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AMENABILITY OF LOCALLY COMPACT GROUPS, SUBSPACES AND SETS OF INVARIANT MEANS

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ABSTRACT

Let G be a locally compact group and let G_d be the algebraic group G with the discrete topology. We prove in this thesis that the set of all left averaging functions is a subspace if and only if G_d is amenable. This settles a problem raised by Emerson, Rosenblatt and Yang, and Wong and Riazi. We also confirm a conjecture of Rosenblatt and Yang in [25] by showing that if all the left averaging functions are right averaging, then G_d is amenable.

Let $LIM(L^\infty(G))$ $[TLIM(L^\infty(G))]$ be the sets of [*topologically*] left invariant means on $L^\infty(G)$. We show that $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ is large by embedding a large set into it. We also show that the cardinalities of $LIM(L^\infty(G))$ and $TLIM(L^\infty(G))$ are equal when G is noncompact and metrizable.

Let $\mathcal{U}$ be the set of functions in $L^\infty(G)$ admitting a unique left invariant mean value. We prove that if $\mathcal{U}$ is a subpace of $L^\infty(G)$, then G is amenable and there is a largest admissible subspace of $L^\infty(G)$ with a unique left invariant mean if and only if G is amenable. This answers two problems raised in Rosenblatt and Yang [25].

Let G be a σ-compact noncompact nondiscrete locally compact group. If G_d is amenable, then we can extend any left invariant mean on $UCB(G)$ to a left invariant functional on $CB(G)$ which is not "topologically left invariant". We also show that if G' is another amenable group, then there is a left invariant mean θ on $CB(G \times G')$ and $f \in CB(G \times G')$ with $\theta(f) = 1$ and $\varphi(f) = 0$ for

any topologically left invariant mean ψ on $CB(G \times G')$. This answers a problem raised in Rosenblatt [24] and also confirms Chou's conjecture in this case.

To my Mom and Dad, and to Wei

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CHAPTER 1

INTRODUCTION

The subject of amenability was initiated by Lebesgue (1904) and Banach when they studied the existence and uniqueness of a positive, finitely additive, translation invariant measure μ on R with $\mu([0,1]) = 1$. In modern language, the problem is concerned with the existence and uniqueness of invariant means on $\ell^\infty(R)$, or the amenability of R. Since Day's work [6] in 1957, this subject has developed very quickly. Numerous characterizations of amenable groups have been obtained (see the recent books of Paterson [19] and Pier [21] for examples). In 1976 Chou [5] proved that for a discrete infinite group G there are $2^{2^{|G|}}$ left invariant means on $\ell^\infty(G)$. More recently, Lau and Paterson [16] have shown that the cardinality of the set of topologically left invariant means on $L^\infty(G)$ is $2^{2^{d(G)}}$ for any noncompact amenable group G, where $d(G) = \min\{|\mathcal{D}| : \mathcal{D}$ is a compact cover of $G\}$ (also see Granirer [12], Klawe [15], Paterson [20] and Yang [33]).

In this thesis, we shall be primarily concerned with the criteria for amenability of locally compact groups and discrete groups in terms of subspaces of $L^\infty(G)$ and the investigation of the set of invariant means. The thesis consists of six chapters.

Chapter 2 contains the definitions and notations used throughout the thesis. We also introduce some properties of locally compact groups which will be used throughout the remainder of the thesis.

Chapter 3 concerns itself with the relationship between the discrete amenability of a locally compact group and the set of left averaging functions. We show that G is amenable as a discrete group if and only if $\mathcal{A}$ is a subspace, where $\mathcal{A}$ is the set of left averaging functions. This provides an answer to problems raised by Emerson [9], Rosenblatt and Yang [25] and Wong and Riazi [32].

In section 3.3, we confirm a conjecture of Rosenblatt and Yang that for any non-amenable group G there is a left averaging function $f \in L^\infty(G)$ which is not right averaging.

In chapter 4, we characterize amenability for a locally compact group and a discrete group in terms of subspaces of $L^\infty(G)$. These will settle two problems raised in Rosenblatt and Yang [25].

In chapter 5, we investigate the sizes of the sets of left invariant means. We show that the size of $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ is very large by embedding $\mathcal{F}_1$ into this set, where

$$\mathcal{F}_1 = \{\theta \in \ell^\infty(\mathbb{N})^* : \theta \geq 0, \|\theta\| = 1 \quad \text{and} \quad \theta(f) = 0 \quad \text{if} \quad f \in \ell^\infty(\mathbb{N}) \\ \text{with} \quad \lim_n f(n) = 0\}.$$

In section 5.3, we prove that $|LIM(L^\infty(G))| = |TLIM(L^\infty(G))|$ for any noncompact locally compact metrizable group G. Also, counterexamples are given to show that this is not the case when G is not metrizable.

In section 5.4, we provide an answer to a problem concerning the relationship between $LIM(L^\infty(G))$ and $TLIM(L^\infty(G))$ raised by Rosenblatt in [22].

Chapter 6 deals with the extension of an invariant mean on $UCB(G)$ to a left invariant functional on $CB(G)$ or $L^\infty(G)$ such that it is not "topologically left invariant".

In section 6.3, we show that this extension is possible for any noncompact nondiscrete σ-compact locally compact group which is amenable as a discrete group.

In section 6.2, we show that there are amenable locally compact groups which are not amenable as discrete groups such that

$$LIM(CB(G)) \neq TLIM(CB(G)).$$

This answers a problem raised in Rosenblatt [24] and this also confirms a conjecture of Chou's in [4] in our case.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

2.1. Introduction.

This chapter is intended to be a reference for the terms and the notations used throughout the thesis. We also include some basic properties of locally compact groups which will be needed in this thesis.

2.2. Notations and Definitions.

Let G be a locally compact group with a fixed left Haar measure λ and let $L^p(G)$ be the associated real Lebesgue spaces $(1 \leq p \leq \infty)$. If G is compact, then we assume that $\lambda(G) = 1$. For each $f \in L^\infty(G)$ and $x \in G$, let ${}_xf \in L^\infty(G)$, the left translation of f by x, be defined by ${}_xf(y) = f(xy)$, $y \in G$. Similarly, we can define $f_x \in L^\infty(G)$, the right translation of f, by $f_x(y) = f(yx)$, $y \in G$.

A subspace S of $L^\infty(G)$ is said to be admissible if it contains the constants and ${}_xf$ for each $f \in S$ and $x \in G$. A functional $m \in S^*$ is called a mean on S if $m(1) = 1$ and $m \geq 0$, i.e. $m(f) \geq 0$ for any $f \in S$ with $f \geq 0$.

For $f \in L^\infty(G)$ and a constant c, we say that f left averages to c if $c \in \|\cdot\|_\infty$-closed convex hull of $\{{}_xf : x \in G\}$. Similarly, we say that f right averages to c if $c \in \|\cdot\|_\infty$-closed convex hull of $\{f_x : x \in G\}$.

The following notations are used throughout this thesis:

$CB(G)$	The space of all bounded real valued continuous functions on G with the supremum norm.
$UCB(G)$	The space of all uniformly continuous bounded real valued functions on G with the supremum norm.
$LIM(S)$	The set of left invariant means on S, i.e. the means m on S with $m(_x f) = m(f)$ for any $f \in S$ and $x \in G$, where S is an admissible subspace of $L^\infty(G)$.
$P(G)$	The set of all $\varphi \in L^1(G)$ with $\varphi \geq 0$ and $\|\varphi\|_1 = 1$.
$\varphi * f$	An element of $L^\infty(G)$ defined by $\varphi * f(x) = \int_G \varphi(t) f(t^{-1}x)dt$, $x \in G$ where $\varphi \in P(G)$ and $f \in L^\infty(G)$.
$TLIM(S)$	The set of topological left invariant means on S, i.e., the means m on S with $m(\varphi * f) = m(f)$ for any $\varphi \in P(G)$ and $f \in S$, where $S = L^\infty(G)$, or $CB(G)$, or $UCB(G)$.
$\mathcal{A}_0$	The set of all $f \in L^\infty(G)$ which left averages to zero.
$\mathcal{A}$	The set of $f \in L^\infty(G)$ which left averages to some constant. Note that we use $\mathcal{A}$ instead of $\mathcal{A}_L$ for notational simplicity.
$\mathcal{A}_R$	The set of $f \in L^\infty(G)$ which right averages to some constant.

S_f The smallest admissible subspace of $L^\infty(G)$ containing f where $f \in L^\infty(G)$.

$|A|$ The cardinality of a set A.

$\mathcal{U}$ The set of all $f \in L^\infty(G)$ with a unique left invariant mean value, i.e. $LIM(S_f) \neq \phi$ and there is a constant c such that $m(f) = c$ for any $m \in LIM(S_f)$.

G_d The algebraic group G with a discrete topological structure.

1_A The characteristic function of the nonempty set A.

F_2 The free group on two generators.

Δ The modular function of the group G.

supp f The support of the function f (or a measure).

H $H =$ span $\{{}_xf - f : x \in G,\ f \in L^\infty(G)\}$.

The space $L^\infty(G)$ is a commutative Banach algebra under pointwise multiplication of functions as the product. Let $\mathcal{D}$ be the maximal ideal space of $L^\infty(G)$ with the Gelfand topology. Then the Gelfand transform $\wedge$ is an isometric isomorphism of $L^\infty(G)$ onto $C(\mathcal{D})$, the algebra of real-valued continuous functions on $\mathcal{D}$ with the supremum norm. If $\theta \in \mathcal{D}$, ${}_x\theta \in \mathcal{D}$ is defined by ${}_x\theta(f) = \theta({}_xf)$ for $f \in L^\infty(G)$ and $x \in G$. For $h \in C(\mathcal{D})$ and $x \in G$, ${}_xh \in C(\mathcal{D})$

is defined by ${}_xh(\theta) = h({}_x\theta)$ for $\theta \in \mathcal{D}$. Then the isomorphism $\wedge$ satisfies the following.

(i) $\hat{1} = 1$,

(ii) $\hat{f} \geq 0$ if and only if $f \geq 0$ $(f \in L^\infty(G))$,

(iii) $\widehat{({}_xf)} = {}_x\hat{f}$ for all $x \in G$ and $f \in L^\infty(G)$.

For each λ-measurable subset M of G, there is a unique open-closed set $\hat{M} \subseteq \mathcal{D}$ such that $\widehat{(1_M)} = 1_{\hat{M}}$ and $\{\hat{M} : M$ is a λ-measurable set in $G\}$ is a basis for the topology of $\mathcal{D}$.

By the Riesz representation theorem, each $\mu \in LIM(L^\infty(G))$ can be identified with a G-invariant probability measure $\hat{\mu}$ on $\mathcal{D}$: $\hat{\mu}(\hat{f}) = \mu(f)$, $f \in L^\infty(G)$. We say that two left invariant means $\mu_1, \mu_2 \in LIM(L^\infty(G))$ are mutually singular if $\hat{\mu_1}$ and $\hat{\mu_2}$ are mutually singular as measures on $\mathcal{D}$ (see [22]).

Let E be a subset of G. E is called a locally null set if $\lambda(E \cap K) = 0$ for every compact set K in G. E is called permanently positive (p.p.) if for any $x_1, x_2, \ldots, x_n \in G$, the set $\bigcap_{i=1}^{n} x_iE$ is not locally null. E is called strictly positive (s.p.) if $U \cap \bigcap_{i=1}^{n} x_iE$ is not locally null for all open sets U and $x_1, x_2, \ldots, x_n \in G$. Note that if G is σ-compact then a set E is p.p. if and only if $\lambda(\bigcap_{i=1}^{n} x_iE) > 0$ for any $x_1, x_2, \ldots, x_n$ and E is s.p. if and only if $\lambda(U \cap \bigcap_{i=1}^{n} x_iE) > 0$ for any open set U in G and $x_1, x_2, \ldots, x_n \in G$.

A function $f \in CB(G)$ with $0 \leq f \leq 1$ is called a permanently near one function if for any $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n \in G$, there is an $x_0 \in G$ such that

$$|1 - {}_{x_i}f(x_0)| < \varepsilon \quad (i = 1, 2, \ldots, n).$$

2.3. Amenability and Properties of Locally Compact Groups.

Let G be a locally compact group. If $LIM(L^\infty(G)) \neq \phi$, we call G an amenable locally compact group. An example of non-amenable group is F_2, the free group on two generators. It is well known that $TLIM(L^\infty(G)) \subseteq LIM(L^\infty(G))$, $LIM(UCB(G)) = TLIM(UCB(G))$ and G is amenable if and only if one of the following conditions is true:

(1) $TLIM(L^\infty(G)) \neq \phi$,

(2) $LIM(CB(G)) \neq \phi$,

(3) $LIM(UCB(G)) \neq \phi$.

Also, if G is amenable as a discrete group, then G is amenable. But the converse may fail. For example, the real 3-dimensional orthogonal group $O(3)$ with its usual Lie group topology is compact so it is amenable. But it fails to be amenable as a discrete group because G has a subgroup which is isomorphic to F_2 (see [13] P26).

Følner in [10] proved that G is amenable as a discrete group if and only if G satisfies the following condition.

Given $\varepsilon > 0$ and finite set K in G, there is a finite non-empty set U in G such that

$$|(xU) \cap U| \geq (1-\varepsilon)|U| \quad \text{for } x \in K.$$

This is called the Følner condition.

In this thesis, we will also use the following results.

Proposition 2.3.1. *Let G be a σ-compact nondiscrete locally compact group. Then for any $\varepsilon > 0$ there exists an open dense set $B \subsetneq G$ with $\lambda(B) < \varepsilon$ (see Granirer [11] Proposition 2 or Rudin [26]).*

Proposition 2.3.2. *If G is a noncompact locally compact group, then there is an open and closed σ-compact noncompact subgroup G_0 of G (see [21] Proposition 22.24 and [14] Theorem 5.7).*

Proposition 2.3.3. *Let G be a locally compact group and let S be an admissible subspace of $L^\infty(G)$. If G_d is amenable, then every $m \in LIM(S)$ can be extended to an element of $LIM(L^\infty(G))$ (see Silverman [29]).*

Proposition 2.3.4. *Let $\{G_\gamma : \gamma \in \Gamma\}$ be a family of compact groups and let A_γ be a λ-measurable subset of G_γ for each $\gamma \in \Gamma$. If all but a countable number of the A_γ are equal to G_γ, then $\prod_{\gamma\in\Gamma} A_\gamma$ is measurable and $\lambda(\prod_{\gamma\in\Gamma} A_\gamma) = \prod_{\gamma\in\Gamma} \lambda(A_\gamma)$. If $\lambda(A_\lambda) < 1$ for an uncountable number of the indices λ, then $\lambda(\prod_{\gamma\in\Gamma} A_\lambda) = 0$ (see [14] 13.22).*

Let $M(G)$ be the Banach space of bounded regular Borel measures on G with total variation norm. For $\mu \in M(G)$ and $f \in L^\infty(G)$, we define

$$\mu * f(x) = \int_G f(t^{-1}x)d\mu(t) \quad (y \in G).$$

For $x \in G$ we write δ_x for the point mass at x. If $f \in L^\infty(G)$, then

$$\delta_x * f(y) = f(x^{-1}y) = {}_{x^{-1}}f(y) \quad (y \in G).$$

Let Δ be the modular function of the locally compact group G. It is a continuous function defined on G with the properties:

(i) $\Delta(xy) = \Delta(x)\Delta(y)$, for $x, y \in G$,

(ii) $\Delta > 0$,

(iii) $\Delta(x)\int_G f(t)dt = \int_G f_{x^{-1}}(t)dt$ for $f \in L^1(G)$ and $x \in G$.

It follows that

$$\varphi * {}_x f = \Delta(x)(\varphi_x) * f$$

for $\varphi \in P(G)$, $x \in G$ and $f \in L^\infty(G)$.

As in [14] (P463 (B37)), the following result will be useful for us.

PROPOSITION 2.3.5. *Let X be a subspace of $L^\infty(G)$ and $|f| \in X$ for any $f \in X$. Then for any $m \in X^*$ there exist unique positive $m^+, m^- \in X^*$ such that $m = m^+ - m^-$ and* $\min(m^+, m^-) = 0$, *where*

$$m^+ = \max(m, 0), \qquad m^- = -\min(m, 0)$$

and

$$\max(m,0)(f) = \sup\{m(g) : 0 \leq g \leq f\}$$

$$\min(m,0)(f) = \inf\{m(g) : 0 \leq g \leq f\}$$

for any $f \in X$ with $f \geq 0$.

