University of Alberta

TRADING LARGE VWAP ORDERS IN DISCRETE TIME

by

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Abstract

This thesis focuses on characterizing an optimal trading strategy for a large trader, who has to buy (or sell) a fixed large volume over a given time period. We propose a model in discrete time, based on VWAP (Volume Weighted Average Price). The objective is to minimize expected deviations between the trader's relative volumes and the market relative volumes at all times.

By applying dynamic programming, we characterize the optimal strategy under three different assumptions on the intraday market volumes: i.i.d. volumes, general independent volumes and independent Gamma distributed volumes. The optimal strategy under the last assumption is meaningful and explicit. For three exemplary Chinese stocks, we present its good data fit and illustrate the improved performance (reduced deviations to the market relative volumes) compared with the empirical strategy, which is one of the most popular and efficient VWAP strategies in the financial industry.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Algorithmic trading means executing orders automatically through computer-based predefined algorithms. It is widely used by investment banks and other large traders. According to data[1] in 2009, 73% of the volume in all US equities was executed by algorithms. The percentage is still increasing year by year.

Due to its popularity, it attracts the attentions of researchers. Generally, these kinds of orders are very large so that even a small gain from a better strategy can lead to a large profit. Therefore, researchers are interested in finding good execution strategies. Among the published papers, there exist several criteria to define a good strategy. The first model in this area is proposed by Bertsimas and Lo [3], who set up a discrete-time model to minimize the expected transaction costs with price impact. In a seminal paper, Almgren and Chriss [2] extend the previous work by taking risk into consideration. They minimize both the transaction costs and the volatility risk. Schied and Schöneborn [18] characterize the optimal strategy to maximize the utility function. Many other related papers are similar to these models, but use different

[1] Source: advancedtrading.com

models for price dynamics or price impact. We refer to Gatheral and Schied [9] and Gökay, Roch and Soner [10] for a detailed exposition of research progress in algorithmic trading.

In investment banks, traders often execute the order to meet or even beat a benchmark set by their clients or themselves. The benchmark can be used to evaluate the quality of trading on a selected interval, such as one trading day. The simplest benchmark is the opening price, called arrival price benchmark. However, it is often not adequate to use the price at only one point to evaluate the trading over a whole day. An alternative is TWAP, which stands for Time Weighted Average Price. It is the average price over a selected interval, just like the payoff of an Asian option. Usually, it is more appropriate than the arrival price benchmark, since it considers all prices during the trading period. However, it does not consider the trading volume, which is the key factor to calculate the transaction cost. Currently, VWAP (Volume Weighted Average Price) is the most popular benchmark in the financial industry. It is the sum of prices in all periods weighted by the corresponding relative volumes. Suppose there are n trading periods, and we execute $(u_i)_{i=1,\dots,n}$ shares at corresponding prices $(p_i)_{i=1,\dots,n}$. Then, the VWAP is defined by

$$\text{VWAP} = \sum_{i=1}^{n} \frac{u_i}{\sum_{j=1}^{n} u_j} p_i,$$

where $\frac{u_i}{\sum_{j=1}^{n} u_j}$ is the weight corresponding to p_i. We call it the relative volume in period i. VWAP takes both the prices and trading volumes into consideration. As Frei and Westray [7] mention, VWAP is a good benchmark because it is simple to calculate, considered as fair benchmark and encourages to split large order into smaller ones.

Our goal in this thesis is to characterize and examine optimal strategies to help traders meet the VWAP benchmark. However, future volumes are random and may deviate from historical volumes. Therefore, a predictable strategy cannot exactly meet the market VWAP. We try to minimize expected deviations between the trader's VWAP and the market VWAP in all periods, for all possible stock prices p_i. Similar to the illustration in Bialkowski et al. [4, 5], if there were no price impact and a trader could keep her trading pattern the same as that of the market, the trader's VWAP would match the market VWAP. However, the terminal market volume is unknown, hence needs to be estimated or approximated in some way, which leads to tracking errors. In our case, we minimize the expected deviations between the traders' relative volumes and the market relative volumes at all times. Suppose there are n trading periods. The trader's volume in period i is denoted by u_i, and the corresponding market volume (excluding the trader's volume) is y_i. Therefore, the trader's and market relative volume in period i are given by $\frac{u_i}{\sum_{j=1}^n u_j}$ and $\frac{u_i+y_i}{\sum_{j=1}^n u_j+\sum_{j=1}^n y_j}$, respectively. We aim to minimize

