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Optimization of Risk and Dividends Under Taxes and Costs for a Firm with Debt Liability and Constraints on Risk Reduction

by

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Abstract

This thesis addresses the problem of maximizing the expected total discounted dividend payouts, up to the bankruptcy time, for a company with debt liability and/or constraints on risk control and under the consideration of taxes and costs. The obtained stochastic control problem is transformed into a quasi-variational inequality (QVI) equation, which is solved explicitly, leading to a complete description of the optimal return function and optimal policy.

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List of Symbols

$\in$	an element of
$\int$	an integral
$\sum$	the sum of
$\mid$	...such that...
$\forall$	for all; for any; for each
$\{\mid\}$	the set of ... such that ...
$x := y$	x is defined to be another name for y
$f : X \to Y$	the function f maps the set X into the set Y
$\mathbb{R}$	real numbers
$f'(x)$	the derivative of the function f at the point x
$f''(x)$	the second derivative of the function f at the point x
$(\Omega, \mathcal{F}, \mathbb{P})$	a probability space
Ω	a non-empty set; the "sample space"
$\mathcal{F}$	a sigma-algebra of subsets of Ω
$\mathbb{P}$	probability measure on $\mathcal{F}$
$1_{condition}$	an indication of the condition
$\sup_{x \in S} x$	the supremum of a set S
$\inf_{x \in S} x$	the infimum of a set S
$\lim_{x \to a} f(x)$	the limit of the function $f(x)$ as x approaches to a
F^{-1}	the inverse function of $F(x)$
$max\{\}$	the maximum value of the set
$min\{\}$	the minimum value of the set

Chapter 1

Introduction

Actuarial science, like mathematical finance, was born in the very beginning of the last century. Precisely, it was founded by the thesis of Filip Lundberg in 1903, just three years after the foundation of mathematical finance was set up by Bachelier in 1900. This latter area then fell in the total forgetfulness for half a century, while actuarial sciences continued advancing through the works of Cramer, Essher, and many other mathematicians. Fortunately, in about the second half of the last century, mathematical finance has come to a big jump due to the works of Paul Samuelson, Black, Scholes, and Merton.

During the last decade, there has been an upsurge interest in the interplay between the two areas. This interplay creates new challenging mathematical problems and enhances the emergence of new markets with their implied new types of risk and new financial products. The risk for an insurance model takes on the form of reinsurance. Reinsurance is one of the risk-management tools that permits insurance companies to deal efficiently with risk. Some of the reasons for the need of reinsurance are outlined below.

- Need for protection against adverse fluctuations that may incur in the course of business
- Need to increase the capacity of the company by offering more services to its clients
- Financial distress due to unexpected changes in premium collection or profit
- Limitation of the impact of excessively large claims and large number of claims, especially for insurance company.

In conclusion, the main reason for reinsurance is the desire to diminish the impact of risks that come from large claims. The first model for the reserve process of an insurance company is called the Cramer-Lundberg model and is given by

$$X(t) = x + pt - \sum_{i=1}^{N_t} U_i. \qquad (1.0.1)$$

Here x is the initial capital, p is the premium rate, and U_i is the size of the i^{th} claim which is a random variable. The claims are assumed to be independently identically distributed and they arrive at a Poissonian rate. This model (1.0.1) represents the case when the company assumes full risk and does not divert any part of the risk to another company. To take into account of this situation, reinsurance should be used, and the model takes the following form

$$X(t) = x + apt - \sum_{i=1}^{N_t} aU_i, \qquad (1.0.2)$$

where $a \in [0, 1]$. There are different types of reinsurance. The two main types are proportional (or quota-share) reinsurance and non-proportional reinsurance. Proportion insurance was introduced by Gerber (1970) and Buhlmann(1970). For this type of reinsurance, the sharing of risk between the ceding company and the reinsurer is determined at issue (i.e. a coverage against a fixed percentage of losses). The first insurer that transfers (part of) his risk is called a cedant. The sharing of the risk for a nonproportional reinsurance between the cedent and the reinsurer is done in a more sophisticated way. Some examples for this kind of reinsurance are excess-of-loss and stop-loss reinsurance.

Notice that the model (1.0.2) fits one kind of reinsurance, the proportional reinsurance, only. In general, we have the following: If a risk U_i is too dangerous (for instance, if U_i has a large variance), then the insurer may want to transfer part of the risk U_i to another insurer. Often the reinsurance company does the same, i.e. it passes part of its own risk to a third company, and so on. By passing on parts of risks, large risks are split into a number of smaller portions taken up by different risk carriers. This procedure of risk exchange makes large claims less dangerous to the individual insurers, while the total risk remains the same.

A reinsurance contract specifies the part $U_i - h(U_i)$ of the claim amount U_i, which has to be compensated by the reinsurer, after taking off the retained amount $h(U_i)$. Here $h : \mathbb{R}_+ \to \mathbb{R}_+$, the retention function, is assumed to have the following properties:

- $h(x)$ and $x - h(x)$ are increasing,
- $0 \leq h(x) \leq x$ and, in particular, $h(0) = 0$.

It is reasonable to suppose that both the retention function $h(x)$ and the compensation function $k(x) = x - h(x)$ are increasing, i.e. with the growing claim size, both parts contribute more. In practice, retention functions are often continuous or even locally smooth, but we do not require such properties in this section. Possible choices of retention functions $h(x)$ are

- $h(x) = ax$ for the proportional contract, where $0 < a \leq 1$ is the proportion of risk,
- $h(x) = \min\{a, x\}$ for the stop-loss contract, where $a > 0$ is the retention level.

The models (1.0.1), (1.0.2) and other similar extended Cramer-Lundberg models corresponding to a retention function h seems not to fit some economic frameworks, like when we are dealing with a big portfolio. This is one of the reasons to consider the limiting models of the Cramer-Lundberg models. This limit leads to the diffusion models for insurance and/or reinsurance. Furthermore, the diffusion models explain a part of the interplay between finance and insurance. It was introduced by Dayananda (1970) and Whittle (1983), see also Emanuel et al (1975), Hojgaard and Taksar (1998b, 1998c), Iglehart (1969), Taksar (2000), and the references therein.

Concerning the role of the optimization technology in actuarial science, we would like to recall a part of the speech made by K. Borch, in 1967, to the Royal Statistical Society of London: "*The theory of control processes seems to be "tailor-made" for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed, if actuaries and engineers had realized that they were studying the same problems and joined forces over* 50 *years ago. A little reflection should teach us that "highly specialized" problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated problems.*" This speech explains how the control theory was pointed out as an important theory for the problem in insurance and finance.

According to modern finance, the objective of a firm is always to maximize the value of the shareholders by maximizing the value of the firm. The idea is that the firm should always maximize its value by investing on projects with high returns instead of paying out the dividends that are subject to double taxation. However, in the practical world, many shareholders prefer constant payments of dividends from the firm, instead of buying and selling the stocks, which involves transaction costs.

In this thesis, we consider a model of a financial corporation where the surplus follows a diffusion process and whose objective is to maximize the total expected discounted dividend payouts to the shareholders up to the bankruptcy time. Here, taxes and costs are taken into consideration. This model was first considered by Jeanblanc-Picqué (1995) for the case of dividend payouts without risk control. Then it was extended by Cadenillas et al (2005) so that the surplus is a controlled diffusion process. The control consists of the risk and the time and the amount for the dividend payments. Here, our model further extends this model to the case where the risk control can take on values between any positives numbers $\alpha \leq \beta$ instead of 0 and 1, and the company has debt liability, such as amortization of bonds, loan or mortgage.

This thesis is organized as follows. In the second chapter, a rigorous mathematical formulation of the problem and the transformation of the control problem into a quasi-variational inequality (QVI) equation will be presented. The third chapter concentrates on the construction and calculation of a smooth solution to the QVI stated in Chapter 2, with the consideration of nonzero debt liability, in another words, $\delta > 0$. This chapter provides a slight generalization that allows the model to take the liability factor into account, while assuming the risk control to take values between 0 and 1. The fourth chapter addresses the problem of determining the value function and the optimal policy for the case where the debt liability rate is zero and the risk control lies between α and β. Next, the fifth chapter investigates the interplay between the constraints on the risk control and the debt liability. Finally, a conclusion will be presented.

Chapter 2

The Mathematical Model and Preliminaries

In this chapter, we will start by providing the mathematical model and the framework that the main core of this thesis focuses on. Then we will develop some mathematical/statistical toolbox that are necessary for the coming analysis.

There is no doubt that the major risk for a financial corporation is essentially resulted from the uncertainty in the markets. A mathematical model for this phenomena can be resulted by considering the set $(\Omega,\ \mathcal{F},\ \mathbb{P})$. Here, Ω represents the set of all possible scenarios for the market, (i.e. the sample space), $\mathcal{F}$ is a σ-field that models the whole information about the market under consideration from the beginning up to the horizon time of investment T, here, $T = \infty$. Since the information about the market evolves in time, this can be precisely modelled by an increasing family of σ-fields $(\mathcal{F}_t)_{0\leq t\leq T}$, where $\mathcal{F}_T = \mathcal{F}$. For technical reasons, we assume that this family $(\mathcal{F}_t)_{0\leq t\leq T}$ is right-continuous and complete. All these elements lead to a filtered space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{0\leq t\leq T},\ \mathbb{P})$. On this space, we suppose a Wiener or Brownian motion process $(W_t)_{t\geq 0}$ is given (and exists).

In the absence of control, the reserve is governed by a Brownian motion with a drift μ and a diffusion coefficient σ. In the models of the behavior of an insurance company, the risk control takes on the form of *proportional* reinsurance. Mathematically, proportional reinsurance corresponds to simultaneously decrease the drift and diffusion coefficient by multiplying both quantities by the same factor $a(t) \in [0, 1]$.

Now we are ready to define mathematically the economic model that we will undertake in this thesis. We consider a financial corporation which has liquid assets, $X = \{X(t); t \geq 0\}$ follows the following dynamic

$$X(t) = x + \int_0^t (\mu a_s - \delta) ds + \int_0^t \sigma a_s \, dW_s - \sum_{n=1}^{\infty} 1_{\{\tau_n < t\}} \xi_n. \qquad (2.0.1)$$

Here below are the explanations of the exogenous parameters of the model.

- x (a positive number) is the initial reserve of the company.
- μ is the profit rate, which is the difference between the earning and the expected payments per unit of time, of the company.
- σ represents the diffusion coefficient (a constant volatility that is greater than zero).
- $\delta \geq 0$ is the liability rate (a constant liability rate of payment of the firm's debt, such as the bond liability, mortgage, and loan amortization.)
- $a = (a_t)_{t \geq 0}$ is an adapted process that represents the proportion of risk such that $\alpha \leq a_s(w) \leq \beta$, $\forall\, w \in \Omega$ and $\forall$ t $\in [0, \infty)$.
- τ_i is the i^{th} stopping time at which the company will pay dividend of size ξ_i. Thus, $\{\tau_i;\ i = 1, 2, ...\}$ is an increasing sequence of random variable.

Definition 2.0.1. *A triple*

$$\pi := (u, \mathcal{T}, \xi) = (u; \tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots) \qquad (2.0.2)$$

is called an admissible control or an admissible policy if

$$u : \Omega \times [0, \infty) \mapsto [\alpha, \beta]$$

is an $\{\mathcal{F}_t\}_{t \geq 0}$-adapted process, $\tau_i, i = 1, 2, \ldots$ is a stopping time with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, $0 \leq \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ a.s., and the random variable $\xi_i, i = 1, 2, \ldots$ is $\mathcal{F}_{\tau_i}$-measurable with $0 \leq \xi_i \leq X(\tau_i -)$. The class of all admissible controls is denoted by $\mathcal{A}(x)$.

The last condition means that when the dividends are distributed, the total amount withdrawn cannot exceed the reserve available at that time. In the sequel, we will call τ_i, $i = 1, 2, \ldots$, the times of intervention.
The time of bankruptcy is defined by

$$\tau \equiv \tau^{\pi} := \inf\{t \geq 0 : \ X(t) = 0\}.$$

Since we deal with an optimization problem on the time interval $[0, \tau)$, we can assume without the loss of generality that $X(t)$ vanishes for $t \geq \tau$ and

$$X(t) = \begin{cases} x + \int_0^t (\mu u(s) - \delta) ds + \int_0^t \sigma u(s) \, dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}} \xi_n & \text{if } 0 \leq t < \tau \\ 0 & \text{if } t \geq \tau. \end{cases} \qquad (2.0.3)$$

Define the function $g : [0, \infty) \mapsto (-\infty, \infty)$ by

$$g(\eta) \ := \ -K + k\eta, \qquad g(0) = 0, \tag{2.0.4}$$

where $K \in (0, \infty)$ and $k \in (0, 1)$ are constants. The constant K represents a fixed set-up cost, which is incurred each time when dividends are paid out (no matter how much). As in Cadenillas, Sarkar and Zapatero (2004), we interpret the quantity $1 - k$ as the tax rate at which the dividends are taxed. Thus, if η is the amount of liquid assets withdrawn, then the net amount of money that the shareholder receives is $-K + k\eta$.
With each admissible control $\pi = (u, \mathcal{T}, \xi)$, we associate a performance functional J defined by

$$J(x;\pi) \ := \ E_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n} g(\xi_n) I_{\{\tau_n<\tau\}}\right], \tag{2.0.5}$$

which represents the total expected present value of the dividends received by the shareholder until the time of bankruptcy. The objective is to address the following problem in different situations

Problem 2.0.1. *(i) Compute the value function V defined by*

$$\begin{aligned} V(x) \ &:= \ \sup\{J(x;\pi);\ \pi \in \mathcal{A}(x)\} \\ &= \ \sup_{\pi\in\mathcal{A}(x)} E_x\left[\sum_{n=1}^{\infty} e^{-\lambda\tau_n} g(\xi_n) I_{\{\tau_n<\tau\}}\right]. \end{aligned} \tag{2.0.6}$$

(ii) Find the optimal admissible policy, that is the control $\pi^* = (u^*, \mathcal{T}^*, \xi^*) \in \mathcal{A}(x)$ satisfying

$$V(x) \ = \ J(x;\pi^*). \tag{2.0.7}$$

The next chapters are devoted to finding the value function and the optimal policies of our problem in different cases. In this section, we will establish the quasi-variational inequalities associated with this stochastic control problem and derive the properties of the value function.

Proposition 2.0.1. *The value function V defined in (2.0.6) satisfies for every $x \in [0, \infty)$:*

$$V(x) \ \le \ k\left(x + \frac{\beta\mu + \delta}{\lambda}\right). \tag{2.0.8}$$

Proof. The proof of this proposition mimics the proof of Proposition 3.1 in Cadenillas, Choulli, Taksar and Zhang (2005). □

For a function $\phi : [0, \infty) \mapsto \mathbf{R}$, we define the *maximum utility operator* M by

$$M\phi(x) \ := \ \sup\Big\{\phi(x - \eta) + g(\eta) :\ \eta > 0, x \ge \eta\Big\}. \tag{2.0.9}$$

where g is given by (2.0.4). Define the operator $\mathcal{L}^u$ by

$$\mathcal{L}^u\psi(x) := \frac{1}{2}\sigma^2u^2\frac{d^2\psi(x)}{dx^2} + (\mu u - \delta)\frac{d\psi(x)}{dx} - \lambda\psi(x). \tag{2.0.10}$$

Next, we derive the inequalities that the value function V should satisfy. Suppose that for each initial position x, there exists an optimal policy. If the process starts at x and is controlled by the optimal policy, then the expected utility associated with this optimal policy is $V(x)$. On the other hand, suppose that the payment of dividends occurs at time 0. If the amount of the liquid assets used to pay dividends is η, then the initial reserve decreases from x to $x-\eta$. If the optimal policy is followed afterwards, then the total expected utility associated with this policy is $-K+k\eta+V(x-\eta)$. If the initial pay out η is chosen to be the one which brings the maximal value to $-K+k\eta+V(x-\eta)$, then the total expected utility under such a policy would be equal to $MV(x)$. Since the first policy is optimal, its associated expected utility is greater than or equal to that of the expected utility associated with the second policy. Hence, $V(x) \geq MV(x)$, with equality being true if x is the position process of optimal intervention.

A standard application of the dynamic programming principle (e.g., Hojgaard and Taksar (1998a, 1998b, 1999)) yields $\max_{u\in[\alpha,\beta]} \mathcal{L}^u V(x) = 0$ in the continuation region, that is, in the region where it is not optimal to intervene. These heuristic arguments enable one to get inequalities that the value function must satisfy. We formalize this intuition in the next two definitions and theorem.

Definition 2.0.2. *A function $v : [0,\infty) \mapsto [0,\infty)$ satisfies the quasi-variational inequalities of the control problem (QVI hereafter) if for every $x \in [0,\infty)$ and $u \in [\alpha,\beta]$:*

$$\mathcal{L}^u v(x) \leq 0, \tag{2.0.11}$$

$$v(x) \geq Mv(x), \tag{2.0.12}$$

$$\Big(v(x) - Mv(x)\Big)\Big(\max_{u\in[\alpha,\beta]} \mathcal{L}^u v(x)\Big) = 0, \tag{2.0.13}$$

and

$$v(0) = 0. \tag{2.0.14}$$

We observe that a solution v of the QVI (see Definition 2.0.2) splits $(0,\infty)$ into two regions: a *continuation region*

$$C := \left\{x \in (0,\infty) : v(x) > Mv(x) \text{ and } \max_{u\in[\alpha,\beta]} \mathcal{L}^u v(x) = 0\right\}$$

and an *intervention region*

$$\Sigma := \left\{x \in (0,\infty) : v(x) = Mv(x) \text{ and } \max_{u\in[\alpha,\beta]} \mathcal{L}^u v(x) \leq 0\right\}.$$

Given a solution v to the QVI (2.0.11)-(2.0.14), we define the following policy associated with this solution.

