University of Alberta

Control Design of Switching Systems by Backstepping

by

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Your file Votre référence
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To my mom and dad
for their moral and financial support
over the many, many
years it took for this degree

Abstract

Switching systems are a particular class of hybrid system consisting of several dynamical subsystems with rules that determines the switching among them. Most references only dealt with continuous switching systems, especially the switching systems with subsystems of same "order". This assumption is restrictive and is not based on theoretical foundations but on mathematical convenience. This thesis proposes a backstepping approach to design stabilizing controllers for a broad class of nonlinear switching systems, which consist of strict feedback subsystems. By switching between sub-controllers, the backstepping-based controller can stabilize the unstable switching system or achieve asymptotic tracking. Meanwhile, the Lyapunov functions in quadric form are obtained simultaneously in the procedure of controller design. Based on the multiple Lyapunov function theorem, some sufficient conditions are given to guarantee the stability of closed-loop switching systems. Finally several examples are given and discussed to illustrate the theoretical results for both same-order and multiple-order switching systems.

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Contents

Chapter 1

Introduction

Switching systems have attracted a great deal of attention in recent years [1, 2, 3, 4, 5]. This is mainly because switching systems constitute a special class of hybrid dynamical systems that take specific and simple forms and thus have numerous applications. There are many practical problems where physical or complexity constraints limit the available choice of controllers and control action must be determined by switching among a finite set of given control laws. Typical real-life examples include car transmission systems, process control systems, mobile robots, etc. We are interested in developing a new, general methodology for designing controllers for a broad class of nonlinear switching systems, especially for those of multiple orders.

1.1 Background

Generally speaking, a switching system can be viewed as a family of continuous-time dynamical subsystems with a rule to determine the switching between them. They have numerous applications in control of mechanical systems, automotive systems, aircraft and air traffic control, switching power converters, and many others. Some examples of such systems are discussed in [6], [7] and [8]. Another source of motivation for studying switching systems comes from the rapidly developing area of switching control. Control techniques based on switching between different controllers have been applied extensively in recent years, particularly in the adaptive context, where they have been shown to achieve stability and improve transient responses.

An overview of general results and ideas regarding switching dynamical systems is given in [1]. The paper surveys recent developments in three basic problems regarding switching dynamics: stability for arbitrary switching signals, stability for slow switching signals, and construction of stabilizing switching signals. These three problems are very general and address fundamental issues concerning stability and design of switching systems. Some theoretical results have been obtained lately for input-to-state stability (ISS) of switching systems in [2] and [3]. Sufficient conditions for switching systems to be input-to-state stable are given under the assumption that all subsystems are input-to-state stable in [9]. A cycle analysis method is used to derive a switching controller for the integral-input-to-state stabilization (iISS) of switching nonlinear systems in [4]. Optimal control problems for both continuous-time and discrete-time switching systems are formulated and investigated in [10]; this paper formulates optimal control problems for switching systems and proposes some solution methods. Both continuous-time and discrete-time switching systems are considered. A two stage optimization method and a dynamic programming (DP) approach are studied in detail. Nonlinear switching systems with state dependent dwell-time are discussed in [11]; this paper analyzes the asymptotic convergence of nonlinear switching systems in the presence of disturbances and discusses the two important cases of locally exponentially stable and feedforward systems. An interesting LMI method for controller design of switching systems is discussed in [12]; the paper establishes a relationship between Lyapunov stability and topological properties of autonomous switching systems.

To solve the control problem for nonlinear switching systems, Lyapunov function theorems are studied and discussed frequently. As is pointed out in [2], the existence of a common ISS Lyapunov triple implies that the switching system is ISS for arbitrary switching, and similarly for iISS. Converse Lyapunov theorems for ISS and iISS of nonlinear switching systems are proved in [3]. Their proofs are based on existing converse Lyapunov theorems for ISS and iISS of nonlinear systems, and on the association of the switching system with a nonlinear system with inputs and disturbances that take values in a compact set. Also, multiple Lyapunov functions are introduced to analyze Lyapunov stability of switching and hybrid systems in [13]. A hybrid nonlinear control methodology for a broad class of switching nonlinear systems with input constraints is proposed in [14]. The key feature of the proposed methodology is the integrated synthesis, via multiple Lyapunov functions, of "lower-level" bounded nonlinear controllers and "upper-level" switching laws that orchestrate the transitions between the constituent dynamical modes of the switching system and their respective controllers. However, these papers fail to propose a general method of controller design for nonlinear switching systems. Thus, though some theorems have been established in previous papers to verify the stability of switching systems such as the common Lyapunov function theorem and the multiple Lyapunov function theorem, controller design for nonlinear switching systems is still an open problem, which is the focus of this thesis.

Switching systems can be classified as either time-dependent or state-dependent systems. Under some conditions, each of them can be transformed into the other [5]. Switching systems can also be classified as continuous and non-continuous switching systems. Continuous switching systems are those where whenever switching occurs between two subsystems, the "states" of both systems are identical. This assumption is restrictive and is not based on theoretical foundations but on mathematical convenience. In particular, with this assumption, all subsystems forming the switching system must be of the same "order". Though some results have been established to verify the stability of switching systems of the same order, such as the common Lyapunov function theorem and the multiple Lyapunov function theorem, switching systems of multiple orders are virtually unexplored and very few results are available in the literature. In addition, controller design for nonlinear switching systems is also a challenging problem. In this thesis, we are interested in developing a methodology for designing controllers for a broad class of nonlinear switching systems, which could be of either same order or multiple orders.

The main tool used in the controller design is the so-called backstepping procedure, one of the most powerful approaches of controller design for nonlinear systems. We propose a new backstepping approach that is capable to stabilize nonlinear switching systems. Our procedure is constructive and permits obtaining Lyapunov functions in the same quadratic form for all subsystems of the switching system.

1.2 Thesis Organization

The rest of this thesis is organized as follows.

In Chapter 2, we first introduce switching systems, especially those of multiple orders. The stability and control of switching systems are discussed with corresponding Lyapunov function theorems. Then the classical backstepping approach for nonlinear systems is introduced at the end of this chapter.

In Chapter 3, we analyze the classical backstepping procedure first. Then we propose a new backstepping procedure to design switching controllers, which can stabilize switching systems of both same and multiple orders. Also in our design procedure, we can obtain Lyapunov functions in the same quadratic form for all subsystems. Thus the stability of closed-loop switching systems in either the time-dependent or state-dependent cases is guaranteed by the Lyapunov function theorem.

In Chapter 4, we prove that the proposed backstepping approach can stabilize nonlinear switching systems if certain sufficient conditions are satisfied. Then several examples are given and discussed in detail. The simulation results show that the proposed backstepping approach can be applied successfully for controller design of nonlinear switching systems.

In Chapter 5, we extend the backstepping design procedure, so that the backstepping-based controller enables the output of a switching system track the expected signal, which could be any first-order differentiable signals. Several examples are given to illustrate the validity of the proposed backstepping approach.

Finally, conclusions of the research are drawn in Chapter 6.

Chapter 2

Switching Systems

In this chapter, some basic definitions and classification of switching systems are given. Then, the stability of switching systems in both same-order and multiple-order cases will be discussed and analyzed in detail.

2.1 Basic Concepts of Switching Systems

By "switching systems", we mean hybrid dynamical systems consisting of a family of continuous-time subsystems and a rule that orchestrates the switching between them. Recently, the study of switching systems has received a great deal of attention. Switching systems have become a rapidly developing area of intelligent control, an important source of motivation for this study. This is mainly because switching systems constitute a special class of hybrid dynamical systems, which have broad applications and take specific and simple forms. The issue of stability of switching systems is considered to be of great importance and therefore has been studied extensively.

Definition 2.1 *Consider a switching system that consists of the following switched subsystems*

$$\dot{x}_i(t) = f_i(x_i(t)), \qquad i \in \{1, ..., N\} \tag{2.1}$$

where $x_i(t) \in R^{n_i}$, with given rules for switching amongst them. We add the following rules:

1. *Each f_i is globally Lipschitz continuous, which guarantees the existence and uniqueness of solutions of each differential equation.*
2. *The i's are picked in such a way that there are finite switches in finite time.*

According to the above definition, we see that a switching system can be viewed as several continuous-time subsystems with (isolated) discrete switching events. A switching system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. This represents a significant departure from hybrid systems, especially at the analysis stage. In switching control design, specifics of the switching mechanism are of greater importance, although typically we will still characterize and exploit only essential properties of the discrete behavior. Having remarked for the purpose of motivation that switching systems can arise from hybrid systems, we henceforth choose switching systems as our focus of study and will generally make no explicit reference to the above connection.

After giving a universal formal definition of a switching system, we now describe several specific categories of systems which will be our main objects of interest. Switching events in switching systems can be classified into

- Autonomous (uncontrolled) versus controlled
- State-dependent versus time-dependent
- Continuous versus non-continuous

Of course, one can have combinations of several types of switching. We now briefly discuss all these possibilities.

a Autonomous (uncontrolled) versus controlled

Switching systems are of "variable structure" or "multi-model"; they are a simple model of (the continuous portion) of hybrid systems. The particular i at any given time may be chosen by some "higher process," such as a controller, computer, or human operator, in which case we say that the system is *controlled*. It may also be a function of time or state or both and receive no effect from outside, in which case we say that the system is *autonomous*.

b State-dependent versus time-dependent

- State-dependent switching

  Suppose that the continuous state space is partitioned into a finite or infinite number of operating regions by means of a family of switching surfaces, or guards. In each of these regions, a continuous-time dynamical system is given. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a reset map. In the simplest case, this is a map whose domain is the union of the switching surfaces and whose range is the entire state space, possibly excluding the switching surfaces (more general reset maps can also be considered, as explained below). We call this kind of switching systems *state-dependent* switching systems. In summary, the system is specified by

  - a family of switching surfaces and the resulting operating regions;
  - a family of continuous-time subsystems, one for each operating region;
  - a reset map.

  In Figure 2.1, the thick curves denote the switching surfaces, the thin curves with arrows denote the continuous portions of the trajectory, and the dashed lines symbolize the jumps. The instantaneous jumps of the continuous state are sometimes referred to as impulse effects. A special case is when such impulse effects are absent, i.e., the reset map is the identity. This means that the state trajectory is continuous everywhere, although in general it is not differentiable when it passes through a switching surface. Most references restrict their attention to systems with no impulse effects. However, many of the results and techniques that we will discuss do generalize to systems with impulse effects.

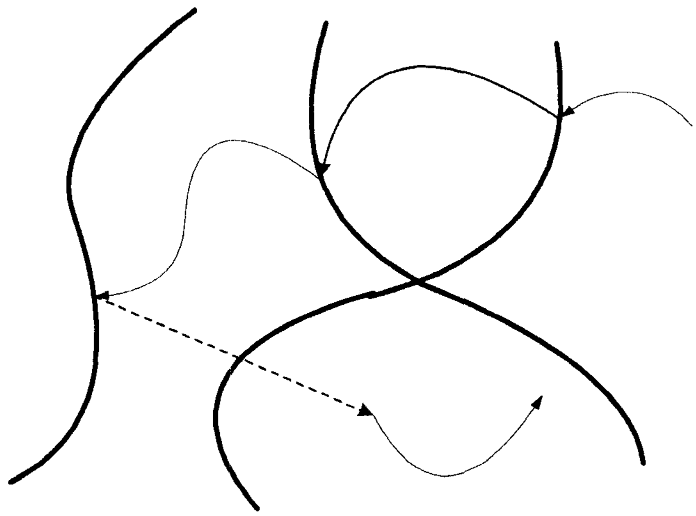

Figure 2.1: Example of a state-dependent switching system

- Time-dependent switching

  Given the family P, we consider the switching system

$$\dot{x}(t) = f_s(x(t), u(t)) \tag{2.2}$$

  where $x(t) \in R^n, u \in L^m_{\infty,e}$ and s is a switching signal, i.e., it is a piecewise constant function $s : [0, +\infty) \to \Gamma$; we will denote by S the family of the switching signals of a given switching system. Associated with each $s \in S$ there is a sequence of real numbers $0 = t_0 < t_1 < ... < t_k < ...$ and a sequence of indexes $\sigma_0, \sigma_1, ..., \sigma_k, ...$ such that $s(t) = \sigma_k$ for all $t_k \leq t < t_{k+1}$. This is a typical example of time-dependent switching system. An example of such a switching signal for the case $S = \{1, 2, 3\}$ is depicted in Figure 2.2.

Note that it is actually difficult to make a formal distinction between state-dependent and time-dependent switching. Each of them can be transferred into the other one. Thus, without losing generality, most of the following work is done on the basis of time-dependent systems.

c Continuous versus non-continuous

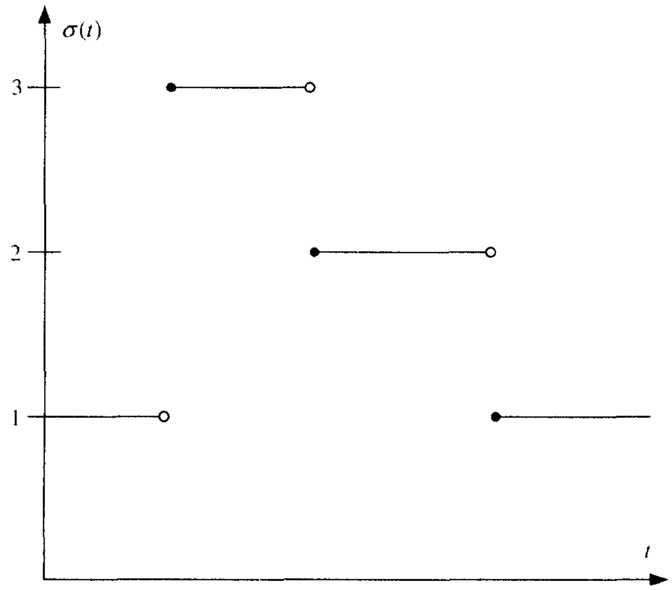

Figure 2.2: Example of a time-dependent switching system

Switching systems can also be classified as continuous and non-continuous systems. Continuous switching systems are those where whenever switching occurs between two subsystems, the "states" of both systems are identical. If that is not the case, then the switching system is called non-continuous. Also according to the values of the n_i, the non-continuous switching systems can be classified into the following two categories.

Case 2.1 *If all of the n_i's are equal, $i \in \{1, \ldots, N\}$, the switching system will be said to be of the same order. Virtually all references dealing with switching systems in the literature consider same-order systems.*

Case 2.2 *If some $i \neq j$, $i, j \in \{1, \ldots, N\}$, we have that $n_i \neq n_j$, the switching system consists of subsystems with different orders.*

In Case 1, we can use established theorem such as the common Lyapunov theorem to study their stability. In this thesis, our main interest is in switching systems of multiple orders. Of course, the result can also be used in switching systems of the same order (Case 1). In Case 2, some of system states only exist in certain subsystems and corresponding time intervals. Thus, the classical definitions and theorems of stability are not applicable. As a consequence, we need to redefine stability for this type of switching systems.

Before introducing the definition for stability for general switching systems, we need to discuss and classify system states and subsystems which play a very important role in this kind of switching systems.

The states of switching systems can be classified into the following categories.

1. Common states. These are the states which exist in all subsystems. They may be continuous or discontinuous at the switching instance. However, they will lose differentiability when switching in most cases.

2. General states. General states do not exist in all of subsystems. They only exist in some of subsystems. In the other subsystems, they are inactive. Thus the reset values of such states when they are activated is a very important issue, which will affect the stability directly. The example in chapter 3 will show this point clearly.

Though the states can be classified into the above two categories, they have effect on each other and determine the stability of switching systems together. This makes the analysis of such switching systems more complex.

In addition to the system states, it is necessary to classify the subsystems forming a switching system into one of the following two categories.

1. Provisional subsystems: These are the subsystems which are only activated before a certain time t_s. This means that when $t > t_s$, they will never be

activated again. Thus some of the general states, which only belong to these provisional subsystems also disappear after t_s.

2. Common subsystems: Aside from provisional subsystems the other subsystems are called common subsystems. After a certain time, all provisional subsystems are no longer activated and the system switches among common subsystems.

A switching system may have no provisional subsystems, but must have at least one common subsystem. Also we can see that, as to the stability of switching systems, common subsystems play a more important role. Compared with the same-order switching system, the multiple-order switching systems show many complex properties because of its own structure. In the following part, we will investigate stability issues for both types of switching systems separately.

2.1.1 Same-order Switching Systems

A same-order switching system is a switching system defined in Definition 1.1 with the equal n_i, $i \in \{1, ..., N\}$. Its stability can be defined as follows:

Definition 2.2 *We will say that a same-order switching system is uniformly asymptotically stable if there exist a positive constant δ and a class $\mathcal{KL}$ function β such that for all switching signals σ, the solutions of 2.1 with $|x(0)| \leq \delta$ satisfy the inequality*

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall t \geq 0 \tag{2.3}$$

If the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some c,$\lambda > 0$, so that the above inequality takes the form

$$|x(t)| \leq c|x(0)|e^{-\lambda t} \qquad \forall t \geq 0 \tag{2.4}$$

then the system (2.1) is called uniformly exponentially stable. If the inequalities (2.3) and (2.4) are valid for all switching signals and all initial conditions, we obtain *global uniform asymptotic stability (GUAS) and global uniform exponential stability (GUES), respectively.*

Equivalent definitions can be given in terms of properties of solutions. The term "uniform" is used here to describe uniformity with respect to switching signals. This is not to be confused with the more common usage which refers to uniformity with respect to the initial time for time-varying systems.

Traditional Lyapunov's stability theorem has a direct extension which provides a basic tool for studying uniform stability of the same-order switching systems. This extension is obtained by requiring the existence of a single Lyapunov function whose derivative along solutions of all systems in the family of system (2.1) satisfies suitable inequalities. We are particularly interested in obtaining a Lyapunov condition for GUAS. To do this, we must take special care in formulating a counterpart of the Lyapunov inequality which ensures a uniform rate of decay.

Given a positive definite continuously differentiable (C^1) function V: $R^n \longrightarrow R$, we will say that it is a common Lyapunov function for the family of system (2.1) if there exists a positive definite continuous function W: $R^n \longrightarrow R$ such that we have

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x) \qquad \forall x, \forall p \in \{1, ..., N\}. \tag{2.5}$$

The following result will be used in the next chapters.

Theorem 2.1 *If all systems in the family (2.1) share a radially unbounded common Lyapunov function, then the switching system is GUAS.*

This theorem is well know and can be derived in the same way as the standard Lyapunov stability theorem. The main point is that the rate of decrease of V along solutions, given by (2.5), is not affected by switching, hence asymptotic stability is uniform with respect to σ.

The common Lyapunov function theorem is a very useful theorem for the stability of switching systems. However, It is not easy to find a common Lyapunov function for all of the subsystems in a switching system. Sometime this function may not exist. In this case, one can try to investigate stability of the switching system using the multiple Lyapunov function theorem.

Theorem 2.2 *Let a switching system be a finite family of globally asymptotically stable systems, and let V_p, $p \in \mathcal{P}$ be a family of corresponding radially unbounded Lyapunov functions. Suppose that there exists a family of positive definite continuous functions W_p, $p \in \mathcal{P}$ with the property that for every pair of switching times (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ and $\sigma(t_k) \neq p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)). \tag{2.6}$$

Then the switching system is globally asymptotically stable.

