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Aspects of Field Theoretic Limits of String Theory

by

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The effort to understand the universe is one of the very few things that lifts human life a little above the level of farce, and gives it some of the grace of tragedy.

Steven Weinberg

What song the sirens sang, or what name Achilles assumed when he hid himself among women, although puzzling questions are not beyond all conjecture.

Thomas Browne

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Aspects of Field Theoretic Limits of String Theory** submitted by **Kirk Kaminsky** in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Abstract

Herein we examine aspects of field theoretic limits of string theory. First, we explore at the field-theoretical level the role of the dynamical dilaton in the vortices associated with the spontaneous breaking of a pseudo-anomalous $U(1)$ symmetry present in certain heterotic string compactifications. Treating the anomaly as a controlled perturbation, we find that the four-dimensional dilaton unavoidably diverges at the vortex core, which signals the presence of a strong-coupling regime, the failure of the classical effective theory, and given the role of the dilaton in string theory, implies that such objects are intrinsically quantum mechanical.

We next examine the quantum dynamics of spontaneous symmetry breaking in the context of noncommutative field theories, which naturally arise on the world-volumes of D-branes in string theory, in the presence of a constant Kalb-Ramond two-form background. We first study the noncommutative linear sigma model with a global $O(N)$ symmetry in the broken phase, where we find conflicts between renormalization and Goldstone symmetry realization leading to the violation of Goldstone's theorem at the quantum level for $N>2$. To

investigate possible group dependence, we compare noncommutative linear sigma models with a U(N) global symmetry, where we find that due to purely noncommutative commutator interactions consistent only with U(N) symmetry, no violation of Goldstone's theorem occurs at one-loop for potentials consistent with a possible noncommutative gauging of the model. To examine possible representation dependence, we repeat the calculations for the adjoint representation U(2) and O(4) global models, and find that the former is consistent with Goldstone's theorem at one-loop if we only include trace invariants consistent with noncommutative gauge invariance, while the latter exhibits violations of the kind seen for the fundamental representation O(N) model for N>2. These results are corroborated with string theory arguments, four-point amplitudes, and gauge theory calculations.

Preface

The results in this thesis were obtained over the course of the author's PhD program at the University of Alberta between 1996 and 2001. The presentation of the work herein is in accordance with the "Paper Format" regulations of the Faculty of Graduate Studies and Research of the University of Alberta, and is based on the following published papers:

- B.A. Campbell & K. Kaminsky, *Anomalous U(1) Vortices and the Dilaton*, Phys. Rev. **D62**, 126001 (2000), hep-th/0001104.
- B.A. Campbell & K. Kaminsky, *Noncommutative Field Theory and Spontaneous Symmetry Breaking*, Nucl. Phys. **B581**, 240 (2000), hep-th/0003137.
- B.A. Campbell & K. Kaminsky, *Noncommutative Linear Sigma Models*, Nucl. Phys. **B606**, 613 (2001), hep-th/0102022.

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What a five years it has been. I never envisioned myself here writing this.

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Chapter 1

Introduction

1.1 Why string theory?

Other than the rhetorical "Why not", string theory is the only, and therefore best, candidate humanity possesses for a consistent unification of the fundamental interactions including gravity, and quantum mechanics. The more one delves into this beautiful and vast framework, the more one acquires the impression that we are discovering structure and not merely inventing it as mathematical fancy: the self-consistencies, the so-called "string miracles", the lack of any free dimensionless parameters, and even the relatively recent revelation that string theory is not even 'merely' a theory of strings, provocatively suggest the presence of a unique and powerful structure.

However, insofar as physics is an empirical science, we must eventually make contact with the structures that underpin our less tentative understanding of the universe (such as the standard model of particle physics), try to explain the puzzles these frameworks have left unanswered, and hopefully make new predictions that are experimentally testable; a formidable task indeed. The central criticism of string theory has traditionally been that it does not make contact with experiment, or *uniquely* single out our low energy 'approximations' to the universe. Or, as Feynman articulated in indirect reference to the massive space of possible vacua and hence the pragmatic incalculability of the theory, "I don't like that they're not calculating anything"[1]. However, the principal problem with string theory lies not in that it is a theory which naturally lives near the Planck scale (and so probably forever beyond *direct* experimental access), but in that we do not understand what conceptual paradigms underlie it, or even what the correct fundamental degrees of freedom are. As Polchinski eloquently summarized, "String theory is smarter than we are. It knows what spacetime is, and we don't, and we have to figure out how to ask it"[2]. Furthermore, while in principle everything about the low-energy world is fixed by the theory, we do not understand how the dynamics of the theory selects out the vacuum.

These issues stem primarily from the fact that string theory was discovered historically through a first-quantized, perturbative formulation, and until we have a nonperturbative definition of the theory, we really cannot hope to accomplish the task discussed above, or answer Feynman's objection. While the discovery of such things as string dualities, D-branes, and the AdS-CFT correspondence has significantly advanced our understanding of nonperturbative string dynamics, we cannot as yet answer the question "What is string theory?". To summarize, the main problems in string theory are theoretical, and until we can solve them, we cannot address phenomenological or experimental issues.

Nonetheless, since string theories are *a priori* capable of making contact with low-energy physics, and predicting new kinds of phenomena through their field-theoretical limits, we can view some the issues discussed above as, in some sense, ones of uniqueness: because we do not know how string theory dynamically selects the vacuum, we cannot make unique or specific predictions, but we can still build models from string theory, make *generalized* predictions, and suggest new phenomena that are falsifiable through experiment. In this thesis we will examine certain aspects of such field-theoretical limits of string theories.

1.2 Why this thesis?

To illustrate the argument just considered, as well the path from string theory to field theoretic models, consider the problems we will study in this thesis. First, we will examine the role of the dilaton in anomalous $U(1)$ vortices. Superstring theories naturally reside in ten spacetime dimensions, so we must compactify six spacelike dimensions in order to make contact with a rather obvious phenomenological fact. Certain classes of compactifications of string theory possess a $U(1)$ symmetry with apparently anomalous matter content. Since this anomaly depends only on the massless content of the theory, and the massive states in string theory are on order of the Planck mass, we should be able to analyze the structure from a field-theoretic limit of the corresponding string theory involving only massless modes of the string. Furthermore, since the underlying string theory is perturbatively consistent, there should be a stringy mechanism to cancel such an anomaly. Indeed, the famous Green-Schwarz anomaly cancellation mechanism involving the two-form Kalb-Ramond field [3], has a four-dimensional remnant that cancels the anomaly. However, the mechanism generates a supersymmetry breaking Fayet-Illipolous D-term at one-loop in the string expansion, and potentially destabilizes the vacuum [4]. In turn, it is often possible to assign a scalar charged under this $U(1)$ symmetry a vacuum expectation value (VEV) to cancel the D-term, thereby restoring supersymmetry and spontaneously breaking the $U(1)$ symmetry in a process called vacuum restabilization. Now on the other hand, generically breaking a $U(1)$ symmetry can lead to topologically stable defects known as vortices, or in a cosmological setting, cosmic strings. In this scenario, the usual Higgs and gauge fields of the system are also coupled to the dilaton, and model-independent axion originating from the universal superfield multiplet in string theory, and in the second chapter we will investigate the consequences of such additional couplings, focusing in particular upon the dynamics of the dilaton.

The second component of this thesis deals with a very different field-theoretical limit of string theory, but which also directly involves the two-form Kalb-Ramond field. One of the deepest aspects of string theory is that spacetime itself is not fundamental, but rather emerges from the two-dimensional conformal field theory of the worldsheet that propagating strings map out; specifically the coordinates of spacetime become bosonic fields living on the worldsheet. Thus one expects that the properties of spacetime are predicted by string theory and in fact, in certain limits, spacetime itself is predicted to be noncommutative. In particular, in the presence of D-branes (nonperturbative, dynamical, solitonic objects on which open strings can end) and a constant two-form Kalb-Ramond background, the conformal field theory becomes topological in a certain scaling limit, and the associated correlators between the fields corresponding to the embedding coordinates now imply the associated spacetime is noncommutative. This limit is a field-theoretical one, which suggests the study of noncommutative field theories with the structure inherited from string theory. Surprising results are found for even the simplest of scalar field theories where, despite the induced infinite nonlocality, the resultant theory is formally renormalizable (with modified combinatorics), the potentially dangerous nonlocal divergences regulated by the noncommutativity of spacetime. However, this regulator couples the external momenta of a given amplitude to the dimensionful noncommutative parameter in such a way that after the ultraviolet cutoff is removed, new infrared divergences in the external momenta are introduced, effectively modifying the infrared part of the spectrum. This result, plus the modified combinatorics, strongly suggest we study spontaneous symmetry breaking in this context where massless Goldstone modes arise in the commutative case, and whose consistent renormalization depends on delicate graphwise cancellations. These studies are the subject of chapters 3 and 4.

In the remainder of this chapter, we will develop the minimal formalism required to demonstrate how the field-theoretic models we study arise from string theory. Because of the massive scope of the material involved (by a crude estimate, there are more than twenty thousand papers on the subject of string theory), we will be extremely cursory in the treatment, focusing on only what we need later. Our approach primarily follows that of the textbooks by Polchinski [5]. Other useful general references include [6]-[10].

1.3 String and conformal theory basics

In this section we will discuss the gauge fixed worldsheet action for the string, its first quantization and basic conformal field theory techniques, emphasizing only the aspects we need for later.

1.3.1 The Polyakov action and its symmetries

Consider first the classical action for a relativistic point particle, defined by its spacetime coordinate positions X^μ, which map out a world-line parametrized by τ as it propagates through spacetime. The natural action

$$S = -m\int ds = -m\int d\tau\sqrt{-\dot{X}^\mu \dot{X}_\mu}. \tag{1.1}$$

proportional to the invariant length of the worldline, is manifestly τ reparametrization and Poincaré invariant but is difficult to quantize because of the presence of the square root. Consider the addition of an auxiliary world-line metric $\eta(\tau)$, and the action

$$S = -\frac{m}{2}\int d\tau\Big(\eta^{-1}\dot{X}^\mu \dot{X}_\mu - \eta m^2\Big). \tag{1.2}$$

By varying with respect to η, and then eliminating it from the action, we recover (1.1), which implies the two forms are classically equivalent. The advantage of the latter is that it is easy to quantize.

Now consider the analogous construction for the propagation of a string in D spacetime dimensions, which now maps out a worldsheet parametrized by $(\sigma,\tau) = (\sigma^1,\sigma^2)$. Requiring again that the action depends only on the spacetime embedding, and not on a particular parametrization, the natural (Nambu-Goto) action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma\sqrt{\det\Big(\partial_a X^\mu\ \partial_b X_\mu\Big)}, \tag{1.3}$$

where a, b take the values σ or τ, is proportional to the area of the worldsheet. Analogous to the point particle case, by introducing an auxiliary (Euclidean) metric on the worldsheet, g_{ab}, we can introduce the classically equivalent Polyakov action

$$S_P[X,g] = \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma g^{1/2} g^{ab}\partial_a X^\mu \partial_b X_\mu\,, \tag{1.4}$$

which again is amenable to quantization. The symmetries of the action are:

a) Worldsheet reparametrization (or diffeomorphism) invariance

$$\begin{aligned} X'^{\mu}(\tau',\sigma') &= X^{\mu}(\tau,\sigma) \\ \frac{\partial\sigma'^{c}}{\partial\sigma^{a}}\frac{\partial\sigma'^{d}}{\partial\sigma^{b}}g'_{cd}(\tau',\sigma') &= g_{ab}(\tau,\sigma) \end{aligned} \tag{1.5}$$

for coordinate transformations $\sigma'^{a}(\tau,\sigma)$.

b) local Weyl invariance

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= X^{\mu}(\tau,\sigma) \\ g'_{ab}(\tau,\sigma) &= \exp(2\omega(\tau,\sigma))g_{ab}(\tau,\sigma) \end{aligned} \tag{1.6}$$

for arbitrary $\omega(\tau,\sigma)$.

c) D-dimensional Poincaré invariance

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= \Lambda^{\mu}{}_{\nu}X^{\nu}(\tau,\sigma) + a^{\mu} \\ g'_{ab}(\tau,\sigma) &= g_{ab}(\tau,\sigma). \end{aligned} \tag{1.7}$$

The parameter α' in (1.4) with dimensions $(Mass)^{-2}$, is called the Regge slope, and is proportional to the inverse tension of the string. Its significance will be seen later. The Weyl symmetry, unique to the string (as opposed to the membrane, ...) because of the elementary identity $\det(aA) = a^{n}\det(A)$ applied to $g^{1/2}g^{ab}$, is central to string physics. In particular, we have three independent worldsheet metric components, and three local symmetries (two diffeomorphism, and one Weyl) with which to put the metric into any canonical form we wish (say δ_{ab}, the unit gauge), at least locally. Explicitly, using the Weyl transform of the worldsheet Ricci scalar which is easily shown to be

$$g'^{1/2}R'^{(2)} = g^{1/2}(R^{(2)} - 2\nabla^{2}\omega), \tag{1.8}$$

we can set $R'^{(2)}$ to zero by locally solving the potential problem $2\nabla^2\omega = R^{(2)}$. But in two dimensions, the vanishing of the Ricci scalar implies the vanishing of the Riemann tensor because the symmetries of the latter imply $R^{(2)}_{abcd} = 1/2(g_{ac}g_{bd} - g_{ad}g_{bc})R^{(2)}$. Thus the metric is (locally) flat, and so diffeomorphically equivalent to the unit metric.

There exist diff $\times$ Weyl transformations that leave the metric in unit gauge, and so are not fixed by our gauge choice: the conformal transformations. Introducing a complex coordinate on the worldsheet, $z = \sigma^1 + i\sigma^2$, so that $ds^2 = dzd\bar{z}$, consider *holomorphic* transformations of z, $z' = f(z)$, and a general Weyl transform, under which the metric transforms as $ds'^2 = \exp(2\omega)|\partial_z f|^{-2}dz'd\bar{z}'$. Thus by choosing $\omega = \log|\partial_z f|$, we remain in the unit gauge. While most of this freedom is eaten when one considers the worldsheet globally, locally this conformal invariance gives rise to conserved currents and Ward identities which are crucial to the consistency of the theory, and the subject of the next subsection.

Incidentally (1.8) also shows that the two-dimensional Einstein-Hilbert action

$$\chi = \frac{1}{4\pi}\int_M d^2\sigma g^{1/2}R^{(2)}, \tag{1.9}$$

possesses the symmetries of the Polyakov action, since its variation is a total derivative [using in addition $g^{1/2}\nabla_a v^a = \partial_a(g^{1/2}v^a)$], and so we must include it in the theory. This term is a topological invariant, since the metric field equation $R^{(2)}_{ab} = (1/2)g_{ab}R^{(2)}$ automatically holds in two dimensions, implying that (1.9) is invariant under continuous metric deformations. In fact, it is the Euler characteristic of the worldsheet. Thus, when we quantize, it has the effect of relatively weighting the path integral by $e^{-\lambda\chi}$ with respect to worldsheet topology. This leads directly to the first quantized expansion of string theory as a sum over worldsheets of different topologies.

1.3.2 Operator product expansions

Assume now that we gauge fix the action (1.4) to the form

$$S = \frac{1}{2\pi\alpha'}\int d^2z \partial X^\mu \bar{\partial} X_\mu, \tag{1.10}$$

where we have expressed it in terms of complex coordinates $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$, and canonical derivatives $\partial_z \equiv \partial = (\partial_1 - i\partial_2)/2$, $\partial_{\bar{z}} \equiv \bar{\partial} = (\partial_1 + i\partial_2)/2$ with $\partial_z z = 1$, etc. Vectors v^a are similarly defined, $v^z = v^1 + iv^2$, $v^{\bar{z}} = v^1 - iv^2$ and are lowered with the metric with components $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$, corresponding to the unit metric in the σ^a coordinates. Finally, an oft used result is the divergence or Green's theorem in complex coordinates

$$\int_R d^2z(\partial_z v^z + \partial_{\bar{z}} v^{\bar{z}}) = i\oint_{\partial R}(v^z d\bar{z} - v^{\bar{z}} dz). \tag{1.11}$$

The field equation derived from (1.10) is simply

$$\partial\bar{\partial}X^\mu(z, \bar{z}) = 0, \tag{1.12}$$

which implies that $\partial X^\mu(z)$ and $\bar{\partial}X^\mu(\bar{z})$ are respectively holomorphic and anti-holomorphic, as the notation suggests. Quantum mechanical expectation values for operators $\mathcal{F}[X]$ are defined as usual by the path integral

$$\langle \mathcal{F}[X] \rangle = \int [dX] \exp(-S) \mathcal{F}[X], \tag{1.13}$$

and are not normalized with respect to $\langle 1 \rangle$. From the spacetime perspective, this makes it manifest that we are doing first quantized string theory, quantizing a single string, and not string *field* theory. From the worldsheet perspective, the X^μ are quantum fields living on the worldsheet, and the path integral defines a two-dimensional quantum (and conformal) field theory.

In the quantum theory, the field equation is encoded as an operator equation, or equivalently as the expectation value $\left\langle \partial\bar{\partial}X \ldots \right\rangle = 0$, where the dots denote insertions away from z. More generally

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} [\exp(-S) X^\nu(z',\bar{z}')] \\ &= \eta^{\mu\nu} \left\langle \delta^2(z-z',\bar{z}-\bar{z}') \right\rangle + \frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} \left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \right\rangle, \end{aligned} \tag{1.14}$$

so that $\partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') = -\pi\alpha'\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')$ holds as an operator equation. Solving this Green's function motivates the definition of *conformal* normal ordering as

$$\begin{aligned} :X^\mu(z,\bar{z}): &= X^\mu(z,\bar{z}) \\ :X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2): &= X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2} \eta^{\mu\nu} \log|z_{12}|^2, \end{aligned} \tag{1.15}$$

with $z_{ij} = z_i - z_j$, so that the property

$$\partial_1 \bar{\partial}_1 : X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) := 0 \tag{1.16}$$

holds as a consequence of $\partial\bar{\partial}\log|z|^2 = 2\pi\delta^2(z,\bar{z})$, which in turn follows from (1.11). This definition is most elegantly extended to arbitrary numbers of fields through the definition

$$\begin{aligned} :\mathcal{F}[X]: &= \exp\left(\frac{\alpha'}{4} \int d^2z_1 d^2z_2 \log|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)} \frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)} \right) \mathcal{F}[X] \\ &\equiv \exp(\hat{A}) \mathcal{F}[X], \end{aligned} \tag{1.17}$$

which has the effect of summing over all ways of choosing one or more pairs of fields inside the product and replacing each with the subtraction $\frac{\alpha'}{2}\eta^{\mu_i\mu_j}\log|z_{ij}|^2$, with the factorial from the exponential cancelling the number of ways the functional derivatives can act. This is invertible in the obvious way, where instead we replace each pair with the contraction (propagator). From this definition, we arrive at the extremely useful expression for the product of normal ordered operators in terms of the normal ordering of the product:

$$\begin{aligned} :\mathcal{F}::\mathcal{G}: &= \exp\left(-\frac{\alpha'}{2} \int d^2z_1 d^2z_2 \log|z_{12}|^2 \frac{\delta}{\delta X^\mu_F(z_1,\bar{z}_1)} \frac{\delta}{\delta X_{G\mu}(z_2,\bar{z}_2)} \right) :\mathcal{F}\mathcal{G}: \\ &\equiv \exp(-\hat{B}) \exp(\hat{A}) \mathcal{F}\mathcal{G} \end{aligned} \tag{1.18}$$

where the functional derivatives pick up one field from each of $\mathcal{F}$ and $\mathcal{G}$. This amounts to the Wick expansion, as its expansion generates all cross-contractions, and is most easily established inductively on the order of the exponential expansion using the basic relation $\hat{A}(\mathcal{F}\mathcal{G}) = (\hat{A}\mathcal{F})\mathcal{G} + \mathcal{F}(\hat{A}\mathcal{G}) + \hat{B}(\mathcal{F}\mathcal{G})$.

The harmonicity of the two-point definition reflected by (1.16) extends pointwise to N-products of Xs. The point of these definitions is that they render the corresponding operator products finite, so we can then Taylor expand inside the normal ordering as necessary, thereby generating the so-called *operator product expansion* (OPE). This states that the product of local operators can be expanded in a basis of of local operators, with c-number coefficient functions that possibly diverge as the relative position of the two operators vanishes. Schematically

$$\langle \mathcal{A}_i(\sigma_1)\mathcal{A}_j(\sigma_2)\ldots\rangle = \sum_k c_{ij}^k(\sigma_1 - \sigma_2)\,\langle \mathcal{A}_k(\sigma_2)\ldots\rangle\,, \tag{1.19}$$

with the other operators outside of $|\sigma_1 - \sigma_2| = r$ from σ_2. These expansions are used as asymptotic expansions, the singularity structure encoding the basic information of the quantum theory, and as we will see a little later, when the operators are conserved currents, the algebra of the associated charges as well.

Before proceeding, we first apply (1.18) to determine the OPE of the product of two exponential operators in anticipation of later use. Thus with $\mathcal{F} = \exp(ik_1 \cdot X(z,\bar{z}))$, and $\mathcal{G} = \exp(ik_2 \cdot X(0,0))$, applying (1.18) directly yields

$$\begin{aligned} &: e^{ik_1\cdot X(z,\bar{z})} :: e^{ik_2\cdot X(0,0)} := \exp(\frac{\alpha'}{2}k_1\cdot k_2 \log|z|^2) : e^{ik_1\cdot X(z,\bar{z})}e^{ik_2\cdot X(0,0)} : \\ = \;& |z|^{\alpha' k_1\cdot k_2} : e^{i(k_1+k_2)\cdot X(0,0)}[1 + ik_1\cdot(\partial X(0,0)z + i\bar{\partial}X(0,0)\bar{z}) + \ldots] : \end{aligned} \tag{1.20}$$

after Taylor expanding inside the normal ordering to express the result as an OPE.

1.3.3 Ward identities and conformal invariance

Continuous symmetries in field theory yield conserved (Noether) currents and Ward identities, the latter of which constrain operator products. Under an infinitesimal continuous symmetry transformation of a field ϕ, denoted by $\delta\phi(\sigma)$, the quantity $[d\phi]\exp(-S)$ is by assumption invariant. Generalizing the transformation to $\rho(\sigma)\delta\phi(\sigma)$ modifies this quantity, but since it still represents just a change of variables, the path integral itself is invariant. Expanding $[d\phi']\exp(-S')$ to first order, with an operator insertion $\mathcal{A}(\sigma_0)$ inside the variation region on which ρ has compact support, yields the operator relation

$$\delta\mathcal{A}(\sigma_0) + \frac{\epsilon}{2\pi i}\int_R d^d\sigma g^{1/2}\nabla_a j^a(\sigma)\mathcal{A}(\sigma_0) = 0, \tag{1.21}$$

where j^a is the Noether current coupled to the gradient of the ρ. Going to two flat dimensions, and invoking the divergence theorem (1.11) yields via the residue theorem

$$\mathrm{Res}_{z\to z_0} j(z)\mathcal{A}(z_0, \bar{z}_0) + \bar{\mathrm{Res}}_{\bar{z}\to\bar{z}_0}\tilde{j}(\bar{z})\mathcal{A}(z_0, \bar{z}_0) = \frac{1}{i\epsilon}\delta\mathcal{A}(z_0, \bar{z}_0), \tag{1.22}$$

where $j = j_z(z)$ and $j = \tilde{j}(\bar{z})$ are respectively holomorphic and antiholomorphic.

We consider two examples of central importance. Consider the X^μ theory, and the spacetime translation $\delta X^\mu(\sigma) = \epsilon a^\mu$. Under $\delta X^\mu = \epsilon\rho(\sigma)a^\mu$ we have $\delta S \propto \epsilon a_\mu \int d^2\sigma \partial^a X^\mu \partial_a \rho$, whence the Noether current $j^\mu_a = \frac{i}{\alpha'}\partial_a X^\mu$, the integral (over σ) of which is just the spacetime momentum of the string. Specifically, using our master formula (1.18), we have the OPE

$$j^\mu(z) \; : e^{ik\cdot X(0,0)} : \sim \frac{k^\mu}{2z} : e^{ik\cdot X(0,0)} :, \tag{1.23}$$

where $\sim$ means equality up to nonsingular terms, consistent with (1.22).

Next conside the worldsheet translation $\delta\sigma^a = \epsilon v^a$, under which invariance implies $\delta X^\mu = -\epsilon v^a \partial_a X^\mu$. After some manipulation via integration by parts, the corresponding Noether current, read from $\delta S \propto \int d^2\sigma \partial^a \rho j_a(\sigma)$, is

$$j_a = iv^b T_{ab} \quad , \quad T_{ab} = -\frac{1}{\alpha'} : \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab}\partial_c X^\mu \partial^c X_\mu : \tag{1.24}$$

with T_{ab} defining the worldsheet energy-momentum tensor.

The OPEs with T_{ab} determine much of the structure of the theory, as we shall now see. First note that T_{ab} is traceless: $T^a_a = 0$. In a quantum field theory this implies the theory is scale-invariant, though such invariance can potentially be broken by anomalies. In complex coordinates this is expressed as $T_{z\bar{z}} = 0$. The usual conservation law $\partial^a T_{ab} = 0$ then implies that $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$, so that $T_{zz} \equiv T(z)$ and $T_{\bar{z}\bar{z}} \equiv \tilde{T}(\bar{z})$ are respectively holomorphic and antiholomorphic. For our theory, these components are

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \quad , \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : . \tag{1.25}$$

The OPE of T (or $\tilde{T}$) with X from our master formula (1.18) is

$$T(z)X^\mu(0,0) \sim \frac{1}{z}\partial X^\mu(0) \quad , \quad \tilde{T}(\bar{z})X^\mu(0,0) \sim \frac{1}{\bar{z}}\bar{\partial} X^\mu(0). \tag{1.26}$$

Combining this with the fact that tracelessness implies the conservation of $j(z) = iv(z)T(z)$, $\tilde{j}(\bar{z}) = iv(z)^*\tilde{T}(\bar{z})$ for *any* holomorphic $v(z)$, implies via the Ward identity (1.22) the transformation

$$\delta X^\mu = -\epsilon v(z)\partial X^\mu - \epsilon v(z)^* \bar{\partial} X^\mu. \tag{1.27}$$

which is the infinitesimal form of the conformal transformation:

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) \quad , \quad z' = f(z). \tag{1.28}$$

Thus the theory is conformally invariant up to possible anomalies.

The OPE of T with a general operator $\mathcal{A}(z,\bar{z})$, which by holomorphicity takes the form of a Laurent expansion

$$T(z)\mathcal{A}(0,0) \sim \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \mathcal{A}^n(0,0), \tag{1.29}$$

is determined by infinitesimal conformal transformations $[z' = z + \epsilon v(z)]$, since a residue in $v(z)T(z)\mathcal{A}(0,0)$ appears when the term z^n in the expansion of $v(z)$ multiplies the z^{-n-1} term in the $T\mathcal{A}$ OPE; whence the Ward identity (1.22) yields

$$\delta\mathcal{A}(z,\bar{z}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \left[\partial^n v(z) \mathcal{A}^{(n)}(z,\bar{z}) + c.c.\right]. \tag{1.30}$$

For a basis of local operators which are rigid scale transformation eigenstates $\mathcal{A}'(z',\bar{z}') = \xi^{-h}\bar{\xi}^{-\tilde{h}}\mathcal{A}(z,\bar{z})$, and for the worldsheet translations, a straightforward calculation determines part of this OPE:

$$T(z)\mathcal{A}(0,0) = \ldots + \frac{h}{z^2}\mathcal{A}(0,0) + \frac{1}{z}\partial\mathcal{A}(0,0) + \ldots. \tag{1.31}$$

The numbers $(h,\tilde{h})$ are called the conformal weights of $\mathcal{A}$; the sum determines the scaling dimension, the difference the behaviour under two-dimensional rotations.

The subset of operators which transform under *arbitrary* conformal transformations as $\mathcal{O}'(z',\bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\tilde{h}}\mathcal{O}(z,\bar{z})$, are called *primary* operators, and have OPEs with T of the form (1.31) with no further singular terms. Primary operators with weights $(h,\tilde{h}) = (1,1)$ will be important later.

Applying these considerations to T itself yields the TT OPE, which via (1.18) and Taylor expansion about $z = 0$ is

$$T(z)T(0) \sim \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0). \tag{1.32}$$

Thus T is not primary, because of the *central charge* term proportional to D (denoted c in an arbitrary CFT). This is central to string theory, and when combined with the reparametrization ghost CFT (which arises by virtue of our gauge fixing), determines the dimension of spacetime.

1.3.4 The OPE-algebra correspondence

The (classical) conformal invariance of (1.10) allows us to switch between conformally related worldsheet coordinate systems. For the canonical interpretation of the theory we have the $w = \sigma^1 + i\sigma^2$ frame, which we have been calling z. For *closed* strings we take the spatial coordinate σ^1 periodic, identifying $\sigma^1 \sim \sigma^1 + 2\pi$, while for the *open* string we take $0 \leq \sigma^1 \leq \pi$. For both, the (Euclidean) time coordinate runs over the reals. For closed strings we obtain an infinite cylinder, while for open strings we obtain the infinite strip. The second system, in which most calculations are done, is defined by

$$z = \exp(-iw) = \exp(-i\sigma^1 + \sigma^2). \tag{1.33}$$

In this system time runs radially, the origin corresponding to the infinite past, the 'point at infinity' the infinite future. This transformation maps the closed string cylinder onto the complex plane, and the open string strip onto the upper half-plane.

Consider the closed string and a holomorphic operator $\mathcal{O}(z)$, with conformal weight h with the Laurent expansion

$$\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}}. \tag{1.34}$$

The coefficients are determined as usual by contour integrals (about the origin counterclockwise)

$$\mathcal{O}_m = \oint_C \frac{dz}{2\pi i z} z^{m+h} \mathcal{O}(z). \tag{1.35}$$

In the w-frame the Laurent coefficients become Fourier coefficients, the extra factor of z^{-h} in the definition cancelling the conformal transformations of the corresponding operator.

For the specific case of the holomorphic component of the energy-momentum tensor $T(z)$, where $h = 2$, the coefficients are labelled as L_m and are called the the *Virasoro* generators for reasons that will become clear.

Now consider the quantum theory, and imagine cutting open the path integral on circles of constant time in the radial frame. The assumed holomorphicity of the integrands in (1.35) allows us to arbitrarily deform the contours C about the origin, and in particular the integrals are invariant under radial (time) rescalings (translations), whence the generators are conserved charges. Pursuing this further leads to the equivalence between the OPEs of currents and the (anti)commutator algebra of the associated charges. For two charges $Q_{1,2}\{C\} = \oint_C dz/(2\pi i) j_{1,2}(z)$ associated with holomorphic currents, consider an equal-time contour C_2 about $z = 0$ associated with a time t_2, and two radial deformation contours: C_1 displaced forward in time $t_1 > t_2$, and C_3 displaced backwards in time $t_3 < t_2$. Since the operator ordering defined when we cut open a path integral is time (and hence radial) ordering, by deforming the *difference* of the contours C_1 and C_3 about a *fixed* point z_2 on C_2 into a contour encircling z_2 (denoted by C_{z_2}), we have successively

$$\begin{aligned}
[Q_1, Q_2]\{C_2\} &= \oint_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) \oint_{C_2} \frac{dz_2}{2\pi i} j_2(z_2) - \oint_{C_2} \frac{dz_2}{2\pi i} j_2(z_2) \oint_{C_3} \frac{dz_1}{2\pi i} j_1(z_1) \\
&= \oint_{C_2} \frac{dz_2}{2\pi i} \left[\oint_{|z_1|>|z_2|} - \oint_{|z_1|<|z_2|} \right] \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2) \\
&= \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2) \\
&= \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2),
\end{aligned} \tag{1.36}$$

where the residue is determined by the OPE of the two currents, assumed here to be bosonic; otherwise, we pick up the anticommutator in the second line. We use this very important concept later in the section on spacetime noncommutativity.

An important, simple, and fun example illustrating the utility of this construction, is the determination of the algebra corresponding to the Virasoro generators L_m, with currents $j_m = z^{m+1}T(z)$. Thus from (1.35),

$$\begin{aligned}
[L_m, L_n] &= \oint_{C_2} \frac{dz_2}{2\pi i} \left[\oint_{|z_1|>|z_2|} - \oint_{|z_2|>|z_1|} \right] \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} T(z_1)T(z_2) \\
&= \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} z_2^{n+1} \left[z_2^{m+1} + (m+1)z_2^m z_{12} + \frac{m^2+m}{2} z_2^{m-1} z_{12}^2 + \right. \\
&\quad \left. \frac{m^3-m}{6} z_2^{m-2} z_{12}^3 + \dots \right] \left[\frac{c}{2z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}} + \dots \right] \\
&= \oint_{C_2} \frac{dz_2}{2\pi i} \left[\frac{c(m^3-m)}{12} z_2^{m+n-1} + 2(m+1) z_2^{m+n+1} T(z_2) + z_2^{m+n+2} \partial T(z_2) \right] \\
&= \frac{c}{12}(m^3-m)\delta_{m,-n} + \oint_{C_2} \frac{dz_2}{2\pi i} \left\{ (m-n) z_2^{m+n-1} T(z_2) + \partial \left[z_2^{m+n+2} T(z_2) \right] \right\} \\
&= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}. \qquad (1.37)
\end{aligned}$$

This is the famous Virasoro algebra, which encodes the infinite dimensional conformal symmetry of the system, with a central charge term reflecting a quantum mechanical anomaly. For the closed string there is a second copy of the Virasoro algebra for $\bar{L}_m$.

1.3.5 Mode expansions

We now determine the spectrum of the simple *free* theory we have been studying, focusing first on the closed string. The OPEs of T and $\bar{T}$ with ∂X are

$$T(z)\partial X^\mu(0) \sim \frac{\partial X(0)}{z^2} + \frac{\partial^2 X^\mu(0)}{z} \quad , \quad \bar{T}(\bar{z})\partial X(0) \sim 0, \qquad (1.38)$$

after Taylor expanding about 0. Thus ∂X is primary with conformal weights $(1, 0)$. Similarly $\bar{\partial} X$ is also primary with conformal weights $(0, 1)$. Since they are (anti)holomorphic, we can thus write Laurent expansions as

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} \quad , \quad \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}, \qquad (1.39)$$

with the normalizations chosen to obtain canonical results in the following. Thus, inverting these relations, we have

$$\alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) \quad , \quad \tilde{\alpha}^\mu_m = -\sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^m \bar{\partial} X^\mu(\bar{z}), \tag{1.40}$$

while partial integration yields

$$X^\mu(z, \bar{z}) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \left[-(\alpha^\mu_0 \log z + \tilde{\alpha}^\mu_0 \log \bar{z}) + \sum_{m \neq 0} \frac{1}{m} \left(\frac{\alpha^\mu_m}{z^m} + \frac{\tilde{\alpha}^\mu_m}{\bar{z}^m} \right) \right] . \tag{1.41}$$

The associated spacetime momentum is given by $p^\mu \equiv (1/\sqrt{2\alpha'})(\alpha^\mu_0 + \tilde{\alpha}^\mu_0)$, and is the 'charge' associated with the Noether current of spacetime translations.

For a *noncompact* spatial dimension μ, single-valuedness implies $\alpha^\mu_0 = \tilde{\alpha}^\mu_0$, but in anticipation of our discussion of T-duality and D-branes, we will consider the general case, where it is convenient to also write the expansion in the w-frame for the zero modes:

$$X^\mu(\sigma, \tau) = x^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 + \tilde{\alpha}^\mu_0)\tau - \sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 - \tilde{\alpha}^\mu_0)\sigma + \ldots \tag{1.42}$$

where the ellipsis refers to the 2π periodic oscillator exponentials.

Via the contour argument presented in the last subsection,we can compute the mode operator commutation relations from the $\partial X(z_1)\partial X(z_2)$ OPE:

$$\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= \frac{2}{\alpha'} \oint_{C_2} \frac{dz_2}{2\pi} \left[\oint_{|z_1|>|z_2|} - \oint_{|z_1|<|z_2|} \right] \frac{dz_1}{2\pi} z_1^m \partial_1 X^\mu(z_1) z_2^n \partial_2 X^\nu(z_2) \\
&= -\oint_{C_2} \frac{dz_2}{2\pi} \oint_{C_{z_2}} \frac{dz_1}{2\pi} z_2^n \left[z_2^m + m z_2^{m-1}(z_1 - z_2) + \ldots \right] \frac{\eta^{\mu\nu}}{(z_1 - z_2)^2} \\
&= -i \oint_{C_2} \frac{dz_2}{2\pi} m z_2^{m+n-1} \eta^{\mu\nu} \\
&= m\delta_{m,-n}\eta^{\mu\nu}.
\end{aligned} \tag{1.43}$$

Similarly $[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m,-n}\eta^{\mu\nu}$, and from the XX OPE, we have

$$[x^\mu, p^\nu] = \frac{2}{\alpha'} \oint_{-C_z} \frac{dz'}{2\pi} X^\mu(z, \bar{z}) \partial' X^\nu(z') = \frac{\eta^{\mu\nu}}{2\pi} \oint_{C_z} \frac{dz'}{z' - z} = i\eta^{\mu\nu}, \tag{1.44}$$

being careful with the contour orientation. These familiar commutators reflect the *first* quantized string programme we are developing.

In the Hilbert space formalism of string theory, these operators act on the vacuum to produce the spectrum; i.e. we start with a momentum eigenstate $|0;k\rangle$ annihilated by all lowering modes α_n^μ for $n > 0$, and then generate the spectrum by applying all combinations of raising operators α_n^μ for $n < 0$.

In terms of modes, the Virasoro generators are expressed as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathcal{N}(\alpha^\mu_{m-n}\alpha_{\mu n}) + a^X\delta_{m,0}. \tag{1.45}$$

where $\mathcal{N}(\cdot)$ denotes ordinary creation-annihilation normal ordering, and a^X a normal-ordering constant for L_0 which is determined from the Virasoro algebra (i.e. without resort to ζ-regularization) to be zero for this CFT. By grouping $p^\mu(x^\mu)$ with the lowering (raising) operators, and keeping track of the commutators in the time-ordered product $X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)$, it is easy to show that the two forms of normal ordering ($\mathcal{N}(\cdot)$ and $:\cdot:$) coincide, the time-ordering $|z_1| > |z_2|$ required for convergence. This relationship (which does not hold in general) is useful in calculating scattering amplitudes.

Now consider the open string. Natural boundary conditions for the open string, following from the variation of (1.10), imply that no spacetime momentum can flow off the the ends of the string: $\partial_\sigma X^\mu(0,\tau) = \partial_\sigma X^\mu(\pi,\tau) = 0$, in the w-frame. In the z-frame this condition becomes $\partial X^\mu = \bar{\partial}X^\mu$ for $\mathrm{Im}(z) = 0$, which couples holomorphic and antiholomorphic modes, whence $\alpha_m^\mu = \tilde{\alpha}_m^\mu$. The normalization for p^μ is now $\alpha_0^\mu = (2\alpha')^{1/2}p^\mu$, and the mode expansion reads

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \log|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha_m^\mu}{m}(z^{-m}+\bar{z}^{-m}). \tag{1.46}$$

1.3.6 Vertex operators and physical states

In conformal field theory there is a bijective correspondence between the Hilbert space of states and the space of local operators. To evaluate the path integral in the w-frame, we would have to specify a boundary condition on the string as $\sigma^2 = \mathrm{Im}(w) \to -\infty$, corresponding to the definition of an initial state for the string. In the z-frame, $\mathrm{Im}(w) = -\infty$ is mapped to the origin, so the boundary condition is equivalent to specifying a local operator at the origin with the quantum numbers of the initial state. This trivially works in the opposite direction, and we will argue in the next subsection that this extends naturally to scattering amplitudes involving many strings, where each asymptotic state is mapped to a local operator on the worldsheet.

First consider the unit operator at the origin, corresponding to the state $|1\rangle$, and consider acting on it with the conserved charges $\alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^m \partial X^\mu(z)$, where the contour circles the origin. Since there is no operator at the origin, there is no OPE to compute, and ∂X^μ is holomorphic everywhere inside C. Thus for $m \geq 0$ we have

$$\alpha^\mu_m|1\rangle \simeq \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) = 0, \tag{1.47}$$

where we have used $\simeq$ to denote the isomorphism between operator and state. Since $p^\mu \sim \alpha^\mu_0$, this state has no momentum. On the other hand for α^μ_{-m} with $m > 0$ we obtain

$$\alpha^\mu_{-m}|1\rangle \simeq \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^{-m} \partial X^\mu(z) = \sqrt{\frac{2}{\alpha'}} \frac{i}{(m-1)!} \partial^m X^\mu(0), \tag{1.48}$$

via the Cauchy integral formula. Therefore, for this CFT, we identify $|1\rangle \equiv |0;0\rangle$.

To obtain a state with nonzero momentum, consider first acting on the vacuum with the operator $:\exp(ik\cdot X):$ at the origin. Then, applying the momentum operator p^μ, and using the OPE (1.23) we obtain

$$\begin{aligned} p^\mu : e^{ik\cdot X} : |0;0\rangle &\simeq \frac{2}{\alpha'}\oint_C \frac{dz}{2\pi}\, \partial X^\mu(z) : e^{ik\cdot X(0,0)} : \\ &= \oint_C \frac{dz}{2\pi i}\frac{k^\mu}{z} : e^{ik\cdot X(0,0)} : \simeq k^\mu : e^{ik\cdot X} : |0,0\rangle, \end{aligned} \tag{1.49}$$

while α^μ_m for $m > 0$ still annihilate the state. This leads us to identify $|0;k\rangle \simeq$ $: \exp(ik\cdot X) :$. The exponential shifts the (center of mass) momentum of the string, while keeping all of the nonzero frequency oscillators in their ground state.

The correspondence continues to hold inductively when we apply α^μ_{-m} to a general state $|\mathcal{A}\rangle$ built from $|0;0\rangle$. To prove this, consider

$$\begin{aligned} \alpha^\mu_{-m} : \mathcal{A}(0,0) : &= \sqrt{\frac{2}{\alpha'}}\oint_C \frac{dz}{2\pi} z^{-m}\partial X^\mu(z) : \mathcal{A}(0,0) : \\ &= \sqrt{\frac{2}{\alpha'}}\oint_C \frac{dz}{2\pi} z^{-m}\left[: \partial X^\mu(z)\mathcal{A}(0,0) : + \sum_{n=0}^{\infty}\frac{\mathcal{A}^{(n)}(0,0)}{z^{n+1}}\right] \\ &= \; : \alpha^\mu_{-m}\mathcal{A}(0,0) : \end{aligned} \tag{1.50}$$

for $m > 0$, and an arbitrary conformally normal-ordered operator $: \mathcal{A}(0,0) :$. Thus we can carry out the same operations as in (1.47)-(1.49), on an arbitrary state built from α^μ_{-m} excitations. As well, all of the above considerations apply in the obvious way for the charges $\tilde{\alpha}^\mu_m$, with $\partial \to \bar{\partial}$, and which represent right-moving excitations of a closed string.

This brings us to the issue of physical states, the proper treatment of which would require us to develop BRST invariance for the string. We will settle for a heuristic argument instead to arrive at the conditions we need. We have argued that asymptotic string states correspond to local operator insertions (called vertex operators) on the worldsheet. In the next subsection we will argue that it is somewhat irrelevant whether we consider freely propagating strings or interacting strings: in both cases the asymptotic states are conformally mapped onto worldsheets (Riemann surfaces) of a given topology. Since the actual points on the worldsheet to which these states are mapped are immaterial, to be democratic (i.e. consistent with quantum mechanics), we should therefore integrate the operators over all possible insertions on the worldsheet. That is to say physical asymptotic states will correspond to $\int d^2z\mathcal{V}(z,\bar{z})$ insertions in the (gauge-fixed) path integral. This quantity must be conformally invariant, and since the measure transforms as

$$d^2z' = \frac{\partial(z',\bar{z}')}{\partial(z,\bar{z})}d^2z = \frac{dz'}{dz}\frac{d\bar{z}'}{d\bar{z}}d^2z \tag{1.51}$$

under *arbitrary* conformal transformations $z' = f(z)$ (i.e. as a $(-1,-1)$ primary), we therefore require that $\mathcal{V}(z,\bar{z})$ transform as a $(1,1)$ *primary*.

This is a powerful restriction which yields much information about the nature of string theory. Consider the closed string states $V_1 \equiv: \exp(ik\cdot X) :$ and $V_2 = f_{\mu\nu} : \partial X^\mu\bar{\partial}X^\nu \exp(ik\cdot X) :$. The OPEs of T with these states are

$$\begin{aligned} T(z)V_1(0,0) &\sim \frac{\frac{\alpha' k^2}{4}V_1(0,0)}{z^2} + \frac{\partial V_1(0,0)}{z} \\ T(z)V_2(0,0) &\sim -\frac{i\alpha' k^\mu f_{\mu\nu}}{z^3} : \bar{\partial}X^\nu e^{ik\cdot X}(0,0) : + \frac{1+\frac{\alpha' k^2}{4}}{z^2}V_2(0,0) + \frac{1}{z}\partial V_2(0,0), \end{aligned} \tag{1.52}$$

and similarly for the OPEs of $\tilde{T}$ with these operators. V_1 and V_2 have weights $(1,1)$ iff $m^2 = -k^2 = -4/\alpha'$, and $k^2 = 0$ respectively. Furthermore, V_2 is primary iff $k^\mu f_{\mu\nu} = 0$.

Thus V_1, having a negative mass squared, is a tachyon, and V_2 is a set of massless states, with a physical transversality condition. The latter decomposes into three different Lorentz multiplets: the 'graviton' $g_{\mu\nu}$ (for $f_{\mu\nu}$ symmetric and traceless), the two-form tensor field $B_{\mu\nu}$ (for $f_{\mu\nu}$ antisymmetric), and the dilaton ϕ (the trace piece of $f_{\mu\nu}$). The transversality condition ensures that unphysical modes are removed from the system, and reflects a deep connection between worldsheet and spacetime physics. The appearance of a massless spin two particle (the graviton) in the spectrum also tells us that string theory should be a theory that automatically contains gravity, a viewpoint we will see again a little later. Furthermore, this thesis is, roughly speaking, a study of some possible low-energy consequences of the form field $B_{\mu\nu}$.

The higher states of the string correspond to vertex operators with more factors of ∂X^μ and $\bar\partial X^\nu$. By again using (1.18), and the expression for T, we can show that the state corresponding to

$$: \left(\prod_i \partial^{m_i} X^{\mu_i}\right)\left(\prod_j \bar\partial^{n_j} X^{\mu_j}\right) e^{ik\cdot X} : \qquad (1.53)$$

has conformal weights $(\frac{\alpha' k^2}{4} + \sum_i m_i, \frac{\alpha' k^2}{4} + \sum_j n_j)$, and hence mass squared $m^2 = \frac{4}{\alpha'}(\sum m_i - 1) = \frac{4}{\alpha'}(\sum n_j - 1)$. This gives us both a physical connection between spacetime spin and mass (more derivatives yielding higher rank Lorentz representations with correspondingly higher masses), and a physical interpretation of the parameter α': it sets the scale of the massive string states (or equivalently the length scale at which states become stringy). In string perturbation theory, $(\alpha')^{-1/2}$ is naturally near the Planck scale so that the graviton interacts with Newtonian strength in the weak coupling limit.

Finally, we briefly note that the considerations of this subsection apply to the open string, where the state-operator isomorphism maps asymptotic states to the worldsheet boundary, the real axis in the z-frame. Then, recalling that there is only one set of modes for the open string, its first excited state $\epsilon_\mu \alpha^\mu_{-1}|0;k\rangle$ has conformal weight one, iff $k^2 = 0$, and is primary iff $\epsilon \cdot k = 0$. The latter condition is again the familiar transversality requirement of a massless vector particle.

1.3.7 String 'interactions' and scattering amplitudes

Thus far, we have been considering the propagation of free strings and now we wish to consider the interactions of strings, considering for clarity mainly closed strings. It turns out that the only symmetry preserving interactions between (weakly-coupled) strings are those already contained in the 'sum over world-sheets'. Specifically, interactions arise from the global topology of the world-sheet as it is embedded in spacetime, while locally the worldsheet looks the same as the free case; put another way, any part of a scattering amplitude diagram looks locally like the propagation of a free string. Time-slicing the worldsheet in a given frame then implies that the basic interaction for a closed string is one where a single closed string breaks (or decays) into two closed strings by local pinching as in figure (1.1), or the time reversed process.