Definition 2.0.3. *The control* $\pi^v = (u^v, \mathcal{T}^v, \xi^v)$ $= (u^v; \tau_1^v, \tau_2^v, \ldots, \tau_n^v, \ldots; \xi_1^v, \xi_2^v, \ldots, \xi_n^v, \ldots)$ *is called the QVI-control associated with v if the associated state process X^v given by (2.0.1) satisfies*

$$P\left\{u^v(t) \neq \arg\max_{u\in[0,1]} \mathcal{L}^u v(X_t^v),\ X_t^v \in \mathcal{C}\right\} = 0, \tag{2.0.15}$$

$$\tau_1^v := \inf\{t \geq 0 : v(X^v(t)) = Mv(X^v(t))\} \tag{2.0.16}$$

$$\xi_1^v := \arg \sup_{\eta>0, \eta\leq X^v(\tau_1^v)} \left\{v\left(X^v(\tau_1^v) - \eta\right) + g(\eta)\right\} \tag{2.0.17}$$

and, for every $n \geq 2$:

$$\tau_n^v := \inf\{t > \tau_{n-1} : v(X^v(t)) = Mv(X^v(t))\} \tag{2.0.18}$$

$$\xi_n^v := \arg \sup_{\eta>0, \eta\leq X^v(\tau_n^v)} \left\{v\left(X^v(\tau_n^v) - \eta\right) + g(\eta)\right\}, \tag{2.0.19}$$

with $\tau_0^v := 0$, $\xi_0^v := 0$.

Under this control the intervention takes place whenever v and Mv coincide, and the amount of the liquid assets withdrawn at these times $\{\tau_i, i = 0, 1, \ldots\}$ is determined from the solution to the one-dimensional optimization problem associated with the operator Mv.

Theorem 2.0.1. *Let $v \in C^1((0,\infty))$ be a solution of the QVI (2.0.11)-(2.0.14). Suppose there exists $U > 0$ such that v is twice continuously differentiable on $(0, U)$ and v is linear on $[U, \infty)$. Then, for every $x \in (0, \infty)$:*

$$V(x) \leq v(x). \tag{2.0.20}$$

Furthermore, if the QVI-control $(u^v, \mathcal{T}^v, \xi^v)$ associated with v is admissible, then v coincides with the value function and the QVI control associated with v is the optimal policy, i.e.,

$$V(x) = v(x) = J(x; u^v, \mathcal{T}^v, \xi^v). \tag{2.0.21}$$

Proof. For the proof of this theorem see Cadenillas et al. (2005). □

Chapter 3

Effect of debt liability on risk and dividend payouts

This chapter addresses the problem of optimal risk and dividend when the liability rate is nonzero and there are no constraints on the risk control. Precisely, we will attempt to solve the problem (2.0.1) with $\alpha = 0$, $\beta = 1$ and $\delta > 0$. Hence, we will start by constructing a smooth candidate for the solution to (2.0.11)-(2.0.14). Let

$$x_D := \inf\{x \geq 0 : v(x) = Mv(x)\}, \tag{3.0.1}$$

then for $x < x_D$, (2.0.13) becomes

$$\max_{a\in} \mathcal{L}^a v(x) = \max \left\{ \frac{1}{2}\sigma^2 a^2 v''(x) + (a\mu - \delta)v'(x) - \lambda v(x) \right\} = 0. \tag{3.0.2}$$

$$\max_{a\in[0,1]} \mathcal{L}^a v(x) = \max_{a\in[0,1]} \left\{ \frac{1}{2}\sigma^2 a^2 v''(x) + (a\mu - \delta)v'(x) - \lambda v(x) \right\} = 0. \tag{3.0.2}$$

While for $x \geq x_D$, (2.0.13) is equivalent to

$$v(x) = Mv(x).$$

The solution for this last equation is given by Cadenillas et al. (2005) and described as follows:

Proposition 3.0.2. *For $x \geq x_D$, we have:*

$$v(x) = v(\tilde{x}) + k(x - \tilde{x}) - K, \tag{3.0.3}$$

where $\tilde{x} = \inf\{x < x_D \mid v'(x) = k\}$.

Proof. See Cadenillas et al. (2005). □

Thus, now we will deal with the equation (3.0.2). It is clear that the maximizer of $\mathcal{L}^a v(x)$ over the whole nonnegative real line is given by

$$a(x) := \arg\max_{a\geq 0} \mathcal{L}^a v(x) = \begin{cases} -\frac{\mu v'(x)}{\sigma^2 v''(x)} & \text{for } v''(x) \neq 0 \\ \infty & \text{for } v''(x) = 0. \end{cases} \tag{3.0.4}$$

Thus, on the set $\{x < x_D \mid 0 \leq a(x) \leq 1\}$, we have

$$\mathcal{L}^{a(x)} v(x) = -\frac{\mu^2 v'(x)^2}{2\sigma^2 v''(x)} - \delta v'(x) - \lambda v(x) = 0, \tag{3.0.5}$$

or equivalently by inserting (3.0.4) into (3.0.5) and divide the equation by $v(x)$ will result in

$$\frac{\mu^2 v'(x)^2}{2\sigma^2 v''(x) v(x)} + \delta \frac{v'(x)}{v(x)} + \lambda = 0. \tag{3.0.6}$$

Consider the following function and its derivative

$$l(x) = \frac{v(x)}{v'(x)}, \qquad l'(x) = 1 - \frac{v''(x)v(x)}{(v'(x))^2}.$$

By inserting the above expressions of $l(x)$ and $l'(x)$ into (3.0.6), we get

$$\frac{\mu^2}{2\sigma^2}\left(\frac{1}{1-l'(x)}\right) + \frac{\delta}{l(x)} + \lambda = 0, \tag{3.0.7}$$

which leads to

$$\frac{\mu^2 + 2\lambda\sigma^2}{2\lambda\sigma^2} = \frac{l(x) + \frac{\delta}{\lambda}}{l(x) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}} l'(x). \tag{3.0.8}$$

Therefore, we derive

$$\begin{aligned} c_1 + \frac{\mu^2 + 2\lambda\sigma^2}{2\lambda\sigma^2} x &= \int \frac{l(x) + \frac{\delta}{\lambda}}{l(x) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}} l'(x) dx \\ &= l(x) + \left(\frac{\delta}{\lambda} - \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right) \ln\left(l(x) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right) \\ &= F(l(x)), \end{aligned} \tag{3.0.9}$$

where c_1 is a constant to be determined later, and

$$F(y) = y + \left(\frac{\delta\mu^2}{\lambda(\mu^2 + 2\lambda\sigma^2)}\right) \ln\left(y + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right). \tag{3.0.10}$$

Remark that F is continuously differentiable on $\left(\frac{-2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}, \infty\right)$, and it is strictly increasing. Therefore, F^{-1} (its inverse function) exists and then we can write

$$\frac{v'(x)}{v(x)} = \frac{1}{F^{-1}\left(c_1 + \frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}x\right)}, \tag{3.0.11}$$

or equivalently,

$$\ln v(x) = c' + \int \frac{dx}{F^{-1}\left(c_1 + \frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}x\right)}. \tag{3.0.12}$$

Consider the following change of variable

$$z = F^{-1}\left(c_1 + \frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}x\right). \tag{3.0.13}$$

Then, a simple calculus shows that

$$\ln v(x) = c' + \gamma \ln\left(z^{\frac{1}{\gamma}}\left(z + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{-\frac{\mu^2}{2\lambda\sigma^2}}\right), \tag{3.0.14}$$

where

$$\gamma = \frac{2\lambda\sigma^2}{\mu^2+2\lambda\sigma^2} > 0. \tag{3.0.15}$$

[Indeed we have

$$\int \frac{dx}{F^{-1}\left(c_1 + \frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}x\right)} = \int \frac{2\lambda\sigma^2}{2\lambda\sigma^2+\mu^2}\frac{F'(z)}{z}dz,$$

where

$$\begin{aligned}\int \frac{F'(z)}{z}dz &= \ln(z) + \frac{\mu^2}{2\lambda\sigma^2}\left(\ln(z) - \ln\left(z + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)\right)\\ &= \ln\left(z^{\frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}}\left(z + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{-\frac{\mu^2}{2\lambda\sigma^2}}\right).]\end{aligned}$$

Then due to (3.0.14) and (3.0.13), we set

$$v(x) = c\left(F^{-1}\left(c_1 + \frac{x}{\gamma}\right)\right)\left(F^{-1}\left(c_1+\frac{x}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}, x < x_D, \tag{3.0.16}$$

where $c = e^{c'} > 0$. Thanks to $v(0) = 0$, we calculate

$$c_1 = F(0). \tag{3.0.17}$$

Since $\arg\max_{a\in[0,1]} \mathcal{L}^a v(x)$ should be between 0 and 1, we need to investigate the monotonicity of $a(x)$, which is given by

Lemma 3.0.1. *For $x < x_D$, we have*

$$a(x) = \frac{2\lambda}{\mu} F^{-1}\left(F(0) + \frac{\mu^2 + 2\lambda\sigma^2}{2\lambda\sigma^2} x\right) + \frac{2\delta}{\mu}. \tag{3.0.18}$$

In particular, $a(0) = \frac{2\delta}{\mu}$, and $a(x)$ is an increasing function.

Proof. Recall that the expression of $a(x)$ in terms of $v'(x)$ and $v''(x)$ is given by

$$a(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}, \tag{3.0.19}$$

and $v(x)$ is described by (3.0.16). By differentiating (3.0.16) twice and simplifying the resulting expression, we derive

$$v''(x) = -\frac{\mu^2}{2\lambda\sigma^2} \frac{v'(x)}{F^{-1}\left(F(0) + \frac{x}{\gamma}\right) + \frac{\delta}{\lambda}}. \tag{3.0.20}$$

This leads to (3.0.18), and in particular, $a(0) = \frac{2\delta}{\mu}$. Since F^{-1} increases, $a(x)$ increases. This completes the proof of the lemma. □

Lemma 3.0.1 allows us to calculate explicitly the sets $\{x < x_D \mid a(x) > 1\}$ and $\{x < x_D \mid 0 \le a(x) \le 1\}$. This description depends heavily on whether δ, the liability rate, exceeds $\frac{\mu}{2}$ or not.

3.1 The case of $\delta \ge \frac{\mu}{2}$

Due to Lemma 3.0.1, $a(x)$ is increasing and $1 \le a(0) \le a(x)$, $\forall\ x \ge 0$. Therefore, in this case of $\delta \ge \frac{\mu}{2}$, we have

$$\{x < x_D \mid a(x) \ge 1\} = [0, x_D),$$

and

$$\max_{a \in [0,1]} \mathcal{L}^a v(x) = \mathcal{L}^1 v(x) = \frac{1}{2}\sigma^2 v''(x) + (\mu - \delta) v'(x) - \lambda v(x) = 0 \tag{3.1.21}$$

subject to

$$v(0) = 0. \tag{3.1.22}$$

The general solution for (3.1.21)-(3.1.22) is given by $c(e^{r_+ x} - e^{r_- x})$, for $x < x_D$. Therefore, we derive our candidate for the QVI (2.0.11)-(2.0.14) as follows

$$v(x) = \begin{cases} c(e^{r_+ x} - e^{r_- x}) & 0 \le x < x_D \\ v(\tilde{x}) + k(x - \tilde{x}) - K & x \ge x_D, \end{cases} \tag{3.1.23}$$

where

$$r_{\pm} = \frac{-(\mu-\delta) \pm \sqrt{(\mu-\delta)^2 + 2\sigma^2\lambda}}{\sigma^2}. \tag{3.1.24}$$

Here c, $\tilde{x}$, and x_D are the parameters to be determined using the smooth fit of v.

It is clear, from (3.1.23), that for $\delta \geq \mu$, we have $v''(x) > 0$ for every $x < x_D$. Then necessarily $x_D = \tilde{x} = 0$ (indeed on $[0, x_D)$, v' is increasing and hence $v'(x) = k$ has at most one root, this implies either $\tilde{x}$ does not exist or $\tilde{x} = x_D = 0$). Therefore, in this case of $\delta \geq \mu$, the problem (2.0.1) has the following solution. The company declares bankruptcy at the beginning and distributes the cash reserve to the shareholders as dividends. Hence, the optimal return function is given by

$$v(x) = g(x), \; x \geq 0.$$

In the remaining part of this section, we assume that $\delta < \mu$. Hence, $x_D > 0$. Consider the following function

$$H_\delta(x) = r_+ e^{r_+ x} - r_- e^{r_- x}, \text{ for } x \geq 0. \tag{3.1.25}$$

Then (3.1.23) can be written as

$$v'(x) = \begin{cases} cH_\delta(x) & 0 \leq x < x_D \\ k & x \geq x_D. \end{cases} \tag{3.1.26}$$

The purpose of this function H_δ is to determine $\tilde{x}$ and x_D. In fact, $\tilde{x}$ and x_D are the first and the second roots of

$$H_\delta(x) = \frac{k}{c} \tag{3.1.27}$$

It is obvious that H_δ is strictly convex, continuous, and attains its minimal value at $\hat{x}$, root of $H'_\delta(x) = 0$, given by

$$\hat{x} = \frac{1}{r_+ - r_-} ln\left(\frac{r_-}{r_+}\right)^2 > 0. \tag{3.1.28}$$

That is,

$$H_{min} = H(\hat{x}) = \min_{x \geq 0} H(x). \tag{3.1.29}$$

Also, $H_\delta(\infty) = \infty$.

- If $c > \frac{k}{H_{min}}$, then (3.1.27) has no solution.
- If $c < \frac{k}{H_\delta(0)}$, then (3.1.27) has only one solution, the one that corresponds to $x_D(c)$. This leads to conclude that $\tilde{x}(c)$ does not exists.

Therefore, for the equation (3.1.27) to have two roots, we need to choose c such that

$$c_0 := \frac{k}{H_\delta(0)} \leq c \leq \frac{k}{H_{min}} =: c_1. \tag{3.1.30}$$

Then for $c \in [c_0, c_1]$, we define

$$I(c) = \int_{\tilde{x}(c)}^{x_D(c)} (k - cH(x))dx. \tag{3.1.31}$$

Now, we would like to find $c \in [c_0, c_1]$ such that

$$I(c) = K. \tag{3.1.32}$$

Notice that for $c = \frac{k}{H_{min}}$, $\tilde{x}(c) = x_D(c) = \hat{x}$, which corresponds to the case where there is no cost, i.e. $K = 0$. Hence, $I\left(\frac{k}{H_{min}}\right) = 0$. It is clear that $c \to \tilde{x}(c)$ is increasing and $c \to x_D(c)$ is decreasing. Thus, $c \to I(c)$ is decreasing and

$$I(\hat{c}) = \int_0^{x_D(\hat{c})} (k - \hat{c}H_\delta(x))dx = K_{max}(\delta), \tag{3.1.33}$$

where

$$\hat{c} = \frac{k}{r_+ - r_-}. \tag{3.1.34}$$

Then if $K > K_{max}(\delta)$, then the equation $I(c) = K$ has no solution. Therefore, $K_{max}(\delta)$ is the upper bound for K such that the QVI (2.0.11)-(2.0.14) has a continuous solution.
For $0 \leq K \leq K_{max}(\delta)$, then there exists $\tilde{c} \in [\frac{k}{r_+ - r_-}, \frac{k}{H_{min}}]$, which is a root of

$$I(c) = K. \tag{3.1.35}$$

3.2 The case of $0 \leq \delta < \frac{\mu}{2}$

Due to Lemma 3.0.1, we have $0 \leq a(0) = \frac{\mu}{2} < 1$ and $a(x)$ is strictly increasing, then there exists $x_1 > 0$ such that $a(x_1) = 1$, and for any $x \geq x_1$, $a(x) \geq 1$. Then x_1 can be calculated using

$$a(x_1) = \frac{2\lambda}{\mu} F^{-1}\left(F(0) + \frac{\mu^2 + 2\lambda\sigma^2}{2\lambda\sigma^2} x_1\right) + \frac{2\delta}{\mu} = 1.$$

This leads to

$$x_1 = \frac{2\lambda\sigma^2}{2\lambda\sigma^2 + \mu^2}\left[F\left(\frac{\mu}{2\lambda}\left(1 - \frac{2\delta}{\mu}\right)\right) - F(0)\right] > 0. \tag{3.2.36}$$

For $0 \le x < x_1$, $v(x)$ is the solution of $\mathcal{L}^{a(x)}v(x) = 0$ and is given by

$$v(x) = c\left(F^{-1}\left(F(0)+\frac{x}{\gamma}\right)\right)\left(F^{-1}\left(F(0)+\frac{x}{\gamma}\right)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}. \tag{3.2.37}$$

For $x_1 \le x < x_D$, $v(x)$ is the solution of $\mathcal{L}^1 v(x) = 0$ and is given by

$$v(x) = c_1 e^{r_+(x-\tilde{x})} + c_2 e^{r_-(x-\tilde{x})}. \tag{3.2.38}$$

While for $x \ge x_D$,

$$v(x) = v(\tilde{x}) + k(x-\tilde{x}) - K. \tag{3.2.39}$$

These pieces can be summarized into the following

$$v(x) = \begin{cases} c\left(F^{-1}\left(F(0)+\frac{x}{\gamma}\right)\right)\left(F^{-1}\left(F(0)+\frac{x}{\gamma}\right)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}, & 0 \le x < x_1 \\ c_1 e^{r_+(x-x_1)} + c_2 e^{r_-(x-x_1)}, & x_1 \le x < x_D \\ v(\tilde{x}) + k(x-\tilde{x}) - K, & x \ge x_D. \end{cases} \tag{3.2.40}$$

Consider the following quantities

$$A_1 = \left(F^{-1}\left(F(0)+\frac{x_1}{\gamma}\right)\right)\left(F^{-1}\left(F(0)+\frac{x_1}{\gamma}\right)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1} \tag{3.2.41}$$

and

$$B_1 = \left(F^{-1}\left(F(0)+\frac{x_1}{\gamma}\right)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}. \tag{3.2.42}$$

Then using the smooth fit of v and v' at the point x_1, we get the constants c_1 and c_2 in terms of c, A_1, B_1 as follows:

$$\begin{cases} c_1 = c\frac{B_1 - r_- A_1}{r_+ - r_-} =: ca_1 \\ c_2 = c\frac{r_+ A_1 - B_1}{r_+ - r_-} =: cb_1. \end{cases} \tag{3.2.43}$$

Consider the following function

$$H_\delta(x) = \begin{cases} \left(F^{-1}\left(F(0)+\frac{x}{\gamma}\right)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}, & 0 \le x < x_1 \\ a_1 r_+ e^{r_+(x-x_1)} + b_1 r_- e^{r_-(x-x_1)}, & x \ge x_1. \end{cases} \tag{3.2.44}$$

Then $H_\delta(x)$ is convex, $H_\delta(0) = \left(\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}$, $H_\delta(\infty) = \infty$, and

$$v'(x) = \begin{cases} cH_\delta(x) & 0 \le x < x_D \\ k & x \ge x_D. \end{cases} \tag{3.2.45}$$

Now we need to characterize the parameters c, $\tilde{x}$, and x_D. We will start by calculating $\tilde{x}$ and x_D in terms of c using the equation

$$H_\delta(x) = \frac{k}{c}. \tag{3.2.46}$$

Precisely, we will show that for some ranges of c, the equation (3.2.46) has two solutions, $\tilde{x}(c)$ and $x_D(c)$. To this end, remark that H_δ is strictly convex and there exists a unique $\hat{x} \in (0, \infty)$ such that

$$H'_\delta(\hat{x}) = 0,$$

and $H'_\delta < 0$ on $(0, \hat{x})$, and $H'_\delta > 0$ on $(\hat{x}, \infty)$. Then denote

$$\hat{H}_\delta = H_\delta|_{(0,\hat{x})}, \; \bar{H}_\delta = H_\delta|_{(\hat{x},\infty)},$$

where $H_\delta|_I$ is the restriction of H_δ on the interval I. Thus, $\hat{H}_\delta$ and $\bar{H}_\delta$ are one-to-one functions that are decreasing and increasing, respectively. Therefore,

$$\tilde{x}(c) = \hat{H}_\delta^{-1}\left(\frac{k}{c}\right), \; x_D(c) = \bar{H}_\delta^{-1}\left(\frac{k}{c}\right). \tag{3.2.47}$$

As a result, $\tilde{x}(c)$ and $x_D(c)$ are calculated in terms of the parameter c, and they are increasing and decreasing, respectively, in the variable c. Notice that

$$H_{min} = \min_{x \geq 0} H_\delta(x) = H_\delta(\hat{x}),$$

and remark the following

- If $c > \frac{k}{H_{min}}$, then $\tilde{x}(c)$ and $x_D(c)$ do not exist.
- If $c < \frac{k}{H_\delta(0)}$, then $\tilde{x}(c)$ does not exist, while $x_D(c)$ exists.

