_{x=0}^{y} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-1}\frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}dy,\ k^* \ge 1,\ \eta > 0 .$$

(without loss of generality $\theta = 1$, $\tau = 0$)

Let

$$I_n'(\eta,k^*) = \int_{y=0}^{\infty} \{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1} e^{-y}y^{\nu}dy \quad \text{for} \quad \nu = k^*\eta-1.$$

Now setting $y = \nu t$, $dy = \nu dt$

$$I_n'(\eta,k^*) = \int_{t=0}^{\infty} \{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1} e^{-\nu t}(\nu t)^{\nu}\nu dt$$

$$= \nu^{\nu+1} \int_{t=0}^{\infty} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$$

where

$$\vartheta(\nu,t) = [\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx]^{n-1}.$$

and where there is no loss in generality in assuming $\nu > 0$ (i.e. $k^*\eta > 1$) since we are interested in $\lim_{k^*\to\infty} I_n'(\eta,k^*)$.

For $0 \le t \le \alpha$, $\vartheta(\nu,0) \le \vartheta(\nu,t) \le \vartheta(\nu,\alpha)$ and $0 \le \vartheta(\nu,t) \le 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. in t as $\nu \to \infty$ (i.e. as $k^* \to \infty$). We may write $\int_0^{\infty} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt = \int_0^{1} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$

$$+ \int_1^{\infty} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt.$$

For each of the integrals on the right hand side, the dominant part occurs in the neighborhood of the point when $\ell n\ t-t$ is maximum, i.e. $t = 1$. Following Copson (1967, p. 36 ff), since $0 \le \vartheta(\nu,t) \le 1$ for all ν and t and $\vartheta(\nu,t)$ is continuous and increasing in t there is a number $t_0 (0<t_0 \le 1)$ such that

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0) \int_0^1 e^{\nu(\ln t-t)}dt , \quad 0 \le t_0 \le 1 .$$

We cannot have $t_0 = 0$ since $\vartheta(\nu,0) = 0$ while

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt > \int_{.5}^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$> \frac{1}{2}\vartheta(\nu, \frac{1}{2})e^{-\nu(\frac{1}{2} -\ln 2)} > 0$$

and

$$\int_1^\infty \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \lim_{\alpha\to\infty} \int_1^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$= \lim_{\alpha\to\infty} \vartheta(\nu,t_1)\int_1^\alpha e^{\nu(\ln t-t)}dt, \quad 1 \le t_1 \le \alpha.$$

Thus

$$\int_0^\infty \vartheta(\nu,t)e^{\nu(\ln t-t)}dt \sim \int_0^\infty e^{\nu(\ln t-t)}dt$$

$$\sim 1 \cdot \frac{\Gamma(\nu+1)}{\nu^{\nu+1}} \text{ as } \nu \to \infty$$

(see Appendix II). As a result, we may write

$$I'_n(\eta,k^*) \sim \nu^{\nu+1}\cdot 1\cdot\frac{\Gamma(\nu+1)}{\nu^{\nu+1}} = \Gamma(\nu+1) \quad \text{as} \quad \nu\to\infty$$

and

$$u(n;n,\eta,k^*) = \frac{I'_n(\eta,k^*)}{\Gamma(k^*\eta)} \quad \text{where} \quad \nu = k^*\eta-1$$

$$\to 1 \quad \text{as} \quad \nu\to\infty \quad \text{(i.e. as} \quad k^*\to\infty).$$

For $u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n-1$ we have

$$0 \le u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_{y=0}^{\infty}\left\{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}$$

$$\left\{1-\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r}\frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy$$

$$= \binom{n-1}{r-1}\frac{I'_r(\eta,k^*)}{\Gamma(k^*\eta)}$$

where

$$I'_r(\eta,k^*) = \int_{t=0}^{\infty}\left\{\int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}\left\{1-\int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r}e^{-\nu t}\nu^{\nu+1}t^{\nu}dt,$$

$\nu = k^*\eta-1$, $y = \nu t$, $dy = \nu dt$ and again we assume $\nu > 0$. Therefore

$$I_r'(\eta,k^*) = \nu^{\nu+1}\int_0^\infty \vartheta(\nu,t)e^{\nu(\ln t - t)}dt$$

$$= \nu^{\nu+1} \lim_{\alpha\to\infty} \int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt .$$

Now $\vartheta(\nu,t) = \{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-1}\{1 - \int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-r} \leq 1$ and as $k^* \to \infty$, $\nu \to \infty$ and $\vartheta(\nu,t) \to 0$ almost everywhere on $[0,\infty]$. Since $\vartheta(\nu,t)$ is continuous and bounded for all t

$$\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0)\int_0^\alpha e^{\nu(\ln t-t)}dt \qquad 0 \leq t_0 \leq \alpha$$

$$\leq \vartheta(\nu,t_0)\int_0^\infty e^{\nu(\ln t-t)}dt = \vartheta(\nu,t_0)\frac{\Gamma(\nu+1)}{\nu^{\nu+1}} .$$

Therefore $0 \leq u(r;n,\eta,k^*) \leq \frac{\binom{n-1}{r-1}}{\Gamma(k^*\eta)}\nu^{\nu+1}\lim_{\alpha\to\infty}\frac{\vartheta(\nu,t_0)\Gamma(\nu+1)}{\nu^{\nu+1}} = \binom{n-1}{r-1}$ $\lim_{\alpha\to\infty}\vartheta(\nu,t_0)$. But $\binom{n-1}{r-1}\lim_{\alpha\to\infty}\vartheta(\nu,t_0) \to 0$ as $\nu\to\infty$ (i.e. as $k^*\to\infty$) and hence $u(r;n,\eta,k^*) \to 0$ as $\nu\to\infty$ for $r = 1,2,\ldots,n-1$.

Theorem 3.2.3: The exchangeable model with (at most) one spurious observation based on the gamma distribution with a change in shape parameter is outlier-prone completely on the right.

Proof: We need to show $\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$.

Since $P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*)$,

$$\lim_{k^*\to\infty} P(E|\eta,k^*) = \sum_{i=1}^{n} \lim_{k^*\to\infty} I_r(\eta,k^*).$$

From Lemma 3.2.1, $0 \le I_r(\eta,k^*) \le u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n$.

By Theorem 3.2.2, $\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 0 & , r = 1,2,\ldots,n-1 \\ 1 & , r = n \end{cases}$.

Thus $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0$, $r = 1,2,\ldots,n-1$.

and $\lim_{k^*\to\infty} P(E|\eta,k^*) = \lim_{k^*\to\infty} I_n(\eta,k^*)$.

Now we need only show that $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ in order for this model to be outlier-prone completely on the right.

But

$$I_n(\eta,k^*) = (n-1)!\int_E \prod_{i\neq n} f(x_{(i)};\eta)f(x_{(n)};k^*\eta)dx_{(1)} \cdots dx_{(n)}$$

$$= (n-1)!\int_E \prod_{i\neq n} \frac{e^{-x_{(i)}}x_{(i)}^{\eta-1}}{\Gamma(\eta)} \frac{e^{-x_{(n)}}x_{(n)}^{k^*\eta-1}}{\Gamma(k^*\eta)} dx_{(1)} \cdots dx_{(n)}$$

where E is such that $0 < x_{(1)} < x_{(2)} < \cdots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Let $y_1,\ldots,y_n$ denote the order statistics $x_{(1)},\ldots,x_{(n)}$, respectively. Then we may write

$$I_n(\eta,k^*) = (n-1) \iint_{0<y_1<y_n<\infty} \left\{\int_0^{\frac{ky_1+y_n}{k+1}} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx - \int_0^{y_1} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-2}$$

$$\cdot \frac{e^{-y_1}y_1^{\eta-1}}{\Gamma(\eta)} \frac{e^{-y_n}y_n^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy_1\,dy_n$$

$$= \int_{y_n=0}^{\infty} \left[\int_{y_1=0}^{y_n} (n-1)\left\{F\left(\frac{ky_1+y_n}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)\,dy_1\right] f(y_n;k^*\eta)\,dy_n$$

where $f(y;\eta) = \frac{e^{-y}y^{\eta-1}}{\Gamma(\eta)}$, $y > 0$. Then

$$I_n(\eta,k^*) = \frac{1}{\Gamma(k^*\eta)} \int_{y_n=0}^{\infty} \vartheta(y_n)e^{-y_n}y_n^{k^*\eta-1}\,dy_n \quad \text{where}$$

$\vartheta(y_n) = \int_{y_1=0}^{y_n} (n-1)\left\{F\left(\frac{ky_1+y_n}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)\,dy_1$. Setting $y_n = \nu t$ where $\nu = k^*\eta-1$, we obtain

$$(3.2.1) \qquad I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \int_{t=0}^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}\,dt$$

$$= \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_{t=0}^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}\,dt$$

where $\vartheta(\nu,t) = \vartheta(\nu t) = \int_{y_1=0}^{\nu t} (n-1)\left\{F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right\}^{n-2} f(y_1;\eta)\,dy_1$.

As $k^* \to \infty$, $\nu \to \infty$ and the major contribution to this integral in 3.2.1 occurs in the neighborhood of $t = 1$. Also $0 \le \vartheta(\nu,t) \le 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. on $[0,\infty)$ as $\nu \to \infty$ (i.e. as $k^* \to \infty$), since

$$\lim_{\nu\to\infty} \left[F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right] = 1-F(y_1;\eta) \text{ a.e. on } [0,\infty).$$

If $t = 0$, $\vartheta(\nu,t) = (n-1)\int_0^{\nu t} \left[F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right]^{n-2} f(y_1;\eta)dy_1$ and $\vartheta(\nu,0) = 0$. But $\vartheta(\nu,t)$ is nonnegative, bounded, increasing and absolutely continuous in t. Thus $\exists t_0 \in [0,\infty) \ni . \int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$ $= \vartheta(\nu,t_0) \int_0^\alpha e^{\nu(\ln t-t)}dt$. On the other hand for any $0 < \xi < 1$

$$\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt > \int_\xi^1 \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$> \vartheta(\nu,\xi)e^{\nu(\ln \xi-\xi)}(1-\xi)$$

$$> 0.$$

Therefore $t_0 \neq 0$.

Thus $\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt \sim \int_0^\infty e^{\nu(\ln t-t)}dt = \frac{\Gamma(\nu+1)}{\nu^{\nu+1}}$ as $\nu \to \infty$

and

$$I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_0^\alpha \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$\sim \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{\nu^{\nu+1}} \text{ as } \nu \to \infty \text{ (i.e. as } k^* \to \infty).$$

Since $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ and $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0$, $r = 1,2,\ldots,n-1$ and

$P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*)$,

$$\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$$

and the family is outlier-prone completely on the right.

Thus for either a scale or shape change, the exchangeable model with at most one spurious observation based on the gamma distribution is outlier-prone completely on the right.

3.3 Detectio[illegible]ers

Mount an[illegible]) considered a general model, assuming $X_1,\ldots,$[illegible] are [illegible]t $n-1$ of them are distributed with distrib[illegible]n fu[illegible]n (d.f) $F(x)$ and one is distributed with d.f. $G(x)$, wh[illegible] F [illegible]d G are stochastically ordered, i.e. $G < F$. *A priori*, ea[illegible] has probability $\frac{1}{n}$ of being the spurious observation distribute[illegible]G. Let $\Psi(x) = \frac{dG}{dF}$. If $\Psi(x)$ is monotone increasing, then $u(1;n,$[illegible]$< u(2;n,k^*) < \ldots < u(n;n,k^*)$ where $u(i;n,k^*) = P(X_{(i)}$ is the spuri[illegible] observation in a sample of size n with k^* the coefficient o[illegible]puriosity).

<u>CASE I: Scale change</u>: Consider a situation where $n-1$ observations are from GAM($\theta,$[illegible]) and one is from GAM($k^*\theta,\eta,0$), $k^* > 0$. If $k^* = 1$ we have homogen[illegible]ta. Now

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}}$$

and

$$\Psi'(x) = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}}\left(\frac{k^*-1}{\theta k^*}\right) \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 \end{cases}$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; if $k^* < 1$, Ψ is monotone decreasing and $X_{(1)}$ has maximum probability of being spurious.

CASE II: Shape change: If n-1 observations come from GAM($\theta,\eta,0$) and one is from GAM($\theta,k^*\eta,0$), $k^* > 0$, assuming the exchangeable model,

$$\Psi(x) = \frac{dG}{dF} = \frac{x^{k^*\eta-\eta}\Gamma(\eta)}{\theta^{k^*\eta-\eta}\Gamma(k^*\eta)}$$

and

$$\Psi'(x) = \frac{\Gamma(\eta)\eta(k^*-1)x^{k^*\eta-\eta-1}}{\Gamma(k^*\eta)\theta^{k^*\eta-\eta}} \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 . \end{cases}$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; for $0 < k^* < 1$, $\Psi(x)$ and $u(r;n,k^*)$ are monotone decreasing and $X_{(1)}$ has maximum probabilty of being spurious.

This now shows that the spurious observation resulting from a scale or shape change is most likely to occur at the sample extremes i.e. it tends to show up as an outlier.

3.4 Estimation for Gamma parameters

3.4.1 Standard Estimators

The gamma distribution is a member of the exponential class, consequently $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \ln X_i\right)$ are complete sufficient statistics for (θ, η).

For known shape η, the MLE $\hat{\lambda}_{MLE}$ of $\lambda = \frac{1}{\theta}$ is $\frac{\eta}{\bar{x}}$ which is biased but consistent and asymptotically $N\left(\lambda, \frac{\lambda^2}{n\eta}\right)$. Thus the UMVUE for θ (η known) is $\frac{\bar{x}}{\eta}$.

For unknown η, we obtain

$$\hat{\theta}_{MLE} = \frac{\bar{x}}{\hat{\eta}}$$

$$g(\hat{\eta}_{MLE}) = \ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE}) - \ln \bar{x} + \ln \tilde{x} = 0$$

where $\tilde{x} = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ and $\psi(z)$ is Euler's psi function i.e.

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \int_0^\infty \frac{\frac{e^{-t}}{t} - e^{-zt}}{1-e^{-t}}\, dt = -\gamma + \frac{1}{z} + z \sum_{i=1}^{\infty} [i(i+z)]^{-1}$$ where

$\gamma = .5772157$ (Euler's constant). These equations must be solved iteratively (see Choi and Wette (1969)). For large $\hat{\eta}_{MLE}$, Linhart (1965) suggested approximating $\ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE})$ by $(2\hat{\eta}_{MLE} - 1/3)^{-1}$ thus

$$\hat{\eta}_{MLE} = \{(\ell n\ \bar{x} - \ell n\ \tilde{x})^{-1} + 1/3\}1/2.$$

Moment estimators give

$$\hat{\eta}_M = \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{\sum_{i=1}^{n} n(X_i - \bar{X})^2}$$

$$\hat{\theta}_M = \frac{\sum_{i=1}^{n} X_i}{n\eta_M} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} X_i} .$$

These have lower asymptotic efficiency than MLE's but Lilliefors (1971) showed that for $n \leq 20$, $\eta \geq 2$, the MSE's of the moment estimators are close to those of the MLE's. Both the MLE's and the moment estimators are biased. To approximate zero bias and smaller MSE, Lilliefors suggested the following corrections:

<u>TABLE 3.4.1</u> Lilliefors' improved estimators of η and $\lambda = 1/\theta$

	Based on M.L.E.	Based on Moment Estimators
$\hat{\hat{\eta}}$	$\frac{\hat{\eta}_{MLE}}{1+3/n}$	$\frac{\hat{\eta}_M}{1+2/n} - 5/3$
$\hat{\hat{\lambda}} = \frac{\hat{1}}{\theta}$	$\frac{n\hat{\eta}_{MLE}}{\left(1+\frac{3}{n}\right)\sum_{i=1}^{n} X_i}$	$\left\{\frac{n\hat{\eta}_M}{(1+2/n)} - 5/3\right\} \frac{1}{\sum_{i=1}^{n} X_i}$

Thom (1968) has given estimators very similar to the MLE's for $\eta > 1$.

$$\eta^* = \frac{1 + \sqrt{1 + \frac{4M}{3}}}{4M}$$

$$\theta^* = \frac{\bar{x}}{\eta^*}$$

where $M = \ln(\bar{x}/\tilde{x}) = -\ln S_1$ and S_1 is sufficient for η and independent of θ. Bain and Englehardt (1975) have a chi-square approximation for M valid for all η (θ acts as a nuisance parameter).

$$2n\eta Mc \sim \chi^2_{\nu df}$$

where $W = 2n\eta M$, $c = 2\,\frac{E(W)}{Var(W)} = \frac{nw_1(\eta)-w_1(n\eta)}{nw_2(\eta)-w_2(n\eta)}$ and $\nu = 2\,\frac{[E(W)]^2}{Var(W)} =$ $[nw_1(\eta)-w_1(n\eta)]c$ and $w_1(z) = 2z\{\ln z-\psi(z)\}$, $w_2(z) = 2z\{z\psi'(z)-1\}$ where $\psi(z)$ is the psi function. As $\eta \to 0$, $W \xrightarrow{d} \chi^2_{2(n-1)df}$.

$$w_1(\eta) \doteq 1 + \frac{1}{1+6\eta}$$

$$w_2(\eta) \doteq \begin{cases} 1 + \frac{1}{1+2.5\eta}, & 0 < \eta < 2 \\ 1 + \frac{1}{3\eta}, & \eta \geq 2 \end{cases}$$

$$\frac{\nu}{n-1} \doteq 1 + \frac{1}{(1+4.3\eta)^2}$$

3.4.2 Estimators suggested for use

Case I: Scale Change

Assuming the exchangeable model where n-1 observations are distributed as $GAM(\theta,\eta,0)$ and one is distributed as $GAM(\theta k^*,\eta,0)$, η known, $k^* \geq 1$, we have for $k^* = 1$, homogeneous data and the best linear unbiased estimator (BLUE) for θ is $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ and $E(\hat{\theta}_{MLE}) = \theta$ and $MSE(\hat{\theta}_{MLE}) = Var(\hat{\theta}_{MLE}) + \{Bias(\hat{\theta}_{MLE})\}^2 = \frac{\theta^2}{n\eta}$. For the heterogeneous case $(k^*>1)$, $E_{het}(\hat{\theta}_{MLE}) = \frac{1}{n\eta}\sum_{i=1}^{n}\left(\frac{n-1}{n}\theta\eta + \frac{1}{n}k^*\theta\eta\right)$ $= \frac{\theta}{n}(n-1+k^*)$ and $Bias_{het}(\hat{\theta}_{MLE}) = \frac{\theta(k^*-1)}{n}$. As $k^* \to \infty$, this bias $\to \infty$. Now $MSE_{het}(\hat{\theta}_{MLE}) = Var_{het}(\hat{\theta}_{MLE}) + \{Bias_{het}(\hat{\theta}_{MLE})\}^2$ and

$$Var_{het}(\hat{\theta}_{MLE}) = \frac{1}{(n\eta)^2}\sum_{i=1}^{n}\left\{\frac{n-1}{n}\theta^2\eta + \frac{1}{n}(k^*\theta)^2\eta\right\}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1\}.$$

therefore
$$MSE_{het}(\hat{\theta}_{MLE}) = \frac{\theta^2}{n^2\eta}(k^{*2}+n-1) + \frac{\theta^2(k^*-1)^2}{n^2}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1+\eta(k^*-1)^2\}$$

As $k^* \to \infty$, $MSE_{het}(\hat{\theta}_{MLE}) \to \infty$ and hence $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ proves to be a poor estimator of θ.

If we consider a random sample of size n where $n-m$ observations are from GAM($\theta,\eta,0$) and m are from GAM($k^*\theta,\eta,0$), $k^* > 1$ (i.e. same shape but a change in scale parameter) and a priori each subset of m observations is equally likely to be the "outlier subset", then the likelihood

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\theta^{\eta}\}^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{k^*\theta}} \prod_{x_i \in I} x_i^{\eta-1}}{\{\Gamma(\eta)(k^*\theta)^{\eta}\}^{m}}$$

$$= \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} e^{-[\sum_{x_i \notin I} \frac{x_i}{\theta} + \sum_{x_i \in I} \frac{x_i}{\theta_1}]}}{\{\Gamma(\eta)\}^n \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

for $k^*\theta = \theta_1$ where $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $I \in \mathcal{T}$, the collection of all possible combinations of m observations out of n. (I represents the subset of m spurious observations). To maximize $L(\underline{x}|\theta,\eta,k^*,I)$ for $\theta,\eta > 0$, $I \in \mathcal{T}$, $k^* > 1$ we use the fact that $\Psi(x) = \frac{dG(x)}{dF(x)}$ is monotone increasing for $k^* > 1$ and hence $\max_{I \in \mathcal{T}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $I = \hat{I} = (x_{(n-m+1)}, \ldots, x_{(n)})$. Therefore

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} e^{-\{\sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}\}}}{\{\Gamma(\eta)\}^n \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

and

$$K(\underline{x}|\theta,\eta,k^*,\hat{I}\} = \ln L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

$$= C + (\eta-1)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} - \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}$$

$$- n\ln \Gamma(\eta) - \eta(n-m)\ln \theta - \eta m \ln \theta_1 \; .$$

Assuming

i) θ, θ_1 and η are unknown, we obtain:

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n} \ln x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - \{(n-m)\ln \theta + m\ln \theta_1\}$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta^2} - \frac{\eta(n-m)}{\theta}$$

$$\frac{\partial K}{\partial \theta_1} = \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1^2} - \frac{\eta m}{\theta_1} \; .$$

Setting the above three equations equal to zero, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)}$$

$$\hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{\eta(n-m)}$$

$$h(\hat{\eta}) = \sum_{i=1}^{n} \ln x_{(i)} - \frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + n\ln \hat{\eta}$$

$$- (n-m)\ln \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m} - m\ln \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m} = 0.$$

ii) for the case of known η, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)} \quad \text{and} \quad \hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m\eta}$$

which are trimmed means.

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$.

$$L(\underline{y};\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i\notin I} y_i + \sum_{i\in I} y_i/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\eta m}}$$

and

$$L(\underline{y};\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\eta m}}$$

and

$$K(\underline{y};\eta,k^*,\hat{I}) = \ln L(\underline{y};\eta,k^*,\hat{I})$$

$$= C' + (\eta-1)\sum_{i=1}^{n} \ln y_i - \{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}$$

$$- n \ln \Gamma(\eta) - \eta m \ln k^*.$$

Then

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n} \ln y_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - m \ln k^*$$

$$\frac{\partial K}{\partial k^*} = \sum_{i=n-m+1}^{n} \frac{y_{(i)}}{k^{*2}} - \frac{m\eta}{k^*}$$

and thus

$$\hat{k}^* = \frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m\eta}$$

and

$$\frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - m \ln \hat{\eta} = \sum_{i=1}^{n} \ln y_i - m \ln \left\{\frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m}\right\} .$$

iv) for the case of known k*

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n} \ln x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - (n-m)\ln\theta - m\ln(k^*\theta)$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta^2} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{k^*\theta^2} - \frac{\eta(n-m)}{\theta} - \frac{\eta m}{\theta} .$$

Setting $\frac{\partial K}{\partial \eta} = 0$ and $\frac{\partial K}{\partial \theta} = 0$, we obtain

$$\hat{\eta}\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)} + \frac{1}{k^*} \sum_{i=n-m+1}^{n} x_{(i)}}{n}$$

$$\ln\hat{\theta} + \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n} \ln x_i - m\ln k^*}{n} .$$

Case II. Shape change

Assuming n-1 observations distributed as $GAM(\theta,\eta,0)$ and one distributed as $GAM(\theta,k^*\eta,0)$, $k^* \geq 1$, we have, for heterogeneous data ($k^*>1$),

$$E_{het}(\hat{\theta}_{MLE}) = E\left(\frac{\bar{X}}{\eta}\right) = \frac{\theta}{n\eta} \sum_{i=1}^{n} \left(\frac{n-1}{n}\theta\eta + \frac{1}{n}\theta k^*\eta\right)$$

$$= \theta + \frac{\theta(k^*-1)}{n} .$$

As $k^* \to \infty$, bias $\to \infty$. Also

$$\text{Var}_{het}(\hat{\theta}_{MLE}) = \text{Var}\left(\frac{\bar{X}}{\eta}\right) = \frac{1}{n^2\eta^2}\sum_{i=1}^{n} \text{Var}(X_i)$$

$$= \frac{1}{n^2\eta^2}\sum_{i=1}^{n}\left(\frac{n-1}{n}\theta^2\eta + \frac{1}{n}\theta^2 k^*\eta\right)$$

$$= \frac{\theta^2}{n^2\eta}(n+k^*-1).$$

Then

$$\text{MSE}_{het}(\hat{\theta}_{MLE}) = \text{Var}_{het}(\hat{\theta}_{MLE}) + \{\text{Bias}(\hat{\theta}_{MLE})\}^2$$

$$= \frac{\theta^2(n+k^*-1)}{n^2\eta} + \frac{\theta^2\eta(k^*-1)^2}{\eta n^2}$$

$$= \frac{\theta^2\{n+k^*-1+\eta(k^*-1)^2\}}{n^2\eta}.$$

As $k^* \to \infty$, $\text{MSE}_{het}(\hat{\theta}_{MLE}) \to \infty$ and hence $\hat{\theta} = \frac{\bar{x}}{\eta}$ proves to be a poor estimator of θ (We have assumed η known).