It is possible to obtain less conservative stability conditions involving multiple Lyapunov functions. In particular, one can relax the requirement that each V_p must decrease on the intervals on which the pth system is active, provided that the admissible growth of V_p on such intervals is bounded in a suitable way. Impulse effects can also be incorporated within the same framework.

2.1.2 Multiple-order Switching Systems

In the case of multiple-order switching systems, since extra states only exist in some of subsystems, we can not use the traditional definition of the stability of switching systems. Here, we state the definition of the equilibrium point and stability of the multiple-order switching systems, based on the traditional definitions.

Definition 2.3 *Consider an autonomous switching system (2.1), suppose the equilibrium point of the common subsystems $f_j(x)$ $(j = 1...m)$ of the switching system* *is* x_{ej} $(j = 1...m)$, *which satisfies*

$$f_j(x_{ej}) = 0 \tag{2.7}$$

Because each subsystem space is a subspace of the system space, each of equilibrium points of subsystems is a subset of systems states. Then we say that the union of the elements in these subsets is the equilibrium point of the switching system.

Remark 2.1 *The definition shows the following result: after a certain time, if all provisional subsystems disappear, the system switches among common subsystems. If states of system trajectories converge to* x_e, *the states of system will stay at* x_e, *no matter how the system switches. This definition is consistent with what we have defined for switching systems of same order.*

Remark 2.2 *In definition 1.3,* x_e *is different from the classical definition of equilibrium point, since* x_e *may not be in any of the subsystem spaces. However, it still exists in the system state space. The following simple example shows this point.*

Example 2.1 *Consider a switching system with two subsystems, which are both common subsystems. The first subsystem has three states* x_1, x_2 *and* x_3 *and* $f_1(2, 3, 5) = 0$. *The second subsystem has two states* x_3 *and* x_4 *and* $f_2(5, 7) = 0$. *Then according to the above definition we say that the equilibrium point* x_e *of this switching systems is the combination of these two equilibrium points:*

$$x_e = U_1 \cup U_2 = [x_{10}, x_{20}, x_{30}] \cup [x_{30}, x_{40}] = [2, 3, 5, 7] \tag{2.8}$$

After defining the equilibrium point of switching systems, we give the definitions of stability and convergence for general switching systems.

Definition 2.4 *The equilibrium point* $x = x_e$ *of the system (2.1) is said to be uniformly stable if for any given* $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_m] > 0, \exists\ t_1$ *and a certain region D* *in the space of initial subsystem such that*

$$x(0) \in D \Rightarrow ||x_i(t) - x_{ei}|| < \epsilon_i \quad \forall t \geq t_1 \tag{2.9}$$

Otherwise, the equilibrium point is said to be unstable.

Definition 2.5 *The equilibrium point $x = x_e$ of the system (2.1) is said to be convergent if for any given $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_m] > 0, \exists\ t_1$ and a certain region D in the state space of initial subsystem such that*

$$x(0) \in D \Rightarrow \lim_{t \to \infty} x_i(t) = x_{ei} \quad i = 1, ..., m \tag{2.10}$$

Definition 2.6 *The equilibrium point $x = x_e$ of the system (2.1) is said to be asymptotically stable if it is both stable and convergent. Also if the convergent region D is the same as the whole state space of initial subsystems, then we can say that the equilibrium point is globally asymptotically stable.*

Remark 2.3 *What makes above definitions different from previous ones is that the initial state $x(0)$ is not required to be close to the equilibrium x_e. This requirement is inapplicable to the switching systems of multiple orders since the initial state $x(0)$ may be in different system spaces from x_e.*

Remark 2.4 *The term "uniform" is used here to describe uniformity with respect to switching signals. It is totally different from the uniformity for LTV systems, which is with respect to time.*

2.1.3 Stability and Control of Switching Systems

In [5], Liberzon formulates several basic questions regarding the stability of switching systems. These questions are still the main topics for the study of switching systems. In this thesis, we will discuss some of them and derive some sufficient conditions for the stability of switching systems.

Problem A. Find conditions that guarantee that the switching system is asymptotically stable for any switching signal.

One situation in which Problem A is of great importance is when a given plant is being controlled by means of switching among a family of stabilizing controllers, which is designed for a specific task. If each stabilizing controller can stabilize the corresponding subsystem in the plant, can the multi-controller guarantees the stability of the closed-loop switching system? Unfortunately, the answer is negative.

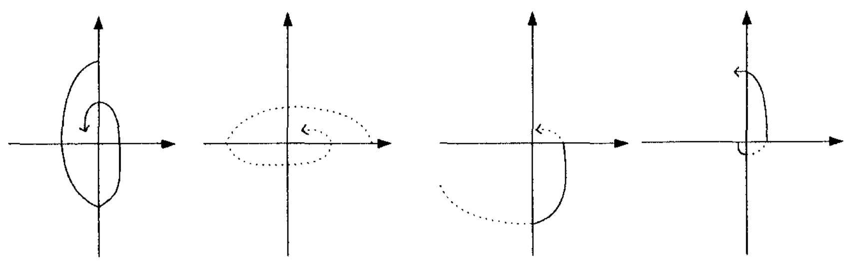

Figure 2.3: Different switching systems with same stable subsystems

Consider the following system where $\mathcal{P} = \{1, 2\}$ and $x \in R^2$, so that we are switching between two systems in the plane. First, suppose that the two individual subsystems are asymptotically stable, with trajectories as shown on the left in Figure 2.3. For different choices of the switching signal, the switching systems might be asymptotically stable or unstable (these two possibilities are shown in Figure 2.3 on the right).

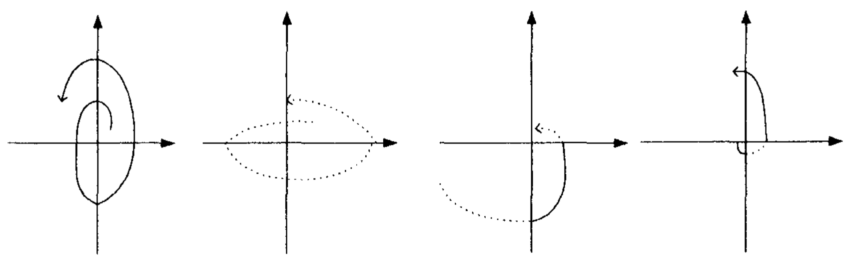

Figure 2.4: Different switching systems with same unstable subsystems

Similarly, Figure 2.4 illustrates the case when both individual subsystems are unstable. Again, the switching system may be either asymptotically stable or unstable, depending on a particular switching signal.

From these two examples, the following facts can be deduced:

- Unconstrained switching may destabilize a switching system even if all individual subsystems are stable.
- It may be possible to stabilize a switching system by means of suitably constrained switching even if all individual subsystems are unstable.

The above example shows that Problem A is not trivial in the sense that it is possible to get instability by switching between asymptotically stable systems. (However, there are certain limitations as to what types of instability are possible in this case. For example, it is easy to see that the trajectories of such a switching systems cannot escape to infinity in finite time.) If this happens, one may ask whether the switching system will be asymptotically stable for certain useful classes of switching signals. This leads to the following problem.

Problem B. Identify those classes of switching signals for which the switching system is asymptotically stable.

Since it is often unreasonable to exclude constant switching signals of the form $\sigma(t) \equiv p$, Problem B will be considered under the assumption that all individual subsystems are asymptotically stable. Basically, we will find that stability is ensured if the switching is sufficiently slow.

Several theorems have been established to show that slow switching can stabilize the switching system under certain conditions. The simplest way that they use to specify slow switching is to introduce a number $\tau > 0$ and restrict the class of admissible switching signals to signals with the property that the interval between any two consecutive switching times is no smaller than τ. This number τ is sometimes called the dwell time (because σ "dwells" on each of its values for at least τ units of time). It is a fairly well-known fact that when all subsystems are asymptotically stable, the linear switching system is globally asymptotically stable if the dwell time τ is large enough. In fact, The required lower bound on τ can be explicitly calculated from the parameters of the individual subsystems. For details, see [20] (Lemma 2].

The other tool is the so-called average dwell time. For each $T > 0$, let $N_\sigma(T)$ denote the number of discontinuities of a given switching signal σ on the interval $[0, T]$. We will say that σ has the average dwell time property if there exist two nonnegative numbers a and b such that for all $T > 0$, we have $N_\sigma(T) \leq a + bT$. The study of average dwell-time switching signals is motivated by the following considerations. Stability problems for switching systems arise naturally in the context of switching control. Switching control techniques employing a dwell time have been successfully applied to linear systems with imprecise measurements or modeling uncertainty. In the nonlinear setting, however, such methods are often unsuitable because of the possibility of finite escape time. Namely, if a "wrong" controller has to remain in the loop with an imprecisely modeled system for a specified amount of time, the solution to the system might escape to infinity before we switch to a different controller (of course, this will not happen if all the controllers are stabilizing, but when the system is not completely known, such an assumption is not realistic).

An alternative to dwell-time switching control of nonlinear systems is provided by the so-called hysteresis switching and its scale-independent versions, which were introduced and applied to control of uncertain nonlinear systems in [21] and [22]. When the uncertainty is purely parametric and there is no measurement noise, switching signals generated by scale-independent hysteresis have the property that the switching stops in finite time, whereas in the presence of noise under suitable assumptions they can be shown to have the average dwell time property.

One reason for the increasing popularity of switching control design methods is that sometimes it is actually easier to find a switching controller performing a desired task than to find a continuous one. In fact, there are situations where continuous stabilizing controllers do not exist, which makes switching control techniques especially suitable. In the context of multi-controller systems mentioned in the last question, it might happen that none of the individual controllers stabilize the plant, yet it is possible to find a switching signal that results in an asymptotically stable switching system. We thus formulate the following problem.

Problem C. Construct a switching signal that makes the switching system asymptotically stable.

Of course, if at least one of the individual subsystems is asymptotically stable, the above problem is trivial (just keep $\sigma(t) \equiv p$ where p is the index of this stable subsystem). Therefore, in the context of Problem C, it will be understood that none of the individual subsystems are asymptotically stable. This problem is more of a design problem than a stability problem, but the previous discussion illustrates that all three problems are closely related.

As to the real plant, we cannot control or change the switching sequence or time in most cases. Thus it makes more practical sense for us to design multiple controllers which can stabilize not only the individual subsystems, but also the whole plan without changing switching condition directly. In this thesis, we will design backstepping-based feedback multi-controller which can stabilize the individual subsystems. In the same time, the Lyapunov theorem will guarantee the stability of the closed-loop plant for unconstrained switching, if some additional conditions are satisfied.

2.2 Classical Backstepping Approach

As we know, backstepping is one of the most powerful tools for stabilizing nonlinear systems. By constructing Lyapunov functions, we can design state-feedback controllers for nonlinear systems using backstepping.

2.2.1 Lyapunov Stability Theorem

The main step of backstepping approach is to construct Lyapunov functions for nonlinear systems. Then the following Lyapunov stability theorem will guarantee the stability of nonlinear systems.

Theorem 2.3 *Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, $V : D \to R$ be a continuously differentiable function such that*

(i) $V(0) = 0$,

(ii) $V(x) > 0 \quad \text{in } D - \{0\}$,

(iii) $\dot{V}(x) \leq 0 \quad \text{in } D - \{0\}$,

thus $x = 0$ is stable.

This theorem implies that a sufficient condition for the stability of the equilibrium point $x = 0$ is that there exists a continuously differentiable-positive definite function $V(x)$ such that $\dot{V}(x)$ is negative semi-definite in a neighborhood of $x = 0$. Also if we replace the third condition in the above theorem as follows:

(iii) $\dot{V}(x) < 0 \quad \text{in } D - \{0\}$,

the origin is asymptotically stable.

In other words, the theorem says that asymptotic stability is achieved if the conditions of the above theorem are strengthened by requiring $\dot{V}(x)$ to be negative definite, rather than semi-definite.

The aim of the backstepping approach is to stabilize the nonlinear system by constructing a Lyapunov function.

2.2.2 Integrator Backstepping

Let's consider a system of the form

$$\dot{x} = f(x) + g(x)\xi, \quad (2.11)$$

$$\dot{\xi} = u. \quad (2.12)$$

Here $x \in R^n, \xi \in R$, and $[x, \xi] \in R^{n+1}$ is the state of the system (2.11)-(2.12). The function $u \in R$ is the control input and functions $f, g : D \longrightarrow R^n$ are assumed to be smooth. As will be seen shortly, the importance of this structure is that it can be considered as a cascade connection of the subsystems (2.11) and (2.12). We will make the following assumptions:

(i) The function $f(\cdot) : R^n \longrightarrow R^n$ satisfies $f(0) = 0$. Thus, the origin is an equilibrium point of the subsystem $\dot{x} = f(x)$.

(ii) Consider the subsystem (2.11). Viewing the state variable ξ as an independent "input" for this subsystem, we assume that there exists a state feedback control law of the form

$$\xi = \phi(x), \quad \phi(0) = 0, \quad (2.13)$$

and a Lyapunov function $V_1 : D \rightarrow R^+$ such that

$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x}[f(x) + g(x) \cdot \phi(x)] \leq -V_a(x) \leq 0 \quad \forall x \in D \quad (2.14)$$

where $V_a(\cdot) : D \rightarrow R^+$ is a positive semidefinite function in D.

According to these assumptions, the system (2.11)-(2.12) consists of the subsystem (2.11), for which a known stabilizing law already exists, augmented with a pure integrator (the subsystem (2.12)). More general classes of systems are considered in the next section. We now endeavor to find a state feedback law to asymptotically stabilize the system (2.11)-(2.12).

By adding and subtracting $g(x)\phi(x)$ to the subsystem (2.11), we obtain the equivalent system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(x) + g(x)[\xi - \phi(x)], \qquad (2.15) \\ \dot{\xi} &= u. \end{aligned}$$

Define

$$z = \xi - \phi(x), \qquad (2.16)$$

$$\dot{z} = \dot{\xi} - \dot{\phi}(x) = u - \dot{\phi}(x), \qquad (2.17)$$

where

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi].$$

This change of variables can be seen as "backstepping" $-\phi(x)$ through the integrator. Defining

$$v = \dot{z}$$

the resulting system is

$$\dot{x} = f(x) + g(x)\phi(x) + g(x)z, \qquad (2.18)$$

$$\dot{z} = v. \qquad (2.19)$$

The above two steps are important for the following reasons:

(i) By construction, the system (2.18)-(2.19) is equivalent to the system (2.11)-(2.12).

(ii) The system (2.18)-(2.19) is, once again, the cascade connection of two subsystems. However, the subsystem (2.18) incorporates the stabilizing state feedback law $\phi(\cdot)$ and is thus asymptotically stable when the input is zero. This feature will now be exploited in the design of a stabilizing control law for the overall system (2.11)-(2.12).

To stabilize the system (2.18)-(2.19), the traditional backstepping approach is to choose a Lyapunov function candidate of the form

$$V = (x, \xi) = V_1(x) + \frac{1}{2}z^2. \tag{2.20}$$

We have that

$$\begin{aligned} \dot{V} &= \frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x) + g(x)z] + z\dot{z} \\ &= \frac{\partial V_1}{\partial x}f(x) + \frac{\partial V_1}{\partial x}g(x)\phi(x) + \frac{\partial V_1}{\partial x}g(x)z + zv. \end{aligned} \tag{2.21}$$

And then choose

$$v = -(\frac{\partial \phi}{\partial x}g(x) + kz), \quad k > 0, \tag{2.22}$$

we obtain

$$\begin{aligned} \dot{V} &= \frac{\partial V_1}{\partial x}f(x) + \frac{\partial V_1}{\partial x}g(x)\phi(x) - kz^2 \\ &= \frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x)] - kz^2 \\ &\leq -V_a(x) - kz^2. \end{aligned} \tag{2.23}$$

It then follows by (2.17) that the origin $x = 0, z = 0$ is asymptotically stable. Moreover, since $z = \xi - \phi(x)$ and $\phi(0) = 0$ by assumption, the result also implies that the origin of the original system $x = 0, \xi = 0$ is also asymptotically stable. If all the conditions hold globally and V_1 is radially unbounded, then the origin is globally asymptotically stable. Finally, noticing that, according to (2.17), the stabilizing state feedback law is given by

$$u = \dot{z} + \dot{\phi}, \tag{2.24}$$

we obtain the input

$$u = \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi] - \frac{\partial V_1}{\partial x} g(x) - k[\xi - \phi(x)] \tag{2.25}$$

with the associated Lyapunov function

$$V = V(x, \xi) = V_1(x) + \frac{1}{2}(\xi - \phi(x))^2. \tag{2.26}$$

2.2.3 Strict Feedback Systems

Next consider nonlinear systems in a more general form

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3 \\ &\vdots \\ \dot{x}_{k+1} &= f_k(x_1, x_2, \cdots, x_{k+1}) + g_k(x_1, x_2, \cdots, x_{k+1})u \end{aligned} \tag{2.27}$$

where $x \in R^n$ and f_i, g_i are smooth, for all $i = 1, \cdots k$. Systems of this form are called strict feedback systems because the nonlinearities f, f_i, and g_i depend only on the variables $x_1, x_2, \cdots$ that are fed back. Strict feedback systems are also called triangular systems. We begin our discussion considering the special case where the x_1 system is of order one (equivalently, $k = 1$ in the system defined above):

$$\dot{x}_1 = f(x_1) + g(x_1)x_2 \tag{2.28}$$

$$\dot{x}_2 = f_1(x_1, x_2) + g_1(x_1, x_2)u. \tag{2.29}$$

Assuming that the subsystem (2.28) satisfies assumptions (i) and (ii) of the backstepping procedure, we now endeavor to stabilize (2.28)-(2.29). This system reduces to the integrator backstepping in the special case where $f_1(x_1, x_2) \equiv 0$, $g_1(x_1, x_2) \equiv 1$. To avoid triviality we assume that this is not the case. If $g_1(x_1, x_2) \neq 0$ over the domain of interest, then we can define

$$u = \phi(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}[u_1 - f_1(x_1, x_2)]. \tag{2.30}$$

Substituting (2.30) into (2.28) we obtain the modified system

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u_1 \end{aligned} \tag{2.31}$$

which is of the form (2.11)-(2.12). It then follows that, using (2.25), the stabilizing control law and associated Lyapunov function are

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{\frac{\partial \phi}{\partial x}[f(x_1) + g(x_1)x_2] \\ &\quad -\frac{\partial V_1}{\partial x_1}g(x_1) - k_1[x_2 - \phi(x_1)] - f_1(x_1, x_2)\},\ k_1 > 0 \end{aligned} \tag{2.32}$$

$$V_2(x_1, x_2) = V_1(x) + \frac{1}{2}[x_2 - \phi(x_1)]^2. \tag{2.33}$$

For the system with three states

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3 \\ \dot{x}_3 &= f_2(x_1, x_2, x_3) + g_2(x_1, x_2, x_3)u \end{aligned} \tag{2.34}$$

which can be seen as a special case of (2.28) with

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_2 = x_3, u = u, f = \begin{bmatrix} f + gx_2 \\ f_1 \end{bmatrix}, g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, f_1 = f_2, g_1 = g_2, \tag{2.35}$$

with these definitions, and using the control law and associated Lyapunov function (2.32)-(2.33), we have that a stabilizing control law and associated Lyapunov function are as follows:

$$\begin{aligned} u &= \phi_2(x_1, x_2, x_3) = \frac{1}{g_2(x_1, x_2, x_3)}\{\frac{\partial \phi_1}{\partial x_2}(f + gx_2) + \\ &\quad \frac{\partial \phi_1}{\partial x_2}(f_1(x) + g_1(x)x_3) - k_2[x_3 - \phi_1] - f_2(x_1, x_2, x_3)\},\ k_2 > 0 \end{aligned} \tag{2.36}$$

$$
\begin{aligned}
V_3 &= V_2(x) + \frac{1}{2}[x_3 - \phi_1(x_1, x_2)]^2 \\
&= V_1(x) + \frac{1}{2}[x_2 - \phi(x_1)]^2 + \frac{1}{2}[x_3 - \phi_1(x_1, x_2)]^2.
\end{aligned}
\tag{2.37}
$$

From the above result we see that the final Lyapunov function (2.26) consists of $V_1(x), \xi$ and $\phi(x)$. Therefore, if we want to provide the Lyapunov function in a unified form for several systems, we have to find a common input $\phi(x)$ which can stabilize all systems. This is a very difficult, even impossible task in most cases. Thus care must be exercised when applying this technique to switching systems.

