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# QUOTIENT-UNIVERSAL SPACES

BY

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A THESIS
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled QUOTIENT-UNIVERSAL SPACES submitted by RENEE SIROIS-DUMAIS in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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# ABSTRACT

Given a family $F$ of topological spaces, we ask the question : does there exist a quotient-universal space for $F$, that is, does there exist an element in $F$ of which every other element of $F$ is a quotient ? The families we are interested in include the metric spaces, the first-countable spaces and (especially) the sequential spaces. A very natural quotient-universal metric space is known for sequential spaces of cardinality $c$ ; we show that this cannot be generalized for cardinality $\aleph_0$ or any cardinal $\kappa$ for which $\kappa^{\aleph_0} > \kappa$ . In fact we obtain the two following results :

- $c$ can be characterized among cardinals as the smallest for which there is a quotient-universal metric space for sequential spaces of that cardinality.
- there is a quotient-universal metric space for sequential spaces of cardinality $\kappa$ if and only if $\kappa^{\aleph_0} = \kappa$.

$2^c$ non-homeomorphic countable sequential spaces are constructed, which are all quotients of separable metric spaces but most of which are not quotients of any countable metric space. The introduction of the notions of quasi-first-countability and weakly-quasi-first-countability enables us to solve the problem of which countable sequential spaces are quotients of countable metric spaces.

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Renée Sirois-Dumais (1977)

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# INTRODUCTION

Certain families of topological spaces contain a member which is of special interest because every other member can be obtained from it in some way; such a space is called universal.Two kinds of universal spaces have been considered so far : those that contain a homeomorphic copy of all other members of the family considered and those of which any other member of the family can be obtained as a quotient. We are interested in spaces of the latter sort, and we will call them, to avoid any confusion, quotient-universal spaces.

One of the first examples of quotient-universal spaces to appear in the literature is the Cantor set C which is universal for the class of compact metric spaces. Indeed, Alexandroff and Urysohn proved in 1929 [1] that every compact metric space is a continuous image of C and hence also a quotient of C (since a continuous map between compact Hausdorff spaces is also a quotient map) . Another example of a quotient-universal space is the unit interval I in R which is universal for the family of compact metric,connected,locally connected spaces,also known as Peano spaces ; the well-known Hahn-Mazurkiewicz theorem (see [10] ) shows that every such space is a continuous image of I and hence, once again, a quotient of I .

The results just mentioned can be considered answers in specific instances to a question which in full generality reads : "Given a family F of topological spaces, does there exist some

$X \in F$ such that every $Y \in F$ is a quotient of $X$ ?" We propose in this thesis, then, to consider this question for various families $F$. Obviously, we must impose some cardinality restriction on the members of $F$ and the results will then often turn out to be results on cardinal numbers.

In Chapter I, we look at families consisting of metric spaces and first-countable spaces. For cardinality $c$, it is known that the disjoint union of $c$ copies of a convergent sequence is quotient-universal for metric spaces of that cardinality ; a similar result holds for cardinality $\kappa$ whenever $\kappa^{\aleph_0} = \kappa$ but we show that it fails for cardinality $\aleph_0$. However we show that for each cardinal $\kappa$, there is a quotient-universal space for metric spaces of that cardinality and this space acts also as a universal space for first-countable spaces.

Chapter II deals with the family of countable sequential spaces. For cardinality $c$, the space mentioned above is quotient-universal for sequential spaces of that cardinality and again a similar situation holds for any cardinality $\kappa$ such that $\kappa^{\aleph_0} = \kappa$. The universal spaces in these cases have the additional property of being metric. We want to know if a similar result can be obtained in the countable case. Since countable metric spaces do have a quotient-universal space, the question is then reduced to : is any countable sequential space a quotient of some countable metric space ? We show that the answer is no by constructing a large family of countable sequential spaces most of which cannot be the quotient of any metric space of cardinality less than $c$. This leads to a characterization of $c$ among cardinals as the smallest one for which there is a metric space quotient-universal for sequential spaces of that cardinality. The same family of countable spaces enables us to answer a question of Michael and Stone in [6] concerning quotients of the set $P$ of irrational numbers.

In chapter III, we finish our study of sequential spaces with the cardinalities $\kappa > \aleph_0$ for which $\kappa^{\aleph_0} = \kappa$. Universal spaces for cardinalites $\kappa$ with $\kappa^{\aleph_0} = \kappa$ have the additional property of being metric ; a construction analogous to that of Chapter II shows that no such universal space exists for cardinalities $\kappa$ such that $\kappa^{\aleph_0} > \kappa$, leading to the characterization of cardinals $\kappa$ for which $\kappa^{\aleph_0} = \kappa$ as precisely the ones for which there is a quotient-universal metric space for sequential spaces of that cardinality.

The realization (in Chapter II) that not every countable sequential space is quotient of a countable metric space led to the problem of characterizing topologically those that are. To solve that problem, Chapter IV introduces and studies "quasi-first-countable" and "weakly-quasi-first-countable" spaces. Some of the results obtained are that countable spaces are hereditarily quotient images of some countable metric space if and only if they are quasi-first countable and countable spaces are quotient images of some countable metric space if and only if they are weakly-quasi-first-countable spaces.

We mention once and for all that all spaces considered are Hausdorff spaces.

Also, any definition, proposition or theorem of topology which is used without special introduction or reference can be found in Willard [10]:

# CHAPTER I

## Metric and First-Countable Spaces

### 1. Quotient maps.

**Definition 1** : A map f from a topological space X onto a topological space Y is called a *quotient map* if for every subset U of Y, U is open if and only if $f^{-1}(U)$ is open.

**Definition 2** : A quotient map $f : X \to Y$ is called *hereditarily quotient* if for any subset A of Y, the restriction map $f: f^{-1}(A) \to A$ is a quotient map.

Hereditarily quotient maps have also been called in the literature "pseudo-open maps" [2]. The reason is made clear by the following theorem.

**Theorem 1** : Let X and Y be two topological spaces and f a continuous map from X onto Y. Then f is hereditarily quotient if and only if for any $y_0 \in Y$ and A, a subset of X, A is a neighborhood of $f^{-1}(y_0)$ in X if and only if f (A) is a neighborhood of $y_0$ in Y.

**Proof** : See [3].

Hereditarily quotient maps are obviously quotient maps while the converse is not true (for an example, see section 1 in Chapter II ). Closed continuous maps ("images of closed sets are closed") and open continuous maps ("images of open sets are open") are particular cases of hereditarily quotient maps.

## 2. Metric spaces

For any metric space $X$ let $S(X)$ be the disjoint union of the convergent sequences of $X$. It is easily seen that the natural map from $S(X)$ onto $X$ (mapping points onto themselves) is a quotient map. Hence if $S$ denotes a convergent sequence, say $S = \{\frac{1}{n} ; n \in N\} \cup \{0\}$ and if $\kappa$ is a cardinal such that $\kappa^{\aleph_0} = \kappa$, the disjoint union of $\kappa$ copies of $S$ provides a quotient-universal space for metric spaces of cardinality less than or equal to $\kappa$ (since such spaces have no more than $\kappa^{\aleph_0} = \kappa$ convergent sequences ). In fact, we will see in Chapter II that this space is universal for an even larger class of spaces.

Hence, in particular , the family of metric spaces of cardinality $c$ (and the same holds for cardinality $2^c$ ) has a quotient-universal space. This leads to the following questions for the countable case :

- is the disjoint union of $\aleph_0$ copies of $S$ (which we will denote by $m(\aleph_0)$ ) a quotient-universal space for countable metric spaces ?

- if not, is some other countable metric space universal ?

The first question could be rephrased as : is it possible to determine the topology of any countable metric space using only countably many of its convergent sequences ? One could hardly expect this to be true and in fact, countable metric spaces are rarely quotients of $m(\aleph_0)$ as the following theorem shows :

Theorem 2 : If $X$ is a countable metric space, the following are equivalent :

(a) $X$ is a quotient of $m(\aleph_0)$.

(b) $X$ is hereditarily locally compact.

(c) $X$ is a subspace of $m(\aleph_0)$.

Proof : (a) → (b). Let us denote by $S^m$ the m-th copy of $S$ in $m(\aleph_0)$. Let $A$ be a subset of $X$ and $y_0$ a non-isolated point of $A$. We want to show that $y_0$ has a compact neighborhood in $A$. Let $\Gamma$ be the set of indices $m$ for which $f^{-1}(y_0)$ contains the limit point of $S^m$ and $f^{-1}(A)$ contains a subsequence of $S^m$.

Let $F = (\cup_{m\in\Gamma} S^m) \cap f^{-1}(A)$. Notice that $\Gamma$ is not empty since $y_0$ is non-isolated. We claim that for some $C \subset F$, where $C$ contains a tail of each $S^m \cap f^{-1}(A)$, $m \in \Gamma$, $f(C)$ is compact.

For suppose not ; let $y_1 \in f(F)$ be such that $d(y_1,y_0)<1$ ; let $x_1 \in F$ be such that $f(x_1) = y_1$ and let $\sigma(1) \in \Gamma$ be such that $x_1 \in S^{\sigma(1)}$. Now since none of the $f(C)$, for $C$ as described above, are compact, we can find $y_2 \in f(F - \cup\{S^m; m \in \Gamma, m \le \sigma(1)\})$ such that $d(y_2,y_0) < \min\{\frac{1}{2}, d(y_1,y_0)\}$ ; let $x_2 \in F$ be such that $f(x_2) = y_2$ and let $\sigma(2) \in \Gamma$ be such that $x_2 \in S^{\sigma(2)}$. Similarly, we find $y_n$ $x_n$, $\sigma(n)$ such that

- $y_n \in f(F - \cup\{S^m; m \in \;, m \le \sigma(n-1)\})$
- $d(y_n,y_0) < \min\{\frac{1}{n}, d(y_{n-1},y_0)\}$
- $f(x_n) = y_n$
- $x_n \in S^{\sigma(n)}$ .

