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Properties of Refinable Functions and Subdivision Schemes

BY

Shurong Zhang ©

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ABSTRACT

This thesis deals with three problems: convergence rates of subdivision schemes, spectral properties of a transition operator associated with a multivariate refinement equation, and the computation of the smoothness order of refinable functions.

Subdivision schemes consist of a class of numerically stable, easily implemented algorithms for the generation of parameterized curves and surfaces. There are many papers concerned with the convergence of subdivision schemes. Here we investigate the convergence rates of subdivision schemes and propose a method to construct modified subdivision schemes which will accelerate the convergence rates. We study the univariate case in Chapter 2 and multivariate case in Chapter 3.

In Chapter 4 of this thesis, we investigate the spectral properties of the transition operator T_a and apply these properties to the study of the approximation and smoothness properties of the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}^s} \phi(M\cdot - \alpha)$. In particular, we provide a detailed analysis of the spectrum of the transition operator associated with a box spline. The results are then applied to interpolatory subdivision schemes induced by box splines.

The last problem we study in this thesis is the computation of the smoothness order of refinable functions. By using the theory established in Chapter 4, we propose a method to calculate the smoothness order of refinable functions by finding the spectrum of T_b restricted to different invariant subspaces. Special attention is given to computing the smoothness order of symmetric refinable functions. Numerical computations are presented to support our theoretical analysis.

To my parents and my family

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CHAPTER 1

Introduction

In this thesis, we will study refinable functions and subdivision schemes. As outlined in the following sections, we will study the convergence rates of subdivision schemes in Chapter 2 for the univariate case, and in Chapter 3 for the multivariate case. In Chapter 4, we will investigate the spectral properties of the transition operator associated with a multivariate refinement equations. As an application, we discuss the numerical procedures for computing the smoothness order of refinable functions in Chapter 5.

1.1 Convergence Rates of Subdivision Schemes

The basic question addressed in Chapter 2 & 3 of the thesis is the convergence rates of subdivision schemes.

Subdivision methods originated in the geometric problem of smoothing the corners of a given polyhedral surface. They offered efficient ways of displaying curves and surfaces as well as methods to design particular shapes. Subdivision schemes also play an important role in wavelet analysis. The theory of convergence of subdivision schemes has been investigated in a general setting by Micchelli and Prautzsch [42] [44]. They approached the problem through the study of control point transformation matrices which define the basic subdivision schemes. The uniform convergence of subdivision schemes for the univariate case has been studied by Dyn, Gregory and Levin [20]. They analyzed the convergence of control polygons to a $\mathbb{C}^0$ curve in terms of the convergence to zero of a derived scheme for the differences $f_{i+1}^k - f_i^k$. A kind of specific interpolatory scheme was proposed and studied by Dubuc in [19]. Deslauriers and Dubuc investigated symmetric interpolatory subdivision scheme and provided a general construction method in [18]. Cavaretta, Dahmen, and Micchelli investigated subdivision schemes systematically in [1]. It was proved in [1] that if the solution to the refinement equation associated with mask a is continuous and has stable shifts, then the subdivision scheme associated with mask a converges uniformly. Jia discussed the L_p cases, $1 \leq p \leq \infty$, in [35] for the univariate case, and characterized the convergence of subdivision schemes in L_p in terms of the joint spectral radius of two matrices associated with the corresponding mask. Other properties of the limit function of a subdivision scheme were also investigated. Multivariate refinement equations and subdivision schemes were studied in the joint work of Han and Jia in [26]. The L_p-convergence of a multivariate subdivision scheme was characterized there in terms of the p-norm joint spectral radius of a collection of matrices associated with the refinement mask.

There are many papers concerned with the convergence of subdivision schemes, see [2] for a survey. It is quite natural to pose the following question: how do we describe the convergence rates of a subdivision scheme? There are several authors who have worked on this topic. Dahmen and Micchelli in [7] showed that in general, the coefficients of the refined control nets of a box spline surface converge to the surface at the rate of the refinement, that is, in a linear order. Dahmen, Dyn and Levin also investigated the convergence rates of subdivision schemes for box spline surfaces in [6]. They pointed out that the subdivision schemes can attain a quadratic convergence rate under certain conditions. Prautzsch in [43] studied the acceleration of the convergence rate of subdivision schemes for box splines by exploiting the quadrature formulas. It was shown in [16] by C. de Boor, K. Höllig and S. Riemenschneider that the subdivision scheme of box splines converges to the box spline surface at a quadratic rate if the box spline functions are continuously differentiable. The convergence rates of subdivision schemes for box splines, a special class of subdivision schemes, were investigated in the existing literatures.

Generally, we know that the subdivision scheme converges to the corresponding function in linear order. Our purpose is to investigate the relationship between the convergence rate of subdivision schemes and the properties of the solution to the refinement equation with general masks so that we can explore modified subdivision schemes to achieve faster algorithms of generating the refinable function. We take a new approach to this problem by using quasi-interpolation and apply this method to refinable functions with general mask. We first investigate the convergence rates of subdivision schemes if the corresponding refinable functions are continuous. The construction of modified subdivision scheme is established under certain conditions such that the convergence rate attained is as high as the smoothness order of the refinable functions. Specifically, we construct an explicit scheme such that the subdivision scheme can attain quadratic order.

In Chapter 2, we discuss the univariate case; in Chapter 3, we investigate the multivariate case. These two chapters are based on [49].

1.2 Spectral Properties of the Transition Operator Associated with a Multivariate Refinement Equation

The second problem we discuss in this thesis is spectral properties of the transition operator associated with a multivariate refinement equation.

Let a be an element in $\ell_0(\mathbb{Z}^s)$. The transition operator T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta) \quad \alpha \in \mathbb{Z}^s, \tag{1.2.1}$$

where $v \in \ell_0(\mathbb{Z}^s)$. The subdivision operator S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) u(\beta) \quad \alpha \in \mathbb{Z}^s, \tag{1.2.2}$$

where $u \in \ell(\mathbb{Z}^s)$. We introduce a bilinear form on the pair of linear spaces $\ell_0(\mathbb{Z}^s)$

and $\ell(\mathbb{Z}^s)$ as follows:

$$\langle u, v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha)v(\alpha) \quad u \in \ell(\mathbb{Z}^s), v \in \ell_0(\mathbb{Z}^s). \tag{1.2.3}$$

Then $\ell(\mathbb{Z}^s)$ is the dual space of $\ell_0(\mathbb{Z}^s)$ with respect to this bilinear form. It is easily seen that

$$\langle S_a u,\, v\rangle = \langle u,\, T_a v\rangle \quad \forall u \in \ell(\mathbb{Z}^s),\ v \in \ell_0(\mathbb{Z}^s).$$

Hence, S_a is the algebraic adjoint of T_a with respect to the bilinear form given in (1.2.3). The subdivision and transition operators play an important role in our study of approximation and smoothness properties of the solution to the refinement equation.

In Chapter 4, we will investigate the properties of the transition operators T_a. Regarding this topic, Deslauries and Dubuc [18] discussed the spectral properties of the transition operator and applied those properties to their study of interpolatory subdivision schemes for $s = 1$. For the multivariate case ($s > 1$), the subdivision operator was introduced by Cavaretta, Dahmen, and Micchelli [1] in their investigation of stationary subdivision schemes. In [24], Goodman, Micchelli, and Ward established a spectral radius formula for the subdivision operator.

Spectral properties of the continuous refinement operator were investigated by Goodman, Micchelli, and Ward in [25]. Spectral properties of the subdivision and transition operators associated to a vector refinement equation were studied by Jia, Riemenschneider, and Zhou [39], Goodman, Jia, and Micchelli [23].

In [27], Han and Jia showed that the transition operator T_a has only finitely many nonzero eigenvectors. Jia studied the subdivision and transition operators associated with a refinement equation and dilation matrix $2I$ in [37]. Eigenvalues and invariant subspaces of the operators are investigated. The spectral properties of the subdivision and transition operators are then used to study approximation and smoothness properties of the solution to the refinement equation. The spectral properties of the subdivision operator and the transition operator with a general dilation matrix M were also studied by Jia [29], and the results were applied to the study of approximation properties of a refinable function. In [30] Jia analyzed the smoothness of refinable functions in terms of their masks. He characterized the optimal smoothness of a multivariate refinable function in terms of the spectral radius of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace.

Chapter 4 focuses on the spectral properties of the transition operator associated with a multivariate refinement equation and their applications to the study of the approximation and smoothness properties of the corresponding refinable function. Suppose the dilation matrix M has eigenvalues $\sigma_1, \sigma_2, \cdots, \sigma_s$. Write σ for the s-tuple $(\sigma_1, \cdots, \sigma_s)$. For a multi-index $\mu = (\mu_1, \cdots, \mu_s)$, define $\sigma^\mu := \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$. We shall show that the spectrum of the transition operator T_a contains $\{\sigma^{-\mu} : |\mu| \leq k\}$, provided ϕ has accuracy k. This gives an upper bound for the accuracy of ϕ in terms of the refinement mask a. We shall also investigate invariant subspaces of the subdivision and transition operators. We prove a necessary and sufficient condition for a finite dimensional shift-invariant subspace of polynomials restricted to integers to be invariant under the subdivision operator. Furthermore, we clarify the relationship among the spectra of the transition operator restricted to different invariant subspaces. As a special class of refinable functions with respect to the dilation matrix $M = 2I$, the spectrum of the transition operators of box splines on the three-direction mesh is found. This result is then applied to interpolatory subdivision schemes induced by box splines. In particular, we find a way to greatly simplify the computation of the smoothness order of the refinable functions which are convolutions of box spline with refinable distributions.

The results in Chapter 4 is based on the joint research with Jia in [40].

1.3 Computations of the Smoothness Order of Refinable Functions

We will investigate the computations of the smoothness order of refinable functions in Chapter 5.

Wavelets are generated from refinable functions. If the refinable mask is given, the question of how to calculate the smoothness order of the corresponding refinable function is an important topic in the study of wavelets.

The binary case where $s = 1$ and $M = (2)$ has been studied in some papers. Eirola in [22] established a formula for the critical exponent of ϕ, under the condition that $\tilde{a}(e^{i\xi}) \neq 0$, where ϕ denotes the normalized solution of the refinement equation with mask a, $\phi \in L_2(\mathbb{R})$, and $\tilde{a}(e^{i\xi})$ denotes the symbol of mask a. The L_p cases were investigated by Villemoes in [48]. Cohen and Daubechies [3] studied the regularity of refinable functions for the case where the refinement mask is not necessarily finitely supported.

The results in both [22] and [48] rely on factorization of the symbol of the mask. In the multivariate case, however, the symbol of the refinement mask is often irreducible. Jia studied the smoothness properties of the general multivariate functions on Sobolev spaces in [30] and provided a conclusive characterization for the smoothness of a multivariate refinable function in terms of the refinement mask and the dilation matrix. He characterized the optimal smoothness of a multivariate refinable functions in terms of the spectral radius of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace. Some estimate of the smoothness of multivariate refinable functions were obtained by Cohen and Daubechies [4], by Goodman, Micchelli, and Ward [25], and Shen [46, 45].

In the multivariate case, there is no general numerical procedure to compute the smoothness order of the refinable functions if the mask is given. In [46], the authors constructed a class of bivariate interpolatory subdivision schemes and established the regularity criteria in terms of the refinement mask. In order to give a general numerical procedure in terms of the refinable mask, we intend to propose a procedure to identify the spectral radius of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace based on the relationship among the spectra of transition operator restricted to different invariant subspaces established in Chapter 4. So that we can calculate the smoothness order of refinable functions. Several examples are provided to illustrate this method.

We observe that usually, the mask has certain symmetric properties. For instance, the bivariate interpolatory subdivision schemes in [46] have symmetry with respect to the origin. The interpolatory refinement masks in [26] enjoy full symmetry. We intend to employ the symmetries of the refinement mask, so that we can shrink the matrix size of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace substantially. Based on the relationship of the transition operator restricted to different invariant subspaces established, we can compute the spectral radius of the corresponding transition operator restricted on the finite dimensional invariant subspace. Then we can obtain the estimate of smoothness order of corresponding refinement functions. In this way, we can overcome the difficulty mentioned in [46]. Numerical examples are provided.

CHAPTER 2

Convergence Rates of Subdivision Schemes

2.1 Introduction

Subdivision schemes play an important role in computer graphics and wavelet analysis. There has been an intensive study of subdivision schemes (see, e.g., [1], [18], [20], [35]). These papers were mainly concerned with the convergence of subdivision schemes.

In this chapter, we study rates of the convergence of subdivision schemes. We show that the convergence rates of a subdivision scheme can attain the same order as the smoothness order of the corresponding refinable function under certain conditions.

Let $L_\infty(\mathbb{R})$ denote the Banach space of all functions f on $\mathbb{R}$ such that $\|f\|_\infty < \infty$, where $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. We use $\mathbb{C}(\mathbb{R})$ to denote the space of all continuous functions on $\mathbb{R}$.

Let $a = (a(j))_{j\in\mathbb{Z}}$ be a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$. Consider the mapping Q_a from $L_\infty(\mathbb{R})$ to $L_\infty(\mathbb{R})$ given by

$$Q_a f = \sum_{j\in\mathbb{Z}} a(j) f(2\cdot - j), \quad f \in L_\infty(\mathbb{R}). \tag{2.1.1}$$

Let ϕ be a compactly supported function in $C(\mathbb{R})$. For $n = 1, 2, \cdots$, let $f_n := Q_a^n \phi$. Each f_n can be expressed as

$$f_n = \sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n \cdot - j), \tag{2.1.2}$$

where a_n is a finitely supported sequence on $\mathbb{Z}$, and is independent of the choice of ϕ. Indeed, a_n can also be computed by an iteration scheme. Applying the operator Q_a to both sides of (2.1.2), we obtain

$$f_{n+1} = \sum_{j\in\mathbb{Z}} a_n(j) \Big[\sum_{k\in\mathbb{Z}} a(k)\phi(2^{n+1} \cdot - 2j - k) \Big].$$

It follows that

$$a_{n+1}(l) = \sum_{j\in\mathbb{Z}} a(l-2j) a_n(j), \quad l \in \mathbb{Z}.$$

This motivates us to introduce the linear operator S_a on $\ell_\infty(\mathbb{Z})$ given by

$$S_a \lambda(i) = \sum_{j\in\mathbb{Z}} a(i-2j)\lambda(j), \quad i \in \mathbb{Z},$$

where $\lambda \in \ell_\infty(\mathbb{Z})$. The operator S_a is called the **subdivision operator** associated with a (see [1]). Thus, the coefficient sequences a_n, $n = 1, 2, \ldots$ in (2.1.2), can be computed by

$$a_0 = \delta$$
$$a_n = S_a a_{n-1} = S_a^n \delta,\ n = 1, 2, \ldots.$$

More generally, if $\lambda \in \ell_\infty(\mathbb{Z})$, then

$$\sum_{j\in\mathbb{Z}} \lambda(j) Q_a^n \phi(\cdot - j) = \sum_{k\in\mathbb{Z}} S_a^n \lambda(k) \phi(2^n \cdot - k).$$

In particular, if ϕ is the hat function ϕ_0:

$$\phi_0(x) = \begin{cases} 1 + x & \text{if } -1 \le x < 0, \\ 1 - x & \text{if } 0 \le x < 1, \\ 0 & \text{elsewhere} \end{cases}$$

then the iteration scheme

$$f_n = Q_a^n \phi_0, \quad n = 0, 1, 2, \ldots ,$$

is called the **subdivision scheme** associated with the mask a. Clearly, (2.1.2) implies that $f_n(j/2^n) = a_n(j)$ for all n and j. This subdivision scheme is said to **converge in the L_∞-norm**, if there is a function $f \in C(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

Consequently, $f = Q_a f$. In other words, the limit function f satisfies the **refinement** equation

$$f = \sum_{j \in \mathbb{Z}} a(j) f(2 \cdot - j). \qquad (2.1.3)$$

Moreover, the function f is compactly supported. The sequence a is called the **refinement mask**, and any function f satisfying the refinement equation is called a **refinable function.**

If a is a mask with $\sum_{j \in \mathbb{Z}} a(j) = 2$, then it is known (see [1], [11], [12]), that there is a unique compactly supported distribution f satisfying $\hat{f}(0) = 1$ and the refinement equation (2.1.3), where $\hat{f}$ is the Fourier transform of f :

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x) e^{-ix\zeta} dx, \qquad \zeta \in \mathbb{R}.$$

This distribution f is said to be the **normalized solution** to the refinement equation with mask a.

We say that the shifts of a function $f \in L_\infty(\mathbb{R})$ are **stable** if there are two positive constants C_1 and C_2 such that

$$C_1 \|\lambda\|_\infty \le \Big\| \sum_{j \in \mathbb{Z}} \lambda(j) f(\cdot - j) \Big\|_\infty \le C_2 \|\lambda\|_\infty, \quad \forall \lambda \in \ell_\infty(\mathbb{Z}).$$

The concept of stability plays an important role in study of the subdivision schemes. See [38] for a characterization of stability.

Let a be a finitely supported sequence such that $\sum_{j \in \mathbb{Z}} a(j) = 2$. Let f be the normalized solution to the refinement equation with mask a. Cavaretta, Dahmen, and Micchelli showed in [1] that if f is a continuous function with stable shifts, then the subdivision scheme associated with a converges uniformly.

The main purpose of this chapter is to investigate the convergence rates of subdivision schemes. Some existing literatures ([16], [5], [6], [47]) regarding this question have only investigated the convergence rates of subdivision schemes for box splines and of control polygons for splines. It was proved, under some additional assumptions, that the convergence rates attain quadratic order.

We take a new approach to this problem by using quasi-interpolation and by applying this method to refinable functions with general masks. Under certain conditions, we shall prove that the convergence rates of subdivision schemes become as high as the smoothness order of the refinable functions.

This chapter is organized as follows. In section 2.2, we discuss the convergence rates for some concrete examples. Section 2.3 is devoted to the proofs of the main results on convergence rates of subdivision schemes. In section 2.4, we construct an explicit scheme with quadratic convergence rates and work out some examples to demonstrate applications of our results. In section 2.5, we discuss the convergence rates of subdivision schemes for refinable functions with unstable shifts.

2.2 Numerical Examples

In this section, we examine one example to see what happens to the convergence rates of subdivision schemes.

Example 2.1 *The Daubechies scaling functions ϕ_N with $N = 6$ and $N = 10$ (see [10] p. 195).*

Let ϕ_6 and ϕ_{10} denote these functions. Then these functions have the following properties

$$\widehat{\phi}_6(0) = 1, \quad \widehat{\phi}'_6(0) \neq 0;$$

$$\widehat{\phi}_{10}(0) = 1, \quad \widehat{\phi}'_{10}(0) \neq 0.$$

Moreover, ϕ_{10} lies in $C^3(\mathbb{R})$ and ϕ_6 lies in $C^2(\mathbb{R})$. Let us define

$$error(n) = \|a_n - a_{n-1}\|_\infty,$$

$$ratio(n) = \frac{error(n)}{error(n-1)}.$$

Subdivision schemes are used to generate these two scaling functions, and the following tables of data show that the subdivision schemes for ϕ_6 and ϕ_{10} converge to these two functions in linear order.

Daubechies Compactly Supported Scaling Function N=6

n	*error(n)*	*ratio(n)*
1	*0.600297054932177*	
2	*0.347047586588837*	*0.578126418807081*
3	*0.171110571423673*	*0.493046423707377*
4	*0.086194516753744*	*0.503735777612022*
5	*0.043288335699455*	*0.502216815289144*
6	*0.021695403803355*	*0.501183597216144*
7	*0.010843801551611*	*0.499820222287622*
8	*0.005422054038466*	*0.500014133664688*

9	0.002711104831332	0.500003927310901
10	0.001355551113321	0.50000995059612

Daubechies Compactly Supported Scaling Function N=10

n	*error(n)*	*ratio(n)*
1	0.850474423373509	
2	0.470087769733873	0.552735927876883
3	0.237009625567442	0.504181646119292
4	0.118523317397780	0.500078075369449
5	0.059374794339912	0.500954543314438
6	0.029701050151112	0.500229945749907
7	0.014852170512117	0.500055399947119
8	0.007426260556445	0.500011803013233
9	0.003713145152272	0.500002002959906
10	0.001856572644454	0.500000018362004

∎

From this example, we see that, in general, the ordinary subdivision scheme can attain only linear convergence, regardless of the smoothness order of the corresponding refinable function. Thus, it is desirable to accelerate the convergence rates. This will be achieved by choosing a suitable initial function ρ instead of ϕ_0.

2.3 Main Results

We use quasi-interpolation to investigate the convergence rates of subdivision schemes. First of all, we need two lemmas.

For a nonnegative integer k, we denote by Π_k the set of all polynomials of degree less than or equal to k. By $\delta_{0\mu}$ we denote the sequence on $\mathbb{Z}$ given by $\delta_{0\mu} = 0$ if $\mu \neq 0$ and $\delta_{0\mu} = 1$ if $\mu = 0$. Let $W_\infty^k(\mathbb{R})$, $1 \leq k \leq \infty$, denote the Sobolev space of all functions f on $\mathbb{R}$ such that $\|f\|_{k,\infty} < \infty$, where

$$\|f\|_{k,\infty} := \sum_{j=0}^{k} \|f^{(j)}\|_\infty.$$

For a nonnegative integer μ and a function f on $\mathbb{R}$, we use $D^\mu f$ to denote the μth order derivative of f. The **modulus of continuity** of a function f in $C(\mathbb{R})$ is defined by

$$\omega(f,h) := \sup_{|y|\leq h} \|f - f(\cdot - y)\|_\infty, \quad h > 0.$$

Lemma 2.2 *Suppose that ρ is a compactly supported continuous function on $\mathbb{R}$ satisfying*

$$D^\mu \hat{\rho}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall \mu = 0, 1, \ldots, k-1, \; k \geq 1.$$

Then

$$\sum_{j\in\mathbb{Z}} p(j)\rho(\cdot - j) = p \quad \forall p \in \Pi_{k-1}.$$

If $f \in C(\mathbb{R})$, then

$$\sup_{x\in\mathbb{R}} \left| f(x) - \sum_{j\in\mathbb{Z}} f(hj)\rho(x/h - j) \right| \leq C\omega(f, h),$$

where C is a positive constant. Moreover, for every function $f \in W^k_\infty(\mathbb{R})$,

$$\Big\| f - \sum_{j\in\mathbb{Z}} f(hj)\rho(\cdot/h - j)\Big\|_\infty \le Ch^k \|f^{(k)}\|_\infty.$$

The first conclusion of this lemma was proved by Schoenberg in [47]. The second conclusion can be proved like Theorem 2.1 by de Boor and Fix in [14].

Lemma 2.3 *If $\phi \in W^k_\infty(\mathbb{R})$ is refinable, then*

$$D^\mu \hat{\phi}(2\pi\beta) = 0, \quad \textit{for all } \beta \in \mathbb{Z}\backslash\{0\},\ 0 \le \mu \le k.$$

See ([1], p. 158) for the proof.

The following result gives an estimate for the convergence rate of the subdivision scheme in terms of the modulus of continuity of the corresponding refinable function.

Theorem 2.4 *If the shifts of ϕ are stable and ϕ is continuous, then*

$$\|\phi_n - \phi\|_\infty \le C\omega(\phi,\ 1/2^n).$$

Proof: Since $\hat{\phi}(0) = 1$, then by Lemma 2.2 and Lemma 2.3, we have

$$\Big\| \phi - \sum_{j\in\mathbb{Z}} \phi(j/2^n)\phi(2^n\cdot - j)\Big\|_\infty \le C\omega(\phi,\ 1/2^n).$$

But

$$\phi = \sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n\cdot - j).$$

It follows that

$$\Big\|\sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n\cdot-j)-\sum_{j\in\mathbb{Z}}\phi(j/2^n)\phi(2^n\cdot-j)\Big\|_\infty\le C\omega(\phi,\ 1/2^n).$$

Using the stability of the shifts of ϕ, we obtain

$$\Big|a_n(j)-\phi(j/2^n)\Big|\le C\omega(\phi,\ 1/2^n),\quad \forall j\in\mathbb{Z}.$$

Since

$$\phi_n=\sum_{j\in\mathbb{Z}} a_n(j)\phi_0(2^n\cdot-j),$$

we have

$$\begin{aligned}
&\|\phi_n-\phi\|_\infty=\Big\|\sum_{j\in\mathbb{Z}} a_n(j)\phi_0(2^n\cdot-j)-\phi\Big\|_\infty\\
&\le\Big\|\sum_{j\in\mathbb{Z}}(a_n(j)-\phi(j/2^n))\phi_0(2^n\cdot-j)\Big\|_\infty+\Big\|\phi-\sum_{j\in\mathbb{Z}}\phi(j/2^n)\phi_0(2^n\cdot-j)\Big\|_\infty\\
&\le \sup_{j\in\mathbb{Z}}\Big|a_n(j)-\phi(j/2^n)\Big|+C\omega(\phi,\ 1/2^n)\\
&\le 2C\omega(\phi,\ 1/2^n).
\end{aligned}$$

This completes the proof of this theorem. ∎

Corollary 2.5 *If $\phi\in W^1_\infty(\mathbb{R})$ has stable shifts, then*

$$\|\phi_n-\phi\|_\infty\le C(\frac{1}{2^n}).$$

For a compactly supported distribution ϕ on $\mathbb{R}$ and a sequence $b\in\ell(\mathbb{Z})$, the semi-convolution of ϕ with b is defined by

$$\phi *' b := \sum_{\alpha \in \mathbb{Z}} \phi(\cdot - \alpha) b(\alpha).$$

We use quasi-interpolation to investigate the convergence rates if the refinable functions have higher order of smoothness.