CHAPTER 3

AMENABILITY OF LOCALLY COMPACT GROUP AND LEFT AVERAGING FUNCTIONS

3.1. Introduction.

Emerson proved in [9] that if G is a locally compact group the following are equivalent:

(a) G is amenable,

(b) $\mathcal{N}_p(G)$ is closed under addition, where

$$\mathcal{N}_p(G) = \{f \in L^\infty(G) : \inf_{\varphi \in P(G)} \|\varphi * f\|_\infty = 0\},$$

(c) $d(\varphi_1 * P(G), \varphi_2 * P(G)) = 0$ for any $\varphi_1, \varphi_2 \in P(G)$, where

$$d(\varphi_1 * P(G), \varphi_2 * P(G)) = inf\{\|\varphi_1 * \varphi - \varphi_2 * \psi\|_1 : \varphi, \psi \in P(G)\}.$$

He asked whether the condition (b) could be replaced by

(b′) $\mathcal{N}_D = \{f \in L^\infty(G) : \inf_{\mu \in D} \|\mu * f\|_\infty = 0\}$ is closed under addition, where D is the set of all finite discrete positive measures on G of weight one and $\mu * f(x) = \int_G f(t^{-1}x)d\mu(t)$. Since

$$\int_G f(t^{-1}x)d\mu(t) = \sum_{i=1}^{n} \lambda_i f(x_i x)$$

for some $\lambda_i > 0$, $x_i \in G$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have $\mathcal{N}_D = \mathcal{A}_0$.

Later Wong and Riazi in [32] (Theorem 4.1) showed that G is amenable if and only if $\mathcal{N}_I(UCB(G))$ is closed under addition, where $\mathcal{N}_I(X)$ denotes the set $\{f \in X : \inf_{\mu\in D} \|\mu * f\|_\infty = 0\}$ for any admissible subspace X of $L^\infty(G)$. It is easy to see that $\mathcal{N}_I(X) = X \cap \mathcal{A}_0$ and if $\mathcal{N}_I(L^\infty(G))$ is closed under addition, then so is $\mathcal{N}_I(UCB(G))$. The converse remains open (see [32] P494).

It is clear that $\mathcal{A}$ is closed under addition if and only if $\mathcal{A}$ is a subspace. Rosenblatt and Yang in [25] proved that if G_d is amenable then $\mathcal{A}$ is a subspace and if G is not amenable then $\mathcal{A}$ is not a subspace. They asked whether $\mathcal{A}$ forms a subspace when G is amenable but not amenable as a discrete group.

In section 3.2, we show that $\mathcal{A}$ is a subspace if and only if $\mathcal{A}_0$ is a subspace. Then we use a characterization theorem of Chou's to prove that $\mathcal{A}_0$ is a subspace if and only if G_d is amenable. This answers the problem raised by Emerson, Rosenblatt and Yang, Wong and Riazi completely.

In the study of the relation between left and right averaging functions, Rosenblatt and Yang in [25] conjectured that $\mathcal{A} \sim \mathcal{A}_R \neq \phi$ for any non-amenable locally compact group. They showed that if a discrete group G contains F_2, then $\mathcal{A} \sim \mathcal{A}_R \neq \phi$ (see [25] Theorem 2.6). In section 3.3, we improve our main theorem of section 3.2 and then we apply this theorem to confirm the conjecture of Rosenblatt and Yang.

3.2. Locally Compact Group Which are Amenable as Discrete Groups and the Set $\mathcal{A}$.

In this section, we are going to prove a criterion for a locally compact group to be amenable as a discrete group. This answers an open problem raised in Emerson [9], Rosenblatt and Yang [25] and Wong and Riazi [32].

In [23], Rosenblatt proved the following (see proposition 3.4 of [23] and also [2]).

THEOREM. *For a σ-compact locally compact group G, G_d is amenable if and only if for each $\theta \in \mathcal{D}$, the subspace $\{\rho_\theta(f) : f \in L^\infty(G)\}$ of $\ell^\infty(G)$ has a left invariant mean, where $\rho_\theta(f) \in \ell^\infty(G)$ is defined by $\rho_\theta(f)(x) = \theta({}_x f)$, $x \in G, f \in L^\infty(G)$.*

Our proof of Theorem 3.2.3 depends on this theorem of Rosenblatt's. We first establish some lemmas.

LEMMA 3.2.1. *For any locally compact group G, the following statements are equivalent:*

(i) $\mathcal{A}$ is a subspace,

(ii) $\mathcal{A}_0$ is a subspace,

(iii) $\mathcal{A}_0 = \bar{H}$.

PROOF: (i) $\Rightarrow$ (ii) If $\mathcal{A}$ is a subspace, let $f_1, f_2 \in \mathcal{A}_0 \subseteq \mathcal{A}$. There is a number c such that $f_1 + f_2$ left averages to c. By Theorem 1.5 of [25], G is amenable. Since this has not appeared yet, we give the proof here for completeness. If G is not amenable, then $L^\infty(G) = \bar{H}$. Since G is infinite, there exists a subset A in G such that both A and A^c are p.p. (see Lemma 4.2.2). Thus, 1_A does not left average. It is easy to see that $\mathcal{A}$ is closed. So some $F \in H$

does not left average. But each ${}_xf - f \in \mathcal{A}_0$ since $lim_n \frac{1}{n}\sum_{k=1}^{n} {}_{x^k}({}_xf - f) = 0$. Therefore $\mathcal{A}$ is not a subspace which is impossible. Let $m \in$ LIM $(L^\infty(G))$, then $m(f_1) = m(f_2) = 0$ since $f_1,\ f_2 \in \mathcal{A}_0$. Therefore $c = m(f_1 + f_2) = 0$, i.e. $f_1 + f_2 \in \mathcal{A}_0$.

(ii) $\Rightarrow$ (iii) For any $x \in G$ and $f \in L^\infty(G)$, since

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} {}_{x^k}(f - {}_xf) = 0,$$

$f - {}_xf \in \mathcal{A}_0$. Hence if $\mathcal{A}_0$ is a subspace, then $H \subseteq \mathcal{A}_0$. It is easy to see that $\mathcal{A}_0$ is closed. So $\bar{H} \subseteq \mathcal{A}_0$. Let $f \notin \bar{H}$, then there is $m \in$ LIM $(L^\infty(G))$ with $m(f) \neq 0$. So $f \notin \mathcal{A}_0$. (See [14] (B37) and p. 236), i.e. $\mathcal{A}_0 \subseteq \bar{H}$.

(iii) $\Rightarrow$ (ii) is clear. To prove (ii) $\Rightarrow$ (i), let f_i left average to c_i. Then $f_i - c_i$ left averages to 0 ($i = 1, 2$). If $\mathcal{A}_0$ is a subspace, then $(f_1 - c_1) + (f_2 - c_2) \in \mathcal{A}_0$, i.e. $f_1 + f_2$ left averages to $c_1 + c_2$.

□

THEOREM 3.2.2. *For any locally compact group G and $\theta \in \mathcal{D}$, $S_\theta = \{\rho_\theta(f) : f \in L^\infty(G)$ is a simple function $\}$ is a subspace of $\ell^\infty(G)$ with the following properties:*

(i) $|F| \in S_\theta$ for any $F \in S_\theta$,

(ii) If $m \in S_\theta^*$ is left invariant, then there are nonnegative left invariant functionals m^+, $m^- \in S_\theta^*$ such that $m = m^+ - m^-$ where $m^+ = \max(m, 0)$, $m^- = -\min(m, 0)$ (see [14] (B34)).

PROOF: (i) If $f = \sum_{i=1}^{n} a_i 1_{E_i}$ is a measurable simple function, then $|\rho_\theta(f)| = \rho_\theta(|f|)$ i.e. $|\rho_\theta(f)| \in S_\theta$. Indeed, for any $x \in G$, since $\{x^{-1}E_i : i = 1, 2, \ldots, n\}$ are pairwise disjoint, there is at most one i with $\theta({}_x 1_{E_i}) \neq 0$. Hence

$$|\rho_\theta(f)(x)| = |\theta\Big(\sum_{i=1}^{n} a_i ({}_x 1_{E_i})\Big)| = |\sum_{i=1}^{n} a_i \theta({}_x 1_{E_i})| = \sum_{i=1}^{n} |a_i| \theta({}_x 1_{E_i})$$
$$= \theta\Big(\sum_{i=1}^{n} |a_i| {}_x 1_{E_i}\Big) = \theta(|f|)(x).$$

(ii) Since $m \in S_\theta^*$ is bounded, there are nonnegative $m^+,\ m^- \in S_\theta^*$ such that $m = m^+ - m^-$, $m^+ = \max(m, 0)$ and $m^- = -\min(m, 0)$ by (i) and (B.37) of [14].

Let $F_0 \in S_\theta$ and $F_0 \geq 0$. Since

$$m^+(F_0) = \sup\{m(F) : 0 \leq F \leq F_0,\ F \in S_\theta\},$$

if $x \in G$ and $0 \leq F \leq F_0$, then $0 \leq {}_xF \leq {}_xF_0$ and $m({}_xF) = m(F)$. Hence $m^+(F_0) = m^+({}_xF_0)$. Since for any $F \in S_\theta$, there are $F^+,\ F^- \in S_\theta$ such that $F^+ \geq 0,\ F^- \geq 0$ and $F = F^+ - F^-$, m^+ is left invariant. Similarly, we can prove that m^- is left invariant.

□

We are now ready to answer a question raised by Rosenblatt and Yang in [25] (Remark 3 after Theorem 1.5).

THEOREM 3.2.3. *For any locally compact group G, $\mathcal{A}$ is a subspace if and only if G_d is amenable.*

PROOF: It is well known that if G_d is amenable, then $\mathcal{A}$ is a subspace (see [25], Theorem 1.1).

Let $\mathcal{A}$ be a subspace. By Lemma 3.2.1, $\mathcal{A}_0 = \bar{H}$. Since any countable subgroup of G is contained in a σ-compact, open and closed subgroup of G (see [21], Proposition 22.24), it suffices to show that, for any σ-compact, open and closed subgroup G_0 of G, $(G_0)_d$ is amenable by (D) and (F) of [6], p. 516. By Rosenblatt's theorem above, it suffices to show that the subspace $\{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}$ of $\ell^\infty(G_0)$ has a left invariant mean for each $\theta_0 \in \mathcal{D}_0$, where $\mathcal{D}_0$ is the maximal ideal space of $L^\infty(G_0)$.

Let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be the set of all the left cosets of G_0 in G and $G_0 \in \{x_\alpha G_0 : \alpha \in \Lambda\}$. For each $f \in L^\infty(G_0)$, let $\mathring{f}$ be defined by $\mathring{f}(x_\alpha x) = f(x)$ for each $x \in G_0$, $\alpha \in \Lambda$. Since $G = \bigcup_{\alpha\in\Lambda} x_\alpha G_0$, $\mathring{f}$ is a function on G. Claim: $\mathring{f} \in L^\infty(G)$. Let $c \in \mathbb{C}$ and K be any compact subset of G. Since G_0 is open, there are only finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $K \cap x_{\alpha_i} G_0 \neq \phi$ $(i = 1, 2, \ldots, n)$. Hence

$$\{x \in G : \mathring{f} > c\} \cap K = \bigcup_{i=1}^{n} K \cap x_{\alpha_i} G_0 \cap \{x \in G : \mathring{f}(x) > c\}$$

is λ-measurable since f is measurable on G_0. Therefore $\mathring{f} \in L^\infty(G)$ (see [14] (11.31) and (15.8)).

Let $\theta_0 \in \mathcal{D}_0$ be given and $\theta \in \mathcal{D}$ be defined by $\theta(f) = \theta_0(f \mid G_0)$ for any $f \in L^\infty(G)$. We first show that there is a left invariant mean m_θ on S_θ where $S_\theta = \{\rho_\theta(f) : f \in L^\infty(G)$ is a simple function$\}$. Let $H_\theta =$ span$\{{}_xF - F : x \in G,\ F \in S_\theta\}$ and $h_\theta \in H_\theta$. Then there is a $h \in H$ such that $h_\theta = \rho_\theta(h)$.

Claim: $\|1-h_\theta\|_\infty \geq 1$. Let $\varepsilon > 0$. Since $h \in H \subseteq \mathcal{A}_0$, there are $\lambda_i > 0$ and $x_i \in G$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\|\sum_{i=1}^{n} \lambda_i ({}_{x_i}h)\|_\infty < \varepsilon/2$. Then

$$\|\sum_{i=1}^{n} \lambda_i ({}_{x_i}\rho_\theta(h))\|_\infty = \|\rho_\theta\Big(\sum_{i=1}^{n} \lambda_i ({}_{x_i}h)\Big)\|_\infty \leq \|\rho_\theta\| \ \|\sum_{i=1}^{n} \lambda_i ({}_{x_i}h)\|_\infty \leq \varepsilon/2.$$

It follows that

$$\|1-h_\theta\|_\infty \geq \|\sum_{i=1}^{n} \lambda_i ({}_{x_i}(1-h_\theta))\|_\infty = \|1 - \sum_{i=1}^{n} \lambda_i ({}_{x_i}\rho_\theta(h))\|_\infty \geq 1 - \varepsilon/2.$$

Since ε was arbitrary, $\|1-h_\theta\|\infty \geq 1$. Hence $\|1-h_\theta\|_\infty \geq 1$ for any $h_\theta \in H_\theta$. Now by Hahn-Banach theorem we can find $M \in S_\theta^*$ such that $M(1) = 1$ and $M(h_\theta) = 0$ for any $h_\theta \in H_\theta$, i.e. M is left invariant. By Theorem 3.2.2, M^+ is a nonnegative left invariant functional on S_θ. Put $m_\theta = \frac{M^+}{M^+(1)}$, then m_θ is a left invariant mean on S_θ.

Let $S_{\theta_0} = \{\rho_{\theta_0}(f) : f \in L^\infty(G_0)$ is a simple function$\}$. By using m_θ, we can define a left invariant mean on S_{θ_0} as following. For each $\rho_{\theta_0}(f) \in S_{\theta_0}$, define $m_{\theta_0}(\rho_{\theta_0}(f)) = m_\theta(\rho_\theta(\overset{\circ}{f}))$. Then m_{θ_0} is well defined. Indeed, for simple functions $f_1,\ f_2 \in L^\infty(G_0)$, let $\rho_{\theta_0}(f_1) = \rho_{\theta_0}(f_2)$ i.e. $\rho_{\theta_0}(f) = 0$ where $f = f_1 - f_2$. Let $f = \sum_{i=1}^{n} a_i 1_{E_i}$. Note that for each $x \in G$, there is at most one i such that $\rho_{\theta_0}(1_{E_i})(x) \neq 0$. Hence we have $\rho_{\theta_0}(1_{E_i})(x) = 0$ for $i = 1, 2, \ldots, n$ and $x \in G_0$. Let $1 \leq i \leq n$ be given,

$$\rho_\theta(\overset{\circ}{1}_{E_i})(x_{\alpha_0}x) = \rho_\theta\big({}_{x_{\alpha_0}x}1_{(\cup_{\alpha\in\Lambda} x_\alpha E_i)}\big) = \theta\big(1_{(\cup_{\alpha\in\Lambda} x^{-1}x_{\alpha_0}^{-1}x_\alpha E_i)}\big)$$

$$= \theta_0\big(1_{(\cup_{\alpha\in\Lambda} x^{-1}x_{\alpha_0}^{-1}x_\alpha E_i)} \mid G_0\big) = \theta_0\big(1_{x^{-1}E_i}\big) = 0$$

for any $x \in G_0$ and $\alpha_0 \in \Lambda$. Hence $\rho_\theta(\mathring{1}_{E_i}) = 0$ $(i = 1, 2, \ldots, n)$ i.e. $\rho_\theta(\mathring{f}) = 0$ and

$$m_{\theta_0}(\rho_{\theta_0}(f_1)) - m_{\theta_0}(\rho_{\rho_0}(f_2)) = m_\theta(\rho_\theta(\mathring{f}_1) - m_\theta(\rho_\theta(\mathring{f}_2))$$
$$= m_\theta(\rho_\theta(\mathring{f})) = 0$$

so $m_{\theta_0}(\rho_{\theta_0}(f_1)) = m_{\theta_0}(\rho_{\theta_0}(f_2))$. It is clear that m_{θ_0} is linear. To see that m_{θ_0} is left invariant, it suffices to show that for any $x \in G_0$ and measurable set E of G_0

$$_x\rho_\theta(\mathring{1}_E) = \rho_\theta((_x\mathring{1}_E)). \qquad (*)$$

Indeed, $(*)$ implies that

$$m_{\theta_0}(_x\rho_{\theta_0}(1_E)) = m_{\theta_0}(\rho_{\theta_0}(_x1_E)) = m_\theta(\rho_\theta((_x\mathring{1}_E))) = m_\theta(_x\rho_\theta(\mathring{1}_E))$$
$$= m_\theta(\rho_\theta(\mathring{1}_E)) = m_{\theta_0}(\rho_{\theta_0}(1_E))$$

for any $x \in G_0$ and measurable subset E of G_0. Hence $m_{\theta_0}(_x\rho_{\theta_0}(f)) = m_{\theta_0}(\rho_{\theta_0}(f))$ for any $x \in G_0$ and simple function $f \in \mathrm{L}^\infty(G_0)$. To prove $(*)$, note that

$$\rho_\theta((_x\mathring{1}_E)) = \rho_\theta\big(1_{\cup_{\alpha\in\Lambda}x_\alpha x^{-1}E}\big) = \theta_0\big(1_{\cup_{\alpha\in\Lambda}x_\alpha x^{-1}E} \mid G_0\big) = \theta_0(1_{x^{-1}E})$$
$$_x\rho_\theta(\mathring{1}_E) = \rho_\theta\big(_x1_{\cup_{\alpha\in\Lambda}x_\alpha E}\big) = \theta_0\big(1_{\cup_{\alpha\in\Lambda}x^{-1}x_\alpha E} \mid G_0\big) = \theta_0\big(1_{x^{-1}E}\big).$$