The following chapter contains a procedure to design a backstepping-based controller for switching systems of both same and multiple orders. Following this procedure, we can not only design a controller to stabilize switching systems, but also obtain Lyapunov functions simultaneously for each of subsystems. These Lyapunov functions are in a unified quadratic form which can satisfy the condition of the multiple Lyapunov theorem in most cases.

Chapter 3

New Backstepping Approach

As introduced in the previous chapter, backstepping is one of the most powerful approaches for controller design of nonlinear systems. However, care must be exercised when applying this technique to switching systems, since the individual Lyapunov functions are not in a unified form. In this chapter, we will propose a new backstepping procedure that can not only stabilize the individual subsystems, but also generate the individual Lyapunov functions in the strict quadratic form. As a consequence, the proposed backstepping can be used to design controller for nonlinear switching systems of both same order and multiple orders.

3.1 Classical Form of Backstepping for Switching Systems

Backstepping has been successfully applied to controller design for many years. By constructing Lyapunov functions, we can use the backstepping procedure introduced in the first chapter to design stabilizing controllers for nonlinear systems. When designing the backstepping-based controller, we obtain the Lyapunov function for the nonlinear systems simultaneously, which guarantees closed-loop stability of the origin. Thus it is natural to consider whether we can design the individual backstepping-based controller for each of subsystems and combine these subcontrollers as a synchronous switched controller to stabilize the nonlinear switching system. Unfortunately, the answer is NO.

Consider the example given in Figure 2.4. By choosing certain switching rule, the switching system can be unstable even if all of subsystems are stable. Thus, though the individual backstepping-based controller can stabilize each of subsystems, it can still not guarantee the stabilization of the switching system. Even if we can obtain the individual Lyapunov function for each subsystem using backstepping, we can still not say that the switching system is stable since the Lyapunov functions are different and are not in the strict quadratic form in most cases. We can use the following example to show this.

Consider a simple switching system consisting of the following two subsystems,

$$\begin{aligned} \dot{x}_1 &= -2x_1 - 4x_2 \\ \dot{x}_2 &= 2x_1 x_2^2 - 3u \end{aligned} \tag{3.1}$$

and,

$$\begin{aligned} \dot{x}_1 &= -5x_1^2 + 2x_2 \\ \dot{x}_2 &= 6x_2^2 + 2u. \end{aligned} \tag{3.2}$$

Assume this switching system is time-dependent with period of 0.5s and initial state $(-2, 2)$. We see that two subsystems are both unstable without input u. Thus we need to design two controllers to respectively stabilize two subsystems. Here we use the classical backstepping approach and obtain two state-feedback controllers for the above two subsystems.

As to subsystem 1, consider the first equation. We can choose

$$\phi(x) = x_1 \tag{3.3}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{3.4}$$

Then consider the first two equations, we can obtain by using (2.32)

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{\frac{\partial \phi}{\partial x}[f(x_1) + g(x_1)x_2] \\ &\quad -\frac{\partial V_1}{\partial x_1}g(x_1) - k_1[x_2 - \phi(x_1)] - f_1(x_1, x_2)\},\ k_1 > 0 \\ &= \frac{(-2x_1 - 4x_2) + 4x_1 - k_1(x_2 - x_1) - 2x_1x_2^2}{-3} \\ &= \frac{-4x_1 + 6x_2 + 2x_1x_2^2}{3},\ k_1 = 2 \end{aligned} \tag{3.5}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - x_1)^2. \tag{3.6}$$

As to subsystem 2, consider the first equation. We can choose

$$\phi(x) = \frac{5}{2}x_1^2 - x_1 \tag{3.7}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{3.8}$$

Then consider the first two equations, we can obtain by using (2.32)

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{\frac{\partial \phi}{\partial x}[f(x_1) + g(x_1)x_2] \\ &\quad -\frac{\partial V_1}{\partial x_1}g(x_1) - k_1[x_2 - \phi(x_1)] - f_1(x_1, x_2)\},\ k_1 > 0 \\ &= \frac{(5x_1^3 - 1)(-5x_1^2 + 2x_2) - 2x_1 - k_1(x_2 - \frac{5}{2}x_1^2 + x_1) - 6x_2^2}{2} \\ &= -\frac{25x_1^3}{2} + 5x_1x_2 + 5x_1^2 - 2x_2 - 2x_1 - 3x_2^2,\ k_1 = 2 \end{aligned} \tag{3.9}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \frac{5}{2}x_1^2 + x_1)^2. \tag{3.10}$$

Figure 3.1 and Figure 3.2 show the trajectories of system states and the Lyapunov function respectively. From the figures, we see that although each individual subsystem of the switching system is stabilized by the backstepping-based controllers, the overall switching system is not stabilized by the multi-controller. The reason is that the Lyapunov functions of two subsystems are different and are not in the strict quadratic form.

3.2 Improved Backstepping Approach

From the analysis in the previous section, we can see that the term $\phi(x)$ in the constructed Lyapunov function prevents us from finding Lyapunov functions in a

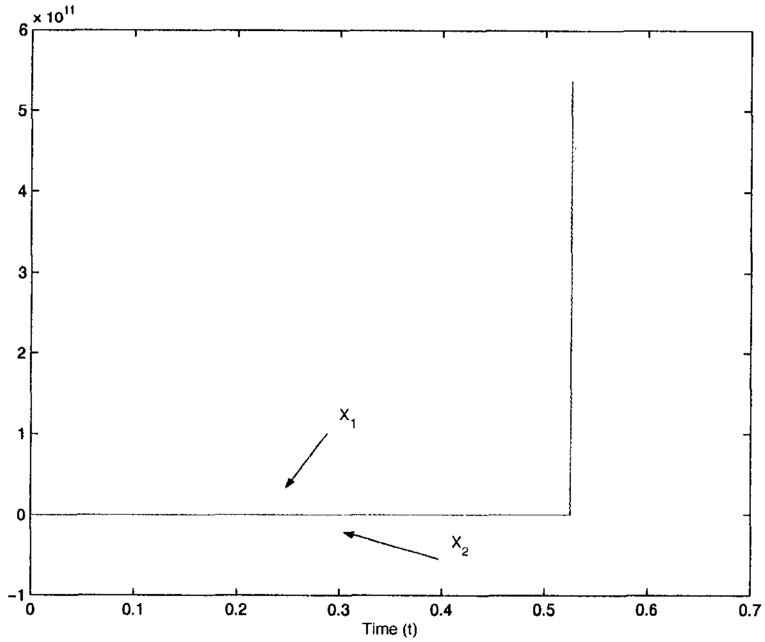

Figure 3.1: System trajectory with respect to time

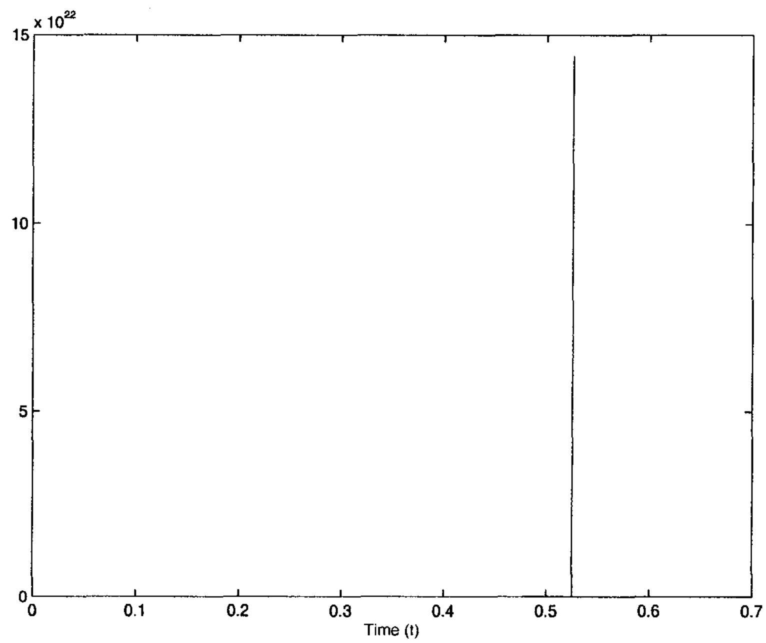

Figure 3.2: Trajectory of the Lyapunov function

unified form for different nonlinear systems. Thus we can consider to select another form of Lyapunov functions.

3.2.1 Integrator Backstepping

Consider the system (2.11)-(2.12) in the first chapter, which still satisfies the following assumptions (see Figure 3.3).

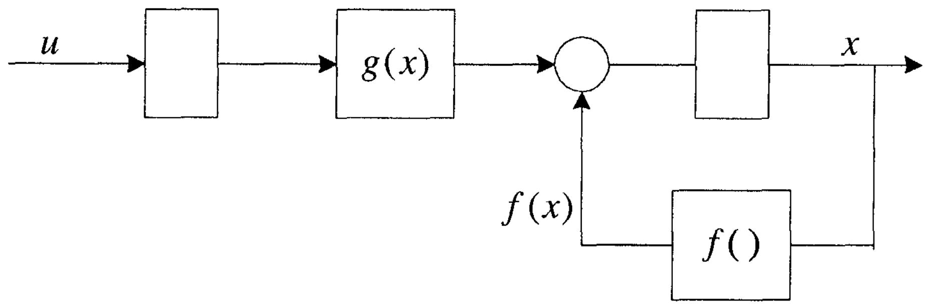

Figure 3.3: The original nonlinear system

(i) The function $f(\cdot) : R^n \longrightarrow R^n$ satisfies $f(0) = 0$. Thus, the origin is an equilibrium point of the subsystems $\dot{x} = f(x)$.

(ii) Consider the subsystems (2.11). Viewing the state variable ξ as an independent "input" for this subsystem, we assume that there exists a state feedback control law of the form

$$\xi = \phi(x), \quad \phi(0) = 0, \tag{3.11}$$

and a Lyapunov function $V_1 : D \rightarrow R^+$ such that

$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x}[f(x) + g(x) \cdot \phi(x)] \leq -V_a(x) \leq 0 \quad \forall x \in D \tag{3.12}$$

where $V_a(\cdot) : D \rightarrow R^+$ is a positive semidefinite function in D.

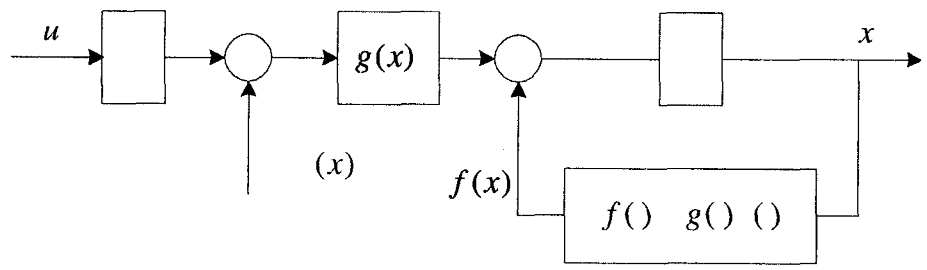

Figure 3.4: Modified system after introducing $-\phi(x)$

By adding and subtracting $g(x)\phi(x)$ to the subsystem (2.11) (Figure 3.4), we obtain the equivalent system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\phi(x) + g(x)[\xi - \phi(x)], \\ \dot{\xi} &= u.\end{aligned} \tag{3.13}$$

Define

$$z = \xi - \phi(x), \tag{3.14}$$

$$v = \dot{z} = \dot{\xi} - \dot{\phi}(x) = u - \dot{\phi}(x), \tag{3.15}$$

where

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi].$$

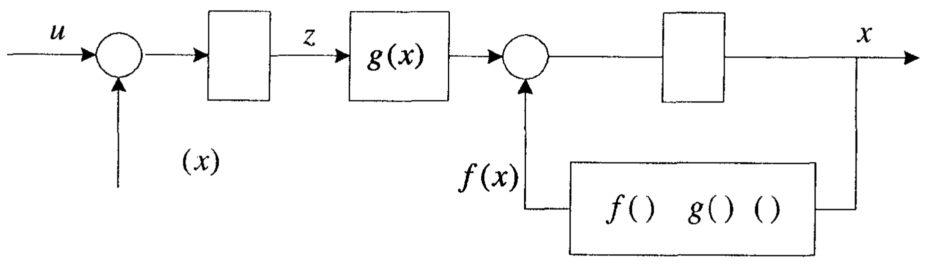

Figure 3.5: "Backstepping" of $-\phi(x)$

This change of variables can be seen as "backstepping" $-\phi(x)$ through the integrator, as shown in Figure 3.5. Defining

$$v = \dot{z} \tag{3.16}$$

the resulting system is

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(x) + g(x)z \\ \dot{z} &= v \end{aligned} \tag{3.17}$$

which is shown in Figure 3.6.

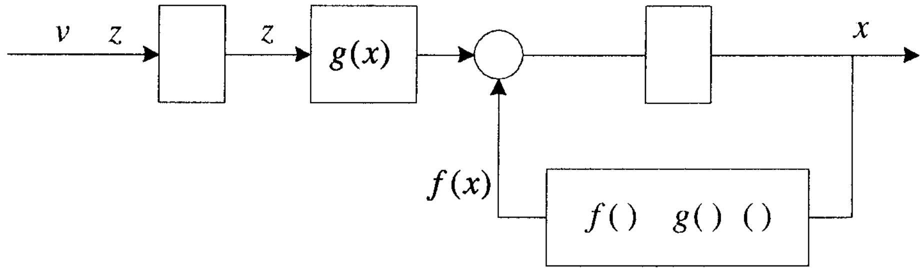

Figure 3.6: The final system after the change of variables

The above steps are the same as the classical backstepping. Next, to stabilize the system (2.18), the classical backstepping approach choose a Lyapunov function candidate of the form

$$V = (x, \xi) = V_1(x) + \frac{1}{2}z^2. \tag{3.18}$$

Since the variable z is actually the function of $\phi(x)$, we cannot obtain the unified individual Lyapunov functions for all of subsystems in a nonlinear switching system.

Here, instead of using (2.20), now we select the following Lyapunov function

$$V = V(x, \xi) = V_1(x) + \frac{1}{2}(z + \phi(x))^2. \tag{3.19}$$

Then, we have that

$$\begin{aligned}\dot{V} &= \frac{\partial V_1}{\partial x}[f(x)+g(x)\phi(x)+g(x)z]+[z+\phi(x)][\dot{z}+\dot{\phi}(x)] \\ &= \frac{\partial V_1}{\partial x}[f(x)+g(x)\phi(x)+g(x)z]+[z+\phi(x)][v+\dot{\phi}(x)].\end{aligned} \tag{3.20}$$

We can choose

$$v = \frac{-k(z+\phi(x))^2 - \frac{\partial V_1}{\partial x}g(x)z}{z+\phi(x)} - \dot{\phi}(x), \qquad k > 0. \tag{3.21}$$

Thus

$$\begin{aligned}\dot{V} &= \frac{\partial V_1}{\partial x}[f(x)+g(x)\phi(x)+g(x)z] \\ &\quad +[z+\phi(x)]\frac{-k(z+\phi(x))^2 - \frac{\partial V_1}{\partial x}g(x)z}{z+\phi(x)} \\ &= \frac{\partial V_1}{\partial x}[f(x)+g(x)\phi(x)] - k(z+\phi(x))^2 \\ &\leq -V_a(x) - k(z+\phi(x))^2, \qquad k>0.\end{aligned} \tag{3.22}$$

Then it follows that the origin ($x=0, z=0$) is asymptotically stable. Moreover, since $z = \xi - \phi(x)$ and $\phi(0) = 0$ by assumption, the result also implies that the origin of the original system (2.11)-(2.12) with $x = 0$ and $\xi = 0$ is also asymptotically stable.

Since

$$u = \dot{z} + \dot{\phi}, \tag{3.23}$$

we obtain that

$$u = -\frac{\partial V_1}{\partial x}g(x) + \frac{\partial V_1}{\partial x}g(x)\frac{\phi(x)}{\xi} - k\xi, \tag{3.24}$$

and the corresponding Lyapunov function

$$V = V_1(x) + \frac{1}{2}\xi^2. \tag{3.25}$$

3.2.2 Chain of Integrators

In the previous part we proposed a new backstepping approach for systems with a state of the form $[x, \xi]$, $x \in R^n, \xi \in R$, under the assumption that a stabilizing law u exists. Now we extend this approach to high-order nonlinear systems.

We consider the following nonlinear system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \qquad (3.26)\\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{k-1} &= \xi_k \\
\dot{\xi}_k &= u.
\end{aligned}
$$

Backstepping design for this class of systems can be approached using successive iterations of the procedure used in the previous section. To simplify our notation, we consider, without loss of generality, a third order system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \qquad (3.27)\\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= u
\end{aligned}
$$

and proceed to design a stabilizing control law. We first consider the first two "subsystems"

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \qquad (3.28)\\
\dot{\xi}_1 &= \xi_2 \qquad (3.29)
\end{aligned}
$$

and assume that $\xi_1 = \phi(x_1)$ is a stabilizing control law for the system

$$\dot{x} = f(x) + g(x)\phi(x). \qquad (3.30)$$

Moreover, we also assume that V_1 is the corresponding Lyapunov function for this subsystem. The second-order system (3.28)-(3.29) can be seen as having the form (2.11)-(2.12) with ξ_2 considered as an independent input. We can asymptotically stabilize this system using the control law (3.24) and the associated Lyapunov function V_1:

$$u = -\frac{\partial V_1}{\partial x} g(x) + \frac{\partial V_1}{\partial x} g(x) \frac{\phi(x)}{\xi} - k\xi, \tag{3.31}$$

and the corresponding Lyapunov function

$$V_2 = V_1(x) + \frac{1}{2}\xi^2. \tag{3.32}$$

We now iterate this process and view this third-order system given by the first three equations as a more general version of (2.11)-(2.12) with

$$x = \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \xi = \xi_2, f = \begin{bmatrix} f(x) + g(x)\xi_1 \\ 0 \end{bmatrix}, g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.33}$$

Applying the backstepping algorithm once more, we obtain the stabilizing control law:

$$\begin{aligned} u &= -\frac{\partial V_2}{\partial x} g(x) + \frac{\partial V_2}{\partial x} g(x) \frac{\phi(x)}{\xi_2} - k\xi_2, \qquad k > 0 \qquad (3.34) \\ &= -[\frac{\partial V_2}{\partial x}, \frac{\partial V_2}{\partial \xi_1}][0,1]^T + [\frac{\partial V_2}{\partial x}, \frac{\partial V_2}{\partial \xi_1}][0,1]^T \frac{\phi(x,\xi_1)}{\xi_2} - k\xi_2, \qquad k > 0 \\ &= -\frac{\partial V_2}{\partial \xi_1} + \frac{\partial V_2}{\partial \xi_1} \frac{\phi(x,\xi_1)}{\xi_2} k\xi_2, \qquad k > 0. \end{aligned}$$

The composite Lyapunov function is

$$\begin{aligned} V_3 &= V_2(x) + \frac{1}{2}\xi_2^2 \qquad (3.35) \\ &= V_1(x) + \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2. \end{aligned}$$

Following the above procedure, we can obtain stabilizing control laws and corresponding Lyapunov functions for third-order systems. The procedure for nth-order systems is entirely analogous. The Lyapunov function for nth-order systems is in the unified form as follows:

$$V_n = V_1(x_1) + \sum_{i=2}^{n} \frac{1}{2}x_i^2 = V_1(x) + \frac{1}{2}x_2^2 + \cdots \frac{1}{2}x_n^2. \tag{3.36}$$

3.2.3 Strict Feedback Systems

Next consider the nonlinear system in a more general form

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2, \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3, \\ &\vdots \\ \dot{x}_{k+1} &= f_k(x_1, x_2, \cdots, x_k) + g_k(x_1, x_2, \cdots, x_k)u \end{aligned} \tag{3.37}$$

where $x \in R^n$, $\xi_i \in R^n$, and f_i, g_i are smooth, for all $i = 1, \cdots k$.