$$E\left[\sum_{i=1}^{n}\left(\frac{u_i}{\sum_{j=1}^n u_j}-\frac{u_i+y_i}{\sum_{j=1}^n u_j+\sum_{j=1}^n y_j}\right)^2\right]. \tag{1.1}$$

While there is a huge literatures on using an arrival price benchmark, there are less papers related to VWAP. Konishi [14] first investigates the optimal trading under the VWAP benchmark when the stock price is a Brownian motion. He derives a static optimal execution strategy of a VWAP order to minimize the expected squared execution error. McCulloch and Kazakov [16] extend the work by applying the quadratic hedging theory when the stock price is a semimartingale. They derive a dynamic mean-variance VWAP trading strategy. Bialkowski et al. [4, 5] decompose the trading volume into two parts to model the dynamics of intraday volume, which leads to a significant reduction of the execution risk in VWAP, compared with a static VWAP strategy. Humphery-Jenner [12] defines a dynamic VWAP strategy which allows traders to utilize random news during the trading. Kakade et al. [13] study competitive algorithms for VWAP trading in an online learning model. Fuh et al. [8] present cross-boundary, relative rank and hybrid strategies as alternatives for VWAP trading. Pemy [17] finds an optimal strategy to maximize the trading VWAP for a large seller when the stock prices follow a geometric Brownian motion, by applying the stochastic control method with resource constraints. Bouchard and Dang [6] apply the stochastic target approach to the VWAP guaranteed contract.

The most recent work is done by Frei and Westray [7], who initiated this thesis. They introduce a Gamma bridge to describe the relative volumes, and characterize an explicit optimal strategy to minimize both the mean and variance of the order slippage with respect to VWAP. In contrast to [7], we also take the trader's own volume into consideration when calculating relative market volumes per period, see (1.1). As discussed in Hu [11], the VWAP cost excluding the trader's execution overstates costs, compared with that including their execution. Especially, if the order is 20% of the market volume, it overestimates costs by 25%. Our setting is meaningful for large VWAP orders ($>20\%$ of daily volume when scaled to a day). For tractability, we consider a discrete-time model and minimize the deviations of the relative volumes rather than VWAP directly. This makes the analysis independent of the stock price dynamics.

As mentioned before, the order size in algorithmic trading is often large. Generally, these orders cannot be liquidated immediately without adverse impact on the market prices, which increases the transaction costs substantially. To reduce the price impact, traders have to slice the orders into small pieces.

There are three kinds of price impact investigated in the previous literature; see Alfonsi, Schied and Slynko[1]. The *temporary impact* only affects the current trading period. The *permanent impact* shifts the price permanently. The *transient impact* affects the current trading period seriously and will last in the following several periods, but will eventually disappear. As explained above, we do not specify the price dynamics in our model, but only measure the deviations in relative volumes. Since price impact is typically linked to the relation between the trader's and the remaining market volume, the minimization of (1.1) automatically reduces the price impact. Additionally, we investigate in Section 5 how a linear price impact affects the trading decisions.

The crucial point in this thesis is that, when calculating the market relative volumes, we also consider the trader's own volumes, which is important for large orders. Simultaneously, this significantly increases the difficulty of the optimization problem. We study a tractable problem formulated by minimizing expected deviations between the trader's and market relative volumes rather than directly the trader's and market VWAP. Through dynamic programming, we characterize a sequence of optimal trading strategies. It allows us to utilize updated market information. Moreover, we also consider the case when there is linear price impact. Through adjusting the coefficient of the linear price impact, we can fit our model to any kind of orders, even very huge orders. To obtain more explicit strategies, we introduce the Gamma distribution to describe the market volumes in discrete time. This results in explicit strategies, whose performance we analyze statistically.