Also m_{θ_0} is nonnegative. Indeed, let $f = \sum_{i=1}^{n} a_i 1_{E_i} \in L^\infty(G_0)$ be a simple function and $\rho_{\theta_0}(f) \geq 0$. Suppose that for each i_0 there exist $x \in G_0$ such that $\theta_0({}_x1_{E_{i_0}}) \neq 0$ (otherwise take $a_{i_0} = 0$). Since $x^{-1}E_i \cap x^{-1}E_j = \phi$ $(i \neq j)$, $\theta_0({}_x1_{E_j}) = 0$ if $j \neq i_0$. Hence

$$\rho_{\theta_0}(f)(x) = \sum_{i=1}^{n} \rho_{\theta_0}(a_{ix}1_{E_i}) = a_{i_0} \geq 0.$$

For each i, note that $\rho_\theta(\mathring{1}_{E_i}) = \rho_\theta(1_{\cup_{\alpha\in\Lambda} x_\alpha E_i}) \geq 0$, so $m_\theta(\rho_\theta(\mathring{1}_{E_i})) \geq 0$ $(i = 1, 2, \ldots, n)$. Therefore

$$m_{\theta_0}(\rho_{\theta_0}(f)) = \sum_{i=1}^{n} a_i m_{\theta_0}(\rho_{\theta_0}(1_{E_i})) = \sum_{i=1}^{n} a_i m_\theta(\rho_\theta(\mathring{1}_{E_i})) \geq 0.$$

It is clear that $m_{\theta_0}(1) = 1$. Hence m_{θ_0} is a left invariant mean on S_{θ_0}.

Since S_{θ_0} is dense in $\{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}$, we can extend m_{θ_0} to $\{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}$ such that m_{θ_0} is a bounded functional. It is easy to see that m_{θ_0} is nonnegative and left invariant, i.e. m_{θ_0} is a left invariant mean on $\{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}$.

□

Let X be a left invariant subspace of $L^\infty(G)$ and D denote the set of all finite convex combinations of Dirac measures. As in [32], we denote the set $\{f \in X : \inf\{\|\mu * f\|_\infty : \mu \in D\} = 0\}$ by $\mathcal{N}_1(X)$ (see [32] p.480 and p.491). It is clear that $\mathcal{N}_1(X) = X \cap \mathcal{A}_0$. See [14] 20.9.

COROLLARY 3.2.4. *$\mathcal{N}_1(L^\infty(G))$ is closed under addition if and only if G_d is amenable.*

PROOF: Since $\mathcal{N}_1(L^\infty(G)) = \mathcal{A}_0$, the Corollary is true by Lemma 3.2.1 and Theorem 3.2.3.

□

REMARK 1. Wong and Riazi in [32], p. 493 proved that for any locally compact group G, G is amenable if and only if $\mathcal{N}_1(UCB(G))$ is closed under addition where $UCB(G)$ is the set of all the uniformly continuous functions on G. Since there are amenable locally compact groups which are not amenable as discrete groups, we have answered the problem raised by Wong and Raizi in [32], p.494 (Remark 3).

REMARK 2. Recall that $P(G) = \{\varphi \in L^1(G) : \varphi \geq 0 \text{ and } \|\varphi\|_1 = 1\}$. Emerson proved in [9] that G is amenable if and only if $\{f \in L^\infty(G) : \inf\{\|\varphi * f\|_\infty : \varphi \in P(G)\} = 0\}$ is closed under addition. This corollary provides an answer to his problem in replacing $P(G)$ by D ([9], p.187).

COROLLARY 3.2.5. (GRANIRER [11] AND RUDIN [26]). *If G is not discrete and G_d is amenable, then*

$$LIM(L^\infty(G)) \neq TLIM(L^\infty(G))$$

where $TLIM(L^\infty(G))$ is the set of all topological left invariant means on $L^\infty(G)$.

PROOF: By Proposition 1.2 and the last proposition of [11], we can find an open dense subset V in G such that $m(1_V) < 1$ for all $m \in TLIM(L^\infty(G))$. By Theorem 3.2.3, $\mathcal{A}_0 = \bar{\bar{H}}$. We can see that $\|1_V - h\|_\infty \geq 1$ for any $h \in \bar{\bar{H}}$. Indeed, if there is $h \in \bar{H}$ such that $\|1_V - h\|_\infty < 1-\epsilon$ for some $\epsilon > 0$, then we can find $x_i \in G$ and $\lambda_i > 0 (i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\| \sum_{i=1}^{n} \lambda_{i x_i} h\|_\infty \leq \epsilon/2$. It follows that

$$\|1_V - h\|_\infty \geq \|\sum_{i=1}^{n} \lambda_{i x_i}(1_V - h)\|_\infty \geq \|\sum_{i=1}^{n} \lambda_{i x_i} 1_V\|_\infty - \|\sum_{i=1}^{n} \lambda_{i x_i} h\|_\infty \geq 1 - \epsilon/2$$

which is impossible. Hence $\|1_V - h\|_\infty \geq 1$ for any $h \in H$. By Hahn-Banach theorem there is a functional $M \in L^\infty(G)^*$ such that $M(1_V) = 1$ and $M(h) = 0$ for any $h \in H$. Then $m = \frac{M^+}{M^+(1)} \in LIM(L^\infty(G))$ such that $m(1_V) = 1$, i.e. $m \notin TLIM(L^\infty(G))$ (see proposition 2.3.5).

□

3.3. The Existence of Left Averaging Functions Which are not Right Averaging.

In this section, we are going to confirm a conjecture of Rosenblatt and Yang in [25] that there is an $f \in L^\infty(G)$ such that f left averages but f does not right average when G_d is not amenable. Theorem 3.3.3 gives another characterization of locally compact groups which are amenable as discrete groups (see Theorem 3.2.3).

LEMMA 3.3.1.

(a) *For $f \in L^\infty(G)$, let $\check{f} \in L^\infty(G)$ be defined by $\check{f}(x) = f(x^{-1})$ $(x \in G)$. Then $f \in \mathcal{A}$ if and only if $\check{f} \in \mathcal{A}_R$.*

(b) $\mathcal{A} \subseteq \mathcal{A}_R$ *if and only if* $\mathcal{A} = \mathcal{A}_R$.

PROOF: (a) For any $x \in G$ and $t \in G$,

$$({}_x\check{f})(t) = f(xt^{-1}) = \check{f}(tx^{-1}) = \check{f}_{x^{-1}}(t).$$

Hence $({}_x\check{f}) = \check{f}_{x^{-1}}$. It is easy to see that the following are equivalent:

(i) $f \in \mathcal{A}$ (ii) there is a constant c such that $c \in \|.\|_\infty$-closed convex hull of $\{{}_xf : x \in G\}$

(iii) $c \in \|.\|_\infty$-closed convex hull of $\{({}_x\check{f}) : x \in G\} = \|.\|_\infty$–closed convex hull of $\{\check{f}_{x^{-1}}:\ x \in G\}$ (iv) $\check{f} \in \mathcal{A}_R$.

(b) Let $\mathcal{A} \subseteq \mathcal{A}_R$. If $f \in \mathcal{A}_R$ then $\check{f} \in \mathcal{A}$ by (a) since $(\check{\check{f}}) = f$. Hence $\check{f} \in \mathcal{A} \subseteq \mathcal{A}_R$ and $f \in \mathcal{A}$ by (a) again. Therefore $\mathcal{A} = \mathcal{A}_R$.

□

LEMMA 3.3.2. *$\mathcal{A}_0$ is a subspace if and only if for any $f \in \mathcal{A}_0$, $\sum_{i=1}^n \lambda_{i\,x_i}f \in \mathcal{A}_0$ for any $x_i \in G$, $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \lambda_i = 1$.*

PROOF: Suppose that $\mathcal{A}_0$ is a subspace. If $f \in \mathcal{A}_0$, then ${}_xf \in \mathcal{A}_0$ for any $x \in G$. Let $x_i \in G$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$, then $\lambda_{i\,x_i}f \in \mathcal{A}_0$ $(i = 1, 2, \ldots, n)$. So $\sum_{i=1}^n \lambda_{i\,x_i}f \in \mathcal{A}_0$ since $\mathcal{A}_0$ is a subspace.

Conversely, let $\sum\limits_{i=1}^n \lambda_{i\,x_i}f \in \mathcal{A}_0$ for any $f \in \mathcal{A}_0$, $x_i \in G$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum\limits_{i=1}^n \lambda_i = 1$. Since $\mathcal{A}_0 \subseteq \bar{H}$ always holds (see the proof of Lemma 3.2.1 (ii) $\rightarrow$ (iii)) and $\mathcal{A}_0$ is closed, it suffices to show that $\mathcal{A}_0 \supseteq H$. First, for $f_1 \in L^\infty(G)$ and $x_1 \in G$, $x_1f - f \in \mathcal{A}_0$. Let $F = \sum\limits_{i=1}^{n-1} a_i({}_{x_i}f_i - f_i) \in \mathcal{A}_0$

for any constants a_i, $x_i \in G$ and $f_i \in L^\infty(G)$ $(i = 1, 2, \ldots, n-1)$, $n \geq 2$. If $x_n \in G$ and $f_n \in L^\infty(G)$, the claim is that $\sum_{i=1}^{n} a_i({}_{x_i}f_i - f_i) \in \mathcal{A}_0$ where a_n is a nonzero constant. Indeed, let $\varepsilon > 0$. Since ${}_{x_n}f_n - f_n \in \mathcal{A}_0$, there are $\lambda_k > 0, y_k \in G$ $(k = 1, 2, \ldots, N)$ such that $\sum_{k=1}^{N} \lambda_k = 1$ and

$$\|\sum_{k=1}^{N} \lambda_{k y_k}({}_{x_n}f_n - f_n)\|_\infty < \frac{\varepsilon}{2|a_n|}.$$

But $\sum_{k=1}^{N} \lambda_{k y_k} F \in \mathcal{A}_0$ by hypothesis. So there are $w_\ell > 0$ and $z_\ell \in G$ $(\ell = 1, 2, \ldots, L)$ such that $\sum_{\ell=1}^{L} w_\ell = 1$ and

$$\|\sum_{\ell=1}^{L} w_{\ell z_\ell}(\sum_{k=1}^{N} \lambda_{k y_k} F)\|_\infty < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}
&\|\sum_{\ell=1}^{L} w_{\ell z_\ell}[\sum_{k=1}^{N} \lambda_{k y_k}\Big(\sum_{i=1}^{n} a_i({}_{x_i}f_i - f_i)\Big)]\|_\infty \\
&\leq \|\sum_{\ell=1}^{L} w_{\ell z_\ell}\Big(\sum_{k=1}^{N} \lambda_{k y_k} F\Big)\|_\infty + \\
&\qquad + |a_n| \sum_{\ell=1}^{L} w_\ell \|\sum_{k=1}^{N} \lambda_{k y_k}({}_{x_n}f_n - f_n)\|_\infty \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

By induction, $\mathcal{A}_0 \supseteq H$.

□

THEOREM 3.3.3. *For a locally compact group G, G_d is amenable if and only if $\sum_{i=1}^{n} \lambda_{i x_i} f \in \mathcal{A}_0$ for any $f \in \mathcal{A}_0$, $\lambda_i > 0$ and $x_i \in G$ with $\sum_{i=1}^{n} \lambda_i = 1$.*

PROOF: This is a direct consequence of Theorem 3.2.3 and Lemma 3.3.2.

□

The next lemma is in Rosenblatt and Yang [25]. For the sake of completeness, we give the proof.

LEMMA 3.3.4. *If $f \in L^\infty(G)$ left averages to c_1 and right averages to c_2, then $c_1 = c_2$, where c_1 and c_2 are constants.*

PROOF: Let $\varepsilon > 0$. Then there exist $\lambda_i > 0$ and $x_i \in G$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\|c_1 - \sum_{i=1}^{n} \lambda_i ({}_{x_i}f)\|_\infty < \varepsilon$. Since f right averages to c_2, $\sum_{i=1}^{n} \lambda_i ({}_{x_i}f)$ also right averages to c_2. Hence there exist $y_k \in G$ and $a_k > 0$ ($k = 1, 2, \ldots, N$) with $\sum_{k=1}^{N} a_k = 1$ and $\|c_2 - \sum_{k=1}^{N} a_k \left(\sum_{i=1}^{n} \lambda_i ({}_{x_i}f)\right)_{y_k} \|_\infty < \varepsilon$. Therefore

$$|c_1 - c_2| \le \| \sum_{k=1}^{N} a_k \Big(\sum_{i=1}^{n} \lambda_i ({}_{x_i}f) \Big)_{y_k} - c_1 \|_\infty + \|c_2 - \sum_{k=1}^{N} a_k \Big(\sum_{i=1}^{n} \lambda_i ({}_{x_i}f \Big)_{y_k} \|_\infty$$

$$< \varepsilon + \sum_{k=1}^{N} a_k \| \Big(c_1 - \sum_{i=1}^{n} \lambda_i ({}_{x_i}f \Big)_{y_k} \|_\infty < 2\varepsilon.$$

Since ε is arbitrary, $c_1 = c_2$.

□

The following theorem confirms the conjecture of Rosenblatt and Yang [25]:

THEOREM 3.3.5. *If $\mathcal{A} \subseteq \mathcal{A}_R$, then G_d is amenable.*

PROOF: By Theorem 3.3.3, it suffices to show that $\sum_{i=1}^{n} \lambda_{i x_i} f \in \mathcal{A}_0$ for any $f \in \mathcal{A}_0$, $x_i \in G$ and $\lambda_i > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^{n} \lambda_i = 1$. Since $\mathcal{A}_0 \subseteq$

$\mathcal{A} \subseteq \mathcal{A}_R$, $f \in \mathcal{A}_R$ and it right averages to 0 by Lemma 3.3.4. Hence for any $\varepsilon > 0$, there are $\beta_k > 0$, $y_k \in G$ $(k = 1, 2, \ldots, N)$ such that $\sum_{k=1}^{N} \beta_k = 1$ and $\| \sum_{k=1}^{N} \beta_k f_{y_k} \|_\infty < \varepsilon$. Note that for each $i = 1, 2, \ldots, n$,

$$\|_{x_i}\Big(\sum_{k=1}^{N} \beta_k f_{y_k}\Big)\|_\infty = \| \sum_{k=1}^{N} \beta_k ({}_{x_i}f)_{y_k} \|_\infty < \varepsilon.$$

Hence

$$\begin{aligned} \| \sum_{k=1}^{N} \beta_k \Big(\sum_{i=1}^{n} \lambda_{i\,x_i} f\Big)_{y_k} \|_\infty &= \| \sum_{i=1}^{n} \lambda_i \Big(\sum_{k=1}^{N} \beta_k ({}_{x_i}f)_{y_k}\Big) \|_\infty \\ &\le \sum_{i=1}^{n} \lambda_i \| \sum_{k=1}^{N} \beta_k ({}_{x_i}f)_{y_k} \|_\infty < \varepsilon, \end{aligned}$$

i.e. $\sum_{i=1}^{n} \lambda_{i\,x_i} f$ right averages to 0. By lemma 3.3.1, $\sum_{i=1}^{n} \lambda_{i\,x_i} f \in \mathcal{A}$. Hence $\sum_{i=1}^{n} \lambda_{i\,x_i} f$ left averages to 0 by Lemma 3.3.4 again. Therefore, $\sum_{i=1}^{n} \lambda_{i\,x_i} f \in \mathcal{A}_0$.