Theorem 2.6 *Let ϕ be the normalized solution of the refinement equation with mask a. Suppose that $\phi \in W^k_\infty(\mathbb{R})$ and the shifts of ϕ are stable. Let ρ be a compactly supported continuous function satisfying the following conditions:*

$$D^\mu \hat{\rho}(2\beta\pi) = D^\mu \hat{\phi}(2\beta\pi), \ \beta \in \mathbb{Z}, \ \mu = 0, 1, 2, \ldots, k-1.$$

Define

$$\phi_n = \sum_{j \in \mathbb{Z}} S^n_a \delta(j) \rho(2^n \cdot - j).$$

Then

$$\|\phi - \phi_n\|_\infty \leq C(\frac{1}{2^n})^k.$$

Proof: Since $\phi \in W^k_\infty(\mathbb{R})$, by Lemma 2.3 we have

$$D^\alpha \hat{\phi}(2\beta\pi) = 0, \quad \alpha = 0, 1, \ldots, k-1, \ \beta \in \mathbb{Z}\backslash\{0\}.$$

Since ϕ is compactly supported, we can find a finitely supported sequence b such that $\psi := \phi *' b$ satisfies

$$D^{\mu}\hat{\psi}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}, \quad \mu = 0, 1, \ldots, k-1, \ \forall \beta \in \mathbb{Z}.$$

By Lemma 2.2, we have

$$\Big\| \phi - \sum_{j \in \mathbb{Z}} \phi(j/2^n)\psi(2^n \cdot -j) \Big\|_{\infty} \leq C(\frac{1}{2^n})^k. \tag{2.3.1}$$

We define $\tau = \rho *' b$. Since

$$D^{\mu}\hat{\rho}(2\beta\pi) = D^{\mu}\hat{\phi}(2\beta\pi), \ \forall \beta \in \mathbb{Z}, \ \mu = 0, 1, \ldots, k-1,$$

τ satisfies the following conditions

$$(D^{\mu}\hat{\tau})(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}, \quad \mu = 0, 1, \ldots, k-1, \ \forall \beta \in \mathbb{Z}.$$

Furthermore, the following estimate is also valid:

$$\Big\| \phi - \sum_{j \in \mathbb{Z}} \phi(j/2^n)\tau(2^n \cdot -j) \Big\|_{\infty} \leq C(\frac{1}{2^n})^k. \tag{2.3.2}$$

It follows from $\psi = \phi *' b$ that

$$\psi(2^n x - j) = \sum_{\ell=0}^{k-1} b(\ell)\phi(2^n x - \ell - j).$$

Hence,

$$\begin{aligned}
&\phi(x) - \sum_{j\in\mathbb{Z}} \phi(j/2^n)\psi(2^n x - j) \\
&= \sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n x - j) - \sum_{j\in\mathbb{Z}} \phi(j/2^n) \sum_{\ell=0}^{k-1} b(\ell)\phi(2^n x - j - \ell) \\
&= \sum_{j\in\mathbb{Z}} (a_n(j) - c_n(j))\phi(2^n x - j),
\end{aligned}$$

where

$$c_n(j) := \sum_{\ell=0}^{k-1} b(\ell)\phi(\frac{j-\ell}{2^n}), \quad j \in \mathbb{Z}.$$

Therefore, by the stability of the shifts of ϕ, we obtain

$$\|a_n - c_n\|_\infty \le C(\frac{1}{2^n})^k. \tag{2.3.3}$$

Since ρ is a continuous function with compact support, (2.3.3) implies

$$\Big\|\phi_n - \sum_{j\in\mathbb{Z}} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty = \Big\|\sum_{j\in\mathbb{Z}} [a_n(j) - c_n(j)]\rho(2^n\cdot - j)\Big\|_\infty \le C(\frac{1}{2^n})^k. \tag{2.3.4}$$

In light of (2.3.2) and (2.3.4), we have

$$\|\phi_n - \phi\|_\infty \le \Big\|\phi - \sum_{j\in\mathbb{Z}} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty + \Big\|\phi_n - \sum_{j\in\mathbb{Z}} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty \le C(\frac{1}{2^n})^k.$$

This completes the proof. ∎

If we choose ρ to be the hat function, then we have the following corollary:

Corollary 2.7 *Suppose $\phi \in W^2_\infty(\mathbb{R})$ is refinable and has stable shifts. If $\hat{\phi}(0) = 1$, and $\hat{\phi}'(0) = 0$, then*

$$\|\phi - \phi_n\|_\infty \leq C(\frac{1}{2^n})^2.$$

In particular, the above estimate is valid if ϕ is symmetric.

Proof: Form Lemma 2.3, we have

$$\hat{\phi}(2k\pi) = \hat{\phi}_0(2k\pi) = 0, \quad \hat{\phi}'(2k\pi) = \hat{\phi}'_0(2k\pi) = 0,$$

for $k \in \mathbb{Z}\backslash\{0\}$. Since $\hat{\phi}_0(0) = 1$, $\hat{\phi}'_0(0) = 0$, it follows from Theorem 2.6 that

$$\|\phi - \phi_n\|_\infty \leq C(\frac{1}{2^n})^2.$$

∎

2.4 Applications

Let us consider some more examples.

Example 2.8 *The Coiflet scaling function ϕ with $K = 4$ (see p. 261 of [10]).*

The mask of this function is:

n	a(n)	n	a(n)
-8	0.0006309610	4	0.0278136400
-7	-0.0011522249	5	0.0177358370
-6	-0.0051945240	6	-0.0107563190
-5	0.0113624590	7	-0.0040010129
-4	0.0188672350	8	0.0026526659
-3	-0.0574642340	9	0.0008955945
-2	-0.0396526490	10	-0.0004165006
-1	0.2936673900	11	-0.0001838298
0	0.5531264500	12	0.0000440804
1	0.3071573300	13	0.0000220829
2	-0.0471127390	14	-0.0000023049
3	-0.0680381270	15	-0.0000012622

This function is symmetric and lies in $C^2(\mathbb{R})$. Hence, by Corollary 2.7, the convergence rate of the subdivision scheme should attain quadratic order. When using the subdivision scheme to generate this function ϕ, we see from the following table of data that the convergence rate of the subdivision scheme of this function does indeed attain quadratic order.

Coiflet Scaling Functions $K = 4$

n	error(n)	ratio(n)
1	0.10596367136144	

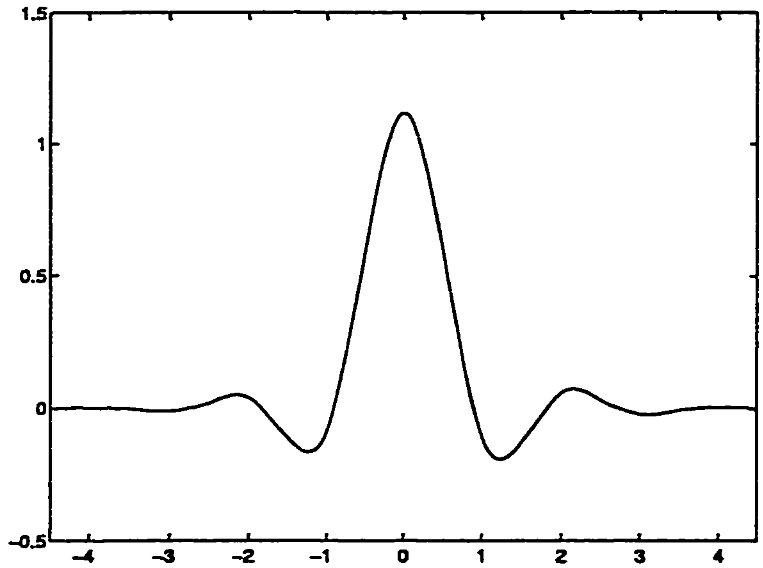

Figure 2.1: Coiflet Scaling Function $K = 4$ Generated by Subdivision Scheme

2	*0.037303359970 08*	*0.352039142196 55*
3	*0.01101803240551*	*0.29536300253771*
4	*0.00297898575879*	*0.27037366102712*
5	*0.00077602528342*	*0.26049982989386*
6	*0.00019817023902*	*0.25536569910070*
7	*0.00005010944722*	*0.25286060849215*
8	*0.00001260457856*	*0.25154096198294*
9	*0.00000316289857*	*0.25093251239353*
10	*0.00000079661251*	*0.25186154279125*

The plot of this scaling function generated by the ordinary scheme is shown in Figure 2.1.

∎

In the discussion of the previous section, in order to accelerate the rates of convergence of subdivision schemes, we need to calculate $D^\mu\hat{\phi}(0)$ where ϕ is the corresponding refinable function. Generally, ϕ has no closed form. The following lemma gives a formula to compute $D^\mu\hat{\phi}(0)$ in terms of its mask where $\mu = 1, 2$. In fact, we can get $D^\mu\hat{\phi}(0)$ for any μ in this way.

Lemma 2.9 *Suppose ϕ is a refinable function with mask a. If $\hat{\phi}(0) = 1$ and $\sum_{j\in\mathbb{Z}} a(j) = 2$, then*

$$\hat{\phi}'(0) = -\frac{i}{2}\sum_{j\in\mathbb{Z}} ja(j),$$

$$\hat{\phi}''(0) = -\frac{1}{6}\Big[\sum_{j\in\mathbb{Z}} j^2a(j) + \big(\sum_{j\in\mathbb{Z}} ja(j)\big)^2\Big].$$

Proof: Since $\phi = \sum_{j\in\mathbb{Z}} a(j)\phi(2\cdot - j)$, we take Fourier transforms of both sides,

$$\hat{\phi}(\xi) = \frac{1}{2}\sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi/2}\hat{\phi}(\xi/2).$$

Then, taking the first and second derivatives on both sides of the above identity, we have

$$\hat{\phi}'(\xi) = -\frac{i}{4}\sum_{j\in\mathbb{Z}} ja(j)e^{-ij\xi/2}\hat{\phi}(\xi/2) + \frac{1}{4}\hat{\phi}'(\xi/2)\sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi/2},$$

$$\hat{\phi}''(\xi) = -\frac{1}{8}\sum_{j\in\mathbb{Z}} j^2a(j)e^{-ij\xi/2}\hat{\phi}(\xi/2) - \frac{i}{4}\hat{\phi}'(\xi/2)\sum_{j\in\mathbb{Z}} ja(j)e^{-ij\xi/2} + \frac{1}{8}\hat{\phi}''(\xi/2)\sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi/2}.$$

Letting $\xi = 0$, we obtain

$$\hat{\phi}'(0) = -\frac{i}{2}\sum_{j\in\mathbb{Z}} ja(j),$$

and

$$\hat{\phi}''(0) = -\frac{1}{6}\Big[\sum_{j\in\mathbb{Z}} j^2 a(j) + \big(\sum_{j\in\mathbb{Z}} j a(j)\big)^2\Big].$$

This completes the proof. ∎

Using Lemma 2.9, we can construct an explicit scheme which attains quadratic convergence order.

Theorem 2.10 *Suppose that $\phi \in W^2_\infty(\mathbb{R})$ has stable shifts. For each $n = 1, 2, \ldots$, let c_n be the sequence on $\mathbb{Z}$ given by*

$$c_n(j) = a_n(j) + i\hat{\phi}'(0)(a_n(j) - a_n(j-1)) \quad j \in \mathbb{Z},$$

where

$$\hat{\phi}'(0) = -\frac{i}{2}\sum_{j\in\mathbb{Z}} j a(j).$$

Let ϕ_n be the function given by

$$\phi_n = \sum_{j\in\mathbb{Z}} c_n(j)\phi_0(2^n \cdot - j).$$

Then

$$\|\phi - \phi_n\|_\infty \le C\left(\frac{1}{2^n}\right)^2.$$

Proof: Let $\rho := (1 - i\hat{\phi}'(0))\phi_0 + i\hat{\phi}'(0)\phi_0(\cdot - 1)$. Then $\hat{\rho}(0) = 1$ and $\hat{\rho}'(0) = \hat{\phi}'(0)$. By Theorem 2.6 we have

$$\left\|\phi - \sum_{j\in\mathbb{Z}} a_n(j)\rho(2^n \cdot -j)\right\|_\infty \leq C(\frac{1}{2^n})^2.$$

Hence

$$\left\|\phi - \sum_{j\in\mathbb{Z}} c_n(j)\phi_0(2^n \cdot -j)\right\|_\infty \leq C(\frac{1}{2^n})^2,$$

where

$$c_n(j) = (1 - i\hat{\phi}'(0))a_n(j) + i\hat{\phi}'(0)a_n(j-1).$$

This completes the proof. ∎

Since the Daubechies scaling function ϕ_6 lies in $C^2(\mathbb{R})$, we use the modified scheme of Theorem 2.10 to generate this function. The following data table shows that the convergence rate of the modified subdivision scheme attains quadratic order. The plot of the scaling function ϕ_6 generated by the modified scheme in [0, 6] is shown in Figure 2.2.

Daubechies Compactly Supported Scaling Function $N = 6$ by Using the Modified Scheme

n	error(n)	ratio(n)
1	1.28332076464445	
2	0.39421909995542	0.30718672277125
3	0.11275797464216	0.28602869484232
4	0.02933304238854	0.26014162174892

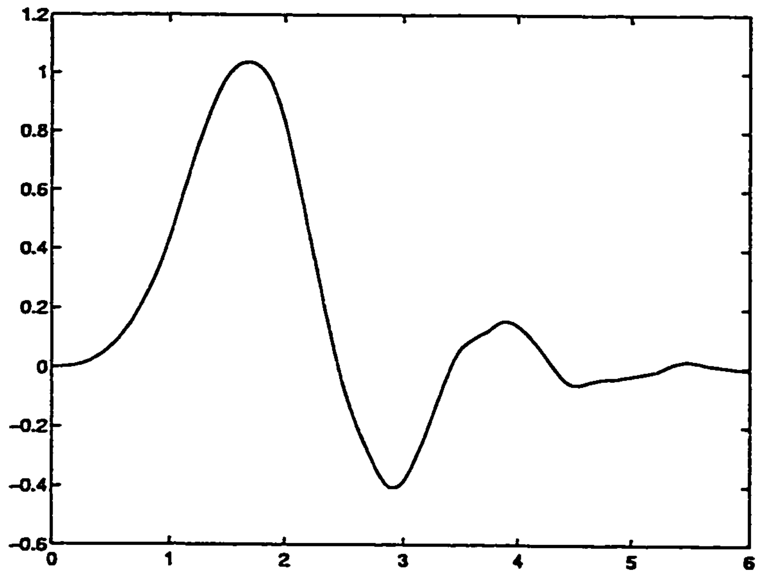

Figure 2.2: **Daubechies Compactly Supported Scaling Function $N = 6$ Generated by Modified Scheme**

5	0.00855539153727	0.29166396802452
6	0.00236111038657	0.27597923207597
7	0.00061593351080	0.26086603756442
8	0.00015900313025	0.25814982862199
9	0.00004040363165	0.25410588823943
10	0.00001034255359	0.25598079100019
11	0.00000264592951	0.25582942257449
12	0.00000068137812	0.25751937736998

On the other hand, since Daubechies scaling function ϕ_{10} has a smoothness order of three, from Theorem 2.6, we can choose a proper function

$$\rho = M_3(\cdot)b(0) + M_3(\cdot - 1)b(1) + M_3(\cdot - 2)b(2),$$

where M_3 is the cubic B-spline and

$$b(0) = -\tfrac{1}{6}(\sum_j j^2 a(j) + (\sum_j ja(j))^2) + \frac{5}{2}\sum_j ja(j) - 5$$
$$b(1) = -\tfrac{1}{12}(\sum_j j^2 a(j) + (\sum_j ja(j))^2) - \sum_j ja(j) + \frac{7}{4}$$
$$b(2) = -\tfrac{1}{12}(\sum_j j^2 a(j) + (\sum_j ja(j))^2) - \frac{3}{2}\sum_j ja(j) + \frac{17}{4}$$

so that

$$\hat{\rho}(0) = \hat{\phi}_{10}(0) = 1, \quad D\hat{\rho}(0) = D\hat{\phi}_{10}(0), \quad D^2\hat{\rho}(0) = D^2\hat{\phi}_{10}(0).$$

Here the B-splines are defined as:

$$M_1(x) = \begin{cases} 1 & \text{for } x \in [0,1], \\ 0 & \text{for } x \in I\!R \backslash [0,1], \end{cases}$$
$$M_n(x) = \int_0^1 M_{n-1}(x-t)dt, \quad \text{for } n \geq 2.$$

The following table of data shows us that the convergence rate attains cubic order when $\sum_{j\in\mathbb{Z}} S_a^n \delta(j)\rho(\cdot - j)$ is used to approximate ϕ_{10}.

Daubechies Compactly Supported Scaling Function $N = 10$ By Using The Modified Scheme

n	error(n)	ratio(n)
1	0.06715925929284	
2	0.01069205551909	0.15920448843053
3	0.00159795221905	0.14945229345255
4	0.00021711224919	0.13586904952582

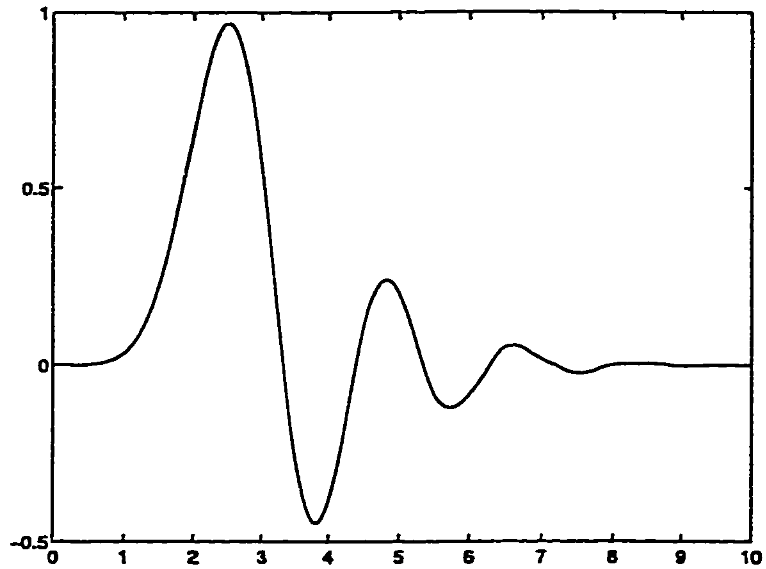

Figure 2.3: **Daubechies Compactly Supported Scaling Function** $N = 10$ **Generated by the Modified Scheme**

5	0.00002831518418	0.13041725782694
6	0.00000362257127	0.12793740796356
7	0.00000046482767	0.12831429262674
8	0.00000005832976	0.12548684978242
9	0.00000000736893	0.12633225303859
10	0.00000000093224	0.12650954751911

In Figure 2.3, the plot of the scaling function of ϕ_{10} generated by the modified scheme is displayed in [0, 10].

Figure 2.4 and Figure 2.5 are the error graphs, where err1(x) and err2(x) denote the errors of the subdivision schemes and the modified schemes respectively. From these two figures, the reader can see the great improvement obtained by using the modified scheme.

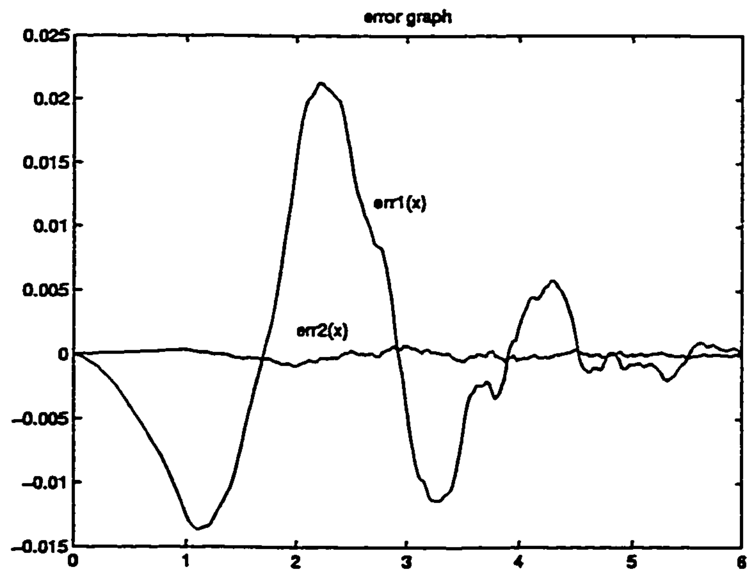

Figure 2.4: **Error Figure of Daubechies Wavelet Scaling Function** $N = 6$

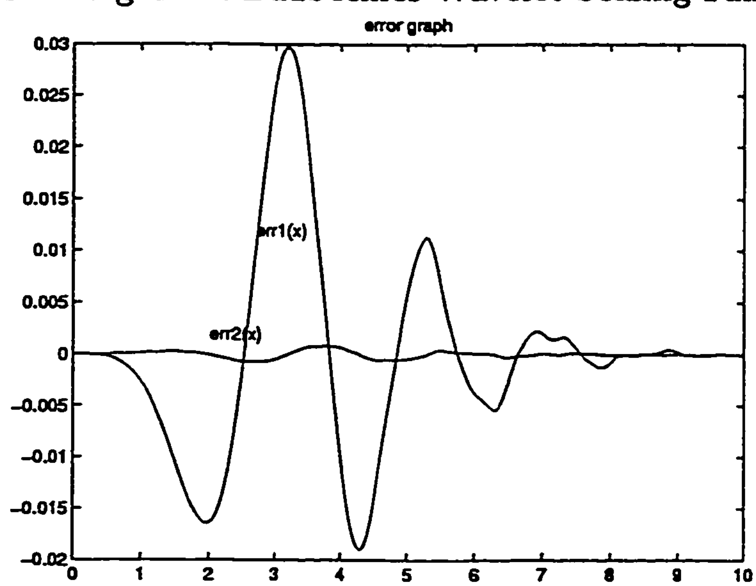

Figure 2.5: **Error Figure of Daubechies Wavelet Scaling Function** $N = 10$

2.5 Extension to the Non-stable Case

In the previous sections, we considered the convergence rates of subdivision schemes for refinable functions with stable shifts. In this section, we extend our results to the non-stable case.

Our starting point is the following lemma, which was established in §5 of [35].

Lemma 2.11 *Suppose a is a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$. Let ϕ be a nontrivial distribution such that*

$$\phi = \sum_{j\in\mathbb{Z}} a(j)\phi(2\cdot -j).$$

Then there exists a finitely supported sequence b with $\sum_{j\in\mathbb{Z}} b(j) = 2$ such that any nontrivial solution ψ of the refinable equation

$$\psi = \sum_{j\in\mathbb{Z}} b(j)\psi(2\cdot -j)$$

has linearly independent shifts and

$$\phi = \sum_{k\in\mathbb{Z}} c(k)\psi(\cdot - k)$$

for some finitely supported sequence c. Consequently, ϕ and ψ have the same smoothness.

Using this lemma, we shall prove the following theorem.

Theorem 2.12 *Suppose $\phi \in W^k_\infty(\mathbb{R})$ satisfies the refinement equation with mask a and $\hat{\phi}(0) = 1$. Let $\phi = c *' \psi$ where c is a finitely supported sequence and ψ is a refinable function with linearly independent shifts. Let ρ be a compactly supported continuous function satisfying the following conditions:*

$$D^\mu \hat{\rho}(2\pi\beta) = D^\mu \hat{\psi}(2\pi\beta),\ \beta \in \mathbb{Z},\ \mu = 0, 1, \cdots, k-1.$$

Then

$$\Big\| \phi - \sum_{j \in \mathbb{Z}} S_b^n c(j) \rho(2^n \cdot - j) \Big\|_\infty \le C(\frac{1}{2^n})^k.$$

Proof: From Lemma 2.11, we see that $\psi \in W^k_\infty$ has stable shifts. By Theorem 2.6, we have

$$\Big\| \sum_{j \in \mathbb{Z}} S_b^n \delta(j) \rho(2^n \cdot - j) - \psi \Big\|_\infty \le C(\frac{1}{2^n})^k.$$

From ([1] p.21) we have

$$S_b^n c(k) = \sum_{j \in \mathbb{Z}} c(j) S_b^n \delta(k - 2^n j).$$

Since c is a finitely supported sequence, we have

$$\begin{aligned}
&\Big\| \phi - \sum_{j \in \mathbb{Z}} S_b^n c(j) \rho(2^n \cdot - j) \Big\|_\infty \\
&= \Big\| \sum_{j \in \mathbb{Z}} c(j) \psi(\cdot - j) - \sum_{j \in \mathbb{Z}} \rho(2^n \cdot - j) \sum_{\ell \in \mathbb{Z}} c(\ell) S_b^n \delta(j - 2^n \ell) \Big\|_\infty \\
&= \Big\| \sum_{j \in \mathbb{Z}} c(j) \big(\psi(\cdot - j) - \sum_{\ell \in \mathbb{Z}} S_b^n \delta(\ell) \rho(2^n(\cdot - j) - \ell) \big) \Big\|_\infty \le C(\frac{1}{2^n})^k.
\end{aligned}$$

The proof of Theorem 2.12 is complete. ∎

Example 2.13 *Let a be the sequence given by its symbol*

$$\tilde{a}(z) = 2^{1-k}(1+z)^k(z^6 - z^3 + 1),$$

where k is a positive integer. Let ϕ be the normalized solution to the refinement equation with mask a. It has been shown in ([2, p.163]) that the subdivision scheme associated with a converges uniformly if and only if $k \geq 2$. Moreover, the shifts of ϕ are not stable. Let $\tilde{c}(z) = (z^6 + z^3 + 1)/3$, then

$$\tilde{b}(z) := \tilde{a}(z)\frac{\tilde{c}(z)}{\tilde{c}(z^2)} = 2^{1-k}(1+z)^k.$$

The normalized solution ψ to the refinement equation with mask b is a B-spline of order k. Clearly, the shifts of ψ are linearly independent. Moreover,

$$\phi = \psi *' c = (\psi(\cdot) + \psi(\cdot - 3) + \psi(\cdot - 6))/3.$$

Let us consider $k = 4$. In this case, $\tilde{b}(z) = \frac{(1+z)^4}{2^3}$, $\tilde{c}(z) = (z^6 + z^3 + 1)/3$. Hence ψ is the B-spline of order 4 and both ψ and $\phi \in W^3_\infty(\mathbb{R})$. In the computation, we choose ρ to be

$$\rho(x) = d(0)M_3(x) + d(1)M_3(x-1) + d(2)M_3(x-2),$$

where $M_3(x)$ is the B-spline of order 3, such that

$$\hat{\rho}(0) = \hat{M_3}(0), \quad \hat{\rho}'(0) = \hat{M_3}'(0), \quad \hat{\rho}''(0) = \hat{M_3}''(0).$$

It follows that

$$\rho(x) = \frac{5}{12}M_3(x) + \frac{2}{3}M_3(x-1) - \frac{1}{12}M_3(x-2).$$

Applying Theorem 2.12 to ϕ, we get the following tables of data which conform to Theorem 2.12.