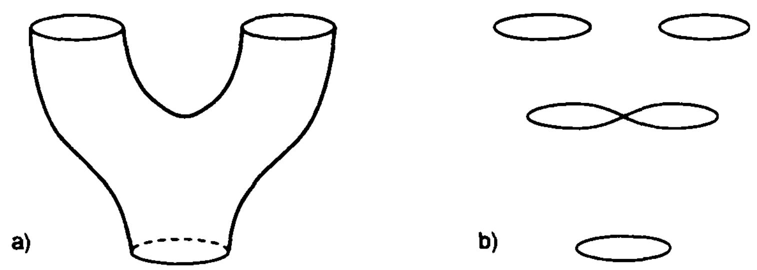


Figure 1.1: a) Decay of one closed string into two, b) as seen in a given Lorentz frame. (From [11].)

A freely propagating closed string will quantum mechanically divide and rejoin any number of times, with each such combination effectively adding a loop (or handle) to the worldsheet. Such changes in worldsheet topology (n-multiple connectedness) correspond to stringy quantum corrections.

The fact that interactions emerge from the topology of the worldsheet, has the (very desirable) effect of spreading out the interaction in spacetime so that different Lorentz frames see the different apparent interaction points. Heuristically speaking, this smearing of the interaction in spacetime is responsible for the well-behaved ultraviolet behaviour for which string theory is so famous.

In the last subsection, via the state-operator correspondence, we argued that an asymptotic state of a freely propagating string could be conformally mapped to a point on the worldsheet, where we insert a vertex operator which carries the quantum numbers of that state. Now consider the (tree-level) 2+2 scattering of closed strings, as in figure 1.2(a).

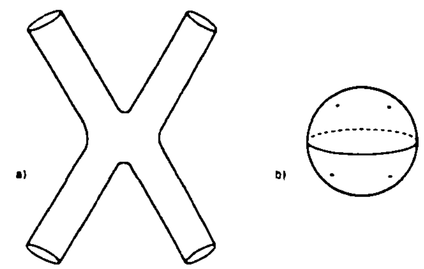


Figure 1.2: a) 4-point scattering of closed strings at tree level, and b) the conformally equivalent worldsheet. (From [8].)

By applying the state-operator correspondence to each external state, we can map each semi-infinite cylinder of the w-frame $[\mathrm{Re}(w) \in [0, 2\pi], -t < \mathrm{Im}(w) < 0]$ into the unit disk in the z-frame, with the boundary loop corresponding to a small circle of radius $\exp(-t)$. Now the worldsheet looks like that of figure 1.2(b). Taking t to infinity (corresponding to asymptotic, on-shell external states), these circles degenerate to points, and the worldsheet becomes (topologically equivalent to) a sphere, with vertex operator insertions.

As discussed above for the free propagation case, by adding handles (loops) to the worldsheet (thereby changing its topology), we can obtain stringy quantum corrections to such a scattering amplitude, the first (corresponding topologically to the torus) of which is shown here in figure (1.3).

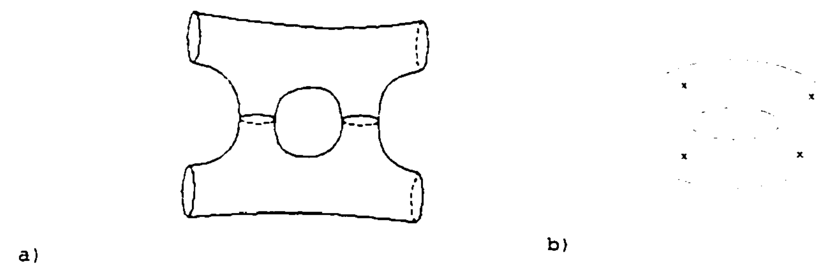


Figure 1.3: a) 4-point scattering of closed strings at one-loop, and b) the conformally equivalent worldsheet. (From [8].)

We noted in our discussion of the Polyakov action that the term $\chi = \int d^2\sigma\sqrt{g}R^{(2)}$ was consistent with the symmetries, but was a topological invariant in two-dimensions. Specifically, it is the Euler characteristic of the worldsheet, and is equal to $\chi = 2(1-g)$ for an integer g called the genus, which corresponds to the number of handles. As a consequence of 19th century mathematics, this integer completely classifies the possible topologies for compact, connected, oriented surfaces without boundary (relevant to the oriented closed string). Thus we have given a precise meaning to the phrase 'sum over worldsheets', as a sum over genus (for this type of string). The other cases (with boundaries, or unoriented) follow without too much added difficulty (each essentially adds an integer to the classification). However, for later reference we note that the relevant surfaces for (oriented) open string scattering amplitudes are topologically equivalent to the disk (at tree level), and the annulus (at one-loop), with asymptotic states mapped to vertex operators on the boundaries.

This leads us finally to the prescription for calculating amplitudes in string theory. For each asymptotic string state, we insert the corresponding vertex operator into the gauge-fixed path integral, which is weighted by the Polyakov action. After integrating the vertex operators over the worldsheet, integrating the worldsheet moduli τ_j present for $g \geq 1$ corresponding to conformally inequivalent surfaces (i.e. metric parameters that cannot be removed by the local symmetries), and summing over worldsheet genus, the resulting CFT correlator describes a spacetime scattering amplitude:

$$S_{j_1 \ldots j_N}(k_1, \ldots, k_N) = \sum_{g=0}^{\infty} \int [dX] e^{-S_X - \lambda(\chi - N)} \prod_j d\tau_j \prod_{i=1}^{N} \int d^2 z_i V_i(z_i, \bar{z}_i) \qquad (1.54)$$

where we have ignored the Fadeev-Popov ghosts arising from a proper gauge-fixing of the Polykov action, and which we will briefly consider momentarily. From (1.54), it is evident that 1) χ effectively sets the string coupling of the theory, and that 2) this is *free* field theory. The case of $g = 1$, corresponding to the torus yields particularly interesting consistency conditions on string theory in the form of modular invariance, as we will see.

We close by pointing out a fundamental limitation of this discussion. In order to invoke conformal invariance to map external states onto points on the worldsheet, we had to take the string sources to infinity ($t \to \infty$), thereby limiting us to a discussion of on-shell S-matrix elements. Another way to see this is to note that as per the previous subsection, the vertex operators were conformally invariant only if they transformed as $(1, 1)$ primaries, which in turn fixed the states to lie on their mass shell. This differs from ordinary field theory, where Green's functions are also defined off-shell, and where we can compute finite time transitions. It is not known in general how to ask string theory off-shell questions, the difficulty being linked to the fact that we do not possess a second-quantized definition of string theory.

1.3.8 The bc CFT

While the X CFT that we have been studying hitherto will be sufficient for our discussion on spacetime noncommutativity, a brief look at the so-called bc CFT involving *anticommuting* fields is useful for understanding the origin of the critical dimension, and for the discussion of superstring theories in the next section.

When we gauge fixed the symmetries of the Polyakov action, we ignored the Fadeev-Popov determinants that would arise in the associated worldsheet quantum field theory. Consider the gauge choice $g_{z\bar{z}} = \exp[2\omega(\sigma)]$, $g_{zz} = g_{\bar{z}\bar{z}} = 0$, and the infinitesimal coordinate transformations $z \to \xi^z$, $\bar{z} \to \xi^{\bar{z}}$, under which the metric changes by $\delta g_{zz} = 2\nabla_z \xi^{\bar{z}}$, $\delta g_{\bar{z}\bar{z}} = 2\nabla_{\bar{z}}\xi^z$. Using the standard functional integral representation for determinants in terms of anticommuting fields, the associated Fadeev-Popov determinant which implements the gauge-fixing is thus

$$\begin{aligned} J &= \det\left(\frac{\delta g_{zz}}{\delta\xi}\right)\cdot\det\left(\frac{\delta g_{\bar{z}\bar{z}}}{\delta\xi}\right) = \det(\nabla_z)\cdot\det(\nabla_{\bar{z}}) \\ &= \int [db_{zz}][dc^{\bar{z}}][db_{\bar{z}\bar{z}}][dc^{\bar{z}}]\exp\left(\frac{1}{2\pi}\int d^2z b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}} + b_{zz}\nabla_{\bar{z}}c^z\right) \\ &= \int [db_{zz}][dc^{z}][db_{\bar{z}\bar{z}}][dc^{\bar{z}}]\exp\left(\frac{1}{2\pi}\int d^2z b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}} + b_{zz}\partial_{\bar{z}}c^z\right), \qquad (1.55) \end{aligned}$$

where we have absorbed factors of two into field definitions. Also, we have used the fact that covariant $\bar{z}$ derivatives of tensors with only z indices reduce to ordinary derivatives, and vice-versa.

Thus we are led to consider the bc CFT described by the action

$$S = \frac{1}{2\pi}\int d^2z\; b\bar{\partial}c, \qquad (1.56)$$

which is (classically) conformally invariant for anticommuting b and c transforming as conformal tensors of weights $(\lambda, 0)$ and $(1-\lambda, 0)$ respectively. This follows because the integrand and measure transform as

$$d^2z'\, b'\bar{\partial}'c' = \frac{dz'}{dz}\frac{d\bar{z}'}{d\bar{z}} d^2z \left(\frac{dz'}{dz}\right)^{-\lambda} b \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-1} \bar{\partial}\left[\left(\frac{dz'}{dz}\right)^{\lambda-1} c\right] = d^2z\, b\bar{\partial}c \qquad (1.57)$$

under conformal transformations $z' = f(z)$. For the gauge-fixing application above, the index structure implies $\lambda = 2$.

The operator equations of motion from (1.56) are given by $\bar{\partial}c(z) = \bar{\partial}b(z) = 0$, so that b and c are holomorphic, while $\bar{\partial}b(z)c(0) = 2\pi\delta^2(z,\bar{z})$ (from a path integral insertion with c). This last equation implies the normal ordered product

$$: b(z_1)c(z_2) := b(z_1)c(z_2) - \frac{1}{z_1 - z_2} \qquad (1.58)$$

obeys the equations of motion using $\bar{\partial}(z^{-1}) = 2\pi\delta^2(z,\bar{z})$ derived from (1.11). Two straightforward calculations yield the energy-momentum tensor

$$T(z) = (1-\lambda) : (\partial b)c : -\lambda : b\partial c :, \qquad (1.59)$$

and the TT OPE

$$T(z)T(0) \sim \frac{1 - 3(2\lambda - 1)^2}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}. \qquad (1.60)$$

Thus the central charge of this theory is $c = -3(2\lambda - 1)^2 + 1$, which is -26 for $\lambda = 2$. The total central charge for the combined X and bc systems is thus $D - 26$, and so vanishes in 26 spacetime dimensions, the so-called critical dimension of bosonic string theory.

Finally, we note that for $\lambda = 1/2$, where $c = 1$, (1.56) can be written as

$$S = \frac{1}{4\pi}\int d^2z\, \psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2, \qquad (1.61)$$

where we have taken $b = \psi$, $c = \bar{\psi}$, and then split $\psi = (1/\sqrt{2})(\psi_1 + i\psi_2)$. This worldsheet fermion CFT arises immediately in the *super*string theories we now consider.

1.4 Supersymmetry, SCFTs, and superstrings

In the previous section we developed some of the fundamentals of what is called *bosonic* string theory. Unfortunately, it suffers from (at least) two drawbacks. First, the presence of a tachyonic mode in the spectrum signals that the theory corresponds to an expansion about an unstable vacuum. Secondly, there are no spacetime fermions. Both of these problems are resolved in string perturbation theory by the addition of worldsheet fermions ψ^μ (as in the so-called Neveu-Ramond-Schwarz formalism) to the Polyakov action (1.4) in such a way that the conformal symmetry of the bosonic string is enlarged to the superconformal symmetry of the superstring. Our treatment is completely cursory; our interest lies in understanding the terminology that occurs later.

1.4.1 SCFT

We have just seen that the purely holomorphic anticommuting bc theory has a conformally invariant splitting (for $\lambda = 1/2$) to two real worldsheet fermions. We can similarly construct a purely antiholomorphic anticommuting CFT. We can apply these facts to write down a supersymmetric generalization of the conformally gauge-fixed action (1.10) in a complex coordinate as

$$S = \frac{1}{4\pi}\int d^2z \frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \tag{1.62}$$

for D holomorphic, and antiholomorphic worldsheet fermions. We have the basic OPEs $X^\mu(z,\bar{z})X^\nu(0,0) \sim -(\alpha'/2)\eta^{\mu\nu}\log|z|^2$, $\psi^\mu(z)\psi^\nu(0) \sim \eta^{\mu\nu}/z$, and $\tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(0) \sim \eta^{\mu\nu}/\bar{z}$. The worldsheet supersymmetry is encoded in the worldsheet supercurrents

$$T_F(z) = i\sqrt{\frac{2}{\alpha'}} : \psi^\mu(z)\partial X_\mu(z) : \quad , \quad \tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}} : \tilde{\psi}^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z}) : \tag{1.63}$$

because for an arbitrary holomorphic anticommuting parameter $\eta(z)$, the OPEs and the Ward identity (1.22) imply that the currents $j^{\eta}(z) = \eta(z)T_F(z)$ and $\tilde{j}^{\bar{\eta}}(\bar{z}) = \bar{\eta}(\bar{z})\tilde{T}_F(\bar{z})$ generate superconformal transformations

$$\begin{aligned} \delta X^{\mu}(z,\bar{z}) &\propto -\eta(z)\psi^{\mu}(z) - \eta(z)^{*}\tilde{\psi}^{\mu}(\bar{z}) \\ \delta\psi^{\mu}(z) &\propto \eta(z)\partial X^{\mu}(z) \\ \delta\tilde{\psi}^{\mu}(\bar{z}) &\propto \eta(z)^{*}\bar{\partial}X^{\mu}(\bar{z}). \end{aligned} \tag{1.64}$$

It seems paradoxical that we have added anticommuting worldsheet fermions that carry spacetime *vector* indices, but recall that they are just internal symmetry indices from the worldsheet perspective. It turns out that the zero modes of ψ^{μ} and $\tilde{\psi}^{\mu}$ satisfy the (spacetime) gamma matrix algebra, and the zero modes of T_F yield the spacetime Dirac equation when applied to physical states .

For this system the worldsheet energy-momentum tensor (which we now denote by T_B) is

$$T_B = -\frac{1}{\alpha'} : \partial X^{\mu}\partial X_{\mu}(z) : -\frac{1}{2} : \psi^{\mu}\partial\psi_{\mu}(z) : \tag{1.65}$$

and hence we have the OPEs

$$\begin{aligned} T_B(z)T_B(0) &\sim \frac{3D}{4z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \\ T_B(z)T_F(0) &\sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \\ T_F(z)T_F(0) &\sim \frac{D}{z^3} + \frac{2}{z}T_B(0), \end{aligned} \tag{1.66}$$

which imply the closure of the corresponding algebra, since only T_B and T_F (or c-numbers) appear in the OPE singular terms. From the T_BT_F OPE we conclude that the T_F is primary with conformal weights $(3/2, 0)$, while the T_BT_B OPE implies that the central charge for the system is $c = 3D/2$. Finally, the T_FT_F OPE implies that the commutator of two superconformal transformations is a conformal transformation. Similarly, we have an antiholomorphic copy of the algebra, and thus in total have a so-called $(1,1)$ superconformal algebra, where in general $(N,\tilde{N})$ denotes the number of $(3/2,0)$ and $(0,3/2)$ currents.

The bc system of Fadeev-Popov ghosts also has a superconformal extension via the superconformal ghost system, which now involves a *commuting* $\beta\gamma$ system in addition to the bc system, and arises from the gauge-fixing of the local supersymmetric extension of the Polyakov action (which we have not discussed) to the superconformal gauge (which we took as our starting point). Specializing to the case of interest in string theory we have

$$S_{FP} = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma), \tag{1.67}$$

with conformal weights $h_b = 2$, $h_c = -1$, $h_\beta = 3/2$, $h_\gamma = -1/2$. This system has a central charge of -15, so the total central charge of the combined matter and ghost SCFTs is $3D/2 - 15$. Requiring this to vanish, so that the theory remains superconformally invariant at the quantum level, implies that $D = 10$. This is the famous critical dimension of superstring theory.

1.4.2 Ramond and Neveu-Schwarz sectors

In addition to an internal $O(D-1,1)$ symmetry, the fermionic part of the action (1.62) has a $\mathbb{Z}_2$ symmetry under which $\psi^\mu \to -\psi^\mu$, or $\tilde{\psi}^\mu \to -\tilde{\psi}^\mu$. Consequently, in the w-frame where we must have invariance under the periodic identification $w \simeq w + 2\pi$, we can choose two different boundary conditions:

$$\psi^\mu(w+2\pi) = +\psi^\mu(w)\ (R),\ \ \psi^\mu(w+2\pi) = -\psi^\mu(w)\ (NS). \tag{1.68}$$

These are called Ramond (R) and Neveu-Schwarz (NS) fermions respectively. We apply the same boundary conditions for all μ in order to maintain maximal

Lorentz invariance. Similar considerations hold for $\tilde{\psi}^\mu$. For a closed string, we therefore have four sectors labelled R-R, R-NS, NS-R, and NS-NS. Superstring consistency will require that certain states survive from each sector to form the full spectrum. For the open string, where ψ^μ and $\tilde{\psi}^\mu$ are linked via the string endpoints, we can have $\psi^\mu(0,\sigma_2) = \pm\tilde{\psi}^\mu(0,\sigma_2)$, and $\psi^\mu(\pi,\sigma_2) = \pm\tilde{\psi}^\mu(\pi,\sigma_2)$. ψ and $\tilde{\psi}$ can be combined to form a single field with $\sigma_1 \in [0, 2\pi]$ via $\psi^\mu(w) = \tilde{\psi}^\mu(2\pi - w)$, for which the boundary conditions now imply the presence of an R and an NS sector for the open string.

The mode expansion for a periodic holomorphic field in the w-frame is

$$\psi^\mu(w) = i^{-1/2} \sum_{n\in\mathbb{Z}+\nu} \psi^\mu_n e^{inw}, \tag{1.69}$$

where $\nu = 0$ in the R sector, and $1/2$ in the NS sector. Using the fact that the conformal weight of ψ is $1/2$, we can transform to the z-frame using $\psi^\mu(z) = (\partial_z w)^{1/2}\psi^\mu(w)$, whence

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}. \tag{1.70}$$

Thus the NS sector fermions are single-valued, while the R sector fermions have a $\mathbb{Z}_2$ branch cut. From the OPE-algebra correspondence, these modes obey the anticommutation relation

$$\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}. \tag{1.71}$$

The spectrum generated by a single set of NS modes is simple because there are no zero modes, so the ground state is singled out as the state for which

$$\psi^\mu_r|0\rangle_{NS} = 0 \ , \ r > 0, \tag{1.72}$$

while the ψ^μ_r with $r < 0$ act as raising operators to generate the spectrum. It can be shown that the NS vacuum state is a Lorentz singlet and since, as discussed, the raising operators are spacetime *bosons* (despite being anticommuting objects on the worldsheet), all the states in the NS spectrum are spacetime bosons.

The R ground state on the other hand is degenerate; (1.71) implies the zero modes ψ_0^μ satisfy the Dirac gamma algebra if we define $\Gamma^\mu \simeq \sqrt{2}\psi_0^\mu$. As a consequence the R ground state must be a spacetime (Dirac) spinor with $2^{D/2}$ components, and which we can label by $|\mathbf{s}\rangle_R$: $\mathbf{s}$ is a vector of eigenvalues $(\pm 1/2)$ of raising and lowering operators of the Lorentz group $O(D-1,1)$. By the same argument we applied to NS spectrum, the states then generated by acting on the R ground state with the raising operators (ψ_r^μ, with $r < 0$) are therefore all spacetime fermions.

Finally, we introduce an operator called the worldsheet fermion number denoted alternately by $\exp(i\pi F)$ or $(-1)^F$, which counts the number of fermionic modes in a state modulo two: to be precise

$$\{(-1)^F, \psi_r^\mu\} = 0. \tag{1.73}$$

This defines the operator only up to an overall sign, which we fix by defining the (massless) vector in the NS spectrum, $\psi_{-1/2}^\mu|0;k\rangle$, to have $(-1)^F = +1$ [6].

1.4.3 Type I and II Superstrings

Let us first discuss the Gliozzi-Scherk-Olive (GSO) projection [12]. Start with a bosonic state $|\phi\rangle$ that will remain in the spectrum, such as the massless vector of the NS spectrum. Then, as discussed, $\psi_{-r_1}^{\mu_1} \dots \psi_{-r_n}^{\mu_n}|\phi\rangle$ is bosonic for any n. For odd n however, this is disconcerting (though not in actual conflict with the spin-statistics theorem [6]), so we would like to keep only states with even n, or equivalently, those for which $(-1)^F = +1$. This is the GSO projection. It has the virtue of projecting out the tachyon, and yielding spacetime supersymmetry (which is obscure in this formalism). For closed strings, we can separately apply GSO projections to the left and the right moving modes. Finally, we note that modular invariant partition functions at one-loop *require* GSO projections to avoid divergences in one-loop string amplitudes.

Now reconsider the NS sector of the open string, or the left moving closed string. The lowest mode $|0;k\rangle_{NS}$ is tachyonic, has $(-1)^F = -1$, and so is projected out by the GSO projection. The next state, $\psi^{\mu}_{-1/2}|0;k\rangle$, is a massless spacetime boson, and has $(-1)^F = 1$. In ten dimensions (the critical dimension of the superstring), *massless* states are classified by their $SO(8)$ representations (the little group). Thus this state corresponds to the eight transverse polarizations that form the vector representation of $SO(8)$, which we denote $\mathbf{8}_v$. In the R sector, we have seen that the states correspond to spacetime fermions; the ground states can be shown to be massless, and form the two inequivalent irreducible spinor representations of $SO(8)$: the $\mathbf{8}_s$ and $\mathbf{8}_c$, with $(-1)^F = 1$ and $(-1)^F = -1$ respectively.

We thus have four sectors labelled by NS±, and R±.[1] Closed string states are tensor products of left and right moving states; the massless states are obtained by taking one left and one right moving state subject to a level matching mass-shell condition. Since the NS− state is tachyonic, and the R-NS and NS-R spectra are identical, we have (in terms of irreducible representations) the massless closed string states shown in table (1.1).

For closed strings, a careful analysis of the consistency conditions coming from

[1] Due to the different vacuum energies of the NS and R ground states relative to the bosonic ground state, the usual bosonic oscillator excitations, α^{μ}_{-n} do not contribute to the *massless* spectra of the theories in this subsection.

Sector	SO(8) spin	tensor decomposition	dimension
(NS+,NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	[0] + [2] + (2)	$\mathbf{1} + \mathbf{28} + \mathbf{35}$
(NS+,R+)	$\mathbf{8}_v \times \mathbf{8}_s$		$\mathbf{8}_c + \mathbf{56}_s$
(NS+,R-)	$\mathbf{8}_v \times \mathbf{8}_c$		$\mathbf{8}_s + \mathbf{56}_c$
(R+,R+)	$\mathbf{8}_s \times \mathbf{8}_s$	$[0] + [2] + [4]_+$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_+$
(R+,R-)	$\mathbf{8}_s \times \mathbf{8}_c$	[1] + [3]	$\mathbf{8}_v + \mathbf{56}_t$
(R-,R-)	$\mathbf{8}_c \times \mathbf{8}_c$	$[0] + [2] + [4]_-$	$\mathbf{1} + \mathbf{28} + \mathbf{35}_-$

Table 1.1: Irrep decomposition of products of SO(8) irreps for closed strings at the massless level. (From [5].)

level matching, OPE locality (R sector branch cuts introduce branch cuts in the OPEs of certain pairs of vertex operators), and one-loop modular invariance (invariance of the string path integral under 'large' coordinate transformations) yields two supersymmetric theories: the type IIA and IIB (oriented) superstrings. The former is defined by the GSO projections $(-1)^F = +1$, $(-1)^{\tilde{F}} = (-1)^{1-2\nu}$ [see below (1.69)], while the latter is defined by taking GSO projections $(-1)^F = (-1)^{\tilde{F}} = +1$.

Thus, from table (1.1) we deduce the massless content for the IIA theory as $[0] + [2] + (2) + [1] + [3] + \mathbf{8}_s + \mathbf{8}_c + \mathbf{56}_s + \mathbf{56}_c$. The first three correspond to the gravity multiplet that we found for the bosonic string (the dilaton, the Kalb-Ramond antisymmetric tensor field, and the graviton respectively). The next two R-R states correspond to a one-form (vector) field and three-index antisymmetric tensor field. The final four states are fermionic, and correspond to two Majorana-Weyl gravitinos (the **56**s), and two spin 1/2 fermions (the **8**s) of opposite chiralities. Since spacetime parity interchanges the **8**s and the **56**s (the other states are invariant) the theory is nonchiral. The massless content of the chiral IIB theory is $[0] + [2] + (2) + [0] + [2] + [4]_+ + \mathbf{8}_c^2 + \mathbf{56}_s^2$. In addition to the gravity multiplet, we have R-R scalar, R-R two-form, and R-R self-dual four-form fields, as well as two gravitinos and two spin 1/2 fermions of the same chirality. These theories both have two gravitinos (the origin of the term type II), which indicates the presence of an $N = 2$ local spacetime supersymmetry. Furthermore, the two theories differ only in their massless spectra.

This brings us finally to the type I unoriented, open plus closed superstring which can be found by first taking the IIB superstring, and gauging world-sheet parity. Among other things, this projects out one of the gravitinos, and so one of the supersymmetries. The result is the type I unoriented closed string theory, which is by itself inconsistent. On the other hand, open strings must couple to closed strings; one way to see this is to consider a process in which two open strings touch endpoints to form one open string, and an interaction where the endpoints of a single open string touch to form a closed string. Since the interaction is local, to forbid the second interaction would require some nonlocal constraint on the dynamics, which would spoil the consistency of the theory. As discussed above, the massless spectra for an open string can consist of NS+ and a R+ (or R-) spectrum. We can add so-called Chan-Paton factors to the string endpoints, and so introduce a gauge group under which the NS+ and R+ (or R-) open string states are charged. However such states fill out the representation of an N=1 supersymmetry vector multiplet, and so can consistently couple only to the unoriented closed string theory, which in turn fixes the open string sector to be unoriented, and pins down the choice of R sector. The result is a massless sector that consists of $N = 1$, $D = 10$ super Yang-Mills coupled to $N = 1$, $D = 10$ supergravity. Finally, both one-loop and spacetime anomaly conditions single out the gauge group $SO(32)$.

1.4.4 Heterotic Strings

The type II theories discussed in the previous section have the drawback that they do not possess (spacetime) gauge groups in ten dimensions, and cannot generate large enough gauge groups to contain the standard model by compactification. A careful analysis of other SCFTs yields two more possibilities for consistent string theories with spacetime supersymmetry, and non-Abelian gauge groups in ten dimensions, one of which has compactifications which closely resemble the standard model; these are the heterotic string theories [13]. Again, anticipating our interest in field-theoretic limits, we will focus on the massless spectra.

These theories are so-named because they combine the left-moving (holomorphic) modes of the (closed) bosonic string, with the right moving (antiholomorphic) modes of the type II superstring. This seems paradoxical because these theories have different critical dimensions, but since they are closed strings, the left and right moving modes are decoupled, so there is no *a priori* contradiction. We can either consider the extra left-moving modes to be toroidally compactified on a length scale of order $\sqrt{\alpha'}$ for a literal interpretation, or take the stringy (i.e. CFT) perspective and simply regard the extra modes as an internal CFT required for conformal invariance and keep the same number of dimensions on both sides: ten. Adopting the latter view, we first have the worldsheet fields

$$X^\mu(z,\bar{z}) \;,\; \tilde{\psi}^\mu(\bar{z}) \;,\; \mu = 0 \ldots 9 \tag{1.74}$$

which give rise to a CFT with central charge $(c,\tilde{c}) = (10,15)$, while the bc conformal ghosts on the left side and the superconformal (bc and $\beta\gamma$) ghosts on the right side contribute $(-26,-15)$. Therefore, to obtain a conformally invariant theory at the quantum level, we must add a holomorphic (matter) CFT with central charge $(16,0)$. From our discussion of the X and bc (i.e.

$\bar{w}$) CFTs, we can take either 16 left-moving bosons or 32 left-moving fermions, which give rise to the so-called bosonic and fermionic constructions respectively; we will describe the latter. Thus we introduce holomorphic worldsheet fermions $\lambda^A(z)$ for $A = 1 \ldots 32$, and so obtain the total matter action

$$S = \frac{1}{4\pi}\int d^2z \frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \lambda^A \bar{\partial}\lambda^A + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu. \tag{1.75}$$

This action has a $(N, \tilde{N}) = (0, 1)$ superconformal symmetry, and an internal (global) $SO(9,1) \times SO(32)$, where the $SO(32)$ acts on the λ^A. As before, we have to specify boundary conditions for the fields and GSO projections. Since we do not apply Poincaré invariance to the *internal* λ^A, w-frame periodicity of the worldsheet energy-momentum tensor (T_B) requires only that λ^A be periodic up to an arbitrary $O(32)$ rotation, under which $\lambda^A(w + 2\pi) = O^{AB}\lambda^B(w)$. This complicates the analysis, but it turns out there are only two theories that are spacetime supersymmetric, tachyon free, and modular invariant.

The first is obtained by imposing the right moving GSO projection $(-1)^{\tilde{F}} = 1$, the left-moving periodicities $\lambda^A(w + 2\pi) = \pm\lambda^A(w)$ for all A, and the left-moving GSO projection $(-1)^F = 1$ on the left-moving fermion number. The right moving side, being the same as the type II string, has $\mathbf{8}_v + \mathbf{8}_s$ at the massless level. The left moving massless states are obtained from $\alpha^i_{-1}|0\rangle_{NS}$, and $\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{NS}$, where the index i refers to the transverse coordinates. Under $SO(8) \times SO(32)$, these transform respectively as $(\mathbf{8}_v, \mathbf{1})$ and $(\mathbf{1}, [2]) = (\mathbf{1}, \mathbf{496})$, since the latter is antisymmetric under $A \leftrightarrow B$ [and hence forms the adjoint representation of $SO(32)$]. Combining left and right movers, we thus have at the massless level 1) the $N = 1$ supergravity multiplet from $(\mathbf{8}_v, \mathbf{1}) \times (\mathbf{8}_v + \mathbf{8}_s) = (\mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}) + (\mathbf{8}_c, \mathbf{1})$, consisting of the usual dilaton, $B_{\mu\nu}$ field and graviton, plus a gravitino and a Majorana-Weyl fermion; and 2) the $N = 1$ vector supermultiplet in the adjoint of $SO(32)$ from $(\mathbf{1}, \mathbf{496}) \times (\mathbf{8}_v + \mathbf{8}_s) =$

$(\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}_s, \mathbf{496})$, corresponding to a gauge field and its gaugino.

The second heterotic string is obtained by dividing up the λ^A into two sets of 16, each with independent w-frame periodicity conditions (which thus lead to four sectors labelled by NS-NS$'$, etc.), and imposing the GSO projections $(-1)^{F_1} = (-1)^{F_1'} = (-1)^{\tilde{F}} = 1$, where the first two refer to the two sets of λs. Again the right side massless states are $\mathbf{8}_v + \mathbf{8}_s$. The left R-R$'$ sector produces no massless states, while the NS-NS$'$ sector has massless states $\alpha^i_{-1}|0\rangle_{NS,NS'}$ and $\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{NS,NS'}$, for A *and* B both from one set of λs. The periodicity conditions on λ^A break the internal $SO(8) \times SO(32)$ to $SO(8) \times SO(16) \times SO(16)$, under which we have $(\mathbf{8}_v, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{120})$, where the $\mathbf{120}$ is the adjoint of $SO(16)$. Meanwhile the NS-R$'$ and R-NS$'$ sectors yield (after GSO projection) the massless states $(\mathbf{1}, \mathbf{128}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{128})$. Tensoring these with the right-moving $\mathbf{8}_v$, we obtain massless vectors transforming as $\mathbf{120} + \mathbf{128}$ for each $SO(16)$. Since consistency requires massless spacetime vectors to transform in the adjoint, we finally arrive at the group E_8, under which the adjoint decomposes as the $\mathbf{120} + \mathbf{128}$ of $SO(16)$. Thus the full gauge group will be $E_8 \times E_8$, though only $SO(16) \times SO(16)$ is manifest in this construction. Putting it all together, under $SO(8) \times E_8 \times E_8$, we thus obtain the now familiar $N = 1$ supergravity multiplet: $(\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}, \mathbf{1}) + (\mathbf{8}_c, \mathbf{1}, \mathbf{1})$, and an $N = 1$ $E_8 \times E_8$ vector supermultiplet, $(\mathbf{8}_v, \mathbf{248}, \mathbf{1}) + (\mathbf{8}_s, \mathbf{248}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{1}, \mathbf{248}) + (\mathbf{8}_s, \mathbf{1}, \mathbf{248})$.

The main reason that the $E_8 \times E_8$ heterotic string generated so much interest historically, was that if we embed the spin connection of the compactification manifold in one of the E_8s (and preserve an unbroken supersymmetry in four dimensions), we naturally break $E_8 \times E_8$ down to $SU(3) \times E_6 \times E_8$, and E_6 is a favorite candidate for grand unified model building.

1.5 Low-energy field theory limits

We have seen that the spectrum of a string theory consists of a finite number of massless states, and an infinite tower of massive states proportional to $M_s = (\alpha')^{-1/2}$. In weakly coupled string perturbation theory, where we can calculate graviton scattering, it emerges that this parameter must be near the Planck scale in order for gravity to interact with Newtonian strength. Therefore if one wishes to obtain a description of string theory at low-energies in order to discuss phenomenology, it should be sufficient to study the behaviour of the massless modes; in effect by constructing a Wilsonian effective action for these *fields* in which the infinite tower of massive string states have been integrated out. This is analogous to the Fermi theory that emerges at low energies by integrating out the massive W-boson in standard electroweak theory.

Since an exact effective action for the massless fields is prohibitively difficult to obtain, in practice we construct a derivative expansion, corresponding to a suppression by powers of E/M_s, where E is the energy scale of an interaction. In the spirit of Wilson, this should represent an increasingly accurate approximation in the infrared (although the resultant field theory is *a priori* nonrenormalizable because we have sacrificed the ultraviolet finiteness of string theory originating from the infinite tower of states/finite size of the string). Another way of thinking about this is to note that formally, the low-energy limit of string theory corresponds to sending α' to zero, in effect sending the massive states to infinity and decoupling them from the massless spectrum. Of course, strictly speaking we cannot send a dimensionful parameter to zero, but we will see that we can give this a precise meaning. In this section, we will therefore describe three different techniques for arriving at low energy, field-theoretic limits of string theory.

1.5.1 Limits from scattering amplitudes

The most direct way of constructing field theoretic limits from string theory is to compute scattering amplitudes of massless string states, as represented by their vertex operators (using the prescription we outlined in section 1.3), and then write down a Lagrangian density for fields corresponding to these states that can reproduce these amplitudes. Since we are restricted to computing on-shell S-matrix elements in string theory, this fixes the effective action only up to terms which vanish by the equations of motion (and field redefinitions).

Specifically, we start by writing down a field theory Lagrangian with the kinetic terms appropriate for the massless modes, $\mathcal{L}_{2pt}$. Then one writes down $\mathcal{L}_{3pt}$ that can reproduce the three point string amplitudes, which allows us to connect the effective action coupling constants to the string theory parameters (the string coupling constant, and α'). The four point amplitudes are slightly more involved. Unitarity ensures that the massless poles are generated by the tree graphs of $\mathcal{L}_{3pt}$, while the remainder is due to massive particle exchange, the effects of which we are integrating out into $\mathcal{L}_{4pt}$, and which are expanded in powers of the external momenta (and hence powers of $\sqrt{\alpha'}$) to generate *local* four point vertices in $\mathcal{L}_{4pt}$. We then similarly proceed to arbitrary order to generate the n-point contributions to the action.

Space prohibits us from presenting more detail of these constructions, so we refer the reader to the references, the early ones of which include [14] (zero slope limit), [15] (open string amplitudes and the Yang-Mills action), [16] (graviton amplitudes and the Einstein-Hilbert action), [17] (superstring effective actions), and [13], [18] (heterotic string effective actions).

1.5.2 Limits from nonlinear sigma models

We have already seen examples of the deep connections between worldsheet and spacetime physics, and in this subsection we briefly consider one of the most dramatic: the emergence of spacetime effective actions from conformal (or Weyl) invariance of strings propagating in nontrivial backgrounds [19], [20]. For simplicity, we restrict our attention to closed, oriented, bosonic strings.

When we discussed the Polyakov action (1.4), we implicitly assumed that the strings were propagating in a flat Minkowski background. Now consider coupling a general background spacetime metric, which we denote by $G_{\mu\nu}(X)$, to the Polyakov action as

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma g^{1/2} g^{ab} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \,. \tag{1.76}$$

To make contact with what we already know, consider a spacetime which is a perturbation about Minkowski space, where may expand the metric as $G_{\mu\nu}(X) = \eta_{\mu\nu} + \epsilon\chi_{\mu\nu}(X)$, and the worldsheet path integrand as

$$\exp(-S_\sigma) = \exp(-S_P)\left[1 - \frac{1}{4\pi\alpha'}\int_M d^2\sigma g^{1/2} g^{ab}\epsilon\chi_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \; + \; O(\epsilon^2)\right]. \tag{1.77}$$

We recognize the term of order ϵ (in conformal gauge) as the graviton vertex operator with $\chi_{\mu\nu}(X) \propto f_{\mu\nu}\exp(ik\cdot X)$, so the action (1.76) just corresponds to an insertion of a coherent state of gravitons.

When we studied the massless modes of the closed bosonic string theory in section 1.3.6, we found that the graviton was just the traceless, symmetric piece of a general two (spacetime) index tensor. This suggests we should also include backgrounds for the other massless fields (namely the antisymmetric two-form tensor and the dilaton, denoted respectively by $B_{\mu\nu}$, Φ). The correct prescription turns out to be

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma g^{1/2}\left[\left(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\right]. \tag{1.78}$$

where ϵ^{ab} is just the antisymmetric tensor (density) on the worldsheet, and as before $R^{(2)}$ is the *worldsheet* Ricci scalar. It is easy to check that the expected spacetime gauge invariances are respected by this action (which includes invariance under $\delta B_{\mu\nu} = \partial_\mu \xi_\nu(X) - \partial_\nu \xi_\mu(X)$, the gauge invariance of a two-form antisymmetric tensor potential).

For historical reasons, models such as (1.78) involving field-dependent kinetic terms are called nonlinear sigma models, the action defining a two-dimensional *interacting* worldsheet quantum field theory. This is treated perturbatively by expanding X^μ about a classical solution x_0^μ as $X^\mu = x_0^\mu + Y^\mu$, and expanding the metric in Riemann normal coordinates as

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{1}{3}\mathbf{R}_{\mu\lambda\nu\kappa}(x_0)Y^\lambda Y^\kappa - \frac{1}{6}D_\rho \mathbf{R}_{\mu\lambda\nu\kappa}Y^\rho Y^\lambda Y^\kappa + \ldots \tag{1.79}$$

where $\mathbf{R}_{\mu\lambda\nu\kappa}$ refers to the *spacetime* Riemann curvature tensor. The coupling constants of the interaction terms thus involve derivatives of the metric, which are of order R_c^{-1}, the characteristic radius of curvature of the spacetime manifold. Thus the effective dimensionless coupling of the worldsheet quantum field theory is $\sqrt{\alpha'}R_c^{-1}$. If this is small, then not only is the (worldsheet) quantum mechanical perturbation theory useful, but is also precisely the regime where we expect to be able to use a low-energy effective field theory in the first place! Thus our restriction to massless modes is a self-consistent one.

Recalling our discussion in section 1.3.4 on the connection between conformal invariance and the tracelessness of the worldsheet energy-momentum tensor, one can show using the sigma model perturbation theory just described, that for general backgrounds, (1.78) is not conformally (strictly speaking Weyl) invariant at the quantum level, but instead we have [5]

$$T^a_{\ a} \propto \frac{1}{2\alpha'}\left(\beta^G_{\mu\nu}g^{ab} + i\beta^B_{\mu\nu}\epsilon^{ab}\right)\partial_a X^\mu \partial_b X^\nu + \frac{1}{2}\beta^\Phi R^{(2)}, \tag{1.80}$$

where the beta functions up to $O(\alpha')$ (two derivatives) are

$$\begin{aligned} \beta^G_{\mu\nu} &= \alpha'\mathbf{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu^{\ \lambda\kappa} + O(\alpha'^2) \\ \beta^B_{\mu\nu} &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha' H_{\lambda\mu\nu}\nabla^\lambda\Phi + O(\alpha'^2) \\ \beta^\Phi &= \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha'^2). \end{aligned} \tag{1.81}$$

In the above, $H_3 = dB_2$ is the field strength associated with $B_{\mu\nu}$.

Since Weyl (and hence conformal) invariance are crucial to the consistency of string theory, these beta functions must vanish. The miracle is that all three of the resultant field equations can arise from Euler-Lagrange variation of the *spacetime* effective action (assuming $D = 26$)

$$\mathbf{S} = \frac{1}{2\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\left[\mathbf{R} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha')\right]. \tag{1.82}$$

We can cast this in standard Einstein-Hilbert form by performing a spacetime Weyl transformation on the metric $G_{\mu\nu}$.

At this point it is natural to point out the central role the dilaton plays in string theory. We note that, as per our discussions of (1.9) in sections 1.3.1 and 1.3.7, if we shift the VEV of $\Phi \to \Phi + \Phi_0$, then (1.78) transforms as $S_\sigma \to S_\sigma + 2\Phi_0(1-g)$, where g is the worldsheet genus. Thus, such a shift in the dilaton corresponds to a shift in the coupling constant of the *string* perturbation expansion; i.e. the dilaton VEV, $e^{-2\Phi_0}$ *is* the string coupling constant. Similarly, the coupling κ in (1.82) is not physical since it can be redefined by a shift in dilaton VEV: there are no free dimensionless parameters in string theory.

1.5.3 Limits from supersymmetry

The previous two methods for determining low-energy effective actions are direct, stringy, and unfortunately, difficult. However, in the case of the five superstring theories we discussed in the previous section, the constraint of local supersymmetry in ten dimensions (along with gauge invariance) is powerful enough to completely determine the low-energy effective action, up to field redefinitions.

For our purposes in the next chapter, all that we need is the effective action for the bosonic massless states we previously found for the heterotic string theories:

$$S_{het,bos} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[\mathbf{R} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|\tilde{H}_3|^2 + \frac{\kappa_{10}^2}{g_{10}^2}Tr_V(|F_2|^2)\right]. \tag{1.83}$$

Here subscripts denote the rank of the antisymmetric tensor to which they refer. κ_{10} and g_{10} are gravitational and gauge couplings respectively (and are related to α' through $\kappa_{10}^2/g_{10}^2 = \alpha'/4$), and the trace is performed in the vector representation for $SO(32)$, or normalized to $\frac{1}{30}\mathrm{Tr}_a(t^a t^b)$ for $E_8 \times E_8$.

We recognize the first two terms in (1.83) already from (1.82), and the fourth term simply represents the kinetic terms for the $SO(32)$ or $E_8 \times E_8$ gauge fields of the heterotic string. Finally, the third term in (1.83), the kinetic term for the B_2, is a modification of the third term in (1.82) required to consistently couple the super Yang-Mills action to the supergravity [6], [21]:

$$\tilde{H}_3 = dB_2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(A_1 \wedge dA_1 + \frac{2}{3}A_1 \wedge A_1 \wedge A_1\right), \quad \delta B_2 = \frac{\kappa_{10}^2}{g_{10}^2}Tr_V(\lambda dA_1). \tag{1.84}$$

In (1.84) we have written a nontrivial gauge transformation law for B_2, which is surprising since it is neutral with respect to the gauge group. This transformation is required for Yang-Mills gauge invariance nonetheless, and plays an important role in superstring anomaly cancellation; the subject to which we now turn.

1.6 Anomalies and anomalous U(1)s

When a classical symmetry or conservation law is violated by quantization, that symmetry is said to be anomalous. While anomalies in global symmetries can be phenomenologically desirable (as in the breaking of global scale invariance in massless QCD for example), anomalies in gauge symmetries are fatal because unphysical longitudinal modes no longer decouple from amplitudes, so that unitarity is violated.

Anomalies have the peculiar property that they can be understood simultaneously as ultraviolet effects (where a quantum mechanical regulator consistent with the symmetry cannot be defined), and infrared effects (because they depend only on the massless spectrum of the theory). The latter perspective allows us to analyze potential anomaly structure from a field-theoretical view. Furthermore it turns out that anomalies in local conservation laws can arise only in parity violating amplitudes, because standard Pauli-Villars regularization involving massive regulator fields can always be used for parity-conserving amplitudes.

In four-dimensions, anomalies arise from triangle diagrams with chiral fermion fields propagating in a loop coupled to three external currents. For $U(1)$ charges q, the anomalous amplitudes themselves involve $F \wedge F$ (and $R \wedge R$), and are proportional to $\sum_L q^3$ (pure gauge anomaly) and $\sum_L q$ (mixed anomaly), where the sums are taken over all massless left-handed fermions that can circulate in the loop. Thus, we require that these sums vanish for an anomaly free theory in four-dimensions. For example, when we include the contributions of a generation of *both* quarks (including colour) and leptons, potential anomalies involving the $U(1)_Y$ hypercharge cancel, suggesting structure beyond the standard model in which quarks and leptons appear in a single multiplet.

1.6.1 Anomalies and the Green-Schwarz mechanism

It turns out that in general, anomalies in D dimensions can be conveniently encoded in terms of a *formal* $D+2$ form $I_{D+2}(R_2, F_2)$ [6]. Here, F_2 is the gauge field strength written as a matrix-valued two-form $F_2 = F_2^a t_r^a$, for a representation r (relevant to gauge and mixed anomalies), and R_2 is the gravitational analogue of F_2 (relevant to gravitational and mixed anomalies); specifically it is the Riemann curvature tensor in the tetrad formalism, $R_{\mu\nu}^{ab}$, with two $O(D-1,1)$ tangent space indices a, b to be contracted with the adjoint representation matrices of $O(D-1,1)$, T^{ab}, to make the matrix-valued two-form R_2. Locally, I_{D+2} can be written as $I_{D+2} = dI_{D+1}$ (since it is closed), where in turn $\delta I_{D+1} = dI_D$ under a gauge transformation. The anomaly corresponding to I_{D+2} is given by $\int d^D x I_D(F_2, R_2)$, which is proportional to the anomalous variation of the path integral under gauge transformations. The condition for anomaly cancellation can then be shown to correspond to the vanishing of I_{D+2}.

For the ten-dimensional heterotic theories, which have the massless content of $N=1$, $D=10$ supergravity and $N=1$, $D=10$ super Yang-Mills, I_{12} receives contributions from the gravitino **56**, the neutral $\mathbf{8}_c$ Majorana-Weyl fermion (the chiral fermions from the supergravity sector), and the gaugino $\mathbf{8}_s$ in the adjoint of the gauge group (the chiral fermion from the Yang-Mills sector). Denoting adjoint representation traces for the gauge field strength two-form F_2 by Tr_a, traces over the tangent space indices of the curvature two form R_2 by tr, and omitting explicit wedge products so that $A_m \wedge B_n \equiv A_m B_n$, the total anomaly polynomial can be shown to be equal to [6], [5]

$$\begin{aligned} I_{12} &= \frac{1}{1440}\left\{-\mathrm{Tr}_a(F_2^6) + \frac{1}{48}\mathrm{Tr}_a(F_2^2)\mathrm{Tr}_a(F_2^4) - \frac{1}{14400}[\mathrm{Tr}_a(F_2^2)]^3\right\} \\ &\quad + \frac{n-496}{725760}\mathrm{tr}(R_2^6) + \frac{1}{768}Y_4 X_8, \end{aligned} \tag{1.85}$$

where

$$
\begin{aligned}
Y_4 &= \mathrm{tr}(R_2^2) - \frac{1}{30}\mathrm{Tr}_a(F_2^2) \\
X_8 &= \mathrm{tr}(R_2^4) + \frac{[\mathrm{tr}(R_2^2)]^2}{4} - \frac{\mathrm{Tr}_a(F_2^2)\mathrm{tr}(R_2^2)}{30} + \frac{\mathrm{Tr}_a(F_2^4)}{3} - \frac{[\mathrm{Tr}_a(F_2^2)]^2}{900}
\end{aligned}
\tag{1.86}
$$

The first two terms cannot be be written in the factorized form that the third one has been written, and must vanish separately for the theories to be anomaly free. The vanishing of the second term implies the gauge group has 496 generators, while, as we will see, the cancellations among the components of the first term further restrict the gauge group to be $SO(32)$, $E_8 \times E_8$, $U(1)^{496}$, or $E_8 \times U(1)^{248}$. No known string theories correspond to the latter two gauge groups, so we will focus on the first two. This leaves the factorized (also called reducible) term $Y_4 X_8$, so written in anticipation of the following discussion.