□

CHAPTER 4

AMENABILITY OF LOCALLY COMPACT GROUPS AND THE SUBSPACES OF $L^\infty(G)$

4.1. Introduction.

In this chapter, we provide answers to two problems raised by Rosenblatt and Yang in [25]. These will give characterizations of amenable groups in terms of various subspaces of $L^\infty(G)$.

Let G be a locally compact group. It is well known that if G_d is amenable, then

$$\mathcal{U} = \|.\|_\infty\text{-closed span } \left\{ {}_x f - f : x \in G,\ f \in L^\infty(G) \right\} \cup C$$

where C is the set of all the constants (see [25]).

Rosenblatt and Yang in [25] proved that if G is discrete and contains F_2, then $\mathcal{U}$ is not a subspace. They asked if $\mathcal{U}$ is a subspace when G is not amenable. Our theorem 4.2.4 answers this problem completely. Corollary 4.2.5 will also give a criterion for amenability of a discrete group.

The existence and uniqueness of the left invariant mean on $L^\infty(G)$ have been discussed in many papers . As is well-known in most cases the size of $LIM(L^\infty(G))$ is either very large or empty. It is natural to ask whether there exists a largest admissible subspace S_M of $L^\infty(G)$ with a unique left invariant mean (see [25] P5 problem (d)). It is proved in [25] that such a space does not exist for any discrete group containing F_2. Our theorem 4.3.1 answers this problem completely. It gives also a criterion for amenabililty of a locally compact group.

4.2. The functions with a unique invariant mean value.

Let G be a locally compact group. We will show in this section that $\mathcal{U}$ is not a subspace when G is not amenable.

We need the following lemma, probably known, for which we were unable to find a reference.

Recall a set E of G is called a permanently positive subset (P.P.) if $\bigcap_{i=1}^{n} x_i E$ is not locally null for any $x_1, x_2, \ldots, x_n \in G$.

LEMMA 4.2.1. *Let E be a P.P. set in G. Then there exists $m \in LIM(S_{1_E})$ with $m(1_E) = 1$.*

PROOF: Put $m(1) = m({}_x 1_E) = 1$ for any $x \in G$ and linearly extend m to S_{1_E}. Then m is well defined. Indeed, let $h = \alpha_0 + \sum_{i=1}^{n} \alpha_{i\,x_i} 1_E = 0$ ($x_i \in G$, $i = 1, 2, \ldots, n$). Since $\bigcap_{i=1}^{n} x_i^{-1} E$ is not locally null and $h = \sum_{i=0}^{n} \alpha_i$ on $\bigcap_{i=1}^{n} x_i^{-1} E$, $\sum_{i=0}^{n} \alpha_i = 0$. Hence $m(h) = \sum_{i=0}^{n} \alpha_i = 0$. Similarly, if $h \geq 0$ and $h \in S_{1_E}$, then $m(h) \geq 0$. Since m is nonnegative and $m(1) = 1$, m is bounded. Therefore $m \in LIM(S_{1_E})$.

□

LEMMA 4.2.2. *For any infinite locally compact group G, there is a subset A in G such that both A and A^c are P.P. sets, where $A^c = G \sim A$.*

PROOF: If G is discrete or G is σ-compact and nondiscrete, then there is a subset A in G such that both A and A^c are P.P. sets in G (see [25], p 5 and [22], Proposition 3.4). Let G be non-σ-compact and nondiscrete. We can find an open and closed σ-compact subgroup G_0 of G (see [21], Proposition 22.24). Let A_0 be a subset of G_0 such that both A_0 and $A_0^c = G_0 \sim A_0$ are P.P. sets in G_0. Suppose $\{x_\alpha G_0 : \alpha \in \Lambda\}$ is all the left cosets of G_0 in G. Let $A = \bigcup_{\alpha\in\Lambda} x_\alpha A_0$. Since for any compact subset K of G, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup_{i=1}^{n} x_\alpha G_0$, $K \cap A = \bigcup_{i=1}^{n} x_{\alpha_i} A_0 \cap K$ is measurable. Hence A is measurable (see [14] (11.31) and (15.8)). For any $g_i \in G$ $(i = 1, 2, \ldots, n)$, since $G = \bigcup_{\alpha\in\Lambda} G_0 x_\alpha^{-1}$, there is $\alpha_i \in \Lambda$ such that $g_i = y_i x_{\alpha_i}^{-1}$ for some $y_i \in G_0$ $(i = 1, 2, \ldots, n)$. Hence

$$\bigcap_{i=1}^{n} g_i A \supseteq \bigcap_{i=1}^{n} y_i x_{\alpha_i}^{-1} x_{\alpha_i} A_0 = \bigcap_{i=1}^{n} y_i A_0.$$

Since A_0 is a P.P. set in G_0, A is a P.P. set in G. Similarly, $A^c = \bigcup_{\alpha\in\Lambda} x_\alpha A_0^c$ is a P.P. set in G, where $A_0^c = G_0 \sim A_0$.

□

LEMMA 4.2.3. *If $\mathcal{U}$ is a subspace, then $\mathcal{U} \supseteq H_s$ where $H_s = span\{_x f - f : x \in G,\ f \in L^\infty(G)$ is a simple function$\}$.*

PROOF: It suffices to show that for each measurable subset E of G and $x \in G$, $_x 1_E - 1_E \in \mathcal{U}$. Assume that G is infinite. By Lemma 4.2.2, we can find a subset A in G such that both A and A^c are P.P. sets. Put $E_A = (E \cap A) \cup A^c$, then E_A is a P.P. set. By Lemma 4.2.1, $LIM(S_{1_{E_A}}) \neq \phi$, $LIM(S_{1_{A^c}}) \neq \phi$. Put $\xi_1 = {_x 1_{E_A}} - 1_{E_A}$ and $\xi_2 = {_x 1_{A^c}} - 1_{A^c}$, then $S_{\xi_1} \subseteq S_{1_{E_A}}$ and $S_{\xi_2} \subseteq S_{1_{A^c}}$.

Hence $LIM(S_{\xi_i}) \neq \phi$ $(i = 1,2)$. Also $\xi_1, \xi_2 \in \mathcal{A}_0$, hence $m(\xi_i) = 0$ for all $m \in LIM(S_{\xi_i})$ $(i = 1,2)$, i.e. $\xi_1, \xi_2 \in \mathcal{U}$. Since $\mathcal{U}$ is a subspace and

$$\xi_1 - \xi_2 = ({}_x1_{E_A} - 1_{E_A}) - ({}_x1_{A^c} - 1_{A^c}) = {}_x1_{E\cap A} - 1_{E\cap A},$$

${}_x1_{E\cap A} - 1_{E\cap A} \in \mathcal{U}$. Similarly, ${}_x1_{E\cap A^c} - 1_{E\cap A^c} \in \mathcal{U}$. Note that

$$({}_x1_{E\cap A} - 1_{E\cap A}) + ({}_x1_{E\cap A^c} - 1_{E\cap A^c}) = {}_x1_E - 1_E.$$

Hence ${}_x1_E - 1_E \in \mathcal{U}$.

□

The following answers question (b) in the Remark following Corollary 1.4 of [25].

THEOREM 4.2.4. *If $\mathcal{U}$ is a subspace, then G is amenable.*

PROOF: By Lemma 4.2.3, $\mathcal{U} \supseteq H_s$. If G is not amenable, $\bar{H}_s = \bar{H} = L^\infty(G)$. For each $f_0 \in L^\infty(G)$, there exist $f_n \in H_s$ $(n = 1,2,\dots)$ such that $f_n \to f_0$ in $\|\cdot\|_\infty$. Let

$$H_{f_n} = \text{span}\{{}_xf_n - f_n : x \in G\} \quad (n = 0,1,2,\dots).$$

Since $f_n \in H_s \subseteq \mathcal{U}$, $LIM(S_{f_n}) \neq \phi$. So

$$1 = m_{f_n}(1 - h_{f_n}) \leq \|1 - h_{f_n}\|_\infty \quad (n = 1,2,\dots)$$

for $m_{f_n} \in LIM(S_{f_n})$ and $h_{f_n} \in H_{f_n}$. If $h_{f_0} = \sum_{i=1}^{m} \alpha_i({}_{x_i}f_0 - f_0) \in H_{f_0}$ for $\alpha_i \in \mathbb{C}$ and $x_i \in G$ $(i = 1,2,\dots,m)$, put $h_{f_n} = \sum_{i=1}^{m} \alpha_i({}_{x_i}f_n - f_n)$, then $h_{f_n} \in$

H_{f_n} $(n = 1, 2, \dots)$ and $h_{f_n} \to h_{f_0}$ in $\|\cdot\|_\infty$. Hence $\|1 - h_{f_n}\|_\infty \to \|1 - h_{f_0}\|_\infty$ i.e. $\|1 - h_{f_0}\|_\infty \geq 1$ for any $h_{f_0} \in H_{f_0}$. Since G is not amenable, by [9] Theorem 2.12, there exists $f_0 \in UCB(G)$, the set of uniformly continuous function on G, and $t_i,\ s_i \in G$ $(i = 1, 2, \dots, N)$ such that

$$\sum_{i=1}^{N} {}_{t_i}f_0 - {}_{s_i}f_0 = \sum_{i=1}^{N} ({}_{t_i}f_0 - f_0) + (f_0 - {}_{s_i}f_0) \geq 1$$

i.e. there exists $h_{f_0} \in H_{f_0}$ such that $h_{f_0} \geq 1$. Take $h^*_{f_0} = (2\|h_{f_0}\|_\infty)^{-1} h_{f_0}$ then $h^*_{f_0} \in H_{f_0}$ and

$$(2\|h_{f_0}\|_\infty)^{-1} \leq h^*_{f_0} \leq 1/2$$

so $\|1 - h^*_{f_0}\|_\infty < 1$ which is impossible.

□

Open Problem: Does the amenability of G imply that $\mathcal{U}$ is a subspace? (This is the case when G is discrete (see [25], Theorem 1.1)).

COROLLARY 4.2.5. *If G is a discrete group, the following statements are equivalent*

(a) *G is amenable,*

(b) *$\mathcal{U}$ is a subspace,*

(c) *$\mathcal{A}$ is a subspace,*

(d) *$\mathcal{A}_0 = \bar{H}$,*

(e) *$\mathcal{U}_0 = \bar{H}$ where $\mathcal{U}_0 = \{f \in \mathcal{U} : m(f) = 0$ for all $m \in LIM(S_f)\}$.*

PROOF: By Theorem 3.2.3 and Lemma 3.2.1, (a) ⇔ (c) ⇔ (d). By Theorem 4.3.4 and Theorem 1.1 of [25], (a) ⇔ (b).

It is clear that if $\mathcal{U}_0$ is a subspace, then $\mathcal{U}$ is a subspace. Hence (e) $\Rightarrow$ (b). To see that (d) $\Rightarrow$ (e), let $f \notin \bar{H}$, there exists $m \in LIM(L^\infty(G))$ such that $m(f) \neq 0$, i.e. $f \notin \mathcal{U}_0$ and $\mathcal{U}_0 \subseteq \bar{H}$. Since $LIM(L^\infty(G)) \neq \phi$ by Theorem 3.2.3, $\bar{H} = \mathcal{A}_0 \subseteq \mathcal{U}_0$. Theorefore $\mathcal{U}_0 = \bar{H}$.

□

4.3. The largest admissible subspace with a unique invariant mean.

Now we come to discuss the existence of the largest admissible subspace of $L^\infty(G)$ with a unique left invariant mean.

THEOREM 4.3.1. *There is a largest admissible subspace S_M in $L^\infty(G)$ with a unique left invariant mean if and only if G is amenable. In this case, $S_M = \bar{H} + \mathbb{C}$.*

PROOF: Suppose that such S_M exists. Note that if $\xi \in \mathcal{A}_0$ and $LIM(S_\xi) \neq \phi$, then $LIM(S_\xi)$ is a singleton. Using this fact and the same proof of Lemma 4.2.3, we have $S_M \supseteq H_s$ (see Lemma 4.2.3 for H_s). Let $m \in LIM(S_M)$, we can extend m to $\bar{S}_M$ such that $m \in LIM(\bar{S}_M)$ which is [illegible] a singleton. Hence S_M is closed and $S_M \supseteq \bar{H}_s = \bar{H}$, i.e. $S_M \supseteq \bar{H} + \mathbb{C}$. Fo[illegible] $\in G$ and $f \in L^\infty(G)$, since ${}_xf - f \in \mathcal{A}_0$, $m({}_xf - f) = 0$, i.e. $m(h) = 0$ f[illegible] $\in H$. Therefore

$$1 = m(1 - h) \leq \|1 - h\|_\infty$$

for any $h \in H$. Hence G is amenable.

Conversely, let G be amenable. Take $S_M = \bar{H} + \mathbb{C}$. [illegible] admissible subspace S with a unique left invariant mean m be given. For each $f \in S$, $f -$

$m(f) \in \bar{H}$, i.e. $S \subseteq \bar{H} + \mathbb{C}$. Indeed, if $f - m(f) \notin \bar{H}$, there exists $M \in LIM(L^\infty(G))$ such that $M(f - m(f)) \neq 0$. Note that $M|S \in LIM(S)$ and $M(f) = M|S(f) \neq m(f)$ which is impossible. For any $m \in LIM(S_M)$, since ${}_xf - f \in \mathcal{A}_0$ $(x \in G,\ f \in L^\infty(G))$, $m(h) = 0$ for any $h \in \bar{H}$. Consequently, S_M must have a unique left invariant mean.

□

CHAPTER 5

THE SIZES OF THE SETS OF INVARIANT MEANS ON $L^\infty(G)$

5.1. Introduction.

The size of $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ was first studied by Granirer [11] and Rudin [26]. They showed independently that

$$LIM(L^\infty(G)) \sim TLIM(L^\infty(G)) \neq \phi$$

if G is nondiscrete and amenable as a discrete group. Earlier Stafney obtained a similar result for second countable abelian groups in [31] (Chapter 3). Chou in [2] and Skantharajah in [30] (Proposition 1) improved this result by showing that if G has a closed normal subgroup H such that G/H is nondiscrete and amenable as a discrete group then $LIM(L^\infty(G)) \neq TLIM(L^\infty(G))$. Rosenblatt in [23] proved the following.

THEOREM (ROSENBLATT). *Let G be a σ-compact locally compact group. If G is nondiscrete and amenable as a discrete group, then there are at least $2^{\mathfrak{c}}$ mutually singular elements of $LIM(L^\infty(G))$ each of which is singular to any element of $TLIM(L^\infty(G))$. In particular,*

$$|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| \geq 2^{\mathfrak{c}}.$$

In section 5.2, we use the axiom of choice and proposition 3.4 of Rosenblatt in [22] to divide a "small"open dense subset of G into infinitely many pairwise disjoint p.p. sets. Then we apply the technique used in Chou [3] to embed a large set $\mathcal{F}_1$ into $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$. This shows that the size of $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ is large and removes the condition of σ-compactness for the above theorem of Rosenblatt (also see [19] chapter 7).

The study of the size of $LIM(L^\infty(G))$ was initiated in Banach [1], Day [6] and Granirer [12]. Chou in [5] proved that $|LIM(\ell^\infty(G))| = 2^{2^{|G|}}$ for any discrete amenable infinite group. Lau and Paterson in [16] showed that $|TLIM(L^\infty(G))| = 2^{2^{d(G)}}$, where $d(G)$ is the smallest possible cardinality of a covering of G by compact sets. Since any topologically left invariant mean is a left invariant mean, $|LIM(L^\infty(G))| \geq 2^{2^{d(G)}}$. The general problem of what the cardinality of $LIM(L^\infty(G))$ is remains open (see [19] chapter 7 and Yang [33]).

