UNIVERSITY OF ALBERTA

SEISMIC WAVES IN VISCOELASTIC MEDIA

by

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ABSTRACT

The effects of anelasticity on the propagation of seismic waves through a layered homogeneous isotropic medium are examined. The theory of linear viscoelasticity is used to model the anelasticity. This shows that plane waves propagating through viscoelastic media are generally inhomogeneous i.e they have different direction of propagation and attenuation (Buchen (1971), Borcherdt (1973)). The angle between the propagation and the attenuation vectors is called the attenuation angle γ. In previous studies on computation of synthetic seismograms for viscoelastic media, using a ray theory, an arbitrary value of γ was chosen leading to different raypaths for different choices of γ. It is possible to remove this arbitrariness by using a stationary ray approach. This consists of computing the viscoelastic ray parameter for which the phase function is stationary. The ray parameter being a function of γ and θ, the propagation angle, the unique value of this ray parameter leads to unique values for these two angles. The coordinates of the raypath associated to this computed ray parameter are complex numbers. This introduces the concept of complex rays. However reflection and transmission coefficients, geometrical spreading and ray synthetic seismograms can be computed using this procedure without concern for the details of complex raypath coordinates. All the results

obtained are for P-SV waves and frequency independent loss factors. The reflection and transmission coefficients exibit some differences in phase and in certain cases in amplitude too with the results obtained for perfectly elastic media. These differences mainly occur in the vicinity of critical incidences. The computations of synthetic seismograms show that the difference in amplitude between the elastic and anelastic cases before encountering any critical incidence is mainly due to the dissipation of the energy of the particle motion along the ray path. The elastic and anelastic geometrical spreadings are practically identical even after any critical incidence.

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CHAPTER 1

INTRODUCTION

Studies of seismic records indicate that the propagation of seismic waves through the earth is affected by the effects of anelasticity. It is then necessary to include these effects in the computation of synthetic seismograms. To model the anelasticity the theory of linear viscoelasticity is used. It assumes that an anelastic material has "memory" i.e the state of the material at a given time depends on its history up to the given time. This theory allows the calculation of the loss factors for each different model. The loss factor $\frac{1}{Q}$ characterises the anelasticity (or the internal friction) of a material. Using the theory of linear viscoelasticity the wave amplitude A is proportional to $E^{1/2}$, E being the peak strain energy stored in the volume of the material, and $\frac{1}{Q}$ can be defined as

$$\frac{1}{Q(\omega)} = -\frac{1}{\Pi}\frac{\Delta A}{A}$$

from which the amplitude variations due to absorption can be determined.

A lot of research on seismic waves propagating in viscoelastic media has already been carried out (Buchen

(1971), Borcherdt (1973), Krebes (1980)) and the results are briefly restated in chapters 2 and 3. They introduce the concept of inhomogeneous plane waves. These waves have an amplitude which varies along the wavefront and they are characterised by different direction of propagation and attenuation. The angle between the propagation and attenuation vectors is called γ the attenuation angle and can vary from -90° to $+90^{\circ}$. Consequently in order to trace a ray for a specified horizontal distance between the source and receiver in a linear viscoelastic medium two angles must be determined: the propagation angle.θ and the attenuation angle γ. For a perfectly elastic medium only θ needs to be known. In previous methods of computing synthetic seismograms for plane layered viscoelastic media, using a ray theory approach, the value of γ was chosen arbitrarily. Knowing γ the propagation angle θ is determined from Snell's law and two point ray tracing. This means that the raypath is not unique and different values of γ produce different raypaths which is a contrast with the elastic case. The main purpose of this thesis is to investigate the use of a method which determines the initial value of the attenuation angle, for the computation of synthetic seismograms in viscoelastic media. This method called the stationary ray method was developed by Hearn and Krebes (1990) and is presented in chapter 4. It is based on Fermat's principle and consists of computing the anelastic ray parameter for which the phase function is stationary.

New expressions for the geometrical spreading and for the reflection and transmission coefficients for an interface formed by two viscoelastic media are calculated using this stationary ray method (chapters 5 and 6). These expressions are then used to compute simple synthetic seismograms for P-SV waves in the case of a point source at the surface (chapter 7). For convenience, in all the numerical computations the loss factor is assumed to be frequency independent.

All the results are compared with the analogous results obtained for the elastic case in order to observe and to better understand the effects of anelasticity on the propagation of seismic waves.

CHAPTER 2

PROPAGATION OF VISCOELASTIC PLANE WAVES

A review of the basic steps of the development of the theory of viscoelastic plane waves propagating in a homogeneous isotropic linear viscoelastic medium is given in this chapter (Borcherdt 1973,1977; Silva 1976; Buchen 1971(a)).

For such a medium the general form of the stress-strain relation is

$$\sigma_{ij}(t) = \delta_{ij}\int_{-\infty}^{t} \lambda(t - \tau)\,\frac{de_{kk}(\tau)}{d\tau}\,d\tau + 2\int_{-\infty}^{t} \mu(t - \tau)\frac{de_{ij}(\tau)}{d\tau}\,d\tau$$

$$= \delta_{ij}\,\lambda(t) * d\,e_{kk}(t) + 2\,\mu(t) * de_{ij} \tag{2.1}$$

(elastic case: $\sigma_{ij} = \lambda\, e_{kk}\, \partial_{ij} + 2\,\mu\, e_{ij}$)

where $\sigma_{ij}(t)$ and $e_{ij}(t)$ are the time dependent stress and strain tensors and $\lambda(t)$ and $\mu(t)$ are the time dependent Lame parameters. The symbol * denotes the Stieltjes convolution (see Fung(1965)). The strain tensor $e_{ij}(t)$ can be written

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{2.2}$$

where u_i, i=1, 2, 3 is the particle displacement or displacement vector. The Einstein summation convention being used in all the equations

$$e_{kk} = e_{11} + e_{22} + e_{33} = \nabla \cdot \vec{u} \tag{2.3}$$

represents the dilation: the relative change in volume due to strain state. Knowing the bulk modulus K(t) is given by

$$K(t) = \lambda(t) + \frac{2}{3}\mu(t) \tag{2.4}$$

Equation (2.1) can be broken up into bulk and shear components:

$$\sigma_{kk}(t) = 3K(t) * de_{kk}(t)$$

and (2.5)

$$\sigma_{ij}(t) = 2\mu(t) * de_{ij}(t) \text{ for } i \neq j$$

The same calculations are also performed in the perfect elastic case. The stress-strain relation can be rewritten:

$$\sigma_{ij}(t) = \delta_{ij}K(t) * de_{kk}(t) + 2\mu(t) * de_{ij} - \delta_{ij}\frac{2}{3}\mu(t) * de_{kk}(t) \tag{2.6}$$

The equation of motion for infinitesimal motion is given by

$$\sigma_{ij,j} = \rho\ddot{u}_i = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{2.7}$$

where ρ is the density of the medium. Substituing (2.1) into (2.7) yields the following equation

$$[\lambda(t) + \mu(t)] * d\,(\nabla(\nabla\cdot\vec{u})) + \mu(t) * d\,(\nabla^2\vec{u}) = \rho\ddot{\vec{u}} \tag{2.8}$$

The convolutions make the representation in the time domain quite tedious. The Fourier transform of equation (2.8) is then taken (see Appendix 1) and gives

$$(\Lambda + M)\nabla(\nabla\cdot\bar{\vec{u}}) + M\nabla^2\bar{\vec{u}} = -\rho\omega^2\bar{\vec{u}} \tag{2.9}$$

where

$$\bar{\vec{u}} = \int_{-\infty}^{\infty}\vec{u}\, e^{-i\omega t}\, dt$$

$$\Lambda = \Lambda(\omega) = i\omega\int_0^{\infty}\lambda(t)\, e^{-i\omega t}\, dt \tag{2.10}$$

$$M = M(\omega) = i\omega\int_0^{\infty}\mu(t)\, e^{-i\omega t}\, dt$$

Introducing $\bar{\vec{u}}$, the transformed displacement potential in terms of Helmholtz's relation

$$\bar{\vec{u}} = \nabla\bar{\Phi} + \nabla\times\bar{\vec{\Psi}} \quad \text{with} \quad \nabla\cdot\vec{\Psi} = 0 \tag{2.11}$$

and inserting (2.11) into (2.9) lead to the Helmholtz equations for the P and S potentials Φ and Ψ:

$$\nabla^2 \overline{\Phi} + k^2_p \overline{\Phi} = 0$$
$$\nabla^2 \overline{\Psi} + k^2_s \overline{\Psi} = 0 \qquad (2.12)$$

where

$$k^2_p = \frac{\omega^2}{\alpha^2} = \frac{\omega^2 \rho}{\left(\Lambda(\omega) + 2M(\omega)\right)}$$
$$k^2_s = \frac{\omega^2}{\beta^2} = \frac{\omega^2 \rho}{M(\omega)} \qquad (2.13)$$

The velocities of P and S waves, respectively α and β become complex and frequency dependent. A general solution of the Helmholtz equation

$$\nabla^2 F + k^2 F = 0 \qquad (2.14)$$

has the following form

$$F = F_0 \, e^{i(\vec{k} \cdot \vec{r})} \qquad (2.15)$$

This can be applied to P waves with $k=k_p$ and $F=\Phi$ and to S waves with $k=k_s$ and $F=\Psi$. The final solution is then given by

$$F = F_0 \, e^{-\vec{A}\cdot\vec{r}} \, e^{i\vec{P}\cdot\vec{r}} \qquad (2.16)$$

where

$$\vec{k} = \vec{P} + i\vec{A} \qquad (2.17)$$

$\vec{P}$ is the propagation vector. It is perpendicular to the planes of constant phase ($\vec{P}\cdot\vec{r}$ = constant). The phase velocity i.e the velocity of the propagation of the planes of constant phase.is given by

$$\vec{v} = \frac{\omega}{Re\{k\}} \hat{P} \qquad (2.18)$$

where $\hat{P}$ is a unit vector in the direction of $\vec{P}$.

$\vec{A}$ is the attenuation vector. It is perpendicular to the planes of constant amplitude ($\vec{A}\cdot\vec{r}$ = constant) and it is such that $e^{-\vec{A}\cdot\vec{r}}$ represents the spatial decay of the wave.

Using equation (2.17) the square of $|\vec{k}|$ is determined and equal to

$$k^2 = \vec{k}\cdot\vec{k} = P^2 - A^2 + 2i\vec{P}\cdot\vec{A} \qquad (2.19)$$

where

$$\vec{P}\cdot\vec{A} = PA \cos\gamma \qquad (2.20)$$

γ is the angle between $\vec{P}$ and $\vec{A}$ and it is called the attenuation angle. When $\gamma = 0$ ($\vec{P}$ and $\vec{A}$ are parallel) the viscoelastic plane wave is homogeneous. When $\gamma \neq 0$ ($\vec{P}$ and $\vec{A}$ are not parallel) the viscoelastic plane wave is inhomogeneous meaning that its amplitude varies along a wavefront. γ has to be between $-\frac{\Pi}{2}$ and $\frac{\Pi}{2}$ so that the amplitude of the wave never increases in the direction of propagation.

P and A have now to be expressed in terms of the medium properties. Writing

$$k^2 = \mathrm{Re}\left(k^2\right) + i\mathrm{Im}\left(k^2\right) = P^2 - A^2 + 2iPA\cos\gamma \quad (2.21)$$

the following expressions are obtained

$$|P| = \frac{1}{2}\left(\mathrm{Re}\left(k^2\right) + \left\{\left[\mathrm{Re}\left(k^2\right)\right]^2 + \frac{\left[\mathrm{Im}(k)^2\right]^2}{\cos^2\gamma}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$(2.22)$$

$$|A| = \frac{1}{2}\left(-\mathrm{Re}\left(k^2\right) + \left\{\left[\mathrm{Re}\left(k^2\right)\right]^2 + \frac{\left[\mathrm{Im}(k)^2\right]^2}{\cos^2\gamma}\right\}^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

(Borcherdt (1973))

The loss factor for P and S are given by

$$Q_S^{-1} = \frac{M_I}{M_R}$$

and (2.23)

$$Q_P^{-1} = \frac{(\Lambda + 2M)_I}{(\Lambda + 2M)_R}$$

Starting from

$$k^2 = \frac{\omega^2}{v^2} \tag{2.24}$$

where

$$v^2 = \frac{M(\omega)}{\rho} \quad \text{for S waves}$$

and (2.25)

$$v^2 = \frac{\Lambda + 2M}{\rho} \quad \text{for P waves}$$

an expression of k^2 in terms of Q^{-1} and v_H, the homogeneous wave speed, is determined:

$$k^2 = \frac{\omega^2}{v^2} = \frac{\omega^2}{v_H^2} \frac{2}{\left[1 + \sqrt{1 + Q^{-2}}\right]} \left(1 + iQ^{-1}\right) \tag{2.26}$$

where

$$v_H = \frac{\omega}{Re(k)} \tag{2.27}$$

(2.26) can be used for P waves taking $k=k_P$, $v_P=v_{HP}$ and $Q^{-1}=Q_P^{-1}$ and for S waves taking $k=k_S$, $v_H=v_{HS}$ and $Q^{-1}=Q_S^{-1}$. The complex frequency dependent wave speed is then given by

$$v^2 = v_H^2 \frac{\left[1 + \sqrt{1 + Q^{-2}}\right]}{2\left(1 + Q^{-2}\right)} \left(1 - iQ^{-1}\right) \tag{2.28}$$

and the final expressions of P^2 and A^2 are

$$P^2 = \frac{\omega^2}{v_H^2}\left(\frac{\left[1 + \sqrt{1 + \frac{Q^{-2}}{\cos^2\gamma}}\right]}{\left[1 + \sqrt{1 + Q^{-2}}\right]}\right)$$

$$A^2 = \frac{\omega^2}{v_H^2}\left(\frac{\left[-1 + \sqrt{1 + \frac{Q^{-2}}{\cos^2\gamma}}\right]}{\left[1 + \sqrt{1 + Q^{-2}}\right]}\right) \qquad (2.29)$$

for P and S waves.

For a non-dissipative medium $\mathrm{Im}(k^2)=0$. From equation (2.19) it is deduced that $\mathrm{Im}(k^2)=0$ when $\vec{A}=0$ or $\gamma=\pm\frac{\Pi}{2}$. Conversely, if $\vec{A}=0$ or $\gamma=\pm\frac{\Pi}{2}$ the medium is non-dissipative. Consequently the following statement is also valid: the medium is dissipative if and only if $\vec{A}\neq 0$ and $\gamma\neq\pm\frac{\Pi}{2}$. These results then show that the only type of inhomogeneous plane wave propagating in a non-dissipative medium (i.e $\vec{A}\neq 0$ and $\gamma=\pm\frac{\Pi}{2}$) does not propagate in a dissipative medium, and vice versa. This difference in the nature of inhomogeneous waves in the two types of material shows that the theories of elasticity and viscoelasticity are different.

Equations (2.18) and (2.22) show that the phase speed of an inhomogeneous wave (i.e $0 < |\gamma| \leq \frac{\Pi}{2}$) is lower than the one for a homogeneous wave

$$|v_{IH}| < |v_H| \qquad (2.30)$$

and that the maximum attenuation of an inhomogeneous wave is greater than that for an homogeneous wave

$$|A_{IH}| > |A_H| \tag{2.31}$$

Finallly we can say that in viscoelastic medium, steady state harmonic plane waves take the form

$$\exp\left(i\omega(\tau - t)\right) \tag{2.32}$$

which is the form of the solution of the wave equation. (2.32) can be rewritten as

$$\exp\left[-\vec{A}\cdot\vec{r}\right] \exp\left[i\left(\vec{P}\cdot\vec{r} - \omega t\right)\right] \tag{2.33}$$

τ is then the complex phase function and it is given by

$$\tau = \frac{\vec{k}\cdot\vec{r}}{\omega} \tag{2.34}$$

only propagation in the x-z plane is assumed. Consequently the wave vector $\vec{k} = (k_x, k_z) = \vec{P} + i\vec{A}$ has to satisfy

$$k^2 = k_x^2 + k_z^2 = \frac{\omega^2}{v^2} = P^2 - A^2 + 2iPA \tag{2.35}$$

where v is the complex frequency dependent wave speed. For a specified value of γ, $\gamma \neq \frac{\Pi}{2}$, and knowing v_H and Q, P, the phase velocity and A are determined uniquely using

(2.29). The choice of γ for a ray is discussed later in the thesis.

SNELL'S LAW IN VISCOELASTIC MEDIUM

In this chapter Snell's law in viscoelastic media is going to be demonstrated for one type of plane waves: SH waves. The final result is of course valid for the other type of plane waves: P and SV waves.

The incident, reflected and transmitted waves at a boundary separating two viscoelastic media V and V' are shown in figure 1. A rectangular system of coordonates $(\hat{x}, \hat{y}, \hat{z})$ is chosen such that the space occupied by medium V is described by $z \geq 0$. The expressions of the complex wave vector $\vec{k}$ for the incident, reflected and transmitted waves are

$$\begin{aligned} \vec{k}(inc.) &= k(inc.)_x\hat{x} + k(inc.)_z\hat{z} \\ \vec{k}(ref.) &= k(ref.)_x\hat{x} + k(ref.)_z\hat{z} \\ \vec{k}(tran.) &= k(tran.)_x\hat{x} + k(tran.)_z\hat{z} \end{aligned} \tag{3.1}$$

The incident and reflected waves being in the same medium

$$k(inc.)_x = k(ref.)_x = k_x \quad \text{and} \quad k(inc.)_z = -k(ref.)_z = -d_\beta \tag{3.2}$$

hence

$$\begin{aligned} \vec{k}(inc.) &= k_x\hat{x} - d_\beta\hat{z} \\ \vec{k}(ref.) &= k_x\hat{x} + d_\beta\hat{z} \end{aligned} \tag{3.3}$$

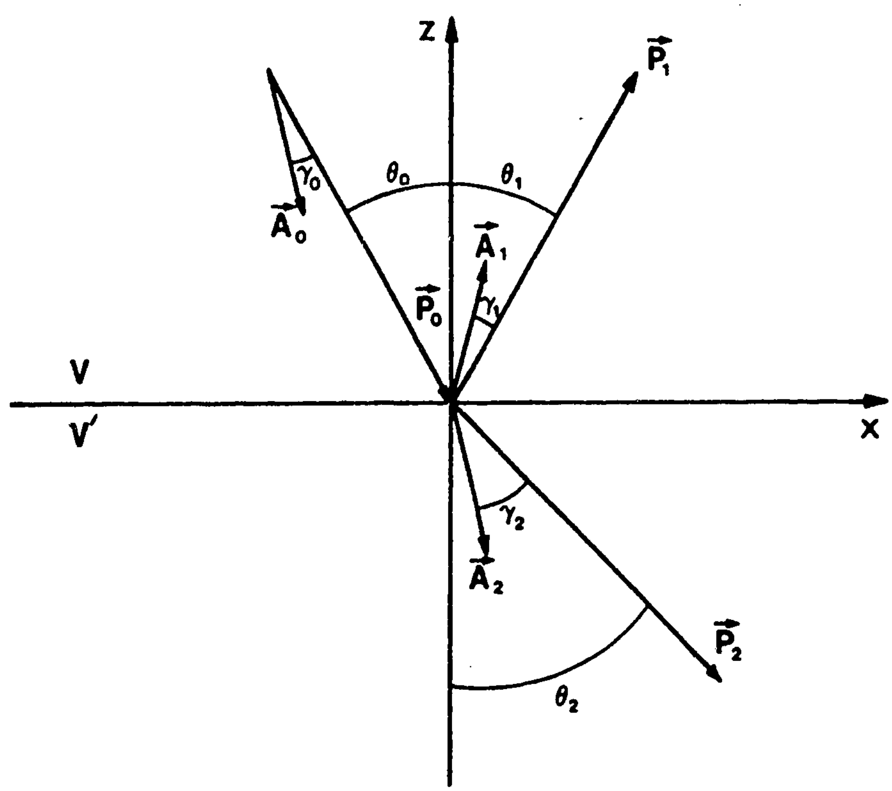


Figure 1: Incident, reflected and transmitted rays at a boundary separating two anelastic media

where

$$d_\beta = + \text{p.v.} \sqrt{k^2 - k_x^2}\,, \quad k^2 = k^2_{inc.} = k^2_{ref} \tag{3.4}$$

For the transmitted wave

$$\vec{k'} = k'_x \hat{x} - d'_\beta \hat{z} \tag{3.5}$$

where

$$d'_\beta = + \text{p.v.} \sqrt{k'^2 - k'^2_x}\,, \quad \vec{k'} = \vec{k}_{(trans)}\,, \quad k_{(trans)x} = k'_x \tag{3.6}$$

The symbol p.v. means the principal value of the complex square root, i.e the complex square root for which the argument that the root makes in the complex plane lies between -90° and +90°. To set up the reflection refraction problem which leads to Snell's law, the media V and V' are in welded contact with a common plane boundary. As it was stated earlier, SH case is only considered consequently the particle motions are purely transverse parallel to the boundary. The welded contact between V and V' is specified by requiring that the stress and the displacement across the boundary are continuous, hence at z=0

$$\vec{u} = \vec{u'}$$

and (3.7)

$$\sigma_{32} = \sigma'_{32} \qquad (\sigma_{31} = \sigma_{33} = 0)$$

Borcherdt (1977) demonstrated that for SH steady state harmonic plane waves, the displacement field is given by

$$= \operatorname{Re}\left(i(\vec{k}\times\vec{B})\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

$$= |D|\; e^{-\vec{A}\cdot\vec{r}} \cos\left(\vec{P}\cdot\vec{r} - \omega t + \arg(D)\right) \hat{x}_2 \qquad (3.8)$$

where

$$\vec{B} = z_1\hat{x}_1 + z_3\hat{x}_3 \qquad (3.9)$$

$$\vec{\psi} = \vec{B}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad (3.10)$$

and

$$D = (z_3\hat{x}_1 - z_1\hat{x}_3)\cdot(\vec{A} - i\vec{P}) \qquad (3.11)$$

z_1 and z_3 are arbitrary complex numbers chosen such that $\nabla\cdot\vec{\psi} = 0$. $\hat{x}_1$ and $\hat{x}_3$ are real unit vectors in x_1 and x_3 directions. $\vec{P}$ and $\vec{A}$ are in x_1-x_3 or x-z plane. The particle motion is linear and perpendicular to the plane of P and A (Figure 2). Using equation (3.8) and considering the real and imaginary parts, u and u' can be written

$$\vec{u} = D_{inc}\; e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}\,\hat{y} + D_{ref}\; e^{i(\vec{k}_{ref}\cdot\vec{r}-\omega t)}\,\hat{y}$$

and (3.12)

$$\vec{u'} = D_{tran}\; e^{i(\vec{k}_{tran}\cdot\vec{r}-\omega t)}\,\hat{y} = D'\; e^{i(\vec{k'}\cdot\vec{r}-\omega t)}\,\hat{y}$$

The expressions of σ_{32} and σ'_{32} are obtained with the equation of the stress tensor for steady state harmonic case:

$$\sigma_{ij} = \Lambda(\omega)\,\theta\,\delta_{ij} + 2\,M(\omega)\,e_{ij} \qquad (3.13)$$

where

$$\theta = e_{kk} = \nabla\cdot\vec{u} \qquad (3.14)$$

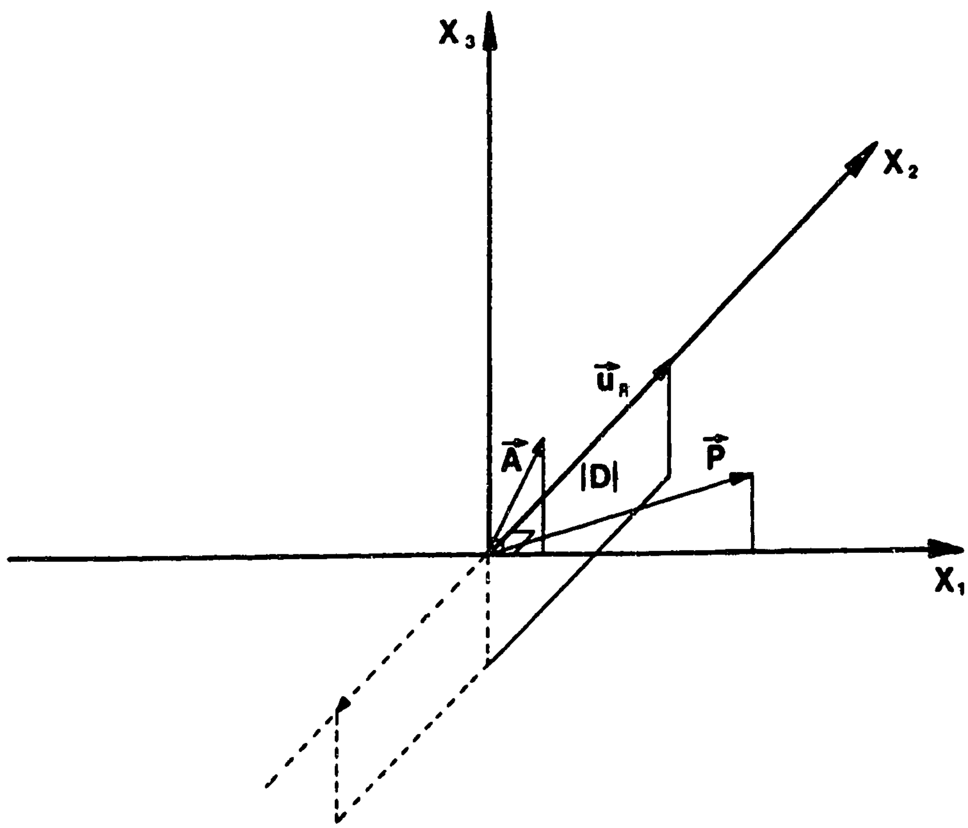


Figure 2: Parameters for SH waves particle motion

(3.13) is obtained by inserting $\vec{u} = U \exp^{-i\omega t}$ which is the steady state harmonic condition on the displacement into (2.1). As previously stated, the displacement of the particles for the SH case is given by

$$\vec{u} = \nabla \times \vec{\Psi} \tag{3.15}$$

where $\vec{\Psi}$ is given in (3.10) and the dilation θ is such that

$$\theta = \nabla \cdot \vec{u} = \nabla \cdot \left(\nabla \times \vec{\Psi}\right) = 0 \tag{3.16}$$

hence

$$\sigma_{ij} = -iMD \left(\delta_{i2}k_j + \delta_{j2}k_i\right) e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)} \tag{3.17}$$

Using (3.17) the expressions σ_{32} and σ_{32}' are

$$\sigma_{32} = -iMD_{inc} \left(k_{3_{inc}}\right) e^{i\left(\vec{k}_{inc}\cdot\vec{r} - \omega t\right)} - iMD_{ref} \left(k_{3_{ref}}\right) e^{i\left(\vec{k}_{ref}\cdot\vec{r} - \omega t\right)}$$

and (3.18)

$$\sigma_{32}' = -iMD' \left(k_3'\right) e^{i\left(\vec{k}'\cdot\vec{r} - \omega t\right)}$$

but

$$k_{3_{inc}} = k_{z_{inc}} = -d_\beta = -k_{3_{ref}} = -k_{z_{ref}}$$

and (3.19)

$$k_3' = k_z' = -d_\beta'$$

(3.18) can be written

$$\sigma_{32} = -iMD_{inc}\, d_\beta\, e^{i\left(\vec{k}_{inc}\cdot\vec{r} - \omega t\right)} + iMD_{ref}\, d_\beta\, e^{i\left(\vec{k}_{ref}\cdot\vec{r} - \omega t\right)}$$

and (3.20)

$$\sigma_{32} = -iMD \; d_{\beta} \, e^{i(k \cdot r - \omega t)}$$

The restrictions implied by equation (3.7) yield these two following relations:

$$D_{inc}e^{ik_{x inc}x} + D_{ref}e^{ik_{x ref}x} = D'e^{ik'_x x}$$
$$Md_{\beta}D_{inc}e^{ik_{x inc}x} - Md_{\beta}D_{ref}e^{ik_{x ref}x} = M'd'_{\beta}D'e^{ik'_x x} \qquad (3.21)$$

The complex amplitudes being independent of the spatial coordinates x, y, z, equation (3.21) implies that

$$k_{x_{inc}} = k_{x_{ref}} = k'_x \qquad (3.22)$$

which leads to the anelastic form of Snell's law. $k_x = P_x + iA_x$ consequently A_x and P_x must be continuous across the boundary and Snell's law is now composed of two parts:

$$P_{x_{inc}} = P_{x_{ref}} = P_{x_{tran}} = k_{x_R} = P_x$$

and (3.23)

$$A_{x_{inc}} = A_{x_{ref}} = A_{x_{tran}} = k_{x_I} = A_x$$

Referring to figure 1, (3.23) can be written as

$$P_1 \sin\theta_1 = P_2 \sin\theta_2 = P' \sin\theta' = k_{x_R}$$

$$A_1 \sin(\theta_1 - \gamma_1) = A_2 \sin(\theta_2 - \gamma_2) = A' \sin(\theta' - \gamma') = k_{x_I} \qquad (3.24)$$

The ray parameter p which is now complex can then be expressed as

$$p = \frac{k_x}{\omega} = \left(\frac{P_x}{\omega}\right) + i\left(\frac{A_x}{\omega}\right) = \frac{\sin \alpha}{v} \qquad (3.25)$$

where α is the complex angle between $\vec{k}$ and the vertical z axis and v is the complex frequency dependent wave speed. (Re $\alpha \neq \theta$). Snell's law implies that p has the same value for the incident, reflected and transmitted waves, then for a ray of m segments propagating through a sequence of flat horizontal homogeneous anelastic layers (Figure 3)

$$p = \frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \frac{\sin \alpha'}{v'} = \ldots\ldots\ldots \qquad (3.26)$$

Two new expressions of $\vec{P}$ and $\vec{A}$ can also be written using equation (3.25) for a wave travelling in the positive x and z directions

$$\begin{aligned} \vec{P} &= \mathrm{Re}(\omega p)\,\hat{x} + \mathrm{Re}\left(\omega\sqrt{\frac{1}{v^2} - p^2}\right)\hat{z} \\ \vec{A} &= \mathrm{Im}(\omega p)\,\hat{x} + \mathrm{Im}\left(\omega\sqrt{\frac{1}{v^2} - p^2}\right)\hat{z} \end{aligned} \qquad (3.27)$$

This means that if v_H the homogeneous phase speed and Q the quality factor are known and a value of p is determined (see next chapter), $\vec{P}$ and $\vec{A}$ can be calculated using (2.28) and (3.27). θ the angle of propagation and γ the angle of attenuation can also be calculated.

For the case excluded in chapter 2, i.e $|\gamma| = \frac{\Pi}{2}$ corresponding to the case of an inhomogeneous elastic plane wave, Q is infinite, the complex wave speed v is such that

$$v^2 = v_H^2 \tag{3.28}$$

and consequently, knowing a value for p and using (3.27), $\vec{P}$ and $\vec{A}$ can then be calculated.

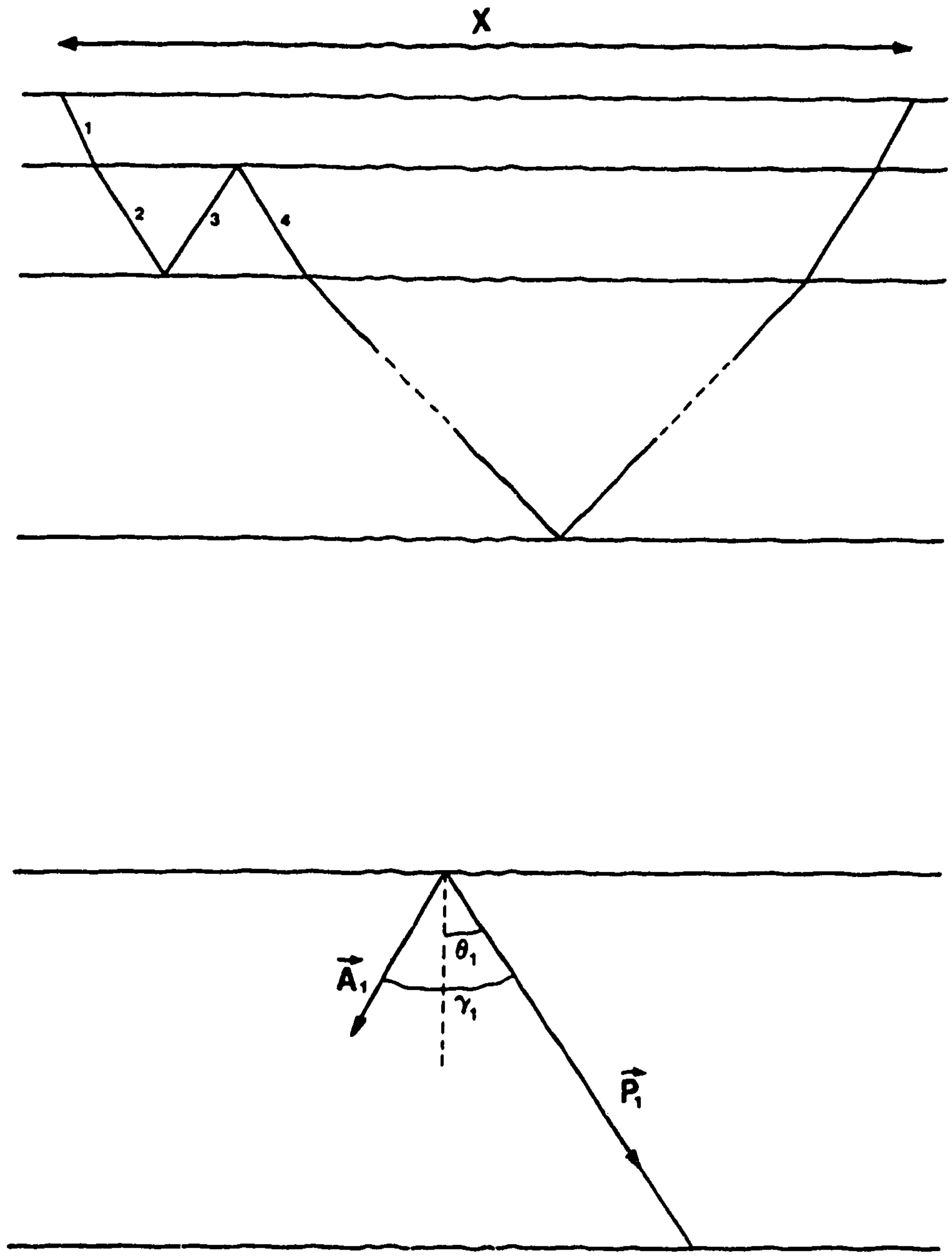


Figure 3 top: A typical ray of m segments

bottom: A blowup of the first ray segment

CHAPTER 4

STATIONARY RAY

It is clearly stated in chapters 2 and 3 that in order to trace the ray corresponding to a plane wave propagating from the source through viscoelastic layers to the receiver, the propagation angle θ and the attenuation angle γ for the first ray segment have to be determined for a given epicentral distance. In the perfect elasticity case, only θ needs to be determined. Previous work on the computation of ray synthetic seismograms for anelastic media seemed to indicate that there was no specific physical rule in the choice of γ. Authors simply assigned arbitrary values to the initial attenuation angle (Krebes and Hron 1980, Kelamis et al 1983, Bourbie and Gonzalez-Serrano 1983) such that γ lies between -90° and $+90^{\circ}$. From this value of γ and for a given offset the initial propagation angle, the raypath, the travel time and the amplitude can be determined. This clearly means that in contrast with the elastic case there is no unique raypath since different values of γ produce different raypaths with different propagation angles. Consequently this approach is not satisfactory.

In this chapter the method discussed by Krebes (1990) for calculating the initial attenuation angle is treated. It is based on Fermat's principle: the proper value of γ is the one which generates the ray with the smallest travel time. This method consists of calculating the value of the anelastic ray parameter (a complex number) for which the complex phase function (equation 2.34) is stationary. The anelastic ray parameter being a function of both angles γ and θ, the initial values of these two angles can be obtained from this ray parameter giving a unique ray path. This method was previously used by Buchen (1971) in Lamb's problem for an SH line source in a viscoelastic media. Rays determined by this method are called "stationary rays" because their ray parameter makes the complex phase function stationary. Such rays are "complex rays", i.e the spatial coordinates of points on their raypaths are complex numbers. The case of a complex ray propagating through a stack of viscoelastic layers and the case of a complex ray propagating through a viscoelastic medium such that the velocity varies continuously with the depth are also mentioned in this chapter.

4.1 Ray Tracing In Viscoelastic Media:

Let us recall the complex phase function defined in chapter 2 as

$$\tau = \frac{\vec{k}\cdot\vec{r}}{\omega} = \frac{k_x \cdot x}{\omega} + \frac{k_z \cdot z}{\omega} \qquad (4.1)$$

For a viscoelastic homogeneous medium, τ can be rewritten using (3.25). The new expression is

$$\tau = px + z\sqrt{\frac{1}{v^2} - p^2} \qquad (4.2)$$

where p is the complex ray parameter, v the complex frequency dependent wave speed and x the horizontal distance between source and receiver. Consequently for a ray of m segments propagating through a stack of flat horizontal homogeneous layers (figure 3), τ is given by

$$\tau = pX + \sum_{j=1}^{m} h_j\sqrt{\frac{1}{v^2} - p^2} \qquad (4.3)$$

where X is the offset, i.e the horizontal distance between source and the receiver located at the end-points of the ray, h_j and v_j are the layer thickness and the complex wave speed of the j^{th} layer. τ is then a function of p, the ray parameter. The real part of τ is the actual travel time of the ray and the imaginary part is associated to the attenuation due to absorption. Using (2.24), (3.25) and (3.4) it is easy to show that

$$d_\beta = +p.v.\sqrt{k^2 - k_x^2} = \omega\left(+p.v.\sqrt{\frac{1}{v^2} - p^2}\right) \qquad (4.4)$$

In chapter 3 the way of choosing the sign of d_β and d'_β for the transmitted ray is described. Consequently the same rule is applied for the sign of the complex square root of equation (4.3).

The commonly approach used in the computation of a raypath and the ray amplitude in viscoelastic medium is now discussed. To calculate τ for a given offset X the ray parameter p has to be determined. The old approach assumed that, for all j, the propagation vector $\vec{P}_j$ was parallel to the j^{th} ray segment, consequently X is given by

$$X = \sum_{j=1}^{m} x_j = \sum_{j=1}^{m} h_j \frac{\sin \theta_j}{\cos \theta_j} = \sum_{j=1}^{m} h_j \, \omega \, \frac{Re(p)}{P_j} \frac{1}{\sqrt{1 - \dfrac{\omega^2 Re(p)^2}{P_j^2}}} \qquad (4.5)$$

All the angles θ_j and γ_j for $j \geq 2$ can be calculated using Snell's law (3.24) once γ_1 and θ_1 have been determined. If a value of γ_1 is chosen arbitrarily and the medium parameters (density, v_H, Q) are given, the take-off angle θ_1 is determined by solving the equation (4.5). Knowing P_1, A_1, θ_1 and γ_1, p can be calculated using (3.25) and consequently τ for the corresponding ray is known using equation (4.3). This approach was used by some authors to produce synthetic seismograms in viscoelastic media. For instance Krebes and Hron (1980) compared ray synthetic seismograms for several values of γ_1. It is unsatisfactory because different values of γ_1 produce different values of θ_1 and consequently different travel times, all satisfying raypath between source and receiver which is unlike the elastic case. Futhermore the real travel time, the attenuation, the reflection and transmission coefficients and the geometrical spreading are all function of γ. The variations of the geometrical spreading as a function of γ have been treated by Krebes and Hearn (1985). Different choices of γ would then result in different waveforms and different arrival times for a same pulse.

There exists another approach based on a more nearly wave-method analysis (Richards, 1984; Hearn and Krebes, 1990). The anelastic wave field at the receiver is expressed as an integral over p. This integral is evaluated by the method of the steepest descent to obtain the ray approximation. This method gives a unique value for p which is the saddle point of the phase function τ in the p integral. This value of p is the solution of the equation $\left(\frac{d\tau}{dp}\right) = 0$. Differentiating equation (4.3) over p and equating the expression obtained to 0 gives

$$X = \sum_{j=1}^{m} \frac{p\ v_j\ h_j}{\sqrt{1 - p^2\ v_j^2}} = \sum_{j=1}^{m} h_j \tan\alpha_j = \sum_{j=1}^{m} x_j \qquad (4.6)$$

where α_j is the complex incidence angle for the j^{th} ray segment, i.e the complex angle between k_j and the vertical. Equation (4.6) is similar to equation (4.5) but the complex quantities α_j, v_j, p in (4.6) replace the real quantities numbers, the sum in equation (4.6) has to give the real number X. Equation (4.6) then represents a system of two equations

$$\sum_{j=1}^{m} \mathrm{Re}(x_j) = X$$

$$\sum_{j=1}^{m} \mathrm{Im}(x_j) = 0 \tag{4.7}$$

which has two unknowns θ_1 and γ_1. Solving the equation (4.6) gives a unique value for p and consequently a unique value for θ_1 and for γ_1. Then using equation (4.3) τ can be determined. A new expression for τ is obtained by inserting (4.6) into (4.3):

$$\tau = \sum_{j=1}^{m} \frac{h_j}{v_j \sqrt{1 - p^2 v_j^2}} \tag{4.8}$$

Equation (4.8) has the same form as the equation for the travel time in the elastic case but v_j and p are now complex quantities. In this method there is no arbitrairy choice of γ_1, γ_1 and θ_1 have to satisfy equation (4.6).

Richard (1984) discussed Lamb's problem for anelastic media and showed that the correct raypath is calculated using the saddle point of τ. Moreover Hearn (1985) showed that in a treatment of the problem of a point source in a viscoelastic medium lying over a sequence of viscoelastic layers, the saddle point used to approximate the transmitted wave field integral giving the field at a point below the stack of layers is the saddle point given by the solution of equation (4.6).

Numerical computations have already been performed by Hearn and Krebes (1990) for primary rays. They show that the stationary rays have smallest real travel time and highest value for Im(τ) compared to raypaths generated by several arbitrary choices of value for γ_1. Stationary rays then satisfy the Fermat's least time principle

4.2 Complex Rays:

As already mentioned in chapter 4, stationary rays are complex rays. This is demonstrated as follow: rewritting equation (4.8) with $p = \frac{\sin \alpha_j}{v_j}$ yields

$$\tau = \sum_{j=1}^{m} \frac{s_j}{v_j} \tag{4.9}$$

where $s_j = \frac{h_j}{\cos \alpha_j}$. Since v_j is a complex quantity the j^{th} ray segment has a complex arc length s_j. The travel time of the ray is given by Re(τ). Equation (4.6) shows that the horizontal distance traversed by the j^{th} ray

$$x_j = h_j \tan \alpha_j \tag{4.10}$$

is also a complex quantity for complex α_j. If the sum of equation (4.6) is performed up to a number n lower than m, the x^{th} coordinate of the n^{th} segment is complex. The ray starts at the source in a real space then goes into complex 3-D space in which the Cartesien coordinates x, y, z are complex as it propagates and ends up at the receiver in real space. This only represents a convenient mathematical way of describing the wave motion. There is no physically real interpretation of the propagation.

In the case of a heterogeneous medium, the phase function τ is equal to

$$\tau(s) = \tau_0(s_0) + \int_{s_0}^{s} \frac{ds}{v(s)} \tag{4.11}$$

where s is the arc length which is generally complex and v the complex frequency dependent wave speed. The analysis of equation (4.11) is similar to the one in inhomogeneous elastic case and yields

$$\frac{dx}{dz} = \pm \frac{p\ v(z)}{\sqrt{1 - p^2 v(z)^2}} \tag{4.12}$$

$$\frac{d\tau}{dz} = \pm \frac{1}{v(z)\sqrt{1 - p^2 v(z)^2}} \tag{4.13}$$

The (+) sign corresponds to the [illegible]
of the ray path if the z-axis is directed to the bottom. The integrals of equations (4.12) and (4.13) give x the horizontal coordinate and τ as functions of depth z. Expressions of $\frac{dx}{dz}$ and $\frac{d\tau}{dz}$ in a heterogeneous viscoelastic medium have the same form as the ones obtained in the elastice case (Hron, ART 1984) except that p and v are now complex quantities.

Two cases of complex rays have been investigated by Krebes (1990). The first one is the propagation of a complex ray through a sequence of homogeneous viscoelastic layers and the second one is the propagation of a complex ray through a medium in which the velocity varies continuously with depth.

In the first case for a given offset X the expressions obtained by integrating equations (4.12) and (4.13) are identical to equations (4.6) and (4.8) as long as

$$\int_{z_{j-1}}^{z_j} dz = z_j - z_{j-1} = \pm h_j, \qquad j = 1, 2.... \tag{4.14}$$

z_j is the z coordinate of the endpoint ot the j^{th} segment and z_0 is the z coordinate of the source with $x(z_0) = 0$. h_j being a real quantity this implies that $Im(z_j)=Im(z_{j-1})$ for all j. The source and receiver lying in the real space, i.e $Im(z_0)=Im(z_m)$, we then obtain

$$Im(z_j) = Im(z_{j-1}) = 0, \; j = 1, 2 \ldots. \qquad (4.15)$$

Consequently the endpoints of each ray segment have real z coordinates. The z coordinates of the other points on the raypath are generally complex. Equations (4.6) and (4.12) also show that the x coordinates of all the points on the raypath except for the source and receiver are complex. For a ray which is symmetric about an axis down its middle, equation (4.6) can be written as

$$X = \sum_{j=1}^{m} \frac{p\, v_j\, h_j}{\sqrt{1 - p^2 v_j^2}} = 2 \sum_{j=1}^{m/2} \frac{p\, v_j\, h_j}{\sqrt{1 - p^2 v_j^2}} \qquad (4.16)$$

consequently $\frac{X}{2}$ is real and this ray has three points in the real space: the source and the receiver points and the midpoint.

In the second case, a vertically inhomogeneous viscoelastic medium in which the velocity varies continuously with depth is considered i.e

$$v(z) = v_0 + az \qquad (4.17)$$

v and v_0 are the complex frequency dependent wave speeds respectively at a depth z and at the surface and 'a' the velocity gradient. Based on the symmetry of the problem it is assumed that there exists a turning point located at a elastic case). At this turning point dz=0 and consequently $\frac{dx}{dz} = \infty$. This implies, using equation (4.12) that $pv(\tilde{z})=1$. The integration of equation (4.12) using the conditions $pv(\tilde{z})=1$, $x(0)=0$ (at the source) and $x(0)=X$ (at the receiver) to determine the constants of integration for downgoing and upgoing parts of the raypath gives

$$x(z) = \frac{1}{2}X \pm \frac{1}{pa}\sqrt{1 - p^2v(z)^2} \tag{4.18}$$

where + corresponds to the upgoing part and - to the downgoing one and where

$$X = \frac{2}{pa}\sqrt{1 - p^2v_0^2} \tag{4.19}$$

Then using equation (4.19), a solution for p is determined

$$p = \frac{1}{v_0\sqrt{1 + \left(\frac{Xa}{2v_0}\right)^2}} \tag{4.20}$$

Equation (4.18) can be squared to obtain

$$\left(x - \frac{X}{2}\right)^2 + \left(z + \frac{v_0}{a}\right)^2 = \left(\frac{1}{pa}\right)^2 \tag{4.21}$$

This is the equation of a circle of radius $\frac{1}{pa}$ and of centre located at $\left(\frac{X}{2}, -\frac{v_0}{a}\right)$. To obtain a new expression of x as a function of z equations (4.17) and (4.20) have to be inserted into (4.19) to yield

$$x(z) = \frac{1}{2}X \pm \frac{v_0}{a}\left(\sqrt{1 + \left(\frac{aX}{2v_0}\right)^2} - \left(1 + \frac{a}{v_0}z\right)^2\right) \qquad (4.22)$$

Equation (4.21) implies that $\frac{1}{pa}$ and $\frac{v_0}{a}$ must be real. The velocity v being dependent on the depth z, z is also a real quantity. From equation (4.22) it is deduced that x is real and consequently the whole ray is in the real space. Knowing that $\frac{a}{v_0}$ and z are real and using equation (4.17) gives

$$\frac{v(z)}{v_0} = 1 + \frac{a}{v_0} z \qquad (4.23)$$

which is a real quantity. This implies that the argument is independent of z, i.e the phase angle of v(z) is equal to the phase angle of v_0 and consequently using equation (2.28) Q is independent of z. This example is quite particular since the concept of a complex ray leads to a ray which lies in the real space.

These two examples show the nature of stationary rays in two very common cases in seismology.

The stationary ray method discussed in this chapter is then going to be used to compute synthetic seismograms for P-SV waves in layered viscoelastic media since it eliminates the arbitraryness in the choice of γ and obeys Fermat's principle. The complex phase function τ being calculated with the value of the ray parameter p, solution of equation (4.6), the other factors affecting the ray amplitude such as geometrical spreading and reflection/transmission coefficients also have to be calculated at the same value of p. Unfortunately the formulae already developed to evaluate these factors are inappropriate since they were obtained with a non-stationary ray approach. The determination of new expressions for geometrical spreading and reflection/transmission coefficients using a stationary ray approach is the object of chapters 5 and 6.