For example, for the system with two states

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2, \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)u. \end{aligned} \tag{3.38}$$

If $g_1(x_1, x_2) \neq 0$ over the domain of interest, then we can define

$$u = \phi(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}[u_1 - f_1(x_1, x_2)]. \tag{3.39}$$

Substituting (3.39) into (3.38) we obtain the modified system

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2, \\ \dot{x}_2 &= u_1, \end{aligned} \tag{3.40}$$

which is of the form (2.11)-(2.12). It then follows that, using (3.24), the stabilizing control law and associated Lyapunov function are

$$\begin{aligned} u = \phi_1(x_1, x_2) &= \frac{1}{g_1(x_1, x_2)}\{-\frac{\partial V_1}{\partial x_1}g(x_1) \\ &+ \frac{\partial V_1}{\partial x_1}g(x_1)\frac{\phi(x_1)}{x_2} - k_1x_2 - f_1(x_1, x_2)\},\ k_1 > 0 \end{aligned} \tag{3.41}$$

$$V = V_1(x) + \frac{1}{2}x_2^2. \tag{3.42}$$

For the system with three states

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + g(x_1)x_2, \qquad (3.43)\\
\dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3,\\
\dot{x}_3 &= f_2(x_1, x_2, x_3) + g_2(x_1, x_2, x_3)u,
\end{aligned}$$

which can be seen as a special case of (3.38) with

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_2 = x_3, u = u, f = \begin{bmatrix} f + gx_2 \\ f_1 \end{bmatrix}, g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, f_1 = f_2, g_1 = g_2. \tag{3.44}$$

With these definitions, and using the control law and associated Lyapunov function (3.41)-(3.42), we have that a stabilizing control law and associated Lyapunov function for this systems are as follows:

$$\begin{aligned}
u &= \phi_2(x_1, x_2, x_3) = \frac{1}{g_2(x_1, x_2, x_3)}\{-\frac{\partial V_2}{\partial x_2}g_2(x_1, x_2) \qquad (3.45)\\
&+\frac{\partial V_2}{\partial x_2}g_2(x_1, x_2)\frac{\phi_1(x_1, x_2)}{x_3} - k_2x_3 - f_2(x_1, x_2, x_3)\}, \; k_2 > 0
\end{aligned}$$

$$V_3 = V_2(x) + \frac{1}{2}x_3^2 = V_1(x_1) + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \tag{3.46}$$

From the above analysis, we can see that, as to the system (3.37) with $k+1$ states, we can build the following Lyapunov function with the input obtained from the above procedure.

$$V_{k+1} = V_k(x_1) + \frac{1}{2}x_{k+1}^2 = V_1(x_1) + \sum_{i=2}^{k+1}\frac{1}{2}x_i^2 \tag{3.47}$$

3.2.4 Example of Proposed Backstepping

Now I will use the following example to verify the validity of the proposed backstepping approach.

Consider the following nonlinear system:

$$\dot{x}_1 = 2x_1^2 - 3x_1^3 - 2x_2 \tag{3.48}$$
$$\dot{x}_2 = 2x_1x_2 + 2u.$$

Consider the first equation and view the "state" x_2 as an independent input for the first equation. Then we can try to find a state feedback law $u_1 = \phi(x)$ to stabilize the origin $x = 0$. Here we can choose

$$\phi(x) = x_1^2 \tag{3.49}$$

and define

$$\begin{aligned} V_1(x_1) &= \frac{1}{2}x_1^2 \\ \Rightarrow \dot{V}_1(x_1) &= 2x_1^3 - 3x_1^4 - 2x_1x_2 = -3x_1^4 \le x_1^4. \end{aligned} \tag{3.50}$$

Then consider the two equations, we can obtain by using (3.41)

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{-\frac{\partial V_1}{\partial x_1}g(x_1) \\ &\quad + \frac{\partial V_1}{\partial x_1}g(x_1)\frac{\phi(x_1)}{x_2} - k_1x_2 - f_1(x_1, x_2)\} \\ &= \frac{1}{2}\{-x_1(-2) + x_1(-2)\frac{x_1^2}{x_2} - k_1x_2 - 2x_1x_2)\} \\ &= x_1 - \frac{x_1^3}{x_2} - x_2 - x_1x_2, \quad k = 2. \end{aligned} \tag{3.51}$$

The Lyapunov function associated with this control law is:

$$V_2 = V_1(x) + \frac{1}{2}x_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{3.52}$$

Also, we can use the classical backstepping approach to design the control law. We can choose

$$\phi(x) = x_1^2 \tag{3.53}$$

and define

$$
\begin{aligned}
V_1(x_1) &= \frac{1}{2}x_1^2 \qquad (3.54)\\
\Rightarrow \dot{V}_1(x_1) &= 2x_1^3 - 3x_1^4 - 2x_1x_2 = -3x_1^4 \le x_1^4.
\end{aligned}
$$

Then consider the two equations, we can obtain by using (2.32)

$$
\begin{aligned}
u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{\frac{\partial\phi}{\partial x}[f(x_1) \qquad (3.55)\\
&\quad +g(x_1)x_2] - \frac{\partial V_1}{\partial x_1}g(x_1) - k_1[x_2 - \phi(x_1)] - f_1(x_1, x_2)\},\ k_1 > 0\\
&= \frac{1}{2}\{2x_1(2x_1^2 - 3x_1^3 - 2x_2) - x_1(-2) - k_1(x_2 - x_1^2) - 2x_1x_2\}\\
&= x_1 + x_1^2 + 2x_1^3 - 3x_1^4 - 3x_1x_2 - 2x_2, \qquad k = 2.
\end{aligned}
$$

Now we can use SIMULINK to verify and compare the performance of two controllers. Here, we choose the initial states as (10, 15). The trajectories of system states are shown respectively in Figure 3.7 and Figure 3.9. The trajectories of Corresponding Lyapunov functions are shown respectively in Figure 3.8 and Figure 3.10.

From the above results, we can see that both backstepping-based controllers can stabilize the nonlinear system (3.48) around the equilibrium point. Both Lyapunov functions goes back to origin rapidly. Thus the example shows that the proposed backstepping approach can stabilize the unstable nonlinear system.

3.3 Controller Design based on Backstepping

In this part, we will first design the backstepping-based feedback controller for switching systems. Then, an example will be given to illustrate its validity.

3.3.1 Backstepping-based Feedback Controller

Now let us consider the switching systems consisting of strict feedback subsystems. Following the above procedure, we can design the corresponding feedback controller

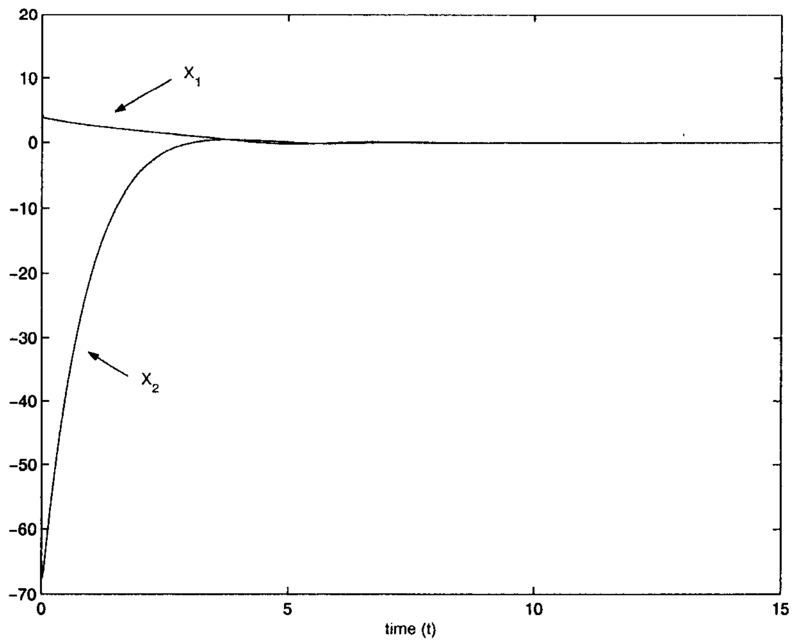

Figure 3.7: Trajectory of system using the proposed backstepping

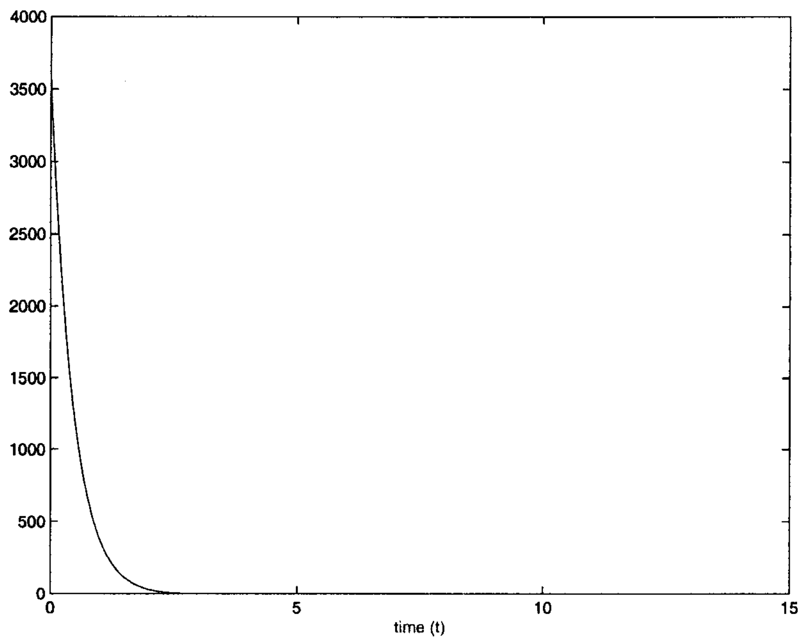

Figure 3.8: The Lyapunov function of system using the proposed backstepping

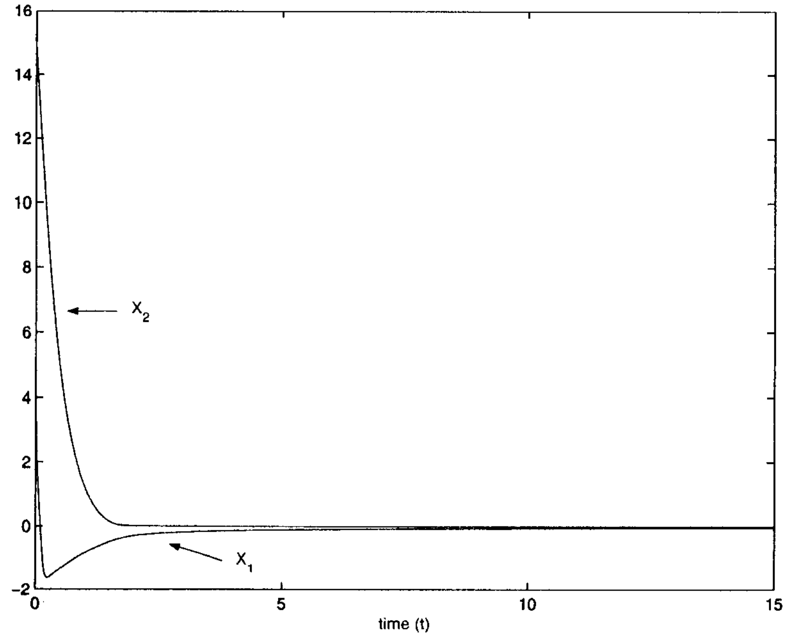

Figure 3.9: Trajectory of system using the classical backstepping

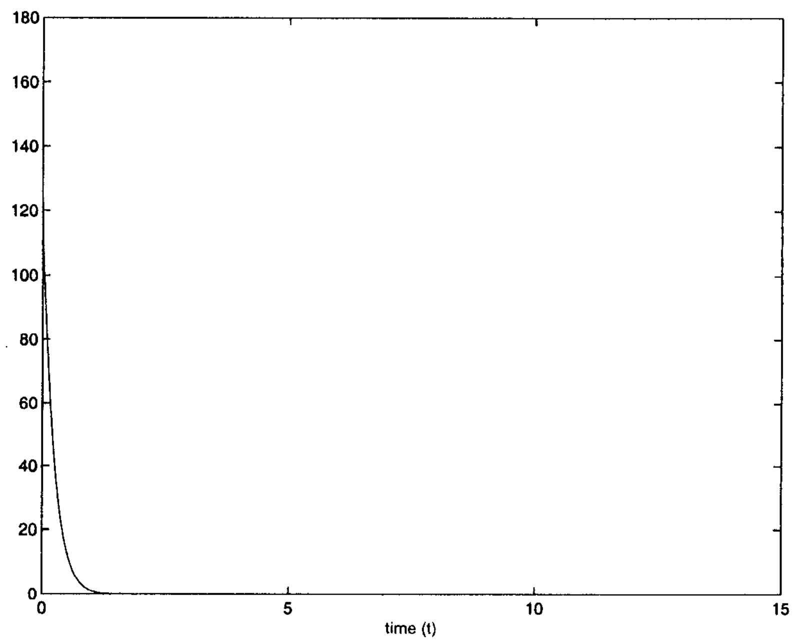

Figure 3.10: The Lyapunov function using the classical backstepping

for each of nonlinear or linear subsystems in a switching system. All controllers are companied with the Lyapunov functions in the form of (3.47). If we can find the same $V_1(x)$ for all of subsystems, the subsystems will have the same Lyapunov function. Then we can use the above procedure to design controllers for each of subsystems and combine these sub-controllers into a synchronous switched controller for the switching system.

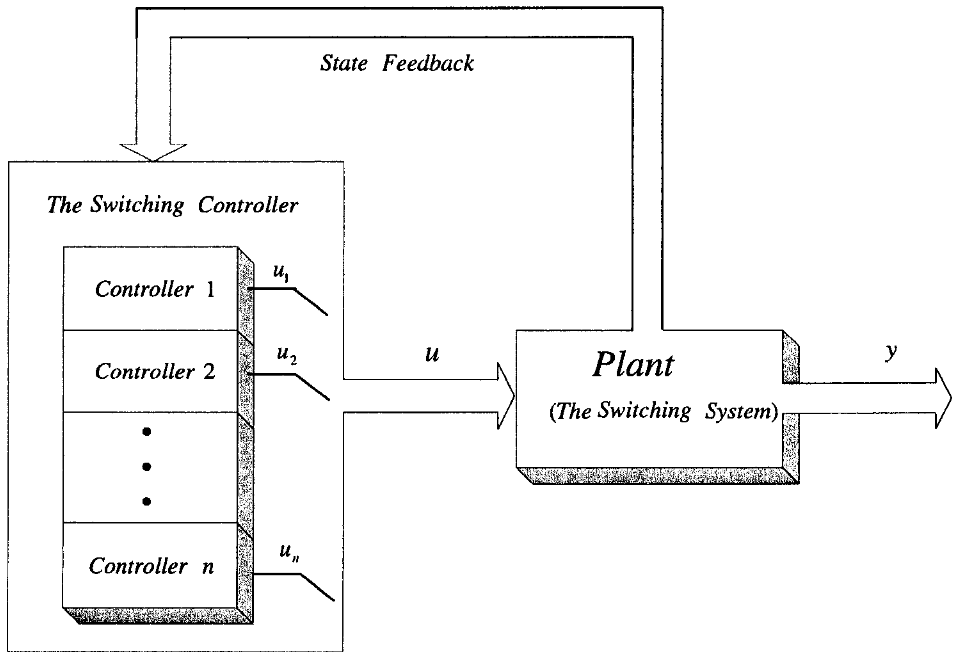

Figure 3.11: Multicontroller architecture

The prototypical architecture for such a switching system with multicontroller is shown in Figure 3.11. We have two assumptions:

- All individual subsystems have the origin as a common equilibrium point: $f_p(0) = 0, p \in P$.
- The synchronous multicontroller is capable to switch at the same time when system switches.

Clearly, a necessary condition for (asymptotic) stability under arbitrary switching is that all individual subsystems are (asymptotically) stable. Indeed, if the pth subsystem is unstable, the switching system will be unstable if we choose this subsystem as the final subsystem. This condition is satisfied by our second assumption because each subsystem is stabilized by the corresponding backstepping sub-controller. However, as we discussed in the first chapter, stability of all individual subsystems is not sufficient for stability of the nonlinear switching system. Fortunately, the proposed backstepping guarantees that this kind of multi-controller can stabilize both the same-order and multiple-order switching systems if certain conditions are satisfied. We will discuss these sufficient conditions in detail in the next chapter. The following part of this chapter will give an example to illustrate the validity of this kind of controllers.

3.3.2 Example

Consider the simple switching system (3.1)-(3.2), which can not be stabilized by the classical form of backstepping. Now we use the proposed backstepping approach to obtain the controller again.