Therefore, the parameters $\tilde{x}(c)$ and $x_D(c)$ defined in (3.2.47) exist if and only if

$$c_0 := \frac{k}{H_\delta(0)} \leq c \leq c_1 := \frac{k}{H_{min}}.$$

Now we need to determine the parameter c. To this end, we denote

$$I(c) = \int_{\tilde{x}(c)}^{x_D(c)} (k - cH_\delta(x))dx, \qquad \text{for } \; c_0 \leq c \leq c_1. \tag{3.2.48}$$

Notice that $I(c)$ is continuously decreasing, $I(c_1) = 0$, and $\max_{c_0 \leq c \leq c_1} I(c) = I(c_0) =: K_{max}(\delta)$. As a result, if $K > K_{max}(\delta)$, then the QVI (2.0.11)-(2.0.14) has no smooth solution, and for any $0 \leq K \leq K_{max}(\delta)$, there exists $\tilde{c} \in [c_0, c_1]$, which is a root of

$$I(c) = K. \tag{3.2.49}$$

3.3 A smooth optimal return function and optimal policies

By comparing the expressions for v given by (3.1.23) and (3.2.40), we can conclude that the equation (3.2.40) can hold for both cases. Indeed, if we redefine x_1 by

$$x_1 = \frac{2\lambda\sigma^2}{2\lambda\sigma^2 + \mu^2}\left[F\left(\frac{\mu}{2\lambda}\left(1 - \frac{2\delta}{\mu}\right)^+\right) - F(0)\right] \geq 0, \tag{3.3.50}$$

where $z^+ = max(z, 0)$ for all $z \in \mathbb{R}$. Then $x_1 = 0$ for $\frac{2\delta}{\mu} \geq 1$ and (3.2.40) is reduced to (3.1.23). Consider

$$\begin{aligned} a^*(x) &= \min(a(x), 1) \\ &= \begin{cases} \frac{2\lambda}{\mu}F^{-1}\left(F(0) + \frac{\mu^2+2\lambda\sigma^2}{2\lambda\sigma^2}x\right) + \frac{2\delta}{\mu} & \text{if } x < x_1 \\ 1 & \text{if } x \geq x_1. \end{cases} \end{aligned} \tag{3.3.51}$$

Now we are ready to state the main result in this chapter.

Theorem 3.3.1. *Suppose that c is a root of (3.2.49). Let x_1, $\widetilde{x}(c)$ and $x_D(c)$ and c_1 and c_2 be given by (3.3.50), (3.2.47) and (3.2.43) respectively. Then the function v described by (3.2.40) is continuously differentiable on $(0, \infty)$ and is twice continuously differentiable on $(0, x_D) \cup (x_D, \infty)$. This function is a solution to the QVI (2.0.11)-(2.0.14) subject to the growth condition (2.0.8).*

Proof. It is clear that v is twice continuously differentiable on $(0, x_D(c))$, since it coincides with

$$c\int_0^x H(y)dy, \qquad x \geq 0,$$

which is twice differentiable on $(0, +\infty)$ due to the choice of a_1 and b_1 as in (3.2.43).

It is obvious that v is linear on the set $(x_D(c), +\infty)$, and from the calculation of c, $\widetilde{x}(c)$ and $x_D(c)$ in the previous section, we deduce that v is continuously differentiable at x_D and then continuously differentiable on $(0, +\infty)$.

We can easily prove that for $x < x_D$, we have

$$Mv(x) = \begin{cases} v(x) - K, & \text{for } x \leq \widetilde{x} \\ v(\widetilde{x}) + k(x - \widetilde{x}) - K, & \text{for } x \geq \widetilde{x}. \end{cases}$$

Since

$$K = I(c) > \int_{\widetilde{x}}^x (k - v'(y))dy,$$

we deduce that

$$v(\widetilde{x}) + k(x - \widetilde{x}) - K < v(x),$$

and hence $Mv(x) < v(x)$ for any $x < x_D$. Due to $\max_{0\le a\le 1} \mathcal{L}^a v(x) = 0$, for any $x < x_D$, we conclude that (2.0.11)-(2.0.14) are satisfied for any $x < x_D$.

Due to $\frac{v''(x_D-)}{c} = H'_\delta(x_D) > H'_\delta(\hat{x}) = 0$, $v''(x_D) = 0$ and the continuity of v and v' at the point x_D, we have

$$\lim_{\substack{x \longrightarrow x_D \\ x < x_D}} \mathcal{L}^a v(x) = \frac{a^2\sigma^2}{2} v''(x_D-) + (\mu a - \delta)v'(x_D-) - \lambda v(x_D-) > \mathcal{L}^a v(x_D).$$

This implies that for any $x \ge x_D$,

$$0 = \max_{0\le a\le 1} \mathcal{L}^a v(x_D-) \ge \max_{0\le a\le 1} \mathcal{L}^a v(x_D) \ge \max_{0\le a\le 1} \mathcal{L}^a v(x).$$

This combined with $Mv(x) = v(x)$ for any $x \ge x_D$ prove that the QVI (2.0.11)-(2.0.14) are satisfied for any $x \ge x_D$. This completes the proof of the theorem. □

The next theorem identifies the optimal policy, and shows that the solution to the QVI constructed above is the value function.

Theorem 3.3.2. *Suppose that c is a root of (3.2.49) and $\tilde{x}(c)$ and $x_D(c)$ are given by (3.2.47). Let $a^*(x)$ be given by (3.3.51). Then the control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, \dots, \tau_n^*, \dots; \xi_1^*, \xi_2^*, \dots, \xi_n^*, \dots)$$

defined by

$$u_t^*(t) = a^*(X_t^*), \; 0 \le t \le T \tag{3.3.52}$$

$$\tau_1^* := \inf\{t \ge 0 : X^*(t) = x_D(c)\} \tag{3.3.53}$$
$$\xi_1^* := x_D(c) - \tilde{x}(c) \tag{3.3.54}$$

and for every $n \ge 2$:

$$\tau_n^* := \inf\{t > \tau_{n-1} : X^*(t) = x_D(c)\} \tag{3.3.55}$$
$$\xi_n^* := x_D(c) - \tilde{x}(c), \tag{3.3.56}$$

where X^* is the solution to the stochastic differential equation

$$X^*(t) = X^*(0) + \int_0^t (\mu u^*(X^*(s)) - \delta)ds + \int_0^t \sigma u^*(X^*(s))dW_s - (x_D(c) - \tilde{x}(c)) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}}, \tag{3.3.57}$$

is the QVI control associated with the function v defined by (3.2.40). This control is optimal and the function v coincides with the value function. That is,

$$V(x) = v(x) = J(x; \pi^*) = J(x; u^*, \mathcal{T}^*, \xi^*). \tag{3.3.58}$$

Proof. In view of Theorem 3.3.1, the function v defined by (3.2.40) satisfies all the conditions in Theorem 2.0.1. From Definition 2.0.3 and the discussion in previous sections, we know that the control π^* defined in (3.3.52)-(3.3.56) is the control associated with v. In addition, according to Definition 2.0.1, the control π^* is admissible. Therefore, with the application of Theorem 2.0.1, we conclude that v is the value function and π^* is the optimal policy. □

3.4 Economic interpretation and numerical examples

We start this section by pointing out some features that remain robust with respect to the nonzero liability rate.

- The optimal dividend policy is always of a threshold type with the threshold level being equal to $x_D(\widetilde{c})$, where $\tilde{c}$ is a root of 3.2.49. Precisely, as soon as the level of the cash reserve reaches the level $x_D(\widetilde{c})$, the firm should distribute $x_D(\widetilde{c}) - \widetilde{x}(\widetilde{c})$ as dividend to the shareholder.
- The maximum business activity is always attained prior to the time the dividend distributions occur (i.e. $x_1 < x_D(\widetilde{c})$).
- As it is noticed in the previous section, we have $x_D(c_1) = \widetilde{x}(c_1) = \hat{x}$. This corresponds to $K = 0$, as $I(c_1) = 0$, the case of no costs. We can also state the following claim: The bigger the cost K is, the later the dividends are paid out to the shareholder. Mathematically, we can prove that if $K_1 < K_2$, then $x_D(\tilde{c}_1) < x_D(\tilde{c}_2)$, where $\tilde{c}_1$ and $\tilde{c}_2$ are roots of (3.2.49) for K replaced by K_1 and K_2 respectively.

The remaining part of this section consists of addressing the effects of a nonzero liability rate δ, which are multifold and can be summarized as follows.

- A nonzero liability rate (i.e. $\delta > 0$) imposes an upper bound on the costs K, which is denoted by $K_{max}(\delta)$. Precisely, for having a smooth optimal return function, the costs K never exceed $K_{max}(\delta)$ which is finite for a positive δ, while for the case of no debt liability (i.e. $\delta = 0$), there is no upper bound on K. In fact, we can show that $\lim\limits_{\delta \longrightarrow 0} K_{max}(\delta) = +\infty$.

Since we have

$$K_{max}(\delta) = \int_0^{x_D(\hat{c})} (k - \hat{c}H(x))dx, \ \hat{c} = \frac{k}{H_\delta(0)} \tag{3.4.59}$$

so $\lim\limits_{\delta \to 0^+} K_{max}(\delta) = \infty$ if and only if

$$\lim_{\delta \to 0^+} x_D(\hat{c}) = \infty, \tag{3.4.60}$$

so it is good enough to prove the latter limit.
We have $\hat{c} = \frac{k}{H(0)}$, and $\lim_{\delta \to 0^+} \hat{c} = \lim_{\delta \to 0^+} \frac{k}{H_\delta(0)} = 0^+$. Also, we have $\hat{c} = \frac{k}{H_\delta(x_D(\hat{c}))}$ or equivalently $\frac{k}{\hat{c}} = H_\delta(x_D(\hat{c}))$, and so

$$\lim_{\delta \to 0^+} \frac{k}{\hat{c}} = \lim_{\delta \to 0^+} H_\delta(x_D(\hat{c})) = \infty \tag{3.4.61}$$

From (3.2.44), we know that $H(x)$ goes to infinity if and only if x goes to infinity. Therefore, the limit (3.4.61) is true if and only if $x_D(\hat{c}) \to \infty$ as $\delta \to 0^+$, which proves (3.4.60).

This explains the interplay and direct effect of liability on the costs. Moreover the quantity $K_{max}(\delta)$ is decreasing when δ increases. This fact can be shown theoretically, but we prefer to show it through the following numerical examples instead.

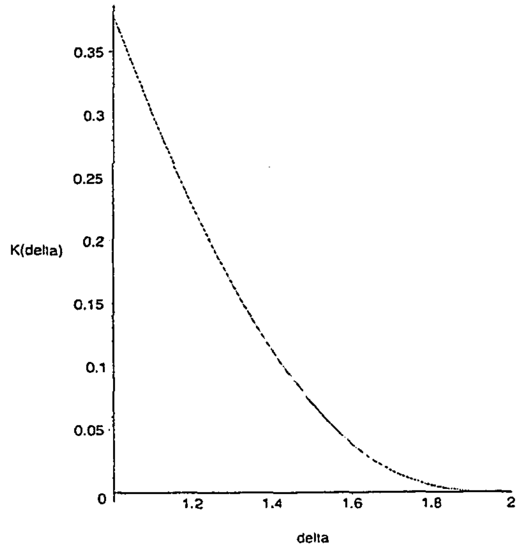

Figure 3.1: The Relationship between δ and $K_{max}(\delta)$.
$K_{max}(\delta)$ is the maximum cost permitted and is given by (3.1.33).
Example for the case $\delta \geq \frac{\mu}{2}$ with $\mu = 2, \sigma = 1, \lambda = 1, k = 0.5$.

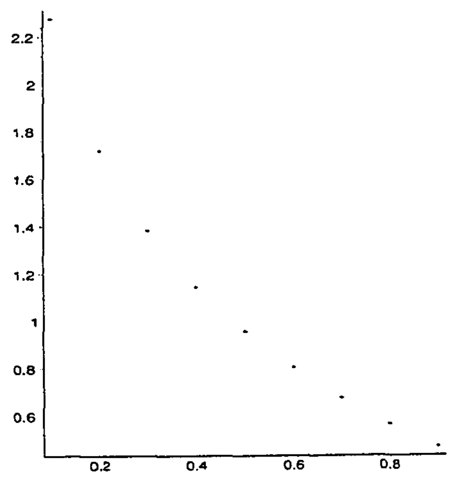

Figure 3.2: The Relationship between δ and $K_{max}(\delta)$.
Example for the case $0 \leq \delta < \frac{\mu}{2}$ with $\mu = 2, \sigma = 1, \lambda = 1, k = 0.5$.

- The second main effect of nonzero liability rate is on the dividend payouts threshold $x_D = x_D(\widetilde{c})$. In fact we can conclude that the bigger the liability rate is, the earlier the dividends are paid out. Below are numerical examples illustrating this fact.

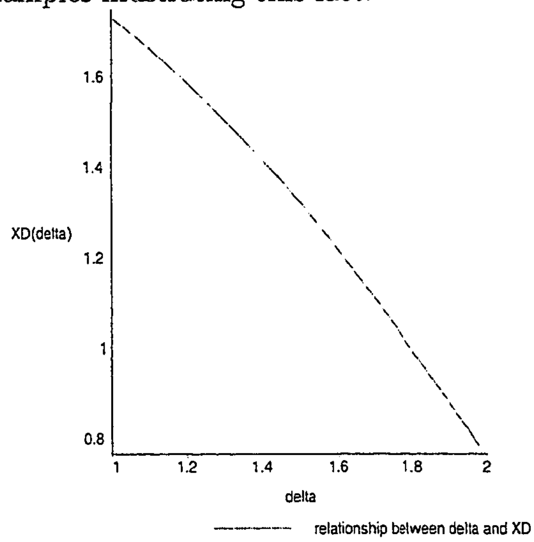

Figure 3.3: The Relationship between δ and $x_D(\delta)$.
Example for the case $\delta \geq \frac{\mu}{2}$ with $\mu = 2, \sigma = 1, \lambda = 1, k = 0.5, K = 0.2$.

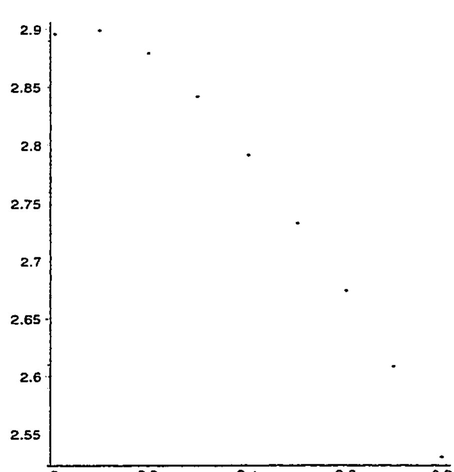

Figure 3.4: The Relationship between δ and $x_D(\delta)$.
Example for the case $0 \leq \delta < \frac{\mu}{2}$ with $\mu = 2, \sigma = 1, \lambda = 1, k = 0.5, K = 0.5$.

- The third clear impact of a nonzero liability rate concentrates on the quantity of optimal risk the company assumes, a_t^* and its relationship to the reserve process X_t^*. For this aspect we can point out two main remarks as follows.

  1) The first remark is concerned with the way the optimal risk depends on the reserve process. Indeed when $\delta = 0$ it was shown in Cadenillas et al. (2005) that this optimal risk is linear with respect to the reserve process up to the dividend threshold $x_D(\tilde{c})$, while it is strictly convex for the case of $0 < \delta < \frac{\mu}{2}$ (smaller but nonzero debt liability rate) and constant equal to the maximum risk allowed, when $\delta \geq \frac{\mu}{2}$ (big enough liability rate). Furthermore, we have

$$a_{\delta_1}^*(x) \geq a_{\delta_2}^*(x) \geq a_0^*(x), \qquad \forall \ \ \delta_1 \geq \delta_2 \geq 0, \ \forall \ \ x \geq 0.$$

  2) Recall from the previous section that when $\delta > 0$, we have

$$a_\delta^*(0) = \min(1, \frac{2\delta}{\mu}) > 0 = a_0^*(0).$$

  This leads to conclude that for the case when $\delta > 0$, it is optimal for the company to gamble on higher potential profits in order to get out of the "bankruptcy zone" as fast as possible, even at the expense of assuming higher risk. In other words, the company starts with more aggressive business activities than that of the case when $\delta = 0$. These activities are done in a proportional manner with respect to δ for relatively smaller values of δ. The following figure illustrates these two remarks.