Consider now the exchangeable model where $n-m$ observations are from $GAM(\theta,\eta,0)$ and m observations are from $GAM(\theta,k^*\eta,0)$, $k^* \geq 1$ (i.e. possible change in shape; scale constant). Then

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\}^{n-m}(\theta^{\eta})^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{\theta}} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(k^*\eta)\}^{n-m}(\theta^{k^*\eta})^{m}}$$

$$= \frac{1}{\binom{n}{m}} \frac{e^{-T/\theta} \prod_{x_i \notin I} x_i^{\eta-1} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^{m}(\theta^{\eta})^{n-m+k^*m}}$$

where $T = \sum_{i=1}^{n} x_i$, $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $I \in \mathcal{T}$, the collection of all possible subsets of m observations out of n. To obtain MLE's, note that $\Psi(x) = \frac{dG}{dF}$ is monotone increasing and, by Kale (1975b), we know $\max_{I \in \mathcal{T}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $\hat{I} = I(\theta,\eta,k^*$ fixed$)$ where $\hat{I} = (x_{(n-m+1)},\ldots,x_{(n)})$. Thus

$$\max_{\substack{k^*>1 \\ I \in \mathcal{T}}} L(\underline{x}|\theta,\eta,k^*,I) = \max_{k^* \geq 1} L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

and

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{e^{-T/\theta} \prod_{i=1}^{n-m} x_{(i)}^{\eta-1} \prod_{i=n-m+1}^{n} x_{(i)}^{k^*\eta-1}}{\binom{n}{m} \{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^{m}(\theta^{\eta})^{n-m+k^*m}} .$$

Using $\eta_1 = k^*\eta$ ($> \eta$ since $k^* > 1$)

$$K(\underline{x}|\theta,\eta,\eta_1,\hat{I}) = \ln L(\underline{x}|\theta,\eta,\eta_1,\hat{I})$$

$$= C - \frac{T}{\theta} + (\eta-1) \sum_{i=1}^{n-m} \ln x_{(i)} + (\eta_1-1) \sum_{i=n-m+1}^{n} \ln x_{(i)}$$

$$- (n-m)\ln \Gamma(\eta) - m \ln \Gamma(\eta_1) - (n-m)\eta \ln \theta - m\eta_1 \ln \theta$$

and

i) assuming θ, η, and η_1 unknown, we obtain

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{m\eta_1}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - (n-m) \ln \theta$$

$$\frac{\partial K}{\partial \eta_1} = \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{m\Gamma'(\eta_1)}{\Gamma(\eta_1)} - m \ln \theta \ .$$

Now $\frac{\partial K}{\partial \theta} = 0$ implies $\hat{\theta} = \dfrac{T}{(n-m)\hat{\eta} + m\hat{\eta}_1}$ and

$$\frac{\partial K}{\partial \eta} = 0 \text{ implies } \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + \ln \hat{\theta} = \frac{\sum_{i=1}^{n-m} \ln x_{(i)}}{n-m} \text{ and}$$

$$\frac{\partial K}{\partial \eta_1} = 0 \text{ implies } \frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} + \ln \hat{\theta} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} \ .$$

ii) for the case of known shape parameter η,

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta} \text{ and}$$

$$\frac{\partial K}{\partial k^*} = \eta \sum_{i=n-m+1}^{n} \ln x_{(i)} - mn \frac{\Gamma'(k^*\eta)}{\Gamma(k^*\eta)} - m\eta \ln \theta$$

and setting these equal to zero, we obtain

$$\hat{\theta} = \frac{T}{\eta(n-m+mk^*)} \quad \text{and}$$

$$\ell(\hat{k}^*) = \frac{\Gamma'(\hat{k}^*\eta)}{\Gamma(\hat{k}^*\eta)} + \ln \hat{\theta} - \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} = 0 .$$

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$ and hence obtain

$$\frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n-m} \ln y_{(i)}}{n-m}$$

and

$$\frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} = \frac{\sum_{i=n-m+1}^{n} \ln y_{(i)}}{m} .$$

iv) for the case of known k^*

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - \frac{mk^*\Gamma'(k^*\eta)}{\Gamma(k^*\eta)}$$

and thus $\hat{\theta} = \frac{T}{\eta\{n-m(1-k^*)\}}$ and

$$\sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - \frac{mk^*\Gamma'(k^*\hat{\eta})}{\Gamma(k^*\hat{\eta})} = 0.$$

CHAPTER IV

The Lognormal Distribution

We now examine the lognormal distribution as a life-testing distribution that is a competitor to the gamma distribution. We shall show that the exchangeable model involving the lognormal family of distributions indexed by the shape parameter σ is outlier-resistant completely on the right. We shall then determine that as heterogeneity increases the spurious observation tends to appear as an outlier and we shall determine where it will most likely appear. We shall also consider estimation in the presence of this outlier.

4.1 Characteristics of the Lognormal Distribution.

If a random variable X has the lognormal distribution with location parameter μ and shape parameter σ, then the probability density function of X is given by

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sigma x\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}(\ln x-\mu)^2\}, & x>0,\ \sigma>0,\ -\infty<\mu<\infty \\ 0 & \text{, elsewhere} \end{cases}$$

We shall write $X \sim \Lambda(\mu,\sigma)$. The following table summarizes some some of its properties:

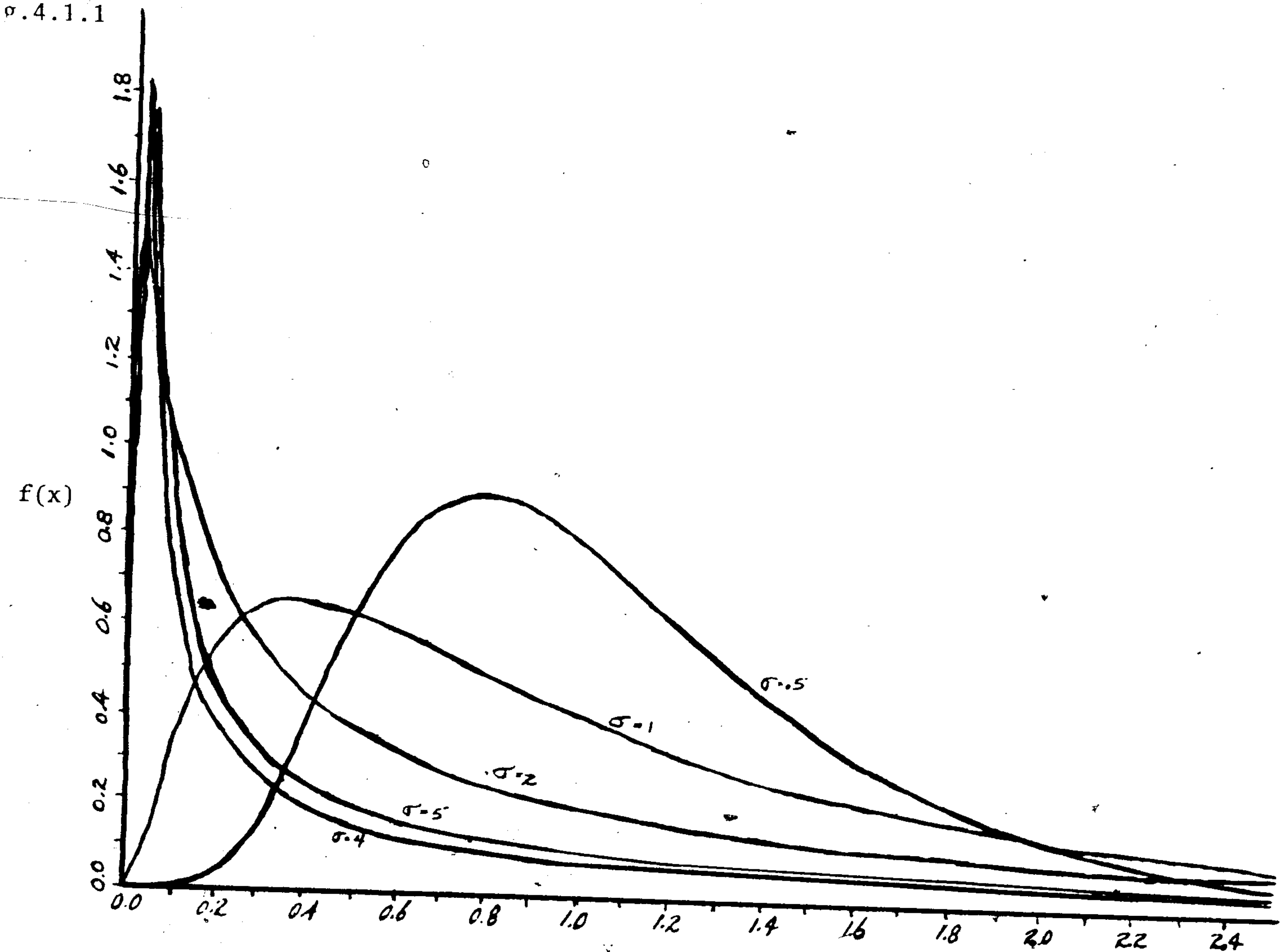
Table 4.1.1

mean	$e^{\mu+\frac{1}{2}\sigma^2}$
variance	$e^{\sigma^2}(e^{\sigma^2}-1)e^{2\mu}$
mode	$e^{\mu-\sigma^2}$
median	e^{μ}
coefficient of variation	$(e^{\sigma^2}-1)^{1/2}$
skewness	$(e^{\sigma^2}+2)(e^{\sigma^2}-1)^{1/2}$
kurtosis	$\omega^4+2\omega^3+3\omega^2-6,\ \omega = e^{\sigma^2}$

The lognormal distribution has monotone likelihood ratio (MLR) in $T_1(x) = \ln x$, σ known and in $T_2(x) = (\ln x-\mu)^2$, μ known but not

MLR in x for μ known and nonzero. $T_1 = \sum_{i=1}^{n} \ln x_i$ is a complete sufficient statistic for μ (σ known) and $(T_2 = \sum_{i=1}^{n} (\ln x_i)^2, \quad T_1 = \sum_{i=1}^{n} \ln x_i)$ are jointly complete sufficient statistics for (μ, σ^2). $T_3 = \sum_{i=1}^{n} (\ln x_i - \mu)^2$ is sufficient for σ^2 (μ known). The lognormal distribution can assume shapes from severely right-skewed to essentially symmetrical.

Fig.4.1.1



Lognormal p.d.f. for various values of σ (μ =0)

If $X \sim \Lambda(\mu, \sigma)$ then $Y = \ln X \sim N(\mu, \sigma^2)$.

For some data, the lognormal distribution is a competitor to the gamma distribution. The lognormal p.d.f. is unimodal, vanishes at $x = 0$ and its mode is at $x = e^{-\sigma^2} < 1$. As σ^2 increases, the mode converges to zero. From the above graph where $\sigma = 2$, the lognormal p.d.f. appears monotonically decreasing for almost all $x > 0$ and when $\sigma = 5$ the mode is virtually zero. This may be compared to the gamma distribution with shape parameter $\eta < 1$. Here the density is infinite at zero and monotonically decreasing thereafter. The lognormal distribution has a non-monotonic failure rate.

Neyman and Scott (1971) document data from rain-making experiments where the data is nonzero rainfall per experimental unit (an experimental day or storm). Here the distributions are reverse J-shaped with long "tails" and frequently display substantial "outliers". They suggest that an outlier-prone distribution such as the gamma or lognormal might appropriately model this data.

Nelson (1977) points out the usefulness of the lognormal distribution for approximating distributions of input variables such as costs, sales, market shares, etc. required for Monte Carlo simulations of business decisions. This distribution has been used by Howard and Dodson (1961) and Peck (1961) to study semiconductor devices and by Goldwaithe (1961) in small-particle statistical economics and biology. Singpurwalla and Keubler (1966) used it to study the lifetimes of high speed steel drills since the failure rate first increases and then decreases, indicating the drill could resharpen itself and prolong its life. It has wide applicability to reliability, especially maintainablity and fracture problems. If $X_1 < X_2 < \ldots < X_n$ denote sizes of a fatigue crack at successive stages of its growth and X_0 the initial size of the crack, assuming a "proportional effect model" for growth of the crack (Kao (1965)), this implies crack growth at stage i, $X_i - X_{i-1}$, is randomly proportional to the size of the crack, X_{i-1} (i.e. $X_i - X_{i-1} = \pi_i X_{i-1}$, $i = 1,2,\ldots,n$ where π_i are independent), and the item fails when crack size reaches X_n. It can be shown that $\ell n\ X_n$ is asymptotically normal and hence X_n is lognormal. This model is also true for the distribution of oil pool sizes by the same argument on how the pools are initially formed.

4.2 Outlier-proneness of the exchangeable model with the Lognormal distribution

Neyman and Scott (1971) have demonstrated that for i.i.d.r.v.'s the family of lognormal distributions indexed by σ is outlier-prone completely on the right. If we now consider the exchangeable model with at most one outlier, we have n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu_1,\sigma)$, $\mu_1 \geq \mu$ (Case I) or n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu,\sigma_1)$, $\sigma_1 \geq 0$ (Case II).

Case I: Scale change

We first consider the exchangeable model based on the lognormal distribution with possible change in μ. Then $f(x;\underline{\theta})$ is $\Lambda(\mu,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(\mu_1,\sigma)$ where $\mu_1 = k^*\mu$, $k^* \geq 1$. Kale (1975b) proved that the exchangeable model with (at most) one possible outlier observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely on the right. Thus the model we are considering would be outlier-prone.

Case II: Shape Change

Consider the exchangeable model based on the lognormal distribution with possible change in shape parameter σ. Then $f(x;\underline{\theta})$ is $\Lambda(0,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(0,\sigma_1)$ where $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$. The likelihood may be written as

$$L(\underline{x};\sigma,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\underline{\theta})f(x_r;\underline{\xi}),\ x_i > 0\ ,\ k^* \geq 1,$$

$$\sigma_1^2 = k^*\sigma^2\ ,\ i = 1,2,\ldots,n$$

($k^* \geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\ldots,x_{(n)}) = \frac{1}{n}\, n! \sum_{r=1}^{n} \frac{f(x_r;\sigma_1)}{f(x_r;\sigma)} \prod_{i=1}^{n} f(x_i,\sigma)$$

and

$$g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= \int_{x_{(1)}}^{x_{(n-1)}} \int_{x_{(1)}}^{x_{(n-2)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(x_{(1)},\ldots,x_{(n)})dx_{(2)}\ldots dx_{(n-2)}$$

$$= (n-1)! \sum_{r=1}^{n} h_r(x_{(1)},x_{(n-1)},x_{(n)})$$

where $h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{S_{n-3}} \frac{f(x_r;\sigma_1)}{f(x_r;\sigma)} \prod_{i=1}^{n} f(x_i;\sigma)dx_2 \ldots dx_{n-2}$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \ldots < x_{(n-2)} < x_{(n-1)}$.

For $r = 2,\ldots,n-2$

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma)}{(r-2)!(n-r-2)!}$$

(continued)

$$\cdot \int_{x_{(1)}}^{x_{(n-1)}} \left[F(x_{(r)};\sigma)-F(x_{(1)};\sigma)\right]^{r-2}\left[F(x_{(n-1)};\sigma)-F(x_{(r)};\sigma)\right]^{n-r-2}$$

$$f(x_{(r)};\sigma_1)dx_{(r)} \ ,$$

while

$$h_1(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma_1)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma) \ \frac{\left[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)\right]^{n-3}}{(n-3)!} \ ,$$

$$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma_1) \ f(x_{(n)};\sigma) \ \frac{\left[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)\right]^{n-3}}{(n-3)!} \ ,$$

and

$$h_n(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma_1) \ \frac{\left[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)\right]^{n-3}}{(n-3)!} \ .$$

Thus, letting $t(y) = f(y;\sigma)$ and $s(y) = f(y;\sigma_1)$

$$(4.2.1) \quad . \quad . \quad . \quad . \qquad g(x_{(1)}, x_{(n-1)}, x_{(n)})$$

$$= (n-1)!\,\{t(x_{(1)})t(x_{(n-1)})t(x_{(n)})$$

$$\times \sum_{r=2}^{n-2} \int_{x_{(1)}}^{x_{(n-1)}} \frac{[T(x_{(r)})-T(x_{(1)})]^{r-2}[T(x_{(n-1)})-T(x_{(r)})]^{n-r-2}s(x_{(r)})}{(r-2)!(n-r-2)!}\,dx_{(r)}$$

$$+[s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+t(x_{(1)})t(x_{(n-1)})s(x_{(n)})] \frac{[T(x_{(n-1)})-T(x_{(1)})]^{n-3}}{(n-3)!}\} .$$

In 4.2.1, we may replace the letter of integration in the definite integrals by z and use the fact that

$$(n-1)! \sum_{r=2}^{n-2} \frac{\alpha^{r-2}\beta^{n-r-2}}{(r-2)!(n-r-2)!} = \frac{(n-1)!}{(n-4)!}(\alpha+\beta)^{n-4}$$

to give

$$g(x_{(1)}, x_{(n-1)}, x_{(n)}) = \frac{(n-1)!}{(n-3)!}\{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)}) \times$$

$$[T(x_{(n-1)})-T(x_{(n)})]^{n-4}[S(x_{(n-1)})-S(x_{(1)})]$$

$$+ [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)}) \qquad \text{(continued)}$$

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]^{n-3}\}$$

$$= \frac{(n-1)!}{(n-3)!}\{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}\{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)})\times$$

$$[S(x_{(n-1)})-S(x_{(1)})] + [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]\}.$$

Let

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^{\infty}\int_0^{x_{(n-1)}}\int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and let $x = x_{(1)}$ and $y = x_{(n-1)}$.

Theorem 4.2.1: If $p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$ then, for non-negative random variables,

$$p \leq 1-k(n-1)(n-2)\int_0^{\infty} t(y)\int_0^{y} S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t are, respectively, the distribution function and probability density function of the non-spurious observations.

Proof:

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^\infty g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!} \int_0^\infty \int_0^{x_{(n-1)}} \{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}$$

$$\{(n-3)t(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}][S(x_{(n-1)})-S(x_{(1)})]$$

$$+ (s(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})s(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})t(x_{(n-1)})[1-S\{(k+1)x_{(n-1)}-kx_{(1)}\}])$$

$$\times [T(x_{(n-1)})-T(x_{(1)})]\}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!} \{I_1+I_2+I_3+I_4 - (I_5+I_6+I_7+I_8)\}.$$

Let $x = x_{(1)}$, $y = x_{(n-1)}$

$$I_1 = \int_0^{\infty}\int_0^{y} (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\{S(y)-S(x)\}dxdy$$

$$= (n-3)\int_0^{\infty} S(y)t(y)\int_0^{y} t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$-(n-3)\int_0^{\infty} t(x)S(x)\int_x^{\infty} t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= (n-3)\int_0^{\infty} S(y)t(y)\frac{\{T(y)\}^{n-3}}{(n-3)} dy - (n-3)\int_0^{\infty} S(x)t(x)\frac{\{1-T(x)\}^{n-3}}{(n-3)} dx$$

$$= \int_0^{\infty} t(x)S(x)[\{T(x)\}^{n-3}-\{1-T(x)\}^{n-3}]dx \quad .$$

$$I_2 = \int_0^{\infty}\int_0^{y} \{T(y)-T(x)\}^{n-3}s(x)t(y)dxdy$$

$$= \int_0^{\infty} s(x)\int_x^{\infty} \{T(y)-T(x)\}^{n-3}t(y)dydx$$

$$= \int_0^{\infty} \frac{s(x)}{(n-2)} \{1-T(x)\}^{n-2}dx.$$

Integrating by parts where $u = \{1-T(x)\}^{n-2}$ $\quad dv = s(x)dx$

$du = -(n-2)t(x)\{1-T(x)\}^{n-3}dx \quad v = S(x)$

then

$$I_2 = \int_0^\infty S(x)t(x)\{1-T(x)\}^{n-3}dx.$$

$$I_3 = \int_0^\infty\int_0^y \{T(y)-T(x)\}^{n-3}t(x)s(y)dxdy$$

$$= \int_0^\infty s(y) \int_0^y \{T(y)-T(x)\}^{n-3}t(x)dxdy$$

$$= \int_0^\infty s(x) \frac{\{T(x)\}^{n-2}}{(n-2)} dx \quad .$$

Integrating by parts where $u = \{T(x)\}^{n-2}$ $dv = s(x)dx$

$du = (n-2)\{T(x)\}^{n-3}t(x)dx$ $v = S(x)$

$$I_3 = \frac{1}{n-2} - \int_0^\infty S(x)t(x)\{T(x)\}^{n-3}dx \quad .$$

$$I_4 = \int_0^\infty\int_0^y t(x)t(y)\{T(y)-T(x)\}^{n-3}dx$$

$$= \int_0^\infty t(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$= \int_0^\infty t(x) \frac{\{1-T(x)\}^{n-2}}{(n-2)} dx$$

$$= \frac{1}{(n-1)(n-2)} \quad .$$

Therefore $\quad I_1 + I_2 + I_3 + I_4 = \frac{1}{(n-2)} + \frac{1}{(n-1)(n-2)}$

$$= \frac{n}{(n-1)(n-2)} \quad .$$

$$I_5 = \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\, T\{(k+1)y-k(x)\}\{S(y)-S(x)\}dxdy$$

$$\geq \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}T(y)\{S(y)-S(x)\}dxdy$$

$$= \int_0^\infty (n-3)t(y)T(y)S(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$- \int_0^\infty (n-3)t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$- \int_0^\infty (n-3)t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \int_0^\infty t(x)T(x)S(x)\{T(x)\}^{n-3}dx - \int_0^\infty \frac{(n-3)}{(n-2)}\, t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore

$$I_5 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx - \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx \; .$$

$$I_7 \geq \int_0^\infty \int_0^y t(x)s(y)T(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(y)T(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(x)T(x)\,\frac{\{T(x)\}^{n-2}}{(n-2)}\,dx$$

$$= \int_0^\infty s(x)\,\frac{\{T(x)\}^{n-1}}{(n-2)}\,dx \; .$$

Integrating by parts where $u = \{T(x)\}^{n-1}$ $\quad dv = s(x)dx$

$du = (n-1)t(x)\{T(x)\}^{n-2}dx \quad v = S(x)$

$$I_7 \geq \frac{-1}{n-2} - \frac{(n-1)}{(n-2)}\int_0^\infty t(x)S(x)\{T(x)\}^{n-2}\,dx \; .$$

$$I_8 \geq \int_0^\infty \int_0^y t(x)t(y)S(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty \frac{t(y)S(y)}{(n-2)}\{T(y)\}^{n-2}dy$$

$$= \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx.$$

$$I_6 = \int_0^\infty t(y)\int_0^y s(x)\{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\}dxdy \quad .$$

Integrating by parts where

$$u = \{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\} \qquad dv = s(x)dx$$

$$du = \{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}\{-kdx\} \qquad v = S(x)$$

$$+ \quad (n-3)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}\{-t(x)\}dx$$

$$I_6 = k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}t(x)dxdy$$

$$\geq k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy \quad .$$

But

$$(n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy$$

$$= (n-3)\int_0^\infty t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$+ (n-3)\int_0^\infty t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore $I_5 + I_6 + I_7 + I_8 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$

$$- \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx + \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

(continued)

$$- \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$+ k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ \frac{(n-3)}{(n-2)} \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx + \frac{1}{(n-2)}$$

$$- \frac{(n-1)}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx \quad + \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$$

Therefore

$$p \leq (n-1)(n-2)\Big[\frac{n}{(n-1)(n-2)} - \frac{1}{(n-2)}$$

$$- k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy\Big]$$

$$= 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)[T(y)-T(x)]^{n-3}t\{(k+1)y-kx\}dxdy \ .$$

Theorem 4.2.2: The exchangeable model with (at most) one spurious observation based on the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables,

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation, and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation. If, for all $k > 0$, $n > 2$ $\sup p < 1$ then the family is outlier-resistant completely on the right.

It is then sufficient to show that

$$(4.2.2) \quad H = k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Let $s_1 = \dfrac{\ln x}{\sigma}$ $\qquad s_2 = \dfrac{\ln y}{\sigma}$ $\qquad v = \dfrac{\sigma}{\sigma_1}$

$ds_1 = \dfrac{dx}{x\sigma}$, $\qquad ds_2 = \dfrac{dy}{\sigma y}$

If ϕ and Φ respectively denote the standardized normal p.d.f. and d.f. then expression (4.2.2) becomes

$$k(n-1)(n-2)\int_{-\infty}^{\infty} \phi(s_2)\int_{-\infty}^{s_2} \Phi(vs_1)\{\Phi(s_2)-\Phi(s_1)\}^{n-3}\phi\{\frac{\ln[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}$$

$$\times \frac{e^{\sigma s_1} ds_1 ds_2}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}$$

$$\geq k(n-1)(n-2)\int_3^5 \phi(s_2)\int_0^2 \frac{\Phi(vs_1)e^{\sigma s_1}}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}\{\Phi(s_2)-\Phi(s_1)\}^{n-3}$$

$$\times \phi\{\frac{\ln[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}ds_1 ds_2$$

In this region, $s_2 \geq 3$, $s_1 \geq 0$, $s_2 \geq 1 + s_1$. Therefore $\Phi(vs_1) \geq 1/2$ for all v(i.e. for all σ_1), $e^{\sigma s_1} \geq 1$,

$$\Phi(s_2) - \Phi(s_1) \geq \Phi(3) - \Phi(2) \text{ and } e^{-\sigma s_2} \geq e^{-5\sigma}$$

Then

$$H \geq \frac{k(n-1)(n-2)}{2(k+1)}\{\Phi(3) - \Phi(2)\}^{n-3}\frac{1}{2}e^{-5\sigma}\, 2\int_3^5 \phi(s_2)\phi(s_2+\frac{\ln(k+1)}{\sigma})ds_2 .$$

Let $u = \frac{\ln(k+1)}{\sigma}$. Then $u > 0$ since $k > 0$ and

$$\int_3^5 \phi(s_2)\phi(s_2+u)ds_2 = \frac{1}{\sqrt{2\pi}}\int_3^5 \frac{1}{\sqrt{2\pi}} e^{-1/2[\frac{s_2+u/2}{1/\sqrt{2}}]^2 - u^2/4} ds_2$$

$$= \frac{e^{-u^2/4}}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{\sqrt{2}(3+u/2)}^{\sqrt{2}(5+u/2)} \phi(z)dz$$

$$= \frac{e^{-u^2/4}}{2\sqrt{\pi}} \left[\Phi\{\sqrt{2}(5+u/2)\} - \Phi\{\sqrt{2}(3+u/2)\}\right]$$

$$> 0.$$

Then for all $k > 0$, $n > 2$, $v = \sigma/\sigma_1 > 0$, $H > 0$ and $p = 1-H < 1$ and therefore $\sup p < 1$. Thus the exchangeable model with (at most) one spurious observation involving the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

4.3 Detection of Outliers

Case I: Scale change.

If $n-1$ observations are from $\Lambda(\mu,\sigma)$ and one observation is from $\Lambda(\mu_1,\sigma)$, $\mu_1 = k^*\mu$, $k^* \gtrless 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ell n\, x-\mu_1}{\sigma}\right)^2}}{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ell n\, x-\mu}{\sigma}\right)^2}}$$

$$= \exp\left[-\frac{1}{2\sigma^2}\{(\ell n\, x-\mu_1)^2-(\ell n\, x-\mu)^2\}\right]$$

$$= \left[\exp\left\{-\frac{(\mu_1^2-\mu^2)}{2\sigma^2}\right\}\right] x^{\frac{\mu_1-\mu}{\sigma^2}}$$

and $$\Psi'(x) = \left(\frac{\mu_1-\mu}{\sigma^2}\right) x^{\frac{\mu_1-\mu}{\sigma^2}-1} \exp\left\{-\frac{1}{2\sigma^2}(\mu_1^2-\mu^2)\right\} .$$

Thus $$\Psi'(x) \begin{cases} > 0 & \text{if } \mu_1 > \mu \\ < 0 & \text{if } \mu_1 < \mu \end{cases} .$$

Consequently $X_{(n)}$ has maximum probability of being the spurious observation if $\mu_1 > \mu$; $X_{(1)}$ has maximum probability of being the spurious observation if $\mu_1 < \mu$.