The thesis is organized as follows. In Section 2, we state the main assumptions and set up the model. Section 3 characterizes the optimal strategies in both two-period and n-period case under three different assumptions on the intraday market volumes. In Section 4, we complete the statistical test and introduce a correlation structure to describe the intraday volumes in the two-period case. Additionally, a linear price impact is taken into consideration in Section 5. We conclude the thesis by discussing future work in Section 6. The Appendices A and B contain auxiliary calculations and MATLAB code used for our results.

Chapter 2

Problem formulation

In this chapter, we state our main assumptions and formulate the problem. Before setting up the model, we first introduce the definition of VWAP. Suppose there are n trading periods, and we execute $(u_i)_{i=1,\ldots,n}$ shares at corresponding prices $(p_i)_{i=1,\ldots,n}$. Then, the VWAP is defined by

$$\text{VWAP} = \sum_{i=1}^{n} \frac{u_i}{\sum_{j=1}^{n} u_j} p_i,$$

where $\frac{u_i}{\sum_{j=1}^{n} u_j}$ is the weight corresponding to p_i. We call it the relative volume in period i.

In our model, we consider the case when we have to buy or sell $X > 0$ shares during n trading periods. Our cumulative traded volume is zero at the beginning, and X at the end. The executed volume u_i in period i is decided at the beginning of period i, with the help of information until that time. This means that $(u_i)_{i=1,\ldots,n}$ needs to be predictable to the market filtration while $(p_i)_{i=1,\ldots,n}$ is adapted. Since we have to execute the entire volume after

n periods, our trading strategy must satisfy

$$\sum_{i=1}^{n} u_i = X.$$

We denote by $(y_i)_{i=1,\dots,n}$ the remaining volume, which is the market volume excluding our volume. The total market volume in period i is $u_i + y_i$. For the empirical part, we will consider different trading days, each grouped in the same n trading periods. The market volumes on different days are supposed to be independent and those in corresponding periods are assumed to be identically distributed. This assumption allows us to use the historical data to estimate the distribution of future market volumes. All assumptions specified for the market volumes in this thesis are based on the relations among the intraday market volumes. For example, in the two-period model, we have both morning volumes and afternoon volumes on days $1, \dots, m$. Morning (Afternoon) volumes on different days are assumed to be independent and identically distributed. The assumptions specified in the thesis are based on the relation between morning volume and afternoon volume.

Our goal is to minimize expected deviation between our VWAP and market VWAP for all possible stock prices p_i. Now, we compare our VWAP and market VWAP,

$$\text{Our VWAP} = \sum_{i=1}^{n} \frac{u_i}{\sum_{j=1}^{n} u_j} p_i,$$

$$\text{Market VWAP} = \sum_{i=1}^{n} \frac{u_i + y_i}{\sum_{j=1}^{n} (u_j + y_j)} p_i.$$

If we assume that we use the same prices in calculating our and the market VWAP, the only differences are the relative volumes. Since we want to keep these two VWAPs close for all possible p_i and we do not specify the process of stock prices, we aim to keep our trading pattern consistent with the market pattern. That is, we can keep our relative volume close to the market relative volume all the time. Our relative volume in period i is equal to $\frac{u_i}{\sum_{j=1}^{n} u_j} = \frac{u_i}{X}$. The market relative volume is $\frac{u_i+y_i}{X+\sum_{j=1}^{n} y_j}$.

This leads to a minimization problem. We choose a quadratic function to measure the closeness, which penalizes both the positive and negative deviations. Since we want to make these two VWAPs close all the time, the objective function should be a sum of deviations in all periods. We notice that there are random variables in the quadratic function, so we take the expectation of the quadratic function. Our aim is to find a sequence of trading strategies $(u_i)_{i=1,\ldots,n}$ which minimize the objective function

$$f(u_1, \ldots, u_n) = E\left[\sum_{i=1}^{n}\left(\frac{u_i}{X} - \frac{u_i + y_i}{X + \sum_{j=1}^{n} y_j}\right)^2\right]$$

subject to $\sum_{i=1}^{n} u_i = X$.