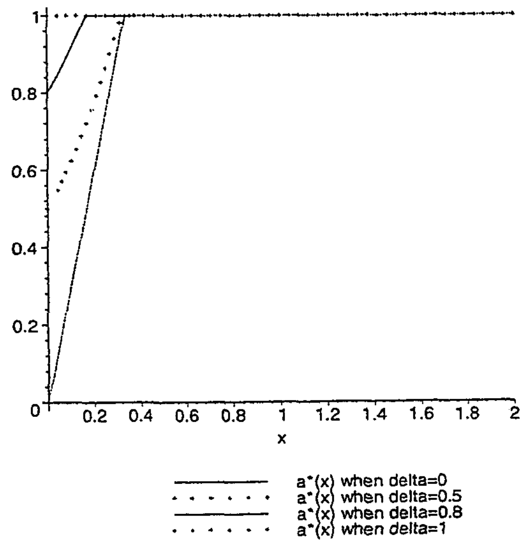

Figure 3.5: Graph for $a_0^*(x)$, $a_{0.5}^*(x)$, $a_{0.9}^*(x)$, and $a_1^*(x)$ with $\mu = 2, \sigma = 1, \lambda = 1$

- Here below are two numerical examples for the function $v'(x)$ when $\delta \geq \frac{\mu}{2}$ and when $0 \leq \delta < \frac{\mu}{2}$ respectively.

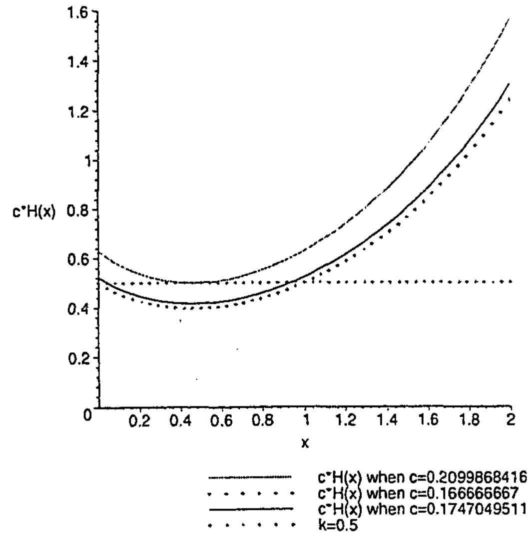

Figure 3.6: The Relationship between x and $c^*H(x)$. Example for the case of $\delta \geq \frac{\mu}{2}$. $\delta = 1, \mu = 2, \sigma = 1, \lambda = 1, K = 0.05, \tilde{c} = 0.1747, \tilde{x}(\tilde{c}) = 0.0496, x_D(\tilde{c}) = 0.9384$

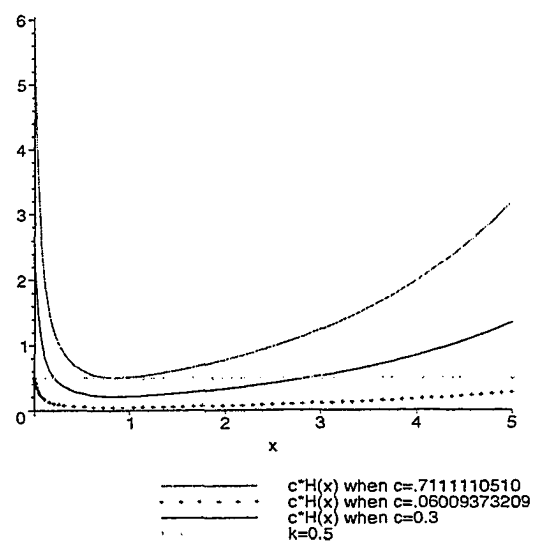

Figure 3.7: The Relationship between x and $c^*H(x)$.
Example for the case of $0 \leq \delta < \frac{\mu}{2}$.
$\delta = 0.125, \mu = 2, \sigma = 1, \lambda = 1, K = 0.5082, \tilde{c} = 0.3, \tilde{x}(\tilde{c}) = 0.2080, x_D(\tilde{c}) = 2.913$

Chapter 4

Constraints on risk control and their impact

This chapter investigates the case where the risk control a is no longer belonging to $[0,1]$ but instead is between α and β, where $0 < \alpha < \beta < +\infty$. These constraints have a clear economic interpretation. In fact, the lower bound means that the company has the obligation to take a minimum of business (like the case of public corporations). While the upper bound of risk β, with $\beta > 1$, represents the case when the company is large enough to take risk from other companies. Therefore, in this context, the QVI (2.0.11)-(2.0.14) takes the following forms

$$\max_{a\in[\alpha,\beta]} \mathcal{L}^a v(x) = 0 \qquad \text{for } 0 \leq x < x_D, \tag{4.0.1}$$

and

$$Mv(x) = v(x) \qquad \text{for } x \geq x_D, \tag{4.0.2}$$

where x_D is defined as in (3.0.1). Denote by

$$a(x) = \arg\max_{a\geq 0} \mathcal{L}^a v(x), \tag{4.0.3}$$

the maximizer of $\mathcal{L}^a v(x)$ over $[0,\infty)$. Then $a(x)$ satisfies $a(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}$ and $\mathcal{L}^{a(x)} v(x) = 0$, on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$. By plugging the first equation into the second, we derive

$$-\frac{\mu^2 v'(x)^2}{2\sigma^2 v''(x)} - \lambda v(x) = 0. \tag{4.0.4}$$

Again, by inserting $a(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}$ or equivalently to $v''(x) = -\frac{\mu v'(x)}{\sigma^2 a(x)}$ into (4.0.4), we get

$$a(x) = \frac{2\lambda v(x)}{\mu v'(x)}, \tag{4.0.5}$$

or by differentiating both terms in (4.0.5), we derive

$$v''(x) = \frac{2\lambda}{\mu}\left(\frac{v'(x)a(x) - v(x)a'(x)}{a(x)^2}\right). \tag{4.0.6}$$

Then by equating $v''(x) = -\frac{\mu v'(x)}{\sigma^2 a(x)}$ with (4.0.6), we get

$$a'(x) = \frac{\mu^2 + 2\lambda\sigma^2}{\mu\sigma^2} \tag{4.0.7}$$

on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$. In turn, we can derive $a(x)$ by integrating (4.0.7) from x_α to x and get

$$a(x) = \frac{\mu^2 + 2\lambda\sigma^2}{\mu\sigma^2}(x - x_\alpha) + \alpha.$$

Thus, $a(x)$ is increasing from α to β on $[x_\alpha, x_\beta)$, where x_α and x_β are roots of $a(x) = \alpha$ and $a(x) = \beta$, respectively. Also, for $x < x_\alpha$, $a(x) < \alpha$, and for $x < x_\beta$, $a(x) < \beta$. Thus, for $\{x < x_D \mid a(x) < x_\alpha\} = [0, x_\alpha)$, we have

$$\max_{a\in[\alpha,\beta]} \mathcal{L}^a v(x) = \mathcal{L}^\alpha v(x) = 0, \tag{4.0.8}$$

which has a solution of the form

$$v(x) = c[e^{r_+(\alpha)x} - e^{r_-(\alpha)x}],\ 0 \leq x < x_\alpha \tag{4.0.9}$$

where

$$r_\pm(z) = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2 z},\ \forall z > 0. \tag{4.0.10}$$

Due to the smooth fit of v' and v'' at the point x_α, and using the equation $a(x_\alpha) = \alpha$, we define x_α by

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} ln\left[\frac{\mu r_-(\alpha) + \alpha\sigma^2 r_-(\alpha)^2}{\mu r_+(\alpha) + \alpha\sigma^2 r_+(\alpha)^2}\right]. \tag{4.0.11}$$

Thanks to (4.0.7), we calculate x_β as

$$x_\beta = x_\alpha + \frac{\mu\sigma^2}{\mu^2 + 2\lambda\sigma^2}(\beta - \alpha). \tag{4.0.12}$$

For $x_\alpha \leq x < x_\beta$, we have $\alpha \leq a(x) \leq \beta$ and then by equating $a(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}$ with (4.0.6) and using the relationship in (4.0.5), we derive

$$v(x) = cv'(x_\alpha-)\left(\frac{\alpha\mu}{2\lambda}\right)\left(\frac{(\mu^2 + 2\lambda\sigma^2)(x - x_\alpha) + \alpha\sigma^2\mu}{\alpha\sigma^2\mu}\right)^\gamma, \tag{4.0.13}$$

where

$$v'(x_\alpha-) = (r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha})$$

and γ is given by (3.0.15). Finally, for $x_\beta \leq x < x_D$,

$$\max_{a\in[\alpha,\beta]} \mathcal{L}^a v(x) = \mathcal{L}^\beta v(x) = 0, \tag{4.0.14}$$

it has a solution of the following form

$$v(x) = c_2 e^{r_+(\beta)(x-x_\beta)} + c_3 e^{r_-(\beta)(x-x_\beta)}, \tag{4.0.15}$$

where $r_\pm(\beta)$ can be calculated using (4.0.10). Since for $x \geq x_D$,

$$v(x) = v(\tilde{x}) + k(x - \tilde{x}) - K, \tag{4.0.16}$$

then by combining these equations ((4.0.9), (4.0.13), (4.0.15) and (4.0.16)), we get

$$v(x) = \begin{cases} c[e^{r_+(\alpha)x} - e^{r_-(\alpha)x}], & 0 \leq x < x_\alpha \\ cv'(x_\alpha -)\left(\frac{\alpha\mu}{2\lambda}\right)\left(\frac{(\mu^2+2\lambda\sigma^2)(x-x_\alpha)+\alpha\sigma^2\mu}{\alpha\sigma^2\mu}\right)^\gamma, & x_\alpha \leq x < x_\beta \\ c_2 e^{r_+(\beta)(x-x_\beta)} + c_3 e^{r_-(\beta)(x-x_\beta)}, & x_\beta \leq x < x_D \\ v(\tilde{x}) + k(x-\tilde{x}) - K, & x \geq x_D. \end{cases} \tag{4.0.17}$$

The constants $c, c_2, c_3, \widetilde{x}$, and x_D are parameters to be determined hereafter. It can be checked that there is a smooth fit of v and v' at $x = x_\alpha$ by using (4.0.5). Remark that $a(x_\alpha) = \alpha$, so we get the relationship of $\frac{v(x_\alpha)}{v'(x_\alpha)} = \frac{\mu\alpha}{2\lambda}$, which gives the equality of $v(x_\alpha -) = v(x_\alpha)$ and $v'(x_\alpha -) = v'(x_\alpha)$. Thanks to the smooth fit of v and v' at $x = x_\beta$, we write c_2 and c_3 in terms of c. To this end, we will solve the system

$$\begin{cases} c_2 + c_3 = cC_\beta, \\ c_2 r_+(\beta) + c_3 r_-(\beta) = cD_\beta, \end{cases}$$

where

$$\begin{aligned} C_\beta &= [r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}]\left(\frac{\alpha\mu}{2\lambda}\right)\left(\frac{\beta}{\alpha}\right)^\gamma, \\ D_\beta &= [r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}]\left(\frac{\beta}{\alpha}\right)^{\gamma-1}. \end{aligned} \tag{4.0.18}$$

The solution to the system above is given by

$$\begin{aligned} c_2 = ce_\beta &:= \frac{D_\beta - r_-(\beta)C_\beta}{r_+(\beta) - r_-(\beta)} \\ c_3 = cf_\beta &:= \frac{r_+(\beta)C_\beta - D_\beta}{r_+(\beta) - r_-(\beta)}. \end{aligned} \tag{4.0.19}$$

We define v' using

$$v'(x) = \begin{cases} cH_{\alpha,\beta}(x) & 0 \leq x < x_D \\ k & x \geq x_D, \end{cases} \tag{4.0.20}$$

where

$$H_{\alpha,\beta}(x) = \begin{cases} r_+(\alpha)e^{r_+(\alpha)x} - r_-(\alpha)e^{r_-(\alpha)x}, & 0 \le x < x_\alpha \\ H_{\alpha,\beta}(x_\alpha -)\left(\frac{(\mu^2+2\lambda\sigma^2)(x-x_\alpha)+\alpha\sigma^2\mu}{\alpha\sigma^2\mu}\right)^{\gamma-1}, & x_\alpha \le x < x_\beta \\ e_\beta r_+(\beta)e^{r_+(\beta)(x-x_\beta)} + f_\beta r_-(\beta)e^{r_-(\beta)(x-x_\beta)}, & x \ge x_\beta. \end{cases} \tag{4.0.21}$$

The only remaining constants to be calculated are $\tilde{x}, x_D$ and c. We will start by determining $\tilde{x}$ and x_D in terms of c using the equation $v'(x) = k$. Indeed, $\tilde{x}$ and x_D represents the first and the second root, respectively, of the equation

$$H_{\alpha,\beta}(x) = \frac{k}{c}. \tag{4.0.22}$$

Remark that $H_{\alpha,\beta}(x)$ is strictly convex and attains its minimal value at $\hat{x}$, which is a unique root of

$$H'_{\alpha,\beta}(\hat{x}) = 0.$$

This root $\hat{x}$ is larger than x_β (i.e. $\hat{x} > x_\beta$), since we can prove that $H'_{\alpha,\beta}(x) < 0$ for $x \le x_\beta$. Then by denoting

$$\hat{H}_{\alpha,\beta} = H_{\alpha,\beta}|_{(0,\hat{x})}, \qquad \bar{H}_{\alpha,\beta} = H_{\alpha,\beta}|_{(\hat{x},\infty)},$$

the restrictions of $H_{\alpha,\beta}$ on $(0,\hat{x})$ and $(\hat{x},\infty)$, respectively, we obtain a decreasing and an increasing one-to-one function, $\hat{H}_{\alpha,\beta}$ and $\bar{H}_{\alpha,\beta}$, respectively. As a result, we calculate $\tilde{x}$ and x_D as follows

$$\tilde{x}(c) = \hat{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right), \qquad x_D(c) = \bar{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right). \tag{4.0.23}$$

Thus, $\tilde{x}(c)$ increases and $x_D(c)$ decreases as c increases.

- If $c > \frac{k}{H_{min}}$, where $H_{min} = H_{\alpha,\beta}(\hat{x})$, then $\tilde{x}(c)$ and $x_D(c)$ do not exists.
- If $c < \frac{k}{H_{\alpha,\beta}(0)}$, then $\tilde{x}(c)$ does not exist.

Therefore, $\tilde{x}(c)$ and $x_D(c)$ defined in (4.0.23) exist if and only if

$$c_0 := \frac{k}{H_{\alpha,\beta}(0)} \le c \le c_1 := \frac{k}{H_{min}}.$$

Now we will determine the parameter c. Consider

$$I(c) = \int_{\tilde{x}(c)}^{x_D(c)} (k - cH_{\alpha,\beta}(x))dx, \text{ for } c_0 \le c \le c_1. \tag{4.0.24}$$

It is obvious that $I(c)$ is decreasing, continuous, and

$$\max_{c_0 \le c \le c_1} I(c) = I(c_0) =: K_{max}(\alpha, \beta).$$

Then in order that the solution to the QVI (2.0.11)-(2.0.14) will be smooth enough as in the hypothesis of Theorem 2.0.1, we need to assume that $K \le K_{max}(\alpha, \beta)$. Thus, for $K \le K_{max}(\alpha, \beta)$, there exists $\tilde{c} \in [c_0, c_1]$, which is a root of

$$I(c) = K. \tag{4.0.25}$$

4.1 A smooth optimal return function and optimal policies

Consider the following function

$$a^*(x) = \min[\max(\alpha, a(x)), \beta]$$
$$= \begin{cases} \alpha & \text{if } 0 \le x < x_\alpha \\ \frac{\mu^2+2\lambda\sigma^2}{\mu\sigma^2}(x - x_\alpha) + \alpha & \text{if } x_\alpha \le x < x_\beta \\ \beta & \text{if } x \ge x_\beta. \end{cases} \tag{4.1.26}$$

Theorem 4.1.1. *Suppose that c is a root of (4.0.25). Let $\widetilde{x}(c)$ and $x_D(c)$, c_2, c_3, x_α and x_β be given by (4.0.23), (4.0.19), (4.0.11) and (4.0.12) respectively. The function v given by (4.0.17) is continuously differentiable on $(0, \infty)$ and is twice continuously differentiable on $(0, x_D(\widetilde{c})) \cup (x_D(\widetilde{c}), \infty)$. This function is a solution to the QVI (2.0.11)-(2.0.14) subject to the growth condition (2.0.8).*

Proof. It is clear that v is twice continuously differentiable on $(0, x_D(c))$, since it coincides with

$$c\int_0^x H_{\alpha,\beta}(y)dy, \qquad x \ge 0,$$

which is twice differentiable on $(0, +\infty)$ due to the choice of a_1 and b_1.
It is obvious that v is linear on the interval $(x_D(c), +\infty)$, and from the calculation of c, $\widetilde{x}(c)$ and $x_D(c)$, we deduce that v is continuously differentiable at x_D and then continuously differentiable on $(0, +\infty)$.
We can easily prove that for $x < x_D$, we have

$$Mv(x) = \begin{cases} v(x) - K, & \text{for } x \le \widetilde{x} \\ v(\widetilde{x}) + k(x - \widetilde{x}) - K, & \text{for } x \ge \widetilde{x}. \end{cases}$$

Since

$$K = I(c) > \int_{\widetilde{x}}^x (k - v'(y))dy,$$

we deduce that

$$v(\widetilde{x}) + k(x - \widetilde{x}) - K < v(x),$$

and hence $Mv(x) < v(x)$ for any $x \leq x_D$. Due to $\max_{\alpha \leq a \leq \beta} \mathcal{L}^a v(x) = 0$, for any $x < x_D$, we conclude that (2.0.11)-(2.0.14) are satisfied for any $x < x_D$.

