UNIVERSITY OF ALBERTA

TORSION FREE SPACE GROUPS

BY

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TO MY WIFE AND MY SON

ABSTRACT

In this thesis, we study several problems concering torsion free space groups which is closely related to the classification of flat manifolds.

In Chapter 1, we introduce the problems we are mainly interested in and the best results we knew about them up to now.

In Chapter 2, we quote some basic results without proofs. They can be easily spotted in standard reference books. We also prove a lemma (Lemma 2.[illegible]) which is important to later chapters.

In Chapter 3, we investigate $\delta(G)$ which is defined to be the minimal dimension of primitive G-manifolds. We calculate $\delta(G)$ for G being $C_{p^2} \times C_p$, $M_3(p)$ and $M(p)$; we also give a lower bound and an upper bound for $\delta(G)$ for G abelian and finally give an upper bound for $\delta(G)$ for G solvable.

In Chapter 4, $n(G)$ is studied for G nilpotent groups. $n(G)$ is defined to be the least positive integer such that for any torsion free space group Γ with point group G, there is a normal subgroup N contained in the translations, such that Γ/N is still torsion free and has dimension $\leq n(G)$. The case when G is a p-group is solved by Cliff and Weiss. For G is nilpotent, $n(G)$ is given in Theorem 4.2.1 by using Cliff-Weiss's method.

In Chapter 5, we try to discuss some $\mathbf{Z}G$-modules through their p-adic correspondence. The main result is a test given in Theorem 5.1.1 and Theorem 5.1.2 that allows us to decide which direct summand of $\mathrm{ind}_C^G \mathbf{Z}_p$ will have nontrivial second cohomology. Then we use it to find $n(G)$ for some non nilpotent groups, e.g. alternating group of 5 letters A_5 and some metabelian groups.

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CHAPTER 1 INTRODUCTION

In this thesis, we are going to study several problems concerning torsion free space groups which are of interest to differential geometers, since they arise as fundamental groups of compact flat Riemannian manifolds, and classify these manifolds up to affine equivalence.

In this Chapter, we introduce the problems we are mainly interested in and the best results we know about them up to now.

In Chapter 2, we quote some basic results without proofs. They can be easily spotted in standard reference books. We also prove a lemma (Lemma 2.10) which is important to later chapters.

In Chapter 3, we investigate $\delta(G)$ which is defined to be the minimal dimension of primitive G-manifolds. We calculate $\delta(G)$ for G being $C_{p^2} \times C_p$, $M_3(p)$ and $M(p)$; we also give a lower bound and an upper bound for $\delta(G)$ for G abelian and finally give an upper bound for $\delta(G)$ for G solvable. We do these by first constructing a primitive lattice for each of above groups to gain an upper bound, and then by discussing $\mathbf{Q} \otimes_{\mathbf{Z}} L$ for any primitive lattice L to get a lower bound.

In Chapter 4, $n(G)$ is studied for G nilpotent groups. The case when G is a p-group is solved by Cliff and Weiss in [7]. For G is nilpotent, $n(G)$ is given in Theorem 4.2.1 by extending Cliff-Weiss's method. For each torsion free space group with point group G, we define a map, with abelian kernel, to a space group whose translation subgroup, consider as a $\mathbf{Z}G$-module, is a quotient of the permutation lattice given by the sum of inductions from non-conjugate subgroups of prime orders multiplied with the complement of Sylow subgroup of that prime in G. Then in the second half, we prove that this is the minimal one. Finally we examine an example to show how the method works.

In Chapter 5, we try to discuss some $\mathbf{Z}G$-modules through their p-adic correspondences. The main result is a test given in Theorem 5.1.1 and Theorem 5.1.2 which allows us to decide which direct summand of $\mathrm{ind}_C^G \mathbf{Z}_p$ will have nontrivial second cohomology. Then we use it to find $n(G)$ for some non nilpotent groups, e.g. alternating group of 5 letters A_5 and some metabelian groups.

§1.1

Let $\mathbf{R}^n$ be the usual n-dimensional Euclidean space,

$$\mathbf{E}_n = \{f \mid f : \mathbf{R}^n \to \mathbf{R}^n \text{ and } f \text{ preserves distance}\}.$$

Then every such f is a composition of a rotation and a translation. So we can identify $\mathbf{E}_n$ with the set

$$\{(m, s) \mid m \in \mathbf{O}_n, s \in \mathbf{R}^n\},$$

where $\mathbf{O}_n$ is the n-dimensional orthogonal group, and $(m, s) \circ x = mx + s$ for $x \in \mathbf{R}^n$. $\mathbf{E}_n$ is a group under the composition :

$$(m, s)(n, t) = (mn, mt + s).$$

We call $\mathbf{E}_n$ the group of *rigid motions* (or *isometries*) of $\mathbf{R}^n$. $\mathbf{E}_n$ has a nice group structure. Let

$$\mathbf{A}_n = \{(I, t) \mid I \text{ the identity of } \mathbf{O}_n, t \in \mathbf{R}^n\},$$

then $\mathbf{A}_n$ is a torsion free abelian subgroup of $\mathbf{E}_n$ and $\mathbf{E}_n/\mathbf{A}_n$ is isomorphic to $\mathbf{O}_n$. Elements of $\mathbf{A}_n$ are called *translations.* $\mathbf{E}_n$ has the topology of $\mathbf{O}_n \times \mathbf{R}^n$. We say that Γ is a *Crystallographic space group* (or *space group* for short) if Γ is a discrete subgroup of $\mathbf{E}_n$ such that the quotient space $\mathbf{R}^n/\sim$ is compact , where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if there is a $g \in \Gamma$ such that $gx = y$. This is equivalent to saying that there exists a closed bounded set B of $\mathbf{R}^n$, such that every $x \in \mathbf{R}^n$ satisfies $x = gy$ for some $y \in B$, $g \in \Gamma$. $A = \Gamma \cap \mathbf{A}_n$ will be called the *translation subgroup* of Γ; Γ/A will be called the *point group.*

For these groups, we have the following:

Bieberbach's First Theorem (see [6], p. 17). Let Γ be a space group as defined above. Then

(i) The point group is finite, and

(ii) the translation subgroup is a lattice (finitely generated free abelian group) which spans $\mathbf{R}^n$. This is equivalent to that Γ contains n linearly independent translations.

Bieberbach's Third Theorem (see [6], p. 40). For each n, there are only finitely many isomorphism classes of n-dimensional space groups.

For example, when $n = 2$, we have 17; $n = 3$, we have 219; $n = 4$, we have 4783 such classes respectively (see [3]).

Among space groups, we are particularly interested in those that are torsion free. These groups sometimes are also called *Bieberbach groups*(see [6]). If Γ is a torsion free space group, then $\mathbf{R}^n/\sim$, the quotient space is a compact connected flat n-dimensional Riemannian manifold, or flat manifold for short. Conversely, an n-dimensional flat manifold has the Euclidean n-space $\mathbf{R}^n$ as its universal covering space and an n-dimensional space group with fixed point free action on $\mathbf{R}^n$ as fundamental group (see [6], [30]). It is this torsion free space group that is our main object of discussion. By Bieberbach's Third Theorem, there are only finite many of them for each fixed dimension n. The several lower dimensional cases are: $n = 2$, there are 2; $n = 3$, there are 10; $n = 4$, there are 74.

Let Γ be a torsion free space group, A be its translation subgroup, then $G = \Gamma/A$, the point group of Γ is finite. So we can find finitely many coset representatives $\{t_g\}$ of A in Γ, such that

$$\Gamma = \cup_{g \in G} t_g A.$$

For any $t_g, t_{g'}$ we have $t_g t_{g'} = t_{gg'} a(g, g')$, with $a(g, g') \in A$. Then we get a map

$$a : G \times G \longrightarrow A.$$

If we pick different coset representatives $\{u_g\}$, $u_g = t_g a_g$, for some $a_g \in A$, then we get another map

$$b : G \times G \longrightarrow A.$$

We will call a, b equivalent. All equivalence classes form an abelian group under $(a + b)(g, g') = a(g, g') + b(g, g')$. This is the usual second cohomology group of G with coefficients in A. It is denoted by $\mathrm{H}^2(G, A)$. By the above discussion, for each torsion free space group with translation subgroup A and point group G, we get an $\alpha \in \mathrm{H}^2(G, A)$. Since Γ is torsion free, for any $C < G$, $\Gamma_C = \cup_{g \in C} t_g A$

is also torsion free, this gives us an $\alpha_C \in \mathrm{H}^2(C, A)$. Denote it by $\mathrm{res}^G_C\alpha$. $\mathrm{res}^G_C\alpha$ is not zero. For otherwise, $t_g t_{g'} = t_{gg'}$ would give a homomorphism from C to Γ_C, contradicting that Γ_C is torsion free. Hence $\mathrm{res}^G_C\alpha$ is not zero for any $C < G$. We will call such α *special* for G. We will say that $\alpha \in \mathrm{H}^2(G, A)$ is special for $g \in G$ if $\mathrm{res}^G_{\langle g \rangle}\alpha$ is special and α is special for a subset $H \subset G$ if it is special for each $g \in H$. We will also say that a lattice M is special for $H \subset G$ if there is $\alpha \in \mathrm{H}^2(G, M)$ such that α is special for H. Now with these notations we can record the following well known result:

Theorem (Zassenhaus [31], [6]). If Γ is the fundamental group of a flat manifold X of dimension n, then Γ is a torsion free space group. That is it contains M, a finite index, normal, free maximal abelian subgroup of rank n. The finite holonomy group G of X is the point group of Γ. It also follows that M is a faithful integral representation of G, or a **Z***G-lattice* which means finitely generated and free as **Z**-module. Furthermore, there is a special $\alpha \in H^2(G, M)$. Conversely given such an G, M and a special $\alpha \in H^2(G, M)$ there exists a corresponding flat manifold X with fundamental group having M as its translation subgroup and G its point group.

It is this fundamental result that makes it possible of changing the discussion of the classification of flat manifolds to the discussion of algebraic objects (integral representation and cohomology). This is because the classification of X is determined by the classification of Γ which in turn is determined by M, G, and α. But unfortunately, given a finite group G, it is not always possible to get all the information about the integral representations of G nor to calculate $H^2(G, M)$.

So people are mainly interested in the following several problems.

§**1.2** $m(G)$

$m(G)$ is defined as the least dimension of flat manifold with holonomy group isomorphic to G. Algebraically, this means, $m(G)$ is the minimal dimension of an integral representation M of G satisfying:

1. M is a faithful integral representation of G,

2. M carries a special class, i.e. there exists an $\alpha \in \mathrm{H}^2(G,M)$ whose restriction to each cyclic subgroup of G is non zero.

Peter Symonds discussed this problem in [25]. His main results are as follows:

Theorem. 1. Let G be a finite abelian group. Factorize G as a direct product of cyclic groups of prime power order; let $\{r_i\}$ be the orders of the factors. Let $a_G =$ number of r_i which are 2; $b_G =$ number of r_i which are odd; $c(G)$ be the minimal dimension of a faithful rational representation of G and l_p be the number of r_i divisible by p^2. Then

$$m(G) \leq c(G) + \max(l_p,\ \min(a_G, b_G), 1).$$

2. If G is solvable then $m(G) \leq |G|$ with equality if and only if G is of prime order.

§**1.3** $\delta(G)$

If G is a finite group, we say that G is *primitive* if G is the holonomy group of a flat manifold X with $b_1(X)$, the first Betti number of X, zero. We call X itself a *primitive G-manifold* in this case. Let

$\delta(G) =$ the minimal dimension of a primitive G-manifold. Algebraically , this is:

$\delta(G)$ is the minimal dimension of an integral representation M of G satisfying:

1. M is a faithful representation;
2. M carries a special class;
3. $M^G = 0$, where $M^G = \{m \in M, hm = m, \forall h \in G\}$.

Hiller, Sah et al [11], [12] discussed $\delta(G)$. They proved that $\delta(G)$ is well defined, or there is at least one torsion free space group of Betti number zero, with point group G, if and only if G has the property that no cyclic Sylow subgroup has a normal complement. They also decided $\delta(G)$ for elementary abelian p-groups, $(\mathbf{Z}/p\mathbf{Z})^k$ and some other special p-groups; Plesken [19] has some different ways of dealing with this problem. He used it to get $\delta(\mathbf{PSL}_2(p))$. Their main results are as follows:

Theorem. 1. If p is a prime, $k > 1$ then

$$\begin{aligned} \delta(\mathbf{Z}_p^k) &= (p-1)(k+p-1); \\ \delta(\mathbf{D}_{2^\alpha}) &= \delta(\mathbf{SD}_{2^\alpha}) = 2^{\alpha-2}+2, \text{ for } \alpha \geq 2. \end{aligned}$$

2. Let $p \geq 5$ be a prime number, then

$$\delta(\mathbf{PSL}_2(p)) = p + \sum_{r \leq 2, r | p+1} \varphi_{r,R}(p+1)(p-1) +$$

$$+\varphi_R(\frac{p-1}{2})(p+1),$$

where $\varphi_R(n) = \varphi(n)/2$ for $n > 2$, $\varphi_{r,R}(n)$ denotes the degree of the maximal real subfield of the r^α-cyclotomic field, r^α is the biggest r-power dividing n.

§1.4 $n(G)$

$n(G)$ is defined to be the least positive integer such that for any torsion free space group Γ with point group G, there exists a normal subgroup N contained in the translations, such that Γ/N is still torsion free and has dimension $\leq n(G)$. The geometric meaning of $n(G)$ is that any compact flat manifold with holonomy group G is a flat toral extension of a flat manifold of dimension $\leq n(G)$. This invariant was first defined and investigated by Vasquez [29]. He proved that $n(G) = 1$, if G has prime order. He did that by using Reiner's classification [21] of indecomposable $\mathbf{Z}G$-lattice for G of prime order. $n(G)$ was then closely investigated by Cliff and Weiss in [7]. For any torsion free space group with point group G, they defined a map, with abelia1 kernel, to a space group whose translation subgroup, consider as a $\mathbf{Z}G$-module, is a permutation lattice given by the sum of inductions from non-conjugate subgroup of prime orders. With this map they were able to prove:

Theorem. Let G be a finite group, $\mathcal{X}$ a set of representatives of the conjugacy classes of subgroups of G of prime order. Then

$$n(G) \leq \sum_{C\in\mathcal{X}} \mid G:C \mid$$

and the equality holds if and only if G is a p-group.

There are also some other approaches of discussing torsion free space groups. For example, focus on specific point groups (See [2] [15] [17] [18].), special modules (See [13].) or specific dimension (See [3] [28].). A survey about some of above problems can be found in Plesken [20] . The detailed description of space groups can also be found in [6] and [30].

CHAPTER 2 SOME BASIC RESULTS

In this chapter we will collect some basic results that will often be used in the later chapters. All G in this chapter are finite groups. All M (if it is not specified) are $\mathbf{Z}G$-lattices.

Let H be a subgroup of G, M be a $\mathbf{Z}H$-module. Define the *induced module* of M from H to G to be $\mathbf{Z}G \bigotimes_{\mathbf{Z}H} M$ and denote it by $\mathrm{ind}_H^G M$.

Suppose that $G = \cup_{u \in T_H} uH$ is the left coset decomposition. Then every element of $\mathbf{Z}G$ has the form $\sum_{g \in G} a_g g = \sum_g a_g u(g) h(g) \in \sum_{u \in T_H} u\mathbf{Z}H$, where $u(g) \in T_H$, $h(g) \in H$. So $\mathbf{Z}G = \sum_{u \in T_H} u\mathbf{Z}H$ since the other direction of inclusion is obvious. Suppose $\sum_{u \in T_H} u\alpha_u = 0$ or $u_0\alpha_{u_0} = \sum_{u \neq u_0} u\alpha_u$ with $\alpha_u \in \mathbf{Z}H$. Write $\alpha_{u_0} = \sum_h b_h h$, we get $u_0\alpha_{u_0} = \sum_h b_h u_0 h$. By moving every term except one to the other side, we can get $c_{g_0} g_0 = \sum_{g \neq g_0} c_g g$. This contradicts to the definition of $\mathbf{Z}G$. Hence the sum is direct. Then

$$\mathrm{ind}_H^G M = \mathbf{Z}G \bigotimes_{\mathbf{Z}H} M = (\bigoplus_{u \in T_H} u\mathbf{Z}H) \bigotimes_{\mathbf{Z}H} M = \bigoplus_{u \in T_H} u \bigotimes M.$$

Therefore each element of $\mathrm{ind}_H^G M$ can be written in the form $\sum_u a_u u \otimes m_u$. In the case $M = \mathbf{Z}$, the trivial $\mathbf{Z}H$-module, we will write an element of $\mathrm{ind}_H^G \mathbf{Z}$ in the form $\sum_{u \in T_H} n_u uH$ with n_u integers.

Induced modules are closely related to permutation lattices. A $\mathbf{Z}G$-lattice M is called a *permutation lattice* if G acts on a $\mathbf{Z}$-basis $B = \{b_1, b_2, \cdots, b_n\}$ of M by permuting them. If furthermore G acts on B transitively, we will call M a *transitive permutation lattice*. In this case, pick any $b_{i_0} \in B$; let $H = \{g \in G \mid gb_{i_0} = b_{i_0}\}$. Then $\mathbf{Z}Hb_{i_0} \cong \mathbf{Z}$ the trivial $\mathbf{Z}H$-module. Suppose $G = \cup_{u \in T_H} uH$ the coset decomposition, then $ub_{i_0} = b_{i(u)} \in B$ and $b_{i(u)} = b_{i(v)}$ if and only if $u = v$. Then by the transitivity we can get

$$M = \bigoplus_{i=1}^{n} \mathbf{Z}Hb_i \cong \mathrm{ind}_H^G \mathbf{Z}Hb_{i_0} \cong \mathrm{ind}_H^G \mathbf{Z}.$$

The following first result we quote here is a very useful result for our later chapters.

Theorem 2.1 (Nakayama Relations or Frobenius Reciprocity [14], p. 87). Let M be a $\mathbf{Z}G$-module, N be a $\mathbf{Z}H$-module. Then we have

$$\mathrm{Hom}_{\mathbf{Z}G}(M, \mathrm{ind}_H^G N) \overset{\pi^*}{\cong} \mathrm{Hom}_{\mathbf{Z}H}(\mathrm{res}_H^G M, N),$$

where $\pi : \mathrm{res}_H^G \mathrm{ind}_H^G N \to N$ is defined by $\pi(\sum_{u \in T_H} n_u uH) = n_1$.

Theorem 2.2 (Shapiro's Lemma [10], p.92). Let H be a subgroup of G, and let M be a H-module. Then

$$\mathrm{H}^n(G, \mathrm{ind}_H^G M) \cong \mathrm{H}^n(H, M) ,$$

for all $n \geq 0$.

From this theorem, it is easy to get that if $g \in G$, then $\mathrm{ind}_{\langle g \rangle}^G \mathbf{Z}$ is special for g, since $\mathrm{H}^2(G, \mathrm{ind}_{\langle g \rangle}^G \mathbf{Z}) \cong \mathrm{H}^2(\langle g \rangle, \mathbf{Z}) \cong \mathbf{Z}/ \mid g \mid \mathbf{Z}$. The following lemma says that $\mathrm{ind}_{\langle g \rangle}^G \mathbf{Z}$ is, in some sense, universal.

Lemma 2.3 (Cliff and Weiss [7]). Let $\alpha \in \mathrm{H}^2(G, M)$ be special for $g \in G$. Then there exists a $\mathbf{Z}G$-homomorphism $h : M \to \mathrm{ind}_{\langle g \rangle}^G \mathbf{Z}$ such that $h^*(\alpha) \in \mathrm{H}^2(G, \mathrm{ind}_{\langle g \rangle}^G \mathbf{Z})$ is also special for g.

Theorem 2.4 (5-Term Sequence [16], p. 355). Let H be a normal subgroup of G and M a G-module. If $\mathrm{H}^i(H,M) = 0$ for $0 < i < j$ then we have the following exact sequence:

$$0 \to \mathrm{H}^j(G/H, M^H) \stackrel{\text{inf}}{\to} \mathrm{H}^j(G,M) \stackrel{\text{res}}{\to} \mathrm{H}^j(H,M)^{G/H}$$

$$\to \mathrm{H}^{j+1}(G/H, M^H) \to \mathrm{H}^{j+1}(G,M) .$$

Theorem 2.5 (Mackey's Decomposition Theorem [9], p.85). Let H, K be subgroups of G and let M be a K-module. Then

$$\mathrm{res}^G_H \mathrm{ind}^G_K M \cong \bigoplus_{g_i \in K\backslash G/H} \mathrm{ind}^H_{K^{g_i}\cap H} \mathrm{res}^K_{K^{g_i}\cap H} M \otimes g_i,$$

where $K\backslash G/H$ denotes an arbitrary transversal of double K, H cosets.

Corollary 2.6 (see [25]). Let H be a normal subgroup of G, S a $\mathbf{Z}H$-lattice and $T = \mathrm{ind}^G_H S$. Then the element of $\mathrm{H}^2(G,T)$ can be special only for elements $g \in G$ for which $\langle\ g\ \rangle \cap H \neq 1$.

Theorem 2.7 (Green Correspondence [14], p. 112). Let R be a principal ideal domain such that Krull-Schmidt holds for modules in $M_R(H)$ for all $H \leq G$, where $M_R(H)$ are the set of all RH-lattices . Let V be a fixed subgroup of G, H a fixed subgroup of G containing $N_G(V)$, the normalizer of V in G. Set

$$\mathcal{X} = \{W \leq G \mid W \leq V^g \cap V, g \in G\backslash H\};$$

$$\mathcal{M} = \{W \leq G \mid W \leq V^g \cap H, g \in G\backslash H\}.$$

Then there is a one to one correspondence between indecomposable modules in $M_R(G)$ with V as vertex and indecomposable modules in $M_R(H)$ with V as vertex, which is characterized as follows:

(i) Let $M \in M_R(G)$ be indecomposable with V as vertex. Then $\mathrm{res}^G_H M$ has a unique indecomposable direct summand $f(M)$ with V as vertex. Furthermore,

$$\mathrm{res}^G_H M \cong f(M) \bigoplus (\bigoplus_i N_i);$$

where the vertices of N_i all lie in $\mathcal{M}$.

(ii) Let $N \in M_R(H)$ be indecomposable with V as vertex. Then $\mathrm{ind}^G_H N$ has a unique indecomposable direct summand $g(N)$ with V as vertex and

$$\mathrm{ind}^G_H N \cong g(N) \bigoplus (\bigoplus_i M_i)$$

where M_i has vertex in $\mathcal{X}$.

Theorem 2.8 (see [4], p. 71). (i) There is a natural isomorphism $\mathrm{H}^0(G, M) \cong M^G$.

(ii) For any exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of G-modules and any integer n there is a natural map

$$\delta: \ \mathrm{H}^n(G, M'') \ \rightarrow \ \mathrm{H}^{n+1}(G, M') \ ,$$

such that the sequence

$$0 \rightarrow \ \mathrm{H}^0(G, M') \ \rightarrow \ \mathrm{H}^0(G, M) \ \rightarrow \ \mathrm{H}^0(G, M'')$$

$$\xrightarrow{\delta} \ \mathrm{H}^1(G, M') \ \rightarrow \ \mathrm{H}^1(G, M) \ \rightarrow \cdots$$

is exact.

(iii) If N is an injective $\mathbf{Z}G$-module then $\mathrm{H}^n(G, N) = 0$ for $n > 0$. In particular, $\mathbf{Q}$ is an injective $\mathbf{Z}G$-module with trivial G-action for any G, so $\mathrm{H}^n(G, \mathbf{Q}) = 0$ for $n > 0$.

Corollary 2.9. Let C be a subgroup of G. Then we have $\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}) \cong \mathrm{H}^1(G, \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z})$. where $\mathbf{Z}$ and $\mathbf{Q}/\mathbf{Z}$ are $\mathbf{Z}C$-modules with trivial actions.

Proof. From the exact sequence of $\mathbf{Z}C$-module with trivial actions

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0,$$

we have $\mathbf{Z}G$-module exact sequence:

$$0 \to \mathrm{ind}_C^G \mathbf{Z} \to \mathrm{ind}_C^G \mathbf{Q} \to \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z} \to 0.$$

Then by 2.8 we have long exact sequence:

$$\begin{aligned} 0 \to\ & \mathrm{H}^0(G, \mathrm{ind}_C^G \mathbf{Z}) \to \mathrm{H}^0(G, \mathrm{ind}_C^G \mathbf{Q}) \to \mathrm{H}^0(G, \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z}) \\ \to\ & \mathrm{H}^1(G, \mathrm{ind}_C^G \mathbf{Z}) \to \mathrm{H}^1(G, \mathrm{ind}_C^G \mathbf{Q}) \to \mathrm{H}^1(G, \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z}) \\ \to\ & \mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}) \to \mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Q}) \to \cdots . \end{aligned}$$

But by 2.2 and 2.8 $\mathrm{H}^i(G, \mathrm{ind}_C^G \mathbf{Q}) = \mathrm{H}^i(C, \mathbf{Q}) = 0$, for $i \geq 1$. So we have the required isomorphism. □

Now we will use above theorems to get a result which will be used in later chapters.