Case II: Shape change

If n-1 observations are from $\Lambda(\mu,\sigma)$ and one observation is from $\Lambda(\mu,\sigma_1)$, $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\sigma}{\sigma_1} \exp\{-\frac{1}{2}(\ell n\ x-\mu)^2(\frac{1}{\sigma_1^2} - \frac{1}{\sigma^2})\}$$

is not monotone in x.

Consider $u(r;n,\sigma,k^*) = P(X_{(r)}$ is the spurious observation) (i.e. $X_{(r)}$ is the order statistic whose distribution has parameter σ_1). Then

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\sigma)\}^{r-1}\{1-F(y;\sigma)\}^{n-r}f(y;\sigma_1)dy$$

$$= \binom{n-1}{r-1}\int_0^\infty \{\Phi(\frac{\ell n\ y-\mu}{\sigma})\}^{r-1}\{1-\Phi(\frac{\ell n\ y-\mu}{\sigma})\}^{n-r}\frac{\exp -\frac{1}{2}(\frac{\ell n\ y-\mu}{\sigma_1})^2}{\sigma_1 y\sqrt{2\pi}}dy$$

where Φ is the standard normal distribution function

$$= \binom{n-1}{r-1}\int_0^\infty \left[\Phi\left\{\left(\frac{\ln\ y-\mu}{\sigma_1}\right)\sqrt{k^*}\right\}\right]^{r-1} \left[1-\Phi\left\{\left(\frac{\ln\ y-\mu}{\sigma}\right)\sqrt{k^*}\right\}\right]^{n-r}$$

$$\frac{\exp -\frac{1}{2}\left(\frac{\ln\ y-\mu}{\sigma_1}\right)^2}{\sigma_1 y\sqrt{2\pi}}\,dy$$

and setting $x = \frac{\ln\ y-\mu}{\sigma_1}$ we have

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_{-\infty}^{\infty} \{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r}\,\frac{\exp\{-\frac{x^2}{2}\}}{\sqrt{2\pi}}\,dx\ .$$

Now

$$0 \le \Phi(x\sqrt{k^*}) \le 1 \quad \text{and} \quad \lim_{k^*\to\infty}\Phi(x\sqrt{k^*}) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}\ .$$

Therefore

$$\lim_{k^*\to\infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\lim_{k^*\to\infty}\left\{\int_0^\infty \{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r}\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right.$$

$$\left. + \int_{-\infty}^{0} \{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r}\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right\}$$

By the dominated convergence theorem

$$\lim_{k^*\to\infty} u(n;n,\sigma,k^*) = \int_0^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = 1/2$$

$$\lim_{k^*\to\infty} u(1;n,\sigma,k^*) = \int_{-\infty}^0 \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = 1/2$$

and for any other $r = 2,\ldots,n-1$

$$\{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r} \to 0 \quad \text{as} \quad k^* \to \infty ,$$

so that, by the dominated convergence theorem,

$$\lim_{k^*\to\infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_{-\infty}^{\infty} 0 \,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = 0,\ r = 2,\ldots,n-1.$$

It thus appears that the spurious observation tends to occur at either $X_{(1)}$ or $X_{(n)}$ with equal probability (approaching 1/2) as the shape parameter of the spurious observation increases.

4.4 Estimation for Lognormal parameters

4.4.1 Standard Estimators

If $X \sim \Lambda(\mu,\sigma)$, M.L.E.'s of μ and σ may be obtained from M.L.E. of $W = \ln X$ since $W \sim N(\mu,\sigma^2)$. Thus the M.L.E.'s of μ and σ are

$$\hat{\mu} = \frac{\sum_{i=1}^{n} W_{(i)}}{n} = \frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (\ln X_{(i)})^2 - n\left(\frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}\right)^2}{n}$$

Since $\hat{\mu}$ and $\hat{\sigma}^2$ are based on complete sufficient statistics, we can obtain best linear estimators $\hat{\mu}$ and $s^2 = \frac{n\hat{\sigma}^2}{n-1}$ and $\frac{n-1}{n+1} s^2$ has minimum MSE among estimators of σ^2 of the form ks^2. The BLIE estimators of μ and σ^2 are $\hat{\mu}$ and $\tilde{\sigma}^2 = \frac{n}{n+1} \hat{\sigma}^2$. Asymptotically, all three estimators are equivalent. However, using the property of MLE's that if $\hat{\gamma}$ is MLE of γ, $g(\hat{\gamma})$ is MLE of $g(\gamma)$, we may obtain M.L.E.'s of the mean and variance of the lognormal to be respectively, $e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}$ and $e^{2\hat{\mu}+\hat{\sigma}^2}(e^{\hat{\sigma}^2}-1)$.

To obtain confidence bounds for parameters of the two-parameter lognormal distributions, we use

$$\hat{\mu} = \bar{W} \sim N(\mu, \sigma^2/n).$$

Thus, for σ^2 known, a $(1-\alpha)$ 100% confidence interval (C.I.) for μ is given by

$$\bar{w} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{w} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} .$$

For σ^2 unknown, $\frac{n\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n} w_i^2 - n\bar{w}^2}{\sigma^2} \sim \chi^2_{n-1}$ df

and $\left(\frac{\bar{w}-\mu}{\sigma/\sqrt{n}}\right) \Big/ \sqrt{\frac{n\sigma^2}{\sigma^2(n-1)}} \sim t_{n-1}$ df

and a two-sided $(1-\alpha)$ 100% C.I. for μ is

$$\bar{w} + t_{\alpha/2,n-1\text{df}} \frac{\hat{\sigma}}{\sqrt{n-1}} \leq \mu \leq \bar{w} + t_{1-\alpha/2,n-1\text{df}} \frac{\hat{\sigma}}{\sqrt{n-1}} .$$

4.4.2 Estimators suggested for use

Case I: Scale change

Consider the estimator $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln X_i}{n}$. Under the homogeneous model, $E(\hat{\mu}) = \mu$ and $\text{Var}(\hat{\mu}) = \frac{\sigma^2}{n}$. Thus its $\text{MSE}(\hat{\mu}) = \frac{\sigma^2}{n}$. Under the exchangeable model with (at most) one possible outlier,

$$E_{het}(\hat{\mu}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n} \mu + \frac{1}{n} \mu_1\right)$$

$$= \mu\{1 + \frac{k^*-1}{n}\}$$

and

$$Var_{het}(\hat{\mu}) = \frac{1}{n^2} \sum_{i=1}^{n} Var_{het.} (\ln X_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left(\frac{n-1}{n} \sigma^2 + \frac{1}{n} \sigma^2\right)$$

$$= \frac{\sigma^2}{n} .$$

We then obtain $MSE_{het}(\hat{\mu}) = Var_{het}(\hat{\mu}) + \{Bias_{het}(\hat{\mu})\}^2$

$$= \frac{\sigma^2}{n} + \{\frac{\mu(k^*-1)}{n}\}^2$$

and, as $k^* \to \infty$, $MSE_{het}(\hat{\mu}) \to \infty$. Thus $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln X_i}{n}$ appears to be a poor estimator in this instance.

Assuming the exchangeable model with m outliers (generally $m \leq 10\%$ of the sample size), we have $n-m$ observations with p.d.f. $f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp - \frac{1}{2\sigma^2}(\ln x-\mu)^2$ and m observations with p.d.f. $f(x;\mu_1,\sigma)$, $\mu_1 \geq \mu$. The joint likelihood may be written as

$$L(\underline{x};\mu,\mu_1,\sigma,I) = \frac{1}{\binom{n}{m}} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum_{x_i \notin I} (\ln x_i - \mu)^2}{\prod_{x_i \notin I} \sigma x_i \sqrt{2\pi}} \right\} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum_{x_i \in I} (\ln x_i - \mu_1)^2}{\prod_{x_i \in I} \sigma x_i \sqrt{2\pi}} \right\}$$

We wish to obtain $\max\limits_{\substack{I \in \mathcal{J} \\ \mu_1 > \mu \\ \sigma > 0}} L(\underline{x};\mu,\mu_1,\sigma,I)$ where $\mathcal{J}$ is the collection of all possible subsets of m outliers in a sample of size n. Kale (1974b) has shown for $\Psi(x) = \frac{dG(x)}{dF(x)}$ monotone increasing in x, $\hat{I} = \{x_{(n-m+1)},\ldots,x_{(n)}\}$ has maximum probability of being the set of spurious observations. For $\mu_1 > \mu$, $\Psi(x)$ is monotone increasing and hence

$$\max_{\substack{I \in \mathcal{J} \\ \mu_1 > \mu \\ \sigma > 0}} L(\underline{x};\mu,\mu_1,\sigma,I) = \max_{\substack{\mu_1 > \mu \\ \sigma > 0}} L(\underline{x};\mu,\mu_1,\sigma,\hat{I}) .$$

Now $K(\underline{x};\mu,\mu_1,\sigma,\hat{I}) = \ln L(\underline{x};\mu,\mu_1,\sigma,\hat{I})$

$$= C - \frac{1}{2\sigma^2} \left\{ \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu_1)^2 \right\}$$

$$- n/2 \ln 2\pi\sigma^2 - \sum_{i=1}^{n} \ln x_i \ .$$

Then $$\frac{\partial K}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)$$

$$\frac{\partial K}{\partial \mu_1} = \frac{1}{\sigma^2} \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu_1)$$

$$\frac{\partial K}{\partial \sigma^2} = \frac{1}{\sigma^4} \left\{ \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu)^2 \right\} - \frac{n}{\sigma^2}$$

and hence $$\hat{\mu}_{het} = \sum_{i=1}^{n-m} \frac{\ln x_{(i)}}{n-m}$$

$$\hat{\mu}_{1_{het}} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m}$$

$$\hat{\sigma}^2_{het} = \frac{\sum_{i=1}^{n-m} (\ln x_{(i)} - \hat{\mu}_{het})^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \hat{\mu}_{1_{het}})^2}{n}$$

Thus it appears that maximum likelihood estimation suggests trimmed means as estimators of μ_1 and μ $(\mu_1 > \mu)$ and a pooled estimator, for σ^2, that utilizes these trimmed means.

Case II. Shape change

Under the homogeneous model, $E(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \mu$ and $Var(\hat{\mu}) = \frac{\sigma^2}{n}$. Under the exchangeable model with at most one possible outlier,

$$E_{het}(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n} \mu + \frac{1}{n} \mu\right) = \mu$$

and

$$Var_{het}(\hat{\mu}) = Var\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var_{het}(\ln x_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left(\frac{n-1}{n} \sigma^2 + \frac{1}{n} \sigma_1^2\right)$$

$$= \frac{\sigma^2(n-1+k^*)}{n^2} .$$

Thus $MSE_{het}(\hat{\mu}) = Var_{het}(\hat{\mu}) + \{Bias_{het}(\hat{\mu})\}^2$

$$= \frac{\sigma^2(n-1+k^*)}{n^2}$$

which tends to infinity as $k^* \to \infty$. Thus $\hat{\mu}$ appears to be a poor estimator of μ.

Since in this instance the spurious observation tends to appear at $X_{(1)}$ or at $X_{(n)}$, an estimator of μ based on the A-rule, W-rule, or S-rule (see Appendix I) would appear preferable to $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$. Similar estimators for σ^2 appear preferable to $s^2_{\ln x}$.

CHAPTER V

The Weibull Distribution

We shall show that for i.i.d.r.v.'s the family of Weibull distributions, indexed by the shape parameter η, is outlier-prone completely on the right. On the other hand, we shall show that the exchangeable model based on the Weibull family of distributions indexed by the shape parameter is outlier-resistant completely on the right. For this exchangeable model, we shall determine which observation is most likely to be the spurious one and find asymptotic limits for these probabilities as heterogeneity increases. Also we shall examine the problem of estimation in the presence of outliers.

5.1 Characteristics of the Weibull Distribution

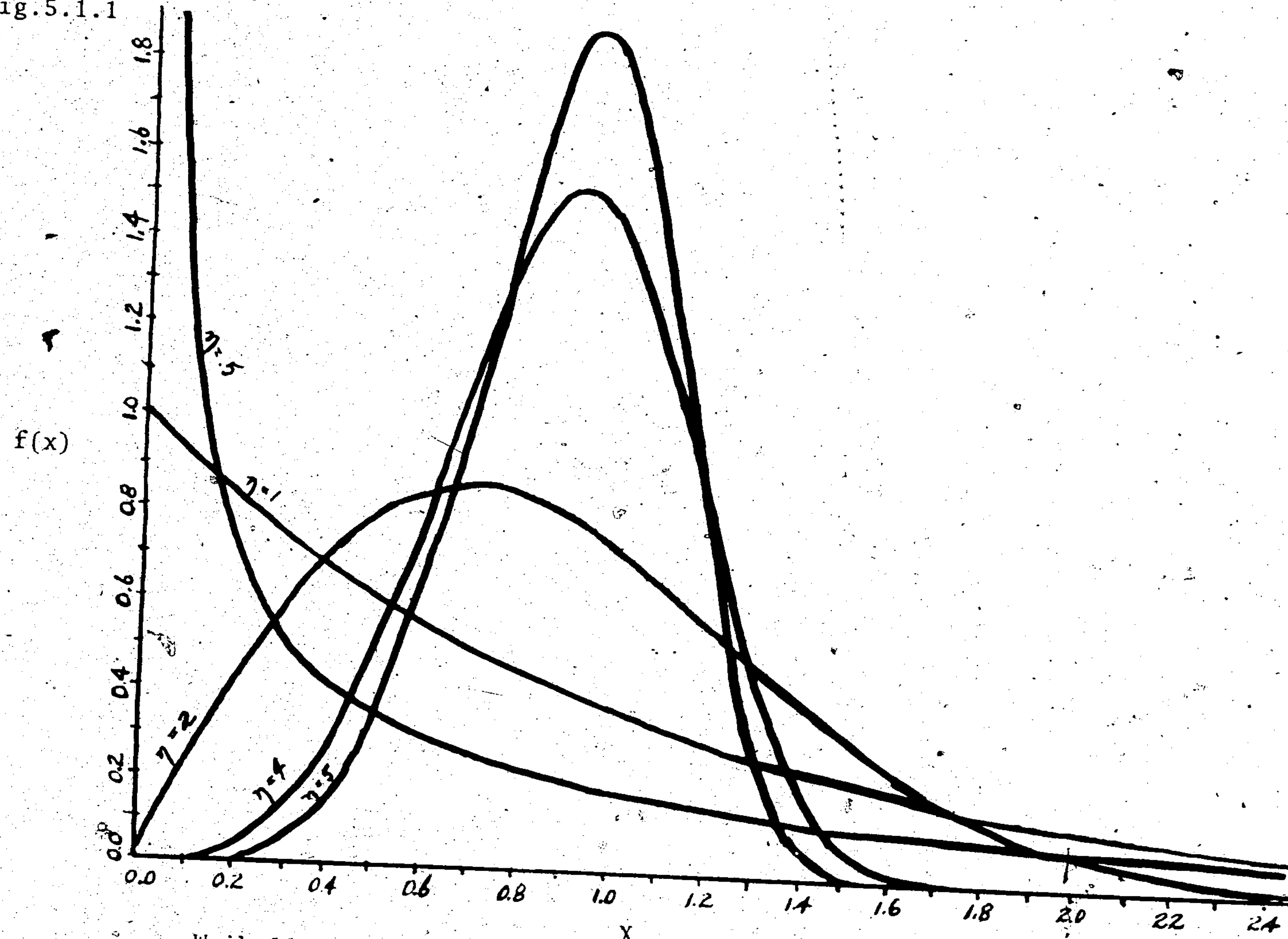
The random variable X is said to have the three-parameter Weibull distribution if the probability density function of X is

$$f(x;\theta,\eta,\tau) = \begin{cases} \frac{\eta}{\theta}\left(\frac{x-\tau}{\theta}\right)^{\eta-1} \exp -\left(\frac{x-\tau}{\theta}\right)^{\eta}, & x > \tau,\ \eta > 0,\ \theta > 0,\ \tau \geq 0 \\ 0 & , \text{otherwise}, \end{cases}$$

giving a d.f. of $F(x;\theta,\eta,\tau) = 1 - \exp\left\{-\left(\frac{x-\tau}{\theta}\right)^{\eta}\right\}$, and hazard rate (intensity) of $h(x;\theta,\eta,\tau) = \frac{\eta(x-\tau)^{\eta-1}}{\theta^{\eta}}$. In this case we write $X \sim WEI(\theta,\eta,\tau)$.

Figure 5.1.1 illustrates the shape of the probability density function $f(x;1,\eta,0)$ for various values of η.

Fig.5.1.1



Weibull p.d.f. for various values of η (θ =1, τ =0)

The Weibull distribution has many applications in life-testing or in problems where a skewed distribution is required. It may be considered to be a generalization of the gamma distribution to allow the hazard rate to depend on a power of X. In contrast to the exponential distribution with constant hazard (failure) rate $h(x;\theta) = \frac{1}{\theta}$ (i.e. $\eta = 1$), the Weibull distribution allows decreasing hazard rates (i.e. $\eta < 1$ implies work-hardened materials) or increasing hazard rates (i.e. $\eta > 1$ implies wearout). Many items, especially nonelectronic parts, exhibit increasing failure rates. Leiblein and Zelen (1956) used it to model ball-bearing failures; Kao (1959) used it to model vacuum-tube failure. Weibull (1951) derived it in the analysis of breaking strengths. Whatever the underlying distribution for positive random variables, the Weibull distribution is one of only two possible limit laws for $\min(X_1,\ldots,X_n)$ as $n \to \infty$. As such it is used to model metal fatigue breaking strength where strength is that at the weakest flaw (e.g. breaking strength of chain, ceramics, lumber, concrete, aircraft parts, etc.). In many applications the location parameter τ is known and without loss of generality we may take $\tau = 0$.

There exists a relationship between $WEI(\theta,\eta,0)$ and the Extreme-Value distribution $EV_I(\xi,b)$ given by $F(y) = 1-\exp\{-\frac{\exp(y-\xi)}{b}\}$, $-\infty < y < \infty$, $-\infty < \xi < \infty$, $b > 0$. If $X \sim WEI(\theta,\eta,0)$ then

$Y = \ln X \sim EV_I(\xi = \ln \theta,\ b = 1/\eta)$. Thus the Weibull distribution competing with the lognormal distribution is similar to the EV_I distribution competing with the normal distribution. For $Y \sim EV_I(\xi,b)$, $E(Y) = \xi - b\gamma$ and $Var(Y) = \frac{\pi^2 b^2}{6}$ where $\gamma = .5772$. (Euler's constant).

For known shape parameter η, we may use the transformation $Y = X^\eta$ to obtain $Y \sim EXP(\theta^\eta) = GAM(\theta^\eta,1,0)$. Outliers here may be handled using methods for the single parameter exponential family (see Chapter II).

5.2. Outlier-proneness of the Weibull and related models

First, let us restrict ourselves to the standard subfamily indexed only by shape parameter η. The probability density function is

$$f(x;\eta) = \begin{cases} \eta x^{\eta-1} e^{-x^{\eta}} , & x > 0 , \quad \eta > 0 \\ 0 , & \text{otherwise} \end{cases} .$$

The corresponding distribution function (d.f.) will be denoted as $F(x;\eta)$ and the family of such distribution functions as $\mathcal{F}$. In this model we consider i.i.d. random variables $X_1,\ldots,X_n$ with order statistics $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$.

Theorem 5.2.1: For i.i.d. random variables the family of Weibull distributions is outlier-prone on the right.

Proof; In order to prove Theorem 5.2.1 we notice that

$$P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)} - X_{(1)}) \cap X_{(1)} > 0\}$$

$$= P\{X_{(1)} > 0\} \cdot P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)} - X_{(1)}) | X_{(1)} > 0\}$$

$$= P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)} - X_{(1)})\} P\{X_{(1)} > 0 | X_{(n)} > X_{(n-1)} + k(X_{(n-1)} - X_{(1)})\} .$$

Now $P\{X_{(1)} > 0\} = 1$ and

$$P\{X_{(1)} > 0 | X_{(n)} > X_{(n-1)} + k(X_{(n-1)} - X_{(1)})\} = 1$$

so that, following Neyman and Scott (1971),

$$P(k,n|\eta) = P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} = P\{X_{(n)} > X_{(n-1)}$$

$$+ k(X_{(n-1)}-X_{(1)}) \mid X_{(1)} > 0\}$$

$$> P\{X_{(n)} > (k+1)X_{(n-1)}\} = Q(k,n|\eta) .$$

Thus it is sufficient to prove the stronger assertion that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$.

$$Q(k,n|\eta) = n\int_0^\infty F^{n-1}\left(\frac{x}{k+1};\eta\right)f(x;\eta)dx$$

$$= n\int_0^\infty \left\{1 - e^{-\left(\frac{x}{k+1}\right)^\eta}\right\}^{n-1} \eta x^{\eta-1}e^{-x^\eta}\,dx$$

Setting $u = 1 - e^{-\left(\frac{x}{k+1}\right)^\eta}$ we obtain

$$Q(k,n|\eta) = n(k+1)^\eta\int_0^1 u^{n-1}(1-u)^{(k+1)^\eta-1}du$$

$$= \frac{n\Gamma(n)\Gamma\{(k+n)^\eta\}(k+1)^\eta}{\Gamma\{n+(k+1)^\eta\}}$$

It then follows that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$ and thus $P(k,n|\eta) \to 1$ and $\Pi_1(k,n|\mathscr{F}) = 1$. Since this result holds for all $k > 0$, $n > 2$, the family of Weibull distributions is outlier-prone on the right.

Consider now the exchangeable model with (at most) one outlier observation. Kale (1975b) has shown that the exchangeable model based on scale parameter families for non-negative random variables with a possible change in scale is outlier-prone completely on the right. This would apply to the exchangeable model based on the Weibull distribution with possible change in the scale parameter θ. Let us consider the other possibility - the exchangeable model involving the Weibull distribution with possible change in the shape parameter η. We shall show that this model is outlier-resistant completely on the right.

Consider the situation where $n-1$ observations are from $WEI(1,\eta,0)$ and one observation is from $WEI(1,k^*\eta,0)$, $0 < k^* \leq 1$ and, a priori, each observation is equally likely to be the spurious one. Then $f(x;\underline{\theta})$ is $WEI(1,\eta,0)$ and $f(x;\underline{\xi})$ is $WEI(1,k^*\eta,0)$ and the likelihood may be written as

$$L(\underline{x};\eta,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\underline{\theta})f(x_r;\underline{\xi}),\ x_i > 0,\ 0 < k^* \leq 1$$

($k^* \leq 1$ since we are considering $X_{(1)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\ldots,x_{(n)}) = \frac{1}{n}\cdot n!\sum_{r=1}^{n}\frac{f(x_r;\eta k^*)}{f(x_r;\eta)}\prod_{i=1}^{n} f(x_i;\eta)$$

and

$g(x_{(1)}, x_{(n-1)}, x_{(n)})$

$$= \int_{x_{(1)}}^{x_{(n-1)}} \int_{x_{(1)}}^{x_{(n-2)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) dx_{(2)} \ldots dx_{(n-2)}$$

$$= (n-1)! \sum_{r=1}^{n} h_r(x_{(1)}, x_{(n-1)}, x_{(n)})$$

where

$$h_r(x_{(1)}, x_{(n-1)}, x_{(n)}) = \int_{S_{n-3}} \frac{f(x_r; \eta_1)}{f(x_r; \eta)} \prod_{i=1}^{n} f(x_i; \eta) dx_{(2)} \ldots dx_{(n-2)}$$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \ldots < x_{(n-2)} < x_{(n-1)}$.
For $r = 2, \ldots, n-2$,

$$h_r(x_{(1)}, x_{(n-1)}, x_{(n)}) = \frac{f(x_{(1)}; \eta) f(x_{(n-1)}; \eta) f(x_{(n)}; \eta)}{(r-2)!(n-r-2)!}$$

$$\int_{x_{(1)}}^{x_{(n-1)}} \{F(x_{(r)}; \eta) - F(x_{(1)}; \eta)\}^{r-2} \{F(x_{(n-1)}; \eta) - F(x_{(r)}; \eta)\}^{n-r-2}$$

$$f(x_{(r)}; \eta_1) dx_{(r)}$$

while

$h_1(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta_1)f(x_{(n-1)};\eta)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta_1)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_n(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta)f(x_{(n)};\eta_1)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

Letting $t(y) = f(y;\eta)$ and $s(y) = f(y;\eta_1)$ we obtain again expression (4.2.2) and, setting

$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and $x = x_{(1)}$ and $y = x_{(n-1)}$, from Theorem 4.2.1

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3} t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t are, respectively, the distribution function and probability density function of the non-spurious observations.

Theorem 5.2.2: The exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3} t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation.