Due to $\frac{v''(x_D-)}{c} = H_{\alpha,\beta}(x_D) > H_{\alpha,\beta}(\hat{x}) = 0$, $v''(x_D) = 0$ and the continuity of v and v' at the point x_D, we have

$$\lim_{\substack{x \longrightarrow x_D \\ x < x_D}} \mathcal{L}^a v(x) = \frac{a^2\sigma^2}{2} v''(x_D-) + (\mu a - \delta)v'(x_D-) - \lambda v(x_D-) > \mathcal{L}^a v(x_D).$$

This implies for any $x \geq x_D$,

$$0 = \max_{0 \leq a \leq 1} \mathcal{L}^a v(x_D-) \geq \max_{0 \leq a \leq 1} \mathcal{L}^a v(x_D) \geq \max_{0 \leq a \leq 1} \mathcal{L}^a v(x).$$

This combined with $Mv(x) = v(x)$ for any $x \geq x_D$ prove that (2.0.11)-(2.0.14) are satisfied for any $x \geq x_D$. This completes the proof of the theorem. □

The next theorem identifies the optimal policy, and shows that the solution to the QVI constructed above is the value function.

Theorem 4.1.2. *Let c be a root of (4.0.25), $\widetilde{x}(c)$ and $x_D(c)$, x_α and x_β, $a^*(x)$ be given by (4.0.23), (4.0.11), (4.0.12) and (4.1.26) respectively. Then the control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots; \xi_1^*, \xi_2^*, \ldots, \xi_n^*, \ldots)$$

defined by

$$u^*(t) := a^*(X^*(t)) \tag{4.1.27}$$

$$\tau_1^* := \inf\{t \geq 0 : X^*(t) = x_D(c)\} \tag{4.1.28}$$

$$\xi_1^* := x_D(c) - \widetilde{x}(c) \tag{4.1.29}$$

and for every $n \geq 2$:

$$\tau_n^* := \inf\{t > \tau_{n-1} : X^*(t) = x_D(c)\} \tag{4.1.30}$$

$$\xi_n^* := x_D(c) - \widetilde{x}(c), \tag{4.1.31}$$

where X^* is the solution to the stochastic differential equation

$$X^*(t) = X^*(0) + \int_0^t \mu a^*(X^*(s))ds + \int_0^t \sigma a^*(X^*(s))dW_s - (x_D(c) - \widetilde{x}(c)) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}}, \tag{4.1.32}$$

is the QVI control associated with the function v defined by (??). This control is optimal and the function v coincides with the value function. That is,

$$V(x) = v(x) = J(x; \pi^*) = J(x; u^*, \mathcal{T}^*, \xi^*). \tag{4.1.33}$$

Proof. In view of Theorem 4.1.1, the function v defined by (4.0.23) satisfies all the conditions in Theorem 2.0.1. From Definition 2.0.3 and the discussion in previous sections, we know that the control π^* defined in (4.1.27)-(4.1.31) is the control associated with v. In addition, according to Definition 2.0.1, the control π^* is admissible. Therefore, applying Theorem 2.0.1, we conclude that v is the value function and π^* is the optimal policy. □

From the above mathematical description of the optimal cash reserve X^*, this stochastic process never vanishes, as long as $X^*(0) = x > 0$. In other words, the firm never goes bankrupt under the optimal policy.

4.2 Economic interpretation and numerical examples

In this section, we will present some numerical examples as well as some economic interpretations highlighting the impact of the constraints.

- Similarly as in Section 3.4 of Chapter 3, the optimal dividend is always a threshold type with the threshold level being equal to $x_D(c)$, where c is a root of (4.0.25). Also, the dividends are paid out only after the company takes the maximum risk allowed β.

- A positive minimum risk allowed α, (i.e. $\alpha > 0$) imposes an upper bound on the costs K. This bound is denoted by $K_{max}(\alpha, \beta)$ and is finite, while

$$\lim_{\alpha \to 0^+} K_{max}(\alpha, \beta) = +\infty. \tag{4.2.34}$$

Proof. Indeed, since we have

$$K_{max}(\delta) = \int_0^{x_D(\hat{c})} (k - \hat{c}H(x))dx, \tag{4.2.35}$$

so $\lim_{\alpha \to 0^+} K_{max}(\alpha, \beta) = \infty$ if and only if

$$\lim_{\alpha \to 0^+} x_D(\hat{c}) = \infty, \tag{4.2.36}$$

so it is good enough to prove the latter limit.

We have $\hat{c} = \frac{k}{H(0)}$, and $\lim_{\alpha \to 0^+} \hat{c} = \lim_{\alpha \to 0^+} \frac{k}{H(0)} = \lim_{\alpha \to 0^+} \frac{k}{r_+(\alpha) - r_-(\alpha)} = 0^+$. Also, we have $\hat{c} = \frac{k}{H(x_D(\hat{c}))}$, or equivalently, $\frac{k}{\hat{c}} = H(x_D(\hat{c}))$, and so

$$\lim_{\alpha \to 0^+} \frac{k}{\hat{c}} = \lim_{\alpha \to 0^+} H(x_D(\hat{c})) = \infty. \tag{4.2.37}$$

From (4.0.21), we know that $H(x)$ goes to infinity if and only if x goes to infinity. Therefore, the limit (4.2.37) is true if and only if $x_D(\hat{c}) \to \infty$.

- The minimal risk is at α, while β is the maximum risk allowed. So we can conclude that for the case when $\delta = 0$, the higher the wealth that the company has, the more likely the company chooses to gamble on higher potential profits at the expense of assuming higher risk, while there is a floor α and ceiling β to how much risk the company can take. The remark is illustrate by the following figure:

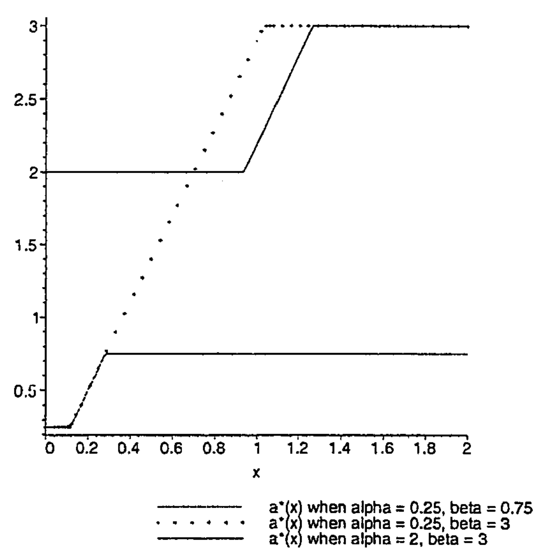

Figure 4.1: Graph for $a^*(x)$ with $\mu = 2, \sigma = 1, \lambda = 1$

• Here below are three numerical examples for the function $cH(x)$.

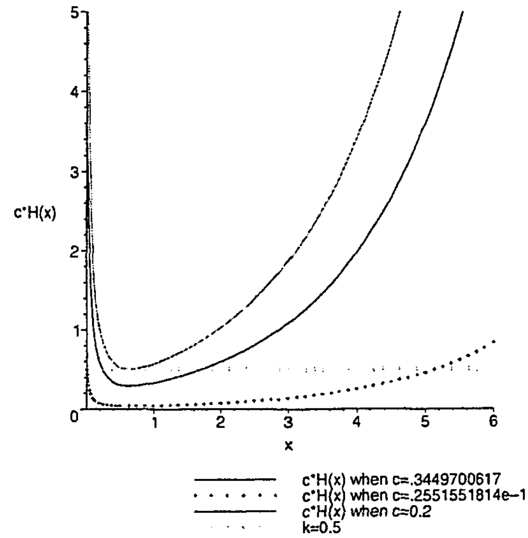

Figure 4.2: The Relationship between x and $c^*H(x)$ for the case with $\alpha = 0.25, \beta = 0.75, \mu = 2, \sigma = 1, \lambda = 1, K = 0.1983751614, \bar{c} = 0.2, \tilde{x}(\bar{c}) = 0.2307325253, x_D(\bar{c}) = 1.704628388$

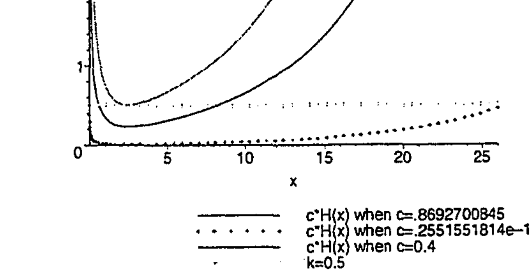

Figure 4.3: The Relationship between x and $c^*H(x)$ for the case with $\alpha = 0.25, \beta = 3, \mu = 2, \sigma = 1, \lambda = 1, K = 3.286378461, \bar{c} = 0.2, \tilde{x}(\bar{c}) = 0.2307325253, x_D(\bar{c}) = 12.88648195$

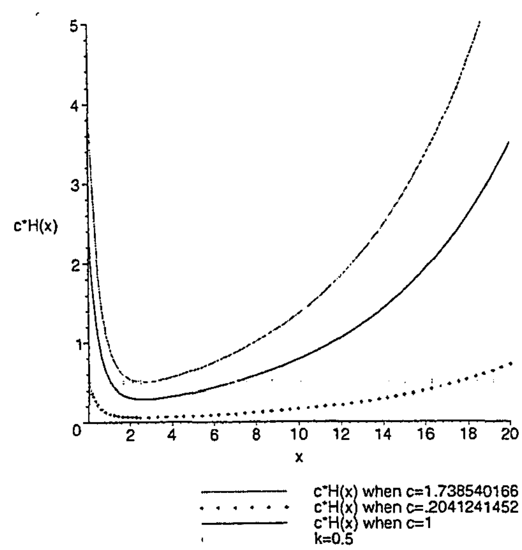

Figure 4.4: The Relationship between x and $c^*H(x)$ for the case with $\alpha = 2, \beta = 3, \mu = 2, \sigma = 1, \lambda = 1, K = 0.8105867317, \tilde{c} = 1, \tilde{x}(\tilde{c}) = 1.04824492, x_D(\tilde{c}) = 7.005935431$

Chapter 5

Interplay between debt liability and constraints

This chapter focuses on the combination of the two features of Chapter 3 and Chapter 4. Precisely, here we will consider a nonzero debt liability rate (i.e. $\delta > 0$) and the risk control a is between α and β ($0 < \alpha < \beta < \infty$). Then the QVI (2.0.11)-(2.0.14) becomes

$$\max_{a\in[\alpha,\beta]} \mathcal{L}^a v(x) = \max_{a\in[\alpha,\beta]} \left\{ \frac{1}{2}\sigma^2 a^2 v''(x) + (a\mu - \delta)v'(x) - \lambda v(x) \right\} = 0, \qquad \forall\, x < x_D, \tag{5.0.1}$$

and

$$v(x) = Mv(x), \text{ for } x \geq x_D, \tag{5.0.2}$$

where x_D is defined by (3.0.1). Recall that the maximizer of $\mathcal{L}^a v(x)$ over $[0, \infty)$ satisfies (3.0.4). Then on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$, we have

$$\mathcal{L}^{a(x)} v(x) = 0.$$

As in Chapter 3, this equation leads to

$$v(x) = c_1 \left(F^{-1}\left(c_0 + \frac{x}{\gamma}\right)\right) \left(F^{-1}\left(c_0 + \frac{x}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right)^{\gamma - 1} \tag{5.0.3}$$

on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$. Here, c_0 and c_1 are constants, and F is given by (3.0.10) (for the details of this implication, see (3.0.5) - (3.0.14)). Then from (5.0.3), on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$, we get

$$a(x) = \frac{2\lambda}{\mu} F^{-1}\left(c_0 + \frac{\mu^2 + 2\lambda\sigma^2}{2\lambda\sigma^2} x\right) + \frac{2\delta}{\mu}. \tag{5.0.4}$$

Then $a(x)$ increases from α to β on $\{x < x_D \mid \alpha \leq a(x) \leq \beta\}$. Let $x_\alpha < x_D$ such that $a(x_\alpha) = \alpha$. Then from (5.0.4), we derive

$$c_0 = -\frac{x_\alpha}{\gamma} + F\left(\frac{\alpha\mu - 2\delta}{2\lambda}\right). \tag{5.0.5}$$

Remark that on $\{x < x_D \mid a(x) < \alpha\}$,

$$\max_{\alpha \leq a \leq \beta} \mathcal{L}^a v(x) = \mathcal{L}^\alpha v(x) = 0. \tag{5.0.6}$$

Then due to $v(0) = 0$, we derive

$$v(x) = c[e^{r_+(\alpha)x} - e^{r_-(\alpha)x}], \text{ for } 0 \leq x < x_\alpha, \tag{5.0.7}$$

where

$$r_\pm(z) = \frac{-(z\mu - \delta) \pm \sqrt{(z\mu - \delta)^2 + 2\sigma^2 z^2 \lambda}}{\sigma^2 z^2}. \tag{5.0.8}$$

Similarly, if $x_\beta \geq x_\alpha$ such that $a(x_\beta) = \beta$, then for $x \geq x_\beta$, we have $a(x) \geq \beta$ and

$$\max_{\alpha \leq a \leq \beta} \mathcal{L}^a v(x) = \mathcal{L}^\beta v(x) = 0.$$

This leads to

$$v(x) = c_2 e^{r_+(\beta)(x - x_\beta)} + c_3 e^{r_-(\beta)(x - x_\beta)}, \text{ for } x_\beta \leq x < x_D. \tag{5.0.9}$$

In summary, by taking into consideration of all the above analysis, we derive our candidate function for a smooth solution of the QVI (2.0.11)-(2.0.14) as follows

$$v(x) = \begin{cases} c[e^{r_+(\alpha)x} - e^{r_-(\alpha)x}], & 0 \leq x < x_\alpha \\ c_1 f(x) \left[f(x) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right]^{\gamma - 1}, & x_\alpha \leq x < x_\beta \\ c_2 e^{r_+(\beta)(x - x_\beta)} + c_3 e^{r_-(\beta)(x - x_\beta)}, & x_\beta \leq x < x_D \\ v(\tilde{x}) + k(x - \tilde{x}) - K, & x \geq x_D, \end{cases} \tag{5.0.10}$$

where

$$f(x) := F^{-1}\left(\frac{x - x_\alpha}{\gamma} + F\Big(\frac{\alpha\mu - 2\delta}{2\lambda}\Big)\right), \ x \geq x_\alpha, \tag{5.0.11}$$

and $x_\alpha, x_\beta, c, c_1, c_2, c_3, \tilde{x}$, and x_D are the parameters to be determined such that v is smooth enough as in the hypothesis of Theorem 2.0.1.
Using the smooth fit of v' and v'' at x_α, we have

$$a(x_\alpha) = \alpha = -\frac{\mu v'(x_\alpha -)}{\sigma^2 v''(x_\alpha -)}.$$

This equation implies

$$e^{r_+(\alpha)-r_-(\alpha)x_\alpha} = \frac{\mu r_-(\alpha) + \alpha\sigma^2 r_-(\alpha)^2}{\mu r_+(\alpha) + \alpha\sigma^2 r_+(\alpha)^2}. \tag{5.0.12}$$

From this equation, we deduce that $x_\alpha > 0$ iff $\frac{\mu r_-(\alpha)+\alpha\sigma^2 r_-(\alpha)^2}{\mu r_+(\alpha)+\alpha\sigma^2 r_+(\alpha)^2} > 1$, which is equivalent to $\delta < \frac{\alpha\mu}{2}$. Hence, we need to distinguish the cases depending on δ.

5.1 The case of $0 \leq \delta < \frac{\alpha\mu}{2}$

In this case, we calculated from (5.0.12) that

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} ln \left[\frac{\mu r_-(\alpha) + \alpha\sigma^2 r_-(\alpha)^2}{\mu r_+(\alpha) + \alpha\sigma^2 r_+(\alpha)^2}\right] > 0. \tag{5.1.13}$$

Then from the equation of $a(x_\beta) = \beta$, where $a(x)$ is given in (5.0.4), we calculate

$$x_\beta = x_\alpha + \gamma \left[F\left(\frac{\beta\mu - 2\delta}{2\lambda}\right) - F\left(\frac{\alpha\mu - 2\delta}{2\lambda}\right)\right] > x_\alpha. \tag{5.1.14}$$

The smooth fit of v at the point x_α leads to

$$c_1 = c\frac{2\lambda}{\alpha\mu - 2\delta}\left(e^{r_+(\alpha)x_\alpha} - e^{r_-(\alpha)x_\alpha}\right)\left(\frac{\alpha\mu - 2\delta}{2\lambda} + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right)^{1-\gamma}.$$

Now we will calculate c_2 and c_3 in terms of c using the smooth fit of v and v' at x_β, so we get the following system

$$\begin{cases} c_2 + c_3 = cC_\beta \\ c_2 r_+(\beta) + c_3 r_-(\beta) = cD_\beta, \end{cases}$$

where

$$C_\beta = \left(e^{r_+(\alpha)x_\alpha} - e^{r_-(\alpha)x_\alpha}\right)\left(\frac{\beta\mu - 2\delta}{\alpha\mu - 2\delta}\right)\left(\frac{\frac{\beta\mu-2\delta}{2\lambda} + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}{\frac{\alpha\mu-2\delta}{2\lambda} + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}\right)^{\gamma-1},$$

and

$$D_\beta = \frac{\left(e^{r_+(\alpha)x_\alpha} - e^{r_-(\alpha)x_\alpha}\right)}{\frac{\alpha\mu-2\delta}{2\lambda}}\left(\frac{\frac{\beta\mu-2\delta}{2\lambda} + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}{\frac{\alpha\mu-2\delta}{2\lambda} + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}\right)^{\gamma-1}.$$

Then a simple calculation shows that this system has the following solution

$$c_2 = ce_\beta := \frac{D_\beta - r_-(\beta)C_\beta}{r_+(\beta) - r_-(\beta)}, \quad c_3 = cf_\beta := \frac{r_+(\beta)C_\beta - D_\beta}{r_+(\beta) - r_-(\beta)}. \tag{5.1.15}$$

This reduces the number of parameters to be calculated and we define v' by

$$v'(x) = \begin{cases} cH_{\alpha,\beta}(x) & 0 \le x < x_D \\ k & x \ge x_D, \end{cases} \tag{5.1.16}$$

where

$$H_{\alpha,\beta}(x) = \begin{cases} r_+(\alpha)e^{r_+(\alpha)x} - r_-(\alpha)e^{r_-(\alpha)x}, & 0 \le x < x_\alpha \\ \frac{2\lambda}{\alpha\mu - 2\delta}\left(e^{r_+(\alpha)x_\alpha} - e^{r_-(\alpha)x_\alpha}\right)\left(\frac{f(x)+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}{\frac{\alpha\mu-2\delta}{2\lambda}+\frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}}\right)^{\gamma-1}, & x_\alpha \le x < x_\beta \\ e_\beta r_+(\beta)e^{r_+(\beta)(x-x_\beta)} + f_\beta r_-(\beta)e^{r_-(\beta)(x-x_\beta)}, & x \ge x_\beta, \end{cases} \tag{5.1.17}$$

with $f(x)$ stated as in (5.0.11).