Using the Ordinary Scheme for $k = 4$

n	error(n)	ratio(n)
1	0.57812500000000	
2	0.53125000000000	0.91891891891892
3	0.09008789062500	0.16957720588235
4	0.03875732421875	0.43021680216802
5	0.01806640625000	0.46614173228346
6	0.00871944427490	0.48263302364852
7	0.00434750318527	0.49859865470840
8	0.00217117369175	0.49940703875877
9	0.00108507182449	0.49976279125574
10	0.00054253695998	0.50000096559046

Using the Modified Scheme for $k = 4$

n	error(n)	ratio(n)
1	0.03645833333334	
2	0.00455729166667	0.12500000000007

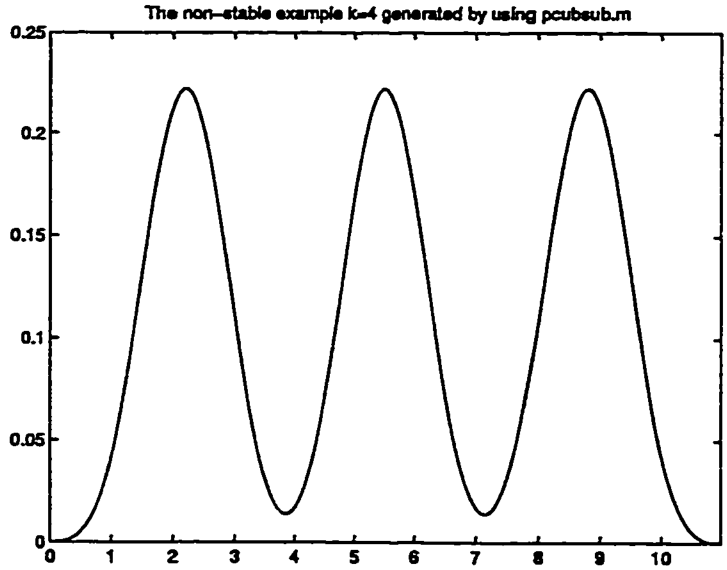

Figure 2.6: Example When $k = 4$ Generated by Modified Scheme

3	*0.00056966145833*	*0.12499999999918*
4	*0.00007120768229*	*0.12500000005266*
5	*0.00000890096029*	*0.12500000042130*
6	*0.000001112620004*	*0.12499999550610*
7	*0.000000013907750*	*0.12500001797559*
8	*0.00000001738469*	*0.12500001797559*
9	*0.00000000217309*	*0.12500021570704*
10	*0.00000000027164*	*0.12500172565333*

The plot of this function generated by the scheme of Theorem 2.12 is shown in Figure 2.6.

∎

CHAPTER 3

On the Convergence Rates of Multivariate Subdivision Schemes

3.1 Introduction

In this chapter, we will investigate the convergence rates of multivariate subdivision schemes.

First we introduce some notation. A multi-index is an s-tuple $\mu = (\mu_1, \cdots, \mu_s)$ with its components being nonnegative integers. We use $\mathbb{Z}_+^s$ to denote the set of multi-indices. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For two multi-indices $\mu = (\mu_1, \cdots, \mu_s)$ and $\nu = (\nu_1, \cdots, \nu_s)$, we write $\nu \leq \mu$ if $\nu_j \leq \mu_j$ for $j = 1, \cdots, s$. Let D_j denote the partial derivative with respect to the jth coordinate. For $\mu = (\mu_1, \cdots, \mu_s)$, D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}dx \quad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$.

Consider the functional equation of the form

$$f = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) f(2 \cdot - \alpha) \tag{3.1.1}$$

where f is an unknown function defined on s-dimensional Euclidean space $\mathbb{R}^s$ and a is a finitely supported sequence on $\mathbb{Z}^s$. The equation is called a **refinement equation**, and the sequence a is called the **refinement mask.** Any function which is a solution to the refinement equation is called a **refinement function.**

In ([1], Chap. 5), it was proved that, if $\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = 2^s$, then there exists a unique compactly supported distribution f satisfying the refinement equation subject to the condition $\hat{f}(0) = 1$.

For $1 \leq k \leq \infty$, let $W^k_\infty(\mathbb{R}^s)$ denote the Sobolev space of all functions f on $\mathbb{R}^s$ such that $\|f\|_{k,\infty} \leq \infty$, where

$$\|f\|_{k,\infty} = \max_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_\infty.$$

In order to solve the refinement equation, we start with a compactly supported function $\phi \in C(\mathbb{R}^s)$ and use the iteration scheme $f_n := Q^n_a \phi$, $n = 0, 1, 2, \cdots$, where

Q_a is the bounded linear operator on $L_\infty(\mathbb{R}^s)$ given by

$$Q_a\phi := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(2\cdot-\alpha), \quad \phi\in C(\mathbb{R}^s). \tag{3.1.2}$$

This iteration scheme is called a **subdivision scheme** (see [1]). Note that in [11] and [12] a subdivision scheme is referred to as a cascade algorithm.

Let ϕ_0 be the function given by

$$\phi_0 := \prod_{j=1}^{s}\chi(x_j), \text{ for } x=(x_1,\cdots,x_s)\in\mathbb{R}^s \tag{3.1.3}$$

where

$$\chi(t)=\begin{cases} 1+t & \text{for } t\in[-1,0) \\ 1-t & \text{for } t\in[0,1] \\ 0 & \text{for } t\in\mathbb{R}\backslash[-1,1]. \end{cases}$$

We say that the subdivision scheme associated with mask a **converges uniformly**, if there is a function $f\in C(\mathbb{R}^s)$ such that $\lim_{n\to\infty}\|Q_a^n\phi_0-f\|_\infty=0$, where ϕ_0 is the function given by (3.1.3).

Let $\ell(\mathbb{Z}^s)$ denote the linear space of all sequences on $\mathbb{Z}^s$, and let $\ell_0(\mathbb{Z}^s)$ denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. By $\ell_\infty(\mathbb{Z}^s)$ we denote the Banach space of all sequences a on $\mathbb{Z}^s$ such that $\|a\|_\infty<\infty$, where $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}^s$.

For a given sequence $a\in\ell_\infty(\mathbb{Z}^s)$, let S_a be the linear operator given by

$$S_a\lambda(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha-2\beta)\lambda(\beta), \quad \alpha\in\mathbb{Z}^s, \tag{3.1.4}$$

where $\lambda \in \ell_\infty(\mathbb{Z}^s)$. The operator S_a is called the **subdivision operator** associated with a (see [1]). By δ we denote the sequence on $\mathbb{Z}^s$ given by $\delta(\alpha) = 1$ for $\alpha = 0$ and $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \backslash \{0\}$. Then for $\phi \in C(\mathbb{R}^s)$ we have

$$Q_a \phi = \sum_{\alpha \in \mathbb{Z}^s} S_a \delta(\alpha) \phi(2 \cdot - \alpha). \tag{3.1.5}$$

By induction on n, it is easily seen that

$$Q_a^n \phi = \sum_{\alpha \in \mathbb{Z}^s} S_a^n \delta(\alpha) \phi(2^n \cdot - \alpha). \tag{3.1.6}$$

In our study of convergence rates, the concept of stability plays an important role. The shifts of a function ϕ in $L_\infty(\mathbb{R}^s)$ are said to be stable if there are two positive constants C_1 and C_2 such that

$$C_1 \|\lambda\|_\infty \le \| \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) \phi(\cdot - \alpha) \|_\infty \le C_2 \|\lambda\|_\infty, \ \forall \lambda \in \ell_0(\mathbb{Z}^s). \tag{3.1.7}$$

It was proved by Jia and Micchelli in [38] that a compactly supported function $\phi \in C(\mathbb{R}^s)$ satisfies the L_∞-stability condition if and only if for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\pi\beta) \neq 0$, i.e., $\sum_{\beta \in \mathbb{Z}^s} |\hat{\phi}(\xi + 2\pi\beta)|^2 \neq 0$.

It was shown in [1] that if the solution to the refinement equation associated with mask a is continuous and has stable shifts, then the subdivision scheme associated with mask a converges uniformly.

In this chapter, we will give some results about the convergence rates of the multivariate subdivision schemes. There are several papers which deal with the convergence rates of subdivision schemes [6], [43], [16]. But all focused on box splines, a kind of special refinable function. In [6], Dahmen, Dyn, and Levin proved that, if a box spline is twice continuous differentiable and has stable shifts, the corresponding subdivision scheme converges to this box spline surface with quadratic order. In [16], a similar result was derived using a different approach. Here we will discuss general refinable functions.

For a compactly supported distribution ϕ on $\mathbb{R}^s$ and a sequence $b \in \ell(\mathbb{Z}^s)$, the semi-convolution of ϕ with b is defined by

$$\phi *' b := \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) b(\alpha).$$

For a nonnegative integer k, we denote by Π_k the linear span of all polynomials of s variables of degree at most k.

The **modulus of continuity** of a function f in $C(\mathbb{R}^s)$ is defined by

$$\omega(f, h) := \sup_{|y| \leq h} \|f - f(\cdot - y)\|_\infty, \quad h > 0.$$

3.2 Main Results

We investigate the convergence rates of subdivision schemes by using quasi interpolation. By $\delta_{0\mu}$ we denote the sequence on $\mathbb{Z}^s$ given by $\delta_{0\mu} = 0$ if $\mu \neq 0$ and $\delta_{0\mu} = 1$ if $\mu = 0$. To obtain our main results, the following two lemmas are needed.

Lemma 3.1 *Suppose ρ is a compactly supported continuous function on $\mathbb{R}^s$. If ρ satisfies the Strang-Fix conditions,*

$$D^\mu \hat{\rho}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad |\mu| \le k-1, \ k \ge 1,$$

where $\mu = (\mu_1, \cdots, \mu_s) \in \mathbb{Z}_+^s$, $\beta = (\beta_1, \cdots, \beta_s) \in \mathbb{Z}_+^s$, then

$$\sum_{j \in \mathbb{Z}^s} p(j)\rho(\cdot - j) = p \quad \forall p \in \Pi_{k-1}. \tag{3.2.1}$$

If $f \in C(\mathbb{R}^s)$, then

$$\sup_{x \in \mathbb{R}^s} \Big| f(x) - \sum_{j \in \mathbb{Z}^s} f(hj)\rho(x/h - j) \Big| \le C\omega(f, h). \tag{3.2.2}$$

Moreover, for every function $f \in W_\infty^k(\mathbb{R}^s)$,

$$\Big\| f - \sum_{j \in \mathbb{Z}^s} f(hj)\rho(\cdot/h - j) \Big\|_\infty \le Ch^k \|f\|_{k,\infty}. \tag{3.2.3}$$

This lemma was proved by de Boor and Jia in [17].

Lemma 3.2 *If $\phi \in W_1^k(\mathbb{R}^s)$ is refinable, then*

$$D^\mu \hat{\phi}(2\pi\beta) = 0, \quad \textit{for all } \beta \in \mathbb{Z}^s \backslash \{0\}, \ |\mu| \le k, \ \mu \in \mathbb{Z}_+^s.$$

See ([1], p. 158) for the proof.

The following result is about the convergence rate of the subdivision scheme for the continuous refinable functions with stable shifts. It describes the convergence rates in terms of the modulus of continuity.

Theorem 3.3 *Let ϕ be the normalized solution of the refinement equation with mask a. If ϕ is continuous and has stable shifts, then*

$$\|\phi_n - \phi\|_\infty \le C\omega(\phi,\ 1/2^n),$$

where $\phi_n = \sum_{j\in\mathbb{Z}^s} a_n(j)\phi_0(2^n\cdot - j)$.

Proof: Since $\hat{\phi}(0) = 1$, then by Lemma 3.1 and Lemma 3.2, we have

$$\Big\|\phi - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\phi(2^n\cdot - j)\Big\|_\infty \le C\omega(\phi,\ 1/2^n).$$

But

$$\phi = \sum_{j\in\mathbb{Z}^s} a_n(j)\phi(2^n\cdot - j).$$

It follows that

$$\Big\|\sum_{j\in\mathbb{Z}^s} a_n(j)\phi(2^n\cdot - j) - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\phi(2^n\cdot - j)\Big\|_\infty \le C\omega(\phi,\ 1/2^n).$$

Using the stability of the shifts of $\phi(x)$, we have

$$\sup_{j\in\mathbb{Z}^s} |a_n(j) - \phi(j/2^n)| \le C\omega(\phi,\ 1/2^n).$$

Moreover,

$$
\begin{aligned}
\|\phi_n - \phi\|_\infty &= \Big\| \sum_{j\in \mathbb{Z}^s} a_n(j)\phi_0(2^n\cdot - j) - \phi(\cdot)\Big\|_\infty \\
&\le \Big\| \sum_{j\in \mathbb{Z}^s} (a_n(j) - \phi(j/2^n))\phi_0(2^n\cdot - j)\Big\|_\infty + \Big\| \phi - \sum_{j\in \mathbb{Z}^s} \phi(j/2^n)\phi_0(2^n\cdot - j)\Big\|_\infty \\
&\le C \sup_{j\in \mathbb{Z}^s} |a_n(j) - \phi(j/2^n)| + C\omega(\phi,\ 1/2^n) \\
&\le 2C\omega(\phi,\ 1/2^n)
\end{aligned}
$$

This completes the proof of the theorem. ∎

Corollary 3.4 *If $\phi \in W^1_\infty(\mathbb{R}^s)$ has stable shifts, then*

$$\|\phi_n - \phi\|_\infty \le C(\frac{1}{2^n}).$$

The next result deals with the convergence rate of a subdivision scheme if the refinable function has higher smoothness order.

Theorem 3.5 *Let ϕ be the normalized solution of the refinement equation with mask a. Suppose $\phi \in W^k_\infty(\mathbb{R}^s)$ has stable shifts. Let ρ be a compactly supported continuous function satisfying the following conditions:*

$$D^\mu \hat{\phi}(2\pi\beta) = D^\mu \hat{\rho}(2\pi\beta), \quad \beta \in \mathbb{Z}^s,\ \mu \in \mathbb{Z}^s_+,\ |\mu| < k. \tag{3.2.4}$$

and let $\phi_n := \sum_{j\in\mathbb{Z}^s} S^n_a\delta(j)\rho(2^n\cdot - j)$. Then

$$\|\phi - \phi_n\|_\infty \le C(\frac{1}{2^n})^k. \tag{3.2.5}$$

Proof: Since $\phi \in W_\infty^k(\mathbb{R}^s)$, by Lemma 3.2 we have $D^\mu \hat{\phi}(2\pi\beta) = 0$, for all $|\mu| \leq k-1$, and all $\beta \in \mathbb{Z}^s \backslash \{0\}$. Since ϕ is compactly supported, we can find a finitely supported sequence b such that $\psi := \phi *' b$ satisfies

$$D^\mu \hat{\psi}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta}, \quad \beta \in \mathbb{Z}^s,\ \mu \in \mathbb{Z}_+^s,\ |\mu| < k.$$

By Lemma 3.1, we have

$$\Big\| \phi - \sum_{j \in \mathbb{Z}^s} \phi(j/2^n)\psi(2^n \cdot - j) \Big\|_\infty \leq C(\frac{1}{2^n})^k. \tag{3.2.6}$$

With the above b, we define $\tau := \rho *' b$. Then τ satisfies

$$D^\mu \hat{\tau}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}, \quad \mu \in \mathbb{Z}_+^s,\ |\mu| < k, \quad \beta \in \mathbb{Z}^s.$$

Furthermore, by Lemma 3.1, we have the following estimate

$$\Big\| \phi - \sum_{j \in \mathbb{Z}^s} \phi(j/2^n)\tau(2^n \cdot - j) \Big\|_\infty \leq C(\frac{1}{2^n})^k. \tag{3.2.7}$$

Since

$$\psi(x) = \sum_{\gamma \in \mathbb{Z}^s} b(\gamma)\phi(x - \gamma), \quad x \in \mathbb{R}^s,$$

and

$$\psi(2^n x - j) = \sum_{\gamma \in \mathbb{Z}^s} b(\gamma)\phi(2^n x - \gamma - j), \quad j \in \mathbb{Z}^s,$$

we have

$$\phi(x) - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\psi(2^n x - j)$$
$$= \sum_{j\in\mathbb{Z}^s} (a_n(j) - c_n(j))\phi(2^n x - j),$$

where $c_n(j) := \sum_{\gamma\in\mathbb{Z}^s} b(\gamma)\phi(j - \gamma/2^n)$. Hence, by the stability of the shifts of ϕ, we have

$$\|a_n - c_n\|_\infty \le C(\frac{1}{2^n})^k. \tag{3.2.8}$$

Since $\hat{\rho}(0) = 1$, we obtain

$$\|\phi_n - \phi\|_\infty \le \Big\|\phi(\cdot) - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty + \Big\|\phi_n - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty.$$

Considering (3.2.8), we have

$$\Big\|\phi_n - \sum_{j\in\mathbb{Z}^s} \phi(j/2^n)\tau(2^n\cdot - j)\Big\|_\infty = \Big\|\sum_{j\in\mathbb{Z}^s} [a_n(j) - c_n(j)]\rho(2^n\cdot - j)\Big\|_\infty \le C(\frac{1}{2^n})^k. \tag{3.2.9}$$

From (3.2.7) and (3.2.9), we can obtain (3.2.5). This completes the proof of this theorem. ∎

Remark: The $\operatorname{supp} b = \{\gamma \in \mathbb{Z}_+^s,\ |\gamma| < k\}$.

Corollary 3.6 *Suppose that* $\phi \in W^2_\infty(\mathbb{R}^s)$ *with stable shifts, and* $\hat{\phi}(0) = 1$, $D_j\hat{\phi}(0) = 0$, $j = 1, \cdots, s$. *In particular, if* ϕ *is symmetric, letting* $\rho = \phi_0$, *then*

$$\|\phi - \phi_n\|_\infty \leq C(\frac{1}{2^n})^2.$$

Proof: Since $\rho = \phi_0 = \prod_{j=1}^s \chi(x_j)$, for $x = (x_1, \cdots, x_s) \in \mathbb{R}^s$, we have

$$\hat{\rho}(\xi) = \prod_{j=1}^s (\frac{1 - e^{-i\xi_j}}{i\xi_j})^2.$$

This implies that $\hat{\rho}(0) = 1$ and

$$D_j\hat{\rho}(0) = 0, \quad j = 1, \cdots, s.$$

We obtain the desired conclusion by applying Theorem 3.5. ∎

3.3 Examples

Let us look at an example.

Example 3.7 *[30] Let a be the sequence on* $\mathbb{Z}^2$ *given by its symbol*

$$\tilde{a}(z) := (1 + z_1) + (1 + z_2) + (1 + z_1z_2)[1 + t + (1 - t)z_1 + (1 - t)z_2 + (1 - t)z_1z_2]/8,$$

where t is a real number. Let ϕ *be the normalized solution of the refinement equation with the mask a corresponding to the parameter t. If we let* $t = \frac{1}{16}$, *then the mask is* *the following matrix:*

$$\begin{pmatrix} 0 & \frac{15}{128} & \frac{1}{4} & \frac{17}{128} \\ \frac{15}{128} & \frac{1}{2} & \frac{81}{128} & \frac{1}{4} \\ \frac{1}{4} & \frac{81}{128} & \frac{1}{2} & \frac{15}{128} \\ \frac{17}{128} & \frac{1}{4} & \frac{15}{128} & 0 \end{pmatrix}.$$

From the result of [30], ϕ has stable shifts, and its L_2 smoothness is 2.5 *Hence the L_∞ smoothness is at least* 1.5. *Using Lemma 2.9, we have*

$$\hat{\phi}(0,0) = \frac{1}{4}\sum_{j_1,j_2} a(j_1,j_2) = 1, \quad D^{(1,0)}\hat{\phi}(0,0) = -\frac{i}{4}\sum_{j_1,j_2} j_1 a(j_1,j_2),$$

$$D^{(0,1)}\hat{\phi}(0,0) = -\frac{i}{4}\sum_{j_1,j_2} j_2 a(j_1,j_2),$$

and

$$\hat{\phi_0}(0) = 1, \quad \hat{\phi_0}'(0) = 0.$$

Now, applying Theorem 3.5 to this ϕ, we choose a function

$$\rho(x_1,x_2) = b(0,0)\phi_0(x_1)\phi_0(x_2) + b(1,0)\phi_0(x_1-1)\phi_0(x_2) + b(0,1)\phi_0(x_1)\phi_0(x_2-1),$$

where ϕ_0 is the hat function and

$$b(1,0) = \frac{1}{4}\sum_{j_1,j_2} j_1 a(j_1,j_2) = \frac{3}{2}, \quad b(0,1) = \frac{1}{4}\sum_{j_1,j_2} j_2 a(j_1,j_2) = \frac{3}{2},$$

$$b(0,0) = 1 - b(1,0) - b(0,1) = -2,$$

such that

$$\hat{\rho}(0,0) = \hat{\phi}(0,0) = 1, \quad D^{(1,0)}\hat{\rho}(0,0) = D^{(1,0)}\hat{\phi}(0,0), \quad D^{(0,1)}\hat{\rho}(0,0) = D^{(0,1)}\hat{\phi}(0,0).$$

$\sum_{j\in Z^2} S_a^n(j)\rho(2^n \cdot -j)$ *is used to approximate the function* ϕ. *The following tables of data demonstrates that the convergence rate is linear if we use the ordinary subdivision scheme, and is nearly quadratic if we choose the above* ρ.

Error Table of Example 3.7 by the Subdivision Scheme

n	error(n)	ratio(n)
1	0.441400000	
2	0.224660840	0.50897335749887
3	0.110506300	0.491880560937990
4	0.054768664	0.495615761273334
5	0.027255653	0.497650499563030
6	0.013605696	0.499188040000360
7	0.006795951	0.499493079957100

Error Table of Example 3.7 by the Modified Scheme

n	error(n)	ratio(n)
1	0.718700000	
2	0.198906437	0.276758643383890
3	0.053381021	0.268372516270050
4	0.014145521	0.264991578186560

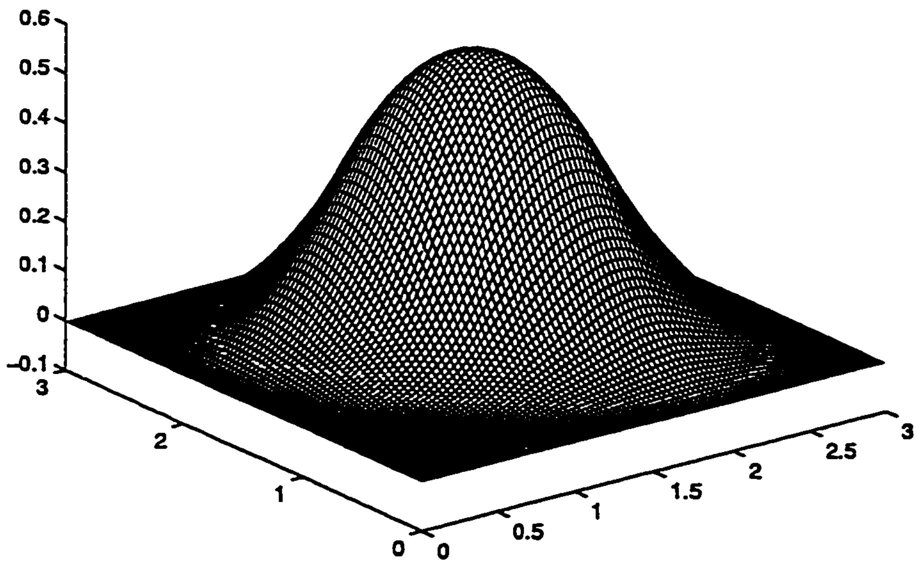

Figure 3.1: **Example When $t = 1/16$ Generated by Modified Scheme**

5	*0.003728967*	*0.263614680576280*
6	*0.000981053*	*0.263089751129447*
7	*0.000257953*	*0.262934826151080*

Plot for this function generated by the modified scheme is shown in Figure 3.1. ∎

CHAPTER 4

Spectral Properties of the Transition Operator Associated with a Multivariate Refinement Equation

4.1 Introduction

In this chapter, we will investigate the spectral properties of the transition operator associated with a multivariate refinement equation and their application to the study of the approximation and smoothness properties of the corresponding refinable function.

A **refinement equation** is a functional equation of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha) \tag{4.1.1}$$

where a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The matrix M is called **dilation matrix**, and the sequence a is called the **refinement mask**. Any function satisfying a refinement equation is called a **refinable function**.

If a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m := |\det M|, \tag{4.1.2}$$

then it is known that there exists a unique compactly supported distribution ϕ satisfying the refinement equation (4.1.1) subject to the condition $\hat{\phi}(0) = 1$. This distribution is said to be the **normalized solution** of the refinement equation (4.1.1). This fact was essentially proved by Cavaretta, Dahmen, and Micchelli in Chapter 5 of [1] for the case in which the dilation matrix is 2 times the $s \times s$ identity matrix I. The same proof applies to the general refinement equation (4.1.1). Throughout this chapter we assume that (4.1.2) is satisfied.

A multi-index is an s-tuple $\mu = (\mu_1, \ldots, \mu_s)$ with its components being nonnegative integers. Define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}, \quad x = (x_1, \cdots, x_s) \in \mathbb{R}^s.$$

We may regard $x_1^{\mu_1} \cdots x_s^{\mu_s}$ as a monomial of total degree $|\mu| := \mu_1 + \cdots + \mu_s$. For a nonnegative integer k, we denote by Π_k the set of all polynomials of degree at most k. A sequence u on $\mathbb{Z}^s$ is called a polynomial sequence, if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of u is the same as the degree of p. For a multi-index $\mu = (\mu_1, \cdots, \mu_s)$, if $p = \sum_\mu c_\mu x^\mu$ is a polynomial, then we use $p(D)$ to denote the differential operator $\sum_\mu c_\mu D^\mu$.

By $\ell(\mathbb{Z}^s)$ we denote the linear space of all sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. For $\alpha \in \mathbb{Z}^s$, we denote by δ_α the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{\alpha\}$. In particular, we write δ for δ_0. The difference operator ∇_j is defined by $\nabla_j u := u - u(\cdot - e_j)$, $u \in \ell(\mathbb{Z}^s)$, where e_j is the jth coordinate unit vector in $\mathbb{R}^s$. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$.