Now we define a subset $M$ of $F$ as follows :

If $i \le \sigma(1)$, $M$ contains the elements $x$ of $S^i \cap f^{-1}(A)$ such that $d(f(x),y_0) < d(y_1,y_0)$.

If $\sigma(1) < i \leq \sigma(2)$, M contains those x of $S^i \cap f^{-1}(A)$ such that $d(f(x),y_0) < d(y_2,y_0)$,

and in general,

if $\sigma(n-1) < i \leq \sigma(n)$, M contains the elements x of $S^i \cap f^{-1}(A)$ such that $d(f(x),y_0) < d(y_n,y_0)$.

The set M is so constructed that f(M) does not contain any of the $y_n$'s. Now, since all our spaces here are metric, the map $f : f^{-1}(A) \to A$ is hereditarily quotient ( see Chapter II, theorem 2); clearly $M \cup f^{-1}(y_0)$ is a neighborhood of $f^{-1}(y_0)$ in $f^{-1}(A)$; therefore by theorem 1, f( M ) is a neighborhood of $y_0$. But this is a contradiction since $(y_n)_{n \in N}$ converges to $y_0$ in A. This provides the compact neighborhood we were looking for. ◊

(b) → (c). If $p_n \to p$ in X, where each $p_n$ is non-isolated, then $X - \{ p_n ; n \in N \}$ is not locally compact since any neighborhood of p would have a sequence with no converging subsequence ( a sequence converging to a suitable $p_n$ ). Hence, if X is hereditarily locally compact, then each point of X has a deleted neighborhood consisting of isolated points. Now if p is a non-isolated point of X, there exists a neighborhood V of p such that V - U is finite for each neighborhood U of p contained in V : simply take V to be any compact neighborhood of p consisting of isolated points. It follows that each non-isolated point p in X has a neighborhood which consists of a sequence converging to p.

(c) → (a). For an example of such a quotient map, if

$A \subset m(\aleph_0)$ and $x_0$ is a fixed point of $A$, map $A$ onto itself and for every $x \in m(\aleph_0) - A$, if $x$ is in a sequence $S^m$ for which the limit point is in $A$, map $x$ onto this limit point, otherwise map $x$ to $x_0$. □

Remark. Among countable metric spaces which are not quotients of $m(\aleph_0)$, the simplest ( in a sense to be made precise shortly ) is the metric wedge product $A$ of countably many converging sequences, that is

$$A = \{a\} \cup \{a_{ij}; i,j \in N\}$$

with the topology in which each $a_{ij}$ is isolated and (basic) neighborhoods of the point $a$ have the form

$$U_n = \{a\} \cup \{a_{ij}; j \geq n\} .$$

This space is the test space for countable metric spaces which are not quotients of $m(\aleph_0)$. In fact, any metric space (countable or not ) is hereditarily locally compact if and only if it contains no copy of $A$. Since $A$ itself is not locally compact, necessity is obvious. For sufficiency, suppose $X$ is not hereditarily locally compact and let $B$ be a non-locally compact subset of $X$ and $p \in B$ a point with no compact neighborhood in $B$. There exists an increasing sequence of integers $(\sigma(n))_{n \in N}$ such that for any $n \in N$, the ring $B(p,\frac{1}{\sigma(n)}) - B(p,\frac{1}{\sigma(n+1)})$ contains countably many elements of $B$, say $(x^n_m)_{m \in N}$. Otherwise, there would be some $n_0 \in N$ such that $B(p,\frac{1}{\sigma(n_0)}) - B(p,\frac{1}{n})$ is finite for every $n > \sigma(n_0)$; but then $B(p,\frac{1}{\sigma(n_0)})$ would be a compact neighborhood of $p$ in $B$. Hence the sequence $(\sigma(n))_{n \in N}$ exists and the sequences $(x^n_m)_{m \in N}$ for $n \in N$

together with p provide the required copy of A .

Theorem 2 shows that very few countable metric spaces are quotients of $m(\aleph_0)$ . Thus, the situation which holds for cardinalities like c , $2^c$ ... does not hold for the countable case. We must then look at the second question : is some other space universal for the countable metric spaces ?

The answer is yes . As a corollary to a result of Sierpinski , every countable metric space is a quotient of the space Q of rational numbers . Specifically ,

Theorem 3 (Sierpinski) : Every countable metric space which is dense-in-itself is homeomorphic to Q .

Proof : see [9] .

Corollary : Every countable metric space is a quotient image of Q .

Proof : If X is a countable metric space , then X × Q is a countable dense-in-itself metric space and hence is homeomorphic to Q. Composition of this homeomorphism with the first projection of the product gives the required quotient map. □

We conclude that for cardinalities κ such that $\kappa^{\aleph_0} = \kappa$ as well as for cardinality $\aleph_0$ , there is a quotient-universal space for metric spaces of that cardinality. It remains to look at the case of cardinalities $\kappa > \aleph_0$ such that $\kappa^{\aleph_0} > \kappa$ . For spaces X of such cardinalities, the space $S(X)$ defined at the beginning of this section will often contain $\kappa^{\aleph_0}$ converging sequences; the usual disjoint union of convergent sequences will then have cardinality bigger than that of the space ($\kappa^{\aleph_0}$) and hence does not qualify as a universal space. But it turns out that those cardinalities still have a universal space and in fact this space will act as a universal space not only for metric spaces but also for first-countable spaces.

## 3. First-countable spaces

We show in this section that first-countable spaces of a given cardinality have a quotient-universal space.

Let $\kappa$ be a cardinal and let $(\omega \times \kappa)^*$ be the following set :

$$(\omega \times \kappa)^* = \{(n,\gamma) ; n \in \mathbb{N},\ \gamma < \kappa\} \cup \{(\omega,\kappa)\} .$$

We give to $(\omega \times \kappa)^*$ the following topology : every point is open except $(\omega,\kappa)$ which has as a neighborhood base the sets :

$$\{(n,\gamma) \ ; \ n \geq n_0 \ , \ \gamma < \kappa \}$$

as $n_0$ runs through the integers. Now let $T(\kappa)$ be the disjoint union of $\kappa$ copies of $(\omega \times \kappa)^*$. We denote the $\alpha$-th copy by $(\omega \times \kappa)^*_\alpha$ so that

$$T(\kappa) = \bigcup_{\alpha<\kappa} (\omega \times \kappa)^*_\alpha$$

and we denote the elements of $(\omega \times \kappa)^*_\alpha$ by $(n,\gamma)_\alpha$ or $(\omega,\kappa)_\alpha$. Clearly the space $T(\kappa)$ is first-countable. We have the following theorem :

Theorem 4 : Every first-countable space $X$ of cardinality less than or equal to $\kappa$ is a quotient of $\mathcal{T}(\kappa)$.

Proof : We may as well assume that $X$ has cardinality $\kappa$ so let $X = \{ x_\alpha ; \alpha < \kappa \}$ be a first-countable space. For each $x_\alpha \in X$, let $( B_\alpha^n )_{n \in N}$ be a countable neighborhood base at $x_\alpha$. We may assume the $B_\alpha^n$'s are decreasing. Now since $|B_\alpha^n| \leq \kappa$, let $f_\alpha^n$ be a map of $\kappa$ onto $B_\alpha^n$. We define $f : \mathcal{T}(\kappa) \to X$ as follows :

$$f((n,\gamma)_\alpha) = f_\alpha^n(\gamma)$$

$$f((\omega,\kappa)_\alpha) = x_\alpha \quad .$$

We show that $f$ is a quotient map. First, $f$ is continuous. For, let $O = \{x_\beta ; \beta \in \Gamma \subset \kappa\}$ be an open set of $X$. To show that $f^{-1}(O)$ is open in $\mathcal{T}(\kappa)$, it suffices to show that $f^{-1}(O)$ contains a neighborhood of any $(\omega,\kappa)_\beta$, $\beta \in \Gamma$. Now since $O$ is open and $x_\beta \in O$, $O$ contains $B_\beta^{n_0}$ for some $n_0 \in N$ and since the $B_\beta^n$'s are decreasing, $O$ contains $B_\beta^n$ for all $n \geq n_0$. Therefore $f^{-1}(O)$ contains $\{ (n,\gamma)_\beta ; n \geq n_0 , \gamma < \kappa \}$. This shows that $f^{-1}(O)$ is open. Therefore $f$ is continuous.

Now let $A \subset X$ be such that $f^{-1}(A)$ is open in $\mathcal{T}(\kappa)$. Say $A = \{ x_\beta ; \beta \in \Gamma \subset \kappa \}$. Since $(\omega,\kappa)_\beta \in f^{-1}(A)$ and $f^{-1}(A)$ is open, for some $n_0$, the set $\{(n,\gamma)_\beta ; n \geq n_0 , \gamma < \kappa \}$ is contained in $f^{-1}(A)$. This implies that $A$ contains $B_\beta^{n_0}$. Therefore $A$ contains a neighborhood of each of its points and hence $A$ is open. □

The space $\mathcal{T}(\kappa)$ is then a quotient-universal space for first-countable spaces of cardinality $\kappa$ . Note that $T(\kappa)$ is in fact a metrizable space . For example , the following metric $d$ is compatible with the topology of $T(\kappa)$ :

$$
\begin{array}{lcll}
d( (n,\gamma)_\alpha , (n',\gamma')_\beta ) & = & 1 & \text{if } \beta \neq \alpha \\
d( (\omega,\kappa)_\alpha , (\omega,\kappa)_\beta ) & = & 1 & \text{if } \beta \neq \alpha \\
d( (\omega,\kappa)_\alpha , (n,\gamma)_\beta ) & = & 1 & \text{if } \beta \neq \alpha \\
d( (n,\gamma)_\alpha , (\omega,\kappa)_\alpha ) & = & \frac{1}{n} & \\
d( (n,\gamma)_\alpha , (m,\gamma')_\alpha ) & = & \frac{1}{m} + \frac{1}{n} &
\end{array}
$$

Therefore $T(\kappa)$ provides also a universal space for metric spaces of cardinality $\kappa$ . We had one already for $\aleph_0$ and $\kappa$ such that $\kappa^{\aleph_0} = \kappa$ but not for cardinalities $\kappa$ with $\kappa^{\aleph_0} > \kappa$ .