As to subsystem 1, consider the first equation. We still choose

$$\phi(x) = x_1, \tag{3.56}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{3.57}$$

Then consider the two equations, we can obtain by using (3.41)

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{-\frac{\partial V_1}{\partial x_1}g(x_1) \\ &\quad +\frac{\partial V_1}{\partial x_1}g(x_1)\frac{\phi(x_1)}{x_2} - k_1x_2 - f_1(x_1, x_2)\},\ k_1 > 0 \\ &= \frac{4x_1 - \frac{4x_1^2}{x_2} - 2x_2 - 2x_1x_2^2}{-3} = -\frac{4}{3}x_1 + \frac{4x_1^2}{3x_2} + \frac{2x_2}{3} + \frac{2x_1x_2^2}{3},\ k_1 = 2 \end{aligned} \tag{3.58}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{3.59}$$

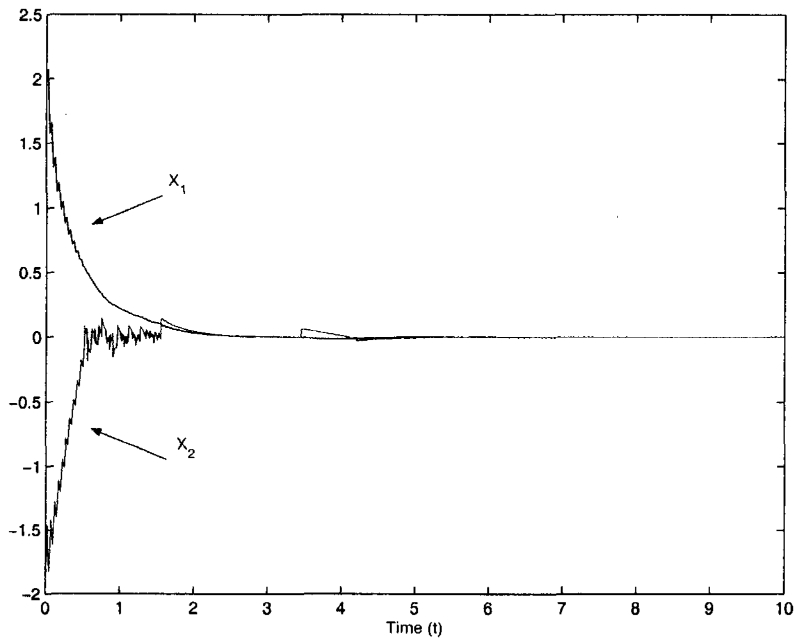

Figure 3.12: System trajectory with respect to time .

As to subsystem 2, consider the first equation. We still choose

$$\phi(x) = \frac{5}{2}x_1^2 - x_1, \tag{3.60}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{3.61}$$

Then consider two equations, we can obtain by using (3.41)

$$\begin{aligned} u &= \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{-\frac{\partial V_1}{\partial x_1}g(x_1) \\ &\quad + \frac{\partial V_1}{\partial x_1}g(x_1)\frac{\phi(x_1)}{x_2} - k_1 x_2 - f_1(x_1, x_2)\},\ k_1 > 0 \\ &= -x_1 + x_1\frac{(\frac{5}{2}x_1^2 - x_1)}{x_2} - x_2 - 3x_2^2 \\ &= -x_1 + \frac{5x_1^3}{2x_2} - \frac{x_1^2}{x_2} - x_2 - 3x_2^2,\ k_1 = 2 \end{aligned} \tag{3.62}$$

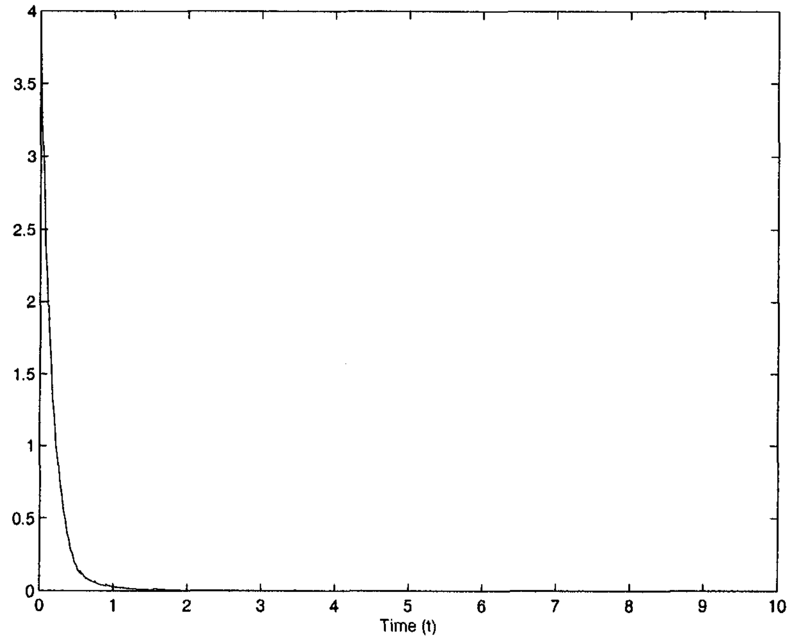

Figure 3.13: The common Lyapunov function.

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{3.63}$$

Figure 3.12 and Figure 3.13 show the trajectories of system states and the Lyapunov function respectively. We see that system states and Lyapunov function go to the origin eventually. Thus, by using the proposed backstepping approach, the switching system can be stabilized by the feedback controller. However, this is only two-order time-dependent simple switching system. Can the proposed backstepping approach stabilize general switching systems, which could be state-dependent or time-dependent, same-order or multiple-order? What additional conditions and constrains should be used to guarantee it? The next chapter will answer these questions.

Chapter 4

Backstepping-based Controller Design for Switching Systems

As introduced in the second chapter, the backstepping approach can not only stabilize nonlinear systems, but also provide corresponding Lyapunov functions. Thus the multi-controller using backstepping can stabilize all of individual subsystems in the switching system. However, this condition is not sufficient to guarantee the stability of switching systems. Therefore, we proposed a new backstepping approach in the third chapter, which can not only stabilize the nonlinear subsystems, but also provide Lyapunov functions in a unified form, or say in the strict quadratic form in most cases. How does this proposed backstepping guarantee the stability of switching systems? Is any additional condition needed? These questions will be answered and discussed in detail in this chapter.

4.1 Same-order Switching Systems

Same-order switching systems are the switching systems with subsystems of the same order. They are analyzed and discussed in many references [8, 4, 16]. Most references use the common Lyapunov theorem to verify the stability of such systems. However, almost none of them can give a general procedure to design controller to stabilize such system. The proposed backstepping and corresponding feedback controller can solve this problem.

Let's recall the common Lyapunov function theorem introduced in the second chapter.

Theorem 4.1 *If all systems in the family (2.1) share a radially unbounded common Lyapunov function, then the switching system is GUAS.*

This theorem is well know and can be derived in the same way as the standard Lyapunov stability theorem.

As talked in the backstepping part, the traditional backstepping can not provide same Lyapunov functions for all subsystems. But the proposed backstepping approach solves this problem.

Theorem 4.2 *As to a continuous same-order switching system with strict feedback subsystems, furnished with an arbitrary switching rule, with the above proposed backstepping approach, if we can find the same* $V_1(x)$ *for all subsystems, then the corresponding backstepping-based controller can stabilize the switching system around the equilibrium point.*

Proof 1 *The proof is straightforward. If we can find same* $V_1(x)$ *for all subsystems, all subsystems in the switching system have the same Lyapunov function (3.47). Then it can be used as the common Lyapunov function of the continuous switching system. We can apply the common Lyapunov function theorem to show that the switching system is (asymptotically) stable.*

Remark 4.1 *The proposed backstepping approach can be used for all same-order switching systems with strict feedback subsystems. The switched rules have no effect on the validity of backstepping-based controllers. Thus this type of controllers can be used for both time-dependent and state-dependent switching systems.*

Remark 4.2 *As to the non-continuous same-order switching systems caused by the jumping, we can still find the common Lyapunov function using proposed backstepping. However, the Lyapunov function will be also discontinuous. The stability of this kind of systems will be discussed in the next part of this chapter with some sufficient conditions.*

4.2 Multiple-order Switching Systems

In this part, the multiple Lyapunov function theorem for multiple-order switching systems will be established and proved first. Then we will introduce sufficient conditions which guarantee the stability of multiple-order switching systems using the proposed backstepping approach.

4.2.1 Multiple Lyapunov Function Theorem

As we know, the common Lyapunov theorem is very useful for the stability analysis of nonlinear systems. The theorem can also be used for stability analysis of switching systems of same order as the above theorem shows. However, as to switching systems of multiple orders, common Lyapunov function is not well defined because some states do not exist in all subsystems. Thus we propose the following multiple Lyapunov theorem for multiple-order switching systems, which shows its own importance for such switching systems.

Theorem 4.3 *Let the switching system (2.1) be a finite family of globally asymptotically stable systems, which may have different orders, and let $p \in \mathcal{P}$ be a family of corresponding radially unbounded Lyapunov functions. If for every pair of switching times (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ and $\sigma(t_k) \neq p$ for $t_i < t_k < t_j$, then the following statements hold*

1. *if*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq 0, \tag{4.1}$$

   then the switching system (2.1) is stable in the sense of Lyapunov.

2. *if we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)) \tag{4.2}$$

   here, W_p, $p \in \mathcal{P}$ is a family of positive definite continuous functions, then the switching system (2.1) is globally uniformly asymptotically stable.

Proof 2 *Suppose the switching system (2.1) consists of m subsystems whose orders are respectively* $r_i, i \in [1, 2, ...m]$. *If the number of switchings between subsystems is finite, then the problem is trivial. In this case, the state trajectory is guaranteed to converge to the origin after the finial switching since every subsystem is asymptotically stable. Now we only need to discuss the case of infinite switchings.*

Suppose, for every subsystem, the initial value of Lyapunov function is $V_{0i}, i \in [1, 2, ..., m]$ *which is a finite positive number. Then we can construct the sphere for each of subsystems. The dimension of the ball is equal to the order of its corresponding subsystem. Let* R_i *be a set of the form* $\{x : V_i(x) \leq V_{0i}\}$ *which is contained in i ball. Suppose at time* $t_j, t_j \in [t_0, \infty\}$, *system is switched to k subsystem. Then the states trajectory will come into the i sphere since (4.1) and stay in this sphere due to the globally asymptotically stability of subsystems until second switching. After the second switching, the trajectory will come into another sphere and stay in that sphere until the next switching. Thus we can see if the system satisfies (4.1). The state trajectory will always belong to one of the bounded spheres, which may have different dimension. Thus the condition (4.1) and asymptotical stability of subsystems guarantee the switching system is stable in the sense of Lyapunov.*

If the switching system satisfies not only (4.1) but also (4.2), due to the finiteness of $\mathcal{P}$, *there exists an index* $q \in \mathcal{P}$ *and an infinite sequence of switching times* $t_{i_1}, t_{i_2}, ...$ *such that* $\sigma(t_{i_j}) = q$. *The sequence* $V(x(t_{i_1})), V(x(t_{i_2})), ...$ *is decreasing and positive and therefore has a limit* $c \geq 0$.

We have

$$\begin{aligned} 0 &= c - c = \lim_{j\to\infty} V_q(x(t_{i_{j+1}})) - \lim_{j\to\infty} V_q(x(t_{i_j})) \\ &= \lim_{j\to\infty} [V_q(x(t_{i_{j+1}})) - V_q(x(t_{i_j}))] \\ &\leq \lim_{j\to\infty} [-W_q(x(t_{i_j}))] \leq 0 \end{aligned} \tag{4.3}$$

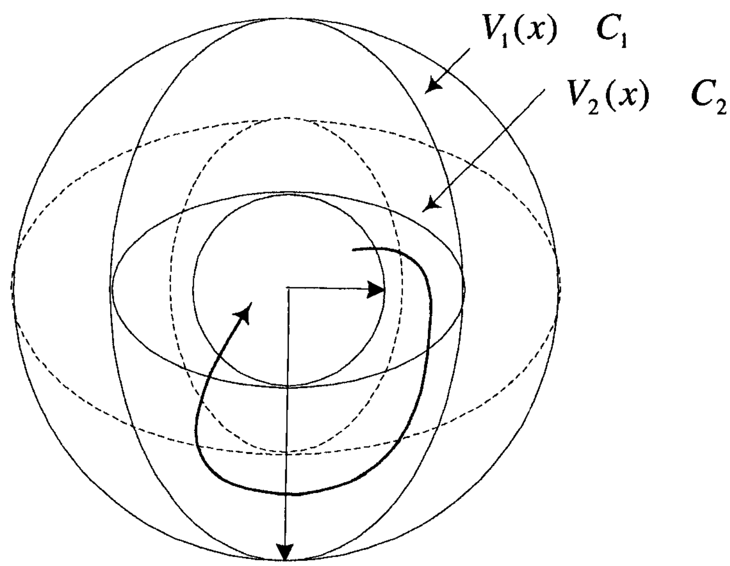

Figure 4.1: Trajectory of the Lyapunov function

Then we can see $W_q(x(t_{i_j})) \to 0$ *as* $j \to 0$. *Since* W_q *is positive definite,* $x(t_{i_j})$ *must converge to zero as* $j \to \infty$. *That means* $x(t) \to 0$ *as* $t \to \infty$. *Thus the switching system is globally uniformly asymptotically stable.*

Remark 4.3 *This theorem can be used for either the same-order or multiple-order switching systems.*

Remark 4.4 *If the subsystems of switching system are only locally asymptotically stable, the switching system is also only locally asymptotically stable. That means, if* $V_{0i}, i \in [1, 2, ..., m]$ *is in the convergent region of each subsystems,* $x(t) \to 0$ *as* $t \to \infty$.

Remark 4.5 *The above theorem only provides a sufficient condition to verify the stability of switching systems. The condition is not necessary since we can easily find a switching system which is GAS but does not satisfy the condition (4.1).*

Remark 4.6 *One of advantages of this theorem is that it can be applied without knowing what happens at the time of switching and where the additional or deficient* *states go to or come from. The theorem requires constructing different Lyapunov functions for the different subsystems and use this theorem to prove the stability of switching systems.*

Remark 4.7 *In the proof of this theorem, we see that the system trajectory is not continuous because of temporary disappearance of some states. This is similar to the jumping in same-order discontinuous switching systems. They are, however, significantly different in nature. Jumping in common switching systems occurs when the state is not continuous at the switching moment. In the case of switching systems of multiple orders, discontinuity in the system space is caused by the disappearance of some states, which do not exist in certain time intervals. Thus it is not suitable to describe the discontinuity in the system space as jumping.*

With the above multiple Lyapunov theorem, we can verify the stability of switching systems of multiple orders. However, as we know, there is no common procedure for us to find Lyapunov functions of nonlinear systems and the multiple Lyapunov theorem sometime is too restrictive for switching systems.

It is important to note that, to apply the above multiple Lyapunov function theorem, one must have some information about the solutions of the system. Namely, one needs to know the values of suitable Lyapunov functions at switching times, which in general requires the knowledge of the state at these times. This is to be contrasted with the classical Lyapunov stability results, which do not require the knowledge of solutions. Of course, in both cases there remains the problem of finding candidate Lyapunov functions. As we will see shortly, multiple Lyapunov function results are useful when the class of admissible switching signals is constrained in a way that makes it possible to ensure the desired relationships between the values of Lyapunov functions at switching times. The proposed backstepping approach will be a good candidate to ensure this condition.

4.2.2 Sufficient Conditions

As to a switching systems of multiple orders, the stability of system is not only determined by the initial values of all states, but also by the reset values of extra states. Thus the proposed backstepping approach cannot guarantee the stability of switching systems of multiple orders. The following theorems provide some sufficient conditions for the stability of closed-loop switching systems.

Theorem 4.4 *Consider a switching system of multiple orders of the form (2.1). If the reset values of extra states are all bounded and within the region of convergence, then the proposed backstepping approach can ensure the closed-loop system stable in the sense of Lyapunov.*

Proof 3 *The proof is straightforward. We can choose (3.47) as the Lyapunov function of the closed-loop switching system. Since we assume the reset values of the extra states are bounded, the value of the Lyapunov function at the switching instances is always bounded. Also the proposed backstepping method guarantees the Lyapunov function monotonically decreasing between two consequent switchings. Thus the values of the Lyapunov function are always bounded. Therefore, the closed-loop system is stable in the sense of Lyapunov.*

Theorem 4.5 *If the values of extra states are kept as their next reset values when they are inactive, then the closed-loop switching system is asymptotically stable.*

Proof 4 *In order to prove this theorem, we can add another assumption that the values of extra states remain as their last ending value when they are not activated. Adding this assumption has no effect on the stability of closed-loop system. Then the switching system can be viewed as the continuous switching system with subsystems of same order. We can choose (3.47) as the common Lyapunov function of* *switching system. Then the proposed backstepping method guarantees the decreasing of Lyapunov function. Thus according to the common Lyapunov function theorem, we can say the closed-loop switching system is asymptotically stable.*

Remark 4.8 *The above theorems provide only sufficient conditions for stability of closed-loop switching systems. It is not difficult to find a stable switching system that does not satisfy the condition of theorems.*

Remark 4.9 *In the proof of theorem 3.5, we assumed that the values of extra states are maintained constant when they are inactive. This assumption enables the system trajectory and that of Lyapunov function continuous so that we can use the common Lyapunov theorem to verify the stability of switching systems. It has no effect on the stability of the closed-loop system, since the inactive extra states have no effect on the other states of the switching system.*

4.3 Examples

In this part, The feedback multi-controllers using the proposed backstepping are designed for both same-order and multiple-order switching systems. The simulation results show the validity of this kind of backstepping-based controllers.

4.3.1 Same-orders Switching Systems

In order to illustrate the backstepping design for general switching systems, we consider a switching system with two subsystems:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -2x_2 - x_3 \\ \dot{x}_3 &= x_1x_2 - 2x_1u \end{aligned} \tag{4.4}$$

and,

$$\begin{aligned}\dot{x}_1 &= -5x_1 - 2x_2 \\ \dot{x}_2 &= -3x_2 - 2x_3 \\ \dot{x}_3 &= 6x_1 + x_2 + 3x_2 u.\end{aligned} \tag{4.5}$$

Assume this switching system is time-dependent with switching period of 0.5s and initial state (-1, 2, 3). We can see that two subsystems are both unstable without input u. Thus we need to design two controllers to respectively stabilize two subsystems. Here we use the proposed backstepping approach and obtain two state-feedback controllers for both of subsystems.

As to subsystem 1, consider the first equation. We can choose

$$\phi(x) = 2x_1 \tag{4.6}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{4.7}$$

Then consider the first two equations, we can obtain by using (3.41)

$$\phi_1(x) = x_1 + (k-2)x_2 - \frac{2x_1^2}{x_2}, \tag{4.8}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{4.9}$$

Then by using (3.45), we can also obtain the final input u

$$\begin{aligned} u &= \phi_2(x) = \frac{x_1x_2 - x_2 + k_2x_3}{2x_1} + \frac{x_1x_2 - 2x_1^2 + (k_1-2)x_2^2}{2x_1x_3} \\ &= \frac{x_1x_2 - x_2 + 8x_3}{2x_1} + \frac{x_1x_2 - 2x_1^2}{2x_1x_3}, \qquad (k_1 = 2, k_2 = 8),\end{aligned} \tag{4.10}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \tag{4.11}$$

As to subsystem 2, consider the first equation. We can choose

$$\phi(x) = 2x_1, \tag{4.12}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{4.13}$$

Then consider the first two equations, we can obtain by using (3.41)

$$\phi_1(x) = x_1 + \frac{(3+k_1)}{2}x_2 - \frac{2x_1^2}{x_2}, \quad k_1 > 0, \tag{4.14}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{4.15}$$

By using (3.45), we can also obtain the final input u

$$\begin{aligned} u &= \phi_2(x) = \frac{-6x_1 + x_2 - k_2x_3}{3x_1} + \frac{4x_1^2 - 2x_1x_2 - (k_1+3)x_2^2}{3x_2x_3} \\ &= \frac{-6x_1 + 2x_2 - 8x_3}{2} + \frac{4x_1^2 - 2x_1x_2 - 6x_2^2}{3x_2x_3}, \qquad (k_1 = 3, k_2 = 8) \end{aligned} \tag{4.16}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \tag{4.17}$$

We then apply these controllers to the nonlinear switching system. In Figure 4.2, we present the trajectories of the closed-loop switching system. From the figure we can see that all system states converge to zero rapidly.