In section 5.3 we prove that for any noncompact locally compact metrizable group G, $|LIM(L^\infty(G))| = |TLIM(L^\infty(G))|$. Since $|TLIM(L^\infty(G))| = 2^{2^{d(G)}}$, this will give us the cardinality of $LIM(L^\infty(G))$ in this case. We also give examples to show that this is not true without the condition of metrizability. Actually, $|LIM(L^\infty(G))|$ can be as large as we want without changing $|TLIM(L^\infty(G))|$ for some amenable locally compact group.

Finally, we answer a problem raised in Rosenblatt [22] on whether any $\theta \in LIM(L^\infty(G))$ which is singular to every $\psi \in TLIM(L^\infty(G))$ can be supported by a small set of G, i.e. a set E with $\lambda(E^{-1}) < 1$ and $\theta(1_E) = 1$. He proved that for any compact group this is true and that for any locally compact group, if $\theta \in LIM(L^\infty(G))$ is supported on such a set E, then θ is singular to every $\psi \in TLIM(L^\infty(G))$.

5.2. The size of the set $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$.

Throughout this section G will denote a locally compact and nondiscrete group. Let G_0 be a noncompact σ-compact open and closed subgroup of G (see [21], Proposition 22.24) and let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be the set of all the left cosets of G_0 in G. Then $G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0$ is a disjoint union.

DEFINITION 5.2.1: Let $\{A_\gamma : \gamma \in \Omega\}$ be a family of λ-measurable subsets of G_0. If $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Omega$, V is an open subset in G_0 and $g_1^{(i)}, g_2^{(i)}, \ldots, g_{m_i}^{(i)} \in G_0$ $(i = 1, 2, \ldots, n)$, the set

$$F = \bigcap_{i=1}^{n} \bigcap_{k=1}^{m_i} g_k^{(i)} A_{\gamma_i} \cap V,$$

the intersection of finite elements of $\{xA_\gamma : x \in G,\ \gamma \in \Omega\}$ and an open set in G_0, is called a *(FI)-form set relative to* $\{A_\gamma : \gamma \in \Omega\}$. If for all (FI)-form sets F relative to $\{A_\gamma : \gamma \in \Omega\}$ we have $\lambda(F) > 0$, we call $\{A_\gamma : \gamma \in \Omega\}$ a *strictly positive (S.P.) family in* G_0.

LEMMA 5.2.2. *If* $\{A_\alpha : \alpha \in \Lambda\}$ *is a S.P. family in* G_0 *and the set* $A = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha$, *then* A *is a* λ*-measurable and S.P. subset of* G.

PROOF: For any compact set K of G, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup_{i=1}^{n} x_{\alpha_i} G_0$ since G_0 is open. Hence $K \cap A = \bigcup_{i=1}^{n} (x_{\alpha_i} A_{\alpha_i}) \cap K$ is λ-measurable. By (11.31) of [14], A is λ-measurable. Given an open set V in G

, there is an $\alpha_0 \in \Lambda$ with $V \cap x_{\alpha_0} G_0 \neq \phi$. Let V_0 be an open set in G_0 such that $V \supseteq x_{\alpha_0} V_0$ and let $x_1, x_2, \ldots, x_n$ be given. For each $1 \leq i \leq n$, there is a $\alpha_i \in \Lambda$ such that $y_i = x_{\alpha_0}^{-1} x_i x_{\alpha_i} \in G_0$. Hence

$$V \cap \bigcap_{i=1}^{n} x_i A = V \cap \bigcap_{i=1}^{n} \bigcup_{\alpha \in \Lambda} x_i x_\alpha A_\alpha \supseteq x_{\alpha_0} V_0 \cap \bigcap_{i=1}^{n} x_i x_{\alpha_i} A_{\alpha_i}$$
$$= x_{\alpha_0}\Big(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}\Big).$$

Also, $\lambda\Big(x_{\alpha_0}\Big(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}\Big)\Big) = \lambda\Big(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}\Big) > 0$ since $\{A_\alpha : \alpha \in \Lambda\}$ is a S.P. family in G_0. Note that $V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}$ is a subset of G_0 which is σ-compact. Hence, $V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}$ is not locally null and so $V \cap \bigcap_{i=1}^{n} x_i A$ is not a locally null set.

□

Let V_0 be an open dense subset of G_0 with $\lambda(V_0) < 1$. Then

$$V = \bigcup_{\alpha \in \Lambda} V_0^{-1} x_\alpha^{-1} = \Big(\bigcup_{\alpha \in \Lambda} x_\alpha V_0\Big)^{-1}$$

is also an open and dense subset in G. Suppose that $V = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha$, then each A_α is an open dense subset in G_0. We shall use Proposition 3.4 of [22] and the axiom of choice to divide V into infinitely many disjoint S.P. subsets as the following.

LEMMA 5.2.3. *For each $\alpha \in \Lambda$, there are subsets $A_\alpha^{(i)}$, $i = 1, 2, \ldots$ in G_0 such that $A_\alpha = \bigcup_{i=1}^{\infty} A_\alpha^{(i)}$ is a disjoint union and for each $i \geq 1$, $\{A_\alpha^{(i)} : \alpha \in \Lambda\}$ is a S.P. family in G_0.*

PROOF: Fix a $\alpha_0 \in \Lambda$. There are disjoint S.P. subsets $A_{\alpha_0}^{(0)}$ and $A_{\alpha_0}^{(1)}$ in G_0 such that $A_{\alpha_0} = A_{\alpha_0}^{(0)} \cup A_{\alpha_0}^{(1)}$ by Proposition 3.4 of [22] since G_0 is σ-compact. Suppose Λ_0 is a subset of Λ with $\alpha_0 \in \Lambda_0$. Set

$$\mathcal{S}_{\Lambda_0} = \{A_\alpha^{(0)} : \alpha \in \Lambda_0,\ A_\alpha^{(0)} \text{ and } A_\alpha^{(1)} \text{ are disjoint subsets of } A_\alpha \text{and}$$

$$A_\alpha = A_\alpha^{(0)} \cup A_\alpha^{(1)} \text{ such that } \{A_\alpha^{(i)} : \alpha \in \Lambda_0\} \quad \text{is a S.P. family } (i = 0, 1)$$

$$\text{and } A_{\alpha_0}^{(0)} \cap A_\alpha^{(1)} = \phi, \quad A_{\alpha_0}^{(1)} \cap A_\alpha^{(0)} = \phi \quad (*) \text{ for all } \alpha \in \Lambda_0\}.$$

Notice that we can have different $\mathcal{S}'_{\Lambda_0}s$ for the same Λ_0. Let $\Lambda_0 = \{\alpha_0\}$. We can see that such $\mathcal{S}_{\Lambda_0}$ exists and $\mathcal{S}_{\Lambda_0} \neq \phi$. Take a partial order in the family of all the nonempty $\mathcal{S}_{\Lambda_0}$ as the following. Put $\mathcal{S}_{\Lambda_0} \leq \mathcal{S}_{\Lambda'_0}$ if and only if $\Lambda_0 \subseteq \Lambda'_0$ and $\mathcal{S}_{\Lambda_0} \subseteq \mathcal{S}_{\Lambda'_0}$ for $\Lambda_0 \subseteq \Lambda$ and $\Lambda'_0 \subseteq \Lambda$. Then it is clear that $\leq$ is a partial order. For each chain $\{\mathcal{S}_{\Lambda_0^{(p)}} : p \in \Sigma\}$, put $\Lambda_0 = \bigcup_{p \in \Sigma} \Lambda_0^{(p)}$, then $\Lambda_0 \subseteq \Lambda$ and $\alpha_0 \in \Lambda_0$. If $\alpha \in \Lambda_0$, then there is $p \in \Sigma$ such that $\alpha \in \Lambda_0^{(p)}$. Let $A_\alpha^{(0)}$ be the same as in $\mathcal{S}_{\Lambda_0^{(p)}}$. Then $A_\alpha^{(0)}$ is well-defined since $\{\mathcal{S}_{\Lambda_0^{(p)}} : p \in \Sigma\}$ is a chain. Also it is clear that $(*)$ is satisfied. Since for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda_0$, there is $p \in \Sigma$ such that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda_0^{(p)}$, both $\{A_\alpha^{(0)} : \alpha \in \Lambda_0\}$ and $\{A_\alpha^{(1)} : \alpha \in \Lambda_0\}$ are S.P. families. Hence $\mathcal{S}_{\Lambda_0} = \{A_\alpha^{(0)} : \alpha \in \Lambda_0\}$ is an upper bound of $\{\mathcal{S}_{\Lambda_0^{(p)}} : p \in \Sigma\}$. By Zorn's Lemma, there is a maximal $\mathcal{S}_{\Lambda_0}$. we claim that $\Lambda = \Lambda_0$. If not, let $\alpha \in \Lambda \sim \Lambda_0$, then there are disjoint S.P. subsets $V_\alpha^{(0)}$ and $V_\alpha^{(1)}$ in G_0 such that $A_\alpha = V_\alpha^{(0)} \cup V_\alpha^{(1)}$. Put

$$A_\alpha^{(0)} = (V_\alpha^{(0)} \cup A_\alpha \cap A_{\alpha_0}^{(0)}) \sim A_{\alpha_0}^{(1)}$$

$$A_\alpha^{(1)} = (V_\alpha^{(1)} \cup A_\alpha \cap A_{\alpha_0}^{(1)}) \sim A_{\alpha_0}^{(0)}$$

then $A_\alpha = A_\alpha^{(0)} \cup A_\alpha^{(1)}$ is a disjoint union and

$$A_\alpha^{(0)} \cap A_{\alpha_0}^{(1)} = \phi, \quad A_\alpha^{(1)} \cap A_{\alpha_0}^{(0)} = \phi.$$

We claim that $\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family. For any (FI)-form set F_0 relative to $\{A_\beta^{(0)} : \beta \in \Lambda_0\}$ and any $g_1, g_2, \dots, g_{m_\alpha} \in G_0$, since $\{A_\beta^{(0)} : \beta \in \Lambda_0\}$ is a S.P. family and $V = \bigcap_{i=1}^{m_\alpha} g_i A_\alpha$ is an open dense subset of G_0, we have

$$\begin{aligned} 0 &< \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(0)} \cap V\Big) = \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(0)} \cap \bigcap_{\ell=1}^{m_\alpha} g_\ell A_\alpha\Big) \\ &\leq \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i (A_{\alpha_0}^{(0)} \cap A_\alpha)\Big) = \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i (A_{\alpha_0}^{(0)} \cap A_\alpha^{(0)})\Big) \\ &\leq \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_\alpha^{(0)}\Big) \end{aligned}$$

i.e. for any (FI)-form set F relative to $\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$, $\lambda(F) > 0$. Therefore $\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family. Similarly, $\{A_\beta^{(1)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family in G_0. Therefore $\mathcal{S}_{\Lambda_0 \cup \{\alpha\}} \geq \mathcal{S}_{\lambda_0}$ and $\mathcal{S}_{\Lambda_0 \cup \{\alpha\}} \neq \mathcal{A}_{\Lambda_0}$ which is a contradiction. Hence $\Lambda = \Lambda_0$.

Suppose for each $\alpha \in \Lambda$, $A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n)}$ is a disjoint union and for each $1 \leq i \leq n$, $\{A_\alpha^{(i)} : \alpha \in \Lambda\}$ is a S.P. family. Also, if $i \neq j$, $A_{\alpha_0}^{(i)} \cap A_\alpha^{(j)} = \phi$ (**) for any $\alpha \in \Lambda$. Note that for each $1 \leq i \leq n$, $A_\alpha^{(i)}$ is a S.P. set in G_0. By Proposition 3.4 of [22] again, there are S.P. sets $A_{\alpha_0}^{(n,0)}$ and $A_{\alpha_0}^{(n,1)}$ in G_0 such that $A_{\alpha_0}^{(n)} = A_{\alpha_0}^{(n,0)} \cup A_{\alpha_0}^{(n,1)}$ is a disjoint union. With the similar order and the argument as above, there is a maximal $\mathcal{S}_{\Lambda_0}$ for every subset Λ_0 of Λ with $\alpha_0 \in \Lambda_0$, where

$$\mathcal{S}_{\Lambda_0} = \{A_\alpha^{(n,0)} : \alpha \in \Lambda_0,\ A^{(n,0)} \text{ and } A_\alpha^{(n,1)} \text{ are disjoint subsets of } A_\alpha^{(n)} \text{ and}$$

$$A_\alpha^{(n)} = A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)} \quad \text{such that } \{A_\alpha^{(n,i)} : \alpha \in \Lambda_0\} \quad \text{is a S.P. family } (i = 0,1)$$

$$\text{and } A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi, \quad A_{\alpha_0}^{(n,1)} \cap A_\alpha^{(n,0)} = \phi \quad (**) \quad \}.$$

Then $\Lambda_0 = \Lambda$. Indeed, if $\alpha \in \Lambda \sim \Lambda_0$, by Proposition 3.4 of [22], there are disjoint S.P. sets $V_\alpha^{(n,0)}$ and $V_\alpha^{(n,1)}$ in G_0 such that $A_\alpha^{(n)} = V_\alpha^{(n,0)} \cup V_\alpha^{(n,1)}$. Put

$$A_\alpha^{(n,0)} = \big(V_\alpha^{(n,0)} \cup A_\alpha^{(n)} \cap A_{\alpha_0}^{(n,0)}\big) \sim A_{\alpha_0}^{(n,1)}$$

$$A_\alpha^{(n,1)} = \big(V_\alpha^{(n,1)} \cup A_\alpha^{(n)} \cap A_{\alpha_0}^{(n,1)}\big) \sim A_{\alpha_0}^{(n,0)}$$

then $A_\alpha^{(n)} = A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)}$ is a disjoint union and

$$A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi, \quad A_{\alpha_0}^{(n,1)} \cap A_\alpha^{(n,0)} = \phi.$$

We claim that $\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family. Let F_0 be a (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0\}$ and $g_1, g_2, \ldots, g_{m_\alpha} \in G_0$. Note that $V = \bigcap_{i=1}^{m_\alpha} g_i A_\alpha$ is open and $F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(n,0)}$ is a (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0\}$. We have

$$0 < \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(n,0)} \cap \bigcap_{\ell=1}^{m_\alpha} g_\ell A_\alpha\Big) \le \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i (A_{\alpha_0}^{(n,0)} \cap A_\alpha)\Big)$$

$$= \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i (A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,0)})\Big) \le \lambda\Big(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_\alpha^{(n,0)}\Big),$$

since $A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n-1)} \cup A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)}$, $A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi$ and $A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(k)} = \phi$ if $k < n$. Hence any (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ has positive measure. Therefore $\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family. Similarly, $\{A_\beta^{(n,1)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is a S.P. family. This contradicts the maximality of S_{Λ_0}. Therefore, for any $\alpha \in \Lambda$, $A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n-1)} \cup A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)}$ satisfies the property (**). The proof follows by induction.

□

LEMMA 5.2.4. *For any nondiscrete locally compact amenable group G, there are S.P. subsets E_n $(n = 1, 2, \ldots\ \ldots)$ in G such that $E_n \cap E_m = \phi$ $(n \neq m)$ and $\psi\left(1_{\bigcup_{n=1}^{\infty} E_n}\right) < 1$ for each $\psi \in TLIM(L^\infty(G))$.*

PROOF: If G is compact, there is an open dense subset V_0 in G with $\lambda(V_0) < 1$ by Proposition 2 of [11]. We can find disjoint S.P. subsets E_n of G such that $V_0 = \bigcup_{n=1}^{\infty} E_n$ by Proposition 3.4 of [22]. Since $TLIM(L^\infty(G)) = \{\lambda\}$, $\psi\left(\bigcup_{n=1}^{\infty} E_n\right) < 1$ for $\psi \in TLIM(L^\infty(G))$. For the noncompact case we use all the notations as in Lemma 5.2.3. Put $E_n = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha^{(n)}$. Then by Lemma 5.2.2 and Lemma 5.2.3 E_n is a S.P. subset in G. Since $\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{\alpha \in \Lambda} x_\alpha V_0\right)^{-1}$ with $\lambda(V_0) < 1$ (see Lemma 5.2.3 for $A_\alpha^{(n)}$ and V_0), $\psi\left(1_{\bigcup_{n=1}^{\infty} E_n}\right) = 0 < 1$ for all $\psi \in TLIM(L^\infty(G))$ by the last proposition of [11].