CHAPTER 5

GEOMETRICAL SPREADING IN VISCOELASTIC MEDIA

Krebes (1980) has treated the problem of body-wave propagation in a homogeneous isotropic linear viscoelsatic medium using asymptotic ray theory. He has then obtained an expression for the geometrical spreading of a viscoelatic wave generated at the surface by a point source. The development closely follows standard applications of asymptotic ray theory to elastic wave propagation (Cerveny and Ravindra (1971) and Hron and Kanasewich (1971)). In this chapter the main steps of Krebes' development, i.e the calculation of the equations required to obtain the amplitude coefficients and the determination of the zeroth order amplitude coefficients for P and S waves are presented here for the sake of completeness. Complex rays, which were introduced in chapter 4 are now used to obtain an expression for the geometrical spreading in a viscoelastic medium.

Let us recall the equation of motion for linear viscoelastic waves in a homogeneous isotropic medium:

$$\left[\lambda(t) + \mu(t)\right] * d\left(\nabla\left(\nabla \cdot \vec{u}\right)\right) + \mu(t) * d\left(\nabla^2\vec{u}\right) = \rho\ddot{\vec{u}} \qquad (5.1)$$

The general solution of equation (5.1) is given by

$$\vec{u}(\vec{r},\ t) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} \vec{W}(\vec{r},\ \omega)\ S(\omega)\ e^{i\omega[t - \tau(\vec{r},\ \omega)]}\ d\omega \qquad (5.2)$$

where $S(\omega)$ is the spectrum of the source pulse and τ the complex phase function already mentioned in chapters 2 and 4. $\vec{W}$ includes quantities related to reflection and transmission at boundaries, the geometrical spreading etc. Asymptotic ray theory is now used in order to solve equation (5.1). $\vec{W}$ has to be expanded as a power series in $\left(\frac{1}{i\omega}\right)$ with coefficients $\vec{W}_n = (\vec{r},\ \omega)$ which are generally frequency dependent for viscoelastic waves. Using Taylor's expansion, $\vec{W} = (\vec{r},\ \omega)$ can be written as

$$\vec{W} = (\vec{r},\ \omega_0 + \Delta\omega) = \sum_{n=0}^{\infty}\left[\frac{\partial^n W}{\partial\omega^n}\right]_{\omega=\omega_0}\frac{(\Delta\omega)^n}{n!} \qquad (5.3)$$

Inserting equation (5.3) into (5.2) yields

$$\vec{u}(\vec{r},\ t) = \frac{1}{\Pi}\ \mathrm{Re}\int_{\omega_1}^{\infty}\sum_{n=0}^{\infty}\vec{W}_n(\vec{r},\ \omega)\ s_n(\xi,\ \omega)\ d\omega \qquad (5.4)$$

where

$$s_n(\xi,\ \omega) = \frac{S(\omega)}{(i\omega)^n}\ e^{i\omega\xi}$$

and (5.5)

$$\xi = t - \tau(\vec{r},\ \omega)\ ^{\dagger}$$

Using the following properties of s_n:

$$s_n' = \frac{ds_n}{d\xi} = s_{n-1}$$

and (5.6)

$$\nabla s_n = -s_{n-1}\nabla\tau$$

expressions for $\nabla(\nabla\cdot\vec{u})$, $\nabla^2\vec{u}$ and $\ddot{\vec{u}}$ are obtained

$$\nabla(\nabla\cdot\vec{u}) = \frac{1}{\Pi}\,\mathrm{Re}\int_{\omega_1}^{\infty}\sum_{n=0}^{\infty}[\nabla(\nabla\cdot\vec{W}_n)s_n - s_{n-1}\nabla\tau(\nabla\cdot\vec{W}_n) + s_{n-2}(\nabla\tau\cdot\vec{W}_n)\nabla\tau - s_{n-1}\nabla(\nabla\tau\cdot\vec{W}_n)]\;d\omega \tag{5.7}$$

$$\nabla^2\vec{u} = \frac{1}{\Pi}\,\mathrm{Re}\int_{\omega_1}^{\infty}\sum_{n=0}^{\infty}[s_{n-2}(\nabla\tau)^2\vec{W}_n - s_{n-1}(2\nabla\tau\cdot\nabla\vec{W}_n + \vec{W}_n\nabla^2\tau) + s_n\nabla^2\vec{W}_n]\;d\omega \tag{5.8}$$

$$\ddot{\vec{u}} = \frac{1}{\Pi}\,\mathrm{Re}\int_{\omega_1}^{\infty}\sum_{n=0}^{\infty}\vec{W}_n s_{n-2}\;d\omega \tag{5.9}$$

Inserting equation (5.7), (5.8) and (5.9) into the equation of motion yields

$$\rho \ \mathrm{Re}\int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} \vec{W}_n s_{n-2} \ d\omega = \mathrm{Re}[(\lambda + \mu) * d\{\int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} [s_{n-2} \ \nabla\tau(\nabla\tau\cdot\vec{W}_n)$$

$$- s_{n-1}(\nabla(\nabla\tau\cdot\vec{W}_n) + \nabla\tau(\nabla\cdot\vec{W}_n)) + s_n \ \nabla(\nabla\cdot\vec{W}_n)] \ d\omega\}$$

$$+ \mu * d\{\int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} [s_{n-2} \ (\nabla\tau)^2\vec{W}_n - s_{n-1}(2\nabla\tau\cdot\nabla\vec{W}_n + \vec{W}_n\nabla^2\tau) + s_n \ \nabla^2\vec{W}_n] \ d\omega\} \tag{5.10}$$

In equation (5.10) the general term has the form

$$1(t) * d\left\{\int_{\omega_1}^{\infty} s_{n-m}(\xi, \ \omega) \ F_n(\vec{r}, \ \omega) \ d\omega\right\} \qquad m = 0, \ 1, \ 2 \tag{5.11}$$

which is equal to

$$\begin{aligned} 1(t) * d\{\ \} &= \int_{-\infty}^{t} 1(t-\eta) \frac{\partial}{\partial\eta}\left\{\int_{\omega_1}^{\infty} s_{n-m}(\eta-\tau, \ \omega) \ F_n(\vec{r}, \ \omega) \ d\omega\right\} d\eta \\ &= \int_{-\infty}^{t} 1(t-\eta) \left\{\int_{\omega_1}^{\infty} \frac{\partial(s_{n-m}(\eta-\tau, \ \omega))}{\partial\eta} F_n(\vec{r}, \ \omega) \ d\omega\right\} d\eta \\ &= \int_{-\infty}^{t} 1(t-\eta) \left\{\int_{\omega_1}^{\infty} s_{n-m-1}(\eta-\tau, \ \omega) \ F_n(\vec{r}, \ \omega) \ d\omega\right\} d\eta \end{aligned} \tag{5.12}$$

where equation (5.5) was used.

Let ϕ be equal to $(t - \eta)$ equation (5.12) can be written as

$$l(t) * d\{\ \} = \int_0^{\infty} l(\phi) \left\{ \int_{\omega_1}^{\infty} s_{n-m-1}(t-\phi-\tau,\ \omega)\ F_n(\vec{r},\ \omega)\ d\omega \right\} d\phi \tag{5.13}$$

Using equation (5.5) again, we can show that

$$s_{n-m-1}(t - \phi - \tau,\ \omega) = s_{n-m}(\xi,\ \omega)\ i\omega e^{-i\omega\phi} \tag{5.14}$$

Equation (5.12) is finally expressed as

$$l(t) * d\{\ \} = \int_0^{\infty} l(\phi) \left\{ \int_{\omega_1}^{\infty} i\omega e^{-i\omega\phi}\ s_{n-m}(\xi,\ \omega)\ F_n(\vec{r},\ \omega)\ d\omega \right\} d\phi$$

$$= \int_{\omega_1}^{\infty} \left\{ i\omega \int_0^{\infty} l(\phi)\ e^{-i\omega\phi}\ d\phi \right\} s_{n-m}(\xi,\ \omega)\ F_n(\vec{r},\ \omega)\ d\omega \tag{5.15}$$

Letting

$$L(\omega) = i\omega \int_0^{\infty} l(t)\ e^{-i\omega t}\ dt$$

as in (2.10) and using (5.15), equation (5.11) is equal to

$$l(t) * d\left\{ \int_{\omega_1}^{\infty} s_{n-m}(\xi,\ \omega)\ F_n(\vec{r},\ \omega)\ d\omega \right\} = \int_{\omega_1}^{\infty} s_{n-m}(\xi,\ \omega)\ L(\omega)\ F_n(\vec{r},\ \omega)\ d\omega \tag{5.16}$$

Consequently using (5.16) and (2.10) we can write for instance

$$\lambda(t) * d\{\int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} [s_{n-2}\nabla\tau(\nabla\tau\cdot\vec{W}_n) - s_{n-1}(\nabla(\nabla\tau\cdot\vec{W}_n) + \nabla\tau(\nabla\cdot\vec{W}_n)) +$$

$$s_n\nabla(\nabla\cdot\vec{W}_n)\ ]\ d\omega\} = \int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} [\Lambda s_{n-2}\nabla\tau(\nabla\tau\cdot\vec{W}_n) -$$

$$\Lambda s_{n-1}(\nabla(\nabla\tau\cdot\vec{W}_n) + \nabla\tau(\nabla\cdot\vec{W}_n)) + \Lambda s_n\nabla(\nabla\cdot\vec{W}_n)\ ]\ d\omega$$

Equation (5.10) then becomes

$$\mathrm{Re}\sum_{n=0}^{\infty}\int_{\omega_1}^{\infty} s_{n-2}(\rho\vec{W}_n)\,d\omega = \mathrm{Re}\sum_{n=0}^{\infty}\{\int_{\omega_1}^{\infty} s_{n-2}(\Lambda + M)\ \nabla\tau(\nabla\tau\cdot\vec{W}_n)\,d\omega$$

$$- \int_{\omega_1}^{\infty} s_{n-1}(\Lambda + M)(\nabla(\nabla\tau\cdot\vec{W}_n) + \nabla\tau(\nabla\cdot\vec{W}_n))\,d\omega + \int_{\omega_1}^{\infty} s_n(\Lambda + M)(\nabla(\nabla\cdot\vec{W}_n))\,d\omega$$

$$+ \int_{\omega_1}^{\infty} s_{n-2}M(\nabla\tau)^2\vec{W}_n\,d\omega - \int_{\omega_1}^{\infty} s_{n-1}M(2(\nabla\tau\cdot\nabla)\vec{W}_n + \vec{W}_n\nabla^2\tau)\,d\omega$$

$$+ \int_{\omega_1}^{\infty} s_n M\nabla^2\vec{W}_n\,d\omega\} \tag{5.17}$$

Another way of writing this equuation is

$$\mathrm{Re}\sum_{n=0}^{\infty}\int_{\omega_1}^{\infty}[s_{n-2}\vec{N}(\tau,\ \vec{W}_n) - s_{n-1}\vec{M}(\tau,\ \vec{W}_n) + s_n\vec{L}(\vec{W}_n)]\,d\omega = 0 \tag{5.18}$$

where

$$
\begin{aligned}
\vec{N}(\tau,\ \vec{W}_n) &= (\Lambda + M)\nabla\tau(\nabla\tau\cdot\vec{W}_n) + \left(M(\nabla\tau)^2 - \rho\right)\vec{W}_n \\
\vec{M}(\tau,\ \vec{W}_n) &= (\Lambda + M)\left(\nabla(\nabla\tau\cdot\vec{W}_n) + \nabla\tau(\nabla\cdot\vec{W}_n)\right) + \\
&\quad M\left(\vec{W}_n\nabla^2\tau + 2(\nabla\tau\cdot\nabla)\vec{W}_n\right) \\
\vec{L}(\vec{W}_n) &= (\Lambda + M)\,\nabla(\nabla\cdot\vec{W}_n) + M\nabla^2\vec{W}_n
\end{aligned}
\tag{5.19}
$$

Defining $\vec{W}_{-1} = \vec{W}_{-2} = 0$, equation (5.18) can be rewritten as

$$
\mathrm{Re}\sum_{n=-2}^{\infty}\int_{\omega_1}^{\infty}\left[s_n(\xi,\ \omega)\left(\vec{N}(\tau,\ \vec{W}_{n+2}) - \vec{M}(\tau,\ \vec{W}_{n+1}) + \vec{L}(\vec{W}_n)\right)\right]d\omega = 0 \tag{5.20}
$$

Equation (5.20) is true if and only if

$$
\vec{N}(\tau,\ \vec{W}_{n+2}) - \vec{M}(\tau,\ \vec{W}_{n+1}) + \vec{L}(\tau,\ \vec{W}_n) = 0 \qquad n = -2,\ -1,\ .. \tag{5.21}
$$

or

$$
\begin{aligned}
&\vec{N}(\tau,\ \vec{W}_0) = 0 \\
&\vec{N}(\tau,\ \vec{W}_1) - \vec{M}(\tau,\ \vec{W}_0) = 0 \\
&\vec{N}(\tau,\ \vec{W}_{n+2}) - \vec{M}(\tau,\ \vec{W}_{n+1}) + \vec{L}(\tau,\ \vec{W}_n) = 0 \qquad n = 0,\ 1,\ 2\ \ldots
\end{aligned}
\tag{5.22}
$$

Using

$$
\vec{N}(\tau,\ \vec{W}_0) = 0
$$

two new equations are obtained

$$\vec{N}(\tau,\ \vec{W}_0)\cdot\nabla\tau = (\Lambda + M)\ \nabla\tau\cdot\nabla\tau\ (\nabla\tau\cdot\vec{W}_0) + (M(\nabla\tau)^2 - \rho)\ (\vec{W}_0\cdot\nabla\tau)$$
$$= [(\Lambda + 2M)\ (\nabla\tau)^2 - \rho]\ (\vec{W}_0\cdot\nabla\tau) = 0 \qquad (5.23)$$

and

$$\vec{N}(\tau,\ \vec{W}_0)\times\nabla\tau = (\Lambda + M)\ (\nabla\tau\times\nabla\tau)\ (\nabla\tau\cdot\vec{W}_0) + (M(\nabla\tau)^2 - \rho)\ (\vec{W}_0\times\nabla\tau)$$
$$= [M\ (\nabla\tau)^2 - \rho]\ (\vec{W}_0\times\nabla\tau) = 0 \qquad (5.24)$$

$\vec{W}_0\cdot\nabla\tau$ and $\vec{W}_0\times\nabla\tau$ are not generally equal to 0 simultaneously. The same note can be made for $(\Lambda + 2M)\ (\nabla\tau)^2 - \rho$ and $(M\ (\nabla\tau)^2 - \rho)$. Consequently the system formed by equations (5.23) and (5.24) has two solutions, i.e

$$(\nabla\tau)^2 = \frac{\rho}{\Lambda + 2M} = \frac{1}{\alpha^2}, \qquad \vec{W}_0\times\nabla\tau = 0$$

and (5.25)

$$(\nabla\tau)^2 = \frac{\rho}{M} = \frac{1}{\beta^2}, \qquad \vec{W}_0\cdot\nabla\tau = 0$$

These are the eikonal equations. As already mentioned in chapter 2, the complex phase function τ is equal to

$$\tau = \frac{\vec{k}\cdot\vec{r}}{\omega} = \frac{\vec{P}\cdot\vec{r}}{\omega} + i\frac{\vec{A}\cdot\vec{r}}{\omega} \qquad (5.26)$$

For either P or S waves $\nabla\tau$ is equal to

$$\nabla\tau = \frac{\vec{k}}{\omega} = \frac{\vec{P}}{\omega} + i\frac{\vec{A}}{\omega}$$

$$(\nabla\tau)^2 = \frac{\vec{k}\cdot\vec{k}}{\omega^2} = \frac{k^2}{\omega^2} \qquad (5.27)$$

hence equation (5.25) and (2.13) are identical. Using the equations (5.22) the amplitude coefficients $\vec{W}_n$ can be calculated. A 0^{th} order approximation is considered, it is then not necessary to know $\vec{W}_n$ for all n. Only the zeroth order amplitude coefficient, i.e $\vec{W}_0$, for P and S waves are calculated.

5.1 P Waves

For P waves

$$\vec{W}_0 = W_0\hat{t}_P \qquad (5.28)$$

where $\hat{t}_P$ is the complex unit vector for P waves and it is equal to

$$\hat{t}_P = \frac{\vec{k}_P}{|k_P|} = \alpha\,\nabla\tau \qquad (5.29)$$

To determine $\vec{W}_0$, $\vec{M}(\tau, \vec{W}_0)\cdot\nabla\tau$ is calculated:

$$\vec{M}(\tau,\ \vec{w}_0)\cdot\nabla\tau = (\Lambda + M)\left(\nabla(\nabla\tau\cdot\vec{w}_0)\cdot\nabla\tau + (\nabla\tau)^2(\nabla\cdot\vec{w}_0)\right) + M\left(\nabla^2\tau(\nabla\tau\cdot\vec{w}_0) + 2((\nabla\tau\cdot\nabla)\vec{w}_0)\cdot\nabla\tau\right) \qquad (5.30)$$

It is known that

$$(\nabla\tau)^2 = \frac{1}{\alpha^2}$$

$$\nabla\cdot\vec{w}_0 = \nabla\cdot(w_0\alpha\nabla\tau) = \alpha\nabla\tau\cdot\nabla w_0 + \alpha w_0\nabla^2\tau$$

$$\vec{w}_0\cdot\nabla\tau = w_0\alpha\ \nabla\tau\cdot\nabla\tau = w_0\alpha(\nabla\tau)^2 = \frac{w_0}{\alpha}$$

$$(\nabla\tau\cdot\nabla)\vec{w}_0 = (\nabla\tau\cdot\nabla)w_0\hat{t}_p = (\nabla\tau\cdot\nabla w_0)\ \hat{t}_p$$

(since the medium is homogeneous)

$$\nabla\tau\cdot((\nabla\tau\cdot\nabla)\vec{w}_0) = \nabla\tau\cdot\hat{t}_p(\nabla\tau\cdot\nabla w_0) = \frac{1}{\alpha}(\nabla\tau\cdot\nabla w_0)$$

hence

$$\begin{aligned}\vec{M}(\tau,\ \vec{w}_0)\cdot\nabla\tau &= (\Lambda + M)\left[\frac{1}{\alpha}(\nabla w_0\cdot\nabla\tau) + \frac{1}{\alpha}w_0\nabla^2\tau + \frac{\nabla w_0}{\alpha}\nabla\tau\right] + M\left[\frac{w_0}{\alpha}\nabla^2\tau + \frac{2}{\alpha}(\nabla\tau\cdot\nabla w_0)\right] \\ &= \frac{(\Lambda + 2M)}{\alpha}\left[w_0\nabla^2\tau + 2(\nabla\tau\cdot\nabla w_0)\right] \\ &= \rho\alpha\left[w_0\nabla^2\tau + 2(\nabla\tau\cdot\nabla w_0)\right] \qquad (5.31)\end{aligned}$$

and then

$$\begin{aligned}\left[\vec{M}(\tau,\ \vec{w}_0)\cdot\nabla\tau\right] w_0 &= \rho\alpha\left[w_0^2\nabla^2\tau + 2(\nabla\tau\cdot\nabla w_0)\right] \\ &= \rho\alpha\left[w_0^2\nabla^2\tau + \nabla\tau\cdot\nabla(w_0)^2\right] \\ &= \rho\alpha\ \nabla\cdot\left[(w_0)^2\nabla\tau\right] \qquad (5.32)\end{aligned}$$

$\vec{N}(\tau,\ \vec{w}_n)\cdot\nabla\tau$ also has to be calculated

$$\begin{aligned}\vec{N}(\tau,\ \vec{w}_n)\cdot\nabla\tau &= (\Lambda + M)(\vec{w}_n\cdot\nabla\tau)(\nabla\tau)^2 + \left[M(\nabla\tau)^2 - \rho\right](\vec{w}_n\cdot\nabla\tau) \\ &= (\vec{w}_n\cdot\nabla\tau)\left[(\Lambda + M)(\nabla\tau)^2 + M(\nabla\tau)^2 - \rho\right] \\ &= \left[\frac{(\Lambda + 2M)}{\alpha^2} - \rho\right](\vec{w}_n\cdot\nabla\tau) = 0 \qquad (5.33)\end{aligned}$$

The second equation of (5.22) which is

$$\vec{M}(\tau, \vec{W}_1) - \vec{M}(\tau, \vec{W}_0) = 0$$

implies that

$$\vec{M}(\tau, \vec{W}_0) \cdot \nabla\tau = 0 \tag{5.34}$$

and consequently

$$\nabla \cdot \left[(W_0)^2 \nabla\tau\right] = 0 \tag{5.35}$$

The concept of a ray tube about a central ray can now be introduced. It is defined as a very narrow pencil of rays connecting two wavefronts of the central ray at different times t_0 and t with $t_0 < t$ (Figure 4) ΔS is the length along the ray path in the direction of $\vec{P}$ between the two wavefront surfaces $d\sigma(S_0)$ and $d\sigma(S)$. ΔS and the distances between the rays bounding the ray tube are infinitesimal quantities. Using Gauss' divergence theorem when the integration over the volume V of the ray tube is performed leads to

$$\int_V \nabla \cdot \left[(W_0)^2 \nabla\tau\right] dv = \int_\sigma (W_0)^2 \nabla\tau \cdot d\vec{\sigma} = 0 \tag{5.36}$$

where $d\sigma$ is an element of area on the ray tube. Knowing that $\nabla\tau$ is equal to

$$\nabla\tau = \frac{1}{\omega}\left(\vec{P} + i\vec{A}\right) = \frac{1}{\omega}\left(P\hat{P} + iA\hat{A}\right) \qquad (5.37)$$

where $\hat{P}$ and $\hat{A}$ are real unit vectors and

$$\hat{P}\cdot d\vec{\sigma} = d\sigma, \qquad \hat{A}\cdot d\vec{\sigma} = d\sigma\cos\gamma \qquad (5.38)$$

the following equation is determined

$$\begin{aligned}\int_\sigma (W_0)^2\nabla\tau\cdot d\vec{\sigma} &= \int_\sigma (W_0)^2\frac{P}{\omega}\hat{P}\cdot d\vec{\sigma} + i\int_\sigma (W_0)^2\frac{A}{\omega}\hat{A}\cdot d\vec{\sigma} \\ &= \frac{P}{\omega}\left(\left[(W_0)^2 d\sigma\right]_S - \left[(W_0)^2 d\sigma\right]_{S_0}\right) \\ &\quad + i\frac{A\cos\gamma}{\omega}\left(\left[(W_0)^2 d\sigma\right]_S - \left[(W_0)^2 d\sigma\right]_{S_0}\right) = 0 \end{aligned} \qquad (5.39)$$

Equation (5.39) implies that

$$\left(\left[(W_0)^2 d\sigma\right]_S - \left[(W_0)^2 d\sigma\right]_{S_0}\right) = 0 \qquad (5.40)$$

Consequently

$$W_0(S) = \sqrt{\frac{d\sigma(S_0)}{d\sigma(S)}}\, W_0(S_0) \qquad (5.41)$$

Equation (5.41) allows to calculate W_0 at any point S on the ray if W_0 is known at S_0. It has the same form as the corresponding result obtained in the elastic case (Cerveny

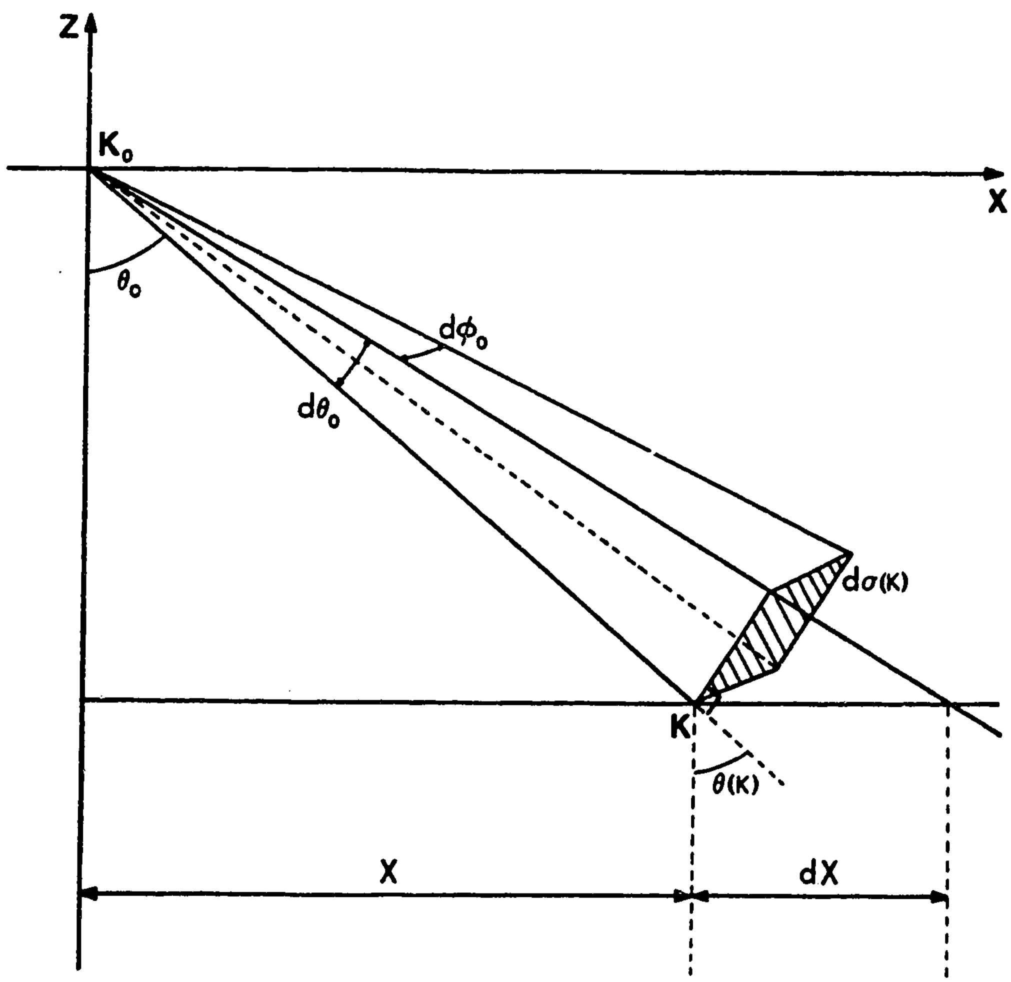


Figure 4: Ray tube between K_0 and K for a homogeneous medium

and Ravindra (1971)). This was expected because it is a purely geometrical result describing the geometrical spreading of a wave. The attenuation term is taken into account by the imaginary part of τ, or in other words by the expression of the geometrical spreading (see last paragraph of this chapter).

5.2 S Waves

For S waves

$$\vec{W}_0 = W_0^2\hat{t}_2 + W_0^3\hat{t}_3 \tag{5.42}$$

where $\hat{t}_2$ and $\hat{t}_3$ are complex unit vectors. $\hat{t}_2$ and $\hat{t}_3$ are given by

$$\hat{t}_2 = \hat{y}\times\hat{k}_S , \qquad \hat{t}_3 = \hat{y} \tag{5.43}$$

where $\hat{y} = \hat{n}$, the real unit vector perpendicular to $\vec{P}$ and $\vec{A}$ (see chapter 3). The vectors $\hat{t}_P$, $\hat{t}_2$, $\hat{t}_3$ are orthogonal. To determine $\vec{W}_0$, $\vec{M}(\tau, \vec{W}_0)\cdot\hat{t}_i$, i = 2, 3 has to be calculated. $\nabla\tau$ having the same direction as $\hat{t}_p$,

$$\hat{t}_i \cdot \nabla\tau = 0 \qquad i = 2,\ 3$$

and (5.44)

$$\vec{W}_0 \cdot \nabla\tau = 0 \qquad \text{for S waves}$$

Consequently

$$\vec{M}(\tau,\ \vec{W}_0) \cdot \hat{t}_i = M\left[2\left((\nabla\tau \cdot \nabla)\vec{W}_0\right) \cdot \hat{t}_i + \vec{W}_0 \nabla^2\tau \cdot \hat{t}_i\right] \qquad i = 2,\ 3 \tag{5.45}$$

but

$$\begin{aligned} \vec{W}_0 \cdot \hat{t}_i &= \left(W_0^2 \hat{t}_2 + W_0^3 \hat{t}_3\right) \cdot \hat{t}_2 + \left(W_0^2 \hat{t}_2 + W_0^3 \hat{t}_3\right) \cdot \hat{t}_3 \\ &= W_0^2 + W_0^3 = W_0^i \end{aligned} \tag{5.46}$$

and

$$\begin{aligned} \left\{(\nabla\tau \cdot \nabla)\vec{W}_0\right\} \cdot \hat{t}_i &= \left(\nabla\tau \cdot \nabla W_0^2\right)\hat{t}_2 \cdot \hat{t}_i + \left(\nabla\tau \cdot \nabla W_0^3\right)\hat{t}_3 \cdot \hat{t}_i \\ &= \left(\nabla\tau \cdot \nabla W_0^2\right) + \left(\nabla\tau \cdot \nabla W_0^3\right) \\ &= \left(\nabla\tau \cdot \nabla W_0^i\right) \qquad i = 2,\ 3 \end{aligned} \tag{5.47}$$

Equation (5.45) can then be written as

$$\vec{M}(\tau,\ \vec{W}_0) \cdot \hat{t}_i = M\left[W_0^i \nabla^2\tau + 2\left(\nabla\tau \cdot \nabla W_0^i\right)\right] \tag{5.48}$$

and

$$\begin{aligned} \left[\vec{M}(\tau,\ \vec{W}_0) \cdot \hat{t}_i\right] W_0^i &= M\left[\left(W_0^i\right)^2 \nabla^2\tau + 2\left(\nabla\tau \cdot \nabla\left(W_0^i\right)^2\right)\right] \\ &= M\left[\nabla \cdot \left(\nabla\tau\, \nabla\left(W_0^i\right)^2\right)\right] \end{aligned} \tag{5.49}$$

For S waves

$$(\nabla\tau)^2 = \frac{1}{\beta^2} = \frac{\rho}{M} \tag{5.50}$$

hence

$$\vec{N}(\tau, \vec{w}_n) = (\Lambda + M)(\vec{w}_n \cdot \nabla\tau)\nabla\tau + \left[M(\nabla\tau)^2 - \rho\right]\vec{w}_n$$
$$= (\Lambda + M)(\vec{w}_n \cdot \nabla\tau)\nabla\tau \tag{5.51}$$

and

$$\vec{N}(\tau, \vec{w}_n)\cdot\hat{t}_i = 0 \qquad i = 2,\ 3 \tag{5.52}$$

Consequently, knowing that

$$\vec{N}(\tau, \vec{w}_1) - \vec{M}(\tau, \vec{w}_0) = 0$$

from equation (5.22) implies that

$$\vec{M}(\tau, \vec{w}_0)\cdot\hat{t}_i = 0 \qquad i = 2,\ 3 \tag{5.53}$$

and

$$\nabla\left[(w_0^i)^2\nabla\tau\right] = 0 \tag{5.54}$$

Equation (5.54) has the same form as equation (5.35) for P waves. Introducing the concept of ray tube again and using the same development as for the P wave case leads to

$$w_0^i(s) = \sqrt{\frac{d\sigma(s_0)}{d\sigma(s)}}\, w_0^i(s_0) \qquad i = 2,\ 3 \tag{5.55}$$

which once again has the same form as the corresponding result in the elastic case (Cerveny and Ravindra (1971)).

5.3 Geometrical Spreading

An expression for the geometrical spreading of a ray propagating through a layered homogeneous isotropic linear viscoelastic medium is now calculated. As mentioned at the begining of this chapter the notion of complex ray is used to calculate the geometrical spreading for this case. Although it is difficult to give physical interpretations to complex rays; an expression for the geometrical spreading can [illegible]ed without concern for the fact that the raypath c[illegible]tes are complex quantities. For instance in cha[illegible]r 4, the complex phase function τ is determined for a given complex ray, knowing its complex ray parameter p. Re(τ) is the real travel time and Im(τ) is the attenuation due to absorption. The same approach is used in the calculation of the geometrical spreading in linear viscoelastic media.

A ray of m segments propagating through a sequence of flat horizontal homogeneous layers is considered (Figure 5). O_j is the endpoint of the j^{th} ray segment at an interface. As demonstrated in paragraph (5.1) and (5.2) the amplitude coefficient at any point S on the raypath can be calculated using

$$W_0(S) = \sqrt{\frac{d\sigma(S_0)}{d\sigma(S)}}\, W_0(S_0) \qquad (5.56)$$

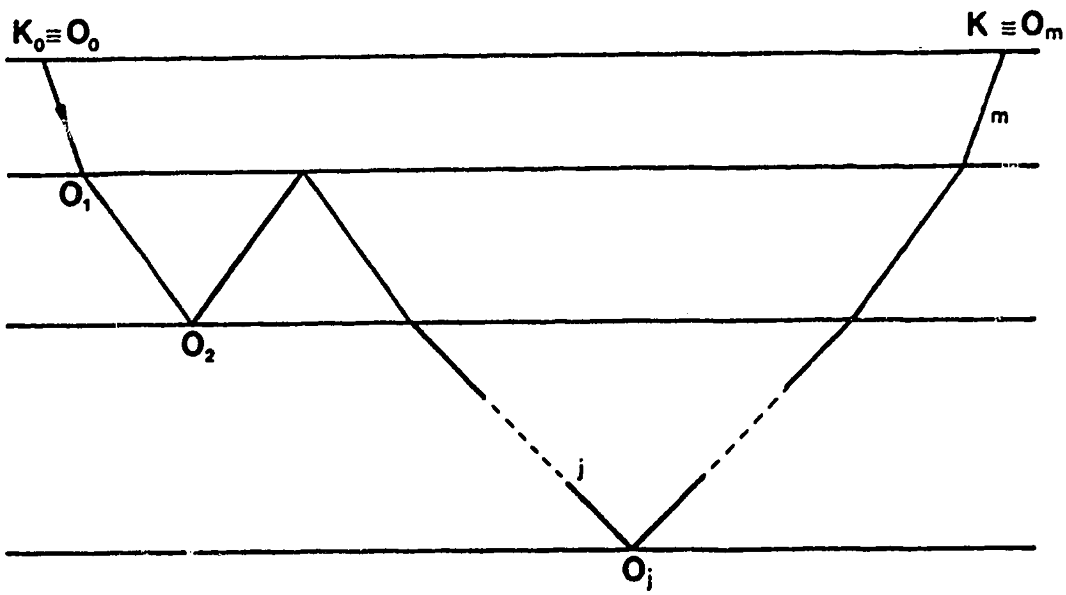


Figure 5: A typical ray in homogeneous layered medium

In the case of figure 5, the amplitude coefficient at the endpoint K is equal to

$$W(K) = \sqrt{\frac{d\sigma(K_0)}{d\sigma(O_1)}} \times R_1 \times \sqrt{\frac{d\sigma'(O_1)}{d\sigma(O_2)}} \times R_2 \ldots \ldots\ldots R_{m-1} \times \sqrt{\frac{d\sigma'(O_{m-1})}{d\sigma(O_m)}} \; W(K_0) \qquad (5.57)$$

where R_j, $j=1, 2, \ldots m-1$ is the reflection/transmission coefficient at O_j, the prime refers to quantities associated with reflected or transmitted wave at O_j and the unprimed quantities are associated with the incident wave at O_j. K_0 is considered to be at a unit distance from the source and $O_m = K$. Equation (5.57) can be written as

$$W(K) = \frac{1}{L(K,\; K_0)} \prod_{j=1}^{m-1} R_j W(K_0) \qquad (5.58)$$

where

$$L(K,\; K_0) = \sqrt{\frac{d\sigma(K)}{d\sigma(K_0)}} \prod_{j=1}^{m-1} \sqrt{\frac{d\sigma(O_j)}{d\sigma'(O_j)}} \qquad (5.59)$$

$L(K, K_0)$ is the geometrical spreading factor of the ray.

It now seems appropriate to introduce the notion of complex rays in the calculation of $L(K, K_0)$. To evaluate (5.59) an expression of $\frac{d\sigma(K)}{d\sigma(K_0)}$ has to be determined. As mentioned earlier it is dificult to visualise complex rays and to give them a physical interpretation. Figures 4 and 6 represent the ray tube and the change in the cross sectional area of the ray tube at an interface for real situation in real space but all the calculations will be performed with complex quantities defined in chapter 4. Consequently all the results obtained are in general complex. Starting with the cross-sectional area at K the expression of $d\sigma(K)$ in the complex plane is

$$d\sigma(K) = (\cos\alpha(K)\, dx)\left(x\, d\phi_0\right) \tag{5.60}$$

where x is the complex is the complex horizontal distance traversed by the ray, ϕ_0 is the complex azimuthal coordinate at K_0 and $\alpha(K)$ is the complex angle between the complex wave vector atK and the vertical z-axis. x is dependent on z and $\alpha_0 = \alpha(K_0)$ which is the complex angle α at the source. Hence the expression of the complex quantity dx is

$$dx = \frac{\partial x}{\partial \alpha_0}\, d\alpha_0 + \frac{\partial x}{\partial z}\, dz \tag{5.61}$$

From figure 4 we see that $dz = 0$. We then assume that $dz = 0$ for the complex ray too. The cross-section of a spherical wave at the source i.e $d\sigma_0 = d\sigma(K_0)$, in the complex space is given by $d\sigma_0 = \sin\alpha_0\, d\phi_0\, d\alpha_0$. Consequently

$$\frac{d\sigma(K)}{d\sigma(K_0)} = \frac{x\left(\dfrac{\partial x}{\partial \alpha_0}\right)\cos\alpha(K)\, d\alpha_0\, d\phi_0}{\sin\alpha_0\, d\alpha_0\, d\phi_0} = \frac{x\left(\dfrac{\partial x}{\partial \alpha_0}\right)\cos\alpha(K)}{\sin\alpha_0} \tag{5.62}$$

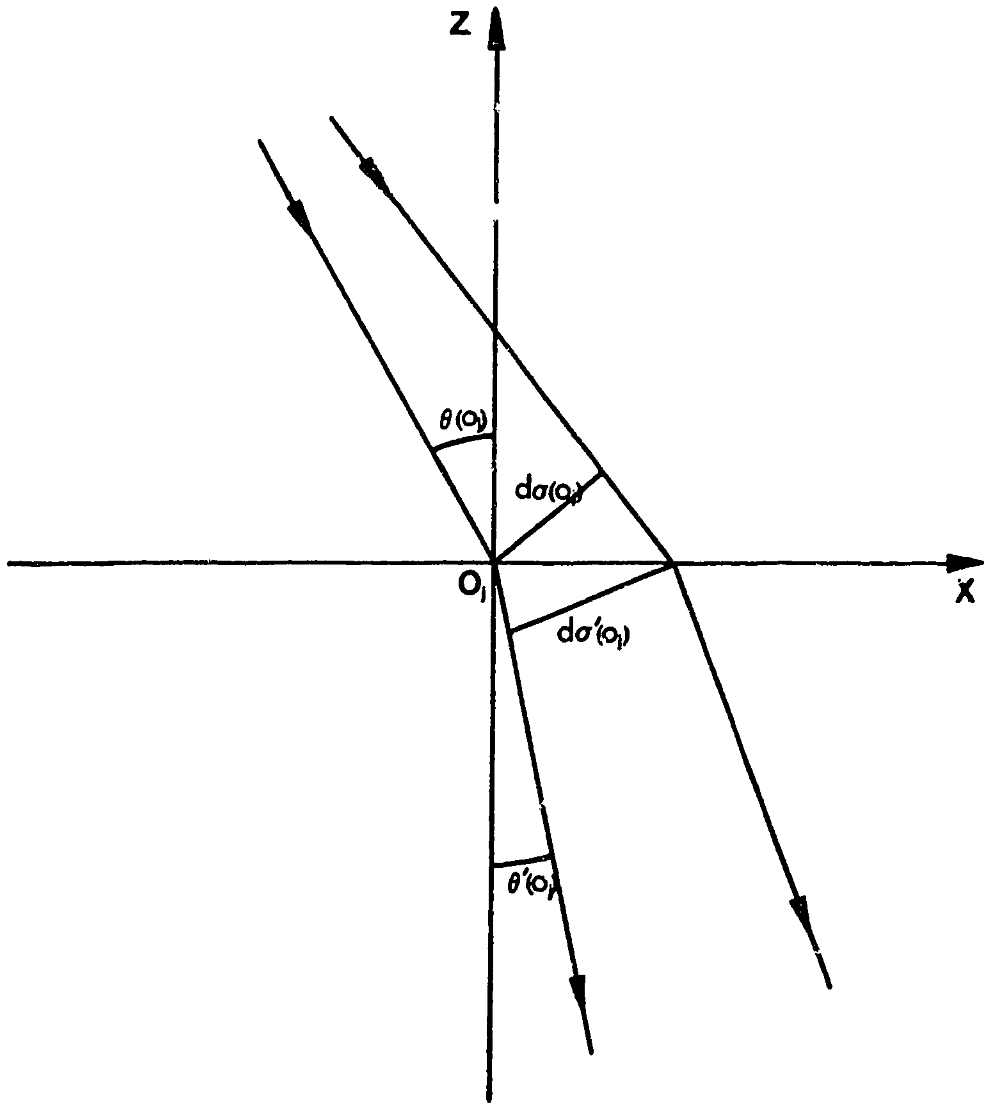


Figure 6: Change in the cross-sectional area of the ray tube at an interface

From Figure 6, drawn in real space, we can see that

$$\frac{d\sigma_R(O_j)}{d\sigma_R'(O_j)} = \frac{\cos\ \theta(O_j)}{\cos\ \theta'(O_j)}$$

where θ and θ' are the real angle of incidence and transmission and $d\sigma_R(O_j)$ and $d\sigma_R'(O_j)$ are the cross-sectional areas for the incident and transmitted real rays. It is reasonable to assume that the same property exists for complex rays with the complex angle of incidence and of transmission $\alpha(O_j)$ and $\alpha'(O_j)$, consequently we can write

$$\frac{d\sigma_R(O_j)}{d\sigma_R'(O_j)} = \frac{\cos\ \alpha(O_j)}{\cos\ \alpha'(O_j)} \tag{5.63}$$

The medium is homogeneous hence $\alpha'(O_j) = \alpha(O_j+1)$. Using this fact and equation (5.59) yield

$$\prod_{j=1}^{m-1}\sqrt{\frac{d\sigma_R(O_j)}{d\sigma_R'(O_j)}} = \sqrt{\frac{\cos\ \alpha(O_1)}{\cos\ \alpha(O_m)}} = \sqrt{\frac{\cos\ \alpha_0}{\cos\ \alpha(K)}} \tag{5.64}$$

Consequently, inserting equation (5.62) and (5.64) into (5.59) gives

$$L(K,\ K_0) = \sqrt{x\,\frac{\partial x}{\partial \alpha_0}\,\cot\alpha_0} \tag{5.65}$$

The complex epicentral distance is given by

$$x = \sum_{j=1} x_j \qquad (5.66)$$

where

$$x_j = h_j \tan\alpha_j \qquad (5.67)$$

as already seen in equation (4.6): h_j is the thickness of the layer containing the j^{th} ray segment, x_j is the complex horizontal distance traversed by the j^{th} ray segment and α_j is the complex incidence angle for the j^{th} ray segment.

$\frac{\partial x}{\partial \alpha_0}$ has to be calculated. It is known that

$$\tan\alpha_j = \frac{x_j}{h_j} = \frac{k_x}{k_{z_j}} \qquad (5.68)$$

where

$$k_{z_j} = +p.v.\sqrt{k^2 - k_x^2} \qquad \text{(P, SV, or SH waves)}$$

and using equation (3.22)

$$k_x = k_j \sin\alpha_j = k_0 \sin\alpha_0 \qquad (5.69)$$

The subscript '0' refers to quantities at K_0 and the '+' sign was chosen (see chapter 3). Consequently x_j is equal to

$$x_j = \frac{h_j\, k_0 \sin\alpha_0}{k_{z_j}} \qquad (5.70)$$

and

$$\frac{\partial x_j}{\partial \alpha_0} = h_j k_0 \left[\frac{k_{z_j} \cos\alpha_0 - \left(\frac{\partial k_{z_j}}{\partial \alpha_0}\right) \sin\alpha_0}{(k_{z_j})^2} \right] \tag{5.71}$$

$\left(\frac{\partial k_{z_j}}{\partial \alpha_0}\right)$ must be evaluated:

$$\left(\frac{\partial k_{z_j}}{\partial \alpha_0}\right) = \left(\frac{\partial \sqrt{\left(k_j^2 - k_x^2\right)}}{\partial \alpha_0}\right)$$
$$= \frac{1}{2k_{z_j}} \left(\frac{\partial k_j^2}{\partial \alpha_0} - \frac{\partial k_x^2}{\partial \alpha_0}\right) \tag{5.72}$$

Knowing that $k_j^2 = \frac{\omega^2}{v_j^2}$ and using equation (2.28) showing that v_j is independent of α_0 yields

$$\left(\frac{\partial k_{z_j}}{\partial \alpha_0}\right) = -\frac{k_x}{k_{z_j}} \frac{\partial k_x}{\partial \alpha_0} = -\frac{k_x}{k_{z_j}} k_0 \cos\alpha_0 \tag{5.73}$$

Equation (5.71) can be rewritten as

$$\frac{\partial x_j}{\partial \alpha_0} = \frac{h_j k_0}{k_{z_j}} \left[\cos\alpha_0 + \frac{k_0^2 \cos\alpha_0 \sin^2\alpha_0}{(k_{z_j})^2} \right] \tag{5.74}$$

and finally $\frac{\partial x}{\partial \alpha_0}$ is equal to

$$\frac{\partial x}{\partial \alpha_0} = \sum_{j=1}^{m} \frac{\partial x_j}{\partial \alpha_0} = k_0 \cos\alpha_0 \left(\sum_{j=1}^{m} \frac{h_j}{k_{z_j}} + \left(k_0^2 \sin^2\alpha_0\right) \sum_{j=1}^{m} \frac{h_j}{(k_{z_j})^3} \right) \tag{5.75}$$

Using equation (2.35) it is easy to show that

and

$$k_0^2 = \frac{\omega^2}{v_0^2} \tag{5.77}$$

hence

$$\frac{\partial x}{\partial \alpha_0} = k_0 \cos\alpha_0 \left(\sum_{j=1}^{m} \frac{v_j h_j}{\omega \cos\alpha_j} + \left(k_0^2 \sin^2\alpha_0\right) \sum_{j=1}^{m} \frac{v_j^3 h_j}{\omega^3 \cos^3\alpha_j} \right)$$

$$= k_0 \cos\alpha_0 \left(\sum_{j=1}^{m} \frac{v_j h_j}{\omega \cos\alpha_j} + \frac{\sin^2\alpha_0}{v_0^2} \frac{1}{\omega} \sum_{j=1}^{m} \frac{v_j^3 h_j}{\cos^3\alpha_j} \right)$$

$$= \frac{k_0 \cos\alpha_0}{\omega} \left[\sum_{j=1}^{m} \frac{h_j v_j}{v_0^2 \cos^3\alpha_j} \left(v_0^2 \cos^2\alpha_j + v_j^2 \sin^2\alpha_0\right) \right] \tag{5.78}$$

Equation (5.78) can be simplifyed. Snell's law in anelastic media shows that

$$v_j \sin\alpha_0 = v_0 \sin\alpha_j \tag{5.79}$$

consequently

$$\frac{\partial x}{\partial \alpha_0} = \frac{k_0 \cos\alpha_0}{\omega} \left[\sum_{j=1}^{m} \frac{h_j v_j}{\cos^3\alpha_j} \right] \tag{5.80}$$

Using equations (5.76), (5.70) and (5.66), an expression for x is obtained

$$x = \sum_{j=1}^{m} \frac{h_j k_0 \sin\alpha_0 v_j}{\omega \cos\alpha_j} = \frac{k_0 \sin\alpha_0}{\omega} \left[\sum_{j=1}^{m} \frac{h_j v_j}{\cos\alpha_j} \right] \tag{5.81}$$

Finally inserting equations (5.80) and (5.81) into (5.65) yields

$$L(K, K_0) = \sqrt{\frac{k_0 \sin\alpha_0}{\omega}\left[\sum_{j=1}^{m}\frac{h_j v_j}{\cos\alpha_j}\right]\frac{k_0 \cos\alpha_0}{\omega}\left[\sum_{j=1}^{m}\frac{h_j v_j}{\cos^3\alpha_j}\right]\frac{\cos\alpha_0}{\sin\alpha_0}}$$

$$L(K, K_0) = \frac{\cos\alpha_0}{v_0}\sqrt{\left[\sum_{j=1}^{m}\frac{h_j v_j}{\cos\alpha_j}\right]\left[\sum_{j=1}^{m}\frac{h_j v_j}{\cos^3\alpha_j}\right]} \qquad (5.82)$$

The expression of the geometrical spreading $L(K, K_0)$ for viscoelastic media obtained using the complex ray method has the same form as the one given by Cerveny and Ravindra (1971) for perfectly elastic media. The difference is that the velocities and the angles are complex and frequency dependent in the viscoelastic case. Krebes and Hearn (1990) mentioned that the ray theory formula for the geometrical spreading, obtained using the complex ray method has to be identical in form to that for perfectly elastic medium due to the elastic-viscoelastic principle. This principle states that exact solutions for viscoelastic problems can be obtained by replacing appropriate parameters from exact solution of elastic problems with complex frequency-dependent ones in the frequency domain.