Remark . The fact that $T(\aleph_0)$ is a universal space for countable metric spaces while $m(\aleph_0)$ is not illustrates the superiority of nets over sequences in the description of topologies ; indeed it shows that it is not always possible to determine the topology of a countable metric space by giving countably many of its convergent sequences while it is always possible to do so by giving countably many of its convergent nets ( $(\omega \times \kappa)^* - \{(\omega,\kappa)\}$ acts as a directed set ) .

# CHAPTER II

## Countable Sequential Spaces

### 1. Sequential and Fréchet spaces.

The sequential and the Fréchet spaces are two classes of topological spaces for which the topology can be determined in some way by the convergent sequences. More precisely,

**Definition 1.** A topological space $X$ is said to be sequential if a subset $A$ of $X$ is open if and only if every sequence converging to a point of $A$ is eventually in $A$.

**Definition 2.** A topological space $X$ is said to be Fréchet if for any subset $A \subset X$, a point is in the closure of $A$ if and only if there exists a sequence in $A$ converging to this point.

Sequential and Fréchet spaces have been investigated by S.P. Franklin in [3] and [4]. He identified those spaces as being the quotients of metric spaces and the hereditary quotients of metric spaces respectively.

Clearly, every first-countable space is both sequential and Fréchet; one of the simplest examples of a non-first-countable Fréchet space is the quotient of the disjoint union of countably many convergent sequences obtained by identifying the limit points.

A Fréchet space is always sequential but the converse is not true , as the following example shows . Franklin shows in [4] that a sequential space is Fréchet if and only if it is hereditarily sequential.

Example . Let $f : Q \to Q$ be the following map :

$$f(x) = x \quad \text{if } x \neq n \text{ for all } n \in N$$
$$f(n) = \frac{1}{n}$$

Let $Q^o = f(Q)$ and let us consider on $Q^o$ the quotient topology induced by $f$ . With this topology , $Q^o$ is a sequential space which is not Fréchet (and also, $f$ is a quotient map which is not hereditarily quotient ). Indeed,let

$$A = Q \cap \left(\bigcup_{n=1}^{\infty} ]n,n+1[\right).$$

Then $0 \in \overline{A}$ ; however , no sequence of points of $A$ converges to $0$ . For suppose $(x_p)_{p \in N} \subset A$ and $x_p \to 0$ ; then, we see first that for each $n$ , only finitely many of the $x_p$ 's are in $Q \cap ]n,n+1[$ , otherwise we would have a subsequence converging to 0 and lying in $]n,n+1[$ which is impossible ; now , since only finitely many of the $x_p$'s are in $Q \cap ]n,n+1[$ , then

$$]-1,1[ \cup \bigcup_{n=1}^{\infty}(Q \cap ]n,n+1[ - \{x_p ; p \in N\})$$

provides a neighborhood of 0 containing no point of the sequence, contradicting the fact that the sequence converges to 0 .

Theorem 1 below shows that $Q$ is sequential .

Theorem 1 : Every quotient of a sequential space is sequential .

Proof : Let $X$ be a sequential space and $f : X \to Y$ be a quotient map of $X$ onto $Y$ . Let $A$ be a subset of $Y$ such that every sequence converging to a point in $A$ is eventually in $A$. Suppose $x_n \to x_0 \in f^{-1}(A)$. Then, by continuity $f(x_n) \to f(x_0) \in A$ and hence $(f(x_n))_{n \in N}$ is eventually in $A$; this shows that $(x_n)_{n \in N}$ is eventually in $f^{-1}(A)$, and since $X$ is sequential, then $f^{-1}(A)$ is open. Now, $f$ being a quotient map, it follows that $A$ is open. □

The following theorem is proved by Franklin in [3].

Theorem 2 : If $X$ and $Y$ are Hausdorff, $X$ is a Fréchet space and $f : X \to Y$ is a quotient map, then $Y$ is a Fréchet space if and only if $f$ is hereditarily quotient.

## 2. Countable sequential spaces.

For any space $X$, let $S(X)$ be, as in Chapter [illegible], the disjoint union of the convergent sequences of $X$. It is easily seen that the natural map from $S(X)$ onto $X$ (mapping points onto themselves) is a quotient map if and only if $X$ is sequential. Hence, if $\kappa^{\aleph_0} = \kappa$, the disjoint union of $\kappa$ copies of $S$ (we had $S = \{ \frac{1}{n} ; n \in N \} \cup \{ 0 \}$) provides a quotient-universal space for sequential spaces of cardinality less than or equal to $\kappa$, and this universal space has the additional property of being metric.

Then, in particular, the family of sequential spaces of cardinality less than or equal to $c$ (and the same holds for cardinality $2^c$) has a quotient-universal metric space. As in the case of metric spaces , this leads to two questions for the countable case :

- is $m(\aleph_0)$ a universal space for countable sequential spaces ?
- if not , is some other countable metric space universal ?

The answer to the first question is of course , no , since we saw in theorem 2 of Chapter I that even metric spaces were rarely quotients of $m(\aleph_0)$ . Now , is some other countable metric space universal for the countable sequential spaces ? We will show that there is no such space by constructing countable sequential spaces that are not quotients of any countable metric space ( though each of them will be a quotient of a separable metric space ) .

## The spaces $Q_A^*$

For any subset $A$ of $R$ , let $Q_A$ be the following subset of $R^2$ :

$$Q_A = Q \times ( Q - \{ 0 \}) \cup A \times \{ 0 \} .$$

Now let $Q^*$ be the set $Q \times ( Q - \{ 0 \}) \cup \{ e \}$ where $e$ is any point not in $Q \times ( Q - \{ 0 \})$ . Let $Q_A^*$ be the set $Q^*$ with the quotient topology determined by the map of $Q_A$ onto $Q^*$ which is the identity on $Q \times ( Q - \{ 0 \})$ and maps $A \times \{ 0 \}$ onto the point $e$ . In other words , $Q_A^*$ is the space obtained from $Q_A$ by identifying $A \times \{ 0 \}$ to a single point .

The spaces $Q_A^*$ as $A$ runs through the subsets of $R$ are countable sequential spaces . They are actually Fréchet-spaces . However not all of them are quotients of countable metric spaces.

Lemma 1 : Let $X$ and $Y$ be Fréchet spaces and let $f : X \to Y$ be a hereditarily quotient map from $X$ onto $Y$ . If $(y_n)_{n \in N}$ is a sequence in $Y$ such that $y_n \to y_o \in Y$ , then there exists a subsequence $(y_{\sigma(n)})_{n \in N}$ of $(y_n)_{n \in N}$ , points $(x_{\sigma(n)})_{n \in N}$ and $x_o$ in $X$ such that $x_{\sigma(n)} \to x_o$ , $x_o \in f^{-1}(y_o)$ and

$$f(x_{\sigma(n)}) = y_{\sigma(n)} .$$

Proof : Since $f$ is hereditarily quotient , then

$$f : f^{-1}(\{y_n ; n \in N\} \cup \{y_o\}) \to \{y_n ; n \in N\} \cup \{y_o\}$$

is a quotient map . Now suppose that the conclusion of the lemma does not hold . Then if $x_o \in f^{-1}(y_o)$ and $(x_n) \subset f^{-1}(\{(y_n)\} \cup \{y_o\} )$ is a sequence converging to $x_o$ , we must have $f(x_n) = y_o$ for all $n$ greater than a certain integer since we must have $f(x_n) \quad (x_o)$ and since we assumed the conclusion of the lemma to be false . Therefore , $f^{-1}(y_o)$ contains a tail of every sequence of $f^{-1}(\{(y_n)\} \cup \{y_o\})$ converging to an element of $f^{-1}(y_o)$ and hence $f^{-1}(y_o)$ is open in $f^{-1}( \{y_n ; n \in N\} \cup \{y_o\})$ . Since $f$ is a quotient map, this would imply that $\{y_o\}$ is open in $\{y_n ; n \in N\} \cup \{y_o\}$ which is not true . Therefore the lemma holds . □

Theorem 3 : If $Q_A^*$ is a quotient of a countable metric space , then $A$ is an $F_\sigma$ subset of $R$ ( that is , $A$ is a union of countably many closed subsets of $R$ ) .

Proof : Suppose that there is a quotient map $f$ of $M$ onto $Q_A^*$ where $M$ is a countable metric space . For each $p \in A$ , let $\xi_p = (x_{p,1}, x_{p,2}, \ldots)$ be a sequence in $Q \times (Q - \{0\})$ such that

$$|x_{p,n} - (p,0)| < \min \{\frac{1}{n} , |x_{p,n-1} - (p,0)| \} .$$

Let $q$ be the quotient map of $Q_A$ onto $Q_A^*$ . For each $p$ and $n$ let $z_{p,n} = q(x_{p,n})$ and let $\eta_p$ be the sequence $(z_{p,1}, z_{p,2}, \ldots)$ in $Q_A^*$ . Now $\eta_p \to e$ . Hence by the lemma , replacing $\eta_p$ by a subsequence if necessary , we may say that there is some $b_p \in f^{-1}(e)$ and a sequence $\tau_p = (s_{p,1}, s_{p,2}, \ldots)$ in $M$ such that

$$\tau_p \to b_p \quad \text{and}$$

$$f(\tau_p) = \eta_p \quad (\text{that is } f(s_{p,n}) = z_{p,n} \text{ for all } p,n ) .$$

Let $f^{-1}(e) = \{x_n : n \in N\}$ , and for $n \in N$ , let

$$A_n = \{ p \in A ; b_p = x_n \} .$$

We claim that the closure in $R$ of each $A_n$ is contained in $A$ . Suppose not ; then some $A_n$ contains a sequence $(p_i)_{i \in N}$ with no cluster point in $A$ . The sequence $\eta_{p_i}$ converges to $e$ for each $i$ and the sequence $\tau_{p_i}$ converges to $x_n$ for each $i$ . A diagonal sequence $(s_{p_1,n_1}, s_{p_2,n_2}, \ldots)$ with $n_k \geq k$ for each $k$ will then converge to $x_n$ and hence $(z_{p_1,n_1}, z_{p_2,n_2}, \ldots)$ converges to $e$ . Therefore $(x_{p_1,n_1}, x_{p_2,n_2}, \ldots)$ must have a cluster point in $Q_A$ and in fact in $A \setminus \{0\}$ . But

$$x_{p_k,n_k} - (p_k,0) \le x_{p_k,k} - (p_k,0) \le \frac{1}{k} \quad ;$$

So any cluster point of $(x_{p_1,n_1}, \ldots)$ in $A \setminus \{0\}$ would also be a cluster point of $\{(p_1,0) ; (p_2,0), \ldots\}$ which is impossible by choice of the $p_i$ 's . Therefore the closure in R of each $A_n$ is contained in A and hence $A = \bigcup_{n\in N} \bar{A}_n$ which shows that A is an $F_\sigma$ subset of R . □

Corollary : If P is the set of irrationals of the real line , then $Q_P^*$ is a countable sequential space which is not a quotient of any countable metric space .