Lemma 2.10. Let $K_1 < K_2$ be subgroups of G, $G = \cup u K_2$ and $K_2 = \cup v K_1$ be the coset decompositions. Let V be the transfer (see [23], p. 61) from K_2 to K_1. Then $V : K_2 \to K_1/[K_1, K_1]$ is a group homomorphism. Let $\pi : K_1 \to$

$K_1/[K_1, K_1]$ be the natural map, and $\pi^{-1}(V(K_2))$ be the preimage of $V(K_2)$. Define $\varphi : \mathrm{ind}_{K_1}^G \mathbf{Z} \to \mathrm{ind}_{K_2}^G \mathbf{Z}$ by $\varphi(\sum_{u,v} a_{uv} uvK_1) = \sum_u (\sum_v a_{uv}) uK_2$. Then

(i). $\mathrm{H}^2(G, \ker\varphi) \cong \mathrm{Hom}(K_1/\pi^{-1}(V(K_2)), \mathbf{Q}/\mathbf{Z})$;

(ii). $(\ker\varphi)^G = 0$.

Proof. From the exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathrm{ind}_{K_1}^G \mathbf{Z} \longrightarrow \mathrm{ind}_{K_2}^G \mathbf{Z} \longrightarrow 0$$

and $\mathrm{H}^1(G, \mathrm{ind}_{K_2}^G \mathbf{Z}) \cong \mathrm{H}^1(K_2, \mathbf{Z}) = \mathrm{Hom}(K_2, \mathbf{Z}) = 0$ since $\mathbf{Z}$ is torsion free, we have

$$0 \to \mathrm{H}^2(G, \ker\varphi) \to \mathrm{H}^2(G, \mathrm{ind}_{K_1}^G \mathbf{Z}) \xrightarrow{\varphi^*} \mathrm{H}^2(G, \mathrm{ind}_{K_2}^G \mathbf{Z}) .$$

By 2.9 and 2.2,

$$\mathrm{H}^2(G, \mathrm{ind}_{K_1}^G \mathbf{Z}) \cong \mathrm{H}^1(G, \mathrm{ind}_{K_1}^G \mathbf{Q}/\mathbf{Z}) \cong \mathrm{H}^1(K_1, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(K_1, \mathbf{Q}/\mathbf{Z});$$

$$\mathrm{H}^2(G, \mathrm{ind}_{K_2}^G \mathbf{Z}) \cong \mathrm{H}^1(G, \mathrm{ind}_{K_2}^G \mathbf{Q}/\mathbf{Z}) \cong \mathrm{H}^1(K_2, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(K_2, \mathbf{Q}/\mathbf{Z}).$$

So we have

$$\begin{array}{ccccc} 0 \to \mathrm{H}^2(G, \ker\varphi) \to & \mathrm{H}^2(G, \mathrm{ind}_{K_1}^G \mathbf{Z}) & \xrightarrow{\varphi^*} & \mathrm{H}^2(G, \mathrm{ind}_{K_2}^G \mathbf{Z}) \\ & \downarrow\cong & & \downarrow\cong \\ & \mathrm{H}^1(G, \mathrm{ind}_{K_1}^G \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\varphi^*} & \mathrm{H}^1(G, \mathrm{ind}_{K_2}^G \mathbf{Q}/\mathbf{Z}) \\ & \downarrow\cong & & \downarrow\cong \\ & \mathrm{Hom}(K_1, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\psi} & \mathrm{Hom}(K_2, \mathbf{Q}/\mathbf{Z}) \end{array}$$

For any $f \in \mathrm{Hom}(K_1, \mathbf{Q}/\mathbf{Z})$, $f|_{[K_1,K_1]} = 0$ since $\mathbf{Q}/\mathbf{Z}$ is abelian. Define $\hat{f}(g) = \sum_{u,v} f(c_{g,u,v}) u_g v_{g,u} K_1$. where $gu = u_g c_{g,u}$, $c_{g,u} v = v_{g,u} c_{g,u,v}$ with $c_{g,u} \in K_2$, $c_{g,u,v} \in K_1$. By

$$g_1 g_2 uv = g_1 u_{g_2} v_{g_2,u} c_{g_2,u,v} = (u_{g_2})_{g_1} (v_{g_2,u})_{g_1,u_{g_2}} c_{g_1,u_{g_2},v_{g_2,u}} c_{g_2,u,v},$$

it is easy to see that

$$\begin{aligned}
\hat{f}(g_1)+g_1\hat{f}(g_2) &= \sum_{u,v} f(c_{g_1,u,v})u_{g_1}v_{g_1,u}K_1 + g_1\sum_{u,v} f(c_{g_2,u,v})u_{g_2}v_{g_2,u}K_1 \\
&= \sum_{u,v} f(c_{g_1,u,v})u_{g_1}v_{g_1,u}K_1 + \\
&\quad \sum_{u,v} f(c_{g_2,u,v})(u_{g_2})_{g_1}(v_{g_2,u})_{g_1,u_{g_2}}K_1; \\
\hat{f}(g_1g_2) &= \sum_{uv} f(c_{g_1,u_{g_2},v_{g_2,u}}c_{g_2,u,v})(u_{g_2})_{g_1}(v_{g_2,u})_{g_1,u_{g_2}}K_1 \\
&= \sum_{u,v} f(c_{g_1,u_{g_2},v_{g_2,u}})(u_{g_2})_{g_1}(v_{g_2,u})_{g_1,u_{g_2}}K_1 \\
&\quad +\sum_{u,v} f(c_{g_2,u,v})(u_{g_2})_{g_1}(v_{g_2,u})_{g_1,u_{g_2}}K_1.
\end{aligned}$$

So $\hat{f}(g_1g_2)=\hat{f}(g_1)+g_1\hat{f}(g_2)$, or $\hat{f}$ is a one cocycle. Hence

$$[\hat{f}] \in \mathrm{H}^1(G, \mathrm{ind}_{K_1}^G \mathbf{Q}/\mathbf{Z})$$

and it is mapped to f under the isomorphism of Shapiro's Lemma.

$$\varphi^*(\hat{f})(g)=\varphi(\hat{f}(g))=\sum_u(\sum_v f(c_{g,u,v}))u_gK_2=\sum_u f(\prod_v c_{g,u,v})u_gK_2.$$

This implies that $f\in\ker\psi$ if and only if $f(\prod_v c_{g,u,v})=0$ for any $g\in G$ and u. For any $k\in K_2$, let $g=uku^{-1}$. Then $gu=uk$, so $k=c_{g,u}$. The above is equivalent to that $f\,|_{\pi^{-1}(V(K_2))}=0$ since $f\,|_{[K_1,K_1]}=0$. Hence (i).

Let $x\in(\ker\varphi)^G$, if $x=\sum_{u,v}a_{uv}uvK_1$ then a_{uv} are all equal. But $\varphi(x)=\sum_u(\sum_v a_{uv})uK_2=a_{11}(K_2:K_1)\sum_u uK_2=0$, so $a_{uv}=a_{11}=0$ for all u,v. Hence $x=0$, i.e. (ii). □

Corollary 2.11 (CW [7]). The same notations as in Lemma 2.10. We have

$$\varphi^*:\ \mathrm{H}^2(G,\mathrm{ind}_{K_1}^G\mathbf{Z}) \rightarrow \mathrm{H}^2(G,\mathrm{ind}_{K_2}^G\mathbf{Z})$$

is 0 if and only if the transfer from K_2 to K_1 is trivial.

Proof. $V(K_2) = 0$ if and only if that

$$\mathrm{H}^2(G, \ker\varphi) \cong \mathrm{Hom}(K_1/[K_1, K_1], \mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(K_1, \mathbf{Q}/\mathbf{Z}) \cong \mathrm{H}^2(G, \mathrm{ind}_{K_1}^G \mathbf{Z}) ,$$

if and only if φ^* maps everything to zero due to the exactness on the previous page.
□

Corollary 2.12 (Symonds [25]). Let M be an irreducible $\mathbf{Z}G$-lattice for which the action of G factors through a group of a prime order p. Let $K = \ker M$ and let $(G\backslash K)^p$ be the subgroup of G generated by p-th power of elements of G not in K. Then

$$\mathrm{H}^2(G, M) \cong \mathrm{Hom}(K/(G\backslash K)^p[K, K], \mathbf{Q}/\mathbf{Z}).$$

In particular, if G is abelian and L is an irreducible $\mathbf{Z}G$-lattice which is induced from an irreducible $\mathbf{Z}H$-lattice L' for some subgroup H of G, and the action of H on L' factors through a group of prime order p with $\ker L' = K$. Then

$$\begin{aligned}\mathrm{H}^2(G, L) &\cong \mathrm{Hom}(K/(H\backslash K)^p, \mathbf{Q}/\mathbf{Z}) \\ &\cong \begin{cases} \mathrm{Hom}(K/K \cap pG, \mathbf{Q}/\mathbf{Z}) & L \text{ not trivial} \\ \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z}) & L \text{ trivial} \end{cases}\end{aligned}$$

Proof. Let $K_1 = K$, $K_2 = G$ in the lemma. Then

$$\mathrm{H}^2(G, M) \cong \mathrm{H}^2(G, \ker\varphi) \cong \mathrm{Hom}(K/\pi^{-1}(V(G)), \mathbf{Q}/\mathbf{Z})$$

Hence we need only prove that $\pi^{-1}(V(G)) = (G\backslash K)^p[K, K]$ in this case. G/K is cyclic of order p. For any $g \notin K$, we have $G = \cup_{i=0}^{p-1} g^i K$. So $V(g) = \prod_i c_{g,g^i}[K, K] = g^p[K, K]$. That is $(G\backslash K)^p[K, K] \subset \pi^{-1}(V(G))$. On the other hand, for any $h = \pi^{-1}(V(g_0)) \in \pi^{-1}(V(G))$, write $h = \prod_i c_{g^i, g_0} k$ for some $g \not\in K$ and $k \in [K, K]$. By $g_0 g^i = g^i(g^{-i} g_0 g^i)$ and K is normal, we can get that

$$\begin{aligned} h &= g_0 g^{-1} g_0 g^1 g^{-2} g_0 g^2 \cdots g^{-(p-1)} g_0 g^{p-1} k \\ &= (g_0 g^{-1})(g_0 g^{-1}) \cdots (g_0 g^{-1}) g^p k \\ &= (g_0 g^{-1})^p g^p k \in (G \backslash K)^p [K, K] \ \Box \end{aligned}$$

CHAPTER 3 SOME DISCUSSION ABOUT $\delta(G)$

First the following lemma will reduce some cases to the discussion of p-groups.

Lemma 3.0.1. Let $G = \prod_p G_p$, the direct product of Sylow p-subgroup G_p of G. Then $\delta(G) \leq \sum_p \delta(G_p)$

This is the direct consequence of the following theorem which we need one more concept to state. For any $g \in G$, $H < G$, let $g : H \to gHg^{-1}$ be the conjugation map. If F is a resolution for G, then g induces a cochain map $c(g) : \mathrm{Hom}_H(F, M) \to \mathrm{Hom}_{gHg^{-1}}(F, M)$ given by

$$f \to [x \to gf(g^{-1}x)].$$

This in turn, gives us an isomorphism $c(g)^* : \mathrm{H}^n(gHg^{-1}, M) \to \mathrm{H}^n(H, M)$ (see [4], p. 80). If $z \in \mathrm{H}^n(H, M)$ then we set

$$gz = (c(g)^*)^{-1}(z) \in \mathrm{H}^n(gHg^{-1}, M) .$$

z is said *G-invariant* if $\mathrm{res}^H_{H \cap gHg^{-1}} z = \mathrm{res}^{gHg^{-1}}_{H \cap gHg^{-1}} gz$ for all $g \in G$.

Theorem (see [4], p. 84). Let G be a finite group and H a Sylow p-subgroup. For any G-module M and any $n > 0$, res^G_H maps $\mathrm{H}^n(G, M)_{(p)}$, the Sylow p-subgroup of $\mathrm{H}^n(G, M)$, isomorphically onto the set of G-invariant elements of $\mathrm{H}^n(H, M)$. If $H \lhd G$ then

$$\mathrm{H}^n(G, M)_{(p)} \cong \mathrm{H}^n(H, M)^{G/H}.$$

Proof of Lemma 3.0.1. Let M_p be a primitive G_p-lattice. Since G is nilpotent, we can define M_p as a G-module for which the action of G factors through G_p, $G \to G_p$. Then G/G_p acts trivially on M_p. Hence

$$\mathrm{H}^2(G, M_p)_{(p)} \cong \mathrm{H}^2(G_p, M_p)^{G/G_p} = \mathrm{H}^2(G_p, M_p) .$$

The last equation is the result that G/G_p acts trivially on both G_p and M_p. So we can find $\alpha \in \mathrm{H}^2(G, M_p)$ such that α is special for G_p. Then $M = \bigoplus_p M_p$ is a primitive lattice for G. Hence the result. □

Suppose M is a primitive lattice for G, that is, (i) M is a faithful $\mathbf{Z}G$-lattice; (ii) M carries a special class; (iii) $M^G = 0$. Decompose $\mathbf{Q} \otimes_{\mathbf{Z}} M$ in to a sum of irreducible modules, so

$$\mathbf{Q} \otimes_{\mathbf{Z}} M \cong \mathbf{Q} \otimes_{\mathbf{Z}} \bigoplus_i M_i,$$

where each M_i is a $\mathbf{Z}G$-lattice with $\mathbf{Q} \otimes_{\mathbf{Z}} M_i$ irreducible. Moreover no $M_i \cong \mathbf{Z}$ because of (iii). There is a monomorphism $j : \bigoplus_i M_i \to M$ and exact sequence

$$0 \to \bigoplus_i M_i \xrightarrow{j} M \to A \to 0.$$

A is finite since $\mathrm{rank}(\bigoplus_i M_i) = \mathrm{rank}(M)$. Since M is a primitive $\mathbf{Z}G$-lattice, there is a torsion free group Γ satisfying:

$$0 \to M \to \Gamma \to G \to 1.$$

Define $F = \Gamma / j(\bigoplus_i M_i)$ then we have

$$0 \to \bigoplus_i M_i \to \Gamma \to F \to 1.$$

From the fact that Γ is torsion free we can get an $\alpha \in \mathrm{H}^2(F, \bigoplus_i M_i)$, such that α is special for F. F satisfies:

$$0 \to A \to F \xrightarrow{\tau} G \to 1.$$

G acts faithfully on $\bigoplus M_i$. Hence we have the following lemma which is similar to proposition 4.9 of [25].

Lemma 3.0.2. $\delta(G)$ is equal to the least possible rank of faithful $\mathbf{Z}G$-lattice $M = \bigoplus_i M_i$ with M_i irreducible $\mathbf{Z}G$-lattices which has no G fixed points except 0, and such that for some finite abelian extension $F \xrightarrow{p} G$ of G, $\mathrm{H}^2(F, M)$ has a special element. In particular, if G is a p-group, M_i above can be taken to be *standard irreducible*, which means irreducible $\mathbf{Z}G$-lattice induced up from a representation of a subgroup H which factors through the standard representation of a cyclic group on $\mathbf{Z}[\zeta]$, $\zeta^p = 1$.

By the algebraic description of $\delta(G)$, in order to get an upper bound for $\delta(G)$, we need only construct a $\mathbf{Z}G$-lattice M which satisfies all three conditions in the description. In this chapter, we are going to

1. give an upper bound for $\delta(G)$ when G is abelian p-group hence an upper bound for $\delta(G)$ for G abelian by lemma 3.0.1;
2. give a lower bound for $\delta(G)$ when G is abelian p-group; in the cases when G is elementary abelian or $C_p \times C_{p^2}$, we will give $\delta(G)$;
3. finish the discussion about p^3-groups which was started in [12] and finally,
4. give an upper bound for $\delta(G)$ for some solvable groups.

§3.1 An upper bound for $\delta(G)$, G an abelian p-group

Set $G = C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_r}} = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$, $r \geq 2$, where a_i has order p^{k_i}, $k_1 \leq k_2 \leq \cdots \leq k_r$. G has the following elements of prime order:

$$S = \{ \prod_{i=1}^r a_i^{l_i p^{k_i - 1}}, \quad l_i = 0, 1, \ldots, p-1, \text{ at least one } l_i \text{ is nonzero } \} .$$

We will construct some $\mathbf{Z}G$-lattices $\{M_t, t \in I\}$, each of them will be special for a subset S_t of S, and $S = \cup_{t \in I} S_t$. By taking care of the faithfulness at the same time, we can get a primitive lattice.

(i). For each i, $1 \leq i \leq r$, define a_{r+1} to be a_1; k_{r+1} to be k_1. Let $G_i = \prod_{k \neq i+1} \langle a_k \rangle$, $G_i' = \langle (a_{i+1})^{p^{k_{i+1}-1}} \rangle \times G_i$. Define:

$$\varphi_i : \quad \mathrm{ind}_{G_i}^G \mathbf{Z} \to \mathrm{ind}_{G_i'}^G \mathbf{Z}$$

by

$$\varphi_i(\sum_{u=0}^{p^{k_{i+1}}-1} n_u a_{i+1}^u G_i) = \sum_{v=0}^{p^{k_{i+1}-1}-1} (\sum_{u \equiv v \pmod{p^{k_{i+1}-1}}} n_u) a_{i+1}^v G_i',$$

where $G = \bigcup_u a_{i+1}^u G_i$, $G = \bigcup_v a_{i+1}^v G_i'$. Then

$$\ker \varphi_i = \{ \sum_u n_u a_{i+1}^u G_i \mid \sum_{u \equiv v} n_u = 0 \, , v = 0, 1, \ldots, p^{k_{i+1}-1} - 1 \}.$$

We could use lemma 2.10 here to get $\mathrm{H}^2(G, \ker \varphi_i)$. But in order to reveal some more information, we will use following more detailed analysis instead.

First we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}) & \xrightarrow{\delta_G} & \mathrm{H}^1(G, \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z}) \\ \mathrm{res}_C^G \downarrow & & \downarrow \mathrm{res}_C^G \\ \mathrm{H}^2(C, \mathrm{res}_C^G \mathrm{ind}_C^G \mathbf{Z}) & \xrightarrow{\delta_C} & \mathrm{H}^1(C, \mathrm{res}_C^G \mathrm{ind}_C^G \mathbf{Q}/\mathbf{Z}) \end{array} .$$

We need only verify that $\mathrm{res}^G_C\delta_G^{-1} = \delta_C^{-1}\mathrm{res}^G_C$ since both δ are isomorphisms by 2.9. Let τ be the homomorphism in the following exact sequence.

$$0 \to \mathrm{ind}^G_C\mathbf{Z} \to \mathrm{ind}^G_C\mathbf{Q} \xrightarrow{\tau} \mathrm{ind}^G_C\mathbf{Q}/\mathbf{Z} \to 0;$$

$$0 \to \mathrm{res}^G_C\mathrm{ind}^G_C\mathbf{Z} \to \mathrm{res}^G_C\mathrm{ind}^G_C\mathbf{Q} \xrightarrow{\tau} \mathrm{res}^G_C\mathrm{ind}^G_C\mathbf{Q}/\mathbf{Z} \to 0.$$

Then $\mathrm{res}^G_C\delta_G^{-1}(f)(c_1,c_2) = \tau^{-1}(f(c_1,c_2)) - \tau^{-1}f(c_1) - \tau^{-1}(c_1f(c_2))$ for any $f \in \mathrm{H}^1(G,\mathrm{ind}^G_C\mathbf{Q}/\mathbf{Z})$. On the other hand

$$\delta_C^{-1}(\mathrm{res}^G_C)(f)(c_1,c_2) = \tau^{-1}(f(c_1,c_2)) - \tau^{-1}f(c_1) - \tau^{-1}(c_1f(c_2))$$

is obvious. So we can change the discussion from H^2 to H^1.

By Shapiro's Lemma, we have

$$\mathrm{H}^2(G,\mathrm{ind}^G_{G_i}\mathbf{Z}) \cong \mathrm{H}^1(G,\mathrm{ind}^G_{G_i}\mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(G_i,\mathbf{Q}/\mathbf{Z}).$$

Now if $k_i = 1$, we pick $\gamma \in \mathrm{Hom}(G_i,\mathbf{Q}/\mathbf{Z})$, such that $\gamma(a_i) = 1/p$, $\gamma(a_j) = 0$ for $j \neq i, i+1$, let α be the preimage of γ in $\mathrm{H}^2(G,\mathrm{ind}^G_{G_i}\mathbf{Z})$, then $\alpha \mid_{\langle c \rangle} \neq 0$ for any c in the following set:

$$S_i = \{a_i^{t_i} \prod_{u\neq i,i+1} a_u^{p^{k_u-1}t_u} \mid t_i = 1,2,\ldots,p-1, t_u = 0,1,\ldots,p-1\},$$

since if $c = a_i^{t_i}\prod a_u^{p^{k_u-1}t_u}$, $\gamma(c) = t_i/p \neq 0$ in $\mathbf{Q}/\mathbf{Z}$.

Now we prove that $\varphi_i^*(\alpha) \mid_{G_i} = 0$ in $\mathrm{H}^2(G,\mathrm{ind}^G_{G_i'}\mathbf{Z})$. We prove the result about H^1. In the isomorphism

$$\mathrm{H}^1(G,\mathrm{ind}^G_{G_i}\mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(G_i,\mathbf{Q}/\mathbf{Z}),$$

pick $[\gamma']$ to be the preimage of γ in $\mathrm{H}^1(G, \mathrm{ind}_{G_i}^G \mathbf{Q}/\mathbf{Z})$. Suppose that $\gamma'(h_i) = \sum_t n_t a_{i+1}^t G_i$, for $h_i \in G_i$. From the definition of γ, we know that $p\gamma = 0$. Hence $p[\gamma'] = 0$, or $[p\gamma'] = 0$. Then

$$p\gamma'(h_i) = \sum_t p n_t a_{i+1}^t G_i = h_i\alpha - \alpha$$

for some $\alpha \in \mathrm{ind}_{G_i}^G \mathbf{Q}/\mathbf{Z}$. But h_i acts trivially on $\mathrm{ind}_{G_i}^G \mathbf{Q}/\mathbf{Z}$, so $h_i\alpha - \alpha = 0$. Hence $pn_t = 0$, or $n_t \in (1/p)\mathbf{Z}$ for all t.

Next, for any $u = a_{i+1}^s \in \langle a_{i+1} \rangle$, $uh_i = h_i u$, so $\gamma'(uh_i) = \gamma'(h_i u)$, or $\gamma'(u) + u\gamma'(h_i) = \gamma'(h_i) + h_i\gamma'(u) = \gamma'(h_i) + \gamma'(u)$. Hence $\gamma'(h_i) = u\gamma'(h_i)$ for any $u \in \langle a_{i+1} \rangle$. But $u\gamma'(h_i) = a_{i+1}^s \sum_t n_t a_{i+1}^t G_i = \sum_t n_t a_{i+1}^{t+s} G_i$, so $u\gamma'(h_i) = \gamma'(h_i)$ giving $n_t = n_{t+s}$, for any s, i.e. n_t are all equal. Then

$$\varphi_i\gamma'(h_i) = \sum_v (\sum_{t \equiv v} n_t) a_{i+1}^v G' = 0.$$

Hence the result.

Since $\langle a_{i+1} \rangle \cap G_i = 1$. for any $a_{i+1}^v \in \langle a_{i+1} \rangle$,

$$\mathrm{res}_{\langle a_{i+1}^v \rangle}^G \mathrm{ind}_{G_i}^G \mathbf{Z} = \bigoplus \mathrm{ind}_1^{\langle a_{i+1}^v \rangle} \mathrm{res}_1^{G_i} \mathbf{Z}$$

by Mackey's formula, which is projective as $\mathbf{Z}\langle a_{i+1}^v \rangle$-module. Hence $[\gamma']|_{\langle a_{i+1}^v \rangle} = 0$ for any v. This means $\gamma'(a_{i+1}^v) = a_{i+1}^v\alpha - \alpha$ for some $\alpha \in \mathrm{res}_{\langle a_{i+1}^v \rangle}^G \mathrm{ind}_{G_i}^G \mathbf{Z}$.

Now for any $h \in G$, write $h = a_{i+1}^v h_i$, with $a_{i+1}^v \in \langle a_{i+1} \rangle$, $h_i \in G_i$. Then $\gamma'(a_{i+1}^v h_i) = \gamma'(a_{i+1}^v) + a_{i+1}^v \gamma'(h_i)$. So

$$\begin{aligned} \varphi_i\gamma'(a_{i+1}^v h_i) &= \varphi_i\gamma'(a_{i+1}^v) + a_{i+1}^v \varphi_i\gamma'(h_i) = \varphi_i(a_{i+1}^v\alpha - \alpha) \\ &= a_{i+1}^v\varphi_i(\alpha) - \varphi_i(\alpha) = (a_{i+1}^v h_i)\varphi_i(\alpha) - \varphi_i(\alpha). \end{aligned}$$

Hence $\varphi_i^*([\gamma']) = 0$ in $\mathrm{H}^{i+1}(G, \mathrm{ind}_{G_i'}^G \mathbf{Z})$.

By this result and the exact sequence

$$\mathrm{H}^2(G, \ker\varphi_i) \rightarrow \mathrm{H}^2(G, \mathrm{ind}_{G_i}^G \mathbf{Z}) \rightarrow \mathrm{H}^2(G, \mathrm{ind}_{G_i'}^G \mathbf{Z})$$

we can find $\beta_i \in \mathrm{H}^2(G, \ker\varphi_i)$, such that β_i is special for any element of S_i. It is obvious that a_{i+1} acts faithfully on $\ker\varphi_i$.

Finally, $(\ker\varphi_i)^H = 0$. If $x = \sum_u n_u a_{i+1}^u G_i \in (\ker\varphi_i)^G$, then $\sum_{u\equiv v} n_u = 0$ for any $v = 0, 1, \ldots, p^{k_{i+1}-1} - 1$, and $x^{a_{i+1}^t} = x$ for any t. Hence $\sum n_u a_{i+1}^{u+t} G_i = \sum n_u a_{i+1}^u G_i$ for any t, so n_u are all equal. Then $\sum n_u = 0$ gives $pn_u = 0$, or $n_u = 0$ for all u.