CHAPTER 6

REFLECTION AND TRANSMISSION COEFFICIENTS

As already mentioned the problem of the reflection and transmission of harmonic plane waves incident on a plane boundary separating two homogeneous isotropic linear viscoelastic media was treated using a non-stationary ray approaach (Krebes 1980). In this chapter the same problem is treated with a stationary ray approach. This introduces the notion of complex rays in the treatment (see chapter 4). As in chapter 5 the figures enclosed in chapter 6 show the problem in real space but all the calculations are performed with complex quantities from following the concept of analytical continuation from real axis into a complex plane. In the first part, the notations and assumptions used to study the above problem are stated. The second part is concerned with the determination, from the boundary conditions, of the system of 6 equations in order to obtain the reflection and transmission coefficients. This system is then solved in the last part.

6.1 Notations and Assumptions

In 'Introduction to ART in Seismology' (Hron 1984), the problem of reflection and transmission for elastic waves is treated using some notations and assumptions. Similar notations and assumptions can be used in this chapter keeping in mind that we now work with complex rays. This means that the system of the wave vectors are complex. Figure 7 shows the intersection of the plane of incidence of the ray with the boundary Σ in real space. The upper and lower media are both homogeneous isotropic elastic media. Medium 1, i.e the upper medium, into which the real z-axis is oriented along the normal to the interface Σ at the point of incidence O, has Lame parameters λ_1, μ_1 and volume density ρ_1 and contains the incident and reflected P and S waves. Medium 2, i.e the lower medium, has Lame parameters λ_2, μ_2 and volume density ρ_2 and contains the transmitted P and S waves. In the viscoelastic case the situation is similar but both media are homogeneous isotropic linear viscoelastic, Λ_1, M_1 and Λ_2, M_2 are now complex and frequency dependent and are given by equations (2.10). In figure 7 the surface Σ is assumed to be smooth in the vicinity of the point of incidence O allowing the construction of tangent plane intersecting the plane of incidence in the x-axis. The positive orientation of the x-axis is chosen in such a way that

$$\left(\nabla\tau_0\right)\cdot\vec{\varepsilon}_x > 0 \qquad (6.1)$$

$\vec{\varepsilon}_z$ is the unit vector along the z-axis. The direction and orientation of the third Cartesian axis y is given by

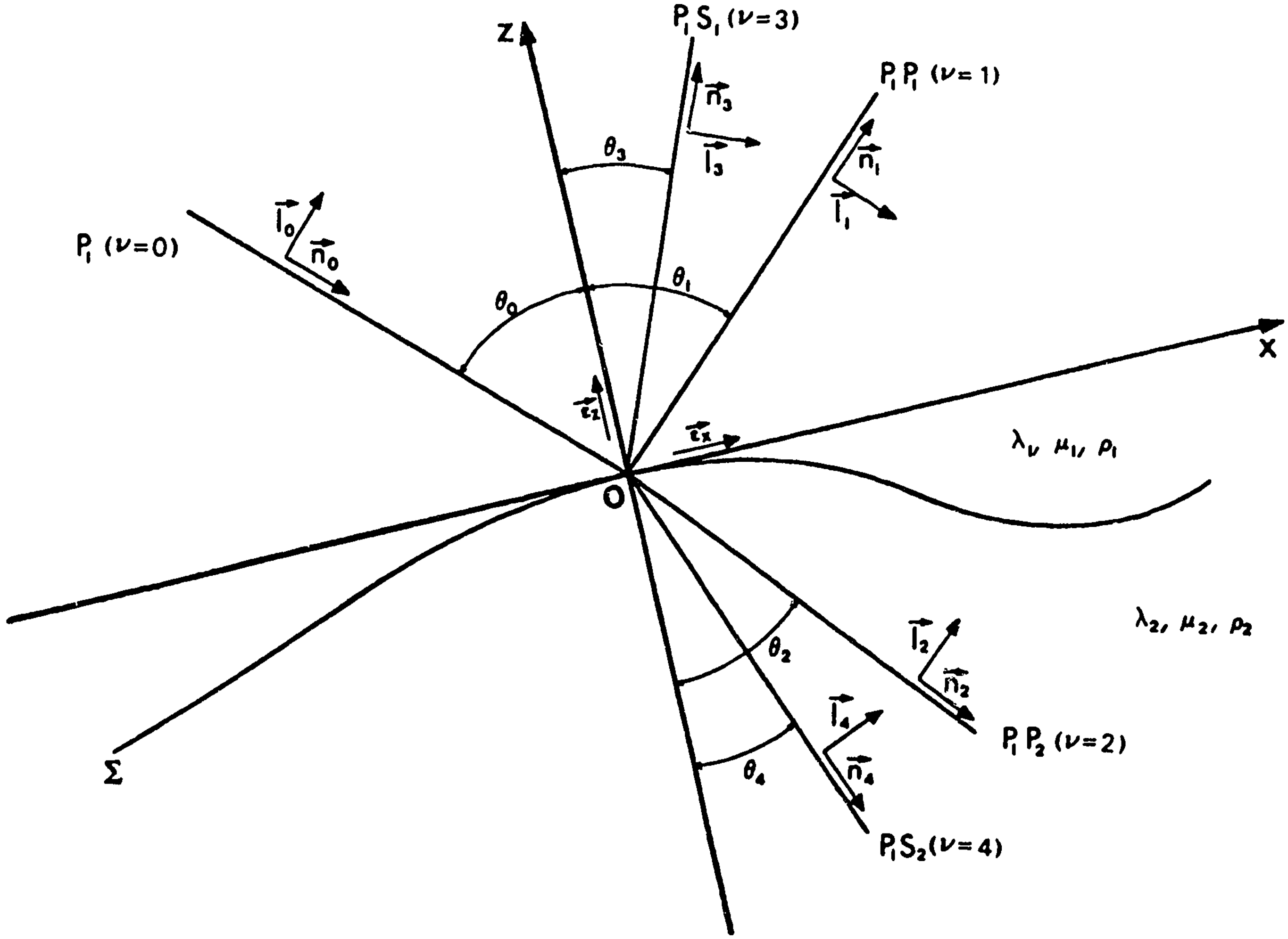


Figure 7: Intersection of the plane of incidence of the ray with the boundary Σ

$$\vec{\varepsilon}_y = \vec{\varepsilon}_z \times \vec{\varepsilon}_x \tag{6.2}$$

The same system of cartesiian axes is used to study the problem of reflection and transmission coefficients in viscoelastic media with complex rays.

In the case of normal incidence the choice of x and y axes is arbitrary as no converted phases are generated upon the incidence of the ray at O (same as elastic case).

Using equations (5.2) and (5.4) a ray series for each of the existing complex rays can be written as

$$\vec{\mathcal{W}}_v(\vec{r}, \omega) = \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega) \, f_n(t - \tau_v(\vec{r}, \omega)) \tag{6.3}$$

where

$$f_n(t - \tau_v(\vec{r}, \omega)) = s_n(\xi, \omega) \tag{6.4}$$

of equation (5.5). The corresponding displacement is then equal to

$$\vec{u}_v(\vec{r}, t) = \frac{1}{\Pi} \operatorname{Re} \int_{\omega_1}^{\infty} \vec{\mathcal{W}}_v(\vec{r}, \omega) \, d\omega \tag{6.5}$$

but we will solve the problem in terms of the complex quantities. The subscript v has the following values:

$v = 0$: incident ray (medium 1)

$v = 1$: reflected P ray (medium.1)

$v = 2$: transmitted P ray (medium 2)

$v = 3$: reflected S ray (medium 1)

$v = 4$: transmitted S ray (medium 2)

In figure 7, two unit vectors $\vec{n}_v$ and $\vec{l}_v$ are defined in the plane of incidence, for each ray. $\vec{n}_v$ is equal to $v_v \nabla \tau_v$ and is along the ray. $\vec{l}_v$ is perpendicular to the ray, i.e $\vec{l}_v \cdot \nabla \tau_v = 0$ so that

$$\vec{l}_v = (-1)^{v+1} \vec{\varepsilon}_y \times \vec{n}_v \tag{6.6}$$

The same conventions can be used for the viscoelastic case. Two complex unit vectors $\vec{q}_v$ and $\vec{r}_v$ are defined for each existing complex ray. $\vec{q}_v$ is the complex unit vector for P waves and is equal to

$$\vec{q}_v = v_v \nabla \tau_v \tag{6.7}$$

where τ_v is the complex travel time for the corresponding ray. $\vec{r}_v$ is orthogonal to the complex ray, i.e $\vec{r}_v \cdot \nabla \tau_v = 0$ and

$$\vec{r}_v = (-1)^{v+1} \vec{\varepsilon}_y \times \vec{q}_v \tag{6.8}$$

$\vec{r}_v$ is the complex unit vector for SV waves. A third vector for SH waves is defined as

$$\vec{m}_v = \vec{m} = \vec{\varepsilon}_y \tag{6.9}$$

Consequently each vector $\vec{W}_v^{(n)}$ can be written as

$$\vec{W}_v^{(n)} = N_v^{(n)}\vec{q}_v + T_v^{(n)}\vec{r}_v + V_v^{(n)}\vec{m} \tag{6.10}$$

where $\vec{W}_v^{(n)}$, $\vec{q}_v$, $\vec{r}_v$ and $\vec{m}$ are complex vectors and where

$N_v^{(n)} = \vec{W}_v^{(n)} \cdot \vec{q}_v$: complex component along the ray

$T_v^{(n)} = \vec{W}_v^{(n)} \cdot \vec{r}_v$: complex component perpendicular to the complex ray lying in the plane of incidence.

$V_v^{(n)} = \vec{W}_v^{(n)} \cdot \vec{m}$: complex component perpendicular to the plane of incidence.

6.2 Boundary Conditions For Viscoelastic Waves In The Case Of Two Solid Media In a Welded Contact

The boundary conditions for two solid homogeneous isotropic linear viscoelastic media in welded contact require the stress and the displacement to be continuous.

The total displacement in media 1 and 2 can be written as

$$\vec{U}_1 = \sum_{v=0,1,3} \vec{u}_v \quad \text{and} \quad \vec{U}_2 = \sum_{v=2,4} \vec{u}_v \tag{6.11}$$

Then the boundary conditions imposed on the displacement vectors are

$$\vec{U}_1 = \vec{U}_2 \qquad (6.12a)$$

and

$$\sigma_{3j}(\vec{U}_1) = \sigma_{3j}\vec{U}_2 \qquad j = 1,\ 2,\ 3 \qquad (6.12b)$$

These boundary conditions must be completed by the boundary condition for the complex phase function τ_v requiring that on the boundary Σ

$$\tau_v = \tau_0 \qquad (6.13)$$

Using the coordinate system defined in paragraph (6.1), $\nabla\tau$ can be written as

$$\nabla\tau = \frac{1}{\vec{v}} = \frac{\partial\tau}{\partial x}\vec{\varepsilon}_x + \frac{\partial\tau}{\partial y}\vec{\varepsilon}_y + \frac{\partial\tau}{\partial z}\vec{\varepsilon}_z \qquad (6.14)$$

hence for

$$\begin{aligned}
v = 0 \quad \nabla\tau_0 &= \frac{1}{v_0}\left(\sin\alpha_0\,\vec{\varepsilon}_x + 0 - \cos\alpha_0\,\vec{\varepsilon}_z\right)\\
v = 1 \quad \nabla\tau_1 &= \frac{1}{v_1}\left(\sin\alpha_1\,\vec{\varepsilon}_x + 0 + \cos\alpha_1\,\vec{\varepsilon}_z\right)\\
v = 2 \quad \nabla\tau_2 &= \frac{1}{v_2}\left(\sin\alpha_2\,\vec{\varepsilon}_x + 0 - \cos\alpha_2\,\vec{\varepsilon}_z\right) \qquad (6.15)\\
v = 3 \quad \nabla\tau_3 &= \frac{1}{v_3}\left(\sin\alpha_3\,\vec{\varepsilon}_x + 0 + \cos\alpha_3\,\vec{\varepsilon}_z\right)\\
v = 4 \quad \nabla\tau_4 &= \frac{1}{v_4}\left(\sin\alpha_4\,\vec{\varepsilon}_x + 0 - \cos\alpha_4\,\vec{\varepsilon}_z\right)
\end{aligned}$$

where

α_0 is the complex angle of incidence

α_1 is the complex angle of reflection for P waves

α_2 is the complex angle of refraction for P waves

α_3 is the complex angle of reflection for S waves

α_4 is the complex angle of refraction for S waves

The sign of the z component was determined using the analogy with the elastic case (Figure 7). The wave speeds v_v's in equations (6.15) are all complex and frequency dependent. Applying Snell's law in anelastic media to our problem yields

$$p = \frac{\sin\alpha_0}{v_0} = \frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = \frac{\sin\alpha_3}{v_3} = \frac{\sin\alpha_4}{v_4} \qquad (6.16)$$

hence

$$\frac{\partial\tau_v}{\partial x} = \frac{\partial\tau_0}{\partial x} \qquad (6.17)$$

Equations (6.15) also give

$$\frac{\partial\tau_v}{\partial y} = 0$$

and

$$\frac{\partial\tau_v}{\partial z} = (-1)^{v+1}\,\frac{\cos\alpha_v}{v_v} \qquad (6.18)$$

$\cos\alpha_v$ is equal to a complex root, the uniqueness of the solution is achieved by implementing the radiation conditions for anelastic media requiring an exponential decay of $f_n(t - \tau_v)$ for a ray going away from the interface. Using equation (3.27) to show the directions of $\vec{P}$ and $\vec{A}$ we can see that in order to obtain this exponential decay the real and imaginary parts of p, the ray parameter and the real part of ξ must be positive. The imaginary part of the

ς being such that the dot product $P \cdot A$ remains positive. A negative value from this dot product would result in an amplitude growth for a ray going away from the contact point between the ray and the interface. This is not physically possible.

The directions of the reflected and transmitted complex rays at O on the boundary are determined using Snell's law for anelastic media. Still using the cartesian coordinate system $\vec{\varepsilon}_x$, $\vec{\varepsilon}_y$, $\vec{\varepsilon}_z$ the complex components of $\vec{W}_\nu^{(n)}$ can be written as

$$
\begin{aligned}
W_{\nu_x}^{(n)} &= N_\nu^{(n)}\sin\alpha_\nu + T_\nu^{(n)}\cos\alpha_\nu \\
W_{\nu_y}^{(n)} &= V_\nu^{(n)} \qquad\qquad \nu = 0,\ 1,\ 2,\ 3,\ 4 \qquad (6.19) \\
W_{\nu_z}^{(n)} &= (-1)^{\nu+1}N_\nu^{(n)}\cos\alpha_\nu + (-1)^{\nu}T_\nu^{(n)}\sin\alpha_\nu
\end{aligned}
$$

The boundary condition requiring the continuity of the displacement across the boundary leads to three equations in terms of amplitude coefficients. Inserting equations (6.5) and (6.11) into equation (6.12a) yields

$$
\sum_{\nu=0,1,3} \frac{1}{\Pi} \operatorname{Re} \int_{\omega_1}^{\infty} \vec{\mathcal{W}}_\nu(\vec{r},\ \omega)\, d\omega = \sum_{\nu=2,4} \frac{1}{\Pi} \operatorname{Re} \int_{\omega_1}^{\infty} \vec{\mathcal{W}}_\nu(\vec{r},\ \omega)\, d\omega \quad (6.20)
$$

Using equation (6.4) gives

$$\sum_{v=0,1,3} \frac{1}{\Pi} \mathrm{Re} \int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v)\, d\omega$$

$$= \tag{6.21}$$

$$\sum_{v=2,4} \frac{1}{\Pi} \mathrm{Re} \int_{\omega_1}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v)\, d\omega$$

For equation (6.21) to be true, we must have

$$\sum_{v=0,1,3} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v) = \sum_{v=2,4} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v) \tag{6.22}$$

for each value of n. Using equation (6.13), equation (6.22) can finally be written as

$$f_n(t - \tau_v) \left[\sum_{v=0,1,3} \vec{W}_v^{(n)}(\vec{r}, \omega) \right] = f_n(t - \tau_v) \left[\sum_{v=2,4} \vec{W}_v^{(n)}(\vec{r}, \omega) \right]$$

$$\sum_{v=0,1,3} \vec{W}_v^{(n)} = \sum_{v=2,4} \vec{W}_v^{(n)} \tag{6.23}$$

Writing the vector $\vec{W}_v^{(n)}$ in terms of its complex x, y, z components lead to the following equations

$$\sum_{v=0,1,3} \left(N_v^{(n)} \sin\alpha_v + T_v^{(n)} \cos\alpha_v\right) = \sum_{v=2,4} \left(N_v^{(n)} \sin\alpha_v + T_v^{(n)} \cos\alpha_v\right) \tag{6.24a}$$

$$\sum_{v=0,1,3} V_v^{(n)} = \sum_{v=2,4} V_v^{(n)} \tag{6.24b}$$

$$\sum_{v=0,1,3} \left((-1)^{v+1} N_v^{(n)} \cos\alpha_v + (-1)^v T_v^{(n)} \sin\alpha_v\right) =$$

$$\sum_{v=2,4} \left((-1)^{v+1} N_v^{(n)} \cos\alpha_v + (-1)^v T_v^{(n)} \sin\alpha_v\right) \tag{6.24c}$$

The same approach is used again for the boundary condition requiring the continuity of the stress tensor components across the boundary in order to obtain three equations in terms of amplitude coefficients. The stress tensor for the steady-state harmonic case in linear viscoelastic media is given by

$$\sigma_{ij} = \Lambda(\omega)\, \nabla \cdot \vec{u}\; \delta_{ij} + 2M(\omega) \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right)\right] \qquad (6.25)$$

(see chapter 3)

Using equations (5.2) and (6.4), $\vec{u}$ can be written † as

$$\vec{u}_v = \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_r(t - \tau)\, d\omega$$

$$= \frac{1}{\Pi} \mathrm{Re} \int_{0}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v)\, d\omega \qquad (6.26)$$

The expressions of σ_{zz}, σ_{xz} and σ_{yz} are then:

$$\sigma_{zz}(\vec{u}) = \Lambda_v(\omega)\, \nabla \cdot \vec{u} + 2M_v \left(\frac{\partial u_z}{\partial z}\right)$$

$$= \Lambda_v \nabla \cdot \left(\frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v)\, d\omega\right) +$$

$$2M_v \frac{\partial}{\partial z}\left(\frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \vec{W}_v^{(n)}(\vec{r}, \omega)\, f_n(t - \tau_v)\, d\omega\right)$$

$$= \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left[\Lambda_v \nabla \cdot \left(\vec{W}_v^{(n)}(\vec{r}, \omega) f_n(t - \tau_v) \right) + 2M_v \frac{\partial \left(W_{v_z}^{(n)} f_n(t - \tau_v) \right)}{\partial z} \right] d\omega$$

$$= \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left[\Lambda_v \left(\nabla f_n \cdot \vec{W}_v^{(n)} + f_n \nabla \cdot \vec{W}_v^{(n)} \right) + 2M_v \left(\frac{\partial f_n}{\partial z} W_{v_z}^{(n)} + f_n \frac{\partial W_{v_z}^{(n)}}{\partial z} \right) \right] d\omega$$

$$= \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=-1}^{\infty} \left[f_n \left(-\Lambda_v \nabla \tau_v \cdot \vec{W}_v^{(n+1)} + 2M_v \frac{\partial \tau_v}{\partial z} W_{v_z}^{(n+1)} + \Lambda_v \nabla \cdot \vec{W}_v^{(n)} + 2M_v \frac{\partial W_{v_z}^{(n)}}{\partial z} \right) \right] d\omega$$

$$\sigma_{zz} = \frac{1}{\Pi} \mathrm{Re} \int_{\omega_1}^{\infty} \sum_{n=-1}^{\infty} f_n \Psi_n \left(\vec{W}_v^{(n+1)}, \vec{W}_v^{(n)} \right) d\omega \qquad (6.27)$$

where Ψ_n is given by

$$\Psi_n \left(\vec{W}_v^{(n+1)}, \vec{W}_v^{(n)} \right) = - \left[N_v^{(n+1)} \left(\frac{\Lambda_v + 2M_v \cos^2 \alpha_v}{v_v} \right) - T_v^{(n+1)} \left(\frac{M_v \sin 2\alpha_v}{v_v} \right) \right] + \left[\Lambda_v \nabla \cdot \vec{W}_v^{(n)} + 2M_v \frac{\partial W_{v_z}^{(n)}}{\partial z} \right] \qquad (6.28)$$

$$\sigma_{xz}(\vec{u}) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left[M_v \left(\frac{\partial \left(W_{v_x}^{(n)} f_n \right)}{\partial z} + \frac{\partial \left(W_{v_z}^{(n)} f_n \right)}{\partial x} \right) \right] d\omega$$

$$= \frac{1}{2\Pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left[-f_{n-1} M_v \left(\frac{\partial \tau_v}{\partial z} W_{v_x}^{(n)} + \frac{\partial \tau_v}{\partial x} W_{v_z}^{(n)} \right) + M_v f_n \left(\frac{\partial W_{v_x}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial x} \right) \right] d\omega$$

$$\sigma_{xz} = \frac{1}{\Pi} \mathrm{Re} \int_{\omega_1}^{\infty} \sum_{n=-1}^{\infty} f_n \chi_n \left(\vec{W}_v^{(n+1)}, \vec{W}_v^{(n)} \right) d\omega \qquad (6.29)$$

where χ_n is given by

$$\chi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) = M_v\left[(-1)^v N_v^{(n+1)}\left(\frac{\sin 2\alpha_v}{v_v}\right) + T_v^{(n+1)}\left(\frac{\cos 2\alpha_v}{v_v}\right)\right] + M_v\left[\frac{\partial W_{v_x}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial x}\right] \qquad (6.30)$$

and finally

$$\sigma_{yz}(\vec{u}) = \frac{1}{2\Pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[M_v\left(\frac{\partial\left(W_{v_y}^{(n)}f_n\right)}{\partial z} + \frac{\partial\left(W_{v_z}^{(n)}f_n\right)}{\partial y}\right)\right] d\omega$$

$$= \frac{1}{2\Pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[-f_{n-1}M_v\left(\frac{\partial\tau_v}{\partial z}W_{v_y}^{(n)} + \frac{\partial\tau_v}{\partial y}W_{v_z}^{(n)}\right) + M_v f_n\left(\frac{\partial W_{v_y}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial y}\right)\right] d\omega$$

$$\sigma_{yz} = \frac{1}{\Pi}\,\mathrm{Re}\int_{\omega_1}^{\infty}\sum_{n=-1}^{\infty} f_n\,\Phi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) d\omega \qquad (6.31)$$

where Φ_n is given by

$$\Phi_n = M_v\left[(-1)^v\frac{\cos\alpha_v}{v_v}V_v^{(n+1)} + \left(\frac{\partial W_{v_y}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial y}\right)\right] \qquad (6.32)$$

The requirement of the continuity of the stress tensor components σ_{xz}, σ_{yz} and σ_{zz} can be written as

$$\sigma_{xz}\left(\sum_{v=0,1,3}\vec{u}_v\right) = \sigma_{xz}\left(\sum_{v=2,4}\vec{u}_v\right) \qquad (6.33a)$$

$$\sigma_{yz}\left(\sum_{v=0,1,3}\vec{u}_v\right) = \sigma_{yz}\left(\sum_{v=2,4}\vec{u}_v\right) \qquad (6.33b)$$

$$\sigma_{zz}\left(\sum_{v=0,1,3}\vec{u}_v\right) = \sigma_{zz}\left(\sum_{v=2,4}\vec{u}_v\right) \qquad (6.33c)$$

implying

$$\int_{\omega_1}^{\infty} \sum_{v=0,1,3} \left(\sum_{n=-1}^{\infty} f_n \, \chi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \;=$$

$$\int_{\omega_1}^{\infty} \sum_{v=2,4} \left(\sum_{n=-1}^{\infty} f_n \, \chi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \qquad (6.34a)$$

$$\int_{\omega_1}^{\infty} \sum_{v=0,1,3} \left(\sum_{n=-1}^{\infty} f_n \, \Phi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \;=$$

$$\int_{\omega_1}^{\infty} \sum_{v=2,4} \left(\sum_{n=-1}^{\infty} f_n \, \Phi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \qquad (6.34b)$$

$$\int_{\omega_1}^{\infty} \sum_{v=0,1,3} \left(\sum_{n=-1}^{\infty} f_n \, \Psi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \;=$$

$$\int_{\omega_1}^{\infty} \sum_{v=2,4} \left(\sum_{n=-1}^{\infty} f_n \, \Psi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right)\right) d\omega \qquad (6.34c)$$

For equations (6.34) to be true, we must have for each value of n

$$\sum_{v=0,1,3} \chi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) = \sum_{v=2,4} \chi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) \qquad (6.35a)$$

$$\sum_{v=0,1,3} \Phi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) = \sum_{v=2,4} \Phi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) \qquad (6.35b)$$

$$\sum_{v=0,1,3} \Psi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) = \sum_{v=2,4} \Psi_n\left(\vec{W}_v^{(n+1)}, \quad \vec{W}_v^{(n)}\right) \qquad (6.35c)$$

Equations (6.35) lead to three other equations:

$$\sum_{v=0,1,3} M_v \left[(-1)^v N_v^{(n+1)}\left(\frac{\sin 2\alpha_v}{v_v}\right) + T_v^{(n+1)}\left(\frac{\cos 2\alpha_v}{v_v}\right) + \left(\frac{\partial W_{v_x}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial x}\right)\right]$$

$$=$$

$$\sum_{v=2,4} M_v \left[(-1)^v N_v^{(n+1)}\left(\frac{\sin 2\alpha_v}{v_v}\right) + T_v^{(n+1)}\left(\frac{\cos 2\alpha_v}{v_v}\right) + \left(\frac{\partial W_{v_x}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial x}\right)\right] \tag{6.36a}$$

$$\sum_{v=0,1,3} M_v \left[(-1)^v \frac{\cos\alpha_v}{v_v} V_v^{(n+1)} + \left(\frac{\partial W_{v_y}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial y}\right)\right]$$

$$=$$

$$\sum_{v=2,4} M_v \left[(-1)^v \frac{\cos\alpha_v}{v_v} V_v^{(n+1)} + \left(\frac{\partial W_{v_y}^{(n)}}{\partial z} + \frac{\partial W_{v_z}^{(n)}}{\partial y}\right)\right] \tag{6.36b}$$

$$\sum_{v=0,1,3} \left[-N_v^{(n+1)}\left(\frac{\Lambda_v + 2M_v\cos^2\alpha_v}{v_v}\right) + T_v^{(n+1)}\left(\frac{M_v \sin 2\alpha_v}{v_v}\right) + \Lambda_v \nabla\cdot\vec{W}_v^{(n)} + 2M_v\frac{\partial W_{v_z}^{(n)}}{\partial z}\right]$$

$$=$$

$$\sum_{v=2,4} \left[-N_v^{(n+1)}\left(\frac{\Lambda_v + 2M_v\cos^2\alpha_v}{v_v}\right) + T_v^{(n+1)}\left(\frac{M_v \sin 2\alpha_v}{v_v}\right) + \Lambda_v \nabla\cdot\vec{W}_v^{(n)} + 2M_v\frac{\partial W_{v_z}^{(n)}}{\partial z}\right] \tag{6.36c}$$

The two sets of equations (6.24) and (6.36) can be combined to give one set of six homogeneous equations for six principal components $N_1^{(n)}$ (reflected P wave); $N_2^{(n)}$ (transmitted P wave); $T_3^{(n)}$, $V_3^{(n)}$ (reflected S waves) and $T_4^{(n)}$, $V_4^{(n)}$ (transmitted S waves). Consequently the reflection and transmission coefficients can be determined by solving this system which is finally written the following way:

$$\sum_{\nu=1}^{2}(-1)^{\nu}\sin\alpha_\nu N_\nu^{(n)} + \sum_{\nu=3}^{4}(-1)^{\nu}\cos\alpha_\nu T_\nu^{(n)}$$

$$=$$

$$\sum_{\nu=0,3,4}(-1)^{\nu+\delta_{\nu 0}+1}\sin\alpha_\nu N_\nu^{(n)} + \sum_{\nu=0}^{2}(-1)^{\nu+\delta_{\nu 0}+1}\cos\alpha_\nu T_\nu^{(n)}$$

$$\sum_{\nu=1}^{2}(-1)^{\nu}\cos\alpha_\nu N_\nu^{(n)} - \sum_{\nu=3}^{4}(-1)^{\nu}\sin\alpha_\nu T_\nu^{(n)}$$

$$=$$

$$\sum_{\nu=0,3,4}(-1)^{\delta_{\nu 0}+1}\cos\alpha_\nu N_\nu^{(n)} + \sum_{\nu=0}^{2}(-1)^{\delta_{\nu 0}}\sin\alpha_\nu T_\nu^{(n)}$$

$$\sum_{\nu=1}^{2} M_\nu\frac{\sin 2\alpha_\nu}{v_\nu}N_\nu^{(n)} + \sum_{\nu=3}^{4} M_\nu\frac{\cos 2\alpha_\nu}{v_\nu}T_\nu^{(n)} = \Phi_1^{(n)}$$

$$\sum_{\nu=1}^{2}(-1)^{\nu}\left(\frac{\Lambda_\nu + 2M_\nu\cos^2\alpha_\nu}{v_\nu}\right)N_\nu^{(n)} + \sum_{\nu=3}^{4}(-1)^{\nu+1}M_\nu\frac{\sin 2\alpha_\nu}{v_\nu}T_\nu^{(n)} = \Phi_2^{(n)}$$

$$\sum_{\nu=3}^{4}(-1)^{\nu}V_\nu^{(n)} = \sum_{\nu=0}^{2}(-1)^{\nu+\delta_{\nu 0}+1}V_\nu^{(n)}$$

$$\sum_{\nu=3}^{4} M_\nu\frac{\cos\alpha_\nu}{v_\nu}V_\nu^{(n)} = \sum_{\nu=0}^{4}(-1)^{\nu+\delta_{\nu 0}+1}M_\nu\left[\frac{\partial W_{\nu_y}^{(n-1)}}{\partial z} + \frac{\partial W_{\nu_z}^{(n-1)}}{\partial y}\right] + \sum_{\nu=0}^{2}(-1)^{\delta_{\nu 0}+1}M_\nu\frac{\cos\alpha_\nu}{v_\nu}V_\nu^{(n)} \quad n = 0, 1, 2, \ldots \quad (6.37)$$

where

$$\Phi_1^{(n)} = \sum_{\nu=0,3,4}(-1)^{\delta_{\nu 0}+1}M_\nu\frac{\sin 2\alpha_\nu}{v_\nu}N_\nu^{(n)} + \sum_{\nu=0}^{2}(-1)^{\delta_{\nu 0}+1}M_\nu\frac{\cos 2\alpha_\nu}{v_\nu}T_\nu^{(n)} + \sum_{\nu=0}^{4}(-1)^{\nu+\delta_{\nu 0}+1}M_\nu\left[\frac{\partial W_{\nu_x}^{(n-1)}}{\partial z} + \frac{\partial W_{\nu_z}^{(n-1)}}{\partial x}\right]$$

and

$$\Phi_2^{(n)} = \sum_{v=0}^{4} (-1)^{v+\delta_{v0}} \left[\Lambda_v \nabla \cdot \vec{W}_v^{(n-1)} + 2M_v \frac{\partial W_{v_z}^{(n-1)}}{\partial z} \right] + \sum_{v=0}^{2} (-1)^{v+\delta_{v0}} \left[M_v \frac{\sin 2\alpha_v}{v_z} T_v^{(n)} \right] + \sum_{v=0,3,4} (-1)^{v+\delta_{v0}+1} \left(\frac{\Lambda_v + 2M_v \cos^2\alpha_v}{v_z} \right) N_v^{(n)} \qquad (6.38)$$

The system (6.37) looks like the one obtained for the elastic case (see chapter 4 of ART, Hron 1984), the diffenrences being that the real quantities θ_v, ν_v, λ_v, μ_v, and the real amplitude coefficient components are replaced by the complex quantities α_v, v_v, Λ_v, M_v, and the complex amplitude coefficient components obtained using a complex ray approach.

As mentioned earlier a plane wave approach is considered. It is then not necessary to calculate $N_v^{(n)}$, $T_v^{(n)}$, and $V_v^{(n)}$ for all n. Only the zeroth order of the theory is considered. This means that the system (6.37) will be solved for n = 0 only.

6.3 Resolution Of The System

Solving the system for n = 0 simplifies the calculations considerably. The differential operations applied to the term $\vec{W}_v^{(-1)}$ are equal to 0 and the system breaks into two separate systems. The first four equations correspond to the P-SV case and the last two to the SH case.

6.3.1 SH waves

By solving the system formed by these last two equations the reflection and transmission coefficients for SH waves can be determined. In this case the incident wave, i.e for $v = 0$ is an SH wave and there is no reflected and transmitted P and SV waves. The system can be written as

$$V_0^{(0)} + V_3^{(0)} = V_4^{(0)}$$

$$\frac{M_0 \cos\alpha_0}{v_0} V_0^{(0)} - \frac{M_3 \cos\alpha_3}{v_3} V_3^{(0)} = \frac{M_4 \cos\alpha_4}{v_4} V_4^{(0)} \qquad (6.39)$$

Using the fact $M_0 = M_3 = M_1$ and $v_0 = v_3$ for medium 1 and $M_4 = M_2$ for medium 2, the solution of the system (6.39) is equal to

$$\frac{V_3^{(0)}}{V_0^{(0)}} = \frac{\rho_1 v_0 \cos\alpha_0 - \rho_2 v_4 \cos\alpha_4}{\rho_1 v_0 \cos\alpha_3 + \rho_2 v_4 \cos\alpha_4} \qquad (6.40)$$

and

$$\frac{V_4^{(0)}}{V_0^{(0)}} = \frac{2\rho_1 v_0 \cos\alpha_0}{\rho_1 v_0 \cos\alpha_3 + \rho_2 v_4 \cos\alpha_4} \qquad (6.41)$$

The reflection and transmission coefficients for SH waves respectively $\frac{V_3^{(0)}}{V_0^{(0)}}$ and $\frac{V_4^{(0)}}{V_0^{(0)}}$ have the same form as those for the elastic case (Aki and Richards, 1980) but v_0, v_4, α_0 and α_4 are now complex quantities. This is not a surprise. Aki and Richards (1980) clearly state that since Snell's law in anelastic media can be written $p = \frac{\sin\alpha}{v}$ where α and v are complex, all the formulae obtained for reflection and transmission coefficients in the elastic case can be applied to the anelastic case by replacing the real elastic constants with their complex analogs defined earlier.

6.3.2 P-SV waves

The zeroth order of the theory is still considered. The system formed by the first four equations of (6.37) has then to be solved in order to obtain the reflection and transmission coefficients for P and SV waves. n being equal to 0, the P and SV waves do not have any additional components consequently N3=N4=T0=T1=T2=0 for an incident P wave and N3=N4=N0=T1=T2=0 for an incident S wave.

Starting with an incident P wave the system obtained is

$$-\sin\alpha_1\frac{N_1}{N_0} - \cos\alpha_3\frac{T_3}{N_0} + \sin\alpha_2\frac{N_2}{N_0} + \cos\alpha_4\frac{T_4}{N_0} = \sin\alpha_0$$

$$\cos\alpha_1\frac{N_1}{N_0} - \sin\alpha_3\frac{T_3}{N_0} + \cos\alpha_2\frac{N_2}{N_0} - \sin\alpha_4\frac{T_4}{N_0} = \cos\alpha_0$$

$$\frac{M_1}{v_1}\sin2\alpha_1\frac{N_1}{N_0} + \rho_3v_3\cos2\alpha_3\frac{T_3}{N_0} + \frac{M_2}{v_2}\sin2\alpha_2\frac{N_2}{N_0} + \rho_4v_4\cos2\alpha_4\frac{T_4}{N_0} = \frac{M_0}{v_0}\sin2\alpha_0$$

$$- \left[\frac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right]\frac{N_1}{N_0} + \rho_3 v_3 \sin 2\alpha_3 \frac{T_3}{N_0} + \left[\frac{\Lambda_2 + 2M_2\cos^2\alpha_2}{v_2}\right]\frac{N_2}{N_0}$$
$$- \rho_4 v_4 \sin 2\alpha_4 \frac{T_4}{N_0} = \left[\frac{\Lambda_0 + 2M_0\cos^2\alpha_0}{v_0}\right] \qquad (6.42)$$

In the case of an incident S wave, the system of four equations becomes

$$-\sin\alpha_1 \frac{N_1}{T_0} - \cos\alpha_3 \frac{T_3}{T_0} + \sin\alpha_2 \frac{N_2}{T_0} + \cos\alpha_4 \frac{T_4}{T_0} = \cos\alpha_0$$
$$- \cos\alpha_1 \frac{N_1}{T_0} + \sin\alpha_3 \frac{T_3}{T_0} - \cos\alpha_2 \frac{N_2}{T_0} + \sin\alpha_4 \frac{T_4}{T_0} = \sin\alpha_0$$

$$\frac{M_1}{v_1}\sin 2\alpha_1 \frac{N_1}{T_0} + \rho_3 v_3 \cos 2\alpha_3 \frac{T_3}{T_0} + \frac{M_2}{v_2}\sin 2\alpha_2 \frac{N_2}{T_0}$$
$$+ \rho_4 v_4 \cos 2\alpha_4 \frac{T_4}{T_0} = \rho_0 v_0 \cos 2\alpha_0$$

$$\left[\frac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right]\frac{N_1}{T_0} - \rho_3 v_3 \sin 2\alpha_3 \frac{T_3}{T_0} - \left[\frac{\Lambda_2 + 2M_2\cos^2\alpha_2}{v_2}\right]\frac{N_2}{T_0}$$
$$+ \rho_4 v_4 \sin 2\alpha_4 \frac{T_4}{T_0} = \rho_0 v_0 \sin 2\alpha_0 \qquad (6.43)$$

Both systems were solved using the method of the determinants. The final expressions of the coefficients for an incidence from the upper medium, i.e medium 1, are given in Appendix 3 with the following convention:

P1P1 = N_1/N_0	P1S1 = T_3/N_0
P1P2 = N_2/N_0	P1S2 = T_4/N_0
S1P1 = N_1/T_0	S1S1 = T_3/T_0
S1P2 = N_2/T_0	S1S2 = T_4/T_0

For an incidence from the lower medium, i.e medium 2, the coefficients are called P2P2, P2S2, P2P1, P2S1, S2P2, S2S2, S2P1 and S2S1 and are also given in Appendix 3.

6.4 Reflection at a Free Surface

The reflection coefficients at a free surface are obtained by vanishing stress at the surface. The system (6.36) then becomes

$$\sum_{v=0,1,3} M_v \left[(-1)^v \left(N_v^{(n)} \frac{\sin 2\alpha_v}{v_v} + T_v^{(n)} \frac{\cos 2\alpha_v}{v_v} \right) + \left(\frac{\partial W_{v_x}^{(n-1)}}{\partial z} + \frac{\partial W_{v_z}^{(n-1)}}{\partial x} \right) \right] = 0 \tag{6.44a}$$

$$\sum_{v=0,1,3} M_v \left[(-1)^v V_v^{(n)} \frac{\cos \alpha_v}{v_v} + \left(\frac{\partial W_{v_y}^{(n-1)}}{\partial z} + \frac{\partial W_{v_z}^{(n-1)}}{\partial y} \right) \right] = 0 \tag{6.44b}$$

$$\sum_{v=0,1,3} \left[N_v^{(n)} \left(\frac{\Lambda_v + 2M_v \cos^2 \alpha_v}{v_v} \right) + T_v^{(n)} \left(\frac{M_v \sin 2\alpha_v}{v_v} \right) + \Lambda_v \nabla \cdot \vec{W}_v^{(n-1)} + 2M_v \frac{\partial W_{v_z}^{(n-1)}}{\partial z} \right] = 0 \tag{6.44c}$$

Still considering the zeroth order of the theory the system (6.44) is only solved for n = 0. Consequently there is still no additional component for P and S waves, the differential term $\frac{\partial W_{v_x}^{(-1)}}{\partial z}$, $\frac{\partial W_{v_z}^{(-1)}}{\partial x}$, $\frac{\partial W_{v_y}^{(-1)}}{\partial z}$, $\frac{\partial W_{v_z}^{(-1)}}{\partial y}$, $\nabla \cdot \vec{W}_v^{(-1)}$ and $\frac{\partial W_{v_z}^{(-1)}}{\partial z}$ are then equal to 0 and the system breaks again into two separate systems: the first and third equations correspond to the P-SV case and the second one to the SH case.

6.4.1 SH waves

Equation (6.44b) can be rewritten

$$M_0\frac{\cos\alpha_0}{v_0}V_0^{(0)} - M_3\frac{\cos\alpha_3}{v_3}V_3^{(0)} = 0 \qquad (6.45)$$

In this case $M_0 = M_3$ and $v_0 = v_3$, the reflection coefficient is then equal to

$$\frac{V_3^{(0)}}{V_0^{(0)}} = \frac{M_0\cos\alpha_0}{v_0} \times \frac{v_3}{M_3\cos\alpha_3} = 1 \qquad (6.46)$$

$\frac{V_3^{(0)}}{V_0^{(0)}}$ at free surface does not depend on α, the complex angle ofincidence, it is constantly equal to 1 like in the elastic case.

6.4.2 P-SV waves

For the P-SV case, the system formed by equations (6.44a) and (6.44c) can be written

$$M_0\left[N_0^{(0)}\frac{\sin2\alpha_0}{v_0} + T_0^{(0)}\frac{\cos2\alpha_0}{v_0}\right] - M_1N_1^{(0)}\frac{\sin2\alpha_1}{v_1} + M_3T_3^{(0)}\frac{\cos2\alpha_3}{v_3} = 0 \qquad (6.47a)$$

$$-N_0^{(0)}\left(\frac{\Lambda_0 + 2M_0\cos^2\alpha_0}{v_0}\right) + T_0^{(0)}\frac{M_0\sin2\alpha_0}{v_0} - N_1^{(0)}\left(\frac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right) + T_3^{(0)}\frac{M_3\sin2\alpha_3}{v_3} = 0 \qquad (6.47b)$$

An incident P wave is first considered, $T_0^{(0)} = 0$ and the system obtained is

$$\frac{M_0 \sin 2\alpha_0}{v_0} - \frac{N_1^{(0)}}{N_0^{(0)}} \frac{M_1 \sin 2\alpha_1}{v_1} - \frac{T_3^{(0)}}{N_0^{(0)}} \frac{M_3 \cos 2\alpha_3}{v_3} = 0 \qquad (6.48a)$$

$$-\left(\frac{\Lambda_0 + 2M_0 \cos^2\alpha_0}{v_0}\right) - \frac{N_1^{(0)}}{N_0^{(0)}} \left(\frac{\Lambda_1 + 2M_1 \cos^2\alpha_1}{v_1}\right) - \frac{T_3^{(0)}}{N_0^{(0)}} \frac{M_3 \sin 2\alpha_3}{v_3} = 0 \qquad (6.48b)$$

Solving the system (6.48) leads to the two reflection coefficients at a free surface:

$$\frac{N_1^{(0)}}{N_0^{(0)}} = \frac{\left(\dfrac{\sin 2\alpha_3 \sin 2\alpha_1 M_1}{v_1 \cos 2\alpha_3} - \dfrac{\Lambda_0 + 2M_1 \cos^2\alpha_0}{v_0}\right)}{\left(\dfrac{\sin 2\alpha_3 \sin 2\alpha_1 M_1}{v_1 \cos 2\alpha_3} + \dfrac{\Lambda_1 + 2M_1 \cos^2\alpha_1}{v_1}\right)} \qquad (6.49)$$

and

$$\frac{T_3^{(0)}}{N_0^{(0)}} = 2\frac{\sin 2\alpha_1 v_3}{v_1 \cos 2\alpha_3} \left[\frac{\dfrac{\Lambda_1 + 2M_1 \cos^2\alpha_1}{v_1}}{\left(\dfrac{\sin 2\alpha_3 \sin 2\alpha_1 M_1}{v_1 \cos 2\alpha_3} + \dfrac{\Lambda_1 + 2M_1 \cos^2\alpha_1}{v_1}\right)}\right] \qquad (6.50)$$

The case of an incident S wave is now investigated, $N_0^{(0)} = 0$ and the system (6.47) becomes

$$\frac{M_0 \cos 2\alpha_0}{v_0} - \frac{N_1^{(0)}}{T_0^{(0)}} \frac{M_1 \sin 2\alpha_1}{v_1} - \frac{T_3^{(0)}}{T_0^{(0)}} \frac{M_3 \cos 2\alpha_3}{v_3} = 0 \qquad (6.51a)$$

$$\frac{M_0 \sin 2\alpha_0}{v_0} - \frac{N_1^{(0)}}{T_0^{(0)}} \left(\frac{\Lambda_1 + 2M_1 \cos^2\alpha_1}{v_1}\right) - \frac{T_3^{(0)}}{T_0^{(0)}} \frac{M_3 \sin 2\alpha_3}{v_3} = 0 \qquad (6.51b)$$

The two reflection coefficients are then equal to

$$\frac{T_3^{(0)}}{T_0^{(0)}} = \frac{\left[\dfrac{\cos 2\alpha_3 v_1}{v_3 \sin 2\alpha_1}\left(\dfrac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right) - \dfrac{M_0 \sin 2\alpha_0}{v_0}\right]}{\left[\dfrac{\cos 2\alpha_3 v_1}{v_3 \sin 2\alpha_1}\left(\dfrac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right) + \dfrac{M_0 \sin 2\alpha_0}{v_0}\right]} \tag{6.52}$$

and

$$\frac{N_1^{(0)}}{T_0^{(0)}} = \frac{\cos 2\alpha_3 v_1}{v_3 \sin 2\alpha_1}\left[\frac{2\dfrac{M_3 \sin 2\alpha_3}{v_3}}{\left(\dfrac{\cos 2\alpha_3 v_1}{v_3 \sin 2\alpha_1}\left(\dfrac{\Lambda_1 + 2M_1\cos^2\alpha_1}{v_1}\right) + \dfrac{M_0 \sin 2\alpha_0}{v_0}\right)}\right] \tag{6.53}$$

6.5 Surface Conversion Coefficients

A receiver located at the earth's surface i.e at the interface of a medium and a vacuum records the disturbance caused by an incident wavefront at that point and the disturbances resulting from the two reflected wavefronts. In our case the medium considered is homogeneous isotropic and anelastic and the x, y and z components of the displacement recorded by the receiver i.e dx, dy, dz are given by the L H S of system (6.24):

$$d_x = \sum_{v=0,1,3} \left(N_v^{(n)} \sin\alpha_v + T_v^{(n)} \cos\alpha_v\right) \tag{6.54a}$$

$$d_y = \sum_{v=0,1,3} V_v^{(n)} \tag{6.54b}$$

$$d_z = \sum_{v=0,1,3} \left((-1)^{v+1} N_v^{(n)} \cos\alpha_v + (-1)^v T_v^{(n)} \sin\alpha_v\right) \tag{6.54c}$$

Once again the zeroth order of the theory is considered hence the first and third equations correspond to the P-SV case and the second one to the SH case.

6.5.1 SH waves

The surface conversion coefficient for an incident SH wave is called CST and equal to $\frac{d_y}{V_0^{(0)}}$. Equation (6.54b) gives

$$d_y = V_0^{(0)} + V_3^{(0)} \tag{6.55}$$

hence

$$CST = 1 + \frac{V_3^{(0)}}{V_0^{(0)}} \tag{6.56}$$

$\frac{V_3^{(0)}}{V_0^{(0)}}$ being equal to 1 (see 6.46) CST is therefore always equal to 2.

6.5.2 P-SV waves

For an incident P wave the surface conversion coefficients are called CPH and CPV respectively for the horizontal and vertical directions of the displacement and are equal to $\frac{d_x}{N_0^{(0)}}$ and $\frac{d_z}{N_0^{(0)}}$. Equations (6.54a) and (6.54c) yield

$$d_x = N_0^{(0)}\sin\alpha_0 + N_1^{(0)}\sin\alpha_1 + T_3^{(0)}\cos\alpha_3 \tag{6.57a}$$

$$d_z = -N_0^{(0)}\cos\alpha_0 + N_1^{(0)}\cos\alpha_1 - T_3^{(0)}\sin\alpha_3 \tag{6.57b}$$

hence

$$CPH = \sin\alpha_1 \left(\frac{N_1^{(0)}}{N_0^{(0)}} + 1\right) + \frac{T_3^{(0)}}{N_0^{(0)}} \cos\alpha_3 \qquad (6.58)$$

and

$$CPV = \cos\alpha_1 \left(\frac{N_1^{(0)}}{N_0^{(0)}} - 1\right) - \frac{T_3^{(0)}}{N_0^{(0)}} \sin\alpha_3 \qquad (6.59)$$

Using expressions (6.49) and (6.50) for $\frac{N_1^{(0)}}{N_0^{(0)}}$ and $\frac{T_3^{(0)}}{N_0^{(0)}}$ CPH and CPV can be determined.