Proof : It is a very well-known fact that P is not an $F_\sigma$ subset of R . See for example [7] .

The condition that A be an $F_\sigma$ subset of R is in fact not only necessary but also sufficient to imply that $Q_A^*$ is a quotient of a countable metric space .

Theorem 3' : If A is an $F_\sigma$ subset of R , then $Q_A^*$ is a quotient of some countable metric space .

Proof : Let $A \subset R$ be such that $A = \bigcup_{n\in N} F_n$ where each $F_n$ is a closed subset of R . Replacing the $F_n$ 's by their intersection with bounded intervals if necessary , we may assume that each $F_n$ is compact . Now , $Q_A^*$ is clearly the quotient of the disjoint union of the $Q_{F_n}^*$ obtained by identifying all copies of a given point of $Q \times (Q - \{0\})$ and identifying all copies of the point $e$. Since a disjoint union of countably many countable metric spaces is a countable metric space, it then suffices to show that $Q_F^*$ is a countable metric space whenever the set $F$ is compact. But since a compact subset of a second-countable metric space clearly has countable character*, then the point $e$ in $Q_F^*$ has a countable base of neighborhoods and therefore $Q_F^*$ is a second-countable regular space and hence a metric space which is also dense-in-itself. The theorem of Sierpinski mentioned in Chapter I implies that $Q_F^*$ is homeomorphic to $Q$. □

We conclude from theorem 3 and its corollary that no countable metric space can be quotient-universal for the countable sequential spaces. We show now that even if we remove the requirement that our universal space be metric, we still have a negative answer.

Let us define an equivalence relation on the subsets of $R$ by:

$$A \simeq B \quad \text{if and only if } Q_A^* \text{ is homeomorphic to } Q_B^* .$$

Theorem 4 : There are $2^c$ equivalence classes for the relation $\simeq$.

Proof : Since any two subsets of $R$ have different families of neighborhoods, all the topologies we determine on $Q^*$ by choice of subsets $A$ of $R$ are distinct. Hence, these are $2^c$ distinct topologies on $Q^*$. Now, if $A \subset R$, to any $B$ equivalent to $A$ there corresponds a homeomorphism between $Q^*_A$ and $Q^*_B$ which is a map from $Q^*$ onto $Q^*$; this correspondence between sets $B$ equivalent to $A$ and maps from $Q^*$ onto $Q^*$ is one-one. Since $Q^*$ is countable, there are $c$ maps from $Q^*$ onto $Q^*$ and therefore there can be at most $c$ sets $B$ equivalent to $A$. Since each equivalence class contains at most $c$ elements, there must be $2^c$ equivalence classes. $\square$

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* A subset $A \subset X$ is said to have countable character in $X$ if there is a countable family $(U_n)$ of open sets containing $A$ such that if $V$ is an open set containing $A$, then for some $n_o$, $A \subset U_{n_o} \subset V$.

Corollary : There are $2^c$ non-homeomorphic spaces $Q^*_A$ as $A$ runs through the subsets of $R$.

Since there can be only $c$ maps from a countable sequential space onto the set $Q^*$, there are at most $c$ spaces $Q^*_A$ that can be quotients of a given countable sequential space. This shows that no countable sequential space can be quotient-universal for the countable sequential spaces.

## 3. Quotients of the space of irrationals.

In [6], Michael and Stone establish tha every metric space which is a continuous image of the set $P$ of irrationals is also a quotient of $P$. The question is raised there whether this result can be extended to non-metrizable regular $T_1$ images of $P$

that are also quotient of some separable metric space : that is , if $X$ is regular and $T_1$ , $X$ is a continuous image of $P$ and $X$ is a quotient image of some separable metric space , then must $X$ also be a quotient of $P$ ? The spaces $Q^*_A$ will provide the negative answer .

First note that every countable space is a continuous image of $P$ since every countable space is a continuous image of the countable discrete space and the countable discrete space is a continuous image of $P$(for example,collapse each interval $]n,n+1[\cap P$ of $P$ to a point ) . Therefore each space $Q^*_A$ is a regular $T_1$ space which is a continuous image of $P$ and also a quotient of some separable metric space (namely $Q_A$ ) . However ,

Theorem 5 : At most $c$ of the $2^c$ spaces $\{Q^*_A ; A \subset R\}$ can be quotients of $P$ .

Proof : Let $S$ be the collection of spaces $Q^*_A$ that are quotients of $P$ . To any $Q^*_A$ in $S$ , we can associate the couple

$$( f^{-1}(e) , f_{/P - f^{-1}(e)} )$$

where $f$ is the quotient map of $P$ onto $Q^*_A$ . This correspondence is clearly one-one . So it suffices to show that this couple can take at most $c$ values ; $f^{-1}(e)$ is a closed subset of $P$ and hence $f^{-1}(e)$ can take only $c$ values ; now, for a given closed subset $A$ of $P$ , if $A = f^{-1}(e)$, then $f_{/P - A}$ is a continuous map between $P - A$ and $Q^*_A - \{e\}$. Since $P - A$ is separable and Hausdorff , such maps are determined by their values on a countable dense set and hence , there can be at most $c$ such maps : Therefore

$( f^{-1}(e) , f_{/P - f^{-1}(e)} )$ can take at most $c$ different values. □

In fact, we have a more precise answer concerning which spaces $Q_A^*$ are quotients of $P$.

Theorem 6 : $Q_A^*$ is a quotient of P if and only if A is an analytic subset of R.

Before we prove this theorem, we give the necessary preliminaries on analytic sets.

Analytic subsets of a given complete separable metric space $M$ are defined to be continuous images of the Borel sets and since every Borel subset of such a space is known to be a continuous image of the set $P$ of irrationals, analytic subsets of $M$ can be thought of as those subsets that are continuous images of $P$. Countable unions and intersections of analytic sets are analytic.

There is an equivalent definition for analytic sets and this is the one we will use to prove the necessity part of our theorem. First, we must define the notion of "A-operation".

Definition. Let $\{A_{k_1k_2\ldots k_n}\}$ be a system of sets defined for each finite sequence $k_1,k_2,\ldots,k_n$ of positive integers. The set

$$R = \bigcup_{k_1k_2\ldots k_n\ldots} \bigcap_{n=1}^{\infty} A_{k_1k_2\ldots k_n}$$

is called the result of the A-operation applied to the system $\{A_{k_1k_2\ldots k_n}\}$. Such a system is called regular if

$A_{k_1k_2\ldots k_{n+1}} \subset A_{k_1k_2\ldots k_n}$ ;

Theorem 7 : A subset A of a complete separable metric space M is analytic if and only if it is the result of the $\mathcal{A}$-operation performed on a regular system of closed subsets of M .

Proof : For a proof of this theorem and more details about analytic sets , see [ 5] or [ 8 ] .

Proof of theorem 6

Sufficiency is easy : if A is analytic , then $Q_A$ (as a union of two analytic subsets of $R^2$ ) is an analytic subset of $R^2$ and hence is a continuous image of P . By the theorem of Michael and Stone , $Q_A$ is then a quotient of P . Since $Q_A^*$ is a quotient of $Q_A$ , $Q_A^*$ is a quotient of P .

Necessity : Let $A \subset R$ and suppose $f : P \to Q_A^*$ is a quotient map . We show that $A \times \{0\}$ is the result of the $\mathcal{A}$-operation applied on some regular system of closed subsets of $R^2$ .

We fix on P a metric with respect to which P is complete and this is the only metric that we will use in the proof .

Let $(B_n)_{n \in N}$ be a family of open sets of diameter less than or equal to 1 which covers P (using the Lindelöf property for example ) . For each $B_n$ , let $(B_{nm})_{m \in N}$ be a family of open sets of P contained in $B_n$ with diameter less than or equal to $\frac{1}{2}$ and which covers $B_n$. Inductively , for each $B_{n_1 n_2 \dots n_k}$ , let $(B_{n_1 n_2 \dots n_k m})_{m \in N}$ be a family of open sets of P such that

$$-B_{n_1 n_2 \dots n_k m} \subset B_{n_1 n_2 \dots n_k}$$

$$-\text{diameter } (B_{n_1 n_2 \dots n_k}) \leq \frac{1}{k+1}$$

$$- \bigcup_{m \in N} B_{n_1 n_2 \ldots n_k m} = B_{n_1 n_2 \ldots n_k} \quad .$$

Let $C = f^{-1}(e)$ .

If $B_{n_1 n_2 \ldots n_k} \cap C \neq \phi$ ,

let $F_{n_1 n_2 \ldots n_k} = Cl_{R^2}(f(B_{n_1 n_2 \ldots n_k} - C)) \cap (R \times \{0\})$ .