Let a be an element in $\ell_0(\mathbb{Z}^s)$. The **transition operator** T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta), \quad \alpha \in \mathbb{Z}^s, \tag{4.1.3}$$

where $v \in \ell_0(\mathbb{Z}^s)$. The **subdivision operator** S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) u(\beta), \quad \alpha \in \mathbb{Z}^s, \tag{4.1.4}$$

where $u \in \ell(\mathbb{Z}^s)$. We introduce a bilinear form on the pair of the linear spaces

$\ell_0(\mathbb{Z}^s)$ and $\ell(\mathbb{Z}^s)$ as follows:

$$\langle u, v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha)v(\alpha), \quad u \in \ell(\mathbb{Z}^s),\ v \in \ell_0(\mathbb{Z}^s). \tag{4.1.5}$$

Then $\ell(\mathbb{Z}^s)$ is the dual space of $\ell_0(\mathbb{Z}^s)$ with respect to this bilinear form. It is easily seen that

$$\langle S_a u, v\rangle = \langle u, T_a v\rangle \quad \forall u \in \ell(\mathbb{Z}^s), v \in \ell_0(\mathbb{Z}^s).$$

Hence, S_a is the algebraic adjoint of T_a with respect to the bilinear form given in (4.1.5).

In [27], Han and Jia showed that the transition operator T_a has only finitely many nonzero eigenvalues. By supp a we denote the set $\{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$. Let

$$\Omega := \left(\sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} a)\right) \cap \mathbb{Z}^s. \tag{4.1.6}$$

We use $\ell(\Omega)$ to denote the linear space of all sequences supported in Ω. It is easily seen that $\ell(\Omega)$ is invariant under T_a. Moreover, if v is an eigenvector of T_a corresponding to a nonzero eigenvalue of T_a, then v must lie in $\ell(\Omega)$. Consequently, any nonzero eigenvalue of T_a must be eigenvalue of the matrix

$$(a(M\alpha - \beta))_{\alpha,\beta\in\Omega}.$$

In particular, T_a has only finitely many nonzero eigenvalues. The spectral radius of T_a, denoted by $\rho(T_a)$, is defined as the spectral radius of the matrix $(a(M\alpha-\beta))_{\alpha,\beta\in\Omega}$.

In [37] and [30], Jia investigated the approximation properties of a refinable function in terms of its refinement mask using the subdivision and transition operators. Let us review some basic results about approximation with shift-invariant spaces.

A compactly supported distribution ϕ on $\mathbb{R}^s$ and a sequence $b \in \ell(\mathbb{Z}^s)$, the semi-convolution of ϕ with b is defined by

$$\phi *' b := \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) b(\alpha).$$

Let $S(\phi)$ denote the linear space $\{\phi *' b : b \in \ell(\mathbb{Z}^s)\}$. We call $S(\phi)$ the **shift-invariant** space generated by ϕ.

For a compactly supported distribution ϕ on $\mathbb{R}^s$ is said to have **accuracy** k, if $S(\phi)$ contains Π_{k-1} (see [28]). If ϕ has accuracy k and $\hat{\phi}(0) \neq 0$, then for any polynomial sequence u of degree at most $k-1$, the semi-convolution $\phi *' u$ is a polynomial of the same degree. Conversely, for any $p \in \Pi_{k-1}$, there exists a unique polynomial sequence u such that $p = \phi *' u$. See ([13], Proposition 1.1) and ([31], Lemma 8.2) for these results. Suppose $1 \leq p \leq \infty$ and ϕ is a compactly supported function in $L_p(\mathbb{R}^s)$ such that $\hat{\phi}(0) \neq 0$. It was proved in [36] that $S(\phi)$ provides approximation order k if and only if ϕ has accuracy k.

Let ϕ be the normalized solution of the refinement equation (4.1.1) with mask a and dilation matrix M. We say that a satisfies the **sum rules** of order k, if

$$\sum_{\beta \in \mathbb{Z}^s} a(\gamma + M\beta) p(\gamma + M\beta) = \sum_{\beta \in \mathbb{Z}^s} a(M\beta) p(M\beta) \quad \forall p \in \Pi_{k-1} \text{ and } \gamma \in \mathbb{Z}^s.$$

It was proved in [37] and [29] that ϕ has accuracy k provided that a satisfies the sum rules of order k. Let

$$V_k := \left\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} p(\alpha) v(\alpha) = 0 \ \forall p \in \Pi_k \right\}.$$

Then a satisfies the sum rules of order k if and only if V_{k-1} is invariant under the transition operator T_a. See the proof in [37].

We denote by $W_2^\nu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ for which

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (1 + |\xi|^\nu)^2 d\xi < \infty.$$

The **smoothness order** $\nu(f)$ of a function $f \in L_2(\mathbb{R}^s)$ is defined by

$$\nu(f) := \sup\{\nu : f \in W_2^\nu(\mathbb{R}^s)\}.$$

Let ϕ be the normalized solution of the refinement equation (4.1.1) with mask a and dilation matrix M. We assume that M is isotropic, i.e., M is similar to a diagonal matrix $\mathrm{diag}\{\sigma_1, \ldots, \sigma_s\}$ with $|\sigma_1| = \ldots = |\sigma_s|$. Let b be the sequence given by

$$b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha + \beta) \overline{a(\beta)} / m \quad \alpha \in \mathbb{Z}^s, \tag{4.1.7}$$

where $m = |\det M|$ and $\bar{a}$ denotes the complex conjugate of a. Suppose a satisfies the sum rules of order k. Then b satisfies the sum rules of order $2k$. Hence V_{2k-1} is invariant under T_b, the transition operator associated to b. In [30], Jia analyzed the smoothness of refinable function in terms of their masks. Let ρ_k denote the spectral radius of $T_b|_{V_{2k-1}}$. Suppose ϕ lies in $L_2(\mathbb{R}^s)$. It was proved in [30] that

$$\nu(\phi) \geq -(\log_m \rho_k)s/2. \tag{4.1.8}$$

If, in addition, $k > -(\log_m \rho_k)s/2$ and the shifts of ϕ are stable, then equality holds in (4.1.8). Note that the shifts of ϕ are stable if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\beta\pi) \neq 0$ (see [38]).

The smoothness analysis of refinable functions is applicable not only to approximation theory and wavelet analysis, but also to numerical solutions of partial differential equations. In this regard the reader is referred to the recent work of Lorentz and Oswald [41].

The following is an outline of this chapter.

Section 4.2 is devoted to a study of the spectrum of the transition operator. Suppose the dilation matrix M has eigenvalues $\sigma_1, \ldots, \sigma_s$. Write σ for the s-tuple $(\sigma_1, \ldots, \sigma_s)$. By convention, for a multi-index $\mu = (\mu_1, \ldots, \mu_s)$ we have

$$\sigma^\mu := \sigma_1^{\mu_1} \ldots \sigma_s^{\mu_s} \quad \sigma^{-\mu} := \sigma_1^{-\mu_1} \ldots \sigma_s^{-\mu_s}.$$

We shall show that the spectrum of the transition operator T_a contains $\{\sigma^{-\mu} : |\mu| < k\}$, provided ϕ has accuracy k. This gives an upper bound for the accuracy of ϕ in terms of the refinement mask a.

In Section 4.3 we shall investigate invariant subspaces of the subdivision and transition operators. We give a necessary and sufficient condition for a subspace of polynomial sequences to be invariant under the subdivision operator. Furthermore, we clarify the relationship among the spectra of the transition operator restricted to different invariant subspaces. In particular, we establish the following formula:

$$\operatorname{spec}(T_a|_{\ell(\Omega)}) = \operatorname{spec}(T_a|_{\ell(\Omega)\cap V_{k-1}}) \cup \{\sigma^{-\mu} : |\mu| < k\},$$

where Ω is given in (4.1.6). Thus, the spectral radius of $T_a|_{V_{k-1}}$ can be found from the spectrum of $T_a|_{\ell(\Omega)}$. This result is significant for calculating the smoothness order of a refinable function in terms of its mask.

Box splines are refinable functions with respect to the dilation matrix $M = 2I_s$, where I_s denotes the $s \times s$ identity matrix. In section 4.4 we shall find explicitly the spectrum of the transition operator associated with a box spline on the three-direction mesh. This result is then applied to interpolatory subdivision schemes induced by box splines. In particular, we find a way to greatly simplify the computation of the smoothness order of refinable functions which are convolution of box splines with refinable distributions.

4.2 The Spectrum of the Transition Operator

The spectrum of a square matrix A is denoted by spec(A), and it is understood to be the *multiset* of its eigenvalues. In other words, multiplicities of eigenvalues are counted and listed in the spectrum of a square matrix. The transpose of a matrix A is denoted by A^T.

Suppose T is a linear mapping on a finite dimensional vector space V over $\mathbb{C}$.

Let $\{v_1, \dots, v_n\}$ be an ordered basis of V. If

$$T(v_i) = b_{1i}v_1 + \cdots + b_{ni}v_n, \quad i = 1, \dots, n,$$

then $(b_{ij})_{1\le i,j\le n}$ is called the matrix representation of T with respect to $\{v_1, \dots, v_n\}$. The spectrum of T is the same as the spectrum of the matrix $(b_{ij})_{1\le i,j\le n}$.

Suppose ϕ is the normalized solution of the refinement equation (4.1.1) with mask a and dilation matrix M. Let $\operatorname{supp}\phi$ denote the support of ϕ. From (4.1.1) we observe that $\phi(x) \neq 0$ implies $\phi(Mx - \alpha) \neq 0$ for some $\alpha \in \operatorname{supp} a$. It follows that

$$x \in M^{-1}(\operatorname{supp} a) + M^{-1}(\operatorname{supp}\phi).$$

Hence we have

$$\operatorname{supp}\phi \subseteq M^{-1}(\operatorname{supp} a) + M^{-1}(\operatorname{supp}\phi).$$

A repeated use of the above relation yields

$$\operatorname{supp}\phi \subseteq \sum_{j=1}^{n} M^{-j}(\operatorname{supp} a) + M^{-n}(\operatorname{supp}\phi), \quad n = 1, 2, \dots.$$

Consequently, we obtain

$$\operatorname{supp}\phi \subseteq \sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} a) \tag{4.2.1}$$

It follows that $\mathbb{Z}^s \cap \operatorname{supp}\phi \subseteq \Omega$, where Ω is the set given in (4.1.6).

Let T_a and S_a be the transition operator and the subdivision operator given in (4.1.3) and (4.1.4), respectively. It is known that S_a is the adjoint of T_a with respect to the bilinear form given in (4.1.5). Let Ω be a nonempty finite subset of $\mathbb{Z}^s$. Suppose $\ell(\Omega)$ is invariant under T_a. By $-\Omega$ we denote the set $\{-\alpha : \alpha \in \Omega\}$. Clearly, $\ell(-\Omega)$ is the dual space of $\ell(\Omega)$ with respect to the bilinear form

$$\langle u, v \rangle_\Omega := \sum_{\alpha \in \Omega} u(-\alpha)v(\alpha), \quad u \in \ell(-\Omega), v \in \ell(\Omega). \tag{4.2.2}$$

Let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ given by

$$Q_\Omega u(\alpha) = \begin{cases} u(-\alpha), & \text{for } \alpha \in -\Omega, \\ 0, & \text{for } \alpha \notin -\Omega, \end{cases} \tag{4.2.3}$$

Then QS_a maps $\ell(-\Omega)$ to $\ell(-\Omega)$. We claim that $(QS_a)|_{\ell(-\Omega)}$ is the algebraic adjoint of $T_a|_{\ell(\Omega)}$. Indeed, for $u \in \ell(-\Omega)$ and $v \in \ell(\Omega)$ we have

$$\langle QS_a u, v \rangle_\Omega = \langle QS_a u, v \rangle = \langle S_a u, v \rangle = \langle u, T_a v \rangle = \langle u, T_a v \rangle_\Omega.$$

This justifies our claim. Consequently, the spectra of $(QS_a)|_{\ell(-\Omega)}$ and $T_a|_{\ell(\Omega)}$ are the same. Moreover, for $u \in \ell(\mathbb{Z}^s)$ and $v \in \ell(\Omega)$ we have

$$\langle QS_a(Qu - u), v \rangle = \langle Qu - u, T_a v \rangle = 0,$$

since $T_a v \in \ell(\Omega)$ and $Qu - u$ vanishes on $\ell(-\Omega)$. Thus, $QS_a(Qu - u)(\alpha) = 0$ for all $\alpha \in \ell(-\Omega)$. But, by the definition of Q, we have $QS_a(Qu - u)(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash (-\Omega)$. This shows $QS_a(Qu - u) = 0$. In other words,

$$QS_a Q = QS_a. \tag{4.2.4}$$

For $u \in \ell(\mathbb{Z}^s)$, we have

$$\sum_{\alpha \in \mathbb{Z}^s} u(\alpha)\phi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S_a u(\alpha)\phi(M \cdot - \alpha). \tag{4.2.5}$$

Indeed, since ϕ satisfies the refinement equation (4.1.1), we obtain

$$\sum_{\alpha \in \mathbb{Z}^s} u(\alpha)\phi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} u(\alpha) \sum_{\beta \in \mathbb{Z}^s} a(\beta)\phi(M \cdot - M\alpha - \beta) = \sum_{\gamma \in \mathbb{Z}^s} w(\gamma)\phi(M \cdot - \gamma),$$

where

$$w(\gamma) = \sum_{\alpha \in \mathbb{Z}^s} a(\gamma - M\alpha)u(\alpha), \quad \gamma \in \mathbb{Z}^s.$$

Hence $w = S_a u$. This verifies (4.2.5).

By $K(\phi)$ we denote the linear space given by

$$K(\phi) := \{u \in \ell(\mathbb{Z}^s) : \phi *' u = 0\}.$$

It follows from (4.2.5) that $K(\phi)$ is invariant under the subdivision operator S_a.

Lemma 4.1 *Let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ given by (4.2.3), where $\Omega = \mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n} L$ for some compact set $L \supseteq \operatorname{supp} a$. If u is a sequence on Z^s such that $p := \phi *' u$ is a nonzero polynomial, then $Qu \notin Q(K(\phi))$.*

Proof: Set

$$G_r := \{(x_1, \ldots, x_s) \in \mathbb{R}^s : |x_1| + \ldots + |x_s| < r\}, \quad r > 0.$$

By (4.2.1), the compact set $\operatorname{supp} \phi$ is disjoint from the closed set $\mathbb{Z}^s \setminus \Omega$; hence there exists some $r > 0$ such that

$$(\operatorname{supp}\phi + G_r) \cap (\mathbb{Z}^s \backslash \Omega) = \emptyset.$$

Suppose $x \in G_r$ and $\alpha \in \mathbb{Z}^s$. Then $\phi(x+\alpha) \neq 0$ implies $x + \alpha \in \operatorname{supp}\phi$. It follows that $\alpha \in \operatorname{supp}\phi + G_r$. Consequently,

$$\alpha \in \mathbb{Z}^s \cap (\operatorname{supp}\phi + G_r) \subseteq \Omega.$$

In other words, $x \in G_r$ and $\alpha \notin \Omega$ imply $\phi(x+\alpha) = 0$. Therefore

$$p(x) = \sum_{\alpha \in \mathbb{Z}^s} u(\alpha)\phi(x-\alpha) = \sum_{\alpha \in \mathbb{Z}^s} u(-\alpha)\phi(x+\alpha) = \sum_{\alpha \in \Omega} u(-\alpha)\phi(x+\alpha), \quad x \in G_r.$$

If $Qu \in Q(K(\phi))$, then there would exist some $w \in K(\phi)$ such that $Qu = Qw$. It follows that $u(-\alpha) = w(-\alpha)$ for all $\alpha \in \Omega$. Thus, for all $x \in G_r$,

$$p(x) = \sum_{\alpha \in \Omega} u(-\alpha)\phi(x+\alpha) = \sum_{\alpha \in \Omega} w(-\alpha)\phi(x+\alpha) = \sum_{\alpha \in \mathbb{Z}^s} w(-\alpha)\phi(x+\alpha) = 0,$$

which is impossible, because p is a nonzero polynomial. This verifies $Qu \notin Q(K(\phi))$.
∎

We are in a position to establish the main result of this section.

Theorem 4.2 *Let ϕ be the normalized solution of the refinement equation with mask a and dilation matrix M. If ϕ has accuracy k, then the spectrum of the transition operator T_a contains $\{\sigma^{-\mu} : |\mu| < k\}$, where $\sigma = (\sigma_1, \ldots, \sigma_s)$ is the s-tuple of the* *eigenvalues of M and μ is any multi-index $\mu = (\mu_1, \ldots, \mu_s)$.*

Proof: Let Ω be the set given in (4.1.6), and let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ as defined in (4.2.3). Since the spectra of $(QS_a)|_{\ell(-\Omega)}$ and $T_a|_{\ell(\Omega)}$ are the same, it suffices to show that the spectrum of $(QS_a)|_{\ell(-\Omega)}$ contains $\{\sigma^{-\mu} : |\mu| < k\}$. For this purpose, we introduce the set

$$W := \{u \in \ell(\mathbb{Z}^s) : \phi *' u \in \Pi_{k-1}\}.$$

By (4.2.5), W is invariant under S_a. Clearly, $Q(W)$ is a subspace of $\ell(-\Omega)$. The theorem will be proved by finding the matrix representation of QS_a with respect to a suitable basis of $Q(W)$.

There exists an invertible matrix $H = (h_{ij})_{1 \le i,j \le s}$ such that HMH^{-1} is a triangular matrix:

$$HMH^{-1} = \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \ldots & \sigma_{ss} \end{bmatrix}.$$

For $i = 1, \ldots, s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, let $l_i(x) := h_{i1}x_1 + \ldots + h_{is}x_s$. Then Hx can be represented as $[l_1(x), \ldots, l_s(x)]^T$. It follows that

$$\begin{bmatrix} l_1(Mx) \\ \vdots \\ l_s(Mx) \end{bmatrix} = HMx = \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \ldots & \sigma_{ss} \end{bmatrix} Hx$$
$$= \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \ldots & \sigma_{ss} \end{bmatrix} \begin{bmatrix} l_1(x) \\ \vdots \\ l_s(x) \end{bmatrix}. \qquad (4.2.6)$$

For simplicity , we write σ_j for σ_{jj}, $j = 1, \ldots, s$. Thus, $\sigma_1, \ldots, \sigma_s$ are the eigenvalues of the matrix M. For two multi-indices $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$, we write $\mu \prec \nu$ if there exists some j, $1 \leq j \leq s$, such that $\mu_j < \nu_j$, and $\mu_{j+1} = \nu_{j+1}, \ldots, \mu_s = \nu_s$.

For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, let p_μ be the polynomial given by

$$p_\mu := l_1^{\mu_1} \cdots l_s^{\mu_s}.$$

Clearly, p_μ ($|\mu| < k$) are linearly independent. With the help of (4.2.6) we obtain

$$\begin{aligned} &p_\mu(Mx) = [l_1(Mx)]^{\mu_1} \ldots [l_s(Mx)]^{\mu_s} \\ &= (\sigma_{11} l_1(x))^{\mu_1} (\sigma_{21} l_1(x) + \sigma_{22} l_2(x))^{\mu_2} \cdots (\sigma_{s1} l_1(x) + \cdots + \sigma_{ss} l_s(x))^{\mu_s} \\ &= \sigma^\mu p_\mu(x) + q_\mu(x), \quad x \in \mathbb{R}^s, \end{aligned}$$

where q_μ is a linear combination of p_ν, i.e., $q_\mu = \sum_\nu c_\nu p_\nu$, where $|\nu| = |\mu|$ and $\nu \prec \mu$. It follows that

$$p_\mu(x) = \sigma^{-\mu} p_\mu(Mx) - \sigma^{-\mu} q_\mu(x), \quad x \in \mathbb{R}^s.$$

A repeated use of the above idea yields

$$p_\mu(x) = \sigma^{-\mu} p_\mu(Mx) + r_\mu(Mx), \tag{4.2.7}$$

where r_μ is a linear combination of p_ν with $|\nu| = |\mu|$ and $\nu \prec \mu$.

By the assumption, ϕ has accuracy k. Thus, for each μ with $|\mu| < k$, the polynomial p_μ lies in $S(\phi)$. Since $\hat{\phi}(0) \neq 0$, there exists a unique polynomial sequence

$u_\mu \in \ell(\mathbb{Z}^s)$ such that

$$p_\mu = \sum_{\alpha \in \mathbb{Z}^s} u_\mu(\alpha)\phi(\cdot - \alpha). \tag{4.2.8}$$

It follows from (4.2.7) and (4.2.8) that

$$p_\mu = \sigma^{-\mu} p_\mu(Mx) + r_\mu(Mx) = \sum_{\alpha \in \mathbb{Z}^s} \left[\sigma^{-\mu} u_\mu(\alpha) + v_\mu(\alpha)\right] \phi(M \cdot - \alpha), \tag{4.2.9}$$

where v_μ is a linear combination of u_ν with $|\nu| = |\mu|$ and $\nu \prec \mu$. On the other hand, we deduce from (4.2.8) and (4.2.5) that

$$p_\mu = \sum_{\alpha \in \mathbb{Z}^s} u_\mu(\alpha)\phi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S_a u_\mu(\alpha)\phi(M \cdot - \alpha).$$

Comparing this equation with (4.2.9), we obtain

$$S_a u_\mu = \sigma^{-\mu} u_\mu + v_\mu + w_\mu,$$

where $w_\mu \in K(\phi)$. By (4.2.4) it follows that

$$QS_a(Qu_\mu) = QS_a u_\mu = \sigma^{-\mu}(Qu_\mu) + Qv_\mu + Qw_\mu. \tag{4.2.10}$$

Let $U := U_0 + \ldots + U_{k-1}$, where each U_j $(j = 0, 1, \ldots, k-1)$ is the linear span of u_μ, $|\mu| = j$. Then $W = U + K(\phi)$. By Lemma 4.1, $Q(U) \cap Q(K(\phi)) = \{0\}$. Hence $Q(W)$ is the direct sum of $Q(U)$ and $Q(K(\phi))$. Moreover, $Q(U)$ is the direct sum of $Q(U_0), \ldots, Q(U_{k-1})$. Choose a basis Y for $Q(K(\phi))$. For each j, the set

$Y_j := \{Qu_\mu : |\mu| = j\}$ is a basis for $Q(U_j)$. The order of this basis is arranged in such a way that Qu_ν precedes Qu_μ when $\nu \prec \mu$. Consequently, $Y \cup Y_0 \cup \ldots \cup Y_{k-1}$ is a basis for $Q(W)$. With respect to this basis, (4.2.10) tells us that QS_a has the following matrix representation:

$$\begin{bmatrix} E & F_0 & F_1 & \ldots & F_{k-1} \\ & E_0 & 0 & \ldots & 0 \\ & & E_1 & \ldots & 0 \\ & & & \ddots & \vdots \\ & & & & E_{k-1} \end{bmatrix},$$

Where each E_j $(j = 0, \ldots, k-1)$ is a triangular matrix with $\sigma^{-\mu}$ $(|\mu| = j)$ being the entries in its main diagonal. We conclude that the spectrum of $(QS_a)|_{Q(W)}$ contains $\{\sigma^{-\mu} : |\mu| < k\}$, as desired. ∎

Example 4.3 *Let M be the matrix*

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and let a be the sequence on $\mathbb{Z}^2$ such that $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \backslash [-1, 1]^2$ and

$$(a(\alpha_1,\alpha_2))_{-2\le\alpha_1,\alpha_2\le 2} = \frac{1}{32}\begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & 10 & 32 & 10 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Let ϕ be the normalized solution of the refinement equation (4.1.1) with mask a and dilation matrix M given as above. Then ϕ has accuracy 4 but does not have accuracy 5.

It can be easily checked that a satisfies the sum rules of order 4. Hence ϕ has accuracy 4. Let us show that ϕ does not have accuracy 5. The matrix M has two eigenvalues $\sigma_1 = 1+i$ and $\sigma_2 = 1-i$, where i denotes the imaginary unit. We have $\operatorname{supp} a \subseteq [-2,2]^2$ and

$$\sum_{n=1}^{\infty} M^{-n}([-2,2]^2) = \{(x_1,x_2)\in \mathbb{R}^2 : |x_1|\le 6, |x_2|\le 6, |x_1-x_2|\le 8, |x_1+x_2|\le 8\}.$$

The set $\Omega := \mathbb{Z}^2 \cap \left(\sum_{n=1}^{\infty} M^{-n}([-2,2]^2)\right)$ has exactly 129 points. Among the 129 eigenvalues of the matrix $A := (a(M\alpha-\beta))_{\alpha,\beta\in\Omega}$ the following are of the form $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for some double-index (μ_1,μ_2) with $\mu_1+\mu_2\le 4$:

$$1,\ 0.5-0.5i,\ 0.5+0.5i,\ -0.5i,\ 0.5,\ 0.5i,$$

$$-0.25-0.25i,\ 0.25-0.25i,\ 0.25+0.25i,\ -0.25+0.25i,$$

$$-0.25,\ 0.25i,\ -0.25i.$$

Since ϕ has accuracy 4, we expect that A has eigenvalues $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for all 10 double-indices (μ_1, μ_2) with $\mu_1 + \mu_2 \le 3$. The above computation confirms our expectation. But A has only three eigenvalues of modulus 0.25. Therefore, by Theorem 4.2, ϕ does not have accuracy 5. ∎

This example is the Example 4.3 taken from [29], where another method was used to achieve the conclusion that the optimal accuracy of ϕ is 4.

4.3 Invariant Subspaces of the Transition Operator

In this section we investigate invariant subspaces of the subdivision and transition operators. We are particularly interested in invariant subspaces of the subdivision operator which consist of polynomial sequences. The results are then applied to smoothness analysis of refinable functions in terms of their masks.

Let Π denote the linear space of all polynomials of s variables. For a compactly supported distribution ϕ on $\mathbb{R}^s$, the intersection $S(\phi) \cap \Pi$ is shift-invariant, i.e., $p \in S(\phi) \cap \Pi$ implies $p(\cdot - \alpha) \in S(\phi) \cap \Pi$ for all $\alpha \in \mathbb{Z}^s$. It was proved in ([33], Theorem 3.1), that a shift-invariant subspace P of Π is D-invariant, that is, $p \in P$ implies all its partial derivatives belong to P.