Let

$$L_1 = \bigoplus_{i=1}^{r} \ker\varphi_i.$$

By above discussion, L_1 is a faithful $\mathbf{Z}G$-lattice, and it is special for all the elements of the form:

$$\{a_i^{t_i} \prod_{u\neq i,i+1} a_u^{p^{k_u-1}t_u} \mid k_i = 1, \text{ i.e. } o(a_i) = p, t_i \neq 0, t_u = 0, 1, \ldots, p-1\}.$$

(ii) For each i with $k_i > 1$, let

$$K_i = \langle\, a_i^{p^{k_i-1}} \,\rangle \times \langle\, a_{i+1}^p \,\rangle \times \prod_{u\neq i,i+1} \langle\, a_u \,\rangle\,, \quad G = \bigcup_{t_i,t_{i+1}} a_i^{t_i} a_{i+1}^{t_{i+1}} K_i,$$

$$K_i' = \langle\, a_i^{p^{k_i-1}} \,\rangle \times \prod_{u\neq i} \langle\, a_u \,\rangle\,, \quad G = \bigcup_{t_i} a_i^{t_i} K_i'.$$

Define:

$$\psi_i : \mathrm{ind}_{K_i}^G \mathbf{Z} \rightarrow \mathrm{ind}_{K_i'}^G \mathbf{Z},$$

$$\psi_i\Big(\sum_{t_i,t_{i+1}} n_{t_i,t_{i+1}} a_i^{t_i} a_{i+1}^{t_{i+1}} K_i\Big) = \sum_{t_i}\Big(\sum_{t_{i+1}} n_{t_i,t_{i+1}}\Big) a_i^{t_i} K_i'.$$

Then

$$\ker\psi_i = \{ \sum_{t_i,t_{i+1}} n_{t_i,t_{i+1}} a_i^{t_i} a_{i+1}^{t_{i+1}} K_i \mid \sum_{t_i+1} n_{t_i,t_{i+1}} = 0, \ \forall t_i \}.$$

Follow the same discussion as in (i), if we pick $\delta_i \in \mathrm{Hom}(K_i, \mathbf{Q}/\mathbf{Z})$ such that $\delta_i(a_i^{p^{k_i-1}}) = 1/p$, $\delta_i(a_{i+1}^p) = \delta_i(a_u) = 0$, $u \neq i, i+1$ we can prove in the same way that $\ker\psi_i$ is special for all elements of the form:

$$\{a_i^{p^{k_i-1}t_i} \prod_{u \neq i,i+1} a_u^{p^{k_u-1}t_u} \mid t_i = 1,2,\ldots,p-1, t_u = 0,1,\ldots,p-1\},$$

and it is easy to see that $(\ker\psi_i)^G = 0$.

Let

$$L_2 = \bigoplus_{k_i > 1} \ker\psi_i,$$

then $L_1 \bigoplus L_2$ is a faithful $\mathbf{Z}G$-lattice and special for all elements of the form:

$$\{ \ \prod_{i=1}^r a_i^{p^{k_i-1}u_i} \mid \text{ at least one } u_i = 0, \text{ and } u_i \text{ are not all } 0 \ \}.$$

Hence the only kind of elements of S left is

$$\{ \ \prod_{i=1}^r a_i^{p^{k_i-1}t_i} \mid t_i = 1,2,\ldots,p-1\}.$$

We next divide the discussion into three cases.

(1). There exists $i < r$, such that $k_i > 1$, $k_{i+1} > 1$.

In this case, $\prod_{i=1}^r a_i^{p^{k_i-1}t_i} \in K_i$, the α picked in (i) is also special for such kind of elements. Hence $L_1 \bigoplus L_2$ is a primitive lattice in this case.

(2). $k_1 = k_2 = \ldots = k_{r-1} = 1$, $k_r > 1$.

We need one more lattice in this case. Let $K = \langle a_r^p \rangle \times \prod_{i=1}^{r-1} \langle a_i \rangle$, $K' = G$. Define $\psi : \mathrm{ind}_K^G \mathbf{Z} \to \mathrm{ind}_{K'}^G \mathbf{Z} = \mathbf{Z}$ by $\psi(\sum_{i=0}^{p-1} n_i a_r^i K) = \sum_{i=0}^{p-1} n_i$.

Then following the discussion before, we can get $\alpha \in \mathrm{H}^2(G, \ker\psi)$, such that α is special for all elements of the form:

$$\{a_r^{p^{k_r-1}t_r} \prod_{i=1}^{r-1} a_i^{t_i} \mid t_i = 1, 2, \ldots, p-1;\ i = 1, 2, \ldots, r\}.$$

Hence $L_1 \bigoplus L_2 \bigoplus \ker\psi$ is a primitive lattice in this case.

(3). $k_i = 1, \quad i = 1, 2, \ldots, r,$ or G is elementary abelian group. In this case, we let $K_j = \langle a_1 a_2^j \rangle \times \prod_{i=3}^{r} \langle a_i \rangle, \quad j = 1, 2, \ldots, p-1, \quad \delta_j : \mathrm{ind}_{K_j}^G \mathbf{Z} \to \mathbf{Z}$ defined by $\delta_j(\sum_u n_u a_1^u K_j) = \sum_u n_u$. Then by the same argument as above, we can get $L_1 \bigoplus \bigoplus_{j=1}^{p-1} \ker\delta_j$ is a primitive lattice of G in this case. Hence we have the following theorem:

Theorem 3.1.1. Let $G = C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_r}}, \quad k_1 \le k_2 \le \cdots \le k_r$, $r \ge 2$. Then

(i). $\delta(G) \le (p + r - 1)(p-1)$, if $k_r = 1$;

(ii). $\delta(C_p \times C_{p^2}) \le (2p+1)(p-1)$;

(iii). $\delta(G) \le r(p-1) + 2p^{k_r-1}(p-1)$, if $k_r > 1, \quad k_{r-1} = 1$;

(iv). $\delta(G) \le \sum_{i=1}^{r} p^{k_i-1}(p-1) + \sum_{k_i>1} p^{k_i-1}(p-1)$, others.

Proof. (i), (iii), (iv) are direct consequences of above argument. In the case of (ii), a_1 acts faithfully on $\ker\psi_2$. So we can drop $\ker\varphi_2$ without losing faithfulness. More precisely,

$\ker\varphi_1$ is special for $\{a_1^{t_1}, t_1 = 1, 2, \cdots, p-1\}$, a_2 acts faithfully on it.

$\ker\psi_2$ is special for $\{a^{pt_2}, t_2 = 1, 2, \cdots, p-1\}$, a_1 acts faithfully on it.

$\ker\psi$ is special for $\{a_1^{t_1} a_2^{pt_2}, t_1 = 1, 2, \cdots, p-1; t_2 = 1, 2, \cdots, p-1\}$.

Hence $\ker\varphi_1 \bigoplus \ker\psi_2 \bigoplus \ker\psi$ is a primitive lattice , i.e. $\delta(C_p \times C_{p^2}) \le (2p+$

$1)(p-1)$. □

Corollary 3.1.2. $\delta(G) < |G|$ for any abelian primitive group G.

Proof. First suppose G is abelian p-group as in Theorem 3.1.1.

i). If $k_r = 1$, $|G| = p^r$; $\delta(G) \leq (p+r-1)(p-1)$, $r \geq 2$. For $r=2$, $p^2 > p^2 - 1$; for $r > 2$, by $a^x \geq a + x, a \geq 2, x \geq 2$, we have $p^r > p^{r-1}(p-1) \geq (p+r-1)(p-1)$. Hence the result.

ii). If $k_r > 1, k_{r-1} = 1$, $|G| = p^{r-1+k_r}$; $\delta(G) \leq r(p-1) + 2p^{k_{r-1}}(p-1)$. By $a^x \geq x + 2(a-1), x \geq 2, a \geq 2$, we can get $p^{r-1+k_r} = p^{k_r-1}p^r \geq p^{k_r-1}(r+2(p-1)) > r(p-1) + 2p^{k_{r-1}}(p-1)$.

iii). In other cases, $|G| = p^{\sum_i k_i}$, $\delta(G) \leq \sum_{i=1}^r p^{k_i-1}(p-1) + \sum_{k_i>1} p^{k_i-1}(p-1)$. $\delta(G)/|G| < 2r/p^r \leq 1$.

Finally, if $G = \prod_p G_p$, then

$$\delta(G) \leq \sum_p \delta(G_p) < \sum_p |G_p| \leq \prod_p |G_p| = |G|. \ \square$$

§3.2 A lower bound for $\delta(G)$, G an abelian p-group

Let G be the same as in §**3.1**, L be a primitive $\mathbf{Z}G$-lattice, $L_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} L$. Then we have

(i) $L_{\mathbf{Q}}$ is a faithful $\mathbf{Q}G$-module;

(ii) $L_{\mathbf{Q}}^{\langle g \rangle} \neq 0$ for any $g \in G$ and $L_{\mathbf{Q}}^G = 0$;

(iii) $L_{\mathbf{Q}} \cong \bigoplus_{d \mid |G|} n_d \mathbf{Q}(\zeta_d)$, where ζ_d is the d-th primitive root of unity and n_d are some nonnegative integers.

These all follow from §1.3. $L_{\mathbf{Q}}$ is a faithful $\mathbf{Q}G$-module since L is a faithful integral representation. (iii) is a direct consequence of the fact that G is abelian since every irreducible $\mathbf{Q}G$-module of G then is of the form $\mathbf{Q}(\zeta_d)$ for some $d \,||\, |G|$. For (ii), $L_{\mathbf{Q}}^G = 0$ since $L^G = 0$, so we need only prove that $L_{\mathbf{Q}}^{\langle g \rangle} \neq 0$ for any $g \in G$. Suppose on the contrary, $L_{\mathbf{Q}}^{\langle g_0 \rangle} = 0$ for some $g_0 \in G$. Then $L^{\langle g_0 \rangle} = 0$. Let $\alpha \in \mathrm{H}^2(G, M)$ be special for G, $\mathrm{res}^G_{\langle g_0 \rangle} \alpha \neq 0$ in particular. But

$$\mathrm{res}^G_{\langle g_0 \rangle} \alpha \in \mathrm{H}^2(\langle g_0 \rangle, \mathrm{res}^G_{\langle g_0 \rangle} L) \cong L^{\langle g_0 \rangle} / h(L) = 0,$$

where $h(L)$ is the image of the map $h : L \to L$ defined by $h(l) = \sum_i g_0^i l$. This is a contradiction. Hence (ii).

Now we try to find minimal $L_{\mathbf{Q}}$ satisfying (i), (ii), (iii).

First, we claim that, if T is a faithful $\mathbf{Q}G$-module of minimal dimension , then by suitable rearrangement of generators of G, we can suppose that

$$T = \bigoplus_{i=1}^{r} \mathbf{Q}(\zeta_{p^{k_i}})$$

with a_j acting faithfully on j-th component and trivially on all other components. That is $a_j \cdot l = \zeta_{p^{k_i}}^{\delta_{ij}} l$,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

Since T is faithful, we can pick $\mathbf{Q}(\zeta_d) \subset T$ irreducible, such that a_r acts on it faithfully. It must be $\mathbf{Q}(\zeta_{p^{k_r}})$, and we can suppose that

$$a_r \cdot l_r = \zeta_{p^{k_r}} l_r \;\; , a_i \cdot l_r = \zeta_{p^{k_r}}^{t_i} l_r \;\; , i < r \;\; , t_i \in \mathbf{Z}.$$

Now let

$$G' = \langle a_r \rangle \times \prod_{k=1}^{r-1} \langle a_{r-k} a_r^{s_{r-k}} \rangle = \langle a'_r \rangle \times \langle a'_{r-1} \rangle \times \cdots \times \langle a'_1 \rangle ,$$

where $a'_r = a_r$, $a'_i = a_i a_r^{s_i}$, such that $s_i + t_i \equiv 0 \pmod{p^{k_r}}, i = 1, 2, \ldots, r-1$. It is easy to see that there are solutions for s_i, and a'_i are still a set of generators of G. Hence $G' = G$. From $s_i + t_i \equiv 0 \pmod{p^{k_r}}$, we can get $s_i \equiv 0 \pmod{p^{k_r - k_i}}$, since $t_i p^{k_i} \equiv 0 \pmod{p^{k_r}}$, $t_i \equiv 0 \pmod{p^{k_r - k_i}}$. Hence $o(a_r^{s_i} a_i) = o(a_i)$, and

$$a'_r \cdot l_r = \zeta_{p^{k_r}} l_r, \; a'_i \cdot l_r = l_r, i < r, \text{ for any } l_r \in \mathbf{Q}(\zeta_{p^{k_r}}) .$$

Now $T / \mathbf{Q}(\zeta_{p^{k_r}})$ is the minimal faithful representation for

$$G / \langle a_r \rangle = \langle a'_{r-1} \rangle \times \cdots \times \langle a'_1 \rangle .$$

By induction on r we can suppose that $T / \mathbf{Q}(\zeta_{p^{k_r}}) \cong \bigoplus_1^{r-1} \mathbf{Q}(\zeta_{p^{k_i}})$ with the required $\langle a'_{r-1} \rangle \times \cdots \times \langle a'_1 \rangle$ action. Hence $T = \bigoplus_{i=1}^r \mathbf{Q}(\zeta_{p^{k_i}})$ and $\langle a'_{r-1} \rangle \times \cdots \times \langle a'_1 \rangle$ acts on it as required. The only thing left is to adjust a_r such that it acts trivially on $\mathbf{Q}(\zeta_{p^{k_i}})$. This can be done by rewriting

$$G = \langle a_r \prod_{i=1}^{r-1} (a')_i^{u_i} \rangle \times \prod_{i=1}^{r-1} \langle a'_i \rangle$$

such that $t_i + u_i \equiv 0 \pmod{p^{k_i}}$.

Denote $L_0 = \bigoplus_{i=1}^r \mathbf{Q}(\zeta_{p^{k_i}})$, then $L_0 \subset L_{\mathbf{Q}}$ by above claim. for any $g = \prod_{i=1}^r a_i^{t_i}$, $t_i \not\equiv 0 \pmod{p^{k_i}}$, we have $L_0^{\langle g \rangle} = 0$. So we need some more modules to assure (ii). Suppose $\mathbf{Q}(\zeta_{p^s})$, $s > 1$ is such a module contained in $L_{\mathbf{Q}}$, $a_i \cdot l = \zeta_{p^s}^{u_i} l$ for any $l \in \mathbf{Q}(\zeta_{p^s})$. Define $\mathbf{Q}(\zeta_p)$ a $\mathbf{Q}G$-module by $a_i \cdot l = \zeta_p^{\bar{u}_i} l$, $(\bar{u}_i \equiv u_i \pmod{p})$ for any $l \in \mathbf{Q}(\zeta_p)$. Then whenever $g \in G$ has nonzero fixed points in $Q(\zeta_{p^s})$, g has nonzero fixed points in $\mathbf{Q}(\zeta_p)$. Hence we need only pick the latter.

For $g = \prod_{i=1}^{r} a_i^{t_i}$, $t_i \not\equiv 0 \pmod{p^{k_i}}$, we can have $\prod_{i=1}^{r} (p^{k_i} - 1)$ such elements. For a given $\mathbf{Q}(\zeta_p)$, suppose $a_i \cdot l = \zeta_p^{u_i} l$ for $l \in \mathbf{Q}(\zeta_p)$, then g has nonzero fixed points in $\mathbf{Q}(\zeta_p)$ if and only if $\sum_{i=1}^{r} t_i u_i \equiv 0 \pmod{p}$. So we need only discuss that for giving $(u_1, u_2, \ldots, u_r)$, how many $(t_1, t_2, \ldots, t_r)$ we can find such that the above equation are satisfied, where $u_i \in (0, 1, \ldots, p-1)$; $t_i \in (1, 2, \ldots, p^{k_i} - 1)$.

If there is only one nonzero u_i, say $u_{i_0} \neq 0$, then all $t_i, i \neq i_0$ can be arbitrary, the equation is actually $u_{i_0} t_{i_0} \equiv 0 \pmod{p}$. this has $(p^{k_{i_0}-1} - 1) \prod_{i \neq i_0} (p^{k_i} - 1)$ solutions. If there are more than one nonzero u_i, say $u_{i_1} \neq 0, u_{i_2} \neq 0$, let all other t_i arbitrary, we will always end in a equation of the form: $u_{i_1} t_{i_1} + u_{i_2} t_{i_2} \equiv b \pmod{p}$.

If $u_{i_2} t_{i_2} - b \equiv 0 \pmod{p}$, we have $p^{k_{i_1}-1} - 1$ of t_{i_1}, $p^{k_{i_2}-1}$ or $p^{k_{i_2}-1} - 1$ of t_{i_2} as solutions.

If $u_{i_2} t_{i_2} - b \not\equiv 0 \pmod{p}$, we have $p^{k_{i_1}-1}$ of t_{i_1}, $p^{k_{i_2}} - p^{k_{i_2}-1}$ or $p^{k_{i_2}} - p^{k_{i_2}-1} + 1$ of t_{i_2} as solutions.

So in any case, the number of solutions of above equation is not greater than $p^{k_{i_1}-1}(p^{k_{i_2}} - p^{k_{i_2}-1} + 1)$. Hence we need at least to add (# of g)/(# of solutions) more $\mathbf{Q}(\zeta_p)$s. But

$$\begin{aligned} \frac{\#\text{ of } g}{\#\text{ of solutions}} &\geq \frac{\prod_{i=1}^{r} (p^{k_i} - 1)}{p^{k_{i_1}-1}(p^{k_{i_2}} - p^{k_{i_2}-1} + 1) \prod_{j \neq i_1, i_2} (p^{k_j} - 1)} \\ &= \frac{(p^{k_{i_1}} - 1)(p^{k_{i_2}} - 1)}{p^{k_{i_1}-1}(p^{k_{i_2}} - p^{k_{i_2}-1} + 1)} \end{aligned}$$

$$> \begin{cases} p-2 & k_{i_1} = k_{i_2} = 1 \\ p-1 & \text{otherwise.} \end{cases}$$

Hence we have the following theorem.

Theorem 3.2.1. Let $G = C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_r}}, \quad k_1 \le k_2 \le \cdots \le k_r, r \ge 2.$ Then

$$\delta(G) \quad \ge \quad \begin{cases} \sum_{i=1}^{r} \varphi(p^{k_i}) + (p-1)^2 & \text{if } k_2 = 1; \\ \sum_{i=1}^{r} \varphi(p^{k_i}) + p(p-1) & \text{if } k_2 > 1. \end{cases}$$

Corollary 3.2.2.

(i) $\delta(C_p^k) = (p+k-1)(p-1);$ see [12]

(ii) $\delta(C_p \times C_{p^2}) = (2p+1)(p-1);$

(iii) let $G = \prod_p G_p$ be an abelian group with

$G_p = \prod_{i=1}^{r_p} C_{p^{k_{p,i}}}, \ k_{p,r_p} > 1$ for all p, $p > 2$, for all p, then

$$\sum_p \sum_i \varphi(p^{k_{p,i}}) \le \delta(G) \le 2 \sum_p \sum_i \varphi(p^{k_{p,i}}).$$

§3.3 p^3-groups

The following are the whole list of groups of order p^3, p prime. $C_p \times C_p \times C_p$, $C_p \times C_{p^2}$, C_{p^3}, $M_3(p)$, and $M(p)$ for $p \ge 3$; D_8 and Q_8 for $p = 2$, Where

$$M_3(p) \quad = \quad \langle\ x, y \mid x^{p^2} = y^p = 1, [x,y] = x^p\ \rangle\ ;$$

$$M(p) \quad = \quad \langle\ x, y, z \mid x^p = y^p = z^p = 1, [x,y] = z, [x,z] = [y,z] = 1\ \rangle\ ;$$

$$D_8 \quad = \quad \langle\ x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1}\ \rangle\ ;$$

$$Q_8 \quad = \quad \langle\ x, y \mid x^2 = y^2, y^{-1}xy = x^{-1}\ \rangle\ .$$

Among them, C_{p^3} is not primitive. $\delta(C_p \times C_p \times C_p)$ and $\delta(C_p \times C_{p^2})$ have been given by corollary 3.2.2. $\delta(D_8) = 4, \delta(Q_8) = 7$ can be found in [12]. So we need only discuss $M_3(p)$ and $M(p)$.

Theorem 3.3.1. $\delta(M_3(p)) = \delta(M(p)) = 2p(p-1)$.

(i) $\delta(M_3(p)) \leq 2p(p-1)$.

$M_3(p)$ has $p+2$ non-trivial irreducible **Q**-representations. The representations $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i, 0 \leq i \leq p$ of degree $p-1$ which factor through a cyclic group of order p and are determined by the condition: $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_0$ is trivial on x and $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i$ is trivial on $x^i y$, for all i, $1 \leq i \leq p$. The representation $T_{\mathbf{Q}} = \mathrm{ind}^G_{\langle x^p, y \rangle} R_{\mathbf{Q}}$, with $R_{\mathbf{Q}}$ irreducible of degree $p-1$ on $\langle x^p, y \rangle = C_p \times C_p$ and $\ker(R_{\mathbf{Q}}) = \langle y \rangle$, is faithful. Furthermore there is an $\alpha \in \mathrm{H}^2(M_3(p), T)$ such that $\alpha \mid_{\langle y \rangle} \neq 0$, or α is special for y (see [12]). But, since

$$xyx^{-1} = yx^p, x^2yx^{-2} = yx^{2p}, \ldots, x^{p-1}yx^{-p+1} = yx^{(p-1)p},$$

α is also special for $yx^{pi}, i = 1, 2, \ldots, p-1$. In $M_3(p)$, all the elements of order p are $\langle x^p, y \rangle - \{1\}$, Hence we need only find a lattice T_1 such that there is an $\alpha_1 \in \mathrm{H}^2(M_3(p), T_1)$ which is special for x^p. In order to construct such a lattice, we use Corollary 2.11.

Let $G = M_3(p)$, $C = \langle x^p \rangle$, $H = \langle x^p \rangle \times \langle y \rangle$. Then the transfer from H to C is trivial, by Corollary 2.11 and the exact sequence

$$\mathrm{H}^2(G, \ker \varphi) \rightarrow \mathrm{H}^2(G, \mathrm{ind}^G_C \mathbf{Z}) \xrightarrow{\varphi^*} \mathrm{H}^2(G, \mathrm{ind}^G_H \mathbf{Z}),$$

we can get $\alpha_1 \in \mathrm{H}^2(G, \ker\varphi)$, such that α_1 is special for x^p, since

$$\mathrm{H}^2(G, \mathrm{ind}^G_{\langle x^p \rangle} \mathbf{Z}) \cong \mathrm{H}^2(\langle x^p \rangle, \mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$$

contains β_1 which is special for x^p. Hence our T_1 can be $\ker\varphi$, or $T \bigoplus \ker\varphi$ is a primitive lattice. So $\delta(M_3(p)) \le 2p(p-1)$.

(ii) $\delta(M_3(p)) \ge 2p(p-1)$.

Suppose L is a primitive lattice for $M_3(p)$. Then $L_{\mathbf{Q}} = \mathbf{Q} \bigotimes_{\mathbf{Z}} L$ is a faithful $\mathbf{Q}M_3(p)$-module and $L_{\mathbf{Q}}^{\langle g \rangle} \neq 0$ for any $g \in M_3(p)$. Hence it must contain $T_{\mathbf{Q}} = \mathrm{ind}^{M_3(p)}_{\langle x^p, y \rangle} R_{\mathbf{Q}}$, $R_{\mathbf{Q}}$ is an irreducible representation of $\langle x^p, y \rangle \cong C_p \times C_p$. Now x acts faithfully on $T_{\mathbf{Q}}$, and dim $T_{\mathbf{Q}} = p(p-1)$. Hence $\mathrm{res}^{M_3(p)}_{\langle x \rangle} T_{\mathbf{Q}} \cong \mathbf{Q}(\zeta_{p^2})$ with $x \cdot l = \zeta_{p^2} l$ for $l \in \mathbf{Q}(\zeta_{p^{k_2}})$. That implies $T_{\mathbf{Q}}^{\langle x \rangle} = 0$. Similarly we can get the $T_{\mathbf{Q}}^{\langle x^i y \rangle} = 0$, for all i such that $(i, p) = 1$ since $x^i y$ also has order p^2 and it acts on $T_{\mathbf{Q}}$ faithfully. Now pick any $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i$, $0 \le i \le p$, then we have

$$\begin{array}{ccc} M_3(p) & \rightarrow & \mathbf{Q} \bigotimes_{\mathbf{Z}} S_i \\ \downarrow & \nearrow & \\ \langle g \rangle & & \end{array}$$

where g has order p. So $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i \cong \mathbf{Q}(\zeta_p)$, or $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i \cong \bigoplus_k \mathbf{Q}_k, \mathbf{Q}_k \cong \mathbf{Q}$ with the trivial action of $M_3(p)$ on it by factoring through $\langle g \rangle$. It can not be the second case because $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i$ is irreducible. So $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i \cong \mathbf{Q}(\zeta_p)$, and hence $x \cdot l = \zeta^{k_1} l, y \cdot l = \zeta^{k_2} l$, with $k_1 i + k_2 \equiv 0 \pmod p$. Since for the fixed k_1, k_2 of above, $k_1 j + k_2 \not\equiv 0 \pmod p$, for any $j \not\equiv i \pmod p$, $(\mathbf{Q} \bigotimes_{\mathbf{Z}} S_i)^{\langle x^i y \rangle} = 0$ for $j \not\equiv i \pmod p$. Hence we need $\mathbf{Q} \bigotimes_{\mathbf{Z}} S_0$ to assure that $L_{\mathbf{Q}}^{\langle x \rangle} \neq 0$; we need $\mathbf{Q}\bigotimes_{\mathbf{Z}} S_i$ to assure that $L_{\mathbf{Q}}^{\langle x^i y \rangle} \neq 0, \quad i = 1, 2, \ldots, p-1$. So

$$L_{\mathbf{Q}} \supseteq T_{\mathbf{Q}} \bigoplus \mathbf{Q}\bigotimes_{\mathbf{Z}} S_0 \bigoplus \mathbf{Q}\bigotimes_{\mathbf{Z}} S_1 \bigoplus \cdots \bigoplus \mathbf{Q}\bigotimes_{\mathbf{Z}} S_{p-1},$$

or $\dim L_{\mathbf{Q}} \geq 2p(p-1)$.