For an incident S wave the same reasoning is applied. The surface conversion coefficients are called CSH and CSV and areequal to $\frac{d_x}{T_0^{(0)}}$ and $\frac{d_z}{T_0^{(0)}}$. Equations (6.54a) and (6.54c) now yield

$$d_x = T_0^{(0)}\cos\alpha_0 + N_1^{(0)}\sin\alpha_1 + T_3^{(0)}\cos\alpha_3 \qquad (6.60a)$$

$$d_z = T_0^{(0)}\sin\alpha_0 + N_1^{(0)}\cos\alpha_1 - T_3^{(0)}\sin\alpha_3 \qquad (6.60b)$$

hence

$$CSH = \cos\alpha_3 \left(1 + \frac{T_3^{(0)}}{T_0^{(0)}}\right) + \frac{N_1^{(0)}}{T_0^{(0)}} \sin\alpha_1 \qquad (6.61)$$

$$CSV = \sin\alpha_3 \left(1 - \frac{T_3^{(0)}}{T_0^{(0)}}\right) + \frac{N_1^{(0)}}{T_0^{(0)}} \cos\alpha_1 \qquad (6.62)$$

Using expressions (6.62) and (6.63) for $\frac{N_1^{(0)}}{T_0^{(0)}}$ and $\frac{T_3^{(0)}}{T_0^{(0)}}$ CSH and CSV can also be determined.

6.6 Examples of Anelastic Coefficients

In order to have an idea of how the anelastic reflection and transmission coefficients look some examples of anelastic reflection and transmission amplitudes and phases are plotted.

The expressions of the coefficients were obtained using the form $e^{i\omega(t - \tau)}$ for steady state harmonic plane waves. To be consistent with chapters 2 and 3 the computations were performed using the form $e^{i\omega(\tau - t)}$. The formulae for the coefficients are still valid. The final results are simply complex conjugate to the results which would have been obtained using $e^{i\omega(t - \tau)}$.

Figure 8 contains all the plots of the sixteen coefficients of the P-SV case for interface A (table 1), the top graph being the amplitude the bottom graph the phase. Curve #1 represents the elastic case (all the Q^{-1} are set equal to 0) and curve #2 represents the anelastic case. The phase graphs show that elastic and anelastic coefficients have opposite phase after the first critical angle. As far as the amplitude is concerned, differences between elastic and anelastic cases are observed only in the vicinity of the critical angle and for only four coefficients: P1P1, S1P1, S1S1 and S1S2. Figure 9 contains the graphs of eight coefficients for interface B (table 1). This second interface is formed with two media which are much more anelastic. The eight coefficients represent all the possibilities for an incident ray from the upper medium in the P-SV case. Curves #1 and 2 represent respectively the elastic and the anelastic cases. Curve #3 in the amplitude graph has the same parameters as curve #2 except for QP2 and QS2 which are now set equal to 80. Differences in amplitude and phase between elastic and anelastic cases are again observed but they are now more obvious and they start shortly before the critical angle, especially with an incident P wave. The phase graphs show that the transition from sub-critical to super-critical phases is much smoother in the anelastic case. The same comment can be made for sub and super critical amplitudes especially for an incident P wave. This would tend to indicate that the influence of critical angle for the linear viscoelastic case is not as sharp as for the elastic one. Krebes (1980) reached the same conclusions for incident SH waves.

Curve #3 shows greater differences than curve #2 when they are both compared to the elastic case. This would show that the difference in amplitude between elastic and viscoelastic coefficients is dependent on the contrast of Q between the two media composing the interface. Krebes (1983) showed for SH waves that the anelastic coefficients have same amplitudes as the elastic coefficients when the anelastic media forming the interface have equal quality factors.

The last comment is about the amplitude of the anelastic coefficient S1S1. After the critical angle for S1S2 the amplitude decreases down to a minimum then increases to finally be very close to 1 for great angles of incidence. This oscillation also seems to be dependent on the contrast of Q. This behaviour was rather unexpected since it is very different from the elastic case. Brekhovskikh (1980) shows that the behaviour of viscoelastic coefficients can be very different from that of elastic coefficients.

Figures 10 and 11 show the effects of elastic and anelastic coefficients on a wavelet. The reasons for having chosen this particular wavelet are explained in chapter 7. Figure 10 contains four cases computed with interface A and figure 11 contains three cases computed with the interface B. For each graph the top left wavelet is the source. The top right one is the elastic response plotted with the same scale. The bottom left wavelet is the elastic response replotted with a new scale and then following are the anelastic responses plotted with this same new scale. The meaning of the numbers is the same as for the graphs of the coefficients. The wavelets were computed at the angles of incidence corresponding to the greatest differences in amplitude and phase between elastic and anelastic cases.

TABLE 1

Interfaces

		VP (km/s)	VS (km/s)	DEN (g/cc)	QP	QS
A	Medium 1	4.2	2.4	2.1	67	30
	Medium 2	6.1	3.5	2.6	100	45
B	Medium 1	1.85	0.3	1.92	20	20
	Medium 2	2.19	0.9	2.30	40	40

VP: elastic wave velocity for P waves

VS: elastic wave velocity for S waves

DEN: density of the medium

QP: quality factor for P waves

QS: quality factor for S waves

Figure 8: P-SV coefficients obtained with interface A.

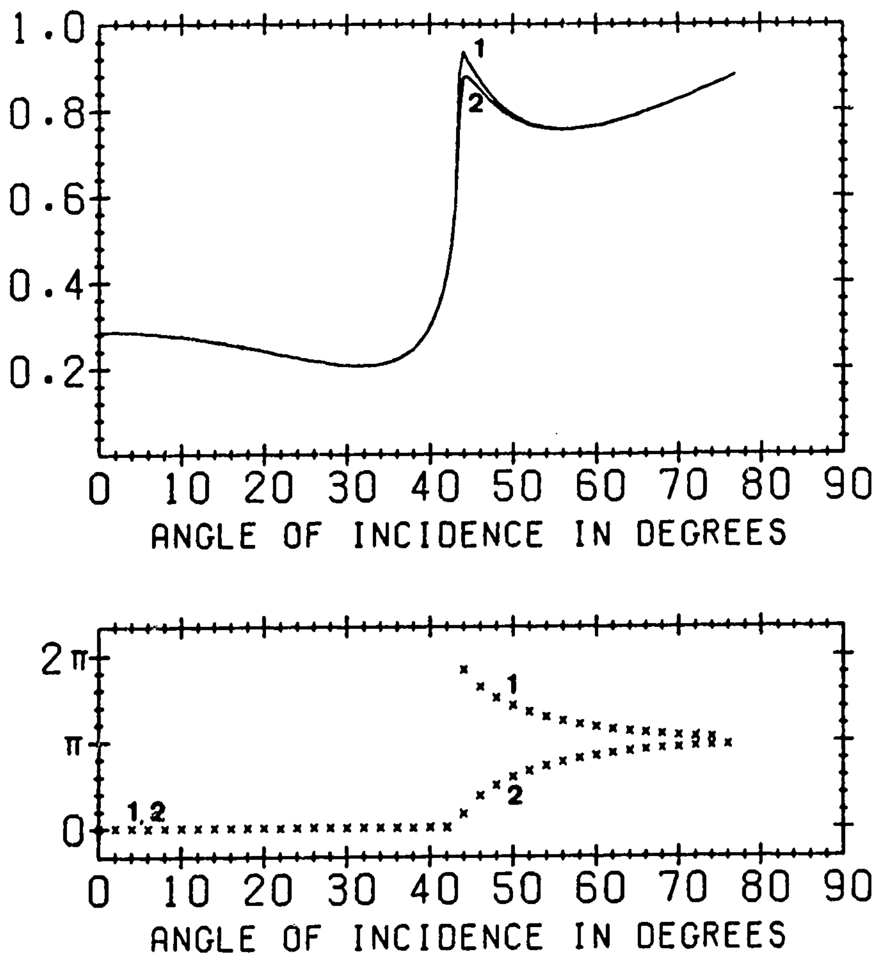
Curve #1 represents the elastic case.

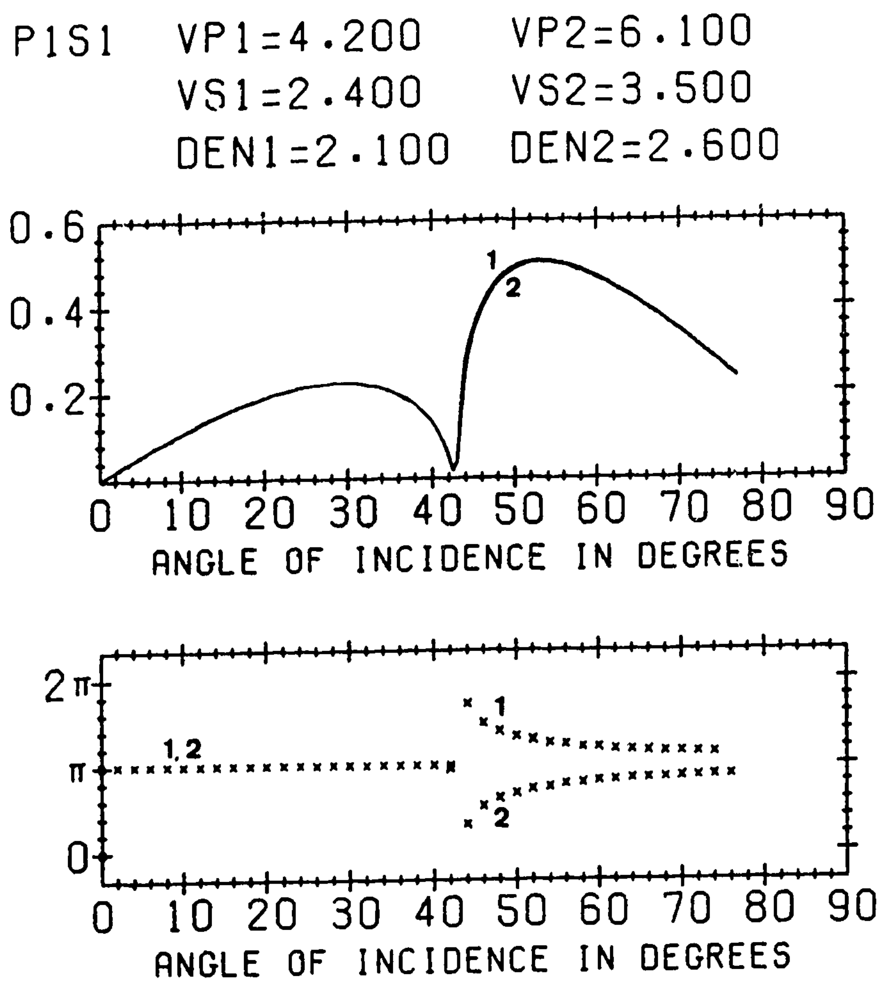
Curve #2 represents the anelastic case.

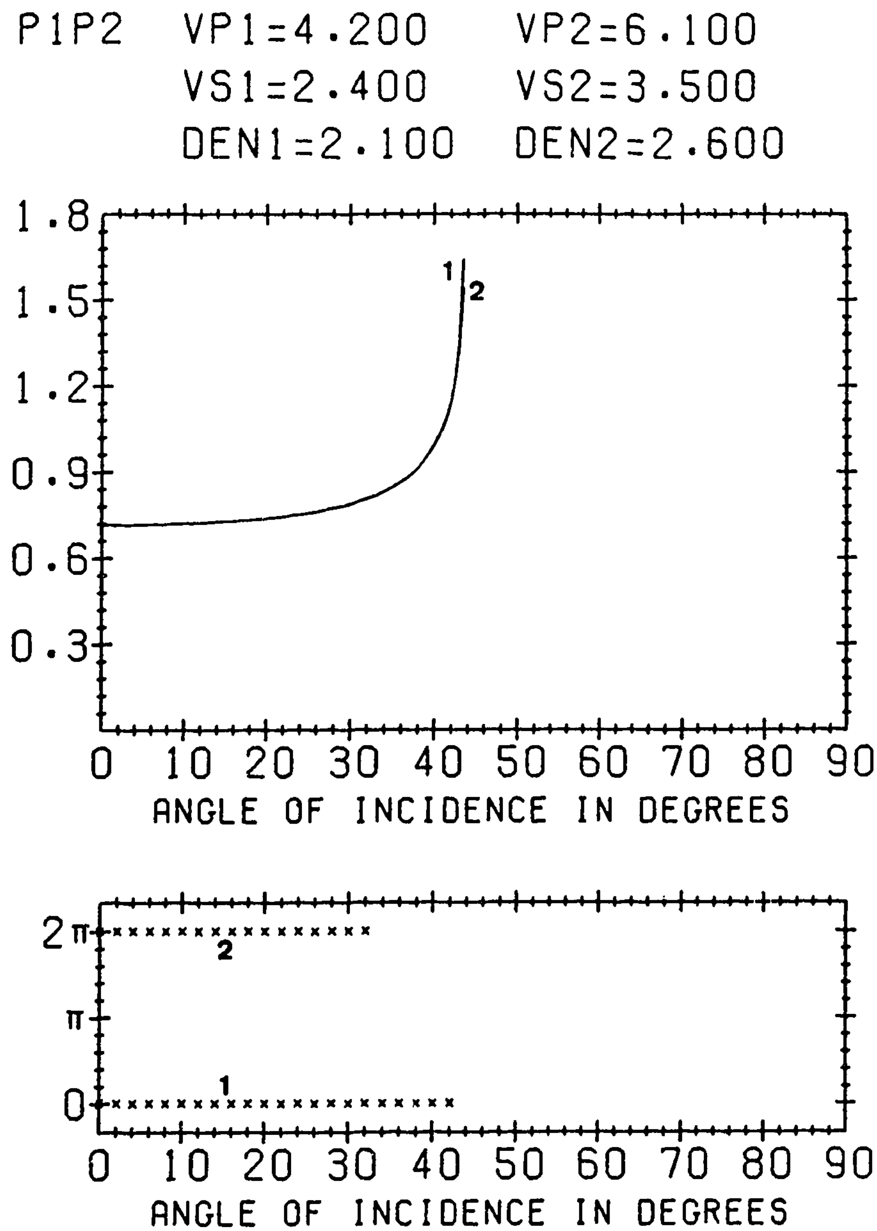
VP1, VS1, DEN1 are parameters of medium 1.

VP2, VS2, DEN2 are parameters of medium 2.

P1P1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600




P1S1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
0.6
0.4
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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DEN1=2.100 DEN2=2.600
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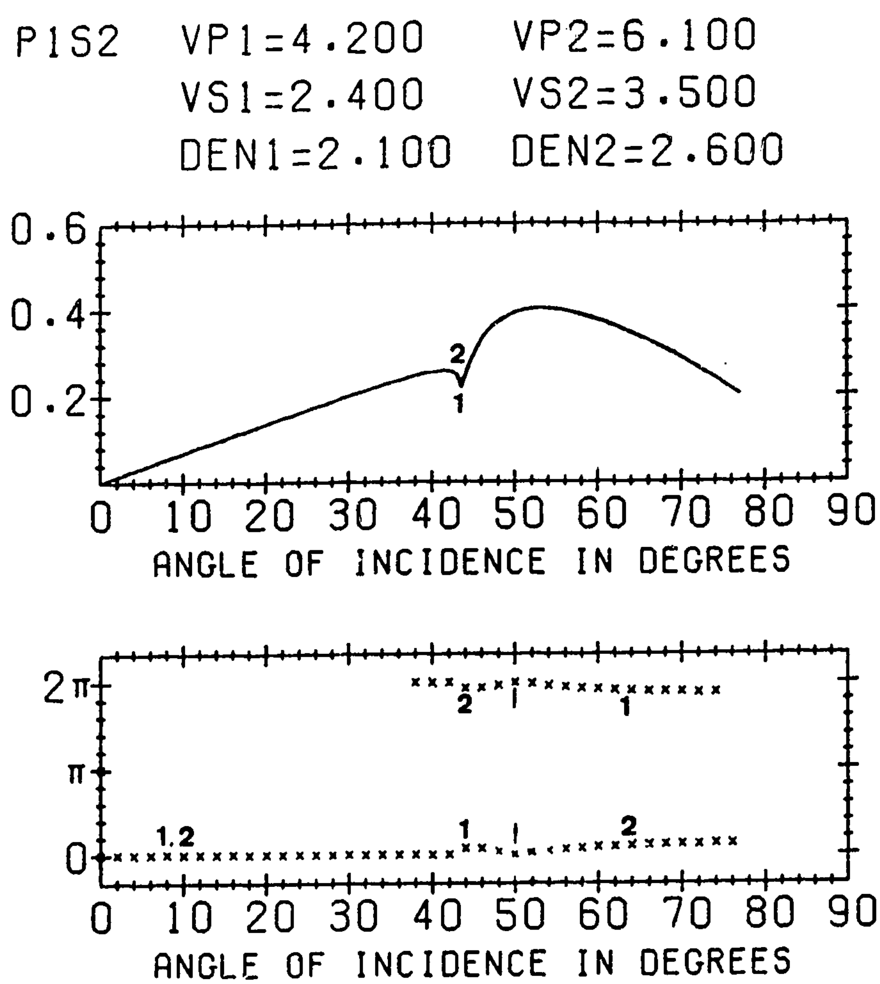
1.5
1.2
0.9
0.6
0.3
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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VS2=3.500
DEN1=2.100
DEN2=2.600
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0.4
0.2
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
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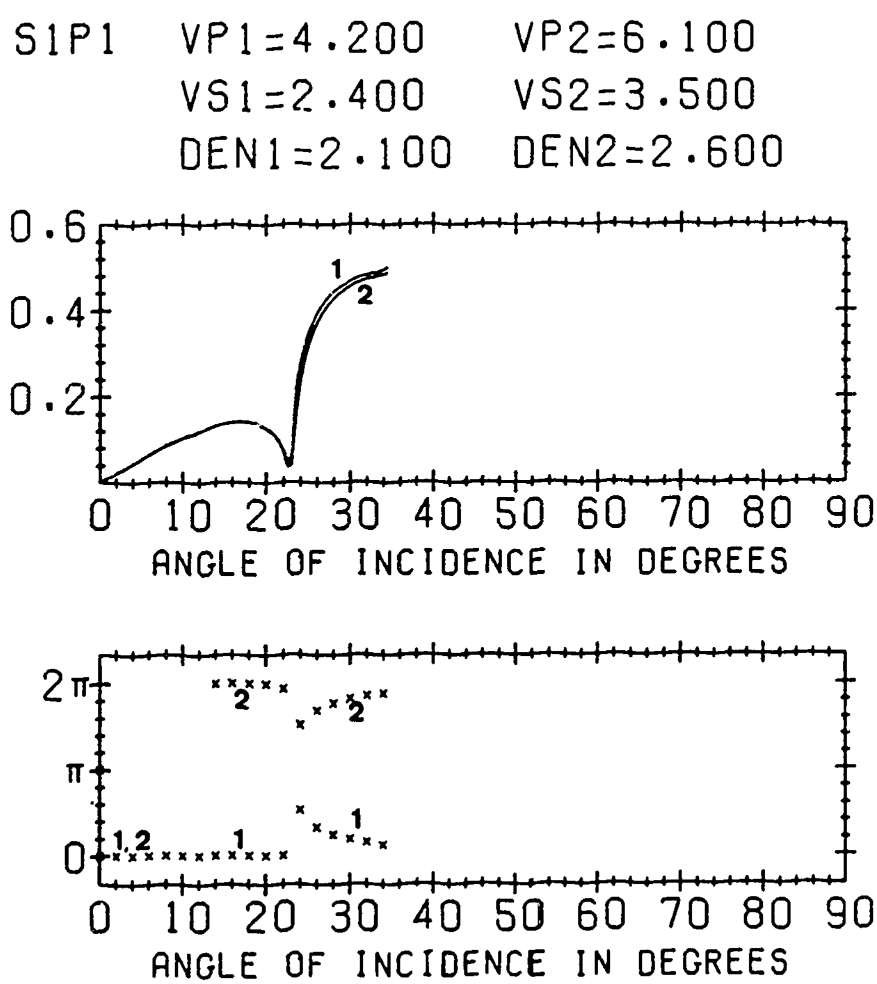
π
0
2
1
1.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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0.4
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
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π
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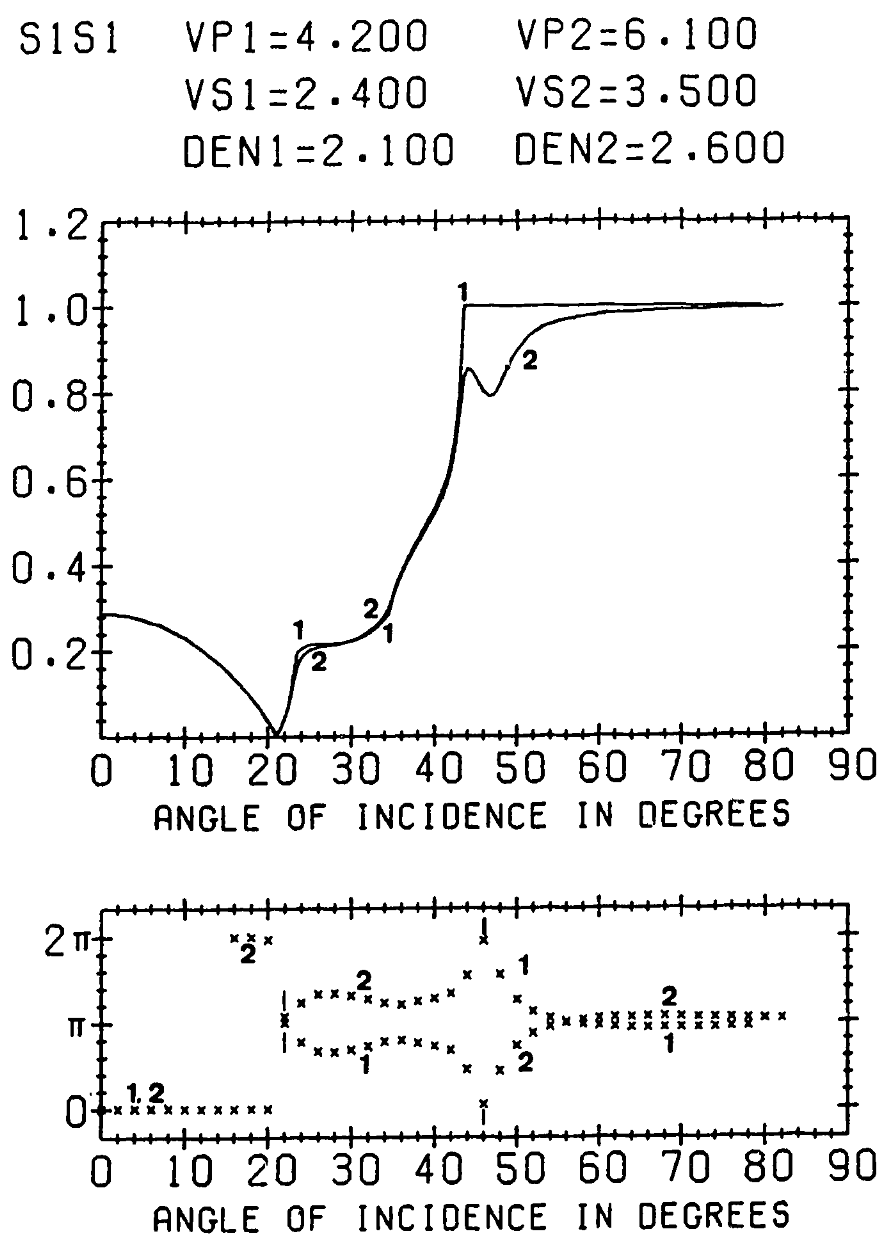
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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DEN1=2.100 DEN2=2.600
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0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

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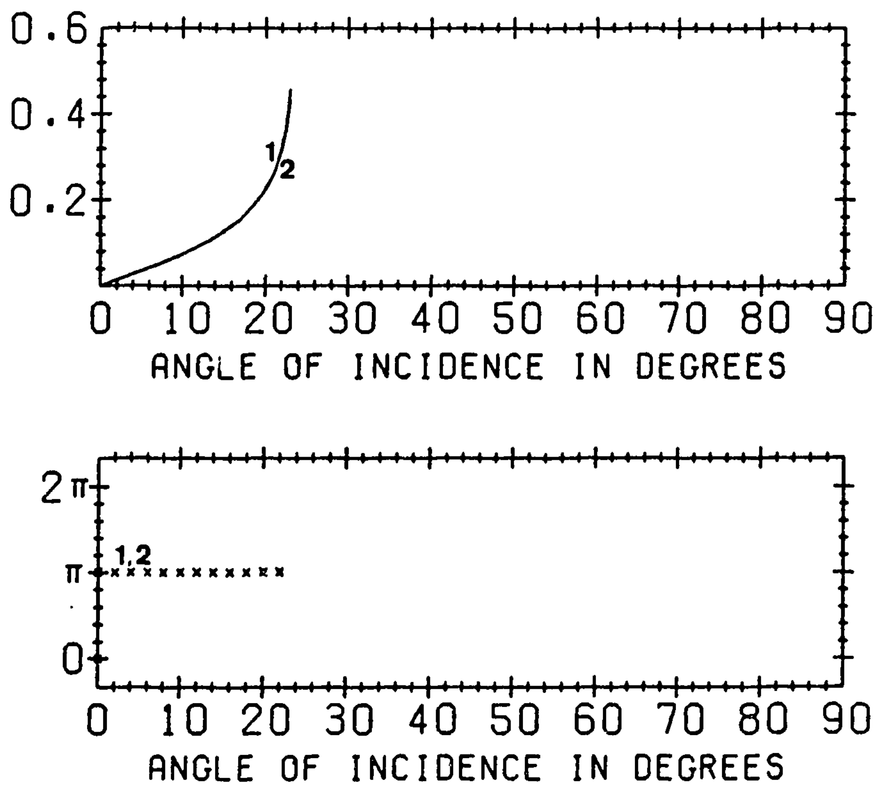
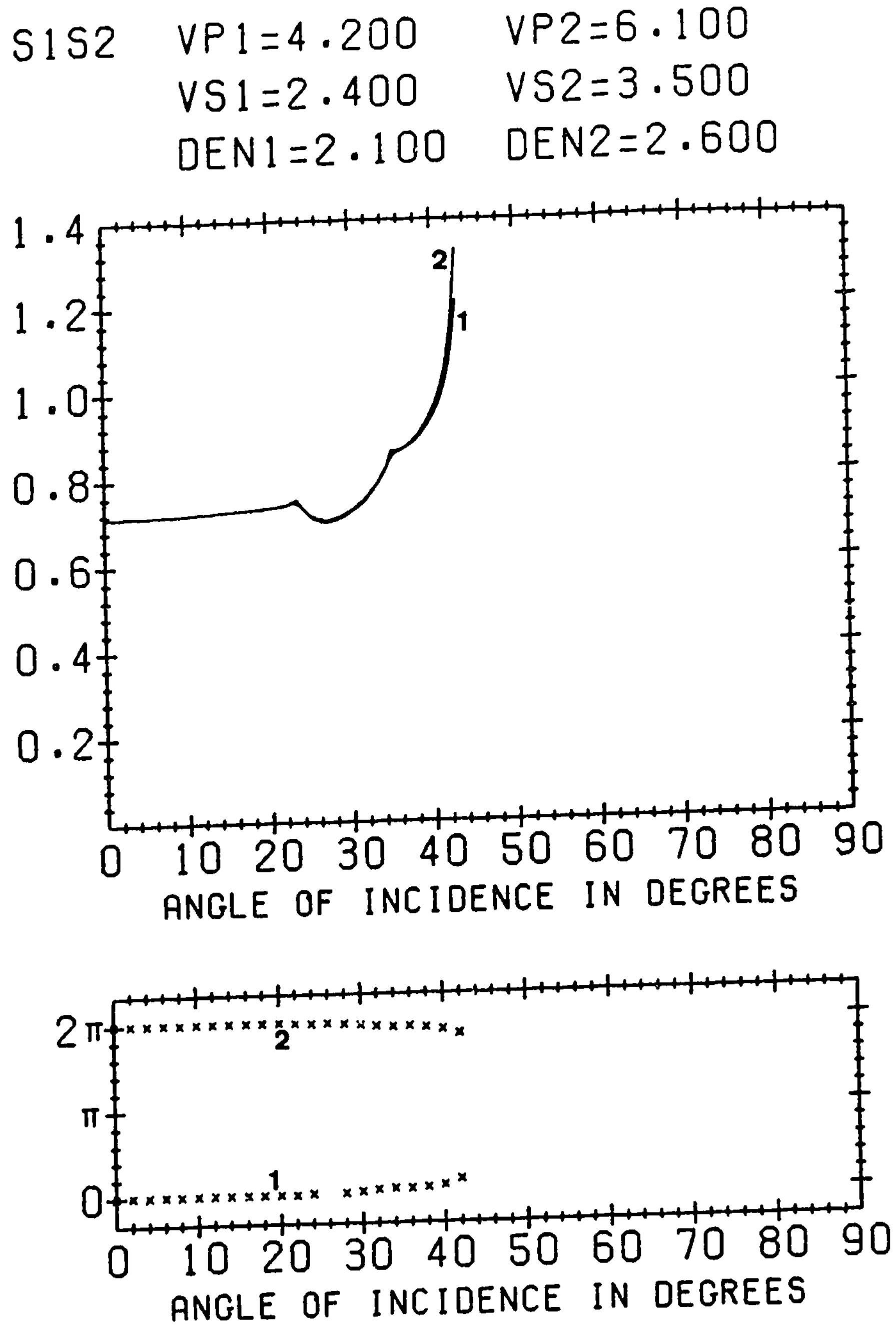
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600

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DEN1=2.100 DEN2=2.600
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1.2
1.0
0.8
0.6
0.4
0.2
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
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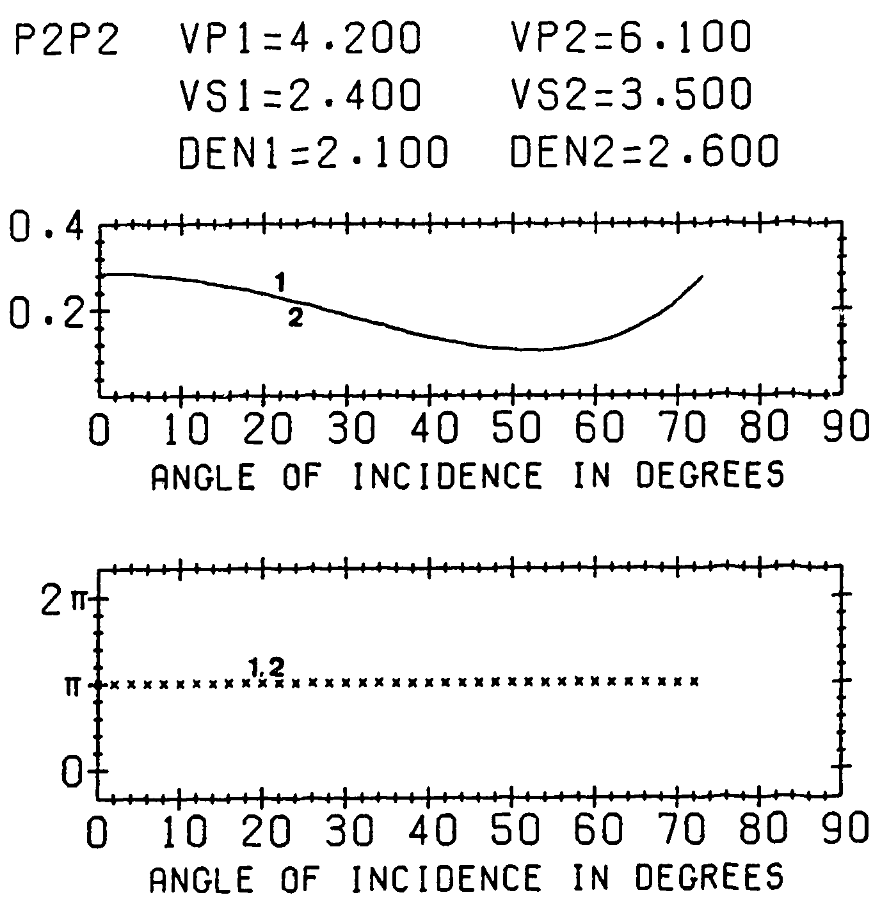
π
0
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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DEN1=2.100
DEN2=2.600
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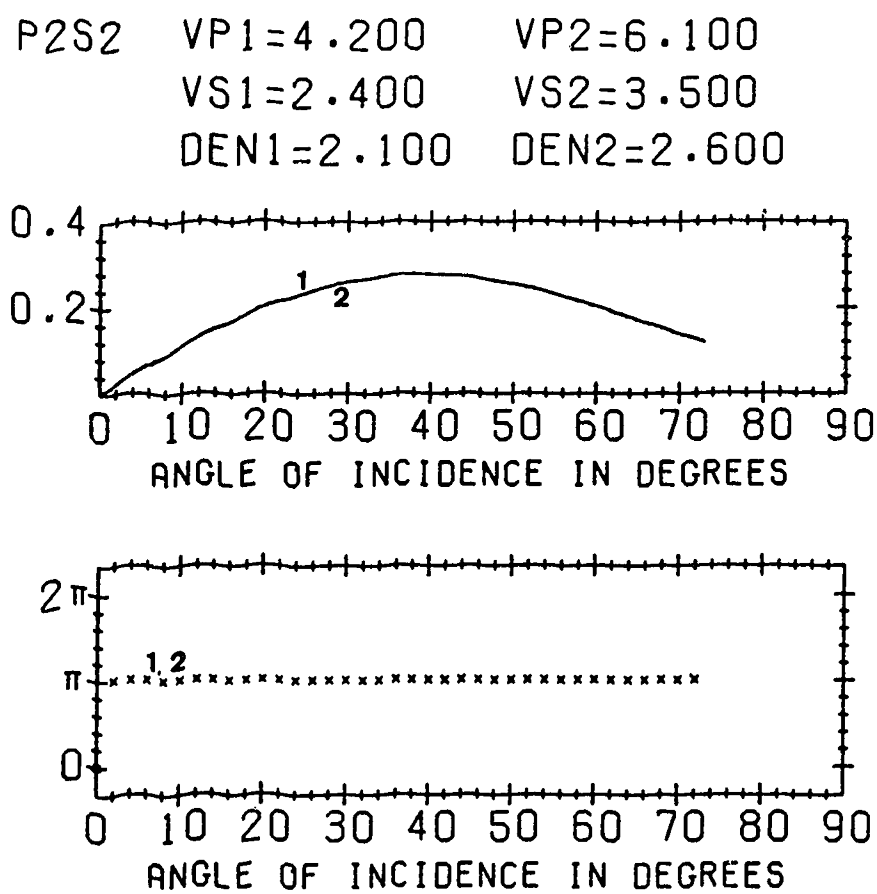
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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DEN1=2.100 DEN2=2.600
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0.2
1
2
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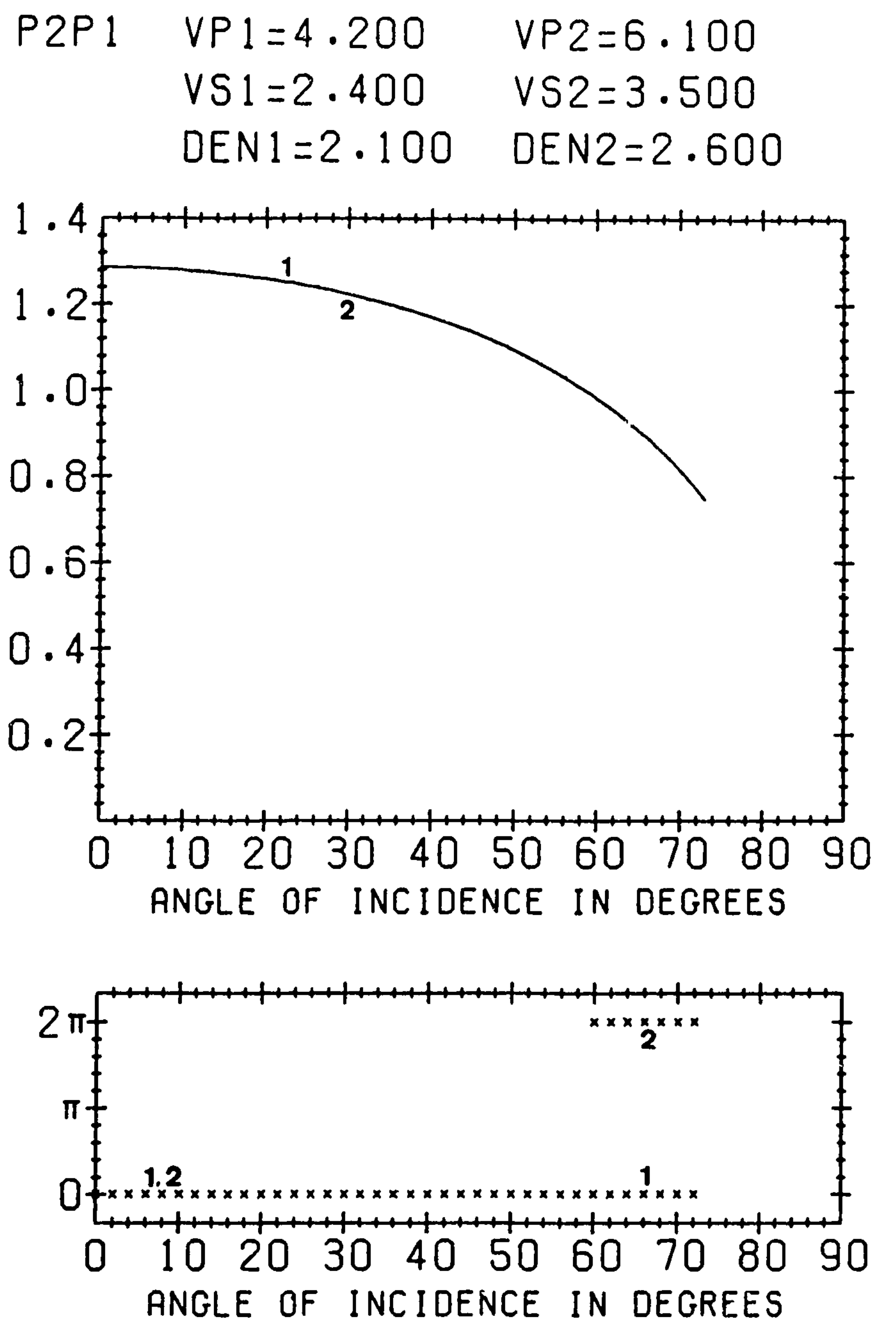
ANGLE OF INCIDENCE IN DEGREES
2π
π
1,2
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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DEN1=2.100 DEN2=2.600
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1.2
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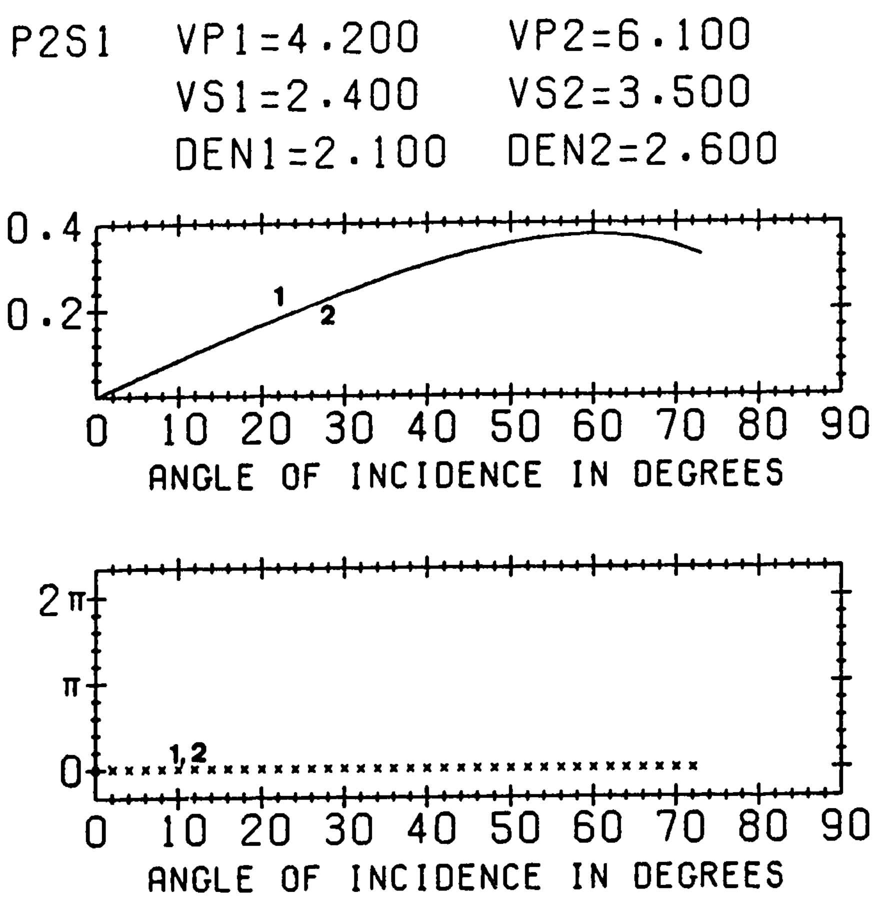
0.8
0.6
0.4
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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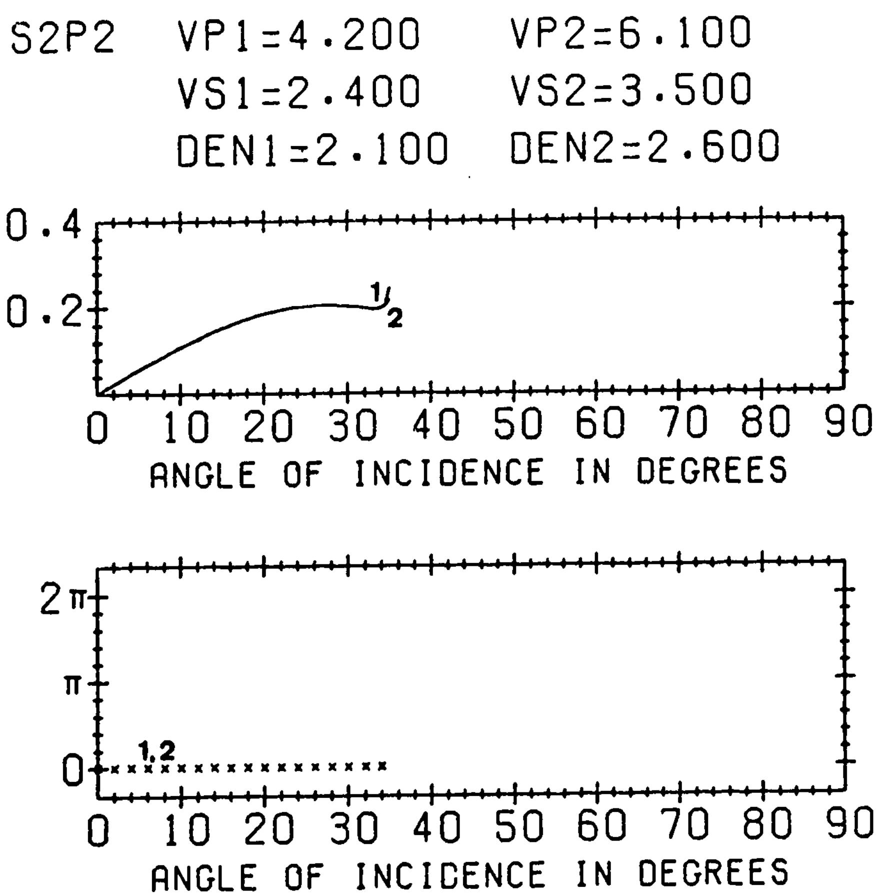
VP2=6.100
VS1=2.400
VS2=3.500
DEN1=2.100
DEN2=2.600
0.4
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
0.4
0.2
1/2
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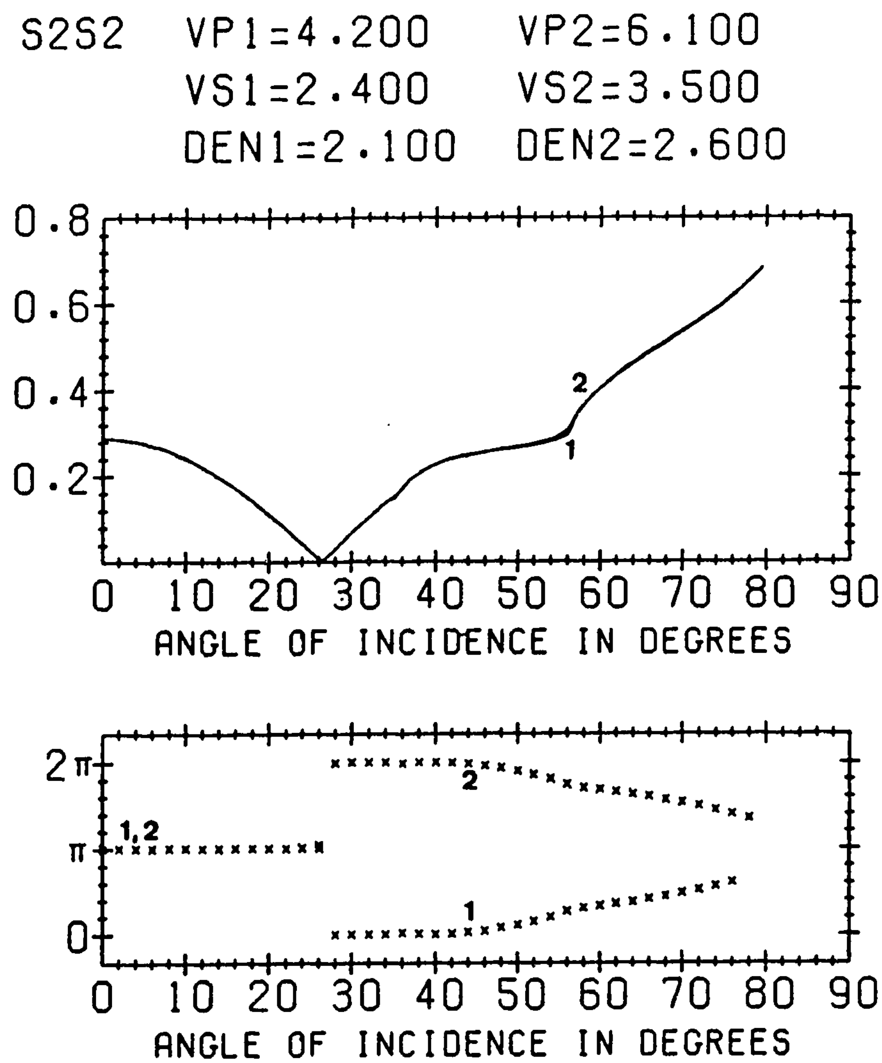
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
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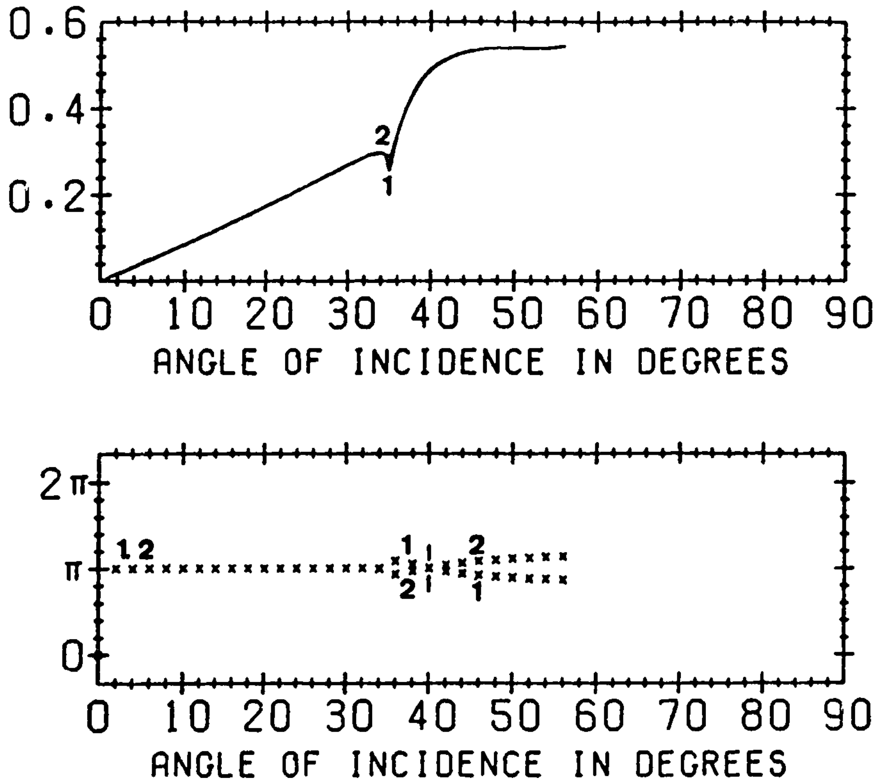
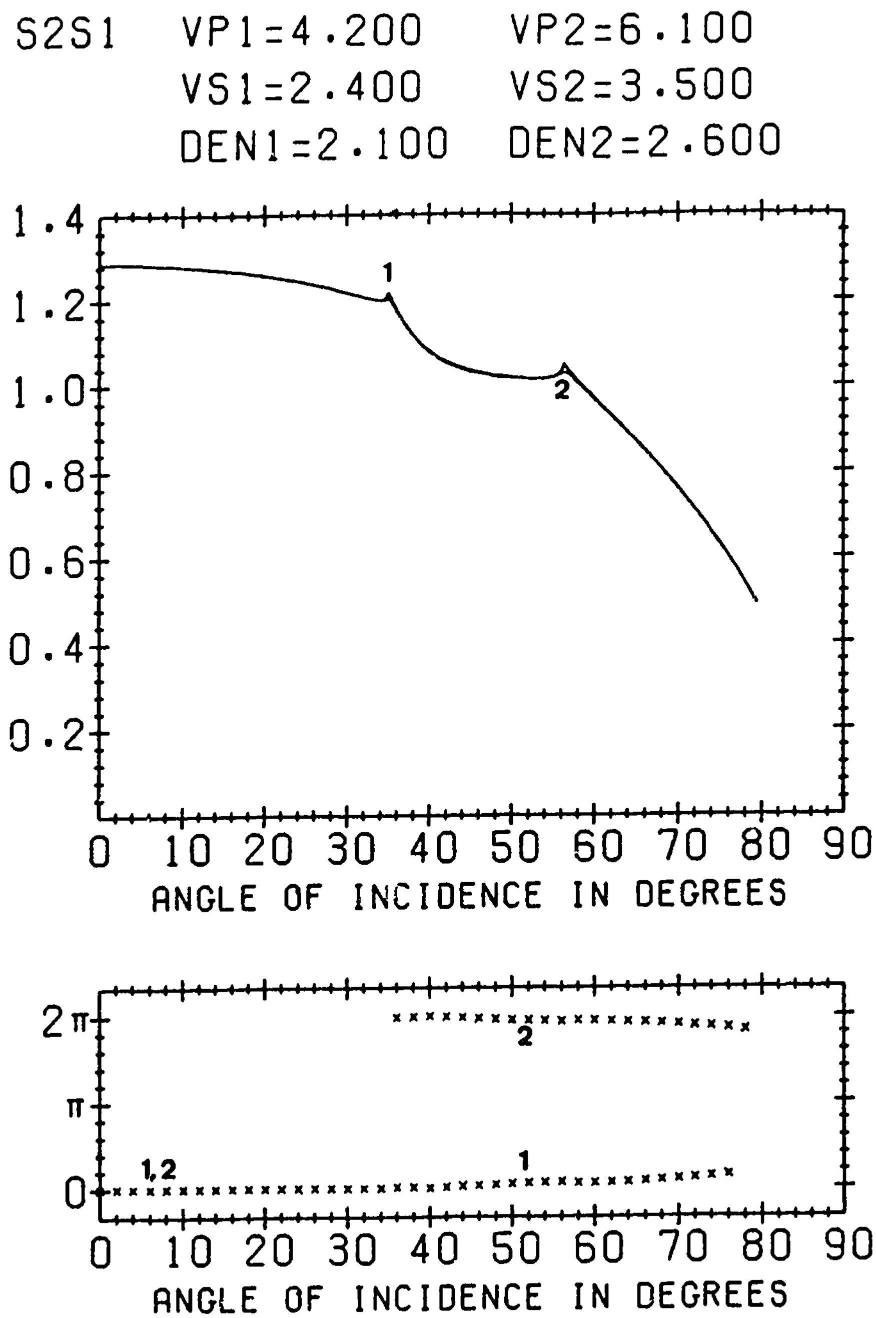
0.6
0.4
0.2
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
2
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