The trajectory of Lyapunov function (4.11) is shown in Figure 4.3. It also converges to zero rapidly. Though the trajectory loses differentiability at switching

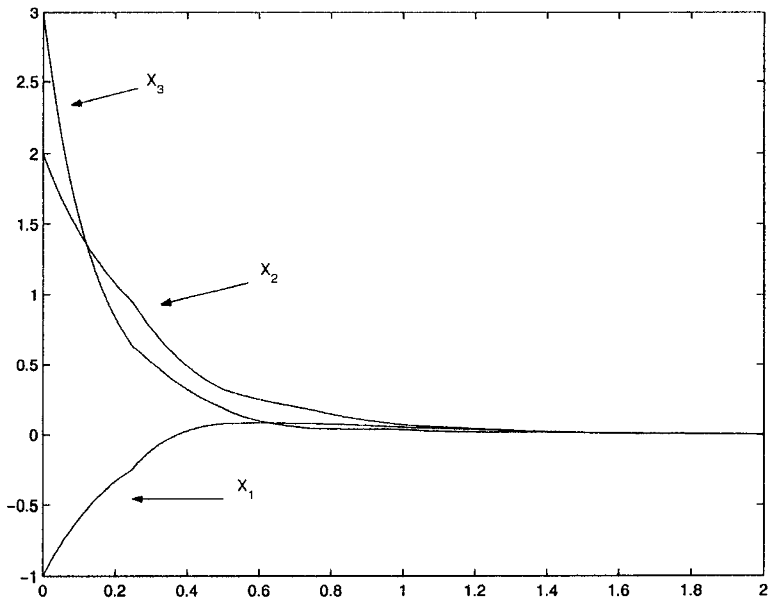

Figure 4.2: System trajectories with respect to time.

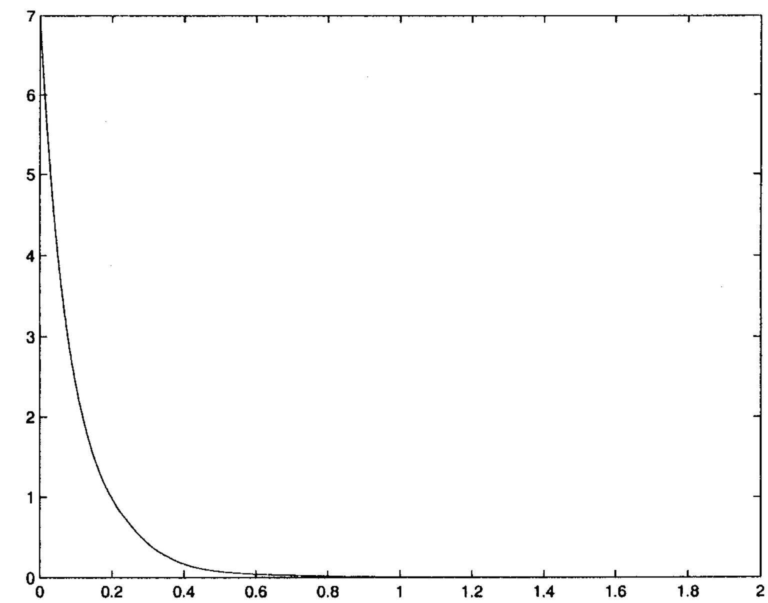

Figure 4.3: The common Lyapunov function.

instances, it is strictly decreasing since it is in the strictly quadratic form. The trajectory of control input u is shown in Figure 4.4. From the trajectories of system states and Lyapunov function, we see that this time-dependent switching system is

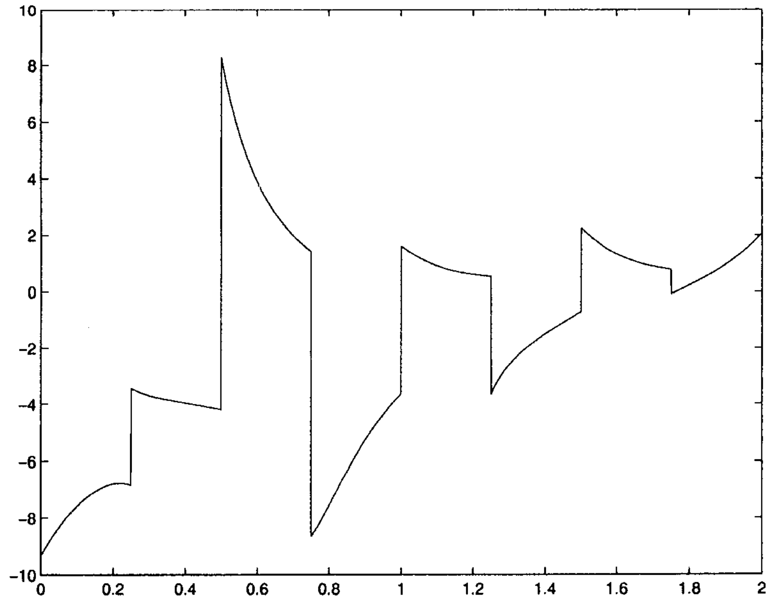

Figure 4.4: The control input

stabilized by the backstepping-based controllers.

The backstepping-based controller can stabilize not only the time-dependent system but also the state-dependent system. We will show this with the following example.

Consider a switching system consisting of two subsystems:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= x_1 + x_2^2 + u \end{aligned} \tag{4.18}$$

and,

$$\begin{aligned} \dot{x}_1 &= -6x_1 + 5x_2 \\ \dot{x}_2 &= 4x_2 - 2x_2 u. \end{aligned} \tag{4.19}$$

Assume this switching system is state-dependent. When $x_2 > 0$, subsystem 1 is activated. Otherwise, subsystem 2 is activated. The initial state is (2, 3). It is obvious that the two subsystems are both unstable under zero input. We need to design the controller to stabilize this system. Following by the procedure in chapter 2, we can obtain two sub-controllers for two subsystems.

As to subsystem 1, the control law is

$$\begin{aligned} u &= \phi_2(x) = -2x_1 - kx_2 - x_2^2 - \frac{4x_1^2}{3x_2} \\ &= -2x_1 - 7x_2 - x_2^2 - \frac{4x_1^2}{3x_2}, \qquad (k_1 = 7). \end{aligned} \tag{4.20}$$

As to subsystem 2, the control law is

$$\begin{aligned} u &= \phi_2(x) = \frac{5x_1 + (k+4)x_2}{2x_2} + \frac{5x_1^2}{x_2^2} \\ &= \frac{5x_1 + 8x_2}{2x_2} + \frac{5x_1^2}{x_2^2} \qquad (k_1 = 4) \end{aligned} \tag{4.21}$$

with respect to the common Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{4.22}$$

Figure 4.5 and Figure 4.6 show the trajectories of system states and the Lyapunov function respectively. We see, even the switching between systems is not regular as that in previous example, the switching system is still stabilized by the backstepping-based controller eventually. The example shows that this backstepping approach can also work for state-dependent switching systems.

Remark 4.10 *This approach can be used for the switching system with the subsystems in the strict feedback form or system that can be converted into this form. Since the common Lyapunov function can be obtained simultaneously while designing backstepping-based controllers, the proposed backstepping is a nice approach for controller design of switching systems with strict feedback subsystems.*

Remark 4.11 *By the proposed backstepping approach, controllers can stabilize the whole switching systems, since Lyapunov functions are always decreasing. This is*

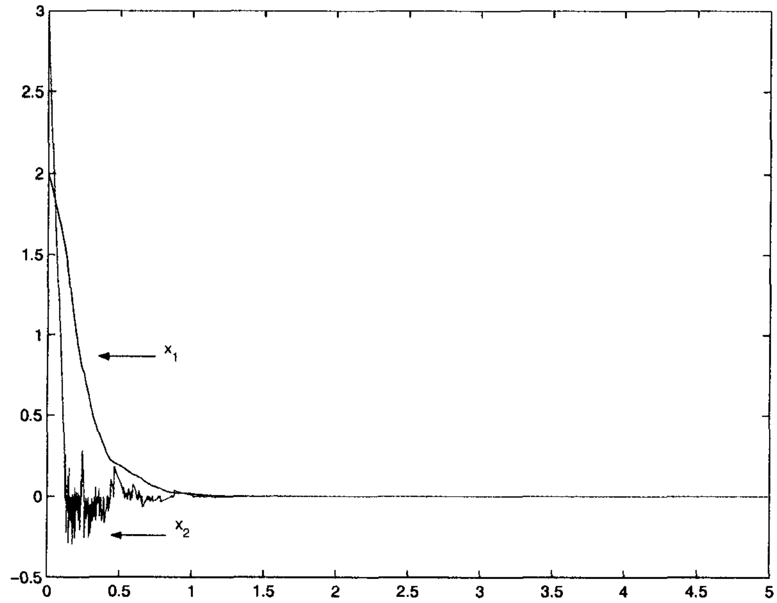

Figure 4.5: System trajectories with respect to time.

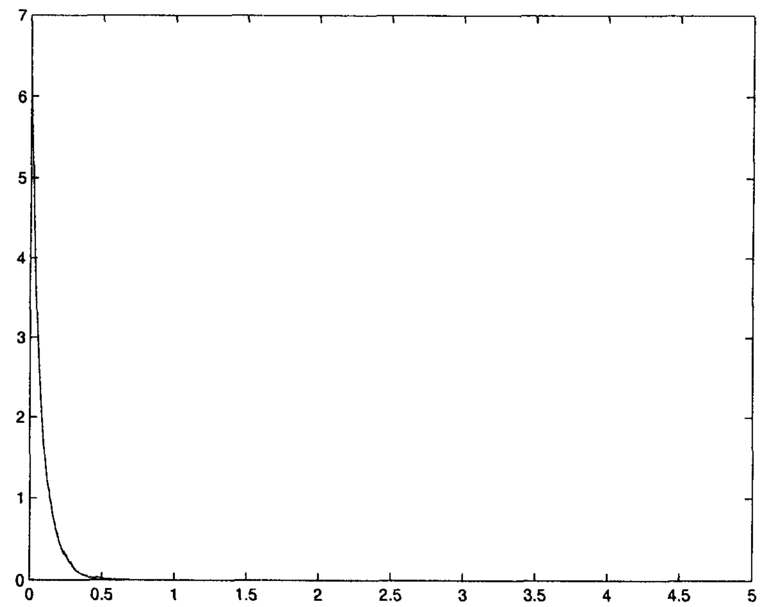

Figure 4.6: The common Lyapunov function.

guaranteed for continuous switching systems by the multiple Lyapunov function theorem. But as to non-continuous switching systems, jumping rule plays an important role. With different jumping rules, the switching system may be either stable, oscil-

lated or unstable.

Remark 4.12 *The trajectories of the Lyapunov functions in Figure 4.3 and Figure 4.6 are actually the combination of those of two subsystems. Thus they may lose the differentiability at switching instances. But benefited from the proposed backstepping method, the common Lyapunov function is in the quadratic form. Thus they are decreasing and converge to zero rapidly.*

4.3.2 Multiple-order Switching Systems

In order to illustrate the backstepping design for general switching systems, we consider a switching system consisting of two subsystems with different orders:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -2x_2 - x_3 \\ \dot{x}_3 &= x_1 + 3x_2 - 2u \end{aligned} \tag{4.23}$$

and,

$$\begin{aligned} \dot{x}_1 &= -6x_1 - x_2 \\ \dot{x}_2 &= -x_1^2 + 6u. \end{aligned} \tag{4.24}$$

Assume this switching system is time-dependent with period of 0.5s and initial state (-3, 6, 2). We see that two subsystems are both unstable without input u. Thus we need to design two controllers to respectively stabilize two subsystems. Here we use the backstepping method and obtain two state-feedback controllers for two subsystems.

As to subsystem 1, consider the first equation. We can choose

$$\phi(x) = 2x_1 \tag{4.25}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2. \tag{4.26}$$

Then consider the first two equations, we can obtain by using (3.41)

$$\phi_1(x) = x_1 + (k-2)x_2 - \frac{2x_1^2}{x_2} \tag{4.27}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{4.28}$$

Then by using (3.45), we can also obtain the final input u

$$\begin{aligned} u &= \phi_2(x) = \frac{x_1 + 2x_2 + k_2x_3}{2} + \frac{x_1x_2 - 2x_1^2 + (k_1-2)x_2^2}{2x_3} \\ &= \frac{x_1 + 2x_2 + 8x_3}{2} + \frac{x_1x_2 - 2x_1^2}{2x_3}, \qquad (k_1 = 2, k_2 = 8) \end{aligned} \tag{4.29}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \tag{4.30}$$

As to the subsystem 2, consider the first equation. We can choose

$$\phi(x) = 2x_1 \tag{4.31}$$

with respect to the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2.$$

Then by using (3.41), we can obtain the final input u

$$u = \phi_1(x) = \frac{1}{6}(x_1 - 2x_2 + x_1^2 - \frac{2x_1^2}{x_2}) \tag{4.32}$$

with respect the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{4.33}$$

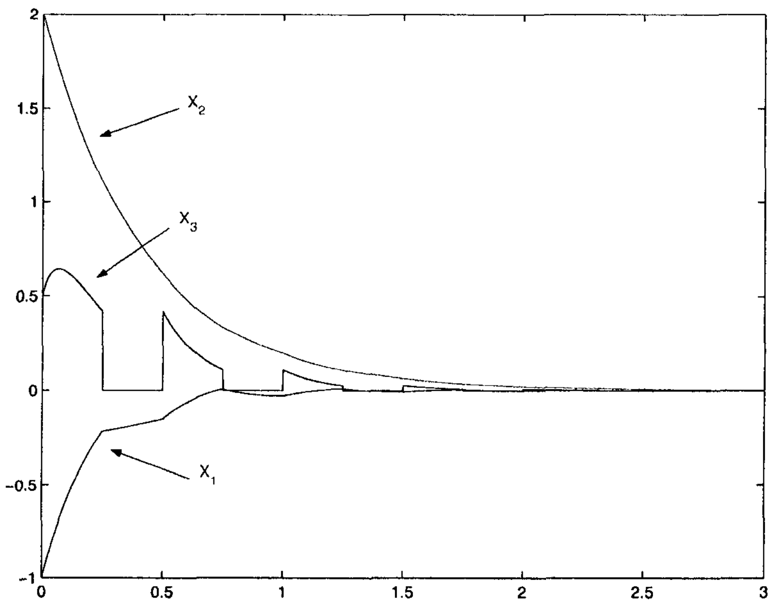

Figure 4.7: System trajectory with respect to time

We then apply these controllers to the nonlinear switching system. First, suppose the value of extra x_3 will be kept when subsystem 1 is not activated. In Figure 4.7, we present the trajectories of the switching system with stabilizing controllers and initial states (-1, 2, 0.5). From Figure 4.7, we see that the system states x_1 and x_2 are continuous but lose differentiability in switching instances. Unlike x_1 and x_2, which belong to both of subsystems, x_3 only belongs to subsystem 1. When subsystem 2 is activated, x_3 has no value. Finally all of system states converge to zero. Thus the system is stabilized by the backstepping-based controller.

The trajectories of two Lyapunov functions are shows in Figure 4.10 and Figure 4.11. We see that both of them converge to zero rapidly. Also since both (4.30) and (4.33) are in the same quadratic form, we can use (4.33) as the common Lyapunov function of two subsystems, which is shown in Figure 4.8. It is actually the combination of two Lyapunov trajectories of subsystem 1 and subsystem 2 in Figure 4.10 and Figure 4.11. Note that although this common Lyapunov function is discontinuous at switching instances, it is always monotonously decreasing between two sequential

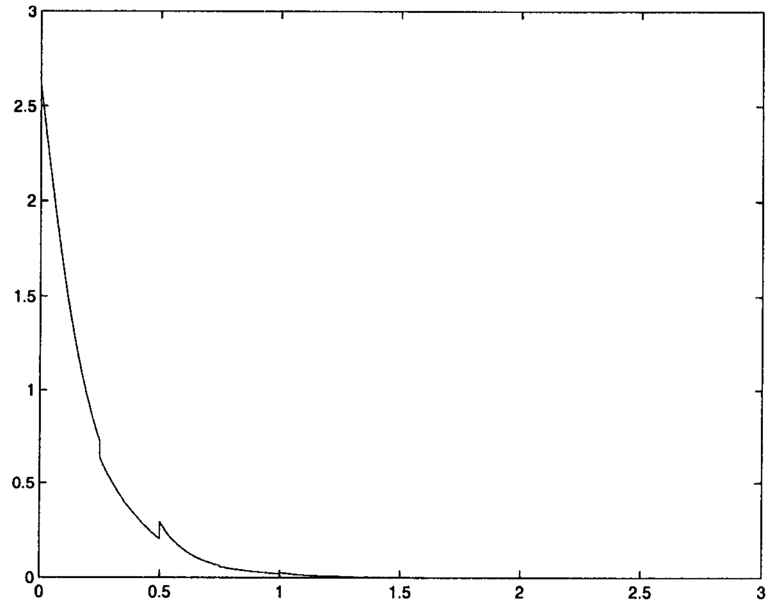

Figure 4.8: The common Lyapunov function

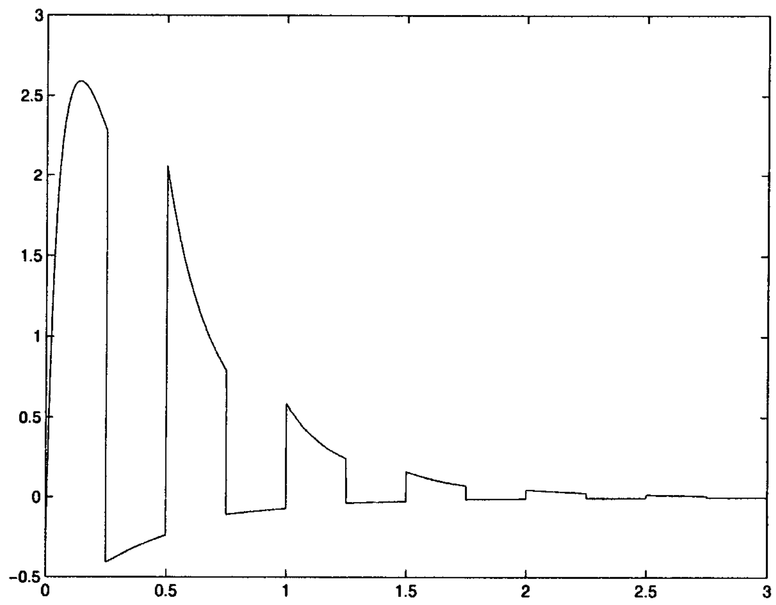

Figure 4.9: The control input

switching events. The trajectory of control input u is shown in Figure 4.9.