One can easily show that H is continuously differentiable, strictly convex, $H'(x) < 0$ for $x \le x_\beta$, and $H'(\infty) = \infty$. Then there exists a unique $\hat{x} > x_\beta$ such that

$$H_{\alpha,\beta}(\hat{x}) = \min_{x\ge 0} H_{\alpha,\beta}(x) =: H_{min}.$$

Also, notice that $\hat{H}_{\alpha,\beta} = H_{\alpha,\beta}|_{(0,\hat{x})}$ (the restriction of $H_{\alpha,\beta}$ on $(0,\hat{x})$) and $\bar{H}_{\alpha,\beta} = H_{\alpha,\beta}|_{(\hat{x},\infty)}$ (the restriction of $H_{\alpha,\beta}$ on $(\hat{x},\infty)$) are one-to-one strictly decreasing and increasing functions, respectively. Then for $c \in [c_0, c_1]$, where

$$c_0 := \frac{k}{H_{\alpha,\beta}(0)}, \qquad c_1 := \frac{k}{H_{min}}, \tag{5.1.18}$$

there exists $\tilde{x}(c) < \hat{x} < x_D(c)$ uniquely such that

$$\tilde{x}(c) = \hat{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right), \qquad x_D(c) = \bar{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right), \tag{5.1.19}$$

and the function

$$I(c) = \int_{\tilde{x}(c)}^{x_D(c)} (k - cH_{\alpha,\beta}(x))dx \tag{5.1.20}$$

is well-defined and continuously decreasing. Hence, for $K \le K_{max}(\alpha,\beta,\delta) = I(c_0)$, there exists $\tilde{c} \in [c_0, c_1]$, which is a root of

$$I(c) = K. \tag{5.1.21}$$

Now if $\delta \ge \frac{\alpha\mu}{2}$, then $x_\alpha = 0$. Then $v(x)$ has the form described in (5.0.3) over $[0, x_\beta)$. Since $v(0) = 0$, then we get $c_0 = F(0)$ and $a(0) = \frac{2\delta}{\mu} \ge \alpha$.

Since $a(x)$ is increasing, we have

$$a(x) \ge a(0) = \frac{2\delta}{\mu} \ge \alpha.$$

Then to completely describe the function, we need to discuss other cases depending whether δ is smaller than $\frac{\beta\mu}{2}$ or not.

5.2 The case of $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$

In this case, the function $a(x)$ takes the following form

$$a(x) = \frac{2\lambda}{\mu} F^{-1}\Big(F(0) + \frac{x}{\gamma}\Big) + \frac{2\delta}{\mu}.$$

Then, for $0 \leq x < x_\beta$ the expression (5.0.3) becomes

$$v(x) = c\left(F^{-1}\left(F(0) + \frac{x}{\gamma}\right)\right)\left(F^{-1}\left(F(0) + \frac{x}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right)^{\gamma-1}, \tag{5.2.22}$$

where

$$x_\beta = \frac{1}{\gamma}\left(F\Big(\frac{\beta\mu - 2\delta}{2\lambda}\Big) - F(0)\right). \tag{5.2.23}$$

These lead to our complete candidate for the solution of the QVI as follows

$$v(x) = \begin{cases} c\left(F^{-1}\left(F(0) + \frac{x}{\gamma}\right)\right)\left(F^{-1}\left(F(0) + \frac{x}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}, & 0 \leq x < x_\beta \\ c_1 e^{r_+(\beta)(x-x_\beta)} + c_2 e^{r_-(\beta)(x-x_\beta)}, & x_\beta \leq x < x_D \\ v(\tilde{x}) + k(x - \tilde{x}) - K, & x \geq x_D. \end{cases} \tag{5.2.24}$$

By using the smooth fit of v and v' at x_β as in the previous section, we get

$$\begin{cases} c_1 + c_2 & = cA_\beta \\ c_1 r_+(\beta) + c_2 r_-(\beta) & = cB_\beta, \end{cases}$$

where

$$A_\beta = \left(F^{-1}\left(F(0) + \frac{x_\beta}{\gamma}\right)\right)\left(F^{-1}\left(F(0) + \frac{x_\beta}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right)^{\gamma-1}$$

and

$$B_\beta = \left(F^{-1}\left(F(0) + \frac{x_\beta}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2 + 2\lambda\sigma^2}\right)^{\gamma-1}.$$

This system has the following solution

$$c_1 = c\frac{B_\beta - r_-(\beta)A_\beta}{r_+(\beta) - r_-(\beta)} =: ca_\beta, \qquad c_2 = c\frac{r_+(\beta)A_\beta - B_\beta}{r_+(\beta) - r_-(\beta)} =: cb_\beta. \tag{5.2.25}$$

We define $v'(x)$ as in (5.1.16)
where

$$H_{\alpha,\beta}(x) = \begin{cases} \left(F^{-1}\left(F(0) + \frac{x}{\gamma}\right) + \frac{2\delta\sigma^2}{\mu^2+2\lambda\sigma^2}\right)^{\gamma-1}, & 0 \le x < x_\beta \\ a_\beta r_+(\beta)e^{r_+(\beta)(x-x_\beta)} + b_\beta r_-(\beta)e^{r_-(\beta)(x-x_\beta)}, & x \ge x_\beta. \end{cases} \tag{5.2.26}$$

$H_{\alpha,\beta}(x)$ is strictly convex and attains its minimal value at the point $\hat{x} > x_\beta$. Denote the restrictions of $H_{\alpha,\beta}$ on $(0, \hat{x})$ and on $(\hat{x}, \infty)$ by $\hat{H}_{\alpha,\beta}$ and $\bar{H}_{\alpha,\beta}$, respectively. These functions are one-to-one functions, and for $c \in [c_0, c_1]$, where

$$c_0 := \frac{k}{H_{\alpha,\beta}(0)}, \qquad c_1 := \frac{k}{H_{\alpha,\beta}(\hat{x})}, \tag{5.2.27}$$

there exists a unique $\tilde{x}(c) < \hat{x} < x_D(c)$ that is given by

$$\tilde{x}(c) = \hat{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right), \qquad x_D(c) = \bar{H}^{-1}_{\alpha,\beta}\left(\frac{k}{c}\right), \tag{5.2.28}$$

and the function $I(c)$ defined by (5.1.20) is well-defined in this case. Furthermore, for $K \le K_{max}(\alpha, \beta, \delta)$, where

$$K_{max}(\alpha, \beta, \delta) := I(c_0),$$

there exists $\tilde{c} \in [c_0, c_1]$, which is a root of (5.1.21).

5.3 The case of $\delta \ge \frac{\mu\beta}{2}$

If $\delta \ge \frac{\beta\mu}{2}$, then in this case, $x_\beta = 0$ and $a(x) \ge a(0) \ge \beta$, for all $x \ge 0$. Therefore, for $0 \le x < x_D$,

$$\max_{a\in[\alpha,\beta]} \mathcal{L}^a v(x) = \mathcal{L}^\beta v(x) = \frac{1}{2}\sigma^2\beta^2 v''(x) + (\beta\mu - \delta)v'(x) - \lambda v(x) = 0. \tag{5.3.29}$$

Thanks to $v(0) = 0$, the general solution of this equation is:

$$v(x) = c(e^{r_+(\beta)x} + e^{r_-(\beta)x}), \text{ for } 0 \le x < x_D, \tag{5.3.30}$$

Recall from the previous chapters that for $x \ge x_D$, we have:

$$v(x) = v(\tilde{x}) + k(x - \tilde{x}) - K. \tag{5.3.31}$$

Thus, by combining (5.3.30) and (5.3.31), we get

$$v(x) = \begin{cases} c(e^{r_+(\beta)x} - e^{r_-(\beta)x}), & 0 \le x < x_D \\ v(\tilde{x}) + k(x - \tilde{x}) - K, & x \ge x_D \end{cases} \tag{5.3.32}$$

If $\delta \geq \beta\mu$ (which is equivalent to $-r_-(\beta) \leq r_+(\beta)$), then necessarily $x_D = 0 = \tilde{x}$ and

$$v(x) = kx - K = g(x), \; x \geq 0.$$

This means that if δ is big enough (exceeds the maximum expected profit rate), then the company should get out of business immediately and declare bankruptcy. The reserve is then distributed as dividends.

For the remaining part of this chapter, we assume that $\delta < \beta\mu$. Then we define

$$v'(x) = \begin{cases} cH_\beta(x) & 0 \leq x < x_D \\ k & x \geq x_D, \end{cases} \tag{5.3.33}$$

with

$$H_\beta(x) = r_+(\beta)e^{r_+(\beta)x} - r_-(\beta)e^{r_-(\beta)x}, \text{ for } x \geq 0. \tag{5.3.34}$$

It is obvious that H_β is strictly convex and attains its minimal value, H_{min}, at the point $\hat{x}$, a root of $H'_\beta(x) = 0$ that can be explicitly calculated in this case and is given by

$$\hat{x} = \frac{ln\left(\frac{r_-(\beta)^2}{r_+(\beta)^2}\right)}{r_+(\beta) - r_-(\beta)}. \tag{5.3.35}$$

With $\hat{H}_\beta := H_\beta|_{(0,\hat{x})}$ and $\bar{H}_\beta := H_\beta|_{(\hat{x},\infty)}$, we denote the restriction of H_β on $(0, \hat{x})$ and on $(\hat{x}, \infty)$, respectively. Then $\tilde{x}$ and x_D can be calculated when they exist, by

$$\tilde{x}(c) = \hat{H}_\beta^{-1}\left(\frac{k}{c}\right), \qquad x_D(c) = \bar{H}_\beta^{-1}\left(\frac{k}{c}\right). \tag{5.3.36}$$

- If $c > \frac{k}{H_{min}} = \frac{k}{H_\beta(\hat{x})}$, then $\tilde{x}(c)$ and $x_D(c)$ do not exist.
- If $c < \frac{k}{H_\beta(0)}$, then $\tilde{x}(c)$ does not exist.

Therefore, $\tilde{x}(c)$ and $x_D(c)$ defined in (5.3.36) exist if and only if

$$c_0 := \frac{k}{H_\beta(0)} \leq c \leq c_1 := \frac{k}{H_\beta(\hat{x})}. \tag{5.3.37}$$

Thus, in this case the function defined by (5.1.20) is well defined, decreasing and continuous. Hence, for any $K \leq K_{max}(\alpha, \beta, \delta)$, there exists $\tilde{c} \in [c_0, c_1]$, which is a root of (5.1.21) in this case.

5.4 A smooth optimal return function and optimal policy

Until now in this chapter, we constructed a candidate function for the solution of the QVI (2.0.11)-(2.0.14). This function v takes different forms depending on the relationship of the debt liability with respect to the minimum and the maximum risk allowed. However, from the whole analysis of the previous sections, we can conclude that the function v defined in (5.0.10) can cover all the other cases by redefining the constants as follows. Let's define

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} ln \left[\frac{\mu r_-(\alpha) + \alpha\sigma^2 r_-(\alpha)^2}{\mu r_+(\alpha) + \alpha\sigma^2 r_+(\alpha)^2} \vee 1 \right] \tag{5.4.38}$$

and

$$x_\beta = \gamma \left[F\left(\frac{(\beta\mu - 2\delta)^+}{2\lambda} \right) - F\left(\frac{(\alpha\mu - 2\delta)^+}{2\lambda} \right) \right]. \tag{5.4.39}$$

Here $y \vee z = max(y, z)$ for any $y, z \in \mathbb{R}$. Then it is easy to see that

- If $\delta \geq \frac{\beta\mu}{2}$, then $\delta \geq \frac{\alpha\mu}{2}$ and consequently from (5.4.38) and (5.4.39), we derive $x_\alpha = x_\beta = 0$. Then (5.1.17), (5.1.18), (5.0.10) coincide with (5.3.34), (5.3.37) and (5.3.32) respectively.
- If $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$, then $x_\alpha = 0 < x_\beta$, and (5.4.39) coincides with (5.2.23). As result (5.1.17), (5.1.18), (5.0.10), coincide with (5.2.26), (5.2.27) and (5.2.24) respectively.
- If $\delta < \frac{\alpha\mu}{2}$, then (5.4.38) and (5.4.39) coincide with (5.1.13) and (5.1.14) respectively.

This explains why the function v and its calculated parameters in the first case still hold as a general candidate solution for the QVI (2.0.11)-(2.0.14).

Now we define the following function

$$a^*(x) = \min(\max(\alpha, a(x)), \beta) = \begin{cases} \alpha & \text{if } 0 \leq x < x_\alpha \\ \frac{2\lambda}{\mu} F^{-1}\left(F\left(\frac{(\alpha\mu - 2\delta)^+}{2\lambda} \right) + \frac{x - x_\alpha}{\gamma} \right) + \frac{2\delta}{\gamma} & \text{if } x_\alpha \leq x < x_\beta \\ \beta & \text{if } x \geq x_\beta. \end{cases} \tag{5.4.40}$$

Theorem 5.4.1. *Suppose that c be a root of (5.1.21) and let x_α, x_β, $\widetilde{x}(c)$ and $x_D(c)$, c_2 and c_3 are given by (5.4.38), (5.4.39), (5.1.19) and (5.1.15) respectively. Then the function v given by (5.0.10) is continuously differentiable* *on $(0,\infty)$ and is twice continuously differentiable on $(0, x_D(c)) \cup (x_D(c), \infty)$. This function is a solution to the QVI (2.0.11)-(2.0.14) subject to the growth condition (2.0.8).*

Proof.

It is clear that v is twice continuously differentiable on $(0, x_D(c))$, since it coincides with

$$c\int_0^x H_{\alpha,\beta}(y)dy, \qquad x \geq 0,$$

which is twice differentiable on $(0,+\infty)$ due to the choice of x_α, e_β and f_β. It is obvious that v is linear on the set $(x_D(c),+\infty)$, and from the calculation of c, $\widetilde{x}(c)$ and $x_D(c)$, we deduce that v is continuously differentiable at x_D and then continuously differentiable on $(0,+\infty)$.

We can easily prove that for $x < x_D$, we have

$$Mv(x) = \begin{cases} v(x) - K, & \text{for } x \leq \widetilde{x} \\ v(\widetilde{x}) + k(x - \widetilde{x}) - K < v(x), & \text{for } x \geq \widetilde{x}. \end{cases}$$

Since

$$K = I(c) > \int_{\widetilde{x}}^x (k - v'(y))dy$$

we deduce that

$$v(\widetilde{x}) + k(x - \widetilde{x}) - K < v(x),$$

and hence $Mv(x) < v(x)$ for any $x \leq x_D$. Due to $\max_{\alpha\leq a\leq\beta} \mathcal{L}^a v(x) = 0$, for any $x < x_D$, we conclude that (2.0.11)-(2.0.14) are satisfied for any $x < x_D$.

Due to $\frac{v''(x_D-)}{c} = H_{\alpha,\beta}(x_D) > H_{\alpha,\beta}(\hat{x}) = 0$, $v''(x_D) = 0$ and the continuity of v and v' at the point x_D, we have

$$\lim_{\substack{x \longrightarrow x_D \\ x < x_D}} \mathcal{L}^a v(x) = \frac{a^2\sigma^2}{2} v''(x_D-) + (\mu a - \delta)v'(x_D-) - \lambda v(x_D-) > \mathcal{L}^a v(x_D).$$

This implies that for any $x \geq x_D$,

$$0 = \max_{0\leq a\leq 1} \mathcal{L}^a v(x_D-) \geq \max_{0\leq a\leq 1} \mathcal{L}^a v(x_D) \geq \max_{0\leq a\leq 1} \mathcal{L}^a v(x).$$

This combined with $Mv(x) = v(x)$ for any $x \geq x_D$ prove that (2.0.11)-(2.0.14) are satisfied for any $x \geq x_D$. This completes the proof of the theorem. □

The next theorem identifies the optimal policy and shows that the solution to the QVI constructed above is the value function.

Theorem 5.4.2. *Suppose that c be a root of (5.1.21) and let $\tilde{x}(c)$, $x_D(c)$ and $a^*(x)$ are given by (5.1.19) and (5.4.40), respectively, then the control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots; \xi_1^*, \xi_2^*, \ldots, \xi_n^*, \ldots)$$

defined by

$$u^*(t) := u^*(X^*(t)) := \begin{cases} \frac{\mu}{\sigma^2(1-\gamma)} X^*(t) & \text{if } 0 \leq X^*(t) \leq x_0 \\ 1 & \text{if } X^*(t) \geq x_0, \end{cases} \tag{5.4.41}$$

$$\tau_1^* := \inf\{t \geq 0 : X^*(t) = x_D(c)\} \tag{5.4.42}$$
$$\xi_1^* := x_D(c) - \tilde{x}(c) \tag{5.4.43}$$

and for every $n \geq 2$:

$$\tau_n^* := \inf\{t > \tau_{n-1} : X^*(t) = x_D(c)\} \tag{5.4.44}$$
$$\xi_n^* := x_D(c) - \tilde{x}(c), \tag{5.4.45}$$

where X^* is the solution to the stochastic differential equation

$$X^*(t) = X^*(0) + \int_0^t (\mu u^*(X^*(s)) - \delta) ds + \int_0^t \sigma u^*(X^*(s)) dW_s - (x_D(c) - \tilde{x}(c)) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}}, \tag{5.4.46}$$

is the QVI control associated with the function v defined by (5.0.10). This control is optimal and the function v coincides with the value function. That is,

$$V(x) = v(x) = J(x; \pi^*) = J(x; u^*, \mathcal{T}^*, \xi^*). \tag{5.4.47}$$

Proof. In view of Theorem 5.4.1, the function v defined by (5.0.10) satisfies all the conditions of Theorem 2.0.1. From Definition 2.0.3 and the discussion in previous sections, we know that the control π^* defined in (5.4.41)-(5.4.45) is the control associated with v. In addition, according to Definition 2.0.1, the control π^* is admissible. Therefore, applying Theorem 2.0.1, we conclude that v is the value function and π^* is the optimal policy. □

From the above mathematical description of the optimal cash reserve X^*, this stochastic process never vanishes, as long as $X^*(0) = x > 0$. In other words, the firm never goes bankrupt under the optimal policy. On the other hand, the optimal dividend policy is always of a threshold type with the threshold level being equal to x_D. Precisely, as soon as the level of the cash reserve reaches the level x_D, the firm should distribute $x_D(c) - \tilde{x}(c)$ as dividend to the shareholder. We also can notice that the maximum business activity is always attained prior to the time the dividend distributions occur.