If, for all $k > 0$, $n > 2$ sup $p < 1$ then the family is outlier-resistant completely on the right. It is then sufficient to show that

$$(5.2.1)\quad J = k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Now

$$t(y) = \begin{cases} \eta y^{\eta-1}e^{-y^\eta}, & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$s(x) = \begin{cases} \eta_1 x^{\eta_1-1}e^{-x^{\eta_1}}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$T(b) = \begin{cases} 1-e^{-b^\eta}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$S(b) = \begin{cases} 1-e^{-b^{\eta_1}}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then the left hand side of expression (5.2.1) becomes

$$(5.2.2)\quad J = k(n-1)(n-2)\int_0^\infty \eta y^{\eta-1}e^{-y^\eta}\int_0^y (1-e^{-x^{\eta_1}})\{(1-e^{-y^\eta})-(1-e^{-x^\eta})\}^{n-3}$$

(continued)

$$\times\ \eta\{(k+1)y-kx\}^{\eta-1}e^{-[(k+1)y-kx]^{\eta}}dxdy.$$

$$J \geq k(n-1)(n-2)\int_3^5 \eta y^{\eta-1}e^{-y^{\eta}}\int_1^2 (1-e^{-x^{\eta_1}})\{(1-e^{-y^{\eta}})-(1-e^{-x^{\eta}})\}^{n-3}$$

$$\eta\{(k+1)y-kx\}^{\eta-1}e^{-[(k+1)y-kx]^{\eta}}dxdy.$$

Now for $\eta > 0,\ k > 0,\ \eta_1 > 0,\ 3 \leq y \leq 5,\ 1 \leq x \leq 2$

$$y^{\eta-1} = \frac{y^{\eta}}{y} \geq \frac{3^{\eta}}{5} > 0$$

$$e^{-y^{\eta}} \geq e^{-5^{\eta}} > 0$$

$$(1-e^{-x^{\eta_1}}) \geq (1-e^{-1^{\eta_1}}) = 1 - \frac{1}{e} > 0 \text{ independently of } \eta_1 > 0.$$

$$\{(1-e^{-y^{\eta}}) - (1-e^{-x^{\eta}})\}^{n-3} = \{e^{-x^{\eta}} - e^{-y^{\eta}}\}^{n-3} \geq \{e^{-2^{\eta}} - e^{-3^{\eta}}\}^{n-3} > 0$$

$$\{(k+1)y-kx\}^{\eta-1} = \frac{\{(k+1)y-kx\}^{\eta}}{\{(k+1)y-kx\}} \geq \frac{\{(k+1)3-2k\}^{\eta}}{\{(k+1)5-k\}} = \frac{(k+3)^{\eta}}{\{4k+5\}} > 0$$

$$e^{-[(k+1)y-kx]^{\eta}} \geq e^{-[(k+1)5-k]^{\eta}} = e^{-[4k+5]^{\eta}} > 0 .$$

Therefore

$$J > k\eta^2(n-1)(n-2)\,\frac{3^{\eta}}{5}\,e^{-5^{\eta}}(1-\frac{1}{e})\{e^{-2^{\eta}}-e^{-3^{\eta}}\}^{n-3}\,\frac{(k+3)^{\eta}}{4k+5}$$

$$\times\ e^{-[4k+5]^{\eta}}2$$

> 0 for all $k > 0,\ n > 2\ ,\ \eta,\ \eta_1 > 0.$

Thus the exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

5.3 Detection of Outliers

Consider a situation where n-1 observations are distributed as WEI$(1,\eta,0)$ and one is distributed as WEI$(1,k^*\eta,0)$, $0 < k \leq 1$ and we have the exchangeable model. If $k^* < 1$ we have exactly one spurious observation. Then $\Psi(x) = \frac{dG}{dF} = k^* x^{k^*\eta-\eta} e^{-(x^{k^*\eta}-x^{\eta})}$ is not monotone. Letting $u(r;n,\eta,k^*) = P(X_{(r)}$ is the spurious observation), we have

$$u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_0^\infty (1-e^{-x^\eta})^{r-1}(e^{-x^\eta})^{n-r} k^*\eta x^{k^*\eta-1} e^{-x^{k^*\eta}} dx$$

and, setting $y = e^{-x^{k^*\eta}}$, we obtain

$$u(r;n,k^*) = \binom{n-1}{r-1}\int_0^1 \left(1-e^{-(\ln 1/y)^{1/k^*}}\right)^{r-1}\left(e^{-(\ln 1/y)^{1/k^*}}\right)^{n-r} dy$$

which is independent of η. Therefore, without loss of generality (w.l.o.g.), we may take $\eta = 1/k^*$, giving

$$u(r;n,k^*) = \binom{n-1}{r-1}\int_0^\infty (1-e^{-x^{1/k^*}})^{r-1}(e^{-x^{1/k^*}})^{n-r} e^{-x} dx$$

and in particular

$$u(1;n,k^*) = \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \frac{1}{(n-1)^{k^*}} \int_0^\infty e^{-v^{1/k^*}} e^{-v/(n-1)^{k^*}} dv$$

$$= \frac{1}{(n-1)^{k^*}} L\left(u = 1/k^*,\ p = \frac{1}{(n-1)^{k^*}}\right)$$

where $L(u,p) = \int_0^\infty e^{-v^k} e^{-pv} dv$ is the Laplace transform of e^{-v^u} (Re $u > 0$).

Now $u(1;n,k^*) = \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$, $n > 1$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(n-1)x^{1/k^*}} dx$$

and setting $t = (n-1)x^{1/k^*}$ we obtain

$$u(1;n,k^*) = \sum_{m=0}^\infty \frac{(-1)^m}{m!} \int_{t=0}^\infty \frac{e^{-t} k^* t^{k^*(m+1)-1}}{(n-1)^{k^*(m+1)}} dt$$

$$= \sum_{m=0}^\infty \frac{(-1)^m k^* \Gamma\{k^*(m+1)\}}{m!(n-1)^{k^*(m+1)}}, \quad n > 1$$

$$(5.3.1) \qquad = \sum_{m=0}^\infty \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}}, \quad n > 1$$

For computational accuracy to the third decimal place, we would require q terms where q is chosen such that $\frac{\Gamma\{k^*(q+1)+1\}}{(q+1)!(n-1)^{k^*(q+1)}} \leq 0.0005$. (Apostol 1964). For fixed n, as $k^* \to \infty$ (heterogeneity increases) q increases; as $k^* \to 1$ (heterogeneity decreases) q decreases. Table 5.3.1 shows q values for some selected (n,k^*) values.

Table 5.3.1 Number of terms required to calculate $u(1;n,k^*)$ accurate to third decimal

		Sample size n			
		5	10	20	50
Coefficient	1/4	5	5	4	4
of spuriosity	1/2	5	4	3	3
k^*	3/4	5	3	3	2
	1	5	3	2	1

We can also develop a recursive formula for $u(r;n,k^*)$ in terms of $u(1;j,k^*)$, $j \leq n$. From equation 5.3.1 we have

$$u(1;n,k^*) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

and, for the case of $n = 1$, we may define $u(1;1,k^*) = 1$. For the special case of $r = n$ we may write

$$u(n;n,k^*) = \int_0^{\infty} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \quad ,$$

$$= \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j (e^{-x^{1/k*}})^j e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k*}} e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-jx^{1/k*}} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{k*\Gamma\{k*(m+1)\}}{j^{k*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \frac{\Gamma\{k*(m+1)+1\}}{j^{k*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k*)$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k*)$$ where $u(1;1,k*) = 1$ by definition.

For $r = 2,3,\ldots,n-1$

$$u(r;n,k*) = \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \int_0^\infty (e^{-x^{1/k*}})^{n-r+j} e^{-x}\, dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-x^{1/k*}(n-r+j)} dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left[\sum_{n=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-r+j)^{k^*(m+1)}} \right]$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) \ .$$

Thus we may write

$$u(r;n,k^*) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} & \text{for } r = 1,\ n \geq 2 \\ 1 \quad \text{if} \quad r = 1,\ n = 1 & \\ \displaystyle\binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) & \text{for } r = 2,\ldots,n \\ 0\ . \qquad \text{otherwise} & \end{cases}$$

(See Appendix IV). Consider now

$$\lim_{k^* \to 1} u(r;n,k^*) = \begin{cases} \displaystyle\int_0^{\infty} \lim_{k^* \to 1} \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx\ , & r = 1 \\ \displaystyle\int_0^{\infty} \lim_{k^* \to 1} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx\ , & r = n \\ \displaystyle\binom{n-1}{r-1} \int_0^{\infty} \lim_{k^* \to 1} \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x} dx\ , & r = 2,\ldots,n-1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

Thus we find that

$$\lim_{k^*\to 1} u(r;n,k^*) = \begin{cases} \int_0^\infty e^{-xn}dx & , r = 1 \\ \int_0^\infty (1-e^{-x})^{n-1}e^{-x}dx & , r = n \\ \binom{n-1}{r-1}\int_0^\infty (1-e^{-x})^{r-1}e^{-x(n-r+1)}dx & , r = 2,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/n & , r = 1 \\ 1/n & , r = n \\ \binom{n-1}{r-1}\dfrac{\Gamma(r)\Gamma(n-r+1)}{\Gamma(n+1)} = 1/n & , r = 2,3,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}$$

The interpretation of this result is that, as $k^* \to 1$ ($k^* = 1$ is the case of homogeneous data), the heterogeneity of the data is decreasing and the probability of any order statistic being spurious approaches $1/n$, i.e. all equally likely.

Now consider $\lim_{k^*\to 0} u(r;n,k^*)$. We have

$$\lim_{k^*\to 0} u(r;n,k^*) = \begin{cases} \int_0^\infty \lim_{k^*\to 0}\left(e^{-x^{1/k^*}}\right)^{n-1}e^{-x}dx & , r = 1 \\ \int_0^\infty \lim_{k^*\to 0}\left(1-e^{-x^{1/k^*}}\right)^{n-1}e^{-x}dx & , r = n \\ \binom{n-1}{r-1}\int_0^\infty \lim_{k^*\to 0}\left(1-e^{-x^{1/k^*}}\right)^{r-1}\left(e^{-x^{1/k^*}}\right)^{n-r}e^{-x}dx & , \\ & r = 2,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\lim_{k^*\to 0} u(1;n,k^*) = \lim_{k^*\to 0} \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m \left(e^{-x^{1/k^*}}\right)^{n-1} dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m! \left|\frac{1}{k^*}\right| (n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m k^*(m+1)\Gamma\{k^*(m+1)\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$\doteq \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} = \frac{(-1)^0}{1!} + \frac{(-1)^1}{2!} + \frac{(-1)^3}{3!} + \ldots$$

$$= 1 - \frac{1}{e} \doteq .63 \; .$$

For $r = n$

$$\lim_{k^*\to 0} u(n;n,k^*) = \lim_{k^*\to 0} \int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \lim_{k^*\to 0} \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \left(e^{-x^{1/k^*}}\right)^j e^{-x} dx \; ,$$

$$= \lim_{k^* \to 0} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx$$

$$= \left\{ \lim_{k^* \to 0} \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx \right\} + \int_0^\infty e^{-x} dx$$

$$= \left\{ \lim_{k^* \to 0} \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!\, j^{k^*(m+1)}} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1)}{(m+1)!} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \left(1 - \frac{1}{e}\right) \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right) \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right) \left\{ \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j - \binom{n-1}{0} (-1)^0 \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right)(-1) + 1$$

$$= \frac{1}{e} \doteq .37 \quad .$$

For $r = 2, 3, \ldots, n-1$,

$$\lim_{k^*\to 0} u(r;n,k^*) = \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x} dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left(e^{-x^{1/k^*}}\right)^{j+n-r} e^{-x} dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j$$

$$\left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(j+n-r)x^{1/k^*}} dx \right\}$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j$$

$$\left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m! \left|\frac{1}{k^*}\right| (j+n-r)^{k^*(m+1)}} \right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \lim_{k^*\to 0} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(j+n-r)^{k^*(m+1)}} \right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \right\}$$

$$= \binom{n-1}{r-1} \left(1-\frac{1}{e}\right) \left\{ \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \right\}$$

$$= \binom{n-1}{r-1} \left(1-\frac{1}{e}\right) (-1+1)$$

$$= 0 .$$

Thus, as $k^* \to 0$ (i.e. one observation with Weibull shape parameter much smaller than the rest), we have

$$\lim_{k^* \to 0} u(r;n,k^*) = \begin{cases} 1-\frac{1}{e} & , \quad r=1 \\ \frac{1}{e} & , \quad r=n \\ 0 & , \quad \text{otherwise} \end{cases}$$

In this situation, heterogeneity is more evident and the probability that the first order statistic is spurious approaches $1 - \frac{1}{e} = .63$, the probability that the last order statistic is spurious approaches $\frac{1}{e} = .37$, and the probability that any other statistic is spurious approaches zero.

5.4 Estimation for Weibull parameters

5.4.1 Standard Estimators

Complete sufficient statistics do not exist for the Weibull distribution. Much of the estimation for the Weibull distribution falls into one of three categories:

i) For M.L.E's, the distributional results are not mathematically tractable hence Monte Carlo simulation is needed to tabulate percentage points. While the M.L.E.'s have optimality properties as n increases, they are not in closed form and require computer solution to obtain

$$\hat{\eta} = \left\{ \frac{\sum_{i=1}^{n} X_i^{\eta} \ln x_i}{\sum_{i=1}^{n} X_i^{\eta}} - \frac{1}{n} \sum_{i=1}^{n} \ln X_i \right\}^{-1} \quad \text{and} \quad \hat{\theta}^{\hat{\eta}} = \frac{\sum_{i=1}^{n} X_i^{\hat{\eta}}}{n} .$$

For $n \geq 100$, $\hat{\eta}$ is asymptotically $N\left(\eta, \frac{.608\eta^2}{n}\right)$ [see Cohen (1965) and Harter and Moore (1965)]. Inference procedures based on M.L.E.'s depend on pivotal quantities $\frac{\hat{\eta}}{\eta}$ and $\hat{\eta} \ln\left(\frac{\hat{\theta}}{\theta}\right)$ [see Thoman, Bain and Antle (1970)].

ii) Best Linear Estimators have properties similar to M.L.E.'s. Mann and Fertig (1973) considered BLIE for the parameter b of EV_I. The BLIE $\tilde{b}_{BLIE} = \sum_{i=1}^{r} c_{i,r,n} X_{(i)}$ is a linear function of the BLUE and a maximal invariant since $\sum_{i=1}^{r} c_{i,r,n} = 0$. (r is the size of the censored sample). The distribution of $\frac{\tilde{b}_{BLIE}}{b}$ is independent of b and ξ. Mann and Fertig (1977) also took Hassanein's (1972) asymptotically unbiased optimum estimators $\hat{b}_k = \sum_{i=1}^{k} b_{i,k} X_{(n_i)}$, $n_i = [\gamma_{i,k} n] + 1$ and obtained, for small samples, the unbiased

$$b^* = \frac{\hat{b}_k}{\bar{b}_{k,n}}$$

where $\bar{b}_{k,n} = E[\sum_{i=1}^{k} b_{i,k} Z_{(n_i)}]$, where $Z_i \sim EV_I(0,1)$ and $\frac{2b^*}{cb} \overset{\bullet}{\sim} \chi^2_{2/c}$ df and the BLIE $b^*_{BLIE} = \frac{b^*}{1+c_{k,n}}$ where $c_{k,n} = Var(\frac{b^*}{b})$.

iii) Simple estimators, such as good linear unbiased estimators (GLUE's) and modified GLUE's, are identical to BLUE's if $r = 2$ and similar but not identical if $r > 2$, where r is the size of the censored sample. They are essentially equivalent to M.L.E.'s for censored sampling but are slightly less efficient for the complete sample case. Bain (1973) suggested using the approximate BLUE

$$\hat{b}_B = \frac{\sum_{i=1}^{r} |y_{(i)} - y_{(r)}|}{nk_{r,n}} = \frac{(r-1)y_{(r)} - \sum_{i=1}^{r-1} y_{(i)}}{nk_{r,n}}$$

where $y_i \sim EV_I(\xi, b)$ and $k_{r,n} = -\frac{1}{n} E\{\sum_{i=1}^{n-1} (w_i - w_r)\}$ where w_i are the order statistics of $EV_I(0,1)$. Then $2nk_{r,n}\frac{\hat{b}_B}{b} \sim \chi^2_{2nk_{r,n}}$ df (if $r/n < 1/2$) and $\hat{\eta}_B = \frac{nk_{r,n}-1}{nk_{r,n}\hat{b}_B}$ is approximately unbiased for η. For complete samples (i.e. $r = n$) , $\hat{b}_B$ has zero asymptotic relative efficiency.

Englehardt and Bain (1973) suggested $\hat{b}_s = \frac{\sum_{i=1}^{r} |y_{(i)}-y_{(s)}|}{nk_{s,r,n}}$ as an improvement where

$$s = \begin{cases} n & 2 \leq n \leq 15 \\ n-1 & 16 \leq n \leq 24 \\ [.892\, n] + 1\,, & n \geq 25 \\ r\,, & r \leq .9n \end{cases} \quad \text{and} \quad r > .9n \text{ (first three cases)}$$

and $h\frac{\hat{b}_s}{b} \overset{\bullet}{\sim} \chi^2_h$ df where $h = \frac{2}{\mathrm{Var}\left(\frac{\hat{b}_s}{b}\right)}$.

For $\frac{r}{n} \to 0$ (very heavy censoring) as $n \to \infty$, $\hat{b}_B$ and M.L.E. $\hat{b}$ appear to agree. Compared to $\hat{b}_B$, $\hat{b}_s$ has relative efficiency ranging from 1 (n=2) to 0.82 (n=25).

Mann and Fertig (1975) converted Bain's estimator to an approximate BLIE. For $n \geq 20$, $\frac{2\frac{\hat{b}_s}{b}}{\ell_{r,n}} \overset{\bullet}{\sim} \chi^2_{2/\ell_{r,n}}$ df where

$\hat{b}_s = \frac{\sum_{i=1}^{r} |y_{(s)}-y_{(i)}|}{nk_{s,r,n}}$ and approximate BLIE $\tilde{b}_{MF} = \frac{\hat{b}_s}{1+\ell_{r,n}}$, where

$\hat{\ell}_{r,n} = \mathrm{Var}\left(\frac{\hat{b}_s}{b}\right) = \frac{1}{nk_{s,r,n}}$. Then $\mathrm{MSE}(\tilde{b}_{MF}) = \frac{\ell_{r,n} b^2}{1+\ell_{r,n}}$. For heavy censoring ($r \ll n$), $\tilde{b}_{MF}$ is closer to $\hat{b}_{MLE}$ then $\hat{b}_s$.

In terms of the Weibull shape parameter η, $a\hat{\eta} = a\left(\frac{1}{\tilde{b}_{MF}}\right)$ is unbiased for η if $a = \frac{h-2}{h+2}$ and $d\hat{\eta} = d\left(\frac{1}{\tilde{b}_{MF}}\right)$ has minimum MSE if $d = \frac{h-4}{h-2}$ where $\frac{h+2}{h} = 1 + \ell_{r,n}$ (i.e. $h = \frac{2}{\mathrm{Var}(\hat{b}_s/b)}$).

Englehardt and Bain (1977a) proposed, as a new simple unbiased estimator for complete samples to replace $\hat{b}_s$,

$$\hat{b}_{EB} = \frac{\left(-\sum_{i=1}^{s} y_{(i)} + \frac{s}{n-s} \sum_{i=s+1}^{n} y_{(i)}\right)}{nk_n}$$

where $s = [qn]$, $0 < q < 1$ is chosen to minimize the asymptotic variance of $\hat{b}_{EB}$. This results in

$$s = [.84n]$$

and

$$k_n = E\left[-\sum_{i=1}^{s} w_{(i)} + \frac{s}{n-s} \sum_{i=s+1}^{n} w_{(i)}\right]/n \ , \quad w_i \sim EV(0,1)$$

$$= -\left[E \sum_{i=1}^{s} w_{(i)} + s\gamma\right]/(n-s)$$

and γ is Euler's constant. Then

$$n \, \mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right) = \frac{V_s + \left(\frac{s}{n-s}\right)U_{s+1} - \frac{s\pi^2}{6}}{(n-s)k_n^2}$$

where $V_s = \mathrm{Var}\left(\sum_{i=1}^{s} w_{(i)}\right)$, $U_{s+1} = \mathrm{Var}\left(\sum_{i=s+1}^{n} w_{(i)}\right)$, $\mathrm{Var}\left(\sum_{i=1}^{n} w_{(i)}\right) = \frac{n\pi^2}{6}$.

To obtain an approximate BLIE (or MLE) we may use $b^*_{EB} = \dfrac{\hat{b}_{EB}}{1+\mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right)}$.

Three other simple estimators have been suggested. Menon (1963) used the transformation $Z = \ln\left(\frac{X}{\theta}\right)^{\eta}$ where $X \sim \mathrm{WEI}(\theta,\eta,0)$ and $\ln X \sim EV_I(\xi,b)$. He suggested $\hat{b}_{MN} = \left(\frac{6s^2_{\ln x}}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{MN} = \widehat{\ln \theta}$ $= \dfrac{\sum_{i=1}^{n} \ln X_i}{n} + \gamma\hat{b}_M$. Then $\hat{\eta}_{MN} = \frac{1}{\hat{b}_M} \overset{\bullet}{\sim} N\left(\eta, \frac{1.1\eta^2}{n}\right)$ and $\hat{\theta}_{MN} = e^{\hat{\xi}_M}$ $\sim N\left(\theta, \frac{1.2b^2\theta^2}{n}\right)$. Dubey (1967) suggested estimating η by using percentiles. Based upon two percentiles,

$$\hat{\eta}_D = \frac{\ln\{-\ln(1-p_1)\} - \ln\{-\ln(1-p_2)\}}{\ln y_{p_1} - \ln y_{p_2}}$$

where $p_1 = 17\%$ and $p_2 = 97\%$ minimize $\mathrm{Var}(\hat{\eta}_D)$ and

$$y_p = \begin{cases} x_{(np)} & \text{if } np \text{ is an integer} \\ x_{[np]+1} & \text{if } np \text{ is not an integer.} \end{cases}$$

Asymptotically $\hat{\eta}_D \stackrel{\bullet}{\sim} N\left(\eta, \frac{\eta^2(.91627479)}{n}\right)$.

Murthy and Swartz (1975) used an approach similar to Dubey but used the EV_I distribution and obtained an estimator of b based upon two order statistics

$$\hat{b}_{MS} = \{\ell n\ T_{(j)} - \ell_n\ T_{(\ell)}\}\ B(N,\ell,j)$$

where $B(N,\ell,j) = \frac{1}{2E(Y)}$ and $Y = \frac{\ell n\ T_{(j)} - \ell n\ T_{(\ell)}}{2b}$. This is MVUE and has relative efficiency w.r.t. the Cramer-Rao lower bound (CRLB) approaching 70%. The following Table 5.4.1 gives the optimum values for j and ℓ.

Table 5.4.1

Optimal Order Statistics for Murthy-Swartz Estimator of $b = \frac{1}{\eta}$

n	j	ℓ
$2 \leq n \leq 5$	n	1
$6 \leq n \leq 10$	n	2
$11 \leq n \leq 16$	n	3
$17 \leq n \leq 23$	n	4
$24 \leq n \leq 26$	n	5

5.4.2 Estimators suggested for use.

Under the exchangeable model with a shape change, we assume n-1 observations are governed by $f(x;\theta,\eta) = \frac{\eta x^{\eta-1}}{\theta^{\eta}} \exp\{-(\frac{x}{\theta})^{\eta}\}$, $x > 0$, $\eta, \theta > 0$ and one observation is governed by $f(x;\theta,k^*\eta)$, $0 < k^* \leq 1$. If $0 < k^* < 1$, we have exactly one spurious observation. We are interested in estimating η (or in the EV_I case, $b = \frac{1}{\eta}$).

From Table 5.4.1, for $2 \leq n \leq 26$, the optimal form of the Murthy-Swartz estimator uses the largest order statistic. Dubey's estimator involves one or both of the smallest and largest order statistics for $2 \leq n \leq 33$. Also $\hat{b}_s$, $2 \leq n \leq 15$, $\hat{b}_B$ and $\hat{b}_{EB}$ all involve these two highly suspect values. Menon's estimator

$$\hat{b}_{MN} = \sqrt{\frac{6s^2_{\ell n\ x}}{\pi^2}}$$ has

$$\begin{aligned} MSE(\hat{b}_{MN}) &= Var(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2 \\ &= \frac{1.1\, b^2}{n} + b^2 O(\frac{1}{n^2}) + \{bO(\frac{1}{n})\}^2, \end{aligned}$$

under the homogeneous model. However, with the above exchangeable model,

$$\begin{aligned} E_{het}(\hat{b}_{MN}) &= \frac{\sqrt{6}}{\pi} \{\sqrt{Var_{\ell n\ x}} + bO(\frac{1}{n})\} \text{ [see Cramer (1946), 27.7.1]} \\ &= \frac{\sqrt{6}}{\pi} \{\sqrt{\frac{\pi^2}{6} b^2 (\frac{n-1}{n}) + \frac{\pi^2}{6}(\frac{b}{k^*})^2 \frac{1}{n}} + bO(\frac{1}{n})\} \end{aligned}$$

$$= b\sqrt{\frac{n-(1-\frac{1}{k^{*2}})}{n}} + bO\left(\frac{1}{n}\right)$$

and as $k^* \to 0$, bias $\to \infty$.

Also $MSE_{het}(\hat{b}_{MN}) = Var_{het}(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2$ and

$$Var_{het}(\hat{b}_{MN}) = \frac{6}{\pi^2}\left\{\frac{\mu_4-\mu_2^2}{4n\mu_2} + O\left(\frac{1}{n^2}\right)\right\} \quad \text{[see Cramer (1946), 27.7.2]}$$

$$= \frac{6}{\pi^2}\left[\frac{3b^2}{2\pi^2}\left\{\frac{\pi^4}{15}\left(\frac{n-1+\frac{1}{k^{*4}}}{n}\right) + \frac{4(2.404)\gamma(n-1+\frac{1}{k^{*3}})(n-1+\frac{1}{k^*})}{n^2}\right.\right.$$

$$\left.\left. + \frac{\pi^2\gamma^2(n-1+\frac{1}{k^{*2}})(n-1+\frac{1}{k^*})^2}{n^3} - \frac{3\gamma^4(n-1+\frac{1}{k^*})^4}{n^4} - \frac{\pi^4(n-1+\frac{1}{k^*})^2}{36n^2}\right\} + b^2 O\left(\frac{1}{n^2}\right)\right]$$

(see Appendix V).

Taking $\lim_{k^*\to 0} MSE_{het}(\hat{b}_{MN})$, we find $MSE(\hat{b}_{MN}) \to \infty$ as $k^* \to 0$. Thus this estimation technique seems poor if an outlier is present. Also $s^2_{\ell n\, x}$ would be strongly affected by the presence of an outlier occurring at $X_{(1)}$ or at $X_{(n)}$ and these are precisely the values where $u(r;n,k^*)$ is largest as $k^* \to 0$. We might obtain a more robust estimator by using trimming, winsorization or semi-winsorization. Thus we might consider

i) $\hat{b}_{RA} = \left(\frac{6\hat{\sigma}_A^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RA} = \hat{\mu}_A + \gamma\hat{b}_{RA}$

ii) $\hat{b}_{RW} = \left(\frac{6\hat{\sigma}_W^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RW} = \hat{\mu}_W + \gamma\hat{b}_{RW}$

iii) $\hat{b}_{RS} = \left(\frac{6\hat{\sigma}_S^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RS} = \hat{\mu}_S + \gamma\hat{b}_{RS}$

where $Y_i = \ell n\ X_i$, $b = \frac{1}{\eta}$, $\xi = \ell n\ \theta$ and $\hat{\sigma}_A^2$, $\hat{\sigma}_W^2$, $\hat{\sigma}_S^2$, $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ are as defined in Appendix I and γ is Euler's constant.