However in the real plant, the extra state, such as x_3, may have very complex initial value when the corresponding subsystem is activated. This value depends on

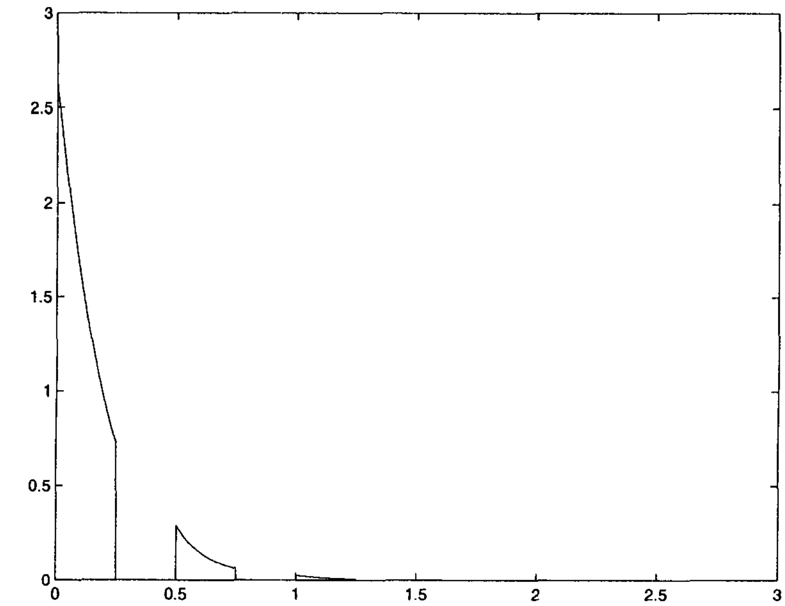

Figure 4.10: The Lyapunov function for subsystem 1

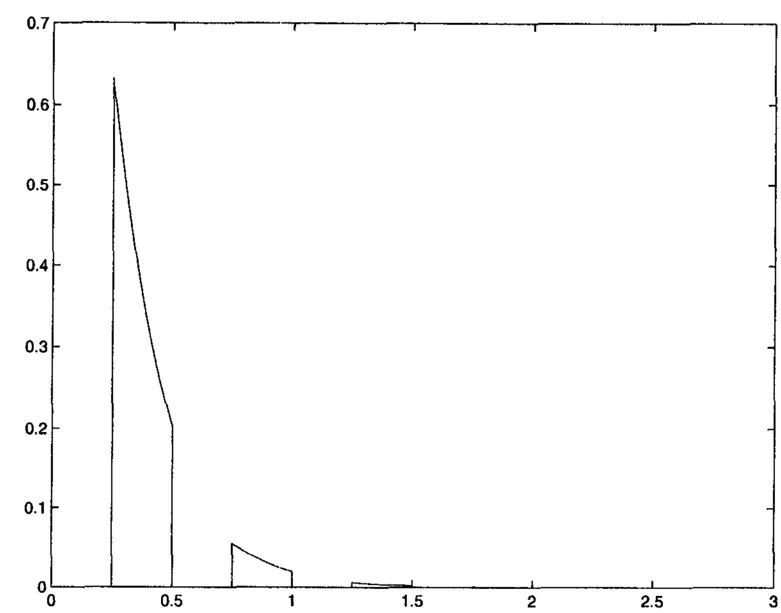

Figure 4.11: The Lyapunov function for subsystem 2

the real system. For example, if we simply choose the initial value of x_2 equal to 0.5 whenever subsystem 1 is activated. We can see x_3 will be oscillated. So do x_1 and x_2. In this situation, the Lyapunov function will oscillate together with system states excited by the initial value of x_3 at the switching moment. Figure 4.12 and Figure 4.13 show this clearly. The trajectory of control input u is shown in Figure 4.14.

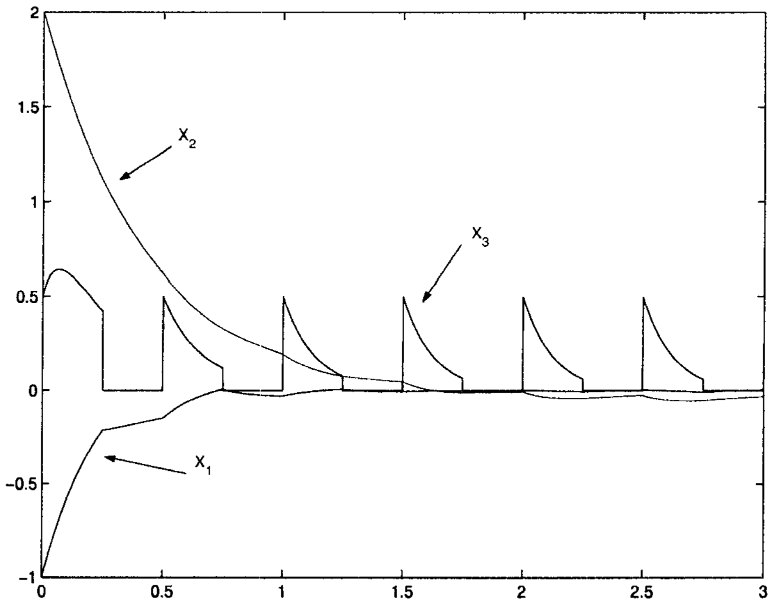

Figure 4.12: System trajectory with respect to time

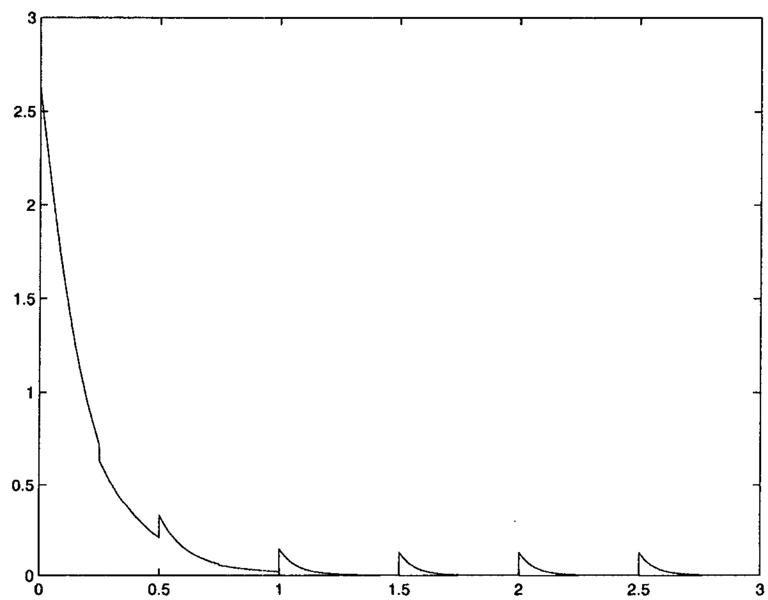

Figure 4.13: The common Lyapunov function

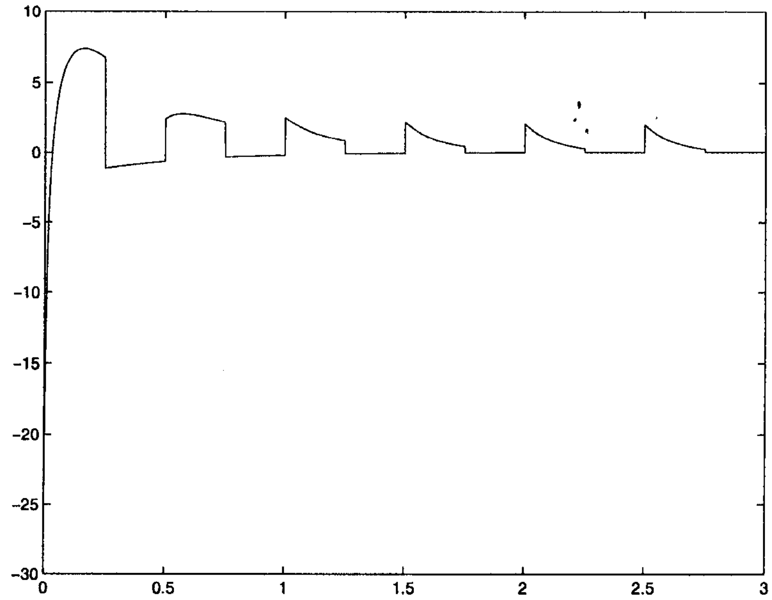

Figure 4.14: The control input

Remark 4.13 *From the above two examples, we see that the proposed backstepping can stabilize the switching systems of multiple orders. Since Lyapunov functions can be obtained simultaneously while designing controllers, the proposed backstepping is a nice approach for controller design of switching systems with subsystems of different orders.*

Remark 4.14 *From Figure 4.9 and Figure 4.14, we see that the control input u is bounded by choosing the suitable $\phi(x)$. When we use the proposed backstepping approach to design controllers for switching systems, we should be careful when choosing $\phi(x)$, so that the control input u is realizable in applications.*

Remark 4.15 *With the backstepping-based feedback controller, the values of Lyapunov functions are always decreasing between two sequential switching events. However, the reset values of additional or deficient states still play an important role here. With different reset values, the switching system may be stable, oscillated or unstable.*

Remark 4.16 *We can also consider (4.30) and (4.33) are the individual Lyapunov functions of subsystem 1 and subsystem 2 respectively. Then we can use the multiple Lyapunov theorem to verify the stability of this switching system. As to this example, from Figure 4.10 and Figure 4.11, we can see that the multiple Lyapunov theorem is satisfied. Thus the whole switching system with the controllers is stable.*

Remark 4.17 *In the last example, if we consider (4.33) as the common Lyapunov function, we can see that this Lyapunov function is not continuous because of the discontinuity of* x_3*, which only has value when subsystem 1 is activated. However, the common Lyapunov function is always monotonously decreasing between two sequential switching events. This can guarantee the stability of switching systems with backstepping controllers in most cases.*

Chapter 5

Backstepping Approach to Tracking Control

In the previous chapters, a new backstepping-based controller has been proposed to stabilize nonlinear switching systems, which could be either same order or multiple orders. With the Lyapunov functions in the strict quadratic form, the backstepping-based controller can stabilize the states of nonlinear switching systems around the equilibrium point. In this chapter, we extend our design procedure, so that the backstepping-based controller enable the outputs of nonlinear switching systems to track the given outputs, which could be any first-order differentiable signal.

5.1 Tracking Control by Backstepping

It is the purpose of this chapter to introduce a new tracking control approach for the class of switching systems (5.1)-(5.2) so that output asymptotic tracking is possible. In the following part of this section, we will first discuss the case of the single nonlinear system with the form (5.1).

Consider the single-input nonlinear system of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2, \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3, \\ &\vdots \\ \dot{x}_{k+1} &= f_k(x_1, x_2, \cdots, x_k, x_{k+1}) + g_k(x_1, x_2, \cdots, x_k, x_{k+1})u \end{aligned} \tag{5.1}$$

and the multiple outputs

$$y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+1} \end{pmatrix}, \tag{5.2}$$

where $x \in R^n$, and f_i, g_i are smooth, for all $i = 1, \cdots k$. Suppose the expected system outputs are

$$y_e(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_{k+1}(t) \end{pmatrix}. \tag{5.3}$$

The control objective is to design the control input u so that the output y can asymptotically track the first-order differentiable trajectories $y_e(t)$, i.e.,

$$\lim_{t \to \infty} (y(t) - y_e(t)) = 0 \tag{5.4}$$

In order to enable the system output to track the given output $y_e(t)$, we need to make each system state $x_j, j \in \{1, 2, ..., k+1\}$ track each given $r_j, j \in \{1, 2, ..., k+1\}$. It is natural to transfer the original system state equations to system error equations. Then, the proposed backstepping method can be used to design a stabilizing controller for the error system. If the backstepping-based controller can stabilize the system error around the origin, the output of the original system can track the given output $r_j(t)$ successfully.

Because the given output is $r_j(t) \;\; j = \{1, \cdots, k+1\}$, the error of output is

$$e_j = x_j - r_j(t), \qquad j \in [1, 2, ..., k+1]. \tag{5.5}$$

Then from the above equations, we can obtain

$$x_j = e_j + r_j(t). \tag{5.6}$$

Subscribe (5.6) to (5.1), we obtain

$$\begin{aligned}
\dot{e}_1 + \dot{r}_1(t) &= f(e_1 + r_1(t)) + g(e_1 + r_1(t))(e_2 + r_2(t)), \\
&\vdots \\
\dot{e}_j + \dot{r}_j(t) &= f_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t)) \\
&\quad + g_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t))(e_{j+1} + r_{j+1}(t)), \\
&\vdots \\
\dot{e}_{k+1} + \dot{r}_{k+1}(t) &= f_k(e_1 + r_1(t), \cdots, e_{k+1} + r_{k+1}(t)) \\
&\quad + g_k(e_1 + r_1(t), \cdots, e_{k+1} + r_{k+1}(t))u.
\end{aligned} \tag{5.7}$$

From (5.7) we can obtain the error equation

$$\begin{aligned}
\dot{e}_1 &= f(e_1 + r_1(t)) - \dot{r}_1(t) + g(e_1 + r_1(t))r_2(t) + g(e_1 + r_1(t))e_2, \\
&\vdots \\
\dot{e}_j &= f_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t)) - \dot{r}_j(t) \\
&\quad + g_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t))r_{j+1}(t) \\
&\quad + g_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t))e_{j+1}, \\
&\vdots \\
\dot{e}_{k+1} &= f_k(e_1 + r_1(t), \cdots, e_{k+1} + r_{k+1}(t)) - \dot{r}_{k+1}(t) \\
&\quad + g_k(e_1 + r_1(t), \cdots, e_{k+1} + r_{k+1}(t))u.
\end{aligned} \tag{5.8}$$

Define

$$\begin{aligned}
\bar{f}_{j-1}(e_1, \cdots, e_j, r_1, \cdots, r_j, \dot{r}_j) &= f_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t)) - \dot{r}_j(t) \\
&\quad + g_{j-1}(e_1 + r_1(t), \cdots, e_j + r_j(t))r_{j+1} \\
j &\in \{1, 2, ..., k\}, \\
\bar{f}_k(e_1, \cdots, e_j, r_1, \cdots, r_{k+1}, \dot{r}_{k+1}) &= f_k(e_1 + r_1(t), \cdots, e_{k+1} + r_{k+1}(t)) - \dot{r}_{k+1}(t).
\end{aligned} \tag{5.9}$$

Then, we obtain the following system

$$\begin{aligned}
\dot{e}_1 &= \bar{f}(e_1, r_1(t), \dot{r}_1(t)) + g(e_1, r_1(t))e_2, \qquad (5.10)\\
\dot{e}_2 &= \bar{f}_1(e_1, e_2, r_1(t), r_2(t), \dot{r}_2(t)) + g_1(e_1, e_2, r_1(t), r_2(t))e_3,\\
&\vdots\\
\dot{e}_{k+1} &= \bar{f}_k(e_1, \cdots, e_{k+1}, r_1(t), \cdots, r_{k+1}(t), \dot{r}_{k+1}(t)) + g_k(e_1, \cdots, e_{k+1}, r_1(t), \cdots, r_{k+1}(t))u.
\end{aligned}$$

Comparing system (5.10) with the previous strict feedback system (5.1), we see that, unlike the strict feedback systems (5.1) that we dealt with before, the above system is nonautonomous because of $r(t)$ and $\dot{r}(t)$. However, let's recall the proposed backstepping approach. The constructed objective Lyapunov function for the system (5.10) is

$$\bar{V}_{k+1} = \bar{V}_k(e_1) + \frac{1}{2}e_{k+1}^2 = \bar{V}_1(e_1) + \sum_{i=2}^{k+1} \frac{1}{2}e_i^2. \qquad (5.11)$$

Notice that the above Lyapunov function is independent of t. Then, the Lyapunov function and the corresponding controller law (5.14) satisfy the nonautonomous Lyapunov function theorem. Thus, the proposed backstepping approach can stabilize both the autonomous system in the form of (5.1) and the nonautonomous system in form of (5.10).

Therefore, we can design the stabilizing controller for system (5.10) using backstepping. The state e_i of the error system (5.10) can be stabilized around the origin. Then the output $y(t)$ of the original system (5.1) can track the given signal $y_e(t)$.

For example, in case of $k = 1$, the control law for the error system (5.10) is

$$\bar{u} = \bar{\phi}_1(e_1, e_2) = \frac{1}{g_1(e_1, e_2)}\{-\frac{\partial \bar{V}_1}{\partial e_1}g(e_1) + \frac{\partial \bar{V}_1}{\partial e_1}g(e_1)\frac{\bar{\phi}(e_1)}{e_2} - k_1 e_2 - \bar{f}_1(e_1, e_2)\},\ k_1 > 0 \qquad (5.12)$$

with the associated Lyapunov function

$$\bar{V}_2 = \bar{V}_1(e_1) + \frac{1}{2}e_2^2. \qquad (5.13)$$

Subscribing (5.5) to (5.12) and (5.13), we have that the final control law and associated Lyapunov function for this systems are

$$u = \phi_1(x_1, x_2) = \frac{1}{g_1(x_1, x_2)}\{-\frac{\partial V_1}{\partial(x_1 - r_1)}g(x_1) + \frac{\partial V_1}{\partial(x_1 - r_1)}g(x_1)\frac{\phi(x_1)}{x_2} \quad (5.14)$$

$$-k_1(x_2 - r_2) - (f_1(x_1, x_2) - \dot{r}_1(t) + g_1(x_1, x_2)r_2(t))\},$$

$$V_2 = V_1(x_1, r_1) + \frac{1}{2}(x_2 - r_2)^2. \quad (5.15)$$

From the above analysis, we see that, as to the system (5.1) with $k+1$ states, we can build the following Lyapunov function with the input obtained from the above procedure.

$$V_{k+1} = V_k(x) + \frac{1}{2}x_{k+1}^2 = V_1(x_1, r_1) + \sum_{i=2}^{k+1}\frac{1}{2}(x_i - r_i)^2. \quad (5.16)$$

In general, we obtain the following theorem.

Theorem 5.1 *The feedback controller (5.14) using the proposed backstepping can guarantee that the SISO or SIMO nonlinear system in the form of (5.1)-(5.2) achieves asymptotic tracking for any first-order differentiable signal.*

Proof 5 *Transfer the system state equations (5.1) to the system error equations (5.10). Then the proposed backstepping method ensures that the transferred system (5.10) is stabilized around the origin for any given first-order differentiable $r_i(t)$, since the constructed Lyapunov function is independent of t. Then the output $y(t)$ of the original system (5.1) can track the given signal $y_e(t)$. Therefore, the closed-loop SISO and SIMO nonlinear system in the form of (5.1)-(5.2) can achieve asymptotic tracking for any first-order differentiable signal.*

Remark 5.1 *The above backstepping approach applies to tracking controller design of nonlinear switching systems in the strict feedback form, especially to those can not be feedback-linearized due to the unstable zero dynamics or the lackness of well defined relative degree.*

5.2 Tracking Theorem for Switching Systems

Consider now the nonlinear switching systems (2.1). If we can find the same $V_1(x)$ for all subsystems, the subsystems will have the Lyapunov functions in the strict quadratic form (5.16). Then we can use the above procedure to design controllers for each of subsystems and combine these sub-controllers to obtain a synchronous switched controller for switching systems. The following theorem guarantees the stability of closed-loop switching systems.