(iii) $\delta(M(p)) \leq 2p(p-1)$.

We have $p+2$ non-trivial standard irreducible $\mathbf{Z}G$-lattices. $S_i, \quad 0 \leq i \leq p$ of rank $p-1$ which factor through a cyclic group of order p and are determined by S_0 is trivial for x; S_i is trivial for $x^i y$ and $T = \mathrm{ind}_{\langle y,z \rangle}^{M(p)} R$, where R is irreducible $\mathbf{Z}\langle y, z\rangle$-lattice of rank $p-1$ with $\ker R = \langle y \rangle$. Representatives of the conjugacy classes of cyclic subgroups are $1, z, x, x^i y, \ 1 \leq i \leq p$.

We have a primitive lattice $T \bigoplus_{i=1}^{p-1} S_i$ for $M(p)$. From Corollary 2.12 we can easily get

$$\mathrm{H}^2(M(p), S_0) \cong \mathrm{Hom}(\langle x, z \rangle, \mathbf{Q}/\mathbf{Z}) \cong C_p \times C_p,$$

$$\mathrm{H}^2(M(p), S_i) \cong \mathrm{Hom}(\langle x^i y, z \rangle, \mathbf{Q}/\mathbf{Z}) \cong C_p \times C_p,$$

$$\mathrm{H}^2(M(p), T) \cong \mathrm{Hom}(\langle y \rangle, \mathbf{Q}/\mathbf{Z}) \cong C_p,$$

so we can find $\alpha_0 \in \mathrm{H}^2(M(p), S_0)$ such that α_0 is special for x and z, $\alpha_i \in \mathrm{H}^2(M(p), S_i)$ such that it is special for $x^i y$, $1 \leq i \leq p-1$, and $\alpha \in \mathrm{H}^2(M(p), T)$ such that it is special for y. So $\delta(M(p)) \leq 2p(p-1)$.

(iv) $\delta(M(p)) \geq 2p(p-1)$.

Suppose L is a primitive lattice of $M(p)$, then

$$\mathbf{Q}\bigotimes_{\mathbf{Z}} L \cong \mathbf{Q}\bigotimes_{\mathbf{Z}} \bigoplus_i M_i \quad M_i \neq \mathbf{Z}, \text{ since } L^{M(p)} = 0,$$

where M_i are some standard irreducible $\mathbf{Z}M(p)$-lattices. By Lemma 3.0.2, $\bigoplus M_i$ is a faithful $\mathbf{Z}M(p)$-lattice and there is a F satisfying:

$$0 \to A \to F \xrightarrow{\tau} M(p) \to 1$$

with A finite abelian and $\bigoplus M_i$ special for F. $M(p)$ acts faithfully on $\bigoplus M_i$, so we need T to get faithfulness. We claim that

$$T = \mathrm{ind}_{\langle y,z \rangle}^{M(p)} R \cong \mathrm{ind}_{\tau^{-1}(\langle y,z \rangle)}^{F} R.$$

Since $F/\tau^{-1}(\langle y,z \rangle) \cong M(p)/\langle y,z \rangle$, we can pick coset representatives of $\langle y,z \rangle$ in $M(p)$ as $\{x^i\}$ and $\tau^{-1}(\langle y,z \rangle)$ in F as $\{\tau^{-1}(x^i)\}$, such that $\tau^{-1}(x^i)\tau^{-1}(x^j) = \tau^{-1}(x^{i+j})$. Then

$$\phi : \sum \tau^{-1}(x^i) r_i \to \sum x^i r_i$$

is a $\mathbf{Z}F$-module isomorphism since for any $f \in F$, $f = \tau^{-1}(x^t)\tau^{-1}(h)$

$$\begin{aligned}
\phi(f \sum \tau^{-1}(x^i) r_i) &= \phi(\sum \tau^{-1}(x^t)\tau^{-1}(h)\tau^{-1}(x^i) r_i) \\
&= \phi(\sum \tau^{-1}(x^{t+i})(\tau^{-1}(h))^{\tau^{-1}(x^i)} r_i) \\
&= \phi(\sum \tau^{-1}(x^{i+t}) h^{x^i} r_i) \\
&= \sum x^{t+i} h^{x^i} r_i = x^t h \sum x^i r_i = f \sum x^i r_i \\
&= f\phi(\sum \tau^{-1}(x^i) r_i).
\end{aligned}$$

Next we will prove that T is not special for $\tau^{-1}(x), \tau^{-1}(x^i y), i = 1, 2, \cdots, p-1$. By the isomorphism

$$\mathrm{H}^2(F,T) \cong \mathrm{H}^1(F, \mathbf{Q} \otimes T/T) \cong \mathrm{H}^1(F, \mathrm{ind}_{\tau^{-1}(\langle y,z \rangle)}^{F} \mathbf{Q} \otimes R/R),$$

we can use H^1 to do it. By Shapiro's Lemma we have

$$\mathrm{H}^1(F, \mathrm{ind}^F_{\tau^{-1}(\langle y,z \rangle)} \mathbf{Q} \otimes R/R) \overset{\varphi^*}{\cong} \mathrm{H}^1(\tau^{-1}(\langle y,z \rangle), \mathbf{Q} \otimes R/R) .$$

Suppose $F = \cup_i \tau^{-1}(x^i)\tau^{-1}(\langle y,z \rangle)$ as above. For any $g \in F$, let $g\tau^{-1}(x^i) = \tau^{-1}(x^{i(g)})C_{g,i}$ with $C_{g,i} \in \tau^{-1}(\langle y,z \rangle)$. If $[f] \in \mathrm{H}^1(\tau^{-1}(\langle y,z \rangle), \mathbf{Q} \otimes R/R)$, the preimage of $[f]$, $[\hat{f}] \in \mathrm{H}^1(F, \mathrm{ind}^F_{\tau^{-1}(\langle y,z \rangle)} \mathbf{Q} \otimes R/R)$ is defined by

$$\hat{f}(g) = \sum_{i=0}^{p-1} f(C_{g,i})(\tau^{-1}(x^{i(g)}))\tau^{-1}(\langle y,z \rangle),$$

Now if $g = \tau^{-1}(x)$, by $\tau^{-1}(x)\tau^{-1}(x^i) = \tau^{-1}(x^{i+1})$ we get $C_{\tau^{-1}(x),i} = 1$. Hence $\hat{f}(\tau^{-1}(x)) = 0$ or $[\hat{f}]$ is not special for $\tau^{-1}(x)$. Therefore T is not special for $\tau^{-1}(x)$. Similarly, we can prove that T is not special for $\tau^{-1}(x^i y)$, $i = 1, 2, \cdots, p-1$.

Since if $i \neq j$, $S_j^{\langle \tau^{-1}(x^i y) \rangle} = 0$, we have

$$\mathrm{H}^2(\langle \tau^{-1}(x^i y) \rangle , \mathrm{res}^F_{\langle \tau^{-1}(x^i y) \rangle} S_j) = 0.$$

That is S_j is not special for $\tau^{-1}(x^i y)$, $i \neq j$. We need S_0 to get special for $\tau^{-1}(x)$; S_1 to get special for $\tau^{-1}(xy)$, $\cdots$, S_{p-1} to get special for $\tau^{-1}(x^{p-1}y)$. Hence

$$\mathrm{rank}(M) \geq \mathrm{rank}(T) + \sum_{i=0}^{p-1} \mathrm{rank}(S_i) = 2p(p-1).$$

§3.4. Some solvable groups

Lemma 3.4.1. Let G be a finite group fitting into

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\tau} B \longrightarrow 1,$$

where both A and B are primitive groups. Then G is a primitive group and $\delta(G) \leq [G:A]\delta(A) + \delta(B)$.

Proof. Let M be a primitive $\mathbf{Z}B$-lattice, N a primitive $\mathbf{Z}A$-lattice. We claim that $T = \mathrm{ind}_A^G N \bigoplus M$ is a primitive $\mathbf{Z}G$-lattice, where G acts on M through $G \to B$.

(i). T is a faithful $\mathbf{Z}G$-lattice since $\mathrm{ind}_A^G N$ is.

(ii). $T^G = 0$. $M^G = 0$ since $M^B = 0$. $(\mathrm{ind}_A^G N)^G \cong N^A = 0$ (see p.86 of [14]). So $M^G = (\mathrm{ind}_A^G N)^G \bigoplus M^G = 0$.

(iii). T carries a special class. From 5-term sequence we have

$$0 \to \mathrm{H}^1(B,M) \to \mathrm{H}^1(G,M) \to \mathrm{H}^1(A,M)^{\,G} \to \mathrm{H}^2(B,M) \to \mathrm{H}^2(G,M)\ .$$

A acts on M trivially, so $\mathrm{H}^1(A,M) \cong \mathrm{Hom}(A,M) = 0$ since M is torsion free. Hence we have

$$0 \longrightarrow \mathrm{H}^2(B,M) \xrightarrow{\tau^*} \mathrm{H}^2(G,M)\ .$$

Let $\beta = \tau^*(\alpha) \in \mathrm{H}^2(G,M)$, where α is special for B. for any $C = \langle\, c \,\rangle <$ G of prime order, suppose $\tau(C) \neq 1$, then $\tau(C) \cong C$; $\mathrm{res}_C^G M \cong \mathrm{res}_{\tau(C)}^B M$. So $\mathrm{H}^2(C, \mathrm{res}_C^G M) \cong \mathrm{H}^2(\tau(C), \mathrm{res}_{\tau(C)}^B M)$. Hence $\beta\ |_C \neq 0$ by the following commutative diagram.

$$\begin{array}{cccc} 0 \longrightarrow & \mathrm{H}^2(B,M) & \longrightarrow & \mathrm{H}^2(G,M) \\ & \downarrow \mathrm{res}_{\tau(C)}^B & & \downarrow \mathrm{res}_C^G \\ & \mathrm{H}^2(\tau(C), \mathrm{res}_{\tau(C)}^B M) & \longrightarrow & \mathrm{H}^2(C, \mathrm{res}_C^G M) \end{array}$$

That is M is special for all elements c of prime order of G such that $\tau(c) \neq 0$.

If $\tau(c) = 0$, then $c \in A$. It is obvious that there exists an $\alpha \in \mathrm{H}^2(G, \mathrm{ind}_A^G N) \cong \mathrm{H}^2(A, N)$ such that $\alpha \mid_{\langle c \rangle} \neq 0$ for all such c. Hence T is a primitive $\mathbf{Z}G$-lattice, this yields the claimed upper bound in the lemma. □

Theorem 3.4.2. Let G be a primitive solvable group. Suppose that

$$1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G$$

with G_i/G_{i+1} primitive abelian. Then $\delta(G) \leq 2\,|G|$.

Proof. By Corollary 3.1.2, $\delta(H) \leq |H|$ for H primitive abelian. From Lemma 3.4.1,

$$\begin{aligned}
\delta(G) &\leq \delta(G/G_1) + |G/G_1|\,\delta(G_1) \\
&\leq \delta(G/G_1) + |G/G_1|\,(\delta(G_1/G_2) + |G_1/G_2|\,\delta(G_2)) \leq \cdots \\
&\leq \delta(G/G_1) + |G/G_1|\,\delta(G_1/G_2) + \cdots + |G/G_{n-1}|\,\delta(G_{n-1}) \\
&\leq |G|\,(1/\,|G_1| + 1/\,|G_2| + \cdots + 1/\,|G_{n-2}| + 1) \\
&\leq |G|\,(1 + 1/2 + 1/2^2 + \cdots) = 2\,|G|.
\end{aligned}$$

where the last inequality is the result of the fact that if $H_1 < H_2$, then $|H_1| \leq 1/2\,|H_2|$. □

CHAPTER 4 $n(G)$ FOR NILPOTENT GROUPS

In [20] (d), Plesken raised the following open problem:

(p3). Determine $n(G)$ for finite groups G which are not prime power order.

In this chapter, we will answer (p3) for nilpotent groups. We do this by first giving an upper bound in §4.1 and then proving that is the required one in §4.2. in §4.3, the final section of this chapter we will look at an example in details to see how our method works.

§4.1

It is proved in Theorem 1 of [7] that for any $\mathbf{Z}G$-lattice M, if $\alpha \in \mathrm{H}^2(G, M)$ is special for G, then there exists a $\mathbf{Z}G$-homomorphism $h : M \to \bigoplus_{C\in\mathcal{X}} \mathrm{ind}_C^G\mathbf{Z}$ such that $h^*(\alpha) \in \mathrm{H}^2(G, \bigoplus \mathrm{ind}_C^G\mathbf{Z})$ is also special for G. Now if we can find a new lattice N and a $\mathbf{Z}G$-homomorphism h', such that $h' : \bigoplus_{C\in\mathcal{X}} \mathrm{ind}_C^G\mathbf{Z} \to N$ satisfies $(h')^*(h^*(\alpha))$ is again special for G, then we can substitute N for $\bigoplus_{C\in\mathcal{X}} \mathrm{ind}_C^G\mathbf{Z}$ and get $n(G) \leq \mathrm{rank}(N)$. The following proposition is the result of this thinking.

Proposition 4.1.1. Let G be a finite group, H, K be subgroups of G satisfying: (i) $K \lhd KH$, (ii) $(|H|, |K|) = 1$. Define $\varphi : \mathrm{ind}_H^G\mathbf{Z} \to \mathrm{ind}_{KH}^G\mathbf{Z}$ by $\varphi(\sum_{u,v} a_{uv}uvH) = \sum_u(\sum_v a_{uv})uKH$, where $G = \cup_u uKH = \cup_{uv} uvH$ are the coset decompositions. Then for any $\alpha \in \mathrm{H}^2(G, \mathrm{ind}_H^G\mathbf{Z})$ such that $\alpha \mid_{\langle x \rangle} \neq 0$ for some $x \in H$, we have $\varphi^*(\alpha) \mid_{\langle x \rangle} \neq 0$.

Proof: By

$$\mathrm{H}^2(G, \mathrm{ind}_H^G \mathbf{Z}) \;\cong\; \mathrm{H}^1(G, \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z}) \;,$$

we can suppose $\alpha \in \mathrm{H}^1(G, \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z})$, $\alpha \mid_{\langle x \rangle} \neq 0$. From Shapiro's Lemma we have

$$\begin{array}{ccc}
\mathrm{H}^1(G, \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\varphi^*} & \mathrm{H}^1(G, \mathrm{ind}_{KH}^G \mathbf{Q}/\mathbf{Z}) \\
\downarrow \mathrm{res}_H^G & & \downarrow \mathrm{res}_{KH}^G \\
\mathrm{H}^1(H, \mathrm{res}_H^G \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z}) & & \mathrm{H}^1(KH, \mathrm{res}_{KH}^G \mathrm{ind}_{KH}^G \mathbf{Q}/\mathbf{Z}) \\
\downarrow \pi_1^* & & \downarrow \pi_2^* \\
\mathrm{H}^1(H, \mathbf{Q}/\mathbf{Z}) & & \mathrm{H}^1(KH, \mathbf{Q}/\mathbf{Z}) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\psi} & \mathrm{Hom}(KH, \mathbf{Q}/\mathbf{Z})
\end{array}$$

where $\pi_1(\sum_{u,v} a_{uv} uvH) = a_{11}$, $\pi_2(\sum_u b_u uKH) = b_1$. Then we need only prove the result at the level of Hom. Pick $f \in \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ such that $f \mid_{\langle x \rangle} \neq 0$, say $f(x) = t_0$. Since $\pi_1^* \mathrm{res}_H^G$ is an isomorphism, it is not difficult to find $(\pi_1^* \mathrm{res}_H^G)^{-1}(f) \in \mathrm{H}^1(G, \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z})$. Suppose

$$guv = u_g g(u) v = u_g v_{g(u)} g(u, v),$$

with $g(u,v) \in H$. Define

$$\hat{f}(g) = \sum_{u,v} f(g(u,v)) u_g v_{g(u)} H.$$

Then $[\hat{f}] \in \mathrm{H}^1(G, \mathrm{ind}_H^G \mathbf{Q}/\mathbf{Z})$ and $\pi_1^* \mathrm{res}_H^G [\hat{f}] = f$.

Now

$$\varphi^*(\hat{f})(g) = \varphi \hat{f}(g) = \varphi \sum_{u,v} f(g(u,v)) u_g v_{g(u)} H$$
$$= \sum_u (\sum_v f(g(u,v))) u_g KH.$$

So

$$\pi_2\varphi\hat{f}(g) = \sum_v f(g(1,v)),\ g \in KH.$$

Hence

$$\psi(f)(x) = \sum_v f(x(1,v)).$$

Since $K \lhd KH$ and $(|H|,|K|) = 1$, we can pick K to be the set of the coset representatives of H in KH. By $xv = (xvx^{-1})x$, we have $x(1,v) = x$. Then $\varphi(f)(x) = |K|\, f(x) = |K|\, t_0 \neq 0$, because $(|K|, o(x)) = 1$. □

Corollary 4.1.2. If G is a finite nilpotent group, $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $\mathcal{X}_{p_i} \subset \mathcal{X}$ containing all subgroups of order p_i, then

$$n(G) \leq \sum_{i=1}^{r} \sum_{C \in \mathcal{X}_{p_i}} |\, G_{p_i} : C \,|,$$

where $|G_{p_i}| = p_i^{\alpha_i}$ is the Sylow p_i-subgroup.

Proof. For $C \in \mathcal{X}_{p_i}$, pick $H = C$, and $K = G_{p_i'}$ in proposition 4.1.1, where $G_{p_i'}$ is the complement of G_{p_i} in G. Then $\text{rank}(\text{ind}_{KH}^{G}\mathbf{Z}) = |\, G_{p_i} : C \,|$, hence the result. □

From now on in this chapter, we will always suppose that the group $G = \prod_{i=1}^{r} G_{p_i}$ is nilpotent, $|G_{p_i}| = p_i^{\alpha_i}$. Let's index p_i in the way such that

$$|\, \mathcal{X}_{p_1} \,| \leq |\, \mathcal{X}_{p_2} \,| \leq \cdots \leq |\, \mathcal{X}_{p_r} \,|,$$

and $\alpha_i \leq \alpha_{i+1}$ if $|\, \mathcal{X}_{p_i} \,| = |\, \mathcal{X}_{p_{i+1}} \,|$, where $|\, \mathcal{X}_{p_i} \,|$ means the cardinality of the set $\mathcal{X}_{p_i}$. Suppose $\mathcal{X}_{p_i} = \{C_{i1}, C_{i2}, \cdots, C_{it_i}\}$, then $t_1 \leq t_2 \leq \cdots \leq t_r$. For each C_{ij}, let $\Delta_{ij} = G_{p_i'} C_{ij}$. Write $G = \cup_u u(ij)\Delta_{ij}$, and let $\hat{\Delta}_{ij} = \sum_u u(ij)\Delta_{ij} \in$

$(\mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})^G$. Define T to be the submodule of the permutation module $\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ generated by

$$\{\hat{\Delta}_{ij} + \hat{\Delta}_{rj} \mid \ i = 1, 2, \cdots, r-1; \ j = 1, 2, \cdots, t_i\}.$$

Define

$$\pi : \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \to (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T$$

to be the natural map. Then $(\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T$ is finitely generated since $\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ is; and it is a free abelian group since for any $x \in (\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T$, k an integer, $kx = 0$ if and only if $kx \in T$, the latter is equivalent to $x \in T$ and so $x = 0$. Hence $(\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T$ is still a $\mathbf{Z}G$-lattice for which we have

Proposition 4.1.3. Suppose $\alpha = \sum_{i,j} \alpha_{ij}$, $\alpha_{ij} \in \mathrm{H}^2(G, \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})$ is the image of the nonzero element of $\mathrm{H}^2(G, \mathrm{ind}_{C_{ij}}^G \mathbf{Z})$ under the map defined in Prop. 4.1.1. Then α_{ij} has order p_i, $\alpha_{ij}\,|_{C_{ij}} \neq 0$ and α is special for G. Let π be defined as above, then $\pi^*(\alpha) \in \mathrm{H}^2(G, (\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T)$ is still special for G.

Proof. Let $D_j = \{i, \ i \in \{1, 2, \cdots, r\}, \ t_i \geq j\}$, $j = 1, 2, \cdots, t_{r-1}$, T_j be the submodule of the permutation module $\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ generated by $\{\hat{\Delta}_{ij} + \hat{\Delta}_{rj}\}$, $i \in D_j$, $j = 1, 2, \cdots, t_{r-1}$. Then $T = \bigoplus_{j=1}^{t_{r-1}} T_j$, and let $D_j = \{r\}, T_j = 0, j = t_{r-1} + 1, \cdots, t_r$, we have

$$(\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T \cong \bigoplus_j (\bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j.$$

So we need only prove that if $\alpha_j \in \mathrm{H}^2(G, \bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})$ is special for C_{ij}, $i \in D_j$, $\pi^*(\alpha_j) \in \mathrm{H}^2(G, (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j)$ is also special for C_{ij}, $i \in D_j$.

From the exact sequence

$$0 \to \bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \to \bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q} \to \bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z} \to 0,$$

we have the exact sequence

$$0 \to (\bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j \to (\bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q})/T_j' \to (\bigoplus_{i \in D_j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})/T_j'' \to 0,$$

where T_j' and T_j'' are sumodules of $\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}$ and $\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}$ generated by $\{q(\hat{\Delta}_{ij}+\hat{\Delta}_{rj}), i \in D_j, q \in \mathbf{Q}\}$ and $\{\bar{q}(\hat{\Delta}_{ij}+\hat{\Delta}_{rj}), i \in D_j, \bar{q} \in \mathbf{Q}/\mathbf{Z}\}$ respectively. Hence

$$\mathrm{H}^2(G, (\textstyle\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j) \;\cong\; \mathrm{H}^1(G, (\textstyle\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})/T_j'') \;.$$

By the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(G, \bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}) & \stackrel{\pi^*}{\longrightarrow} & \mathrm{H}^2(G, (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^1(G, \bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}) & \stackrel{\pi^*}{\longrightarrow} & \mathrm{H}^1(G, (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})/T_j'') \end{array}$$

we can change the discussion to the first cohomologies.

Pick $[\beta] \in \mathrm{H}^1(G, \bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})$, write $[\beta] = \sum_{i \in D_j} [\beta_i]$ with $[\beta_i] \in \mathrm{H}^1(G, \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})$ being the image of the nonzero element of $\mathrm{H}^1(G, \mathrm{ind}_{C_{ij}}^G \mathbf{Q}/\mathbf{Z})$ under the map in 4.1.1. Then $[\beta_i]\,|_{C_{ij}} \neq 0$ and $[\beta_i]$ has order p_i. Then $\pi^*([\beta_i])$ has order either 1 or p_i. Hence $\pi^*([\beta_i])\,|_{C_{kj}} = 0$ if $k \neq i$. Therefore $\pi^*([\beta])\,|_{C_{ij}} \neq 0$ if and only if $\pi^*([\beta_i])\,|_{C_{ij}} \neq 0$. Now we prove that $\pi^*([\beta_i])\,|_{C_{ij}} \neq 0$.

Let $C_{ij} = \langle\, c \,\rangle$, suppose $\beta_i(c) = \sum_u a_{u(ij)} u(ij) \Delta_{ij}$. If $\pi^*([\beta_i])\,|_{C_{ij}} = 0$, then $\pi\beta_i(c) = \overline{\sum_u a_{u(ij)} u(ij) \Delta_{ij}} = c\overline{\alpha} - \overline{\alpha}$, for some $\overline{\alpha} \in (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})/T_j''$.

Write

$$\alpha = \sum_i \sum_u b_{u(ij)} u(ij)\Delta_{ij},$$

then

$$\sum_u a_{u(ij)} u(ij)\Delta_{ij} = c\sum_l \sum_u b_{u(lj)} u(lj)\Delta_{lj} - \sum_l \sum_u b_{u(lj)} u(lj)\Delta_{lj} + T_\alpha,$$

where $T_\alpha \in T_j''$. Compare the coefficients of $u(ij)\Delta_{ij}$, we get

$$\sum_u a_{u(ij)} u(ij)\Delta_{ij} = c\sum_u b_{u(ij)} u(ij)\Delta_{ij} - \sum_u b_{u(ij)} u(ij)\Delta_{ij} + T_\alpha(i),$$

with $T_\alpha(i) = a_1\hat{\Delta}_{ij}$. So $\beta_i(c) = c\alpha_0 - \alpha_0 + T_\alpha(i)$, $\alpha_0 \in \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}$. By choosing another representative of $[\beta_i]$, we can suppose that $\beta_i(c) = T_\alpha(i)$. Then $\pi\beta_i(c) = \overline{T_\alpha(i)}$. $\pi^*([\beta_i])\,|_{C_{ij}} = 0$ implies $\overline{T_\alpha(i)} = c\overline{\alpha}' - \overline{\alpha}'$ for some $\overline{\alpha}' \in (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z})/T_j''$. So $T_\alpha(i) = c\alpha' - \alpha' + T_{\alpha'}$.