If $B_{n_1 n_2 \ldots n_k} \cap C = \phi$ ,

define $F_{n_1 n_2 \ldots n_k} = \phi$ .

The family $\{F_{n_1 n_2 \ldots n_k} ; k \in N , n_i \in N\}$ is a regular family of closed sets of $R \times \{0\}$ . We claim that $A \times \{0\}$ is the result of the A-operation performed on the system $\{F_{n_1 n_2 \ldots n_k}\}$ of sets , that is

$$A \times \{0\} = \bigcup_{n_1 n_2 \ldots n_k \ldots} \bigcap_{k=1}^{\infty} F_{n_1 n_2 \ldots n_k} \quad .$$

1. $A \times \{0\} \subset \bigcup_{n_1 n_2 \ldots n_k \ldots} \bigcap_{k=1}^{\infty} F_{n_1 n_2 \ldots n_k}$ .

Let $x_0 \in A \times \{0\}$ . Let $(x_n)_{n \in N}$ be a sequence of $Q \times (Q - \{0\})$ converging to $x_0$ . Then , in $Q_A^*$ , $x_n \to e$ . Since $f : P \to Q_A^*$ is hereditarily quotient , there exist $(y_{\sigma(n)})_{n \in N} \subset P$ and $y_0 \in C = f^{-1}(e)$ such that

$$- y_{\sigma(n)} \to y_0 \quad \text{and}$$

$$- f(y_{\sigma(n)}) = x_{\sigma(n)}$$

(by the lemma 1 of this chapter ) . Since the family $(B_n)_{n \in N}$ covers P , let $n_1$ be such that $y_0 \in B_{n_1}$ . Inductively , if $n_1, n_2, \ldots n_k$ have been chosen so that $y_0 \in B_{n_1 n_2 \ldots n_k}$ , then find $n_{k+1}$ such that $y_0 \in B_{n_1 n_2 \ldots n_{k+1}}$ by using the fact that the family $(B_{n_1 \ldots n_k m})_{m \in N}$ covers $B_{n_1 n_2 \ldots n_k}$ . Now each $B_{n_1 n_2 \ldots n_k}$ is open and contains $y_0 \in C$; hence it contains a tail of the sequence $(y_{\sigma(n)})$ and since $x_{\sigma(n)} \to x_0$

in $R^2$ , then

$$x_o \in Cl_{R^2}(f(B_{n_1n_2\ldots n_k} - C)) \cap (R \times \{0\}) .$$

Hence for this particular choice of $n_1, n_2, \ldots, n_k, \ldots$ ,

$$x_o \in \bigcap_{k=1}^{\infty} F_{n_1n_2\ldots n_k} .$$

2. $\bigcup_{n_1n_2\ldots n_k\ldots} \bigcap_{k=1}^{\infty} F_{n_1n_2\ldots n_k} \subset A \times \{0\}$ .

Let $x_o \in \bigcap_{k=1}^{\infty} F_{n_1n_2\ldots n_k}$ and suppose $x_o \notin A \times \{0\}$ . We seek a contradiction . We have

$$x_o \in Cl_{R^2}(f(B_{n_1n_2\ldots n_k} - C)) \cap (R \times \{0\}) .$$

Therefore in each $f(B_{n_1n_2\ldots n_k} - C)$ , there is a sequence converging to $x_o$ in the sense of $R^2$ . Say $(x_n^k)_{n \in N} \subset f(B_{n_1n_2\ldots n_k} - C)$ and $x_n^k \to x_o$ . Since $f(B_{n_1n_2\ldots n_k} - C) \subset f(B_{n_1n_2\ldots n_{k-1}} - C)$ , by taking some kind of a diagonal sequence , we can get a sequence $(x_n)_{n \in N}$ such that

- $x_n \to x_o$ (in $R^2$) and
- $x_n \in f(B_{n_1n_2\ldots n_k} - C)$ for all $n \geq k$ .

(For example , take

$$x_1 \in (x_n^1)_{n \in N} \text{ with } d(x_1, x_o) < 1$$

$$x_2 \in (x_n^2)_{n \in N} \text{ with } d(x_2, x_o) < \frac{1}{2}$$

$$\ldots$$

$$x_p \in (x_n^p)_{n \in N} \text{ with } d(x_p, x_o) < \frac{1}{p} \quad \ldots )$$

Now since $x_k \in f(B_{n_1n_2\ldots n_k} - C)$, let $y_k \in B_{n_1n_2\ldots n_k} - C$ be such that $f(y_k) = x_k$ . Then $(y_n)_{n \in N}$ is a sequence in $P$ such that $y_n \in B_{n_1n_2\ldots n_k}$ for all $n \geq k$ (since $B_{n_1n_2\ldots n_{k+1}} \subset B_{n_1n_2\ldots n_k}$ ) . Since the diameter of $B_{n_1n_2\ldots n_k}$ is less than or equal to $\frac{1}{k}$ , then $(y_n)_{n \in N}$ is a Cauchy sequence . P being complete with the metric considered , let $y_0 \in P$ be such that $y_n \to y_0$ . Since f is continuous , $f(y_n) \to f(y_0)$ . But $f(y_n) = x_n$ and $(x_n)_{n \in N}$ does not converge in $Q_A^*$ since $x_n \to x_0$ in the sense of $R^2$ and $x_0 \notin A \times \{0\}$ . Therefore we get a contradiction .

We finally conclude that $A \times \{0\}$ is an analytic subset of $R^2$ and hence A is an analytic subset of R . ☐

# CHAPTER III

## Sequential Spaces of Higher Cardinalities

We have established in the previous chapter that there is no countable metric space which is quotient-universal for the countable sequential spaces while we know that such a space exists for the corresponding problem in cardinality c . We now look at the same question for cardinalities less than c if the continuum hypothesis does not hold . And finally , to complete the study we will look at cardinalities bigger than c for which the disjoint union of convergent sequences of a space has cardinality bigger than that of the space and hence does not qualify as a universal space .

### 1. Sequential spaces of cardinality $\kappa$ with $\aleph_0 < \kappa < c$ .

The spaces $Q_A^*$ are as constructed in the previous chapter.

Theorem 1 : $Q_A^*$ is a quotient of a metric space of cardinality $\kappa < c$ if and only if A is a union of $\kappa$ closed sets of R .

Proof : The proof can be copied exactly from the proofs of theorems 4 4' of Chapter II except that $f^{-1}(e)$ will now be written as $\{x_\alpha; \alpha < \kappa\}$ instead of $\{x_n; n \in N\}$ and the sets $A_n$ will be replaced by the corresponding sets $A_\alpha'$, $\alpha < \kappa$, with

$$A_\alpha = \{ p \in A ; b_p = x_\alpha \} .$$

Proposition : There exists a subset $A_O$ of $R$ which is not a union of less than $c$ closed subsets of $R$ .

Proof : There are $c$ uncountable closed sets in $R$ ; so let $\{C_\alpha \; ; \; \alpha < c\}$ be those sets . Let $p_1$ and $q_1$ be any two distinct elements of $C_1$ . If $p_\beta$ and $q_\beta$ have been chosen for $\beta < \alpha$ such that $p_\beta \varepsilon C_\beta$ , $q_\beta \varepsilon C_\beta$ and all of them are distinct , then since $\{p_\beta , q_\beta \; ; \; \beta < \alpha\}$ has cardinality less than $c$ and since any uncountable closed set of $R$ has cardinality $c$ , we can pick $p_\alpha \varepsilon C_\alpha$, $q_\alpha \varepsilon C_\alpha$ such that $p_\alpha \neq q_\alpha$ and

$$p_\alpha \not\varepsilon \{p_\beta , q_\beta \; ; \; \beta < \alpha\}$$
$$q_\alpha \not\varepsilon \{p_\beta , q_\beta \; ; \; \beta < \alpha\} \quad .$$

Let $A_O = \{p_\alpha \; ; \; \alpha < c\}$ . If $A_O$ were a union of $\kappa$ closed sets of $R$ ( $\kappa < c$) , then one of them would have to be one of the $C_\alpha$'s . But it is impossible since for any $\alpha < \kappa$ , $q_\alpha \varepsilon C_\alpha$ and $q_\alpha \not\varepsilon A_O$ . □

Corollary : $Q^*_{A_O}$ is not a quotient of any metric space of cardinality less than $c$ .

These results lead to the following theorem .

Theorem 2 : $c$ can be characterized among cardinals as the smallest one for which there is a metric space quotient-universal for sequential spaces of that cardinality .

## 2. Sequential spaces of cardinality $\kappa > c$ .

We already know that if $\kappa^{\aleph_0} = \kappa$ , the disjoint union of $\kappa^{\aleph_0} = \kappa$

copies of $S = \{\frac{1}{n} ; n \in N\} \cup \{0\}$ acts as a quotient-universal space for sequential spaces of that cardinality. So we study the case $\kappa^{\aleph_0} > \kappa$.

For $\kappa > c$ with $\kappa^{\aleph_0} > \kappa$, let $D(\kappa)$ be the discrete space of cardinality $\kappa$ and let $B(\kappa)$ be the product of countably many copies of the discrete space, that is

$$B(\kappa) = \prod_{n \in N} ( D(\kappa) )_n .$$

Being a countable product of metric spaces, $B(\kappa)$ is metric. It has cardinality $\kappa^{\aleph_0}$ and it has a dense subset of cardinality $\kappa$. Indeed, let $x_0$ be a fixed element of $D(\kappa)$ and let $K(\kappa)$ be the set of all points of $B(\kappa)$ for which all but finitely many coordinates are equal to $x_0$. This set is dense in $B(\kappa)$ since any element $(y_1, y_2, \ldots, y_n, \ldots)$ is the limit of the sequence

$$\begin{gathered}(y_1, x_0, x_0, \ldots, x_0, \ldots)\\ (y_1, y_2, x_0, \ldots, x_0, \ldots) \qquad \ldots\\ (y_1, y_2, \ldots, y_n, x_0, \ldots x_0, \ldots)\\ \ldots\end{gathered}$$

of elements of $K(\kappa)$. Furthermore, $K(\kappa)$ has clearly cardinality $\kappa$.