The difficulties of realizing this goal are as follows: firstly, in contrast to [7], we consider the trader's own volume when calculating the market relative volumes, which significantly increases the difficulty of the minimization problem. Secondly, the future market volumes are unknown. When we make our decision u_i, we have no information about the precise value of y_i, but we can use historical data to estimate it. Thirdly, it is a dynamic programming problem. For an n-period case, we should make decisions for $n-1$ times and adjust our strategy based on updated market information.

Chapter 3

Main results

In this chapter, we discuss three cases when intraday market volumes are independent. We characterize the optimal trading strategy in each case. Each case consists of two parts: a two-period model and an n-period model. In Section 3.1, we assume that the intraday market volumes are independent and identically distributed. The optimal strategy is a TWAP strategy, which means splitting the order evenly over time. The case with general independent volumes is investigated in Section 3.2. Section 3.3 is a special case of Section 3.2, where market volumes are assumed to be Gamma distributed with same scale parameter.

The two-period model is equivalent to dividing a trading day into morning and afternoon sessions. We decide at the beginning of the day how many shares should be executed in the morning and how many in the afternoon. The reason why we discuss the two-period case individually is that there are lunch breaks in some Asian markets, such as the Shanghai Stock Exchange, Tokyo Stock Exchange, and Hong Kong Stock Exchange. The lunch break naturally divides a trading day into two parts. The two-period model corresponds exactly to the trading pattern in these Asian markets.

In the n-period model, we divide a trading day into n periods with the same length. The number of trading periods varies with different markets, even though the lengths of a trading period in each market are the same, since the trading hours in different markets are not the same. If we divide a trading day into 5-minute intervals, there are 102, 78 and 48 trading periods per day for the UK, US and China market, respectively.

3.1 Solution for general i.i.d. volumes

In this section, market volumes are independent and identically distributed. That is, the market volumes in every trading period have the same distribution type and same parameters. However, it is not necessary to specify the distribution type or parameters.

3.1.1 Two-period case

This case is the simplest one. We only need to make one decision on our trading strategy. At time zero, we decide to buy (or sell in the case of a sell order) u_1 shares in period one, where u_1 is deterministic. Since our total volume is fixed, our strategy in period two is automatically determined, namely, $X - u_1$. Hence, we aim to minimize the objective function

$$f(u_1) = E\left[\left(\frac{u_1}{X} - \frac{u_1 + y_1}{X + y_1 + y_2}\right)^2 + \left(\frac{X - u_1}{X} - \frac{X - u_1 + y_2}{X + y_1 + y_2}\right)^2\right].$$

We notice that y_1 and y_2 are i.i.d. The objective function is equivalent to the one where in the numerator of $\frac{X-u_1+y_2}{X+y_1+y_2}$, y_2 is replaced by y_1,

$$f(u_1) = E\left[\left(\frac{u_1}{X} - \frac{u_1+y_1}{X+y_1+y_2}\right)^2 + \left(\frac{X-u_1}{X} - \frac{X-u_1+y_1}{X+y_1+y_2}\right)^2\right].$$

For this function, we just need to find the optimal value of u_1, which enables the objective function to achieve its minimum. To this aim, take the derivative of f with respect to u_1, and set it equal to zero, which gives,

$$\begin{aligned} f'(u_1) =& E\left[2\left(\frac{u_1}{X} - \frac{u_1+y_1}{X+y_1+y_2}\right)\left(\frac{1}{X} - \frac{1}{X+y_1+y_2}\right)\right. \\ &\left. + 2\left(\frac{X-u_1}{X} - \frac{X-u_1+y_1}{X+y_1+y_2}\right)\left(-\frac{1}{X} + \frac{1}{X+y_1+y_2}\right)\right] = 0. \end{aligned}$$

By simplifying it, we get

$$E\left[\left(\frac{1}{X} - \frac{1}{X+y_1+y_2}\right)^2(-X+2u_1)\right] = 0.$$

Since market volumes are strictly positive, $\left(\frac{1}{X} - \frac{1}{X+y_1+y_2}\right)^2$ is always nonzero. Therefore, the above equation is true if and only if $-X+2u_1 = 0$. We obtain

$$u_1^* = u_2^* = \frac{X}{2}.$$

That is, in the two-period i.i.d. case, our optimal strategy is buying $\frac{X}{2}$ shares in each period. It is a TWAP strategy, which is exactly what we expected. The reason is, under the i.i.d. assumption, the distributions of market volumes in each period are the same, so that we minimize the deviations to the relative market volumes by splitting our order equally.