Now write $\alpha' = \sum_l \sum_u c_{u(lj)} u(lj)\Delta_{lj}$. Then c acts on $\{u(lj)\Delta_{lj}\}_l$, $l \in D_j$ with lengths of orbits either $|C_{ij}|$ or 1. If there is one orbit having length one, then c acts trivially on that term. That term will not appear in $c\alpha' - \alpha'$. Since $c\alpha' - \alpha' = T_\alpha(i) - T_{\alpha'}$ having equal coefficients for all $\{u_{lj}\Delta_{lj}\}$ with l fixed, every term with the same l will not appear. Now for any l such that $l \neq i$, if all the lengths of orbits are $|C_{ij}| = p_i$, then $p_i \mid [G : \Delta_{lj}]$, a contradiction. Hence at least one of them must have length one. Then c acts trivially on all terms by the above argument. That is $c\alpha' - \alpha' \in \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}$. But then $T_{\alpha'} = T_\alpha(i) - (c\alpha' - \alpha') \in \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}$. This gives $T_{\alpha'} \in (\mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}) \cap T_j'' = 0$, or $T_\alpha(i) = c\alpha' - \alpha'$. It is obvious that we can pick $\alpha' \in \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}$. Then $[\beta_i]\,|_{C_{ij}} = 0$, contradiction. Hence $\pi^*([\beta_i])\,|_{C_{ij}} \neq 0$. $\square$

This implies

Theorem 4.1.4.

$$\begin{aligned} n(G) &\leq \sum_{i=1}^{r} \sum_{C \in \mathcal{X}_{p_i}} \mid G_{p_i} : C \mid - \operatorname{rank}(T) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{t_i} \mid G_{p_i} : C_{ij} \mid - \sum_{i=1}^{r-1} t_i. \end{aligned}$$

§4.2

We will prove in this section that Theorem 4.1.4 gives the exact $n(G)$. We do this by proving that the lattice we got is the minimum one in the sense that if M is another lattice and

$$\psi : (\bigoplus_{i,j} \operatorname{ind}_{\Delta_{ij}}^{G} \mathbf{Z})/T \rightarrow M$$

is a $\mathbf{Z}G$-homomorphism with nonzero kernel, then $\psi^*(\beta)$ is not special for a $\beta \in \mathrm{H}^2(G, (\bigoplus_{i,j} \operatorname{ind}_{\Delta_{ij}}^{G} \mathbf{Z})/T)$ which is special. The main idea is to reduce to something that contradicts the following result which is the main result, theorem 2 of [7].

Theorem. Let G be a finite p-group, $\mathcal{X}$ a set of representatives of the conjugacy classes of subgroups of G of prime order. $S = \bigoplus_{C \in \mathcal{X}} \operatorname{ind}_C^G \mathbf{Z}$, $\beta = \sum_{C \in \mathcal{X}} \beta_C \in \bigoplus_{C \in \mathcal{X}} \mathrm{H}^2(G, \operatorname{ind}_C^G \mathbf{Z})$ is special for G. Then (S, β) is minimal, in the sense that if $e : S \rightarrow S'$ is any $\mathbf{Z}G$-homomorphism with non-zero kernel, then $e^*(\beta)$ is not special.

Theorem 4.2.1. Let G be as in Theorem 4.1.4. $\mid G \mid = \prod_i p_i^{\alpha_i}$ then

$$n(G) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} \mid G_{p_i} : C_{ij} \mid - \sum_{i=1}^{r-1} t_i.$$

Proof of Theorem 4.2.1. We will first prove the cases that $\alpha_i > 1, i = 1, 2, \cdots, r$. Let $\beta' \in H^2(G, \bigoplus \mathrm{ind}_{C_{ij}}^G \mathbf{Z})$ be special for G and β be its image in $H^2(G, (\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T)$ under the composition of the map in Proposition 4.1.1 and the map in Proposition 4.1.3, which is still special for G. Let

$$e : (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T \to M$$

be a $\mathbf{Z}G$-homomorphism with non zero kernel and $e^*(\beta)$ special. By Theorem 1 of [7], we have a $\mathbf{Z}G$-homomorphism $h : M \to \bigoplus \mathrm{ind}_{C_{ij}}^G \mathbf{Z}$. Combining with Proposition 4.1.1 and 4.1.3, we can get a $\mathbf{Z}G$-homomorphism

$$d : M \xrightarrow{h} \bigoplus \mathrm{ind}_{C_{ij}}^G \mathbf{Z} \xrightarrow{\varphi} \bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \xrightarrow{\pi} (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T,$$

so that $d^*(e^*(\beta))$ is special. Let $f = de$ be the composition. We have $f^*(\beta)$ special and $\ker f \neq 0$. We will show through a series of lemmas that this is impossible.

Lemma 4.2.2. f above can be lifted to $\hat{f}$, where

$$\hat{f} : \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \to \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$$

is a $\mathbf{Z}G$-homomorphism such that the following diagram commutes.

$$\begin{array}{ccccc}
\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} & \xrightarrow{\pi} & (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T & \cong & \bigoplus_j (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j \\
\downarrow \hat{f} & & \downarrow f & & \downarrow f \\
\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} & \xrightarrow{\pi} & (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T & \cong & \bigoplus_j (\bigoplus_i \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T_j
\end{array}$$

Proof. From the following diagram,

$$\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \xrightarrow{\pi} (\bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T \xrightarrow{e} M \xrightarrow{h} \bigoplus \mathrm{ind}_{C_{ij}}^G \mathbf{Z}$$

$$\xrightarrow{\varphi} \bigoplus \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \xrightarrow{\pi} (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})/T,$$

we can see that $\hat{f}$ can be taken to be $\varphi h e \pi$, since then $\hat{f}\pi = (\pi e h \varphi)\pi = \pi(e h \varphi \pi) = \pi e d = \pi f$. □

Write

$$\hat{f} = \sum_{(i,j),(k,l)} \kappa_{(i,j)} \hat{f}_{(i,j)(k,l)} \lambda_{(k,l)}$$

with

$$\lambda_{(k,l)} : \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^{G} \mathbf{Z} \to \mathrm{ind}_{\Delta_{kl}}^{G} \mathbf{Z}$$

the projections,

$$\kappa_{(i,j)} : \mathrm{ind}_{\Delta_{ij}}^{G} \mathbf{Z} \to \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^{G} \mathbf{Z}$$

the embeddings and $\hat{f}_{(i,j)(k,l)} \in \mathrm{Hom}_{\mathbf{Z}G}(\mathrm{ind}_{\Delta_{kl}}^{G} \mathbf{Z}, \mathrm{ind}_{\Delta_{ij}}^{G} \mathbf{Z})$. For subgroups C and D of G, let $[CxD] : \mathrm{ind}_{C}^{G} \mathbf{Z} \to \mathrm{ind}_{D}^{G} \mathbf{Z}$ be given by

$$[CxD](tC) = \sum_{u \in U} tuxD \in \mathrm{ind}_{D}^{G} \mathbf{Z}, \quad t \in T_C$$

where CxD is the disjoint union $\cup_{u \in U} uxD$. If there is no confusion, we will also use $[CxD]$ to mean an element of $\mathrm{End}_{\mathbf{Z}G}(\bigoplus_{H \in \mathcal{Y}} \mathrm{ind}_{H}^{G} \mathbf{Z})$, by having it annihilate $\mathrm{ind}_{H}^{G} \mathbf{Z}$ for $H \in \mathcal{Y}$, $H \neq C$. We know that $\mathrm{Hom}_{\mathbf{Z}G}(\mathrm{ind}_{\Delta_{kl}}^{G} \mathbf{Z}, \mathrm{ind}_{\Delta_{ij}}^{G} \mathbf{Z})$ has a basis $\{[\Delta_{kl} x \Delta_{ij}] \mid$ x runs over the double coset representatives $\}$ (see [14], p.177). But when $k \neq i$, $\Delta_{kl} x \Delta_{ij} = G$ for any x. Hence it has basis $[\Delta_{kl} 1 \Delta_{ij}]$ in this case.

Now let

$$\begin{aligned} \hat{f}_1 &= \sum_{k=i} \kappa_{(i,j)} \hat{f}_{(i,j)(k,l)} \lambda_{(k,l)} = \sum_{i,j,l} \kappa_{(i,j)} \hat{f}_{(i,j)(i,l)} \lambda_{(i,l)}; \\ \hat{f}_2 &= \sum_{k \neq i} \kappa_{(i,j)} \hat{f}_{(i,j)(k,l)} \lambda_{(k,l)} = \sum_{k \neq i} a_{klij} [\Delta_{kl} 1 \Delta_{ij}]. \end{aligned}$$

Then $\hat{f} = \hat{f}_1 + \hat{f}_2$.

Lemma 4.2.3. Let G be a p-group, $\langle c \rangle = C < G$ having order p. Suppose that $G \neq C$, then $V_{G/C}(g) = 1$ for any $g \in G$ of order p, where $V_{G/C}$ is the transfer map from G to C.

Proof. Since G is a p-group, $N_G(C) = C_G(C)$ and it strictly contains C. From $V_{G/C}(g) = V_{C_G(C)/C}V_{G/C_G(C)}(g)$, it suffices to show that $V_{C_G(C)/C}(g) = 1$ for any $g \in C_G(C)$ of order p. If $g = c$, $C_G(C) = \cup_{u \in U} uC$ the coset decomposition, then $cu = uc$ for any $u \in U$. Hence $V_{C_G(C)/C}(c) = c^{[C_G(C):C]} = 1$. If $g \neq c$, then $C_G(C) > K = \langle g \rangle \times C$. By Lemma in section 4 of [7], $V_{K/C} = 1$. So $V_{C_G(C)/C} = V_{K/C}V_{C_G(C)/K} = 1$. □

Lemma 4.2.4. If $\beta \in \mathrm{H}^2(G, \bigoplus_{i,j} \mathrm{ind}_{C_{ij}}^G \mathbf{Z})$ is special for G, φ is as in 4.1.1, $\varphi^*(\beta) \in \mathrm{H}^2(G, \bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})$ is the image of β, which is still special, then $\hat{f}_1^*(\varphi^*(\beta))$ is special for G.

Proof. Let $\sum_u a_{u(kl)} u(kl)\Delta_{kl} \in \mathrm{ind}_{\Delta_{kl}}^G \mathbf{Z}$.

$$[\Delta_{kl} 1 \Delta_{ij}] : \sum_u a_{u(kl)} u(kl)\Delta_{kl} \to (\sum_{u(kl)} a_{u(kl)}) \sum_{u(ij)} u(ij)\Delta_{ij}, \quad k \neq i.$$

Suppose $\varphi_{kl} = \varphi|_{\mathrm{ind}_{C_{kl}}^G \mathbf{Z}}$, where φ is as in 4.1.1 Then we have

$$\mathrm{ind}_{C_{kl}}^G \mathbf{Z} \xrightarrow{\varphi_{kl}} \mathrm{ind}_{\Delta_{kl}}^G \mathbf{Z} \xrightarrow{[\Delta_{kl} 1 \Delta_{ij}]} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}.$$

This implies

$$\mathrm{H}^2(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Z}) \xrightarrow{\varphi_{kl}^*} \mathrm{H}^2(G, \mathrm{ind}_{\Delta_{kl}}^G \mathbf{Z}) \xrightarrow{[\Delta_{kl} 1 \Delta_{ij}]^*} \mathrm{H}^2(G, \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}) .$$

We will prove that if $\beta_{kl} \in \mathrm{H}^2(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Z})$ is special for C_{kl}, then

$$[\Delta_{kl} 1 \Delta_{ij}]^* \varphi_{kl}^*(\beta_{kl})|_C = 0$$

for any C of prime order.

We will again prove things in H^1 and we will denote the maps by the same notations. We have

$$\mathrm{H}^1(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Q}/\mathbf{Z}) \xrightarrow{\varphi_{kl}^*} \mathrm{H}^1(G, \mathrm{ind}_{\Delta_{kl}}^G \mathbf{Q}/\mathbf{Z}) \xrightarrow{[\Delta_{kl} 1 \Delta_{ij}]^*} \mathrm{H}^1(G, \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Q}/\mathbf{Z}) .$$

Let $\gamma_{kl} \in \mathrm{H}^1(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Q}/\mathbf{Z})$ be special for C_{kl}. Since γ_{kl} has order dividing $|C_{kl}|$, $[\Delta_{kl} 1 \Delta_{ij}]^* \varphi_{kl}^*(\gamma_{kl}) |_C = 0$ for any $C = C_{uv}$, $u \neq k$. So it remains only to prove for the case when $C = C_{ks}$. By Shapiro's Lemma

$$\mathrm{H}^1(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(C_{kl}, \mathbf{Q}/\mathbf{Z})$$

Let $C = \langle c \rangle$, $G = \cup_{u \in U} u \Delta_{kl}$, $\Delta_{kl} = \cup_{v \in V} v C_{kl}$, and $guv = u_g v_{u(g)} g(u,v)$ with $g(u,v) \in C_{kl}$. Then $f(c) = i / \mid C_{kl} \mid$, $(i, \mid C_{kl} \mid) = 1$ for any non zero $f \in \mathrm{Hom}(C_{kl}, \mathbf{Q}/\mathbf{Z})$. The preimage of f, $[\hat{f}] \in \mathrm{H}^1(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Q}/\mathbf{Z})$ is given by

$$\hat{f}(g) = \sum_{u,v} f(g(u,v)) u_g v_{u(g)} C_{kl}.$$

So if g has order $|C_{kl}|$,

$$\begin{aligned}
[\Delta_{kl} 1 \Delta_{ij}]^* \varphi_{kl}^*(\hat{f})(g) &= [\Delta_{kl} 1 \Delta_{ij}]^* \sum_u (\sum_v f(g(u,v))) u_g \Delta_{kl} \\
&= (\sum_{u,v} f(g(u,v))) \sum_{u(ij)} u(ij) \Delta_{ij} \\
&= f(T_{G/C_{kl}})(g) \sum_{u(ij)} u(ij) \Delta_{ij} \\
&= f(T_{C_G(C_{kl})/C_{kl}} T_{G/C_G(C_{kl})}(g)) \sum_{u(ij)} u(ij) \Delta_{ij} = 0.
\end{aligned}$$

The last equation is the result of Lemma 4.2.3. Hence whenever $k \neq i$, we have

$$[\Delta_{kl} 1 \Delta_{ij}]^* \varphi^*(\beta_{kl}) |_C = 0 \text{ for any } C \text{ of prime order.}$$

Hence $\hat{f}*(\varphi^*(\beta)) |_C = \hat{f}_1^*(\varphi^*(\beta)) |_C + \hat{f}_2^*(\varphi^*(\beta)) |_C = \hat{f}_1^*(\varphi^*(\beta)) |_C$, since $\hat{f}_2^*$ is a linear combination of $[\Delta_{kl}1\Delta_{ij}]^*$ with $k \neq i$. Therefore $\hat{f}_1^*(\varphi^*(\beta))$ is special as required. □

Since G is nilpotent, we can pick the set of coset representatives of Δ_{ij} in G the same as the coset representatives of C_{ij} in G_{p_i}. We will identify them and denote by U_{ij}. Define

$$\delta_i : \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \to \bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}, \text{ by } \delta_i(u\Delta_{ij}) = uC_{ij}.$$

Then we have

$$\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} = \bigoplus_j \mathrm{ind}_{C_{ij}G_{p_i'}}^G \mathbf{Z} \overset{\delta_i}{\cong} \bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}$$

with $G_{p_i'}$ acting trivially on $\bigoplus \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}$. For any $t \in U_{ij}$,

$$[C_{ij}xC_{ik}](tC_{ij}) = \sum_{u \in U} tuxC_{ik}$$

where $C_{ij}xC_{ik}$ is the disjoint union $\cup_{u \in U} uxC_{ik}$. On the other hand, $\Delta_{ij} = C_{ij}G'_{p_i}$, $\Delta_{ik} = C_{ik}G'_{p_i}$, So we have the double coset decomposition $G = \cup_x \Delta_{ij}x\Delta_{ik} = \cup_x C_{ij}G'_{p_i}xC_{ik}G'_{p_i}$ with x runs over the double coset representatives of C_{ij} and C_{ik} in G_{p_i}. Thus

$$\delta_i[\Delta_{ij}x\Delta_{ik}]\delta_i^{-1}(tC_{ij}) = \delta_i[\Delta_{ij}x\Delta_{ik}](t\Delta_{ij}) = \delta_i(\sum_{v \in V} tvx\Delta_{ik}) = \sum_{u \in U} tuxC_{ik}$$

Hence we get an isomorphism

$$\delta_i^* : \mathrm{End}(\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}) \longrightarrow \mathrm{End}(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})$$

by

$$[\Delta_{ij}x\Delta_{ik}] \xrightarrow{\delta_i^*} [C_{ij}xC_{ik}],$$

where $\delta_i^*(f) = \delta_i f \delta_i^{-1}$. Let $\hat{f}_1 = \sum_{i=1}^r \hat{f}_{p_i}$ with

$$\hat{f}_{p_i} = \sum_{j,l} \kappa_{(i,j)} \hat{f}_{(i,j)(i,l)} \lambda_{(i,l)} \quad (\text{ see } 4.2.2).$$

Write $\beta = \sum_{k,l} [\beta_{kl}]$ with $[\beta_{kl}] \in \mathrm{H}^2(G, \mathrm{ind}_{C_{kl}}^G \mathbf{Z})$. Then for any $k \neq i$, we have

$$G \times G \xrightarrow{\beta_{kl}} \mathrm{ind}_{C_{kl}}^G \mathbf{Z} \xrightarrow{\varphi_{kl}} \mathrm{ind}_{\Delta_{kl}}^G \mathbf{Z} \xrightarrow{\kappa_{(i,j)} \hat{f}_{(i,j)(i,l)} \lambda_{(i,l)}} 0,$$

since $\lambda_{i,j}$ projects only (i,j)-th component. Hence

$$\hat{f}_1^*(\varphi^*(\beta)) = \sum_{i,k,l,} \hat{f}_{p_i}^*(\varphi^*([\beta_{kl}])) = \sum_i \hat{f}_{p_i}^*(\varphi^*(\sum_l [\beta_{i,l}]).$$

Since $\hat{f}_{p_i}^*(\varphi^*(\sum_l [\beta_{i,l}])$ has order p_i, it can only be special for subgroups of order p_i. Then since $\hat{f}_1^*(\varphi^*(\beta))$ is special for G, we get $\hat{f}_{p_i}^*(\varphi^*(\beta))$ is special for all $C_{ij}, j = 1, 2, \cdots, t_i$.

Let $u_{ik} : \mathrm{ind}_{\Delta_{ik}}^G \mathbf{Z} \to \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ be embeddings and $v_{ik} : \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \to \mathrm{ind}_{\Delta_{ik}}^G \mathbf{Z}$ be projections. Define

$$\bar{f}_{p_i} : \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z} \longrightarrow \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$$

by $\bar{f}_{p_i} = \sum_{k,l} u_{ik} \hat{f}_{(il)(ik)} v_{il}$. Then $\bar{f}_{p_i} \in \mathrm{End}(\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})$. If $x \in \bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$, then $\hat{f}_{p_i}(x)$ and $\bar{f}_{p_i}(x)$ are the same except that they are in $\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ and $\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z}$ respectively. So $\bar{f}_{p_i}^*(\varphi^*(\sum_l \beta_{i,l}))$ is special for $C_{ij}, j = 1, 2, \cdots, t_i$. Then $(\delta_i^*)^*(\bar{f}_{p_i}^*(\varphi^*(\sum_j \beta_{ij})))$ is special for all $C_{ij}, j = 1, 2, \cdots, t_i$. So

$$\mathrm{res}_{G_{p_i}}^G (\delta_i^*)^*(\bar{f}_{p_i}^*(\varphi^*(\sum_j \beta_{ij})))$$

is special for G_{p_i} if we think $\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}$ as G_{p_i}-module. Let I be the kernel of the map

$$\phi : \mathrm{End}_{\mathbf{Z}G_{p_i}}(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}) \longrightarrow \mathrm{End}(\ \mathrm{H}^2(G_{p_i}, \bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})\ )$$

defined by $\phi(f) = f^*$. By Cliff-Weiss's Lemma 3 ([7]),

$$(\delta_i^*)(\bar{f}_{p_i}) \equiv \sum_j a_j[C_{ij}1C_{ij}] \pmod{I},$$

where $(a_j, p_i) = 1$. This is because the following elements are all in I.

(i) $[C_{ij}xC_{ik}]$ where $j \neq k$ or $j = k$, $x \notin N_{G_{p_i}}(C_{ij})$;

(ii) $[C_{ij}xC_{ij}] - [C_{ij}1C_{ij}]$ where $x \in N_{G_{p_i}}(C_{ij})$;

(iii) $p_i[C_{ij}1C_{ij}]$.

Lemma 4.2.5. If $t \in (\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})^{G_{p_i}} \backslash p_i(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})^{G_{p_i}}$, then

$$\delta_i^*(\bar{f}_{p_i})(t_i) \not\equiv 0 \pmod{p_i(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})^{G_{p_i}}}.$$

Proof. Let I_1 be the linear span of the elements in (i) ,(ii) and (iii). Then again modulo I_1, every element of $\operatorname{End}_{\mathbf{Z}G_{p_i}}(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})$ is congruent to an element of the form $\sum_j a_j[C_{ij}1C_{ij}]$ where $(a_j, p_i) = 1$. Hence the index of I_1 in $\operatorname{End}_{\mathbf{Z}G_{p_i}}(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})$ is $p_i^{t_i}$. But the index of I in $\operatorname{End}_{\mathbf{Z}G_{p_i}}(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})$ is $p_i^{t_i}$ from the proof of Theorem 2 in [7]. Therefore $I = I_1$. Then, we need only prove that $g(t) \in p_i(\bigoplus_j \operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z})^{G_{p_i}}$ for any $g \in I$ since $[C_{ij}1C_{ij}]$ is identity on $\operatorname{ind}_{C_{ij}}^{G_{p_i}} \mathbf{Z}$ and zero on others. It is enough if we can prove that elements in (i) to (iii) satisfy this property.

Elements in (iii) are obvious. For elements in (ii), write $t = \sum_j b_j t_j$ with $t_j = \sum_u uC_{ij} \in (\operatorname{ind}_{C_{ij}}^{G_{p_I}} \mathbf{Z})^{G_{p_i}}$. Then

$$([C_{ij}xC_{ij}] - [C_{ij}1C_{ij}])(\sum_j b_j t_j) = b_j([C_{ij}xC_{ij}] - [C_{ij}1C_{ij}])t_j$$

$$= b_j(\sum_u uxC_{ij} - \sum_u uC_{ij}) = 0.$$

For elements in (i)

$$[C_{ij}xC_{ij}]\sum_j b_jt_j = b_j[C_{ij}xC_{ij}]t_j = b_j\sum_u\sum_{v\in V} uvxC_{ij}$$

$$= b_j\sum_v\sum_u uvxC_{ij} = b_j\sum_u uC_{ij}\sum_v 1 = \mid V \mid b_j\sum_u uC_{ij},$$

where $\mid V \mid = \mid C_{ij} : C_{ij}\cap x^{-1}C_{ij}x \mid = p_i$. Hence the result. Similarly we can prove that $[C_{ij}xC_{ik}]\sum_j b_jt_j \in p_i(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}}\mathbf{Z})^{G_{p_i}}$. □

Corollary 4.2.6. $\bar{f}_{p_i}(t) \not\equiv 0 \pmod{p_i(\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z})^{G}}$ if

$$t \in (\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z})^{G}\backslash p_i(\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z})^{G}.$$

Proof. Since δ_i is an isomorphism, we have

$$\delta_i(t) \in (\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}}\mathbf{Z})^{G_{p_i}}\backslash p_i(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}}\mathbf{Z})^{G_{p_i}}.$$

Hence

$$\delta_i^*(\bar{f}_{p_i})(\delta_i(t)) \notin p_i(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}}\mathbf{Z})^{G_{p_i}}.$$

That is $\delta_i(\bar{f}_{p_i}(t)) \notin p_i(\bigoplus_j \mathrm{ind}_{C_{ij}}^{G_{p_i}}\mathbf{Z})^{G_{p_i}}$. Hence $\bar{f}_{p_i}(t) \notin p_i(\bigoplus_j \mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z})^{G}$. □

Now we are in the position to complete the proof of Theorem 4.2.1. Recall that $\hat{f}$ satisfies:

(i). $\hat{f}(T) \subseteq T$;

(ii). $\hat{f}^*(\beta)$ is special if $\beta \in \mathrm{H}^2(G, \bigoplus_{i,j}\mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z})$ is special;

(iii). There exists an $x \in (\bigoplus_{i,j}\mathrm{ind}_{\Delta_{ij}}^{G}\mathbf{Z}) \setminus T$, such that $\hat{f}(x) \in T$.

(i) is due to that $\hat{f}$ is a lifting. (ii) follows from the fact that $f^*\pi^*(\beta)$ is special. (iii) is the result of the assumption that $\ker f \neq 0$.

We divide the discussion into two cases according to the property of x in (iii).

(1) $x \in (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})^G$.