To illustrate the general principle, let us first consider the pure gauge anomaly in ten dimensions, which is proportional to $\mathrm{Tr}_a(F_2^6)$. On the other hand, consider the Chern-Simons interaction

$$S' = \delta_{gs} \int B_2 \mathrm{Tr}(F_2^4), \tag{1.87}$$

where δ_{gs} is a number, wedge products between forms are again implicit, and where we do not specify for the moment the representation in which we perform the trace. Recall from the previous section that under gauge transformations, B_2 was assigned a nontrivial gauge transformation law (so that the supergravity could be consistently coupled to the super Yang-Mills theory), whence

$$\delta S' \propto \int \mathrm{Tr}(\lambda dA_1)\mathrm{Tr}(F_2^4), \tag{1.88}$$

using the fact that F_2 itself is gauge-invariant by construction. Therefore, by taking $I_D \propto \mathrm{Tr}(\lambda dA_1)\mathrm{Tr}(F_2^4)$ [whence $I_{D+1} \propto \mathrm{Tr}(A_1 F_2)\mathrm{Tr}(F_2^4)$, and $I_{D+2} \propto$

$\mathrm{Tr}(F_2^2)\mathrm{Tr}(F_2^4)]$, we can cancel an anomaly of the form $\mathrm{Tr}(F_2^2)\mathrm{Tr}(F_2^4)$ by the correct choice of δ_{gs}. Similarly a Chern-Simons term of the form $\int B_2[\mathrm{Tr}(F_2^2)]^2$ can cancel an anomaly polynomial of the form $[\mathrm{Tr}(F_2)]^3$. This is the simplest form of the so-called Green-Schwarz mechanism [3].

Since the pure gauge anomaly $\mathrm{Tr}_a(T_2^6)$ is not of this factorized form, this does not seem to help us. However, for $SO(N)$ groups there exist relations between invariants in different representations, and in particular

$$\mathrm{Tr}_a(t^6) = (N-32)\mathrm{Tr}_V(t^6) + 15\mathrm{Tr}_V(t^2)\mathrm{Tr}_V(t^4) \tag{1.89}$$

for any linear combination t of group generators. Thus precisely for $SO(32)$, we can cancel the anomaly with the Green-Schwarz mechanism. Similarly, for the group E_8, we have $\mathrm{Tr}_a(t^6) = [\mathrm{Tr}_a(t^2)]^3/7200$, and holds for $E_8 \times E_8$ as well. (For a single E_8 factor with only 248 generators, we cannot cancel the second term in (1.85) proportional to $\mathrm{tr}(R_2^6)$, which essentially involves the adjoint of $SO(10)$, and as such has no factorization under (1.89); consequently there is a gravitational anomaly for a single E_8.)

Returning now to the general anomaly, we generalize the Chern-Simons interaction (1.87) to

$$\int B_2 X_8(F_2, R_2), \tag{1.90}$$

and now include a Lorentz Chern-Simons term ω_{3L} (in addition to the Yang-Mills Chern Simons term we wrote earlier) in the field strength for B_2:

$$\tilde{H}_3 = dB_2 - c_Y\omega_{3Y} - c_L\omega_{3L} \tag{1.91}$$

where $\omega_{3L} = \omega_1 d\omega_1 + (2/3)\omega_1^3$, and ω_1 is the spin connection[2]. Then the leading

[2]This additional Chern-Simons term did not appear earlier in the low-energy action because it corresponds to a higher derivative correction.

order transformation law for B_2 becomes

$$\delta B_2 = c_V \mathrm{Tr}(\lambda dA_1) + c_L \mathrm{Tr}(\beta d\omega_1), \tag{1.92}$$

while $\delta A_1 = d\lambda$, and $\delta\omega_1 = d\beta$. With this generalization we can thus cancel an anomaly polynomial I_{12} of the form

$$[c_V \mathrm{Tr}(F_2^2) + c_L \mathrm{tr}(R_2^2)] X_8(F_2, R_2), \tag{1.93}$$

which is precisely the form of the third (and remaining) term in (1.85) for $c_V = 1 = -c_L$. The careful reader will note that because the gauge transformation law for B_2 was originally defined in terms of the vector representation, we are implicitly using the $SO(N)$ relation $\mathrm{Tr}_a(t^2) = (N-2)\mathrm{Tr}_V(t^2)$. Thus the normalization defined for δB_2 in the $E_8 \times E_8$ case (where there is no vector representation) is uniform with the $SO(32)$ definition. Finally, by using $Tr_a(t^4) = (N-8)Tr_V(t^4) + 3Tr_V(t^2)Tr_V(t^2)$ [again for $SO(N)$], in addition to the second and sixth order relations we have written, we can show that the first term in (1.85) cancels for $SO(32)$ as claimed, by expressing everything in terms of the vector representation. A similar argument holds for $E_8 \times E_8$.

To summarize, for the gauge groups $E_8 \times E_8$ and $SO(32)$, the irreducible part of the total anomaly polynomial vanishes, and the remaining factorizable part can be cancelled with a Green-Schwarz counterterm. These are precisely the gauge groups we claimed are singled out for the heterotic string theories by the requirements of modular invariance, and spacetime supersymmetry. The connection between these two ideas (modular invariance of the *worldsheet* and the absence of *spacetime* gauge and gravitational anomalies), as well as the direct verification of the presence of the Green-Schwarz counterterms in the one-loop string amplitudes was obtained in [22]. The presence of these counterterms in the type I theory, was of course earlier obtained by Green and Schwarz [3].

Let us note that the term 'Green-Schwarz anomaly cancellation mechanism' is in some sense a misnomer, because these "counterterms" are really already present in the string theory amplitudes (see below) without any additional input. They are so named because they are not part of canonical low-energy $\mathcal{N} = 1$, $D = 10$ supergravity plus super Yang-Mills system; in fact we might even take this as a hint that $\mathcal{N} = 1$ local supersymmetry in ten-dimensions *implies* string theory, since these theories are anomalous without the Green-Schwarz counterterms.

Before we leave this very technical section, let us consider this mechanism from the perspective of Feynman diagrams. Analogous to the triangle graph in four dimensions, the basic anomalous graph in ten-dimensions is the one-loop hexagon graph with chiral fermions in the loop coupled to six external currents. The Green-Schwarz "counterterm" then amounts to a cancelling *tree-level* graph involving the exchange of a $B_{\mu\nu}$ field between two vertices containing two and four fermions. The apparent paradox that a tree-level graph can cancel the anomalous variation from a loop diagram is resolved by either a careful study of the dilaton dependence in the vertices of the tree graph, or the observation that both graphs in string theory arise from the same topology but in different limits of moduli space.

1.6.2 Pseudo-anomalous U(1) symmetries

It is of obvious interest to ask what four-dimensional ramifications these ten-dimensional considerations might have. While a general discussion of geometric or CFT compactifications lies outside the scope of this introduction, it suffices to note for our purposes in the next chapter that many compactifications of heterotic string theory to four-dimensions which preserve an unbroken $\mathcal{N} = 1$

spacetime supersymmetry may also possess a $U(1)$ symmetry with *apparently* anomalous fermion content. In standard field theory this would imply that under a $U(1)$ gauge transformation the effective action is not invariant, but picks up an anomalous variation proportional to $\mathrm{Tr}(Q_{U(1)})F \wedge F$. But we expect that string theory, which is anomaly-free in ten-dimensions, has a four-dimensional remnant of the Green-Schwarz mechanism which cancels such an anomaly. Indeed this is the case, and furthermore is linked to the issue of the vacuum stability of string theory.

Let us consider a specific example due to Dine, Seiberg, and Witten [4] for the heterotic $SO(32)$ superstring compactified on a so-called Calabi-Yau manifold [23]. If we wish to compactify string theory on a manifold, *and* preserve precisely one spacetime supersymmetry, we are naturally led to these objects as follows [6]. An unbroken supersymmetry Q, is a conserved charge which annihilates the vacuum $|\Omega\rangle$, or equivalently satisfies $\langle\Omega|\{Q, U\}|\Omega\rangle = 0$ (using the the hermicity of Q) for all operators U. Since this automatically holds for all bosonic U by Lorentz invariance, we can restrict our attention to fermionic U. But then $\{Q, U\}$ is just the supersymmetric variation $\delta_Q U$, so in the classical limit we have $\delta_Q U = 0$ for all fermionic fields as the condition for an unbroken supersymmetry. In particular, the gravitino variation leads us to consider compact six-manifolds admitting a covariantly constant spinor field. Since the spin connection on a six-manifold is in general an $SO(6) \simeq SU(4)$ gauge field, the existence of a covariantly constant spinor implies that the *holonomy* (parallel transport of a field around a closed curve) group of the compact manifold is $SU(3)$. Manifolds with metrics of $SU(3)$ holonomy are equivalent to manifolds with Ricci-flat, Kähler metrics. Explicit examples of the latter are difficult to construct due to the Ricci-flat condition, but a crucial existence theorem conjectured by Calabi and proven by Yau states that any Kähler manifold with so-called vanishing first Chern class, admits a unique Ricci-flat metric, and so a metric of $SU(3)$ holonomy. Examples of these Calabi-Yau manifolds are easy to find, and the existence theorem does the rest. Finally, we note that all of the Hodge numbers $h^{r,s}$ (the complex dimension of the Dolbeault cohomology groups $H^{r,s}_{\bar{\partial}}$, which characterize the topological nontriviality of the manifold) for a six (real) dimensional Calabi-Yau manifold are determined from just $h^{1,1}$ and $h^{2,1}$.

Returning to the example, if we embed the $SU(3)$ spin connection of the Calabi-Yau manifold into the gauge group $SO(32)$, the latter is broken to $SU(3) \times SO(26) \times U(1)$. Recalling that the massless modes charged under the $SO(32)$ are the gauge fields and the gauginos (both in the 496 dimensional adjoint representation), we have the decomposition

$$(\mathbf{8},1)_0 + (1,\mathbf{325})_0 + (1,1)_0 + (\mathbf{3},\mathbf{26})_1 + (\mathbf{3},1)_{-2} + (\bar{\mathbf{3}},\mathbf{26})_{-1} + (\bar{\mathbf{3}},1)_2. \quad (1.94)$$

where the subscripts denote the $U(1)$ charges. Only the last four are charged under the $U(1)$ (which can be anomalous), so consider the massless four-dimensional states to which they give rise. Wave operators in ten-dimensions break into noncompact and internal pieces for scalars and fermions respectively as

$$\nabla_M\nabla^M = \partial_\mu\partial^\mu + \nabla_m\nabla^m \ , \ \Gamma_M\nabla^M = \Gamma_\mu\partial^\mu + \Gamma_m\nabla^m, \quad (1.95)$$

where $M = 0,\ldots 9$, $\mu = 0\ldots 3$ and $m = 4\ldots 9$. Thus massless fields in four dimensions correspond to the modes of the massless fields in ten-dimensions that are also zero modes of the internal operators $\nabla_m\nabla^m$ or $\Gamma^m\nabla^m$, the number of which are determined topologically by the Hodge numbers of the Calabi-Yau manifold. Specifically, the last four states in (1.94) give rise to $h^{1,1}$ massless four-dimensional fields transforming as 26_{+1} under the surviving $SO(26) \times U(1)$, $h^{1,1}$ as 1_{-2}, $h^{2,1}$ as 26_{-1}, and $h^{2,1}$ as 1_{+2}. This applies to both the original gauge fields [which under $SO(8)$ spin $\times SO(32)$ transformed as $(\mathbf{8}_V, \mathbf{496})$], *and* their supersymmetric gaugino partners [transforming as $(\mathbf{8}_s, \mathbf{496})$]. Thus if $h^{2,1} \neq h^{1,1}$, the latter gives rise to a four-dimensional massless fermion spectrum is both chiral, and anomalous under the $U(1)$ [since $\mathrm{Tr}(Q_{U(1)}) \neq 0$].

As suggested, an (apparent) anomaly in the four-dimensional effective theory is cancelled by a four-dimensional remnant of the Green-Schwarz mechanism. Here is how it works. Recall that a Green-Schwarz counterterm could take the form (1.87): $B \wedge F \wedge F \wedge F \wedge F$. In compactifying to four-dimensions, we can take the six spacetime indices on three of the Fs to correspond to internal dimensions, so that in four dimensions this term gives rise to a term $B \wedge F$, with a proportionality factor $\int_K F \wedge F \wedge F$ (K is the compact manifold) that can be computed in string theory [25]. Recalling the form of the gauge transformation of B, we see that this term could therefore cancel an anomalous variation of the effective action proportional to $F \wedge F$.

On the other hand, the components of $B_{\mu\nu}$ with indices tangent to the four noncompact spacetime dimensions always yield a pseudoscalar called the (model-independent) axion a, via the (differential forms) dualization $\partial_\mu a \sim \epsilon_{\mu\nu\lambda\tau}\partial^\nu B^{\lambda\tau}$. This combines with the dilaton ϕ to form a scalar component of a chiral superfield $S = \phi + ia$, called the universal superfield. The kinetic terms for $B_{\mu\nu}$ and ϕ in (1.83) imply (see chapter 2, or [5]) that the Kähler potential for S at string tree level is $-\log(S + S^*)$. But now the $U(1)_A$ gauge transformation assigned to B to cancel an anomaly proportional to δ_{gs}, under which $S \rightarrow S + 2i\delta_{gs}\Lambda$, implies that at one-loop the Kähler potential is modified to $K_S = -\log(S + S^* - 4\delta_{gs}V)$ in order to maintain gauge invariance (V is the vector superfield for the pseudoanomalous $U(1)_A$ gauge symmetry).

In addition to generating potentials for the dilaton and the scalars charged under the $U(1)_A$ symmetry, an important consequence of this modified Kähler potential is the generation of a Fayet-Ilopoulos [24] D-term at one-loop in the string expansion corresponding to the $U(1)_A$ and proportional to $\mathrm{Tr}(Q_{U(1)})$, which potentially breaks the spacetime supersymmetry and destabilizes the vacuum by allowing tachyons. While nonrenormalization theorems protect the tree level superpotential (so that one might expect the perturbative stability of supersymmetric vacua), this is an important potential exception. Fortunately, the full D-term includes contributions from the charged fields, which in the known cases can be assigned vacuum expectation values to cancel the Fayet-Iliopoulos term. The net effect is to restore supersymmetry (albeit at a shifted vacuum), and spontaneously break the $U(1)_A$ pseudoanomalous symmetry.

This brings us directly to the topic studied in chapter 2: when we spontaneously break a $U(1)$ symmetry, we generally induce topologically stable solutions called Nielsen-Olesen vortices [28]. In the stringy framework described here, such potential vortices are also coupled to the axion and the dilaton. While the role of the axion had been previously emphasized, the dilaton was frozen to its asymptotic vacuum expectation value, which explicitly breaks the supersymmetry we sought to restore. The study of the full system is the subject of chapter 2, to which the reader may now turn.

Finally, let us note that the above low-energy field-theoretical arguments can be explicitly confirmed by one-loop string theory calculations; in particular the explicit generation of the Fayet-Iliopoulos term (and its finite Green-Schwarz coefficient), and some of the one-loop induced masses were calculated in [25], [26] and [27].

1.7 Spacetime noncommutativity from strings

Thus far our discussion of string theory has been entirely within perturbation theory: small numbers of weakly coupled strings. This is related to the fact that our starting point, the Polyakov path integral, involved the quantization of the coordinates of a string (first quantization) and not fields of strings. Furthermore, interactions emerged naturally from the sum over topologies where no special interaction points occurred. This is completely dissimilar to point particle field theory, which if developed in a first quantized formalism as a sum over particle histories, requires the *ad hoc* introduction of special interaction points or vertices, constrained by unitarity.

On the other hand, quantum field theory (as a path integral sum of field configurations) contains many phenomena which are intrinsically nonperturbative, and which play a central role in our understanding of the vacuum. While it is possible to develop a string *field* theory for bosonic open strings, this approach has not yet been particularly fruitful in yielding new insights into string theory beyond perturbation theory. Nevertheless, much progress has been made in recent years towards understanding what kinds of nonperturbative phenomena occur in string theory (see [5]) from the study of string dualities, which relate different string theories and different vacua to each other. While we do not have the space to examine these vast developments, for our purposes, we note that at the center of this nonperturbative understanding are the so-called Dirichlet branes (or D-branes for short) [29], [5]: nonperturbative dynamical objects on which open strings can end. Furthermore, when we turn on a constant, background $B_{\mu\nu}$ field on a D-brane, string theory naturally predicts spacetime noncommutativity on the brane [30]. In this section we briefly develop this construction in order to motivate our work in chapters 3 and 4.

1.7.1 D-branes via T-duality

The emergence of D-branes in string theory can fortunately be understood via a symmetry called T-duality, which is visible in perturbation theory. This is the approach we follow, our goal merely to see how these objects necessarily appear.

First however, consider an ordinary free massless scalar ϕ in D dimensions compactified on a circle of radius R along the direction corresponding to x^d ($d = D - 1$). We can then freely expand the dependence of ϕ on x^d as

$$\phi(x^M) = \sum_n \phi_n(x^\mu) \exp(inx^d/R), \tag{1.96}$$

where $M = 0 \dots d$, $\mu = 0 \dots d-1$. The momentum in the compact direction is therefore quantized as $p_d = n/R$, and the wave equation for ϕ, $\partial^M \partial_M \phi = 0$, becomes an infinite set of equations for the modes ϕ_n:

$$\partial^\mu \partial_\mu \phi_n(x^\mu) = \frac{n^2}{R^2} \phi_n(x^\mu). \tag{1.97}$$

This is the simplest example of a *Kaluza-Klein* tower of states: for energies small compared to R^{-1}, physics is $D-1$ dimensional and we only see the ϕ_0 mode, whereas for energies above R^{-1}, the effects of the massive modes become visible.

Now consider bosonic string theory in $D = 26$, compactified on a circle along X^{25}, so that we identify $X^{25} \simeq X^{25} + 2\pi nR$, $n \in \mathbb{Z}$. This has two effects. First, since the operator $\exp(2\pi i R p_{25})$ which translates string states around the compact dimension must leave states invariant, the center of mass momentum is again quantized: $p_{25} = n/R$, $n \in \mathbb{Z}$. The second, intrinsically stringy effect, corresponds to a closed string *winding* around the compact dimension:

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R w \ , \ w \in \mathbb{Z}. \tag{1.98}$$

Otherwise the world-sheet action, and hence the OPEs and worldsheet energy-momentum tensor, are unchanged by these worldsheet solitons.

Now recall the mode expansion for the closed string (1.42):

$$X^\mu(\sigma,\tau) = x^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau - \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + (\text{oscillators}). \qquad (1.99)$$

Under $\sigma \to \sigma + 2\pi$, X^{25} changes by $2\pi\sqrt{\alpha'/2}(\alpha_0^{25} - \tilde{\alpha}_0^{25})$, while the Noether momentum in the 25 direction is given by $p_{25} = (1/\sqrt{2\alpha'})(\alpha_0^{25} + \tilde{\alpha}_0^{25})$, so that (1.98) and Kaluza-Klein momentum quantization respectively imply

$$\alpha_0^{25} - \tilde{\alpha}_0^{25} = wR\sqrt{\frac{2}{\alpha'}} \quad , \quad \alpha_0^{25} + \tilde{\alpha}_0^{25} = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}}. \qquad (1.100)$$

Inverting this we have

$$p_{25,L} \equiv \sqrt{\frac{2}{\alpha'}}\alpha_0^{25} = \frac{n}{R} + \frac{wR}{\alpha'} \quad , \quad p_{25,R} \equiv \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^{25} = \frac{n}{R} - \frac{wR}{\alpha'}. \qquad (1.101)$$

Recalling our discussion of string state masses below (1.53), and the state-operator correspondence, the mass spectrum is now given by

$$\begin{aligned} m^2 &= \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) = \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\tilde{N}-1) \\ &= \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \end{aligned} \qquad (1.102)$$

where N and $\tilde{N}$ represent the total number of left and right moving oscillator excitations. (This also implies $\tilde{N} - N = nw$.)

There are two limits of interest: $R \to \infty$ and $R \to 0$. In the former, corresponding to the decompactification of X^{25}, the winding states become infinitely massive while the compact momentum spectrum approaches a continuum. In the latter, the compact momentum states become very massive, but now the winding number states become light and approach a continuum, because it costs little energy to wrap a string around a small circle. The surprise is that these two limits are identical, since the spectrum is invariant under

$$R \to R' = \frac{\alpha'}{R} \quad , \quad n \leftrightarrow w, \qquad (1.103)$$

which takes $p_L^{25} \to p_L^{25}$, and $p_R^{25} \to -p_R^{25}$. This last fact, combined with the harmonic separation $X^{25}(z,\bar{z}) = X_L^{25}(z) + X_R^{25}(\bar{z})$, suggests the definition of a *dual* coordinate

$$X'^{25}(z,\bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z}), \tag{1.104}$$

which has the same OPEs and energy-momentum tensor as X^{25} (because the sign change always occurs in pairs in these objects). The only change to the CFT induced by this rewriting, is the spectrum of the dual coordinate is now that of the $R' = \alpha'/R$ theory.

This equivalence is called T-duality and implies, at least in string perturbation theory, that the minimum length scale is set by $\sqrt{\alpha'}$. Putative physics at smaller scales always has a dual description at a larger distance scale. This is obviously completely unlike point particle physics, and is tied to the existence of the winding modes admitted by an extended object such as the string.

Now let us consider this discussion in the context of open strings. Recalling our mode expansion (1.46) which couples left and right moving modes by the Neumann boundary conditions at the endpoints, we again can separate $X^{25}(z,\bar{z})$ into its holomorphic and antiholomorphic components:

$$\begin{aligned} X^{25}(z) &= \frac{x^{25}}{2} + C - i\alpha' \log(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^{25}}{m z^m} \\ X^{25}(\bar{z}) &= \frac{x^{25}}{2} - C - i\alpha' \log(\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^{25}}{m \bar{z}^m} \end{aligned} \tag{1.105}$$

for a constant C. As above, consider the theory written in terms of the dual coordinate $X'^{25} = X^{25}(z) - X^{25}(\bar{z})$:

$$\begin{aligned} X'^{25}(z,\bar{z}) &= 2C - i\alpha' p^{25} \log\left(\frac{z}{\bar{z}}\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^{25}}{m}\left(z^{-m} - \bar{z}^{-m}\right) \\ &= 2C - 2\alpha' p^{25}\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^{25}}{m} \mathrm{e}^{-m\tau}\left(\mathrm{e}^{im\sigma} - \mathrm{e}^{-im\sigma}\right). \end{aligned} \tag{1.106}$$

Recalling that the endpoints of the open string correspond to $\sigma = 0$ or π, (1.106) implies that they are τ-independent in this direction; that is they cannot move along X'^{25}. The Neumann conditions in the X^{25} coordinate for the endpoints have become Dirichlet conditions in the dual X'^{25} coordinate, so in effect the open string endpoints are restricted to the 24-dimensional hypersurface.

We can easily extend these considerations to the toroidal compactification of $25 - p$ dimensions. $X^{25}, \ldots, X^{p+1}$, and, by adding Chan-Paton factors to the string endpoints, to the case of multiple D-branes. The string endpoints will then be restricted to a $(p+1)$-dimensional hypersurface, called a Dp-brane. As discussed earlier, since open strings necessarily couple to closed strings, and hence gravity, we expect that these objects are not rigid, but dynamical, fluctuating in shape and position. For example, massless states arise for non-winding open strings states with endpoints on the same Dp-brane as $\alpha^{\mu}_{-1}|k, ii\rangle$ and $\alpha^{m}_{-1}|k, ii\rangle$ ($|k, ij\rangle$ denotes an open string state with momentum k, and Chan-Paton quantum numbers i, j), which correspond respectively to a $U(1)$ gauge-field tangent to the hyperplane and $25 - p$ scalars corresponding to the transverse fluctuations of the brane. When N Dp-branes coincide, this gauge symmetry is enhanced to $U(N)$.

1.7.2 D-branes in background B-fields

Following the approach of Seiberg and Witten [30], we now examine the spacetime effects of a constant $B_{\mu\nu}$ NS-NS background on a Dp-brane in flat space. We will now denote flat *spacetime* metric by $g_{\mu\nu}$ to free up the symbol G; since we work in a flat worldsheet gauge there is no ambiguity relative to earlier notation, where g denoted the worldsheet metric.

Consider the usual bosonic worldsheet action (in a flat worldsheet gauge), with a constant $B_{\mu\nu}$ of full rank with respect to the Dp-brane[3], as in (1.78):

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_M \left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - 2\pi i\alpha'\epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right) \\ &= \frac{1}{4\pi\alpha'}\int_M g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - \frac{i}{2}\int_{\partial M} B_{\mu\nu}X^\mu \partial_t X^\nu, \end{aligned} \qquad (1.107)$$

after using the divergence theorem, and where we have (essentially) taken out a factor of α' out of $B_{\mu\nu}$ for later convenience. Here ∂_t denotes the tangent derivative to the worldsheet boundary ∂M. The boundary conditions the equations of motion imply for μ along the D-brane are

$$g_{\mu\nu}\partial_n X^\nu + 2\pi i\alpha' B_{\mu\nu}\partial_t X^\nu = 0 \text{ on } \partial M, \qquad (1.108)$$

with ∂_n the normal derivative to ∂M. For $B = 0$, these are the usual Neumann conditions for an open string, whereas in the limit $B \to \infty$ (or $g \to 0$) for the spatial directions along the brane, these conditions become Dirichlet. In this limit, each boundary of the worldsheet can be thought of as being attached to a D0-brane within the Dp-brane.

Since we are considering open strings, the relevant worldsheet at the classical level is the disk, which can be conformally mapped to the upper-half plane with the real axis as the worldsheet boundary. Then for the 'z'-coordinate of this description, $z = x + iy$, for which $\partial_t \propto \partial/\partial x$ and $\partial_n \propto -\partial/\partial y$ on $y = 0$, (1.108) becomes

$$g_{\mu\nu}(\partial - \bar{\partial})X^\nu + 2\pi\alpha' B_{\mu\nu}(\partial + \bar{\partial})X^\nu = 0 \text{ on } \mathrm{Im}(z) = 0. \qquad (1.109)$$

The problem of finding the two-dimensional Green's function or propagator sat-

[3]The components of B transverse to the D-brane can be gauged away [as below (1.78)], but in the presence of the D-brane they give rise to physical effects as we shall see.

isfying (1.109) was solved in [31]:

$$\begin{aligned}\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle &= -\alpha'\left[g^{\mu\nu}\log|z-z'| - g^{\mu\nu}\log|z-\bar{z}'| + \right.\\ &\left. + G^{\mu\nu}\log|z-\bar{z}'|^2 + \frac{\theta^{\mu\nu}}{2\pi\alpha'}\log\left(\frac{z-\bar{z}'}{\bar{z}-z'}\right)\right], \quad (1.110)\end{aligned}$$

where we take the branch cut of the fourth term in the lower half plane (for single-valuedness), and where

$$\begin{aligned}G^{\mu\nu} &= \left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}\\ G_{\mu\nu} &= g_{\mu\nu} - (2\pi\alpha')^2(Bg^{-1}B)_{\mu\nu}\\ \theta^{\mu\nu} &= -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}. \quad (1.111)\end{aligned}$$

For open strings, whose vertex operators are inserted on the worldsheet boundary, we restrict z and z' to the reals τ and τ', for which (1.110) reduces to

$$\langle X^\mu(\tau)X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau'), \quad (1.112)$$

where ϵ is the ± 1 step function. $G_{\mu\nu}$ is called the open string metric.

The surprise comes when we consider the consequences of the second term in (1.112), using the tools of conformal field theory we developed in section 1.3. Recall that the propagator (Green's function) determined the basic XX OPE, and the OPEs of operators built from Xs (including conserved currents). The OPE-algebra correspondence then determined the commutator algebra of operators when the operator ordering corresponded to time ordering. Thus, by interpreting τ as time, and displacing forward and backwards in time, (1.112) implies

$$[X^\mu(\tau), X^\nu(\tau)] = i\theta^{\mu\nu}. \quad (1.113)$$

Thus X^μ define a noncommutative spacetime. Note that the components $\theta^{\mu\nu}$ are constants since B and g are, and that the mass scale of this noncommutativity is set by $1/\sqrt{||\theta||}$. Furthermore, since the combinatorics of normal ordering defined by (1.18) are unchanged [(1.18) is modified only by the replacement of $(\alpha'/2)\log|z|^2$ by the new propagator (1.112)], the normal ordered product of two vertex operators $e^{ip\cdot X}(\tau)$ and $e^{iq\cdot X}(\tau')$ is now

$$: e^{ip\cdot X}(\tau) : \cdot : e^{iq\cdot X}(\tau') := (\tau - \tau')^{2\alpha' G^{\mu\nu} p_\mu q_\nu} e^{-\frac{i}{2}\theta^{\mu\nu} p_\mu q_\nu \epsilon(\tau)} : e^{i(p+q)\cdot X}(\tau') + \dots : \tag{1.114}$$

which is the analogue of (1.20) for the propagator (1.112).

It turns out particularly useful to consider the zero-slope ($\alpha' \to 0$) limit, taken in such a way that G and θ (the parameters to which open strings are sensitive) are kept fixed, rather than g and B (the closed string parameters). Specifically, we scale α' as $\epsilon^{1/2}$, and $g_{\mu\nu}$ as ϵ for (μ, ν along the brane), holding everything else fixed. In the limit $\epsilon \to 0$, where G and θ remain finite, several things happen. The boundary propagator further simplifies to $\langle X^\mu(\tau) X^\nu(0)\rangle = \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau)$, and the worldsheet theory for the X^μ corresponding to directions along the brane becomes topological, since the first term in (1.107) vanishes. Furthermore, the normal ordered product of the tachyon vertex operators now simplifies to

$$: e^{ip\cdot X}(\tau) : \cdot : e^{iq\cdot X}(\tau') := e^{-\frac{i}{2}\theta^{\mu\nu} p_\mu q_\nu \epsilon(\tau)} : e^{i(p+q)\cdot X}(\tau') + \dots : \tag{1.115}$$

which immediately generalizes, essentially as a consequence of our old friend (1.18), to the product of general functions

$$: f(x(\tau)) : \cdot : g(x(0)) :=: e^{\frac{i}{2}\epsilon(\tau)\theta^{\mu\nu}\frac{\partial}{\partial x^\mu(\tau)}\frac{\partial}{\partial x^\nu(0)}} f(x(\tau)) g(x(0)) : . \tag{1.116}$$

We may now take the (finite) $\tau \to 0$ coincidence limit to obtain finally

$$: f(x) : \cdot : g(x) :\equiv: f(x) * g(x) :=: e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial \xi^\nu}\frac{\partial}{\partial \zeta^\mu}} f(x+\xi) g(x+\zeta)|_{\xi=\zeta=0} : . \tag{1.117}$$

This is the so-called star product of functions, which we will see emerge directly from (1.113) in chapter 3, and is the basis for everything we do in chapters 3 and 4 of this thesis.

The $\alpha' \to 0$ limit just discussed is called the Seiberg-Witten scaling or decoupling limit [30]; so-called because we have decoupled the closed string modes (including gravity) from the open ones. But when we recall that an $\alpha' \to 0$ limit is really a *field-theoretical* limit, it suggests that we therefore study noncommutative field theories with the noncommutative structure defined by (1.113). Furthermore, if we take the notion of a noncommutative spacetime as a phenomenological possibility, it then becomes imperative to study the compatibility of such a structure with standard field-theoretical constructs as spontaneous symmetry breaking.[4]. This brings us to the topics we elucidate upon in chapters 3 and 4, to which the reader may now turn, having seen the stringy motivation for spacetime noncommutativity.

[4]In fact we will see in chapter 3 that there are other, more specific motivations coming from the study of simple noncommutative scalar field theories [32].

Bibliography

[1] P.C.W Davies & J. Brown, *Superstrings: A Theory of Everything?*, (Cambridge University Press, Cambridge, 1988).

[2] J. Polchinski, *String Duality - A Colloquium*, Rev. Mod. Phys **58**, 1245 (1996), hep-th/9607050.

[3] M.B. Green & J.H. Schwarz, *Anomaly cancellations in supersymmetric d=10 gauge theory and superstring theory*, Phys. Lett. **B149**, 117 (1984).

[4] M. Dine, N. Seiberg & E. Witten, *Fayet-Iliopoulos Terms in String Theory*, Nucl. Phys. **B289**, 589 (1987).

[5] J. Polchinski, *String Theory*, two volumes, (Cambridge University Press, Cambridge, 1998).

[6] M. Green, J.H. Schwarz & E. Witten, *Superstring Theory*, two volumes, (Cambridge University Press, Cambridge, 1987).

[7] D. Lüst & S. Theisen, *Lectures in String Theory*, (Springer-Verlag, Berlin, 1989).

[8] E. Kiritsis, *Introduction to Superstring Theory*, (Leuven University Press, 1998)

[9] M. Peskin, *Introduction to String and Superstring Theory. 2.*, TASI 86 Lectures, SLAC-PUB-4251 (1986).

[10] L. J. Dixon, *Introduction to Conformal Field Theory and String Theory*, TASI 89 Lectures, SLAC-PUB-5149 (1989).

[11] J. Polchinski, *What is String Theory?*, 1994 Les Houches Lectures, hep-th/9411028.

[12] F. Gliozzi, J. Scherk, & D. Olive, *Supersymmetry, Supergravity theories, and the dual spinor model*, Nucl. Phys. **B122**, 253 (1977).

[13] D. Gross, J. Harvey, E. Martinec & R. Rohm, *Heterotic string theory (I). The free heterotic string*, Nucl. Phys. **B256**, 253 (1985).
D. Gross, J. Harvey, E. Martinec & R. Rohm, *Heterotic string theory (II). The interacting heterotic string*, Nucl. Phys. **B267**, 75 (1986).

[14] J. Scherk, *Zero slope limit of the dual resonance model*, Nucl. Phys. **B31**, 222 (1971).

[15] A. Neveu & J. Scherk, *Connection between Yang-Mills fields and dual models*, Nucl. Phys. **B36**, 155 (1972).

[16] T. Yoneya, *Connection of dual models to electrodynamics and gravidynamics*, Prog. Theor. Phys. **51**, 1907 (1974).
J. Scherk & J.H. Schwarz, *Dual models for non-hadrons*, Nucl. Phys. **B81**, 118 (1974).

[17] M.B. Green, J.H. Schwarz & L. Brink, *N=4 Yang Mills and N=8 supergravity as the limit of string theories*, Nucl. Phys. *B198*, 474 (1982).

[18] N. Cai & C.A. Nuñez, *Heterotic string covariant amplitudes and low energy effective action*, Nucl. Phys. **B287**, 279 (1987).

D. Gross & J. Sloan. *The quartic effective action for the heterotic string*, Nucl. Phys. **B291**, 41 (1987).

[19] L. Alvarez-Gaumé, D.Z. Freedman & S. Mukhi, *The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model*, Ann. Phys. **134**, 85 (1981).
D. Friedan, *Nonlinear sigma models in* $2+\epsilon$ *dimensions*, Ann. Phys. **163**, 318 (1985).

[20] C. Lovelace, *Strings in curved space*, Phys. Lett. **B135**, 75 (1984).
E.S. Fradkin & A. A. Tseytlin, *Quantum string theory effective actions*, Nucl. Phys. **B261**, 1 (1985).
C. Callan, D. Friedan, E. J. Matinec, & M. J. Perry, *Strings in Background Fields*, Nucl. Phys. **B262**, 593 (1985).
A. Sen, *Equations of motion for the heterotic string theory from the conformal invariance of the sigma model*, Phys. Rev. Lett. **55**, 1846 (1985).

[21] G.F. Chapline & N.S. Manton, *Unification of Yang-Mills theory and supergravity in ten dimensions*, Phys. Lett. **120B**, 105 (1983).

[22] W. Lerche, B.E. Nilsson and A.N. Schellekens, *Heterotic String Loop Calculation Of The Anomaly Cancelling Term*, Nucl. Phys. **B289**, 609 (1987).

[23] P. Candelas, G. Horowitz, A. Strominger & E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. **B258**, 46 (1985).

[24] P. Fayet & J. Iliopoulos, *Spontaneously broken supergauge symmetries and Goldstone spinors*, Phys. Lett. **B51**, 461 (1974).

[25] J. Atick, L. Dixon & A. Sen, *String Calculation of Fayet-Iliopoulos D-terms in arbitrary supersymmetric compactifications*, Nucl. Phys. **B292**, 109 (1987).

[26] M. Dine, I. Ichinose & N. Seiberg, *F terms and D terms in string theory*, Nucl. Phys. **B293**, 253 (1987).

[27] M. Dine & C. Lee, *Fermion masses and Fayet-Iliopoulos terms in string theory*, Nucl. Phys. **B336**, 317 (1990).

[28] H.B. Nielsen & P. Olesen, *Vortex line models for dual strings*, Nucl. Phys. **B61**, 45 (1973).

[29] J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** 4724 (1995). hep-th/9510017.
S. Chaudhuri, C. Johnson & J. Polchinski, *Notes on D-branes*, (1996). hep-th/9602052 .
J. Polchinski, *TASI lectures on D-branes*, (1996). hep-th/9611050.

[30] V. Schomerus, *D-Branes and deformation quantization*, JHEP **9906**, 030 (1999). hep-th/9903205. N. Seiberg & E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999). hep-th/9908142.

[31] A. Abouelsaood, C.G. Callan, C.R. Nappi & S.A. Yost, *Open strings in background gauge fields*, Nucl. Phys. **B280**, 599 (1987).
C.G. Callan, C. Lovelace, C.R. Nappi & S.A. Yost, *String Loop corrections to beta functions*, Nucl. Phys. **288**, 525 (1987).

[32] S. Minwalla, M. Van Raamsdonk & N.Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002**, 020 (2000). hep-th/9912072.

Chapter 2

Anomalous U(1) vortices and the dilaton

2.1 Introduction

Many four-dimensional compactifications of superstring theory [1, 2, 3] which preserve an unbroken N=1 spacetime supersymmetry also possess a U(1) gauge symmetry with apparently anomalous content for the massless fermions of the associated gauge charge. The apparent anomalies of these U(1) gauge groups are cancelled by a four-dimensional remnant of the Green-Schwarz mechanism [4], as originally argued by Dine, Seiberg, and Witten [5, 6, 7].

These authors noted that while the superpotential is not renormalized in either string or sigma model perturbation theory (so that solutions of the string equations at lowest order remain solutions to all orders and the vacuum remains perturbatively stable), vacuum degeneracy can still be lifted if a compactification contains a gauge group with an unbroken U(1) subgroup, by generating a Fayet-Iliopoulos [8] D-term. By assumption such a term is not present at tree level in the loop or sigma-model expansion, so the question arises as to whether it is possible to generate it radiatively in perturbation theory. It turns out that it can arise only at one loop in the string-loop expansion, and then only if the U(1) is anomalous (since the term is proportional to the trace over the U(1) charges of the left-handed massless fermions [6]).

In fact many string compactifications have precisely such an anomalous U(1), with an explicit example being furnished by Dine, Seiberg and Witten for the SO(32) heterotic string. They argue that the anomalies induced by such a U(1) are cancelled by assigning the model-independent axion a nontrivial U(1) gauge variation, corresponding to the remnant of the underlying ten-dimensional Green-Schwarz anomaly cancellation mechanism. Supersymmetrically, the model independent axion is paired with the dilaton [whose vacuum expectation value (VEV) sets the string-loop coupling constant] to form the scalar component of a chiral multiplet, whose modified (due to the anomaly cancellation and gauge invariance) Kahler potential now yields the Fayet-Iliopoulos term. The effect of this induced Fayet-Iliopoulos D-term, generically, is to break spacetime supersymmetry as a one-loop effect in the string loop expansion. However, the full D-term also includes contributions from charged scalars in the theory. In the known cases some of these scalars can acquire VEVs to cancel the Fayet-Iliopoulos D-term thereby restoring supersymmetry by spontaneously breaking the U(1) symmetry in a process referred to as vacuum restabilization.

It has recently been argued that in heterotic $E_8 \times E_8$ [as opposed to heterotic $SO(32)$] compactifications, that the axion involved in the anomaly cancellation is a model-dependent axion originating from internal modes of the Kalb-Ramond form field B_{ij}, with $i, j = 4 \ldots 9$. (The essence of this argument dates back to Distler and Greene [9].) Such axionic modes appear paired with an internal

Kahler form zero mode to form the scalar components of complex moduli T_i, which describe the size and shape of the compactification manifold. However as Dine, Seiberg, and Witten had noted [5], if we assign one of the model-dependent axions a nontrivial gauge transformation to cancel the anomaly, and then proceed as in the model-independent case, we again get mass and tadpole terms that now appear at string *tree* level because there is no longer the dilaton (and hence string-loop) dependence that occurs in the model-independent case. These terms are by assumption absent in the classical, massless limit of string theory. The other way of saying this [9] is that the U(1) is not a symmetry of the world-sheet construction, and hence is not a symmetry of the low-energy effective theory describing the (classical) string vacuum. Furthermore there is no Fayet-Iliopoulos term generated in this case, so spacetime supersymmetry is not spontaneously broken and the vacuum destabilized. Thus, henceforth, we will work within the usual framework of Dine, Seiberg, and Witten [5] and consider anomaly cancellation via the dilaton/model-independent axion, or S multiplet.

On the other hand, it is well known that the breaking of a U(1) symmetry can give rise to topological defects known as Nielsen-Olesen vortices [10], which may appear in a cosmological context as cosmic strings [11]. Binétruy, Deffayet, and Peter [12] analyzed the vortices arising from such anomalous U(1) scenarios and concluded that there exist configurations of the axion such that some of these vortices can be local gauge strings, whereas, for other choices of the axion configuration the vortices are global [11]. However, in order to arrive at their final model, they freeze the dilaton to its (asymptotic) VEV while leaving the axion dynamical. Since the dilaton and model-independent axion form the scalar component of a chiral superfield, this ansätz explicitly breaks supersymmetry as they acknowledge. Since vacuum restabilization perturbatively restores supersymmetry in the resulting low-energy effective theory, an analysis of the vortex solutions of this effective theory should retain the fields required by the supersymmetry. In this chapter we present such an analysis, and examine the structure of the anomalous U(1) vortex including the dilaton as a dynamical field.

In order to treat the dilaton, axion, and anomaly in a systematic way, we show that the anomaly can be treated in the low-energy effective Lagrangian, and in the field equations, as a perturbation about the Abelian Higgs model and Nielsen-Olesen equations respectively. The dimensionless Green-Schwarz coefficient δ_{gs} will be considered as the perturbation parameter; in the simplified model of [12], wherein a single scalar accomplishes the vacuum restabilization, supersymmetry restoration, and U(1) breaking, this parameter is of order 10^{-3}. Then, looking for static, axially symmetric (vortex) solutions of the field equations using the standard ansatz for the Higgs (scalar) and gauge fields, we show that the axion is only θ dependent (as [12] obtain), and the dilaton is only r dependent given the assumed time-independent, cylindrical symmetry of the fields. The axion field equation effectively decouples (we still obtain the asymptotically converging solution of [12] for the axion, plus the others corresponding to global axionic strings), and we obtain ordinary differential equations for the dilaton, Higgs modulus, and the nontrivial component of the gauge field.

Corrections to a constant dilaton appear only at $O(\delta_{gs})$; at zeroth order we simply obtain the usual Nielsen-Olesen equations for the Higgs and gauge field. Using a parametrization for the solutions to the Nielsen-Olesen equations correct at the asymptotic limits $r \to \infty$, and $r \to 0$, we obtain the first order correction to the dilaton. We find that the correction necessarily diverges logarithmically to positive infinity as $r \to 0$ as a direct consequence of the $r \to \infty$ boundary condition and the two-dimensional nature of the problem. We also show this is not an artifact of the parametrization of the Nielsen-Olesen solutions, but is only dependent on these asymptotic regimes. This divergence reflects a transition to a (heterotic) strong-coupling regime and hence a failure of the effective theory as a classical limit (since the large dilaton field means large quantum effects). Finally, we check the consistency of this result outside of δ_{gs} perturbation theory by examining an exact solution to the large-dilaton limit of the full dilaton field equation, which involves exponential dilaton self-couplings, and the axion contribution, neither of which is visible in the first order δ_{gs} perturbation theory.

2.2 The model Lagrangian

In this section we will construct the effective field-theoretic action that will be the basis for the rest of this chapter. We will only consider the model-independent framework (and hence the model-independent axion), in which the details of the compactification from ten to four dimensions are not important.

The four-dimensional low energy limit of an N=1 supersymmetric compactification of heterotic string theory is an N=1 supersymmetric field theory for the massless fields. In ten dimensions, the massless fields in the pure supergravity sector are the dilaton Φ, the antisymmetric tensor field B_{MN}, the graviton g_{MN}, and their fermionic superpartners, while in the Yang-Mills sector we have the massless gauge fields of $E_8 \times E_8$ or $SO(32)$, and their gaugino superpartners. Independently of the compactification scheme to four dimensions, the antisymmetric tensor field B_{MN} yields via dualization the universal or model-independent axion a, which combines with the four-dimensional dilaton to form the scalar component of a chiral superfield denoted by S. Starting from the bosonic ten dimensional effective action for the heterotic string [2],

$$S_{het} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,(-G)^{\frac{1}{2}}e^{-2\Phi}\left[R + 4\,\partial_M\Phi\partial^M\Phi - \frac{1}{2}|\tilde{H}_3|^2 - \frac{\alpha'}{4}Tr_V(|F_2|^2)\right], \tag{2.1}$$

[with $\tilde{H}_3 = dB_2 - \frac{\alpha'}{4}\mathrm{Tr}_V(A_1 \wedge dA_1 + 2A_1 \wedge A_1 \wedge A_1/3)$, and where subscripts denote the rank of the associated tensor, so that for example F_2 is the field strength of the gauge fields A_1, etc.], dimensional reduction to four dimensions, Weyl transformation from the string to Einstein metric, and Poincaré duality yield [2]:

$$S_{4D,het} = \int d^4x\,(-G_{(4)})^{\frac{1}{2}}\left\{\frac{1}{2\kappa_4^2}\left[R^{(4)} - 2\frac{\partial_\mu S\partial^\mu S^*}{(S+S^*)^2}\right] - \frac{1}{4g_4^2}\left[e^{-2\Phi_4}F^{a\mu\nu}F^a_{\mu\nu} - a\,F^{a\mu\nu}\tilde{F}^a_{\mu\nu}\right]\right\} + \ldots \tag{2.2}$$

where the ellipsis represents compactification-dependent terms involving for example the other T-like moduli of the orbifold or Calabi-Yau manifold, threshold corrections, and the scalars (matter fields) coming from the ten-dimensional gauge fields. Here, $g_4^2 = \kappa_4^2/\alpha'$, $S = e^{-2\Phi_4} + ia$, $\Phi_4 = \Phi_{10} + det(G_{MN})$, and a is the dualization of the Kalb-Ramond field strength [2], define the four-dimensional gauge coupling and the four-dimensional dilaton, and axion multiplet. Also, all gauge field references now refer to the surviving 4-dimensional gauge group. Thus we see that a in fact has the required axion-like coupling, and that the dilaton VEV sets the four-dimensional gauge coupling.

On the other hand the purely bosonic sector of the general supergravity form [1],[2],[3] is:

$$\frac{\mathcal{L}_{bos}}{(-G)^{\frac{1}{2}}} = \frac{1}{2\kappa^2}R - K_{,\bar{i}j}D_\mu\phi^{i*}D^\mu\phi^j - \frac{1}{4}Re(f_{ab}(\phi))F^a_{\mu\nu}F^{b\mu\nu} + \frac{1}{8}Im(f_{ab}(\phi))\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^b_{\rho\sigma} + \frac{1}{2}Re(f_{ab}(\phi))D^aD^b + \ldots \tag{2.3}$$

where $K(\phi,\phi^*)$ is the Kahler potential, f_{ab} is the gauge kinetic function, D^a is the supergravity D-term (included in anticipation of what is to come), and ...

represent omitted superpotential terms. Thus comparing (2.2) and (2.3) we see that the string action yields a supergravity action with a dilaton-axion Kahler potential given by $-\log(S+S^\dagger)$, and that the gauge kinetic function is simply given by $f_{ab} = \frac{\delta_{ab}}{g_4^2}S$.