Table 5.4.2-Computation of means and variances and winsorized means and variances for 25 samples of size five based on $EV_I(0,1/n)$ with one spurious observation present (see Appendix I)

n = 5.00000 k^* = 0.10000

Weibull (V)

0.102121E+01	0.981289E+00	0.125494E+01	0.812820E+00	0.222729E+01
0.803152E+00	0.937152E+00	0.805811E+00	0.590656E+00	0.370809E+01
0.882887E+00	0.100282E+01	0.123995E+01	0.941884E+00	0.423470E+00
0.837319E+00	0.105347E+01	0.757929E+00	0.780034E+00	0.423634E-01
0.927337E+00	0.877620E+00	0.782281E+00	0.123292E+01	0.127538E+01
0.881348E+00	0.858771E+00	0.100320E+01	0.107197E+01	0.841635E-01
0.820470E+00	0.106828E+01	0.120829E+01	0.889250E+00	0.493110E-02
0.105857E+01	0.897129E+00	0.897488E+00	0.848710E+00	0.367857E+00
0.138243E+01	0.725353E+00	0.925986E+00	0.679848E+00	0.359345E+00
0.813881E+00	0.107227E+01	0.764280E+00	0.938452E+00	0.387794E+00
0.937215E+00	0.895044E+00	0.548735E+00	0.310820E+00	0.134010E+01
0.105094E+01	0.915291E+00	0.121399E+01	0.110028E+01	0.590449E-02
0.624547E+00	0.931728E+00	0.121240E+01	0.110820E+01	0.252715E+01
0.985510E+00	0.780880E+00	0.122693E+01	0.101766E+01	0.577123E+00
0.809748E+00	0.860884E+00	0.703416E+00	0.113378E+01	0.902794E+00
0.549408E+00	0.991488E+00	0.115158E+01	0.138753E+01	0.116588E+00
0.752757E+00	0.989805E+00	0.107318E+01	0.118550E+01	0.111090E+01
0.468859E+00	0.941908E+00	0.706789E+00	0.840323E+00	0.222891E-01
0.116367E+01	0.727470E+00	0.124621E+01	0.564784E+00	0.340384E+00
0.125350E+01	0.113521E+01	0.399613E+00	0.874926E+00	0.916387E+00
0.738252E+00	0.982388E+00	0.894012E+00	0.104101E+01	0.485553E+01
0.104429E+01	0.106242E+01	0.984392E+00	0.785419E+00	0.344395E-01
0.104512E+01	0.118195E+01	0.101539E+01	0.100000E+01	0.281081E-01
0.764350E+00	0.923114E+00	0.864647E+00	0.661185E+00	0.301828E-02
0.902853E+00	0.810298E+00	0.773216E+00	0.904088E+00	0.261505E-01

Extreme Value (W)

0.209857E-01	-0.188888E-01	0.227088E+00	-0.207491E+00	0.800785E+00
-0.219212E+00	-0.649100E-01	-0.989251E-01	-0.526522E+00	0.130998E+01
-0.377159E-01	0.261685E-02	0.215074E+00	-0.598728E-01	0.859273E+00
-0.647313E-01	0.520921E-01	-0.277166E+00	-0.287636E+00	-0.316147E+01
-0.754381E-01	-0.130542E+00	-0.271840E+00	0.209382E+00	0.243242E+00
-0.413474E+00	-0.154584E+00	0.319303E-02	0.684987E-01	-0.247499E+01
-0.197878E+00	0.841838E-01	0.189202E+00	-0.108085E-01	-0.531219E+01
0.550264E-01	-0.108555E+00	-0.108156E+00	-0.164038E+00	-0.100006E+01
0.323842E+00	-0.321097E+00	-0.758960E-01	-0.385742E+00	-0.102347E+01
-0.488019E+00	0.697750E-01	-0.268847E+00	-0.888867E-01	-0.100023E+01
-0.644428E-01	-0.110882E+00	-0.598318E+00	-0.118854E+01	0.292745E+00
0.496879E-01	-0.885127E-01	0.193914E+00	0.955844E-01	-0.813204E+01
-0.470729E+00	-0.707178E-01	0.192598E+00	0.102741E+00	0.827094E+00
-0.145959E-01	-0.288510E+00	0.209514E+00	0.175057E-01	-0.589889E+00
-0.945874E-01	-0.150051E+00	-0.351808E+00	0.125958E+00	-0.102261E+00
-0.598914E+00	-0.854804E-02	0.141132E+00	0.313007E+00	-0.214930E+01
-0.284013E+00	-0.306602E-01	0.706298E-01	0.144831E+00	0.105167E+00
0.757452E+00	-0.598498E-01	-0.347052E+00	-0.173889E+00	-0.380368E+01
0.151577E+00	-0.318183E+00	0.220108E+00	-0.571348E+00	-0.107768E+01
0.225939E+00	0.126819E+00	-0.917260E+00	-0.133615E+00	-0.873161E-01
-0.302117E+00	-0.177689E-01	-0.112037E+00	0.401939E-01	0.160050E+01
0.433347E-01	0.605533E-01	-0.157307E-01	-0.267331E+00	-0.336855E+01
0.441359E-01	0.150088E+00	0.152692E-01	0.324247E-05	-0.357170E+01
-0.268730E+00	-0.800028E-01	-0.145434E+00	-0.413722E+00	-0.580307E+01
-0.102195E+00	-0.210354E+00	-0.257197E+00	-0.100829E+00	-0.364389E+01

Sample	$\bar{w}$	s^2_w	$\bar{w}_{(2,1)}$	$s^2_{(2,1)w}$	$\bar{w}_{(5,1)}$	$s^2_{(5,1)w}$	$\bar{w}_{(4,5)}$	$s^2_{(4,5)w}$	$\bar{w}_{(1,5)}$	$s^2_{(1,5)w}$
1	0.16450	0.15035	0.20222	0.12232	0.38615	0.16614	0.04976	0.03365	-0.03716	0.03269
2	0.08008	0.50581	0.14154	0.43149	0.44738	0.62335	-0.19490	0.03839	-0.28722	0.05101
3	-0.14783	0.17009	0.01205	0.01353	0.06704	0.01874	-0.19033	0.14056	-0.36270	0.20699
4	-0.74778	1.84136	-0.17302	0.02488	-0.10507	0.02849	-0.77115	1.79737	-1.39049	2.62154
5	-0.00896	0.04982	0.02322	0.03502	0.09798	0.03423	-0.01173	0.04585	-0.10790	0.03893
6	-0.59407	1.14017	-0.18177	0.05137	-0.08517	0.04206	-0.80733	1.11905	-1.10297	1.59083
7	-1.05350	5.68732	-0.03064	0.02841	0.04678	0.02601	-1.07850	5.61277	-2.15378	8.32210
8	-0.26516	0.17550	-0.08795	0.00809	-0.05414	0.01045	-0.29779	0.15470	-0.47817	0.22923
9	-0.29867	0.24253	-0.16913	0.09212	-0.02721	0.11598	-0.37682	0.16031	-0.58614	0.18758
10	-0.35060	0.17637	-0.24815	0.06247	-0.13859	0.05780	-0.37768	0.15158	-0.56460	0.18046
11	-0.32997	0.32032	-0.21592	0.14626	-0.03771	0.13469	-0.40149	0.23456	-0.62222	0.29241
12	-0.97628	5.40733	0.03243	0.01490	0.08891	0.01378	-0.99595	5.35172	-2.04147	7.98428
13	0.13620	0.26025	0.21620	0.17086	0.41576	0.22684	-0.01070	0.07769	-0.14337	0.09826
14	-0.12576	0.08688	-0.07312	0.04495	0.02509	0.04078	-0.18316	0.08298	-0.27660	0.07610
15	-0.11463	0.02894	-0.07428	0.01315	-0.01916	0.01790	-0.18866	0.01219	-0.21010	0.01718
16	-0.46052	1.00894	-0.15045	0.18055	0.03198	0.14238	-0.49490	0.94837	-0.95298	1.26919
17	0.00113	0.02963	0.05180	0.00635	0.08684	0.00527	-0.00674	0.02712	-0.08458	0.03564
18	-1.02840	2.47696	-0.41916	0.10583	-0.27963	0.08516	-1.05122	2.42430	-1.77716	3.46716
19	-0.31911	0.28784	-0.21784	0.14707	-0.05955	0.13245	-0.33281	0.27030	-0.57866	0.27480
20	-0.15709	0.20265	-0.00036	0.02762	0.07155	0.02951	-0.17691	0.18563	-0.38573	0.24509
21	0.24176	0.59379	0.27977	0.54932	0.62228	0.80038	-0.07031	0.02066	-0.13877	0.02619
22	-0.70955	2.22678	-0.08930	0.02721	-0.02372	0.01953	-0.71299	2.22021	-1.39537	-3.25816
23	-0.67244	2.63022	0.04190	0.00398	0.07192	0.00634	-0.69363	2.68888	-1.41680	3.86992
24	-1.34218	6.23479	-0.26432	0.02319	-0.19758	0.02054	-1.35572	6.19435	-2.48680	9.17369
25	-0.86289	2.42150	-0.18555	0.00625	-0.15429	0.00554	-0.86317	2.42098	-1.57150	3.58214

We could seek M.L.E's of θ and η under the exchangeable model where

$$L(\underline{x};k^*,\eta,\theta) = \frac{k^*\eta^n}{n\theta^{\eta(n+k^*-1)}} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\eta}$$

$$\prod_{i=1}^{n} x_i^{\eta-1}\left\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^\eta}\right\}$$

and

$$K(\underline{x};k^*,\eta,\theta) = \ell n\, L(\underline{x};k^*,\eta,\theta)$$

[illegible] $k^* + n\,\ell n\,\eta - \ell n\, n - \eta(n+k^*-1)\,\ell n\,\theta - \sum_{i=1}^{n}(x_i/\theta)^\eta$

$$+ \sum_{i=1}^{n}(\eta-1)\,\ell n\, x_i + \ell n\left\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^\eta}\right\}.$$

Then

$$\frac{\partial K}{\partial \theta} = \frac{-\eta(n+k^*-1)}{\theta} + \sum_{i=1}^{n}\frac{\eta\, x_i^\eta}{\theta^{\eta+1}}$$

(continued)

$$+\frac{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{k^*\eta \frac{x_i^{k^*\eta}}{\theta^{k^*\eta+1}} - \frac{\eta x_i^{\eta}}{\theta^{\eta+1}}\}}{\sum_{i=1}^{n} x_i^{\eta(k^*+1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}}$$

$$\frac{\partial K}{\partial \eta} = \frac{n}{\eta} - (n+k^*-1)\ln\theta - \sum_{i=1}^{n} (\frac{x_i}{\theta})^{\eta} \ln(\frac{x_i}{\theta}) + \sum_{i=1}^{n} \ln x_i$$

$$+ [\sum_{i=1}^{n} x_i^{\eta(k^*-1)} (\ln x_i)(k^*-1) e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}$$

$$+ x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-k^*(\frac{x_i}{\theta})^{k^*\eta} \ln(\frac{x}{\theta}) + (\frac{x_i}{\theta})^{\eta} \ln(\frac{x_i}{\theta})\}]$$

$$[\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}]^{-1}$$

$$\frac{\partial K}{\partial k^*} = \frac{1}{k^*} - \eta \ln\theta + [\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}]^{-1}$$

$$\left[\sum_{i=1}^{n} [x_i^{\eta(k^*-1)} (\ln x_i) \eta e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\right.$$

$$\left. + x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-\eta (x_i/\theta)^{k^*\eta} \ln(\frac{x_i}{\theta})\}]\right]$$

The above equations appear mathematically intractable.

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Appendix I: The A-, W-, and S-Rules and Premium/Protection.

We adopt the following notation:

$$z_i = y_i - \bar{y} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad s^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}$$

$$\bar{y}_{(1)} = \frac{\sum_{i=2}^{n} y_{(i)}}{n-1} \qquad \bar{y}_{(n)} = \frac{\sum_{i=1}^{n-1} y_{(i)}}{n-1} \qquad \bar{y}_{(j,k)} = \frac{\left(\sum_{i=1}^{n} y_{(i)} + y_{(j)} - y_{(k)}\right)}{n}$$

$$s^2_{(1)} = \frac{\sum_{i=2}^{n} \left(y_{(i)} - \bar{y}_{(1)}\right)^2}{n-2} \qquad s^2_{(n)} = \frac{\sum_{i=1}^{n-1} \left(y_{(i)} - \bar{y}_{(n)}\right)^2}{n-2}$$

$$s^2_{(j,k)} = \frac{1}{(n-1)} \left\{ \sum_{i=1}^{n} \left(y_{(i)} - \bar{y}_{(j,k)}\right)^2 + \left(y_{(j)} - \bar{y}_{(j,k)}\right)^2 - \left(y_{(k)} - \bar{y}_{(j,k)}\right)^2 \right\}.$$

Anscombe's Rule (A-Rule):

$$\mu_A = \begin{cases} \bar{y} & \text{if } |z_{(n)}| < Cs \text{ and } |z_{(1)}| < Cs \\ \bar{y}_{(1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma^2_A = \begin{cases} Ds^2 & \text{if } z^2_{(1)} < Ks^2 \text{ and } z^2_{(n)} < Ks^2 \\ Ds^2_{(1)} & \text{if } z^2_{(1)} \geq Ks^2 \text{ and } z^2_{(1)} > z^2_{(n)} \\ Ds^2_{(n)} & \text{if } z^2_{(n)} \geq Ks^2 \text{ and } z^2_{(n)} \geq z^2_{(1)} \end{cases} .$$

Winsorization (W-Rule):

$$\mu_W = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \bar{y}_{(2,1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n-1,n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma_W^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ D\max[s_{(2,1)}^2, s_{(n,1)}^2] & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ D\max[s_{(1,n)}^2, s_{(n-1,n)}^2] & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

Semi-Winsorization (S-Rule):

$$\mu_S = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \dfrac{(n-1)\bar{y}_{(1)}+(\bar{y}-Cs)}{n} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \dfrac{(n-1)\bar{y}_{(n)}+(\bar{y}+Cs)}{n} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma_S^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ \dfrac{D}{n-1}\{(n-2)s_{(1)}^2 + Ks^2\} & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ \dfrac{D}{n-1}\{(n-2)s_{(n)}^2 + Ks^2\} & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

If we adopt the premium-protection approach suggested by Anscombe (1960) we define

$$\text{Premium} = \frac{\text{MSE(new estimator)}-\text{MSE(old estimator)}}{\text{MSE(old estimator)}}$$

assuming homogeneous data

and

$$\text{Protection} = \frac{\text{MSE(old estimator)}-\text{MSE(new estimator)}}{\text{MSE(old estimator)}}$$

assuming spurious values(s) are present.

Appendix II: Asymptotic approximation for $\Gamma(\nu+1)$

Consider $\Gamma(\nu+1) = \int_0^\infty e^{-u}u^\nu du$. Setting $u = \nu t$, $du = \nu dt$ and $\Gamma(\nu+1) = \int_0^\infty e^{-\nu t}(\nu t)^\nu \nu dt = \nu^{\nu+1}\int_0^\infty e^{\nu(-t+\ell n\ t)}dt$. This last integral is of the form $\int_0^\infty \vartheta(t)e^{\nu h(t)}dt$ where ν is a large positive constant and $\vartheta(t)$ and $h(t)$ are real and continuous in $[0,\infty)$. Now $h(t) = -t + \ell n\ t$ which has a single maximum at $t = 1$ with $h'(1) = 0$, $h''(1) = -1$. Applying the Laplace approximation to the two intervals $0 \leq t \leq 1$ and $1 \leq t < \infty$ we obtain the following:

$$\int_0^\infty e^{\nu(-t+\ell n\ t)}dt = \int_0^1 e^{\nu(-t+\ell n\ t)}dt + \int_1^\infty e^{\nu(-t+\ell n\ t)}dt .$$

Since $h(t)$ has a maximum at $t = 1$, the main contributions to the two integrals on the right occur in the neighborhood of $t = 1$. We may rewrite $\int_0^1 e^{\nu(-t+\ell n\ t)}dt = \int_0^{1-\varepsilon} e^{\nu(-t+\ell nt)}dt + \int_{1-\varepsilon}^1 e^{\nu(-t+\ell n\ t)}dt$ and $\int_1^\infty e^{\nu(-t+\ell n\ t)}dt = \int_1^{1+\delta} e^{\nu(-t+\ell n\ t)}dt + \int_{1+\delta}^\infty e^{\nu(-t+\ell n\ t)}dt$. Setting $x^2 = h(1) - h(t)$, $2xdx = -h'(t)dt$ and expanding $h(t)$ in a Taylor's Series, $h(t) = h(1) + h'(1)(t-1) + h''(\xi)\frac{(t-1)^2}{2!}$, $1 < \xi < 1+\delta$, we obtain $x^2 = h(1) - h(t) = -h''(\xi)\frac{(t-1)^2}{2}$ and

$$h'(t) = h''(\xi)(t-1) \quad .$$

Now

$$\frac{2x}{h'(t)} = \frac{2\sqrt{-h''(\xi)\,\frac{(t-1)^2}{2}}}{h''(\xi)(t-1)}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\,h''(\xi)}}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\,h''(1)}}$$

Thus

$$\int_{t=1}^{1+\delta} e^{\nu(-t+\ln t)}dt = \int_{x=0}^{\tau} e^{\nu(-1-x^2)}\left\{\frac{-2x}{h'(t)}\right\} dx$$

$$= e^{-\nu}\int_{\tau}^{0} e^{-\nu x^2}\frac{-1}{\sqrt{-\frac{1}{2}\,h''(1)}}\,dx$$

$$= e^{-\nu}\sqrt{2}\int_0^{\tau} e^{-\nu x^2}\,dx \;.$$

But $\int_0^{\tau} e^{-\nu x^2}\,dx \doteq \int_0^{\infty} e^{-\nu x^2}\,dx$ since the major contribution to this latter integral occurs in the neighborhood of $x = 0$. Thus

$$\int_1^{1+\delta} e^{\nu(-t+\ell n\ t)}dt \doteq e^{-\nu}\sqrt{2}\int_0^{\infty} e^{-\nu x^2}dx$$

$$= e^{-\nu}\sqrt{2}\left(\frac{1}{2}\sqrt{\frac{\pi}{\nu}}\right)$$

$$= e^{-\nu}\sqrt{\frac{\pi}{2\nu}} \quad .$$

Similarly we can show $\int_{1-\varepsilon}^{1} e^{\nu(-t+\ell n\ t)}dt \doteq e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$ and thus

$$\Gamma(\nu+1) \doteq \nu^{\nu+1}\left\{\int_{1-\varepsilon}^{1} e^{\nu(-t+\ell n\ t)}\,dt + \int_1^{1+\delta} e^{\nu(-t+\ell n\ t)}dt\right\}$$

$$= \nu^{\nu+1}\ 2e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$$

$$= e^{-\nu}\nu^{\nu}\sqrt{2\pi\nu} \quad .$$

Appendix III: Evaluation of $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

Theorem: $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Proof: Let $f(v) = \int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

Then $f'(v) = \int_{-\infty}^{\infty} \phi(x)\phi(vx)xdx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+v^2x^2)} xdx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} \frac{\sqrt{1+v^2}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1+v^2)} xdx$$

and setting $u = x\sqrt{1+v^2}$ we obtain

$$f'(v) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= 0.$$

Then $f(v) = c$ for all v.

In particular, if $v = 1$,

$$f(1) = c = \int_{-\infty}^{\infty} \phi(x)\Phi(x)dx = \left.\frac{\Phi(x)^2}{2}\right|_{-\infty}^{\infty} = 1/2.$$

Thus $f(v) = 1/2$ for all v.

i.e. $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Appendix IV

Tables of $u(r;n,k^*) = P(X_{(r)}$ is spurious in a sample of $n|k^*)$ where the exchangeable model of Weibulls with at most one outlier (involving shape change) is assumed.

K = .01050
N = 50

n														
1	1.00000													
2	.62989	.37011												
3	.62721	.00535	.36743											
4	.62565	.00470	.00334	.36632										
5	.62454	.00444	.00273	.00263	.36566									
6	.62368	.00431	.00249	.00205	.00226	.36521								
7	.62297	.00422	.00237	.00182	.00171	.00202	.36487							
8	.62238	.00417	.00229	.00171	.00148	.00151	.00186	.36461						
9	.62186	.00412	.00223	.00164	.00137	.00127	.00138	.00173	.36439					
10	.62141	.00408	.00219	.00159	.00131	.00120	.00110	.00128	.00183	.36421				
11	.62100	.00407	.00216	.00154	.00127	.00110	.00100	.00110	.00120	.00154	.36405			
12	.62063	.00405	.00213	.00151	.00123	.00110	.00095	.00091	.00099	.00110	.00146	.36392		
13	.62030	.00403	.00211	.00148	.00120	.00100	.00092	.00084	.00084	.00095	.00110	.00140	.36381	
14	.61999	.00402	.00210	.00146	.00120	.00100	.00091	.00081	.00075	.00079	.00092	.00100	.00134	.36370
15	.61970	.00401	.00209	.00145	.00110	.00098	.00089	.00080	.00072	.00069	.00078	.00089	.00099	.00129
	.36361													
16	.61943	.00400	.00208	.00144	.00110	.00095	.00087	.00079	.00071	.00064	.00065	.00075	.00086	.00095
	.00125	.36353												
17	.61918	.00399	.00207	.00143	.00110	.00093	.00084	.00078	.00071	.00063	.00059	.00063	.00073	.00084
	.00090	.00120	.36345											
18	.61895	.00398	.00206	.00142	.00110	.00091	.00081	.00076	.00070	.00063	.00057	.00055	.00061	.00072
	.00081	.00086	.00120	.36338										
19	.61873	.00397	.00205	.00142	.00110	.00088	.00079	.00074	.00070	.00063	.00056	.00052	.00053	.00061
	.00071	.00078	.00082	.00120	.36332									
20	.61852	.00396	.00204	.00141	.00110	.00088	.00077	.00071	.00068	.00063	.00057	.00051	.00048	.00051
	.00060	.00070	.00075	.00078	.00110	.36326								
21	.61832	.00396	.00204	.00140	.00110	.00087	.00075	.00069	.00068	.00063	.00058	.00051	.00046	.00046
	.00051	.00060	.00069	.00072	.00075	.00110	.36320							
22	.61814	.00395	.00203	.00140	.00110	.00087	.00074	.00067	.00064	.00062	.00058	.00052	.00046	.00043
	.00044	.00050	.00060	.00067	.00069	.00072	.00110	.36315						
23	.61798	.00395	.00202	.00139	.00110	.00087	.00073	.00065	.00062	.00061	.00058	.00053	.00047	.00042
	.00040	.00043	.00050	.00060	.00066	.00068	.00069	.00110	.36310					
24	.61778	.00394	.00202	.00139	.00110	.00087	.00072	.00064	.00060	.00059	.00057	.00053	.00048	.00042
	.00039	.00038	.00042	.00051	.00060	.00064	.00063	.00067	.00110	.36306				
25	.61762	.00394	.00201	.00138	.00110	.00087	.00072	.00062	.00058	.00057	.00056	.00054	.00049	.00043
	.00039	.00036	.00037	.00042	.00051	.00059	.00062	.00061	.00066	.00110	.36301			
26	.61746	.00394	.00201	.00138	.00110	.00087	.00072	.00062	.00056	.00055	.00054	.00053	.00049	.00044
	.00039	.00036	.00034	.00037	.00043	.00051	.00058	.00060	.00058	.00063	.00100	.36297		
27	.61731	.00394	.00201	.00137	.00110	.00087	.00073	.00061	.00055	.00053	.00053	.00052	.00050	.00046
	.00040	.00036	.00033	.00033	.00036	.00043	.00052	.00058	.00058	.00056	.00061	.00100	.36293	
28	.61717	.00393	.00200	.00136	.00110	.00087	.00073	.00061	.00054	.00051	.00051	.00051	.00050	.00046
	.00041	.00036	.00033	.00031	.00032	.00036	.00044	.00052	.00058	.00055	.00053	.00060	.00100	.36289
29	.61703	.00393	.00200	.00136	.00110	.00087	.00073	.00062	.00053	.00049	.00049	.00049	.00049	.00047
	.00042	.00037	.00033	.00031	.00030	.00032	.00037	.00044	.00052	.00055	.00053	.00051	.00058	.00100
	.36286													
30	.61689	.00393	.00200	.00136	.00100	.00087	.00073	.00062	.00053	.00048	.00047	.00047	.00048	.00047
	.00043	.00039	.00034	.00031	.00029	.00029	.00031	.00037	.00045	.00052	.00054	.00051	.00049	.00057
	.00100	.36282												
31	.61676	.00393	.00200	.00135	.00100	.00087	.00074	.00062	.00053	.00047	.00045	.00045	.00047	.00046
	.00044	.00040	.00035	.00031	.00029	.00028	.00028	.00031	.00038	.00046	.00052	.00052	.00049	.00046
	.00056	.00100	.36279											
32	.61663	.00392	.00199	.00135	.00100	.00088	.00074	.00063	.00053	.00047	.00044	.00044	.00045	.00045
	.00044	.00041	.00036	.00032	.00029	.00027	.00027	.00028	.00032	.00038	.00045	.00051	.00050	.00047
	.00046	.00055	.00100	.36275										
33	.61651	.00392	.00199	.00134	.00100	.00088	.00074	.00063	.00054	.00046	.00043	.00042	.00043	.00044
	.00044	.00041	.00037	.00033	.00029	.00027	.00026	.00026	.00027	.00032	.00039	.00047	.00050	.00048
	.00046	.00046	.00055	.00100	.36272									
34	.61639	.00392	.00199	.00134	.00100	.00085	.00074	.00064	.00054	.00046	.00042	.00040	.00041	.00043
	.00043	.00042	.00038	.00034	.00030	.00027	.00026	.00025	.00025	.00027	.00033	.00040	.00047	.00049
	.00047	.00043	.00043	.00054	.00100	.36269								
35	.61628	.00392	.00199	.00134	.00100	.00085	.00074	.00064	.00054	.00047	.00041	.00039	.00039	.00041
	.00042	.00042	.00039	.00035	.00031	.00028	.00026	.00024	.00024	.00025	.00028	.00033	.00041	.00047
	.00048	.00046	.00041	.00042	.00053	.00100	.36266							