Suppose $x = \sum_{i,j} a_{ij}\hat{\Delta}_{ij}$, since $\hat{\Delta}_{ij} \equiv -\hat{\Delta}_{rj} \pmod T$, we have $x = x_r + t_0$ with $x_r \in (\bigoplus_j \mathrm{ind}_{\Delta_{rj}}^G \mathbf{Z})^G$ and $t_0 \in T$. $x_r \neq 0$ since $x \notin T$. Then since $\hat{f}(T) \subseteq T$, we get $\hat{f}(x_r) \in T$. Thus we can suppose $x \in (\bigoplus_j \mathrm{ind}_{\Delta_{rj}}^G \mathbf{Z})^G$. Then $\hat{f}(x) = \hat{f}_1(x) + \hat{f}_2(x) = \hat{f}_{p_r}(x) + \hat{f}_2(x) \in T$. Suppose $x = \sum_j b_j \hat{\Delta}_{rj}$, $x \notin p_r(\bigoplus_j \mathrm{ind}_{\Delta_{rj}}^G \mathbf{Z})^G$. (Otherwise if p_r^s is the maximal power of p_r dividing x, we can consider $(1/p_r^s)x$ instead.)

$$\begin{aligned}
\hat{f}_2(x) &= \sum_{i \neq r} \kappa_{(i,j)} \hat{f}_{(i,j)(r,l)} \lambda_{(r,l)} (\sum_s b_s \hat{\Delta}_{rs}) \\
&= \sum_{i \neq r} a_{rlij} [\Delta_{rl} 1 \Delta_{ij}] (\sum_s b_s \hat{\Delta}_{rs}) \\
&= \sum_s b_s \sum_{i \neq r} a_{rsij} p_r^{\alpha_r - 1} \hat{\Delta}_{ij} \\
&\equiv -p_r^{\alpha_r - 1} \sum_s b_s \sum_{i \neq r} a_{rsij} \hat{\Delta}_{rj} \pmod T.
\end{aligned}$$

Since $\hat{f}_{p_r}(x) + \hat{f}_2(x) \in T$, $\hat{f}_{p_r}(x) - p_r^{\alpha_r - 1} \sum_s b_s \sum_{i \neq r} a_{rsij} \hat{\Delta}_{rj} \in T$. But every element of T is a linear combination of $\hat{\Delta}_{ij} + \hat{\Delta}_{rj}$, hence

$$\hat{f}_{p_r}(x) = p_r^{\alpha_r - 1} \sum_s b_s \sum_{i \neq r} a_{rsij} \hat{\Delta}_{rj}.$$

So $\hat{f}_{p_r}(x) \equiv 0 \pmod{p_r(\bigoplus_j \mathrm{ind}_{\Delta_{rj}}^G \mathbf{Z})^G}$ since $\alpha_r > 1$. Then

$$\bar{f}_{p_r}(x) \equiv 0 \pmod{p_r(\bigoplus_j \mathrm{ind}_{\Delta_{rj}}^G \mathbf{Z})^G}.$$

This contradicts Corollary 4.2.6.

(2) $x \notin (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})^G$. Then there exists a $g \in G$ such that $gx \neq x$, or $gx - x \neq 0$. But since $\hat{f}(x) \in T \subset (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})^G$, $\hat{f}(gx - x) = g\hat{f}(x) - \hat{f}(x) = 0$.

On the other hand, $[\Delta_{kl} 1 \Delta_{ij}]$ will map $t\Delta_{uv}$ to zero if $(u,v) \neq (k,l)$ and $t\Delta_{ki}$ to $\sum_{u \in U} tu\Delta_{ij} = \hat{\Delta}_{ij}$. So $\mathrm{Im}(\hat{f}_2) \subset (\bigoplus_{i,j} \mathrm{ind}_{\Delta_{ij}}^G \mathbf{Z})^G$ since $\hat{f}_2$ is a linear combination of $[\Delta_{kl} 1 \Delta_{ij}]$ with $i \not\leq k$. Then $\hat{f}_2(gx - x) = 0$. Hence $\hat{f}_1(gx - x) = 0$, or $\ker \hat{f}_1 \neq 0$. But $\hat{f}_1 = \sum_i f_{p_i}$, so there must be an $i \in \{1, 2, \cdots, r\}$, such that $\ker \hat{f}_{p_i} \neq 0$. Then $\ker \bar{f}_{p_i} \neq 0$ and so $\ker \delta_i^*(\bar{f}_{p_i}) \neq 0$. This contradicts Cliff-Weiss's Theorem 2 of [7] again since ($(\delta_i^*)(\bar{f}_{p_i}))^*(\sum_l [\beta_{il}])$ is special for G_{p_i}. This completes the proof of the case when all $\alpha_i > 1$.

Now we will prove the general case. Suppose $G = \prod_{i=1}^r p_i^{\alpha_i}$ with $\alpha_1 = \alpha_2 = \cdots = \alpha_s = 1$, $\alpha_{s+1} > 1, \cdots, \alpha_r > 1$. Let $G_1 = \prod_{i=1}^s G_{p_i}$, $G_2 = \prod_{i=s+1}^r$, then by Theorem 4.1.4,

$$n(G) \leq \sum_{i=1}^{r} \sum_{j=1}^{t_i} | G_{p_i} : C | - \sum_{i=1}^{r-1} t_i = \sum_{i=s+1}^{r} \sum_{j=1}^{t_i} | G_{p_i} : C | - \sum_{i=s+1}^{r-1} t_i = n(G_2).$$

On the other hand, $G = G_1 \times G_2$. Let Γ_2 be an arbitrary torsion free space group with point group G_2 and translation subgroup A_2. For any torsion free space group Γ_1 with point group G_1 and translation subgroup A_1, we can form torsion free space group $\Gamma_1 \times \Gamma_2$ with point group $G_1 \times G_2$ and translation subgroup $A_1 \bigoplus A_2$. By definition there is a normal subgroup N of $\Gamma_1 \times \Gamma_2$ contained in $A_1 \bigoplus A_2$ such that $(\Gamma_1 \times \Gamma_2)/N$ is still torsion free, and has dimension $\leq n(G)$. Then $N \cap \Gamma_2$ is a normal subgroup of Γ_2 and $\Gamma_2 / N \cap \Gamma_2 \cong \Gamma_2 N/N < (\Gamma_1 \times \Gamma_2)/N$. So it is torsion free and has dimension $\leq n(G)$, or $n(G_2) \leq n(G)$. Therefore

$$n(G) = n(G_2) = \sum_{i=s+1}^{r} \sum_{j=1}^{t_i} | G_{p_i} : C | - \sum_{i=S+1}^{r-1} t_i = \sum_{i=1}^{r} \sum_{j=1}^{t_i} | G_{p_i} : C | - \sum_{i=1}^{r-1} t_i.$$

That proves Theorem 4.2.1 finally. □

§4.3. An example

We will look at an easiest nontrivial example in this section to see how the argument in §4.2 works. Let $G = C_4 \times C_9 = \langle x \rangle \times \langle y \rangle$; $C_{11} = \langle x^2 \rangle$, $C_{21} = \langle y^3 \rangle$; $\Delta_{11} = \langle x^2 \rangle C_9$, $\Delta_{21} = \langle y^3 \rangle C_4$. Denote

$$S = \mathrm{ind}^G_{\langle x^2 \rangle}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\langle y^3 \rangle}\mathbf{Z}; \;\; S' = \mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z};$$

$t = \Delta_{11} + x\Delta_{11} + \Delta_{21} + y\Delta_{21} + y^2\Delta_{21}$; and $S'' = S'/\langle t \rangle$, where $\langle t \rangle$ is the submodule of S' generated by t. It is obvious that S'' is still a $\mathbf{Z}G$-lattice. We will prove that S'' is the lattice we need for $n(G)$, or $n(G) = 4$.

First let $\beta \in \mathrm{H}^2(G, S)$ be special, $S \xrightarrow{\varphi} S' \xrightarrow{\pi} S''$ is defined in the natural way. We want to prove that $(\pi\varphi)^*(\beta) \in \mathrm{H}^2(G, S'')$ is again special. By Proposition 4.1.1, we need only prove that if $\beta' \in \mathrm{H}^2(G, S')$ is special, $\pi^*(\beta')$ is also special. We, as usual, will prove things in H^1. Pick

$$[\beta'] \in \mathrm{H}^1(G, \mathrm{ind}^G_{\Delta_{11}}\mathbf{Q}/\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Q}/\mathbf{Z}) \;,$$

write $[\beta'] = [\beta'_{11}] + [\beta'_{21}]$ with $[\beta'_{i1}] \in \mathrm{H}^1(G, \mathrm{ind}^G_{\Delta_{i1}}\mathbf{Q}/\mathbf{Z})$, $i = 1, 2$ and $[\beta'_{11}] \mid_{\langle x^2 \rangle} \neq 0$, $[\beta'_{21}] \mid_{\langle y^3 \rangle} \neq 0$. Suppose $\beta'_{11}(x^2) = n_1\Delta_{11} + n_2x\Delta_{11}, n_1, n_2 \in \mathbf{Q}/\mathbf{Z}$, if $\pi^*([\beta'_{11}]) \mid_{\langle x^2 \rangle} = 0$, then there exists an $\alpha \in S''$, so that $\pi(\beta'_{11}(x^2)) = x^2\alpha - \alpha = 0$ since x^2 acts trivially on S''. Hence $\pi(n_1\Delta_{11} + n_2x\Delta_{11}) = 0$, so $n_1 = n_2 = 0$, contradiction. Similarly $\pi^*([\beta'_{21}])$ is special for $\langle y^3 \rangle$. Combining with Proposition 4.1.1, we get $n(G) \leq 4$. To prove that 4 is the exact number, we follow our usual approach, that is to prove that if $f \in \mathrm{End}_{\mathbf{Z}G}S''$ and $\ker f \neq 0$ then $f^*(\pi\varphi)^*(\beta))$ is not special any more . We prove this in three steps.

(i) rank $(\mathrm{End}_{\mathbf{Z}G} S'') = 4$.

By the familiar isomorphism

$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}G}(M, L) \cong \mathrm{Hom}_{\mathbf{Q}G}(\mathbf{Q} \otimes M, \mathbf{Q} \otimes L),$$

we have rank $(\mathrm{End}_{\mathbf{Z}G} S'') = \dim\, (\mathrm{End}_{\mathbf{Q}G} \mathbf{Q} \otimes S'')$.

$$\mathbf{Q} \bigotimes S' \cong \mathrm{ind}^G_{\Delta_{11}} \mathbf{Q} \bigoplus \mathrm{ind}^G_{\Delta_{21}} \mathbf{Q} \cong \mathbf{Q} \bigoplus \mathbf{Q}_1 \bigoplus \mathbf{Q} \bigoplus \mathbf{Q}(\zeta_3),$$

where $\mathbf{Q}_1$ has the same base set as $\mathbf{Q}$, but with nontrivial G action. So

$$\mathbf{Q} \bigotimes S'' (\cong (\mathbf{Q} \bigotimes S')/\mathbf{Q}) \cong \mathbf{Q} \bigoplus \mathbf{Q}_1 \bigoplus \mathbf{Q}(\zeta_3).$$

Hence $\dim\, (\mathrm{End}_{\mathbf{Q}G} \mathbf{Q} \otimes S'') = 1 + 1 + 2 = 4$.

(ii) For subgroups C, D of G, let $[CxD]$ have the same meaning as in §4.2, $l_1 = [\Delta_{11} 1 \Delta_{11}] + [\Delta_{21} 1 \Delta_{21}]$; $l_2 = [\Delta_{11} 1 \Delta_{11}] + [\Delta_{21} y \Delta_{21}]$; $l_3 = [\Delta_{11} x \Delta_{11}] + [\Delta_{21} 1 \Delta_{21}]$; $l_4 = 2[\Delta_{21} 1 \Delta_{11}] + 3[\Delta_{11} 1 \Delta_{21}]$. Then $l_i \in \mathrm{End}_{\mathbf{Z}G} S', i = 1, 2, 3, 4$. Define $\bar{l}_i : S'' \to S''$ by $\bar{l}_i(\bar{x}) = \overline{l_i(x)}$. Then we have

Lemma 4.3.1. $\bar{l}_i$, $i = 1, \cdots, 4$, form a basis of $\mathrm{End}_{\mathbf{Z}G} S''$.

Proof. First we need to prove that $\bar{l}_i$ is well defined, or $l_i(\langle t \rangle) \subseteq \langle t \rangle$. This is just a routine check. For example, $l_4(t) = 2[\Delta_{21} 1 \Delta_{11}](t) + 3[\Delta_{11} 1 \Delta_{21}](t) = 2[\Delta_{21} 1 \Delta_{11}](\Delta_{21} + y\Delta_{21} + y^2\Delta_{21}) + 3[\Delta_{11} 1 \Delta_{21}](\Delta_{11} + x\Delta_{11}) = 6t$.

Pick a $\mathbf{Z}$-basis of S'' $\{\bar{\Delta}_{11}, x\bar{\Delta}_{11}, \bar{\Delta}_{21}, y\bar{\Delta}_{21}, \}$, let $l = \sum_i a_i l_i$, then l maps this basis to $(a_1\bar{\Delta}_{11} + a_2\bar{\Delta}_{11} + a_3 x\bar{\Delta}_{11} - 2a_4(\bar{\Delta}_{11} + x\bar{\Delta}_{11}),$ $a_1 x\bar{\Delta}_{11} + a_2 x\bar{\Delta}_{11} + a_3\bar{\Delta}_{11} - 2a_4(\bar{\Delta}_{11} + x\bar{\Delta}_{11}),$ $a_1\bar{\Delta}_{21} + a_2 y\bar{\Delta}_{21} + a_3\bar{\Delta}_{21} + 3a_4(\bar{\Delta}_{11} + x\bar{\Delta}_{11}),$ $a_1 y\bar{\Delta}$ [illegible] $a_2(\bar{\Delta}_{11} + x\bar{\Delta}_{11} + \bar{\Delta}_{21} + y\bar{\Delta}_{21}) + a_3 y\bar{\Delta}_{21} + 3a_4(\bar{\Delta}_{11} + x\bar{\Delta}_{11}))$. Hence $l = 0$ [illegible] $a_i = 0, i = 1, \cdots, 4$ and so $\{\bar{l}_i\}$ generates a subgroup of $\mathrm{End}_{\mathbf{Z}G} S''$ [illegible] index. For any $\bar{f} \in \mathrm{End}_{\mathbf{Z}G}S''$, we can find $g \in \langle \bar{l}_i, i = 1, \cdots, 4 \rangle$ such that $k\bar{f} = \bar{g}$ for some $k \in \mathbf{Z}^+$, or $\bar{f} = (1/k)\bar{g}$. Write $(1/k)g = \sum_i a_i l_i = (a_1 + a_2)[\Delta_{11} 1 \Delta_{11}] + a_3[\Delta_{11} x \Delta_{11}] + 3a_4[\Delta_{11} 1 \Delta_{21}] + 2a_4[\Delta_{21} 1 \Delta_{11}] + (a_1 + a_3)[\Delta_{21} 1 \Delta_{21}] + a_2[\Delta_{21} y \Delta_{21}]$. If $(1/k)g \notin \mathrm{End}_{\mathbf{Z}G}S'$, then one of $a_1 + a_2, a_3, 2a_4, 3a_4, a_1 + a_3, a_2$ is not an integer. But in any case, this implies that one of a_i is not an integer. We will prove that then $\sum_i a_i \bar{l}_i \notin \mathrm{End}_{\mathbf{Z}G}S''$. This can be done by looking at the image of $(\bar{\Delta}_{11}, x\bar{\Delta}_{11}, \bar{\Delta}_{21}, y\bar{\Delta}_{21})$ under $\sum_i a_i \bar{l}_i$. Hence $(1/k)g \in \mathrm{End}_{\mathbf{Z}G}S'$, or $\bar{l}_i$ is a basis of $\mathrm{End}_{\mathbf{Z}G}S''$. □

(iii) Now suppose $f : S'' \to S''$ satisfies $f^*((\pi\varphi)^*\beta)$ special and $\ker f \neq 0$, where $\beta \in \mathrm{H}^2(G, \mathrm{ind}^G_{\langle x^2 \rangle}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\langle y^3 \rangle}\mathbf{Z})$ is special for G . Let $\hat{f} : S' \to S'$ be the lifting of f, then from

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S'/\langle t \rangle \\ \downarrow \hat{f} & & \downarrow f \\ S' & \xrightarrow{\pi} & S'/\langle t \rangle \end{array}$$

we get $f^*(\pi^*(\varphi^*(\beta))) = \pi^*\hat{f}^*(\varphi^*(\beta))$. So $\hat{f}^*(\varphi^*(\beta))$ is also special.

Now we will figure out what the kernel of $\hat{f}$ should look like. Write $\hat{f} = a_{11}[\Delta_{11} 1 \Delta_{11}] + a'_{11}[\Delta_{11} x \Delta_{11}] + a_{12}[\Delta_{21} 1 \Delta_{11}] + a_{21}[\Delta_{11} 1 \Delta_{21}] + a_{22}[\Delta_{21} 1 \Delta_{21}] + a'_{22}[\Delta_{21} y \Delta_{21}] + a''_{22}[\Delta_{21} y^2 \Delta_{21}] = f_1 + f_2$ with $f_2 = a_{21}[\Delta_{11} 1 \Delta_{21}] + a_{12}[\Delta_{21} 1 \Delta_{11}]$ and $f_1 = \hat{f} - f_2$. First we will prove that $\ker f_1 = 0$. Suppose not, let $\alpha \in \ker f_1$, $\alpha \neq 0$. Consider

$$[\Delta_{21} 1 \Delta_{11}] : \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z} \to \mathrm{ind}^G_{\Delta_{11}}\mathbf{Z};$$

$$a_1\Delta_{21} + a_2 y \Delta_{21} + a_3 y^2 \Delta_{21} \to (a_1 + a_2 + a_3)(\Delta_{11} + x\Delta_{11}).$$

We will prove that $[\Delta_{21} 1 \Delta_{11}]^* \varphi^*_{21}(\beta_{21}) |_C = 0$ for any C of prime order. But it has order dividing 3, so we need only verify for $C = \langle y^3 \rangle$. We will do it in H^1.

From

$$\mathrm{ind}_{C_{21}}^{G}\mathbf{Q}/\mathbf{Z} \xrightarrow{\varphi_{21}} \mathrm{ind}_{\Delta_{21}}^{G}\mathbf{Q}/\mathbf{Z} \xrightarrow{[\Delta_{21}1\Delta_{11}]} \mathrm{ind}_{\Delta_{11}}^{G}\mathbf{Q}/\mathbf{Z},$$

we get

$$\mathrm{H}^1(G, \mathrm{ind}_{C_{21}}^{G}\mathbf{Q}/\mathbf{Z}) \xrightarrow{\varphi_{21}^*} \mathrm{H}^1(G, \mathrm{ind}_{\Delta_{21}}^{G}\mathbf{Q}/\mathbf{Z}) \xrightarrow{[\Delta_{21}1\Delta_{11}]^*} \mathrm{H}^1(G, \mathrm{ind}_{\Delta_{11}}^{G}\mathbf{Q}/\mathbf{Z}) .$$

By Shapiro's Lemma, $\mathrm{H}^1(G, \mathrm{ind}_{C_{21}}^{G}\mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(C_{21}, \mathbf{Q}/\mathbf{Z})$. For any $h \in \mathrm{Hom}(C_{21}, \mathbf{Q}/\mathbf{Z})$, $h(y^3) = i/3$ for some $i \in \{1, 2\}$. Then the preimage of h, $[\hat{h}] \in \mathrm{H}^1(G, \mathrm{ind}_{C_{21}}^{G}\mathbf{Q}/\mathbf{Z})$ is given by

$$\hat{h}(y^3) = \sum_{u,v} h(y^3)x^u y^v C_{21} = i/3 \sum_{u,v} x^u y^v C_{21}.$$

Hence by the index of C_{21} in Δ_{21} is 4, we get

$$\hat{h}(y^3) \xrightarrow{\varphi_{21}} 4i/3 \sum_{v} y^v \Delta_{21} \xrightarrow{[\Delta_{21}1\Delta_{11}]} 4i \sum_{j=0}^{1} x^j \Delta_{11} = 0.$$

Similarly $[\Delta_{11}1\Delta_{21}]^*\varphi_{11}^*(\beta_{11}) |_C = 0$ for any C of prime order. Hence $\hat{f}^*(\beta) |_C = f_1^*(\beta) |_C$ since $f_2^*(\beta) |_C = a_{21}[\Delta_{11}1\Delta_{21}]^*(\beta_{11}) |_C + a_{12}[\Delta_{21}1\Delta_{11}]^*(\beta_{21}) |_C = 0$. These imply that $f_1^*(\varphi^*(\beta))$ is special since $\hat{f}^*(\varphi^*(\beta))$ is. But

$$\begin{aligned} f_1^*(\varphi^*(\beta)) \quad = \quad & (a_{11}[\Delta_{11}1\Delta_{11}] + a_{11}'[\Delta_{11}x\Delta_{11}])^*(\varphi^*(\beta_{11})) \\ & +(a_{22}[\Delta_{21}1\Delta_{21}] + a_{22}'[\Delta_{21}y\Delta_{21}] + a_{22}''[\Delta_{21}y^2\Delta_{21}])^*(\varphi^*(\beta_{21})). \end{aligned}$$

So

$$(a_{11}[\Delta_{11}1\Delta_{11}] + a_{11}'[\Delta_{11}x\Delta_{11}])^*(\varphi^*(\beta_{11}))$$

is special for C_4;

$$(a_{22}[\Delta_{21}1\Delta_{21}] + a_{22}'[\Delta_{21}y\Delta_{21}] + a_{22}''[\Delta_{21}y^2\Delta_{21}])^*(\varphi^*(\beta_{21}))$$

is special for C_9. But $\ker f_1 \neq 0$ implies that one of

$$a_{11}[\Delta_{11}1\Delta_{11}] + a'_{11}[\Delta_{11}x\Delta_{11}], a_{22}[\Delta_{21}1\Delta_{21}] + a'_{22}[\Delta_{21}y\Delta_{21}] + a''_{22}[\Delta_{21}y^2\Delta_{21}]$$

has non zero kernel. Say for example, $a_{11}[\Delta_{11}1\Delta_{11}] + a'_{11}[\Delta_{11}x\Delta_{11}]$ having non zero kernel. Then we get a map from $\mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \cong \mathrm{ind}^{\langle x\rangle}_{\langle x^2\rangle}\mathbf{Z}$ to itself with non zero kernel and mapping a special element to a special element. This contradicts Theorem 2 of [7]. Hence $\ker f_1 = 0$ as required.

Now if $\alpha \in \ker \hat{f}$, $\alpha \neq 0$, then $f_1(\alpha) = -f_2(\alpha) \neq 0$ since $\ker f_1 = 0$. Since $\mathrm{Im}(f_2)$ is contained in $(\mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z})^G$, $f_2(g\alpha - \alpha) = gf_2(\alpha) - f_2(\alpha) = 0$. If $\alpha \notin (\mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z})^G$, or $g\alpha - \alpha \neq 0$, $\hat{f}(g\alpha - \alpha) = f_1(g\alpha - \alpha) + f_2(g\alpha - \alpha) = f_1(g\alpha - \alpha) \neq 0$. That is $\hat{f}(\alpha) \neq 0$ or $\alpha \notin \ker \hat{f}$. Therefore $\ker \hat{f}$ is generated by elements of $(\mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z})^G$, or elements of the form $\{n_1(\Delta_{11} + x\Delta_{11}) + n_2(\Delta_{21} + y\Delta_{21} + y^2\Delta_{21})\}$. Suppose there is a $w \in \ker \hat{f}$ with $w = n_1(\Delta_{11} + x\Delta_{11}) + n_2(\Delta_{21} + y\Delta_{21} + y^2\Delta_{21}) \neq kt$ for any $k \in \mathbf{Z}$, or $n_1 \neq n_2$. Then by $f\pi(w) = \pi\hat{f}(w) = 0$, we have $\pi(w) \in \ker f$. But $f(\pi(w)) = f(\pi(n_1(\Delta_{11} + x\Delta_{11}) + n_2(\Delta_{21} + y\Delta_{21} + y^2\Delta_{21})) = f((n_1 - n_2)(\bar{\Delta}_{11} + x\bar{\Delta}_{11})$. So $f(\pi(w)) = 0$ implies that $\bar{\Delta}_{11} + x\bar{\Delta}_{11} \in \ker f$. Similarly $\bar{\Delta}_{21} + y\bar{\Delta}_{21} + y^2\bar{\Delta}_{21}$ is in $\ker f$. But then $(\mathrm{Im} f)^G = 0$. This is impossible since G is not primitive. Hence $\ker \hat{f}$ is generated by t or is simply 0. By $\ker f \neq 0$, there exists $\alpha \notin \{kt\}$, such that $\hat{f}(\alpha) \in \{kt\}$. Then $\alpha - g\alpha \in \ker \hat{f}$ for any $g \in G$. So $|G|\,\alpha = kt + \sum_{g\in G} g\alpha \in (\mathrm{ind}^G_{\Delta_{11}}\mathbf{Z} \bigoplus \mathrm{ind}^G_{\Delta_{21}}\mathbf{Z})^G$. Hence $\alpha = m_1(\Delta_{11} + x\Delta_{11}) + m_2(\Delta_{21} + y\Delta_{21} + y^2\Delta_{21})$ with $m_1 \neq m_2$. But then again we can get $(\mathrm{Im} f)^G = 0$. So the only possibility is that $\ker \hat{f} = 0$. But this contradicts that there exists $\alpha \notin \{kt\}$ such that $\hat{f}(\alpha) \in \{kt\}$, since $\hat{f}(k_0 t - k_1\alpha)$ would be zero for some $k_0, k_1 \in \mathbf{Z}$.