Many compactifications of string theory possess gauge groups containing U(1) subgroups. Sometimes the quantum numbers of the massless fermions associated with such a compactification appear to lie in anomalous representations, and hence the U(1) is referred to as anomalous. As Dine, Seiberg, and Witten [5] showed, the Green-Schwarz mechanism of the underlying string theories (which ensures that the string theories themselves are anomaly free) has a four-dimensional remnant which cancels the would-be anomalies associated with U(1), thereby resolving the paradox. Specifically, a U(1) anomaly means that under a U(1) gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$ the effective action is not invariant, but picks up the usual anomalous variation:

$$\delta \mathcal{L}_{eff} = -\frac{1}{2}\delta_{gs}\lambda F^a_{\mu\nu}\tilde{F}^{a\mu\nu}, \tag{2.4}$$

where δ_{gs} is the anomaly coefficient (henceforth we work in the notation of [12]). Since this is of the form of the standard axion coupling term in (2.2), it is clear that this anomalous variation can be cancelled by assigning the axion a nontrivial U(1) variation: namely $a \to a + 2\delta_{gs}\lambda$. In terms of the dilaton/axion superfield S this reads

$$S \to S + 2i\delta_{gs}\Lambda, \tag{2.5}$$

where Λ, a chiral superfield, is the supersymmetric generalization of the gauge transformation parameter λ; i.e. the vector field A gets promoted to a vector superfield V, with gauge transformation $V \to V + i(\Lambda - \Lambda^\dagger)/2$. However, now the Kahler potential for the S is no longer gauge invariant, and must be modified to the gauge invariant form:

$$K = -\log(S + S^\dagger - 4\delta_{gs}V). \tag{2.6}$$

Among other terms this induces a one-loop (in the string loop expansion) Fayet-Iliopoulos term [8]. We can also now add the contributions coming from the (other) scalars charged under the U(1). Specializing to the anomalous U(1) sector of the theory, denoting the four-dimensional dilaton now by $\Phi_4 \to \Psi$, and the scalar (chiral) superfields by $\mathcal{A}_i$with charges X_i and scalar components Φ_i, we can write the effective Lagrangian of our model:

$$\mathcal{L} = \int d^4\theta \left[K(S, S^\dagger) + \mathcal{A}_i^\dagger e^{X_i V}\mathcal{A}_i\right] + \int d^2\theta \frac{1}{4} S W^\alpha W_\alpha + h.c. \tag{2.7}$$

with W^α the spinor (chiral) superfield associated with the field strength of V. While a superpotential for the $\mathcal{A}_i$ could be added, since it must be independent of the dilaton superfield S in perturbation theory, we neglect it for simplicity since we are primarily interested in dilaton and axion dynamics. Using the fact that $(S+S^\dagger - 4\delta_{gs}V)_{bos} = 2e^{-2\Psi} + 2\theta\sigma^\mu\bar{\theta}(2\delta_{gs}A_\mu - \partial_\mu a) + \theta^2\bar{\theta}^2(\partial^\mu\partial_\mu(e^{-2\Psi}/2) - 2\delta_{gs}D)$ in Wess-Zumino gauge, we can expand the Kahler potential term in component form to get:

$$\int d^4\theta K(S, S^\dagger) = -\left[\partial_\mu\Psi\partial^\mu\Psi + \frac{e^{4\Psi}}{4}(\partial^\mu a - 2\delta_{gs}A^\mu)^2 + e^{2\Psi}\delta_{gs}D + \ldots\right]. \tag{2.8}$$

Note that the kinetic terms agree with those found in equation (2.2) since $(\partial_\mu S\ \partial^\mu S^\dagger)/(S+S^\dagger)^2 = \partial^\mu\Psi\ \partial_\mu\Psi + (e^{4\Psi}/4)\ \partial_\mu a\partial^\mu a$. The last term is the Fayet-Iliopoulos term, and is explicitly dependent upon the dilaton. The coupling between the gauge field and the axion is the four dimensional remnant of the Green-Schwarz counterterm [4]. Next, we have:

$$\int d^4\theta\ \mathcal{A}_i^\dagger e^{X_i V}\mathcal{A}_i = -(D_\mu\Phi_i)^\dagger D^\mu\Phi_i - X_i\Phi_i^\dagger\Phi_i D + \ldots \tag{2.9}$$

where $D_\mu\Phi_i = (\partial_\mu - iX_iA_\mu)\Phi_i$. These are simply the minimal kinetic terms for the charged scalars, and the usual D-term contribution to their potential. The final term in (2.7), expanded as usual yields:

$$\frac{1}{4}\int d^2\theta\; SW^{\alpha}W_{\alpha} = -\frac{1}{4}e^{-2\Psi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}aF^{\mu\nu}\tilde{F}_{\mu\nu} + \frac{1}{2}e^{-2\Psi}D^2 + \ldots \qquad (2.10)$$

The F^2, and the $F\tilde{F}$ terms agree of course with (2.2) [or (2.3) with $f_{ab} = \delta_{ab}S$] by construction, and the D^2 term is the last supergravity term from (2.3) . Combining the terms from (2.8), (2.9), and (2.10) as in (2.7) we finally arrive at:

$$\begin{aligned} \mathcal{L}_{bos} &= -\partial_\mu\Psi\partial^\mu\Psi - \frac{e^{4\Psi}}{4}\left(\partial^\mu a - 2\delta_{gs}A^\mu\right)^2 - (D_\mu\Phi_i)^\dagger D^\mu\Phi_i \\ &\quad -\frac{1}{4}e^{-2\Psi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{4}aF^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{e^{2\Psi}}{2}\left(e^{2\Psi}\delta_{gs} + X_i\Phi_i^\dagger\Phi_i\right)^2, \qquad (2.11) \end{aligned}$$

where we have eliminated D by its algebraic equation of motion:

$$D = e^{2\Psi}\left(e^{2\Psi}\delta_{gs} + X_i\Phi_i^\dagger\Phi_i\right). \qquad (2.12)$$

Equation (2.11), with the Planck mass restored everywhere, (which we have implicitly suppressed by setting $\kappa_4 = \alpha' = 1$) and with s instead of $e^{-2\Psi}$ for the dilaton, agrees with the Lagrangian of reference [12]. Notice that throughout, we have been implicitly using the metric convention $(-,+,+,+)$, in accordance with [12], as is evident from the negative signs in front of some of the kinetic terms. This Lagrangian represents the bosonic part of the anomalous U(1) sector of the low energy action of heterotic string theory, and represents our starting point for the analysis of vortex solutions.

2.3 Perturbation scheme and field equations

In string theory the dilaton is the string loop expansion parameter, its vacuum expectation value setting the string coupling constant [1]. As is evident from (2.11), its four dimensional remnant in this model manifestly sets the U(1) *gauge* coupling: $\langle e^{\Psi}\rangle = g$. Since our main interest is in the dilaton, it will be convenient for our purposes to consider variations of the dilaton about its VEV. Thus define $\psi \equiv \Psi - \langle\Psi\rangle$ so that

$$e^{\Psi} \equiv g e^{\psi}. \tag{2.13}$$

We will henceforth refer to ψ as the dilaton. Then $\psi = 0 \leftrightarrow \langle \mathrm{Re}(S)\rangle = 1/g^2$. Inserting this into (2.11), restoring the Planck mass, and rescaling δ_{gs} and a by $1/g^2$ we have

$$\begin{aligned}\mathcal{L}_{eff} &= -M_p^2\partial_\mu\psi\partial^\mu\psi - (D_\mu\Phi_i)^\dagger D^\mu\Phi_i - \frac{e^{-2\psi}}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{a}{4g^2M_p}F_{\mu\nu}\tilde{F}^{\mu\nu} \\ &\quad -M_p^2e^{4\psi}\left(\frac{\partial^\mu a}{2M_p} - \delta_{gs}A^\mu\right)^2 - \frac{g^2e^{2\psi}}{2}\left(\delta_{gs}M_p^2e^{2\psi} + X_i\Phi_i^\dagger\Phi_i\right)^2 .\end{aligned} \tag{2.14}$$

This is invariant under local U(1) gauge transformations [with gauge parameter $\lambda(x^\mu)$] which now read

$$\Phi_i \to e^{iX_i\lambda}\Phi_i \quad , \quad A_\mu \to A_\mu + \partial_\mu\lambda \quad , \quad a \to a + 2M_p\delta_{gs}\lambda. \tag{2.15}$$

As discussed above, the gauge variation of the axion in the $F\tilde{F}$ term cancels the anomalous variation of the Lagrangian due to the (suppressed) fermions. In weakly coupled string theory, the anomaly coefficient δ_{gs} is calculated to be [5]

$$\delta_{gs} = \frac{1}{192\pi^2}\sum_i X_i, \tag{2.16}$$

where the sum is over the U(1) charges of the massless fermions and hence, by supersymmetry, over the charges of the massless bosons. In semi-realistic string models this sum may be large. A particular example furnished by the free-fermionic construction [13] yields $\mathrm{Tr}(Q_X) = 72/\sqrt{3}$, so that $\delta_{gs} \sim 10^{-2}$. Assuming without loss of generality that $\delta_{gs} > 0$, the presence of a single scalar with negative charge can minimize the potential in (2.14) (assuming we assign the other scalars zero VEVs), thereby cancelling the Fayet-Iliopoulos D-term, restoring supersymmetry, and spontaneously breaking the U(1) gauge symmetry. Thus, as in [12], we consider a single Higgs scalar Φ with negative unit charge, effectively ignoring quantum fluctuations of the other scalars about their zero VEVs, and working in the classical limit. This is consistent with ignoring the fermionic contributions.

Then (2.14) essentially becomes an Abelian Higgs model, coupled to the dilaton and axion through the anomaly, which may be viewed as a perturbation. To motivate this perspective, introduce a fictitious scaling parameter α so that

$$\delta_{gs} \to \alpha\delta_{gs}. \tag{2.17}$$

Then, as $\alpha \to 0$, the anomaly is turned off. In order for the spontaneously broken Abelian Higgs model to remain in this limit, the invariance of the term $\delta_{gs} M_p^2 e^{2\varphi}$ in the potential, and in turn the gauge transformation of the axion, imply respectively that M_p and a should scale as

$$M_p \to \alpha^{-\frac{1}{2}} M_p \quad , \quad a \to \alpha^{1/2} a. \tag{2.18}$$

Next we switch to dimensionless variables using the symmetry breaking scale defined by $\delta_{gs}^{1/2} M_p$ [1]

$$\hat{x}^\mu = g\delta_{gs}^{1/2} M_p x^\mu \quad , \quad \hat{\phi} = \frac{\phi}{\delta_{gs}^{1/2} M_p} \quad , \quad \hat{A}^\mu = \frac{A^\mu}{g\delta_{gs}^{1/2} M_p} \quad , \quad \hat{a} = \frac{a}{\delta_{gs} M_p}. \tag{2.19}$$

where we have written $\Phi = \phi e^{i\eta}$, so $(D_\mu\Phi)^\dagger D^\mu\Phi = \partial_\mu\phi\,\partial^\mu\phi + \phi^2\left(\partial_\mu\eta + A_\mu\right)^2$. By design, these dimensionless variables are α invariants as required for a consistent perturbation scheme. Effecting these transformations and dropping the hats, we

[1] As typically $\delta_{gs}^{1/2} < 10^{-1}$, the tension of our vortex solutions, which is set by the scale of the spontaneous U(1) breaking, is below the Planck scale, justifying our neglect of metric back reaction in our analysis of these solutions.

arrive at our final Lagrangian form:

$$
\begin{aligned}
\mathcal{L}'_{eff} &= \frac{-1}{\alpha\delta_{gs}}\partial_\mu\psi\partial^\mu\psi - \partial_\mu\phi\,\partial^\mu\phi - \phi^2(\partial_\mu\eta + A_\mu)^2 \\
&\quad -\frac{e^{-2\psi}}{4}F^{\mu\nu}F_{\mu\nu} - \frac{e^{2\psi}}{2}\left(\phi^2 - e^{2\psi}\right)^2 \\
&\quad +\alpha\delta_{gs}\left[\frac{a}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{e^{4\psi}}{4}\left(\partial^\mu a - 2A^\mu\right)^2\right], \qquad (2.20)
\end{aligned}
$$

where we have rescaled the overall Lagrangian by the factor $M_p^4 g^2 \delta_{gs}^2$. In the limit $\alpha\,\delta_{gs} \to 0$, we identically get the spontaneously broken Abelian Higgs model[2]. Thus, since only the combination $\alpha\delta_{gs}$ appears, setting $\alpha = 1$ (or relabelling $\beta = \alpha\delta_{gs}$), the only remaining parameter is δ_{gs} (or β) which is now to be interpreted as a perturbation parameter.

The field equations derived from (2.20) are

$$
\begin{aligned}
\Box\psi &= \frac{\beta}{2}\left[e^{2\psi}(3e^{2\psi} - \phi^2)(e^{2\psi} - \phi^2) - \frac{e^{-2\psi}}{2}F^{\mu\nu}F_{\mu\nu}\right] \\
&\quad +\frac{\beta^2}{2}e^{4\psi}(\partial^\mu a - 2A^\mu)^2 \qquad (2.21) \\
\Box\phi &= \phi(\partial_\mu\eta + A_\mu)^2 + e^{2\psi}\phi(\phi^2 - e^{2\psi}) \qquad (2.22) \\
0 &= \partial_\mu[\phi^2(\partial^\mu\eta + A^\mu)] \qquad (2.23) \\
\Box a &= 2\partial_\mu A^\mu - \frac{e^{-4\psi}}{2}F_{\mu\nu}\tilde{F}^{\mu\nu} - 4\partial_\mu\psi(\partial^\mu a - 2A^\mu) \qquad (2.24) \\
\partial_\mu(e^{-2\psi}F^{\mu\nu}) &= 2\phi^2(\partial^\nu\eta + A^\nu) + \beta\left[\partial_\mu(a\tilde{F}^{\mu\nu}) - e^{4\psi}(\partial^\nu a - 2A^\nu)\right]. \qquad (2.25)
\end{aligned}
$$

Despite the presence of the dynamical dilation, by differentiating (2.25) with respect to x^ν, and then using (2.23), (2.24), and $\partial_\mu\tilde{F}^{\mu\nu} = 0$, we obtain

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = 0. \qquad (2.26)$$

[2]As we will later show explicitly, in this limit, the dilaton $\psi \to \langle\psi\rangle \equiv 0$, so its gradients vanish identically and the kinetic term poses no problem.

Then, after choosing the Lorentz gauge $\partial_\mu A^\mu = 0$, the axion field equation (2.24) simplifies to

$$\Box a = -4\partial_\mu \psi(\partial^\mu a - 2A^\mu). \tag{2.27}$$

2.4 Vortex ODE's

It is well-known that the spontaneously broken Abelian Higgs model possesses topologically stable vortex solutions sometimes called Nielsen-Olesen vortices [10] (see Shellard and Vilenkin [11] for a complete reference on the subject). These correspond to static, cylindrically symmetrical solutions of the field equations for the Higgs and gauge fields. Specifically, working in cylindrical coordinates (t, r, θ, z) we look for solutions independent of t and z, with the standard vortex ansatz [10],[11] for the Higgs phase and the gauge field

$$\begin{aligned} \eta &= n\theta, \\ A^\mu &= (0, 0, A^\theta(r), 0) \equiv (0, 0, A(r), 0), \end{aligned} \tag{2.28}$$

where n is an integer characterizing the winding number of the vortex. The Higgs field $\Phi = \phi e^{i\eta} \to \langle\phi\rangle e^{i\eta}$ (as $r \to \infty$) defines a representation of the U(1) gauge group space S^1 since from (2.15), $\Phi \to e^{-i\lambda}\Phi$ under a gauge transformation. Thus Φ defines (as $r \to \infty$) a mapping from the boundary S^1 of physical space onto the group space S^1, and so can topologically be classified by an integer n. In the language of homotopy theory $\pi_1(S^1) = \mathbb{Z}$. With these ansätze, the Higgs phase field equation (2.23) can be written as

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}(\frac{n}{r} + A) = 0, \tag{2.29}$$

where we have used $\partial_\mu A^\mu = 0$, and the fact that $\eta = n\theta$ implies $\Box\eta = 0$. Then since in general $A(r) \neq -n/r$, we get

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \Rightarrow \quad \phi = \phi(r). \tag{2.30}$$

This is normally assumed as an ansatz, but this shows it actually follows from the Higgs phase field equation. Then (2.23) is identically satisfied with these forms of η, A, and ϕ. At this point we still have $a = a(r,\theta)$, and $\psi = \psi(r,\theta)$ assuming only static, axial symmetry. However, writing the Higgs modulus equation (2.22) as[3]

$$\begin{aligned} \Box\phi - \phi(\partial_\mu \eta + A_\mu)^2 &= \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \phi(r)\left[\frac{n}{r} + A(r)\right]^2 \\ &\equiv f(r) \\ &= e^{2\psi(r,\theta)}\phi(r)\left[\phi^2(r) - e^{2\psi(r,\theta)}\right] \end{aligned} \tag{2.31}$$

determines ψ algebraically as a function of r alone, so $\psi = \psi(r)$. Furthermore, consider the gauge field equation (2.25) for $\nu = r$, i.e. $\nu = 1$. Since $A^\mu = \delta^{2\mu}A(r)$, only F^{12} and $\tilde{F}^{03}$ are nonzero. Then (2.25) for $\nu = 1$ reads

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left[e^{-2\psi(r)}F^{21}(r)\right] \equiv 0 = 2\phi^2(0+0) + \beta\left[0 - e^{4\psi}(\frac{\partial a}{\partial r} - 0)\right] \quad \Rightarrow \quad \frac{\partial a}{\partial r} = 0, \tag{2.32}$$

so that $a = a(\theta)$. Now $\psi = \psi(r)$, $a = a(\theta)$, and $A = A(r)$ imply in the axion field equation (2.27) that

$$\partial_\mu \psi(\partial^\mu a - 2A^\mu) = 0 \quad \Rightarrow \quad \Box a = \frac{1}{r^2}\frac{d^2 a}{d\theta^2} = 0. \tag{2.33}$$

This fixes

$$a(\theta) = C\theta + D. \tag{2.34}$$

[3]Remember we are always working with metric signature $(-,+,+,+)$ so $\Box = -\frac{\partial^2}{\partial t^2} + \Delta$, etc.

Because a appears only derivatively coupled, we may take without loss of generality $D = 0$. Furthermore, single-valuedness in the physical space requires that C be an integer, so that a represents a mapping from physical space into the gauge group space just as η does (see [12]). The specific axion solution of Binétruy, Deffayet and Peter [12] corresponds to the choice $C = -2n$, where n is the winding number of the Higgs phase[4]. It has the property of rendering the energy-momentum tensor associated with the Lagrangian (2.11), but with a *fixed* dilaton, asymptotically finite. It is clear however, that we can also get the usual global-axionic strings with other choices of the integer C. We will consider the general case for the moment, leaving $C = -2m$ without loss of generality, with m not necessarily equal to n. Effectively this allows the axion and the Higgs phase to have different winding numbers. In a moment it will be clear why the local solution of [12] is special.

Combining what we have learned about the coordinate dependences of the fields, we can now reduce the remaining field equations (2.21), (2.22), and (2.25) to three ordinary differential equations:

$$\begin{aligned} \frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} &= \frac{\beta}{2}\left[e^{2\psi}(3e^{2\psi} - \phi^2)(e^{2\psi} - \phi^2) - e^{-2\psi}\left(\frac{1}{r}\frac{d}{dr}(rA)\right)^2\right] \\ &\quad + 2\beta^2 e^{4\psi}\left(\frac{m}{r} + A\right)^2, \end{aligned} \tag{2.35}$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \phi\left(\frac{n}{r} + A\right)^2 + e^{2\psi}\phi\left(\phi^2 - e^{2\psi}\right), \tag{2.36}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right] = 2\frac{d\psi}{dr}\frac{1}{r}\frac{d}{dr}(rA) + 2\phi^2 e^{2\psi}\left(\frac{n}{r} + A\right) + 2\beta e^{6\psi}\left(\frac{m}{r} + A\right), \tag{2.37}$$

where we have used the following:

$$F_{\mu\nu}F^{\mu\nu} = 2(\nabla \times \vec{A})^2 = 2\left[\frac{1}{r}\frac{d}{dr}(rA)\right]^2,$$

[4] $a = -2n\theta$ in the original variables reads $a = 2\delta_{gs} M_p \eta / X$

$$\begin{aligned}\partial_\mu F^{\mu 2} &= \nabla^2(\vec{A})\Big|_\theta = \nabla^2 A - A/r^2 = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right], \\ \partial_\mu \psi F^{\mu\nu} &= 2\nabla\psi\cdot(\nabla\times\vec{A}). \end{aligned} \tag{2.38}$$

As in the standard Nielsen-Olesen vortices [10],[11] of the Abelian Higgs model, we require that the Higgs modulus approach its vacuum expectation value asymptotically to minimize the potential term, and that the covariant derivative $D_\mu\Phi$ vanish asymptotically (i.e. the gauge field asymptotically becomes a pure gauge) so that the energy (per unit length) of the vortex remains finite. Translated into our language, these conditions read

$$\begin{aligned}\phi(r) &\rightarrow 1 \quad , \quad r\rightarrow\infty, \\ A(r) &\rightarrow \frac{-n}{r} \quad , \quad r\rightarrow\infty. \end{aligned} \tag{2.39}$$

The Higgs 'screening' by the gauge fields prevents the logarithmic divergence of global vortices, so that the terms involving $(\frac{n}{r}+A)^2$ (remnants of the covariant derivative $D_\mu\Phi$) are well behaved under the energy integral. To be precise the energy of the vortex contains the following term[5]

$$\int_0^\infty \left[(\frac{n}{r}+A)\phi\right]^2 rdr, \tag{2.40}$$

which may converge asymptotically if A has the behaviour (2.39), and necessarily diverges logarithmically otherwise. However, once we have fixed the asymptotic behaviour of the gauge field (with respect to the Higgs field), the presence of the axion reintroduces these logarithmic divergences *if* $m\neq n$, because now the $(\frac{m}{r}+A)^2$ term (from the axion kinetic term) in the energy integral is divergent . In the special case that $m=n$, corresponding to the Binétruy et. al result, the

[5] Incidentally, we require $\phi(0)=0$ in order for this integral to be well-behaved for small r, something we will not be able to do for the analogous axion term we are about to discuss. This is precisely the origin of the short-distance log divergence of Binétruy et. al [12].

axion contribution equals the Higgs phase contribution, and again the asymptotic logarithmic divergence coming from the axion kinetic term is avoided by the cancelling contribution of the axion-gauge field coupling. Now however, a short-distance divergence remains since the energy integral $\int(\frac{m}{r}+A)^2 rdr$ is ill-defined for $r \to 0$ (see footnote). Since our primary interest is now in the dilaton, for the remainder of our discussion we consider the $m = n$ case to simplify the equations slightly. We demonstrate in the next section that this will in no way affect any subsequent results.

Before proceeding we now make a convenient change of variables for the gauge field. Define $v(r)$ through

$$A(r) = \frac{-n\left[1 - v(r)\right]}{r}, \tag{2.41}$$

so that

$$v(r) \to 0 \quad , \quad r \to \infty. \tag{2.42}$$

The equations (2.35)-(2.37) now read, denoting r derivatives by primes:

$$\begin{aligned}
\psi'' + \frac{\psi'}{r} &= \frac{\beta}{2}\left[3e^{6\psi} - 4\phi^2 e^{4\psi} + \phi^4 e^{2\psi} - \frac{e^{-2\psi}n^2}{r^2}(v')^2\right] + 2\beta^2 e^{4\psi}\frac{n^2 v^2}{r^2} &(2.43)\\
\phi'' + \frac{\phi'}{r} &= \frac{n^2}{r^2}\phi v^2 + e^{2\psi}\phi(\phi^2 - e^{2\psi}), &(2.44)\\
v'' - \frac{v'}{r} &= 2\psi' v' + 2(\phi^2 e^{2\psi} + \beta e^{6\psi})v. &(2.45)
\end{aligned}$$

We require the dilaton to approach its asymptotic VEV as $r \to \infty$, which, in our language, means

$$\psi \to 0 \quad , \quad r \to \infty \quad (i.e. \langle Re(S)\rangle = \frac{1}{g^2}). \tag{2.46}$$

Now consider the boundary conditions at $r = 0$. In the standard Nielsen-Olesen/Abelian Higgs model [11], the vortex configuration means that ϕ attains the symmetric (false vacuum) state $\phi = 0$ at $r = 0$ (which we argued was necessary for the energy integral to be well-defined), and A remains bounded (more precisely the magnetic field remains bounded). Thus we have

$$\phi(0) = 0 \quad , \quad v(0) = 1. \tag{2.47}$$

This leaves, finally, the boundary condition for the dilaton at $r = 0$. Of course we would like to have the dilaton (VEV) remain bounded in the core, but as we shall now show, this is not possible if $\beta \neq 0$.

2.5 Perturbative expansion and corrections to the dilaton

Throughout this section we will make usage of the following elementary fact of our radial equations:

$$f'' + \frac{f'}{r} = 0 \quad \Rightarrow f(r) = C_1 + C_2 \log(r). \tag{2.48}$$

First, note that if $\beta = 0$, then the dilaton equation (2.43) becomes (2.48), so that the asymptotic condition (2.46) on the dilaton then implies

$$\psi_0(r) \equiv 0 \quad \forall r. \tag{2.49}$$

This of course corresponds to the frozen dilaton. Then the other two equations, (2.44) and (2.45), identically reduce to the Nielsen-Olesen equations of the Abelian Higgs model, as promised:

$$\phi_0'' + \frac{\phi_0'}{r} = \frac{n^2}{r^2}\phi_0 v_0^2 + \phi_0\left(\phi_0^2 - 1\right), \tag{2.50}$$

$$v_0'' - \frac{v_0'}{r} = 2\phi_0^2 v_0, \tag{2.51}$$

with $v_0(0) = 1$, $v_0(\infty) = 0$, $\phi_0(0) = 0$, $\phi_0(\infty) = 1$. We have subscripted the fields with zeros to indicate that these are the zeroth order terms in a perturbation expansion in β, which we now define formally in the obvious way:

$$\psi(r) = \sum_{i=0}^{\infty} \beta^i \psi_i(r) \ , \ \phi(r) = \sum_{i=0}^{\infty} \beta^i \phi_i(r) \ , \ v(r) = \sum_{i=0}^{\infty} \beta^i v_i(r). \tag{2.52}$$

Substituting these into (2.43)-(2.45) yields the following $O(\beta)$ corrections:

$$\psi_1'' + \frac{\psi_1'}{r} = \frac{1}{2}\left[3 - 4\phi_0^2 + \phi_0^4 - \frac{n^2}{r^2}(v_0')^2\right], \tag{2.53}$$

$$\phi_1'' + \frac{\phi_1'}{r} = \frac{n^2}{r^2}\left(\phi_1 v_0^2 + 2\phi_0 v_0 v_1\right) + 2\psi_1\left(\phi_0^3 - 2\phi_0\right) + \phi_1\left(3\phi_0^2 - 1\right), \tag{2.54}$$

$$v_1'' - \frac{v_1'}{r} = 2\psi_1' v_0' + 2v_0\left(2\phi_0\phi_1 + 2\phi_0^2\psi_1 + 1\right) + 2v_1\phi_0^2, \tag{2.55}$$

where we have included the corrections to the Higgs and gauge field for completeness. What really interests us is the first of these equations, (2.53), the first correction to the dilaton. Note that this $O(\beta)$ correction does *not* depend on having chosen the choice of Binétruy et. al. for the axion, since the axion does not enter at this order. This can be seen directly from (2.20) or (2.43). More importantly, this dilaton correction can be calculated from knowledge of only ϕ_0 and v_0; i.e. the Nielsen-Olesen solution for the Higgs and the gauge field.[6]

Unfortunately explicit solutions to the Nielsen-Olesen equations (2.50), (2.51) are not known. However, all we really need is a parametrization of the solutions with the correct behaviour at $r \to \infty$ and at $r \to 0$. The conclusions we will draw will depend only on the asymptotic behaviour of ϕ_0, v_0, *and* in particular the $r \to \infty$ boundary condition on ψ itself.

Thus, first consider the large r behaviour of the Nielsen-Olesen equations (2.50), (2.51). Write ϕ_0 and v_0 as $1 - \delta\phi_0$ and δv_0 respectively, where δ's represent

[6] In fact, it is obvious that the dilaton at any order is determined only by functions of lower order.

deviations with respect to asymptotic values. Then the linearizations of (2.50), (2.51) are

$$\delta\phi_0'' + \frac{\delta\phi_0'}{r} = 2\delta\phi_0 + O(\delta^2), \tag{2.56}$$

$$\delta v_0'' - \frac{\delta v_0'}{r} = 2\delta v_0 + O(\delta^2). \tag{2.57}$$

Note that as per Perivolaropoulos [14] (or Shellard and Vilenkin [11]), since we have the case '$\beta < 4$' (in their notation), we do not need to consider the inhomogeneous term $(\delta v_0)^2/r^2$ in the $\delta\phi_0$ equation, which can dominate a linear term of $O(\delta\phi_0)$ if $\beta > 4$. In this case, the gauge field dictates the falloff of the Higgs field. Our 'β' (not to be confused with the perturbation parameter) is 1, so this usual (strict) linearization applies. The solutions to these linearized equations, with the asymptotic boundary conditions, are in terms of modified Bessel functions:

$$\delta\phi_0 \to K_0(\sqrt{2}r) \to C_\phi \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \quad , \quad r \to \infty, \tag{2.58}$$

$$\delta v_0 \to K_1(\sqrt{2}r) \to C_v \sqrt{r} e^{-\sqrt{2}r} \quad , \quad r \to \infty, \tag{2.59}$$

where C_ϕ, and C_v are constants of order 1. As Perivolaropolous [14] notes, the factor of $1/\sqrt{r}$ is usually neglected in (2.58). We will neglect these $\sqrt{r}$ terms as being negligible with respect to the exponentials when parametrizing a solution of the Nielsen-Olesen equations over the whole range, and later show that this does not affect our results.

Now consider the small r behaviour, this time taking ϕ_0 as $\delta\phi_0$. With $v_0(r \ll 1) \approx 1$ the leading order behaviour of equation (2.50) at small r is

$$\delta\phi_0'' + \frac{\delta\phi_0'}{r} = \frac{n^2\delta\phi_0}{r^2} \quad \Rightarrow \quad \delta\phi_0 = Ar^n \quad , \quad r \ll 1, \tag{2.60}$$

where $A > 0$ (to be determined conveniently in a moment), and where we have discarded the second singular solution. At this point we specialize to the $n = \pm 1$

vortex for simplicity. Then the small r gauge field equation is

$$v_0'' - \frac{\delta v_0'}{r} = 2(\delta\phi_0)^2 v_0 = 2A^2 r^2 v_0, \tag{2.61}$$

with solution

$$v_0 = e^{-Ar^2/\sqrt{2}} \sim 1 - \frac{A}{\sqrt{2}} r^2 + O(r^4) \quad , \quad r \ll 1, \tag{2.62}$$

where again we have discarded the second solution (a positive exponential), which has the wrong behaviour near $r = 0$, and used $v_0(0) = 1$. Combining (2.58), (2.59), (2.60), and (2.62) suggests the following parametrizations of the solutions to the Nielsen-Olesen equations:

$$\phi_0(r) \sim \tanh\left(\frac{r}{\sqrt{2}}\right), \tag{2.63}$$

$$v_0(r) \sim \mathrm{sech}^2\left(\frac{r}{\sqrt{2}}\right), \tag{2.64}$$

which corresponds to setting $A = 1/\sqrt{2}$. They have the following asymptotic behaviour:

$$\phi_0(r) \to \frac{r}{\sqrt{2}} \quad (r \to 0) \quad , \quad \phi_0(r) \to 1 - 2e^{-\sqrt{2}r} \quad (r \to \infty) \tag{2.65}$$

$$v_0(r) \to 1 - \frac{r^2}{2} \quad (r \to 0) \quad , \quad v_0(r) \to 4e^{-\sqrt{2}r} \quad (r \to \infty), \tag{2.66}$$

and are therefore suitable parametrizations that become 'exact' in both r-limits.[7] These are of course the usual solitonic-type forms that qualitatively describe the behaviour of the solutions to (2.50), (2.51) very well, as can be checked by comparing them with the exact numerical calculations.

Inserting (2.63) and (2.64) into the dilaton correction (2.53) yields, after some trigonometric simplification,

$$v_1'' + \frac{v_1'}{r} = \mathrm{sech}^2\left(\frac{r}{\sqrt{2}}\right) + \mathrm{sech}^4\left(\frac{r}{\sqrt{2}}\right) \frac{\left[\mathrm{sech}^2\left(\frac{r}{\sqrt{2}}\right) - \left(1 - \frac{r^2}{2}\right)\right]}{r^2} \equiv f(r). \tag{2.67}$$

[7] A quick numerical check reveals that the error, by construction, is concentrated near $r = 1$ and is bounded above by about 20%.

However, the inhomogeneous right hand side is well approximated globally by the first term $\text{sech}^2(r/\sqrt{2})$. In particular, the dominant asymptotic behaviour as $r \to \infty$ is the same [since the latter term is a correction of $O(\exp(-2\sqrt{2}r))$ coming from the $(v_0')^2$ and the ϕ_0^4 contributions], and is correct to $O(r)$ in the small r limit. [8] Thus we take

$$\psi_1'' + \frac{\psi_1'}{r} \simeq \text{sech}^2\left(\frac{r}{\sqrt{2}}\right) \quad (\to 4e^{-\sqrt{2}r} \text{ as } r \to \infty). \tag{2.68}$$

where we have included the explicit asymptotic behaviour for later usage. The general solution of (2.68) is a particular solution of the inhomogeneous equation, plus the fundamental solution (2.48) with the arbitrary constants chosen to satisfy the boundary conditions. From the theory of ODEs, a particular solution to an inhomogeneous second order ODE, with inhomogeneity $f(r)$ is given by

$$y_p(r) = y_1(r)\int \frac{-y_2(r)f(r)}{W(y_1(r), y_2(r))}dr + y_2(r)\int \frac{y_1(r)f(r)}{W(y_1(r), y_2(r))}dr, \tag{2.69}$$

where $W(y_1, y_2)$ is the Wronksian; $1/r$ in our case. Thus we have the general solution for $\psi_1(r)$

$$\psi_1(r) = \log(r)\int r\ \text{sech}^2\left(\frac{r}{\sqrt{2}}\right)dr - \int r\log(r)\text{sech}^2\left(\frac{r}{\sqrt{2}}\right)dr + C_1 + C_2\log(r), \tag{2.70}$$

with the requirement that $\psi_1(\infty) = 0$. Evaluating the first integral explicitly, and then integrating the second integral by parts using the result just obtained, allows us to bring this to the much more convenient form

$$\psi_1(r) = \int_a^r \left[\sqrt{2}\tanh\left(\frac{x}{\sqrt{2}}\right) - 2\frac{\log\left[\cosh\left(\frac{x}{\sqrt{2}}\right)\right]}{x}\right]dx + C_1 + C_2\log r, \tag{2.71}$$

where we have introduced a lower integration limit a, to be determined momentarily. In order to be able to impose the boundary condition $\psi_1(\infty) = 0$, we need

[8] Alternatively, we do not have to make this truncation, at the price of making the subsequent analysis much more algebraically tedious, without qualitatively changing the result. The point is that it will be the dominant asymptotic behaviour that determines the dilaton behaviour.

to understand the convergence of this integral as a (type I) improper integral. It is easy to show that in fact the integral is logarithmically divergent as $r \to \infty$ since

$$\lim_{r\to\infty} \frac{\sqrt{2}\tanh\left(\frac{r}{\sqrt{2}}\right) - 2\frac{\log\left[\cosh\left(\frac{r}{\sqrt{2}}\right)\right]}{r}}{\frac{1}{r}} = 2\log(2). \tag{2.72}$$

If we rewrite the integrand in terms of exponentials, this limit is made more evident, as well as allowing us to write a closed form expression for the integral. Denoting the integrand by $F(r)$ we have

$$\begin{aligned} F(r) &= \sqrt{2}\left[\frac{1-e^{-\sqrt{2}r}}{1+e^{-\sqrt{2}r}}\right] - \sqrt{2} - \frac{2\log(1+e^{-\sqrt{2}r})}{r} + \frac{2\log(2)}{r} \\ &= 2\sqrt{2}\sum_{n=1}^{\infty}(-1)^n e^{-n\sqrt{2}r} - \frac{2}{r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\sqrt{2}r}}{n} + \frac{2\log(2)}{r}, \end{aligned} \tag{2.73}$$

whence it is clear that the last term yields the logarithmic divergence, whereas the other terms yield obviously convergent integrals. This divergence must be cancelled by the $C_2\log(r)$ term of the homogeneous solution (2.48), by setting $C_2 = -2\log(2)$. This is a necessary condition of being able to impose $\psi_1(\infty) = 0$. Then, pulling the homogeneous solution $-2\log(2)\log(r)$ under the integral to cancel the $2\log(2)/r$ piece, to fully impose the boundary condition we must take the integration limit a to infinity since the integrand is monotonic. Also, we must take the constant homogeneous solution $C_1 = 0$. Putting it all together, we finally have

$$\begin{aligned} \psi_1(r) &= \int_{\infty}^{r}\left[2\sqrt{2}\sum_{n=1}^{\infty}(-1)^n e^{-n\sqrt{2}r} - \frac{2}{r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\sqrt{2}r}}{n}\right]dr \\ &= 2\log(1+e^{-\sqrt{2}r}) + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Ei}_1(n\sqrt{2}r), \end{aligned} \tag{2.74}$$

where we have introduced the exponential integral defined by

$$\mathrm{Ei}_1(x) = \int_1^{\infty}\frac{e^{-xt}}{t}dt. \tag{2.75}$$

It is easy to verify explicitly that this solves the dilaton correction equation (2.68) and satisfies

$$\lim_{r\to\infty} \psi_1(r) = 0. \tag{2.76}$$

However, though we have been able set the dilaton ψ equal to zero at spatial infinity, the dilaton now diverges to $+\infty$ at $r = 0$ since

$$\begin{aligned} \lim_{r\to 0} \psi_1(r) &= \lim_{r\to 0} 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\mathrm{Ei}_1(n\sqrt{2}r) \sim \lim_{r\to 0} -2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\log(r) \\ &= \lim_{r\to 0} -2\log(2)\log(r) \to +\infty, \end{aligned} \tag{2.77}$$

using the fact that

$$\lim_{r\to 0} \frac{\mathrm{Ei}_1(ar)}{-\log(r)} = \lim_{r\to 0} \frac{\frac{-e^{-ar}}{r}}{\frac{-1}{r}} = 1 \quad \forall a > 0. \tag{2.78}$$

How did this come about? This singularity is none other than the one introduced when we were forced to assign a nonzero value to the homogeneous term $C_2\log(r)$ in order to obey the boundary condition at infinity. Thus in order to avoid a logarithmic divergence at infinity, we are forced to introduce one at zero by turning on $\log(r)$. This can be viewed as a direct consequence of the fact that we are dealing with an essentially two-dimensional problem and the two-dimensional Laplace equation.

It is now clear why this result is independent of the parametrizations (2.63), (2.64) and of the truncation made in going to (2.68). The $C_2\log(r)$ homogeneous term is turned on (and effectively shifts the particular solution) if and only if the (unshifted) particular solution integral is asymptotically divergent, which in turn depends only on the dominant asymptotic behaviour of the Nielsen-Olesen solutions. But this is precisely how we chose the parametrization and made the truncation: they have the correct asymptotic behaviour. Conversely, once the $C_2\log(r)$ term is turned on, we now unavoidably have a positive logarithmic divergence at $r = 0$, because the *unshifted* integrand is well behaved near $r = 0$. Again, we chose our parametrization to have the correct small r behaviour of the Nielsen-Olesen solutions.

Finally one might worry in taking, as most authors including Nielsen-Olesen do, the asymptotic behaviour of ϕ_0 as $\exp(-\sqrt{2}r)$ and not $\exp(-\sqrt{2}r)/\sqrt{r}$, that we may have affected the convergence of the unshifted particular integral. This is not the case. Consider taking $\phi_0 \to 1 - \frac{\exp(-\sqrt{2}r)}{\sqrt{r}}$, and $v_0 \to \sqrt{r}\exp(-\sqrt{2}r)$, as (2.58)-(2.59) indicate. Then the dilaton correction (2.53) reads

$$\begin{aligned} v_1'' + \frac{v_1'}{r} &\sim \frac{1}{2}\Big[3 - 4(1 - c_\phi e^{\sqrt{2}r})^2 + (1 - c_\phi e^{-\sqrt{2}r})^4 \\ &\quad -\frac{n^2 c_v^2}{r^2}(\frac{1}{2\sqrt{r}} - \sqrt{2})^2 e^{-2\sqrt{2}r}\Big] \\ &= 2c_\phi \frac{e^{-\sqrt{2}r}}{\sqrt{r}} + O(e^{-2\sqrt{2}r}), \end{aligned} \tag{2.79}$$

where we retain, as per our argument above, only the dominant asymptotic contribution. Excepting the $\sqrt{r}$ factor, this is the same result as our parametrization and truncation (2.68), i.e. the terms linear in $\delta\phi_0$ yield the dominant asymptotic contribution. This is now the exact asymptotic behaviour. Now using (2.69), the bare particular solution is

$$\begin{aligned} v_{1,p}(r) &\sim 2c_\phi\left[\log(r)\int\sqrt{r}e^{-\sqrt{2}r}dr - \int\log(r)\sqrt{r}e^{-\sqrt{2}r}dr\right] \\ &= \int_a^r\left[\frac{2^{-3/4}\sqrt{\pi}\text{erf}(2^{1/4}\sqrt{x})}{x} - \frac{\sqrt{2}e^{-\sqrt{2}x}}{\sqrt{x}}\right]dx\ , \end{aligned} \tag{2.80}$$

after various substitutions, and integration by parts very similar to the previous case. Here $\text{erf}(x) = 2/\sqrt{\pi}\int_0^x \exp(-t^2)dt$. Clearly this integral diverges logarithmically as $2^{-3/4}\sqrt{\pi}\log(a)$ as $a \to \infty$, so once again, in order to impose $v_1(\infty) = 0$, we cancel this with the homogeneous solution $-2^{-3/4}\sqrt{\pi}\log(r)$. Then, pulling it under the integral sign and reversing limits as before to make the sign of the correction clear, we have

$$\psi_1(r) = \int_r^\infty \left\{ \frac{2^{-3/4}\sqrt{\pi}\left[1 - \text{erf}(2^{1/4}\sqrt{x})\right]}{x} + \frac{\sqrt{2}e^{-\sqrt{2}x}}{\sqrt{x}} \right\} dx. \tag{2.81}$$

As can be easily checked, this satisfies (2.79) and $\psi_1(\infty) = 0$. Thus the asymptotic divergence is not avoided by the additional factor of $1/\sqrt{r}$, and we are again forced to turn on the homogeneous $C_2 \log(r)$ solution. We worked with a simpler global parametrization before, so that we could discuss small r behaviour of the solution as well. This completes the argument that our result is independent of our parametrization, and our truncation.

2.6 Discussion

The results of the previous section are perhaps surprising. In fact, this is a rather generic property of solutions to the inhomogeneous equation

$$\psi_1'' + \frac{\psi_1'}{r} = f(r) \tag{2.82}$$

with a vanishing asymptotic boundary condition, and with reasonable assumptions on $f(r)$. As we have seen, the general solution of (2.82) can be written as

$$\begin{aligned} \psi_1(r) &= \log(r)\int rf(r)dr - \int r\log(r)f(r)dr + C_1 + C_2\log(r) \\ &= \log(r)\int_a^r xf(x)dx - \int_b^r x\log(x)f(x)dx, \end{aligned} \tag{2.83}$$

where we have absorbed the homogeneous solution into the particular indefinite integrals by making them definite integrals: the arbitrary constants of the general solution are now the lower, constant, limits of integration. Clearly, we cannot in general impose the boundary condition $\psi_1(\infty) = 0$. A necessary condition for being able to impose this condition is that

$$\lim_{r\to\infty} \int^r x\log(x) f(x) dx \tag{2.84}$$

exists. Unfortunately, this is not quite sufficient [$f(x) = \sin(x^2)/(x\log(x))$ furnishes a counterexample]. However, the *absolute* convergence of the integral (2.84) is sufficient to be able to impose $\psi_1(\infty) = 0$, i.e. if

$$\lim_{r\to\infty} \int^r x\log(x) \left|f(x)\right| dx = K < \infty. \tag{2.85}$$

For if this limit exists, then so does the limit

$$\lim_{r\to\infty} \int^r x \left|f(x)\right| dx. \tag{2.86}$$

Then the squeeze theorem, and the inequalities

$$0 \le \left|\log(r) \int_r^\infty x f(x) dx\right| \le \int_r^\infty \log(r) x \left|f(x)\right| dx \le \int_r^\infty \log(x) x \left|f(x)\right| dx \to 0 \tag{2.87}$$

as $r \to \infty$, imply that

$$\lim_{r\to\infty} \log(r) \int_r^\infty x f(x) dx = 0. \tag{2.88}$$

This establishes the sufficiency of the condition (2.85).

From (2.53), the actual $f(r)$ in which we are interested is determined from the Nielsen-Olesen solutions ϕ_0 and v_0, and the arguments from the previous section establish that this $f(r)$ decays exponentially as $r \to \infty$. Thus we easily satisfy the above sufficient condition allowing us to take $\psi_1(\infty) = 0$.

Now consider the behaviour of $\psi_1(r)$ near $r = 0$, *subsequent* to imposing $\psi_1(\infty) = 0$. We now write the solution (2.83) as

$$\psi_1(r) = \int_r^\infty x\log(x) f(x) dx - \log(r) \int_r^\infty x f(x) dx. \tag{2.89}$$

Remembering that $x\log(x) \to 0$ as $x \to 0^+$, we now demonstrate the inevitable presence of a logarithmic divergence of $\psi_1(r)$ at $r = 0$ as long as $f(r)$ is well-behaved near $r = 0$ and $K \equiv \int_0^\infty xf(x)dx \neq 0$. The sign of the divergence will depend on the sign of K. Explicitly we have

$$\lim_{r\to 0}\psi_1(r) \sim \int_0^\infty x\log(x)f(x)dx - \log(r)\int_0^\infty xf(x)dx \to \mathrm{sgn}(K)\times\infty. \quad (2.90)$$

Note that these integrals exist assuming only, in addition to the previous restrictions on f ensuring improper convergence, that f is defined and say continuous (or Riemann integrable) everywhere on $r \geq 0$, and in particular at 0[9].

Again, because our $f(r)$ from (2.53) is defined and continuous for all $r \geq 0$ because the Nielsen-Olesen solutions are [remember that the term $(v_0')^2/r^2$ in (2.53) is finite as $r \to 0$ as seen in (2.67); in other words the field strength of the Nielsen-Olesen vortex is finite at the core], we have a logarithmic divergence at $r = 0$ as explicitly shown in the previous section. In fact, since our $f(r)$ is explicitly nonnegative [as seen in either (2.67) or its truncation (2.68)], the K defined above is positive, and so the logarithmic divergence is to *positive* infinity at $r = 0$. Again, this was seen explicitly in the last section.

To summarize, we have found that a solution to (2.82) can satisfy $\psi_1(\infty) = 0$, if the limit (2.85) exists. Furthermore, if this limit exists so that we may impose $\psi_1(\infty) = 0$, the solution diverges logarithmically at $r = 0$. Thus $\psi(\infty) = 0$ implies $\psi(0) = \infty$. Since the $f(r)$ relevant to our discussion decays exponentially as $r \to \infty$, and is well behaved at $r = 0$, this provides a general and generic proof of our result. Incidentally, this also shows why our results of the previous section are independent of either the parametrizations to the Nielsen-Olesen solutions or the truncation made in going from (2.67) to (2.68): this general

[9] Of course if f is poorly behaved (say divergent) as $r \to 0$, so that the integral diverges, then already the dilaton diverges without further argument.

behaviour depends only on the behaviour of f as $r \to \infty$ and as $r \to 0$, and our parametrization was chosen to be exact in these limits.