36	.61616	.00392	.00199	.00134	.00100	.00084	.00073	.00064	.00055	.00047	.00041	.00038	.00038	.00039
	.00041	.00041	.00040	.00036	.00032	.00028	.00026	.00024	.00023	.00023	.00024	.00028	.00034	.00042
	.00047	.00047	.00043	.00039	.00041	.00053	.00100	.36263						
37	.61606	.00391	.00199	.00134	.00100	.00084	.00073	.00064	.00055	.00047	.00041	.00037	.00036	.00037
	.00039	.00040	.00040	.00037	.00033	.00029	.00026	.00024	.00023	.00022	.00022	.00024	.00028	.00035
	.00042	.00046	.00045	.00041	.00038	.00041	.00052	.00100	.36260					
38	.61595	.00391	.00199	.00134	.00100	.00083	.00073	.00064	.00056	.00048	.00041	.00037	.00035	.00036
	.00038	.00039	.00040	.00038	.00035	.00030	.00027	.00025	.00023	.00022	.00022	.00022	.00024	.00029
	.00036	.00043	.00046	.00044	.00039	.00037	.00040	.00052	.00100	.36258				
39	.61585	.00391	.00198	.00134	.00100	.00083	.00072	.00064	.00056	.00048	.00041	.00037	.00034	.00034
	.00036	.00038	.00039	.00038	.00035	.00032	.00028	.00025	.00023	.00022	.00021	.00021	.00022	.00024
	.00030	.00037	.00043	.00045	.00042	.00037	.00035	.00039	.00051	.00100	.36255			
40	.61575	.00391	.00198	.00133	.00100	.00082	.00072	.00064	.00057	.00049	.00042	.00037	.00034	.00033
	.00034	.00035	.00038	.00038	.00039	.00033	.00029	.00026	.00023	.00022	.00021	.00021	.00020	.00021
	.00025	.00031	.00038	.00043	.00044	.00040	.00036	.00034	.00039	.00051	.00100	.36252		
41	.61565	.00391	.00198	.00133	.00100	.00082	.00071	.00064	.00057	.00049	.00042	.00037	.00033	.00032
	.00033	.00035	.00037	.00037	.00037	.00034	.00030	.00026	.00024	.00022	.00021	.00020	.00020	.00020
	.00021	.00025	.00032	.00038	.00043	.00043	.00039	.00034	.00033	.00038	.00051	.00100	.36250	
42	.61555	.00391	.00198	.00133	.00100	.00081	.00071	.00064	.00057	.00050	.00043	.00037	.00033	.00031
	.00031	.00033	.00035	.00037	.00037	.00035	.00031	.00027	.00024	.00022	.00021	.00020	.00020	.00019
	.00019	.00022	.00026	.00033	.00039	.00043	.00042	.00037	.00033	.00033	.00038	.00051	.00100	.36248
43	.61546	.00391	.00198	.00133	.00100	.00081	.00070	.00063	.00057	.00050	.00043	.00037	.00033	.00031
	.00030	.00031	.00033	.00035	.00036	.00035	.00032	.00029	.00026	.00022	.00021	.00020	.00020	.00019
	.00019	.00019	.00022	.00027	.00034	.00040	.00042	.00040	.00036	.00031	.00032	.00037	.00050	.00100
	.36245													
44	.61537	.00390	.00198	.00133	.00100	.00081	.00070	.00063	.00057	.00051	.00044	.00038	.00033	.00030
	.00029	.00030	.00032	.00034	.00035	.00036	.00033	.00030	.00026	.00023	.00021	.00020	.00020	.00019
	.00018	.00018	.00019	.00022	.00028	.00034	.00040	.00042	.00039	.00034	.00030	.00031	.00037	.00050
	.00100	.36243												
45	.61528	.00390	.00198	.00133	.00100	.00080	.00069	.00063	.00057	.00051	.00045	.00038	.00033	.00030
	.00029	.00029	.00030	.00032	.00034	.00035	.00034	.00031	.00027	.00024	.00021	.00020	.00019	.00019
	.00018	.00018	.00018	.00019	.00023	.00029	.00035	.00040	.00041	.00037	.00032	.00029	.00031	.00037
	.00050	.00100	.36240											
46	.61519	.00390	.00198	.00133	.00100	.00080	.00069	.00062	.00057	.00051	.00045	.00039	.00034	.00030
	.00028	.00028	.00029	.00031	.00033	.00034	.00034	.00032	.00029	.00025	.00022	.00020	.00019	.00019
	.00018	.00018	.00017	.00017	.00019	.00024	.00030	.00036	.00040	.00040	.00036	.00031	.00028	.00030
	.00036	.00050	.00100	.36238										
47	.61511	.00390	.00198	.00133	.00100	.00080	.00068	.00062	.00057	.00052	.00046	.00039	.00034	.00030
	.00028	.00027	.00028	.00029	.00032	.00033	.00034	.00033	.00030	.00026	.00023	.00020	.00019	.00018
	.00018	.00018	.00017	.00017	.00017	.00020	.00024	.00031	.00037	.00040	.00039	.00034	.00029	.00027
	.00030	.00036	.00049	.00100	.36236									
48	.61503	.00390	.00198	.00133	.00100	.00080	.00068	.00061	.00057	.00052	.00046	.00040	.00034	.00030
	.00027	.00026	.00026	.00028	.00030	.00032	.00033	.00033	.00031	.00027	.00024	.00021	.00019	.00018
	.00018	.00018	.00017	.00017	.00016	.00017	.00020	.00026	.00032	.00037	.00040	.00038	.00033	.00028
	.00026	.00030	.00036	.00049	.00100	.36234								
49	.61495	.00390	.00197	.00133	.00100	.00080	.00067	.00061	.00056	.00052	.00046	.00040	.00035	.00030
	.00027	.00026	.00026	.00027	.00029	.00031	.00033	.00033	.00031	.00029	.00026	.00022	.00019	.00018
	.00018	.00018	.00017	.00017	.00016	.00016	.00017	.00021	.00026	.00033	.00038	.00039	.00037	.00031
	.00027	.00026	.00030	.00036	.00048	.00100	.36232							
50	.61487	.00390	.00197	.00133	.00100	.00080	.00067	.00060	.00056	.00052	.00047	.00041	.00035	.00031
	.00027	.00025	.00025	.00025	.00027	.00030	.00032	.00033	.00032	.00030	.00026	.00023	.00020	.00018
	.00017	.00017	.00017	.00017	.00016	.00015	.00015	.00017	.00021	.00027	.00033	.00038	.00038	.00035
	.00030	.00025	.00026	.00029	.00036	.00049	.00100	.36230						

$K^* = .01100$
N = 50

n														
1	1 00000													
2	62979	37021												
3	62898	00561	36741											
4	62534	00492	00349	36625										
5	62418	00466	00286	00275	36556									
6	62327	00451	00261	00215	00236	36509								
7	62254	00443	00248	00191	00180	00212	36473							
8	62181	00436	00240	00179	00156	00158	00194	36446						
9	62137	00432	00234	00172	00144	00134	00144	00181	36423					
10	62090	00429	00228	00166	00137	00120	00120	00133	00170	36404				
11	62047	00426	00226	00162	00132	00120	00110	00110	00125	00161	36388			
12	62008	00424	00223	00158	00128	00110	00100	00096	00100	00120	00154	36374		
13	61973	00422	00221	00155	00125	00110	00097	00088	00089	00099	00110	00147	36362	
14	61941	00421	00220	00153	00120	00110	00095	00085	00080	00084	00095	00110	00141	36351
15	61911	00420	00219	00152	00120	00100	00092	00084	00076	00074	00081	00092	00100	00137
	36341													
16	61883	00419	00218	00151	00120	00099	00090	00082	00074	00068	00070	00078	00089	00097
	00133	36332												
17	61857	00418	00216	00150	00120	00097	00087	00081	00074	00067	00063	00067	00077	00086
	00093	00129	36324											
18	61832	00417	00216	00149	00120	00085	00085	00079	00073	00066	00060	00059	00065	00075
	00083	00088	00126	36317										
19	61809	00416	00215	00148	00110	00094	00082	00077	00072	00066	00059	00055	00056	00064
	00074	00080	00085	00124	36310									
20	61787	00415	00214	00148	00110	00093	00080	00074	00071	00066	00060	00054	00051	00055
	00063	00072	00076	00081	00120	36303								
21	61767	00415	00213	00147	00110	00092	00079	00072	00069	00065	00060	00054	00049	00049
	00054	00063	00071	00073	00078	00120	36297							
22	61747	00414	00213	00146	00110	00092	00078	00070	00066	00064	00060	00055	00049	00046
	00047	00054	00063	00069	00070	00075	00120	36292						
23	61728	00414	00212	00146	00110	00092	00077	00068	00064	00063	00060	00055	00049	00045
	00043	00046	00054	00062	00067	00067	00072	00120	36287					
24	61710	00413	00211	00145	00110	00092	00077	00067	00062	00061	00059	00056	00050	00045
	00041	00041	00046	00054	00061	00065	00064	00070	00120	36281				
25	61693	00413	00211	00144	00110	00092	00077	00066	00060	00058	00058	00056	00051	00046
	00041	00039	00040	00046	00054	00061	00063	00062	00068	00110	36277			
26	61677	00413	00211	00144	00110	00092	00077	00066	00059	00056	00056	00055	00052	00047
	00042	00038	00037	00039	00046	00054	00060	00060	00059	00066	00110	36272		
27	61661	00412	00210	00143	00110	00092	00077	00065	00058	00054	00054	00054	00052	00048
	00043	00038	00035	00035	00039	00046	00054	00058	00058	00057	00065	00110	36268	
28	61645	00412	00210	00143	00110	00091	00077	00065	00057	00053	00052	00052	00051	00048
	00044	00039	00035	00034	00035	00039	00047	00054	00057	00056	00054	00063	00110	36284
29	61631	00412	00210	00142	00110	00091	00077	00066	00057	00052	00050	00050	00050	00048
	00045	00040	00036	00033	00032	00034	00039	00047	00054	00055	00053	00052	00062	00110
	36280													
30	61617	00411	00209	00142	00110	00091	00077	00066	00057	00051	00048	00048	00049	00048
	00045	00041	00036	00033	00031	00031	00034	00040	00048	00053	00054	00051	00051	00061
	00110	36256												
31	61603	00411	00209	00142	00110	00090	00077	00066	00057	00050	00047	00046	00047	00047
	00046	00042	00037	00033	00031	00029	00030	00034	00041	00048	00052	00052	00049	00049
	00060	00110	36252											
32	61590	00411	00209	00141	00110	00090	00077	00067	00057	00050	00046	00045	00046	00046
	00046	00043	00038	00034	00031	00029	00028	00030	00034	00041	00048	00051	00050	00047
	00048	00050	00110	36248										
33	61577	00411	00209	00141	00110	00089	00077	00067	00057	00050	00045	00043	00044	00045
	00045	00043	00040	00035	00031	00029	00027	00028	00030	00035	00042	00048	00050	00048
	00045	00046	00059	00110	36245									
34	61564	00411	00209	00141	00110	00089	00077	00067	00058	00050	00044	00042	00042	00043
	00044	00043	00040	00036	00032	00029	00027	00028	00027	00030	00035	00043	00048	00049
	00046	00043	00045	00058	00110	36241								
35	61552	00410	00209	00141	00110	00088	00076	00067	00058	00050	00044	00041	00040	00041
	00043	00043	00041	00037	00033	00030	00027	00026	00025	00027	00030	00036	00043	00048
	00048	00044	00042	00044	00058	00110	36238							

36	61540	00410	00208	00141	00110	00088	00076	00067	00059	00051	00044	00040	00039	00040
	00041	00042	00041	00038	00034	00031	00028	00026	00025	00025	00026	00031	00037	00044
	00047	00046	00042	00040	00043	00057	00110	36235						
37	61529	00410	00208	00140	00110	00087	00076	00067	00059	00051	00044	00040	00038	00038
	00040	00041	00041	00039	00036	00032	00028	00026	00024	00024	00024	00026	00031	00038
	00044	00047	00045	00041	00039	00042	00057	00110	36232					
38	61518	00410	00208	00140	00110	00087	00075	00067	00059	00052	00045	00039	00037	00037
	00038	00040	00040	00039	00036	00033	00029	00026	00024	00023	00023	00024	00027	00032
	00039	00044	00046	00043	00039	00038	00042	00057	00110	36229				
39	61507	00410	00208	00140	00110	00086	00075	00067	00060	00052	00045	00039	00036	00035
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26 .49642 .03509 .01848 .01276 .00987 .00813 .00698 .00617 .00557 .00512 .00478 .00451 .00431 .00416
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	00303	00318	00338	00367	00410	00480	00611	00950	31707					
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	00274	00282	00292	00307	00328	00357	00400	00470	00599	00935	31657			
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43	47874	03422	01792	01227	00940	00767	00650	00567	00505	00457	00419	00387	00362	00340
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	00905	31545												
45	47717	03414	01787	01223	00937	00763	00647	00564	00502	00454	00415	00384	00358	00337
	00318	00303	00289	00278	00268	00259	00252	00246	00241	00236	00233	00230	00228	00227
	00227	00227	00228	00231	00235	00240	00247	00255	00267	00283	00303	00333	00375	00444
	00571	00899	31525											
46	47641	03411	01785	01222	00935	00762	00646	00563	00501	00452	00414	00382	00357	00335
	00316	00301	00287	00276	00266	00257	00250	00243	00238	00233	00229	00226	00224	00222
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	00440	00566	00894	31505										
47	47567	03407	01783	01220	00934	00761	00645	00562	00499	00451	00412	00381	00355	00333
	00315	00299	00285	00274	00264	00255	00247	00241	00235	00230	00226	00223	00220	00218
	00217	00217	00217	00218	00220	00223	00227	00232	00239	00248	00260	00276	00297	00326
	00368	00436	00562	00889	31486									
48	47495	03404	01781	01218	00932	00759	00643	00560	00498	00450	00411	00379	00353	00332
	00313	00297	00284	00272	00262	00253	00245	00238	00232	00227	00223	00220	00217	00215
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	00312	00296	00282	00270	00260	00251	00243	00236	00230	00225	00221	00217	00214	00211
	00210	00208	00208	00208	00209	00210	00212	00216	00220	00226	00233	00242	00254	00270
	00291	00320	00362	00430	00555	00879	31449							
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	00310	00294	00280	00268	00258	00249	00241	00234	00228	00223	00218	00214	00211	00208
	00206	00205	00204	00203	00204	00205	00206	00209	00212	00217	00222	00230	00239	00251
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25 .43673 .04914 .02670 .01869 .01458 .01209 .01040 .00926 .00840 .00775 .00725 .00687 .00659 .00639
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	00510	00490	00473	00459	00447	00439	00432	00428	00427	00427	00431	00437	00448	00463
	00484	00514	00556	00620	00724	00918	01418	29243						
37	41750	04751	02574	01791	01387	01140	00974	00855	00766	00697	00642	00597	00561	00530
	00505	00484	00467	00452	00440	00431	00424	00419	00416	00415	00416	00421	00428	00439
	00454	00475	00505	00548	00612	00716	00909	01407	29204					
38	41621	04741	02567	01786	01383	01140	00971	00852	00762	00693	00638	00593	00556	00526
	00500	00479	00461	00446	00434	00424	00416	00410	00406	00404	00404	00406	00411	00419
	00430	00446	00468	00498	00541	00605	00708	00900	01396	29166				
39	41497	04730	02562	01782	01379	01130	00967	00848	00759	00689	00634	00589	00552	00521
	00496	00474	00455	00441	00428	00417	00409	00402	00397	00394	00393	00394	00397	00402
	00410	00422	00438	00460	00490	00533	00597	00700	00891	01386	29130			
40	41376	04720	02556	01777	01375	01130	00964	00845	00755	00686	00630	00585	00548	00518
	00492	00470	00451	00436	00422	00411	00402	00395	00389	00385	00383	00383	00385	00388
	00394	00402	00414	00431	00453	00484	00527	00590	00693	00883	01376	29085		
41	41258	04711	02551	01773	01372	01130	00961	00842	00752	00683	00627	00582	00545	00514
	00488	00466	00447	00431	00417	00406	00396	00389	00382	00378	00375	00374	00374	00378
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	00484	00462	00443	00426	00413	00401	00391	00383	00376	00371	00367	00365	00364	00365
	00368	00372	00379	00388	00401	00417	00440	00471	00514	00577	00679	00868	01357	29027
43	41032	04692	02540	01765	01365	01120	00955	00836	00746	00677	00621	00576	00538	00507
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	00357	00360	00365	00372	00381	00394	00411	00434	00465	00508	00571	00673	00861	01348
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44	40923	04683	02535	01762	01362	01120	00952	00833	00744	00674	00618	00573	00535	00504
	00477	00455	00435	00419	00404	00392	00382	00373	00365	00359	00354	00351	00349	00348
	00348	00350	00353	00358	00365	00375	00388	00405	00428	00458	00502	00566	00667	00855
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46	40713	04666	02526	01754	01355	01110	00947	00828	00739	00669	00613	00568	00530	00498
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47	40612	04657	02521	01751	01353	01110	00944	00826	00736	00666	00611	00565	00528	00496
	00469	00446	00426	00409	00394	00381	00370	00360	00352	00345	00339	00334	00330	00327
	00326	00325	00325	00327	00330	00334	00340	00348	00358	00372	00390	00413	00444	00487
	00550	00650	00836	01315	28876									
48	40513	04649	02516	01748	01350	01110	00942	00823	00734	00664	00608	00563	00525	00493
	00466	00443	00423	00406	00391	00378	00367	00357	00348	00341	00334	00328	00325	00322
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	00464	00441	00421	00403	00388	00375	00363	00353	00345	00337	00330	00325	00320	00317
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	00434	00477	00540	00640	00824	01300	28821							
50	40322	04633	02508	01741	01344	01100	00937	00819	00730	00660	00604	00559	00521	00489
	00462	00438	00418	00401	00385	00372	00360	00350	00341	00333	00327	00321	00316	00312
	00309	00307	00306	00305	00306	00307	00309	00313	00318	00325	00333	00344	00358	00376
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7	.46779	.07260	.04353	.03446	.03239	.03802	.31122							
8	.45767	.07079	.04173	.03202	.02829	.02837	.03490	.30623						
9	.44900	.06935	.04042	.03043	.02601	.02446	.02560	.03257	.30216					
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	.02472	.28563												
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	.01687	.02412	.28403											
18	.40178	.06264	.03542	.02539	.02020	.01708	.01504	.01365	.01270	.01208	.01173	.01160	.01186	.01246
	.01373	.01638	.02358	.28264										
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	.01198	.01328	.01594	.02309	.28136									
20	.39508	.06177	.03487	.02492	.01975	.01661	.01454	.01310	.01207	.01130	.01080	.01050	.01040	.01050
	.01090	.01160	.01288	.01564	.02265	.28016								
21	.39201	.06137	.03462	.02471	.01956	.01642	.01433	.01287	.01182	.01110	.01050	.01020	.00995	.00992
	.01010	.01040	.01120	.01252	.01518	.02224	.27905							
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	.00948	.00967	.01010	.01080	.01219	.01485	.02187	.27801						
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	.00902	.00909	.00932	.00977	.01050	.01190	.01455	.02152	.27703					
24	.38372	.06032	.03398	.02419	.01907	.01594	.01384	.01235	.01130	.01040	.00980	.00934	.00900	.00877
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26	.37884	.05970	.03361	.02389	.01881	.01568	.01358	.01208	.01100	.01010	.00947	.00896	.00858	.00829
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	.00788	.00773	.00765	.00765	.00773	.00792	.00824	.00876	.00968	.01090	.01357	.02037	.27363	
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	.01282	.01947	.27085											
32	.36641	.05815	.03271	.02319	.01819	.01510	.01301	.01150	.01040	.00950	.00881	.00825	.00780	.00743
	.00714	.00690	.00671	.00657	.00648	.00643	.00642	.00646	.00656	.00672	.00697	.00734	.00789	.00873
	.01010	.01266	.01928	.27023										
33	.36460	.05792	.03258	.02310	.01811	.01502	.01294	.01140	.01030	.00942	.00873	.00817	.00771	.00734
	.00703	.00679	.00659	.00644	.00632	.00625	.00622	.00623	.00628	.00639	.00656	.00682	.00720	.00775
	.00858	.00995	.01251	.01909	.26963									
34	.36286	.05770	.03246	.02300	.01803	.01495	.01286	.01140	.01020	.00935	.00865	.00809	.00763	.00725
	.00694	.00668	.00648	.00631	.00619	.00610	.00604	.00603	.00605	.00612	.00624	.00642	.00669	.00707
	.00762	.00846	.00982	.01236	.01891	.26906								
35	.36116	.05748	.03234	.02291	.01795	.01488	.01280	.01130	.01020	.00928	.00858	.00802	.00755	.00717
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36	35953	05728	03222	02283	01787	01481	01273	01120	01010	00922	00852	00795	00748	00710
	00677	00651	00628	00610	00596	00584	00575	00571	00569	00570	00574	00583	00596	00616
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	00960	00921	00888	00860	00836	00816	00800	00787	00777	00770	00766	00765	00768	00774
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	01070	01249	01584	02432	21422									
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	00888	00848	00812	00782	00755	00731	00710	00691	00675	00661	00649	00639	00630	00623
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7	.36864	.10900	.07185	.05896	.05800	.06488	.27069																												
8	.35372	.10460	.06789	.05433	.04885	.04909	.05933	.26221																											
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22	.22609	.07989	.05277	.04075	.03387	.02941	.02630	.02403	.02234	.02106	.02012	.01944	.01901	.01881
	.01886	.01920	.01993	.02125	.02358	.02809	.03959	.19564						
23	.22251	.07874	.05202	.04016	.03336	.02894	.02584	.02358	.02187	.02058	.01959	.01886	.01835	.01805
	.01795	.01808	.01849	.01927	.02061	.02286	.02746	.03886	.19387					
24	.21913	.07765	.05131	.03961	.03289	.02850	.02543	.02316	.02145	.02014	.01913	.01836	.01779	.01741
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	.01660	.01648	.01653	.01678	.01728	.01813	.01952	.02188	.02634	.03755	.19065			
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	.01521	.01492	.01473	.01466	.01471	.01490	.01526	.01585	.01676	.01819	.02054	.02493	.03589	.18644
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32	.19755	.07068	.04681	.03612	.02993	.02585	.02294	.02077	.01909	.01776	.01669	.01581	.01508	.01449
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	.01914	.02344	.03410	.18172										
33	.19536	.06997	.04636	.03577	.02963	.02559	.02271	.02055	.01888	.01755	.01647	.01559	.01486	.01426
	.01376	.01335	.01302	.01276	.01257	.01245	.01239	.01241	.01251	.01270	.01301	.01346	.01412	.01508
	.01652	.01865	.02312	.03371	.18067									
34	.19326	.06928	.04592	.03544	.02935	.02534	[illegible]	.02034	.01867	.01735	.01627	.01539	.01466	.01405
	.01354	.01312	.01277	.01250	.01229	.01214	[illegible]	.01202	.01206	.01218	.01239	.01271	.01317	.01384
	.01481	.01625	.01858	[illegible]	.03333	.17966								
35	.19124	.06863	.04550	.03511	.02908	.02510	.02226	.02013	.01848	.01716	.01609	.01520	.01447	.01385
	.01334	.01291	.01255	.01226	.01203	.01186	.01174	.01170	.01170	.01174	.01187	.01210	.01243	.01291
	.01368	.01465	.01599	.01830	.02252	.03288	.17869							