S2P1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600

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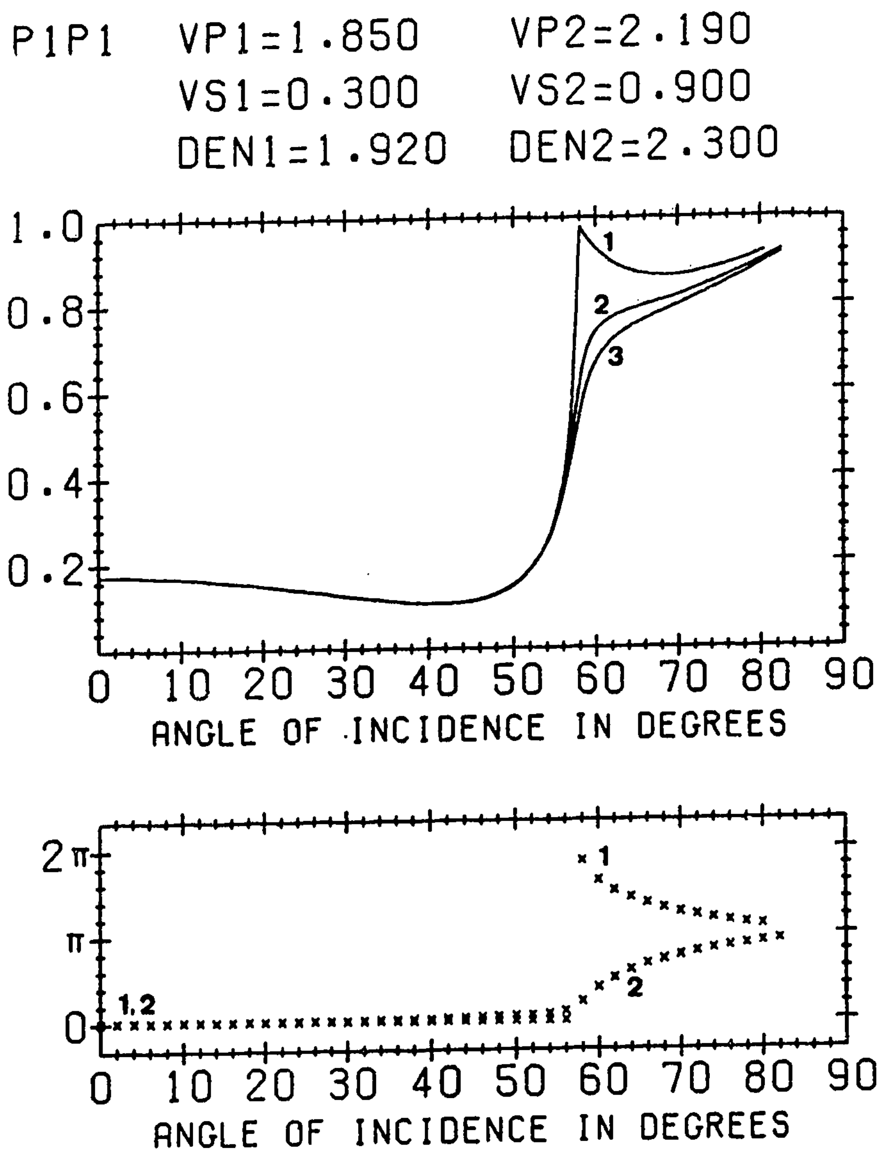
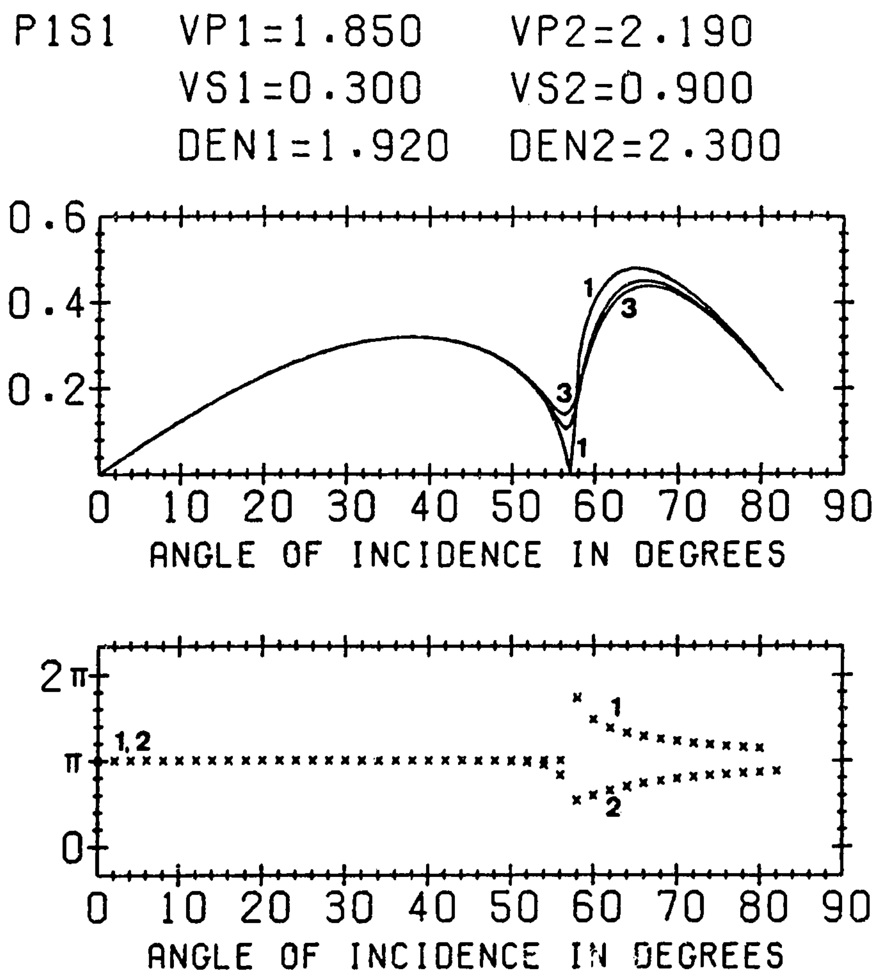
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
1.4
1.2
1.0
0.8
0.6
0.4
0.2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

Figure 9: P-SV coefficients obtained with interface B.
(only incidence from upper medium is considered)
Curve #1 represents the elastic case.
Curve #2 represents the anelastic case.
Curve #3 represents a second anelastic
case: QP2 and QS2 are set equalt to 80
VP1, VS1, DEN1 are parameters of medium 1.
VP2, VS2, DEN2 are parameters of medium 2.


P1P1 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
1.0
0.8
0.6
0.4
0.2
1
2
3
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1
2
1,2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES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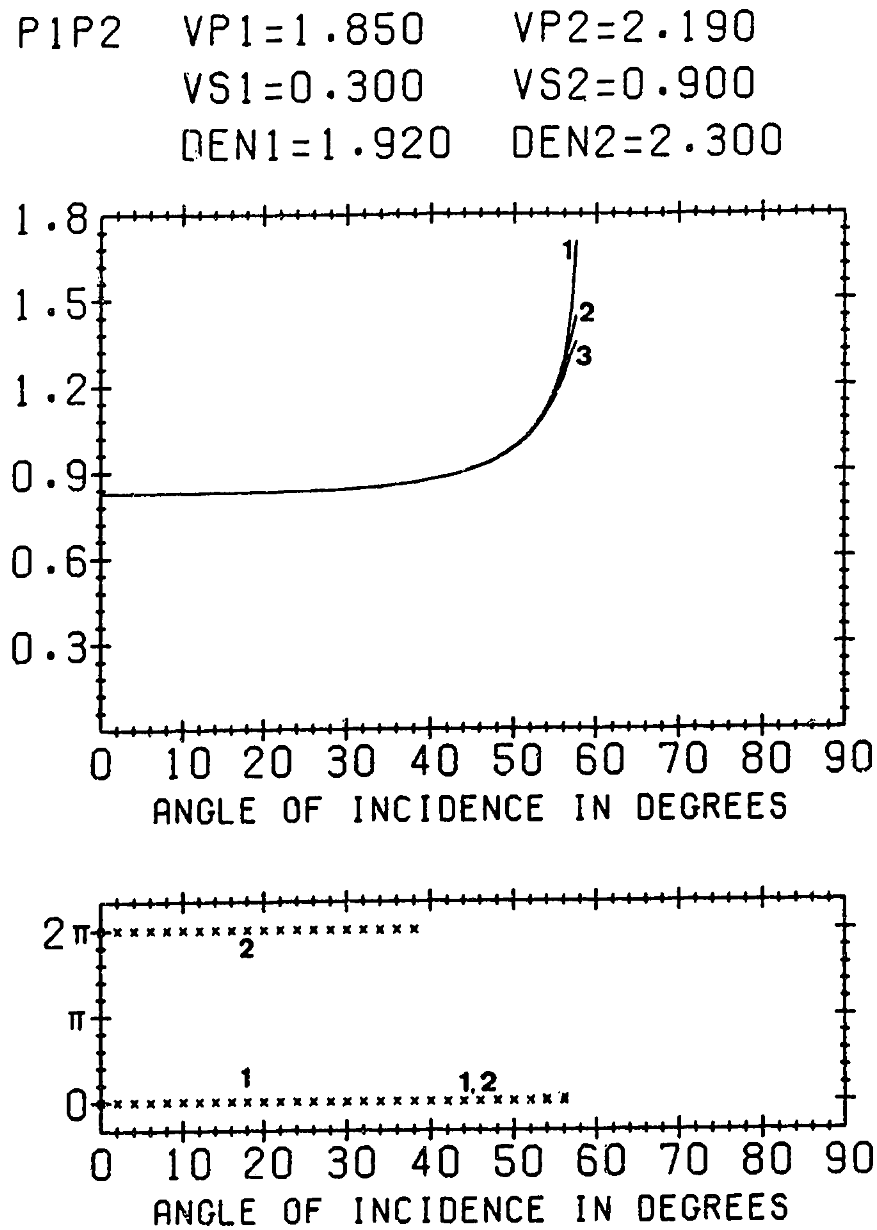
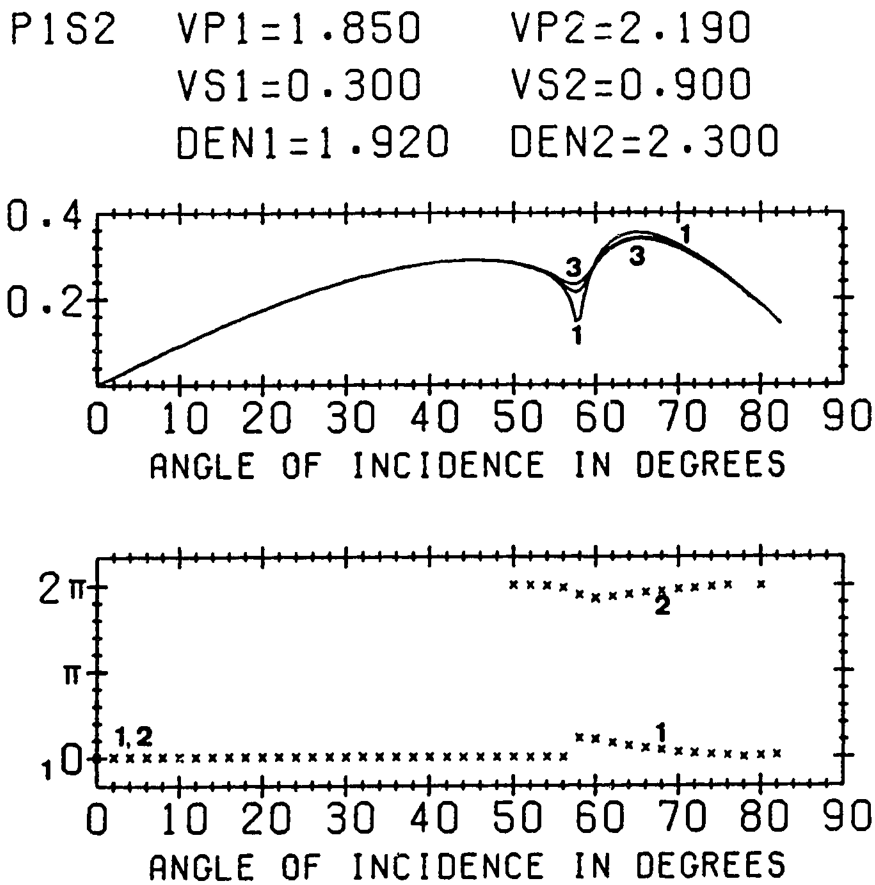
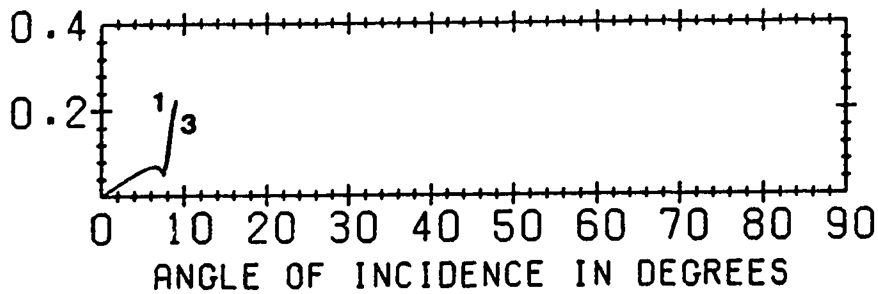
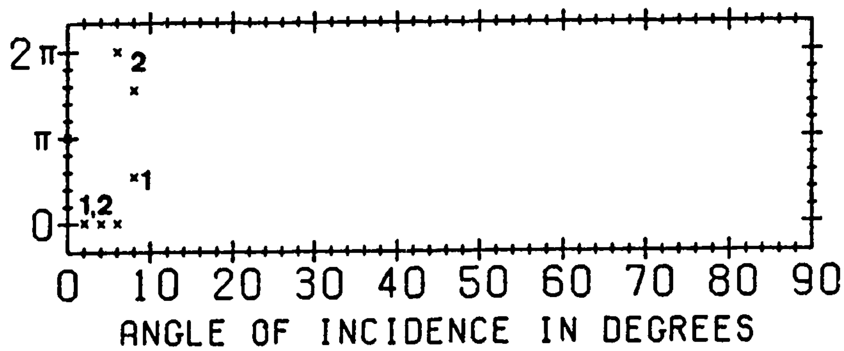
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
0.6
0.4
0.2
1
3
3
1
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

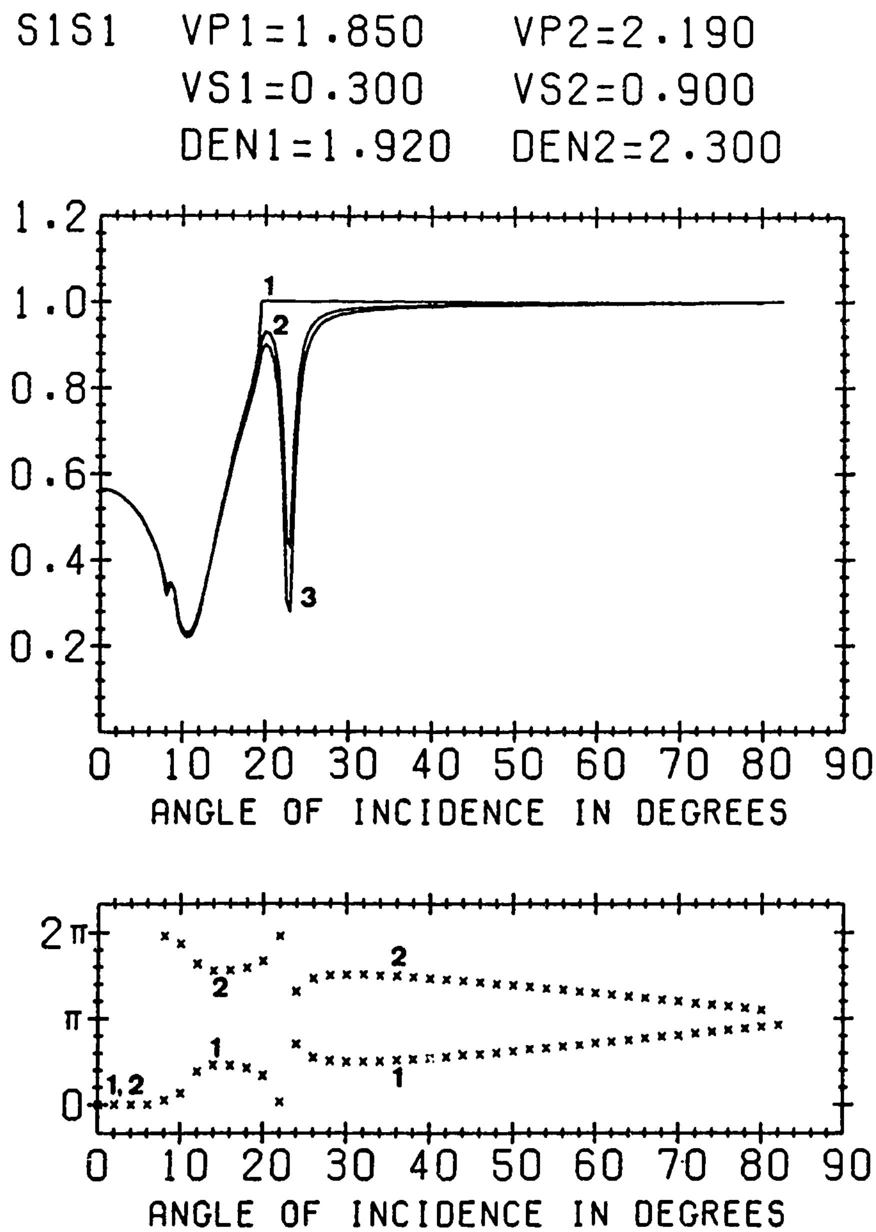

P1P2 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
1.8
1.5
1.2
0.9
0.6
0.3
1
2
3
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
2
1
1,2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


P1S2 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

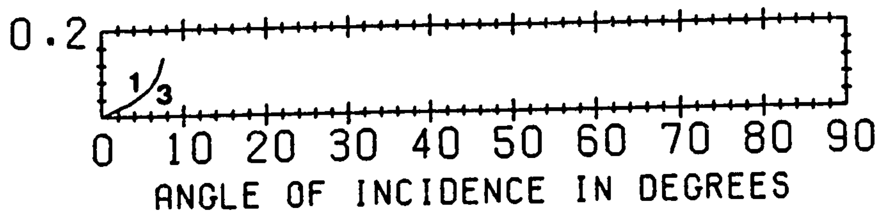
S1P1 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300

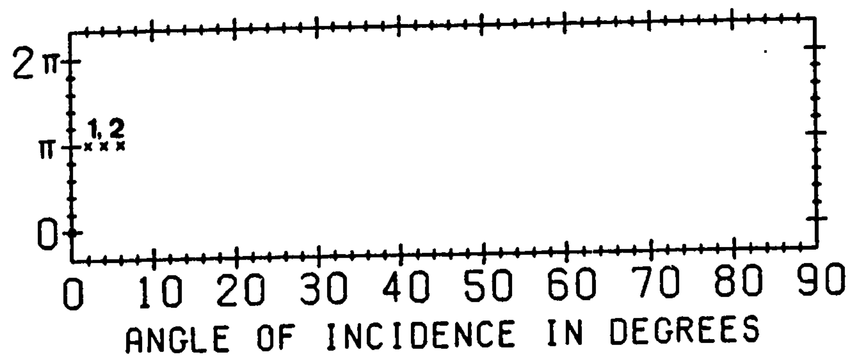




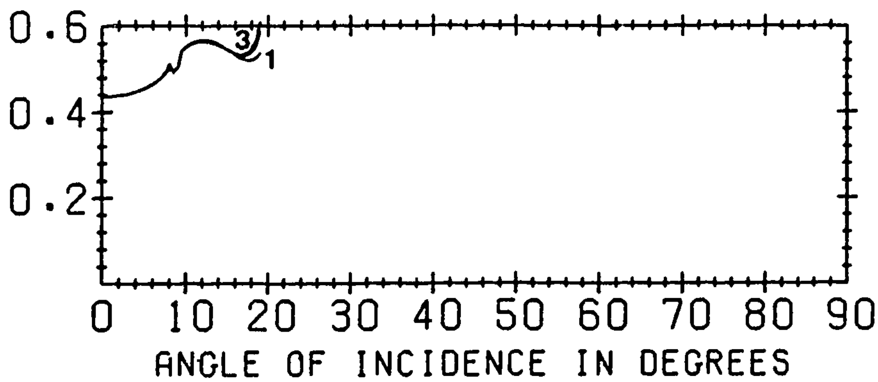

S1S1 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
1.2
1.0
0.8
0.6
0.4
0.2
1
2
3
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
1,2
1
2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

S1P2 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300





S1S2 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300



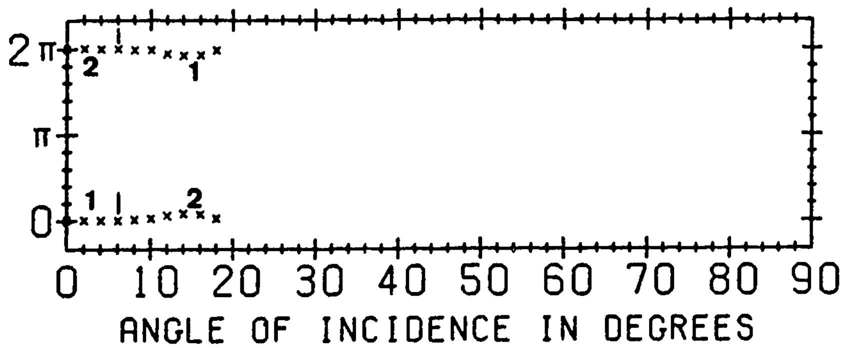


Figure 10: Effects of reflection and transmission coefficients on a wavelet (4 cases computed with interface A).

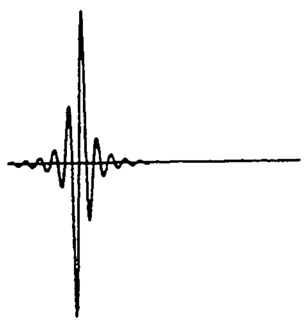
VP1, VS1, DEN1 are parameters of medium 1

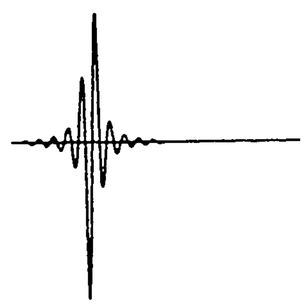
VP2, VS2, DEN2 are parameters of medium 2

(Numbers have the same meaning as in Fig. 8)

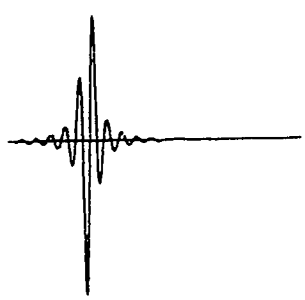
P1P1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
ANGLE OF INCIDENCE= 44.0

S.W.

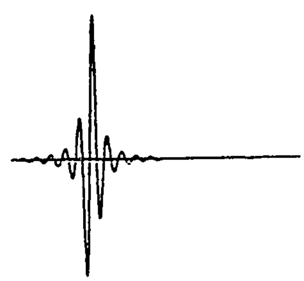




1



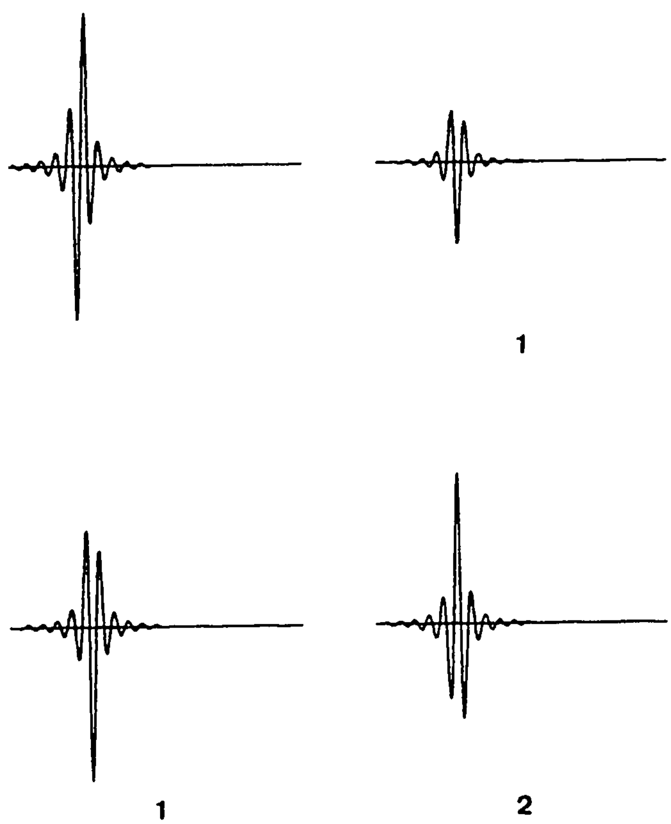
1



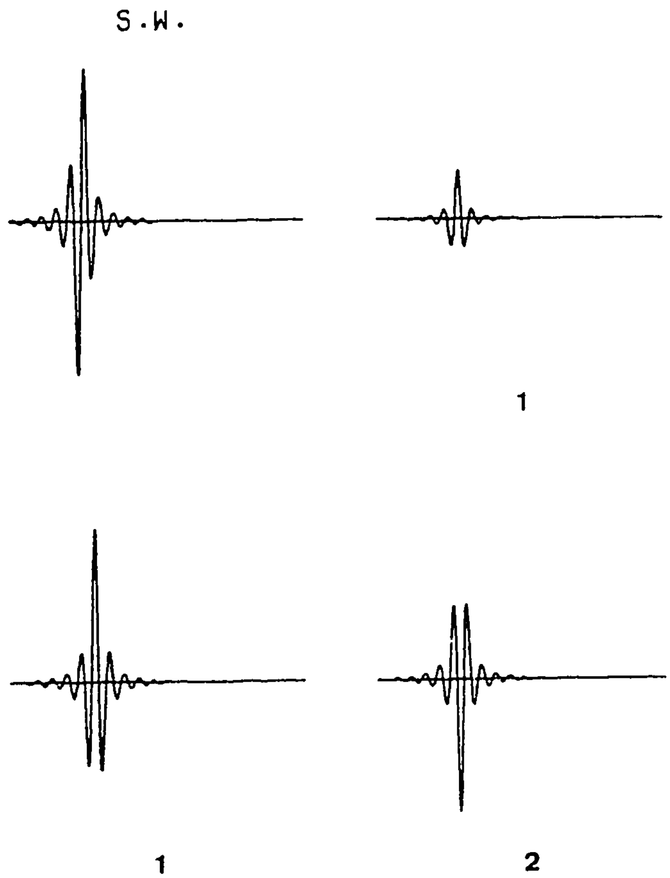
2

P1S1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
ANGLE OF INCIDENCE= 48.0

S.W.



S1P1 VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
ANGLE OF INCIDENCE= 24.0



SISI VP1=4.200 VP2=6.100
VS1=2.400 VS2=3.500
DEN1=2.100 DEN2=2.600
ANGLE OF INCIDENCE= 48.0

S.W.

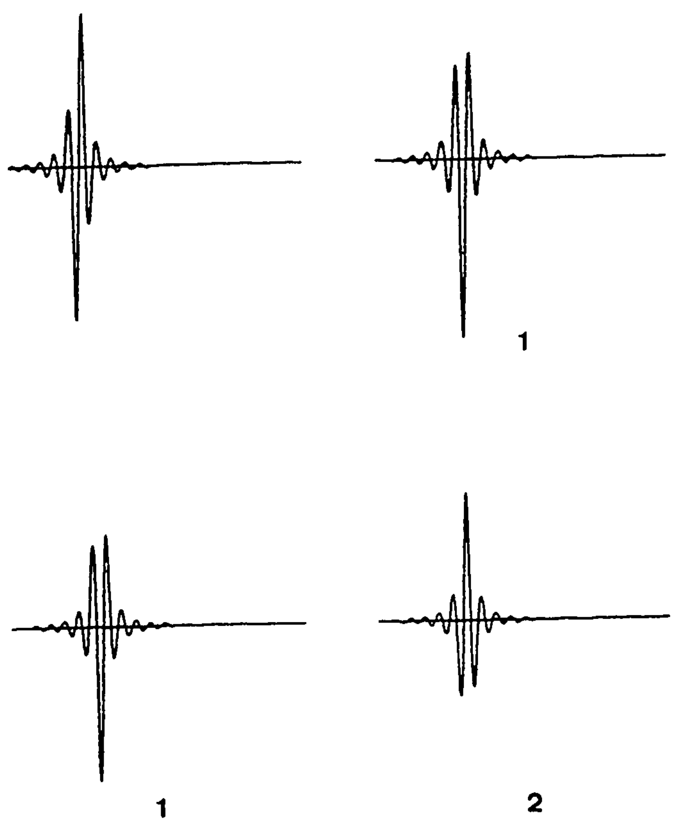


Figure 11: Effects of reflection and transmission coefficients on a wavelet (3 cases computed with interface B).

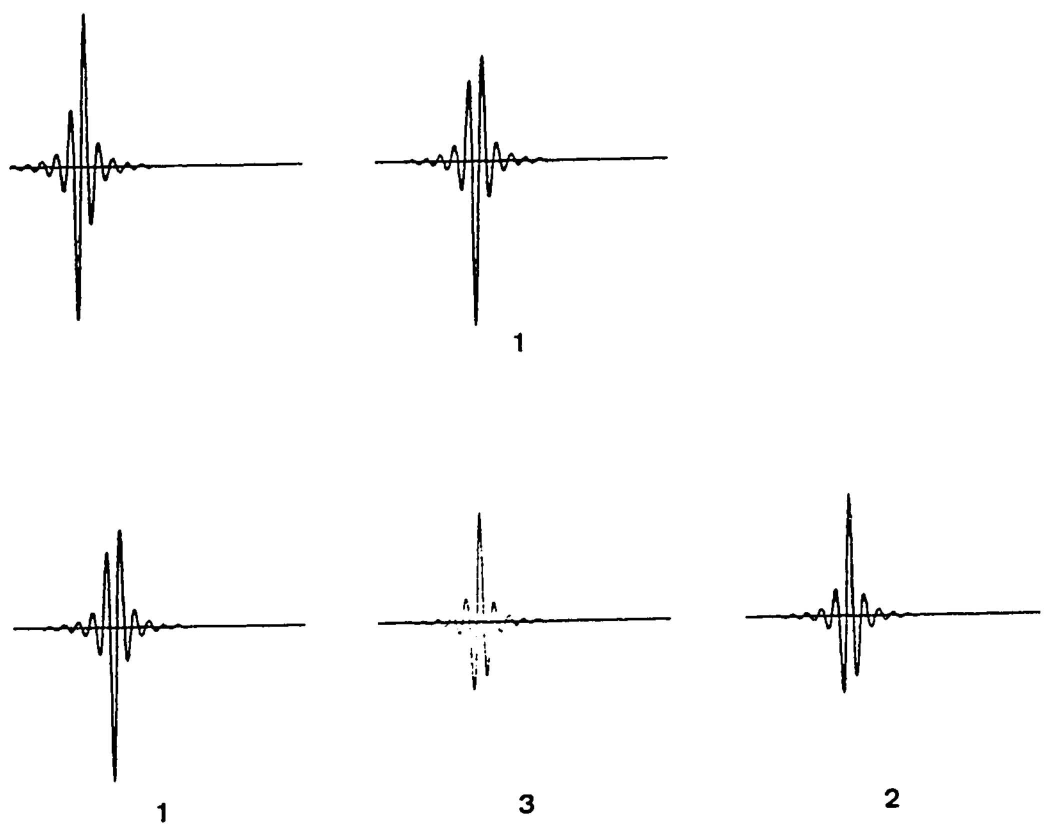
VP1, VS1, DEN1 are parameters of medium 1

VP2, VS2, DEN2 are parameters of medium 2

(Numbers have the same meaning as in Fig. 9)

P1P1 VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
ANGLE OF INCIDENCE= 60.0

S.W.



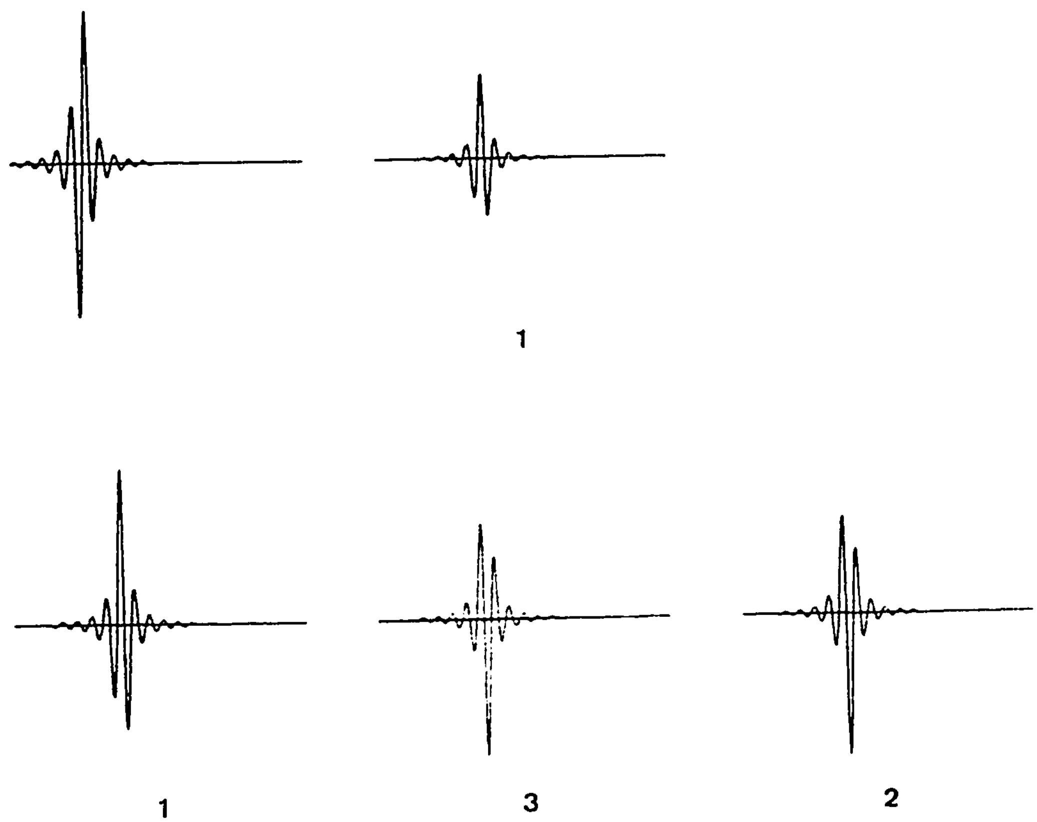
P1S1 VP1=1.850 VP2=2.190

VS1=0.300 VS2=0.900

DEN1=1.920 DEN2=2.300

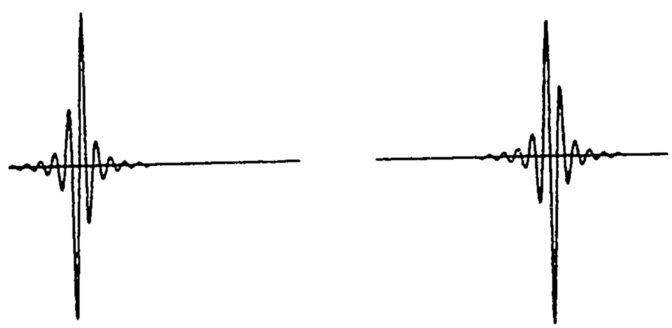
ANGLE OF INCIDENCE= 64.0

S.W.



SISI VP1=1.850 VP2=2.190
VS1=0.300 VS2=0.900
DEN1=1.920 DEN2=2.300
ANGLE OF INCIDENCE= 23.0

S.W.



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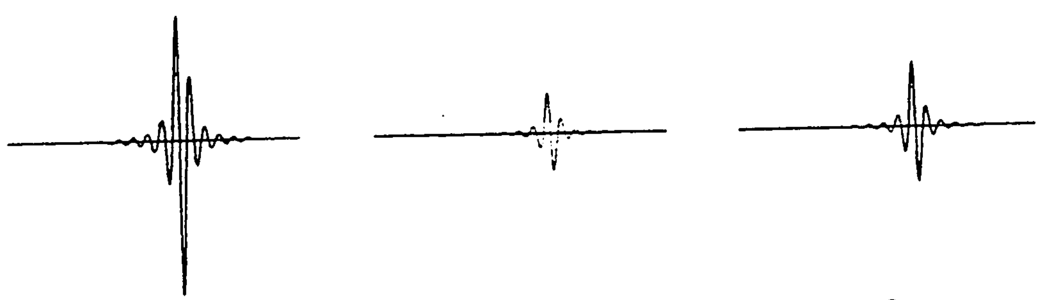


1 3 2

CHAPTER 7

SEISMOGRAMS

In this chapter the theory developed in the previous chapters is used to compute simple synthetic seismograms. The purpose is to evaluate the importance of including viscoelastic geometrical spreading and viscoelastic coefficients in the computation of synthetic seismograms for waves passing through layered linear viscoelastic media.

The zeroth order of the asymptotic ray theory is used. The amplitude of motion of a ray propagating through a model and reaching a receiver is given by an inverse Fourier transform as

$$W = \frac{1}{2\Pi} \int_{-\infty}^{\infty} S(\omega) \, \frac{Y(\omega) \, e^{i\phi(\omega)}}{L(\omega)} \, e^{i\omega(\tau - t)} \, d\omega \qquad (7.1)$$

where $S(\omega)$ is the frequency spectrum of the source pulse $s(t)$, $Y(\omega)\ e^{i\phi(\omega)}$ is the complex product of the anelastic reflection and transmission coefficients for the specified ray, Y being the relative amplitude and ϕ the relative phase and $L(\omega)$ is the geometrical spreading given by (5.82). τ is the complex phase function. Its real part is the arrival time of the ray and its imaginary part gives the attenuation of the ray due to absorption. $W(t)$ can easily be determined using a Fast Fourier Transform (FFT) algorithim, the term S $\frac{Y\, e^{i(\phi + \omega\tau)}}{L}$ having to be equal to zero beyondsome Nyquist frequency.

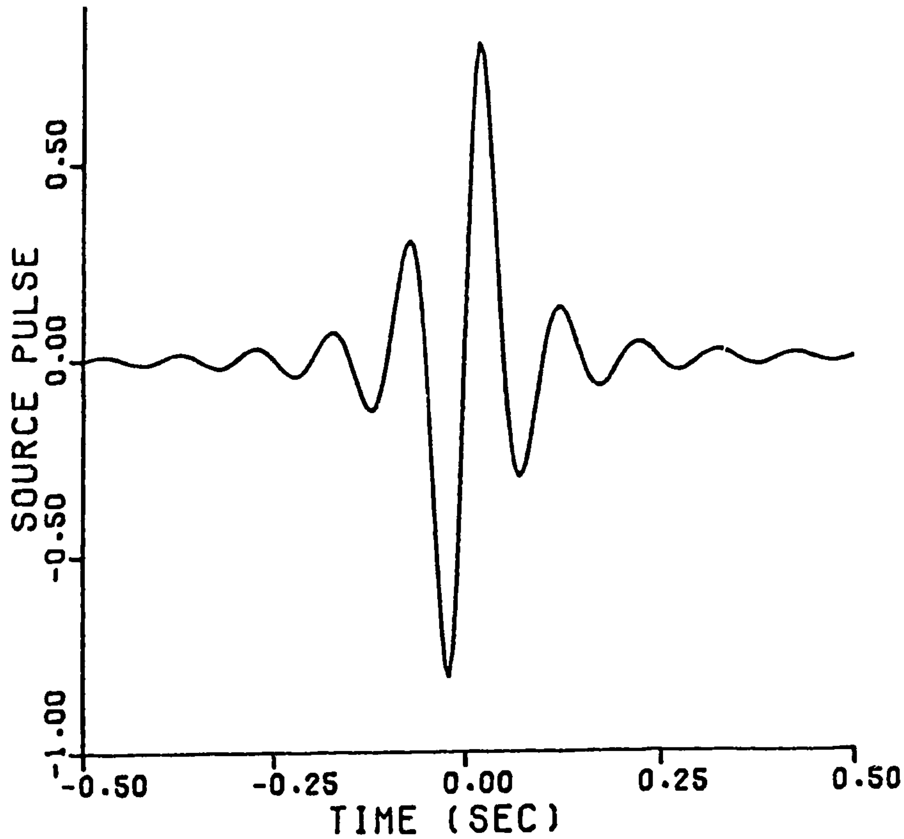
In the computations, Q is assumed to be independent of frequency. Consequently the velocity dispersion is neglected and a constant velocity v_H is used for all frequencies. This assumption introduces an error in the response however this error can be largely reduced by considering a source pulse which exibits a high and narrow peak at a dominant frequency f_0. Most of the contribution to the integral (7.1) will thus come from a narrow interval centered about $\omega = \omega_0 = 2\Pi f_0$. The dispersion in this interval will be negligible. To reduce the errors in arrival time caused by neglecting the velocity dispersion the values of the constant velocity v_H and the quality factor Q used in the computations are the ones calculated at the dominant frequency of the source pulse. Taking into consideration all these factors the source pulse is chosen to be

$$s(t) = \frac{\sin(\omega_0 t)}{1 + \left(\frac{\omega_0 t}{\eta}\right)^2} \qquad (7.2)$$

Figure 12 shows the source pulse and its amplitude spectrum.

The model used to compute synthetic seismograms is a crustal model calculated by Silva (1976) (table 2). Both the point source and the receiver are located at surface. The rays considered are P1P2P3P3P2P1 and S1S2S3S3S2S1. The vertical and horizontal components of the arrivals are computed for elastic (Q^{-1} is set equal to 0 in each layer) and viscoelastic cases (figures 13 and 14).