Theorem 5.2 *As to a continuous switching system consisting of the subsystems with the form (5.1)-(5.2), furnished with an arbitrary switching rule, with the proposed backstepping tracking method, if we can construct the same $V_1(x_1)$ for all subsystems, then the corresponding close-loop switching system with backstepping-based controller can track the expected outputs $y_e(t)$, which could be any first-order differentiable signals.*

Proof 6 *The proof is straightforward. First we transfer all subsystems to error-state form (5.10). Assuming we can find the same $V_1(x_1)$ for all subsystems, there exists the same $\bar{V}_1(\bar{e})$ for all subsystems in the error state form (5.10). Then Theorem 4.2 and Theorem 4.4 guarantee the closed-loop switching systems in the form (5.10) stable around the origin. In the other words, the state errors of original switching system converge to zero asymptotically. Therefore, the original switching system can track the expected outputs $y_e(t)$. The constructed Lyapunov function (5.16) also shows that the proposed controller will ideally achieve asymptotic tracking.*

Remark 5.2 *The above theorem is applicable to any continuous SISO or SIMO switching system consisting of the subsystems in the form (5.1)-(5.2).*

Remark 5.3 *The proposed backstepping approach is independent of the switching* *rules. Thus, it can be used for both time-dependent and state-dependent switching systems.*

Remark 5.4 *The above theorem is based on Theorem 4.2 and Theorem 4.4. Thus the switching system could be either same-order or multiple-order system.*

The following example will illustrate theoretical results for multiple-order switching systems.

5.3 Example

In this part, the following example is given to show the performance of the proposed backstepping-based control for the nonlinear switching systems. More explicitly, consider the switching system with two subsystems of multiple orders.

$$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2, \\ \dot{x}_2 &= -2x_2 - x_3, \\ \dot{x}_3 &= x_1 + 3x_2 - 2u, \end{aligned} \tag{5.17}$$

and,

$$\begin{aligned} \dot{x}_1 &= -6x_1 - x_2, \\ \dot{x}_2 &= -x_1^2 + 6u, \end{aligned} \tag{5.18}$$

with the output

$$y = x_1. \tag{5.19}$$

Assume this switching system is time-dependent with period of 0.5s and initial state $(-1, 2, 0.5)$. The expected output is $r(t)$. Thus we need to design two controllers to solve this tracking problem. Here we use the backstepping method and obtain two state-feedback controllers for each of subsystems. First, we transfer the system equations to error equations. Since

$$\begin{aligned} e_1 &= x_1 - r_1(t), \\ e_2 &= x_2 - r_2(t), \\ e_3 &= x_3 - r_3(t), \end{aligned} \tag{5.20}$$

then, according to (5.7), we obtain the system error equations as follows:

$$\begin{aligned} \dot{e}_1 &= -3e_1 - 3r_1(t) - \dot{r}_1(t) + r_2(t) + e_2, \\ \dot{e}_2 &= -2e_2 - 2r_2(t) - r_3(t) - \dot{r}_2(t) - e_3, \\ \dot{e}_3 &= e_1 + r_1(t) + 3e_2 + 3r_2(t) - \dot{r}_3(t) - 2u. \end{aligned} \tag{5.21}$$

and,

$$\begin{aligned} \dot{e}_1 &= -6x_1 - 6r_1(t) - \dot{r}_1(t) - r_2(t) - e_2, \\ \dot{e}_2 &= -(e_1 + r_1(t))^2 - \dot{r}_2(t) + 6u. \end{aligned} \tag{5.22}$$

As to subsystem 1, consider the first equation. We can choose

$$\bar{\phi}(e_1) = \dot{r}_1(t) + 3r_1(t) - r_2(t) + e_1, \tag{5.23}$$

with respect to the Lyapunov function

$$\bar{V}(e_1) = \frac{1}{2}e_1^2. \tag{5.24}$$

Then considering the first two equations, we can obtain by using (3.41)

$$\bar{\phi}_1(e_1, e_2) = e_1 + (k-2)e_2 - \frac{e_1(\dot{r}_1(t) + 3r_1(t) + e_1 - r_2(t))}{e_2} - 2r_2(t) - r_3(t) - \dot{r}_2(t) \tag{5.25}$$

with respect to the Lyapunov function

$$\bar{V}_2(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2. \tag{5.26}$$

Then considering the third equation, we can obtain the final input u

$$\bar{u} = \bar{\phi}_2(e_1, e_2, e_3) = (e_2 - (e_1 + (k_1 - 2)e_2 - \frac{e_1(\dot{r}_1(t) + 3r_1(t) + e_1 - r_2(t))}{e_2} \tag{5.27}$$
$$-2r_2(t) - r_3(t) - \dot{r}_2(t))/e_3 - k_2 x_3 e_1 - r_1(t) - 3e_2 - 3r_2(t) + \dot{r}_3(t))/(-2) \ ,$$

with respect to the Lyapunov function

$$\bar{V}_3(e_1, e_2, e_3) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2. \tag{5.28}$$

As to subsystem 2, consider the first equation. We can choose

$$\bar{\phi}(e_1) = 2e_1 - 6r_1(t) - \dot{r}_1(t) - \dot{r}_2(t), \tag{5.29}$$

with respect to the Lyapunov function

$$\bar{V}_1(e_1) = \frac{1}{2}e_1^2. \tag{5.30}$$

Then we can obtain the final input u

$$\bar{u} = \bar{\phi}_1(e_1, e_2) = (e_1 + \frac{e_1(\dot{r}_1(t) + 6r_1(t) - 2e_1 + \dot{r}_2(t))}{x_2} - k_1 e_2 + (e_1 + r_1(t))^2 + \dot{r}_2(t))/6 \tag{5.31}$$

with respect to the Lyapunov function

$$\bar{V}_2(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2. \tag{5.32}$$

We can use the above controller to verify the performance of closed-loop switching systems. First, we use the step signal as the tracking signal with the other two states x_2, x_3 stabilized at the origin. Then

$$y_e(t) = 1, \qquad \forall t > 0, \tag{5.33}$$

and

$$\dot{y}_e(t) = \dot{r}_1(t) = 0, \qquad \forall t > 0. \tag{5.34}$$

Design the backstepping controller according to (5.27) and (5.31). In Figure 5.1, we compare the trajectories of the expected output and the system output. The error is shown in Figure 5.2. From above two figures, we see that the output of the

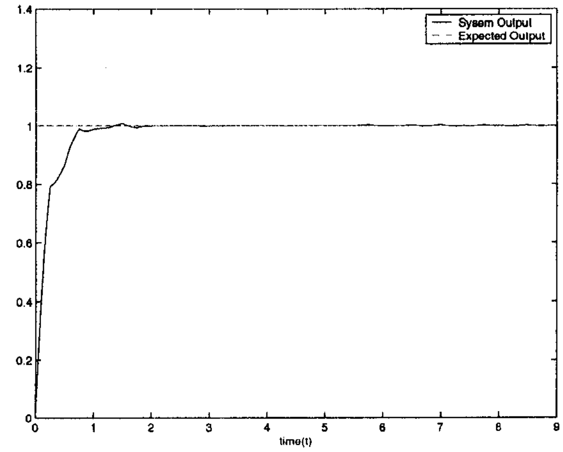

Figure 5.1: Expected and system outputs

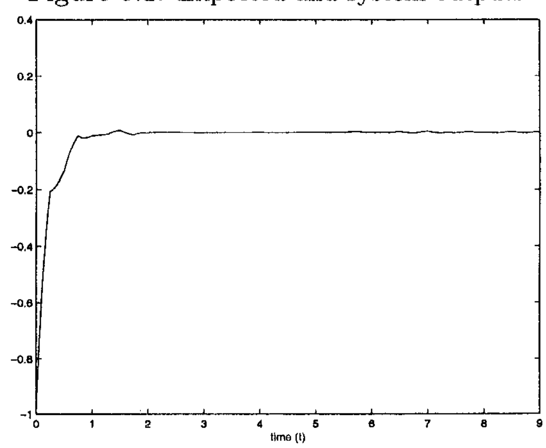

Figure 5.2: Trajectory of output error

switching system can track the expected output successfully.

Consider now the same system and let the tracking signal be a ramp. Then:

$$y_e(t) = t, \qquad \forall t > 0, \tag{5.35}$$

and

$$\dot{y}_e(t) = \dot{r}_1(t) = 1, \qquad \forall t > 0. \tag{5.36}$$

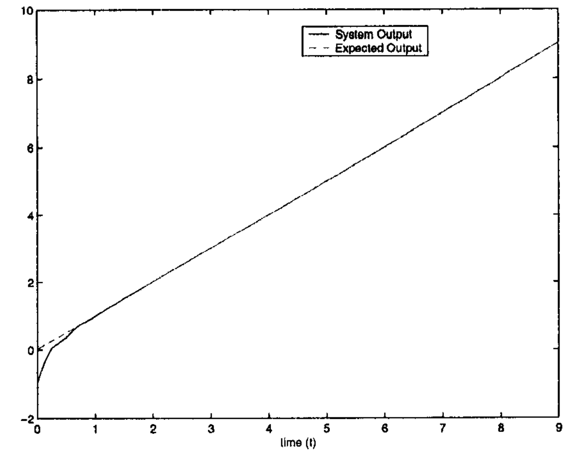

Figure 5.3: Expected and system outputs

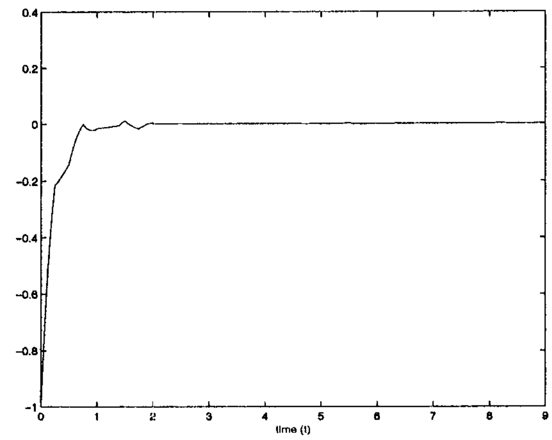

Figure 5.4: Trajectory of output error

We obtain the results shown in Figure 5.3 and Figure 5.4. From the above figures, we see that the output of the switching system can track the given ramp signal.

Consider now the same system and change the tracking signal to a sinusoid. Then

$$y_e(t) = \sin t, \qquad \forall t > 0, \tag{5.37}$$

and

$$\dot{y}_e(t) = \dot{r}_1(t) = \cos t, \qquad \forall t > 0. \tag{5.38}$$

And we obtain the results shown in Figure 5.5 and Figure 5.6:

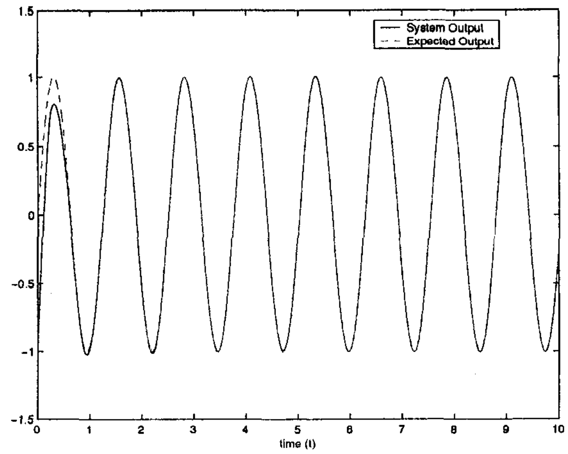

Figure 5.5: Expected and system outputs

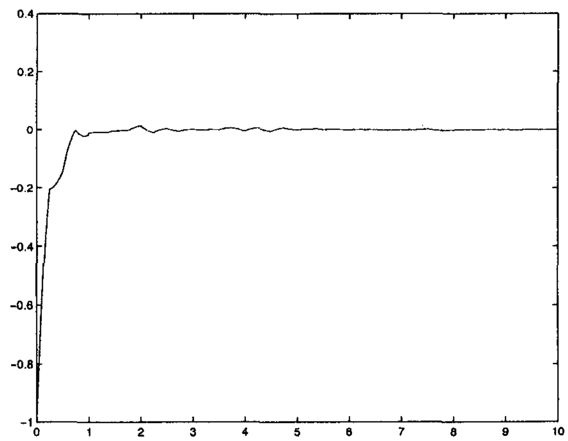

Figure 5.6: Trajectory of output error

The output trajectories are shown in Figure 5.5. We can see that both of them converge to zero very rapidly. Also since both (4.30) and (4.33) are in the same quadratic form, we can use (4.33) as the common Lyapunov function of two subsystems, which is shown in Figure 5.9. It is actually the combination of two Lyapunov trajectories of subspace 1 and subspace 2 in Figure 5.7 and Figure 5.8. Note that although this common Lyapunov function is discontinuous at the switching instance, it is always continuously decreasing between two sequential switching events.

From the above figures, we see that the output of the switching system tracks the given sinusoid.

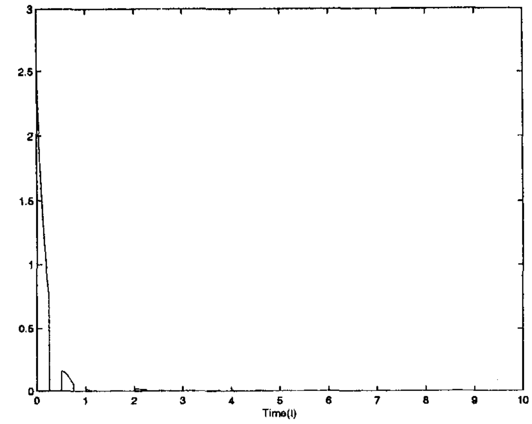

Figure 5.7: The Lyapunov function for subsystem 1 (5.21)

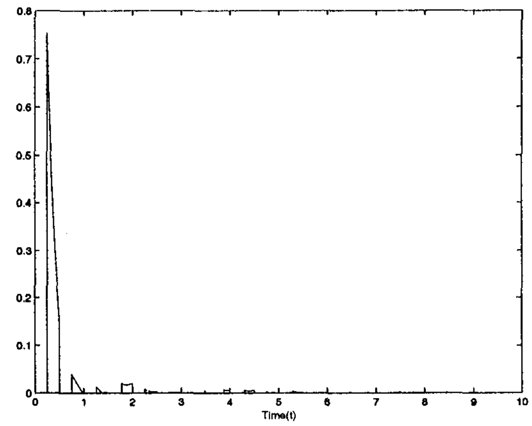

Figure 5.8: The Lyapunov function for subsystem 2 (5.22)

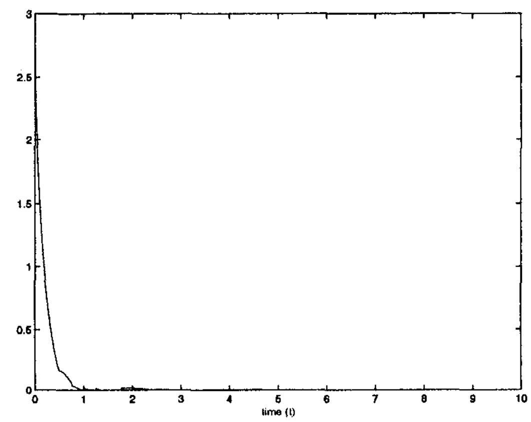

Figure 5.9: The common Lyapunov function

Remark 5.5 *The above example shows that the backstepping method can be used for switching systems of multiple orders. The proposed backstepping method can not only stabilize switched systems around the equilibrium point, but also solve the tracking problem of nonlinear switching systems via system transformation. The backstepping-based controller enables the system output to track any given first-order differentiable signal.*

Chapter 6

Conclusions

Generally speaking, switching systems can be viewed as continuous-time systems with discrete switching events from a certain class. Such systems can be considered as higher-level abstractions of hybrid systems.

In this thesis, we have proposed a method to design controllers for nonlinear switching systems. First, we introduced basic concepts and classification of switching systems, especially those that apply to systems of multiple orders. We then discussed the traditional backstepping approach for controller design of nonlinear systems and use an example to show its shortcoming for nonlinear switching systems. In order to solve this problem, we proposed a new backstepping approach to controller design for nonlinear switching systems. This new approach can stabilize switching systems of both same order and multiple orders. Also in our design procedure, we can obtain Lyapunov functions in the same quadratic form for all of subsystems of switching systems. We then obtained and discussed several sufficient conditions for same-order and multiple-order switching systems. With these conditions, the stability of closed-loop switching systems in either the time-dependent or state-dependent cases is guaranteed by the multiple Lyapunov function theorem for switching systems, which is proposed and proved in this thesis. Also, the proposed controller can achieve asymptotic tracking by system transformation. Finally several examples were given and discussed in detail. The simulation results show that the proposed backstepping approach can be applied successfully for controller design of nonlinear switching systems.

6.1 Switching Systems

In this thesis, we introduced the traditional definitions and some stability theorems for switching systems. It is easy to see that the common and multiple Lyapunov function theorems play important roles for switching systems. However, these stability theorems can only be used for the same-order switching system, whose subsystems have the same order. This assumption is restrictive and is not based on theoretical foundations but on mathematical convenience.

The main objective of this thesis is to extent these results to the case of multiple-order switching system, which is virtually unexplored and very few results are available in the literature. The subsystems in these switching systems have different orders. Thus we can not use the established theorems to verify the stability of such systems. First, we classified the states and subsystems in multiple-order systems, according to their different roles in switching. The definitions of equilibrium point and stability of such system are given and discussed in detail. We can see that the equilibrium point x_e of such systems is different from the classical definition of equilibrium point, since x_e may not be in any of the subsystem spaces. However, it still exists in the system state space.

Based on these definitions, the multiple Lyapunov function theorem for multiple-order systems is proposed and proved. The theorem establishes a powerful tool to design controllers for multiple-order switching systems. The result provides a sufficient condition to verify the stability of switching systems. One of advantages of this theorem is that it can be applied without knowing what happens at the time of switching and where the additional or deficient states go to or come from. The theorem requires constructing different Lyapunov functions for the different subsystems, and then it can be used to prove the stability of switching systems.

6.2 Proposed Backstepping Approach

One of the most powerful methods to design controller for nonlinear systems is the backstepping procedure. By constructing Lyapunov functions, we can design stabilizing controllers for nonlinear systems via the backstepping approach. When designing the backstepping-based controller, we obtain the Lyapunov function for the nonlinear system simultaneously, which guarantees closed-loop stability of the origin. However, since the Lyapunov function obtained by the traditional backstepping method is not in the strict quadratic form. The traditional backstepping cannot stabilize nonlinear switching systems. In the thesis, we proposed a new backstepping procedure to design switched controller, which can stabilize switching systems of both same order and multiple orders. Also in our design procedure, we can obtain Lyapunov functions in the same quadratic form for all of subsystems of switching systems. Thus the stability of closed-loop switching systems in either the time-dependent or state-dependent cases is guaranteed by the Lyapunov function theorem. Also, after system transformation, asymptotic tracking can be achieved by the proposed controller. Several examples given in the thesis show the validity of this method.

6.3 Fulfillment of Thesis Objectives and Future Work

Four examples were given for the theory presented. Two of them are about same-order switching systems and carried to illustrate the proposed backstepping approach can stabilize both the time-dependent and state-dependent cases. The other two examples dealt with multiple-order switching systems. We can see that the backstepping-based controller can stabilize the switching systems if the sufficient conditions given in the second chapter are satisfied.

Because this backstepping approach can stabilize a broad class of nonlinear switching systems, which consist of strict feedback subsystems, this method could have a broad class application. Typical examples in real life include car transmission systems, process control systems, mobile robots etc.

It is hoped that this thesis fulfilled the motivation to provide a solid framework and basis for switching systems of both same and multiple orders, and give a general method to design controllers for nonlinear switching systems, which could be of same or multiple orders.

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