36	18930	06799	04509	03480	02882	02487	02206	01994	01829	01698	01591	01503	01429	01367
	01315	01271	01235	01204	01180	01160	01150	01140	01130	01140	01140	01160	01183	01217
	01266	01334	01431	01575	01805	02225	03264	17775						
37	18743	06738	04470	03450	02857	02465	02186	01975	01812	01681	01574	01486	01412	01350
	01298	01253	01216	01184	01160	01140	01120	01110	01100	01100	01110	01120	01130	01160
	01193	01242	01310	01408	01551	01781	02199	03232	17686					
38	18562	06679	04432	03421	02833	02444	02167	01958	01795	01685	01558	01470	01396	01334
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	01130	01170	01220	01289	01386	01530	01758	02174	03201	17599				
39	18388	06622	04395	03394	02810	02424	02148	01941	01779	01649	01543	01455	01381	01319
	01266	01220	01182	01150	01120	01100	01080	01060	01050	01050	01040	01050	01060	01070
	01080	01110	01150	01199	01268	01365	01508	01736	02150	03171	17516			
40	18220	06567	04360	03367	02788	02405	02131	01924	01763	01634	01529	01441	01367	01305
	01251	01206	01170	01130	01100	01080	01060	01040	01030	01020	01020	01020	01020	01030
	01040	01060	01090	01130	01179	01249	01346	01489	01715	02127	03143	17435		
41	18057	06513	04326	03340	02766	02386	02114	01908	01749	01620	01516	01427	01354	01291
	01238	01192	01150	01120	01090	01060	01040	01030	01010	01000	00996	00993	00994	00999
	01010	01020	01040	01070	01110	01160	01230	01327	01470	01695	02106	03115	17357	
42	17899	06461	04293	03315	02745	02368	02097	01894	01734	01607	01502	01414	01341	01278
	01225	01178	01140	01100	01070	01050	01030	01010	00995	00984	00975	00971	00969	00971
	00977	00987	01000	01020	01050	01090	01140	01212	01310	01452	01676	02084	03089	17282
43	17747	06411	04260	03291	02725	02350	02082	01879	01721	01593	01489	01402	01329	01266
	01212	01170	01130	01090	01060	01030	01010	00994	00979	00966	00957	00950	00947	00946
	00949	00956	00967	00983	01010	01030	01070	01130	01196	01293	01435	01658	02064	03064
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44	17589	06362	04229	03267	02706	02333	02066	01865	01708	01581	01477	01390	01317	01254
	01201	01150	01110	01080	01050	01020	00999	00980	00963	00950	00939	00931	00926	00924
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	03039	17138												
45	17455	06315	04199	03244	02686	02317	02051	01851	01695	01569	01465	01379	01306	01243
	01188	01140	01100	01070	01040	01010	00985	00966	00949	00935	00923	00914	00908	00904
	00903	00905	00910	00919	00931	00949	00972	01000	01040	01090	01160	01261	01402	01624
	02026	03016	17070											
46	17316	06269	04169	03222	02668	02301	02037	01838	01682	01557	01454	01368	01295	01233
	01179	01130	01090	01060	01020	00998	00974	00954	00936	00921	00909	00899	00891	00886
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47	17181	06224	04141	03200	02650	02285	02023	01826	01671	01546	01443	01357	01285	01222
	01170	01120	01080	01050	01010	00987	00963	00942	00924	00908	00895	00884	00876	00870
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	01372	01592	01990	02971	16939									
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	01219	01358	01577	01974	02950	16876								
49	16921	06139	04086	03158	02616	02255	01997	01801	01648	01524	01423	01337	01265	01203
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6	33578	13327	09585	08462	09153	25896								
7	31499	12470	08803	07432	07119	08136	24540							
8	29813	11800	08244	06801	06205	[illegible]244	07411	23481						
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13	24439	09753	06706	05340	04574	04107	03823	03677	03661	03810	04253	05544	20312	
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	02640	02878	03358	04587	18368									
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21	20082	08126	05582	04408	03721	03268	02949	02716	02543	02413	02320	02256	02221	02216
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22	19702	07983	05486	04331	03653	03205	02889	02656	02481	02349	02250	02179	02133	02112
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23	19345	07849	05396	04259	03591	03148	02834	02602	02426	02291	02188	02112	02058	02026
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	01832	02078	02525	03618	16094									
34	16462	06761	04666	03684	03099	02706	02423	02208	02040	01905	01795	01703	01627	01564
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36	.16076	.06615	.04567	.03507	.03034	.02649	.02371	.02159	.01993	.01880	.01750	.01660	.01583	.01519
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	.01415	.01487	.01591	.01743	.01986	.02425	.03496	.15782						
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38	.15720	.06478	.04476	.03536	.02975	.02597	.02323	.02115	.01951	.01819	.01711	.01620	.01544	.01480
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	.01406	.01359	.01318	.01284	.01255	.01230	.01211	.01195	.01184	.01177	.01174	.01177	.01184	.01197
	.01217	.01246	.01285	.01339	.01412	.01516	.01667	.01907	.02339	.03390	.15504			
40	.15388	.06352	.04391	.03470	.02919	.02548	.02279	.02074	.01913	.01782	.01675	.01585	.01509	.01444
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	.01170	.01193	.01222	.01262	.01316	.01390	.01493	.01645	.01883	.02313	.03357	.15418		
41	.15231	.06291	.04351	.03438	.02893	.02525	.02258	.02055	.01894	.01765	.01658	.01568	.01493	.01428
	.01373	.01325	.01283	.01248	.01217	.01191	.01170	.01150	.01140	.01130	.01120	.01120	.01120	.01120
	.01130	.01150	.01170	.01200	.01240	.01295	.01369	.01472	.01623	.01880	.02285	.03326	.15335	
42	.15079	.06233	.04312	.03408	.02867	.02503	.02238	.02036	.01877	.01748	.01642	.01553	.01477	.01413
	.01357	.01309	.01267	.01231	.01200	.01174	.01150	.01130	.01120	.01100	.01100	.01090	.01090	.01090
	.01100	.01110	.01120	.01150	.01178	.01219	.01274	.01349	.01462	.01602	.01838	.02264	.03296	.15255
43	.14932	.06176	.04274	.03379	.02843	.02481	.02218	.02018	.01860	.01732	.01626	.01538	.01462	.01398
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	.01511	.01742	.02157	.03161	.14889									
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5	.34135	.15414	.11850	.11780	.26833									
6	.31326	.14049	.10440	.09322	.10040	.24825								
7	.29151	.13047	.09530	.08167	.07857	.08906	.23340							
8	.27399	.12260	.08874	.07446	.06847	.06892	.08093	.22184						
9	.25946	.11630	.08366	.06933	.06226	.05974	.06203	.07477	.21250					
10	.24713	.11090	.07953	.06540	.05790	.05417	.05350	.05883	.06891	.20473				
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17	.19347	.08791	.06273	.05072	.04361	.03893	.03570	.03344	.03189	.03094	.03056	.03080	.03185	.03419
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	.03283	.03778	.05039	.16929										
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22	.17171	.07859	.05615	.04533	.03881	.03443	.03129	.02896	.02719	.02585	.02483	.02410	.02363	.02342
	.02348	.02385	.02468	.02614	.02870	.03360	.04570	.15954						
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24	.16491	.07567	.05411	.04367	.03737	.03311	.03003	.02773	.02596	.02458	.02350	.02266	.02207	.02165
	.02144	.02142	.02164	.02216	.02307	.02459	.02717	.03202	.04387	.15560				
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	.01940	.01912	.01898	.01899	.01916	.01953	.02016	.02115	.02272	.02530	[illegible]	.04158	.15051	
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	.02003	.02256	.02713	.03809	.14245									
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36	13852	06337	04551	03678	03144	02778	02509	02303	02140	02008	01899	01807	01730	01665
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	01561	01637	01744	01902	02151	02600	03672	13917						
37	13478	06261	04498	03635	03108	02746	02480	02278	02114	01983	01874	01783	01707	01641
	01586	01538	01498	01464	01436	01414	01396	01384	01377	01375	01380	01390	01409	01437
	01476	01531	01607	01714	01872	02120	02588	03630	13816					
38	13311	06188	04447	03595	03073	02715	02452	02250	02090	01969	01851	01761	01684	01619
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	01407	01447	01502	01579	01686	01843	02080	02533	03590	13719				
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40	12995	06049	04350	03518	03007	02657	02399	02201	02043	01915	01808	01718	01642	01577
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	01299	01321	01351	01393	01450	01526	01634	01790	02035	02472	03515	13535		
41	12846	05984	04304	03481	02976	02628	02374	02178	02021	01894	01788	01699	01623	01558
	01502	01453	01411	01375	01343	01316	01294	01275	01261	01250	01242	01239	01240	01245
	01255	01272	01295	01326	01368	01425	01502	01610	01765	02009	02444	03480	13448	
42	12701	05920	04260	03446	02946	[illegible]	[illegible]	02155	02000	01874	01769	01680	01604	01539
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	01184	01192	01204	01222	01247	01280	01323	01380	01458	01565	01719	01960	02390	03414
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	01130	01130	01140	01140	01160	01178	01204	01238	01282	01340	01417	01523	01677	01916
	02341	03352	13128											
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47	12050	05633	04059	03285	02810	02483	02241	02055	01906	01784	01682	01596	01523	01459
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	01638	01875	02295	03295	12983									
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49	11810	05529	03986	03228	02761	02440	02203	02019	01873	01752	01652	01567	01494	01431
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5	32185	16221	12843	12777	25974									
6	29254	14855	11240	10160	10690	23796								
7	27003	13504	10200	08881	08581	09637	22190							
8	25202	12605	09447	08085	07476	07528	08734	20942						
9	23718	11880	08867	07478	06783	06535	06770	08047	19937					
10	22466	11270	08378	07024	06290	05920	05855	06194	07504	19103				
11	21392	10740	07976	06656	05911	05486	05295	05339	05741	07062	18397			
12	20456	10290	07632	06349	05607	05156	04902	04819	04932	05373	06694	17788		
13	19632	09895	07333	06086	05353	04891	04605	04457	04443	04602	05067	06382	17256	
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15	18239	09224	06833	05656	04948	04485	04173	03968	03850	03817	03884	04095	04585	05877
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16	17643	08937	06621	05475	04782	04324	04007	03790	03650	03578	03579	03669	03896	04391
	05669	15988												
17	17101	08675	06429	05313	04634	04181	03864	03640	03486	03390	03352	03377	03485	03723
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18	16605	08436	06253	05165	04501	04054	03738	03510	03347	03237	03174	03161	03204	03325
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19	16148	08215	06091	05030	04380	03940	03626	03396	03227	03107	03029	02991	02986	03054
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22	14971	07646	05675	04685	04073	03654	03351	03122	02948	02814	02712	02638	02592	02570
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26	13732	07044	05236	04323	03755	03363	03076	02856	02683	02546	02437	02349	02280	02228
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27	13466	06914	05142	04246	03688	03302	03018	02801	02630	02493	02383	02294	02223	02167
	02126	02098	02084	02085	02103	02140	02204	02306	02465	02726	03202	04333	13364	
28	13214	06791	05053	04172	03624	03244	02964	02749	02579	02443	02333	02243	02170	02112
	02067	02035	02015	02007	02013	02034	02076	02142	02246	02405	02665	03138	04258	13206
29	12976	06675	04968	04103	03563	03189	02913	02701	02533	02397	02287	02196	02122	02062
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36	11500	05998	04475	03700	03214	02875	02623	02427	02270	02142	02035	01946	01870	01805
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37	11430	05917	04416	03652	03172	02838	02588	02395	02240	02113	02007	01918	01842	01778
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38	11270	05838	04360	03606	03132	02802	02556	02364	02211	02085	01980	01891	01816	01751
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39	11120	05784	04305	03561	03094	02767	02524	02335	02183	02058	01954	01868	01791	01726
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	01478	01509	01551	01608	01685	01794	01953	02198	02636	03656	11880			
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38	.08102	.05061	.04063	.03515	.03154	.02892	.02692	.02532	.02402	.02293	.02201	.02122	.02054	.01996
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	.01912	.01868	.01829	.01796	.01768	.01744	.01724	.01709	.01697	.01690	.01688	.01690	.01698	.01712
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	.01647	.01748	.01890	.02105	.02473	.03283	.08055							
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6	.23988	.15894	.13387	.12542	.13193	.20987								
7	.21605	.14299	.11930	.10880	.10650	.11570	.19058							
8	.19738	.13067	.10850	.09785	.09281	.09342	.10390	.17574						
9	.18228	.12080	.09996	.08932	.08364	.08159	.08377	.09475	.16390					
10	.16977	.11260	.09305	.08277	.07682	.07373	.07324	.07831	.08753	.15417				
11	.15920	.10570	.08728	.07741	.07145	.06790	.06629	.06674	.07036	.08161	.14601			
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13	.14223	.09470	.07813	.06907	.06335	.05957	.05715	.05588	.05579	.05725	.06141	.07248	.13300	
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16	.12360	.08280	.06817	.06015	.05494	.05130	.04870	.04685	.04564	.04503	.04505	.04589	.04795	.05233
	.06293	.11880												
17	.11870	.07937	.06553	.05780	.05275	.04920	.04661	.04472	.04340	.04257	.04224	.04248	.04348	.04564
	.05003	.06046	.11500											
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	.04360	.04798	.05824	.11160										
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21	.10290	.06906	.05709	.05036	[illegible]	.04267	.04025	.03838	.03693	.03581	.03498	.03440	.03409	.03405
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	.03253	.03289	.03366	.03502	.03734	.04161	.05123	.10050						
23	.09672	.06504	.05381	.04747	.04324	.04019	.03787	.03605	.03461	.03347	.03257	.03188	.03140	.03110
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	.02761	.02743	.02738	.02750	.02781	.02837	.02928	.03071	.03303	.03713	.04620	.09235		
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	.02573	.02794	.03176	.04005	.08202									
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	.02232	.02192	.02159	.02132	.02112	.02097	.02088	.02085	.02090	.02103	.02125	.02159	.02208	.02277
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	.02185	.02145	.02111	.02083	.02061	.02043	.02032	.02026	.02028	.02033	.02047	.02071	.02106	.02156
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	02141	02101	02065	02037	02013	01994	01980	01972	01968	01971	01979	01996	02021	02057
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	02100	02059	02024	01994	01968	01949	01933	01922	01916	01915	01919	01929	01947	01973
	02010	02062	02132	02230	02371	02587	02954	03746	07756					
38	06877	04684	03876	03427	03125	02902	02728	02589	02474	02377	02294	02223	02161	02107
	02060	02020	01984	01954	01928	01907	01890	01877	01868	01864	01864	01870	01882	01901
	01929	01967	02018	02089	02187	02327	02541	02905	03689	07656				
39	06756	04584	03811	03370	03073	02853	02683	02546	02432	02337	02256	02185	02123	02070
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	01858	01886	01925	01977	02048	02146	02285	02498	02868	03634	07581			
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	01742	01757	01779	01809	01849	01901	01972	02070	02208	02417	02771	03531	07381	
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	01922	01882	01846	01814	01787	01763	01742	01724	01710	01699	01690	01686	01684	01686
	01693	01704	01720	01742	01773	01813	01866	01937	02034	02172	02379	02730	03483	07296
43	06321	04295	03574	03163	02885	02679	02520	02390	02283	02193	02115	02048	01989	01937
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	01447	01443	01441	01442	01445	01450	01459	01470	01486	01506	01532	01565	01607	01661
	01731	01825	01957	02154	02485	03193	06779							
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	01421	01416	01413	01412	01413	01417	01423	01432	01444	01461	01481	01506	01541	01583
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6	22506	18152	14031	13289	13875	20147								
7	20104	14416	12420	11510	11300	12130	18126							
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9	16734	12010	10280	09358	08860	08680	08878	09857	15344					
10	15497	11130	09514	08632	08111	07837	07795	08074	08071	14336				
11	14458	10400	08879	08036	07516	07202	07059	07102	07430	08427	13494			
12	13570	09769	08339	07536	07027	06698	06507	06445	06543	06900	07890	12776		
13	12801	09225	07875	07109	06614	06282	06067	05954	05947	06081	06456	07434	12160	
14	12130	08749	07469	06738	06259	05930	05704	05563	05502	05533	05691	06077	07040	11620
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18	10100	07310	06247	05630	05215	04916	04692	04523	04399	04313	04265	04255	04283	04394
	04596	04989	05885	09982										
19	09706	07033	06012	05418	05017	04728	04507	04339	04212	04119	04057	04028	04033	04082
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	03895	04011	04219	04607	05471	09379								
21	09022	06547	05600	05047	04672	04397	04187	04024	03895	03796	03720	03669	03640	03637
	03664	03728	03848	04058	04443	05291	09114							
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	03474	03508	03578	03701	03912	04283	05127	08870						
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	03314	03328	03367	03442	03568	03778	04155	04975	08644					
24	08183	05950	05094	04593	04251	03998	03802	03647	03522	03421	03338	03275	03227	03194
	03176	03176	03196	03240	03318	03446	03656	04029	04836	08434				
25	07942	05778	04949	04462	04130	03884	03693	03541	03417	03317	03238	03169	03117	03079
	03055	03045	03051	03076	03124	03205	03334	03543	03912	04705	08238			
26	07717	05618	04813	04341	04017	03777	03591	03442	03320	03221	03139	03072	03018	02977
	02948	02931	02927	02938	02966	03018	03101	03230	03439	03803	04584	08054		
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	02851	02828	02817	02819	02834	02866	02920	03005	03135	03342	03702	04471	07882	
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	02763	02737	02721	02714	02720	02739	02774	02830	02916	03046	03252	03608	04366	07721
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	07568													
30	05950	05069	04348	03923	[illegible]	03414	03244	03107	02994	02889	02819	02752	02695	02648
	02609	02578	02554	02539	[illegible]	02536	02546	02570	02609	02669	02756	02887	03090	03438
	04173	07424												
31	05785	04951	04248	03834	03549	03336	03170	03035	02924	02831	02752	02685	02628	02580
	02540	02508	02482	02464	02452	02449	02454	02468	02494	02536	02596	02684	02815	03016
	03360	04085	07288											
32	05628	04839	04153	03749	03470	03263	03100	02968	02858	02767	02689	02623	02568	02517
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	02947	03287	04003	07159										
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	02417	02383	02354	02331	02314	02303	02299	02301	02311	02329	02358	02403	02465	02554
	02683	02882	03218	03924	07036									
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	02362	02327	02297	02273	02254	02240	02232	02230	02235	02246	02267	02298	02343	02406
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	02310	02274	02244	02218	02198	02182	02172	02166	02167	02173	02186	02208	02241	02288
	02350	02439	02567	02762	03091	03778	06809							

36	06080	04446	03819	03449	03194	03003	02853	02731	02629	02543	02469	02406	02350	02302
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	02214	02178	02147	02120	02097	02079	02066	02055	02049	02048	02052	02061	02078	02102
	02136	02183	02247	02336	02482	02654	02976	03648	06601					
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	01981	02007	02042	02090	02156	02243	02366	02557	02872	03528	06411			
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	02089	02053	02021	01993	01969	01948	01930	01916	01905	01898	01893	01893	01898	01904
	01918	01937	01964	02000	02048	02112	02200	02325	02512	02824	03473	06322		
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	01981	01946	01913	01885	01860	01838	01819	01802	01788	01777	01769	01763	01760	01760
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	01948	01912	01881	01853	01827	01805	01786	01769	01754	01742	01733	01726	01721	01719
	01720	01724	01732	01744	01760	01782	01811	01848	01897	01961	02048	02169	02350	02650
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	01916	01881	01850	01821	01795	01774	01754	01737	01722	01709	01699	01691	01686	01681
	01681	01683	01688	01696	01708	01725	01748	01777	01814	01863	01927	02014	02134	02314
	02611	03226	05926											
46	05077	03724	03205	02899	02687	02528	02402	02299	02212	02139	02075	02019	01969	01925
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	01644	01644	01647	01653	01662	01675	01692	01715	01745	01782	01831	01895	01981	02101
	02279	02573	03183	05856										
47	04988	03667	03156	02855	02647	02490	02366	02265	02179	02107	02043	01988	01939	01896
	01857	01822	01791	01763	01738	01716	01695	01677	01662	01648	01636	01627	01619	01613
	01610	01609	01610	01613	01620	01630	01643	01661	01684	01714	01752	01801	01864	01950
	02069	02245	02537	03141	05787									
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	01920	02038	02213	02503	03100	05721								
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	01802	01768	01737	01710	01685	01662	01642	01623	01607	01593	01580	01569	01560	01553
	01548	01544	01542	01543	01545	01550	01558	01569	01584	01602	01626	01656	01694	01743
	01807	01891	02009	02182	02469	03061	05658							
50	04775	03506	03020	02732	02533	02384	02265	02168	02087	02017	01956	01903	01856	01814
	01777	01743	01712	01684	01660	01637	01617	01598	01582	[illegible]	01554	01543	01533	01526
	01519	01515	01512	01511	01512	01516	01521	01530	01541	[illegible]	01575	01599	01629	01667
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13	.11530 .08949 .07897 .07280 .06871 .06592 .06410 .06314 .06308 .06425 .06748 .07570 .11100
14	.10880 .08451 .07459 .06872 .06479 .06203 .06013 .05892 .05841 .05868 .06006 .06337 .07144 .10550
15	.10310 .08013 .07073 .06515 .06137 .05867 .05674 .05541 .05463 .05443 .05493 .05646 .05982 .06773 .10070
16	.09803 .07624 .06731 .06198 .05835 .05573 .05380 .05241 .05148 .05101 .05104 .05171 .05334 .05672 .06447 .09637
17	.09349 .07276 .06425 .05916 .05567 .05312 .05122 .04980 .04879 .04816 .04791 .04811 .04890 .05080 .05398 .06157 .09253
18	.08939 .06962 .06150 .05662 .05326 .05079 .04892 .04750 .04645 .04571 .04530 .04522 .04555 .04643 .04818 .05154 .05898 .08906
19	.08568 .06677 .05900 .05432 .05109 .04870 .04687 .04546 .04438 .04358 .04306 .04280 .04285 .04329 .04424 .04602 .04935 .05654 .08591
20	.08230 .06418 .05672 .05223 .04911 .04680 .04501 .04362 .04253 .04170 .04111 .04074 .04061 .04077 .04127 .04228 .04407 .04738 .05452 .08304
21	.07921 .06180 .05464 .05031 .04730 .04506 .04333 .04196 .04087 .04002 .03938 .03894 .03870 .03867 .03890 .03947 .04051 .04232 .04559 .05259 .08041
22	.07637 .05962 .05273 .04855 .04565 .04348 .04179 .04044 .03936 .03851 .03784 .03735 .03702 .03688 .03693 .03723 .03784 .03891 .04073 .04396 .05083 .07799
23	.07376 .05761 .05096 .04693 .04412 .04201 .04037 .03905 .03799 .03713 .03645 .03592 .03554 .03531 .03525 .03537 .03572 .03637 .03746 .03927 .04247 .04920 .07575
24	.07133 .05574 .04932 .04543 .04271 .04066 .03906 .03777 .03673 .03587 .03518 .03463 .03421 .03393 .03378 .03378 .03395 .03434 .03502 .03613 .03793 .04109 .04770 .07368
25	.06908 .05401 .04780 .04403 .04140 .03941 .03785 .03659 .03556 .03472 .03402 .03346 .03301 .03269 .03248 .03239 .03245 .03266 .03309 .03379 .03491 .03671 .03982 .04632 .07175
26	.06699 .05239 .04638 .04273 .04018 .03825 .03673 .03550 .03449 .03365 .03295 .03238 .03192 .03157 .03132 .03117 .03114 .03123 .03148 .03193 .03266 .03378 .03557 .03864 .04502 .06995
27	.06503 .05089 .04505 .04152 .03904 .03716 .03568 .03448 .03348 .03266 .03197 .03138 .03092 .03055 .03026 .03007 .02998 .02999 .03012 .03040 .03087 .03161 .03274 .03451 .03756 .04382 .06826
28	.06320 .04947 .04382 .04038 .03797 .03614 .03470 .03352 .03255 .03174 .03105 .03048 .03000 .02961 .02930 .02908 .02894 .02889 .02893 .02910 .02940 .02989 .03064 .03177 .03353 .03653 .04270 .06668
29	.06148 .04814 .04265 .03931 .03697 .03519 .03378 .03263 .03168 .03087 .03020 .02962 .02914 .02874 .02842 .02817 .02800 .02790 .02788 .02796 .02815 .02848 .02899 .02974 .03087 .03262 .03558 .04164 .06519
30	.05986 .04690 .04156 .03831 .03602 .03429 .03292 .03179 .03086 .03007 .02940 .02883 .02835 .02794 .02761 .02734 .02714 .02701 .02694 .02696 .02707 .02728 .02762 .02814 .02890 .03003 .03176 .03468 .04065 .06379
31	.05834 .04572 .04052 .03736 .03513 .03344 .03210 .03100 .03009 .02931 .02865 .02809 .02760 .02719 .02685 .02657 .02635 .02619 .02609 .02606 .02611 .02623 .02647 .02682 .02735 .02812 .02924 .03088 .03384 .03972 .06247
32	.05690 .04461 .03954 .03646 .03429 .03264 .03133 .03026 .02936 .02860 .02795 .02739 .02691 .02650 .02615 .02586 .02562 .02544 .02532 .02525 .02525 .02531 .02546 .02571 .02608 .02661 .02738 .02850 .03021 .03305 .03884 .06122
33	.05554 .04355 .03862 .03561 .03350 .03189 .03061 .02956 .02868 .02793 .02729 .02673 .02626 .02584 .02549 .02519 .02495 .02475 .02461 .02451 .02447 .02449 .02457 .02474 .02500 .02538 .02592 .02669 .02781 .02949 .03230 .03801 .06003
34	.05425 .04256 .03774 .03481 .03274 .03117 .02992 .02889 .02803 .02729 .02666 .02611 .02564 .02523 .02488 .02457 .02432 .02411 .02395 .02383 .02376 .02374 .02378 .02389 .02406 .02434 .02473 .02527 .02604 .02718 .02882 .03160 .03722 .05890
35	.05302 .04161 .03691 .03405 .03203 .03049 .02926 .02826 .02741 .02669 .02607 .02553 .02506 .02465 .02430 .02399 .02373 .02351 .02334 .02320 .02311 .02306 .02307 .02312 .02324 .02343 .02371 .02411 .02466 .02543 .02653 .02819 .03093 .03648 .05783