5.5 Numerical examples

The following features of the nonzero liability rate are worth noticing.

Example 1: The following figures show the relationship between δ and $K_{max}(\delta)$ that depends on the strength of α and β

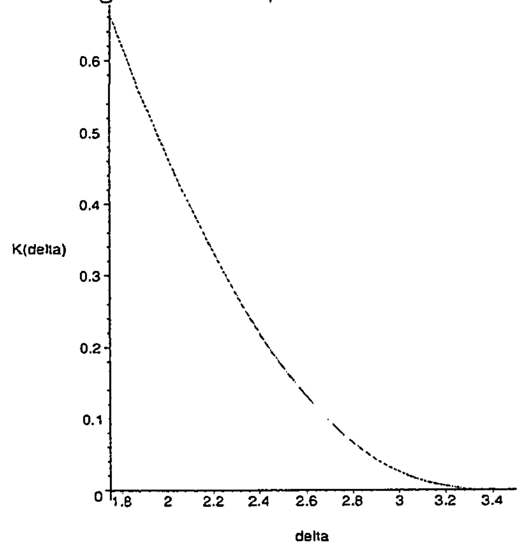

Figure 5.1: The Relationship between δ and $K_{max}(\delta)$. This is the example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \beta = 1.75, k = 0.5$.

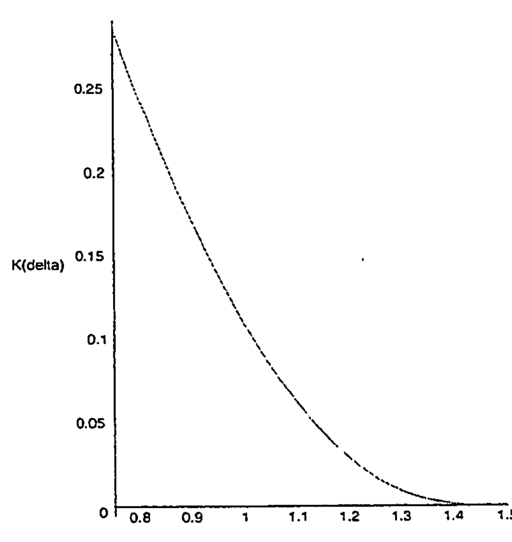

Figure 5.2: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \beta = 0.75, k = 0.5$.

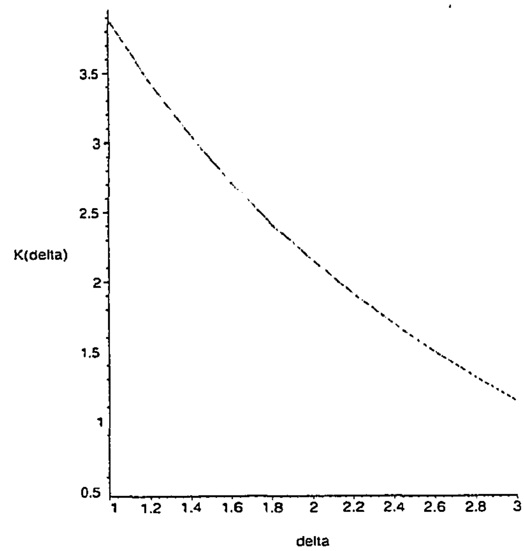

Figure 5.3: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 1, \beta = 3, k = 0.5$.

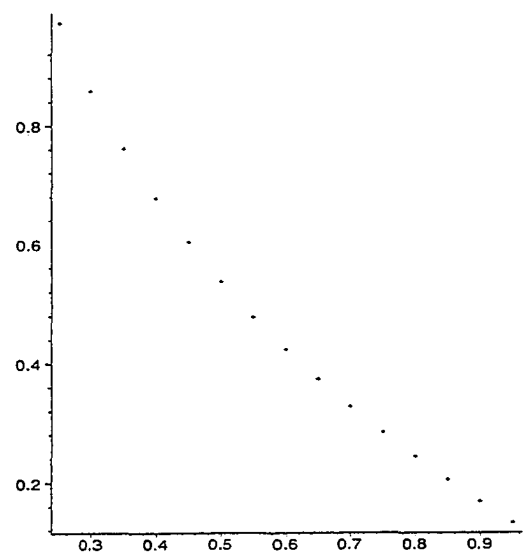

Figure 5.4: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, k = 0.5$.

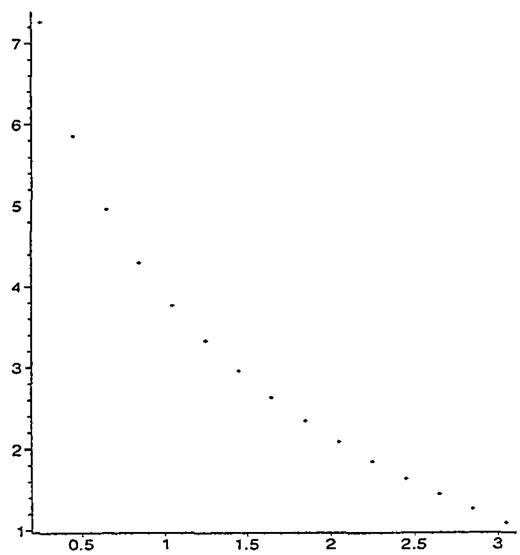

Figure 5.5: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, k = 0.5$.

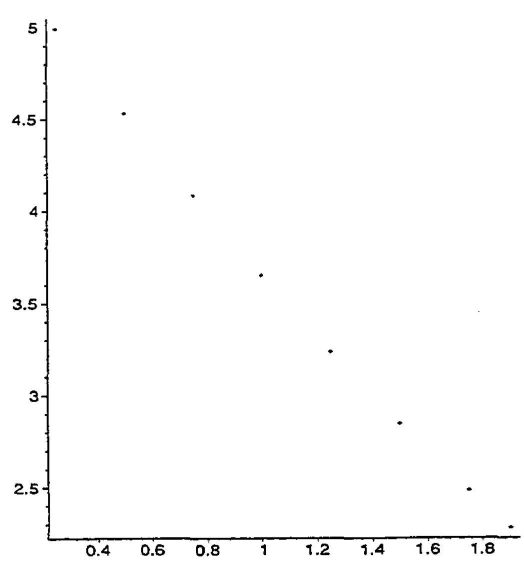

Figure 5.6: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 2, \beta = 3, k = 0.5$.

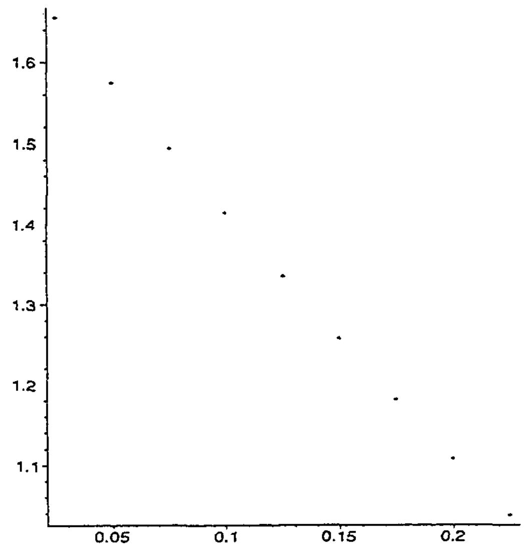

Figure 5.7: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, k = 0.5$.

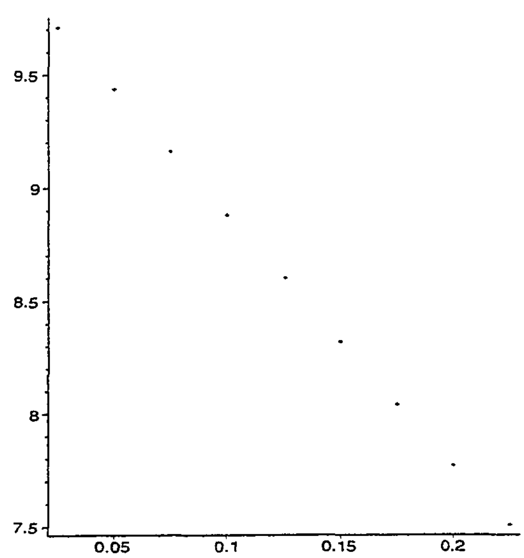

Figure 5.8: The Relationship between δ and $K_{max}(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, k = 0.5$.

Example 2: There is a negative relationship between δ and $x_D(\delta)$ and it is illustrated by the following examples while considering the different intensity of α and β:

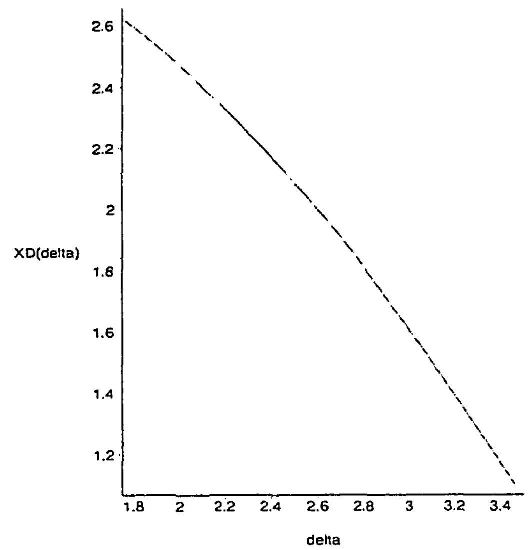

Figure 5.9: The Relationship between δ and $x_D(\delta)$. This is the example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \mu = 2, \beta = 1.75, k = 0.5$, and $K = 0.2$.

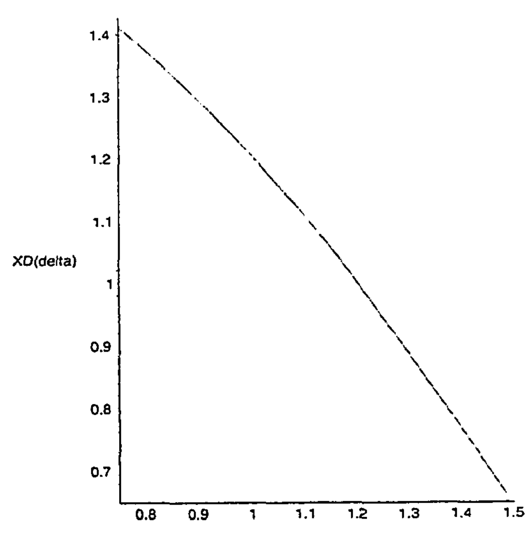

Figure 5.10: The Relationship between δ and $x_D(\delta)$. This is an example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \mu = 2, \beta = 0.75, k = 0.5$, and $K = 0.2$.

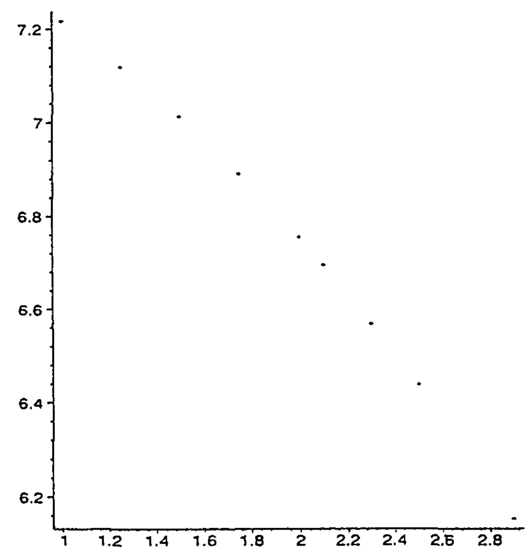

Figure 5.11: The Relationship between δ and $x_D(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 1, \beta = 3, k = 0.5$, and $K = 1$.

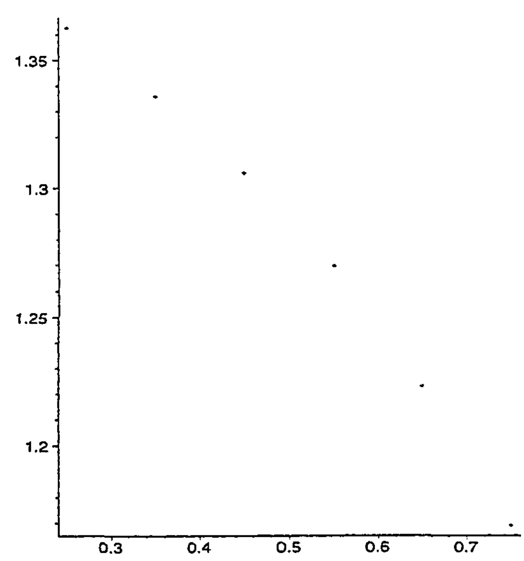

Figure 5.12: The Relationship between δ and $x_D(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, k = 0.5$, and $K = 0.1$

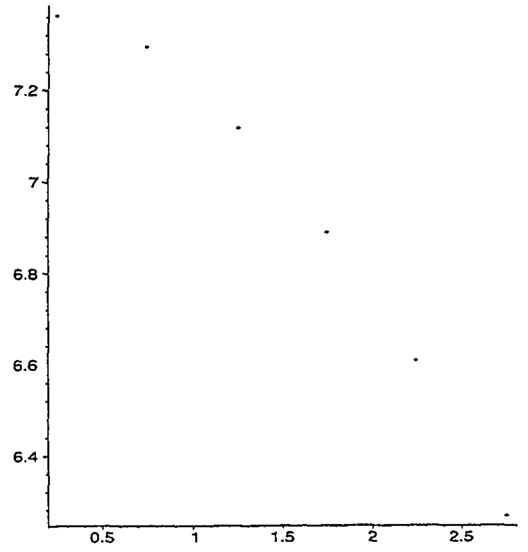

Figure 5.13: The Relationship between δ and $x_D(\delta)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, k = 0.5$, and $K = 1$.

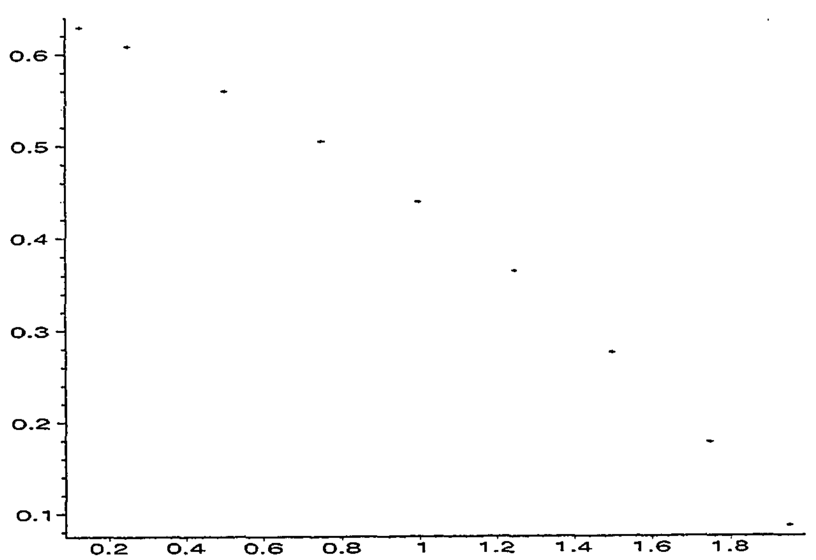

Figure 5.14: The Relationship between δ and $x_D(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 2, \beta = 3, k = 0.5$, and $K = 2$.

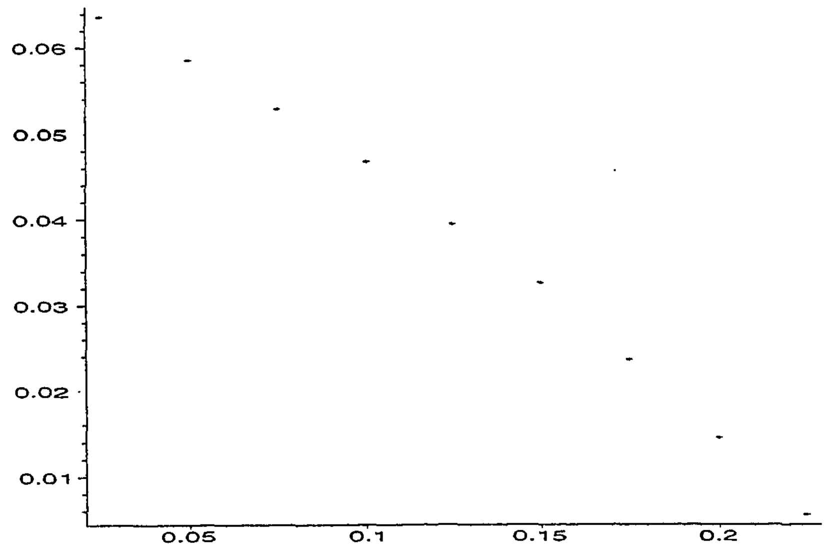

Figure 5.15: The Relationship between δ and $x_D(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, k = 0.5$, and $K = 1$.

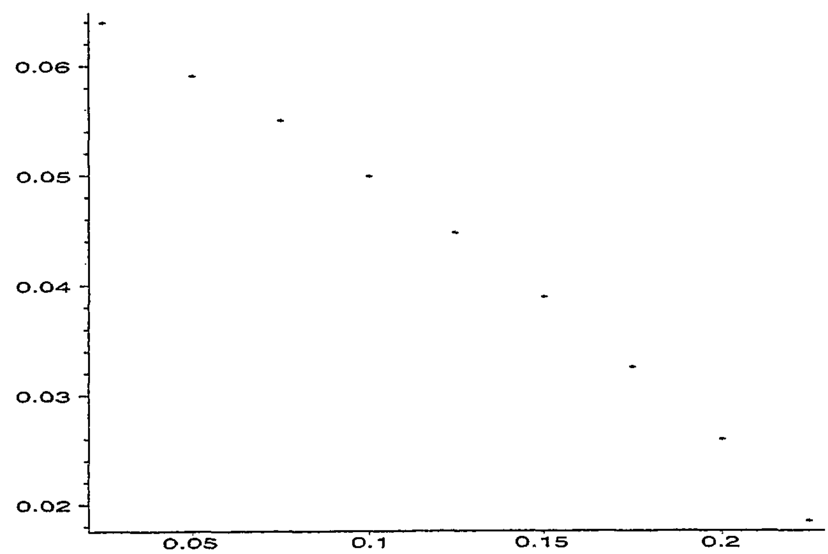

Figure 5.16: The Relationship between δ and $x_D(\delta)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, k = 0.5$ and $K = 7$.