$B(\kappa)$ and $K(\kappa)$ will play the same rôle as $R$ and $Q$ respectively played in the case of cardinalities smaller than $c$. Namely let $K(\kappa)^* = K(\kappa) \times (K(\kappa) - (x_0, x_0, x_0, \ldots, x_0, \ldots)) \cup \{e\}$ where $e$ is a point not in $K(\kappa)$. For a subset $A$ of $B(\kappa)$, let $K(\kappa)_A$ be the set

$$K(\kappa)_A = K(\kappa) \times (K(\kappa) - (x_0, x_0, \ldots, x_0 \ldots)) \cup (A \times (x_0, x_0, \ldots, x_0, \ldots))$$

and let $K(\kappa)^*_A$ be the set $K(\kappa)^*$ with the quotient topology determined by the natural map of $K(\kappa)_A$ onto $K(\kappa)^*$ (mapping points of $K(\kappa) \times (K(\kappa) - (x_0,x_0,\dots,x_0,\dots))$ onto themselves and $A \times (x_0,x_0,\dots)$ onto e).

As A runs through all subsets of $B(\kappa)$, we get a family of $2^{\kappa^{\aleph_0}}$ sequential topologies on $K(\kappa)^*$ and actually those are Fréchet topologies.

We prove, as in the previous case, that not all of those spaces can be quotients of a metric space of cardinality $\kappa$.

Lemma : If K is a compact subset of a metric space X, then K has countable character in X.

Proof : Let us denote by $B(K,\frac{1}{n})$ the set K enlarged by $\frac{1}{n}$, that is

$$B(K,\tfrac{1}{n}) = \{x \in X \,;\, d(x,K) < \tfrac{1}{n}\} \quad .$$

We claim that the sets $B(K,\frac{1}{n})$ form a base of neighborhoods of K, that is, that every open set containing K must contain one of these. Indeed, let U be an open set of X such that $K \subset U$. Since K is compact and and $X - U$ is closed, then $d(K,X-U) \geq \frac{1}{n_0}$ for some $n_0 \in N$. It is then clear that $B(K,\frac{1}{n_0}) \subset U$. □

Theorem 3 : $K(\kappa)^*_A$ is a quotient of a metric space of cardinality less than $\kappa^{\aleph_0}$ if and only if A is a union of less than $\kappa^{\aleph_0}$ compact subsets of $B(\kappa)$.

Proof : Sufficiency : Let $A = \bigcup_{\alpha<\tau} F_\alpha$ with $\tau < \kappa^{\aleph_0}$, and $F_\alpha$ compact. Clearly, $K(\kappa)^*_A$ is a quotient of the disjoint union of all $K(\kappa)^*_{F_\alpha}$'s, $\alpha < \tau$, (mapping all copies of points of $K(\kappa) \times (K(\kappa) - (x_0,x_0,\dots))$

onto themselves and all points of the sets $F_\alpha \times (x_o,x_o,\ldots)$ onto $e$. Hence it suffices to show that if $F$ is compact, $K(\kappa)^*_F$ is a quotient of a metric space of cardinality $\kappa$.

We show in fact that $K(\kappa)^*_F$ is itself metric, using the general metrization theorem. By the lemma above, $F$ has a countable base of neighborhoods in $K(\kappa) \times \left(K(\kappa) - (x_o,x_o,\ldots)\right) \cup F \times (x_o,x_o,\ldots)$; let $(B_n)_{n\in N}$ be those neighborhoods and we may assume that they are decreasing. Now, $K(\kappa)_F$ is metric and hence has a $\sigma$-locally finite base, say $\{\mathcal{U}_n\}_{n\in N}$ where $\mathcal{U}_n$ is a locally finite family. Now, if for any $U \in \mathcal{U}_n$ we define $U^* = U - \{F \times \{0\}\}$ and if we define $\mathcal{U}^*_n = \{U^* ; U \in \mathcal{U}_n\} \cup \{B_n\}$, then $\bigcup_{n\in N} \mathcal{U}^*_n$ is a base for $K(\kappa)^*_F$. It follows for a classical compactness argument that $\mathcal{U}^*_n$ is also locally finite and hence $K(\kappa)^*_F$ is metrizable.

**Necessity :** The proof of the necessity is very much like the proof of the corresponding theorem in the case of cardinality $\aleph_0$ once we realize that in theorem 4 of Chapter II , besides proving that the $A_n$ 's had their closure in $A$ , we could have proved at the same time that $\overline{A}_n$ is also compact . This did not make a difference then since in R , $\sigma$-compact sets and $F_\sigma$ sets are the same . But it does make a difference now and we do want the stronger conclusion about compactness of the corresponding sets . So , as before , for each $p \in A$ , we let $\sigma_p = (x_{p,1},x_{p,2},\ldots)$ be a sequence in $K(\kappa)\times(K(\kappa)-(x_o,x_o,\ldots))$ such that

$$d[x_{p,n},(p,(x_o,x_o,\ldots))] \leq \min\left\{\frac{1}{n}, d[x_{p,n-1},(p,(x_o,x_o,\ldots))]\right\} .$$

Let $q$ be the quotient map of $K(\kappa)_A$ onto $K(\kappa)_A^{\star}$. For each $p,n$, let $z_{p,n} = q(x_{p,n})$ and let $\eta_p$ be the sequence $(z_{p,1}, z_{p,2}, \ldots)$ in $K(\kappa)^{\star}$. Now $\eta_p \to e$. Hence, since $f$ is hereditarily quotient, replacing $\eta_p$ by a subsequence if necessary, we may say that there is some $b_p \in f^{-1}(e)$ and a sequence $\tau_p = (s_{p,1}, s_{p,2}, \ldots)$ in $f^{-1}(e)$ such that

$$- \tau_p \to b_p \qquad \text{and}$$
$$- f(\tau_p) = \eta_p$$

(by lemma 1 of Chapter II). Let $f^{-1}(e) = \{x_\alpha ; \alpha < \tau < \kappa^{\aleph_0}\}$ and for $\alpha < \tau$, let $A_\alpha = \{p \in A ; b_p = x_\alpha\}$.

The sets $A_\alpha$ are the analogue of the sets $A_n$ in the countable case. Now, in order to show that $\overline{A}_n$ was contained in A in the proof of theorem 4 (Chapter II), we supposed there was a sequence in $A_n$ with no cluster point in A and produced a cluster point in A for that sequence, hence a contradiction; it is important to note that we did not need to assume anything about that sequence to start with. Hence, if we take any sequence contained in $A_\alpha$, using the same argument as in theorem 4, we can show that the sequence has a cluster point in A. This shows both that $\overline{A}$ is compact and $\overline{A}_\alpha \subset A$.

Therefore A is a union of $\tau$ $(<\kappa^{\aleph_0})$ compact subsets of $B(\kappa)$. □

Proposition : $B(\kappa)$ is not a union of less than $\kappa^{\aleph_0}$ compact subsets.

Proof : Suppose $B(\kappa) = \bigcup_{\alpha<\tau} F_\alpha$ $(\tau < \kappa^{\aleph_0})$ where $F_\alpha$ is compact for any $\alpha$. Then at least one $F_\alpha$, say $F_{\alpha_0}$, must have a cardinality bigger than $\kappa$. Now since $F_{\alpha_0}$ is compact, then $\pi_n(F_{\alpha_0})$ is finite for any

$n \in N$ (where $\pi_n$ denotes the projection of the product on the n-th coordinate ); but $F_{\alpha_0} \subset \pi_1(F_{\alpha_0}) \times \pi_2(F_{\alpha_0}) \times \ldots \times \pi_n(F_{\alpha_0}) \quad \ldots$ and the set on the right has cardinality $2^{\aleph_0} = c < \kappa$ , which is a contradiction . This shows more generally that no subset of $B(\kappa)$ with cardinality $\kappa^{\aleph_0}$ can be a union of less than $\kappa^{\aleph_0}$ compact sets . □

Corollary : There is no quotient-universal metric space for sequential spaces of cardinality $\kappa$ ( $\kappa^{\aleph_0} > \kappa$ ) .

Proof : The proposition above together with theorem 3 shows that $K(\kappa)^{*}_{B(\kappa)}$ is not the quotient of any metric space of cardinality less than $\kappa^{\aleph_0}$. □

Finally , we can say that the cardinals $\kappa$ such that $\kappa^{\aleph_0} = \kappa$ are characterized among cardinals as precisely those for which there is a quotient-universal metric space for sequential spaces of that cardinality .

# CHAPTER IV

## Quasi- and Weakly-Quasi-First-Countable Spaces

### 1. Quasi-first-countable spaces .

In studying the notion of first-countability , one often gives as an example of a non-first-countable space, the quotient of $m(\aleph_0)$ obtained by identifying the limit points to a point $x_0$ , that is the disjoint union of $\aleph_0$ copies of $\{\frac{1}{n} ; n \in N\} \cup \{0\}$ in which we identify all copies of 0 to a point $x_0$ . However, the way in which this space fails to be first-countable is not too drastic because the space still carries some notion of countability : indeed, at $x_0$ , there are countably many branches on each of which we can fix countably many sets (namely the tails of the sequences ) such that to construct a neighborhood of $x_0$ , it suffices to pick one such set on each branch . Another similar space is the quotient of Q obtained by identifying Z to a single point . Those spaces are much "closer" to first-countability than spaces like $[0,1]^c$ (or for that matter any uncountable product of first-countable spaces) and $Q_p^*$ for example,where $Q_p^*$ is as defined in Chapter II. We will call such spaces "quasi-first-countable spaces" .

Definition : We say that a space X is quasi-first-countable at $x_0 \in X$ if there exist countably many countable families of decreasing subsets of X containing $x_0$ such that a subset V of X is a neighborhood of $x_0$ in $X$ if and only if $V$ contains a member of each family .