Suppose ϕ is a compactly supported distribution on $\mathbb{R}^s$ such that $\hat{\phi}(0) \neq 0$. Let

P be a finite dimensional D-invariant subspace of Π. Then $P \subseteq S(\phi)$ if and only if

$$p(-iD)\hat{\phi}(2\pi\beta) = 0 \quad \forall p \in P \text{ and } \beta \in \mathbb{Z}^s \backslash \{0\}.$$

Suppose $P \subseteq S(\phi)$. Then $u \in P|_{\mathbb{Z}^s}$ implies $p := \phi *' u$ lies in P. Conversely, for each $p \in P$, there exists a unique polynomial sequence $u \in P|_{\mathbb{Z}^s}$ such that $p = \phi *' u$. See [13] and [33] for these results.

Now let ϕ be the normalized solution of the refinement equation with mask a and dilation matrix M, where $\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m = |\det M|$. Let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M\mathbb{Z}^s$, and let Θ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M^T \mathbb{Z}^s$. Recall that a satisfies the sum rules of order k implies ϕ has accuracy k. The converse of this statement is valid under the additional condition that

$$N(\phi) \cap (2\pi(M^T)^{-1}\Theta) = \emptyset, \tag{4.3.1}$$

where

$$N(\phi) := \{\xi \in \mathbb{R}^s : \hat{\phi}(\xi + 2\pi\beta) = 0 \quad \forall \beta \in \mathbb{Z}^s\}.$$

These results can be extended to shift-invariant subspaces of Π. Let P be a finite dimensional shift-invariant subspace of Π. If

$$\sum_{\beta \in \mathbb{Z}^s} a(\gamma + M\beta) p(-\gamma - M\beta) = \sum_{\beta \in \mathbb{Z}^s} a(M\beta) p(-M\beta) \quad \forall p \in P \text{ and } \gamma \in \Gamma, \tag{4.3.2}$$

then $P \subseteq S(\phi)$. Conversely, if $P \subseteq S(\phi)$ and (4.3.1) is valid, then a satisfies the conditions in (4.3.2). The proof is similar to that of Lemma 3.3 in [29].

It was proved in [29] that a satisfies the sum rules of order k if and only if $\Pi_{k-1}|_{\mathbb{Z}^s}$ is invariant under the subdivision operator S_a. In order to extend this result to shift-invariant subspaces of Π, additional work is needed.

Theorem 4.4 *Let M be a dilation matrix, and let a be an element in $\ell_0(\mathbb{Z}^s)$ such that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m = |\det M|$. Suppose P is a finite dimensional shift-invariant subspace of Π. Then $P|_{\mathbb{Z}^s}$ is invariant under S_a if and only if a satisfies the conditions in (4.3.2) and $p \in P$ implies $p(M^{-1}\cdot) \in P$.*

Proof: Suppose $U := P|_{\mathbb{Z}^s}$ is invariant under S_a. Let us first show that $S_a|_U$ is one-to-one. Let u be an element in U such that $S_a u = 0$. Then

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta) = 0 \quad \forall\alpha \in \mathbb{Z}^s. \tag{4.3.3}$$

Suppose $u \neq 0$. Since u is a polynomial sequence, there exists a multi-index μ and a complex number $c \neq 0$ such that $\nabla^\mu u(\beta) = c$ for all $\beta \in \mathbb{Z}^s$. It follows from (4.3.3) that

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)\nabla^\mu u(\beta) = 0 \quad \forall\alpha \in \mathbb{Z}^s.$$

Hence $\sum_{\beta\in\mathbb{Z}^s} a(\alpha-M\beta) = 0$ for all $\alpha \in \mathbb{Z}^s$. This contradicts to the assumption that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m \neq 0$. Therefore, $S_a|_U$ is one-to-one. But U is finite dimensional. Hence $S_a|_U$ is one-to-one and onto.

Next, we show that $p \in P$ implies $P(M\cdot) \in P$. Let $p \in P$. Since $S_a|_U$ is onto, there exists $f \in P$ such that $p|_{\mathbb{Z}^s} = S_a(f|_{\mathbb{Z}^s})$, that is, $p(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha-M\beta)f(\beta)$ for all $\alpha \in \mathbb{Z}^s$. It follows that

$$p(M\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - M\beta)f(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta)f(\alpha-\beta) \quad \forall\alpha\in\mathbb{Z}^s.$$

Let $q(x) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta)f(x-\beta)$, $x\in\mathbb{R}^s$. Since P is shift-invariant, q belongs to P. Thus, q and $p(M\cdot)$ agree on the lattice $\mathbb{Z}^s$. Therefore, we have $p(M\cdot) = q \in P$.

For $p\in P$, let

$$u(\gamma) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)p(-M\beta-\gamma), \quad \gamma\in\mathbb{Z}^s.$$

We claim that u is a polynomial sequence. Indeed, by using Taylor's formula, we obtain

$$p(-M\beta-\gamma) = \sum_{\mu} t_\mu(-M\beta)(-\gamma)^\mu,$$

where $t_\mu := D^\mu p/\mu!$. Since P is D-invariant, $t_\mu \in P$ for every multi-index μ. Set $q_\mu(x) := t_\mu(Mx)$ for $x\in\mathbb{R}^s$. By what has been proved, we have $q_\mu \in P$. Let $u_\mu := q_\mu|_{\mathbb{Z}^s}$. Then for $\gamma\in\mathbb{Z}^s$,

$$\begin{aligned} u(\gamma) &= \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)p(-M\beta-\gamma) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\sum_{\mu} t_\mu(-M\beta)(-\gamma)^\mu \\ &= \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\sum_{\mu} q_\mu(-\beta)(-\gamma)^\mu = \sum_{\beta\in\mathbb{Z}^s}\sum_{\mu} a(\gamma+M\beta)u_\mu(-\beta)(-\gamma)^\mu \\ &= \sum_{\mu}(-\gamma)^\mu \sum_{\beta\in\mathbb{Z}^s} a(\gamma+M\beta)u_\mu(-\beta) = \sum_{\mu} S_a u_\mu(\gamma)(-\gamma)^\mu. \end{aligned}$$

Since U is invariant under S_a, $S_a u_\mu \in U$. Hence u is a polynomial sequence. By the definition of u, we have

$$u(\gamma + M\eta) = \sum_{\beta \in \mathbb{Z}^s} a(M\beta + M\eta + \gamma)p(-M\beta - M\eta - \gamma)$$
$$= \sum_{\beta \in \mathbb{Z}^s} a(M(\beta + \eta) + \gamma)p(-M(\beta + \eta) - \gamma) = u(\gamma)$$

for all $\eta \in \mathbb{Z}^s$ and $\gamma \in \mathbb{Z}^s$. Therefore, u itself must be a constant sequence. This shows that a satisfies the condition in (4.3.2). Consequently, $P \subset S(\phi)$, where ϕ is the normalized solution of the refinement equation with mask a and dilation matrix M.

It remains to prove that $p \in P$ implies $p(M^{-1}\cdot) \in P$. Let $p \in P$. Then there exists a unique $u \in U$ such that $p = \phi *' u$. From (4.2.5) we deduce that

$$p(M^{-1}x) = \sum_{\alpha \in \mathbb{Z}^s} u(\alpha)\phi(M^{-1}x - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} S_a u(\alpha)\phi(x - \alpha).$$

Since U is invariant under S_a, we have $S_a u \in U$. This shows $p(M^{-1}\cdot) \in P$.

Now suppose a satisfies the condition in (4.3.2) and $p \in P$ implies $p(M^{-1}\cdot) \in P$. We wish to show that $U = P|_{\mathbb{Z}^s}$ is invariant under S_a. Let $p \in P$ and $u = p|_{\mathbb{Z}^s}$. We first show that $S_a u$ is a polynomial sequence. Set $q(x) := p(M^{-1}x)$, $x \in \mathbb{R}^s$. By our assumption $q \in P$. An application of Taylor's formula gives

$$q(M\beta) = q(-\alpha + M\beta + \alpha) = \sum_{\mu} q_\mu(-\alpha + M\beta)\alpha^\mu,$$

where $q_\mu = D^\mu p/\mu!$. Since P is D-invariant, we have $q_\mu \in P$ for all multi-index μ. It follows that

$$\begin{aligned}
S_a u(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) p(\beta) = \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) q(M\beta) \\
&= \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) \sum_{\mu} q_\mu(-\alpha + M\beta) \alpha^\mu \\
&= \sum_{\mu} \Big[\sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) q_\mu(-\alpha + M\beta) \Big] \alpha^\mu.
\end{aligned}$$

Since a satisfies the conditions in (4.3.2), $c_\mu := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) q_\mu(\alpha - M\beta)$ is independent of α. Therefore, $S_a u(\alpha) = \sum_\mu c_\mu \alpha^\mu$ for all $\alpha \in \mathbb{Z}^s$. This shows that $S_a u$ is a polynomial sequence.

To finish the proof, we have to show that $S_a u \in U$. We observe that

$$S_a u(M\gamma) = \sum_{\beta \in \mathbb{Z}^s} a(M(\gamma - \beta)) u(\beta) = \sum_{\beta \in \mathbb{Z}^s} a(M\beta) p(\gamma - \beta), \quad \gamma \in \mathbb{Z}^s.$$

Since P is shift-invariant, there exists $f \in P$ such that

$$\sum_{\beta \in \mathbb{Z}^s} a(M\beta) p(\gamma - \beta) = f(\gamma) \quad \forall \gamma \in \mathbb{Z}^s.$$

Let $g(x) := f(M^{-1}x)$, $x \in \mathbb{R}^s$. Then $g \in P$ and

$$S_a u(M\gamma) = f(\gamma) = g(M\gamma) \quad \forall \gamma \in \mathbb{Z}^s.$$

This shows that $S_a u$ and g agree on the lattice $M\mathbb{Z}^s$. But both $S_a u$ and $g|_{\mathbb{Z}^s}$ are polynomial sequences. Therefore, $S_a u = g|_{\mathbb{Z}^s} \in U$. We conclude that U is invariant under S_a.

∎

The following theorem clarifies the relationship among the spectra of the transition operator restricted to different invariant subspaces.

Theorem 4.5 *Let U be a finite dimensional subspace of $\ell(\mathbb{Z}^s)$, and let*

$$V := \Big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} u(-\alpha)v(\alpha) = 0 \ \forall u \in U \Big\}. \tag{4.3.4}$$

Then U is invariant under the subdivision operator S_a if and only if V is invariant under the transition operator T_a. Let Ω be a finite subset of $\mathbb{Z}^s$ such that $\ell(\Omega)$ is invariant under T_a, and let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ as defined in (4.2.3). If U is invariant under S_a, and if $Q|_U$ is one-to-one, then

$$\operatorname{spec}(T_a|_{\ell(\Omega)}) = \operatorname{spec}(T_a|_{\ell(\Omega)\cap V}) \cup \operatorname{spec}(S_a|_U). \tag{4.3.5}$$

In particular, the above relation is valid when $\Omega = \mathbb{Z}^s \cup \sum_{n=1}^{\infty} M^{-n} L$ for some compact set $L \supset \operatorname{supp} a$ and $U = P|_{\mathbb{Z}^s}$ for some finite dimensional shift-invariant subspace P of Π which is invariant under S_a.

Proof: Let $\langle u, v\rangle$ be the bilinear form defined in (4.1.5). Then $v \in V$ if and only if $\langle u, v\rangle = 0$ for all $u \in U$. Suppose U is invariant under S_a. Then for $v \in V$ we have

$$\langle u, T_a v\rangle = \langle S_a u, v\rangle = 0 \quad \forall u \in U.$$

Hence $v \in V$ implies $T_a v \in V$. This shows that V is invariant under T_a.

Choose a basis $\{u_1, \ldots, u_n\}$ for U. Then there exist $v_1, \ldots, v_n \in \ell_0(\mathbb{Z}^s)$ such that $\langle u_j, v_k\rangle = \delta_{jk}$ for $j, k = 1, \ldots, n$, where δ_{jk} stands for the Kronecker sign. It is easily seen that $\ell_0(\mathbb{Z}^s)$ is the direct sum of V and the linear span of $v_1, \ldots, v_n$. Indeed, for any v in $V \cap \operatorname{span}\{v_1, \cdots, v_n\}$, we have $\langle u_i, v\rangle = 0$, for $i = 1, \cdots, n$, since $v \in V$. On the other hand, there exist $\{c_i\}_{i=1}^n$ such that $v = \sum_{i=1}^n c_i v_i$ since $v \in \operatorname{span}\{v_1, \cdots, v_n\}$. Then $\langle u_j, v\rangle = c_j$, for $j = 1, \cdots, n$. Hence all $c_j = 0$, $j = 1, \cdots, n$, which implies that $v = 0$, i.e., $V \cap \operatorname{span}\{v_1, \cdots, v_n\} = \{0\}$. Next, for a given $v \in \ell_0(\mathbb{Z}^s)$, let $c_i := \langle u_i, v\rangle$, $i = 1, 2, \cdots, n$. Then $v - \sum_{i=1}^n c_i v_i \in V$. Hence $\ell_0(\mathbb{Z}^s) \subseteq V + \operatorname{span}\{v_1, \cdots, v_n\}$. Obviously, $V + \operatorname{span}\{v_1, \cdots, v_n\} \subseteq \ell_0(\mathbb{Z}^s)$. We prove that $\ell_0(\mathbb{Z}^s) = V + \operatorname{span}\{v_1, \cdots, v_n\}$.

Suppose V is invariant under T_a. We wish to show that U is invariant under S_a. Let $u \in U$ and $w = S_a u$. Then

$$\langle w, v\rangle = \langle S_a u, v\rangle = \langle u, T_a v\rangle = 0 \quad \forall v \in V.$$

Moreover, with $c_j := \langle w, v_j\rangle$, $j = 1, \ldots, n$, we have

$$\langle w - (c_1 u_1 + \ldots + c_n u_n), v_j\rangle = 0 \quad \forall j = 1, \ldots, n.$$

For any $v \in V$, we have

$$\langle w - (c_1 u_1 + \ldots + c_n u_n), v\rangle = \langle w, v\rangle - \sum_{i=1}^n c_i \langle u_i, v\rangle = 0.$$

Since $\ell_0(\mathbb{Z}^s) = V + \operatorname{span}\{v_1, \cdots, v_n\}$, it follows that $\langle w - (c_1 u_1 + \ldots + c_n u_n), y\rangle = 0$ for all $y \in \ell_0(\mathbb{Z}^s)$, i.e., $w - (c_1 u_1 + \cdots + c_n u_n) = 0$. This shows that $w = c_1 u_1 + \ldots + c_n u_n \in U$. In other words, U is invariant under S_a. This proves the first statement of the theorem.

Now suppose U is invariant under S_a. Choose a basis $\{u_1, \dots, u_r\}$ for U. Since $Q|_U$ is one-to-one, $\{Qu_1, \dots, Qu_r\}$ is a basis for $Q(U)$. We supplement elements $u_{r+1}, \dots, u_n$ in $\ell(-\Omega)$ such that $\{Qu_1, \dots, Qu_r, u_{r+1}, \dots, u_n\}$ forms a basis for $\ell(-\Omega)$. Clearly, $Qu_j = u_j$ for $j = r+1, \dots, n$. Suppose

$$QS_a(Qu_j) = \sum_{k=1}^{n} b_{jk}(Qu_k), \quad \text{for } j = 1, \dots, n. \tag{4.3.6}$$

Let $B := (b_{jk})_{1 \le j,k \le n}$. Then B^T, the transpose of B, is the matrix of the linear mapping $(QS_a)|_{\ell(-\Omega)}$ with respect to the basis $\{Qu_1, \dots, Qu_n\}$. Since U is invariant under S_a, $Q(U)$ is invariant under QS_a in light of (4.2.4). Therefore, $b_{jk} = 0$ for $j = 1, \dots, r$ and $k = r+1, \dots, n$. In other words, B is a block triangular matrix:

$$B = \begin{bmatrix} E & 0 \\ G & F \end{bmatrix},$$

where $E = (b_{jk})_{1 \le j,k \le r}$ and $F = (b_{jk})_{r+1 \le j,k \le n}$. Since $\ell(\Omega)$ is invariant under T_a, by (4.2.4) we have $QS_aQ = QS_a$. Thus, it follows from (4.3.6) that

$$Q(S_a u_j) = QS_a(Qu_j) = \sum_{k=1}^{r} b_{jk}(Qu_k) = Q(\sum_{k=1}^{r} b_{jk} u_k), \quad j = 1, \cdots, r.$$

By our assumption that $Q|_U$ is one-to-one, we obtain

$$S_a u_j = \sum_{k=1}^{r} b_{jk} u_k, \quad j = 1, \dots, r.$$

Therefore, E^T is the matrix of $S_a|_U$ with respect to the basis $\{u_1, \dots, u_r\}$.

Note that $\ell(-\Omega)$ is the dual space of $\ell(\Omega)$ with respect to the bilinear form $\langle u, v\rangle_\Omega$ defined by (4.2.2). Let $\{v_1, \ldots, v_n\}$ be the basis of $\ell(\Omega)$ dual to $\{Qu_1, \ldots, Qu_n\}$, that is,

$$\langle Qu_j, v_k\rangle = \delta_{jk} \quad \text{for } j, k = 1, \ldots, n.$$

Clearly, $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\ell(\Omega) \cap V$. It was proved in Section 2 that $(QS_a)|_{\ell(-\Omega)}$ is the adjoint of $T_a|_{\ell(\Omega)}$ with respect to the bilinear form $\langle u, v\rangle_\Omega$. Consequently, by (4.3.6) we have

$$\begin{aligned}\langle Qu_j, T_a v_k\rangle_\Omega &= \langle QS_a Qu_j, v_k\rangle_\Omega \\ &= \langle \sum_{\ell=1}^{n} b_{j\ell}(Qu_\ell), v_k\rangle_\Omega = b_{jk}, \quad j, k = 1, \ldots, n.\end{aligned}$$

This shows that $B = (b_{jk})_{1\le j,k\le n}$ is the matrix of $T_a|_{\ell(\Omega)}$ with respect to the basis $\{v_1, \ldots, v_n\}$. But $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\ell(\Omega) \cap V$. Hence $F = (b_{jk})_{r+1\le j,k\le n}$ is the matrix of $T_a|_{\ell(\Omega)\cap V}$ with respect to this basis. To summarize, we obtain

$$\text{spec}(T_a|_{\ell(\Omega)}) = \text{spec}(B) = \text{spec}(E) \cup \text{spec}(F) = \text{spec}(S_a|_U) \cup \text{spec}(T_a|_{\ell(\Omega)\cap V}).$$

This verifies (4.3.5).

Finally, suppose $U = P|_{\mathbb{Z}^s}$ for some shift-invariant subspace P of Π and U is invariant under S_a. Theorem 4.4 tells us that $P \subset S(\phi)$, where ϕ is the normalized solution of the refinement equation (4.1.1) with mask a and dilation matrix M. Suppose that $\Omega = \mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n}L$ for some compact set $L \supset \text{supp}\, a$. Let $u \in U$ and $p := \phi *' u$. If $u \neq 0$, then $p \neq 0$; hence $Qu \neq 0$ by Lemma 4.1. This shows that $Q|_U$ is one-to-one. Therefore, (4.3.5) is valid for this case. ∎

4.4 The Transition Operator Associated with a Box Spline

Box splines are refinable functions with respect to the dilation matrix $M = 2I_s$, where I_s denotes the $s \times s$ identity matrix. The reader is referred to the monograph [16] by de Boor, Höllig, and Riemenschneider for a comprehensive study of box splines. In this section we shall find explicitly the spectrum of the transition operator associated to a box splines on the three-direction mesh. The result is then applied to interpolatory subdivision schemes induced by box splines. Finally, we provide a method to simplify the computation of the smoothness order of refinable functions which are convolutions of box splines with refinable distributions.

For an element $a \in \ell_0(\mathbb{Z}^s)$ we use $\tilde{a}(z)$ to denote its symbol:

$$\tilde{a}(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha, \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

The convolution of two sequences a and b in $\ell_0(\mathbb{Z}^s)$ is defined by

$$a * b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - \beta) b(\beta), \quad \alpha \in \mathbb{Z}^s.$$

If $c = a * b$, then

$$\tilde{c}(z) = \tilde{a}(z)\tilde{b}(z), \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

For $r = 1, 2, \ldots$, let a_r be the element in $\ell_0(\mathbb{Z})$ defined by its symbol:

$$\tilde{a}_r(z) = (1+z)^r/2^{r-1}.$$

The B-spline B_r of order r can be viewed as the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}} a_r(\alpha)\phi(2\cdot-\alpha)$. We have $\operatorname{supp} B_r = [0, r]$. Let us find the spectrum of the transition operator T_{a_r} restricted to $\mathbb{Z}\cap[0,r]$. We observe that the linear space $\ell(\mathbb{Z}\cap[0,r])$ is invariant under T_{a_r}. Moreover, $\Pi_{r-1}|_{\mathbb{Z}\cap[0,r-1]}$ has dimension r, the same as $\dim(\Pi_{r-1})$. Hence, by Theorems 4.2 and 4.5, we have

$$\operatorname{spec}(T_{a_r}|_{\mathbb{Z}\cap[0,r-1]}) = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2^{r-1}}\right\}.$$

Furthermore, $T_{a_r}\delta_r = 2^{1-r}\delta_r + w$ for some $w \in \ell(\mathbb{Z}\cap[0,r-1])$. Therefore, we conclude that

$$\operatorname{spec}\big(a_r(2\alpha-\beta)\big)_{0\le\alpha,\beta\le r} = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2^{r-1}}, \frac{1}{2^{r-1}}\right\}.$$

Evidently, the spectrum of the transition operator associated to the tensor product of two B-splines can be easily found.

Now let us consider box splines on the three-direction mesh in $\mathbb{R}^2$. For $r, s, t \ge 1$, let $a_{r,s,t}$ be the element in $\ell_0(\mathbb{Z}^2)$ defined by its symbol:

$$\tilde{a}_{r,s,t}(z_1, z_2) := (1+z_1)^r(1+z_2)^s(1+z_1z_2)^t/2^{r+s+t-2}, \quad (z_1, z_2) \in \mathbb{C}^2.$$

The box spline $B_{r,s,t}$ is defined as the normalized solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a_{r,s,t}(\alpha)\phi(2\cdot - \alpha).$$

The three families of mesh lines are $L_{1k} := \{(k, x_2) : x_2 \in \mathbb{R}\}$, $L_{2k} := \{(x_1, k) : x_1 \in \mathbb{R}\}$, and $L_{3k} := \{(x_1, x_1 + k) : x_1 \in \mathbb{R}\}$, where $k \in \mathbb{Z}$. On each connected component of $\mathbb{R}^2 \setminus \cup_{k \in \mathbb{Z}} (L_{1k} \cup L_{2k} \cup L_{3k})$, $B_{r,s,t}$ agrees with a polynomial of degree at most $r + s + t - 2$. The support of $B_{r,s,t}$ is the hexagon

$$K_{r,s,t} := \{y_1 e_1 + y_2 e_2 + y_3 e_3 : 0 \le y_1 \le r, 0 \le y_2 \le s, 0 \le y_3 \le t\}, \tag{4.4.1}$$

where $e_1 = (1,0), e_2 = (0,1)$, and $e_3 = (1,1)$. Let $\Omega_{r,s,t} := \mathbb{Z}^2 \cap K_{r,s,t}$. Our goal is to find the spectrum of the transition operator associated to the mask $a_{r,s,t}$:

$$T_{r,s,t}v(\alpha) = \sum_{\beta \in \mathbb{Z}^2} a_{r,s,t}(2\alpha - \beta)v(\beta), \quad v \in \ell_0(\mathbb{Z}^2).$$

Also, we use $S_{r,s,t}$ to denote the corresponding subdivision operator. Let

$$\Gamma_{r,s,t} := \mathbb{Z}^2 \cap \left((1/2, 1/4) + K_{r,s,t}\right).$$

We claim that $\ell(\Gamma_{r,s,t})$ is invariant under the transition operator $T_{r,s,t}$. Let $v \in \ell(\Gamma_{r,s,t})$. We observe that $T_{r,s,t}v(\alpha) \neq 0$ implies $2\alpha - \beta \in K_{r,s,t}$ for some $\beta \in \Gamma_{r,s,t}$. It follows that

$$\alpha \in \frac{1}{2}(\Gamma_{r,s,t} + K_{r,s,t}) \subseteq \frac{1}{2}((1/2, 1/4) + K_{r,s,t} + K_{r,s,t}) = (1/4, 1/8) + K_{r,s,t}.$$

It easily seen that

$$Z^2 \cap ((1/4, 1/8) + K_{r,s,t}) = Z^2 \cap ((1/2, 1/4) + K_{r,s,t}) = \Gamma_{r,s,t}.$$

This shows $T_{r,s,t}v \in \ell(\Gamma_{r,s,t})$, thereby verifying our claim.

Let $\mathbb{P}_{r,s,t} := \Pi \cap S(B_{r,s,t})$. It was proved in [15] that

$$\mathbb{P}_{r,s,t} = \{p \in \Pi : D_1^r D_2^s p = 0, D_1^r (D_1 + D_2)^t p = 0, D_2^s (D_1 + D_2)^t p = 0\}.$$

In particular, $\mathbb{P}_{r,s,t} \subseteq \Pi_{r+s+t-2}$. The dimension of $\mathbb{P}_{r,s,t}$ was found in [8]:

$$\dim(\mathbb{P}_{r,s,t}) = rs + st + tr.$$

The shifts of the box spline $B_{r,s,t}$ are locally linearly independent (see [8] and [32]). On the basis of this fact, Dahmen and Micchelli [8] established the following interesting result: Given any data $\{f_\alpha : \alpha \in \Gamma\}$, there exists a unique polynomial p in $\mathbb{P}_{r,s,t}$ such that $p(\alpha) = f_\alpha$ for all $\alpha \in \Gamma_{r,s,t}$ since the dimension of $\mathbb{P}_{r,s,t}$ is the same as the dimension of $\Gamma_{r,s,t}$.

The following theorem provides complete information about the spectrum of the matrix $(a_{r,s,t}(2\alpha - \beta))_{\alpha,\beta \in \Omega_{r,s,t}}$.