Example 3: Here below are some numerical examples for the function $cH(x)$ when $\delta \geq \frac{\beta\mu}{2}$, when $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$, and when $\delta < \frac{\alpha\mu}{2}$, respectively. Recall that $v'(x) = cH(x)$ for $x < x_D$ and $v'(x) = k$ for $x \geq x_D$. Now, we notice the function $cH(x)$ is depended on the liability rate δ, the minimum risk allowed α, and the maximum risk allowed β.

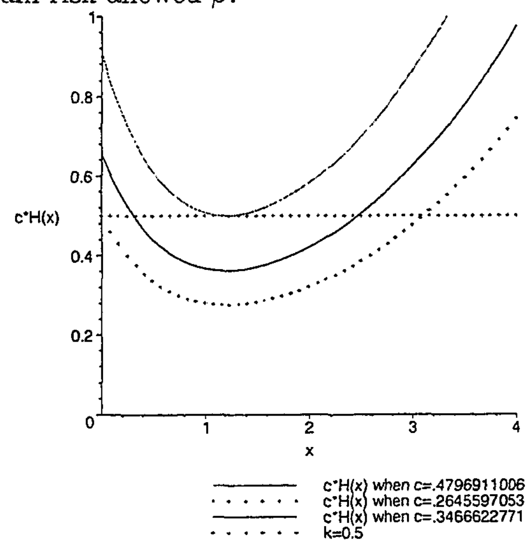

Figure 5.17: The Relationship between x and $c^*H(x)$. This is an example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \beta = 1.75, \delta = 2, k = 0.5, K = 0.2, \tilde{c} = 0.34666, \tilde{x}(\tilde{c}) = 0.31135, x_D(\tilde{c}) = 2.469754044$

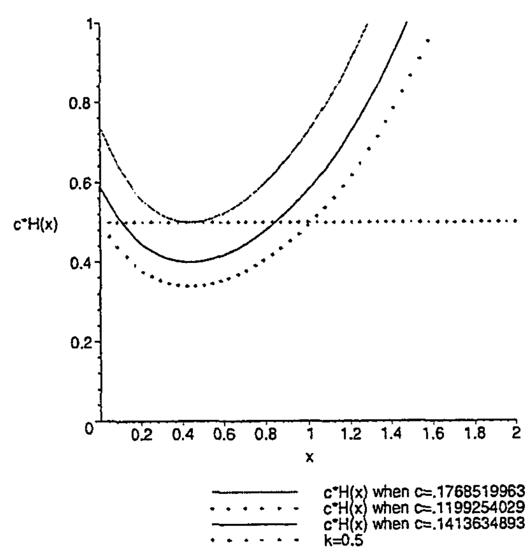

Figure 5.18: The Relationship between x and $c^*H(x)$. This is an example for $\delta \geq \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \beta = 0.75, \delta = 1, k = 0.5, K = 0.05, \tilde{c} = 0.141363, \tilde{x}(\tilde{c}) = 0.10393, x_D(\tilde{c}) = 0.8486$

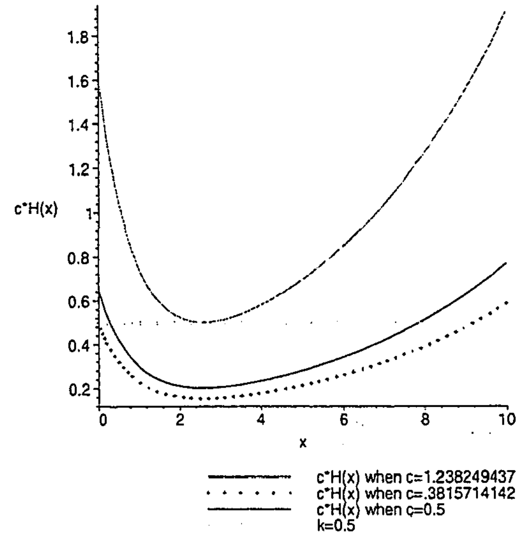

Figure 5.19: The Relationship between x and $c^*H(x)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 1, \beta = 3, \delta = 2, k = 0.5, K = 1.5066, \tilde{c} = 0.5, \tilde{x}(\tilde{c}) = 0.2913, x_D(\tilde{c}) = 7.88556$

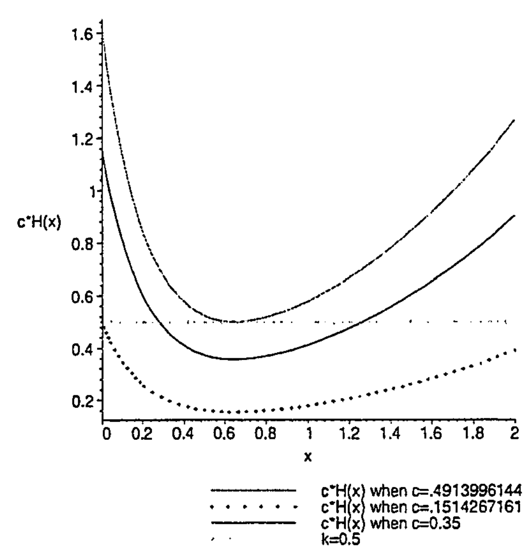

Figure 5.20: The Relationship between x and $c^*H(x)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, \delta = 0.5, k = 0.5, K = 0.0934095, \tilde{c} = 0.35, \tilde{x}(\tilde{c}) = 0.280958, x_D(\tilde{c}) = 1.264935$

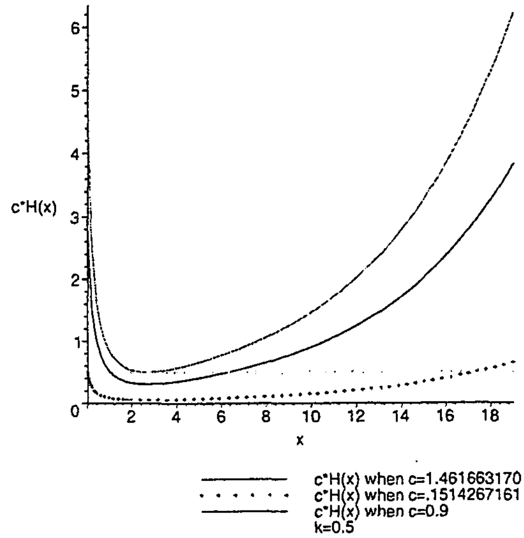

Figure 5.21: The Relationship between x and $c^*H(x)$. This is an example for $\frac{\alpha\mu}{2} \leq \delta < \frac{\beta\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, \delta = 0.5, k = 0.5, K = 0.65507, \tilde{c} = 0.9, \tilde{x}(\tilde{c}) = 1.046477746, x_D(\tilde{c}) = 6.328770913$

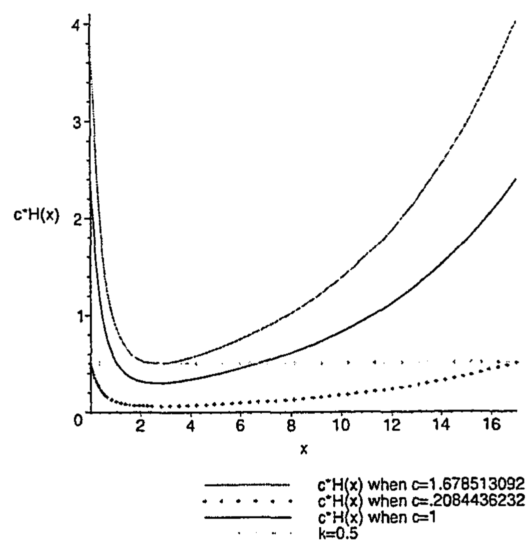

Figure 5.22: The Relationship between x and $c^*H(x)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 2, \beta = 3, \delta = 0.125, k = 0.5, K = 0.7319173298, \tilde{c} = 1, \tilde{x}(\tilde{c}) = 1.087543636, x_D(\tilde{c}) = 6.730404350$

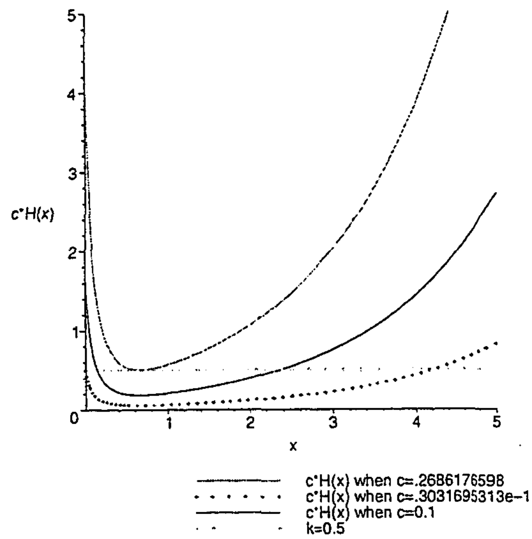

Figure 5.23: The Relationship between x and $c^*H(x)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 0.75, \delta = 0.125, k = 0.5, K = 0.458998034, \bar{c} = 0.1, \bar{x}(\bar{c}) = 0.1358376194, x_D(\bar{c}) = 0.4589998034$

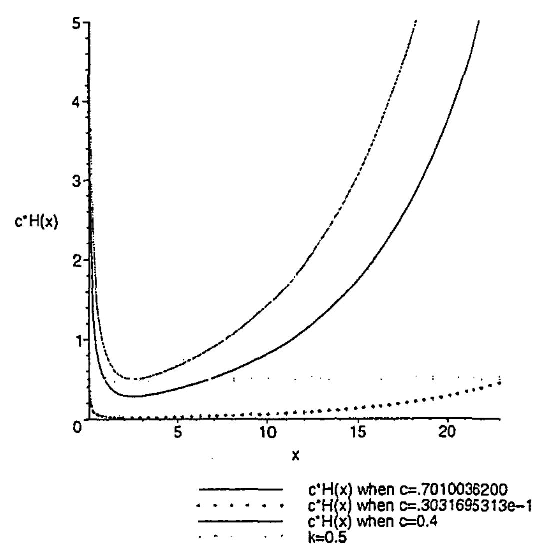

Figure 5.24: The Relationship between x and $c^*H(x)$. This is an example for $\delta < \frac{\alpha\mu}{2}$ with $\sigma = 1, \lambda = 1, \mu = 2, \alpha = 0.25, \beta = 3, \delta = 0.125, k = 0.5, K = 0.8237187325, \bar{c} = 0.4, \bar{x}(\bar{c}) = 0.8558635077, x_D(\bar{c})$ =6.834526155

Example 4: The following three figures illustrates how $a_\delta^*(x)$ depends on δ, α, and β.

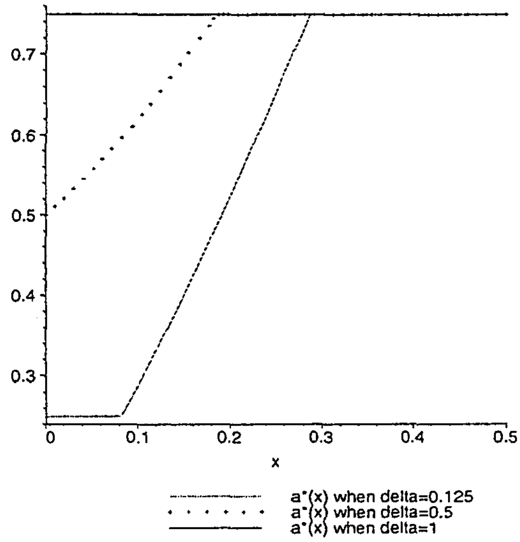

Figure 5.25: Graph for $a_{0.125}^*(x)$, $a_{0.5}^*(x)$, and $a_1^*(x)$ with $\mu = 2, \sigma = 1, \lambda = 1, \alpha = 0.25, \beta = 0.75$

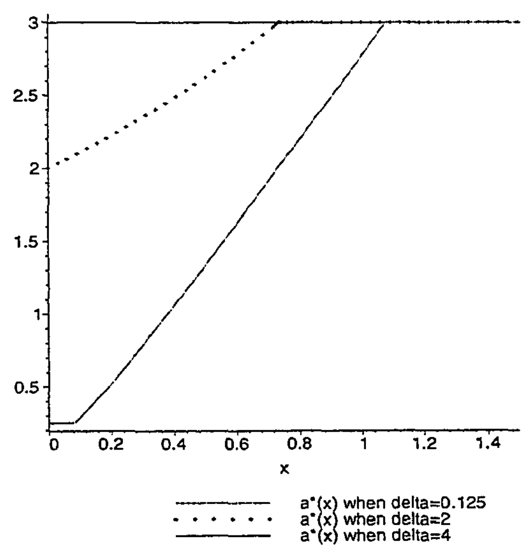

Figure 5.26: Graph for $a_{0.125}^*(x)$, $a_{0.5}^*(x)$, and $a_1^*(x)$ with $\mu = 2, \sigma = 1, \lambda = 1, \alpha = 0.25, \beta = 3$

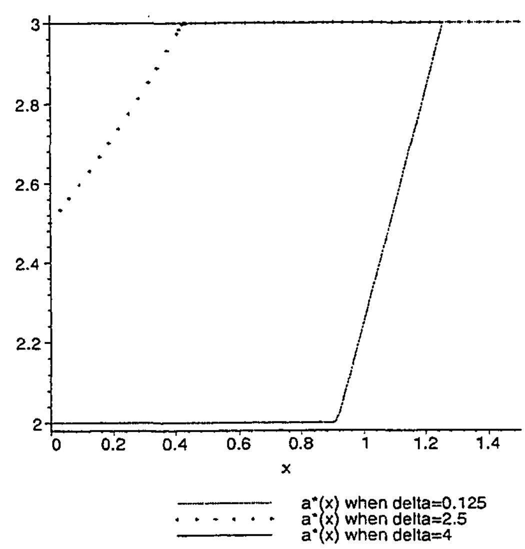

Figure 5.27: Graph for $a^*_{0.125}(x)$, $a^*_{0.5}(x)$, and $a^*_1(x)$ with $\mu = 2, \sigma = 1, \lambda = 1, \alpha = 2, \beta = 3$

Chapter 6

Conclusion

The optimal policies obtained in the previous chapters have a clear economic meaning. The optimal risk control policy is characterized by x_α and x_β, which are the reserve levels. These two levels depend on the minimum risk allowed (α), the maximum risk allowed (β), the liability rate (δ) and the profit rate (μ).

When $\frac{2\delta}{\mu} \leq \alpha$, which means the company has a fairly small debt-profit rate, both the critical reserve levels, x_α and x_β, exist and are positive. The company will minimize the business activity (i.e., take the minimum risk α) when the reserve is below the level x_α. It will gradually increase the business activity when the reserve is between x_α and x_β. Then when the reserve goes beyond the level x_β, the company will maximize its business activity (i.e., take the maximum risk β). This policy also applies similarly to the case when there is no debt in the company. However, this case does not exist if $\alpha = 0$.

For the case when the company has a higher debt-profit ratio, $\alpha < \frac{2\delta}{\mu} < \beta$, $x_\alpha = 0$ and x_β exists and is positive. In this case, the company becomes more aggressive, because company will never take the minimum risk no matter how small its reserve is. It will start with the risk level $\frac{2\delta}{\mu}$ and eventually increase up to the maximum risk level β when the reserve reaches and go above the level x_β. In another words, when the debt rate is high, the company needs to take higher risk to gamble on the risky projects with higher returns in order to get out of the bankruptcy zone as quickly as possible.

The company becomes even more aggressive when $\beta \geq \frac{2\delta}{\mu}$, which means the company has a very high debt compared to its potential profit. In this case, the company always gambles at the maximum allowable risk β, and the two critical levels x_α and x_β are both zero.

Furthermore, the optimal dividend policy always sets a threshold value x_D. In another words, the company should keep it reserve below the critical level x_D

and distribute the excess as dividends. We also observe that $x_\beta < x_D$ all the time, so the maximum business activity is always attained before dividend distributions occur. For all these cases, we notice that when the reserve reaches the threshold level of the dividend pay-outs, the company becomes unnecessary to increase its risk even when the reserve increases.

Finally, if a company has a liability rate that is greater than the maximal expected profit rate, then the company should declare bankruptcy and go out of business immediately, while distributing all its reserve as dividend to the shareholders. The reason for this is that the expected net cash flow is negative in this case, so the company's control policy is irrelevant here.

Notice that if $0 < \alpha < \beta < 1$, the company takes the portion $a^*(x)$ of the risk and transfers the remaining portion, $1 - a^*(x)$, of risk to another company by reinsurance. For the case of $0 < \alpha < 1 < \beta < +\infty$, if $a^*(x) < 1$, then the company takes the portion $a^*(x)$ of the risk and transfers the remaining portion, $1 - a^*(x)$, of risk to another company by reinsurance. If $a^*(x) \geq 1$, then the company will take full risk by not transferring any of its risk to another company and will even take the risk from another company. For the case of $1 < \alpha < \beta < +\infty$, the company takes the portion $a^*(x)$ of the risk, which is always greater than 1, so it means that the company is a very big company, so that it always takes reinsurance for other companies and it will not transfer any portion of the risk to another company.

Both the liability factor and the constraints will impose an upper bound $K_{max}(\alpha, \beta, \delta)$ on the cost variable, K. In contrast, when there is no liability and no constraints, the cost can take any nonnegative value. Thus, the QVI has a solution only when the cost variable does not exceed $K_{max}(\alpha, \beta, \delta)$.

Overall, we notice that there is an interesting interplay between the liability rate and the constraints on the risk control, which in turn, becomes the main factor that affects the decision of the optimal policy.

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