As expected there is no noticeable diffence in travel time between synthetic seismograms computed in the elastic and anelastic models. This is due to the fact that stationary ray path satisfies the least time Fermat's principle (see chapter 4). Considering the ray P1P2P3P3P2P1 the difference in amplitude between elastic and anelastic seismograms is only due to the viscoelasticity of the media traversed. There is no significant difference in phase. Figures 15 and 16 show the logarithm of the amplitude and the phase of the product $\frac{Y e^{i\phi}}{L}$ for the ray P1P2P3P3P2P1 at all epicentral distances X until the ray does not exist anymore. Curve #1 is obtained using elastic geometrical spreading and elastic reflection and transmission coefficients. Curve #2 is determined using anelastic geometrical spreading and anelastic reflection and transmission coefficients. The case formed by elastic geometrical spreading and anelastic reflection and transmission coefficients is represented by curve #3 and curve #4 is obtained with anelastic geometrical spreading and elastic reflection and



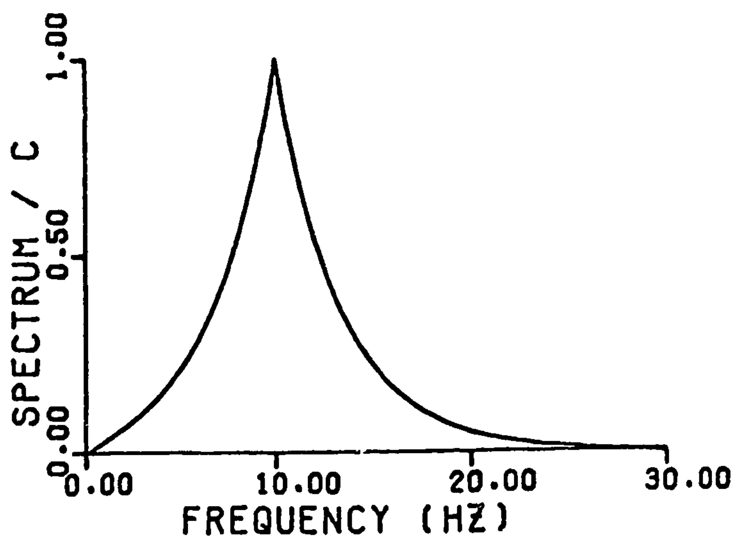


Figure 12: top Source pulse in the time domain
bottom Amplitude spectrum (divided by $C=\Pi\eta/2\omega_0$). (f_0 = 10 hz and η = 3)

TABLE 2

Crustal Model

Layer	VP (km/s)	VS (km/s)	DEN (g/cc)	QP	QS	H (km)
1	4.2	2.4	2.1	67	30	1.4
2	6.1	3.5	2.6	100	45	8.2
3	7.3	4.2	3.0	180	80	12.9
4	7.8	4.5	3.3	∞	∞	∞

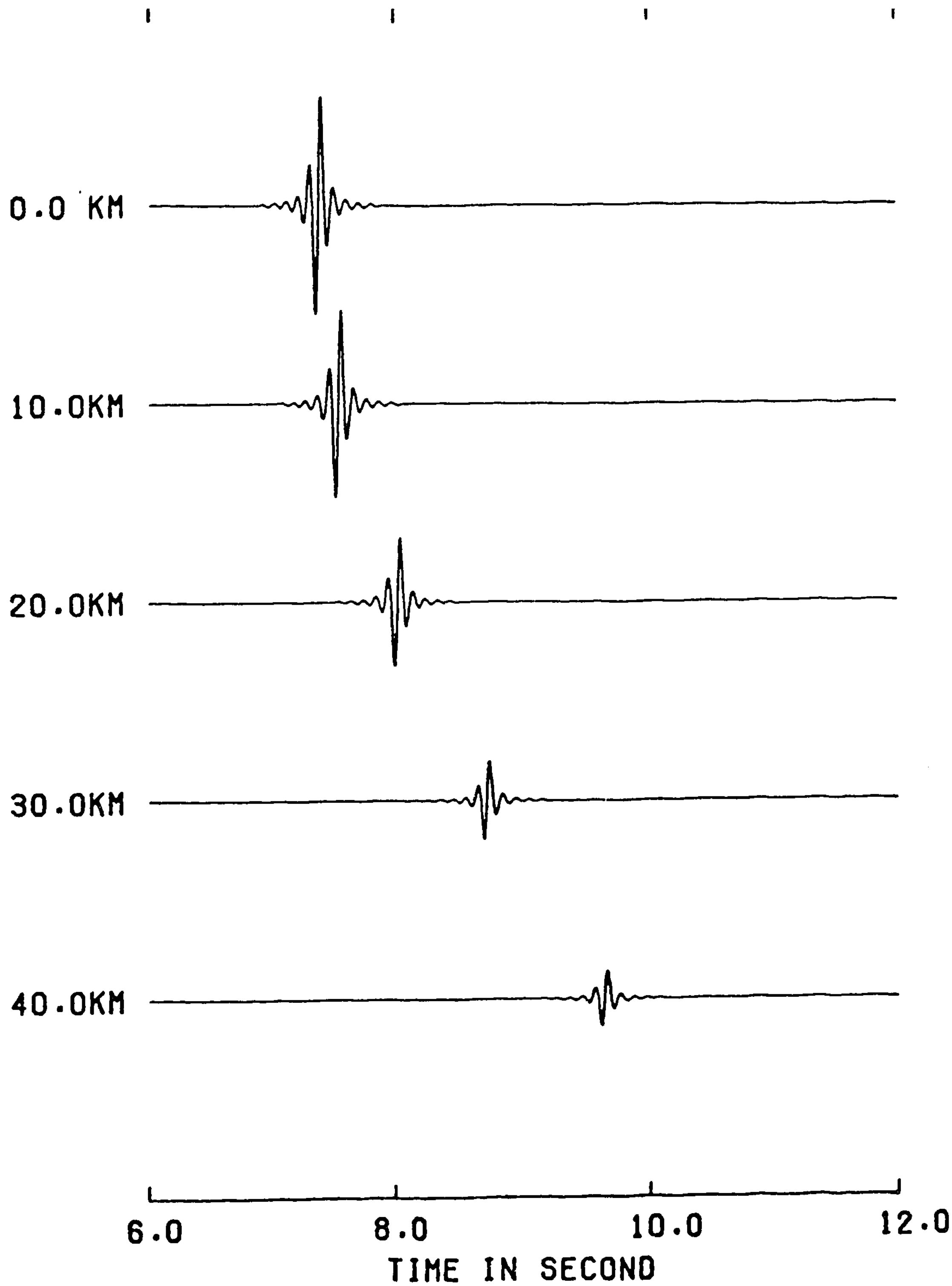
Figure 13: Seismograms obtained for the ray P1P2P3P3P2P1

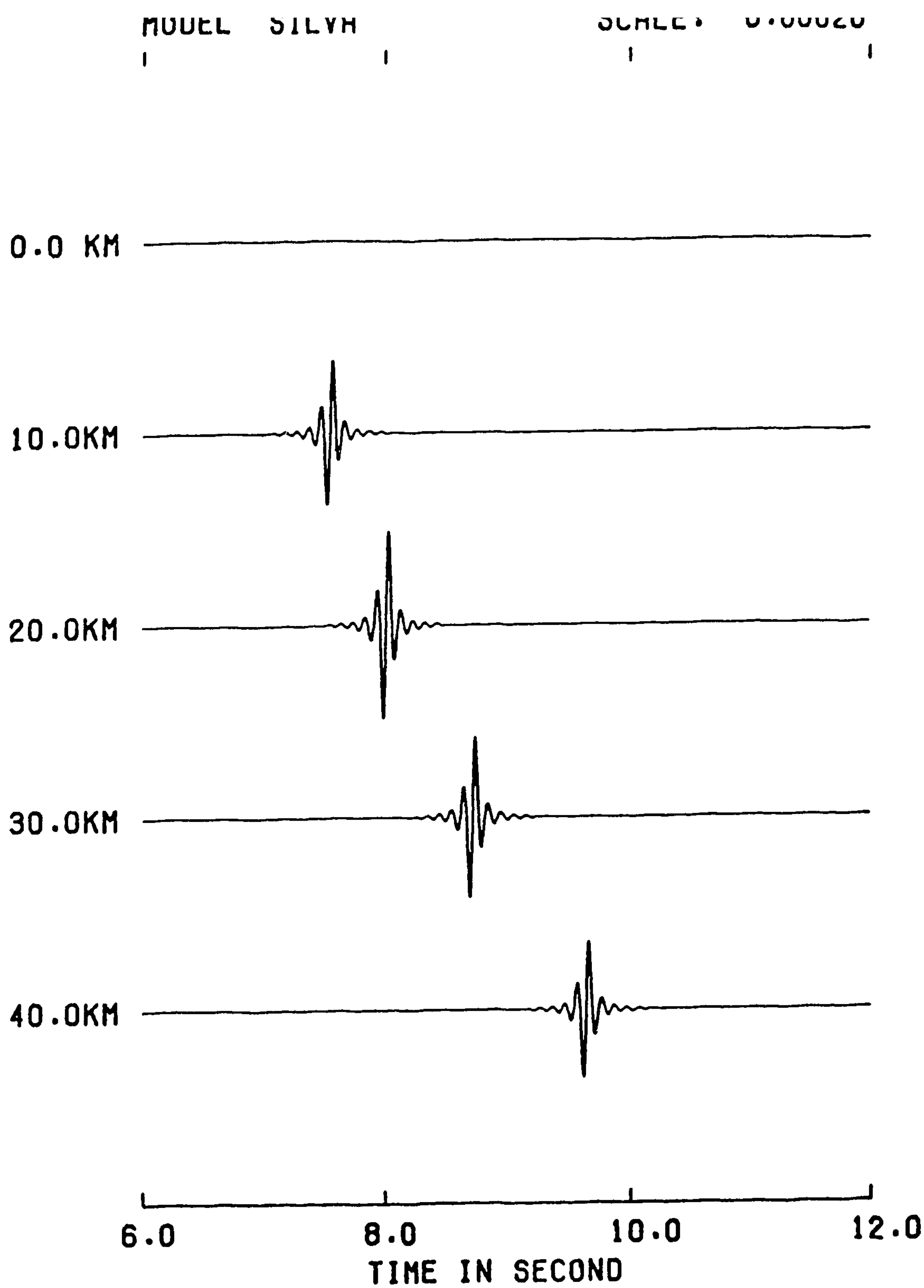
First: vertical component (elastic case)

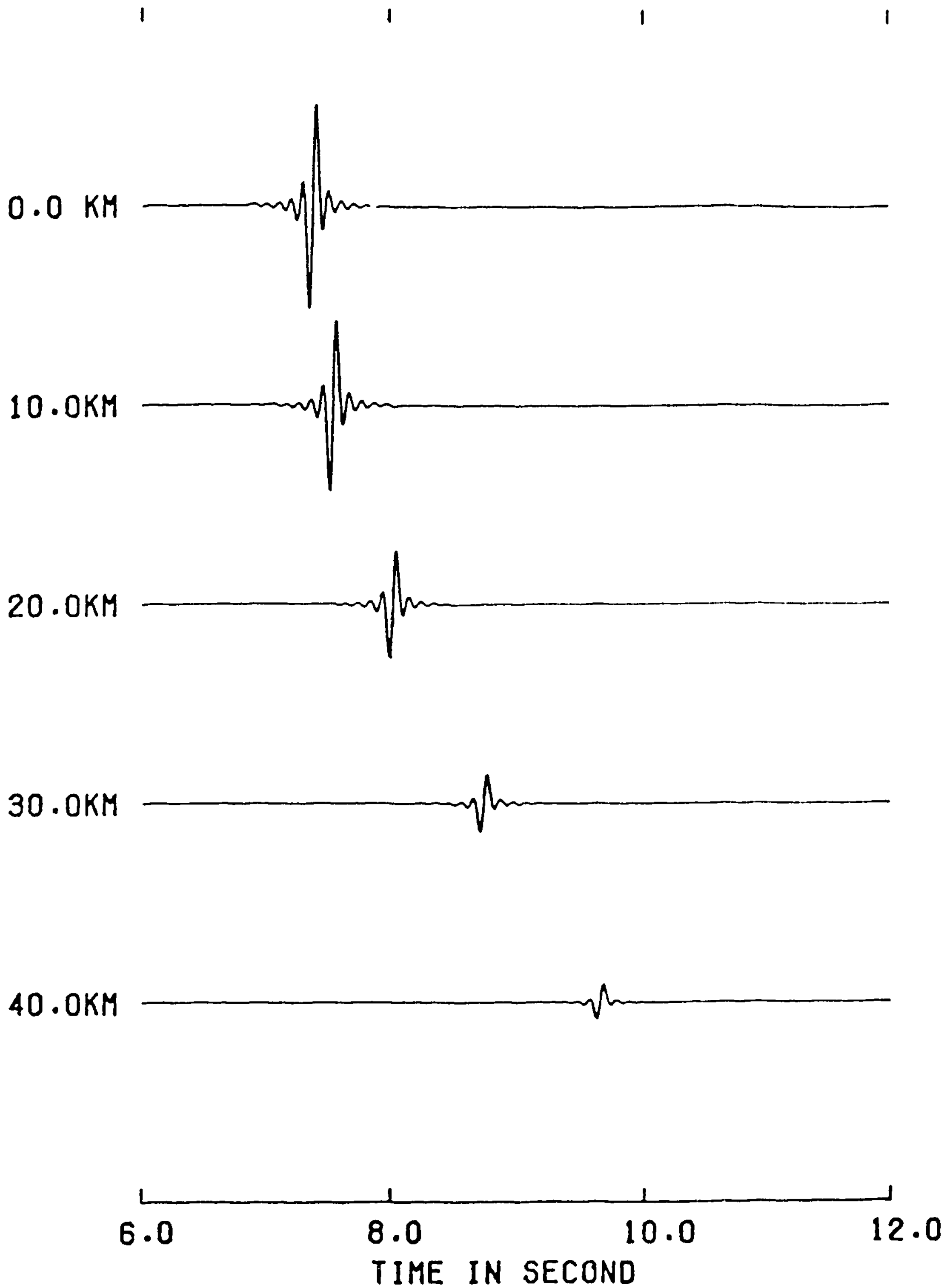
Second: horizontal component (elastic case)

Third: vertical component (anelastic case)

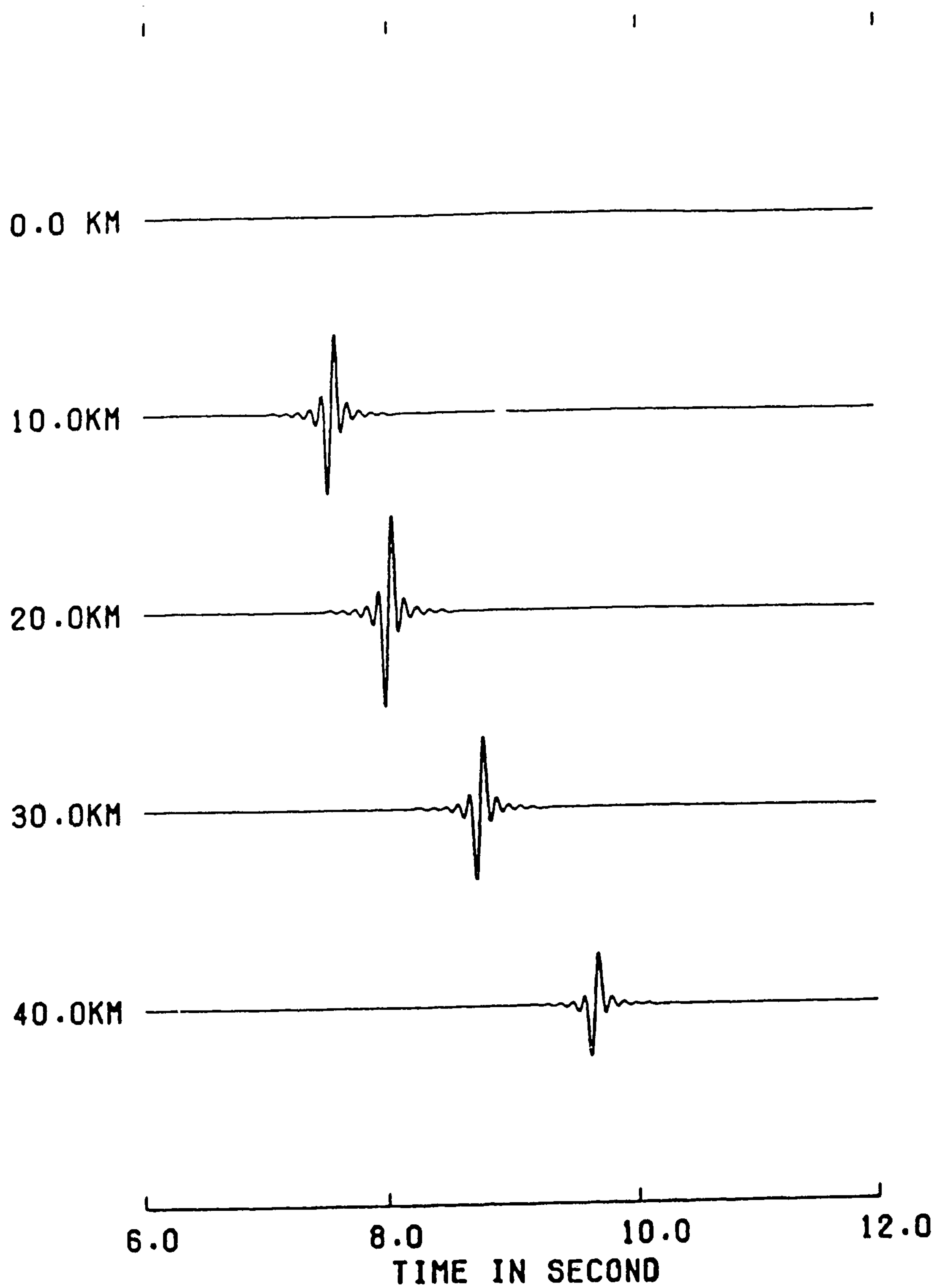
Fourth: horizontal component (anelastic case)


0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
6.0
8.0
10.0
12.0
TIME IN SECOND


0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
6.0
8.0
10.0
12.0
TIME IN SECOND



0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
6.0
8.0
10.0
12.0
TIME IN SECOND



0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
6.0
8.0
10.0
12.0
TIME IN SECOND

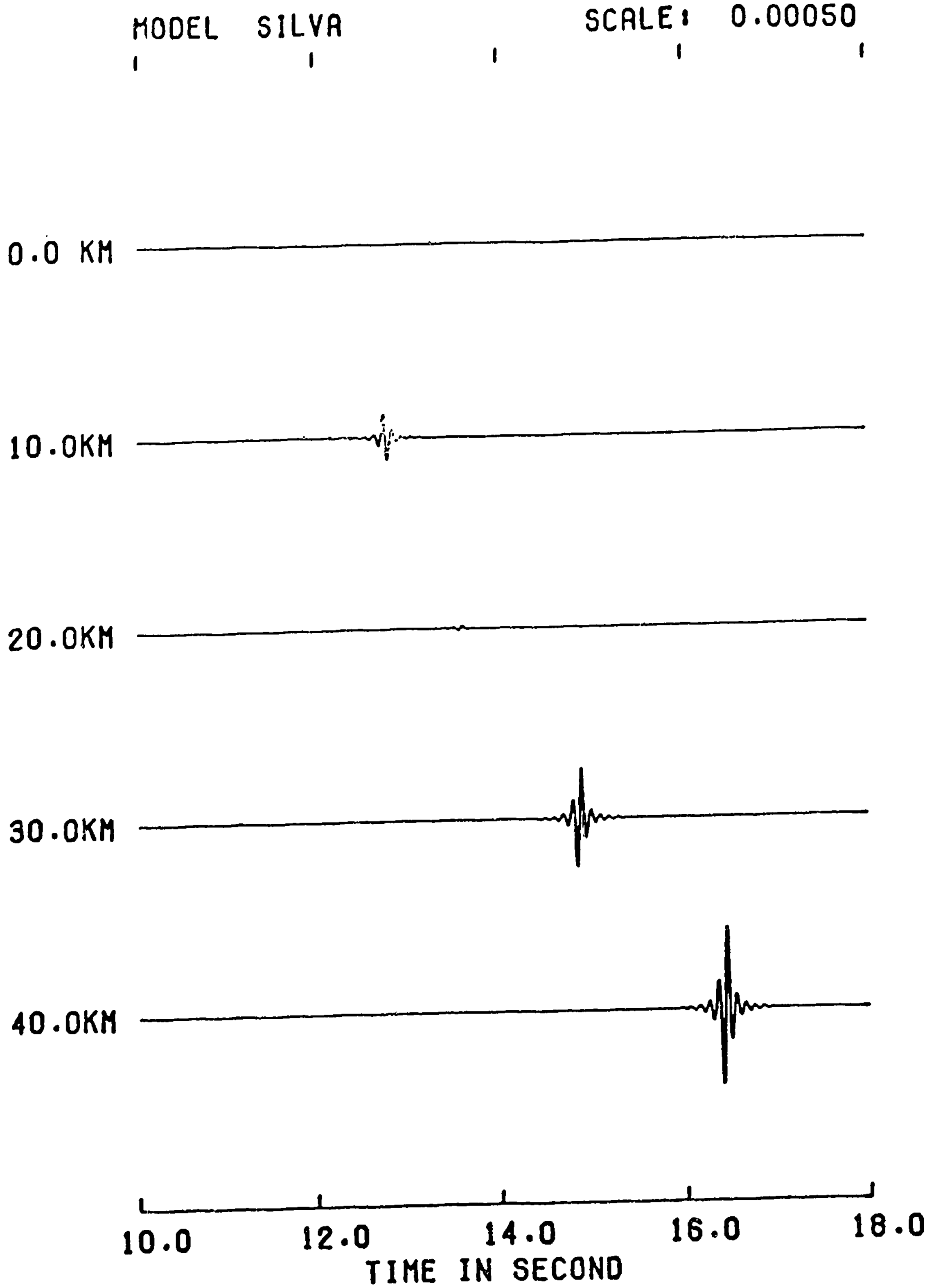
Figure 14: Seismograms obtained for the ray S1S2S3S3S2S1

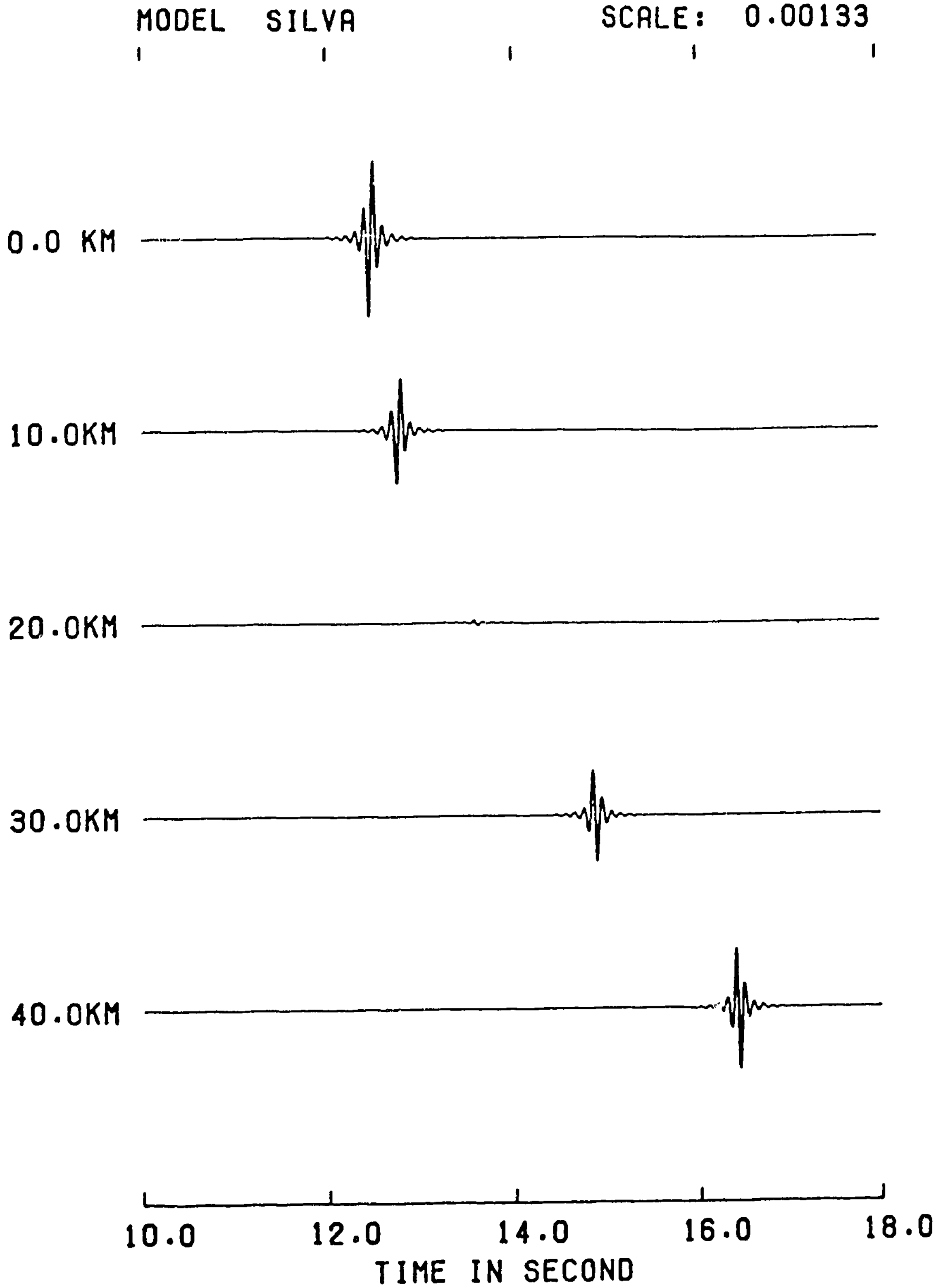
First: vertical component (elastic case)

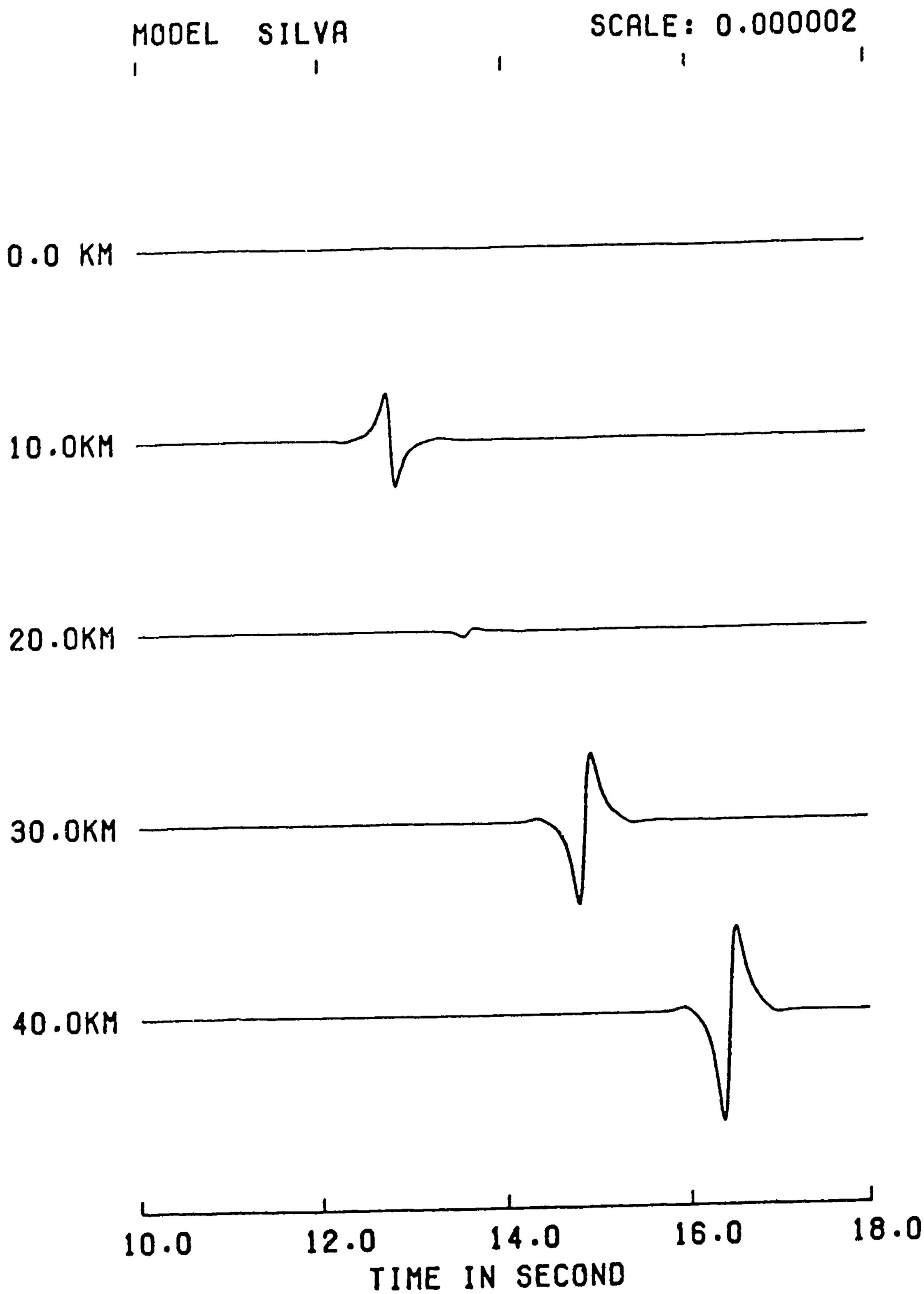
Second: horizontal component (elastic case)

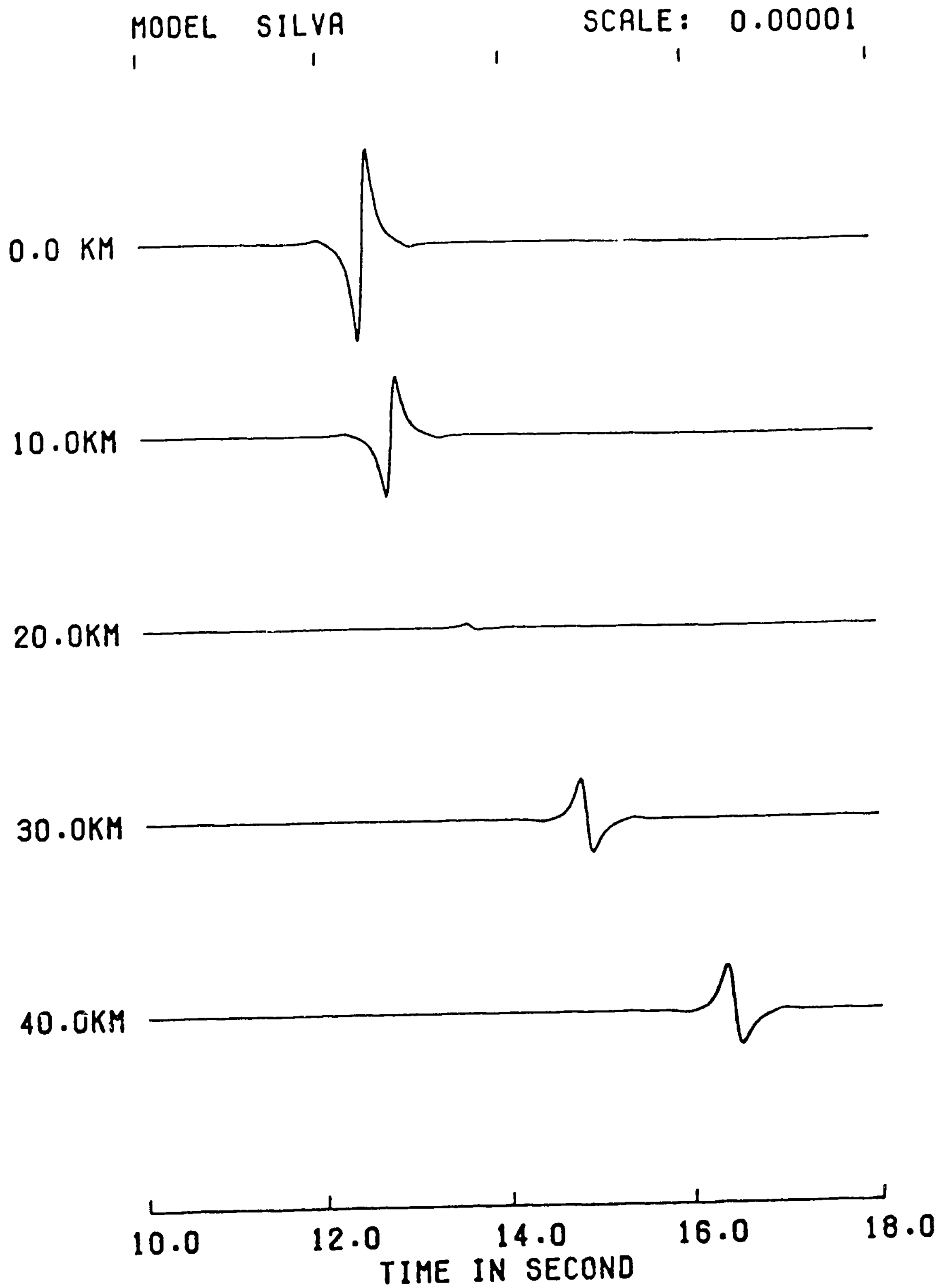
Third: vertical component (anelastic case)

Fourth: horizontal component (anelastic case)


MODEL SILVA
SCALE: 0.00050
0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
10.0
12.0
14.0
16.0
18.0
TIME IN SECOND


MODEL SILVA
SCALE: 0.00133
0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
10.0
12.0
14.0
16.0
18.0
TIME IN SECOND


MODEL SILVA
SCALE: 0.000002
0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
10.0
12.0
14.0
16.0
18.0
TIME IN SECOND


MODEL SILVA
SCALE: 0.00001
0.0 KM
10.0KM
20.0KM
30.0KM
40.0KM
10.0
12.0
14.0
16.0
18.0
TIME IN SECOND

transmission coefficients. The "log amplitude" curves clearly show no difference between all the cases described above until X gets close to . 5 km which corresponds to the critical incidence for P waves between the third layer and the half space. This is true for both vertical and horizontal components. The same comment can also be mentioned for the phase curves. The graphs also show that the differences in amplitude and phase after this critical incidence are mainly due to the anelastic coefficients. Curves #2 and 3 are practically identical to each other in each case and so are curves #1 and 4. This would tend to show that the difference between elastic and viscoelastic geometrical spreadings is negligible even after the critical angle. The ray S1S2S3S3S2S1 is now investigated. The attenuation due to absorption is much stronger in this case since the quality factors for S wave are much lower than for P wave. Hence only the highest peaks of the arrivals can be observed on the seismograms since the contribution of higher frequency components is mostly eliminated (figure 14). Figures 17 and 18 show the logarithm of the amplitude and the phase of the product $\frac{Y e^{i\phi}}{L}$ for the ray S1S2S3S3S2S1. The meaning of the numbers is the same as for the previous case. The "log amplitude" curves again show that the difference in amplitude between elastic and anelastic seismograms is due to the viscoelasticity of the media traversed. The logarithm of the amplitudes of the elastic and anelastic products $\frac{Y e^{i\phi}}{L}$ are practically identical until X gets close to 90 km except for the horizontal component where a difference can be observed at X = 19.7 km. This two values of offset respectively correspond to the critical incidence for the coefficient S1S2 between the third layer and the half space and to a minimum amplitude for the coefficient S1S1 for this same interface. The phase curves show a slight difference between elastic and viscoelastic phases from X = 19.7 km. This difference becoming greater when X approaches 90 km. This is true for both vertical and horizontal components. Comparing curves #2 and 3 on one hand and curves #1 and 4 on the other hand again show that the differences observed in amplitude and phase are mainly due to the viscoelastic coefficients. The logarithm of the amplitude and the phase of the product the product $\frac{Y e^{i\phi}}{L}$ were also computed for the converted ray P1P2P3S3S2S1 (figures 19 and 20). Once again the differences in amplitude and phase start from the vicinity of a critical incidence (X = 54 km corresponds to the critical angle of the coefficient P1P2 between the third layer and the half space) and are due to the viscoelastic coefficients.

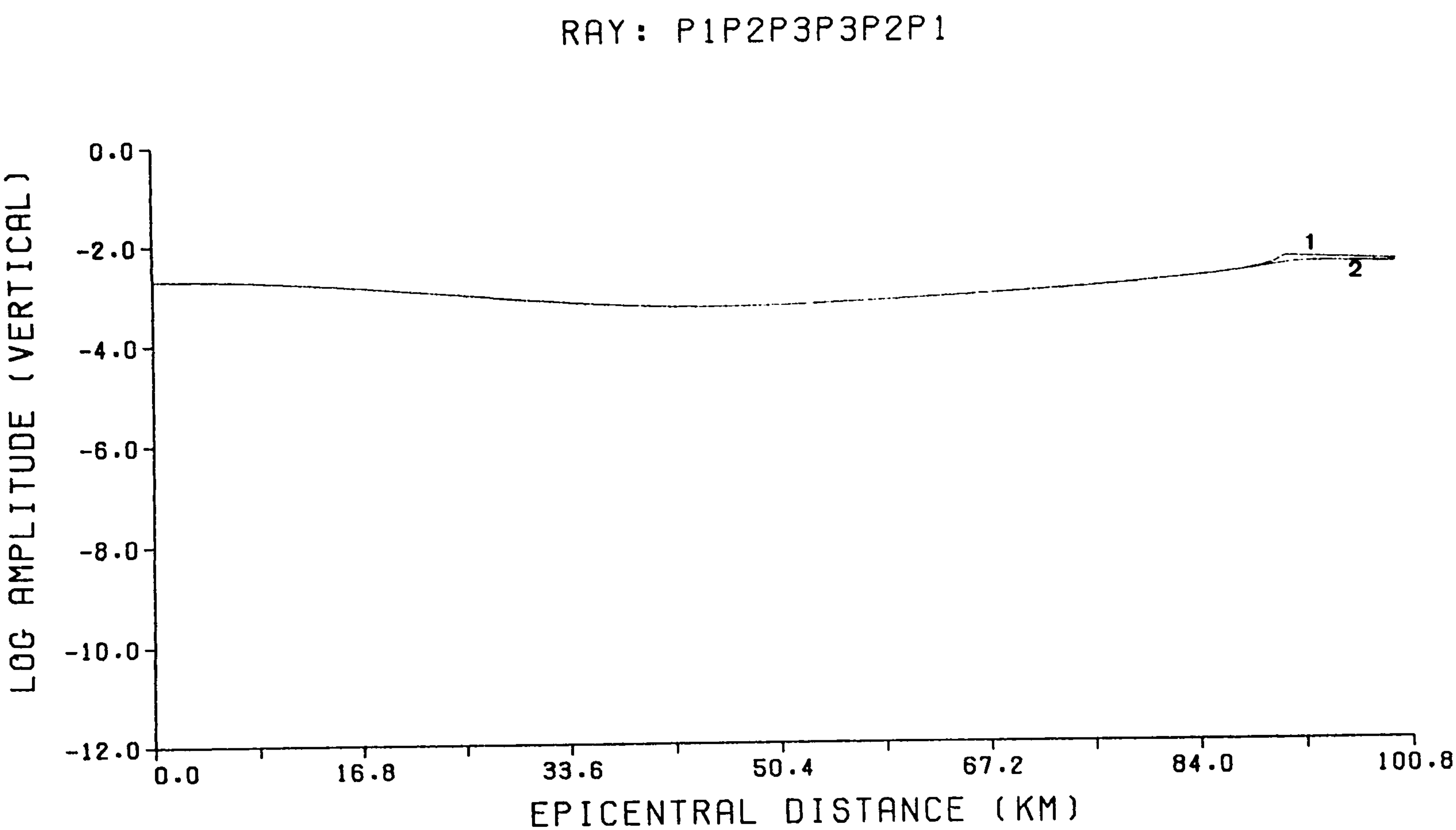
Figure 15: Logarithm of the amplitude of the product $\frac{Y e^{i\phi}}{L}$ for the ray P1P2P3P3P2P1 at all possible epicentral distances.

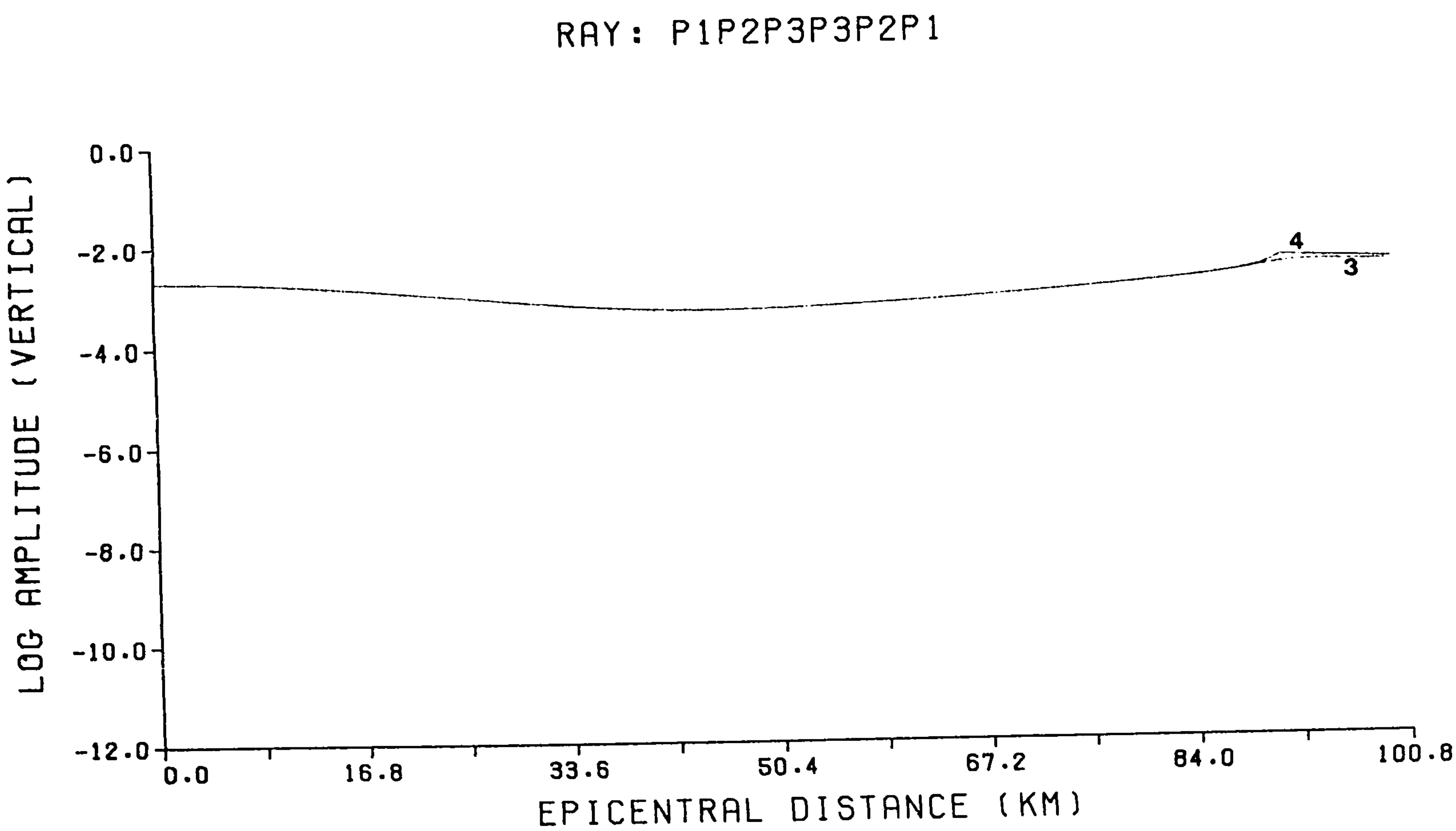
Curve #1: elastic geometrical spreading
elastic coefficients

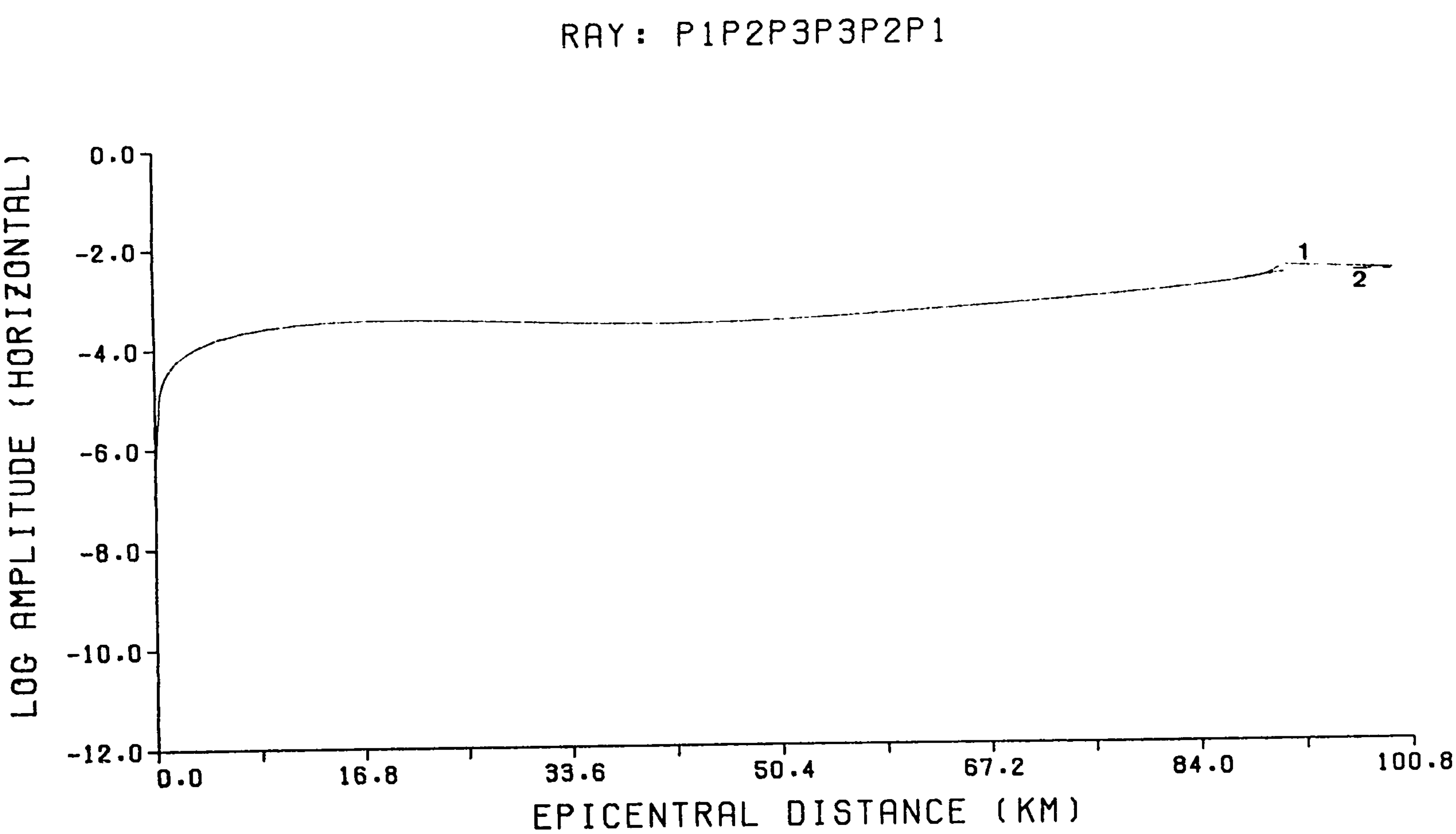
Curve #2: anelastic geometrical spreading
anelastic coefficients

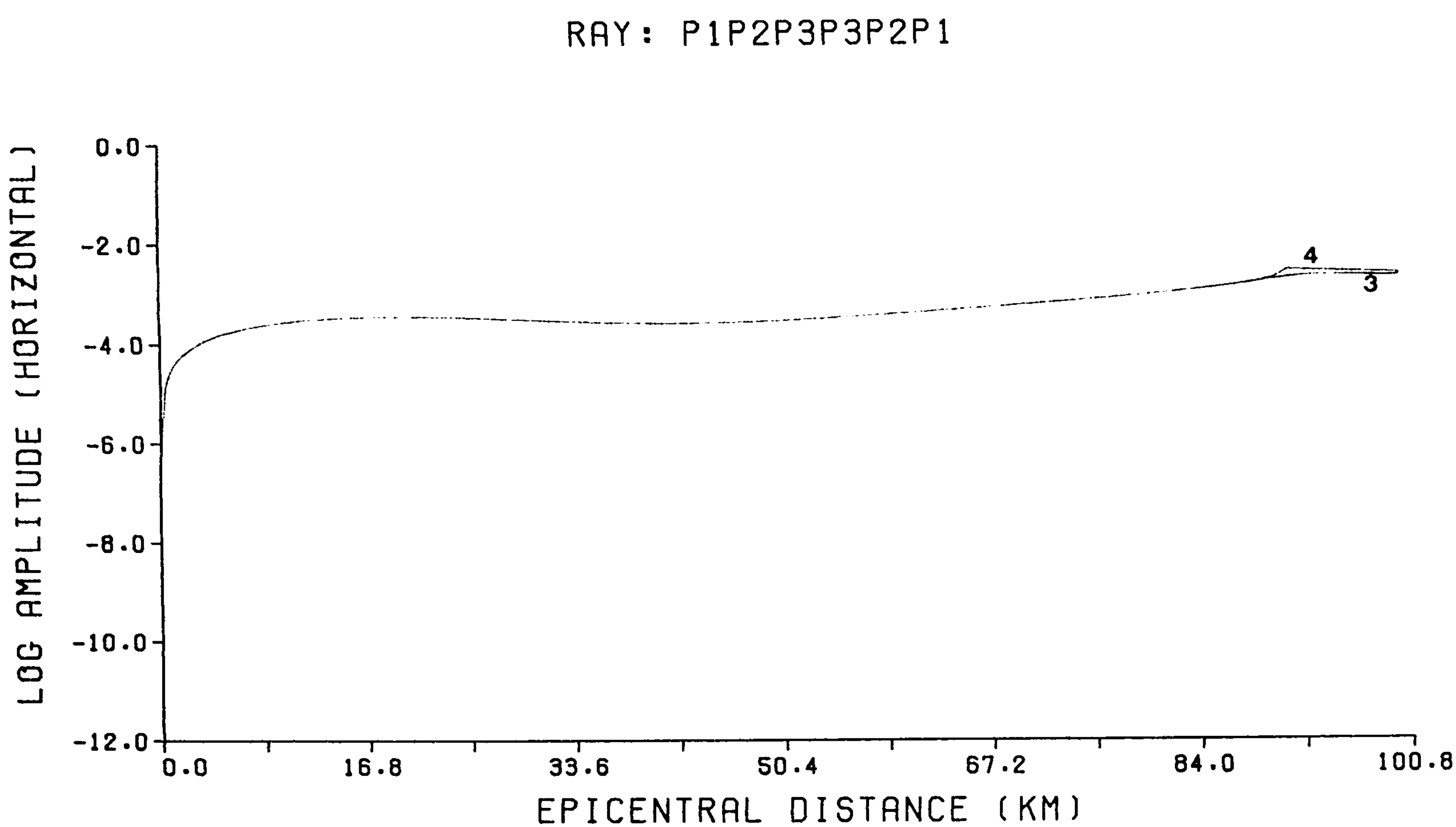
Curve #3: elastic geometrical spreading
anelastic coefficients

Curve #4: anelastic geometrical spreading
elastic coefficients


RAY: P1P2P3P3P2P1
LOG AMPLITUDE (VERTICAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
16.8
33.6
50.4
67.2
84.0
100.8
1
2
EPICENTRAL DISTANCE (KM)