Given that we have now established that this dilaton behaviour is rather generic, one might wonder if this divergent behaviour of the dilaton at the core of the vortex is somehow an artifact of the perturbation theory. In fact, we now expect the full dilaton equation to yield even worse behaviour because of the exponential feedback. As a consistency check of our result, we will briefly examine the full dilaton equation (2.43). If we take the perturbation theory to be valid only for very large r, where the dilaton VEV is still small, so that we are still in a classical and perturbative regime, we know that it starts to run positive as one comes in from spatial infinity. A positive exponential self-coupling acts as a source term that becomes larger and larger as $r \to 0$. So if we equate small r with large ψ, then the dilaton equation (2.43) is dominated by the vacuum Fayet-Iliopoulos term [2] proportional to $e^{6\psi}$ [or $1/(S+S^\dagger)^3$ in the notation of Polchinski], which comes directly from the anomaly cancellation as a two string-loop tadpole [5], so that, approximately

$$\psi'' + \frac{\psi'}{r} \sim \frac{3\beta}{2} e^{6\psi}, \tag{2.91}$$

where we are taking β so small that we can neglect the axion contribution that is otherwise possibly as large (but of the same sign in any case), and where we are assuming that we still have $\phi \to 0$ as $r \to 0$: i.e. the vortex is well defined. An exact solution to (2.91) is given by

$$\psi(r) \sim \frac{-1}{6} \log \left[a_1 r \left(1 - \frac{9\beta}{2a_1} r \right)^2 \right], \tag{2.92}$$

where a_1 is an undetermined constant. For very small β this is essentially the same behaviour as our perturbative calculations. This solution is obviously consistent with the approximation (2.91) to the full dilaton equation (2.43) if we assume that the gauge and Higgs fields still have the boundary values $\phi(0) = 0$, and $v'(0) = 0$.

In any case, we seem to be led to the conclusion that the four-dimensional dilaton in this model starts to grow as we come in from spatial infinity. Since the dilaton VEV in this model sets the anomalous U(1) gauge coupling, we eventually enter a strongly coupled regime where not only the β perturbation theory breaks down, but where it no longer makes sense to ignore quantum and string threshold corrections. In other words, such a vortex is fundamentally a quantum mechanical object. Furthermore, as we have seen, the unavoidable singularities we have encountered are a direct consequence of the effectively *two*-dimensional nature of the vortex system: the solution of the Laplace (or Poisson) equation in two dimensions involves a logarithm which is singular at both $r = 0$ and $r \to \infty$.

Our conclusion then is that anomalous U(1) vortex solutions of heterotic superstring theory, if they are to have the standard asymptotic structure at large radial distances from the vortex core, necessarily generate large dilaton field values within that core signalling the presence of strong coupling and large quantum fluctuations. As such, these vortices can never be adequately described as entirely classical objects: their classical exterior surrounds an interior that is intrinsically quantum mechanical.

Bibliography

[1] M. Green, J.H. Schwarz & E. Witten, *Superstring Theory*, two volumes, (Cambridge University Press, Cambridge, 1987).

[2] J. Polchinski, *String Theory*, two volumes, (Cambridge University Press, Cambridge, 1998).

[3] A. Bailin & D. Love, *Supersymmetric Gauge Field Theory and String Theory*, (Institute of Physics Publishing, Bristol 1996).

[4] M.B. Green & J.H. Schwarz, *Anomaly cancellations in supersymmetric d=10 gauge theory and superstring theory*, Phys. Lett. **B149** (1984), 117.

[5] M. Dine, N. Seiberg & E. Witten, *Fayet-Iliopoulos Terms in String Theory*, Nucl. Phys. **B289**, 589 (1987).

[6] J. Atick, L. Dixon & A. Sen, *String Calculation of Fayet-Iliopoulos D-terms in arbitrary supersymmetric compactifications*, Nucl. Phys. **B292**, 109 (1987).

[7] M. Dine, I. Ichinose & N. Seiberg, *F terms and D terms in string theory*, Nucl. Phys. **B293**, 253 (1987).

[8] P. Fayet & J. Iliopoulos, *Spontaneously broken supergauge symmetries and Goldstone spinors*, Phys. Lett. **B51**, 461 (1974).

[9] J. Distler & B. Greene, *Aspects of (2,0) string compactifications.* Nucl. Phys **B304**, 1 (1988).

[10] H.B. Nielsen & P. Olesen, *Vortex line models for dual strings*, Nucl. Phys. **B61**, 45 (1973).

[11] E. Shellard & A. Vilenkin, *Cosmic Strings and other Topological Defects*, (Cambridge University Press, Cambridge 1994).

[12] P. Binétruy, C. Deffayet & P. Peter, *Global versus local cosmic strings from pseudoanomalous U(1)*, Phys. Lett. **B441**, 52 (1998). hep-ph/9807233.

[13] A. Faraggi & J. Pati, *A family universal anomalous U(1) in string models as the origin of supersymmetry breaking and squark degeneracy*, Nucl. Phys. **B526**, 21 (1998). hep-ph/9712516.

[14] L. Perivolaropoulos, *Asymptotics of Nielsen-Olesen vortices*, Phys. Rev. **D48**, 5961 (1993). hep-ph/9310264.

Chapter 3

Noncommutative field theory and spontaneous symmetry breaking

3.1 Introduction

In this chapter we undertake a perturbative analysis of quantum field theories on noncommutative $\mathbb{R}^n$ which exhibit the spontaneous breaking of a continuous *global* $O(N)$ symmetry. The perturbative expansion of noncommutative field theories has been the subject of much recent investigation. In the original work of Filk [1], it was suggested that despite being (infinitely) nonlocal, these theories exhibit the same divergence structure as their commutative counterparts (due to nonplanar oscillatory damping). More recently, Minwalla, Van Raamsdonk, and Seiberg [2] have shown that the effective cutoff of nonplanar graphs at one-loop in scalar theories replaces a UV divergence (which would ordinarily be cancelled with a counterterm) with an IR divergence in the external momenta:

$$\Lambda_{eff}^2 = \frac{1}{\frac{1}{\Lambda^2} + p \circ p} \to \frac{1}{p \circ p} \quad , \quad \Lambda^2 \to \infty. \tag{3.1}$$

They find that UV and IR limits do not commute, and suggest that UV divergences that persist at higher orders can be interpreted as IR divergences.

We continue these investigations, and examine global spontaneous symmetry breaking in the simplest case of the (noncommutative) $O(N)$ linear sigma model. In particular, we wish to investigate the status of Goldstone's theorem at one-loop for the noncommutative case. As is well known, the renormalizability of spontaneously broken theories is more subtle because the number of counterterm vertices exceeds the number of renormalization parameters (eight and three respectively in the linear sigma model). In the standard commutative case, the renormalizability of the theory, and the persistence of Goldstone's theorem ensuring the masslessness of the classical pions at the quantum level, involve a delicate cancellation between the relevant graphs and the pion propagator counterterm at zero external momentum [5-8]. The pion propagator counterterm is however fixed by the sigma tadpole counterterm, which in turn is fixed by the usual renormalization condition imposed on tadpoles: namely that they vanish so that the VEV of the sigma is not renormalized. Thus, the persistence of masslessness of the pions (Goldstone's theorem) at the quantum level is a genuine prediction of the quantum field theory. We now generalize this standard calculation to the noncommutative case.

We will find that the cancellation in the calculation of the pion mass renormalization is violated already at one loop order, and the 1PI *renormalized* (ie. after adding the counterterm fixed by the tadpole renormalization condition) effective action depends explicitly on the UV cutoff Λ, so the naive continuum limit does not exist; there is no more counterterm freedom to subtract off these divergences. Put another way, turning on the noncommutativity parameter induces UV cutoff dependence in the one-loop corrections to the renormalized pion propagator, and renders renormalization inconsistent with Goldstone symmetry realization. Viewed however as a Wilsonian effective theory where Λ is fixed, the limit of the mass correction as the external momentum is taken to zero still vanishes, so that Goldstone's theorem is satisfied for the Wilsonian action. The difference between the Wilsonian effective theory, and the putative continuum renormalized theory is a direct consequence of the noncommutativity of the UV and IR limits in noncommutative field theories.

The basic idea behind this calculation is that the noncommutativity does *not* affect the tadpole calculation as no external momentum flows into a trilinear tadpole vertex. Insofar as the sigma tadpole measures quantum corrections to the sigma VEV, this suggests that the order parameter for spontaneous symmetry breaking is insensitive to the underlying noncommutativity. Furthermore, since the tadpole counterterm, and the pion propagator counterterm are (essentially) the same (modulo wavefunction renormalization), the latter is therefore unmodified, and *fixed* with respect to renormalization. Now however, the re-weighting of planar graphs (due to noncommutativity) with respect to commutative graphs, and the distinct behaviour of new nonplanar graphs occurring in the one-loop contributions to the pion propagator, lead to an inexact UV cutoff cancellation with the associated counterterm. We emphasize that this cannot be evaded by simply imposing that the propagator counterterm cancel the cutoff dependence of the one-loop planar and nonplanar diagrams (in effect changing the renormalization scheme), because now the sigma tadpole corrections will unavoidably diverge. This is a direct consequence of the relationship between the counterterm vertices in a spontaneously broken quantum field theory. In the following we calculate the one-loop renormalization of the inverse pion propagator in the linear sigma model of noncommutative field theory, and demonstrate the incompatibility of continuum renormalization with spontaneous symmetry breaking.

3.2 Background and Formalism

In this section we introduce the noncommutative geometry and linear sigma model formalism we will need for the calculations of the next section in a reasonably self-contained way. The underlying noncommutative $\mathbb{R}^n$ is labelled by n coordinates satisfying

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad , \quad \theta^{\mu\nu} \in C \tag{3.2}$$

so that the commutators are C-numbers. Let $\mathcal{A}_x$ be the associative algebraic structure naturally defined by (3.2). We are naturally interested in functions defined over the noncommutative spacetime. Following Weyl [9] define

$$O(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k \; e^{ik_\mu \hat{x}^\mu} \; \tilde{f}(k) \tag{3.3}$$

where $\tilde{f}$ is the Fourier transform of f:

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x \; e^{-ik_\mu x^\mu} \; f(x). \tag{3.4}$$

This uniquely associates an operator in $O(f) \in \mathcal{A}_x$ with a function of classical variables: replacing the commuting variables x with operators $\hat{x}$ in a symmetric fashion. The product of two such operators is then defined in the obvious way

$$\begin{aligned} O(f) \cdot O(g) &= \frac{1}{(2\pi)^n} \int d^n k d^n p \; e^{ik_\mu \hat{x}^\mu} e^{ip_\nu \hat{x}^\nu} \; \tilde{f}(k)\tilde{g}(p) \\ &= \frac{1}{(2\pi)^n} \int d^n k d^n p \; e^{i(k_\mu + p_\mu)\hat{x}^\mu - \frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu} \; \tilde{f}(k)\tilde{g}(p). \end{aligned} \tag{3.5}$$

where on the second line we have used the Baker-Campbell-Hausdorff lemma $e^A e^B = e^{A+B-1/2[A,B]+\cdots}$, and the fact that for the canonical structure (3.2), the higher commutators vanishes. This allows us to establish a homomorphism, $O(f) \cdot O(g) = O(f * g)$, between this operator product $(\cdot)$ and the Moyal [10] product $(*)$ of ordinary functions:

$$\begin{aligned} (f * g)(x) &= \frac{1}{(2\pi)^n} \int d^n k d^n p \; e^{i(k_\mu + p_\mu)x^\mu} e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} p_\nu} \; \tilde{f}(k)\tilde{g}(p) \\ &= e^{+\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}} f(y)g(z)|_{y,z \to x}. \end{aligned} \tag{3.6}$$

This induced homomorphism allows us to view the algebra of functions on non-commutative $\mathbb{R}^n$ as the algebra of ordinary functions on commutative $\mathbb{R}^n$ with the Moyal $*$-product instead of the usual pointwise product. In particular, we can study field theories defined by classical actions of the usual form $S = \int d^n x \mathcal{L}[\phi]$ but with the $*$-product of fields.

Now consider the spontaneously broken $O(N)$ linear sigma model. At this point we specialize to four dimensions for the remainder of this paper. Furthermore, our sigma-model conventions will be essentially those of Peskin and Schroeder [11]. The commutative linear sigma model involves a set of N interacting real scalar fields $\phi^i(x)$ with a continuous $O(N)$ internal symmetry. Renormalizability (of the commutative case) in four dimensions implies the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[(\phi^i)^2\right]^2 \tag{3.7}$$

with implicit sums over the internal i index. Note the rescaling of the coupling λ to avoid awkward factors of $\frac{1}{6}$. The *global* $O(N)$ symmetry acts as

$$\phi^i \to R^{ij}\phi^j, \tag{3.8}$$

where R is a spacetime-*constant* $N \times N$ orthogonal matrix. Because the symmetry is global (so R is constant), the noncommutative generalization of this symmetry will manifestly pose no problem with respect to the Moyal product, which explicitly degenerates into the pointwise product if one of the factors is a spacetime constant. In particular we will not need to worry about the intricacies of the noncommutative generalization of gauge invariance (see for example [3]).

For $\mu^2 > 0$, the classical potential is minimized by a *constant* field ϕ_0^i configuration such that

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda} \tag{3.9}$$

and the $O(N)$ symmetry is spontaneously broken. Since this determines only the length of the vector ϕ^i, the rotational invariance allows us to choose coordinates so that

$$\phi_0^i = (0, 0, \ldots, 0, v), \qquad v \equiv \frac{\mu}{\sqrt{\lambda}}. \tag{3.10}$$

Then define $\sigma = \phi_0^n - v$ such that

$$\phi^i = (\pi^k, \sigma), \qquad k = 1, \ldots N-1, \tag{3.11}$$

with $<\sigma>= 0$. The linear sigma-model Lagrangian (3.7) rewritten in terms of these fields, and Wick rotated to Euclidean space (via $x^0 = -ix_E^0$) yields the following (commutative) action:

$$\begin{aligned} S_E &= -\int d^4x \left[\frac{1}{2}\partial_\mu \pi^k \partial^\mu \pi^k + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}(2\mu^2)\sigma^2 + \frac{\lambda}{4}[(\pi^k)^2]^2 \right. \\ &\left. + \frac{\lambda}{4}\sigma^4 + v\lambda\sigma(\pi^k)^2 + \frac{\lambda}{2}(\pi^k)^2\sigma^2 + \lambda v \sigma^3 \right], \end{aligned} \tag{3.12}$$

which reveals explicitly the masslessness of the $N-1$ pions of Goldstone's theorem at tree level. The renormalization counterterm structure determined from the symmetric theory (if the $O(N)$ symmetry is to hold quantum mechanically) is given by:

$$\begin{aligned} -\mathcal{L}_{E,ct} &= \frac{\delta_Z}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}\delta_\mu (\phi^i)^2 + \frac{\delta_\lambda}{4}[(\phi^i)^2]^2 \\ &= \frac{\delta_Z}{2}(\partial_\mu \pi^k)^2 + \frac{\delta_Z}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\delta_\mu + \delta_\lambda v^2)(\pi^k)^2 \\ &+ \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\sigma^2 + (\delta_\mu v + \delta_\lambda v^3)\sigma + \delta_\lambda v \sigma (\pi^k)^2 + \delta_\lambda v \sigma^3 \\ &+ \frac{\delta_\lambda}{4}[(\pi^k)^2]^2 + \frac{\delta_\lambda}{2}\sigma^2(\pi^k)^2 + \frac{\delta_\lambda}{4}\sigma^4. \end{aligned} \tag{3.13}$$

Now consider the noncommutative generalization of this theory. As discussed above, the effect at the Lagrangian level of the noncommutativity is to replace the (implicit) pointwise product of fields with the Moyal $*$-product. However, under the spacetime integral, an elementary but important result is that the quadratic part of the action is identical with commutative theory because $\theta^{\mu\nu}$ is antisymmetric. For example from (3.6),

$$\begin{aligned}(\phi_1 * \phi_2)(x) &= \phi_1(x)\phi_2(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu\phi_1(x)\partial_\nu\phi_2(x) + \ldots \\ &= \phi_1(x)\phi_2(x) + \frac{i}{2}\theta^{\mu\nu}\left[\partial_\mu(\phi_1(x)\partial_\nu\phi_2(x)) - \phi_1(x)\partial_\mu\partial_\nu\phi_2(x)\right] + \ldots \\ &= \phi_1(x)\phi_2(x) + \text{ total derivative} + \ldots. \end{aligned} \quad (3.14)$$

where ... represent the higher terms in θ that behave identically with respect to this calculation. Thus dropping total derivatives, by assuming appropriate asymptotic conditions on the fields, we have

$$\begin{aligned}\int d^4x\ \phi * \phi &= \int d^4x\ \phi \cdot \phi \\ \int d^4x\ \partial\phi * \partial\phi &= \int d^4x\ \partial\phi \cdot \partial\phi \end{aligned} \quad (3.15)$$

Interactions in higher powers of the fields are modified, and yield nontrivial phase factors in the momentum space Feynman rules, as we will see in a moment. There are two possible orderings for the noncommutative generalizations of the quartic terms $\pi^k\pi^k\pi^l\pi^l$, and $\pi^k\pi^k\sigma\sigma$, so we will include both orderings for each interaction in the model with unit total weighting. Thus the most general noncommutative, spontaneously broken linear sigma model action in Euclidean space is given by

$$\begin{aligned}S_{E,nc} = & -\int d^4x\ \Big[\frac{1}{2}\partial_\mu\pi^k\partial^\mu\pi^k + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}(2\mu^2)\sigma^2 \\ & +\frac{\lambda}{4}f(\pi^k * \pi^k) * (\pi^l * \pi^l) + \frac{\lambda}{4}(1-f)(\pi^k * \pi^l) * (\pi^k * \pi^l) \\ & +\frac{\lambda}{4}\sigma * \sigma * \sigma * \sigma + v\lambda\sigma * (\pi^k * \pi^k) + \frac{\lambda}{2}f(\pi^k * \pi^k) * \sigma * \sigma \\ & +\frac{\lambda}{2}(1-f)(\pi^k * \sigma) * (\pi^k * \sigma) + \lambda v\sigma * \sigma * \sigma\Big], \end{aligned} \quad (3.16)$$

and similarly for the counterterm structure. Note that the global $O(N)$ symmetry of the model implies that only a single f can occur. Since we wish to examine the theory at the quantum level, we will need the Feynman rules. In momentum space the Moyal product (3.6) of n fields yields

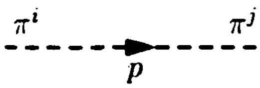
$$\int d^4x\phi_1(x) * \ldots * \phi_n(x) = \int \left(\prod_{i=1}^{n} d^4p_i\right) \delta\left(\sum_{i=1}^{n} p_i\right) e^{-\frac{i}{2}\sum_{i<j} p_i \times p_j} \tilde{\phi}_1(p_1)\cdots\tilde{\phi}_n(p_n) \tag{3.17}$$

where $p_i \times p_j \equiv p_{i\mu}\theta^{\mu\nu}p_{j\nu}$. We will prove this in the appendix to this chapter inductively. Thus the only modification to the Feynman rules is due to the phase factor

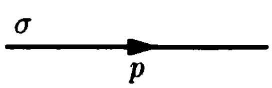
$$V(p_1 \ldots p_n) = e^{-\frac{i}{2}\sum_{i<j} p_i \times p_j} \tag{3.18}$$

at each vertex of n fields with p_i the momentum *into* the vertex from the ith field. Otherwise the Feynman rules will be identical to the linear sigma model. Note that the sum contains $n(n-1)/2$ terms, and by momentum conservation is invariant under cyclic permutations, but not under arbitrary permutations of the momenta. We have both three and four point vertices in the spontaneously broken linear sigma model; a fact that will be crucial for our analysis.

To summarize we write the *symmetrized* Feynman rules for the spontaneously broken noncommutative linear sigma model using (3.16) and (3.18). Throughout solid lines refer to the sigma, and dotted lines refer to the pions. As discussed above, the quadratic terms are not modified, so the propagators are not modified from the commutative case:

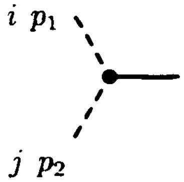


$= \dfrac{\delta^{ij}}{p^2}$

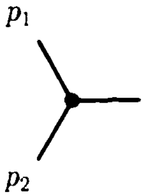


$= \dfrac{1}{p^2 + 2\mu^2}$ (3.19)

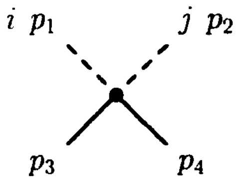
The *symmetrized* vertices (all momenta flow into the vertex) of the theory are now given by:



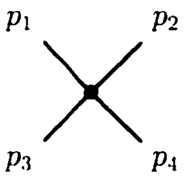
$$= -2v\lambda\delta^{ij}\cos(\frac{p_1 \times p_2}{2}). \tag{3.20}$$



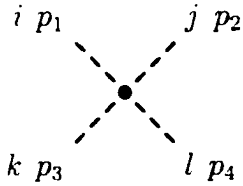
$$= -2v\lambda\left[\cos(\frac{p_1 \times p_2}{2}) + \cos(\frac{p_1 \times p_3}{2}) + \cos(\frac{p_2 \times p_3}{2})\right]. \tag{3.21}$$



$$\begin{aligned} = & -2\lambda\delta^{ij}\left[f\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2})\right. \\ & \left. +(1-f)\cos(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2})\right], \end{aligned} \tag{3.22}$$



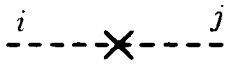
$$\begin{aligned} = & -2\lambda\left[\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) + \cos(\frac{p_1 \times p_3}{2})\cos(\frac{p_2 \times p_4}{2})\right. \\ & \left. + \cos(\frac{p_1 \times p_4}{2})\cos(\frac{p_2 \times p_3}{2})\right]. \end{aligned} \tag{3.23}$$



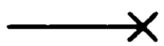
$$\begin{aligned} = & -2\lambda\left[\delta^{ij}\delta^{kl}\left(f\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2})\right.\right. \\ & \left. +(1-f)\cos(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2})\right) \\ & +\delta^{ik}\delta^{jl}\left(f\cos(\frac{p_1 \times p_3}{2})\cos(\frac{p_2 \times p_4}{2})\right. \\ & \left. +(1-f)\cos(\frac{p_1 \times p_2}{2} + \frac{p_3 \times p_4}{2})\right) \\ & +\delta^{il}\delta^{jk}\left(f\cos(\frac{p_1 \times p_4}{2})\cos(\frac{p_2 \times p_3}{2})\right. \\ & \left.\left. +(1-f)\cos(\frac{p_1 \times p_2}{2} + \frac{p_4 \times p_3}{2})\right)\right]. \end{aligned} \tag{3.24}$$

Working with totally symmetrized vertices allows us to capture both planar and nonplanar terms at once, though to make this explicit we will *a posteriori* extract planar and nonplanar parts below.

In the calculations of the next section we will only be interested in one and two point amplitudes, so the counterterms from (3.13) we will need are given by



$$= -\delta^{ij}(\delta_\mu + \delta_\lambda v^2 - \delta_z p^2),$$



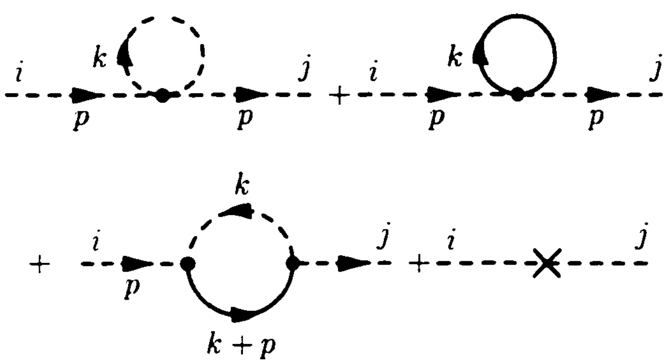
$$= -(\delta_\mu v + \delta_\lambda v^3),$$

$$= -(\delta_\mu + 3\delta_\lambda v^2 - \delta_z p^2). \qquad (3.25)$$

3.3 One-loop quantum level calculations

The $O(N)$ linear sigma model contains three counterterm parameters, and so three renormalization conditions are needed. These are conventionally taken to be conditions specifying the field strength of σ, the 4-σ scattering amplitude at threshold, and the vanishing of the one-point amplitude or vacuum expectation value renormalization of the σ. Everything else, including the masses of both σ and the π's, are predictions of the quantum field theory. In the renormalized perturbation theory sense, if the counterterms can be adjusted order by order to maintain the renormalization conditions, and yield finite predictions for everything else then the theory is perturbatively renormalizable. In the following we will examine the one-loop quantum structure of the noncommutative theory.

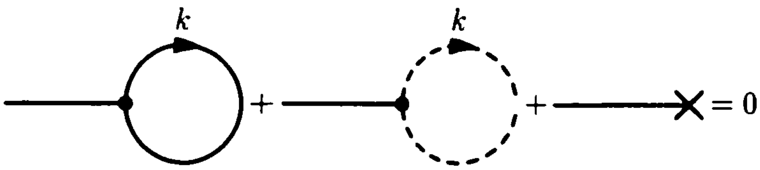
Of course, our results will not depend on this particular renormalization scheme, but since our explicit interest is in Goldstone's theorem at the quantum level for the noncommutative case, and in this scheme the masslessness of the pions is a *prediction* of the quantum field theory, we will employ it in the following.[1] In the ordinary commutative calculation of pion mass renormalization (see [11] for example) the masslessness of the pions at one-loop (Goldstone's theorem) follows from a cancellation of the following graphs at zero external momentum ($p = 0$):



(3.26)

Because the full noncommutative calculation we are about to explore subsumes this as the special case where $\theta \to 0$, we will not do this calculation explicitly: for a demonstration using dimensional regularization, the reader is referred to [11]. We do point out that the counterterm displayed is not arbitrary, but completely fixed (at zero external momentum) by the renormalization condition on the sigma tadpole, as inspection of (3.25) reveals. At one loop the sigma tadpole counterterm is fixed by the requirement that

[1] Actually we will directly use only the renormalization condition on the sigma one-point amplitude: namely that it vanishes. We will consider the general case later.



(3.27)

Now consider the noncommutative case. The first, perhaps surprising, fact is that noncommutativity does *not* alter the tadpole graphs at one-loop. This follows from the elementary observation that while the one-loop tadpole graphs above contain cubic vertices (and so yield nontrivial noncommutative phase terms), momentum conservation dictates that no external momentum flows into the cubic vertices, so that phase factor (3.18) always degenerates to one, or equivalently the arguments of the cosine terms in the symmetrized vertices always vanish.

Thus the sigma tadpole counterterm will not be modified in the noncommutative case. Nonetheless we need to explicitly calculate it from (3.27). Since loop integrals containing phases like (3.18) are most easily evaluated using Schwinger parameters (see for example [12],[13]), we will henceforth use them, coupled with a C^∞ smooth UV momentum cutoff regularization throughout. The basic Schwinger parameter representation is

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)} \tag{3.28}$$

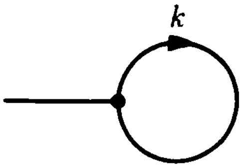
whence

$$\int \frac{d^4k}{k^2+m^2} = \int d\Omega_4 \int_0^\infty d\alpha \int dk k^3 e^{-\alpha(k^2+m^2)} = \pi^2 \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha^2 m^2}. \tag{3.29}$$

Thus high momentum divergences are replaced with small α divergences which are regulated by inserting a factor of $e^{-1/(\alpha\Lambda^2)}$ under the α integral, where Λ is the (fundamental) UV momentum cutoff. The basic integral we will encounter repeatedly is then of the form

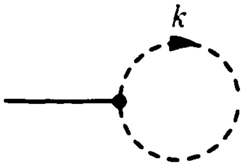
$$\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{x^2}{\alpha m^2}} = \frac{m^2}{x^2} + m^2 \log(x^2) + m^2(2\gamma - 1) + O(x) \tag{3.30}$$

in the limit of small $x(=\frac{m}{\Lambda})$. This is derived in the appendix to this chapter. Thus,



$$= \frac{1}{2}(-6v\lambda)\int\frac{d^4k}{(2\pi^4)}\frac{1}{k^2+m_\sigma^2}$$
$$= -\frac{3\lambda v}{16\pi^2}\left[\Lambda^2 - m_\sigma^2\log(\frac{\Lambda^2}{m_\sigma^2})\right.$$
$$\left.+m_\sigma^2(2\gamma-1)+O(\frac{m_\sigma}{\Lambda})\right], \quad (3.31)$$

where $m_\sigma^2 \equiv 2\mu^2$ is the (tree-level) mass-squared for the sigma. Next,

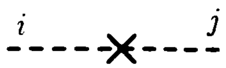


$$= \frac{1}{2}(-2v\lambda\delta^{ij})\int\frac{d^4k}{(2\pi^4)}\frac{\delta^{ij}}{k^2+\xi^2}$$
$$= -\frac{(N-1)\lambda v}{16\pi^2}\left[\Lambda^2 - \xi^2\log(\frac{\Lambda^2}{\xi^2})\right.$$
$$\left.+\xi^2(2\gamma-1)+O(\frac{\xi}{\Lambda})\right], \quad (3.32)$$

where ξ is a small infrared mass for the pion, to be taken to zero later. Thus (3.27) fixes the tadpole counterterm to be

$$= -(\delta_\mu v + \delta_\lambda v^3)$$
$$= \frac{\lambda v}{16\pi^2}\left\{\Lambda^2(N+2) - 3m_\sigma^2\log(\frac{\Lambda^2}{m_\sigma^2}) - \xi^2(N-1)\log(\frac{\Lambda^2}{\xi^2})\right.$$
$$\left.+(2\gamma-1)\left[3m_\sigma^2+\xi^2(N-1)\right]+O(\frac{1}{\Lambda})\right\}. \quad (3.33)$$

As discussed this fixes the pion propagator counterterm at zero momentum (and at one loop). Thus, including the momentum-dependent wave function renormalization term we have



$$= \frac{\lambda\delta^{ij}}{16\pi^2}\left\{\Lambda^2(N+2) - 3m_\sigma^2\log(\frac{\Lambda^2}{m_\sigma^2}) - \xi^2(N-1)\log(\frac{\Lambda^2}{\xi^2})\right.$$

$$+(2\gamma-1)\left[3m_\sigma^2+\xi^2(N-1)\right]+O(\frac{1}{\Lambda})\Big\}+\delta^{ij}\delta_z p^2. \quad (3.34)$$

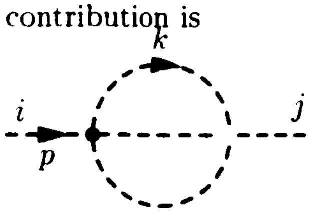
Now consider the one-loop contributions to the pion propagator. In order to consider separately the (noncommuting) [2] UV ($\Lambda \to \infty$), and IR ($p \to 0$) limits, we will not set $p = 0$ *a priori*, but will compute the $\pi\pi$ amplitude $\Gamma^{(2)}_{1PI}$ for an arbitary external momentum p. The first graph in (3.26) has a single quartic vertex (3.24). Denoting the loop momentum by k, (3.24) yields a phase factor of

$$\begin{aligned} & f\left(2\cos^2(\frac{p\times k}{2})+N-1\right)+(1-f)\left(2+(N-1)\cos(p\times k)\right) \\ &= fN+2(1-f)+\left(f+(1-f)(N-1)\right)\cos(p\times k). \end{aligned} \quad (3.35)$$

The constant term is the planar contribution to the noncommutative analogue of the first graph in (3.26), and is explicitly

$$\begin{aligned} &= \frac{1}{2}\left[-2\delta^{ij}\lambda(fN+2(1-f))\right]\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\xi^2} \\ &= -\frac{\lambda\delta^{ij}\left[fN+2(1-f)\right]}{16\pi^2}\Big[\Lambda^2-\xi^2\log(\Lambda^2/\xi^2)+ \\ &\qquad \xi^2(2\gamma-1)+O(\frac{\xi}{\Lambda})\Big]. \end{aligned} \quad (3.36)$$

whereas writing $\cos(k\times p) = \frac{1}{2}(e^{ik\times p}+e^{ip\times k})$, and using the fact that the exponentials are the same under the integral over all momenta, the nonplanar contribution is



$$\begin{aligned} &= \frac{1}{2}\left[-2\lambda\delta^{ij}(f+(1-f)(N-1))\right]\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\times p}}{k^2+\xi^2} \\ &= -\frac{\lambda\delta^{ij}\left[f+(1-f)(N-1)\right]}{16\pi^2}\Big[\Lambda_{eff}^2-\xi^2\log(\Lambda_{eff}^2/\xi^2) \\ &\qquad +\xi^2(2\gamma-1)+O(\frac{\xi}{\Lambda_{eff}})\Big], \end{aligned} \quad (3.37)$$

where

$$\Lambda_{eff}^2 = \frac{1}{\frac{1}{\Lambda^2} + p \circ p} \tag{3.38}$$

and

$$p \circ q \equiv -\frac{p_\mu \theta_{\mu\nu}^2 q_\nu}{4} \tag{3.39}$$

in a basis where $\theta^{\mu\nu}$ is skew-symmetric (the coordinates form pairs of noncommuting coordinates), so that its square is diagonal. This effective cutoff arises because completing the square in the Schwinger integral now yields an inhomogeneous term that is the same form as the fundamental UV cutoff regulator:

$$\begin{aligned} \frac{e^{ik \times p}}{k^2 + m^2} &= \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)+ik\times p} \\ &= \int_0^\infty d\alpha e^{-\alpha(l^2+m^2)} e^{-\frac{1}{4\alpha} p_\rho p^\nu \theta_{\mu\nu} \theta^{\mu\rho}} \\ \rightarrow_{reg} & \int_0^\infty d\alpha e^{-\alpha(l^2+m^2) - \frac{1}{\alpha \Lambda_{eff}^2}}, \end{aligned} \tag{3.40}$$

where $l_\mu = k_\mu - i/(2\alpha)\theta_{\mu\nu}p^\nu$ is a linear change of variables with respect to the k integral. Note the (irrelevant) 1/4 factor in the definition of $p \circ q$ which does not appear in [2].

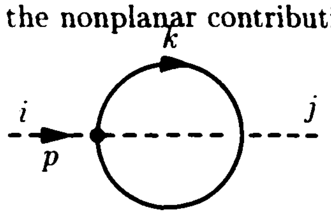
The second diagram in (3.26) is handled similarly. The phase factor from (3.22) is now

$$f + (1 - f)\cos(k \times p) \tag{3.41}$$

Thus the planar contribution to the noncommutative analogue of the second graph in (3.26) is

$$\begin{aligned} &= \frac{1}{2}(-2\lambda\delta^{ij} f) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_\sigma^2} \\ &= -\frac{\lambda\delta^{ij} f}{16\pi^2} \Big[\Lambda^2 - m_\sigma^2 \log(\Lambda^2/m_\sigma^2) + \\ &\qquad m_\sigma^2(2\gamma - 1) + O(\frac{m_\sigma}{\Lambda})\Big] \end{aligned} \tag{3.42}$$

and the nonplanar contribution is



$$= -\lambda\delta^{ij}(1-f)\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\times p}}{k^2+m_\sigma^2}$$

$$= -\frac{\lambda\delta^{ij}(1-f)}{16\pi^2}\Big[\Lambda_{eff}^2 - m_\sigma^2\log(\Lambda_{eff}^2/m_\sigma^2)+ m_\sigma^2(2\gamma-1)+O(\frac{m_\sigma}{\Lambda_{eff}})\Big]. \tag{3.43}$$

The third diagram of (3.26) will yield momentum dependent corrections to the propagator, as the external momentum circulates in the loop. Note also that it does not depend on f since there are no quartic vertices. Instead of using two Schwinger parameters for the two internal propagators, we will combine the propagators using a Feynman parameter, and then use a Schwinger parameter via the identity

$$\frac{1}{(l^2+\Delta^2)^2} = -\frac{d}{d(l^2)}\left[\int_0^\infty d\alpha e^{-\alpha(l^2+\Delta^2)}\right] = \int_0^\infty d\alpha\alpha e^{-\alpha(l^2+\Delta^2)}. \tag{3.44}$$

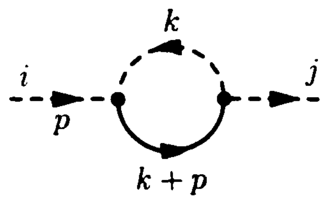
Then we will need (see appendix) the small x expansion:

$$\int\frac{d\alpha}{\alpha}e^{-\alpha\Delta^2-\frac{x^2}{\alpha\Delta^2}} = -\log(x^2)-2\gamma+O(x), \tag{3.45}$$

which makes explicit the logarithmic (as opposed to quadratic) divergence of this type of diagram. Now, since we have two cubic vertices, we pick up two phase factors due to noncommutativity. Denoting by k the pion momentum in the loop, the vertex (3.20) yields for the third diagram of (3.26) the phase

$$\cos(\frac{k\times p}{2})\cos(\frac{-k\times -p}{2}) = \frac{1}{2}(1+\cos(k\times p)). \tag{3.46}$$

Thus planar and nonplanar contributions are weighted equally. The former yields



$$= \frac{1}{2}4v^2\lambda^2\delta^{ij}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\xi^2}\frac{1}{(k+p)^2+m_\sigma^2}$$

$$
\begin{aligned}
&= 2v^2\lambda^2\delta^{ij}\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+\Delta^2)^2}\\
&\rightarrow_{reg} \frac{v^2\lambda^2\delta^{ij}}{8\pi^2}\int_0^1 dx\int_0^\infty\frac{d\alpha}{\alpha}e^{-\alpha\Delta^2-\frac{1}{\alpha\Lambda^2}}\\
&= \frac{-v^2\lambda^2\delta^{ij}}{8\pi^2}\left[\int_0^1\log(\frac{\Delta^2}{\Lambda^2})dx+2\gamma+O(\frac{\Delta}{\Lambda})\right]
\end{aligned}
\tag{3.47}
$$

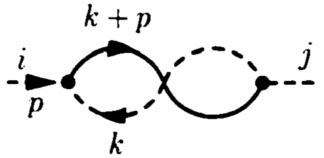
where $\Delta^2 = x(1-x)p^2 + x\xi^2 + (1-x)m_\sigma^2$, $l = k + (1-x)p$, and where in the second line we introduced the Feynman parametrization:

$$
\frac{1}{(k^2+\xi^2)[(k+p)^2+m_\sigma^2])} = \int_0^1 dx\frac{1}{(l^2+\Delta^2)^2}
\tag{3.48}
$$

To proceed, we can take the IR regulator ξ to zero without any difficulty, to simplify the Feynman parameter integral, so that (recalling $v^2 = \mu^2/\lambda$)

$$
(3.47) = \frac{-\lambda m_\sigma^2\delta^{ij}}{16\pi^2}\left[\log\left(\frac{p^2+m_\sigma^2}{\Lambda^2}\right)+\frac{m_\sigma^2}{p^2}\log\left(\frac{p^2+m_\sigma^2}{m_\sigma^2}\right)+2(\gamma-1)+O(\frac{\Delta}{\Lambda})\right].
\tag{3.49}
$$

Noting that $l = k + (1-x)p$ implies that $k \times p = l \times p$, and writing as usual $\cos(k\times p) = \cos(l \times p)$ in exponential form, and noting the symmetry of the terms under integration over all l, the nonplanar contribution to the third graph in (3.26) is



$$
\begin{aligned}
&= 2v^2\lambda^2\delta^{ij}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\xi^2}\frac{e^{ik\times p}}{(k+p)^2+m_\sigma^2}\\
&= 2v^2\lambda^2\delta^{ij}\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{e^{il\times p}}{(l^2+\Delta^2)^2}\\
&= \frac{-\lambda m_\sigma^2\delta^{ij}}{16\pi^2}\left[\log\left(\frac{p^2+m_\sigma^2}{\Lambda_{eff}^2}\right)+\frac{m_\sigma^2}{p^2}\log\left(\frac{p^2+m_\sigma^2}{m_\sigma^2}\right)+2(\gamma-1)+O(\frac{\Delta}{\Lambda_{eff}})\right].
\end{aligned}
\tag{3.50}
$$

Thus the effect of noncommutativity is to re-weight the planar and nonplanar graphs with respect to the commutative graphs (where there is no distinction between planarity and nonplanarity) in the cases where nonplanar graphs are generated, and replace the Λ in the planar graphs with Λ_{eff} in the nonplanar graphs. The one-loop correction to the (inverse) propagator is the sum of the six graphs (3.36), (3.37), (3.42), (3.43), (3.47), (3.50), and the counterterm (3.34). This sum is the noncommutative equivalent of (3.26). Explicitly it is equal to

$$\sum_{1-loop} = \frac{\lambda\delta^{ij}}{16\pi^2}\left\{\left[N(1-f)+f\right]\left(\Lambda^2-\Lambda_{eff}^2\right)+\left(2-f\right)m_\sigma^2\log(\frac{\Lambda_{eff}^2}{\Lambda^2})\right.$$
$$\left.+2m_\sigma^2\left(1-\frac{p^2+m_\sigma^2}{p^2}\log(\frac{p^2+m_\sigma^2}{m_\sigma^2})\right)\right\}]+\delta^{ij}\delta_Z p^2, \quad (3.51)$$

or eliminating Λ_{eff} in favour of Λ, θ, and p

$$\sum_{1-loop} = \frac{\lambda\delta^{ij}}{16\pi^2}\left\{\left[N(1-f)+f\right]\Lambda^2\left(1-\frac{1}{1+\Lambda^2(p\circ p)}\right)\right.$$
$$-\left(2-f\right)m_\sigma^2\log(1+\Lambda^2(p\circ p))$$
$$\left.+2m_\sigma^2\left(1-\frac{p^2+m_\sigma^2}{p^2}\log(\frac{p^2+m_\sigma^2}{m_\sigma^2})\right)\right\}+\delta^{ij}\delta_Z p^2. \quad (3.52)$$

3.4 Discussion

This result is rather remarkable, and some comments are in order. First note that if we take the noncommutativity θ to zero, the explicit dependence on Λ cancels, and we are left with the usual finite commutative result:

$$\sum_{1-loop,comm} = \frac{\lambda\delta^{ij}}{8\pi^2}m_\sigma^2\left[1-\frac{p^2+m_\sigma^2}{p^2}\log(\frac{p^2+m_\sigma^2}{m_\sigma^2})\right]+\delta^{ij}\delta_Z p^2 \quad (3.53)$$

which furthermore vanishes in the $p\to 0$ limit. Thus the mass shift of the pions due to one-loop quantum corrections vanishes; this is just Goldstone's theorem. However, unless we choose $N=2$ (corresponding to the trivial Abelian case) *and* $f=2$,[2] for a strictly nonzero θ, the result is unavoidably UV cutoff dependent

[2] We will see the interpretation of this bizarre ordering choice in the next chapter.

(the freedom to choose δ_Z clearly does not help us because the divergences are not proportional to p^2), and therefore naively divergent (both quadratically and logarithmically) in the continuum ($\Lambda \to \infty$) limit. However, viewed as a Wilsonian action, for which Λ is a fixed, physical parameter, the limit $p \to 0$ exists, and (3.52) becomes zero: again yielding Goldstone's theorem in this ordering of the limits. The conclusion is that not only do UV and IR limits not commute, but the continuum limit of the renormalized theory is inconsistent with Nambu-Goldstone realization of the global symmetry. We emphasize, that this occurs after the renormalization programme has been carried out; we have used up the counterterm freedom in fixing the sigma VEV.

This is not an artifact of the renormalization prescription (of which we have only used the condition imposing nonrenormalization of the sigma VEV), since clearly if we somehow arrange the $\pi\pi$ amplitude to contain no net fundamental Λ dependence, post-renormalization Λ dependence will be shifted into the sigma tadpole. This is a direct consequence of the restrictive counterterm structure present in a spontaneously broken theory.

To summarize, because no external momentum flows into a tadpole vertex, noncommutativity does not affect tadpole graphs, and hence the sigma tadpole renormalization counterterm is not modified. Since tadpole terms reflect shifts in vacuum expectation values of fields, and since the sigma VEV measures the spontaneous symmetry breaking of the theory, colloquially, spontaneous symmetry breaking is blind to the noncommutativity at this level. On the other hand, since the counterterm structure is fixed, this means that the pion propagator counterterm is not modified (modulo wave function renormalization). However, the graphs that contribute to the one-loop quantum corrections of the pion propagator are modified, and the usual cancellation that ensures the pion masses are finite (and zero) after quantum corrections is violated in such a way, that cutoff dependence is restored; so UV and IR limits do not commute. In sum, we conclude that the Nambu-Goldstone realization of the $O(N)$ symmetry (except for the trivial Abelian case) of this model is not perturbatively compatible with the continuum renormalization of the noncommutative theory.

To illuminate the nature of these unusual results, we will explore group and representation dependence in the next chapter [where, for example we find the origin of the unusual $f = 2$ restriction on the $O(2)$ case], while in appendix B to this thesis, we will see how the pathology discovered here manifests itself in four-point amplitudes.

3.5 Appendix for chapter 3

3.5.1 The Feynman rule for noncommutative phases

Proceeding inductively, we have:

$$
\begin{aligned}
&((\phi_1 * \ldots\phi_n) * \phi_{n+1})(x) \\
=& \int \prod_i \frac{d^4p_i}{(2\pi)^2} \lim_{y,z\to x} e^{\frac{i}{2}\theta^{\mu\nu}\partial^y_\mu\partial^z_\nu} \left(e^{i(p_1+\ldots+p_n)\cdot y} e^{ip_{n+1}\cdot z}\right) \\
&\qquad \times e^{-\frac{i}{2}\sum_{i<j}^{n} p_i\times p_j} \tilde{\phi}_1(p_1)\ldots\tilde{\phi}_{n+1}(p_{n+1}) \\
=& \int \prod_i \frac{d^4p_i}{(2\pi)^2} e^{-\frac{i}{2}\theta^{\mu\nu}(p_1+\ldots p_n)_\mu (p_{n+1})_\nu} e^{i(p_1+\ldots+p_{n+1})\cdot x} \\
&\qquad \times e^{-\frac{i}{2}\sum_{i<j}^{n} p_i\times p_j} \tilde{\phi}_1(p_1)\ldots\tilde{\phi}_{n+1}(p_{n+1}) \\
=& \int \prod_i \frac{d^4p_i}{(2\pi)^2} e^{i(p_1+\ldots+p_{n+1})\cdot x} e^{-\frac{i}{2}\sum_{i<j}^{n+1} p_i\times p_j} \tilde{\phi}_1(p_1)\ldots\tilde{\phi}_{n+1}(p_{n+1}) \qquad (3.54)
\end{aligned}
$$

where the induction hypothesis is used in the second line. Thus,

$$V(p_1, ..., p_n) = e^{-\frac{i}{2}\sum_{i<j} p_i \times p_j}. \tag{3.55}$$

3.5.2 A useful integral

In evaluating the loop integrals in this paper, we encountered integrals of the form

$$\mathcal{I}_n(x) \equiv \int_0^\infty \frac{d\alpha}{\alpha^n} e^{-\alpha m^2 - \frac{x^2}{\alpha m^2}}, \tag{3.56}$$

which can be taken as definitions (modulo constants) of the K type modified Bessel functions. Since we are really only interested in the small x (large Λ) behaviour, and in the cases $n = 1, 2$, an elementary treatment suffices to determine dominant behaviour. We split the integration region into $[0, x/m^2]$, and $[x/m^2, \infty)$, and then expand the appropriate exponential on each region. For $n = 2$, after changing variables $\alpha \to 1/\alpha \cdot x^2/m^4$,

$$\begin{aligned}
\mathcal{I}_2(x) &= \frac{m^4}{x^2}\int_0^\infty e^{-\alpha m^2 - \frac{x^2}{\alpha m^2}} d\alpha \\
&= \frac{m^4}{x^2}\left[\int_0^{\frac{x}{m^2}} \left(1 - \alpha m^2 + \frac{\alpha^2 m^4}{2} + O(\alpha^3)\right) \cdot e^{-\frac{x^2}{m^2\alpha}} \quad d\alpha \right.\\
&\quad \left. + \int_{\frac{x}{m^2}}^\infty \left(1 - \frac{x^2}{m^2\alpha} + O(x^4)\right) e^{-\alpha m^2} d\alpha\right] \\
&= \frac{m^2}{x^2}\left[(x - \frac{x^2}{2})e^{-x} - x^2 \mathrm{Ei}_1(x) + O(x^3) + e^{-x} - x^2 \mathrm{Ei}_1(x) + O(x^4)\right] \\
&= \frac{m^2}{x^2}\left[1 - x^2 + 2x^2\log(x) + 2x^2\gamma + O(x^3)\right] \\
&= \frac{m^2}{x^2} + m^2\log(x^2) + m^2(2\gamma - 1) + O(x),
\end{aligned} \tag{3.57}$$

where we have used the exponential integral

$$\mathrm{Ei}_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt \tag{3.58}$$

and its asymptotic expansion near $x = 0$, discovered through

$$\begin{aligned} \frac{d}{dx}[\mathrm{Ei}_1(x)] &= -\int_1^{\infty} e^{-xt}dt = -\frac{e^{-x}}{x} \\ &= -\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-x)^{n-1}}{n!} \end{aligned} \tag{3.59}$$

whence

$$\mathrm{Ei}_1(x) = -\log(x) + C + O(x). \tag{3.60}$$

It can be shown that $C = -\gamma$, but since it must cancel out of all physical amplitudes, we heed it no further. The $n = 1$ case, also used earlier, is handled identically and we omit it here.