□

As in Chou [3], let

$$\mathcal{F}_1 = \{\theta \in \ell^\infty(\mathbb{N})^* : \theta \geq 0,\ \|\theta\| = 1 \quad \text{and} \quad \theta(f) = 0 \quad \text{if}$$
$$f \in \ell^\infty(\mathbb{N}) \quad \text{with} \quad \lim_n \ f(n) = 0\}$$

then $\beta\mathbb{N} \sim \mathbb{N} \subseteq \mathcal{F}_1$ and $|\mathcal{F}_1| = 2^c$. We are going to prove our first main result.

THEOREM 5.2.5. *Let G be a nondiscrete locally compact group which is amenable as a discrete group. Then there exists a positive mapping of $L^\infty(G)$ onto $\ell^\infty(\mathbb{N})$, say π, such that $\|\pi\| = 1$ and its conjugate π^* is a linear isometry of $\ell^\infty(\mathbb{N})^*$ into $L^\infty(G)^*$ with $\pi^*\mathcal{F}_1 \subseteq LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$. Moreover, elements of $\pi^*(\beta\mathbb{N} \sim \mathbb{N})$ are mutually singular and $\pi^*\theta$ is singular to every $\psi \in TLIM(L^\infty(G))$ for any $\theta \in \mathcal{F}_1$.*

PROOF: Let $\{E_n : n = 1, 2, \dots\}$ be the subsets of Lemma 5.2.4. Since G is amenable as a discrete group and E_n is a S.P. subset of G there is a $m_n \in LIM(L^\infty(G))$ such that $m_n(1_{E_n}) = 1$ for each n (see [2] p.48 the proof of (3) $\Rightarrow$ (4)). Define $\pi : L^\infty(G) \to \ell^\infty(\mathbb{N})$ by $\pi(f)(n) = m_n(f)$ for $f \in L^\infty(G)$ and $n \in \mathbb{N}$. Then π is linear and nonnegative. Since $\pi(1) = 1$, and for each $f \in L^\infty(G)$

$$\|\pi(f)\| = \sup_n |m_n(f)| \leq \|f\|_\infty,$$

$\|\pi\| = 1$. For each $F \in \ell^\infty(\mathbb{N})$, define $f(x) = F(n)$ if $x \in E_n$ and $f(x) = 0$ if $x \notin \bigcup_{n=1}^{\infty} E_n$. Then $f \in L^\infty(G)$ and $\pi(f)(n) = m_n(f) = m_n(f \cdot 1_{E_n}) = F(n)$ ($n \in \mathbb{N}$), i.e. $\pi(f) = F$ and $\|f\|_\infty = \|F\|_\infty$. Hence π is onto and π^* is a linear isometry.

For each $\theta \in \mathcal{F}_1$, $\pi^*\theta \in LIM(L^\infty(G))$. Indeed, given $f \in L^\infty(G)$ and $x \in G$, since for each $n \in \mathbb{N}$

$$\pi({}_xf)(n) = m_n({}_xf) = m_n(f) = \pi(f)(n),$$

i.e. $\pi({}_xf) = \pi(f)$, we have

$$\pi^*\theta({}_xf) = \theta(\pi_x f) = \theta(\pi f) = \pi^*\theta(f).$$

Hence $\pi^*\theta$ is left invariant. Since both π and θ are nonnegative, $\pi^*\theta$ is nonnegative. Also, $\pi^*\theta(1) = \theta(\pi(1)) = \theta(1) = 1$, hence $\pi^*\theta \in LIM(L^\infty(G))$. Let $E = \bigcup_{n=1}^{\infty} E_n$, then $\pi(1_E)(n) = m_n(1_E) = 1$ $(n \in \mathbb{N})$, i.e. $\pi(1_E) = 1$. Hence $\pi^*\theta(1_E) = 1$. By Lemma 5.2.4, $\pi^*\theta \notin TLIM(L^\infty(G))$. If G is not compact, then $\pi^*\theta$ is singular to any $\psi \in TLIM(L^\infty(G))$ since $\sup \widehat{\pi^*\theta} \subseteq \hat{E}$ and $\sup \hat{\psi} \subseteq \widehat{G \sim E}$ (see [22], P.35). If G is compact, since $\pi^*\theta(1_{G\sim E}) = 0$ and $\lambda(G \sim E) > 0$, by Proposition 2.4 and Lemma 2.6 of [22], $\pi^*\theta$ is singular to λ. Let $\theta_1,\ \theta_2 \in \beta\mathbb{N} \sim \mathbb{N}$ and $\theta_1 \neq \theta_2$, then $\|\theta_1 - \theta_2\| = 2$ (see [2], page 208). Hence

$$\|\widehat{\pi^*\theta_1} - \widehat{\pi^*\theta_2}\| = \|\pi^*\theta_1 - \pi^*\theta_2\| = \|\theta_1 - \theta_2\| = 2.$$

By the Hahn decomposition theorem, for the signed measure $\mu = \widehat{\pi^*\theta_1} - \widehat{\pi^*\theta_2}$, there are subsets D^+ and D^- of $\mathcal{D}$ such that $\mu \geq 0$ on D^+ and $\mu \leq 0$ on D^-. Also $\mathcal{D} = D^+ \cup D^-$, $D^+ \cap D^- = \phi$. Since $\|\mu\| = 2$, $\|\widehat{\pi^*\theta_1}\| = \|\widehat{\pi^*\theta_2}\| = 1$, and $\|\mu\| = \mu(D^+) - \mu(D^-)$, $\widehat{\pi^*\theta_1}(D^-) = 0$, $\widehat{\pi^*\theta_2}(D^+) = 0$. Hence $\pi^*\theta_1$ and $\pi^*\theta_2$ are mutually singular (see [27], p.134).

□

COROLLARY 5.2.6. *Let G be a nondiscrete locally compact group. If G is amenable as a discrete group, then there is a subset $E \subseteq LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ with $|E| \geq 2^c$ and $\|m_1 - m_2\| = 2$ for any $m_1, m_2 \in E$. In particular,*

$$|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| \geq 2^c.$$

PROOF: As in the proof of Theorem 5.2.5, let $E = \pi^*(\beta\mathbb{N} \sim \mathbb{N})$. Then for any $\theta_1,\ \theta_2 \in \beta\mathbb{N} \sim \mathbb{N}$, $\|\pi^*\theta_1 - \pi^*\theta_2\| = \|\theta_1 - \theta_2\| = 2$. Since $|\beta\mathbb{N} \sim \mathbb{N}| = 2^c$, $|E| \geq 2^c$.

□

Remark. 1. Corollary 5.2.6 removes the condition of σ-compact for Corollary (7.20) of [19].

2. Let V_0 be an open dense subset of G_0 with $\lambda(V_0) < 1$. Then $V = \bigcup_{\alpha\in\Lambda} x_\alpha V_0$ is an open dense subset of G. Since V_0 can be divided into disjoint S.P. subsets $V_0^{(0)}$ and $V_0^{(1)}$, V can be divided into disjoint S.P. subsets $V^{(0)} = \bigcup_{\alpha\in\Lambda} x_\alpha V_0^{(0)}$ and $V^{(1)} = \bigcup_{\alpha\in\Lambda} x_\alpha V_0^{(1)}$, and so on (see Lemma 5.2.2). Therefore we can remove the condition of σ-compactness for Rosenblatt's theorem of Proposition 3.5 of [22].

5.3. The size of the set $LIM(L^\infty(G))$ for a noncompact metrizable locally compact group.

By comparing $|LIM(L^\infty(G))|$ with $|TLIM(L^\infty(G))|$ for a metrizable noncompact locally compact group, we obtain the cardinality of $LIM(L^\infty(G))$ as the following (see [19], Chapter 7).

THEOREM 5.3.1. *If G is a metrizable noncompact locally compact amenable group, then $|LIM(L^\infty(G))| = |TLIM(L^\infty(G))| = 2^{2^{d(G)}}$, where $d(G)$ is the smallest possible cardinality for a covering of G by compact subsets.*

PROOF: Let G_0 be an open and closed σ-compact subgroup of G (see [21], Proposition 22.24) and let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be all the left cosets of G_0 in G. Since G_0 is σ-compact, we can find compact subsets K_n of G_0 such that $K_n \subseteq K_{n+1}$, $K_n \neq K_{n+1}$ $(n = 1, 2, \dots)$ and $G_0 = \bigcup_{n=1}^{\infty} K_n$. Let $E_n = K_n \sim K_{n-1}$ $(n = 1, 2, \dots)$, where we assume that $K_0 = \phi$. Then $E_n \cap E_m = \phi$ if $n \neq m$, E_n is λ-measurable and $\bar{E}_n$ is compact $(n = 1, 2, \dots)$. Since $G_0 = \bigcup_{n=1}^{\infty} E_n$,

$$G = \bigcup_{\alpha\in\Lambda} x_\alpha G_0 = \bigcup_{\alpha\in\Lambda} \bigcup_{n=1}^{\infty} x_\alpha E_n = \bigcup_{(n,\alpha)\in\mathbb{N}\times\Lambda} x_\alpha E_n$$

and

$$x_\alpha E_n \cap x_{\alpha'} E_{n'} = \phi \quad \text{if} \quad (n, \alpha) \neq (n', \alpha').$$

We first show that $d(G) = |\mathbb{N} \times \Lambda|$. Since $\{x_\alpha \bar{E}_n : (n, \alpha) \in \mathbb{N} \times \Lambda\}$ is a compact cover of G, $d(G) \leq |\mathbb{N} \times \Lambda|$. To prove that $d(G) \geq |\mathbb{N} \times \Lambda|$, let $\mathcal{D}$ be a compact cover of G with $|\mathcal{D}| = d(G)$ and let

$$\mathcal{D}_n = \{C \cap (x_{\alpha_i} G_0) : C \in \mathcal{D},\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda \quad \text{with} \quad C \subseteq \bigcup_{i=1}^{n} x_{\alpha_i} G_0$$
$$\text{and} \qquad i = 1, 2, \ldots, n\}.$$

Note that the mapping $C \cap (x_{\alpha_i} G_0) \to (C, x_1, x_2, \ldots, x_n)$ from $\mathcal{D}_n$ to a subset of $\mathcal{D} \times \{0,1\}^n$ is $1-1$, where $x_i = 1,\ x_j = 0\ (j \neq i)$. Hence $|\mathcal{D}_n| \leq |\mathcal{D}| = d(G)$. Since for each $C \in \mathcal{D}$, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $C \subseteq \bigcup_{i=1}^{n} x_{\alpha_i} G_0$, $\bigcup_{n=1}^{\infty} \mathcal{D}_n$ is a compact cover of G and $\left|\bigcup_{n=1}^{\infty} \mathcal{D}_n\right| \leq |\mathcal{D}| = d(G)$. Therefore we can assume that for each $C \in \mathcal{D}$, there is a $\alpha \in \Lambda$ such that $C \subseteq x_\alpha G_0$. For each $\alpha \in \Lambda$, there is $C_\alpha \in \mathcal{D}$ with $C_\alpha \subseteq x_\alpha G_0$. So the mapping $\alpha \to C_\alpha$ is $1-1$ from Λ to a subset of $\mathcal{D}$. Hence $|\mathbb{N} \times \Lambda| = |\Lambda| \leq |\mathcal{D}| = d(G)$.

Since $|LIM(L^\infty(G))| \geq 2^{2^{d(G)}}$ (see [19], p.274) and $LIM(L^\infty(G)) \subseteq L^\infty(G)^*$, to show that $|LIM(L^\infty(G))| = 2^{2^{d(G)}}$, it suffices to show that $|L^\infty(G)^*| \leq 2^{2^{d(G)}}$.

For any subset E of G, let $CB(E)$ be the set of all continuous functions on E. For each $(n, \alpha) \in \mathbb{N} \times \Lambda$, since $x_\alpha \bar{E}_n$ is compact and metrizable, $x_\alpha \bar{E}_n$ is separable and $|CB(x_\alpha \bar{E}_n)| \leq c$. Hence $|\mathcal{F}| \leq c|\mathbb{N} \times \Lambda| = c \cdot d(G)$, where $\mathcal{F} = \bigcup_{(n,\alpha) \in \mathbb{N} \times \Lambda} CB(x_\alpha E_n)$.

Let $f \in L^\infty(G)$ and $(n, \alpha) \in \mathbb{N} \times \Lambda$. By Lusin's theorem (see [27], P.55), for each $k \in \mathbb{N}$, there is $f_{(k,n,\alpha)} \in \mathcal{F}$ such that

$$\lambda\{x \in x_\alpha E_n : f_{(k,n,\alpha)}(x) \neq f(x)\} < \frac{1}{k}.$$

If g is a λ-measurable function on G such that $F = \{x \in G,\ f(x) \neq g(x)\}$ is locally null, then $\lambda(F \cap x_\alpha E_n) = 0$ for any $(n,\alpha) \in \mathbb{N} \times \Lambda$, i.e.

$$\lambda\{x \in x_\alpha E_n : f_{(k,n,\alpha)}(x) \neq g(x)\} < \frac{1}{k}$$

for $(k,n,\alpha) \in \mathbb{N} \times \mathbb{N} \times \Lambda$. So $f_{(k,n,\alpha)}$ is well-defined. Let the mapping ϕ : $L^\infty(G) \to \mathcal{F}^{\mathbb{N}\times\mathbb{N}\times\Lambda}$ be defined by

$$\phi(f) = (f_{(k,n,\alpha)})_{(k,n,\alpha)\in\mathbb{N}\times\mathbb{N}\times\Lambda}.$$

Then ϕ is a $1-1$ mapping from $L^\infty(G)$ to a subset of $\mathcal{F}^{\mathbb{N}\times\mathbb{N}\times\Lambda}$. Indeed, let $g \in L^\infty(G)$ with $f \neq g$. Then there is $\alpha \in \Lambda$ such that $f \neq g$ on $x_\alpha G_0$. Since $x_\alpha G_0 = \bigcup_{n=1}^{\infty} x_\alpha E_n$, there is $n \in \mathbb{N}$ such that $f \neq g$ on $x_\alpha E_n$. Hence there is $k \in \mathbb{N}$ such that

$$\lambda\{x \in x_\alpha E_n : f(x) \neq g(x)\} > \frac{2}{k}$$

so $f_{(k,n,\alpha)} \neq g_{(k,n,\alpha)}$ by the definition of $f_{(k,n,\alpha)}$ and $g_{(k,n,\alpha)}$, i.e. $\phi(f) \neq \phi(g)$. Hence

$$|L^\infty(G)| \leq |\mathcal{F}^{\mathbb{N}\times\mathbb{N}\times\Lambda}| \leq (cd(G))^{d(G)}$$

since $2 \leq cd(G) \leq 2^{d(G)}$, $(cd(G))^{d(G)} = 2^{d(G)}$. Therefore

$$|L^\infty(G)| \leq 2^{d(G)} \quad \text{and} \quad |L^\infty(G)^*| \leq 2^{2^{d(G)}}.$$

$\square$

COROLLARY 5.3.2. *Let G be a σ-compact metrizable locally compact group. If G is nondiscrete and amenable as a discrete group, then*

$$|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| = 2^{\mathfrak{c}}.$$

PROOF: By Corollary 5.2.6, $|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| \geq 2^c$. By Theorem 5.3.1, $|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| \leq 2^c$. Hence $|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| = 2^c$.

□

As in the proof of Theorem 5.3.1, we have the following.

THEOREM 5.3.3. *Let G be a locally compact amenable group. If G is metrizable, then $|LIM(CB(G))| = 1$ when G is compact and $|LIM(CB(G))| = 2^{2^{d(G)}}$ when G is not compact,where $d(G)$ is the smallest possible cardinality for a covering of G by compact sets.*

Unfortunately, Theorem 5.3.1 does not hold without the metrizability.