RAY: P1P2P3P3P2P1
LOG AMPLITUDE (VERTICAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
16.8
33.6
50.4
67.2
84.0
100.8
4
3
EPICENTRAL DISTANCE (KM)


RAY: P1P2P3P3P2P1
LOG AMPLITUDE (HORIZONTAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
16.8
33.6
50.4
67.2
84.0
100.8
1
2
EPICENTRAL DISTANCE (KM)



RAY: P1P2P3P3P2P1
LOG AMPLITUDE (HORIZONTAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
16.8
33.6
50.4
67.2
84.0
100.8
EPICENTRAL DISTANCE (KM)
4
3

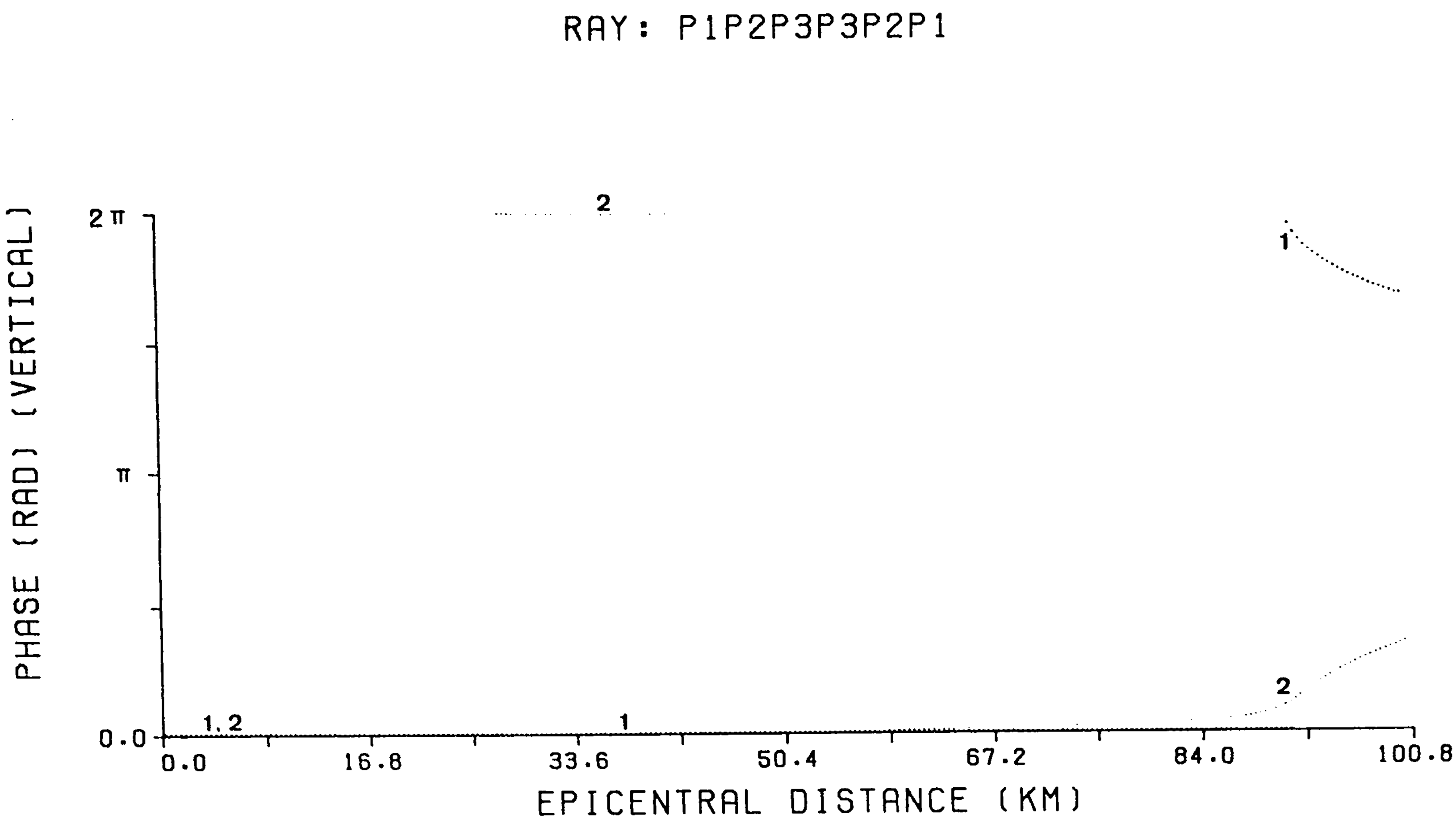
Figure 16: Phase of the product $\frac{Y e^{i\phi}}{L}$ for the ray P1P2P3P3P2P1 [illegible] possible epicentral [illegible]

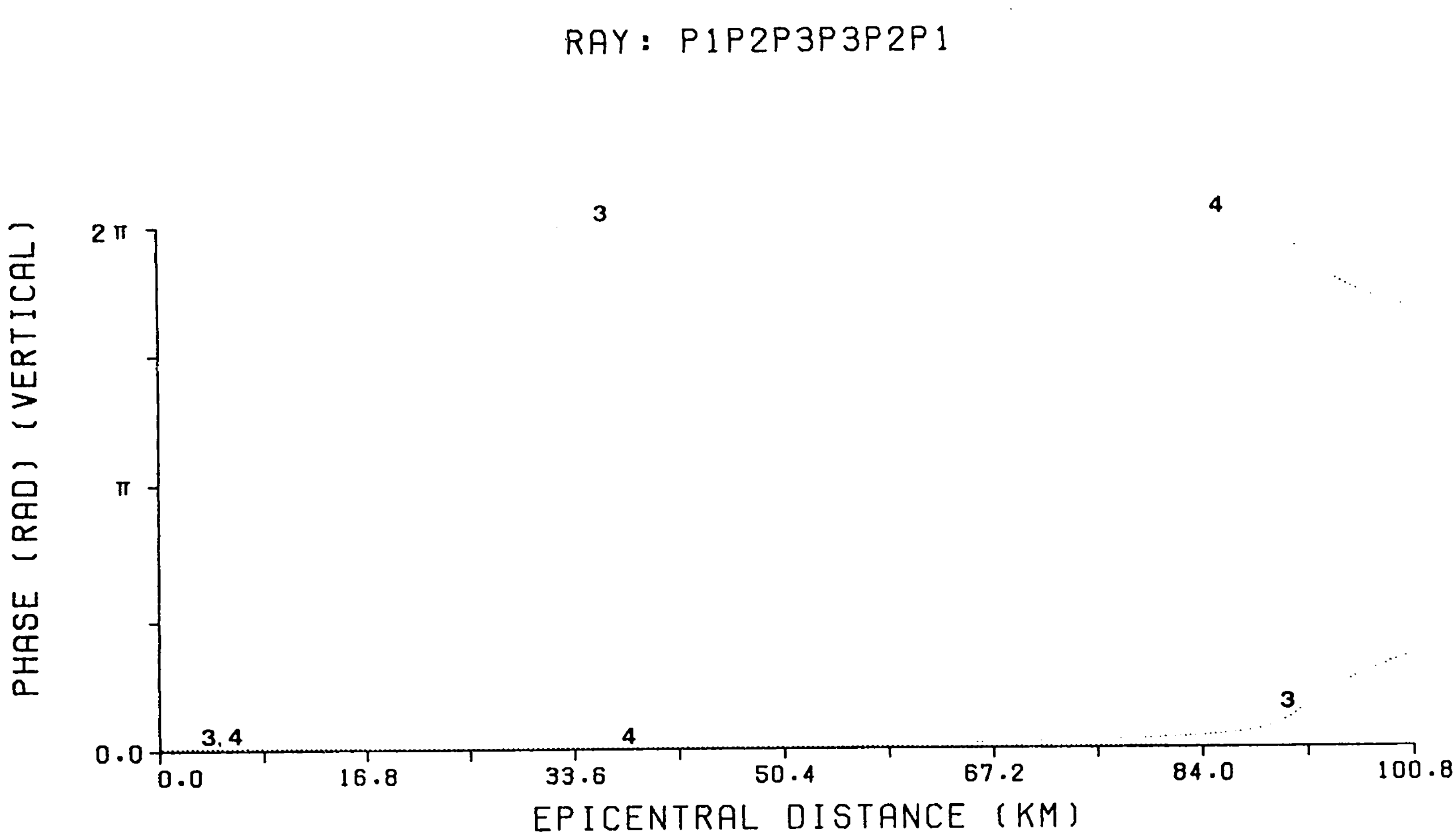
Curve #1: e[illegible] geometrical spreading
el[illegible]c coefficients

Curve #2: anelastic geometrical spreading
anelastic coefficients

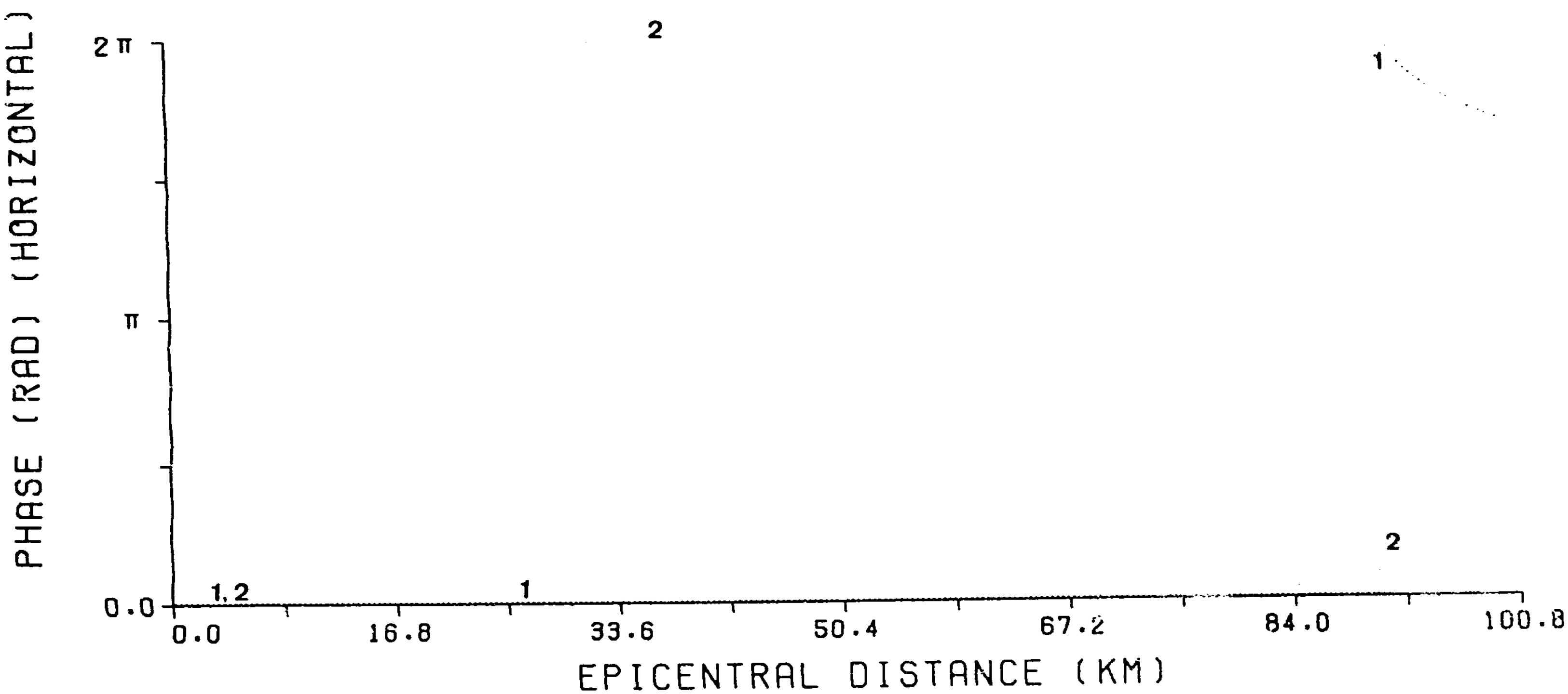
Curve #3: elastic geometrical spreading
anelastic coefficients

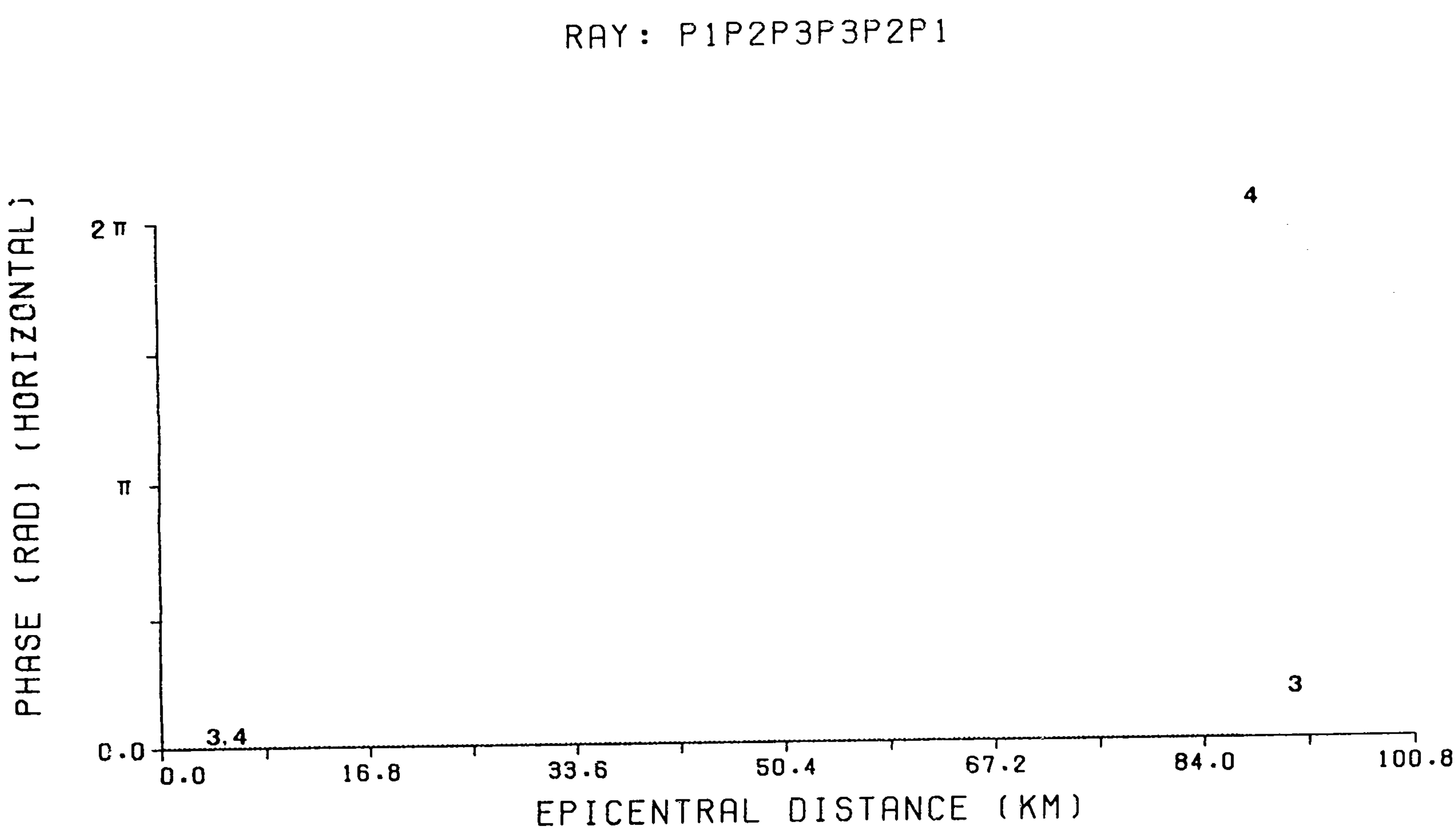
Curve #4: anelastic geometrical spreading
elastic coefficients


RAY: P1P2P3P3P2P1
PHASE (RAD) (VERTICAL)
2π
π
0.0
0.0
16.8
33.6
50.4
67.2
84.0
100.8
EPICENTRAL DISTANCE (KM)
1,2
1
2
1
2


RAY: P1P2P3P3P2P1
PHASE (RAD) (VERTICAL)
2π
π
0.0
3
4
3
3,4
4
0.0
16.8
33.6
50.4
67.2
84.0
100.8
EPICENTRAL DISTANCE (KM)

RAY: P1P2P3P3P2P1





RAY: P1P2P3P3P2P1
PHASE (RAD) (HORIZONTAL)
2π
π
0.0
3.4
4
3
0.0
16.8
33.6
50.4
67.2
84.0
100.8
EPICENTRAL DISTANCE (KM)

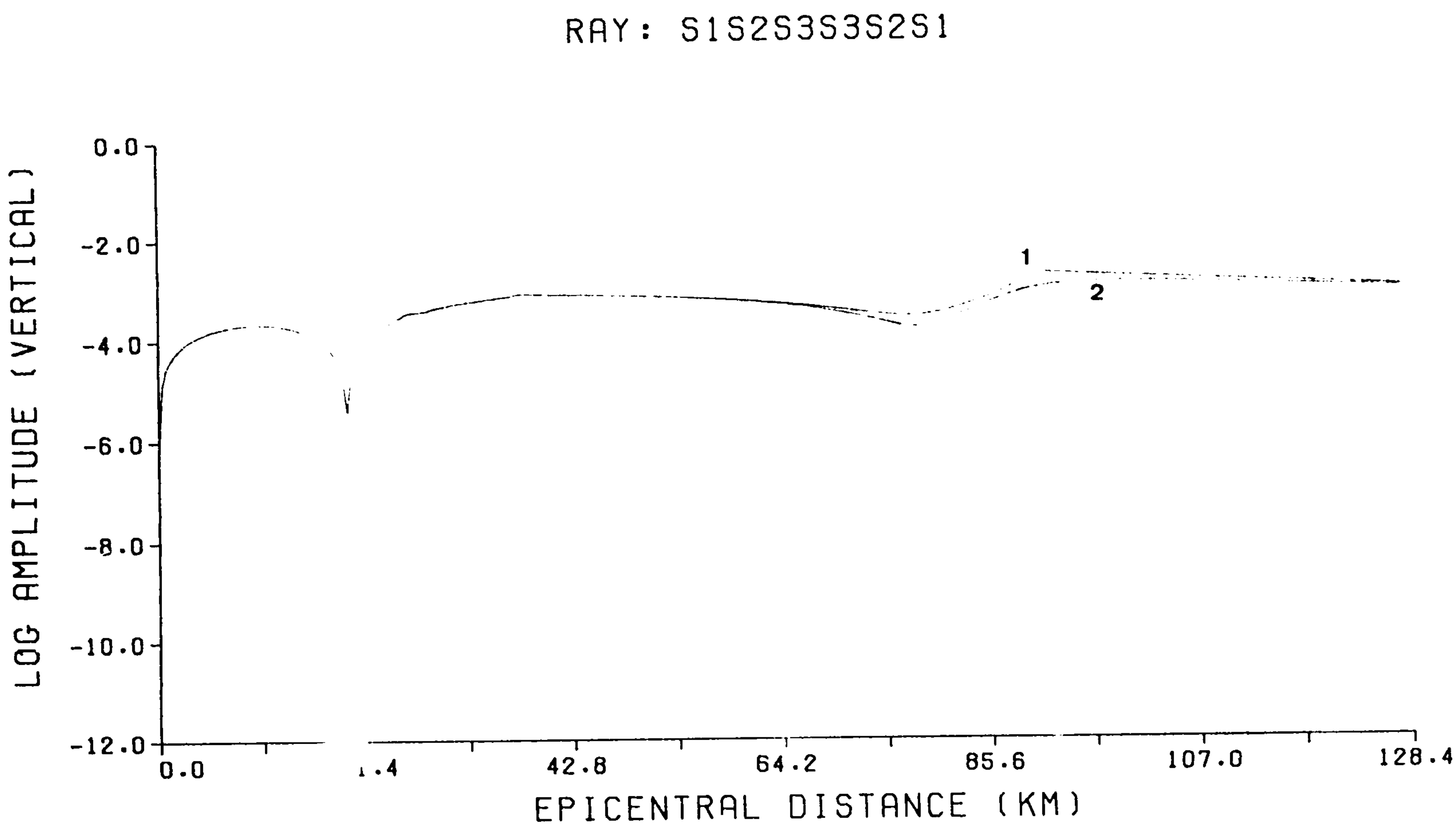
Figure 17: Logarithm of the amplitude of the product $\frac{X e^{i\phi}}{L}$ for the ray S1S2S3S3S2S1 at all possible epicentral distances.

Curve #1: elastic geometrical spreading
elastic coefficients

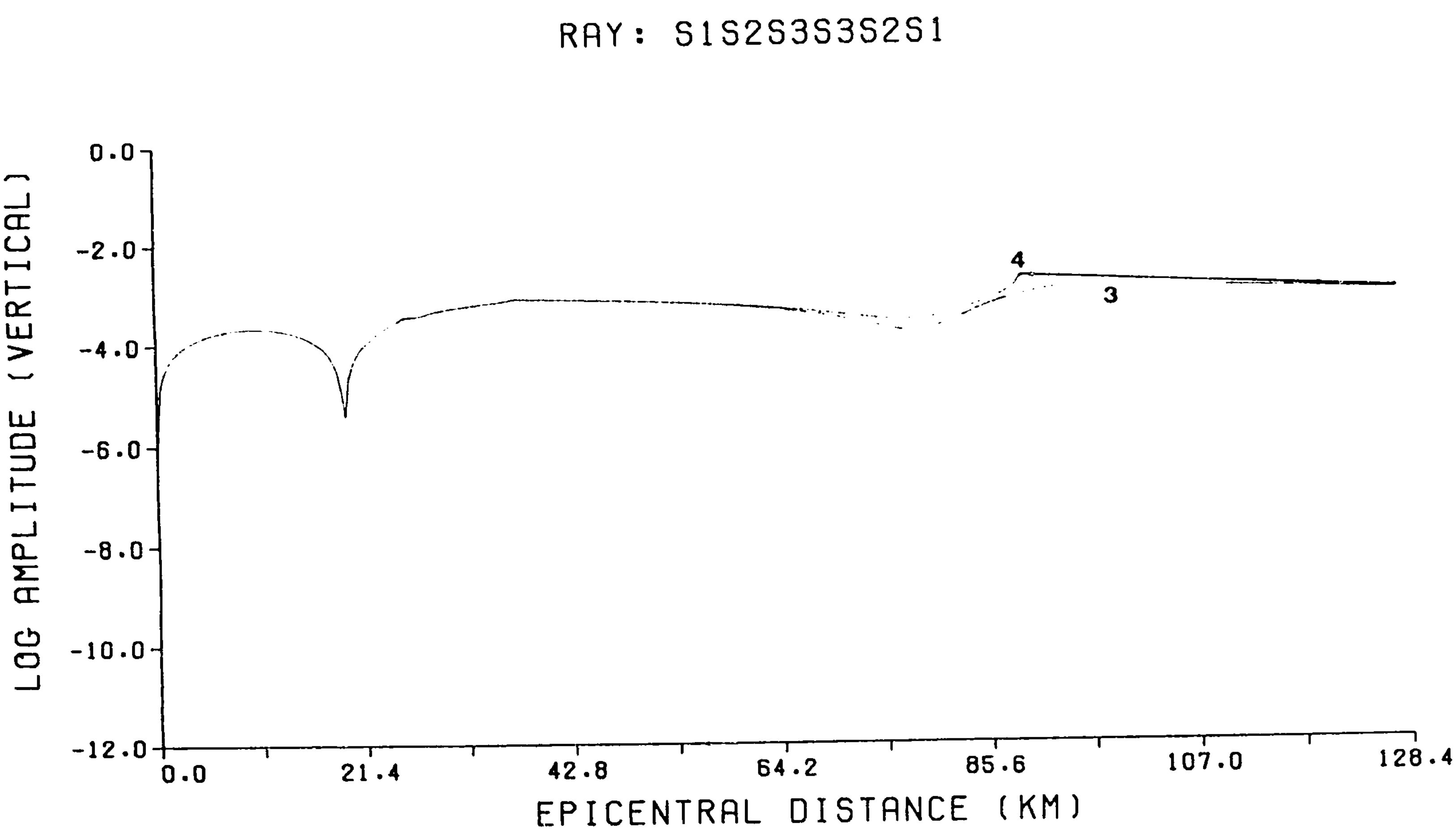
Curve #2: anelastic geometrical spreading
anelastic coefficients

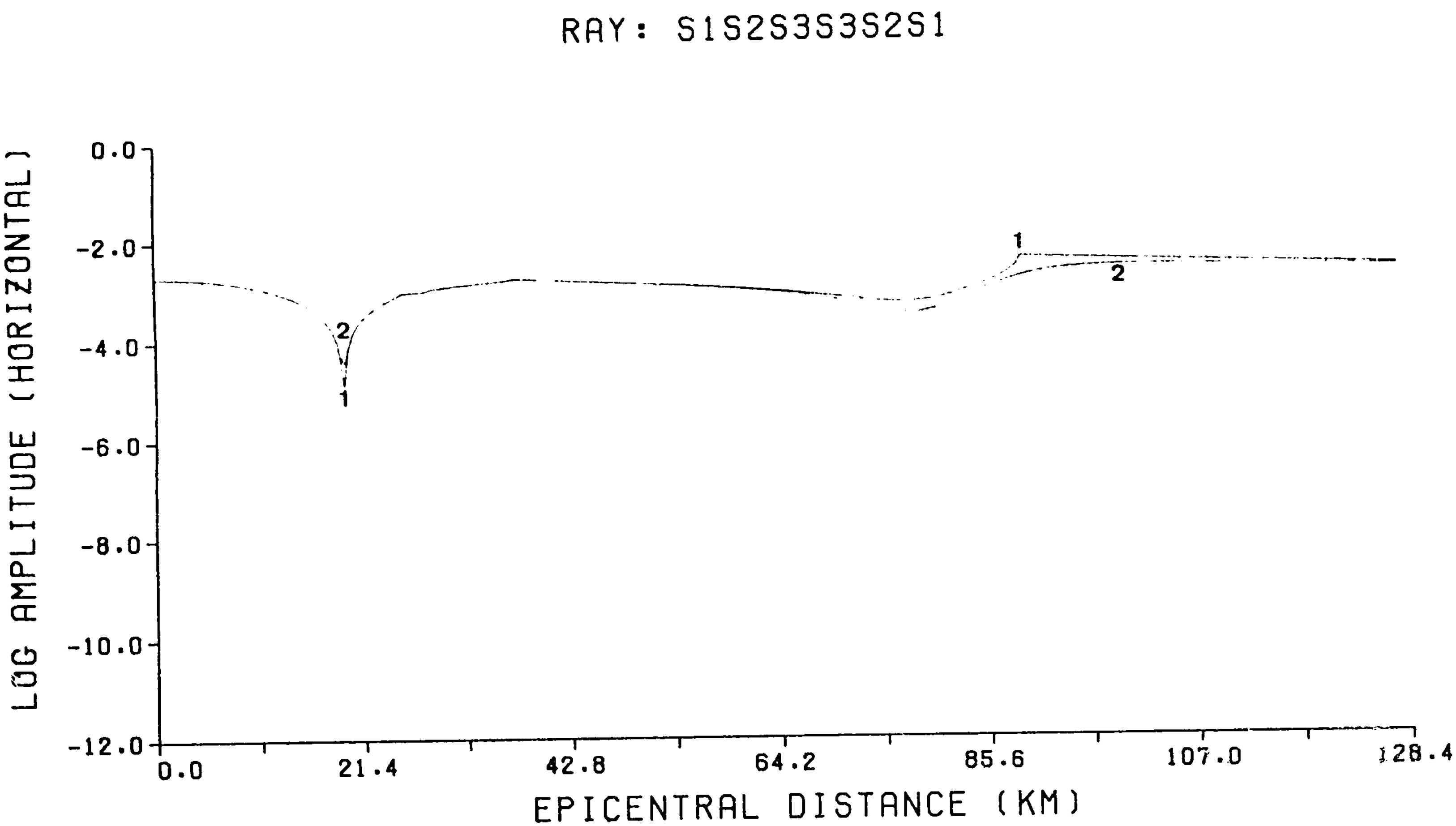
Curve #3: elastic geometrical spreading
anelastic coefficients

Curve #4: anelastic geometrical spreading
elastic coefficients



RAY: S1S2S3S3S2S1
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
LOG AMPLITUDE (VERTICAL)
1
2
0.0
.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)


RAY: S1S2S3S3S2S1
LOG AMPLITUDE (VERTICAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
21.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)
4
3


RAY: S1S2S3S3S2S1
LOG AMPLITUDE (HORIZONTAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
21.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)
1
2
1
2

RAY: S1S2S3S3S2S1

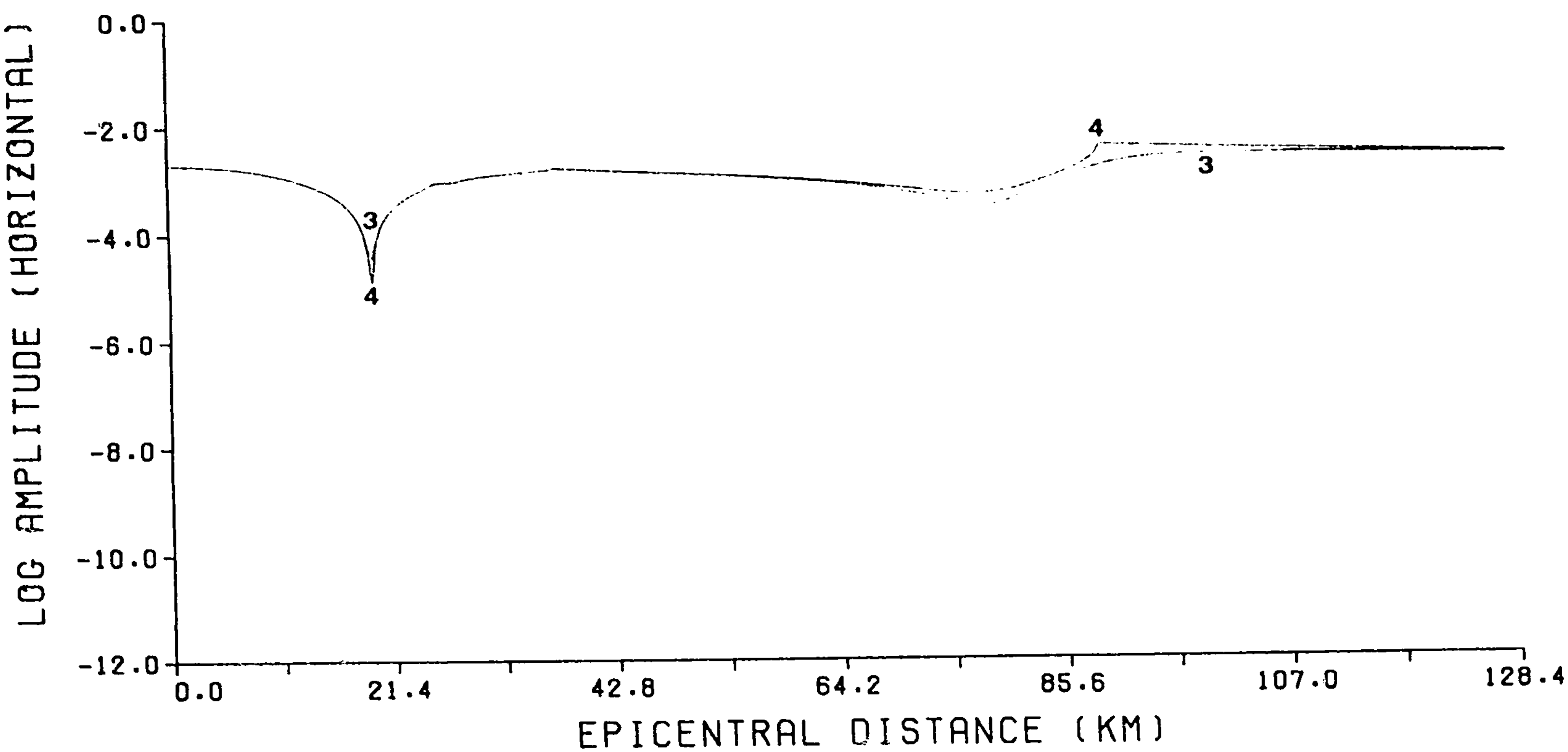


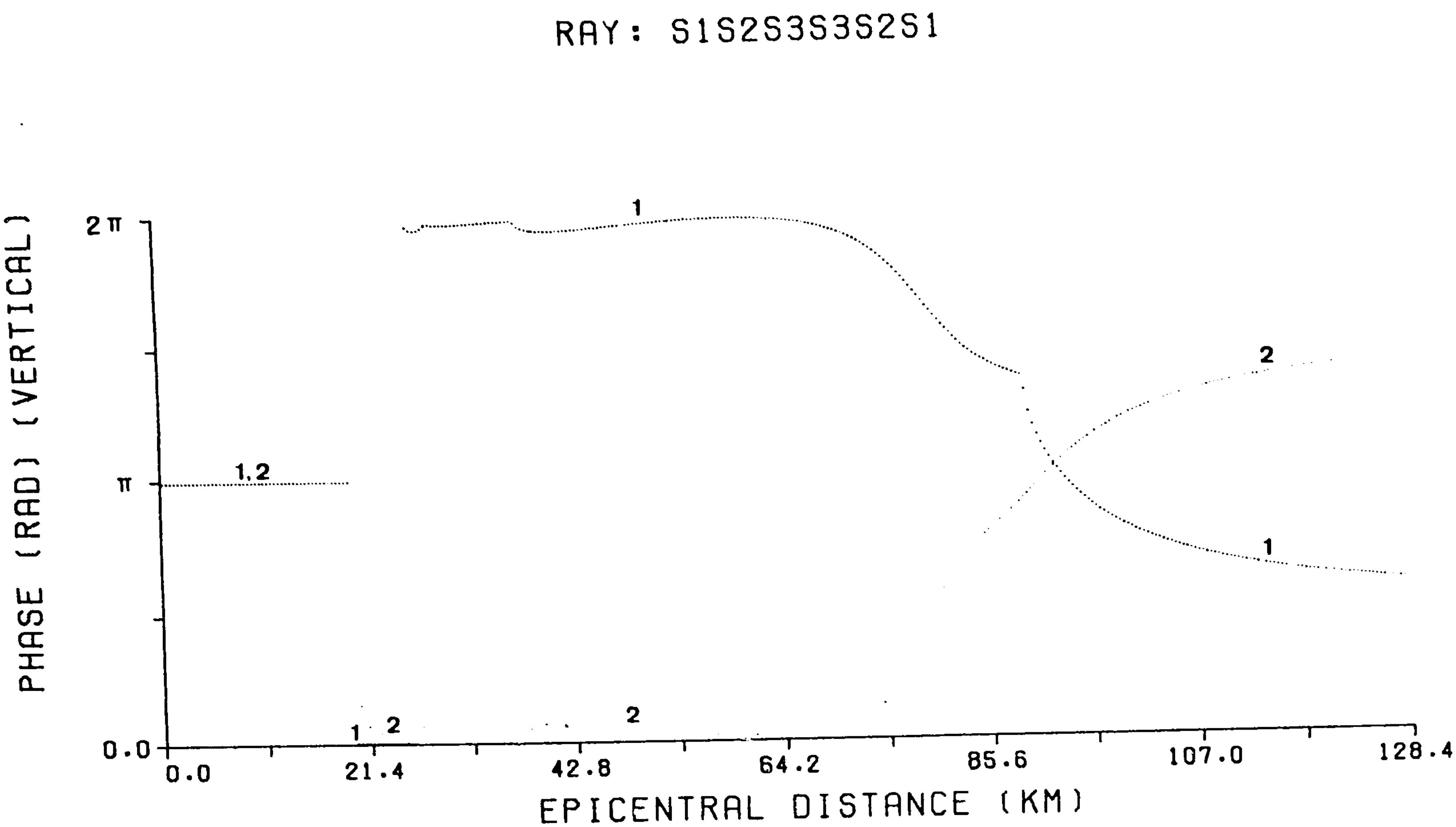
Figure 18: Phase of the product $\frac{Y e^{i\phi}}{L}$ for the ray S1S2S3S3S2S1 at all possible epicentral distances.

Curve #1: elastic geometrical spreading
elastic coefficients

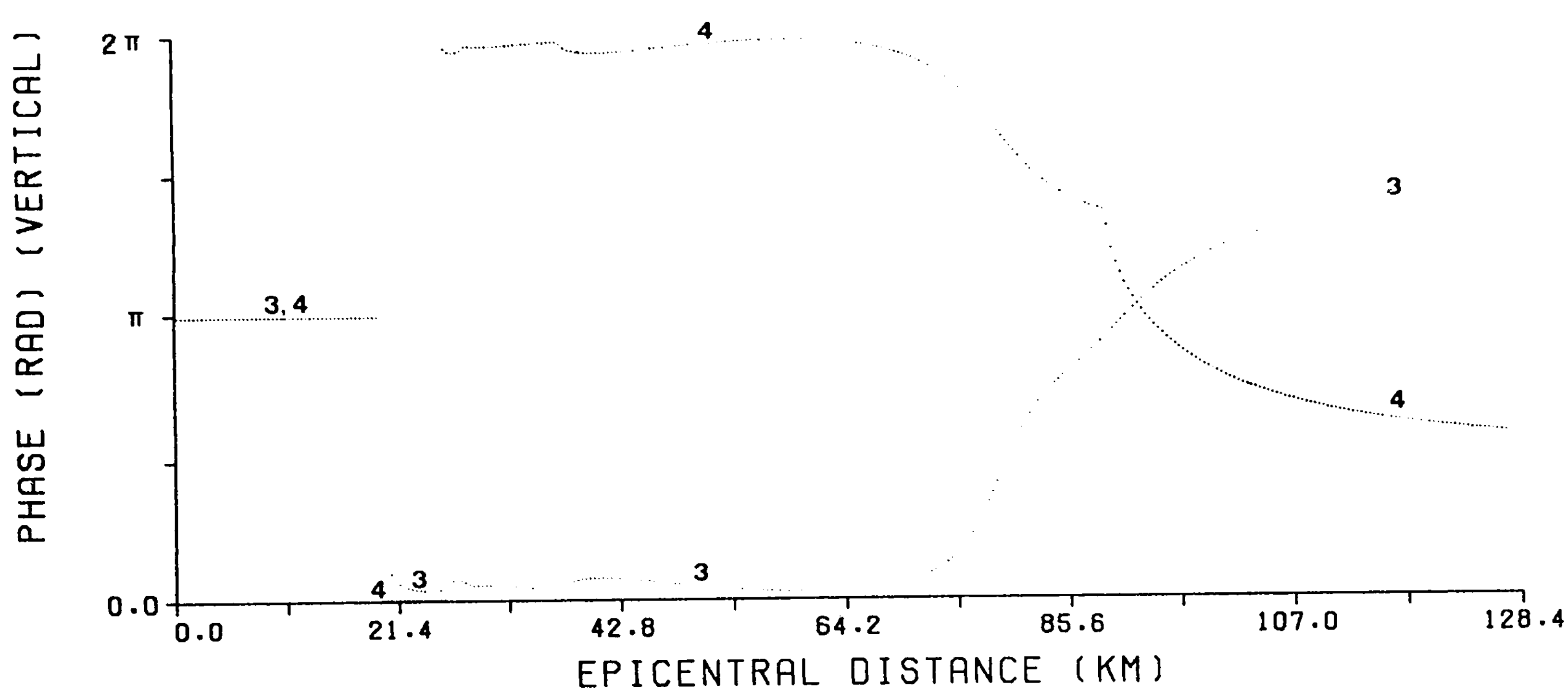
Curve #2: anelastic geometrical spreading
anelastic coefficients

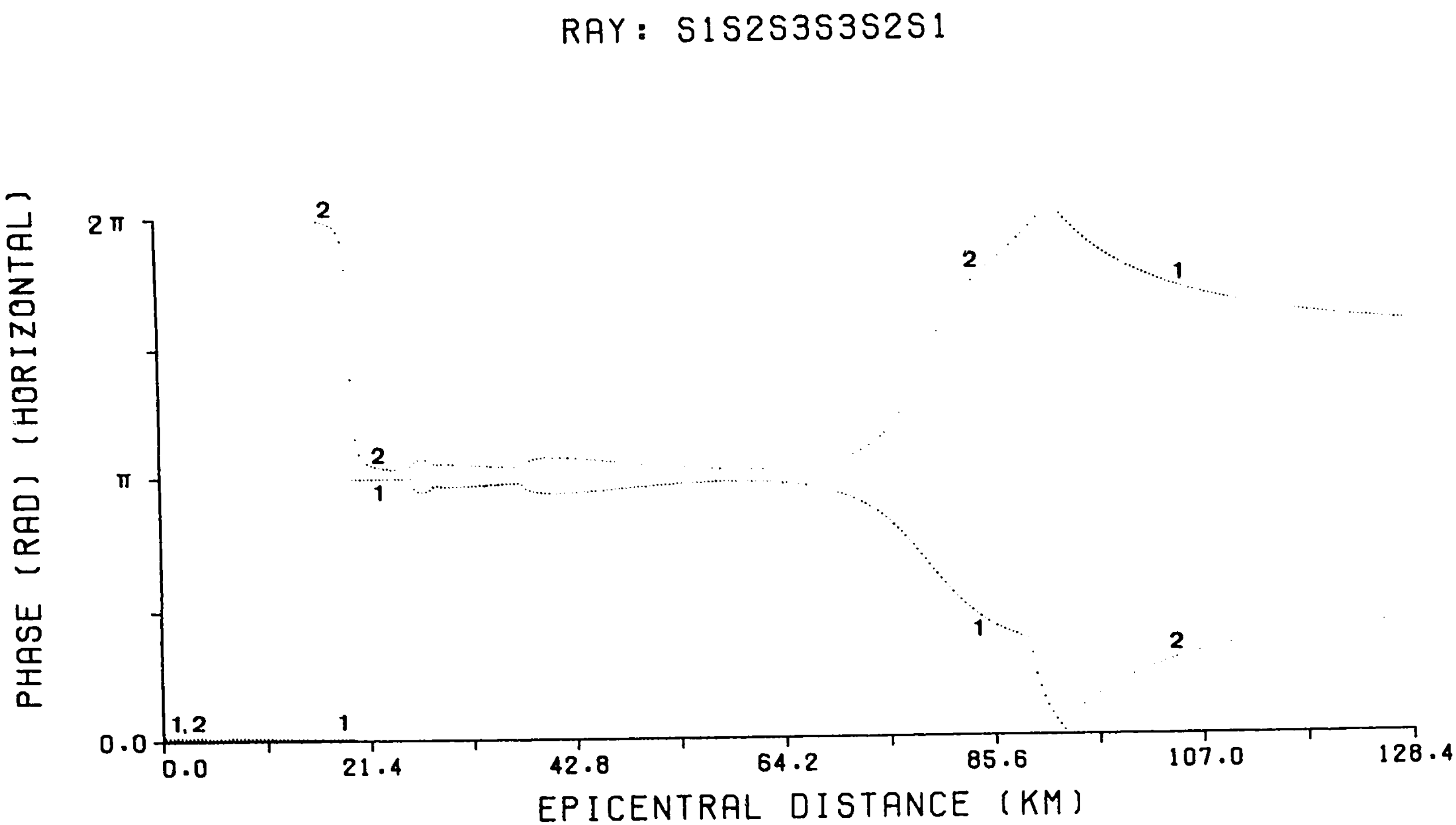
Curve #3: elastic geometrical spreading
anelastic coefficients

Curve #4: anelastic geometrical spreading
elastic coefficients


RAY: S1S2S3S3S2S1
PHASE (RAD) (VERTICAL)
2π
π
0.0
1,2
1
2
1
2
2
1
0.0
21.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)

RAY: S1S2S3S3S2S1




RAY: S1S2S3S3S2S1
PHASE (RAD) (HORIZONTAL)
2π
π
0.0
1,2
1
2
1
2
1
2
1
2
0.0
21.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)

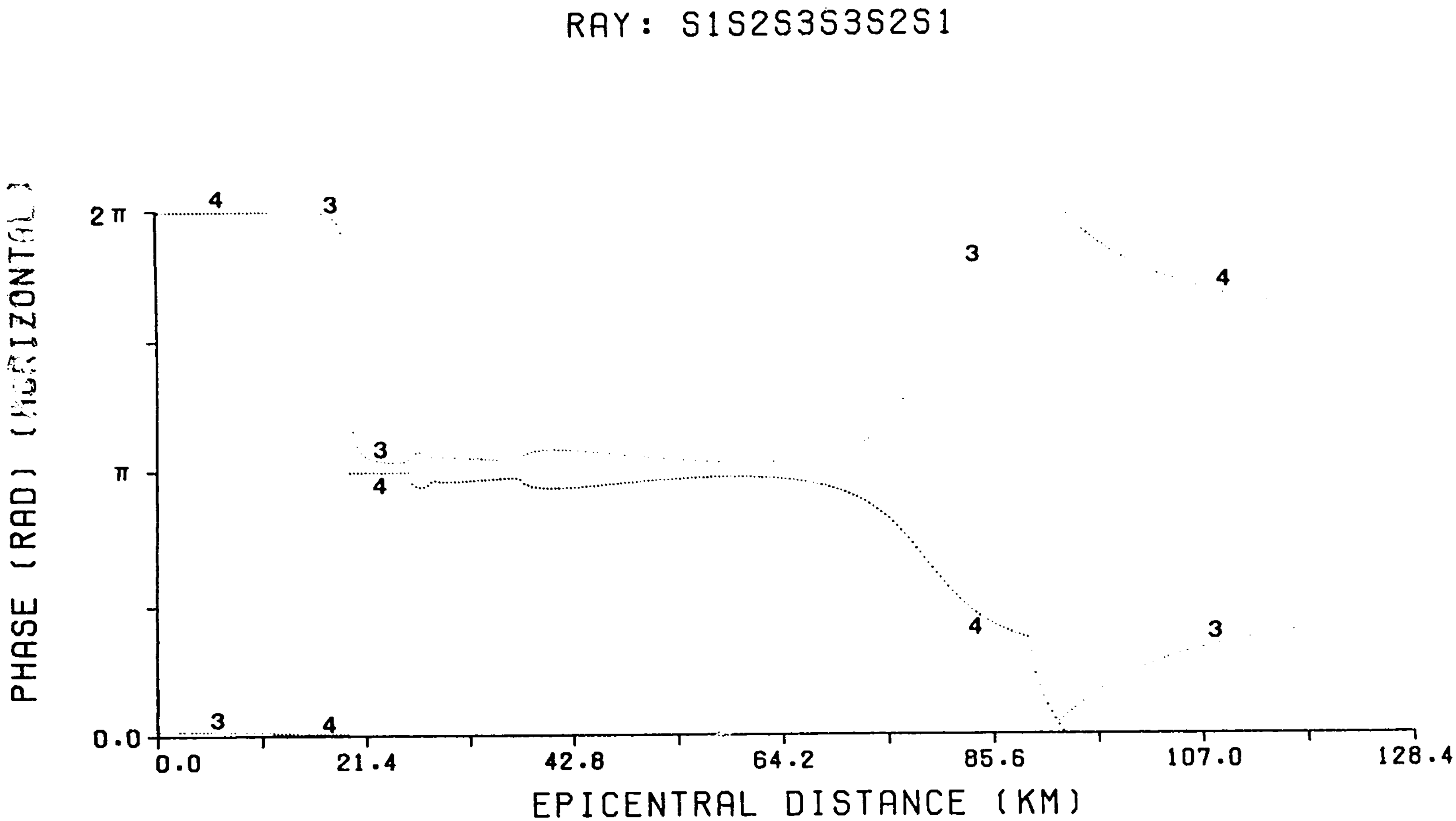

RAY: S1S2S3S3S2S1
PHASE (RAD)
2π
π
0.0
4
3
0.0
21.4
42.8
64.2
85.6
107.0
128.4
EPICENTRAL DISTANCE (KM)

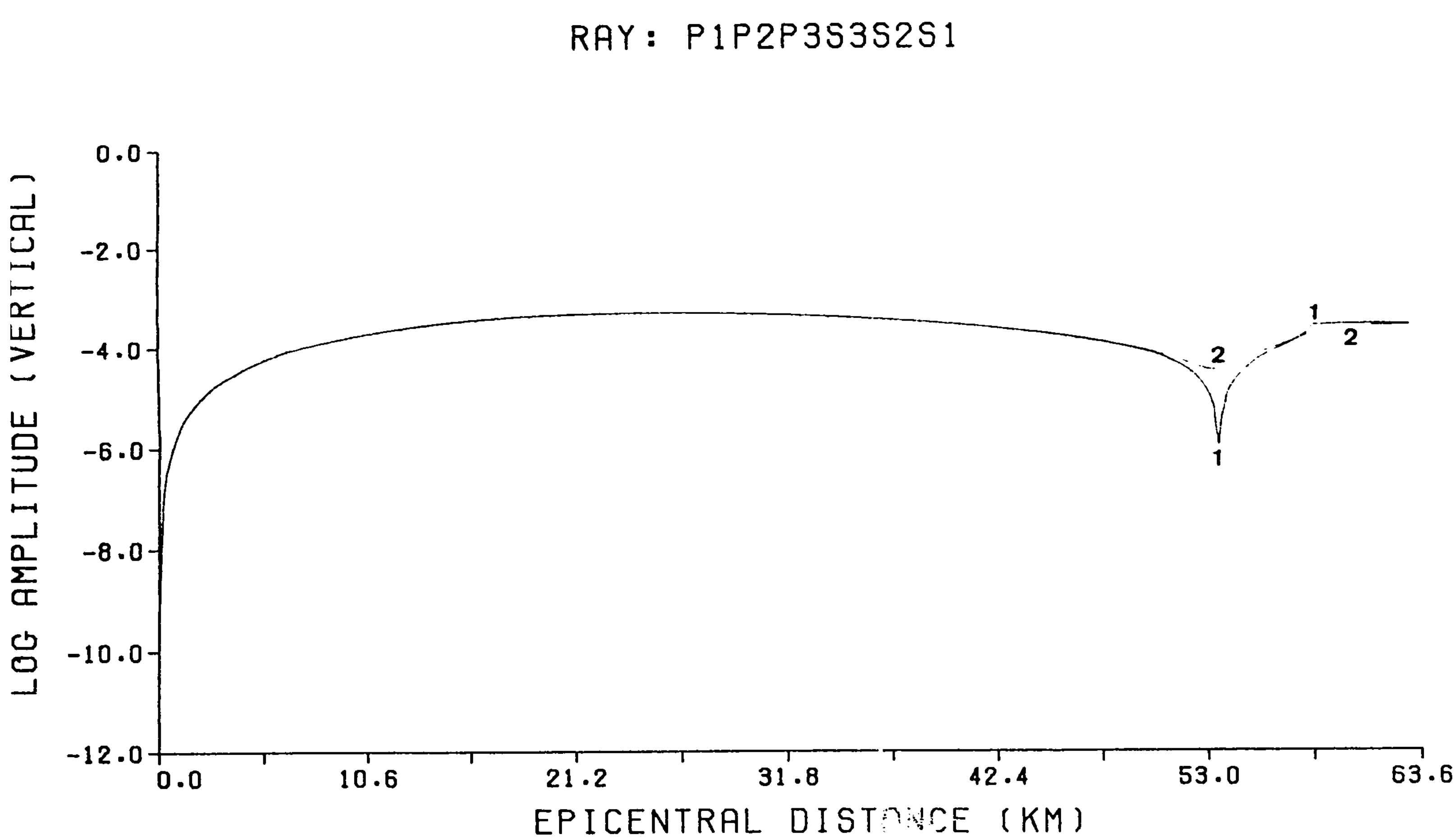
Figure 19: Logarithm of the amplitude of the product $\frac{Y e^{i\phi}}{L}$ for the ray P1P2P3S3S2S1 at all possible epicentral distances.