Bibliography

[1] T. Filk, *Divergences in a field theory on quantum space*, Phys. Lett. **B376** 53, (1996).

[2] S. Minwalla, M. Van Raamsdonk & N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002**, 020 (2000). hep-th/9912072.

[3] J. Madore, S. Schraml, P. Schupp & J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. **C16**, 161 (2000). hep-th/0001203.

[4] M. Van Raamsdonk & N. Seiberg, *Comments on noncommutative perturbative dynamics*, JHEP **0003**, 035 (2000). hep-th/0002186.

[5] B.W. Lee, *Renormalization of the sigma model*, Nucl. Phys. **B9**, 649 (1969).

[6] J.-L. Gervais, B.W. Lee, *Renormalization of the sigma-model (2) fermion fields and regularization*, Nucl. Phys. **B12**, 627 (1969).

[7] K. Symanzik, Lettere al Nuovo Cimento **1**, 10 (1969).

[8] K. Symanzik, *Renormalizable models with simple symmetry breaking. 1. Symmetry breaking by a source term*, Comm. Math. Phys., **16**, 48 (1970).

[9] H. Weyl, Z. Physik **46**, 1 (1927).

[10] J.E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[11] M. Peskin & D. Schroeder, *An Introduction to Quantum Field Theory*, (Addison-Wesley, Reading, Massachusetts, 1995).

[12] C. Itzykson, J. Zuber, *Quantum Field Theory*, (McGraw-Hill, New York, 1980).

[13] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford University Press, Oxford, 1996).

Chapter 4

Noncommutative linear sigma models

4.1 Introduction

Recently field theories on noncommutative spacetime backgrounds have been the subject of intense scrutiny. Part of this motivation stems from the fact that noncommutative $U(N)$ gauge theories arise on D-branes in the presence of a constant NS-NS B-field background, in the zero-slope, field theoretic limit of string theory [1],[2]. A second motivation, independent of string theory, is the question of whether the world we live in is based on a noncommutative spacetime. In order to construct realistic models of particle physics on noncommutative spacetimes, one needs to be sure that noncommutative theories preserve the features that underlie the standard model, including perturbative renormalizability in the presence of spontaneous symmetry breaking[3],[4].

The general scheme for defining field theories with the noncommutative spacetime structure defined by $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$, $\theta^{\mu\nu}$ real, constant and antisymmetric, is to invoke Weyl-Moyal correspondence. This has the effect of replacing the underlying noncommutative spacetime with a commutative spacetime at the expense of replacing the ordinary pointwise product of spacetime dependent functions with an infinitely nonlocal star product. The induced momentum space Feynman rules for interaction vertices associated with a given field theory then involve momentum-dependent phases, which generically split a graph (at least at one-loop) into planar and nonplanar parts. The former are identical to the usual commutative graphs (up to a total phase depending only on the external momenta, and a combinatorial reweighting), and in particular possess the usual divergence structure associated with a commutative quantum field theory. The latter, nonplanar components are explicitly finite (at least at one-loop) because of oscillatory damping due to the phases, and replace an ultraviolet divergence with an infrared divergence in the external momenta [5],[6].

Superficially, as a consequence of the finiteness of nonplanar graphs, and of the similar divergence structure of the planar graphs, one might conclude that the renormalization of noncommutative field theories proceeds as in the commutative theory, because the counterterm structure is formally the same. However, as is well-known, the renormalization of spontaneously broken theories, with either underlying global or gauge symmetries, is more subtle because the number of counterterm vertices exceeds the number of renormalization parameters. As a result, the renormalizability of (commutative) spontaneously broken theories hinges in general on intricate graphwise cancellations [3], [4] order by order in perturbation theory. Thus it is of obvious interest to examine whether or not these cancellations persist in noncommutative field theories.

In the previous chapter we studied the spontaneous symmetry breaking of a global $O(N)$ symmetry in the noncommutative deformation of the linear sigma model with scalars in the fundamental representation. We found that one-point tadpoles of the sigma at one-loop were insensitive to the noncommutativity because no external momentum flows into the trilinear tadpole vertex. Thus the one-point sigma counterterm is identical to the one in the commutative limit, which in turn fixes the pion mass counterterm to be the same as its commutative limit. On the other hand, the planar components of the 1PI graphs contributing to the one-loop pion (inverse) propagator renormalization are re-weighted with respect to the corresponding commutative graphs. As a consequence, there is an unavoidable UV cutoff dependence (for nonzero external momentum) *after* renormalization, signalling the nonexistence of a continuum limit, and noncommuting UV ($\Lambda_{UV} \to \infty$) and IR ($p \to 0$) limits. Specifically we found that the sum of the 1PI graphs and the counterterm contributing to the pion mass renormalization yielded quadratic and logarithmic UV cutoff Λ dependences as:

$$\begin{aligned}\sum_{1-loop} &= \frac{\lambda\delta^{ij}}{16\pi^2}\left\{\Big[N(1-f)+f\Big]\Lambda^2\left(1-\frac{1}{1+\Lambda^2(p\circ p)}\right)\right.\\ &\left. -\Big(2-f\Big)m_\sigma^2\log(1+\Lambda^2(p\circ p))\right\} + \text{finite} * p^2 \qquad (4.1)\end{aligned}$$

respectively. Here N is the dimension of the fundamental of $O(N)$, $p\circ q \equiv -p_\mu\theta^2_{\mu\nu}q_\nu/4$, and f takes into account the two possible quartic orderings for $\pi\pi\pi\pi$ and $\pi\pi\sigma\sigma$ terms:

$$\begin{aligned}&\frac{\lambda}{4}f(\pi^k*\pi^k)*(\pi^l*\pi^l)+\frac{\lambda}{4}(1-f)(\pi^k*\pi^l)*(\pi^k*\pi^l)\\ &+\frac{\lambda}{2}f(\pi^k*\pi^k)*\sigma*\sigma+\frac{\lambda}{2}(1-f)(\pi^k*\sigma)*(\pi^k*\sigma)\subset\mathcal{L} \qquad (4.2)\end{aligned}$$

For nonzero θ and p, the only circumstance under which we can take the continuum limit is when $f = 2$ *and* $N = 2$, where both logarithmic and quadratic dependences on Λ vanish. This corresponds to the *Abelian* $O(2)$ model, and if written in terms of a complex scalar ϕ corresponds precisely to the ordering $\phi^* * \phi * \phi^* * \phi$. Otherwise, the conditions $f = N/(N-1)$ and $f = 2$ required for the cancellations of quadratic and logarithmic dependences on Λ respectively, cannot be simultaneously satisfied, Goldstone's theorem fails at the one-loop level, and the continuum limit of the model fails to exist. Furthermore, a study of the four-point scattering amplitudes (as in appendix B) confirms that the model is nonrenormalizable for $N > 2$. Thus for general $N > 2$ and for *all* possible orderings consistent with the global $O(N)$ symmetry, the noncommutative $O(N)$ linear sigma model does not exist in the continuum limit.

Prima facie, this incompatibility of continuum renormalizability with spontaneous symmetry breaking for $O(N)$ linear sigma models appears to present severe difficulties for attempts to make realistic models of particle physics on noncommutative spacetimes. First, it is clear that models with spontaneously broken *gauge* symmetries must have consistent spontaneously broken global limits (as the gauge couplings vanish): the absence of such a global limit with spontaneous symmetry breaking would preclude its subsequent gauging (at least perturbatively). Second, the standard model of the fundamental interactions (and unified theories which encompass it) depends, for electroweak symmetry breaking, on a complex Higgs doublet. As is well known, resolved into real components, the purely scalar sector of the standard model is $O(4)$ invariant (and not just $SU(2) \times U(1)$ invariant), with the real components in the fundamental representation: our previous results then appear to preclude noncommutative deformation of the standard model. We will argue below that this is *not* necessarily the case. In particular, noncommutative theories with N complex scalars, Φ^i ($i = 1..N, N > 1$), and with $U(N)$ invariant self-interactions, are *not* invariant under an $O(2N)$ symmetry acting on their real components, due to purely noncommutative commutator interactions arising from the noncommutativity of the spacetime. Thus we will first undertake an analysis for the case of a $U(N)$ symmetry group with the scalars in the fundamental representation, choosing the quartic invariant

$\Phi^\dagger\Phi\Phi^\dagger\Phi$. The spontaneous breaking of this group to $U(N-1)$ leaves $N-1$ complex pions, and one real pion. We find that the one-loop 1PI graphs contributing to the mass renormalization of the complex pions, like the one-point tadpoles, do not see the noncommutativity at this order, and so Goldstone's theorem holds. The 1PI one-loop graphs contributing to the mass renormalization of the real pion (which arises through the breaking of the $U(1) \equiv O(2)$ subgroup of the $U(N)$), are split into divergent planar, and finite nonplanar pieces in such a way that Goldstone's theorem holds at one-loop. The essential difference between the $U(N)$ models and the corresponding $O(2N)$ models ($N > 1$) is the presence of the purely noncommutative commutator interactions in the former.

We will also begin to explore how our present and previous, results might depend on the scalar field representation responsible for spontaneously breaking the symmetry. In particular, we consider both an $O(4)$ and a $U(2)$ model, with scalars in the adjoint representation, to see if our previous results depended on our scalars being in the respective fundamentals. For the $U(2)$ model with matter in the adjoint representation, we will find that Goldstone's theorem holds if we include only interactions involving a single trace operator, which we will in turn demonstrate are the only ones consistent with noncommutative gauge invariance in the case that we gauge the $U(2)$ symmetry. In this model Goldstone's theorem holds due to a notable cancellation of a purely noncommutative graph involving couplings to the $U(1)$ component of the field. For the $O(4)$ model with matter in the adjoint, we find violations of Goldstone's theorem at one-loop of the type found in the $O(N)$ fundamental representation studied in the previous chapter. Finally we discuss the implications of these results for model building, and comment on the nature of the IR divergences found by [6] in the context of noncommutative theories with matter in the adjoint representation.

4.2 NC $U(N)$ Linear Sigma model: Fundamental Representation

In this section we examine Goldstone's theorem in the noncommutative deformation of the linear sigma model with a global $U(N)$ symmetry group, and contrast the results with our previously discovered violations of Goldstone's theorem in the $O(N)$ linear sigma model.

The noncommutative $U(N)$ linear sigma model is defined by the Lagrangian density given by

$$\mathcal{L} = \partial_\mu \Phi^\dagger * \partial^\mu \Phi + \mu^2 \Phi^\dagger * \Phi - \lambda \Phi^\dagger * \Phi * \Phi^\dagger * \Phi, \qquad (4.3)$$

where Φ is an N-vector of *complex* fields ϕ_i $(i = 1..N)$, where the star product is defined as usual by $f(x)*g(x) = \exp(i\theta^{\mu\nu}\partial_\mu^y \partial_\nu^z) f(y)g(z)|_{y,z\to x}$, and where we have included the star ordering of the quartic term consistent with noncommutative gauge invariance of a possible gauging of the model (see below). For $\mu^2 > 0$, the symmetry is spontaneously broken to $U(N-1)$. Throughout the remainder of this paper, we will consider only translationally invariant vacua[1]. By an $SU(N)$ transformation, we can rotate the resulting VEV into the last field of Φ, and by a $U(1)$ rotation we can identify this VEV with a constant shift, a, in the real part of this field. Thus we define $\pi_i = \phi_i$ for $i = 1..N-1$, while $\phi_N = (\sigma + a + i\pi_0)/\sqrt{2}$: there are $N-1$ complex Goldstone bosons, and one real Goldstone mode. The minimization of the potential for this configuration implies

$$V(a) = \frac{\mu^2}{2}a^2 - \frac{\lambda}{4}a^4 \longrightarrow a^2 = \frac{\mu^2}{\lambda}. \qquad (4.4)$$

[1] As Gubser and Sondhi have argued [9], more exotic vacua such as stripe phases are possible in noncommutative theories.

Writing (4.3) in terms of these variables yields:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi_0)^2 + \partial_\mu\pi_i^*\partial^\mu\pi_i - \frac{1}{2}(2\mu^2)\sigma^2 - \lambda a\sigma^3 \\ &\quad -\lambda a\pi_0^2\sigma - 2\lambda a\pi_i^*\pi_i\sigma - \lambda\pi_i^*\pi_i\pi_j^*\pi_j - \frac{\lambda}{4}(\sigma^4 + \pi_0^4) \\ &\quad -\lambda\pi_0^2\sigma^2 + \frac{\lambda}{2}\sigma\pi_0\sigma\pi_0 - \lambda\pi_i^*\pi_i(\sigma^2 + \pi_0^2) - \lambda\pi_i^*\pi_i[\sigma, \pi_0] \end{aligned} \tag{4.5}$$

For notational brevity all star products will be suppressed henceforth, unless there is danger of confusion. Furthermore, we will implicitly use the identity

$$\int A_1 * \ldots * A_n = \int A_{\sigma(1)} * \ldots * A_{\sigma(n)}, \tag{4.6}$$

(where $\{\sigma(1)\ldots\sigma(n)\}$ represents any cyclic permutation of $\{1\ldots n\}$), with the understanding that all Lagrangian density terms sit under a spacetime integral. This identity means that quadratic terms in the action, and hence propagators, are identical to their commutative counterparts.

To simplify the discussion relative to the previous chapter, and to emphasize the regulator independent nature of the results, we will not *a priori* impose the vanishing of the tadpole as a renormalization condition. Instead we will include the one-point tadpole contributions, and their counterterm directly in calculating the mass renormalization of the pion. In this completely equivalent language, the two counterterms present cancel each other, up to the wavefunction renormalization, so the sum of the one-particle irreducible (1PI) graphs and the one-point tadpole insertions must be automatically finite up to wavefunction renormalization (and for Goldstone's theorem to hold at one-loop, must vanish in the $p \to 0$ limit)[8]. Furthermore, to exhibit the essentially algebraic nature of the result, we will expand the non-phase part of the integrands about zero-external momentum, in the cases where there are two propagators in the loop using the Taylor expansion

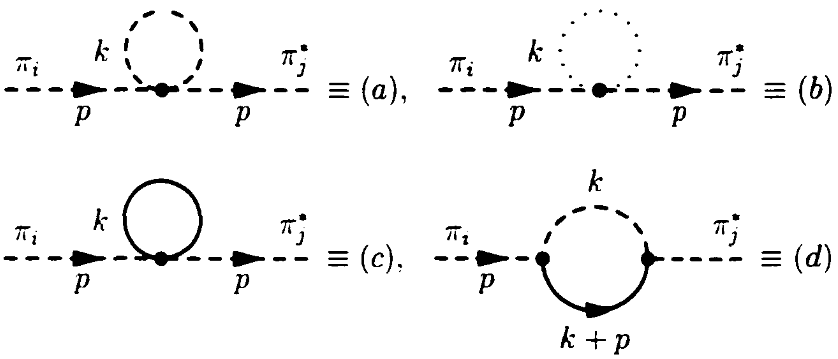
$$\frac{1}{k^2[(p+k)^2 - m^2]} = \frac{1}{k^2(k^2 - m^2)} - p_\mu \frac{2k^\mu}{k^2[k^2 - m^2]^2} + \ldots \tag{4.7}$$

and then note that the p-dependent terms yield finite loop-momentum integrals (for all p), and vanish as $p \to 0$. We then define the momentum integrals

$$I(m^2) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}, \quad I_{\theta,p}(m^2) = \int \frac{d^4k}{(2\pi)^4}\frac{\cos(k\times p)}{k^2-m^2} = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\times p}}{k^2-m^2}. \tag{4.8}$$

where $k \times p = k_\mu \theta^{\mu\nu} p_\nu$.

The vertices for the theory are listed in the appendix to this chapter, and the propagators are the usual ones. Dashes denote complex pions, dots the real pion associated with the sigma, and solid lines denote the sigma. The 1PI one-loop graphs contributing to the mass renormalization of the $N-1$ complex pions are

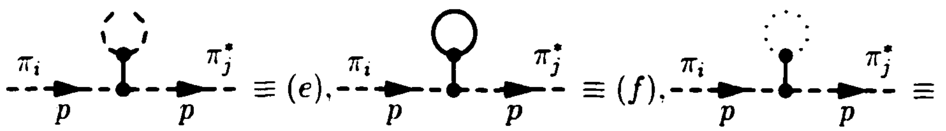


(4.9)

They are given respectively by

$$\begin{aligned}
(a) &= -2i\lambda i \int \frac{d^4k}{(2\pi)^4}\frac{\delta_{kl}\left[\delta^{ij}\delta^{kl}e^0 + \delta^{il}\delta^{jk}e^0\right]}{k^2} = 2N\lambda\delta^{ij}I(0)\\
(b) &= \frac{-2i\lambda i\delta^{ij}}{2}\int \frac{d^4k}{(2\pi)^4}\frac{e^0\cos(0)}{k^2} = \lambda\delta^{ij}I(0)\\
(c) &= \frac{-2i\lambda i\delta^{ij}}{2}\int \frac{d^4k}{(2\pi)^4}\frac{e^0\cos(0)}{k^2-2\mu^2} = \lambda\delta^{ij}I(2\mu^2)\\
(d) &= (-2i\lambda a)^2 i^2\delta^{ik}\delta_{kl}\delta^{lj}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-\frac{1}{2}(k\times p)}e^{-\frac{1}{2}(-p\times -k)}}{[(p+k)^2-2\mu^2]k^2}\\
&= \frac{4\lambda^2a^2}{2\mu^2}\delta^{ij}\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2-2\mu^2}-\frac{1}{k^2}\right] + \delta^{ij}A^\mu p_\mu\\
&= 2\lambda\delta^{ij}\left[I(2\mu^2)-I(0)\right] + \delta^{ij}A^\mu p_\mu,
\end{aligned} \tag{4.10}$$

where $A^\mu(p)$ is finite for all p. Evidently these 1PI graphs do not see the noncommutativity. Meanwhile the one-point tadpoles insertions, as in the previous chapter, also do not see the noncommutativity at one-loop. They are given by



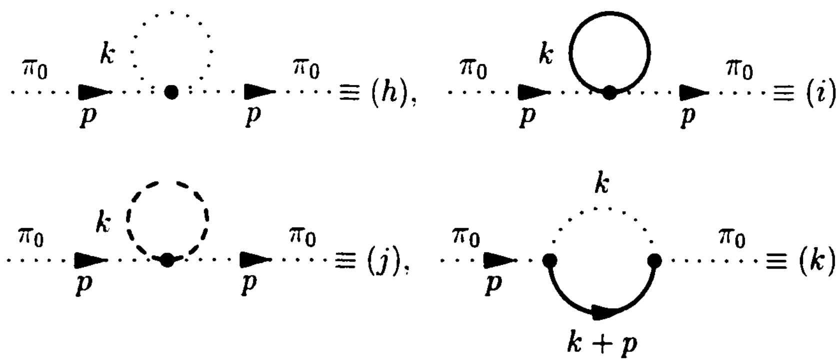
(4.11)

Their values are given by

$$\begin{aligned}
(e) &= (-2i\lambda a\delta^{ij})\frac{i}{-2\mu^2}(-2i\lambda a\delta^{kk})iI(2\mu^2) = -2(N-1)\lambda\delta^{ij}I(0) \\
(f) &= (-2i\lambda a\delta^{ij})\frac{i}{-2\mu^2}(-6i\lambda a)\frac{i}{2}I(2\mu^2) = -3\lambda\delta^{ij}I(2\mu^2) \\
(g) &= (-2i\lambda a\delta^{ij})\frac{i}{-2\mu^2}(-2i\lambda a)\frac{i}{2}I(2\mu^2) = -\lambda\delta^{ij}I(0)
\end{aligned} \tag{4.12}$$

where all noncommutative vertices manifestly collapse.

The sum of these seven graphs is equal to zero (modulo the finite term which itself vanishes as $p \to 0$), independently of a regulator, and of the noncommutativity, whence Goldstone's theorem holds at one-loop; the complex pions undergo no mass renormalization. Now consider the one-loop mass renormalization of π_0. The 1PI graphs contributing are given by



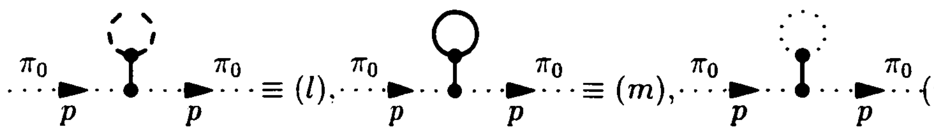
(4.13)

with values given by

$$(h) = \frac{-2i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i\left[2\cos^2(\frac{k\times p}{2})+1\right]}{k^2} = 2\lambda I(0) + \lambda I_{\theta,p}(0)$$

$$
\begin{aligned}
(i) &= \frac{-2i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i\,[2\cos^2(0)-cos(k\times p)]}{k^2-2\mu^2} = 2\lambda I(2\mu^2)-\lambda I_{\theta,p}(2\mu^2) \\
(j) &= -2i\lambda\delta^{kk}\int\frac{d^4k}{(2\pi)^4}\frac{ie^0\cos(0)}{k^2} = 2(N-1)\lambda I(0) \\
(k) &= (-2i\lambda a)^2\int\frac{d^4k}{(2\pi)^4}\frac{i^2\cos^2(\frac{p\times k}{2})}{[(p+k)^2-2\mu^2]k^2} \\
&= \frac{4\lambda^2a^2}{2\mu^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}(1+\cos(k\times p))\left[\frac{1}{k^2-2\mu^2}-\frac{1}{k^2}\right]+B^\mu_\theta p_\mu \\
&= \lambda[I(2\mu^2)-I(0)]+\lambda[I_{\theta,p}(2\mu^2)-I_{\theta,p}(0)]+B^\mu_\theta p_\mu, \qquad (4.14)
\end{aligned}
$$

where $B^\mu_\theta(p)$ is finite for all p. Evidently, the nonplanar contributions due to the noncommutativity cancel between these graphs. For completeness, the one-point tadpole insertions are



(4.15)

with values given by

$$
\begin{aligned}
(l) &= (-2i\lambda a)\frac{i}{-2\mu^2}(-2i\lambda a\delta^{kk})iI(0) = -2(N-1)\lambda I(0) \\
(m) &= (-2i\lambda a)\frac{i}{-2\mu^2}(-6i\lambda a)\frac{i}{2}I(2\mu^2) = -3\lambda I(2\mu^2) \\
(n) &= (-2i\lambda a)\frac{i}{-2\mu^2}(-2i\lambda a)\frac{i}{2}I(0) = -\lambda I(0).
\end{aligned}
\qquad (4.16)
$$

The sum of these seven graphs also vanishes (again modulo the finite p dependent term, which vanishes as $p\to 0$); so that Goldstone's theorem also holds for the neutral pion of this model.

Let us reflect on these results. First, had we included the other ordering of the quartic term $\phi_i^* * \phi_j^* * \phi_i * \phi_j$, we would again find violations of Goldstone's theorem of the type found in the previous chapter. Secondly, we contrast these results with those of the general $O(N)$ model studied in the previous chapter, where we showed violations of Goldstone's theorem at one-loop for *all* orderings consistent with the $O(N)$ global symmetry (except for the trivial Abelian case). The difference here of course is that we are working with a $U(N)$ group, which we now show exhibits crucial algebraic differences with the $O(N)$ case in noncommutative scalar theories.

Matter in the fundamental representation of $O(N)$ is described by a *real* N-vector of fields, which we denote by Ψ. As such, the invariant term $\Psi^T * \Psi$ merely is the sum of squares of the real components. Then, the expansion of the quartic invariant can yield cross-terms only of the form $\psi_i * \psi_i * \psi_j * \psi_j$, or $\psi_i * \psi_j * \psi_i * \psi_j$ corresponding to the two possible star product orderings of such an invariant. Note that no more than two distinct fields can occur.

On the other hand, the fundamental of $U(N)$ is described by a *complex* N-vector of fields, Φ. Now however, the quadratic invariant $\Phi^\dagger * \Phi$, written in terms of real fields picks up the *commutator* of each field's real part with its imaginary part due to the noncommutativity since

$$(R - iI)(R + iI) = R^2 + I^2 + i[R, I]. \tag{4.17}$$

While such commutators in the quadratic term vanish when integrated over spacetime, the quartic invariant now picks up products of such commutators with other fields or commutators which, for $N > 1$, constitute new interactions between real components of two complex ϕ's, not present in, and incompatible with, the $O(2N)$ symmetry.

Let us make this argument manifest. Expanding the quartic term in the $U(N)$ theory in terms of its real components yields

$$\phi_i^* \phi_i \phi_j^* \phi_j = (\phi_{iR} - i\phi_{iI})(\phi_{iR} + i\phi_{iI})(\phi_{jR} - i\phi_{jI})(\phi_{jR} + i\phi_{jI})$$

$$= \phi_{iR}^2\phi_{jR}^2 + \phi_{iR}^2\phi_{jI}^2 + \phi_{iI}^2\phi_{jR}^2 + \phi_{iI}^2\phi_{jI}^2 - [\phi_{iR},\phi_{iI}][\phi_{jR},\phi_{jI}]$$
$$+i(\phi_{iR}^2 + \phi_{iI}^2)[\phi_{jR},\phi_{jI}] + i(\phi_{jR}^2 + \phi_{jI}^2)[\phi_{iR},\phi_{iI}]. \qquad (4.18)$$

We note for $i \neq j$ (which can occur for $N > 1$), the presence of interactions (those involving three or four distinct real fields) which *cannot* occur in the $O(2N)$ case by the general argument above. We emphasize this is a purely noncommutative effect[2]. The presence of these extra, purely noncommutative interactions is responsible for the differing behaviour of the spontaneously broken phase at the quantum level for these models.

To conclude, we have found that one cannot in general spontaneously break a fundamental representation NC $O(N)$ linear sigma model, while one can break a fundamental representation NC $U(N)$ linear sigma model for the noncommutative gauge invariant quartic ordering. This latter theme is one that will arise again in a more dramatic fashion in the adjoint representation model to which we now turn.

4.3 NC $U(2)$ Sigma Model: Adjoint Representation

We now examine the status of Goldstone's theorem in the noncommutative deformation of the linear sigma model with scalars in the adjoint representation of $U(2)$. There are several reasons for this: first, we wish to compare the results for adjoint representation scalars with our results from the previous section for fun-

[2]For $i = j$ (or $N = 1$), the last two terms vanish under the spacetime integral, and the product of commutators merely induces the orderings of the $O(2)$ model studied in the previous chapter with $f = 2$.

damental scalars, in a tractable case. Secondly, adjoint matter naturally arises in noncommutative world-volume theories on D-branes. Thirdly, grand unified theories embedding the standard model commonly rely on adjoints for the first stage of symmetry breaking.

We write the scalars in the adjoint of $U(2)$ as

$$\Phi = \phi_a T^a = \frac{1}{2}\begin{pmatrix} \phi_4 + \phi_3 & \sqrt{2}\phi^* \\ \sqrt{2}\phi & \phi_4 - \phi_3 \end{pmatrix}, \tag{4.19}$$

where T^a are the canonical generators of $U(2)$: $T^a = \sigma^a/2$, for $a = 1, 2, 3$ and $T^4 = I_2/2$. The global $U(2)$ symmetry transformation acts as

$$\Phi \to U\Phi U^\dagger \tag{4.20}$$

and as before does not involve the star product because the symmetry is global. For simplicity we impose invariance under $\Phi \to -\Phi$. The Lagrangian density for the global model we consider is defined by

$$\mathcal{L} = \mathrm{Tr}\left(\partial_\mu \Phi * \partial^\mu \Phi\right) + \mu^2 \mathrm{Tr}\left(\Phi * \Phi\right) - \lambda_1 \mathrm{Tr}\left(\Phi * \Phi * \Phi * \Phi\right) - \lambda_2 \left[\mathrm{Tr}(\Phi * \Phi)\right]^2, \tag{4.21}$$

where we define

$$\begin{aligned} \mathrm{Tr}(\Phi_*^4) &\equiv \Phi^i_j * \Phi^j_k * \Phi^k_l * \Phi^l_i \\ [\mathrm{Tr}(\Phi * \Phi)]^2_* &\equiv \Phi^i_j * \Phi^j_i * \Phi^k_l * \Phi^l_k \end{aligned} \tag{4.22}$$

and where we discuss the remaining, omitted trace invariants and star product orderings at the end of this section.

Let us now consider spontaneous symmetry breaking which occurs for $\mu^2 > 0$ (we take $\lambda_i > 0$). Then Φ acquires a vacuum expectation value, say Φ_0, and since it is a Hermitian (but not necessarily traceless) matrix, we analyze it by diagonalization to the form

$$\Phi_0 = \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{4.23}$$

whence the potential becomes

$$V(a,b) = -\frac{\mu^2}{4}(a^2+b^2) + \frac{\lambda_1}{16}(a^4+b^4) + \frac{\lambda_2}{16}(a^4+2a^2b^2+b^4). \tag{4.24}$$

This is minimized for

$$a^2 = b^2 = \frac{\mu^2}{\frac{\lambda_1}{2}+\lambda_2} \equiv \frac{\mu^2}{\lambda}. \tag{4.25}$$

The states corresponding to $a = b$, which are degenerate in energy with the states corresponding to $a = -b$, and admitted because we are considering $U(2)$ and not simply $SU(2)$, do not reflect spontaneously broken states, because Φ_0 is then proportional to the identity and so manifestly commutes with all of the generators. Furthermore since they correspond to constant shifts in the $U(1)$ component ϕ_4, they are forbidden by the discrete symmetry. On the other hand, the states corresponding to $a = -b$ do yield spontaneously broken vacua, since they do not commute with the T^1 and T^2 generators and reflect a vacuum expectation value for the field ϕ_3.

In notation suggestive of the linear sigma model, we expand around the vacuum $b = -a < 0$ (without any loss of generality), defining σ and π through

$$\Phi' = \frac{1}{2}\begin{pmatrix} \phi_4+\sigma & \sqrt{2}\pi^* \\ \sqrt{2}\pi & \phi_4-\sigma \end{pmatrix} \equiv \Phi - \Phi_0 \tag{4.26}$$

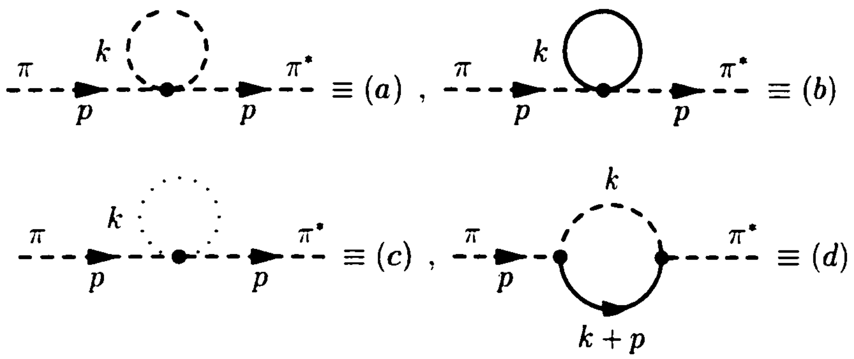
so that $\phi_3 = \sigma + a$. Expanding the scalar potential in terms of these variables yields

$$\begin{aligned} V &= \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\lambda_1 a^2)\phi_4^2 + \frac{\lambda_1+\lambda_2}{2}\pi^*\pi\pi^*\pi + \frac{\lambda_2}{2}\pi^*\pi^*\pi\pi + \\ &\quad \frac{\lambda_1+\lambda_2}{2}(\pi^*\pi+\pi\pi^*)\sigma^2 - \frac{\lambda_1}{2}\pi^*\sigma\pi\sigma + \frac{\lambda_1+\lambda_2}{2}(\pi^*\pi+\pi\pi^*)\phi_4^2 \end{aligned}$$

$$+\frac{\lambda_1}{2}\pi^*\phi_4\pi\phi_4 + a\lambda(\pi^*\pi + \pi\pi^*)\sigma + \frac{\lambda}{4}(\sigma^4 + \phi_4^4) + \lambda a\sigma^3 + \frac{\lambda_1+\lambda_2}{2}\sigma^2\phi_4^2$$
$$+\frac{\lambda_1}{4}\sigma\phi_4\sigma\phi_4 + (\lambda+\lambda_1)a\phi_4^2\sigma +$$
$$\frac{\lambda_1}{2}\left[\pi^*\phi_4\pi\sigma - \pi^*\sigma\pi\phi_4 + a(\pi\pi^*\phi_4 - \pi^*\pi\phi_4)\right] \quad (4.27)$$

using $\lambda = \lambda_1/2 + \lambda_2$.

The symmetrized vertices for this theory are listed in the appendix to this chapter. In the following, solid lines denote the σ, dots denote the ϕ_4, and dashes denote the π. Excluding the purely noncommutative interactions for separate consideration, there are four 1PI graphs contributing to the mass renormalization of the complex pion (Goldstone mode) in this model:



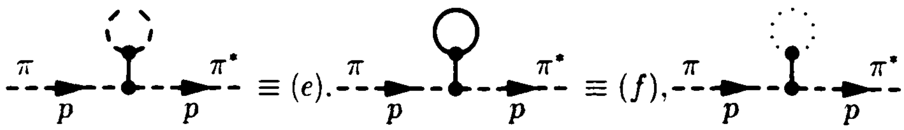
(4.28)

with values given by

$$\begin{aligned}
(a) &= -2i(\lambda_1+\lambda_2)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2} - 2i\lambda_2\int\frac{d^4k}{(2\pi)^4}\frac{i\cos^2(\frac{k\times p}{2})}{k^2}\\
&= (2\lambda_1+3\lambda_2)I(0) + \lambda_2 I_{\theta,p}(0)\\
(b) &= (\lambda_1+\lambda_2)I(2\mu^2) - \frac{\lambda_1}{2}I_{\theta,p}(2\mu^2)\\
(c) &= (\lambda_1+\lambda_2)I(\lambda_1 a^2) + \frac{\lambda_1}{2}I_{\theta,p}(\lambda_1 a^2)\\
(d) &= (-2i\lambda a)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p+k)^2-2\mu^2}\cos^2(\frac{k\times p}{2})\\
&= 4\lambda^2a^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{2\mu^2}\left[\frac{1}{k^2-2\mu^2}-\frac{1}{k^2}\right]\cos^2(\frac{k\times p}{2}) + C^\mu(p)p_\mu\\
&= \lambda\left[I(2\mu^2)-I(0)\right] + \lambda\left[I_{\theta,p}(2\mu^2)-I_{\theta,p}(0)\right] + C^\mu(p)p_\mu, \qquad (4.29)
\end{aligned}$$

where C^μ is finite for all p.

The one-point tadpole contributions are given by



$$\equiv (e), \quad \equiv (f), \quad \equiv (g) \tag{4.30}$$

whose values are respectively given by

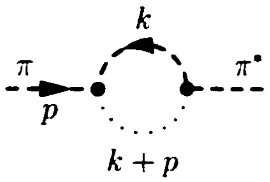
$$\begin{aligned}
(e) &= (-2i\lambda a)^2 \frac{i}{-2\mu^2}\, iI(0) = -2\lambda I(0) \\
(f) &= (-2i\lambda a) \frac{i}{-2\mu^2} (-6i\lambda a) \frac{i}{2} I(2\mu^2) = -3\lambda I(2\mu^2) \\
(g) &= (-2i\lambda a) \frac{i}{-2\mu^2} (-2i(\lambda + \lambda_1)a) \frac{i}{2} I(\lambda_1 a^2) \\
&= -(\lambda + \lambda_1) I(\lambda_1 a^2).
\end{aligned} \tag{4.31}$$

The sum of these seven graphs is given by

$$\begin{aligned}
\sum &= \frac{\lambda_1}{2}\left[I(0) - I_{\theta,p}(0)\right] - \lambda_2\left[I(2\mu^2) - I_{\theta,p}(2\mu^2)\right] - \frac{\lambda_1}{2}\left[I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2)\right] \\
&\quad + C^\mu(p) p_\mu.
\end{aligned} \tag{4.32}$$

In the commutative limit $\theta \to 0$. this degenerates to the finite term $C^\mu(p)p_\mu$ (which itself vanishes as $p \to 0$), so the mass counterterm vanishes and this is a demonstration of Goldstone's theorem for this model. However for nonzero θ, the $I(m^2)$ terms are divergent and require regularization, say by an ultraviolet cutoff Λ. But there is no counterterm freedom to cancel the Λ dependence, so for nonzero p and nonzero θ we cannot take the continuum limit; that is, UV ($\Lambda \to \infty$) and IR ($p \to 0$) limits do not commute.

However. we have (intentionally) neglected a purely noncommutative graph due to the last interaction in (4.27). The purely noncommutative interaction generated by $(\pi\pi^* - \pi^*\pi)\phi_4$ yields a graphical contribution given by



$$
\begin{aligned}
&= (-\lambda_1 a)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{\sin(\frac{p\times k}{2})\sin(\frac{-k\times -p}{2})}{k^2[(p+k)^2 - \lambda_1 a^2]} \\
&= \lambda_1^2 a^2 \int \frac{d^4k}{(2\pi)^4} \frac{\sin^2(\frac{k\times p}{2})}{\lambda_1 a^2}\left[\frac{1}{k^2-\lambda_1 a^2} - \frac{1}{k^2}\right] + D_\theta^\mu(p)p_\mu \\
&= \frac{\lambda_1}{2}\int \frac{d^4k}{(2\pi)^4}\left[1-\cos(k\times p)\right]\left[\frac{1}{k^2-\lambda_1 a^2} - \frac{1}{k^2}\right] + D_\theta^\mu p_\mu \\
&= \frac{\lambda_1}{2}\left[I(\lambda_1 a^2) - I_{\theta,p}(\lambda_1 a^2)\right] - \frac{\lambda_1}{2}\left[I(0) - I_{\theta,p}(0)\right] \\
&\qquad + D_\theta^\mu(p)p_\mu, \qquad (4.33)
\end{aligned}
$$

where again D_θ^μ is finite for all p, and vanishes also in the limit $\theta \to 0$. Rather unexpectedly, this graph, which manifestly vanishes in the commutative limit, and involves the $U(1)$ component of the matter field, cancels the λ_1 pieces in (4.32), leaving behind a residual divergence (for nonzero p) that depends only on the coupling to the $\mathrm{Tr}(\Phi^2)^2$ term in the potential.

However, in the corresponding gauge theory, the term

$$\mathrm{Tr}(\Phi_*^2) * \mathrm{Tr}(\Phi_*^2) \qquad (4.34)$$

is *not* gauge invariant even under the spacetime integral. In fact no term involving the product of more than one trace in the adjoint representation is gauge invariant (even under $\int d^Dx$) in noncommutative theories for $N > 1$. To see this write (4.34) in terms of its internal indices (first choosing the canonical ordering with respect to the star product) and gauge transform:

$$
\begin{aligned}
[\mathrm{Tr}(\Phi_*^2)]_*^2 &= \Phi^i_j * \Phi^j_i * \Phi^k_l * \Phi^l_k \\
&\to (U^i_{i1} * \Phi^{i1}_{j1} * U_j^{\dagger j1} * U^j_{j2} * \Phi^{j2}_{i2} * U_i^{\dagger i2}) \\
&\qquad *(U^k_{k1} * \Phi^{k1}_{l1} * U_l^{\dagger l1} * U^l_{l2} * \Phi^{l2}_{k2} * U_k^{\dagger k2}) \\
&= U^i_{i1} * \Phi^{i1}_{j1} * \Phi^{j1}_{i2} * U_i^{\dagger i2} * U^k_{k1} * \Phi^{k1}_{l1} * \Phi^{l1}_{k2} * U_k^{\dagger k2} \qquad (4.35)
\end{aligned}
$$

The presence of the star product does not allow us to use $U^i_j * U^{\dagger j}_k = \delta^i_k$ on the remaining *local* U and $U^\dagger$ factors which are separated by two factors of Φ, even if we use the cyclicity property of the star product under the spacetime integral. This is to be contrasted with a single internal index trace term (with canonical internal index ordering), and the commutative limit where the ordering of components is immaterial.

It is clear this argument applies both to the other internal index ordering $\Phi^i_j * \Phi^k_l * \Phi^j_i * \Phi^l_k$ (whose gauge transformation does not allow the use of $U * U^\dagger = I$ anywhere), and to any product of (internal index) traces in the adjoint representation. Thus if we forbid $[\mathrm{Tr}(\Phi^2)]^2$ from the scalar potential, by regarding the global theory as the limit of a gauge theory, we have no remaining violation of Goldstone's theorem for this model. Incidentally, this argument also forbids the other terms still allowed by the imposition of the discrete symmetry that we neglected when we wrote the scalar potential for this theory; namely

$$\mathrm{Tr}(\Phi) * \mathrm{Tr}(\Phi^3) \ , \ \mathrm{Tr}(\Phi) * \mathrm{Tr}(\Phi) \ , \ [\mathrm{Tr}(\Phi)]^4_* \ , \ \mathrm{Tr}(\Phi^2_*) * [\mathrm{Tr}(\Phi)]^2_* \tag{4.36}$$

as well as other star product orderings of the $\mathrm{Tr}(\Phi^4)$ term.

An immediate consequence of the preceding argument is that for $U(N)$ gauge theories with adjoint scalar matter, the symmetry breaking pattern is restricted to only one of the two possible patterns that would be allowed by the commutative limit of the theory. Specifically, because noncommutative gauge invariance forbids $\mathrm{Tr}(\Phi^2_*) * \mathrm{Tr}(\Phi^2_*)$, vacuum stability now requires $\lambda_1 > 0$, and thus allows only the breaking pattern $U(N) \to U(n_1) \times U(N - n_1)$ (with $n_1 = N/2$, N even; or $n_1 = (N+1)/2$, N odd)[10], and forbids $U(N) \to U(N-1)$[10].

This argument has another consequence for noncommutative theories in general. As van Raamsdonk and Seiberg [6] demonstrated in considering scalar theories with scalars represented by $N \times N$ matrices, all infrared divergences of the type found in [5] are proportional at one-loop to

$$\mathrm{Tr}(\mathcal{O}_1)\mathrm{Tr}(\mathcal{O}_2) \tag{4.37}$$

where $\mathcal{O}_i$ are operators built out of Φ. Furthermore we have seen above that an operator of this form $(\mathrm{Tr}(\Phi_*^2) * \mathrm{Tr}(\Phi_*^2))$ appearing in the scalar potential, would induce violations of Goldstone's theorem by renormalization effects. However the preceding argument indicates that these are precisely the form of operators that are not gauge invariant in an adjoint representation gauge theory. So if we regard these theories as embedded in a corresponding gauge theory where we must forbid such terms, then we would expect that infrared divergences for $N > 1$[3] of the form observed in [6], no longer appear. To corroborate the claims made in this section, we will show in Appendix A, that if we include such terms classically in a gauge theory, the Higgs particle of the theory acquires a divergent, gauge-dependent mass shift at the one-loop level. This calculation will also illustrate how to compute in a noncommutative gauge theory.

4.4 NC O(4) Sigma Model: Adjoint Representation

In this section we repeat the analysis of the previous section for the noncommutative $O(4)$ sigma model in the adjoint representation; again this will allow us to study, in a simple context, scalar representation (in)dependence of our results on Goldstone renormalization, this time in the context of orthogonal symmetry

[3]For the $N = 1$ case considered in [5], corresponding to a single scalar, the above argument fails, since the index structure becomes degenerate.

groups.

We consider the classical symmetry breaking $O(4) \to U(2)$. Now Φ is a real antisymmetric matrix, whence the vacuum state Φ_0 can be put in standard form

$$\Phi_0 = \frac{a}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.38}$$

The scalar potential is given by

$$V(\Phi) = \frac{\mu^2}{2}\mathrm{Tr}(\Phi_*^2) + \frac{\lambda_1}{4}[\mathrm{Tr}(\Phi_*^2)]_*^2 + \frac{\lambda_2}{4}\mathrm{Tr}(\Phi_*^4), \tag{4.39}$$

where we note that the sign of the quadratic term is opposite that of the $U(2)$ model of the previous section because of the antisymmetry (as opposed to Hermicity) of Φ, and where we have normalized differently for later convenience (we now assume the canonical internal index ordering with respect to the star product as per the conclusions of the previous section). Thus the minimization of the potential with respect to the vacuum (4.38), yields

$$V(\Phi_0) = -\frac{\mu^2}{2}a^2 + \frac{\lambda_1 a^4}{4} + \frac{\lambda_2 a^4}{16} \to a^2 = \frac{4\mu^2}{4\lambda_1 + \lambda_2} = \frac{\mu^2}{\lambda_1 + \frac{\lambda_2}{4}}. \tag{4.40}$$

The suitable parametrization of Φ relevant to a discussion of spontaneous symmetry breaking is given by

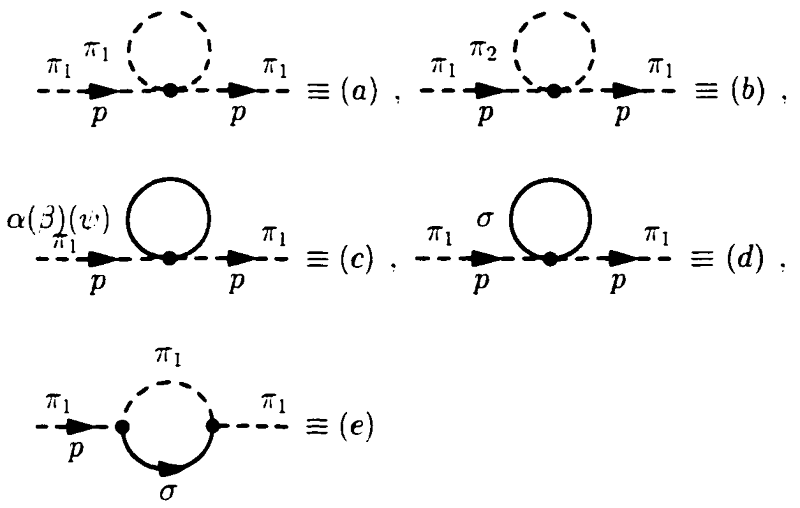
$$\frac{1}{2}\begin{pmatrix} 0 & (\sigma + a) + \psi & \alpha + \pi_1 & \beta + \pi_2 \\ -((\sigma + a) + \psi) & 0 & \pi_2 - \beta & \alpha - \pi_1 \\ -(\alpha + \pi_1) & \beta - \pi_2 & 0 & (\sigma + a) - \psi \\ -(\beta + \pi_2) & \pi_1 - \alpha & \psi - (\sigma + a) & 0 \end{pmatrix}, \tag{4.41}$$

where the σ is the field acquiring the VEV, and π_1, π_2 are the two Goldstone modes. Focusing now on the one-loop mass renormalization of one of the π's, say π_1, the expansion of the potential reads

$$\begin{aligned} V &= \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\lambda_2 a^2/2)\left[\psi^2+\alpha^2+\beta^2\right] + \frac{1}{4}\left(\lambda_1+\frac{\lambda_2}{4}\right)\pi_1^4 + \\ &\left(\frac{\lambda_1}{2}+\frac{\lambda_2}{4}\right)\pi_1^2\left[\pi_2^2+\alpha^2+\beta^2+\psi^2+\sigma^2\right] + \frac{\lambda_2}{8}\Big[-\pi_1\pi_2\pi_1\pi_2 + \\ &\pi_1\alpha\pi_1\alpha + \pi_1\beta\pi_1\beta + \pi_1\psi\pi_1\psi - \pi_1\sigma\pi_1\sigma\Big] + \left(\lambda_1+\frac{\lambda_2}{4}\right)a\sigma\Big[\sigma^2 + \\ &\pi_1^2+\pi_2^2\Big] + \left(\lambda_1+\frac{3\lambda_2}{4}\right)a\sigma\left[\alpha^2+\beta^2+\psi^2\right] + \ldots \end{aligned} \tag{4.42}$$

where the ellipsis represents (four-field) terms that do not contribute to the one-loop mass renormalization of π_1. The Feynman rules for these vertices are in the appendix. Now that we have six distinct fields, we simply use dotted lines to denote the π's, and use solid lines for the other four fields, and instead explicitly label the lines.