Theorem 4.6 *Let r, s, t be positive integers such that $r \le s \le t$. The spectrum of the matrix $(a_{r,s,t}(2\alpha - \beta))_{\alpha,\beta \in \Gamma_{r,s,t}}$ consists of eigenvalues $1/2^j, j = 0, 1, \ldots, r+s+t-2$. The multiplicity μ_j of each eigenvalue $1/2^j$ is given by $\mu_j = j + 1$ for*

$j = 0, 1, \ldots, r+s-1$ and

$$\mu_j = min\{r, r+s+t-j-1\} + min\{s, r+s+t-j-1\},\ j = r+s, \ldots, r+s+t-2.$$

Moreover, the spectrum of the matrix $(a_{r,s,t}(2\alpha - \beta))_{\alpha,\beta\in\Omega_{r,s,t}}$ is the union of the spectrum of $(a_{r,s,t}(2\alpha - \beta))_{\alpha,\beta\in\Gamma_{r,s,t}}$ with the multiset

$$\left\{\frac{1}{2^{r+s+t-j-1}} : j = 1, \ldots, r, j = 1, \ldots, s, j = 1, \ldots, t\right\} \cup \left\{\frac{1}{2^{r+s+t-2}}\right\}.$$

Proof: Let $U := \mathbb{P}_{r,s,t}|_{\mathbb{Z}^2}$, where $\mathbb{P}_{r,s,t} := \Pi \cap S(B_{r,s,t})$, and let

$$V := \left\{v \in \ell_0(\mathbb{Z}^2) : \sum_{\alpha\in\mathbb{Z}^2} u(-\alpha)v(\alpha) = 0\ \forall u \in U\right\}.$$

For each $\gamma \in \Gamma_{r,s,t}$, by Theorem 4 in [8], we can find a unique $q_\gamma \in \mathbb{P}_{r,s,t}$ such that $q_\gamma(\gamma) = 1$ and $q_\gamma(\alpha) = 0$ for all $\alpha \in \Gamma_{r,s,t}\backslash\{\gamma\}$. Set $p_\gamma(x) := q_\gamma(-x)$ for $x \in \mathbb{R}^2$. Then $p_\gamma \in \mathbb{P}_{r,s,t}$. Suppose $v \in \ell(\Gamma_{r,s,t}) \cap V$. Then

$$v(\gamma) = \sum_{\alpha\in\Gamma_{r,s,t}} q_\gamma(\alpha)v(\alpha) = \sum_{\alpha\in\Gamma_{r,s,t}} p_\gamma(-\alpha)v(\alpha) = 0 \quad \forall\gamma \in \Gamma_{r,s,t}.$$

Hence $v = 0$. In other words, $\ell(\Gamma_{r,s,t}) \cap V = \{0\}$. Therefore, by Theorem 4.5 we obtain

$$\text{spec}(T_{r,s,t}|_{\ell(\Gamma_{r,s,t})}) = \text{spec}(S_{r,s,t}|_U).$$

For $j = 0, 1, \ldots$, by $\mathbb{H}_j$ we denote the linear space of homogeneous polynomials of degree j. Let

$$U_j := \{u \in U : B_{r,s,t} *' u \in \mathbb{H}_j\}.$$

Then U is the direct sum of U_j, $j = 0, 1, \ldots, r+s+t-2$. Let $u \in U_j$ and $p := B_{r,s,t} *' u \in \mathbb{H}_j$. By (4.2.5) we obtain

$$p(x) = \sum_{\alpha \in \mathbb{Z}^2} u(\alpha) B_{r,s,t}(x - \alpha) = \sum_{\alpha \in \mathbb{Z}^2} S_{r,s,t} u(\alpha) B_{r,s,t}(2x - \alpha), \quad x \in \mathbb{R}^2.$$

On the other hand, since p is a homogeneous polynomial of degree j, we have

$$p(x) = 2^{-j} p(2x) = \sum_{\alpha \in \mathbb{Z}^2} 2^{-j} u(\alpha) B_{r,s,t}(2x - \alpha), \quad x \in \mathbb{R}^2.$$

But the shifts of $B_{r,s,t}$ are linearly independent. Thus, the above two equations yield $S_{r,s,t} u = 2^{-j} u$ for all $u \in U_j$. In particular, U_j is invariant under $S_{r,s,t}$. Therefore $\operatorname{spec}(S_{r,s,t}|_U)$ consists of eigenvalues $2^{-j}, j = 0, 1, \ldots, r+s+t-2$. Let μ_j denote the multiplicity of the eigenvalue 2^{-j}. The preceding discussion tells us

$$\mu_j = \dim(U_j) = \dim(\mathbb{P}_{r,s,t} \cap \mathbb{H}_j).$$

In order to find $\dim(\mathbb{P}_{r,s,t} \cap \mathbb{H}_j)$ we employ the polynomial space $\mathbb{F}_{r,s,t}$ introduced in [33]. For a triple $\nu = (\nu_1, \nu_2, \nu_3)$ of nonnegative integers, let p_ν be the polynomial given by

$$p_\nu(x_1, x_2) := x_1^{\nu_1} x_2^{\nu_2} (x_1 + x_2)^{\nu_3}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The space $\mathbb{F}_{r,s,t}$ is defined to be the linear span of all polynomials p_ν for which $\nu_1 \le r$, $\nu_2 \le s$, $\nu_3 \le t$ and at most one equality holds. For two polynomials p and q, let

$$\langle p, q \rangle := p(D) q(0).$$

It was proved in ([33], Theorem 4.1) and in ([34], Theorem 4.1) that $\mathbb{F}_{r,s,t}$ is the dual space of $\mathbb{P}_{r,s,t}$ with respect to this bilinear form. We observe that $\langle p, q \rangle = 0$ provided $p \in \mathbb{H}_j, q \in \mathbb{H}_k$, and $j \neq k$. Hence $\mathbb{F}_{r,s,t} \cap \mathbb{H}_j$ is the dual space of $\mathbb{P}_{r,s,t} \cap \mathbb{H}_j$. This shows that

$$\dim(\mathbb{P}_{r,s,t} \cap \mathbb{H}_j) = \dim(\mathbb{F}_{r,s,t} \cap \mathbb{H}_j).$$

Note that $r \leq s \leq t$. When $j = 0, 1, \ldots, r+s-1$, we have $\mathbb{F}_{r,s,t} \cap \mathbb{H}_j = \mathbb{H}_j$; hence $\mu_j = \dim(\mathbb{H}_j) = j+1$. When $j = r+s, \ldots, r+s+t-2$, the union of

$$\{x_1^r x_2^k (x_1 + x_2)^{j-r-k} : \max\{0, j-r-t+1\} \leq k \leq s-1\}$$

and

$$\{x_1^k x_2^s (x_1 + x_2)^{j-s-k} : \max\{0, j-s-t+1\} \leq k \leq r-1\}$$

forms a basis for $\mathbb{F}_{r,s,t} \cap \mathbb{H}_j$. Therefore, for $j = r+s, \ldots, r+s+t-2$ we have

$$\mu_j = \dim(\mathbb{F}_{r,s,t} \cap \mathbb{H}_j) = \min\{r, r+s+t-j-1\} + \min\{s, r+s+t-j-1\}.$$

This proves the first statement of the theorem.

In order to find the spectrum of $(a_{r,s,t}(2\alpha - \beta))_{\alpha,\beta \in \Omega_{r,s,t}}$, we observe that

$$\Omega_{r,s,t} \backslash \Gamma_{r,s,t} = \{(0,0)\} \cup E_1 \cup E_2 \cup E_3,$$

where $E_1 = \{(j, 0) : j = 1, \ldots, r\}, E_2 = \{(0, j) : j = 1, \ldots, s\}$, and E_3 is the set $\{(j, j+s) : j = 1, \ldots, t\}$. In what follows we omit the subscripts r, s, t. Then for $v \in \ell(E_1)$ we have

$$Tv = T_1 v + w$$

where $w \in \ell(\Gamma)$ and $T_1 v \in \ell(E_1)$ is given by

$$T_1 v(j,0) = \sum_{k=1}^{r} a(2j-k,0)v(k,0), \quad j = 1,\ldots,r.$$

But

$$\sum_{k\in\mathbb{Z}} a(k,0) z_1^k = (1+z_1)^r/2^{r+s+t-2}, \quad z_1 \in \mathbb{C}.$$

Thus, from the analysis for the transition operator associated to a B-spline, we obtain

$$\operatorname{spec}\big((a(2j-k,0))_{1\le j,k\le r}\big) = \{1/2^{r+s+t-n-1} : n = 1,\ldots,r\}.$$

This shows that

$$\operatorname{spec}(T|_{\ell(\Gamma\cup E_1)}) = \operatorname{spec}(T|_{\ell(\Gamma)}) \cup \{1/2^{r+s+t-n-1} : n = 1,\ldots,r\}.$$

Similarly, we have

$$\operatorname{spec}(T|_{\ell(\Gamma\cup E_1\cup E_2)}) = \operatorname{spec}(T|_{\ell(\Gamma\cup E_1)}) \cup \{1/2^{r+s+t-n-1} : n = 1,\ldots,s\}.$$

$$\operatorname{spec}(T|_{\ell(\Gamma\cup E_1\cup E_2\cup E_3)}) = \operatorname{spec}(T|_{\ell(\Gamma\cup E_1\cup E_2)}) \cup \{1/2^{r+s+t-n-1} : n = 1,\ldots,t\}.$$

and

$$\operatorname{spec}(T|_{\ell(\Omega)}) = \operatorname{spec}(T|_{\ell(\Gamma\cup E_1\cup E_2\cup E_3)}) \cup \{1/2^{r+s+t-2}\}.$$

The proof of the theorem is complete. ∎

We used MAPLE to compute the eigenvalues of the matrices $(a_{r,s,t}(2\alpha-\beta))_{\alpha,\beta\in\Omega_{r,s,t}}$

for $2 \le r, s, t \le 4$. The computation confirmed the assertion made in Theorem 4.6.

Theorem 4.6 has an interesting application to interpolatory subdivision schemes. For simplicity, we assume that the dilation matrix M is 2 times the identity matrix. An element $a \in \ell_0(\mathbb{Z}^s)$ is called an **interpolatory mask** if it satisfies (4.1.2) and $a(2\alpha) = \delta(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. Let ϕ be the normalized solution of the refinement equation with an interpolatory mask a. If the subdivision scheme associated with the mask a converges uniformly (see [27]), then ϕ is fundamental, i.e., ϕ is continuous and $\phi(\alpha) = \delta(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. In [18] Deslauriers and Dubuc introduced a general method to construct symmetric interpolatory subdivision schemes on $\mathbb{R}$. In [21], Dyn, Levin, and Micchelli analyzed convergence of the so-called butterfly scheme which is induced by the box spline $B_{1,1,1}$. More generally, using convolutions of box splines with distributions, Riemenschneider and Shen [46] constructed a family of bivariate interpolatory subdivision schemes with symmetry. Recently, Han and Jia [26] provided a general way for construction of bivariate interpolatory refinement masks such that the corresponding fundamental and refinable functions attain the optimal approximation order and smoothness order.

Let r, s, t be positive integers. We assume that both $r+t$ and $s+t$ are even integers. Thus, the box spline $B_{r,s,t}$ is symmetric about the point $((r+t)/2, (s+t)/2)$. Its shift $B_{r,s,t}(x_1 - (r+t)/2, x_2 - (s+t)/2)$ $((x_1, x_2) \in \mathbb{R}^2)$ is refinable with the mask h given by its symbol

$$\tilde{h}(z_1, z_2) = z_1^{-(r+t)/2} z_2^{-(s+t)/2} (1+z_1)^r (1+z_2)^s (1+z_1 z_2)^t / 2^{r+s+t-2}.$$

Clearly, h is supported on $(-(r+t)/2, -(s+t)/2) + K_{r,s,t}$, where $K_{r,s,t}$ is the hexagon given in (4.4.1). Let

$$\Gamma := \mathbb{Z}^2 \cap \left[\left(-\frac{r+t-2}{2}, -\frac{s+t-2}{2}\right) + K_{r-1,s-1,t-1}\right].$$

The following result was established by Riemenschneider and Shen [46] for the case $r = s = t \le 8$. We extend their result to the general case.

Theorem 4.7 *There exists a unique sequence c supported on Γ such that the mask a given by $a = h * c$ is interpolatory.*

Proof: We have

$$a(2\alpha) = \sum_{\beta \in \mathbb{Z}^2} h(2\alpha - \beta)c(\beta), \quad \alpha \in \mathbb{Z}^2.$$

The mask a is supported on $(-r-t+1, -s-t+1) + K_{2r-1,2s-1,2t-1}$. It can be easily verified that $\alpha \in \Gamma$ if and only if

$$2\alpha \in \mathbb{Z}^2 \cap \left[(-r-t+1, -s-t+1) + K_{2r-1,2s-1,2t-1}\right].$$

It follows that $a(2\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \backslash \Gamma$. Hence, a is interpolatory if and only if

$$\sum_{\beta \in \Gamma} h(2\alpha - \beta)c(\beta) = \delta(\alpha) \quad \forall \alpha \in \Gamma.$$

Thus, it suffices to show that the matrix $(h(2\alpha - \beta))_{\alpha,\beta\in\Gamma}$ is invertible.

Let T_h be the transition operator associated to h. Then $\ell(\Gamma)$ is invariant under T_h. Let

$$\Omega := \mathbb{Z}^2 \cap \left[\left(-\frac{r+t}{2}, -\frac{s+t}{2}\right) + K_{r,s,t}\right].$$

By Theorem 4.6, all the eigenvalues of $T_h|_{\ell(\Omega)}$ are powers of 1/2. But $\Gamma \subseteq \Omega$, so that the spectrum of $T_h|_{\ell(\Gamma)}$ is contained in spec$(T_h|_{\ell(\Omega)})$. Hence all eigenvalues of $(T_h|_{\ell(\Gamma)})$ are nonzero. This shows that the matrix $(h(2\alpha-\beta))_{\alpha,\beta\in\Gamma}$ is invertible. The proof of the theorem is complete. ∎

The following theorem provides a method to simplify the computation of the smoothness order of refinable functions which are convolutions of box splines $B_{r,r,r}$ with refinable distributions. In what follows we use T^2 to denote the torus

$$\{(z_1,z_2)\in \mathbb{C}^2 : |z_1|=1, |z_2|=1\}.$$

Theorem 4.8 *Let c be an element in $\ell_0(\mathbb{Z}^2)$ such that $\sum_{\alpha\in\mathbb{Z}^2} c(\alpha)=4$, and let a be given by its symbol*

$$\tilde{a}(z)=\left(\frac{1+z_1}{2}\right)^r\left(\frac{1+z_2}{2}\right)^r\left(\frac{1+z_1z_2}{2}\right)^r\tilde{c}(z),\quad z=(z_1,z_2)\in T^2,$$

where r is a positive integer. Let ϕ be the normalized solution of the refinement equation $\phi=\sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\phi(2\cdot-\alpha)$. Write z_3 for z_1z_2. Let a_j $(j=1,2,3)$ be given by

$$\tilde{a}_j(z)=\left(\frac{1+z_j}{2}\right)^r\tilde{c}(z),\quad z\in T^2,$$

and let b_j $(j=1,2,3)$ be given by

$$\tilde{b}_j(z)=|\tilde{a}_j(z)|^2/4,\quad z\in T^2.$$

Let $\rho := \max_{1\le j\le 3}\{\rho(T_{b_j})\}$. *If* $\rho > 1$ *and if the shifts of* ϕ *are stable, then*

$$\nu(\phi) = 2r - \log_4 \rho. \tag{4.4.2}$$

Proof: Let $b \in \ell_0(\mathbb{Z}^2)$ be given by

$$\tilde{b}(z) = |\tilde{a}(z)|^2/4, \quad z \in T^2.$$

By (4.1.8) we have $\nu(\phi) \ge -\log_4 \rho_{2r}$, where $\rho_{2r} := \rho(T_b|_{V_{4r-1}})$. Moreover, if $2r > -\log_4 \rho_{2r}$, and if the shifts of ϕ are stable, then $\nu(\phi) = -\log_4 \rho_{2r}$. For $j = 1, 2, 3$, we use Δ_j to denote the difference operator on $\ell_0(\mathbb{Z}^2)$ given by

$$\Delta_j v := -v(\cdot - e_j) + 2v - v(\cdot + e_j), \quad v \in \ell_0(\mathbb{Z}^2),$$

where $e_1 = (1,0), e_2 = (0,1)$, and $e_3 = (1,1)$. Let V be the linear span of $\Delta_1^r\Delta_2^r\delta_\beta$, $\Delta_2^r\Delta_3^r\delta_\beta$, and $\Delta_1^r\Delta_3^r\delta_\beta$, $\beta \in \mathbb{Z}^2$, and let

$$U := \{u \in \ell(\mathbb{Z}^2) : \langle u, v\rangle = 0 \ \ \forall v \in V\},$$

where $\langle u, v\rangle$ is the bilinear form given in (4.1.5). Then u belongs to U if and only if u satisfies the following system of partial difference equations:

$$\Delta_1^r\Delta_2^r u = 0, \ \Delta_1^r\Delta_3^r u = 0, \ \Delta_2^r\Delta_3^r u = 0.$$

By [9] Proposition 2.1 we have

$$U = \mathbb{P}_{2r,2r,2r}|_{\mathbb{Z}^2}.$$

Note that $\ell(\mathbb{Z}^2)$ is the dual space of $\ell_0(\mathbb{Z}^2)$ with respect to the bilinear form $\langle u, v\rangle$. Suppose $w \in \ell_0(\mathbb{Z}^2)\backslash V$. Then there exists an element $u \in \ell(\mathbb{Z}^2)$ such that $\langle u, v\rangle = 1$ and $\langle u, v\rangle = 0$ for all $v \in V$. This shows

$$V = \{v \in \ell_0(\mathbb{Z}^2) : \langle u, v\rangle = 0 \quad \forall u \in U\}.$$

Since U is invariant under the subdivision operator S_b, V is invariant under the transition operator T_b, by Theorem 4.5. Let $U_k := \Pi_k|_{\mathbb{Z}^2}$. Then we have $U_{4r-1} \subseteq U$ and $V \subseteq V_{4r-1}$.

We observe that $\sum_{n=1}^{\infty} 2^{-n}\operatorname{supp} b$ is contained in the convex hull of $\operatorname{supp} b$. Let Ω be the intersection of $\mathbb{Z}^2$ with the convex hull of $\operatorname{supp} b$. By Theorem 4.5 we have

$$\operatorname{spec}(T_b|_{\ell(\Omega)}) = \operatorname{spec}(T_b|_{\ell(\Omega)\cap V}) \cup \operatorname{spec}(S_b|_U)$$

and

$$\operatorname{spec}(T_b|_{\ell(\Omega)}) = \operatorname{spec}(T_b|_{\ell(\Omega)\cap V_{4r-1}}) \cup \operatorname{spec}(S_b|_{U_{4r-1}}).$$

From the proof of Theorem 4.6 we see that the difference of $\operatorname{spec}(S_b|_U)$ and $\operatorname{spec}(S_b|_{U_{4r-1}})$ consists of eigenvalues 2^{-j} $(j = 4r, \ldots, 6r-2)$ with certain multiplicities. Hence

$$\rho_{2r} = \rho(T_b|_{\ell(\Omega)\cap V_{4r-1}}) = \max\{\rho_V, 2^{-4r}\},$$

where $\rho_V := \rho(T_b|_{\ell(\Omega)\cap V})$. For convenience, we set $\Delta_{j+3} := \Delta_j, j = 1, 2, 3$. In order to find ρ_V, let W_j be the minimal invariant subspace of T_b generated by the sequences $\Delta_{j+1}^r \Delta_{j+2}^r \delta_\beta,\ \beta \in \mathbb{Z}^2$. Then $V = W_1 + W_2 + W_3$, so

$$\rho_V = \max_{1 \le j \le 3}\{\rho(T_b|_{W_j})\}.$$

Let S_a denote the subdivision operator associated to a as defined in (4.1.4). It follows from Theorem 4.1 in [27] and Theorem 3.3 in [30] that

$$\lim_{n\to\infty} \|\nabla_1^r \nabla_2^r S_a^n \delta\|_2^{1/n} = 2\sqrt{\rho(T_b|_{W_3})}.$$

Since $\tilde{a}(z) = 2^{-2r}(1+z_1)^r(1+z_2)^r \tilde{a}_3(z)$, by [35] Theorem 3.3 and [26] Theorem 3.1 we have

$$\lim_{n\to\infty} \|\nabla_1^r \nabla_2^r S_a^n \delta\|_2^{1/n} = 2^{-2r} \lim_{n\to\infty} \|S_{a_3}^n \delta\|_2^{1/n}.$$

But $\tilde{b}_3(z) = |\tilde{a}_3(z)|^2/4,\ z \in T^2$. Hence

$$\lim_{n\to\infty} \|S_{a_3}^n \delta\|_2^{1/n} = 2\sqrt{\rho(T_{b_3})}.$$

The preceding discussion tells us that

$$\rho(T_b|_{W_j}) = 2^{-4r}\rho(T_{b_j})$$

is true for $j = 3$. Clearly, this relation is also valid for $j = 1$ or $j = 2$. It follows that

$$\rho_V = \max_{1 \le j \le 3}\{\rho(T_b|_{W_j})\} = 2^{-4r} \max_{1 \le j \le 3}\{\rho(T_{b_j})\} = 2^{-4r}\rho.$$

By our assumption, $\rho > 1$. Hence $\rho_{2r} = \max\{\rho_V, 2^{-4r}\} = \rho_V > 2^{-4r}$. It follows that $2r > -\log_4 \rho_{2r}$. If, in addition, the shifts of ϕ are stable, then

$$\nu(\phi) = -\log_4 \rho_{2r} = -\log_4 \rho_V = 2r - \log_4 \rho.$$

This verifies (4.4.2). ∎

The following example demonstrates the power of Theorem 4.8 in the computation of the smoothness order of refinable functions which are convolution of box splines with refinable distributions.

Example 4.9 *For $r = 1, 2, \dots$, let h_r be the mask on $\mathbb{Z}^2$ given by its symbol*

$$\tilde{h}_r(z_1, z_2) = z_1^{-r} z_2^{-r} (1 + z_1)^r (1 + z_2)^r (1 + z_1 z_2)^r / 2^{3r-2}.$$

By Theorem 4.7, there exists a unique sequence c_r supported in

$$(1 - r, 1 - r) + K_{r-1,r-1,r-1}$$

such that $a_r := h_r * c_r$ is an interpolatory mask. Let ϕ_r be the normalized solution of the refinement equation

$$\phi_r = \sum_{\alpha \in \mathbb{Z}^2} a_r(\alpha) \phi_r(2 \cdot - \alpha).$$

The smoothness order $\nu(\phi_r)$ was computed in [46] for $r = 2, 3, \dots, 8$. Theorem 4.8 enables us to simplify the computation significantly so that we obtain $\nu(\phi_r)$ for

r	$\nu(\phi_r)$	$\nu(f_r)$
9	5.89529419	6.33524331
10	6.42640635	6.81143594
11	6.17848062	7.28259907
12	6.68092993	7.74953085
13	6.41506309	8.21284369
14	6.89718935	8.67302201
15	6.61823707	9.13045707
16	7.08520104	9.58546997

$r = 9, \dots, 16$ *in the above table.*

The interpolatory mask a_r is obtained by solving the system of linear equations

$$\sum_{\beta \in \Gamma_r} h_r(2\alpha - \beta) c_r(\beta) = \delta(\alpha) \quad \forall \alpha \in \Gamma_r, \tag{4.4.3}$$

where $\Gamma_r := \mathbb{Z}^2 \cap ((1-r, 1-r) + K_{r-1,r-1,r-1})$. For large r, the coefficient matrix $(h_r(2\alpha - \beta))_{\alpha\beta\in\Gamma_r}$ is ill-conditioned. To overcome this difficulty, we use MAPLE to find the exact solution of (4.4.3) for $r = 9, \dots, 16$. By Theorem 4.7 for each $\beta \in \Gamma_r, c_r(\beta)$ is a quotient of one integer divided by a power of 2. The computation confirms our assertion.

In [18], Deslauriers and Dubuc showed that, for each $r = 1, 2 \dots$, there exists a unique interpolatory mask B_r supported on $[1-2r, 2r-1]$ such that B_r is symmetric about the origin and its symbol $\tilde{b}_r(z)$ is divisible by $(1+z)^{2r}$. Let f_r be the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}} b_r(\alpha)\phi(2\cdot - \alpha)$. The smoothness order $\nu(f_r)$ was computed in [22] for $r = 1, 2, \dots, 20$. For the purpose of comparison, we have listed the values of $\nu(f_r)$ $(r = 9, \dots, 16)$ in the above table.

Suppose a is an interpolatory mask supported on the square $[1-2r, 2r-1]^2$ and ϕ is the corresponding refinable function. It was proved in [26] that $\nu(\phi) \leq \nu(f_r)$, provided ϕ has accuracy $2r$. Thus, we say that a is optimal if ϕ has accuracy $2r$ and $\nu(\phi) = \nu(f_r)$. It was shown in [46] that a_r is optimal for $r = 2, \ldots, 8$. However, the above table demonstrates that a_r is not optimal for $r = 9, \ldots, 16$. Moreover, $\nu(\phi_r)$ is not an increasing function of r. In particular, $\nu(\phi_{13}) < \nu(\phi_{10})$ and $\nu(\phi_{15}) < \nu(\phi_{12})$. It seems that $\nu(\phi_{2r})$ and $\nu(\phi_{2r-1})$ are increasing functions of r, respectively. ∎

CHAPTER 5

Computation of the Smoothness Order of Refinable Functions

5.1 Introduction

In this chapter, we are concerned with functional equations of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha) \tag{5.1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, and a is a finitely supported sequence on $\mathbb{Z}^s$. Wavelets are generated from refinable functions. The approximation and smoothness properties of wavelets are determined by the corresponding refinable functions. It is important to know the smoothness order of refinable functions.

If the refinement mask is given, methods of calculating the smoothness order of the corresponding refinable function is an important topic in the study of wavelets. See Eirola [22], Villemoes [48], R. Q. Jia [35, 30], Cohen and Daubechies [4].

For $\nu > 0$, we denote by $W_2^\nu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ for which $\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2(1+|\xi|^\nu)^2 d\xi < \infty$. The **smoothness order** of f is defined by $\nu(f) := \sup\{\nu : f \in W_2^\nu(\mathbb{R}^s)\}$. In this chapter, we propose two numerical procedures to calculate the smoothness order of the refinable function if the refinement mask is given.

In [30], Jia investigated the smoothness properties of multivariate refinable functions in Sobolev spaces. He characterized the optimal smoothness order of a multivariate refinable function in terms of the spectral radius of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace.