We say that $X$ is quasi-first-countable if it is so at each of its points .

Quasi-first-countability at $x_0$ is an intermediate property between having $\chi(x_0,X) = \aleph_0$ and $\chi(x_0,X) = c$ , where $\chi(x_0,X)$ is the smallest cardinality for a neighborhood base at $x_0$ . Not every space with $\chi(x_0,X) = c$ is quasi-first-countable ; for example $Q_P^*$ is a countable not quasi-first-countable space (see theorems below) .

Remark : The notion of quasi-first-countability retains enough of the idea of first-countability to insure "sequentialness" and in fact "Fréchetness" , that is :

Proposition : Every quasi-first-countable space is Fréchet .

Proof : Let $x_0 \in \overline{A}$ and let $(B^n_m)_{m \in N}$ be the families provided by quasi-first-countability at $x_0$ . There exists $n_0 \in N$ such that each $B^{n_0}_m$ meets $A$ ; for otherwise , for any $n \in N$ , we could find $\sigma(n) \in N$ such that

$$B^n_{\sigma(n)} \cap A = \phi \quad .$$

Now by quasi-first-countability $\bigcup_{n\in N} B^n_{\sigma(n)}$ is a neighborhood of $x_0$ and $( \bigcup_{n\in N} B^n_{\sigma(n)} ) \cap A = \phi$ contradicting the fact that $x_0 \in \overline{A}$ , Hence there exists $n_0$ such that $B^{n_0}_m \cap A \neq \phi$ for all $m \in N$ ; let $x_m \in B^n_m \cap A$. Then $(x_m)_{m \in N}$ converges to $x_0$ since any neighborhood of $x_0$ contains one of the $B^n_m$ 's . Therefore the space is Fréchet . ☐

In previous chapters , we saw that some countable sequential spaces were quotients of countable metric spaces while some others were not . The notion of quasi-first-countability and as we will see later, of weakly-quasi-first-countability , came up as we were trying to identify which countable sequential spaces were quotients of a countable metric space by characterizing internally the topology of such spaces . Theorem 1 and more completely theorem 4 give the answer to that problem .

Theorem 1 : A countable space X is quasi-first-countable if and only if it is a hereditarily quotient image of a countable metric space .

Proof : Suppose X is countable and quasi-first-countable . For each x , let $(B_m^{x,n})_{m \in N}$ be the families provided by quasi-first-countability at x . Let $Y^{x,n}$ be the set X provided with the discrete topology except for the point x, which has as a base of neighborhoods the family $(B_m^{x,n})_{m \in N}$ . Then $Y^{x,n}$ is first-countable , countable and regular , and hence metric .

Let Y be the disjoint union of all $Y^{x,n}$ 's for $x \in X$ and $n \in N$ . Then Y is a countable metric space .

Let f be the natural map of Y onto X ( mapping a point onto itself ) . Then f is continuous : for, let 0 be open in X ; let $f^{-1}( 0 ) \cap Y^{x,n} \neq \phi$ . If $x \in 0$ , then since 0 is open, 0 contains one of the $B_m^{x,n}$ 's for some $m \in N$; hence $f^{-1}( 0 )$ contains a neighborhood of x in $Y^{x,n}$ and clearly it contains a neighborhood of each of its other points in $Y^{x,n}$ . Hence $f^{-1}( O ) \cap Y^{x,n}$ is open . Therefore $f^{-1}( O )$ is open and hence f is continuous.

The map f is also a quotient map : for suppose $f^{-1}( O )$ is open in Y , that is $f^{-1}( O ) \cap Y^{x,n}$ is open for all $x \varepsilon X$ and $n \varepsilon N$ . If $x \varepsilon O$ , then since $f^{-1}( O ) \cap Y^{x,n}$ is open , $f^{-1}( O ) \cap Y^{x,n}$ contains a $B^{x,n}_m$ for a certain m ; hence O contains $B^{x,n}_m$ ; this is true for each n and hence , by quasi-first-countability, O is a neighborhood of x . Therefore , O is a neighborhood of each of its points , that is O is open . It follows that f is a quotient map and since X is Hausdorff and Fréchet ( by quasi-first-countability ) , f is hereditarily quotient (see theorem 2 of Chapter II ) .

Now for the converse , let us assume that X is a hereditarily quotient image of a countable metric space M ; let $f : M \to X$ be a quotient map . Let $f^{-1}( x ) = \{x_1,x_2,\ldots,x_n\ldots\}$ . We want to show that X is quasi-first-countable at x ; let $(C^n_m)^{\infty}_{m=1}$ be a countable base of neighborhoods of M at $x_n$ ; let

$$B^n_m = f( C^n_m ) .$$

Then the $(B^n_m)_{m \varepsilon N}$ are the required families . For , if O is a neighborhood of x , then for any n , there exists $\sigma(n) \varepsilon N$ such that

$$C^n_{\sigma(n)} \subset f^{-1}( O )$$

and then $O = f \circ f^{-1}( O )$ contains $B^n_{\sigma(n)}$ for any $n \varepsilon N$ . Conversely , if O contains some $B^n_{\sigma(n)}$ for all $n \varepsilon N$ , then $f^{-1}(O)$

contains $C^n_{\sigma(n)}$ for all n and hence $f^{-1}( O )$ is a neighborhood of $f^{-1}( x )$ . This implies that O is a neighborhood of x since f is hereditarily quotient . $\square$

Remark : The result cannot be improved , in the sense that "hereditarily quotient" cannot be replaced by "quotient" . Indeed , the example given at the beginning of Chapter II of a sequential space not Fréchet is countable and given as a quotient of Q . However , being non-Fréchet , it is not quasi-first-countable .

## 2. Quasi-first-countable spaces as quotients of metric spaces .

In this section , we look at the following question : Can the quasi-first-countable spaces be characterized as the quotients of metric spaces by some particular kind of maps ?

Definition : A map $f : X \to Y$ is said to have countable frontier if for any $y \in Y$ , $fr(f^{-1}(y))$ is countable . (We recall that the frontier of a subset A of a space X is defined as the closure of A in X minus its interior.)

Theorem 2 : The quasi-first-countable spaces are precisely the images of metric spaces by maps that are hereditarily quotient and have countable frontier .

Proof : Suppose X is quasi-first-countable and let $(B^{x,n}_m)_{m \in N}$

be the families provided by this property . Let $Y^{x,n}$ be the set $X$ with the topology in which points are open except for $x$ which has as a base of neighborhoods the family $(B_m^{x,n})_{m \in N}$ . $Y^{x,n}$ is a metric space (for example the following metric is easily seen to be compatible with the topology of $Y^{x,n}$ :

- $d(y,x) = \frac{1}{m}$ where $m$ is the smallest integer such that $y \notin B_m^{x,n}$
- $d(y_1,y_2) = d(y_1,x) + d(x,y_2)$ for $y_1 \neq x$ and $y_2 \neq x$ ) .

We define the space $Y$ to be the disjoint union of all $Y^{x,n}$ 's and we consider the natural map $f : Y \to X$ . Arguments similar to those used in the proof of theorem 1 show that $f$ is a quotient map and hence hereditarily quotient since $X$ is Fréchet . Furthermore , $fr(f^{-1}( x ) )$ is countable since $fr(f^{-1}( x ) )$ is formed by the point $x$ of each $Y^{x,n}$ as $n$ runs through $N$ .

Conversely , let $f : M \to X$ be a hereditarily quotient map that has countable frontier , with $M$ a metric space . Let $fr(f^{-1}( x ) ) = \{x_1,x_2,\ldots,x_n,\ldots\}$ . Let $(C_m^{x,n})_{m \in N}$ be a neighborhood base at $x_n$ and let

$$B_m^{x,n} = f( C_m^{x,n} ) .$$

Again an argument as in theorem 1 shows that the families $(B_m^{x,n})_{m\in N}$ are as required . ☐

A quotient map between a metric space and a quasi-first-countable space does not have to have countable frontier . In fact, in a metric space , as far as neighborhoods are concerned , compact sets and points behave the same way ( we established previously that compact subsets have countable character in metric spaces ) , and hence , one sees easily that in the second part of the proof above, "countable frontier" could be replaced by "$\sigma$-compact frontier" (that is $fr(f^{-1}( x ) )$ is $\sigma$-compact for each $x \in X$ ) . Hence , we also have :

Theorem 2' : The quasi-first-countable spaces are precisely the images of metric spaces by maps that are hereditarily quotient and have $\sigma$-compact frontier .

Still , a quotient map between a metric space and a quasi-first-countable space need not have $\sigma$-compact frontier : for example , take $X$ to be a metric space , $M(X)$ to be the disjoint union of uncountably many copies of $X$ and $f$ to be the natural map of $M(X)$ onto $X$ ( mapping points onto themselves) . However, we do have a result in that direction for quotients that do not involve too many identifications . Namely , let us called a quotient map $f : Y \to X$ a "nice quotient" if for any $x \in X$ there is a neighborhood $V$ of $x$ in $X$ such that $x$ is the only point of $V$ with an inverse image of possibly more than one point, that is the elements x of X for which $f^{-1}( x )$ contains more than one point form a relatively discrete subspace of $X$ . Then we have the following :

Theorem 3 : Let $f : M \to X$ be a "nice quotient map" of a metric space onto a quasi-first-countable space . Then $f$ is hereditarily quotient and has $\sigma$-compact frontier .

Proof : The map $f$ is hereditarily quotient (by theorem 2 of Chapter II ) since $X$ is Hausdorff and Fréchet . Let $x \in X$ be such that $f^{-1}( x )$ contains more than one element . We want to show that $f^{-1}( x )$ has $\sigma$-compact frontier .