THEOREM 5.3.4. *For any cardinal numbers η_1 and η_2, if η_2 is infinite, then there is a locally compact group G such that $|LIM(L^\infty(G))| \geq \eta_1$ and $|TLIM(L^\infty(G))| = 2^{2^{\eta_2}}$. Moreover, there is a compact group G with $|LIM(L^\infty(G))| =$ η_1.*

PROOF: Let S be a compact nondiscrete abelian group and let A and B be S.P. subsets in S such that $\lambda(A) < 1$, $\lambda(B) < 1$ and $A \cap B = \phi$ (see [11], Proposition 2 and [22], Proposition 3.4). Let $G_0 = \prod_{\gamma \in \eta_1} S_\gamma$ where $S_\gamma = S$ for any $\gamma \in \eta_1$. Take a discrete abelian group U with $|U| = \eta_2$. Let $G = U \times G_0$. Then G is a nondiscrete abelian group. Note that G_0 is an open and closed subgroup of G and $\{uG_0 : u \in U\}$ is the set of all cosets of G_0 in G. For each finite subset Δ of η_1 and $\beta \in \eta_1 \sim \Delta$, let $E_{(\Delta,\beta)} = \bigcup_{u \in U} u\Big(\prod_{\gamma \in \eta_1} E_\gamma\Big)$, where

$E_\gamma = B$ if $\gamma = \beta$, $E_\gamma = A$ if $\gamma \in \Delta$ and $E_\gamma = S$ if $\gamma \notin \Delta \cup \{\beta\}$. Since $\prod_{\gamma \in \eta_1} E_\gamma$ is a S.P. subset in G_0 (see [14], 13.22) $E_{(\Delta,\beta)}$ is a S.P. set in G by Lemma 5.2.2. Note that for any $\beta \in \eta_1$, finite subsets Δ_i of $\eta_1 \sim \{\beta\}$ and $x_i \in G$ $(i = 1, 2, \ldots, n)$, $\bigcap_{i=1}^{n} x_i E_{(\Delta_i,\beta)} \supseteq \bigcap_{i=1}^{n} x_i E_{(\Delta,\beta)}$ is not locally null since $E_{(\Delta,\beta)}$ is a S.P. subset, where $\Delta = \bigcup_{i=1}^{n} \Delta_i$. Also, the maximal ideal space $\mathcal{D}$ of G is compact (see the beginning of section 5.2 and [22], p.35). Hence the set

$$D_\beta = \cap\{\widehat{xE}_{(\Delta,\beta)} : \Delta \text{ is a finite subset of } \eta_1 \sim \{\beta\},\ x \in G\}$$

is a nonempty left invariant and closed subset of $\mathcal{D}$. By Proposition 3.4 of [23], there is a left invariant probability measure μ_β on D_β. Hence there is $m_\beta \in LIM(L^\infty(G))$ such that $\hat{m}_\beta = \mu_\beta$ (see the beginning of section 5.2). If $\beta,\ \beta' \in \eta_1$ with $\beta \neq \beta'$, let $\Delta = \{\beta'\},\ \Delta' = \{\beta\}$. Then $E_{(\Delta,\beta)} \cap E_{(\Delta',\beta')} = \phi$ by the definition of $E_{(\Delta,\beta)}$. Hence $D_\beta \cap D_{\beta'} = \phi$ and $m_\beta \neq m_{\beta'}$. Therefore $|LIM(L^\infty(G))| \geq \eta_1$. Also, as in the proof of Theorem 5.3.1, $|U| = d(G)$. Hence $|TLIM(L^\infty(G))| = 2^{2^{d(G)}}$ where $d(G) = \inf\{|\mathcal{D}| : \mathcal{D}$ is a compact cover of $G\}$.

If we take U such that $|U| = 1$ or $G = G_0$, then G is compact and $|LIM(L^\infty(G))| \geq \eta_1$.

□

5.4. An invariant mean which is singular to every element of $TLIM(L^\infty(G))$.

Let $f \in L^\infty(G)$ and $I(f)$ denote the smallest closed left invariant ideal containing f. In [22] Rosenblatt showed that if a subset E of G satisfies

$\lambda(E^{-1}) < 1$, then any $m \in LIM(L^\infty(G))$ with $\ker m \supseteq I(1_{G\sim E})$ is singular to every $\psi \in TLIM(L^\infty(G))$. He asked if the converse is true and he proved that it is for a compact group. Our Theorem 5.4.2 shows that for a class of groups it is not the case. We need a lemma first.

LEMMA 5.4.1. *Let G be a locally compact noncompact group and let G_0 be an open and closed compact subgroup of G. If $\{x_\alpha G_0 : \alpha \in \Lambda\}$ is the set of all left cosets of G_0 in G and V_0 is an open dense subset of G_0, then $\bigcap_{i=1}^n x_i V$ is not locally null and $\lambda\left(x_\alpha G_0 \cap \bigcap_{i=1}^n x_i V\right) \geq \varepsilon_n$ for any $x_1, x_2 \cdots x_n \in G$ and $\alpha \in \Lambda$, where $V = \bigcup_{\alpha \in \Lambda} x_\alpha V_0$ and $\varepsilon_n > 0$ depends on n only.*

PROOF: For each n, the function $\lambda\left(\bigcap_{i=1}^n x_i V_0\right)$ of $(x_1, x_2, \ldots, x_n)$ on the compact space G_0^n is continuous. Since $\lambda\left(\bigcap_{i=1}^n x_i V_0\right) > 0$ for any $(x_1, x_2, \ldots, x_n) \in G_0^n$ and G_0^n is compact, there is $\varepsilon_n > 0$ such that $\lambda\left(\bigcap_{i=1}^n x_i V_0\right) \geq \varepsilon_n$ for any $(x_1, x_2, \ldots, x_n) \in G_0^n$. If $\alpha \in \Lambda$, for each $1 \leq i \leq n$, there is $\alpha_i \in \Lambda$ such that $y_i = x_\alpha^{-1} x_i x_{\alpha_i} \in G_0$. Hence

$$x_\alpha G_0 \cap \bigcap_{i=1}^n x_i V \supseteq x_\alpha G_0 \cap \bigcap_{i=1}^n (x_i x_{\alpha_i} V_0) = x_\alpha \left(\bigcap_{i=1}^n y_i V_0\right).$$

This implies that

$$\lambda\left(x_\alpha G_0 \cap \bigcap_{i=1}^n x_i V\right) \geq \lambda\left(\bigcap_{i=1}^n y_i V_0\right) \geq \varepsilon_n.$$

Also, therefore $\bigcap_{i=1}^n x_i V$ is not locally null.

□

THEOREM 5.4.2. *If G is an abelian locally compact noncompact group which contains an open and closed compact subgroup G_0, then there is an $m \in LIM(L^\infty(G))$ such that m is singular to every $\psi \in TLIM(L^\infty(G))$ and $m(1_E) = 0$ for any subset E of G with $\lambda(E^{-1}) < \infty$. In particular,* $\ker m$ *does not contain $I(1_{G\sim E})$ for any subset E with $\lambda(E^{-1}) < 1$.*

PROOF: For each n, we can find an open dense subset V_{n_0} in G_0 such that $\lambda(V_{n_0}) < \frac{1}{n}$ (see [11], Proposition 2). Put $V_n = \bigcup_{\alpha\in\Lambda} x_\alpha V_{n_0}$, then V_n is an open dense subset in G (see Lemma 5.4.1 for α, Λ and x_α).

For each $x \in G$, let $x = x_{\alpha_0} g_0$ for some $\alpha_0 \in \Lambda$ and $g_0 \in G_0$, then

$$\begin{aligned} 1_{G_0} * 1_{V_n}(x) &= \int_G 1_{G_0}(t) 1_{V_n}(t^{-1}x)dt = \lambda(G_0 \cap xV_n^{-1}) = \lambda(g_0 V_{n_0}^{-1}) \\ &= \lambda(V_{n_0}) < \frac{1}{n} \end{aligned}$$

since G is abelian. Hence for any $\psi \in TLIM(L^\infty(G))$,

$$\psi(1_{V_n}) = \psi(1_{G_0} * 1_{V_n}) \leq \frac{1}{n}.$$

Let I_n be the smallest left invariant ideal of $L^\infty(G)$ containing $1_{G\sim V_n}$ and all the functions of the form 1_A for the subsets A of G with $\lambda(A) < \infty$. Then it is clear that

$$I_n = \operatorname{span}\{f \cdot_x 1_{G\sim V_n} + g \cdot 1_A : f,\ g \in L^\infty(G),\ x \in G,\ A \subseteq G \quad \text{with} \quad \lambda(A) < \infty\}.$$

Then $\bar{I}_n \neq L^\infty(G)$. Indeed, for any $f \in I_n$, there are $g_1, g_2, \cdots, g_m \in G$ and a subset A in G such that $\lambda(A) < \infty$ and

$$|f| \leq \|f\|_\infty \Big(\sum_{i=1}^{m} 1_{G\sim g_i V_n} + 1_A \Big).$$

Let

$$E = \Big(G \sim \Big(\bigcup_{i=1}^{m} G \sim g_i V_n\Big)\Big) \sim A = \bigcap_{i=1}^{m} g_i V_n \sim A.$$

Since there is $\varepsilon_m > 0$ such that $\lambda\Big(x_\alpha G_0 \cap \bigcap_{i=1}^{m} g_i V_n\Big) \geq \varepsilon_m$ for any $\alpha \in \Lambda$ by Lemma 5.4.1, and since G is not compact, E is not locally null. Since $f = 0$ on E, $\|f-1\|_\infty \geq 1$, i.e. $1 \notin \bar{I}_n$. By Proposition 2.5 of [22], there is a $m_n \in LIM(L^\infty(G))$ with $\ker m_n \supseteq \bar{I}_n$. We can assume that $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$. Let m be a w^*-cluster point of net $\{m_n\}$. Then there is a subnet $\{m_{n_\beta}\}$ of $\{m_n\}$ such that

$$m = \lim_\beta m_{n_\beta} \quad \text{in} \quad w^*\text{-topology.}$$

For each n, there is q_n such that $n_\beta \geq n$ for all $\beta \geq q_n$. Hence

$$m(G \sim V_n) = \lim_\beta m_{n_\beta}(G \sim V_n) \leq \lim_{\beta \geq q_n} m_{n_\beta}(G \sim V_n) \leq \lim_{\beta \geq q_n} m_{n_\beta}(G \sim V_{n_\beta}) = 0$$

i.e. $m(V_n) = 1$ and $\operatorname{supp}\hat{m} \subseteq \hat{V}_n$. If $\psi \in TLIM$, then

$$\hat{\psi}(\operatorname{supp}\hat{m}) \leq \hat{\psi}(\hat{V}_n) = \psi(V_n) \leq \frac{1}{n}$$

for any n. So $\hat{\psi}(\operatorname{supp}\hat{m}) = 0$ and m is singular to ψ. If A is a subset of G with $\lambda(A^{-1}) < \infty$, then $\lambda(A) < \infty$. So

$$m(A) = \lim_\beta m_{n_\beta}(A) = 0.$$

Therefore $\ker m$ can not contain $I(1_{G\sim A})$.

□

CHAPTER 6

LEFT INVARIANT MEANS ON $CB(G)$

6.1. Introduction.

Liu and van Rooji in [18] showed that if G is noncompact, nondiscrete and amenable as a discrete group, then $LIM(CB(G)) \neq TLIM(CB(G))$. Rosenblatt in [24] showed the following:

THEOREM(ROSENBLATT). *Assume that G is nondiscrete noncompact σ-compact and amenable as a discrete group. Then there exists $f \in CB(G)$ with $0 \leq f \leq 1$ and $\theta \in LIM(CB(G))$ such that $\theta(f) = 1$ and $\psi(f) = 0$ for any $\psi \in TLIM(CB(G))$.*

And, he asked if discrete amenability is necessary in the theorem. Chou in [4] speculated that if G is noncompact, nondiscrete and amenable, then $LIM(CB(G)) \neq TLIM(CB(G))$ and he also showed in [2] that there exist compact groups which are not amenable as discrete groups such that $LIM(L^\infty(G)) \neq TLIM(L^\infty(G))$. Since there are amenable groups which are not amenable as discrete groups, our Theorem 6.2.1 answers the problem of Rosenblatt above negatively. This also confirms Chou's conjecture in that case.

Let G be a locally compact group such that G_d is amenable. By Day's fixed-point theorem (see[7]), any $\theta \in LIM(UCB(G))$ can be extended to an element of $LIM(L^\infty(G))$. Since θ is topologically left invariant on $UCB(G)$, the problem is whether the extension is also topologically left invariant on

$L^\infty(G)$. A group G is said to be an [IN]-group if it has a compact neighborhood which is invariant under all inner automorphisms. Yang in [34] showed that the extension of any $\theta \in LIM(UCB(G))$ to the space $UCB(G)_\ell$ is unique for any $[IN]$-group G, where $UCB(G)_\ell$ is the space of all left uniformly continuous bounded functions on G. Since we can extend θ to $TLIM(UCB(G)_\ell)$ by $\hat{\theta}(f) = \theta(\varphi * f)$, $(f \in UCB(G)_\ell)$, where $\varphi \in P(G)$ is fixed, the extension of θ to $LIM(UCB(G)_\ell)$ is also topologically left invariant ([13] p27).

Since $LIM(UCB(G))$ is large when G is amenable and non-compact, to find the size of the set $LIM(CB(G)) \sim TLIM(CB(G))$, it is natural to ask the following question (See Rosenblatt [24] P320):

Let G be nondiscrete σ-compact noncompact and amenable as a discrete group and let $\theta \in LIM(UCB(G))$. Does there exist $\psi \in LIM(CB(G)) \sim TLIM(CB(G))$ such that $\psi = \theta$ on $UCB(G)$?

In section 6.3 we show that we can extend θ to a left invariant functional on $CB(G)$ such that it is not "topologically left invariant". We also prove that any $\theta \in LIM(UCB(G))$ can be extended to an element of $LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$. Hence $|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| = |LIM(L^\infty(G))|$, where G is any nondiscrete locally compact group which is amenable as a discrete group.

6.2. Discrete Amenability and the set $LIM(CB(G)) \sim TLIM(CB(G))$.

Let G be a nondiscrete noncompact σ-compact group which is amenable as a discrete group. The function $f \in CB(G)$ in the theorem of Rosenblatt in section 6.1 can be taken as a permanently near one function with the following property:

(p) for any $\varepsilon > 0$ there are $\varphi \in P(G)$ and $f_M \in CB(G)$ such that $\|\varphi * f_M\|_\infty < \varepsilon$ and the support of $f - f_M$ is compact.

Since there are amenable groups which are not amenable as discrete groups and a group is amenable as a discrete group if and only if any of its subgroup is amenable as a discrete group, the following theorem implies that the condition of the discrete amenability in Rosenblatt's theorem of section 6.1 is not necessary.

THEOREM 6.2.1. *Let G_1 be a noncompact σ-compact nondiscrete group which is amenable as a discrete group. If G_2 is any amenable locally compact group and $G = G_1 \times G_2$, then there exists $F \in CB(G)$ with $0 \le F \le 1$ and $\theta \in LIM(CB(G))$ such that $\theta(F) = 1$ and $\psi(F) = 0$ for any $\psi \in TLIM(CB(G))$.*

PROOF: Suppose $f_1 \in CB(G_1)$ is a permanently near one function with property (p) as in Rosenblatt's theorem of section 6.1. Let $F_{f_1} \in CB(G)$ be defined by $F_{f_1}(x, y) = f_1(x)$ for any $(x, y) \in G$ and let

$$H = \operatorname{span}\left\{ {}_{(x,y)}F - F : (x, y) \in G, F \in CB(G) \right\}.$$

Note that for any $(x, y) \in G$ and $F \in CB(G)$,

$$\begin{aligned} {}_{(x,y)}F - F &= \left({}_{(x,y)}F - {}_{(e,y)}F\right) + \left({}_{(e,y)}F - F\right) \\ &= \left[{}_{(x,e)}\left({}_{(e,y)}F\right) - \left({}_{(e,y)}F\right)\right] + \left[{}_{(e,y)}F - F\right] \end{aligned}$$

where e is the group unit of G_1 or G_2. Hence for any $h \in H$, there are $x_i \in G_1, y_i \in G_2$, constant $a_i, F_i \in CB(G)$, and $\bar{F}_i \in CB(G)$ $(i = 1,2,\dots,n)$ such that $h = h_1 + h_2$, where

$$h_1 = \sum_{i=1}^{n} a_i\big({}_{(x_i,e)}F_i - F_i\big), \quad h_2 = \sum_{i=1}^{n} a_i\big({}_{(e,y_i)}\bar{F}_i - \bar{F}_i\big).$$

Then $\|F_{f_1} - h\|_\infty \geq 1$. Indeed, for any $\varepsilon > 0$, by the Fϕlner condition argument, there are $x'_k \in G_1, \lambda_k > 0$ $(k = 1,2,\dots,N)$ with $\sum_{k=1}^{N} \lambda_k = 1$ and

$$\|\sum_{k=1}^{N} \lambda_{k(x'_k,e)} h_1\|_\infty < \varepsilon.$$

Hence

$$\|F_{f_1} - h\|_\infty \geq \|\sum_{k=1}^{N} \lambda_{k(x'_k,e)}(F_{f_1} - h)\|_\infty \geq \|\sum_{k=1}^{N} \lambda_{k(x'_k,e)} F_{f_1} - \sum_{k=1}^{N} \lambda_{k(x'_k,e)} h_2\|_\infty - \varepsilon. \tag{*}$$