Before we go to detail, let us introduce some useful notation.

Let M be an $s \times s$ matrix with its entries in $\mathbb{C}$. We say that M is isotropic if M is similar to diagonal matrix $\mathrm{diag}\{\sigma_1, \ldots, \sigma_s\}$ with $|\sigma_1| = \ldots = |\sigma_s|$. Let $m := |\det M|$.

A sequence u on $\mathbb{Z}^s$ is called a **polynomial sequence** if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of u is the same as the degree of p. For a nonnegative integer k, let Π_k be the linear space of all polynomial sequences of degree at most k, and let

$$V_k := \left\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} p(\alpha)v(\alpha) = 0 \ \ \forall\, p \in \Pi_k \right\}. \tag{5.1.2}$$

We observe that V_k is shift-invariant, that is, $v \in V_k$ implies $v(\cdot - \alpha) \in V_k$ for every $\alpha \in \mathbb{Z}^s$.

Now, we are in a position to state the theorem proved in [30], which is the basis for our numerical procedures.

Theorem 5.1 *Let ϕ be the normalized solution of the refinement equation (5.1.1) with the dilation matrix M and the mask a. Suppose that the dilation matrix M is isotropic. Let $b := a * a^*/m$, where a^* is the sequence given by $a^*(\alpha) = \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. If k is the largest integer such that $S(\phi)$ contains Π_{k-1}, then V_{2k-1} is an invariant subspace of T_b. Moreover, if the shifts of ϕ are stable, then*

$$\nu(\phi) = (-\log_m \rho)s/2, \tag{5.1.3}$$

where ρ is the spectral radius of the linear operator $T_b|_{V_{2k-1}}$.

One might ask how we can select the largest integer k? Well, this k can be determined by checking the order of the so-called sum rules satisfied by the refinement mask. For an $s \times s$ dilation matrix M, let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Let k be a positive integer, an element $a \in \ell_0(\mathbb{Z}^s)$ is said to satisfy the sum rules of order k if for all $p \in \Pi_{k-1}$

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta)p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)p(M\beta+\gamma) \quad \forall\, \gamma \in \Gamma. \tag{5.1.4}$$

The following result proved in [29] gives the relationship between the order of sum rules and the approximation order provided by a refinable function ϕ.

Theorem 5.2 *Let ϕ be the normalized solution of the refinement equation (5.1.1) with the dilation matrix M and mask a. If the refinement mask a satisfies the sum* *rules of order k, then $S(\phi)$ contains Π_{k-1}. Conversely, if $S(\phi)$ contains Π_{k-1}, and if the shifts of ϕ are stable, then a satisfies the sum rules of order k.*

From Theorem 5.1, it turns out that the critical exponent of the normalized solution of (5.1.1) can be computed by finding ρ_k, $\rho_k := \rho(T_b|_{V_{2k-1}})$, where the spectral radius of T_b is restricted to V_{2k-1} if ϕ satisfies the sum rules of order k.

The next question is: how to calculate ρ_k?

Let us review some facts. Suppose that a is an element of $\ell_0(\mathbb{Z}^s)$ and M is a dilation matrix. Let H be a compact set such that $H \supseteq \operatorname{supp} a$ and

$$\Omega := \left(\sum_{n=1}^{\infty} M^{-n} H\right) \cap \mathbb{Z}^s := \left\{\sum_{n=1}^{\infty} M^{-n} h_n : \; h_n \in H \; \forall n \in \mathbb{N}\right\} \cap \mathbb{Z}^s. \tag{5.1.5}$$

Then $\operatorname{supp} \phi \cap \mathbb{Z}^s \subseteq \Omega$ and $\ell(\Omega)$ is invariant under the transition operator T_a. Moreover, for any $v \in \ell_0(\mathbb{Z}^s)$, there exists some integer r such that $T_a^r v \in \ell(\Omega)$, (see [27]). From this fact, we can draw several useful consequences. If $v \in \ell_0(\mathbb{Z}^s)$ is an eigenvector of T_a corresponding to an eigenvalue σ, then $\sigma^r v = T_a^r v \in \ell(\Omega)$ for sufficiently large r. Hence $\sigma \neq 0$ implies $v \in \ell(\Omega)$, and $v \notin \ell(\Omega)$ implies $\sigma = 0$. This shows that T_a only has finitely many nonzero eigenvalues. For an invariant subspace V of T_a, we define the spectral radius of $T_a|_V$ by

$$\rho(T_a|_V) := \rho(T_a|_{\ell(\Omega)\cap V}). \tag{5.1.6}$$

In particular, $\rho(T_a) := \rho(T_a|_{\ell(\Omega)})$. Note that the subdivision operator S_a and the transition operator T_a have the same nonzero eigenvalues (see [36]).

From the above discussion, it follows that we can find ρ_k by finding the spectral radius of $T_b|_{V_{2k-1}\cap\ell(\Omega)}$. Since we know that $\rho(T_b) = \rho(T_b|_{\ell(\Omega)})$, we can represent $T_b|_{\ell(\Omega)}$ in matrix form. By finding the spectrum of the corresponding matrix, we can determine the spectrum of T_b. From the spectrum of T_b, we can find the spectral radius of $T_b|_{V_{2k-1}\cap\ell(\Omega)}$, by picking out the eigenvalues whose corresponding eigenvectors are not in V_{2k-1}. In [46], Riemenschneider and Shen proposed a numerical procedure based on this idea. The first step is to find all eigenvalues and eigenvectors of $T_b|_{\ell(\Omega)}$, and then throw out those eigenvalues whose corresponding eigenvectors are not in V_{2k-1} by checking if the corresponding eigenvector $v = (v_\beta)$ satisfies

$$\max_{0\le|\ell|\le 2k-1} \left|\sum_\beta \beta^\ell v_\beta\right| > 0.$$

In this chapter, we will apply the theory of Chapter 4 to the smoothness analysis of refinable functions in terms of their mask. We will provide a theorem to identify $\rho(T_b|_{V_{2k-1}})$ from the spectrum of $T_b|_{\ell(\Omega)}$. By this technique, we can handle refinement masks of large size. Then we will consider a refinable function possessing a certain symmetric property. By taking advantage of the symmetric property, we can find ρ_k by calculating the spectrum of T_b restricted to a subspace of $W \cap \ell(\Omega)$ where W is an invariant subspace of S_b. This shrinks the matrix size substantially. Numerical examples are provided.

5.2 A Method to Compute the Smoothness Order of Refinable Functions

As we discussed before, the main problem in the computing smoothness order of an refinable function is to find the spectral radius of $T_b|_{V_{2k-1}}$.

In Chapter 4, Theorem 4.5 tells us the relationship among the spectra of the transition operator restricted to different invariant subspaces. That is, if U is a finite dimensional subspace of $\ell(\mathbb{Z}^s)$, we define

$$V := \left\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} u(-\alpha)v(\alpha) = 0 \ \forall u \in U \right\}.$$

Let Ω be a finite subset of $\mathbb{Z}^s$ such that $\ell(\Omega)$ is invariant under T_a. Let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ as defined in (4.2.3). If U is invariant under S_a, and if $Q|_U$ is one-to-one, then we have

$$\operatorname{spec}(T_a|_{\ell(\Omega)}) = \operatorname{spec}(T_a|_{\ell(\Omega)\cap U}) \cap \operatorname{spec}(S_a|_U).$$

The case $U = \Pi_{k-1}|_{\mathbb{Z}^s}$ is of particular interest. Suppose a satisfies the sum rules of order k. Then U is invariant under S_a, by Theorem 4.4. By Theorem 4.2 we have $\operatorname{spec}(S_a|_U) = \{\sigma^{-\mu} : |\mu| < k\}$. Thus, we have the following theorem.

Theorem 5.3 *Let ϕ be the normalized solution of the refinement equation (5.1.1) with a mask a and an isotropic dilation matrix M. Let b be the sequence given by $b := a * a^*/m$, where $a^* = \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. Suppose a satisfies the sum rules of* *order k. Then*

$$spec(T_b|_{\ell(\Omega)}) = spec(T_b|_{\ell(\Omega)\cap V_{2k-1}}) \cup \{\sigma^{-\mu} : 1 \le |\mu| < 2k\}, \qquad (5.2.1)$$

where $\Omega := \mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} b)$.

For the univariate case ($s = 1$), this formula was established by Deslauriers and Dubuc in ([18], Theorem 8.2).

We can find ρ_k from $\operatorname{spec}(T_b|_{\ell(\Omega)})$ by using the formula in (5.2.1). Rounding errors might occur in the computation of the spectrum of $T_b|_{\ell(\Omega)}$. But we know *a* priori that $\sigma^{-\mu}$ ($|\mu| < 2k$) are eigenvalues of $T_b|_{\ell(\Omega)}$. In the list of the computed eigenvalues of $T_b|_{\ell(\Omega)}$, remove the complex number closest to $\sigma^{-\mu}$ for each μ, $|\mu| < 2k$. Then ρ_k is the maximum of the absolute value of the remaining eigenvalues. The following examples illustrate this technique.

Example 5.4 *Let M be the matrix*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and let a be the mask given in Example 4.3. Let us determine the smoothness order of the normalized solution ϕ of the refinement equation with mask a and dilation M.

Let b be the mask computed from a by using $b = a * a^*/m$, where $m = |\det M|$. Then $\operatorname{supp} b \subseteq [-4,4]^2$ and the set $\sum_{n=1}^{\infty} M^{-n}([-4,4]^2)$ is

$$\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \le 12, |x_2| \le 12, |x_1 - x_2| \le 16, |x_1 + x_2| \le 16\}.$$

The set $\Omega := \mathbb{Z}^2 \cap (\sum_{n=1}^{\infty} M^{-n}([-4,4]^2))$ has exactly 481 points. We arrange the 481 eigenvalues of the matrix $\left(b(M\alpha-\beta)\right)_{\alpha,\beta\in\Omega}$ in the order of descending absolute values. The following is a list of the first 22 eigenvalues.

$$1,\ 0.5+0.5i,\ 0.5-0.5i,\ 0.5,\ 0.5i,\ -0.5i,$$

$$-0.25+0.25i,\ -0.25-0.25i,\ 0.25+0.25i,\ 0.25-0.25i,$$

$$-0.25,\ -0.25,\ 0.25i,\ 0.25i,\ 0.25,$$

$$0.1832744177,\ 0.125+0.125i,\ 0.125-0.125i,$$

$$-0.125+0.125i,\ -0.125-0.125i,\ -0.125+0.125i,\ -0.125-0.125i$$

Note that the matrix M has two eigenvalues $\sigma_1 = 1+i$ and $\sigma_2 = 1-i$. In the above list, 21 eigenvalues are of the form $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for double indices (μ_1,μ_2) with $\mu_1+\mu_2 \le 5$. Therefore, $\rho_4 \approx 0.1832744177$. From the results in [27] we know that the subdivision scheme associated to mask a and dilation matrix M converges uniformly. Moreover, the mask a is interpolatory (see Section 4.4). Hence the shifts of ϕ are stable. By (5.1.3) we obtain

$$\nu(\phi) = -\log_2 0.1832744178 \approx 2.44792267.$$

We conclude that $\nu(\phi) \approx 2.44792267$ ∎

Let us look at another example given in [46].

Example 5.5 *For $r = 1, 2, \cdots$, let h_r be the mask on $\mathbb{Z}^2$ given by its symbol*

$$\tilde{h}_r(z_1, z_2) = z_1^{-r} z_2^{-r}(1 + z_1)^r(1 + z_2)^r(1 + z_1 z_2)^r / 2^{3r-2}.$$

By Theorem 4.8, there exists a unique sequence c_r supported in

$$(1 - r, 1 - r) + K_{r-1,r-1,r-1}$$

such that $a_r := h_r * c_r$ is an interpolatory mask. Let ϕ_r be the normalized solution of the refinement equation

$$\phi_r = \sum_{\alpha \in \mathbb{Z}^2} a_r(\alpha)\phi_r(2 \cdot - \alpha).$$

Let $b_r := a_r * a_r^*/4$. We compute the smoothness order $\nu(\phi_r)$, $r = 2, \cdots, 8$ by using Theorem 5.3 and Theorem 5.1. We display the result in Table 5.1.

Since the interpolatory mask a_r is supported on $[1 - 2r, 2r - 1]^2$. Then

$$\operatorname{supp} b_r \subseteq [2 - 4r, 4r - 2]^2, \text{ and } \sum_{n=1}^{\infty} 2^{-n}([2 - 4r, 4r - 2]^2) = [2 - 4r, 4r - 2]^2,$$

i.e., $\Omega := \mathbb{Z}^2 \cap [2 - 4r, 4 - 2r]^2$. We can compute all the eigenvalues of the matrix $(B_r(2\alpha - \beta))_{\alpha,\beta \in \Omega}$. Since b_r satisfies the sum rules of order $2r$, we throw away $2^{-|\mu|}$ for each μ, $|\mu| < 2r$. Then ρ_r is the maximum of the absolute value of the remaining eigenvalues. Since a_r is interpolatory, the shifts of ϕ are stable. We conclude that

$$\nu(\phi_r) = -\log_4 \rho_r, \quad r = 1, 2, \cdots.$$

Table 5.1: Numerical results

r	Size of a_r	Size of b_r	size of $T_{b_r}\|_{\ell(\Omega)}$	$\rho(T_b\|_{\ell(\Omega)\cap V_{2k-1}})$	$\nu(\phi_r)$
2	7×7	13×13	169×169	0.135697788100	2.4407654451
3	11×11	21×21	441×441	0.049027566992	3.1751315103
4	15×15	29×29	841×841	0.020814492581	3.7931339005
5	19×19	37×37	1369×1369	0.009697524658	4.3440838721
6	23×23	45×45	2025×2025	0.004729686883	4.8620198038
7	27×27	53×53	2809×2809	0.002362188508	5.3628300925
8	31×31	61×61	3721×3721	0.001197453657	5.8529072308

We compute $\nu(\phi_r)$, $r = 2, \cdots, 8$ in the above table. ∎

5.3 Computation of the Smoothness Order of Symmetric Refinable Functions

In this section, we will investigate the computation of the smoothness order of a refinable function by taking advantage of the symmetric properties of the refinable function. We focus on the bivariate case, the dilation matrix $M = 2I$ where I is 2×2 identity matrix, and $m := |\det M| = 4$. Usually, we hope the function which we are going to study has some good properties, for example, symmetry, smoothness, etc. In [46], Riemenschneider and Shen claimed that there is a closed connection between smoothness and symmetry, that is, with the increased symmetry comes increased smoothness. Han and Jia [26] provided a general method for the construction of bivariate interpolatory refinement masks such that the corresponding fundamental refinable functions attain the optimal approximation order and smoothness order.

Specifically, these interpolatory refinement masks are minimally supported and enjoy full symmetry. By Theorem 5.3, we can identify ρ_k from the spectrum of $T_b|_{\ell(\Omega)}$. But if the size of the mask a becomes big, it is difficult to carry out the numerical procedure to calculate the spectrum of $T_b|_{\ell(\Omega)}$. Our goal is to find an invariant subspace W of S_b such that the spectrum of $T_b|_{\ell(\Omega)\cap W}$ will contain an eigenvalue whose absolute value is the spectral radius of $T_b|_{V_{2k-1}}$. By considering the matrix representation of T_b on $\ell(\Omega)\cap W$, we can reduce substantially the size of the matrix of $T_b|_{\ell(\Omega)\cap W}$, so that we can carry out the numerical procedure easily. Several examples are explicitly computed.

Let $a \in \ell_0(\mathbb{Z}^2)$ be a refinable mask. Let $b = a * a^*/m$, and

$$\Omega := \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \operatorname{supp} b\right) \cap \mathbb{Z}^2 = \operatorname{supp} b \cap \mathbb{Z}^2.$$

Let T_b denote the transition operator associated with b. Since $\ell(\Omega)$ is invariant under T_b (see [27]), we consider the matrix representation of $T_b|_{\ell(\Omega)}$. We use T to denote the matrix of $T_b|_{\ell(\Omega)}$. It is easy to see that b is symmetric with respect to the origin, i.e., $b(-\alpha) = b(\alpha)$.

As we mentioned before, we consider the matrix representation T of T_b restricted to $\ell(\Omega) \cap W$. Then we find the spectrum of $T_b|_{\ell(\Omega)\cap W}$ by calculating the eigenvalues of matrix T. We identify the spectral radius of $T_b|_{\ell(\Omega)\cap V_{2k-1}}$ from the spectrum of $T_b|_{\ell(\Omega)\cap W}$. The key is that we have to choose W such that the spectrum of $T_b|_{\ell(\Omega)\cap W}$ contains the eigenvalue such that its absolute value is the spectral radius of $T_b|_{\ell(\Omega)\cap V_{2k-1}}$. To do this, we also have to provide criterion to identify the spectral radius of $T_b|_{\ell(\Omega)\cap V_{2k-1}}$ from the spectrum of $T_b|_{\ell(\Omega)\cap W}$.

The following theorem clarifies the relationship among the spectra of the transition operator restricted to different invariant subspace. It can be derived from Theorem 4.5 in Chapter 4.

Theorem 5.6 *Let U be a finite dimensional subspace of $\ell(\mathbb{Z}^2)$ and suppose U is invariant under S_a. Define*

$$V := \{v \in \ell_0(\mathbb{Z}^2) : \langle u, v\rangle = 0, \quad \forall u \in U\} \tag{5.3.1}$$

Let W be a subspace of $\ell(\mathbb{Z}^2)$ such that $W \cap U$ is an invariant subspace of S_a and $W \cap \ell(\Omega)$ is an invariant subspace of T_a, and let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^2)$ to $\ell(-\Omega)$. If $Q|_U$ is one to one, then

$$\operatorname{spec}(T_a|_{\ell(\Omega)\cap W}) = \operatorname{spec}(T_a|_{\ell(\Omega)\cap V\cap W}) \cup \operatorname{spec}(S_a|_{U\cap W}). \tag{5.3.2}$$

Next, we will use the above formula to calculate the smoothness order of a refinable function if the associated mask a has certain symmetric properties. Before proceeding further, we investigate properties which are related to the symmetric properties of refinement masks.

In the following, we assume that the mask a satisfies the sum rules of order k. It follows that $b = a * a^*/m$ satisfies the sum rules of order $2k$. Let $U_k := \Pi_k|_{\mathbb{Z}^2}$. From [35], we know that U_{2k-1} is invariant under S_b and V_{2k-1} is invariant under T_b. From (5.3.2), we will have

$$\operatorname{spec}(T_b|_{\ell(\Omega)\cap W}) = \operatorname{spec}(T_b|_{\ell(\Omega)\cap V_{2k-1}\cap W}) \cup \operatorname{spec}(S_b|_{U_{2k-1}\cap W}). \tag{5.3.3}$$

We define

$$W_2 := \{w \in \ell(\mathbb{Z}^2) : \ w(\alpha) = w(-\alpha), \forall \alpha \in \mathbb{Z}^2\}. \tag{5.3.4}$$

Since $b(\alpha) = \sum_{\beta\in\mathbb{Z}^2} a(\alpha+\beta)\overline{a(\beta)}/m$, then

$$b(-\alpha) = \sum_{\beta\in\mathbb{Z}^2} a(-\alpha+\beta)\overline{a(\beta)}/m = \sum_{\beta\in\mathbb{Z}^2} a(\alpha-\beta)\overline{a(-\beta)}/m = b(\alpha). \tag{5.3.5}$$

Thus, we have the following result.

Lemma 5.7 *If b satisfies $b(\alpha) = b(-\alpha)$ for all $\alpha \in \mathbb{Z}^2$ and satisfies the sum rules of order $2k$, then $W_2 \cap U_{2k-1}$ is invariant under S_b. Moreover, $W_2 \cap \ell(\Omega)$ is invariant under T_b.*

Proof: For $w \in W_2$ and $\alpha \in \mathbb{Z}^2$,

$$\begin{aligned} S_b w(-\alpha) &= \sum_\beta b(-\alpha-2\beta)w(\beta) \\ &= \sum_\beta b(-(\alpha+2\beta))w(\beta) = \sum_\beta b(\alpha+2\beta)w(\beta) \\ &= \sum_\beta b(\alpha-2\beta)w(-\beta) = S_b w(\alpha). \end{aligned}$$

This means that W_2 is invariant under S_b. Since U_{2k-1} is invariant under S_b, so is

$U_{2k-1} \cap W_2$.

For any $w \in W_2 \cap \ell(\Omega)$, $T_b w \in \ell(\Omega)$ since $\ell(\Omega)$ is invariant under T_b. For each $\alpha \in \Omega$, we have

$$T_b w(-\alpha) = \sum_\beta b(-2\alpha - \beta) w(\beta) = \sum_\beta b(2\alpha + \beta) w(\beta)$$
$$= \sum_\beta b(2\alpha - \beta) w(-\beta) = T_b w(\alpha).$$

Hence $W_2 \cap \ell(\Omega)$ is an invariant subspace of T_b. ∎

Consider the case where the mask b has the following symmetric properties:

$$b(\alpha_1, \alpha_2) = b(-\alpha_1, \alpha_2) = b(-\alpha_1, -\alpha_2) = b(\alpha_1, -\alpha_2) \quad \forall \alpha \in \operatorname{supp} b. \tag{5.3.6}$$

Let W_4 be the subspace of $\ell(\mathbb{Z}^2)$ given by

$$W_4 := \{w \in \ell(\mathbb{Z}^2) : \; w(\alpha_1, \alpha_2) = w(-\alpha_1, \alpha_2)$$
$$= w(-\alpha_1, -\alpha_2) = w(\alpha_1, -\alpha_2) \; \forall (\alpha_1, \alpha_2) \in \mathbb{Z}^2\}.$$

Then, we have the following lemma:

Lemma 5.8 *If b satisfies (5.3.6) and the sum rules of order $2k$,, then $W_4 \cap U_{2k-1}$ is invariant under S_b. Moreover, $W_4 \cap \ell(\Omega)$ is invariant under T_b.*

Proof: For $w \in W_4$ and $\alpha \in \mathbb{Z}^2$,

$$S_b w(-\alpha_1, \alpha_2) = \sum_{\beta} b(-\alpha_1 - 2\beta_1, \alpha_2 - 2\beta_2) w(\beta_1, \beta_2)$$
$$= \sum_{\beta} b(\alpha_1 + 2\beta_1, \alpha_2 - 2\beta_2) w(\beta_1, \beta_2)$$
$$= \sum_{\beta} b(\alpha_1 - 2\beta_1, \alpha_2 - 2\beta_2) w(-\beta_1, \beta_2)$$
$$= \sum_{\beta} b(\alpha_1 - 2\beta_1, \alpha_2 - 2\beta_2) w(\beta_1, \beta_2) = S_b w(\alpha_1, \alpha_2).$$

In the same way we can prove that

$$S_b w(-\alpha_1, -\alpha_2) = S_b w(\alpha_1, -\alpha_2) = S_b w(\alpha_1, \alpha_2).$$

Since b satisfies the sum rules of order $2k$, it implies that U_{2k-1} is invariant under S_b. We have proved W_4 is invariant under S_b. So is $U_{2k-1} \cap W_4$.

For any $w \in W_4 \cap \ell(\Omega)$, $T_b w \in \ell(\Omega)$ since $\ell(\Omega)$ is invariant under T_b.

For each $\alpha \in \Omega$, we have

$$T_b w(-\alpha_1, \alpha_2) = \sum_{\beta} b(-2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2) w(\beta_1, \beta_2)$$
$$= \sum_{\beta} b(2\alpha_1 + \beta_1, 2\alpha_2 - \beta_2) w(\beta_1, \beta_2)$$
$$= \sum_{\beta} b(2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2) w(-\beta_1, \beta_2)$$
$$= \sum_{\beta} b(2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2) w(\beta_1, \beta_2) = T_b w(\alpha_1, \alpha_2).$$

Similarly, we can prove that

$$T_b w(-\alpha_1, -\alpha_2) = T_b w(\alpha_1, -\alpha_2) = T_b w(\alpha_1, \alpha_2).$$

Hence $W_4 \cap \ell(\Omega)$ is an invariant subspace of T_b. ∎

Suppose that b satisfies the following symmetric properties: $\forall\ (\alpha_1, \alpha_2) \in \mathbb{Z}^2$

$$\begin{aligned} b(\alpha_1, \alpha_2) &= b(\alpha_2, \alpha_1) = b(-\alpha_1, -\alpha_2) = b(-\alpha_2, -\alpha_2) \\ &= b(-\alpha_2, \alpha_1) = b(-\alpha_1, \alpha_2) = b(\alpha_2, -\alpha_1) = b(\alpha_1, -\alpha_2). \end{aligned} \tag{5.3.7}$$

Let W_8 be the subspace of $\ell(\mathbb{Z}^2)$ given by

$$\begin{aligned} W_8 := \{w \in \ell(\mathbb{Z}^2) : \ & w(\alpha_1, \alpha_2) = w(\alpha_2, \alpha_1) = w(-\alpha_2, \alpha_1) = w(-\alpha_1, \alpha_2) \\ &= w(-\alpha_1, -\alpha_2) = w(-\alpha_2, -\alpha_1) = w(\alpha_2, -\alpha_1) = w(\alpha_1, -\alpha_2)\}. \end{aligned} \tag{5.3.8}$$

Then, we have the following result:

Lemma 5.9 *If b satisfies (5.3.7) and sum rules of order $2k$, then $W_8 \cap U_{2k-1}$ is invariant under S_b. Moreover, $W_8 \cap \ell(\Omega)$ is invariant under T_b.*

The proof of this lemma is similar to Lemma 5.7 and Lemma 5.8. So we will not give the details.

Before we proceed further, let us demonstrate the advantage of using $\ell(\Omega) \cap W_2$, $\ell(\Omega) \cap W_4$, and $\ell(\Omega) \cap W_8$. Suppose $\operatorname{supp} a = [-L, L]^2$. Then $\operatorname{supp} b = [-2L, 2L]^2$ and $\Omega = \operatorname{supp} b \cap \mathbb{Z}^2$. Then the matrix size of $T_b|_{\ell(\Omega)\cap W_2}$ is only about $1/2$ of the size of $T_b|_{\ell(\Omega)}$; the matrix size of $T_b|_{\ell(\Omega)\cap W_4}$ is about $1/4$ of the size of $T_b|_{\ell(\Omega)}$; and the matrix size of $T_b|_{\ell(\Omega)\cap W_8}$ is only about 1/8 of the size of $T_b|_{\ell(\Omega)}$. From the above analysis, we see the substantial reduction of the size of the matrix by restricting T_b to a smaller invariant subspace. In order to exploit this idea in calculating the smoothness order of refinable functions, we have to ensure that the spec$(T_b|_{\ell(\Omega)\cap W})$ contains the eigenvalue such that its absolute value is the spectral radius of $T_b|_{V_{2k-1}\cap\ell(\Omega)}$.