Let $(B^n_m)_{m \in N}$ be the families provided by quasi-first-countability at $x$ ; we may assume that $B^n_m \subset V$ for each $n$ , $m \in N$ . For each $y \in fr(f^{-1}( x ))$ , we can find a sequence $(y_n)_{n \in N}$ in $M - f^{-1}( x )$ such that :

- $y_n \to y$ and
- for some $n_0 \in N$ , $B^{n_0}_m$ contains a tail of $(f(y_n))_{n \in N}$ for all $m \in N$

Indeed , let $(z_n)_{n \in N}$ be any sequence of $M - f^{-1}$ converging to $y$ ; then $(f(z_n))_{n \in N}$ converges to $x$ ; now there exists $n_0 \in N$ such that , for all $m \in N$ ,

$$B^{n_0}_m \cap \{f(z_n);n \in N\} \neq \phi .$$

For otherwise , for each $n \in N$ we could find $\sigma(n) \in N$ such that

$$B^n_{\sigma(n)} \cap \ (z_n);n \in N\} = \phi$$

and then $\bigcup_{n \in N} B^n_{\sigma(n)}$ would be a neighborhood of $x$ not meeting $(f(z_n))_{n \in N}$ contradicting the fact that $(f(z_n))_{n \in N}$ converges to $x$ . Hence there does exist $n_0$ such that

$$B^{n_0}_m \cap \{f(z_n);n \in N\} \neq \phi$$

for all $m \in N$ ; using this and the fact that the $B^{n_0}_m$ 's are decreasing one can construct by induction a subsequence $(z_{\sigma(n)})_{n \in N}$ such that

$$f(z_{\sigma(n)}) \varepsilon B^{n}_{m} .$$

Let $y_n = z_{\sigma(n)}$ . Then $(y_n)_{n \varepsilon N}$ has the required property, that is each $B^{n_0}_{m}$ contains a tail of $(f(y_n))_{n \varepsilon N}$ .

Now , we associate each $y \varepsilon fr(f^{-1}(x))$ **(via this** sequence $(y_n)_{n \varepsilon N}$ ) to the $n_0$ obtained as above . Let $A_{n_0}$ be the set of y 's which are associated to $n_0$ . We clai**m** that $\overline{A_{n_0}}$ is compact , which will show that $fr(f^{-1}(x))$ is $\sigma$-compact .

Since M is metric , it suffices to show that each sequence in $A_{n_0}$ has an accumulation point in $fr(f^{-1}(x))$ . Let $(y^p)_{p \varepsilon N}$ be a sequence in $A_{n_0}$ . For each $y^p$ , let $(y^p_n)_{n \varepsilon N}$ be the sequence considered above , so that

- $(y^p_n) \to y^p$ and
- $B^{n_0}_{m}$ contains a tail of $(f(y^p_n))_{n \varepsilon N}$ for any m .

Since $B^{n_0}_{m}$ contains a tail of $(f(y^m_n))_{n \varepsilon N}$ , we can find $z_m \varepsilon (y^m_n)_{n \varepsilon N}$ such that

- $d(y^m, z_m) < \frac{1}{m}$ and
- $f(z_m) \varepsilon B^{n_0}_{m}$ .

We then get a sequence $(z_m)_{m \varepsilon N}$ such that $(f(z_m))_{m \varepsilon N}$ converges to x since any neighborhood of x contains $B^{n_0}_{m}$ for some m and each $B^{n_0}_{p}$ contains all $f(z_m)$ for $m \geq p$ . Since $(f(z_m))_{m \varepsilon N}$ converges to x and f is hereditarily quotient , replacing $(f(z_m))_{m \varepsilon N}$ by a subsequence if necessary , we can say that there is a sequence $(y_n)_{n \varepsilon N}$ in M and a point $y \varepsilon f^{-1}(x)$ such that

- $y_n \to y$ and
- $f(y_n) = f(z_n)$ .

Now , since the $f(z_m)$ 's are in the sets $B^{n_0}_m$ and hence in $V$ , then we must have $y_n = z_n$ for all $n \in N$ ; but $y_n \to y$; hence $z_n \to y$ . Furthermore , we have $d(z_p, y_p) < \frac{1}{p}$ ; hence we can conclude that $y^p \to y$ . Therefore the sequence $(y^p)_{p \in N}$ has an accumulation point in $fr(f^{-1}( x ))$ and this proves that $\overline{A_{n_0}}$ is compact . □

We deduce from theorem 3 a result obtained before :

Corollary : $Q^*_A$ is a quotient of a countable metric space if and only if $A$ is an $F_\sigma$ subset of $R$ .

Proof : If $A$ is an $F_\sigma$ subset of $R$ , then $A$ is $\sigma$-compact ; hence the quotient map of $Q_A$ onto $Q^*_A$ has $\sigma$-compact frontier and since it is hereditarily quotient , then by theorem 2' of this chapter , $Q^*_A$ is quasi-first-countable . By theorem 1 , $Q^*_A$ is a quotient of a countable metric space .

Conversely , if $Q^*_A$ is a quotient of a countable metric space , then it is also a hereditary quotient (since $Q^*_A$ is Fréchet) and by theorem 1 , $Q^*_A$ is quasi-first-countable . Now by theorem 3 , the quotient map of $Q_A$ onto $Q^*_A$ must have $\sigma$-compact frontier and since $fr(f^{-1}( e )) = A \times \{0\}$ , then $A$ must be $\sigma$-compact . □

## 3. Weakly-quasi-first-countable spaces .

We identified in theorem 1 the countable sequential spaces that are hereditarily quotient images of countable metric spaces .

We want to do the same now for countable sequential spaces that are simply quotients of countable metric spaces .

Definition : A space X is said to be weakly-quasi-first-countable if and only if for all $x \in X$ , there exist countably many countable families of decreasing subsets containing x such that a set O is open if and only if for any $x \in O$ , O contains a member of each family associated to x .

A straightforward argument shows that every weakly-quasi-first-countable space is sequential since if a set contains a tail of any sequence converging to one of its points , it must also contains a member of each family associated to its points ; but weakly-quasi-first-countable spaces need not be Fréchet as is shown by the space given in the beginning of Chapter II as an example of a non-Fréchet sequential space .

Weak quasi-first-countability is the property we were looking for in order to characterize internally by their topologies the quotients of countable metric spaces .

Theorem 4 : A countable space X is a quotient of a countable metric space if and only if it is weakly-quasi-first-countable .

Proof : The proof can be copied on the proof of theorem 1 doing the few necessary changes .

The weakly-quasi-first-countable spaces can be characterized as was done for quasi-first-countable spaces and the proof of theorem 2 can easily be adapted to prove the following theorem :

Theorem 5 : The weakly-quasi-first-countable spaces are precisely the images of metric spaces by quotient maps with countable frontier (or $\sigma$-compact frontier as in theorem 2' ) .

We have defined the notion of weak quasi-first-countability in order to characterize internally the quotients of countable metric spaces and in doing so , we were forced to define it as a global property instead of as a local property as for quasi-first-countability (that is we could not find a way to define it at a point in such a way that if it is satisfied at each point , then it is equivalent to the definition we gave ) . This reflects the fact that "sequentialness" is defined as a global property while "Fréchetness" can be considered a local property ( we could say that X is "Fréchet at x " $x \in \overline{A} \rightarrow$ there exists a sequence in A converging to x , and X is Fréchet if it is so at each of its points ) . The difference between the definitions of weak quasi-first-countability and quasi-first-countability is the exact translation of the difference between sequential spaces and quotient maps on one hand , Fréchet spaces and hereditarily quotient maps on the other hand . It is the difference between topologies given in terms of open sets and topologies given in terms of neighborhoods . Indeed , quotient maps are defined by

" $f^{-1}( U )$ is open if and only if $U$ is open "

while hereditarily quotient maps can be equivalently defined by

"$f^{-1}( U )$ is a neighborhood of $f^{-1}( x )$ if and only if $U$ is a neighborhood of $x$ " .

Similarly , sequential spaces are defined by

"sequentially open sets are open " ,

while Fréchet spaces , as the following proposition shows , could be equivalently defined by

"sequential neighborhoods of $x$ are neighborhoods of $x$ " ,

where one defines a sequential neighborhood of $x$ in a natural way as a set containing a tail of each of the sequences converging to $x$ .

Proposition : $X$ is Fréchet if and only if sequential neighborhoods of $x$ are neighborhoods of $x$ , for any $x \in X$ .

Proof : Suppose $X$ is Fréchet ; let $V$ be a sequential neighborhood of $x$ . We claim that $V$ is a neighborhood of $x$ . Let $(B_\alpha)_{\alpha \in A}$ be a neighborhood base at $x$ . If $V$ is not a neighborhood of $x$ , then for any $\alpha \in A$ , there exists $x_\alpha$ such that

- $x_\alpha \in B_\alpha$ but
- $x_\alpha \notin V$ .

Consider the set $\{x_\alpha ; \alpha \in A\}$ . Clearly ,

$$x \in \overline{\{x_\alpha ; \alpha \in A\}}$$

but no sequence of that set converges to $x$ since such a sequence would have to have a tail in $V$ . This contradicts the fact that $F$ is Fréchet . Hence $V$ is a neighborhood of $x$ .

Conversely , suppose $X$ is not Fréchet . Then there exists some subset $A \subset X$ and a point $x \in X$ such that $x \in \overline{A}$ but no sequence of $A$ converges to $x$ . Then $(X - A) \cup \{x\}$ is a sequential neighborhood of $x$ ; however $[(X - A) \cup \{x\}] \cap A = \phi$ and hence , since $x \in \overline{A}$ , then $(X - A) \cup \{x\}$ is not a neighborhood of $x$ .

Finally , we obtain as a corollary to theorems 2 and 5 :

Corollary : $X$ is quasi-first-countable if and only if $X$ is weakly-quasi-first-countable and Fréchet .

Proof : Necessity is clear . For sufficiency , let $X$ be Fréchet and weakly-quasi-first-countable . By theorem 5 , $X$ is a quotient of some metric space by a map with countable frontier . Now , since $X$ is Fréchet , this map must be hereditarily quotient , and then by theorem 2 , $X$ must be quasi-first-countable .

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