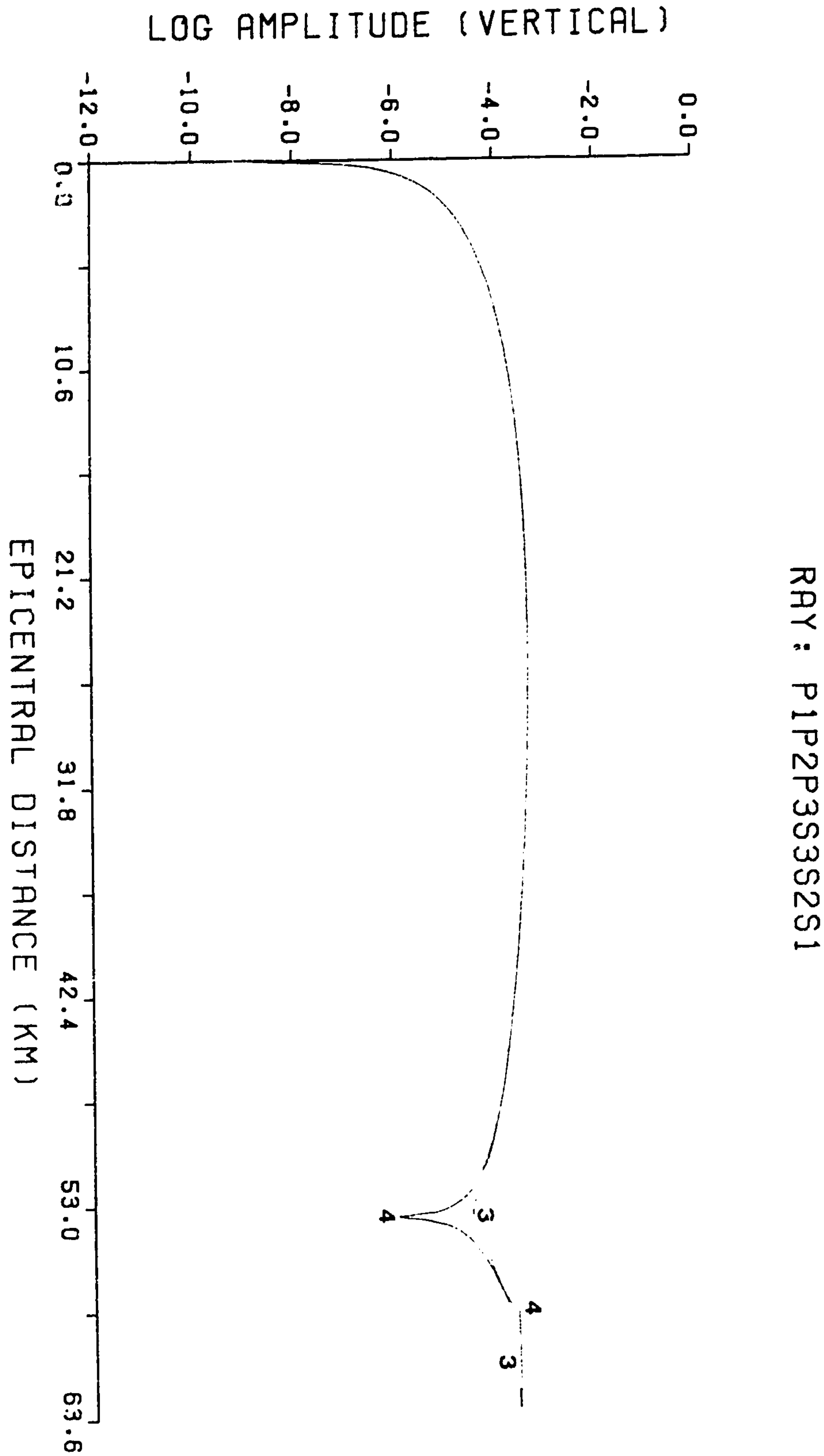
Curve #1: elastic geometrical spreading
elastic coefficients

Curve #2: anelastic geometrical spreading
anelastic coefficients

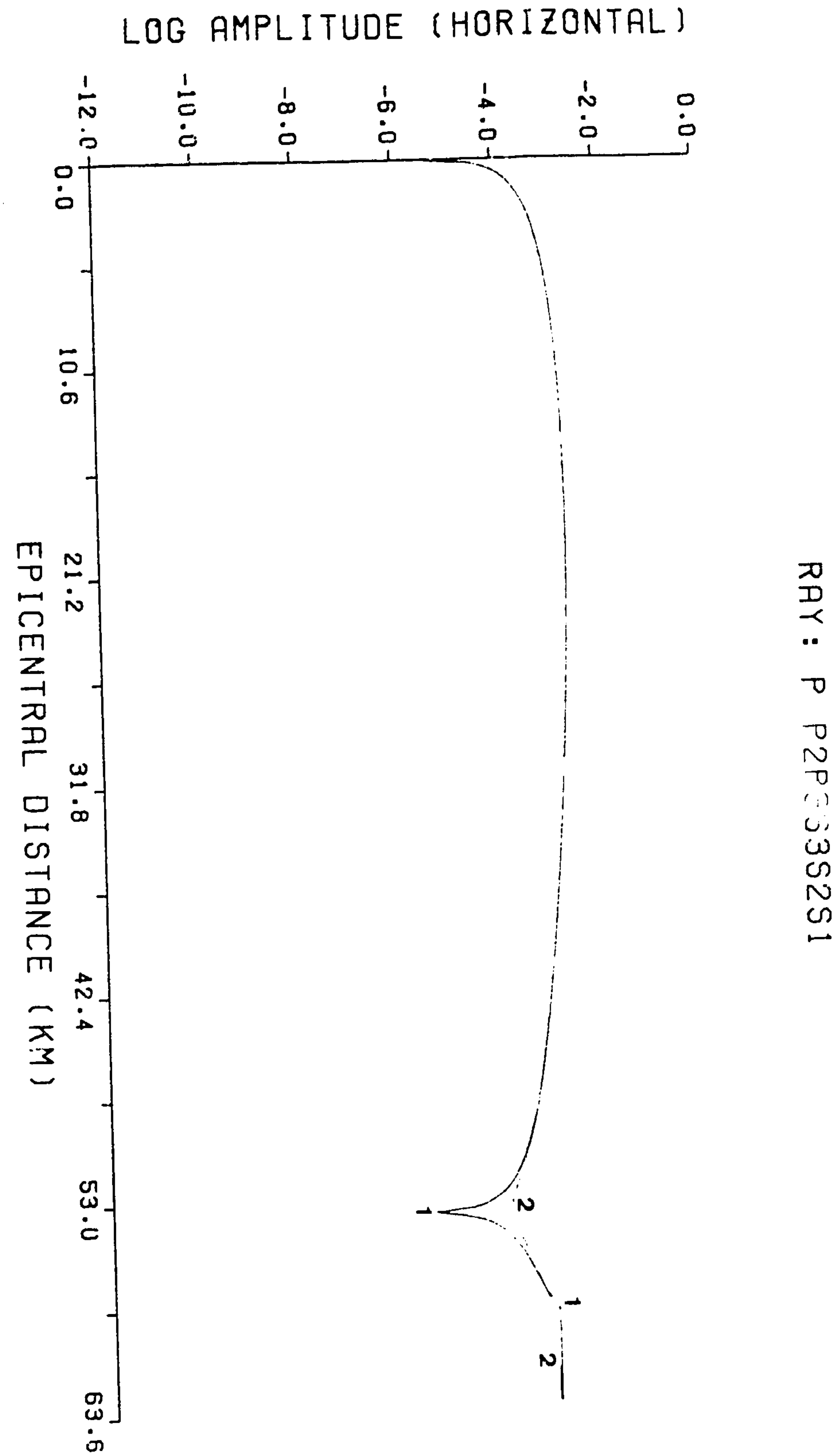
Curve #3: elastic geometrical spreading
anelastic coefficients

Curve #4: anelastic geometrical spreading
elastic coefficients


RAY: P1P2P3S3S2S1
LOG AMPLITUDE (VERTICAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
10.6
21.2
31.8
42.4
53.0
63.6
EPICENTRAL DISTANCE (KM)
1
2
1
2



RAY: P1P2P3S3S2S1
LOG AMPLITUDE (VERTICAL)
-12.0
-10.0
-8.0
-6.0
-4.0
-2.0
0.0
0.0
10.6
21.2
31.8
42.4
53.0
63.6
EPICENTRAL DISTANCE (KM)
4
3
4
3



LOG AMPLITUDE (HORIZONTAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
10.6
21.2
31.8
42.4
53.0
63.6
EPICENTRAL DISTANCE (KM)
RAY: P P2P3S3S2S1
1
2
1
2

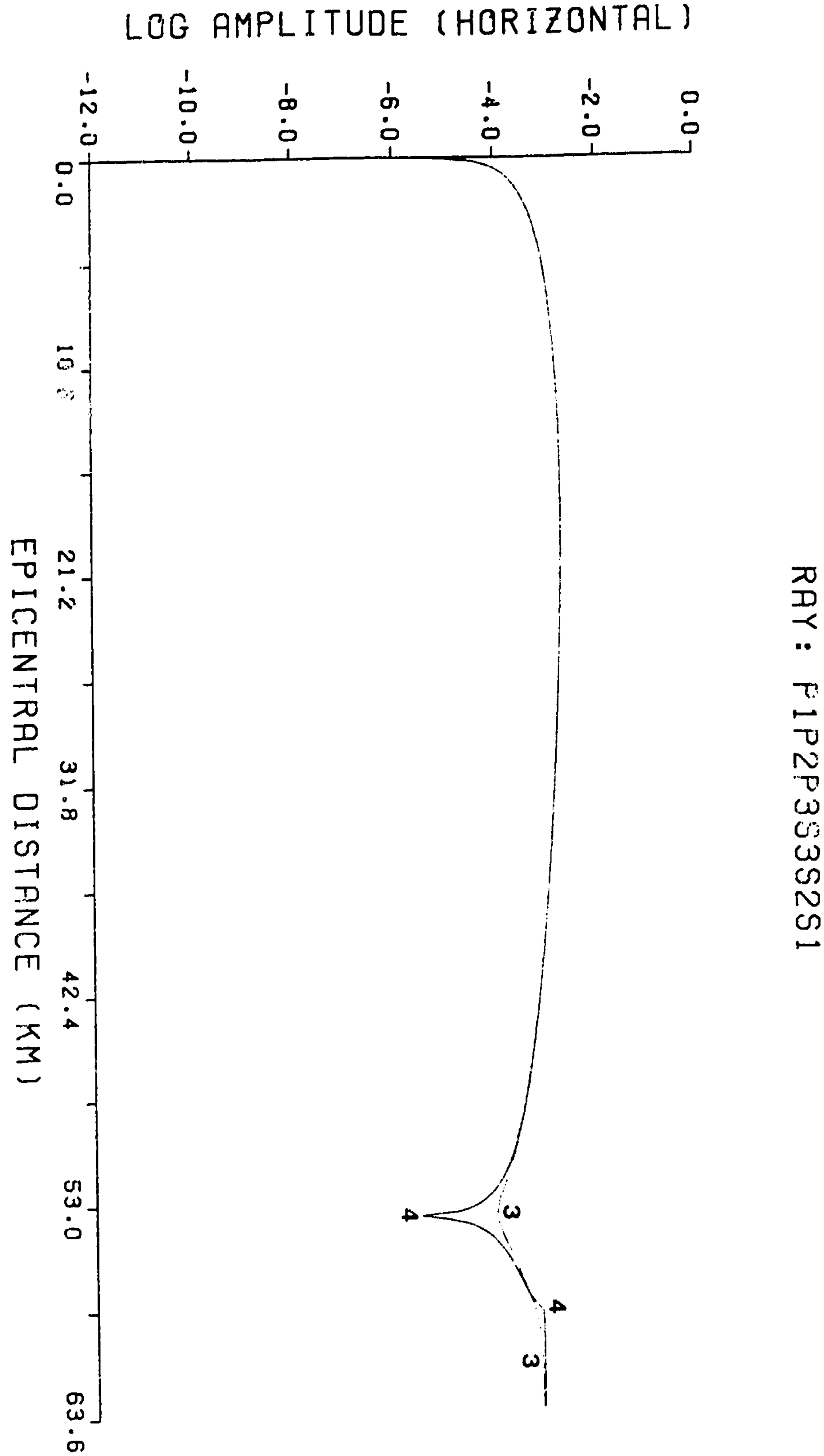

LOG AMPLITUDE (HORIZONTAL)
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
0.0
21.2
31.8
42.4
53.0
63.6
EPICENTRAL DISTANCE (KM)
RAY: F1P2P3S3S2S1
3
4
4
3

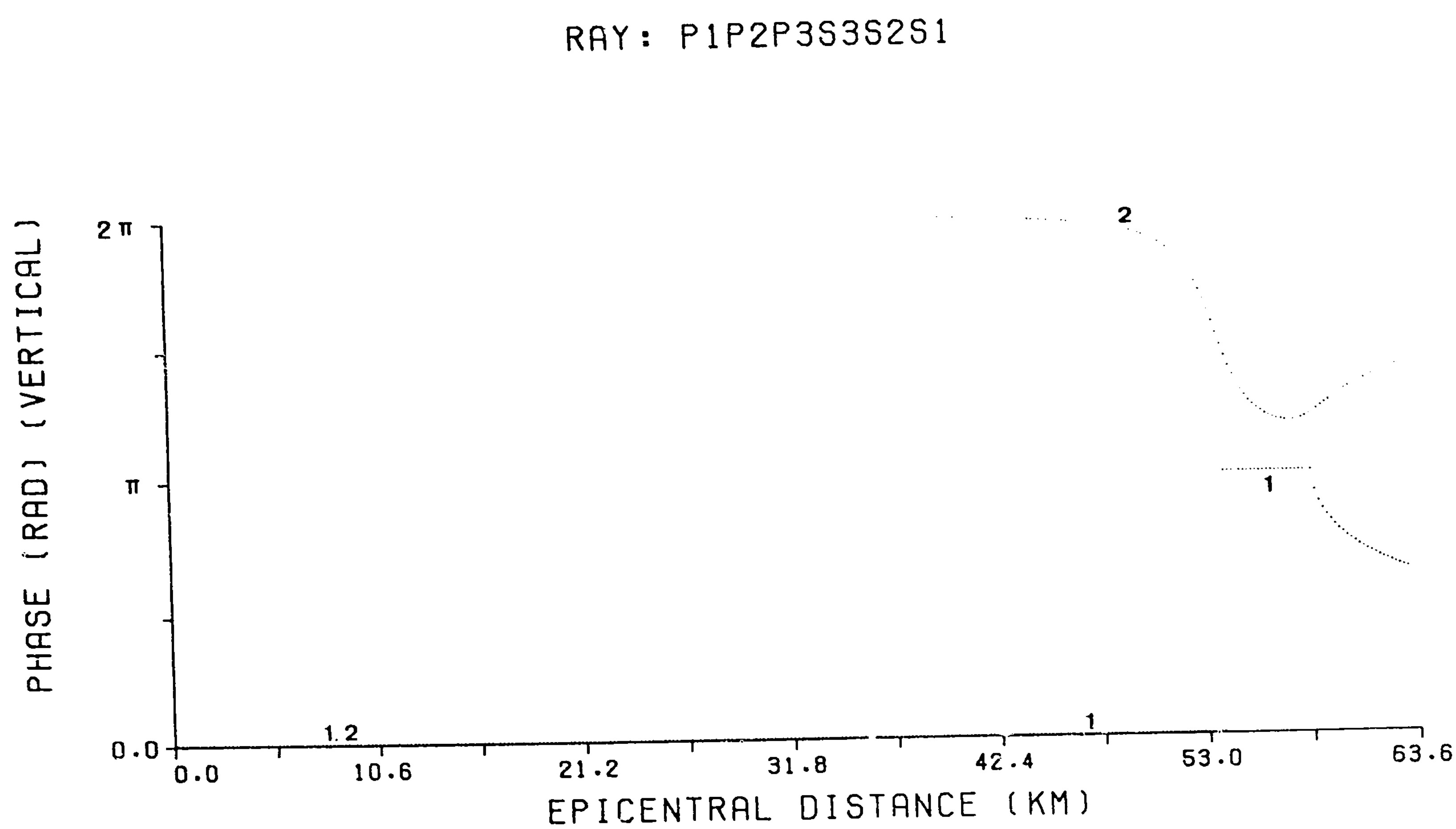
Figure 20: Phase of the product $\frac{Y e^{i\phi}}{L}$ for the ray P1P2P3S3S2S1 at all possible epicentral distances.

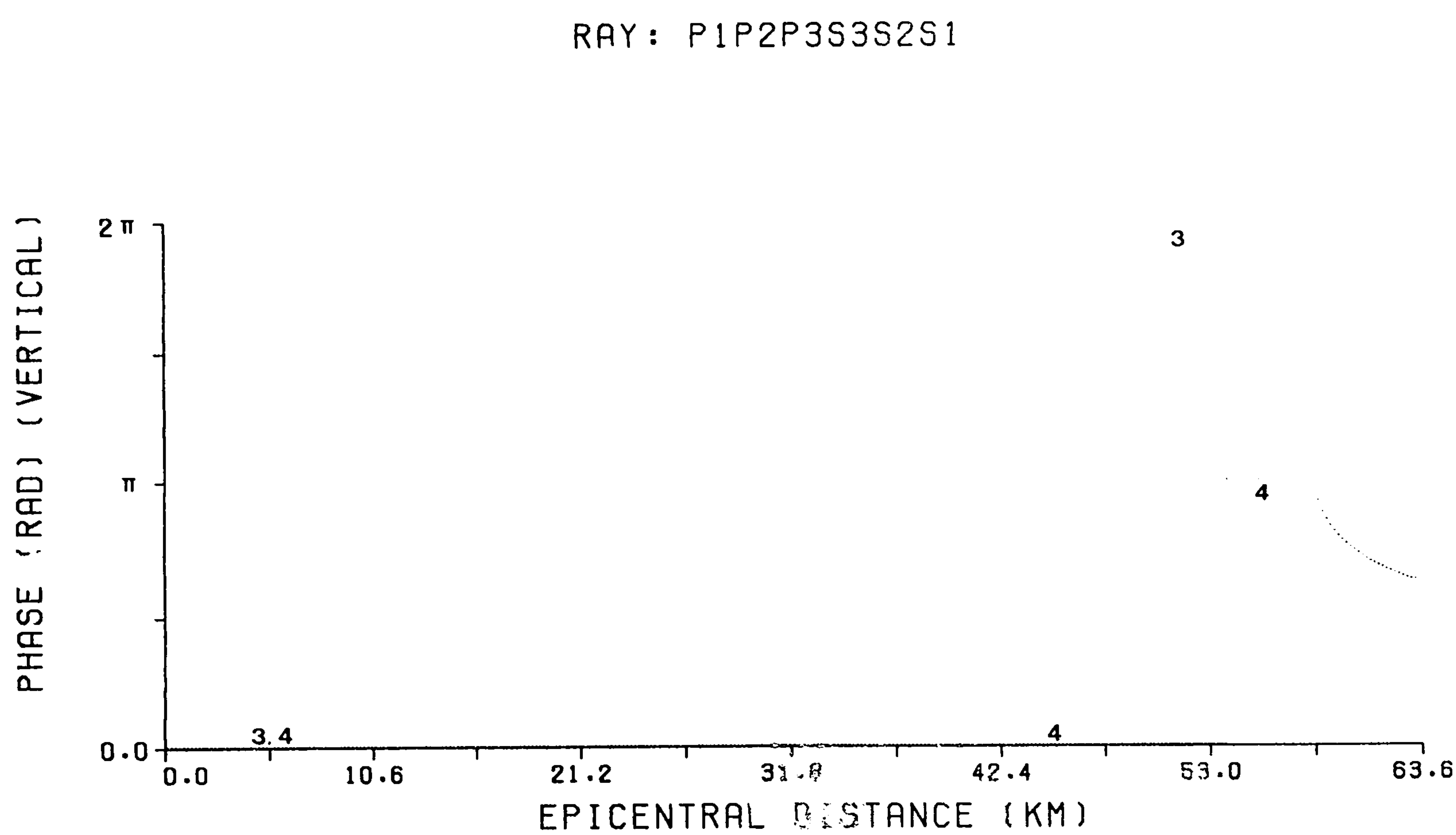
Curve #1: elastic geometrical spreading
elastic coefficients

Curve #2: anelastic geometrical spreading
anelastic coefficients

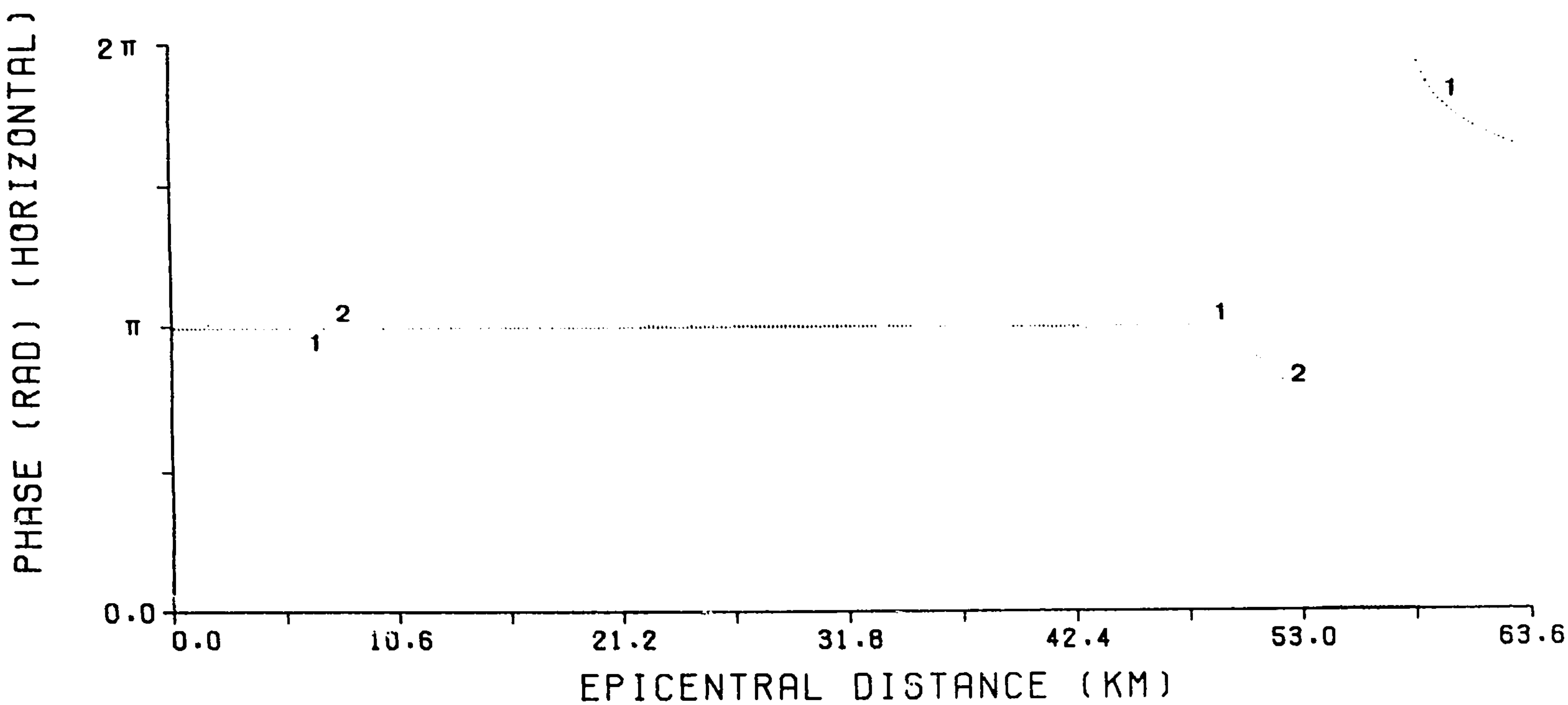
Curve #3: elastic geometrical spreading
anelastic coefficients

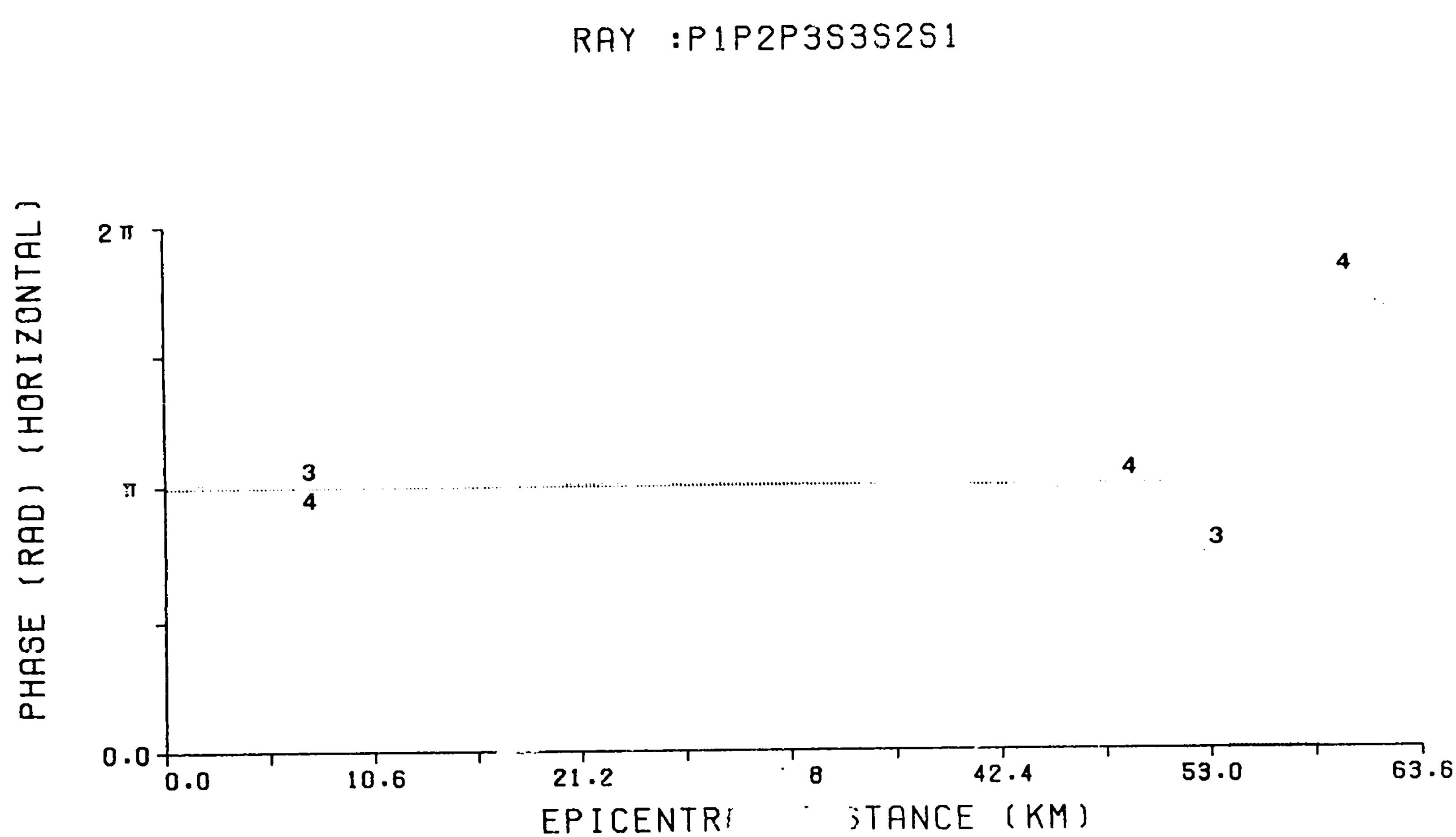
Curve #4: anelastic geometrical spreading
elastic coefficients


RAY: P1P2P3S3S2S1
PHASE (RAD) (VERTICAL)
2π
π
0.0
0.0
10.6
21.2
31.8
42.4
53.0
63.6
1.2
1
2
1
EPICENTRAL DISTANCE (KM)


RAY: P1P2P3S3S2S1
PHASE (RAD) (VERTICAL)
2π
π
0.0
3
4
3.4
4
0.0
10.6
21.2
31.8
42.4
53.0
63.6
EPICENTRAL DISTANCE (KM)

RAY: P1P2P3S3S2S1




RAY :P1P2P3S3S2S1
PHASE (RAD) (HORIZONTAL)
2π
π
0.0
0.0
10.6
21.2
8
42.4
53.0
63.6
3
4
4
3
4
EPICENTR
STANCE (KM)

CHAPTER 8

CONCLUSION

The stationary ray method has been used to compute reflection and transmission coefficients, geometrical spreading and simple synthetic seismograms for P-SV waves in linear viscoelastic media because it satisfies both Snell's law and the least time Fermat's principle. This was not the case with the conventional ray method; γ the attenuation angle had to be chosen arbitrarily consequently the conventional ray method only satisfied Snell's law.

The results obtained for linear viscoelastic reflection and transmission coefficients show differences in phase and for certain coefficients, in amplitude too with their elastic analogs. The phases of elastic and viscoelastic coefficients are opposite to each other from any critical incidence. The diffence in amplitude when it exists occurs in the vicinity of a critical incidence and is dependent on the contrast of Q between the two media forming the interface.

The simple synthetic seismograms were computed with frequency independent loss factors. These computations reveal that the difference in amplitude between wavelets propagating in elastic and viscoelastic media before encountering any critical incidence is mainly due to the anelasticity of the media traversed. The results obtained for the geometrical spreading using Silva model show practically no difference between the elastic and the viscoelastic case even after a critical angle.

The stationary ray method shows the importance of including viscoelastic reflection and transmission coefficients in the computation of synthetic seismograms for waves propagating through layered linear viscoelastic media. Of course more computation using this same method to represent the anelasticity is needed in order to draw final conclusions, especially for the geometrical spreading. The next step would be to produce for several anelastic models synthetic seismograms showing all the arrivals at a receiver, including the multiples, and compare them with real data.

FOOTNOTES

[page 38]

Equation (5.5) contradicts equation (2.18). This is because the calculations in chapter 2 were performed using the form $e^{i\omega(\tau-t)}$ whereas in chapter 5 the form $e^{i\omega(t-\tau)}$ was used. All the previous calculations on geometrical spreading were performed using $e^{i\omega(t-\tau)}$. In order to be consistent this form is used again. The final result for the geometrical spreading is of course valid for both forms.

[page 73]

Since $\vec{u}(\vec{r}, t)$ is the real response of the integral $\frac{1}{2\Pi}\int_{-\infty}^{\infty}$ can be replaced by $\frac{1}{\Pi}\mathrm{Re}\int_{0}^{\infty}$ (see Appendix [illegible]). To avoid divergence problem at $\omega=0$, the lower limit of the integral is replaced by ω_1 where ω_1 is small but greater than zero (see ART, Hron (1984)).

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APPENDIX 1

FOURIER TRANSFORM OF f(t) * dg(t)

f(t) and g(t) are two arbitrary functions of time t. The Stieltjes convolution of f(t) and g(t) is given by

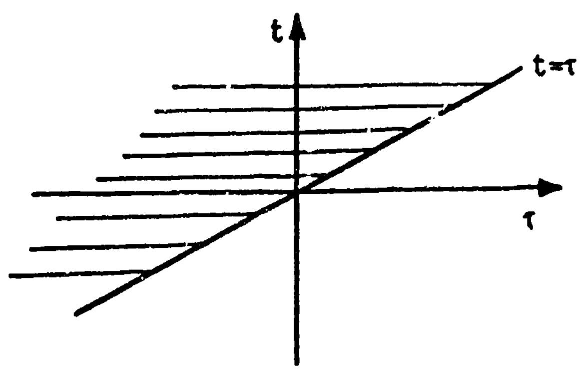
$$f(t) * dg(t) = \int_{-\infty}^{t} f(t-\tau) \frac{dg(\tau)}{d\tau} d\tau \qquad \text{(A 1.1)}$$

Its Fourier transform is equal to

$$\overline{f(\omega) * dg(\omega)} = I = \int_{-\infty}^{\infty} f(t) * dg(\tau) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{t} f(t-\tau) \frac{dg(\tau)}{d\tau} d\tau \right] e^{-i\omega t} dt \qquad \text{(A 1.2)}$$

The integration region is shown on the following graph



Changing the order of integration yields

$$I = \int_{-\infty}^{+\infty} \left[\int_{\tau}^{+\infty} f(t - \tau)\, e^{-i\omega t}\, dt \right] \frac{dg(\tau)}{d\tau}\, d\tau \qquad \text{(A 1.3)}$$

Let $t-\tau = \xi$, the t-integral becomes

$$\int_{\tau}^{+\infty} f(t - \tau)\, e^{-i\omega t}\, dt = e^{-i\omega\tau} \int_{0}^{+\infty} f(\xi)\, e^{-i\omega\xi}\, d\xi$$

$$= e^{-i\omega\tau} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt \qquad \text{(A 1.4)}$$

if $f(t) = 0$ when $t < 0$. Then

$$I = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt \int_{-\infty}^{+\infty} e^{-i\omega\tau} \frac{dg(\tau)}{d\tau}\, d\tau \qquad \text{(A 1.5)}$$

Calculating the τ-integral by part yields

$$\int_{-\infty}^{+\infty} e^{-i\omega\tau} \frac{dg(\tau)}{d\tau}\, d\tau = \left[e^{-i\omega\tau}\, g(\tau)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} g(\tau)\, de^{-i\omega\tau}$$

$$= i\omega \int_{-\infty}^{+\infty} g(\tau)\, e^{-i\omega\tau}\, d\tau \qquad \text{(A 1.6)}$$

Finally

$$I = i\omega \int_{-\infty}^{+\infty} f(t)\ e^{-i\omega t}\ dt \int_{-\infty}^{+\infty} g(\tau)\ e^{-i\omega \tau}\ d\tau$$

$$I = F(\omega) \times \overline{g}(\omega) \qquad \text{(A 1.7)}$$

APPENDIX 2

THE INVERSE FOURIER TRANSFORM FOR A REAL RESPONSE

Considering a real response f(t) at the receiver, f(t) being a real function of time t, we have

$$f(t) = f^*(t) \qquad \text{(A 2.1)}$$

where * means the complex conjugate. From its Fourier transform

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt \qquad \text{(A 2.2)}$$

we obtain

$$\begin{aligned} \overline{f^*}(\omega) &= \int_{-\infty}^{+\infty} f^*(t)\, e^{i\omega t}\, dt \\ &= \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt \\ &= \bar{f}(-\omega) \qquad \text{(A 2.3)} \end{aligned}$$

Consequently we can write

$$\begin{aligned} f(t) &= \frac{1}{2\Pi} \int_{-\infty}^{+\infty} \bar{f}(\omega)\, e^{i\omega t}\, d\omega \\ &= \frac{1}{2\Pi} \left[\int_{-\infty}^{0} \bar{f}(\omega)\, e^{i\omega t}\, d\omega + \int_{0}^{+\infty} \bar{f}(\omega)\, e^{i\omega t}\, d\omega \right] \end{aligned}$$

$$= \frac{1}{2\Pi}\left[\int_0^{+\infty} \bar{f}(-\omega)\, e^{-i\omega t}\, d\omega + \int_0^{+\infty} \bar{f}(\omega)\, e^{i\omega t}\, d\omega\right]$$

$$= \frac{1}{2\Pi}\int_0^{+\infty} \left[\overline{f^*}(\omega)\, e^{-i\omega t} + \bar{f}(\omega)\, e^{i\omega t}\right] d\omega$$

$$= \frac{1}{2\Pi}\int_0^{+\infty} \left[\left(\bar{f}(\omega)\, e^{i\omega t}\right)^* + \bar{f}(\omega)\, e^{i\omega t}\right] d\omega$$

$$= \frac{1}{2\Pi}\int_0^{+\infty} 2\ \mathrm{Re}\left(\bar{f}(\omega)\, e^{i\omega t}\right) d\omega$$

$$= \frac{1}{\Pi}\ \mathrm{Re}\int_0^{+\infty} \bar{f}(\omega)\, e^{i\omega t}\, d\omega \qquad \text{(A 2.4)}$$

If f(t) is a real function, the form $\frac{1}{\Pi}\ \mathrm{Re}\int_0^{+\infty}$ can be used instead of the form $\frac{1}{2\Pi}\int_{-\infty}^{+\infty}$. The integrand of equation (5.2) satisfies (A 2.3) if the various terms are defined approriately for negative frequencies.

APPENDIX 3

FORMULAE OF THE VISCOELASTIC REFELECTION AND TRANSMISSION COEFFICIENTS

$$P1P1 = \frac{NP1 + NP2 + NP3 + NP4}{DP}$$

where

NP1 = X[K1 X(SI2 SI4 + B CO4) + S(CO3 SI4 + SI3 CO4) +
K2 X(CO3 B - SI3 SI2)]

NP2 = Q[P(-SI2 SI4 - B CO4) + S(SI1 SI4 + A CO4) -
K2 X(SI1 B - A SI2)]

NP3 = $\frac{X}{N}$[P(-CO3 SI4 - SI3 CO4) + K1 X(-SI1 SI4 -A CO4) -
K2 X(SI1 SI3 -A CO3)]

NP4 = -R[P(CO3 B- SI3 SI2) + K1 X(SI1 B - SI2 A) +
S(SI1 SI3 - A CO3)]

and

$$DP = DP1 + DP2 + DP3 + DP4$$

where

DP1 = X[K1 X(-SI2 SI4 - B CO4) + S(-CO3 SI4 - SI3 CO4) +
K2 X(CO3 B - SI3 SI2)]

$$DP2 = Q[P(-SI2\ SI4 - B\ CO4) - S(-SI1\ SI4 + A\ CO4) - K2\ X(SI1\ B + A\ SI2)]$$

$$DP3 = \frac{X}{N}[P(-CO3\ SI4 - CO4\ SI3) + K1\ X(-SI1\ SI4 + A\ CO4) - K2\ X(SI1\ SI3 + A\ CO3)]$$

$$DP4 = -R[P(CO3\ B - SI3\ SI2) + K1\ X(SI1\ B + A\ SI2) + S(SI1\ SI3 + A\ CO3)]$$

$$P1S1 = \frac{TP1 + TP2 + TP3 + TP4}{DP}$$

where

$$TP1 = -X[P(-SI2\ SI4 -B\ CO4) + S(SI4\ SI1 + A\ CO4) - K2\ X(SI1\ B - A\ SI2)]$$

$$TP2 = -X[P(-SI2\ SI4 - B\ CO4) - S(-SI1\ SI4 + A\ CO4) - K2\ X(SI1\ B + A\ SI2)]$$

$$TP3 = \frac{X}{N}[P(-SI1\ SI4 - A\ CO4) - P(-SI1\ SI4 + A\ CO4) - K2\ X(SI1\ A + A\ SI1)]$$

$$TP4 = -R[P(SI1\ B - A\ SI2) - P(SI1\ B + A\ SI2) + S(SI1\ A + SI1\ A)]$$

$$P1P2 = \frac{NP5 + NP6 + NP7 + NP8}{DP}$$

where

NP5 = -X[K1 X(SI4 SI1 + A CO4) + P(CO3 SI4 + SI3 CO4) -
K2 X(CO3 A -SI1 SI3)]

NP6 = Q[P(-SI1 SI4 - A CO4) - P(-SI4 SI1 + A CO4) -
K2 X(SI1 A + A SI1)]

NP7 = X[P(-SI4 CO3 - SI3 CO4) + K1 X(-SI1 SI4 + A CO4) -
K2 X(SI1 SI3 + A CO3)]

NP8 = -R[P(CO3 A - SI3 SI1) + K1 X(SI1 A + A SI1) +
P(SI1 SI3 + A CO3)]

$$P1S2 = \frac{TP5 + TP6 + TP7 + TP8}{DP}$$

where

TP5 = -X[-K1 X(SI2 A - B SI1) - S(CO3 A - SI3 SI1) +
P(CO3 B - SI2 SI3)]

TP6 = Q[P(SI2 A - B SI1) - S(SI1 A + A SI1) +
P(SI1 B + A SI2)]

$$TP7 = \frac{X}{N}[P(CO3\ A - SI3\ SI1) + K1\ X(SI1\ A + A\ SI1) + P(SI1\ SI3 + A\ CO3)]$$

$$TP8 = -X[P(CO3\ B - SI3\ SI2) + K1\ X(SI1\ B + A\ SI2) + S(SI1\ SI3 + A\ CO3)]$$

$$S1P1 = \frac{NS1 + NS2 + NS3 + NS4}{DS}$$

where

$$NS1 = Q[K1\ X(SI2\ SI4 + B\ CO4) + S(CO3\ SI4 + SI3\ CO4) + K2\ X(-CO3\ B + SI2\ SI3)]$$

$$NS2 = Q[K1\ X(SI2\ SI4 + B\ CO4) + S(CO3\ SI4 - CO4\ SI3) + K2\ X(-CO3\ B - SI3\ SI2)]$$

$$NS3 = \frac{X}{N}[K1\ X(CO3\ SI4 + SI3\ CO4) - K1\ X(CO3\ SI4 - SI3\ CO4) + K2\ X(-CO3\ SI3 - SI3\ CO3)]$$

$$NS4 = -R[K1\ X(-B\ CO3 + SI3\ SI2) - K1\ X(-B\ CO3 - SI2\ SI3) + S(CO3\ SI3 + SI3\ CO3)]$$

and

$$DS = DS1 + DS2 + DS3 + DS4$$

where

$$DS1 = -X[K1\ X(SI2\ SI4 + B\ CO4) + S(CO3\ SI4 + SI3\ CO4) + K2\ X(-CO3\ B + SI2\ SI3)]$$

$$DS2 = Q[-P(SI2\ SI4 + B\ CO4) + S(SI1\ SI4 - A\ CO4) + K2\ X(-SI1\ B - SI2\ A)]$$

$$DS3 = \frac{X}{N}[-P(CO3\ SI4 + SI3\ CO4) - K1\ X(SI1\ SI4 - A\ CO4) + K2\ X(-SI1\ SI3 - A\ CO3)]$$

$$DS4 = -R[-P(-CO3\ B + SI3\ SI2) + K1\ X(B\ SI1 + A\ SI2) + S(SI3\ SI1 + A\ CO3)]$$

$$S1S1 = \frac{TS1 + TS2 + TS3 + TS4}{DS}$$

where

$$TS1 = -X[K1\ X(SI2\ SI4 + B\ CO4) + S(CO3\ SI4 - SI3\ CO4) + K2\ X(-CO3\ B - SI2\ SI3)]$$

$$TS2 = -Q[-P(SI2\ SI4 + B\ CO4) + S(SI1\ SI4 - A\ CO4) + K2\ X(-SI1\ B - A\ SI2)]$$

$$TS3 = \frac{X}{N}[-P(CO3\ SI4 - SI3\ CO4) - K1\ X(SI1\ SI4 - A\ CO4) + K2\ X(SI1\ SI3 - A\ CO3)]$$

$$TS4 = -R[P(CO3\ B + SI3\ SI2) + K1\ X(SI1\ B + A\ SI2) - S(SI1\ SI3 - A\ CO3)]$$

$$S1P2 = \frac{NS5 + NS6 + NS7 + NS8}{DS}$$

where

NS5 = -X[K1 X(CO3 SI4 - SI3 CO4) - K1 X(CO3 SI4 + SI3 CO4) +
K2 X(CO3 SI3 + SI3 CO3)]

NS6 = Q[-P(CO3 SI4 - SI3 CO4) - K1 X(SI1 SI4 - A CO4) +
K2 X(-SI1 SI3 -A CO3)]

NS7 = Q[-P(CO3 SI4 + SI3 CO4) - K1 X(SI1 SI4 - A CO4) +
K2 X(-SI1 SI3 - A CO3)]

NS8 = -R[-P(CO3 SI3 + SI3 CO3) - K1 X(SI1 SI3 -A CO3) +
K1 X(-SI1 SI3 -A CO3)]

$$S1S2 = \frac{TS5 + TS6 + TS7 + TS8}{DS}$$

where

TS5 = -X[K1 X(SI2 SI3 + B CO3) + S(CO3 SI3 + SI3 CO3) +
K1 X(-B CO3 + SI3 SI2)]

TS6 = Q[-P(SI2 SI3 + B CO3) + S(SI1 SI3 - A CO3) +
K1 X(-B SI1 - A SI2)]

TS7 = $\frac{X}{N}$[-P(CO3 SI3 + SI3 CO3) - K1 X(SI1 SI3 - A CO3) +
K1 X(-SI1 SI3 - A CO3)]

TS8 = -Q[-P(-B CO3 + SI3 SI2) + K1 X(SI1 B + A SI2) + S(SI1 SI3 + A CO3)]

P2P2 = P1P1　　P2S2 = -P1S1

P2P1 = P1P2　　P2S1 = P1S2

S2P2 = S1P1　　S2S2 = -S1S1

S2P1 = S1P2　　S2S1 = S1S2

$$A = RO_1\ V_1 - \frac{2\ RO_1\ V_3^2\ X^2}{V_1}$$

$$B = RO_2\ V_2 - \frac{2\ RO_2\ V_4^2\ \left(\frac{X}{N}\right)^2}{V_2}$$

$$SI1 = \frac{2\ RO_1\ V_3^2\ X\ P}{V_1}$$

$$SI2 = \frac{2\ RO_2\ V_4^2\ X\ S}{V_2\ N}$$

$$SI3 = 2\ RO_1\ V_3\ K1\ X\ Q$$

$$SI4 = 2\ RO_2\ V_4\ K2\ X\ R$$

$$CO3 = RO_1\ V_3\ (1 - 2(K1\ X)^2)$$

$$C04 = RO_2 \; V_4 \; (1 - 2(K2 \; X)^2)$$

$$N = \frac{V_1}{V_2} \qquad K1 = \frac{V_3}{V_1} \qquad K2 = \frac{V_4}{V_1}$$

$$X = \sin\alpha_1$$

$$P = \cos\alpha_1 \qquad S = \cos\alpha_2 \qquad Q = \cos\alpha_3 \qquad R = \cos\alpha_4$$

(RO_n is the density of the n^{th} medium and V is the complex velocity (see chapter 6 for legend))