36	.05186	.04071	.03612	.03332	.03135	.02984	.02884	.02786	.02683	.02612	.02551	.02497	.02451	.02410
	.02375	.02344	.02318	.02295	.02276	.02262	.02251	.02244	.02241	.02243	.02250	.02263	.02284	.02313
	.02353	.02408	.02485	.02595	.02759	.03030	.03577	.05681						
37	.05075	.03985	.03536	.03263	.03070	.02923	.02805	.02709	.02627	.02557	.02497	.02446	.02398	.02358
	.02323	.02292	.02266	.02242	.02223	.02207	.02195	.02186	.02181	.02180	.02184	.02192	.02207	.02228
	.02257	.02298	.02354	.02431	.02539	.02702	.02970	.03508	.05583					
38	.04970	.03903	.03464	.03197	.03008	.02864	.02748	.02654	.02574	.02506	.02447	.02395	.02349	.02309
	.02274	.02243	.02216	.02193	.02173	.02156	.02143	.02133	.02126	.02123	.02123	.02126	.02138	.02153
	.02175	.02206	.02246	.02302	.02379	.02487	.02648	.02912	.03445	.05490				
39	.04869	.03825	.03396	.03134	.02949	.02808	.02695	.02602	.02524	.02456	.02398	.02347	.02302	.02263
	.02228	.02197	.02170	.02146	.02126	.02108	.02094	.02083	.02074	.02069	.02067	.02069	.02075	.02086
	.02102	.02125	.02156	.02197	.02253	.02329	.02437	.02596	.02858	.03384	.05401			
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	.02184	.02153	.02126	.02102	.02081	.02063	.02048	.02036	.02026	.02020	.02016	.02016	.02019	.02026
	.02037	.02054	.02078	.02109	.02151	.02207	.02283	.02389	.02547	.02806	.03325	.05316		
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	.02142	.02111	.02084	.02060	.02039	.02020	.02005	.01992	.01981	.01973	.01968	.01966	.01967	.01971
	.01979	.01991	.02009	.02033	.02065	.02107	.02162	.02238	.02344	.02500	.02756	.03269	.05234	
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	.02101	.02071	.02044	.02020	.01999	.01980	.01964	.01950	.01939	.01930	.01924	.01920	.01919	.01921
	.01926	.01935	.01948	.01966	.01990	.02022	.02064	.02120	.02195	.02300	.02456	.02709	.03215	.05156
43	.04508	.03545	.03149	.02908	.02737	.02607	.02502	.02416	.02343	.02280	.02225	.02177	.02135	.02097
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8	.15822	.12950	.11840	.11240	.10980	.11000	.11630	.14787						
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15	.09230	.07683	.07006	.06588	.06300	.06091	.05938	.05834	.05772	.05756	.05797	.05921	.06189	.06804
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	.02021	.02100	.02215	.02396	.02743	.03903								
49	.03417	.02872	.02633	.02483	.02375	.02290	.02222	.02164	.02115	.02072	.02034	.02001	.01971	.01943
	.01919	.01897	.01876	.01858	.01841	.01826	.01812	.01799	.01788	.01778	.01770	.01762	.01756	.01751
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6	18726	16588	15713	15387	15668	1781[illegible]								
7	16318	14449	13641	13238	13152	135[illegible]	15662							
8	14485	12827	12090	11680	11490	11520	11860	13854						
9	13041	11550	10880	10480	10260	10170	10270	10730	12613					
10	11870	10520	09902	09528	09294	09186	09150	09289	09749	11530				
11	10910	09668	09097	08746	08515	08370	08303	08326	08488	08946	10630			
12	10090	08951	08421	08092	07868	07717	07627	07599	07649	07825	08276	09883		
13	09397	08338	07845	07536	07321	07170	07069	07016	07014	07082	07266	07707	09240	
14	08796	07808	07347	07056	06851	06703	06598	06531	06502	06519	06599	06787	07219	08685
15	08271	07346	06913	06637	06442	06298	06193	06119	06075	06064	06094	06182	06372	06793
	08200													
16	07809	06937	06529	06269	06083	05944	05839	05762	05710	05684	05686	05726	05819	06009
	06420	07772												
17	07398	06575	06189	05942	05764	05631	05528	05450	05393	05358	05344	05356	05402	05500
	05649	06090	07391											
18	07030	06250	05884	05649	05480	05351	05261	05173	05114	05073	05049	05045	05065	05117
	05217	05404	05795	07050										
19	06699	05958	05610	05386	05224	05100	05003	04926	04866	04822	04792	04778	04781	04806
	04862	04964	05149	05530	06743									
20	06400	05693	05361	05148	04993	04874	04779	04704	04644	04598	04564	04543	04536	04545
	04575	04633	04736	04919	05290	06465								
21	06127	05452	05135	04931	04782	04668	04576	04503	04443	04396	04360	04335	04321	04320
	04333	04367	04427	04530	04711	05073	06211							
22	05878	05232	04929	04733	04590	04480	04391	04319	04261	04213	04176	04148	04129	04121
	04124	04142	04178	04240	04343	04521	04875	05979						
23	05649	05030	04739	04551	04414	04307	04222	04152	04094	04046	04008	03978	03957	03944
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	03628	03623	03626	03639	03664	03705	03769	03871	04041	04371	05387			
26	05064	04511	04252	04085	03962	03866	03788	03724	03669	03624	03586	03554	03528	03508
	03493	03485	03483	03489	03504	03530	03572	03637	03738	03904	04227	05218		
27	04896	04363	04113	03951	03832	03739	03664	03601	03548	03504	03466	03434	03407	03386
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28	04738	04224	03983	03826	03711	03621	03548	03487	03436	03392	03356	03323	03296	03274
	03257	03244	03236	03233	03236	03245	03263	03292	03336	03400	03499	03660	03970	04913
29	04593	04085	03861	03710	03598	03511	03440	03381	03331	03288	03251	03219	03192	03170
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	03352	03643	04525											
32	04207	03752	03539	03401	03300	03220	03155	03100	03053	03013	02978	02947	02921	02898
	02878	02862	02849	02838	02831	02827	02827	02831	02840	02854	02876	02907	02952	03015
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33	04092	03651	03444	03310	03211	03134	03070	03017	02971	02932	02898	02868	02841	02819
	02799	02782	02768	02757	02748	02743	02741	02742	02747	02757	02772	02794	02826	02870
	02933	03027	03176	03457	04302									
34	03985	03555	03354	03224	03128	03052	02991	02939	02894	02856	02822	02793	02767	02744
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35	03883	03465	03269	03142	03049	02976	02915	02865	02821	02784	02751	02721	02696	02673
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36	03786	03379	03189	03065	02974	02903	02844	02785	02752	02715	02683	02654	02629	02607
	02587	02570	02555	02542	02531	02523	02517	02513	02511	02512	02517	02524	02536	02553
	02577	02609	02653	02715	02806	02947	03214	04012						
37	03685	03288	03112	02982	02903	02834	02776	02728	02686	02650	02619	02591	02566	02544
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	02489	02513	02546	02589	02650	02739	02879	03142	03824					
38	03808	03221	03040	02922	02838	02768	02712	02665	02624	02589	02558	02530	02506	02484
	02465	02447	02432	02419	02408	02398	02390	02384	02380	02379	02379	02382	02388	02397
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	02616	.02607	.02599	.02592	02586	.02582	02578	.02575	.02574	.02573	.02575	02577	.02582	.02589
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	02522	.02532	02545	.02562	.02585	02617	.02662	.02732	.02857	03187				
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	.02046	.02038	02031	02024	02018	02012	.02007	.02002	.01998	.01995	.01991	.01989	.01986	.01985
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9	.11290	.11160	.11090	.11050	.11030	.11020	.11030	.11080	.11250					
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	.06806													
16	.06389	.06318	.06281	.06255	.06237	.06222	.06211	.06203	.06197	.06194	.06194	.06199	.06209	.06230
	.06272	.06388												
17	.06017	.05950	.05915	.05892	.05874	.05860	.05849	.05841	.05835	.05831	.05829	.05831	.05836	.05847
	.05867	.05908	.06019											
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	.05524	.05545	.05584	.05690										
19	.05391	.05331	.05300	.05278	.05262	.05250	.05240	.05231	.05225	.05220	.05217	.05215	.05216	.05218
	.05225	.05236	.05256	.05295	.05398									
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22	.04653	.04611	.04585	.04566	.04553	.04542	.04532	.04525	.04519	.04514	.04509	.04506	.04504	.04503
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23	.04462	.04413	.04388	.04370	.04357	.04346	.04338	.04330	.04324	.04319	.04315	.04312	.04309	.04308
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	.03963	.03963	.03963	.03965	.03967	.03972	.03979	.03990	.04008	.04041	.04122			
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	.03671	.03670	.03669	.03669	.03670	.03672	.03675	.03680	.03687	.03698	.03715	.03746	.03823	
28	.03673	.03633	.03612	.03598	.03587	.03579	.03571	.03565	.03560	.03555	.03551	.03548	.03545	.03542
	.03540	.03539	.03538	.03538	.03538	.03539	.03541	.03545	.03549	.03557	.03567	.03584	.03614	.03689
29	.03548	.03509	.03489	.03476	.03465	.03457	.03450	.03444	.03438	.03434	.03430	.03427	.03424	.03421
	.03419	.03418	.03416	.03416	.03415	.03416	.03418	.03420	.03423	.03428	.03435	.03446	.03462	.03492
	.03564													
30	.03431	.03394	.03374	.03361	.03351	.03343	.03336	.03330	.03325	.03321	.03317	.03314	.03311	.03308
	.03306	.03304	.03303	.03302	.03302	.03302	.03303	.03304	.03306	.03310	.03315	.03322	.03332	.03348
	.03377	.03448												
31	.03321	.03285	.03267	.03254	.03244	.03236	.03230	.03224	.03219	.03215	.03211	.03208	.03205	.03202
	.03200	.03199	.03197	.03196	.03195	.03195	.03196	.03196	.03198	.03200	.03204	.03209	.03216	.03226
	.03242	.03270	.03339											
32	.03219	.03184	.03166	.03153	.03144	.03136	.03130	.03124	.03120	.03116	.03112	.03109	.03106	.03103
	.03101	.03099	.03098	.03097	.03096	.03095	.03095	.03096	.03097	.03099	.03101	.03105	.03109	.03116
	.03127	.03142	.03170	.03237										
33	.03122	.03088	.03071	.03059	.03050	.03042	.03036	.03031	.03026	.03022	.03019	.03015	.03013	.03010
	.03008	.03006	.03005	.03003	.03002	.03002	.03002	.03002	.03002	.03003	.03006	.03008	.03011	.03016
	.03023	.03033	.03048	.03075	.03140									
34	.03031	.02999	.02982	.02970	.02961	.02954	.02948	.02943	.02938	.02934	.02931	.02928	.02925	.02922
	.02920	.02918	.02917	.02916	.02915	.02914	.02913	.02913	.02913	.02914	.02915	.02917	.02920	.02923
	.02928	.02936	.02946	.02960	.02986	.03050								
35	.02945	.02914	.02897	.02886	.02878	.02871	.02865	.02860	.02855	.02851	.02848	.02845	.02842	.02840
	.02838	.02836	.02834	.02833	.02832	.02831	.02830	.02830	.02830	.02830	.02831	.02832	.02834	.02837
	.02841	.02845	.02852	.02862	.02877	.02902	.02945							

36	02865	02834	02818	02807	02799	02792	02786	02781	02777	02773	02770	02767	02764	02762
	02760	02758	02756	02755	02754	02753	02752	02751	02751	02751	02752	02753	02754	02756
	02758	02762	02767	02774	02784	02798	02823	02864						
37	02788	02758	02742	02732	02724	02717	02712	02707	02703	02699	02696	02693	02690	02688
	02686	02684	02682	02681	02680	02679	02678	02677	02677	02677	02677	02678	02679	02680
	02682	02685	02688	02693	02700	02709	02724	02748	02808					
38	02715	02685	02671	02661	02653	02647	02641	02637	02633	02629	02626	02623	02620	02618
	02616	02614	02612	02611	02610	02609	02608	02607	02607	02606	02606	02607	02607	02609
	02610	02612	02615	02618	02623	02630	02639	02653	02677	02735				
39	02646	02618	02603	02593	02586	02580	02574	02570	02566	02562	02559	02556	02554	02552
	02550	02548	02546	02545	02543	02542	02541	02541	02540	02540	02540	02540	02540	02541
	02542	02543	02546	02548	02552	02557	02563	02572	02586	02610	02657			
40	02581	02553	02539	02529	02522	02516	02511	02506	02502	02499	02496	02493	02491	02489
	02486	02485	02483	02482	02480	02479	02478	02477	02477	02476	02476	02476	02476	02477
	02478	02479	02480	02482	02485	02489	02493	02500	02509	02522	02546	02601		
41	02519	02492	02478	02468	02461	02455	02450	02446	02442	02439	02436	02433	02431	02429
	02427	02425	02423	02422	02420	02419	02418	02417	02417	02416	02416	02416	02416	02416
	02417	02417	02419	02420	02422	02425	02429	02433	02440	02449	02462	02485	02539	
42	02459	02433	02419	02410	02403	02397	02393	02388	02385	02381	02379	02376	02374	02371
	02369	02368	02366	02364	02363	02362	02361	02360	02359	02359	02358	02358	02358	02358
	02359	02359	02360	02361	02363	02365	02368	02371	02376	02382	02391	02404	02427	02480
43	02403	02377	02364	02355	02348	02342	02338	02333	02330	02327	02324	02321	02319	02317
	02315	02313	02312	02310	02309	02308	02307	02306	02305	02304	02304	02303	02303	02303
	02303	02304	02305	02306	02307	02308	02310	02313	02317	02321	02328	02338	02349	02371
	02424													
44	02349	02324	02311	02302	02295	02290	02285	02281	02278	02274	02272	02269	02267	02265
	02263	02261	02260	02258	02257	02256	02255	02254	02253	02252	02252	02251	02251	02251
	02251	02251	02252	02252	02253	02255	02256	02258	02261	02265	02269	02275	02284	02297
	02318	02370												
45	02297	02273	02260	02251	02245	02239	02235	02231	02228	02225	02222	02219	02217	02215
	02213	02211	02210	02208	02207	02206	02205	02204	02203	02202	02202	02201	02201	02201
	02201	02201	02201	02202	02203	02204	02205	02207	02209	02211	02215	02219	02226	02234
	02246	02268	02318											
46	02248	02224	02211	02203	02197	02191	02187	02183	02180	02177	02174	02172	02169	02167
	02166	02164	02162	02161	02160	02158	02157	02156	02155	02155	02154	02154	02153	02153
	02153	02153	02153	02154	02154	02155	02156	02157	02159	02161	02164	02167	02172	02178
	02186	02198	02219	02269										
47	02200	02177	02165	02157	02150	02145	02141	02137	02134	02131	02128	02126	02124	02122
	02120	02118	02117	02115	02114	02113	02112	02111	02110	02109	02109	02108	02108	02107
	02107	02107	02107	02107	02108	02108	02109	02110	02112	02113	02116	02118	02122	02126
	02132	02140	02152	02173	02222									
48	02155	02132	02120	02112	02106	02101	02097	02093	02090	02087	02085	02082	02080	02078
	02076	02075	02073	02072	02071	02069	02068	02067	02067	02066	02065	02064	02064	02064
	02063	02063	02063	02063	02064	02064	02065	02066	02067	02068	02070	02072	02075	02078
	02082	02088	02096	02108	02128	02176								
49	02111	02089	02077	02070	02064	02059	02055	02051	02048	02045	02043	02040	02038	02036
	02035	02033	02031	02030	02029	02028	02027	02026	02025	02024	02023	02023	02022	02022
	02021	02021	02021	02021	02021	02022	02022	02023	02024	02025	02026	02028	02030	02033
	02036	02040	02046	02054	02066	02085	02133							
50	02070	02048	02035	02028	02023	02018	02014	02010	02007	02005	02002	02000	01998	01996
	01994	01993	01991	01990	01989	01987	01986	01985	01985	01984	01983	01982	01982	01981
	01981	01981	01981	01981	01981	01981	01981	01982	01982	01983	01984	01986	01988	01990
	01992	01996	02000	02006	02014	02025	02045	02091						

Appendix V: Evaluation of moments of $EV_I(\xi,b)$

By definition, for $p > 0$

$$\psi(p) = \frac{\Gamma'(p)}{\Gamma(p)} \ .$$

But, for any positive integer k

$$\Gamma^{(k)}(p) = \int_0^\infty x^{p-1} \exp(-x)(\ell n\ x)^k dx$$

and

$$\Gamma^{(k)}(1) = \lambda_1 = \int_0^\infty (\ell n\ x)e^{-x}\ dx \ .$$

In general, define $\lambda_i = \int_0^\infty (\ell n\ x)^i e^{-x}\ dx$. In particular $\lambda_1 = \Gamma^{(1)}(1) = -\gamma = -.5772$. From Jahnke (1960) (p. 12)

$$\psi^{(1)}(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

and, by differentiating successively and setting $z = 1$,

$\psi^{(k)}(z) = (-1)^{k-1}k!\zeta(k+1)$, where $\zeta(x)$ is Riemann's zeta function.

Using values of $\zeta(k)$ [Jahnke (1960, p. 37)], we have

$$\psi^{(1)}(1) = \frac{\pi^2}{6}$$

$$\psi^{(2)}(1) = -2.404$$

$$\psi^{(3)}(1) = \frac{\pi^4}{15}$$

Under the homogeneous model, we obtain

$$\mu_2 = E[\ln X - E(\ln X)]^2 = b^2(\lambda_2-\lambda_1^2)$$

$$\mu_3 = b^3(\lambda_3-3\lambda_2\lambda_1+2\lambda_1^3) = -2.4036b^3$$

$$\mu_4 = b^4(\lambda_4-4\lambda_3\lambda_1+6\lambda_2\lambda_1^2-3\lambda_1^4) = 14.6119b^4$$

(see [Menon (1963)]. Here

$$\lambda_2 = \gamma^2 + \frac{\pi^2}{6}, \quad \lambda_3 = -5.4445, \quad \lambda_4 = 235601.$$

However, under the exchangeable model, using $Y = \ln X$,

$$E_{het}(Y) = \frac{b\gamma}{n}\{n - 1 + \frac{1}{k^*}\}$$

$$E_{het}(Y^2) = \frac{\pi^2 b^2}{6n}\{n - 1 + \frac{1}{k^{*2}}\} + \{\frac{b\gamma}{n}(n - 1 + \frac{1}{k^*})\}^2$$

$$E_{het}(Y^3) = \frac{n-1}{n}(-b)^3\psi^{(2)}(1) + \frac{1}{n}\left(\frac{-b}{k^*}\right)^3\psi^{(2)}(1)$$

$$= \frac{\psi^{(2)}(1)(-b^3)}{n}\left\{n - 1 + \frac{1}{k^{*3}}\right\}$$

$$E_{het}(Y^4) = \frac{n-1}{n}(b^4)\psi^{(3)}(1) + \frac{1}{n}\left(\frac{-b}{k^*}\right)^4\psi^{(3)}(1)$$

$$= \frac{\psi^{(3)}(1)(b^4)}{n}\left\{n - 1 + \frac{1}{k^{*4}}\right\}$$

where $E_{\text{homogeneous}}(Y^r) = -b^r\psi^{(r)}(1)$ [see Patel et al (1976)].

Appendix VI Computer Programs

Program A was used to compute $u(r,n,k^*)$ for the Weibull distribution using the recursive formula from Chapter V.3, p. 103. Extended precision Fortran was used to perform the calculations.

Program A

```
      IMPLICIT LOGICAL*1 (A-Z)
      DIMENSION U(50,50), EFUN(501), UBAR(50)
      REAL*16 U, EFUN, UBAR, SUM, K, C, Y, YINCR
      INTEGER I, J, N, R
C
      WRITE(6,200)
 200  FORMAT(' Enter K (with decimal point) and N
     1       (integer)'/ each followed by a comma.')
      READ(5,100) K, N
 100  FORMAT(F7.5,I2)
      WRITE(7,201) K, N
 201  FORMAT('1K = ',F7.5,/' N = ',I2,/'0')
C
C INITIALIZATIONS
C
      DO 300 I=1,N
        DO 301 J=1,N
          U(J,I) = 0.0Q0
 301    CONTINUE
 300  CONTINUE
C
      YINCR = (1.Q0/500.Q0)
  DO 302 I=1,501
    Y = QFLOAT(I-1)*YINCR
    IF (Y .GT. 1.Q-30) GO TO 3021
    EFUN(I) = 0.0Q0
    GO TO 302
1   EFUN(I) = QEXP(-(QABS(QLOG(1.Q0/Y)))**(1.Q0/K))
  CONTINUE
CALCULATE VALUES OF U-BAR (FIRST COLUMN OF U).
USE SIMPSON'S (1/3) RULE FOR THE INTEGRATION.
  DO 310 R=1,N
    IF (EFUN(1) .LT. 1.Q-70) EFUN(1) = 1.Q-70
    SUM = EFUN(1)**(R-1)
    DO 311 J=1,249
      IF (EFUN(2*J) .LT. 1.Q-70) EFUN(2*J) = 1.Q-70
      IF (EFUN(2*J+1) .LT. 1.Q-70) EFUN(2*J+1) = 1.Q-70
      SUM = SUM + 4.Q0*EFUN(2*J)**(R-1)
1                + 2.Q0*EFUN(2*J+1)**(R-1)
311     CONTINUE
        SUM = SUM + 4.Q0*EFUN(500)**(R-1) + EFUN(501)**(R-1)
        UBAR(R) = SUM*(.002Q0/3.Q0)
310   CONTINUE
C
C SET 1ST COLUMN OF U EQUAL TO UBAR
C
      DO 315 I=1,N
        U(I,1) = UBAR(I)
315   CONTINUE
```

```
      WRITE(7,250) U(1,1)
  250 FORMAT(14('0',F8.5)/,3(3X,14(1X,F8.5)/)/)
C
C  CALCULATE THE REST OF U AND WRITE IT OUT.
C
      DO 320 I=2,N
        DO 321 R=2,I
          SUM = 0.Q0
          DO 322 J=1,R
            CALL COMB(R-1,J-1,C)
            SUM = SUM + C*((-1)**(J-1))*UBAR(I-R+J)
  322     CONTINUE
          CALL COMB(I-1,R-1,C)
          U(I,R) = C*SUM
  321   CONTINUE
        WRITE(7,250) (U(I,R),R=1,I)
  320 CONTINUE
      STOP
      END
      SUBROUTINE COMB(I,J,C)
      IMPLICIT REAL*16 (A-H,K,O-Z)
      REAL*16 K(15)
C
      DO 300 IK=1,15
        K(IK) = 0.Q0
  300 CONTINUE
C
      LSTOP = MIN0(I-J,J)
      L=1
      PN=QFLOAT(L)
      PD=QFLOAT(L)
      IF (LSTOP .LE. 0) GO TO 3999
      DO 3000 L=1,LSTOP
        A=QFLOAT(L)
        PD=A*PD
        M=I-L+1
        A=QFLOAT(M)
        PN=A*PN
        DO 3001 IK=1,15
          K(IK) = K(IK)+1
 3001   CONTINUE
        IF (K(1) .NE. 2.Q0) GO TO 3101
          FACT=2.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(1) = 0.Q0
 3101   CONTINUE
        IF (K(2) .NE. 3.Q0) GO TO 3102
          FACT=3.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(2) = 0.Q0
```

```
3102   CONTINUE
       IF (K(3) .NE. 5.Q0) GO TO 3103
         FACT=5.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(3) = 0.Q0
3103   CONTINUE
       IF (K(4) .NE. 7.Q0) GO TO 3104
         FACT=7.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(4) = 0.Q0
3104   CONTINUE
       IF (K(5) .NE. 11.Q0) GO TO 3105
         FACT=11.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(5) = 0.Q0
3105   CONTINUE
       IF (K(6) .NE. 13.Q0) GO TO 3106
         FACT=13.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(6) = 0.Q0
3106   CONTINUE
       IF (K(7) .NE. 17.Q0) GO TO 3107
         FACT=17.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(7) = 0.Q0
3107   CONTINUE
       IF (K(8) .NE. 19.Q0) GO TO 3108
         FACT=19.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(8) = 0.Q0
3108   CONTINUE
       IF (K(9) .NE. 23.Q0) GO TO 3109
         FACT=23.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(9) = 0.Q0
3109   CONTINUE
       IF (K(10) .NE. 29.Q0) GO TO 3110
         FACT=29.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(10) = 0.Q0
```

```
3110   CONTINUE
       IF (K(11) .NE. 31.Q0) GO TO 3111
         FACT=31.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(11) = 0.Q0
3111   CONTINUE
       IF (K(12) .NE. 37.Q0) GO TO 3112
         FACT=37.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(12) = 0.Q0
3112   CONTINUE
       IF (K(13) .NE. 41.Q0) GO TO 3113
         FACT=41.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(13) = 0.Q0
3113   CONTINUE
       IF (K(14) .NE. 43.Q0) GO TO 3114
         FACT=43.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(14) = 0.Q0
3114   CONTINUE
       IF (K(15) .NE. 47.Q0) GO TO 3115
         FACT=47.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(15) = 0.Q0
3115   CONTINUE
3000 CONTINUE
3999 CONTINUE
     C=PN/PD
     RETURN
     END
```

Program B was used to generate 25 random samples each of size 5 with one outlier, starting from a uniform (0,1) distribution. The samples of Weibulls are printed out as X's and transformed to W's, samples of EV_I, by a ℓn transformation. For each sample $\bar{w}$ and s_w^2 are computed. Also for each sample the smallest observation is replaced by the second smallest and by the largest and the new means $\bar{w}_{(2,1)}$, $\bar{w}_{(n,1)}$ and variances $s^2_{(2,1)}$, $s^2_{(n,1)}$ are computed. Also the largest observation is replaced by the second largest observation and by the smallest observation and again means $\bar{w}_{(n-1,n)}$, $\bar{w}_{(1,n)}$ and variances $s^2_{(n-1,n)}$, $s^2_{(1,n)}$ are computed for each sample.

Program B

```
C     This program computes Weibull and EV variables,
C     every fifth one spurious.
C     It prints out samples of size 5
C     along with mean and variance.
C     For each sample it also computes
C     the Winsorized mean and variance
C     by replacing the smallest
C     observation or the largest observation.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  X(125), W(125), W21(125), W51(125),
       W45(125), W15(125)
      DIMENSION WB(25), WV(25), W21B(25), W21V(25),
        W51B(25), W51V(25)
      DIMENSION W45B(25), W45V(25), W15B(25), W15V(25)
      REAL RAN(125)
      INTEGER ISEED(2)
      INTEGER*2 HSEED(3)
      REAL*8 DSEED
      EQUIVALENCE (ISEED(2),HSEED(1))
C
      WRITE(6,200)
 200  FORMAT(' Enter BETA and K')
      READ(5,100) BETA, C
 100  FORMAT(2F12.5)
      WRITE(7,201) BETA, C
 201  FORMAT('1       BETA = ',F12.5,'    K = ',F7.5)
      BETAC = BETA*C
      NSIZE = 5
      NSAMP = 25
      NTOTAL = NSIZE*NSAMP
C
C Get random number depending on time of day into DSEED
C
      CALL TIME(4,0,ISEED)
      HSEED(3) = HSEED(1)
      HSEED(1) = HSEED(2)
      HSEED(2) = HSEED(3)
      DSEED = DABS(DFLOAT(ISEED(2)))
C
C Call *IMSL routine GGUBS to get NTOTAL random
C   numbers into RAN
C
      NR = NTOTAL
      CALL GGUBS(DSEED, NR, RAN)
C
```

```
C Compute Weibull r.v.'s X and W
C
      DO 301 K=1,NTOTAL
        FACT = BETA
        IF (MOD(K,5) .EQ. 0)    FACT = BETAC
        X(K) = DBLE(-ALOG(RAN(K)))**(1.DO/FACT)
        W(K) = DLOG(X(K))
 301  CONTINUE
C
C Print out X and W
C
      WRITE(7,202)
 202  FORMAT('-              X'/)
      WRITE(7,203) X
 203  FORMAT(25(3X,5(1X,E13.6)/))
      WRITE(7,204)
 204  FORMAT('0              W')
      WRITE(7,203) W
C
C Compute Means and Variances for each sample of size
C NSIZE in X and W
C
      K = 1
      DO 310 KK=1,NSAMP
        CALL STAT(W, K, NSIZE, WB(KK), WV(KK))
        K = K+NSIZE
 310  CONTINUE
C
C Find the smallest, second smallest, largest, and second
C largest entry in each of the samples in W.
C
      K = 1
      DO 320 KK=1,NSAMP
        WMIN1 = 1.D75
        WMAX1 = -1.D75
        DO 321 K1=1,NSIZE
          IF (W(K) .GE. WMIN1)  GO TO 3211
            WMIN2 = WMIN1
            WMIN1 = W(K)
            KMIN = K
            GO TO 3212
 3211     CONTINUE
          IF (W(K) .GE. WMIN2)  GO TO 3212
            WMIN2 = W(K)
 3212     CONTINUE
```

```
C
            IF (W(K) .LE. WMAX1)  GO TO 3216
              WMAX2 = WMAX1
              WMAX1 = W(K)
              KMAX = K
              GO TO 3217
 3216       CONTINUE
            IF (W(K) .LE. WMAX2)  GO TO 3217
              WMAX2 = W(K)
 3217       CONTINUE
            K = K + 1
 321      CONTINUE
C
C Replace the smallest and largest entries with
C the second smallest
C and largest and with the second largest and
C smallest entries to
C form the altered samples W(21), W(51), W(45), and W(15).
C Recompute the mean and variance for each of
C the altered samples.
C
          K = K - NSIZE
          DO 322 K1=1,NSIZE
            W21(K) = W(K)
            IF (K .EQ. KMIN)  W21(K) = WMIN2
            W51(K) = W(K)
            IF (K .EQ. KMIN)  W51(K) = WMAX1
            W45(K) = W(K)
            IF (K .EQ. KMAX)  W45(K) = WMAX2
            W15(K) = W(K)
            IF (K .EQ. KMAX)  W15(K) = WMIN1
            K = K + 1
 322      CONTINUE
 320    CONTINUE
C
        K = 1
        DO 330 KK=1,NSAMP
          CALL STAT(W21, K, NSIZE, W21B(KK), W21V(KK))
          CALL STAT(W51, K, NSIZE, W51B(KK), W51V(KK))
          CALL STAT(W45, K, NSIZE, W45B(KK), W45V(KK))
          CALL STAT(W15, K, NSIZE, W15B(KK), W15V(KK))
          K = K + NSIZE
 330    CONTINUE
        WRITE(7,225)
```

```
 225  FORMAT('$**$FORMAT=FMTL1 FONTNEXTIMAGE=
        1200.MEDIUM.9.FIXED.LANDSCAPE.1')
      WRITE(7,250)
 250  FORMAT('1 Sample     WBAR
               WVAR       W(21)BAR W(21)VAR   W(51)
     BAR W(51)VAR   W(45)BAR W(45)VAR
      W(15)BAR W(15)VAR
      DO 350 J=1,NSAMP
C
C Reassign these so the WRITE statement will
C fit on one line.
C
        W1 = WB(J)
        W2 = WV(J)
        W3 = W21B(J)
        W4 = W21V(J)
        W5 = W51B(J)
        W6 = W51V(J)
        W7 = W45B(J)
        W8 = W45V(J)
        W9 = W15B(J)
        W10 = W15V(J)
        WRITE(7,255) J, W1, W2, W3, W4, W5, W6,
            W7, W8, W9, W10
 255    FORMAT(6X,I2,2X,5('  ',2(F8.5,1X)),' ')
 350  CONTINUE
      STOP
      END
      SUBROUTINE STAT(X, KK, NSIZE, XBAR, XVAR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(125)
      KSTOP = KK + NSIZE - 1
      SUM = 0.0D0
      SSQ = 0.0D0
      DO 300 K=KK,KSTOP
        SUM = SUM + X(K)
        SSQ = SSQ + (X(K)**2)
 300  CONTINUE
      XBAR = SUM/DFLOAT(NSIZE)
      XVAR = (SSQ - ((SUM**2)/DFLOAT(NSIZE)))/
             (DFLOAT(NSIZE) - 1
      RETURN
      END
```