Let us review some known results. For $j = 1, \ldots, s$, let Δ_j denote the difference operator on $\ell_0(\mathbb{Z}^s)$ given by

$$\Delta_j u := 2u - u(\cdot - e_j) - u(\cdot + e_j), \quad u \in \ell_0(\mathbb{Z}^s).$$

We use δ to denote the sequence on $\mathbb{Z}^2$ given by $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^2\backslash\{0\}$. Jia [30] proved the following theorem:

Theorem 5.10 *Suppose that the normalized solution ϕ of the refinable equation lies in $L_2(\mathbb{R}^2)$. Let $b := a * a^*/m$ and, for a positive integer k, let*

$$\rho := max\{\rho(T_b|_{Y_j}) : \ j = 1, 2\},$$

where Y_j is the minimal T_b-invariant subspace generated by $\Delta_j^k \delta$. Then

$$\nu(\phi) \geq \nu := -\log_m \rho.$$

Moreover, if $k > \nu$, and if the shifts of ϕ are stable, then $\nu(\phi) = \nu$.

We will choose Ω in such a way that $\Delta_j^k \delta \in \ell(\Omega)$, $j = 1, 2$. Since $\Delta_1^k \delta + \Delta_2^k \delta \in W_8$

and $W_8 \subseteq W_4 \subseteq W_2$, then

$$\Delta_1^k\delta + \Delta_2^k\delta \in \ell(\Omega) \cap W_\ell, \; \ell = 2, 4, 8. \tag{5.3.9}$$

Let $W(\Delta_1^k\delta + \Delta_2^k\delta)$ to denote the minimal invariant subspace of T_b generated by $\Delta_1^k\delta + \Delta_2^k\delta$. Hence

$$W(\Delta_1^k\delta + \Delta_2^k\delta) \subseteq \ell(\Omega) \cap W_\ell \cap V_{2k-1}, \quad \ell = 2, 4, 8. \tag{5.3.10}$$

The shift operator τ^β on $\ell(\mathbb{Z}^2)$ is defined by

$$\tau^\beta u = u(\cdot - \beta), \quad u \in \ell(\mathbb{Z}^2).$$

An element $v \in \ell_0(\mathbb{Z}^2)$ induces the Laurent polynomial $\tilde{v}(z) = \sum_{\alpha \in \mathbb{Z}^2} v(\alpha) z^\alpha$, which in turn induces the difference operator

$$\tilde{v}(\tau) = \sum_{\alpha \in \mathbb{Z}^2} v(\alpha)\tau^\alpha.$$

The following theorem clarifies the relationship between $\rho(T_b|_W)$ and $\rho(T_b|_{Y_j})$, $j = 1, 2$.

Theorem 5.11 *Let $W = W(\Delta_1^k\delta + \Delta_2^k\delta)$ denote the minimum invariant subspace of T_b generated by $\Delta_1^k\delta + \Delta_2^k\delta$,, and Y_1 and Y_2 the minimum invariant subspaces of T_b generated by $\Delta_1^k\delta$ and $\Delta_2^k\delta$ respectively. If a has the following properties:*

$$\begin{aligned} a(\alpha_1,\alpha_2) &= a(-\alpha_1,\alpha_2) = a(-\alpha_1,-\alpha_2) = a(\alpha_1,-\alpha_2) \\ &= a(\alpha_2,\alpha_1) = a(-\alpha_2,\alpha_1) = a(-\alpha_2,-\alpha_1) = a(\alpha_2,-\alpha_1) \end{aligned} \tag{5.3.11}$$

then

$$\rho(T_b|_W) = \rho(T_b|_{Y_1}) = \rho(T_b|_{Y_2}). \tag{5.3.12}$$

Proof: Since $b = a * a^*/m$, then b possesses the same symmetry as a does. Let $v_j = \nabla_j^k \delta$ and $w_j = \Delta_j^k \delta$, $j = 1,2$. Then $\tilde{v}_j(z) = (1-z_j)^k$ and $\tilde{w}_j(z) = (1-z_j)^{2k}$, $j = 1,2$. Let $\tilde{w}_1(\tau)b_n$ denote $\Delta_1^k b_n$, $\tilde{w}_2(\tau)b_n$ denote $\Delta_2^k b_n$, while $b_n = S_b^n \delta$. Let $\tilde{v}_1(\tau)a_n$ denote $\nabla_1^k a_n$, $\tilde{v}_2(\tau)a_n$ denote $\nabla_2^k a_n$, while $a_n = S_a^n \delta$. Note that the symbol of $\tilde{v}_1(\tau)a_n$ is $\tilde{v}_1(z)\tilde{a}_n(z)$, $\tilde{v}_2(\tau)a_n$ is $\tilde{v}_2(z)\tilde{a}_n(z)$. And also, the symbol of $\tilde{w}_1(\tau)b_n$ is $\tilde{w}_1(z)\tilde{b}_n(z)$, $\tilde{w}_2(\tau)b_n$ is $\tilde{w}_2(z)\tilde{b}_n(z)$. Moreover,

$$\tilde{w}_1(z)\tilde{b}_n(z) = |\tilde{v}_1(\tau)a_n(z)|^2 \geq 0$$

$$\tilde{w}_2(z)\tilde{b}_n(z) = |\tilde{v}_2(\tau)a_n(z)|^2 \geq 0.$$

Consider

$$\begin{aligned} &\tfrac{1}{(2\pi)^2}\textstyle\int_{[0,2\pi]^2}\big(\tilde{w}_1(e^{i\xi})\tilde{b}_n(e^{i\xi}) + \tilde{w}_2(e^{i\xi})\tilde{b}_n(e^{i\xi})\big)\,d\xi \\ &= \tfrac{1}{(2\pi)^2}\textstyle\int_{[0,2\pi]^2}\big(|\tilde{v}_1(e^{i\xi})\tilde{a}_n(e^{i\xi})|^2 + |\tilde{v}_2(e^{i\xi})\tilde{a}_n(e^{i\xi})|^2\big)\,d\xi \\ &= \|\tilde{v}_1(\tau)a_n\|_2^2 + \|\tilde{v}_2(\tau)a_n\|_2^2. \end{aligned}$$

Since $\tilde{w}_1(e^{i\xi})\tilde{b}_n(e^{i\xi}) \geq 0$, $\tilde{w}_2(e^{i\xi})\tilde{b}_n(e^{i\xi}) \geq 0$, for all $\xi \in \mathbb{R}^2$, it follows that

$$\begin{aligned}
&(\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n(0) \le \|(\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n\|_\infty \\
&\le \tfrac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (\tilde{w}_1(e^{i\xi}) + \tilde{w}_2(e^{i\xi}))\tilde{b}_n(e^{i\xi})d\xi \\
&= (\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n(0).
\end{aligned}$$

From [30], Lemma 3.2, we have

$$\begin{aligned}
&(\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n(0) = T_b^n(w_1 + w_2)(0) \\
&\le \|T_b^n(w_1 + w_2)\|_\infty \le \|(\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n\|_\infty \\
&= (\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n(0).
\end{aligned}$$

Since $W = W(\Delta_1^k\delta + \Delta_2^k\delta)$ is the minimal invariant subspace of T_b generated by $\Delta_1^k\delta + \Delta_2^k\delta$, we obtain

$$\begin{aligned}
\rho(T_b|_W) &= \lim_{n\to\infty} \|T_b^n(\Delta_1^k\delta + \Delta_2^k\delta)\|_\infty^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \Big((\tilde{w}_1(\tau) + \tilde{w}_2(\tau))b_n(0)\Big)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} (\|\tilde{v}_1(\tau)a_n\|_2^2 + \|\tilde{v}_2(\tau)a_n\|_2^2)^{\frac{1}{n}}.
\end{aligned} \tag{5.3.13}$$

Since a satisfies (5.3.11), then $\|\nabla_1^k a_n\|_2^2 = \|\nabla_2^k a_n\|_2^2$. Hence

$$\rho(T_b|_W) = \lim_{n\to\infty} (2\|\tilde{v}_1(\tau)a_n\|_2^2)^{\frac{1}{n}} = \rho(T_b|_{Y_1}).$$

This completes the proof of the theorem. ∎

From Theorem 5.10, we have $\rho(T_b|_{\ell(\Omega)\cap V_{2k-1}}) = \max\{\rho(T_b|_{Y_1}), \rho(T_b|_{Y_2})\}$.

From Theorem 5.11 and $W \subseteq \ell(\Omega) \cap V_{2k-1} \cap W_\ell$, where $\ell = 2, 4, 8$, we obtain

$$\rho(T_b|_{\ell(\Omega)\cap V_{2k-1}}) = \rho(T_b|_W) \leq \rho(T_b|_{\ell(\Omega)\cap V_{2k-1}\cap W_\ell}) \leq \rho(T_b|_{\ell(\Omega)\cap V_{2k-1}}),$$

for $\ell = 2, 4, 8$. Hence

$$\rho(T_b|_W) = \rho(T_b|_{\ell(\Omega)\cap V_{2k-1}\cap W_\ell}), \quad \ell = 2, 4, 8.$$

We can calculate the smoothness order of ϕ by considering the spectrum of $T_b|_{\ell(\Omega)\cap W_\ell}$, $\ell = 2, 4, 8$, since the following formula is valid:

$$\operatorname{spec}(T_b|_{\ell(\Omega)\cap W_\ell}) = \operatorname{spec}(T_b|_{\ell(\Omega)\cap W_\ell\cap V_{2k-1}}) \cup \operatorname{spec}(S_b|_{U_{2k-1}\cap W_\ell}), \quad \ell = 2, 4, 8.$$

Let us investigate the $\operatorname{spec}(S_b|_{U_{2k-1}\cap W})$ where $W = W_2$, W_4, W_8. Since

$$U_{2k-1} = \operatorname{span}\{x_1^{\mu_1} x_2^{\mu_2} : 0 \leq \mu_1 + \mu_2 \leq 2k-1\},$$

then

$$W_2 \cap U_{2k-1} = \operatorname{span}\{x_1^{\mu_1} x_2^{\mu_2} : \mu_1 + \mu_2 = \text{even}, 0 \leq \mu_1 + \mu_2 \leq 2k-1\};$$

$$W_4 \cap U_{2k-1} = \operatorname{span}\{x_1^{\mu_1} x_2^{\mu_2} : \mu_1 = \text{even}, \mu_2 = \text{even}, 0 \leq \mu_1 + \mu_2 \leq 2k-1\};$$

$$W_8 \cap U_{2k-1} = \operatorname{span}\{x_1^{\mu_1} x_2^{\mu_2} + x_1^{\mu_2} x_2^{\mu_1} : \mu_1 = \text{even}, \mu_2 = \text{even}, 0 \leq \mu_1 + \mu_2 \leq 2k-1\}.$$

The following lemmas describe the spectra of S_b restricted to different invariant subspaces.

Theorem 5.12 *Let ψ be the normalization solution of the refinement equation with mask b and dilation matrix $2I$. Suppose ψ has accuracy $2k$. If b satisfies (5.3.5), then*

$$\operatorname{spec}(S_b|_{U_{2k-1}\cap W_2}) = \left\{2^{-|\mu|} : \mu = (\mu_1, \mu_2), |\mu| = \text{even}, 0 \le |\mu| \le 2k-1\right\}; \tag{5.3.14}$$

If b satisfies (5.3.6), then

$$\operatorname{spec}(S_b|_{U_{2k-1}\cap W_4}) = \left\{2^{-|\mu|} : \mu = (\mu_1, \mu_2), \mu_1 = \text{even}, \mu_2 = \text{even}, 0 \le |\mu| \le 2k-1\right\}; \tag{5.3.15}$$

If b satisfies (5.3.7), then

$$\operatorname{spec}(S_b|_{U_{2k-1}\cap W_8}) = \left\{2^{-|\mu|} : \mu = (\mu_1, \mu_2), \mu_1 = \text{even}, \mu_2 = \text{even}, \mu_1 \le \mu_2, 0 \le |\mu| \le 2k-1\right\}. \tag{5.3.16}$$

Proof: We shall prove (5.3.14). (5.3.15) and (5.3.16) can be proved by the same approach.

Since b satisfies (5.3.5), then $\psi(x) = \psi(-x)$ for $x \in \mathbb{R}^2$. Let p_μ be a polynomial given by

$$p_\mu(x) = x_1^{\mu_1} x_2^{\mu_2}, \quad \mu_1 + \mu_2 = \text{even,and } 0 \le |\mu| \le 2k-1, x \in \mathbb{R}^2.$$

Clearly, p_μ ($|\mu| < 2k, \mu_1 + \mu_2 = \text{even}$) are linearly independent. Since

$$p_\mu(2x) = 2^{\mu_1+\mu_2} x_1^{\mu_1} x_2^{\mu_2} = 2^{\mu_1+\mu_2} p_\mu(x), \quad x \in \mathbb{R}^2,$$

we obtain

$$p_\mu(x) = 2^{-(\mu_1+\mu_2)} p_\mu(2x), \quad x \in \mathbb{R}^2. \tag{5.3.17}$$

By the assumption, ψ has accuracy $2k$. Thus, for each μ with $|\mu| < 2k$, $|\mu| =$ even, the polynomial p_μ lies in $S(\psi)$. Since $\hat{\psi}(0) \neq 0$, there exists a unique polynomial sequence $u_\mu \in \ell(\mathbb{Z}^2)$ such that $p_\mu = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha)\psi(\cdot - \alpha)$. We claim that u_μ has the same symmetry as p_μ does. Indeed, since $p_\mu(x) = p_\mu(-x)$ and $\psi(x) = \psi(-x)$, we have

$$p_\mu(-x) = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha)\psi(-x-\alpha) = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha)\psi(-(x+\alpha)) = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(-\alpha)\psi(x-\alpha),$$

i.e.,

$$p_\mu(x) = 1/2(p_\mu(x) + p_\mu(-x)) = 1/2 \sum_{\alpha\in\mathbb{Z}^2} \big(u_\mu(\alpha) + u_\mu(-\alpha)\big)\psi(x-\alpha).$$

Hence, $u_\mu(\alpha) = u_\mu(-\alpha)$, i.e., $u_\mu \in W_2 \cap U_{2k-1}$. It follows from (5.3.17) that

$$p_\mu(x) = 2^{-|\mu|} p_\mu(2x) = \sum_{\alpha\in\mathbb{Z}^2} 2^{-|\mu|} u_\mu(\alpha)\psi(2x-\alpha). \tag{5.3.18}$$

On the other hand, we deduce from $p_\mu = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha)\psi(\cdot - \alpha)$,

$$\begin{aligned} p_\mu(x) &= \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha)\psi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha) \sum_{\beta\in\mathbb{Z}^2} b(\beta)\psi(2x-2\alpha-\beta) \\ &= \sum_{\gamma\in\mathbb{Z}^2} \psi(2x-\gamma) \sum_{\alpha\in\mathbb{Z}^2} u_\mu(\alpha) b(\gamma-2\alpha) = \sum_{\gamma\in\mathbb{Z}^2} S_b u_\mu(\gamma)\psi(2x-\gamma). \end{aligned} \tag{5.3.19}$$

Let $K(\psi) := \{u \in \ell(\mathbb{Z}^2) : \psi *' u = 0\}$. Comparing (5.3.18) and (5.3.19), we obtain

$$S_b u_\mu = 2^{-|\mu|} u_\mu + w_\mu, \quad w_\mu \in K(\phi). \tag{5.3.20}$$

By Lemma 5.7, we know that $W_2 \cap U_{2k-1}$ is invariant under S_b. This means that $w_\mu \in W_2 \cap U_{2k-1}$. Since w_μ is a polynomial sequence, there exists a multi-index γ and a complex number $c \neq 0$, where $|\gamma| \leq 2k-1$, such that $\nabla^\gamma w_\mu(\beta) = c$ for all $\beta \in \mathbb{Z}^2$. Because $w_\mu \in K(\psi)$, it implies that

$$\sum_{\alpha \in \mathbb{Z}^2} w_\mu(\alpha)\psi(x-\alpha) = 0. \tag{5.3.21}$$

It follows from (5.3.21) that

$$\sum_{\alpha \in \mathbb{Z}^2} \nabla^\gamma w_\mu(\alpha)\psi(x-\alpha) = 0.$$

Hence $\sum_{\alpha \in \mathbb{Z}^2} \psi(x-\alpha) = 0$ for all $x \in \mathbb{R}^2$. This contradictions to the assumption that $\hat{\psi}(0) \neq 0$. Hence $w_\mu = 0$.

We want to show that polynomial sequences u_μ ($|\mu| = \text{even}, 0 \leq |\mu| \leq 2k-1$) are linear independent. Considering that $\sum_\mu c_\mu u_\mu = 0$, where c_μ is a complex number, we have

$$\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu \sum_{\alpha \in \mathbb{Z}^2} u_\mu(\alpha)\psi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^2} \psi(x-\alpha) \sum_\mu c_\mu u_\mu(\alpha) = 0.$$

It implies that $c_\mu = 0$ since the p_μ ($|\mu| = \text{even}, 0 \leq |\mu| \leq 2k-1$) are linear independent. We prove that the polynomial sequences u_μ ($|\mu| = \text{even}, 0 \leq |\mu| \leq 2k-1$) are the base of $W_2 \cap U_{2k-1}$. Hence we obtain (5.3.14). ∎

Next, we will use the ideas discussed in this section to calculate the smoothness order of refinable functions possessing symmetric properties. Let us look at a family of interpolatory refinement masks constructed in [26]. In [26], Han and Jia proposed a general way for the construction of bivariate interpolatory refinement masks such that the corresponding fundamental and refinable functions attain the optimal approximation order and smoothness order. These interpolatcry refinement masks are minimally supported and enjoy full symmetry. We will use the theory discussed in the previous section to compute the smoothness order of the refinable functions corresponding to the given masks in [26]. Since the masks possess symmetric properties given in (5.3.5), (5.3.6), and (5.3.7), we will compute the spectra of $T_b|_{W_2 \cap \ell(\Omega)}$, $T_b|_{W_4 \cap \ell(\Omega)}$, and $T_b|_{W_8 \cap \ell(\Omega)}$. By picking out the spectra of $T_b|_{W_2 \cap U_{2k-1}}$, $T_b|_{W_4 \cap U_{2k-1}}$, and $T_b|_{W_8 \cap U_{2k-1}}$ from the spectra of $T_b|_{W_2 \cap \ell(\Omega)}$, $T_b|_{W_4 \cap \ell(\Omega)}$, and $T_b|_{W_8 \cap \ell(\Omega)}$, we can determine the spectral radius of $T_b|_{W_2 \cap \ell(\Omega) \cap V_{2k-1}}$, $T_b|_{W_4 \cap \ell(\Omega) \cap V_{2k-1}}$, and $T_b|_{W_8 \cap \ell(\Omega) \cap V_{2k-1}}$. The following tables display the computation results. From these tables, we also can see the corresponding matrix size shrinks dramatically.

NOTE: "X", in Table 5.2, Table 5.3, Table 5.4, and Table 5.5, means that MATLAB cannot handle a matrix of that size.

Table 5.2: Numerical results					
Mask of a	Size of a	Size of b	Size of $T_b\|_{\ell(\Omega)}$	$\rho(T_b\|_{\ell(\Omega)\cap V_{2k-1}})$	Smoothness
$g2$	7×7	13×13	169×169	0.135697790000	2.44076543
$g3$	11×11	21×21	441×441	0.049027566993	3.17513151
$g4$	15×15	29×29	841×841	0.020814492583	3.79313390
$g5$	19×19	37×37	1369×1369	0.009697524682	4.34408387
$g6$	23×23	45×45	2025×2025	0.004729686899	4.86201980
$g7$	27×27	53×53	2809×2809	0.002362190206	5.36282957
$g8$	31×31	61×61	3721×3721	0.001197424203	5.85292497
$g9$	35×35	69×69	4761×4761	X	
$g10$	39×39	77×77	5929×5929	X	
$g11$	43×43	85×85	7225×7225	X	
$g12$	47×47	93×93	8649×8649	X	

Table 5.3: Numerical results			
Mask of a	Size of $T_b\|_{\ell(\Omega)\cap W_2}$	$\rho(T_b\|_{\ell(\Omega)\cap W_2\cap V_{2k-1}})$	Smoothness
$g2$	85×85	0.135697788100	2.44076544
$g3$	221×221	0.049027566993	3.17513151
$g4$	421×421	0.020814492582	3.79313390
$g5$	685×685	0.009697524680	4.34408387
$g6$	1013×1013	0.004729686897	4.86201980
$g7$	1405×1405	0.002362190320	5.36282953
$g8$	1861×1861	0.001197423176	5.85292559
$g9$	2381×2381	0.000613569351	6.33524298
$g10$	2965×2965	0.000317050682	6.81149944
$g11$	3613×3613	0.000165047791	7.28241427
$g12$	4325×4325	X	

Table 5.4: Numerical results

Mask of a	Size of $(T_b\|_{\ell(\Omega)\cap W_4})$	$\rho(T_b\|_{\ell(\Omega)\cap W_4\cap V_{2k-1}})$	Smoothness
$g2$	49×49	0.135697790000	2.44076543
$g3$	121×121	0.049027566993	3.17513151
$g4$	225×225	0.020814492581	3.79313390
$g5$	361×361	0.009697524681	4.34408387
$g6$	529×529	0.004729686847	4.86201980
$g7$	729×729	0.002361903321	5.36291719
$g8$	961×961	0.001197425101	5.85292443
$g9$	1225×1225	0.000613578261	6.33523251
$g10$	1521×1521	0.000317113728	6.81135602
$g11$	1849×1849	0.000164904914	7.28303899
$g12$	2209×2209	0.000085744994	7.75479401

Table 5.5: Numerical results

Mask of a	Size of $(T_b\|_{\ell(\Omega)\cap W_8})$	$\rho(T_b\|_{\ell(\Omega)\cap W_8\cap V_{2k-1}})$	Smoothness
$g2$	28×28	0.135697788100	2.44076544
$g3$	66×66	0.049027566993	3.17513151
$g4$	120×120	0.020814492524	3.79313390
$g5$	190×190	0.009697524680	4.34408387
$g6$	276×276	0.004729686837	4.86201981
$g7$	378×378	0.002362190292	5.36282954
$g8$	496×496	0.001197423725	5.85292526
$g9$	630×630	0.000613569509	6.33524279
$g10$	780×780	0.000317095407	6.81139769
$g11$	946×946	0.000165017194	7.282548012
$g12$	1128×1128	0.000085914583	7.753368720

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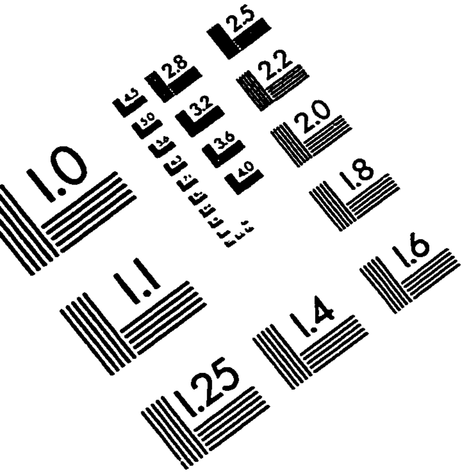
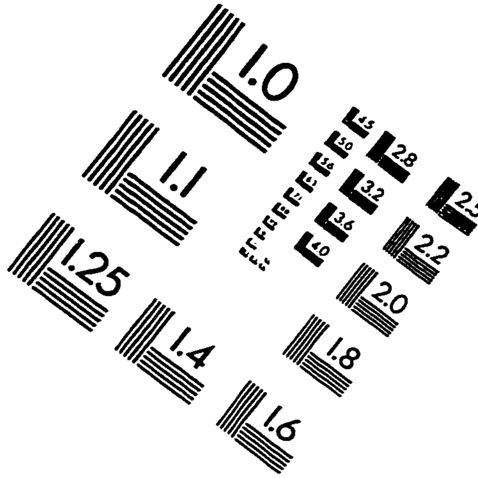
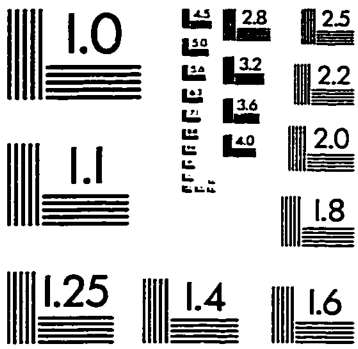
IMAGE EVALUATION TEST TARGET (QA-3)







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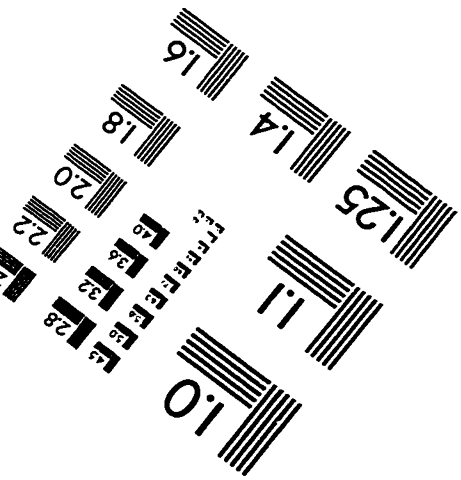

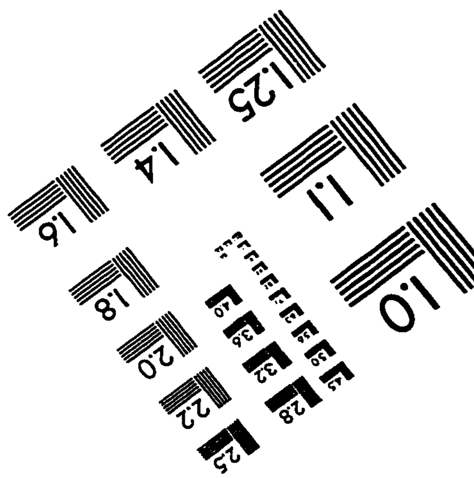

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