The 1PI graphs contributing are



(4.43)

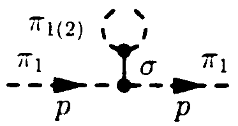
and are given respectively by

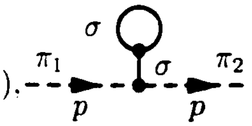
$$\begin{aligned} (a) &= -2i\left(\lambda_1+\frac{\lambda_2}{4}\right)\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1+\cos^2(\frac{k\times p}{2})}{k^2} = (\lambda_1+\frac{\lambda_2}{4})\left[2I(0)+I_{\theta,p}(0)\right] \\ (b) &= \left(\lambda_1+\frac{\lambda_2}{2}\right)I(0) - \frac{\lambda_2}{4}I_{\theta,p}(0) \end{aligned}$$

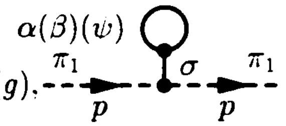
$$
\begin{aligned}
(c) &= 3\left[\left(\lambda_1+\frac{\lambda_2}{2}\right)I(\lambda_2 a^2/2)+\frac{\lambda_2}{4}I_{\theta,p}(\lambda_2 a^2/2)\right]\\
(d) &= \left(\lambda_1+\frac{\lambda_2}{2}\right)I(2\mu^2)-\frac{\lambda_2}{4}I_{\theta,p}(2\mu^2)\\
(e) &= \left[-2i\left(\lambda_1+\frac{\lambda_2}{4}\right)a\right]^2 i^2\int\frac{d^4k}{(2\pi)^4}\frac{\cos^2(\frac{p\times k}{2})}{k^2[(p+k)^2-2\mu^2]}\\
&= \frac{2\left(\lambda_1+\frac{\lambda_2}{4}\right)^2a^2}{2\mu^2}\int\frac{d^4k}{(2\pi)^4}[1+\cos(p\times k)]\left[\frac{1}{k^2-2\mu^2}-\frac{1}{k^2}\right]+D^\mu p_\mu\\
&= \left(\lambda_1+\frac{\lambda_2}{4}\right)\left[I(2\mu^2)-I(0)\right]+\left(\lambda_1+\frac{\lambda_2}{4}\right)\left[I_{\theta,p}(2\mu^2)-I_{\theta,p}(0)\right]\\
&\quad +D^\mu p_\mu. \qquad (4.44)
\end{aligned}
$$

where D^μ is finite for all p, and where the factor of three in the third graph originates from having three species of particle with the same contribution.

The one-point tadpoles contributions are



$\equiv (f)$,


$\equiv (g)$,


$\equiv (h)$ (4.45)

with values given by

$$
\begin{aligned}
(f) &= 2\times -2i\left(\lambda_1+\frac{\lambda_2}{4}\right)a\frac{i}{-2\mu^2}(-2i)\left(\lambda_1+\frac{\lambda_2}{4}\right)a\frac{i}{2}I(0)\\
&= -2\left(\lambda_1+\frac{\lambda_2}{4}\right)I(0)\\
(g) &= -2i\left(\lambda_1+\frac{\lambda_2}{4}\right)a\frac{i}{-2\mu^2}(-6i)\left(\lambda_1+\frac{\lambda_2}{4}\right)a\frac{i}{2}I(2\mu^2)\\
&= -3\left(\lambda_1+\frac{\lambda_2}{4}\right)I(2\mu^2)\\
(h) &= 3\times -2i\left(\lambda_1+\frac{\lambda_2}{4}\right)a\frac{i}{-2\mu^2}(-2i)\left(\lambda_1+\frac{3\lambda_2}{4}\right)a\frac{i}{2}I(\frac{\lambda_2a^2}{2})\\
&= -3\left(\lambda_1+\frac{3\lambda_2}{4}\right)I(\frac{\lambda_2a^2}{2})
\end{aligned}
\qquad (4.46)
$$

where the overall factors of two and three in the first and third graphs respectively again come from the multiplicity of particle species with the same contribution.

Thus the total one-loop contribution to the mass renormalization of the π_1 (or π_2) in this model is

$$\begin{aligned}\Sigma &= \frac{\lambda_2}{4}[I(0) - I_{\theta,p}(0)] - \frac{3\lambda_2}{4}\left[I(\frac{\lambda_2 a^2}{2}) - I_{\theta,p}(\frac{\lambda_2 a^2}{2})\right] \\ &\quad -\lambda_1\left[I(2\mu^2) - I_{\theta,p}(2\mu^2)\right] + D^\mu p_\mu. \end{aligned} \tag{4.47}$$

Unlike the $U(2)$ adjoint representation model, there is no purely noncommutative graph that saves us for either quartic invariant, and so again we cannot take the continuum limit ($\Lambda_{UV} \to \infty$) and Goldstone's theorem fails for this model.

4.5 Discussion

Before concluding, a final calculational note is in order. Since Goldstone's theorem is really an algebraic, and not a regulator dependent result, we have emphasized the former throughout the present chapter. In this language, the two conditions from the previous chapter ($N = 2$ and $f = 2$) which emerged from the vanishing of quadratic and logarithmic divergences, would instead arise from the separate cancellations involving the $I(2\mu^2)$ and $I(0)$ type terms. The calculations in the previous chapter had the advantage, however, of demonstrating how to explicitly compute the nonplanar pieces.

To summarize: in noncommutative field theory, $U(N)$ ($N > 1$) linear sigma models with complex scalars in the fundamental representation, do not have $O(2N)$ global invariance due to noncommutative commutator interactions between the real components, which vanish in the commutative limit. As a result of these commutator interactions, noncommutative linear $U(N)$ sigma models with fundamental matter can be continuum renormalized while preserving Nambu-Goldstone symmetry realization, at least at one-loop. This contrasts with our previous results, where we demonstrated that for noncommutative linear $O(N)$ sigma models with fundamental matter, continuum renormalization is inconsistent with Nambu-Goldstone symmetry realization already at one loop (except for the degenerate Abelian case $O(2) \equiv U(1)$).

To investigate possible scalar representation dependence of these contrasting results, we have considered linear sigma models with adjoint matter. For the adjoint $U(2)$ linear sigma model, we again find that Nambu-Goldstone symmetry realization survives at one-loop, provided we drop interaction terms (and star product orderings) which would be inconsistent with the gauging of the symmetry: noncommutative restrictions on the allowed operators in a $U(N)$ gauge theory Lagrangian also act to restrict the allowed symmetry breaking patterns. For the adjoint $O(4)$ linear sigma model, we find violations of Nambu-Goldstone symmetry realization at one-loop order, as in the fundamental $O(N)$ models. These results suggest that the difference in behaviour is determined by the symmetry group, as opposed to the scalar representation thereof.

Some of the results in this paper are suggested by D-brane dynamics. Consider the coincidence of N D3-branes in type IIB string theory, in a constant NS-NS background. It describes a $U(N)$ gauge theory in the decoupling limit. The separation of $k < N$ D3-branes from the other branes spontaneously breaks the $U(N)$ gauge symmetry down to $U(N-k) \times U(k)$, the global limit ($g_{YM} \to 0$) of which is the process described in section 3. On the other hand, since orthogonal groups are realized on branes by orientifold projections, which project out the NS-NS field responsible for noncommutativity on the brane, we expect that similar constructions with $O(N)$ groups should encounter difficulties, and our arguments bear this out.

Our results for the noncommutative linear $U(N)$ sigma models now open the possibility of building models of the elementary particles and their interactions based on noncommutative non-Abelian theories with spontaneous symmetry breaking. Clearly, to make particle physics models, it is necessary that the spontaneous symmetry breaking be consistent with the renormalization not just of the global limits of these theories, but also with their gaugings; we see two reasons to be sanguine on this point, at least at one-loop. First, the gauging of the $U(1)(= O(2))$ model [7] is consistent with spontaneous symmetry breaking for precisely the star orderings uniquely picked out[4] by our previous calculation of the Goldstone violating effects in the general noncommutative $O(N)$ fundamental linear sigma model. Second, in our treatment of the non-Abelian $U(2)$ model with adjoint scalars, violations of Nambu-Goldstone symmetry realization vanish when one restricts to the subset of couplings which would be allowed, were the symmetry to be gauged: so the limited evidence suggests that global theories may be a good guide to the behaviour of the local theories, much as in the case of commutative field theories[3], [4].

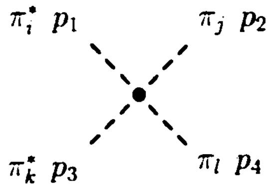
However, to go from models to actual theories would require demonstration of all-order consistency of continuum renormalization of noncommutative theories with spontaneous symmetry breaking. While failure of Nambu-Goldstone symmetry breaking can be demonstrated at one-loop, demonstrating consistency requires an all order analysis; this remains a major open issue in this field.

[4] to make the anomalous effects vanish, which can happen only in the Abelian $O(2)$ case.

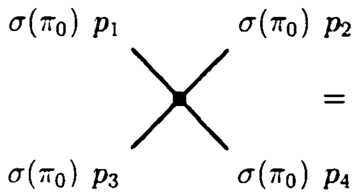
4.6 Appendix for chapter 4

All momenta flow into the vertices.

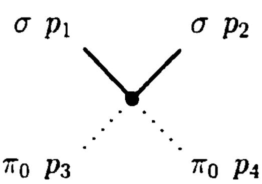
4.6.1 Scalar potential Feynman rules, $U(N)$ fundamental



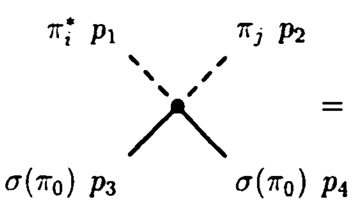
$$= -2i\lambda\left[\delta^{ij}\delta^{kl}e^{-\frac{i}{2}(p_1\times p_2+p_3\times p_4)}+\delta^{il}\delta^{jk}e^{+\frac{i}{2}(p_1\times p_2+p_3\times p_4)}\right] \tag{4.48}$$



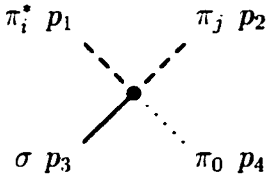
$$= -2i\lambda\left[\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2}) + \cos(\frac{p_1\times p_3}{2})\cos(\frac{p_2\times p_4}{2}) + \cos(\frac{p_1\times p_4}{2})\cos(\frac{p_2\times p_3}{2})\right] \tag{4.49}$$



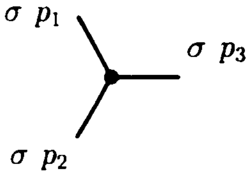
$$= -2i\lambda\left[2\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2}) - \cos(\frac{p_1\times p_3}{2}+\frac{p_2\times p_4}{2})\right] \tag{4.50}$$



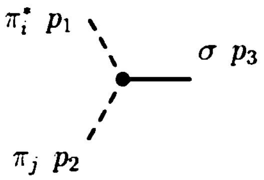
$$= -2i\lambda\delta^{ij}e^{-\frac{i}{2}(p_1\times p_2)}\cos(\frac{p_3\times p_4}{2}) \tag{4.51}$$



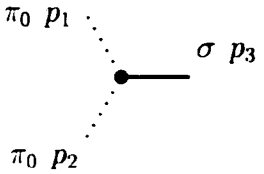
$$= \; -2i\lambda\delta^{ij}e^{-\frac{i}{2}(p_1\times p_2)}\sin(\frac{p_3\times p_4}{2}) \quad (4.52)$$



$$= \; -2i\lambda a\Big[\cos(\frac{p_1\times p_2}{2}) + \cos(\frac{p_1\times p_3}{2}) + \cos(\frac{p_2\times p_3}{2})\Big] \quad (4.53)$$

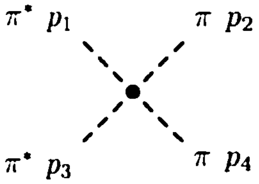


$$= \; -2i\lambda a\delta^{ij}e^{-\frac{i}{2}(p_1\times p_2)} \quad (4.54)$$

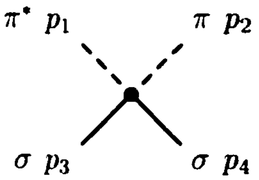


$$= \; -2i\lambda a\cos(\frac{p_1\times p_2}{2}) \quad (4.55)$$

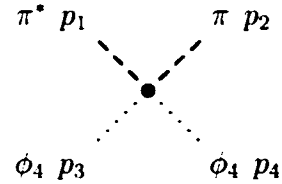
4.6.2 Scalar potential Feynman rules, $U(2)$ adjoint



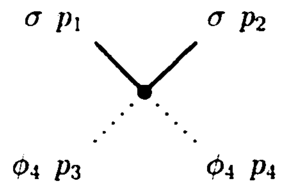
$$= \; -2i(\lambda_1+\lambda_2)\cos(\frac{p_1\times p_2}{2}+\frac{p_3\times p_4}{2}) -2i\lambda_2\cos(\frac{p_1\times p_3}{2})\cos(\frac{p_2\times p_4}{2}) \quad (4.56)$$



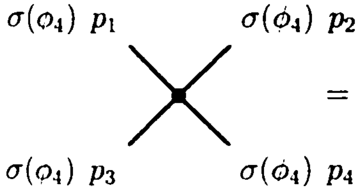
$$= \; -2i(\lambda_1+\lambda_2)\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2}) +i\lambda_1\cos(\frac{p_1\times p_3}{2}+\frac{p_2\times p_4}{2}) \quad (4.57)$$



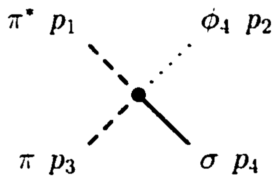
$$= -2i(\lambda_1+\lambda_2)\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2}) -i\lambda_1\cos(\frac{p_1\times p_3}{2}+\frac{p_2\times p_4}{2}) \tag{4.58}$$



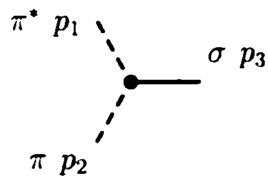
$$= -2i(\lambda_1+\lambda_2)\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2}) -i\lambda_1\cos(\frac{p_1\times p_3}{2}+\frac{p_2\times p_4}{2}) \tag{4.59}$$



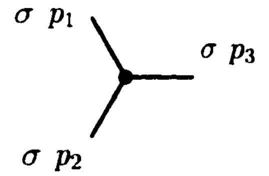
$$= -2i\lambda\left[\cos(\frac{p_1\times p_2}{2})\cos(\frac{p_3\times p_4}{2})+ \cos(\frac{p_1\times p_3}{2})\cos(\frac{p_2\times p_4}{2}) +\cos(\frac{p_1\times p_4}{2})\cos(\frac{p_2\times p_3}{2})\right] \tag{4.60}$$



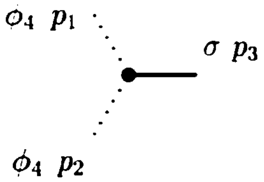
$$= -\lambda_1\sin(\frac{p_1\times p_2}{2}+\frac{p_3\times p_4}{2}) \tag{4.61}$$



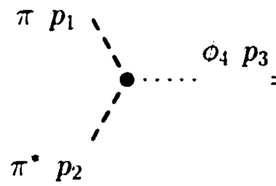
$$= -2i\lambda a\cos(\frac{p_1\times p_2}{2}) \tag{4.62}$$



$$= -2i\lambda a\left[\cos(\frac{p_1 \times p_2}{2}) + \cos(\frac{p_1 \times p_3}{2}) + \cos(\frac{p_2 \times p_3}{2})\right] \quad (4.63)$$

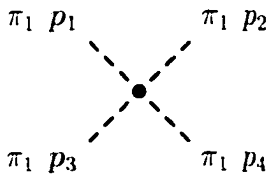


$$= -2i(\lambda + \lambda_1)a\cos(\frac{p_1 \times p_2}{2}) \quad (4.64)$$

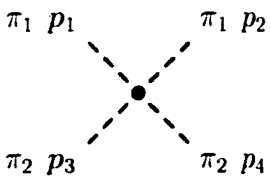


$$= -\lambda_1 a\sin(\frac{p_1 \times p_2}{2}) \quad (4.65)$$

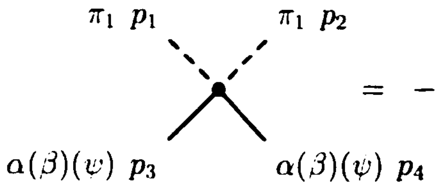
4.6.3 Scalar potential Feynman rules, O(4) adjoint (partial)



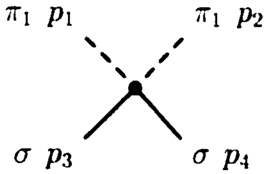
$$= -2i(\lambda_1 + \frac{\lambda_2}{4})\left[\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) + \cos(\frac{p_1 \times p_3}{2})\cos(\frac{p_2 \times p_4}{2}) + \cos(\frac{p_1 \times p_4}{2})\cos(\frac{p_2 \times p_3}{2})\right] \quad (4.66)$$



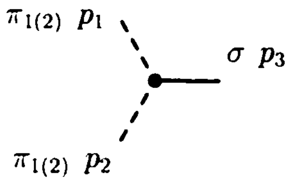
$$= -2i(\lambda_1 + \frac{\lambda_2}{2})\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) + \frac{i\lambda_2}{2}\cos(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}) \quad (4.67)$$



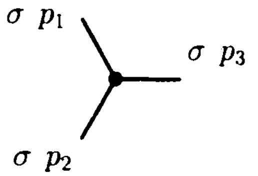
$$= -2i(\lambda_1 + \frac{\lambda_2}{2})\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) - \frac{i\lambda_2}{2}\cos(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}) \quad (4.68)$$



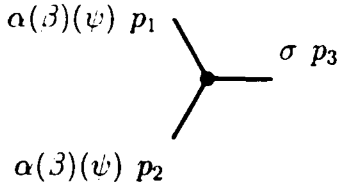
$$= -2i(\lambda_1 + \frac{\lambda_2}{2})\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) + \frac{i\lambda_2}{2}\cos(\frac{p_1 \times p_3}{2} + \frac{p_2 \times p_4}{2}) \quad (4.69)$$



$$= -2i(\lambda_1 + \frac{\lambda_2}{4})a\cos(\frac{p_1 \times p_2}{2}) \quad (4.70)$$



$$= -2i(\lambda_1 + \frac{\lambda_2}{4})a\left[\cos(\frac{p_1 \times p_2}{2}) + \cos(\frac{p_1 \times p_3}{2}) + \cos(\frac{p_2 \times p_3}{2})\right] \quad (4.71)$$



$$= -2i(\lambda_1 + \frac{3\lambda_2}{4})a\cos(\frac{p_1 \times p_2}{2}) \quad (4.72)$$

Bibliography

[1] V. Schomerus, *D-Branes and deformation quantization*, JHEP **9906**, 030 (1999). hep-th/9903205.

[2] N. Seiberg & E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999). hep-th/9908142.

[3] B.W. Lee, *Renormalization of the sigma model*, Nucl. Phys. **B9**, 649 (1969).
J.-L. Gervais, B.W. Lee, *Renormalization of the sigma-model (2) fermion fields and regularization*, Nucl. Phys. **B12**, 627 (1969).
K. Symanzik, Lettere al Nuovo Cimento **1**, 10 (1969).
K. Symanzik, *Renormalizable models with simple symmetry breaking. 1. Symmetry breaking by a source term*, Comm. Math. Phys., **16**, 48 (1970).

[4] M. Veltman, *Perturbation theory of massive Yang-Mills fields*, Nucl. Phys. **B7**, 637 (1968).
G. 't Hooft, *Renormalizable Lagrangians for massive Yang-Mills fields*, Nucl. Phys. **B35**, 167 (1971).
G. 't Hooft & M. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. **B44** 189 (1972).
G. 't Hooft & M. Veltman, *Combinatorics of gauge fields*, Nucl. Phys. **B50** 318 (1972).

[5] S. Minwalla, M. Van Raamsdonk & N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002**, 020 (2000). hep-th/9912072.

[6] M. Van Raamsdonk & N. Seiberg, *Comments on noncommutative perturbative dynamics*, JHEP **0003**, 035 (2000). hep-th/0002186.

[7] F. Petriello, *The Higgs mechanism in noncommutative gauge theories*, hep-th/0101109.

[8] T.-P. Cheng & L.-F. Li, *Gauge Theory of Elementary Particle Physics*, (Oxford University Press, Oxford, 1984).

[9] S.S. Gubser & S.L. Sondhi, *Phase structure of non-commutative scalar field theories*, to appear in Nucl. Phys. B. hep-th/0006119.

[10] L.-F. Li, *Group theory of spontaneously broken gauge symmetries*, Phys. Rev. **D9** 1723 (1974).

Chapter 5

Conclusions

String theory is an intensely active branch of theoretical physics, with the most ambitious goal of all of science: to understand the fundamental structures of the universe. However, as discussed at the outset, because physics is an experimental and phenomenological science, string theory must therefore ultimately make contact with the real world in order for it to be seriously accepted by the broader scientific community.

We have reviewed how string theory is, in fact, a potentially vast predictive framework: there are no free dimensionless parameters in the theory, so in principle everything is already determined by the theory once the dynamics that select the vacuum are understood. Furthermore, even perturbative string theory makes such dramatic predictions as the number of spacetime dimensions we live in, and sets the minimum length scale below which the theory has an alternate description at larger length scales. We have also seen that under fairly trivial assumptions, string theory even predicts that spacetime itself becomes noncommutative. As an example of a more particle physics-like 'prediction' that we unfortunately did not have space to explore. the mystery of why nature chose to replicate itself with essentially three identical generations of particles (at low energies at least) is in principle solved in string theory: the net number of chiral generations minus antichiral fermion generations is determined as a *topological* property of the compactification (Calabi-Yau) manifold. It is hard to envision a more elegant solution to this perplexing problem that was succinctly summarized by I.I. Rabi's famous remark as "Who ordered that?". Of course this is an 'in principle' resolution because again, until we understand how vacua are dynamically selected. we do not know *which* Calabi-Yau manifold vacuum is the one we might inhabit. But it illustrates the point that *a priori*, string theory is capable of making predictions. if not specific (because of our lack of understanding about nonperturbative string theory), then general, as the topics explored in this thesis indicate.

We saw how string theory compactifications to four-dimensions can predict the presence of a pseudo-anomalous $U(1)$ symmetry. which is spontaneously broken. and hence can generically give rise to vortex configurations. In chapter 2 we studied such string theory motivated vortices, which now include couplings to the dilaton and the model-independent axion (which is the four-dimensional remnant of the universal $B_{\mu\nu}$ field). We showed how the anomaly remnants and the stringy couplings to the dilaton and axion can be treated as a perturbation about the the standard Nielsen-Olesen vortex system. Then, after reducing the system to a set of coupled ordinary differential equations for the Higgs modulus, gauge field *and* the dilaton, we solved the system for the dilaton at first order. We found that the dilaton inexorably diverged in the effective theory as one approaches the vortex core, essentially as a consequence of the two-dimensional nature of the vortex system. This was confirmed with generalized arguments, and a self-consistent nonperturbative solution. Since the dilaton vacuum expectation value sets the strength of quantum corrections in string theory (and specifically the strength of the gauge coupling in this model), such divergent behaviour signals the breakdown of the classical effective description of the system, and the emergence of a strongly-coupled, intrinsically quantum mechanical regime. Such string-motivated objects, if they are to have a classical exterior, necessarily surround a core that is quantum mechanical.

We also saw how the universal $B_{\mu\nu}$ antisymmetric tensor field of string theory can predict the emergence of spacetime noncommutativity of a specific form, which has its clearest interpretation in the field-theoretic or zero-slope limit of string theory. Since previous studies of noncommutative field theories had indicated unusual ultraviolet-infrared mixings in even the simplest of such theories, it was natural to undertake an analysis of spontaneous symmetry breaking in this framework, where in commutative case, *massless* Goldstone particles are predicted and renormalizability hinges on delicate graphwise cancellations. This was the subject of chapters 3 and 4, as well as the corroborative appendices. In chapter 3 we studied the most general noncommutative $O(N)$ linear sigma model in the spontaneously broken phase. In particular we studied a key signature of spontaneous symmetry breaking at the quantum level: the mass renormalization of the tree-level massless pions. We found that because one-point functions are insensitive at one-loop to the noncommutativity, while the 1PI corrections to the inverse pion propagator do see the noncommutativity in a nontrivial way, the cancellations present in the commutative case that ensure that pions remain massless at the quantum level are violated for all $N > 2$ and all star product orderings of the potential terms. Consequently, the continuum renormalization of the model is in conflict with Nambu-Goldstone symmetry realization for $N > 2$, a conclusion also borne out by studying the four-point functions of the theory, as in the second appendix. The case of $N = 1$ for the particular star ordering corresponding to $f = 2$ led to no conflict with renormalization and Goldstone's theorem.

This takes us directly to the studies in chapter 4, where we investigated group and representation dependence of these results. We found that for the $U(N)$ linear sigma model, again with matter in the fundamental representation, that there was no conflict between renormalization and Goldstone symmetry realization, at least at one-loop for the star product ordering consistent with a possible noncommutative gauging of the model. For $N = 1$, this ordering corresponds precisely with the one consistent $O(N)$ model: $O(2)$ with $f = 2$. We explained the differing behaviour of the two models as arising from the presence of purely noncommutative commutator interactions that are consistent with $U(N)$, but not $O(2N)$, symmetries. When we then studied representation dependence, we found that the $U(2)$ model with adjoint representation matter could only be consistently renormalized in the spontaneously broken phase and Goldstone's theorem retained if we forbade multiple trace terms in the scalar potential, which are incompatible with noncommutative gauge invariance, even though we had not imposed *a priori* local gauge invariance on our Lagrangian. (The noncommutative sickness of such terms was corroborated in the first appendix, where we showed that in the corresponding putative gauge theory, the physical Higgs field would acquire a divergent, gauge-dependent, on-shell mass shift at one-loop proportional to the coupling of a multiple trace term.) In particular, we found that in the $U(2)$ model, a purely noncommutative graph played a key role in ensuring that the remaining potential divergences were cancelled. Finally, we studied the $O(4)$ adjoint representation noncommutative linear sigma model in the broken phase, and showed that as in the $O(N)$ fundamental case, there was no way to consistently renormalize the theory and preserve Goldstone's theorem; this time there was no purely noncommutative graph to save us.

Thus we were led to conclude that $U(N)$ symmetry groups can be used in models of spontaneous symmetry breaking in a noncommutative context, while $O(N)$ models cannot. Furthermore, we saw that the consistency of the spontaneously broken $U(N)$ models at the quantum level is intimately connected with (noncommutative) gauge invariance, even though the we had not imposed noncommutative gauge invariance at the outset. These claims are consistent with string theory, in which the orientifold projection on a D-brane required to obtain orthogonal gauge groups projects out the B-field responsible for the spacetime noncommutativity. This supports the contention that noncommutative field theories 'know' about string theory.

Finally we note that these disparate studies are related by the fact that they are both directly connected to the $B_{\mu\nu}$ field that is present in all perturbative string theories. In the first instance, the $B_{\mu\nu}$ field is responsible for the anomaly cancellation, and sits in the same supermultiplet that the dilaton does. In the second instance, a constant B-field on a Dp-brane leads directly to spacetime noncommutativity. So in conclusion, we have seen in this thesis some *possible* consequences of the $B_{\mu\nu}$ field in string theory on low-energy physics; that is, we have studied specific aspects of the field-theoretic limits of string theory.

Appendix A

Corroboration by gauge theory

In this appendix, we investigate the consequences of noncommutativity-induced gauge *non*invariance (of terms like $[\mathrm{Tr}(\Phi * \Phi)]_*^2$) on the renormalization of physical quantities: we study the one-loop corrections to the physical Higgs mass for a noncommutative (NC) $U(2)$ adjoint representation Higgs model. We emphasize that we are looking for a pathology that terms like $[\mathrm{Tr}(\Phi*\Phi)]_*^2$ induce, that single trace terms like $\mathrm{Tr}(\Phi_*^4)$ do not. This will also serve to illustrate how noncommutative gauge calculations are approached. We will define the R_ξ gauge fixing for the theory so that we may attempt to quantize it, and will study the simplest quantity which must be gauge-independent: the *on-shell* mass renormalization of the physical Higgs particle (the σ of our model). An excellent reference for the methodology used here is given in the original article by Appelquist, Carazzone, Goldman and Quinn: *Renormalization and gauge independence in spontaneously broken gauge theories*, Phys. Rev. **D8:6**, 1747 (1973).

The NC $U(2)$ gauge transformations on the scalar field Φ in the adjoint representation read

$$\Phi \to U_2 * \Phi * U_2^\dagger. \qquad \text{(A.1)}$$

Expanding these gauge transformations to linear order in the gauge parameter yields

$$\lambda_a T^a = \frac{1}{2}\begin{pmatrix} \lambda_4 + \lambda_3 & \sqrt{2}\lambda^* \\ \sqrt{2}\lambda & \lambda_4 - \lambda_3 \end{pmatrix}, \tag{A.2}$$

whence the components of Φ defined in (4.19) have infinitesimal transformations

$$\delta\phi \sim \frac{i}{4}\Big[\{\lambda, \phi_3\} + [\lambda, \phi_4] + [\lambda_4, \phi] - \{\lambda_3, \phi\}\Big] \tag{A.3}$$

$$\delta\phi^* \sim \frac{i}{4}\Big[[\lambda^*, \phi_4] - \{\lambda^*, \phi_3\} + \{\lambda_3, \phi^*\} + [\lambda_4, \phi^*]\Big] \tag{A.4}$$

$$\delta\phi_3 \sim \frac{i}{4}\Big[[\lambda_3, \phi_4] + [\lambda_4, \phi_3] + \{\lambda^*, \phi\} - \{\phi^*, \lambda\}\Big] \tag{A.5}$$

$$\delta\phi_4 \sim \frac{i}{4}\Big[[\lambda_3, \phi_3] + [\lambda_4, \phi_4] + [\lambda^*, \phi] + [\lambda, \phi^*]\Big], \tag{A.6}$$

where λ_i are not to be confused with the scalar potential coupling constants; they will never appear in the same discussion.

We introduce gauge fields in the adjoint of NC $U(2)$ as

$$\mathcal{A}^\mu = \frac{1}{2}\begin{pmatrix} A_4^\mu + A_3^\mu & \sqrt{2}A^{\mu *} \\ \sqrt{2}A^\mu & A_4^\mu - A_3^\mu \end{pmatrix}. \tag{A.7}$$

The gauge transformation of $\mathcal{A}_\mu$ is given by

$$\mathcal{A}_\mu \to U_2 * \mathcal{A}_\mu * U_2^\dagger - \frac{i}{g}(\partial_\mu U_2) * U_2^\dagger \tag{A.8}$$

and the gauge-invariant (under the spacetime integral) field strength is given by

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A} - ig[A_\mu, A_\nu]_*, \tag{A.9}$$

which generates the usual kinetic term $-(1/2)\mathrm{Tr}(\mathcal{F}^{\mu\nu} * \mathcal{F}_{\mu\nu})$.

This allows us to build the covariant derivative for Φ in the usual way for a field in the adjoint representation. Defining

$$D_\mu \Phi = \partial_\mu \Phi - ig[\mathcal{A}_\mu, \Phi]_*, \tag{A.10}$$

the correctly normalized kinetic term for Φ now reads

$$\mathrm{Tr}(D_\mu\Phi * D^\mu\Phi) = \mathrm{Tr}(\partial_\mu\Phi\partial^\mu\Phi) - 2ig\mathrm{Tr}(\partial_\mu\Phi * [\mathcal{A}^\mu, \Phi]_*) - g^2\mathrm{Tr}([\mathcal{A}^\mu, \Phi]_* * [\mathcal{A}_\mu, \Phi]_*) \tag{A.11}$$

using $\int \mathrm{Tr}(A * [B, C]_*) = \int \mathrm{Tr}([B, C]_* * A)$.

Due to the noncommutativity (and exacerbated by the fact that we have a non-Abelian gauge group), the complete expansion of some of these objects in component form yields a large number of terms, especially in the $\mathrm{Tr}([\mathcal{A}, \Phi][\mathcal{A}, \Phi])$ term: because of our stated intention to study the mass renormalization of the Higgs at one-loop, we will only exhibit the terms we will later need.

Next, consider as usual translationally invariant vacua, where the discussion of spontaneous symmetry breaking (at the classical level) proceeds exactly as in chapter 4. Thus, defining as before $\phi_3 = \sigma + a$ and $\pi = \phi$, we have successively

$$\mathrm{Tr}(\partial_\mu\Phi\partial^\mu\Phi) = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\phi_4)^2\right] + \partial_\mu\pi\partial^\mu\pi^*, \tag{A.12}$$

$$\begin{aligned} -2ig\mathrm{Tr}(\partial_\mu\Phi[\mathcal{A}^\mu, \Phi]) &= -\frac{ig}{2}\Big\{\partial_\mu\sigma\Big[[A_3^\mu, \phi_4] + [A_4^\mu, \sigma] + \{A^{*\mu}, \pi\} - \{\pi^*, A^\mu\}\Big] \\ &+ \partial_\mu\phi_4\Big[[A_3^\mu, \sigma] + [A_4^\mu, \phi_4] + [A^{*\mu}, \pi] + [A^\mu, \pi^*]\Big] \\ &+ \partial_\mu\pi\Big[[A_4^\mu, \pi^*] + [A^{*\mu}, \phi_4] + \{A_3^\mu, \pi^*\} - \{A^{*\mu}, \sigma\}\Big] \\ &+ \partial_\mu\pi^*\Big[[A_4^\mu, \pi] + [A^\mu, \phi_4] - \{A_3^\mu, \pi\} + \{A^\mu, \sigma\}\Big] \\ &+ 2a(\partial_\mu\pi^* A^\mu - \partial_\mu\pi A^{*\mu})\Big\}, \end{aligned} \tag{A.13}$$

(the last two terms are the ones the R_ξ gauges are engineered to cancel) and

$$\begin{aligned} &-g^2\mathrm{Tr}([\mathcal{A}^\mu, \Phi][\mathcal{A}_\mu, \Phi]) \\ &= \frac{-g^2}{4}\Big\{A_3^\mu[\sigma, A_{3\mu}]\sigma + A_4^\mu[\sigma, A_{4\mu}]\sigma + A_3^\mu[\phi_4, A_{3\mu}]\phi_4 + A_4^\mu[\phi_4, A_{4\mu}]\phi_4 \\ &+ A_3^\mu[\sigma, A_{4\mu}]\phi_4 + A_3^\mu[\phi_4, A_{4\mu}]\sigma + A_4^\mu[\phi_4, A_{3\mu}]\sigma + A_4^\mu[\sigma, A_{3\mu}]\phi_4 \end{aligned}$$

$$
\begin{aligned}
&+\left[2A^\mu\phi_4 A^*_\mu\phi_4 - \phi_4^2(A^*_\mu A^\mu + A_\mu A^{*\mu})\right] + 2(\phi_4 A^\mu\sigma A^*_\mu - \sigma A^\mu\phi_4 A^*_\mu)\\
&+\phi_4\sigma[A^\mu, A^*_\mu] + \sigma\phi_4[A^\mu, A^*_\mu] + 4a\phi_4[A^\mu, A^*_\mu]\Big\}\\
&+g^2\left\{\frac{1}{4}\left[2A^\mu\sigma A^*_\mu\sigma + \sigma^2(A^*_\mu A^\mu + A_\mu A^{*\mu})\right]\right.\\
&\left.+a\sigma(A^\mu A^*_\mu + A^*_\mu A^\mu) + a^2 A^*_\mu A^\mu\right\} + \ldots \qquad \text{(A.14)}
\end{aligned}
$$

where we have written the terms which survive in the commutative limit in (A.14) at the end, and where the ellipsis represents four-field terms involving π and π^* that do not contribute to the one-loop corrections to the inverse Higgs/sigma propagator. The last term in (A.14) gives the complex gauge field A^μ its properly weighted mass $M \equiv ag$. The Feynman rules we will need are displayed at the end of this appendix.

The construction of the R_ξ gauge fixing proceeds as in the commutative case, because the gauge-fixing function is linear in the fields, so its Gaussian weighted insertion into the Lagrangian density is at most quadratic in the fields: by design, it is to cancel $A - \pi$ mixing terms. Explicitly, we take

$$
\mathcal{G} \equiv \frac{1}{2\sqrt{\xi}}\begin{pmatrix} G_4 + G_3 & \sqrt{2}G^* \\ \sqrt{2}G & G_4 - G_3 \end{pmatrix}, \qquad \text{(A.15)}
$$

with

$$
\begin{aligned}
G = \partial_\mu A^\mu - ig\xi a\pi \quad &, \quad G^* = \partial_\mu A^{*\mu} + ig\xi a\pi^*\\
G_3 = \partial_\mu A_3^\mu \quad &, \quad G_4 = \partial_\mu A_4^\mu, \qquad \text{(A.16)}
\end{aligned}
$$

so the contribution to the Lagrangian density is given by

$$
\begin{aligned}
\mathcal{L}_{gf} = -\mathrm{Tr}(\mathcal{G}^2) \;&=\; -\frac{1}{2\xi}\left[(\partial_\mu A_3^\mu)^2 + (\partial_\mu A_4^\mu)^2\right] - \frac{1}{\xi}(\partial_\mu A^\mu)(\partial_\nu A^{*\nu})\\
&\quad + iag(\partial_\mu\pi^* A^\mu - \partial_\mu\pi A^{*\mu}) - \xi a^2 g^2\pi\pi^* \qquad \text{(A.17)}
\end{aligned}
$$

after integrating by parts the mixing terms and dropping the total derivative.

This gives the usual gauge-dependent mass term to the would-be Goldstone mode, and signals that the π is now unphysical.

Recalling that even in the commutative Abelian Higgs model, the ghosts are coupled to the physical Higgs field, we now construct the piece of the ghost Lagrangian we need. The σ couples to the ghosts corresponding to the complex vector field A^μ. To each *real* gauge field (more precisely to each real gauge fixing function), corresponds a *complex* ghost. Defining the infinitesimal gauge transformations for the real and imaginary components of A^μ through $(A^\mu + A^{*\mu})/\sqrt{2}$ and $i(A^{*\mu} - A^\mu)/\sqrt{2}$ respectively, we obtain

$$\begin{aligned} \delta A_1^\mu &= \frac{1}{4}\left[\{\lambda_3, A_2^\mu\} - \{\lambda_2, A_3^\mu\} + \text{commutators}\right] + \frac{1}{2g}\partial^\mu \lambda_1 \\ \delta A_2^\mu &= \frac{1}{4}\left[\{\lambda_1, A_3^\mu\} - \{\lambda_3, A_1^\mu\} + \text{commutators}\right] + \frac{1}{2g}\partial^\mu \lambda_2, \end{aligned} \tag{A.18}$$

where the omitted commutators will not contribute to the linear object $(\delta G)/(\delta\lambda)$ needed to construct the Fadeev Popov ghosts below. The infinitesimal gauge transformations of the real and imaginary components of π, are similarly obtained. Thus for our purposes, we need only note the presence of two complex ghosts denoted by c_1 and c_2, corresponding to the gauge field A^μ (or the gauge-fixing function G) and the ghost-ghost-Higgs couplings associated with these ghosts, which we now fix. We first decompose G

$$G = \partial_\mu A^\mu - ig\xi a\pi \rightarrow \begin{cases} G_1 = \partial_\mu A_1 + g\xi a\pi_2 \\ G_2 = \partial_\mu A_2 - g\xi a\pi_1 \end{cases}. \tag{A.19}$$

Then from (A.3), and (A.18) we obtain

$$\frac{\delta\pi_1}{\delta\lambda_2} = -\frac{1}{2}(\sigma + a)\,, \quad \frac{\delta\pi_2}{\delta\lambda_1} = \frac{1}{2}(\sigma + a)\,, \quad \frac{\delta A_1^\mu}{\delta\lambda_1} = \frac{\delta A_2^\mu}{\delta\lambda_2} = \frac{1}{2g}\partial^\mu. \tag{A.20}$$

Combining these, we get the derivatives

$$\frac{\delta G_1}{\delta\lambda_1} = \frac{\delta G_2}{\delta\lambda_2} = \frac{1}{2g}\partial_\mu\partial^\mu + \frac{1}{2}\xi a g(\sigma + a) \tag{A.21}$$

which yield the desired interactions (modulo a factor of $2g$, which we absorb into the definition of the ghosts themselves to obtain canonical kinetic terms), plus canonical ghost kinetic and mass terms:

$$\bar{c}_1 \left[-\partial_\mu \partial^\mu - \xi g^2 a(\sigma + a)\right] c_1 + (1 \leftrightarrow 2). \tag{A.22}$$

Of course there are other interactions between ghosts and gauge fields, [because we have a (NC) non-Abelian group] and arising from other possible derivatives that we will not require.

In commutative gauge theories, the ghost terms are written schematically as $\bar{c}(\delta G/\delta\lambda)c$, which includes terms of the form $\bar{c}\sigma c$. We have two possible noncommutative orderings for such terms, and as in the BRST treatment of unbroken noncommutative gauge theories we must include both orderings, symmetrically weighted. In the calculation we report below, we will find that in the absence of such symmetric weightings of the ghost orderings, the gauge-dependence of the renormalized theory would be more severe. Thus the final piece of the Fadeev-Popov ghost contribution to the Lagrangian density that we will require is given by

$$\mathcal{L}_{FPG} = \bar{c}_1 \left[-\partial_\mu \partial^\mu - \xi M^2\right] c_1 - \frac{\xi M^2}{2a}\left[\bar{c}_1 \sigma c_1 + \bar{c}_1 c_1 \sigma\right] + (1 \leftrightarrow 2) + \ldots \tag{A.23}$$

where we use $M = ag$, and where the ellipsis denotes the aforementioned ghost-ghost-gauge couplings we do not need here.

Putting it all together, the classical R_ξ gauge-fixed Lagrangian density is given by

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}\mathrm{Tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) + \mathrm{Tr}(D_\mu\Phi D^\mu\Phi) + \mu^2\mathrm{Tr}(\Phi^2) - \lambda_1\mathrm{Tr}(\Phi^4) - \lambda_2\left[\mathrm{Tr}(\Phi^2)\right]^2 \\ & + \mathcal{L}_{gf} + \mathcal{L}_{FPG},\end{aligned} \tag{A.24}$$

where star products are implicit. We now discuss the one-loop on-shell mass renormalization of the physical Higgs, which we have been denoting by σ.

We generate the counterterms for this model by rescaling the fields and parameters according to

$$\begin{aligned} \mathcal{A}_\mu &\to \sqrt{Z_3}\mathcal{A}_\mu \ , \ \Phi \to \sqrt{Z}\Phi \ , \ g \to \frac{1}{\sqrt{Z_3}} g \\ \mu^2 &\to \frac{Z_\mu}{Z}\mu^2 \ , \ \lambda_1 \to \frac{Z_{\lambda_1}}{Z^2}\lambda_1 \ , \ \lambda_2 \to \frac{Z_{\lambda_2}}{Z^2}\lambda_2 \end{aligned} \tag{A.25}$$

applied to the symmetric Lagrangian (the gauge fixing terms. and the Fadeev-Popov terms are assumed to be written in terms of renormalized fields). The only terms out of the counterterm Lagrangian we need are the terms associated with σ. and σ^2:

$$\begin{aligned} &\left[Z_\mu\lambda - Z_{\lambda_1}\frac{\lambda_1}{2} - Z_{\lambda_2}\lambda_2\right]a^3\sigma + \frac{1}{2}\left[(Z_\mu - 1)\mu^2 - \frac{3}{2}(Z_{\lambda_1} - 1)\lambda_1 a^2 \right. \\ &\left. - 3(Z_{\lambda_2} - 1)\lambda_2 a^2 + p^2(Z - 1)\right]\sigma^2 \subset \mathcal{L}_{ct}. \end{aligned} \tag{A.26}$$

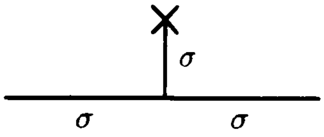
At this point we note that Z_μ/Z must be a gauge-independent quantity, since it represents the (symmetric) mass renormalization of the model. In the loop expansion,

$$\frac{Z_\mu}{Z} = \frac{1 + Z_\mu^{(1)} + O(\hbar^2)}{1 + Z^{(1)} + O(\hbar^2)} = 1 + (Z_\mu^{(1)} - Z^{(1)}) + O(\hbar^2) \tag{A.27}$$

which means that $Z_\mu - Z$ must be gauge-independent to lowest nontrivial order. This quantity is proportional to the on-shell mass renormalization of the physical Higgs σ as follows. Without imposing a condition on the one-point Higgs amplitude (which is itself gauge-dependent and divergent, but unphysical), there are two types of one-loop quantum corrections to the inverse Higgs propagator: the usual 1PI self-energy graphs (and their counterterm), and one-point Higgs tadpole insertions (and their counterterm). From (A.26) these two counterterms are

$$= i\left[(Z_\mu - 1)\mu^2 - \frac{3}{2}(Z_{\lambda_1} - 1)\lambda_1 a^2 - 3(Z_{\lambda_2} - 1)\lambda_2 a^2\right.$$

$$+p^2(Z-1)\Big]$$



$$= -3ia^2\left[Z_\mu\lambda - \frac{\lambda_1}{2}Z_{\lambda_1} - \lambda_2 Z_{\lambda_2}\right]. \tag{A.28}$$

Using $\lambda \equiv \lambda_1/2 + \lambda_2$, the sum of these two counterterms equals

$$\sum_{ct} = -i\left[2\lambda a^2(Z_\mu - 1) - p^2(Z-1)\right], \tag{A.29}$$

which, when evaluated on the mass-shell of the Higgs ($p^2 = 2\mu^2 = 2\lambda a^2$) equals

$$\sum_{ct}(p^2 = 2\mu^2) = -2\lambda a^2 i\left[Z_\mu - Z\right]. \tag{A.30}$$

Thus, denoting $\Pi(p^2)$ as the sum of all 1PI and one-point tadpole corrections to the inverse σ propagator, performing on-shell mass subtraction means that

$$\Pi(2\mu^2) = 2\lambda a^2 i\left[Z_\mu - Z\right], \tag{A.31}$$

and so should be gauge-invariant by the previous argument.

We now come to the main calculation where we will show that the divergent (i.e. cutoff-dependent[1]) part of this graphical sum is not gauge-invariant even when evaluated on-shell. Because the nonplanar parts of these one-loop graphs are finite, in essence we will only be examining the now 're-weighted' planar parts of these graphs. Since we are discarding the finite nonplanar pieces (which become divergent themselves as $\theta \to 0$), we will separately keep track of the commutative values of the relevant graphs as a double check at the end. Finally, we will proceed as far as possible algebraically [by getting momentum independent pieces separately from $O(p^2)$ pieces] in order to see how most of the ξ (i.e. gauge parameter) dependence is still cancelled. The presence of divergent wave-function

[1]Of course, we will be using dimensional regularization which respects gauge symmetries.

renormalizations due to momentum-dependent vertices (i.e. outside of the non-commutative phases) will not allow us to completely and conveniently carry this out, so we will express all remaining divergences in terms of the dimensional pole at $D = 4$.