A REMARK ABOUT CHAPTER 4

Dr. A. Weiss suggested an easier way to prove the main result of this Chapter , namely Theorem 4.2.1 after reading my thesis. The main idea are sketched below.

Lemma A.(see [R], P. 55 & 70) Let V be a $\mathbf{Q}G$-module. For a finite number of primes $\{p_1, p_2, \cdots, p_r\}$, let there be given a full $\mathbf{Z}_{p_i}G$-lattice $X(p_i)$ in V. Then there is a full $\mathbf{Z}G$-lattice N in V such that $N_{p_i} = X(p_i)$ for all i.

Lemma B. Let G be as in Theorem 4.2.1. Let M be a $\mathbf{Z}G$-lattice, then M is special for G if and only if $M_{p_i} = \mathbf{Z}_{p_i} \otimes_{\mathbf{Z}} M$ is special for G_{p_i} for $i = 1, 2, \cdots, r$

Proof.

$$\mathrm{H}^2(G, M) \cong \bigoplus_i \mathrm{H}^2(G, \mathbf{Z}_{p_i} \otimes M) \text{ (see [CR], P. 529).}$$

By

$$0 \to G'_{p_i} \to G \to G_{p_i} \to 0,$$

we have

$$\cdots \to \mathrm{H}^1(G_{p_i}, M_{p_i}) \to \mathrm{H}^2(G'_{p_i}, M_{p_i}) \to \mathrm{H}^2(G, M_{p_i}) \to \mathrm{H}^2(G_{p_i}, M_{p_i}) \to \cdots.$$

But $\mathrm{H}^2(G'_{p_i}, M_{p_i}) = 0$, so $\mathrm{H}^2(G, M_{p_i})$ is a p_i-group. Hence it is the Sylow p_i-subgroup of $\mathrm{H}^2(G, M)$. Then it is well known that

$$0 \to \mathrm{H}^2(G, M_{p_i}) \stackrel{\text{res}}{\to} \mathrm{H}^2(G_{p_i}, M_{p_i})$$

and

$$\mathrm{H}^2(G_{p_i}, M_{p_i}) \stackrel{\text{cor}}{\to} \mathrm{H}^2(G, M_{p_i}) \to 0$$

are both exact(see [W], P. 92). But $\mathrm{cor}^G_{G_{p_i}} \mathrm{res}^G_{G_{p_i}} z = (G : G_{p_i}) z$ and $(G : G_{p_i})$ is invertible in M_{p_i}. Hence it is a isomorphism. So we have

$$\mathrm{H}^2(G, M) \cong \bigoplus_i \mathrm{H}^2(G_{p_i}, M_{p_i}) .$$

Now it is easy to see that the Lemma is true. □

Proof of Theorem 4.2.1. Let G, G_{p_i}, $\mathcal{X}$, and $\mathcal{X}_{p_i}$ be the same as in §4.1. Define

$$L_{p_i} = \mathrm{inf}^G_{G_{p_i}} \bigoplus_{C \in \mathcal{X}_{p_i}} \mathrm{ind}_C^{G_{p_i}} \mathbf{Z}$$

with $G_{p_i'}$ acting trivially on it. Let $L'_{p_i} = L_{p_i}/L^G_{p_i}$; $l_{p_i} = \mathrm{rank}_{\mathbf{Z}}(L^G_{p_i})$ and $l = \max(l_{p_i})$. Let

$$M_{p_i} = \mathbf{Z}_{p_i}^{l-l_{p_i}} \bigoplus (\mathbf{Z}_{p_i} \otimes_{\mathbf{Z}} L_{p_i}) \bigoplus \bigoplus_{j \neq i} \mathbf{Z}_{p_i} \otimes_{\mathbf{Z}} L'_{p_j}.$$

Then

$$\mathbf{Q}_{p_i} \otimes_{\mathbf{Z}_{p_i}} M_{p_i} \cong \mathbf{Q}^l_{p_i} \bigoplus (\bigoplus_{j=1}^r \mathbf{Q}_{p_i} \otimes L'_{p_j}).$$

The characters of $\mathbf{Q}_{p_i} \otimes_{\mathbf{Z}_{p_i}} M_{p_i}$ are the same for all i and afforded by a $\mathbf{Q}G$-module. Hence by Lemma A, There exists a $\mathbf{Z}G$-lattice M such that $\mathbf{Z}_{p_i} \otimes M \cong M_{p_i}$ for all i. Then we claim that M is the lattice we needed in Theorem 4.2.1.

(i). M is special for G . M_{p_i} is obviously special for G_{p_i} because of the direct summand

$$\mathbf{Z}_{p_i} \bigotimes L_{p_i} \cong \bigoplus_{C \in \mathcal{X}_{p_i}} \mathrm{ind}_C^{G_{p_i}} \mathbf{Z}_{p_i}.$$

Hence M is special for G by Lemma B.

(ii). M is minimum. Pick $[\alpha] = \sum_i [\alpha_i]$ with $[\alpha_i] \in \mathrm{H}^2(G, \mathbf{Z}_{p_i} \bigotimes L_{p_i})$ such that $[\alpha_i]$ is special for G_{p_i}. Then $[\alpha]$ is special for G. Suppose now that

T is a $\mathbf{Z}G$-lattice and $M \xrightarrow{e} T$ is any $\mathbf{Z}G$-homomorphism such that $e^*([\alpha])$ is still special. Then we will prove that $\ker e = 0$. Hence M is minimum. Since $\mathbf{Z}_{p_i} \bigotimes L_{p_i}$ is a direct summand of M_{p_i}, we have

$$\mathbf{Z}_{p_i} \bigotimes L_{p_i} \xrightarrow{\varphi_i} M_{p_i} \xrightarrow{1 \otimes e} \mathbf{Z}_{p_i} \bigotimes T,$$

where φ_i is the embedding. By Cliff and Weiss's result, $\ker \varphi_i(1 \otimes e) = 0$. For otherwise, $\mathrm{Im}(1 \otimes e)(\varphi_i)$ cannot be special for G_{p_i}, which contradicts to the fact that $e^*([\alpha])$ is still special. Therefore χ_T contains all $\chi_{L'_{p_i}}$, $i = 1, 2, \cdots, r$ and l copies of identity. But all $\chi_{L'_{p_i}}$ are different from each other as G'_{p_i} acts trivially on L'_{p_i}. Hence e is into. □

REFERENCES FOR THE REMARK

[R] I. Reiner, "Maximal orders," Academic Press, New York, 1975.

[CR] C. W. Curtis, I. Reiner, "Representation theory of finite groups and associative algebras," Interscience, New York, 1962.

[W] E. Weiss, "Cohomology of groups," Academic Press, New York, 1969.

CHAPTER 5 SOME LOCAL ANALYSIS

§5.1

Let G be a finite group, C be a subgroup of G of prime order p. We have seen that $\mathbf{Z}G$-lattice $\mathrm{ind}_C^G\mathbf{Z}$ is often used in the above discussion. Since

$$\mathrm{H}^2(G,\mathrm{ind}_C^G\mathbf{Z}) \cong \mathrm{H}^2(C,\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z},$$

there exists an $\alpha \in \mathrm{H}^2(G,\mathrm{ind}_C^G\mathbf{Z})$ such that $\alpha \neq 0$ when it is restricted to C. This is what we need in constructing torsion free space groups. For this kind of construction, Professor Plesken introduced a new way of dealing with it in [19]. To understand his idea, we need some notations. A *G-extension* (R,ϵ) consists of a group R and an epimorphism $\epsilon : R \to G$. The G-extension (R,ϵ) is called *crystallographic* respectively *p-adic*, if $\ker\epsilon$ is free of finite rank over $\mathbf{Z}$ respectively over the ring $\mathbf{Z}_p$ of p-adic integers. When G also acts faithfully on $\ker\epsilon$, R is the *crystallographic* respectively a *p-adic space group*. Plesken's main result can be formulated as follows:

Theorem. Let $(R(p),\epsilon(p))$ be p-adic G-extensions for each prime p dividing $|G|$. Then a crystallographic G-extension (R,ϵ) of minimal dimension exists such that the p-adic completion (R_p,ϵ_p) of (R,ϵ) maps onto $(R(p),\epsilon(p))$, where

$$R_p = \varprojlim R/p^n \ker\epsilon \text{ for } n \to \infty.$$

If $(R(p),\epsilon(p))$ is p-torsion free (no elements of p-power order) for each p, then R is torsion free.

This result makes it possible to change the construction of a torsion free crystallographic space group to the construction of some p-adic space groups. For the latter it is natural to ask that how $\mathrm{ind}_C^G \mathbf{Z}_p$ will decompose and if $\mathrm{ind}_C^G \mathbf{Z}_p = \bigoplus M_i$, which M_i will carry the nontrivial second cohomology that we need. Since

$$\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}_p) \cong \bigoplus \mathrm{H}^2(G, M_i) \cong \mathbf{Z}_p/p\mathbf{Z}_p,$$

there must be only one M_{i_C} such that $\mathrm{H}^2(G, M_{i_C}) \cong \mathbf{Z}/p\mathbf{Z}$ and all others are zero. So we can substitute M_{i_C} for $\mathrm{ind}_C^G \mathbf{Z}_p$ in constructing p-adic space groups. Dr. W. Plesken suggested that at most circumstances, it might be the Scott module that carries the non-trivial cohomology. The Scott module in $\mathrm{ind}_C^G \mathbf{Z}_p$ is the unique summand of $\mathrm{ind}_C^G \mathbf{Z}_p$ containing the trivial submodule (see [14] or [5]). We will prove the following theorem 5.1.2 in §5.2; theorem 5.1.1 follows easily from theorem 5.1.2. Then give some applications of them in §5.3.

Theorem 5.1.1. Let M_1 be the summand which satisfies $\mathrm{H}^2(G, M_1) \neq 0$ Then M_1 is the Scott-module in $\mathrm{ind}_C^G \mathbf{Z}_p$ if and only if $C_G(C) = N_G(C)$, the normalizer of C in G is equal to the centralizer of C in G.

Theorem 5.1.2. Let C , G, and M_1 be as in Theorem 5.1.1. Let $N = N_G(C)$, and let M_0 be the $\mathbf{F}_p N/C$-module of $\mathbf{F}_p$-dimension 1, whose N/C-action comes from the action of N on C. Let M_0' be the $\mathbf{Z}_p N/C$-module which is the projective cover of M_0, and let $\hat{M_0}'$ be the inflation of M_0' to a $\mathbf{Z}_p N$-module. Then M_1 is isomorphic to the Green correspondent of $\hat{M_0}'$.

§5.2. Proof of Theorem 5.1.2

(1). Suppose $C \lhd G$; $|C| = p$; $C = \langle c \rangle$; $H = C_G(C)$ the centralizer of C in G. Then it is easy to see that

Lemma 5.2.1. 1. $H \supseteq G_p$, the Sylow p-subgroup of G; 2. $H \lhd G$.

Proof. For the proof of 1, picking any $x \in G_p$, $o(x) = p^t$, suppose $x^{-1}cx = c^j$, $(j,p) = 1$, then $c = x^{-p^t}cx^{p^t} = c^{j^{p^t}}$. So $j^{p^t} \equiv 1 \pmod p$. But $j^{p^t} \equiv j \pmod p$ for any $t \geq 0$. Hence $j \equiv 1 \pmod p$, or $x^{-1}cx = c$. That is $x \in H$. For the proof of 2, picking $h \in H$, $g \in G$, then $h^{-1}ch = c$ and

$$(g^{-1}hg)^{-1}c(g^{-1}hg) = g^{-1}h^{-1}gcg^{-1}hg = g^{-1}h^{-1}c^jhg = g^{-1}c^jg = c,$$

hence $g^{-1}hg \in H$. □

Now define $\mathbf{F}_pG$-module $M_0 \cong \mathbf{F}_p$ with the G-action

$$g \cdot 1 = k_g \text{ if and only if } c^g = c^{j_g}, \text{ such that } k_g j_g \equiv 1 \pmod p.$$

Extending it linearly, we get a G-action. This is because that $c^{g_2g_1} = (c^{j_{g_2}})^{g_1} = c^{j_{g_2}j_{g_1}}$, or $j_{g_2g_1} \equiv j_{g_2}j_{g_1} \pmod p$; but $g_2g_1 \cdot 1 = k_{g_2g_1} \cdot 1$, $k_{g_2g_1}j_{g_2g_1} \equiv 1$. So $k_{g_2g_1}j_{g_2}j_{g_1} \equiv 1$, $(k_{g_2g_1} - k_{g_2}k_{g_1})j_{g_2}j_{g_1} \equiv 0$, or $k_{g_2g_1} \equiv k_{g_2}k_{g_1} \pmod p$.

Since H acts trivially on C, H acts trivially on M_0, we get a $\mathbf{F}_p(G/H)$-module M_0 with the action $\bar{g} \cdot 1 = g \cdot 1$. But $(|G/H|, p) = 1$, hence every irreducible $\mathbf{F}_p(G/H)$-module is a direct summand of $\mathbf{F}_p(G/H) \cong \mathrm{ind}_H^G\mathbf{F}_p$. Then $M_0 \mid \mathrm{ind}_H^G\mathbf{F}_p$. This is still true if we think both sides as G-module in the natural way (inflation). Write $\mathrm{ind}_C^G\mathbf{F}_p = \mathrm{ind}_H^G\mathrm{ind}_C^H\mathbf{F}_p$. Let N be the Scott-module of $\mathrm{ind}_C^H\mathbf{F}_p$. Since N contains a submodule that is isomorphic to $\mathbf{F}_p$ as H-module, $\mathrm{ind}_H^G\mathbf{F}_p$ can be identified with a submodule of $\mathrm{ind}_H^G N$. Under this identification, there is a component of $\mathrm{ind}_H^G N$, say M_0', such that M_0 is a submodule of M_0'. Let $\hat{M_0}'$ be the lifting of M_0' to $\mathbf{Z}_pG$-module. Then $\hat{M_0}' \mid \mathrm{ind}_C^G \mathbf{Z}_p$ and we have

Proposition 5.2.2.

$$\mathrm{H}^2(G, \hat{M_0}') \neq 0.$$

Proof of Proposition 5.2.2.

Lemma 5.2.3. If $\mathrm{ind}_C^G \mathbf{Z}_p = \bigoplus_i M_i$, then $\mathrm{ind}_C^G \mathbf{Q}_p \cong \bigoplus_i (\mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i)$ and $\mathrm{ind}_C^G \mathbf{Q}_p/\mathbf{Z}_p \cong \bigoplus_i (\mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i)/M_i$.

Proof: By the exact sequence of $\mathbf{Z}_pG$-module

$$0 \to \mathrm{ind}_C^G \mathbf{Z}_p \to \mathrm{ind}_C^G \mathbf{Q}_p \to \mathrm{ind}_C^G \mathbf{Q}_p/\mathbf{Z}_p \to 0,$$

we get

$$\mathrm{ind}_C^G \mathbf{Q}_p/\mathbf{Z}_p \cong \mathrm{ind}_C^G \mathbf{Q}_p/\mathrm{ind}_C^G \mathbf{Z}_p \cong \bigoplus (\mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i)/\bigoplus M_i.$$

Define

$$\varphi : \bigoplus \mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i \to \bigoplus (\mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i)/M_i$$

by $\varphi((a_1, a_2, \cdots, a_r)) = (\bar{a_1}, \cdots, \bar{a_r})$, It is obvious that it is onto and $\ker\varphi = \bigoplus M_i$.
□

Now by

$$0 \to M_i \to \mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i \to (\mathbf{Q}_p \bigotimes_{\mathbf{Z}_p} M_i)/M_i \to 0,$$

we can get

$$\mathrm{H}^2(G, M_i) \;\cong\; \mathrm{H}^1(G, \bar{M_i}) \;\text{ with } \bar{M_i} = (\mathbf{Q}_p \bigotimes M_i)/M_i.$$

By 5-term sequence, we have

$$0 \to \; \mathrm{H}^1(G/C, (\bar{M_i})^C) \;\to\; \mathrm{H}^1(G, \bar{M_i}) \;\to\; \mathrm{H}^1(C, \bar{M_i})^{\;G}$$

$$\to\; \mathrm{H}^2(G/C, \bar{M_i}) \;\to\; \mathrm{H}^2(G, \bar{M_i})\;.$$

Since

$$\mathrm{H}^i(G/C, \mathrm{ind}_C^G \mathbf{Q}_p/\mathbf{Z}_p) \;\cong\; \mathrm{H}^{i+1}(G/C, \mathrm{ind}_C^G \mathbf{Z}_p)$$

$$\cong\; \mathrm{H}^{i+1}(G/C, \mathbf{Z}_p(G/C)) \;= 0 \text{ for } i \geq 0,$$

we get

$$\mathrm{H}^1(G, \bar{M_i}) \;\cong\; \mathrm{H}^1(C, \bar{M_i})^{\;G} \cong \mathrm{Hom}(C, \bar{M_i})^G,$$

where if $f : C \to \bar{M_i}$ is a homomorphism of abelian groups, then $g \cdot f : C \to \bar{M_i}$ is the homomorphism

$$g \cdot f(x) = g f(g^{-1} x g).$$

Now we prove that

$$\mathrm{Hom}(C, (\mathbf{Q}_p \bigotimes \hat{M}_0')/\hat{M}_0') \neq 0.$$

By the definition of M_0, there exists $t_0 \in M_0 \subseteq M_0'$, such that $g \cdot t_0 = k_g$, if and only if $t^g = t^{j_g}$ with $k_g j_g \equiv 1 \pmod{p}$. Pick $\hat{t}_0 \in \hat{M}_0'$, $\hat{t}_0 \to t_0$ under the map $\hat{M}_0' \to \hat{M}_0'/p\hat{M}_0' \cong M_0'$. Then

$$g \cdot \hat{t}_0 \equiv k_g \hat{t}_0 \pmod{p\hat{M}_0}.$$

Let $\alpha = (1/p) \otimes \hat{t}_0 \in (\mathbf{Q}_p \bigotimes \hat{M}_0')/\hat{M}_0'$, then

$$g \cdot \alpha = (k_g/p) \otimes \hat{t}_0 = k_g((1/p) \otimes \hat{t}_0) = k_g \alpha.$$

Define

$$f : C \to (\mathbf{Q}_p \bigotimes \hat{M}_0')/\hat{M}_0'$$

by $f(c) = \alpha$. Then for any $g \in G$

$$g \cdot f = g \cdot f(g^{-1}cg) = gf(c^{j_g}) = j_g g \cdot \alpha = j_g k_g \alpha = \alpha.$$

Hence

$$f \in \mathrm{Hom}(C, (\mathbf{Q}_p \bigotimes \hat{M}_0')/\hat{M}_0')^G,$$

or

$$\mathrm{Hom}(C, (\mathbf{Q}_p \bigotimes \hat{M}_0'/\hat{M}_0')^G \neq 0.$$

Hence the result. □

Proposition 5.2.4. M_0' is the projective cover of M_0 as $\mathbf{F}_p(G/C)$-module.

This is the direct result of the following lemma.

Lemma 5.2.5.(see [8], p. 395 & 401) Let G be a finite group, $N = \mathrm{rad}(\mathbf{F}_p G)$; $r(N) = \{a \in \mathbf{F}_p G : Na = 0\}$; let e be a primitive idempotent of $\mathbf{F}_p G$. Then:

1. $r(N)e$ is the unique minimal submodule of $\mathbf{F}_p Ge$;

2. $\mathbf{F}_p Ge/Ne \cong r(N)e$.

Proof of proposition 5.2.4. Since $\mathrm{ind}_C^G \mathbf{F}_p \cong \mathbf{F}_p(G/C)$, and $M_0' \mid \mathbf{F}_p(G/C)$, there exists a primitive idempotent e of $\mathbf{F}_p(G/C)$ such that $M_0' = \mathbf{F}_p(G/C)e$.

M_0 is a simple $\mathbf{F}_p(G/C)$-module. It must be embedded as the unique simple module of $(\mathbf{F}_p(G/C))e$. Then by $(\mathbf{F}_p(G/C))e$ is the projective cover of $(\mathbf{F}_p(G/C))e/Ne \cong M_0$, we can get the result, where $N = \mathrm{rad}(\mathbf{F}_p(G/C))$. □

(2). If $C < G$, $N_G(C)$ is the normalizer of C in G, write $\mathrm{ind}_C^{N_G(C)}\mathbf{Z}_p = \bigoplus_i M_i$ with $\mathrm{H}^2(N_G(C), M_0) \neq 0$. Let $\mathrm{ind}_{N_G(C)}^G M_0 = g(M_0) \bigoplus_i N_i$ with $g(M_0)$ the Green correspondence of M_0. Then

$$\mathrm{H}^2(G, g(M_0)) \neq 0.$$

Proof: $\mathrm{H}^2(G, \mathrm{ind}_{N_G(C)}^G M_0) \cong \mathrm{H}^2(N_G(C), M_0) \neq 0$. All N_i have vertices in

$$\mathcal{X} = \{w \leq G \mid w \leq C^g \cap C, g \in G \setminus N_G(C)\} = \{1\}.$$

Hence $\mathrm{H}^2(G, N_i) = 0$. This gives $\mathrm{H}^2(G, g(M_0)) \neq 0$. □

Corollary 5.2.6. The Scott-module of $\mathrm{ind}_C^G\mathbf{Z}_p$ contains nontrivial cohomology if and only if $C_G(C) = N_G(C)$.

Proof: Combine (1) and (2). □

§5.3. Applications of Theorem 5.1.1 and 5.1.2

We will use Theorem 5.1.1 to discuss $n(G)$ and $\delta(G)$ for $G = A_5$, the alternating group on 5 letters. Let χ_1, χ_2, χ_3, χ_4 and χ_5 be all irreducible characters over $\mathbf{C}$ with $\dim(\chi_1) = 1$, $\dim(\chi_2) = \dim(\chi_3) = 3$, $\dim(\chi_4) = 4$ and $\dim(\chi_5) = 5$ (see [24]).

(5.3.1). $n(G) \leq 16$.

By [7], for any $\mathbf{Z}G$-lattice M such that there exists $\alpha \in \mathrm{H}^2(G, M)$ which is special for G. We have a map

$$\varphi : M \to \bigoplus_{C \in \mathcal{X}} \mathrm{ind}_C^G \mathbf{Z},$$

such that $\varphi^*(\alpha)$ is still special. Now suppose that $\mathrm{ind}_C^G \mathbf{Z}_p = \bigoplus M_i'$ with M_{i_C}' carries the nontrivial cohomology. Assume that there is a $\mathbf{Z}G$-lattice M_{i_C} which is a homomorphic image of $\mathrm{ind}_C^G \mathbf{Z}$, such that $M_{i_C}' \cong \mathbf{Z}_p \bigotimes_{\mathbf{Z}} M_{i_C}$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}) & \to & \mathrm{H}^2(G, M_{i_C}) & \to & \mathrm{H}^2(C, M_{i_C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}_p) & \to & \mathrm{H}^2(G, M_{i_C}') & \to & \mathrm{H}^2(C, M_{i_C}')
\end{array}$$

The leftmost vertical arrow is an isomorphism, since by Shapiro's Lemma the left hand cohomology groups are both isomorphic to C. We want to show that the image of the non zero element of $\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z})$ in the top right group $\mathrm{H}^2(C, M_{i_C})$ is not 0; this is true, since the image of the non zero element of $\mathrm{H}^2(G, \mathrm{ind}_C^G \mathbf{Z}_p)$ in the bottom right group $\mathrm{H}^2(C, M_{i_C}')$ does not vanish. Then we can extend the above map as follows:

$$M \xrightarrow{\varphi} \bigoplus_{C \in \mathcal{X}} \mathrm{ind}_C^G \mathbf{Z} \xrightarrow{\psi} \bigoplus_{C \in \mathcal{X}} M_{i_C},$$

It is easy to see that $(\psi\varphi)^*(\alpha)$ is still special. Then we have

$$n(G) \leq \sum_{C \in \mathcal{X}} \mathrm{rank}(M_{i_C}).$$

Now let $G = A_5$.

i. $p = 5$, $C_5 = \langle\ (12345)\ \rangle$; $N(C_5) = \langle\ (12345), (25)(34)\ \rangle$.

It is easy to see that $\mathrm{ind}_{C_5}^{N(C_5)}\mathbf{F}_5 = \mathbf{F}_5 \bigoplus \mathbf{F}_5'$ with (25)(34) acts nontrivially on $\mathbf{F}_5'$. By Theorem 5.1.1 $\mathbf{F}_5'$ contains nontrivial cohomology for $N(C_5)$. Now consider

$$\mathrm{ind}_{N(C_5)}^{G}\mathbf{F}_5' = g(\mathbf{F}_5') \bigoplus_i M_i$$

with $g(\mathbf{F}_5')$ the Green correspondence of $\mathbf{F}_5'$. Since all M_i are projective $\mathbf{F}_5G$-modules and Sylow 5-group of G has order 5, we have $5 \mid \dim_{\mathbf{F}_5}(M_i)$. So $\dim(g(\mathbf{F}_5')) = 1$ (with one M_i having dimension 5), or $= 6$ (without M_i). If $\dim(g(\mathbf{F}_5')) = 1$ then $g(\mathbf{F}_5') \cong \mathbf{F}_5$ with trivial G-action. This contradicts to $(\mathrm{ind}_{N(C_5)}^{G}\mathbf{F}_5')^G = 0$. Hence

$$\dim(g(\mathbf{F}_5')) = 6.$$

Now we will find the character χ of the lifting of $g(\mathbf{F}_5')$. The lifting of $\mathrm{ind}_{N(C_5)}^{G}\mathbf{F}_5'$ is $\mathrm{ind}_{N(C_5)}^{G}\mathbf{Z}_5'$. Let ϕ denote the character of $N(C_5)$-module $\mathbf{Z}_5'$, then we have

$$\chi(x) = \mathrm{ind}_{N(C_5)}^{G}\phi(x) = (n/h_x) \sum_{w \in C_x \cap N(C_5)} \phi(w),$$

where $n = [G : N(C_5)] = 6; h_x = |C_x|$. Then it is easy to verify that $\chi = \chi_2 + \chi_3$. Hence it can also afforded by a $\mathbf{Z}G$-lattice.

ii. $p = 3$.