Note that

$$\sum_{k=1}^{N} \lambda_{k(x'_k,e)} h_2 = \sum_{i=1}^{n} a_i \Big[{}_{(e,y_i)}\big(\sum_{k=1}^{N} \lambda_{k(x'_k,e)} \bar{F}_i\big) - \big(\sum_{k=1}^{N} \lambda_{k(x'_k,e)} \bar{F}_i\big)\Big] = \sum_{i=1}^{n} a_i\big({}_{(e,y_i)}T_i - T_i\big), \tag{**}$$

where $T_i = \sum_{k=1}^{N} \lambda_{k(x'_k,e)} \bar{F}_i$. Since f_1 is a permanently near one function and G_2 is amenable, there is an $x_0 \in G_1$ such that

$$|1 - {}_{x'_k}f_1(x_0)| < \varepsilon \quad (k = 1,2,\dots,N) \tag{***}$$

and $m_2 \in LIM(CB(G_2))$. For any $F \in CB(G)$, let $F^{(x_0)} \in CB(G_2)$ be defined by $F^{(x_0)}(y) = F(x_0, y)$ for any $y \in G_2$. Then by $(*), (**)$ and $(***)$

$$\|F_{f_1} - h\|_\infty \geq \|\sum_{k=1}^{N} \lambda_{k(x'_k,e)} F_{f_1} - \sum_{i=1}^{n} a_i\big({}_{(e,y_i)}T_i - T_i\big)\|_\infty - \varepsilon$$

$$\geq \|\Big(\sum_{k=1}^{N} \lambda_{k(x'_k,e)}, F_{f_1}\Big)^{(x_0)} - \Big(\sum_{i-1}^{n} a_i(_{(e,y_i)}T_i - T_i)\Big)^{(x_0)}\|_\infty - \varepsilon$$

$$= \|\sum_{k=1}^{N} \lambda_{k x'_k} f_1(x_0) - \sum_{i=1}^{n} a_i\Big(_{y_i}(T_i)^{(x_0)} - (T_i)^{(x_0)}\Big)\|_\infty - \varepsilon$$

$$\geq \|1 - \sum_{i=1}^{n} a_i\Big(_{y_i}(T_i)^{(x_0)} - (T_i)^{(x_0)}\Big)\|_\infty - 2\varepsilon$$

$$\geq m_2\Big(1 - \sum_{i=1}^{n} a_i\big(_{y_i}(T_i)^{(x_0)} - (T_i)^{(x_0)}\big)\Big) - 2\varepsilon = 1 - 2\varepsilon.$$

Therefore $\|F_{f_1} - h\|_\infty \geq 1$ for any $h \in H$.

Let $\theta \in LIM(CB(G))$ such that $\theta(F_{f_1}) = 1$ (see section 2.2.5). Notice that for any $\psi \in TLIM(CB(G))$, $\psi(F_{f_1}) = 0$. Indeed, for any $\varepsilon > 0$, let $f_M \in CB(G_1)$ and $\varphi_1 \in P(G_1)$ such that the support of $f_M - f_1$ is compact and $\|\varphi_1 * f_M\|_\infty < \varepsilon$. Take an $\varphi_2 \in P(G_2)$. Then φ defined by $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ for $(x,y) \in G$ is an element of $P(G)$. Also, for any $(x,y) \in G$,

$$|\varphi * F_{f_M}(x,y)| = |\int_G \varphi_1(t_1)\varphi_2(t_2) f_M({t_1}^{-1}x)dt_1dt_2| = |\varphi_1 * f_M(x)| < \varepsilon,$$

where $F_{f_M} \in G$ is defined by $F_{f_M}(x,y) = f_M(x)$ for $(x,y) \in G$. So $\|\varphi * F_{f_M}\|_\infty < \varepsilon$. Since the support of $f_M - f_1$ is compact in G_1, the support of $F_{f_M} - F_{f_1}$ is contained in $C \times G_2$ for some compact subset C of G_1. Also, G_1 is not compact, hence $m(F_{f_M}) = m(F_{f_1})$ for any $m \in LIM(CB(G))$. Therefore

$$\psi(F_{f_1}) = \psi(F_{f_M}) = \psi(\varphi * F_{f_M}) < \varepsilon$$

for any $\psi \in TLIM(CB(G))$ i.e. $\psi(F_{f_1}) = 0$.

□

COROLLARY 6.2.2. *Let G be as in Theorem 6.2.1. Then*

$$LIM(CB(G)) \neq TLIM(CB(G))$$

Remark. This confirms Chou's conjecture in this case.

6.3. Extension of Invariant Means on $UCB(G)$.

We start with a lemma. It is a key lemma for our main theorem in this section.

LEMMA 6.3.1. *Let G be a locally compact group and let $f \in CB(G)$ be a permanently near one function with $0 \leq f \leq 1$. If G_d is amenable and for any $x_1, x_2, \ldots, x_n \in G$ and $\varepsilon > 0$, there are $\varphi \in P(G)$ and $f_M \in CB(G)$ such that $f - f_M$ has a compact support and $\|\varphi_{x_i} * f_M\|_\infty < \varepsilon$ $(i = 1, 2, \ldots, n)$, then*

$$\inf\left\{\|a_f - u\|_\infty : u \in UCB(G), a_f \in \|\cdot\|_\infty\text{-closed convex of}\{_x f : x \in G\}\right\} = \varepsilon_0 \geq 1/2.$$

PROOF: Assume that $\varepsilon_0 < 1/2$. Then there are $\lambda_i > 0$, $x_i \in G$ $(i = 1, 2, \ldots, n)$ and $u \in UCB(G)$ such that $\sum_{i=1}^n \lambda_i = 1$ and $a_f = \sum_{i=1}^n \lambda_{i\,x_i} f$ with $\|a_f - u\|_\infty < 1/2$. By Theorem 1.6 of [24], there is an $m \in LIM(CB(G))$ with $m(f) = 1$. Therefore $m(a_f) = 1$ and

$$m(u) = m(u - a_f) + m(a_f) \geq 1 - \|u - a_f\|_\infty > 1/2 \quad (*).$$

On the other hand, for any $\varepsilon > 0$, there is an $\varphi \in P(G)$ such that $\|\varphi_{x_i} * f_M\|_\infty < \varepsilon b^{-1}$, where $b = \max_{1\leq i\leq n} \Delta(x_i)$. Also, $\varphi *_{x_i} f = \Delta(x_i)\varphi_{x_i} * f$ ($i = 1, 2, \ldots, n$). Hence

$$\|\varphi * a_{f_M}\|_\infty = \|\sum_{i=1}^{n} \lambda_i \varphi * ({}_{x_i} f_M)\|_\infty = \|\sum_{i=1}^{n} \lambda_i \Delta(x_i)\varphi_{x_i} * f_M\|_\infty$$

$$\leq \sum_{i=1}^{n} \lambda_i |\Delta(x_i)| \|\varphi_{x_i} * f_M\|_\infty < \varepsilon.$$

Since $u \in UCB(G)$, by Lemma 2.2 of [13] (P27), we have

$$m(u) = m(\varphi * u) = m(\varphi * (u - a_f)) + m(\varphi * a_f)$$

$$\leq \|u - a_f\|_\infty + m(\varphi * (a_f - a_{f_M})) + m(\varphi * a_{f_M})$$

$$\leq \|u - a_f\|_\infty + m(\varphi * (a_f - a_{f_M})) + \varepsilon,$$

where $a_{f_M} = \sum_{i=1}^{n} \lambda_{i\,x_i}(f_M)$. Since the support of $a_f - a_{f_M} = a_{f-f_M}$ is compact, $a_{f-f_M} \in UCB(G)$, where $a_{f-f_M} = \sum_{i=1}^{n} \lambda_{i\,x_i}(f - f_M)$. Also, G is noncompact and the support of $a_f - a_{f_M}$ is compact, we have

$$m(\varphi * (a_f - a_{f_M})) = m(a_f - a_{f_M}) = 0.$$

It follows that

$$m(u) \leq \|u - a_f\|_\infty + \varepsilon \leq 1/2 + \varepsilon$$

for any $\varepsilon > 0$, which is a contradiction to $(*)$.

$\square$

Now we can prove our main theorem of this section.

THEOREM 6.3.2. *Let G be a noncompact nondiscrete locally compact group. If G is σ-compact and amenable as a discrete group, then there is an $f \in CB(G)$ with $0 \le f \le 1$ such that any $\theta \in LIM(UCB(G))$ can be extended to a left invariant functional m on $CB(G)$ with $m(f) > 0$, $m(\varphi * f) = 0$ and $\psi(f) = 0$ for any $\psi \in TLIM(CB(G)$ and $\varphi \in P(G)$.*

PROOF: Let $f \in CB(G)$ be a permanently near one function with $0 \le f \le 1$ and for any $\varepsilon > 0$, $x_i \in G$ $(i = 1, 2, \ldots, n)$, there exist $\varphi \in P(G)$ and $f_M \in CB(G)$ with the property that $\|\varphi_{x_i} * f_M\|_\infty < \varepsilon (i = 1, 2, \ldots, n)$ and the support of $f - f_M$ is compact (**) (see [24] Proposition 1.5, Lemma 1.3 and its proof). Set

$$H = \text{span}\{{}_x f - f : x \in G, f \in CB(G)\}$$

$$H_\theta = \text{ span } \{u - \theta(u) : u \in UCB(G)\}$$

$$A = \text{ convex}\{1, f\}.$$

Then A is compact. Claim for each $f_r = r + (1-r)f \in A$ $(0 \le r \le 1)$, $h \in H$ and $h_\theta \in H_\theta$,

$$\|f_r - h - h_\theta\|_\infty \ge 1/4. \qquad (*)$$

If $r \ge 1/2$, we can extend θ to $CB(G)$ such that $\theta \in LIM(CB(G))$ since G is amenable as a discrete group (see Chapter 2, Proposition 2.2.3). Suppose $\theta \in TLIM(CB(G))$. Then $\theta(f) = 0$ by the property (**). Hence

$$1/2 \le r\theta(1) = \theta(f_r - h - h_\theta) \le \|f_r - h - h_\theta\|_\infty,$$

where $\theta(h) = \theta(h_\theta) = 0$. It follows that (*) is true. If $r < 1/2$, then $1 - r \geq 1/2$. Suppose (*) false in this case. Then there is an $\varepsilon > 0$ such that $\|f_r - h - h_\theta\|_\infty < 1/4 - \varepsilon$. Since $h \in \mathcal{A}_0$, there are $\lambda_i > 0, x_i \in G$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\|\sum_{i=1}^n \lambda_{i x_i} h\|_\infty < \varepsilon$ (see Theorem 3.2.3). Put $u = \sum_{i=1}^n \lambda_{i x_i} h_\theta - r$. Then $u \in UCB(G)$. By Lemma 6.2.1, we have

$$\begin{aligned} &\|\sum_{i=1}^n \lambda_{i x_i}(f_r - h - h_\theta)\|_\infty \\ &\geq \|r - \sum_{i=1}^n (1-r)\lambda_{i x_i} f - \sum_{i=1}^n \lambda_{i x_i} h_\theta\|_\infty - \varepsilon \\ &= \|(1-r)a_f - u\|_\infty - \varepsilon \\ &\geq (1-r)1/2 - \varepsilon \geq 1/4 - \varepsilon, \end{aligned}$$

where $a_f = \sum_{i=1}^n \lambda_{i x_i} f$. On the other hand,

$$\|\sum_{i=1}^n \lambda_{i x_i}(f_r - h - h_\theta)\|_\infty \leq \sum_{i=1}^n \lambda_i \|_{x_i}(f_r - h - h_\theta)\|_\infty < 1/4 - \varepsilon$$

which is impossible. Let $B = \overline{H + H_\theta}^{\|\cdot\|_\infty}$. By Hahn-Banach theorem ([28] P58), there exists $\Lambda \in CB(G)^*$ and $r_1, r_2 \in R$, such that for any $x \in A$ and $y \in B$,

$$\Lambda(x) < r_1 < r_2 < \Lambda(y).$$

Since B is a subspace, $\Lambda(y) = 0$ on B. So Λ is a left invariant functional on $CB(G)$. Let $m = \Lambda/\Lambda(1)$. Since $1,\ f \in A$, $\Lambda(1) < 0$ and $\Lambda(f) < 0$. Hence $m(1) = 1$ and $m(f) > 0$. For any $u \in UCB(G)$, $u - \theta(u) \in H_\theta$. We have

$$m(u - \theta(u)) = m(u) - \theta(u)m(1) = 0$$

i.e. m is an extension of θ and $m(f) > 0$. Let $\varphi \in P(G)$. Since $M(f) = 0$ for any $M \in TLIM(CB(G))$ (see [24] Proposition 1.5), and if $m^+(1) \neq 0$ then $\frac{M^+}{M^+(1)} \in LIM(UCB(G))$ which can be extended to an element of $TLIM(CB(G))$, $m^+(\varphi * f) = 0$. Similarly, $m^-(\varphi * f) = 0$. Therefore $m(\varphi * f) = m^+(\varphi * f) - m^-(\varphi * f) = 0$ for any $\varphi \in P(G)$.

□

Now we consider any nondiscrete locally compact group G. Let G be noncompact and let G_0 be a σ-compact noncompact open and closed subgroup of G (see Proposition 2.2.2). Suppose $\{x_\alpha G_0 : \alpha \in \Lambda\}$ is a complete set of left cosets of G_0 in G and V_0 is an open dense subset of G_0 with $\lambda(V_0) < 1$ (see [11] Proposition 2). Put

$$V = \Big(\bigcup_{\alpha\in\Lambda} x_\alpha V_0\Big)^{-1}.$$

Then we have the following lemma which has been proved by Granirer in [11].

LEMMA 6.3.3. *For any $\varepsilon > 0$ there is an $\varphi \in P(G)$ such that $\|\varphi * f\|_\infty < \varepsilon$ where $f = 1_V$.*

THEOREM 6.3.4. *For any noncompact nondiscrete locally compact group G, if G is amenable as a discrete group and $\theta \in LIM(UCB(G))$, then there exists $\psi \in LIM(L^\infty(G)) \sim TLIM(L^\infty(G))$ such that $\psi = \theta$ on $UCB(G)$. In particular,*

$$|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| \geq |LIM(UCB(G))| = 2^{2^{d(G)}},$$

where $d(G)$ is the smallest possible cardinality of a covering of G by compact subsets of G.

PROOF: Let $f = 1_V$ and let S_U be the smallest left invariant subspace of $L^\infty(G)$ containing f and $UCB(G)$. For $u \in UCB(G)$, constants a_i and $x_i \in G(i = 1, 2, \ldots, n)$, define

$$\psi(u + \sum_{i=1}^{n} a_{i x_i} f) = \theta(u) + \sum_{i=1}^{n} a_i.$$

Then ψ is well defined. Indeed, let $u + \sum_{i=1}^{n} a_{i x_i} f = 0$. Suppose $x_0 \in G$ and $u(x_0) + \sum_{i=1}^{n} a_i > 0$. Then there is an open set U in G such that $u + \sum_{i=1}^{n} a_i > 0$ on U. Since $u + \sum_{i=1}^{n} a_{i x_i} f = u + \sum_{i=1}^{n} a_i$ on $U \cap \bigcap_{i=1}^{n} x_i{}^{-1} V$, $u + \sum_{i=1}^{n} a_{i x_i} f > 0$ on $U \cap \bigcap_{i=1}^{n} x_i{}^{-1} V$ which is impossible. Therefore $u + \sum_{i=1}^{n} a_i = 0$ and $\psi(u + \sum_{i=1}^{n} a_{i x_i} f) = 0$. Similarly, we can show that ψ is nonnegative. Hence $\psi \in LIM(S_U)$. Since G_d is amenable, ψ can be extended to an element of $LIM(L^\infty(G))$.

□

Remark 1. This theorem improves theorem(7.20) of [19].

Remark 2. In theorem 6.3.2, we only can extend $\theta \in LIM(UCB(G))$ to a left invariant functional on $CB(G)$ which is not topologically left invariant. We do not know whether this functional can be a mean on $CB(G)$.

COROLLARY 6.3.5. *Under the condition of Theorem 6.3.4,*

$$|LIM(L^\infty(G)) \sim TLIM(L^\infty(G))| = |LIM(L^\infty(G))|.$$

Proof: It is a direct consequence of Theorem 6.3.4.

□

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