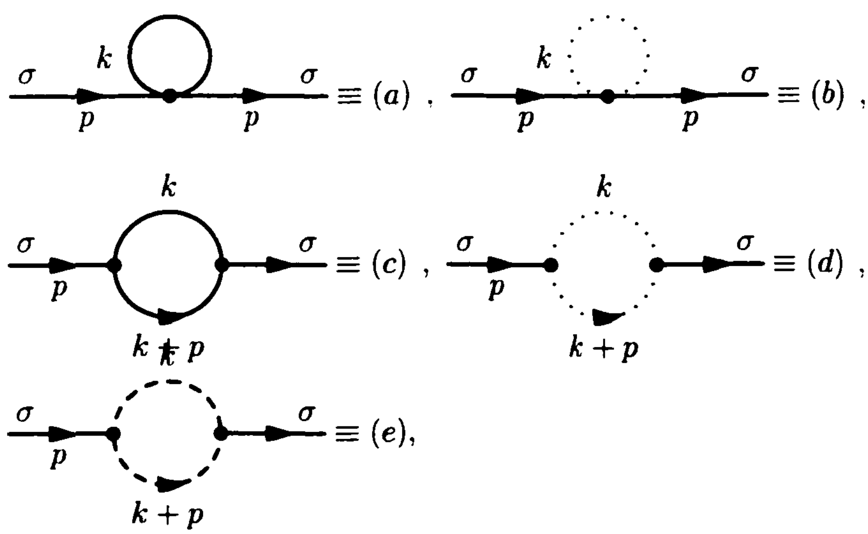
We will handle the purely noncommutative graphs at the end; first we will calculate the 1PI graphs that survive in the commutative limit. We use the definition

$$I_n(m^2) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - m^2)^n}, \tag{A.32}$$

which will be useful for the momentum-independent pieces, the identity $\Gamma(2 - D/2) = (1 - D/2)\Gamma(1 - D/2) = -\Gamma(1 - D/2) +$ finite, near $D = 4$, and the dimensional regularization formulae

$$\begin{aligned}
\int \frac{d^Dk}{(2\pi)^D} \frac{(k^2)^2}{(k^2 - \Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{D(D+2)}{4} \frac{\Gamma(n - D/2 - 2)}{\Gamma(n)} \Delta^{D/2-n+2} \\
\int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2 - \Delta)^n} &= \frac{(-1)^{n-1} i}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n - D/2 - 1)}{\Gamma(n)} \Delta^{D/2-n+1} \\
\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \Delta^{D/2-n}.
\end{aligned} \tag{A.33}$$

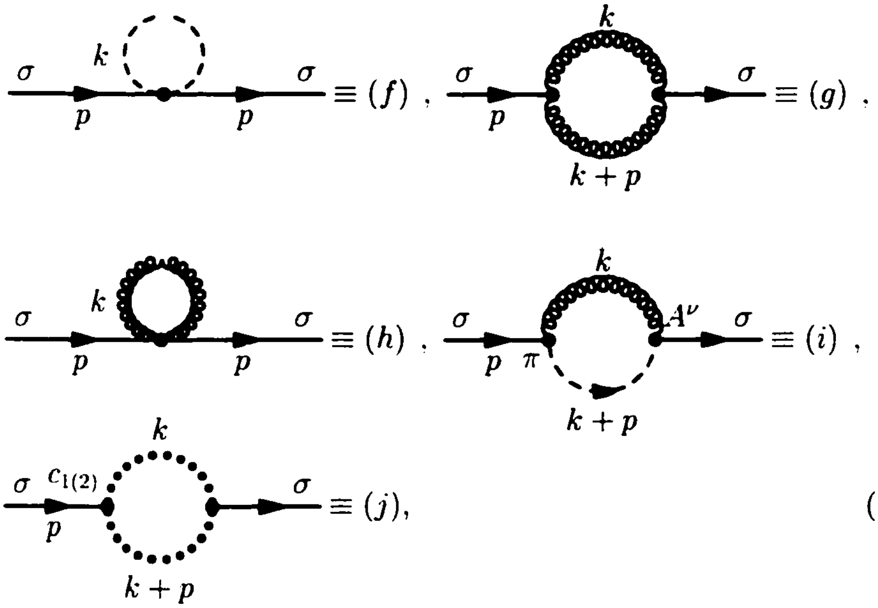
The first five graphs contributing to the one-loop mass renormalization of the Higgs.



(A.34)

are either manifestly gauge-independent, (a)-(d), or have no *divergent* gauge-dependent piece, (e), in either the noncommutative or commutative theories.

The remaining 1PI graphs possess gauge-dependent divergent pieces,



(A.35)

and have values in the noncommutative theory as

$$
\begin{aligned}
(f) &= \int \frac{d^D k}{(2\pi)^D}\left[-2i(\lambda_1+\lambda_2)+i\lambda_1\cos(p\times k)\right]\frac{i}{k^2-\xi M^2} \\
&= 2(\lambda_1+\lambda_2)I_1(\xi M^2)+\text{finite} \\
&= \frac{2i(\lambda_1+\lambda_2)(\xi M^2)}{(4\pi)^2}\Gamma(2-D/2)+\text{finite}, \qquad (A.36)\\
(g) &= (2iag^2)^2(-i)^2\int\frac{d^D k}{(2\pi)^D}\cos^2(\frac{p\times k}{2})\left\{\frac{D-2k^2/M^2+k^4/M^4}{(k^2-M^2)^2}\right. \\
&\quad \left.+\frac{2(k^2/M^2-k^4/M^4)}{(k^2-M^2)(k^2-\xi M^2)}+\frac{k^4/M^4}{(k^2-\xi M^2)^2}\right\}+\text{finite} \\
&= 2g^4a^2\left[(D-1)I_2(M^2)+\xi^2 I_2(\xi M^2)\right]+\text{finite}, \qquad (A.37)\\
(h) &= ig^2(-i)\int\frac{d^D k}{(2\pi)^D}\left[1+\cos(p\times k)\right]\left\{\frac{D-k^2/M^2}{k^2-M^2}+\frac{k^2/M^2}{k^2-\xi M^2}\right\} \\
&= g^2\left[(D-1)I_1(M^2)+\xi I_1(\xi M^2)\right]+\text{finite}, \qquad (A.38)\\
(i) &= (ig)^2 i(-i)\int\frac{d^D k}{(2\pi)^D}\cos^2(\frac{p\times k}{2})\frac{(-2p-k)_\mu(-2p-k)_\nu}{(p+k)^2-\xi M^2}\times
\end{aligned}
$$

$$
\begin{aligned}
&\left\{\frac{g^{\mu\nu}-k^\mu k^\nu/M^2}{k^2-M^2}+\frac{k^\mu k^\nu}{k^2-\xi M^2}\right\} \\
=&\ -\frac{g^2}{2}\int\frac{d^Dk}{(2\pi)^D}\left\{\frac{k^2-k^4/M^2}{(k^2-M^2)(k^2-\xi M^2)}+\frac{k^4/M^2}{(k^2-\xi M^2)^2)}\right\} \\
&-\frac{1}{2}\frac{ig^2p^2}{(4\pi)^2}(3-\xi)+\text{finite} \\
=&\ -\frac{g^2}{2}\left[\xi I_1(\xi M^2)+\xi^2M^2I_2(\xi M^2)\right]-\frac{1}{2}\frac{ig^2p^2}{(4\pi)^2}(3-\xi)\Gamma(2-D/2)+\text{finite},
\end{aligned}
\tag{A.39}
$$

$$
\begin{aligned}
(j) =&\ 2(-1)\left(-i\frac{\xi M^2}{a}\right)^2 i^2\int\frac{d^Dk}{(2\pi)^D}\frac{\cos^2(\frac{p\times k}{2})}{[(p+k)^2-\xi M^2](k^2-M^2)} \\
=&\ -g^4a^2\xi^2I_2(\xi M^2)+\text{finite}.
\end{aligned}
\tag{A.40}
$$

A few comments are in order about the above computations. By power-counting, graph (i) has momentum-dependent (outside of the noncommutative phase) σ-π-A^* and σ-π^*-A vertices, which yield divergent contributions at $O(p^2)$; i.e. divergent wavefunction renormalization. To evaluate it we used the following identity (established with a Feynman parameter, and symmetric integration) :

$$
\begin{aligned}
&\frac{d}{dp^2}\left(\int\frac{d^Dk}{(2\pi)^D}\left\{\frac{(2p+k)^2-(2p\cdot k+k^2)^2/M^2}{[(p+k)^2-\xi M^2](k^2-M^2)}\right.\right. \\
&\left.\left.+\frac{(2p\cdot k+k^2)^2/M^2}{[(p+k)^2-\xi M^2](k^2-\xi M^2)}\right\}\right)=\frac{i}{(4\pi)^2}\Gamma(2-D/2)(3-\xi)+\text{finite}.
\end{aligned}
\tag{A.41}
$$

We then Taylor expand as usual the planar part of the graph in powers of p^2. As well, there is a 'crossed' graph to (i) with the σ-π-A^* and σ-π^*-A vertices switched with respect to the external lines; equivalently, the gauge charge circulates in the opposite direction. It has the identical value as (i), so we will account for it by a factor of two below. The ghost graph (h) includes a factor of two for the two sets of ghosts, and an overall minus sign for the ghost statistics. It is also now evident that if we had not included both ghost-ghost-Higgs orderings, the noncommutative phase at each vertex of (j) would cancel, and the ghost graphs would coincide with their commutative counterparts. Clearly, the ghost graph is used to cancel divergent, momentum-independent, gauge-dependent contributions coming from (A.37) and (A.39), whose divergent pieces in the noncommutative case are half those of their commutative counterparts. Thus, if we were not to introduce both orderings symmetrically weighted, we would (already) obtain a manifest gauge-dependence proportional to $\xi^2 g^2 M^2$.

The values for graphs (f)-(j) in the commutative theory are given by

$$(f)' = 2\lambda I_1(\xi M^2) = \frac{2i(\frac{\lambda_1}{2} + \lambda_2)\xi M^2}{(4\pi)^2}\Gamma(2 - D/2) + \text{finite}, \quad \text{(A.42)}$$

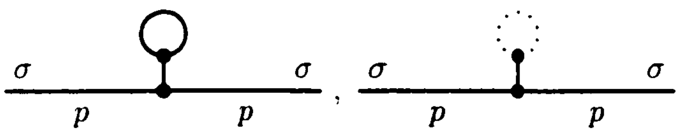
$$(g)' = 4g^4 a^2 \left[(D-1)I_2(M^2) + \xi^2 I_2(\xi M^2)\right] + \text{finite}, \quad \text{(A.43)}$$

$$(h)' \equiv 2g^2 \left[(D-1)I_1(M^2) + \xi I_1(\xi M^2)\right], \quad \text{(A.44)}$$

$$(i)' = -g^2 \left[\xi I_1(\xi M^2) + \xi^2 M^2 I_2(\xi M^2)\right] - \frac{ig^2 p^2}{(4\pi)^2}(3 - \xi)\Gamma(2 - D/2) + \text{finite}, \quad \text{(A.45)}$$

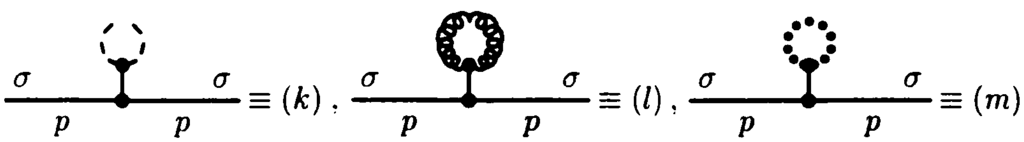
$$(j)' = -2g^4 a^2 \xi^2 I_2(\xi M^2) + \text{finite}. \quad \text{(A.46)}$$

We have four purely noncommutative 1PI graphs which we will add later. Let us now consider the one-point tadpole contributions, which, as usual are the same for both the noncommutative and commutative theories. There are two gauge-independent tadpoles:



(A.47)

which do not concern us. The gauge-dependent one-point tadpole graphs are



(A.48)

with values

$$
\begin{aligned}
(k) &= -6\lambda I_1(\xi M^2) = \frac{-6i\lambda}{(4\pi)^2}(\xi g^2 a^2)\Gamma(2-D/2) + \text{finite}, && \text{(A.49)}\\
(l) &= -6g^2\left[(D-1)I_1(M^2) + \xi I_1(\xi M^2)\right], && \text{(A.50)}\\
(m) &= +6\xi g^2 I_1(\xi M^2). && \text{(A.51)}
\end{aligned}
$$

The gauge-dependence between the last two graphs explicitly cancels, so the only one-point tadpole correction which remains in our calculation is (A.49).

This completes the list of graphs that survive in the commutative limit. Thus, let us first check that the sum of divergent gauge-dependent contributions for the commutative theory graphs vanishes on the Higgs mass-shell as required. Adding (A.42), (A.43), (A.44), 2× (A.45), (A.46), and (A.49)-(A.51) we obtain

$$
\begin{aligned}
\Pi_{div,c}(p^2,\xi) &= 2\Gamma\lambda\xi M^2 + 4g^4a^2\xi^2 I_2(\xi M^2) + 2g^2\xi I_1(\xi M^2) - 2g^2\left[\xi I_1(\xi M^2)\right.\\
&\quad \left.+\xi^2 M^2 I_2(\xi M^2)\right] + 2\Gamma\xi g^2 p^2 - 2g^4a^2\xi^2 I_2(\xi M^2) - 6\Gamma\lambda g^2 a^2\\
&= 2\xi\Gamma g^2(p^2 - 2\lambda a^2)\\
&\to 0 \quad \text{as} \quad p^2 \to 2\lambda a^2, && \text{(A.52)}
\end{aligned}
$$

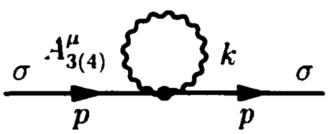
where Γ is shorthand for $i\Gamma(2-D/2)/(16\pi^2)$.

Repeating this calculation for the noncommutative theory, by adding the gauge-dependent, divergent pieces from (A.36), (A.37), (A.38), 2× (A.39), (A.40), (A.49)-(A.51) we get

$$
\begin{aligned}
\sum_{div,nc}(p^2,\xi) &= 2(\lambda_1+\lambda_2)\Gamma\xi M^2 + 2g^4a^2\xi^2 I_2(\xi M^2) + g^2\xi I_1(\xi M^2)\\
&\quad -g^2\left[\xi I_1(\xi M^2) + \xi^2 M^2 I_2(\xi M^2)\right] + \Gamma\xi g^2 p^2 - g^4a^2\xi^2 I_2(\xi M^2)\\
&\quad -6\Gamma\lambda g^2 a^2\\
&= \xi\Gamma g^2(p^2 - \lambda_1 a^2 - 4\lambda_2 a^2)\\
&\to -2\xi\Gamma g^2\lambda_2 a^2 \quad \text{as} \quad p^2 \to 2\lambda a^2. && \text{(A.53)}
\end{aligned}
$$

Thus, although most of the ξ cancellation persists, since the divergent parts of several of the graphs are simply halved with respect to their commutative counterparts, (A.36) is split differently than the p^2 wavefunction renormalization piece in (A.39), and (A.49), the one-point tadpole, is not split at all.

However, we still have to add the contributions from purely noncommutative graphs, i.e. graphs that disappear in the commutative limit. There are four of them, although only one will contribute. First the graphs with the gauge fields A_3^μ and A_4^μ as 1PI tadpoles disappear in dimensional regularization because they involve massless propagators:

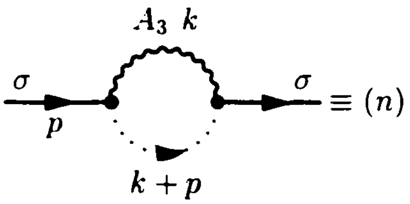


$= 0.$ (A.54)

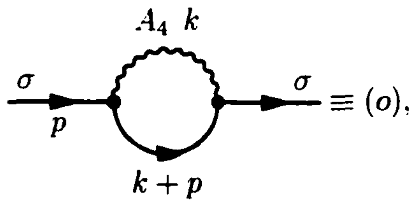
The last two graphs we need to consider both involve the following integral (again evaluated by introducing a Feynman parameter, and symmetrically integrating)

$$\begin{aligned}&\int \frac{d^Dk}{(2\pi)^D}\frac{(2p+k)^2+(\xi-1)(2p\cdot k+k^2)^2/k^2}{[(p+k)^2-m^2]k^2}\\ &= \frac{i}{(4\pi)^2}\Gamma(2-D/2)\left[\xi m^2+(3-\xi)p^2\right]+\text{finite},\end{aligned} \tag{A.55}$$

which like (A.39) have divergent wavefunction renormalization contributions, and originate from the matter covariant derivative. Thus we have



$\equiv (n)$,



$\equiv (o),$ (A.56)

with values given by

$$\begin{aligned}(n) &= g^2\int\frac{d^Dk}{(2\pi)^D}\sin^2\left(\frac{p\times k}{2}\right)\frac{(-p-k-p)_\mu(p+k+p)_\nu i(-i)}{[(p+k)^2-\lambda_1 a^2]k^2}\times\\ &\times\left[g^{\mu\nu}+(\xi-1)\frac{k^\mu k^\nu}{k^2}\right]\end{aligned}$$

$$= -\frac{g^2}{2}\int \frac{d^Dk}{(2\pi)^D}\left[1-\cos(p\times k)\right]\frac{(2p+k)^2+(\xi-1)(2p\cdot k+k^2)^2/k^2}{[(p+k)^2-\lambda_1 a^2]k^2}$$

$$= -\frac{ig^2}{2(4\pi)^2}\Gamma(2-D/2)\left[\xi\lambda_1 a^2+(3-\xi)p^2\right]+\text{finite}, \qquad (A.57)$$

$$(o) = g^2\int \frac{d^Dk}{(2\pi)^D}\sin^2(\frac{p\times k}{2})\frac{(-p-k-p)_\mu(p+k+p)_\nu i(-i)}{[(p+k)^2-2\lambda a^2]k^2}\times$$

$$\times\left[g^{\mu\nu}+(\xi-1)\frac{k^\mu k^\nu}{k^2}\right]$$

$$= -\frac{ig^2}{2(4\pi)^2}\Gamma(2-D/2)\left[\xi(2\lambda a^2)+(3-\xi)p^2\right]. \qquad (A.58)$$

In the notation used above, the sum of the divergent, gauge-dependent pieces from these two graphs is

$$\sum_{\xi-dep,div}(\text{pure noncomm}) = \frac{\xi\Gamma g^2}{2}\left[(p^2-\lambda_1 a^2)+(p^2-2\lambda a^2)\right]$$

$$\to \xi\Gamma g^2\lambda_2 a^2 \quad \text{as} \quad p^2\to 2\lambda a^2, \qquad (A.59)$$

which is not enough to cancel the residual piece in (A.53), although yet again, depends only on the coupling λ_2 (and g) and not λ_1. Thus the sum of all gauge-dependent pieces evaluated on the Higgs mass-shell is

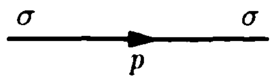
$$\Pi_{\xi-dep,noncomm}(p^2=2\lambda a^2) = -\xi\Gamma g^2\lambda_2 a^2+\text{finite} \qquad (A.60)$$

where $\Gamma = i\Gamma(2-D/2)/(16\pi^2)$.

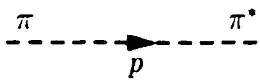
This signals gauge-dependence in the on-shell mass renormalization of the Higgs in this model and confirms directly, at the quantum level, that terms like $[\mathrm{Tr}(\Phi_\star^2)]_\star^2$ are pathological in noncommutative gauge theories.

A.1 Gauge sector Feynman rules

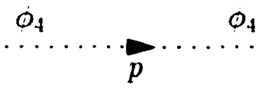
The scalar potential Feynman rules are identical to those of chapter 4. As usual, all momenta flow into interaction vertices.



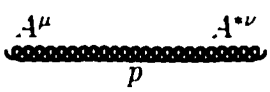
$$= \frac{i}{p^2 - 2\lambda a^2 + i\epsilon} \qquad \text{(A.61)}$$



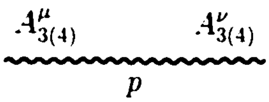
$$= \frac{i}{p^2 - \xi M^2 + i\epsilon} \qquad \text{(A.62)}$$



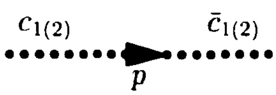
$$= \frac{i}{p^2 - \lambda_1 a^2 + i\epsilon} \qquad \text{(A.63)}$$



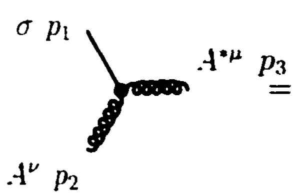
$$= -i\left[\frac{g_{\mu\nu} - k_\mu k_\nu/M^2}{k^2 - M^2 + i\epsilon} + \frac{k_\mu k_\nu/M^2}{k^2 - \xi M^2 + i\epsilon}\right] \qquad \text{(A.64)}$$



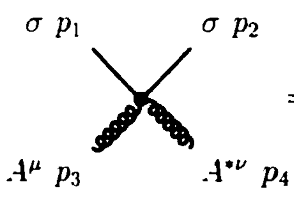
$$= \frac{-i}{k^2 + i\epsilon}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right] \qquad \text{(A.65)}$$



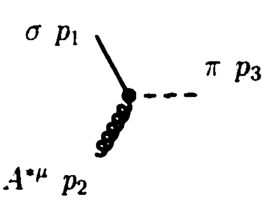
$$= \frac{i}{p^2 - \xi M^2 + i\epsilon} \qquad \text{(A.66)}$$



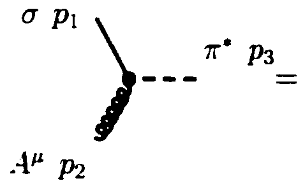
$$= 2iag^2 g_{\mu\nu} \cos(\frac{p_2 \times p_3}{2}) \qquad \text{(A.67)}$$



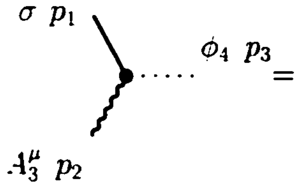
$$= ig^2 g_{\mu\nu}\left[\cos(\frac{p_1 \times p_2}{2})\cos(\frac{p_3 \times p_4}{2}) + \cos(\frac{p_1 \times p_4}{2} + \frac{p_2 \times p_3}{2})\right] \qquad \text{(A.68)}$$



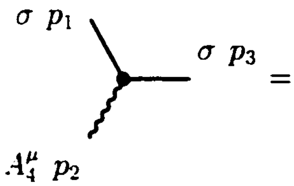
$$= ig(p_3 - p_1)_\mu \cos(\frac{p_1 \times p_2}{2}) \qquad \text{(A.69)}$$



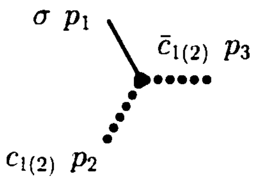
$$= \quad ig(p_1 - p_3)_\mu \cos(\frac{p_1 \times p_2}{2}) \tag{A.70}$$



$$= \quad g(p_3 - p_1)_\mu \sin(\frac{p_1 \times p_2}{2}) \tag{A.71}$$



$$= \quad g(p_3 - p_1)_\mu \sin(\frac{p_1 \times p_2}{2}) \tag{A.72}$$



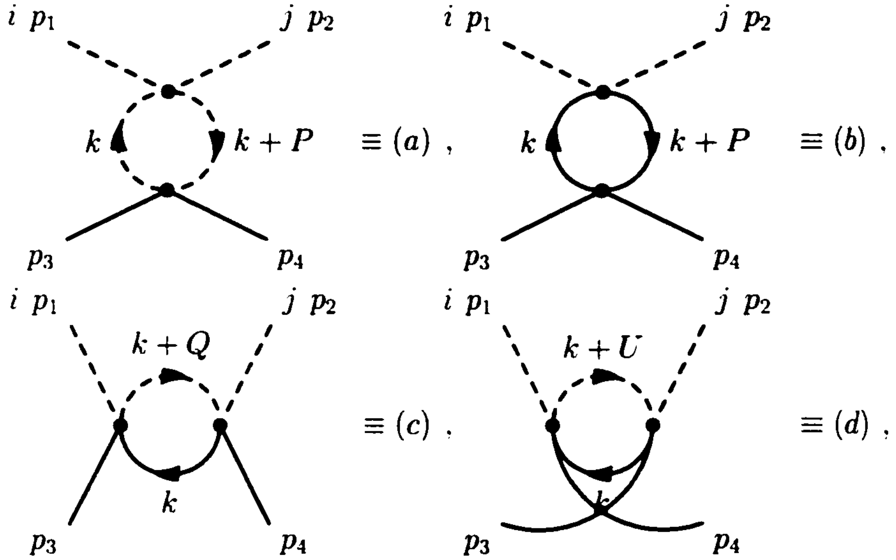
$$= \quad -\frac{i\xi M^2}{a} \cos(\frac{p_2 \times p_3}{2}) \tag{A.73}$$

Appendix B

Corroboration by four-point functions

The restrictions to $N = 2$ and $f = 2$ in the noncommutative $O(N)$ linear sigma model that we found in chapter 3 are quite intriguing, although we found a natural interpretation of them in chapter 4, by rewriting of the theory in terms of a $U(1)$ symmetry. It is interesting to see how these restrictions emerge by studying the four-point scattering amplitudes of the noncommutative $O(N)$ linear sigma model. Not only will this corroborate the claims made in chapter 3, it will also serve as an example of how to compute higher point functions in a noncommutative theory (which are more challenging algebraically than their commutative counterparts), and further illuminate the connection these results have with spontaneous symmetry breaking.

So let us return to the $O(N)$ model of chapter 3, and consider the 1PI contributions to the $\pi - \pi - \sigma - \sigma$ amplitude at one-loop, which has four graphical contributions



(B.1)

where all external momenta flow into the vertices, $P = p_1 + p_2$, $Q = p_1 + p_3$, and $U = p_1 + p_4$. Graph (d) is merely the crossing of (c) under $p_3 \leftrightarrow p_4$. One immediate condition for renormalizability requires that the sum of the divergent contributions of these graphs must be proportional to the tree level vertex.

All four graphs are logarithmically divergent, their respective divergences originating from the zero-frequency component of the total phase; i.e. the terms independent of θ, as in chapter 3. To this end, we need isolate only the divergent parts of the diagrams and so now introduce several convenient notations and identities that will simplify the algebraic task. First define

$$i \wedge j \equiv p_i \wedge p_j \equiv \frac{p_i \times p_j}{2} \, , \; c(\cdot) \equiv \cos(\cdot) \, , \; s(\cdot) \equiv \sin(\cdot) \, , \; c(i) \equiv c(p_i \wedge k). \quad \text{(B.2)}$$

Momentum conservation and antisymmetry of $\wedge$ imply

$$\begin{aligned}
p_1 \wedge p_2 + p_3 \wedge p_4 &= p_2 \wedge p_3 + p_4 \wedge p_1 \quad , \quad p_1 \wedge p_4 + p_3 \wedge p_2 = p_4 \wedge p_3 + p_2 \wedge p_1 \\
p_1 \wedge p_3 + p_2 \wedge p_4 &= p_3 \wedge p_2 + p_4 \wedge p_1 \quad , \quad p_1 \wedge p_4 + p_2 \wedge p_3 = p_4 \wedge p_2 + p_3 \wedge p_1 \\
p_1 \wedge p_2 + p_4 \wedge p_3 &= p_2 \wedge p_4 + p_3 \wedge p_1 \quad , \quad p_1 \wedge p_3 + p_4 \wedge p_2 = p_3 \wedge p_4 + p_2 \wedge p_1
\end{aligned} \quad \text{(B.3)}$$

where we have grouped pairs of equalities that are the negatives of each other, and so equal as arguments of a cosine. Then, in order to extract the 'constant' part of a product of trigonometric functions we have the equivalences

$$
\begin{aligned}
c(1)c(2)c(3)c(4) &= c(1)c(2)c(3)c(1+2+3) \simeq c(1)c(2)c^2(3)c(1+2) \simeq \frac{1}{8} \\
c(1)c(2)c(3)s(4) &= -c(1)c(2)c(3)s(1+2+3) \simeq -c(1)c(2)c^2(3)s(1+2) \simeq 0 \\
c(1)c(2)s(3)s(4) &= -c(1)c(2)s(3)s(1+2+3) \simeq -\frac{1}{8} \\
c(1)s(2)s(3)s(4) &= -c(1)s(2)s(3)s(1+2+3) \simeq -c(1)s(2)s^2(3)c(1+2) \simeq 0 \\
s(1)s(2)s(3)s(4) &= -s(1)s(2)s(3)s(1+2+3) \simeq -s(1)s(2)s^2(3)c(1+2) \simeq \frac{1}{8}
\end{aligned}
\tag{B.4}
$$

which in turn imply

$$
\begin{aligned}
c(1+2)c(1-2) &= [c(1)c(2)-s(1)s(2)][c(1)c(2)+s(1)s(2)] \simeq \frac{1}{4}-\frac{1}{4}=0 \\
s(1+2)s(1-2) &= [s(1)c(2)+s(2)c(1)][s(1)c(2)-s(2)c(1)] \simeq \frac{1}{4}-\frac{1}{4}=0 \\
c(1+2)s(1-2) &= [c(1)c(2)-s(1)s(2)][s(1)c(2)-c(1)s(2)] \simeq 0
\end{aligned}
\tag{B.5}
$$

and

$$
\begin{aligned}
c(P\wedge k)c(1)c(2) &= [c(1)c(2)-s(1)s(2)]c(1)c(2) \simeq \frac{1}{4} \\
c(P\wedge k)s(1)s(2) &= [c(1)c(2)-s(1)s(2)]s(1)s(2) \simeq -\frac{1}{4}.
\end{aligned}
\tag{B.6}
$$

Now consider diagram (a) whose value is

$$
\begin{aligned}
(a) = {} & \frac{(-2\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\delta^{kl}}{[k^2+\xi^2][(k+P)^2+\xi^2]} \times \\
& [fc(-k\wedge(k+P))c(p_3\wedge p_4)+(1-f)c(-k\wedge p_3+(k+P)\wedge p_4)] \times \\
& \Big\{\delta^{ij}\delta^{kl}[fc(p_1\wedge p_2)c(k\wedge(-k-P))+(1-f)c(p_1\wedge k-p_2\wedge(k+P))] \\
& +\delta^{ik}\delta^{jl}[fc(p_1\wedge k)c(p_2\wedge(-k-P))+(1-f)c(p_1\wedge p_2-k\wedge(k+P))] \\
& +\delta^{il}\delta^{jk}[fc(p_1\wedge(k+P))c(p_2\wedge k)+(1-f)c(p_1\wedge p_2-(k+P)\wedge k)]\Big\}
\end{aligned}
\tag{B.7}
$$

The divergence of this graph is proportional to $(N-1)A+B+C$, where A,B,C are the three trigonometric polynomials obtained from multiplying the terms inside the first square bracket by the each of the three sets of terms inside the curly bracket of (B.7) respectively. Inspection reveals that $C = B(p_1 \leftarrow p_2)$.

Then by using the notation and algebraic tricks above we have successively

$$
\begin{aligned}
A &\sim \frac{f^2}{2}c(1\wedge 2)c(3\wedge 4) \\
&\quad +(1-f)^2[c(3)c(p_4\wedge(k-p_3)) + s(3)s(p_4\wedge(k-p_3))]\times \\
&\quad\quad \times[c(1)c(p_2\wedge(k+p_1)) + s(1)s(p_2\wedge(k+p_1))] \\
&\quad +f(1-f)\,\{c(P\wedge k)c(3\wedge 4)\,[c(1)c(p_2\wedge(k+p_1)) + s(1)s(p_2\wedge(k+p_1))] \\
&\quad +c(1\wedge 2)c(P\wedge k)\,[c(3)c(p_4\wedge(k-p_3)) + s(3)s(p_4\wedge(k-p_3))]\} \\
&\sim \frac{f^2}{2}c(1\wedge 2)c(3\wedge 4) + (1-f)^2\times \\
&\quad \{c(1)c(3)\,[c(4)c(4\wedge 3)+s(4)s(4\wedge 3)]\,[c(2)c(2\wedge 1)-s(2)s(2\wedge 1)] \\
&\quad +c(3)s(1)\,[c(4)c(4\wedge 3)+s(4)s(4\wedge 3)]\,[s(2)c(2\wedge 1)+c(2)s(2\wedge 1)] \\
&\quad +s(3)c(1)\,[s(4)c(4\wedge 3)-c(4)s(4\wedge 3)]\,[c(2)c(2\wedge 1)-s(2)s(2\wedge 1)] \\
&\quad +s(3)s(1)\,[s(4)c(4\wedge 3)-c(4)s(4\wedge 3)]\,[s(2)c(2\wedge 1)+c(2)s(2\wedge 1)]\} \\
&\quad +f(1-f)\,\{c(P\wedge k)c(3\wedge 4)c(1)\,[c(2)c(2\wedge 1)-s(2)s(2\wedge 1)] \\
&\quad\quad +c(P\wedge k)c(3\wedge 4)s(1)\,[s(2)c(2\wedge 1)+c(2)s(2\wedge 1)] \\
&\quad\quad +c(1\wedge 2)c(P\wedge k)c(3)\,[c(4)c(4\wedge 3)+s(4)s(4\wedge 3)] \\
&\quad\quad +c(1\wedge 2)c(P\wedge k)s(3)\,[s(4)c(4\wedge 3)-c(4)s(4\wedge 3)]\} \\
&\sim \frac{f^2}{2}c(1\wedge 2)c(3\wedge 4) + \frac{(1-f)^2}{8}\,[c(1\wedge 2)c(3\wedge 4)+s(1\wedge 2)s(3\wedge 4) \\
&\quad -c(1\wedge 2)c(3\wedge 4) - s(1\wedge 2)s(3\wedge 4) - c(1\wedge 2)c(3\wedge 4) \\
&\quad -s(1\wedge 2)s(3\wedge 4) + c(1\wedge 2)c(3\wedge 4) + s(1\wedge 2)s(3\wedge 4)] \\
&\quad +\frac{f(1-f)}{4}c(1\wedge 2)c(3\wedge 4)[1-1+1-1] \\
&= \frac{f^2}{2}c(p_1\wedge p_2)c(p_3\wedge p_4).
\end{aligned}
\tag{B.8}
$$

Next we have

$$
\begin{aligned}
B \sim\ & f^2 c(3\wedge 4)c(P\wedge k)c(1)\left[c(2)c(2\wedge 1) - s(2)s(2\wedge 1)\right] \\
& +(1-f)^2\left[c((p_3-p_4)\wedge k)c(4\wedge 3) - s((p_3-p_4)\wedge k)s(4\wedge 3)\right] \\
& \quad \times \left[c(1\wedge 2)c(P\wedge k) - s(1\wedge 2)s(P\wedge k)\right] \\
& +f(1-f)\left\{c(P\wedge k)c(3\wedge 4)\left[c(1\wedge 2)c(P\wedge k) - s(1\wedge 2)s(P\wedge k)\right]\right. \\
& \quad +c(1)\left[c(2)c(2\wedge 1) - s(2)s(2\wedge 1)\right]\left[c((p_3-p_4)\wedge k)c(4\wedge 3)\right. \\
& \quad \left.\left. -s((p_3-p_4)\wedge k)s(4\wedge 3)\right]\right\} \\
\sim\ & \frac{f^2}{4}c(1\wedge 2)c(3\wedge 4) + 0 + f(1-f)\left[\frac{1}{2}c(1\wedge 2)c(3\wedge 4) + 0 + 0\right] \\
=\ & \left[\frac{f^2}{4} + \frac{f(1-f)}{2}\right] c(p_1\wedge p_2)c(p_3\wedge p_4). \qquad \text{(B.9)}
\end{aligned}
$$

whence it follows that

$$
C = B(p_1 \leftrightarrow p_2) = \left[\frac{f^2}{4} + \frac{f(1-f)}{2}\right] c(p_1\wedge p_2)c(p_3\wedge p_4). \qquad \text{(B.10)}
$$

Thus combining (B.8), (B.9) and (B.10), the total divergence in (B.7) is proportional to

$$
\mathrm{Div}(a) \sim 2\lambda^2\delta^{ij}\left[\frac{(N-1)f^2 + f(2-f)}{2}\right]\cos(p_1\wedge p_2)\cos(p_3\wedge p_4). \qquad \text{(B.11)}
$$

Now consider the second diagram (b) which is equal to

$$
\begin{aligned}
(b)\ =\ & \frac{(-2\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}}{(k^2+m_\sigma^2)\left[(P+k)^2+m_\sigma^2\right]}\times \\
& \times\left[fc(1\wedge 2)c(k\wedge(-k-P)) + (1-f)c(p_1\wedge k - p_2\wedge(k+P))\right]\times \\
& \left[c(k\wedge(k+P))c(3\wedge 4) + c(3)c((k+P)\wedge p_4) + c(4)c((k+P)\wedge p_3)\right].
\end{aligned}
$$

(B.12)

The divergence in this graph is proportional to

$$\begin{aligned}
(b) \propto\ & \frac{f}{2}c(1\wedge 2)c(3\wedge 4) \\
& + fc(1\wedge 2)c((p_3+p_4)\wedge k)c(3)[c(4)c(4\wedge 3)+s(4)s(4\wedge 3)] \\
& + fc(1\wedge 2)c((p_3+p_4)\wedge k)c(4)[c(3)c(3\wedge 4)+s(3)s(3\wedge 4)] \\
& + (1-f)c((p_1-p_2)\wedge k)c(1\wedge 2) - s((p_1-p_2)\wedge k)s(1\wedge 2)] \\
& \quad \times c(3\wedge 4)c(P\wedge k) \\
& + (1-f)[c((p_1-p_2)\wedge k)c(1\wedge 2) - s((p_1-p_2)\wedge k)s(1\wedge 2)] \\
& \quad \times c(3)[c(4)c(4\wedge 3)+s(4)s(4\wedge 3)] \\
& + (1-f)[c((p_1-p_2)\wedge k)c(1\wedge 2) - s((p_1-p_2)\wedge k)s(1\wedge 2)] \\
& \quad \times c(4)[c(3)c(3\wedge 4)+s(3)s(3\wedge 4)] \\
\sim\ & \frac{f}{2}c(1\wedge 2)c(3\wedge 4) + \left(\frac{f}{4}+\frac{f}{4}\right)c(1\wedge 2)c(3\wedge 4) + 0 \\
& + (1-f)c(1\wedge 2)c(3\wedge 4)c(3)c(4)[c(1)c(2)+s(1)s(2)] \\
& + (1-f)s(1\wedge 2)s(3\wedge 4)c(3)s(4)[s(1)c(2)-c(1)s(2)] \\
& + (1-f)c(1\wedge 2)c(3\wedge 4)c(3)c(4)[c(1)c(2)+s(1)s(2)] \\
& - (1-f)s(1\wedge 2)s(3\wedge 4)s(3)c(4)[s(1)c(2)-c(1)s(2)] \\
\sim\ & fc(1\wedge 2)c(3\wedge 4) + (1-f)c(1\wedge 2)c(3\wedge 4)\left[\frac{1}{8}-\frac{1}{8}+\frac{1}{8}-\frac{1}{8}\right] + \\
& + (1-f)s(1\wedge 2)s(3\wedge 4)\left\{-\frac{1}{8}-\left(-\frac{1}{8}\right)-\left[-\frac{1}{8}-\left(-\frac{1}{8}\right)\right]\right\} \\
\sim\ & fc(1\wedge 2)c(3\wedge 4).
\end{aligned} \tag{B.13}$$

Thus we have

$$\mathrm{Div}(b) \sim 2\lambda^2\delta^{ij} f\cos(p_1\wedge p_2)\cos(p_3\wedge p_4). \tag{B.14}$$

Diagram (c) has no symmetry factor and is equal to

$$\begin{aligned}
(c) =\ & (-2\lambda)^2 \int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ik}\delta^{kj}}{(k^2+m_\sigma^2)\left[(Q+k)^2+\xi^2\right]} \times \\
& [fc(p_1\wedge(k+Q))c(3) + (1-f)c(1\wedge 3-(Q+k)\wedge k)] \\
& [fc((Q+k)\wedge p_2)c(4) + (1-f)c((Q+k)\wedge(-k)+2\wedge 4)]\,.
\end{aligned} \tag{B.15}$$

Its divergence is proportional to

$$
\begin{aligned}
(c) \;\propto\; & f^2 c(3)c(4)[c(1)c(1\wedge 3) - s(1)s(1\wedge 3)][c(2)c(2\wedge 4) + s(2)s(2\wedge 4)] \\
& +(1-f)^2[c(1\wedge 3)c((p_1+p_3)\wedge k) + s(1\wedge 3)s((p_1+p_3)\wedge k)] \\
& \qquad \times[c(2\wedge 4)c((p_2+p_4)\wedge k) - s(2\wedge 4)s((p_2+p_4)\wedge k)] \\
& +f(1-f)\{c(3)[c(1)c(1\wedge 3) - s(1)s(1\wedge 3)] \\
& \qquad \times[c(2\wedge 4)c((p_2+p_4)\wedge k) - s(2\wedge 4)s((p_2+p_4)\wedge k)] \\
& \quad +c(4)[c(2)c(2\wedge 4) + s(2)s(2\wedge 4)] \\
& \qquad \times[c(1\wedge 3)c((p_1+p_3)\wedge k) + s(1\wedge 3)s((p_1+p_3)\wedge k)]\} \\
\sim\; & \frac{f^2}{8}[c(1\wedge 3)c(2\wedge 4) + s(1\wedge 3)s(2\wedge 4)] \\
& +(1-f)^2 c(1\wedge 3)c(2\wedge 4)[c(1)c(3) - s(1)s(3)][c(2)c(4) - s(2)s(4)] \\
& -(1-f)^2 s(1\wedge 3)s(2\wedge 4)[s(1)c(3) + s(3)c(1)][s(2)c(4) + s(4)c(2)] \\
& +f(1-f)\{c(1\wedge 3)c(2\wedge 4)c(1)c(3)[c(2)c(4) - s(2)s(4)] \\
& \quad +s(1\wedge 3)s(2\wedge 4)s(1)c(3)[s(2)c(4) + s(4)c(2)] \\
& \quad +c(1\wedge 3)c(2\wedge 4)c(2)c(4)[c(1)c(3) - s(1)s(3)] \\
& \quad +s(1\wedge 3)s(2\wedge 4)s(2)c(4)[s(1)c(3) + s(3)c(1)]\} \\
\sim\; & \frac{f^2}{8}c(1\wedge 3 - 2\wedge 4) + \frac{(1-f)^2}{2}\{c(1\wedge 3)c(2\wedge 4) + s(1\wedge 3)s(2\wedge 4)\} \\
& +\frac{f(1-f)}{2}[c(1\wedge 3)c(2\wedge 4) - s(1\wedge 3)s(2\wedge 4)] \\
=\; & \left[\frac{f^2}{8} + \frac{(1-f)^2}{2}\right] c(1\wedge 3 - 2\wedge 4) + \frac{f(1-f)}{2}c(1\wedge 3 + 2\wedge 4). \qquad \text{(B.16)}
\end{aligned}
$$

That is,

$$
\begin{aligned}
\mathrm{Div}(c) \;\sim\; & 4\lambda^2\delta^{ij}\left\{\left[\frac{f^2}{8} + \frac{(1-f)^2}{2}\right]\cos(p_1\wedge p_3 - p_2\wedge p_4)\right. \\
& \left. +\frac{f(1-f)}{2}\cos(p_1\wedge p_3 + p_2\wedge p_4)\right\}. \qquad \text{(B.17)}
\end{aligned}
$$

Since the fourth diagram, (d), is just (c) with p_3 and p_4 interchanged, we immediately obtain

$$\begin{aligned}
\mathrm{Div}(d) &\sim 4\lambda^2\delta^{ij}\left\{\left[\frac{f^2}{8}+\frac{(1-f)^2}{2}\right]\cos(p_1\wedge p_4-p_2\wedge p_3)\right.\\
&\quad\left.+\frac{f(1-f)}{2}c(p_1\wedge p_4+p_2\wedge p_3)\right\}\\
&= 4\lambda^2\delta^{ij}\left\{\left[\frac{f^2}{8}+\frac{(1-f)^2}{2}\right]\cos(p_1\wedge p_4-p_2\wedge p_3)\right.\\
&\quad\left.+\frac{f(1-f)}{2}c(p_1\wedge p_3+p_2\wedge p_4)\right\}. \qquad \text{(B.18)}
\end{aligned}$$

using $\cos(x)=\cos(-x)$, and the identities (B.3). Furthermore, we note that by eliminating and then restoring p_4, we can obtain

$$\begin{aligned}
&\cos(p_1\wedge p_3-p_2\wedge p_4)+\cos(p_1\wedge p_4-p_2\wedge p_3)\\
&= c(p_1\wedge p_3+p_2\wedge(p_1+p_3))+c(p_1\wedge(-p_2-p_3)-p_2\wedge p_3)\\
&= c(p_1\wedge p_3+p_2\wedge p_3-p_1\wedge p_2)+c(p_1\wedge p_3+p_2\wedge p_3+p_1\wedge p_2)\\
&= 2c(p_1\wedge p_3+p_2\wedge p_3)c(p_1\wedge p_2)\\
&= 2c(p_1\wedge p_2)c((p_1+p_2+p_3)\wedge p_3)\\
&= 2\cos(p_1\wedge p_2)\cos(p_3\wedge p_4). \qquad \text{(B.19)}
\end{aligned}$$

This now allows us to add the divergences from all four graphs, (B.11), (B.14), (B.17), (B.18), to finally arrive at

$$\begin{aligned}
\sum\mathrm{Div} &\propto \left[\frac{N+3}{2}f^2+2(1-f)\right]\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)\\
&\quad+2f(1-f)\cos(p_1\wedge p_3+p_2\wedge p_4), \qquad \text{(B.20)}
\end{aligned}$$

whence, comparing with the tree-level form (3.22), we require

$$\begin{aligned}
&\left[\frac{N+3}{2}f^2+2(1-f)\right]\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)\\
&\quad+2f(1-f)\cos(p_1\wedge p_3+p_2\wedge p_4)\\
&= C\left[f\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)+(1-f)\cos(p_1\wedge p_3+p_2\wedge p_4)\right] \qquad \text{(B.21)}
\end{aligned}$$

(where C is an arbitrary constant) for renormalizability. Remembering that $f = 1$ leads to problems in the pion two-point function, we arrive at the system

$$\frac{N+3}{2}f^2 + 2(1-f) = Cf \ , \ 2f = C \tag{B.22}$$

or

$$(N-1)f^2 - 4f + 4 = 0. \tag{B.23}$$

Solving for f in terms of $N \geq 2$ (and integer of course) yields

$$f = \frac{4 \pm \sqrt{16 - 16(N-1)}}{2(N-1)} = \frac{2}{N-1} \pm \frac{2\sqrt{2-N}}{N-1}. \tag{B.24}$$

Now for $N > 2$, f is necessarily complex which conflicts with the Hermicity of the Lagrangian density. Thus we require $N = 2$, which yields $f = 2$, in complete support of the results of chapter 3.

Furthermore, if we repeat the calculations of this section for the easier $4 - \sigma$ amplitude, it is not difficult to show that the one-loop divergences are then proportional to

$$\text{Div} \propto \lambda^2 \left[4 + (N-1)f^2\right] \left\{\cos(p_1 \wedge p_2)\cos(p_3 \wedge p_4) + (p_2 \leftrightarrow p_3) + (p_2 \rightarrow p_4)\right\}. \tag{B.25}$$

As one might expect, this amplitude in isolation is automatically consistent for all N and f, analogous to the unbroken phase, or ϕ^4 theory; our restrictions emerge from an analysis of the (putative) spontaneously broken phase, which is why we studied the $\pi - \pi - \sigma - \sigma$ amplitude in this appendix, and the pion propagator in chapter 3. To see this connection from yet another view, remember that the limited counterterm structure in the spontaneously broken phase also requires consistency *between* the $4-\sigma$ and $\pi-\pi-\sigma-\sigma$ amplitudes [for renormalizability or at least quantum mechanical consistency with the $O(N)$ global symmetry], so that comparing (B.20) and (B.25) with (3.16), we require

$$[4 + (N-1)f^2]f = (N+3)f^2 + 4(1-f), \tag{B.26}$$

which again leads to the solutions $N = 2$, $f = 2$ (after again excluding $f = 1$ on the basis of results of chapter 3). Finally, with a moment's thought, one can convince oneself that the remaining four-point function (the four-pion amplitude) will also yield the same conditions without further calculation.

Thus, we again conclude that a *global* $O(N)$ symmetry is incompatible with continuum quantization in the spontaneously broken phase, except for the case $N = 2$, $f = 2$, which we saw in chapter 4 precisely corresponds to a $U(1)$ symmetry (in disguise) with the star product ordering consistent with the gauging of that symmetry.