$C_3 = \langle (123) \rangle ; N(C_3) = \langle (123), (12)(45) \rangle$. Again

$$\mathrm{ind}_{C_3}^{N(C_3)}\mathbf{F}_3 \cong \mathbf{F}_3 \bigoplus \mathbf{F}_3'$$

with $\mathbf{F}_3'$ carries the nontrivial cohomology. Let

$$\mathrm{ind}_{N(C_3)}^{G}\mathbf{F}_3' = g(\mathbf{F}_3') \bigoplus_i M_i$$

with $g(\mathbf{F}_3')$ the Green correspondence of $\mathbf{F}_3'$ and M_i projective G-module.

We need the following lemma to change the discussion to algebraically closed field case.

Lemma 5.3.1(Noether [8]). Let $\bar{\mathbf{F}}_p$ be the algebraic closure of $\mathbf{F}_p$. Let V and W be $\mathbf{F}_p$-modules. Then $\bar{\mathbf{F}}_p \bigotimes V \cong \bar{\mathbf{F}}_p \bigotimes W$ if and only if $V \cong W$.

We have (everything comes from Serre [24]) 4 irreducible $\bar{\mathbf{F}}_3 G$-modules V_1, V_2, V_3, V_4 with dim $V_1 = 1$, dim $V_2 = \dim V_3 = 3$, dim $V_4 = 4$. We also have the Cartan matrix:

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

and hence $P(V_1)$ (the projective cover of V_1) $= 2V_1 \bigoplus V_4$, dim $P(V_1) = 6$; $P(V_2) = V_2$; $P(V_3) = V_3$ and $P(V_4) = V_1 \bigoplus 2V_4$, dim $P(V_4) = 9$. Write

$$\mathrm{ind}_{N(C_3)}^G \bar{\mathbf{F}}_3 = g(\bar{\mathbf{F}}_3') \bigoplus \bar{M}_i.$$

Again by $(\mathrm{ind}_{N(C_3)}^G \bar{\mathbf{F}}_3')^G = 0$, we have that $\bar{M}_i \neq P(V_1), P(V_4)$. In order to decide whether $\bar{M}_i = P(V_2), P(V_3)$ or not , we need the following Lemma of Robinson's [22].

Lemma 5.3.2. Let $\mathbf{F}$ be an algebraically closed field. Let H be a subgroup of G and let V and W be irreducible $\mathbf{F}G$-modules and $\mathbf{F}H$-modules respectively. Let $P(M)$ be the projective cover of M. Then the multiplicity of $P(V)$ as a direct summand of $\mathrm{ind}_H^G W$ is equal to the multiplicity of $P(W)$ as a direct summand of $\mathrm{res}_H^G V$.

Now let $G = A_5; \mathbf{F} = \bar{\mathbf{F}}_3; N(C_3) \cong S_3$. Then we have two irreducible $\bar{\mathbf{F}}_3 S_3$-modules, say $W_1(= \bar{\mathbf{F}}_3)$ and $W_2(= \bar{\mathbf{F}}'_3)$. Both have dimension 1. $P(W_1) = 3W_1 + W_2$; $P(W_2) = W_1 + W_2$. So dim $P(W_1) = 4$; dim $P(W_2) = 2$.

By $\mathrm{res}^G_{N(C_3)} V_2 = P(W_2) + W_1$ and Lemma 5.3.2, we get $P(V_2) \mid \mathrm{ind}^G_{N(C_3)} \bar{\mathbf{F}}'_3$. Similarly we have $P(V_3) \mid \mathrm{ind}^G_{N(C_3)} \bar{\mathbf{F}}'_3$. Now it is easy to calculate that $\chi_{g(\bar{\mathbf{F}}'_3)} = \chi(V_4)$. Hence $g(\bar{\mathbf{F}}'_3) = \bar{\mathbf{F}}'_3 \bigotimes V_4$. Then we have

$$\bar{\mathbf{F}}'_3 \bigotimes \mathrm{ind}^G_{N(C_3)} \mathbf{F}'_3 \cong \mathrm{ind}^G_{N(C_3)} \bar{\mathbf{F}}'_3 \cong \bar{\mathbf{F}}'_3 \bigotimes V_4 \bigoplus \bar{\mathbf{F}}'_3 \bigotimes (V_2 \bigoplus V_3).$$

So by Noether's Lemma we get

$$\mathrm{ind}^G_{N(C_3)} \mathbf{F}'_3 \cong V_4 \bigoplus (V_2 \bigoplus V_3)$$

with $g(\mathbf{F}'_3) = V_4$.

iii. $p = 2$.

$C_2 = \langle\ (12)(34)\ \rangle$, $N(C_2) = \{1, (12)(34), (13)(24), (14)(23)\}$, $\mathrm{ind}^{N(C_2)}_{C_2} \mathbf{F}_2$ is indecomposable. The Scott-module of $\mathrm{ind}^G_{C_2} \mathbf{F}_2$ will carry the cohomology.

There are 4 irreducible $\bar{\mathbf{F}}_2 A_5$-modules V_1, V_2, V_3, V_4 with dim $V_1 = 1$, dim $V_2 = 2$, dim $V_3 = 2$ and dim $V_4 = 4$. The projective covers of them are respectively $P(V_1) = 4V_1 + 2V_2 + 2V_3$, $P(V_2) = 2V_1 + 2V_2 + V_3$, $P(V_3) = 2V_1 + V_2 + 2V_3$ and $P(V_4) = V_4$ with dim $P(V_1) = 12$, dim $P(V_2) = 8$, dim $P(V_3) = 8$ and dim $P(V_4) = 4$.

$\mathbf{F}_2 C_2$ has only one irreducible module $\bar{\mathbf{F}}_2$, $P(\bar{\mathbf{F}}_2) = \bar{\mathbf{F}}_2 C_2$. By the same argument as above we can get that $P(V_2) \mid \mathrm{ind}^G_{C_2} \bar{\mathbf{F}}_2$, $P(V_3) \mid \mathrm{ind}^G_{C_2} \bar{\mathbf{F}}_2$ and $P(V_4)^2 \mid \mathrm{ind}^G_{C_2} \bar{\mathbf{F}}_2$. Hence dim $g(\mathbf{F}_2) = 30 - 8 - 8 - 8 = 6$ by Lemma 5.3.1.

Let $g(\hat{\mathbf{F}}'_5)$, $g(\hat{\mathbf{F}}'_3)$ and $g(\hat{\mathbf{F}}_2)$ be the lifting of the correspondent ones. Then it is not difficult to calculate that

$$\begin{aligned} \chi_{g(\hat{\mathbf{F}}'_5)} &= \chi_2 + \chi_3; \\ \chi_{g(\hat{\mathbf{F}}'_3)} &= \chi_4; \\ \chi_{g(\hat{\mathbf{F}}_2)} &= \chi_1 + \chi_5. \end{aligned}$$

Each of them is afforded by a $\mathbf{Q}G$-module. Hence by a $\mathbf{Z}G$-module. let's denote them by V_5, V_3 and V_2 respectively, where the subindex indicate the corespondent prime. That gives

$$n(A_5) \leq 16.$$

If we use a different way to deal with C_2 as follows:

$$\mathrm{ind}_{C_2}^{A_5}\mathbf{Z} \xrightarrow{\phi} \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z},$$

where C_5 is a subgroup such that $N_G(C_5) = C_5C_2$. It is not difficult to prove that if $\alpha \in \mathrm{H}^2(A_5, \mathrm{ind}_{C_2}^{A_5}\mathbf{Z})$ is special for C_2, then $\phi^*(\alpha)$ is still special for C_2. Hence we can get the same result

$$n(A_5) \leq 16.$$

(5.3.2.) $n(G) \geq 16$.

We prove this in the way that if M is a $\mathbf{Z}A_5$-lattice and

$$\varphi : V_2 \bigoplus V_3 \bigoplus V_5 \to M$$

is a $\mathbf{Z}A_5$-module homomorphism with nonzero kernel, then $\varphi^*(\beta)$ is not special for some $\beta \in \mathrm{H}^2(G, \bigoplus_i V_i)$ special.

By the discussion above , we have a $\mathbf{Z}A_5$-homomorphism

$$d : M \to \bigoplus \mathrm{ind}_{C_i}^{A_5}\mathbf{Z} \xrightarrow{\psi} \bigoplus V_i$$

such that $d^*(\varphi^*(\beta))$ is special. Let f be the composition of φ and d, then we have $f : \bigoplus V_i \to \bigoplus V_i$ satisfying (i) $f^*(\beta)$ special and (ii) $\ker f \neq 0$. We will show that this is impossible.

Let's pick $\alpha \in \mathrm{H}^2(G, \bigoplus \mathrm{ind}_{C_i}^{A_5}\mathbf{Z})$ special and $\beta = \psi^*(\alpha)$. Write $\alpha = \alpha_2 + \alpha_3 + \alpha_5$, $\beta = \beta_2 + \beta_3 + \beta_5$ with α_i and β_i special for C_i. Since $\chi_{V_2 \bigoplus V_3 \bigoplus V_5} = (\chi_1 + \chi_5) + \chi_4 + (\chi_2 + \chi_3)$, $\chi_{f(V_2 \bigoplus V_3 \bigoplus V_5)} = \sum \chi_i$ with at least one χ_i missing since $\ker f \neq 0$. It is easy to see that $f^*(\beta_i)$ is not special for C_j if $i \neq j$ from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^2(A_5, \mathrm{ind}_{C_i}^{A_5}\mathbf{Z}) & \to & \mathrm{H}^2(A_5, V_i) & \to & \mathrm{H}^2(A_5, f(V_i)) \\
\downarrow \mathrm{res}_{C_j}^{A_5} & & \downarrow \mathrm{res}_{C_j}^{A_5} & & \downarrow \mathrm{res}_{C_j}^{A_5} \\
\mathrm{H}^2(C_j, \mathrm{res}_{C_j}^{A_5}\mathrm{ind}_{C_i}^{A_5}\mathbf{Z}) & \to & \mathrm{H}^2(C_j, \mathrm{res}_{C_j}^{A_5} V_i) & \to & \mathrm{H}^2(C_j, \mathrm{res}_{C_j}^{A_5} f(V_i))
\end{array}$$

since $\mathrm{H}^2(C_j, \mathrm{res}_{C_j}^{A_5}\mathrm{ind}_{C_i}^{A_5}\mathbf{Z}) = 0$. Hence we c[illegible]t miss $\chi_2 + \chi_3$, χ_4 or $\chi_1 + \chi_5$. If we missed χ_5, $f(V_2)$ would be a trivial A_5-module. But then $\mathrm{H}^2(A_5, f(V_2)) = 0$ since A_5 is perfect. It cannot be special for C_2. So the only possibility is that χ_1 is missing. We will use that $V_2 \cong \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z}$. From

$$0 \to \ker \varphi \to \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z} \to \mathbf{Z} \to 0,$$

we have (by taking the dual)

$$0 \to \mathbf{Z} \xrightarrow{\hat{\varphi}} \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z} \to (\ker \varphi)^* \to 0,$$

where $(\ker\varphi)^*$ is the dual of $\ker\varphi$. Hence $f(V_2) \cong (\ker\varphi)^*$. Since $\mathrm{H}^2(A_5, \mathbf{Z}) = 0$, $\mathrm{H}^2(C_2, \mathbf{Z}) \cong C_2$, we have

$$\begin{array}{ccccc} 0 & \rightarrow & \mathrm{H}^2(A_5, \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z}) & \stackrel{f^*}{\rightarrow} & \mathrm{H}^2(A_5, (\ker\varphi)^*) \\ \downarrow & & \downarrow \mathrm{res}_{C_2}^{A_5} & & \downarrow \mathrm{res}_{C_2}^{A_5} \\ C_2 & \stackrel{(\hat\varphi)^*}{\rightarrow} & \mathrm{H}^2(C_2, \mathrm{res}_{C_2}^{A_5}\mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Z}) & \stackrel{f^*}{\rightarrow} & \mathrm{H}^2(C_2, \mathrm{res}_{C_2}^{A_5}(\ker\varphi)^*) \end{array}$$

Hence to prove $\mathrm{res}_{C_2}^{A_5} f^*(\beta_2) = 0$, amounts to prove that $f^*\mathrm{res}_{C_2}^{A_5}\beta_2 = 0$, or $\mathrm{res}_{C_2}^{A_5}\beta_2 \in \mathrm{im}(\hat\varphi)^*$. We again change the discussion to H^1.

$$\begin{array}{ccccc} 0 & \rightarrow & \mathrm{H}^1(A_5, \mathrm{ind}_{C_5}^{A_5}\mathbf{Q}/\mathbf{Z}) & \stackrel{f^*}{\rightarrow} & \mathrm{H}^1(A_5, \mathbf{Q}\bigotimes(\ker\varphi)^*/(\ker\varphi)^*) \\ \downarrow & & \downarrow \mathrm{res}_{C_2}^{A_5} & & \downarrow \mathrm{res}_{C_2}^{A_5} \\ C_2 & \stackrel{(\hat\varphi)^*}{\rightarrow} & \mathrm{H}^1(C_2, \mathrm{res}_{C_2}^{A_5}\mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Q}/\mathbf{Z}) & \stackrel{f^*}{\rightarrow} & \mathrm{H}^1(C_2, \mathrm{res}_{C_2}^{A_5}\mathbf{Q}\bigotimes(\ker\varphi)^*/(\ker\varphi)^*) \end{array}$$

Now pick $\alpha \in \mathrm{H}^1(C_2, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z})$, $\alpha \neq 0$. Then $\alpha(c_2) = 1/2$, $C_2 = \langle c_2 \rangle$.

$$(\hat\varphi)^*(\alpha)(c_2) = \varphi\alpha(c_2) = \varphi(1/2) = 1/2\sum_u uC_5C_2.$$

Now let's examine the image of $\mathrm{res}_{C_2}^{A_5}$.

$$\begin{array}{ccc} \mathrm{H}^1(A_5, \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Q}/\mathbf{Z}) & \stackrel{\mathrm{res}_{C_2}^{A_5}}{\rightarrow} & \mathrm{H}^1(C_2, \mathrm{res}_{C_2}^{A_5}\mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Q}/\mathbf{Z}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^1(C_5C_2, \mathbf{Q}/\mathbf{Z}) & & \mathrm{H}^1(C_2, \bigoplus \mathrm{ind}_{C_2\cap(C_5C_2)^t}^{C_2}\mathrm{res}_{C_2\cap(C_5C_2)^t}^{C_5C_2}\mathbf{Q}/\mathbf{Z}) \\ \downarrow= & & \downarrow= \\ \mathrm{Hom}(C_5C_2, \mathbf{Q}/\mathbf{Z}) & \rightarrow & \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z}) \bigoplus \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z}) \end{array}$$

The last equality on the right hand side is the result that

$$\mathrm{H}^1(C_2, \mathrm{ind}_{C_2\cap(C_5C_2)^t}^{C_2}\mathrm{res}_{C_2\cap(C_5C_2)^t}^{C_5C_2}\mathbf{Q}/\mathbf{Z}) = 0$$

if $C_2 \cap (C_5C_2)^t = 1$, or $\mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z})$ if $C_2 \cap (C_5C_2)^t = C_2$. But $C_2 \cap (C_5C_2)^t \neq 1$ if and only if $C_2^t \cap C_5C_2 \neq 1$, this in turn if and only if $t \in N_{A_5}(C_2) = \{1, (12)(34), (13)(24), (14)(23)\}$ if $C_2 = \langle\ (12)(34)\ \rangle$. They belong to two different double coset of C_2 and C_5C_2 in A_5. Hence the result.

For nonzero $\alpha' \in \mathrm{Hom}(C_5C_2, \mathbf{Q}/\mathbf{Z})$, $\alpha'(c_5^i c_2^j) = \alpha'(c_2^j) = 0$ if $j = 0$ and $1/2$ if $j = 1$. Let the preimage of α' in $\mathrm{H}^1(A_5, \mathrm{ind}_{C_5C_2}^{A_5}\mathbf{Q}/\mathbf{Z})$ be $[\hat{\alpha'}]$, then

$$\hat{\alpha'}(g) = \sum_{u \in T_{C_5C_2}} \alpha'(V_{g,u}) u_g C_5C_2,$$

where $T_{C_5C_2}$ is a set of coset representatives of C_5C_2 in A_5 and $gu = u_g V_{g,u}$ with $V_{g,u} \in C_5C_2$. Then

$$\mathrm{res}_{C_2}^{A_5} \hat{\alpha'}(c_2) = \sum_{u \in T_{C_5C_2}} \alpha'(V_{c_2,u}) u_{c_2} C_5C_2.$$

So we need only prove that $\alpha'(V_{c_2,u}) = 1/2$ for all u. We will prove this in two ways.

1. Let $C_5 = \langle\ (12345)\ \rangle$, $C_2 = \langle\ (25)(34)\ \rangle = \langle\ c\ \rangle$. Then $C_2 < N_{A_5}(C_5) = C_5C_2$.

$C_5C_2 = \{1, (12345), (14253), (13524), (15432), (25)(34), (21)(35), (54)(31), (24)(15)\}$.

We can pick

$$T_{C_5C_2} = \{1, (123), (124), (125), (152), (154)\}$$

Then

$$\begin{aligned} (25)(34)1 &= 1(25)(34), \quad V_{c1} = (25)(34); \\ (25)(34)(123) &= (154)(25)(34), \quad V_{c(123)} = (25)(34); \end{aligned}$$

$$
\begin{aligned}
(25)(34)(124) &= (124)(13524)^{(25)(34)}(25)(34), \\
& \qquad V_{c(124)} = (13524)^{(25)(34)}(25)(34); \\
(25)(34)(125) &= (125)(25)(35), \quad V_{c(125)} = (25)(34); \\
(25)(34)(152) &= (125)(25)(34), \quad V_{c(152)} = (25)(34); \\
(25)(34)(154) &= (123)(25)(34), \quad V_{c(154)} = (25)(34).
\end{aligned}
$$

$V_{cu} \notin C_5$ for all $u \in T_{C_5C_2}$, so $\alpha'(V_{cu}) = 1/2$ for all u as required.

2. It is easy to get that $\alpha'(V_{c_2,1}) = 1/2$. On the other hand, let $c' \in N_{A_5}(C_2)\backslash C_2$ and suppose $c' \in T_{C_5C_2}$. Then $c_2c' = c'c_2$, so $\alpha'(V_{c_2,c'}) = \alpha'(c_2) = 1/2$. Hence $\hat{\alpha}'(c_2)$ is neither in $C_2 \bigoplus 0 \cong \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z}) \bigoplus 0$ since $1/2c'C_5C_2$ is not in it nor $0 \bigoplus C_2 \cong 0 \bigoplus \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z})$ since $1/2C_5C_2$ is not in it. So it must be the generator of $C_2 \bigoplus C_2 \cong \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z}) \bigoplus \mathrm{Hom}(C_2, \mathbf{Q}/\mathbf{Z})$ which is $1/2 \sum_u uC_5C_2$. Hence the result.

(5.3.3). $\delta(A_5)$.

We already had lattices which are special for (abc) and $(abcde)$ in (5.3.2). Hence we just need one for $(ab)(cd)$. By

$$0 \to \ker\rho \to \mathrm{ind}_{N(C_5)}^{A_5}\mathbf{Z} \xrightarrow{\rho} \mathbf{Z} \to 0$$

and $\mathrm{H}^2(A_5, \mathbf{Z}) = 0$, we can get that $\ker\rho$ is the one we need. Hence $\delta(A_5) \leq 5 + 6 + 4 = 15$ which is the exact number of $\delta(A_5)$. (See [12] [19].)

(5.3.4). $n(G)$ for some metcyclic groups.

Let G be a semi-direct product of C_p and C_q with C_p, C_q cyclic groups of prime order p and q respectively. From Proposition 4.1.1, we have

$$\varphi : \mathrm{ind}_{C_q}^{G}\mathbf{Z} \to \mathrm{ind}_{G}^{G}\mathbf{Z} = \mathbf{Z},$$

such that if $\alpha \in \mathrm{H}^2(G, \mathrm{ind}_{C_q}^G \mathbf{Z})$ is special for C_q then $\varphi^*(\alpha)$ is still special for C_q. On the other hand, $\mathrm{ind}_{C_p}^G \mathbf{F}_p \cong \mathbf{F}_p C_q \cong \mathbf{F}_p \bigoplus \mathbf{F}_p(\zeta_q)$. By the Theorem 5.1.1, $\mathbf{Z}_p[\zeta_q]$ will carry the second cohomology since $N_G(C_p) \neq C_G(C_p)$. It is easy to see that $\chi_{\mathbf{Z}_p[\zeta_q]}$ can be also afforded by a $\mathbf{Z}G$-module. Hence $n(G) \leq q$. Follow the same discussion as in (5.3.2) we can get that $n(G) = q$.

REFERENCES

[1] L. Auslander, M. Kuranishi, *On the holonomy group of locally Euclidean space,* Ann. Math. 65 (1957), 411-415.

[2] N. Broderick, G. Maxwell, *Crystallography of Coxeter groups II,* J. Algebra, 29 (1974), 587-599.

[3] H. Brown, R. Bülow , J. Neubüser, H. Wondratschek and H. Zassenhauss, "Crystallographic groups of four-dimensional space," John Wiley, New York, 1978.

[4] K. Brown, " Cohomology of groups", Springer Verlag, New York, 1982.

[5] D. Burry, *Scott modules and lower defect groups*, Comm. Algebra 10 (1982), 1855-1872.

[6] L. S. Charlap, "Bieberbach groups and flat manifolds," Springer-Verlag, New York, 1986.

[7] G. Cliff, A. Weiss, *Torsion free space groups and permutation lattices for finite groups*, Contemporary Math. V93 (1989), 123-132.

[8] C. W. Curtis, I. Reiner, "Representation theory of finite groups and associative algebras," Interscience, New York, 1962.

[9] W. Feit, "The representation theory of finite groups," North Holland, Amsterdam, 1982.

[10] K. W. Gruenberg, "Cohomological topics in group theory," Lecture notes in Math. 143, Springer-Verlag, Berlin, 1970.

[11] H. Hiller , C.H. Sah, *Holonomy of flat manifolds with* $b_1 = 0$, I, Quarterly J. Math. Oxford 37 (1986), 177-187.

[12] H. Hiller, Z. Marciniak, C.H. Sah, A. Szczepański, *Holonomy of flat mani-*

folds with $b_1 = 0$, II Quarterly J. Math. Oxford 38 (1987), 213-220.

[13] G. Hiss , A. Szczepański, *On torsion free crystallographic groups,* J. Pure and Applied Algebra 74 (1991), 39-56.

[14] P. Landrock, "Finite group algebras and their modules," London Math. Sco. Lecture Note Series 84, Cambridge University Press, Cambridge, 1983.

[15] Fubin Li, *The crystallographic group of infinite Coxeter groups,* J. Algebra 146 (1992), 190-204.

[16] S. MacLane, "Homology", Springer-Verlag, Berlin, 1963.

[17] G. Maxwell, *The crystallography of Coxeter groups,* J. Algebra, 35 (1975), 159-177.

[18] G. Maxwell, *Compact Euclidean space-form,* J. Algebra, 44 (1977), 191-195.

[19] W. Plesken, *Minimal dimensions for flat manifolds with prescribed holonomy,* Math. Ann. 284 (1989) 477-486.

[20] W. Plesken, *[illegible]ations of representation theory*, Progress in Math. Vol. 95, (1991) 477-496.

[21] I. Reiner, *Integral representations of cyclic groups of prime order,* Proc. Amer. Math. Sco. 8 (1957), 142-146.

[22] G. R. Robinson, *On projective summands of induced modules*, J. of Algebra 122 (1989), 106-111.

[23] W. R. Scott, "Group theory", Dover Publications, Inc. New York 1987.

[24] J. P. Serre, "Linear representations of finite groups", Springer, New York, 1977.

[25] P. Symonds, " Flat manifolds," Ph. D. thesis, 1987, Cambridge University.

[26] P. Symonds, *Localization in the classification of flat manifolds*, Pacific J. Math. V127, No. 2 (1987), 389-399.

[27] A. Szczepański, *Euclidean space forms with first Betti number equals to zero*, Q. J. Math. Oxf. II. Ser. 36(1985), 489-493.

[28] A. Szczepański, *Five dimensional Bieberbach groups with trivial center*, Manuscripta. Math. 68 (1990), 191-208.

[29] A. T. Vasquez. *Flat Riemannian manifolds*, J. Differential Geometry 4 (1970), 367-382.

[30] J. A. Wolf, "Spaces of constant curvature," MacGraw-Hill, New York, 1968.

[31] H. Zassenhaus, *Beweis eines Satzes über diskrete Gruppen,* Abh. Math. Semin. Univ. Hamburg 21 (1938), 289-312.