3.1.2 n-period case

In this part, we assume there are n trading periods per day. We do not need to make all decisions at the beginning of the trading day. Indeed, we can decide u_j at the beginning of period j. Therefore, it is a dynamic programming problem. Before solving it, we would like to clarify five important ingredients for the dynamic programming in this problem. Firstly, *state variables*, which are the information we need to make our decision. These include the number j of the current trading period, our total volume W_j before the current period, and the market total volume Z_j before the current period. Secondly, *control variables* are our decision variables. We decide to execute u_j shares in period j. Thirdly, *randomness*. In this model, future market volumes are unknown. We model future market volumes as random variables. Fourthly, the *objective function*. Since we want to minimize the expected deviations all the time, the objective function is the sum of expected deviations in each trading period. The last one is the *law of motion*. It summarizes the relations between state variables in the neighboring periods. The relations between W_{j+1} and W_j, Z_{j+1} and Z_j are given by

$$W_{j+1} = W_j + u_j, \quad Z_{j+1} = Z_j + y_j.$$

Our goal is to find the optimal strategy $\{u_1^*, u_2^*, ..., u_n^*\}$ in every trading period. Based on the property of dynamic programming, $\{u_i^*, ..., u_n^*\}$ is also optimal for the remaining program starting at the beginning of period i, $1 < i \leq n$. This property can be summarized by the following Bellman equation, which is the key to solving this dynamic programming problem,

$$V(j, W_j, Z_j) = \min_{u_j} E\left[\left(\frac{u_j}{X} - \frac{u_j + y_j}{X + \sum_{\ell=j}^{n} y_\ell + Z_j}\right)^2 + V\left(j+1, W_j + u_j, Z_j + y_j\right)\right]$$

subject to $0 \le u_j \le X - W_j$.

It is reasonable to add this constraint, since it is not sensible to trade in the opposite direction (selling during a buy order or buying during a sell order). That is, our strategy cannot be negative. Moreover, before we make the decision on u_j, we have already executed W_j shares and our total number of shares is X at the end. Therefore, u_j cannot be greater than our remaining shares, $X - W_j$. With this Bellman equation, we can start the dynamic programming.

In period n, the value function only has the first term. It is obvious that we have to execute all the remaining shares to complete our task. Therefore, it is reasonable that we start at the beginning of period $n-1$. The value function then equals

$$V\left(n-1, W_{n-1}, Z_{n-1}\right) = \min_{u_{n-1}} f(u_{n-1}, n-1, W_{n-1}, Z_{n-1}),$$

$$f(u_{n-1}, n-1, W_{n-1}, Z_{n-1}) = E\left[\left(\frac{u_{n-1}}{X} - \frac{u_{n-1} + y_{n-1}}{X + y_{n-1} + y_n + Z_{n-1}}\right)^2 + \left(\frac{X - W_{n-1} - u_{n-1}}{X} - \frac{X - W_{n-1} - u_{n-1} + y_n}{X + y_{n-1} + y_n + Z_{n-1}}\right)^2\right].$$

Take the derivative of $f\left(u_{n-1}, n-1, W_{n-1}, Z_{n-1}\right)$ with respect to u_{n-1}, and set it equal to zero, which gives

$$\frac{\partial f(u_{n-1}, n-1, W_{n-1}, Z_{n-1})}{\partial u_{n-1}} = E\left[\left(\frac{1}{X} - \frac{1}{X + y_{n-1} + y_n + Z_{n-1}}\right)\right.$$
$$\left.\times \left(\frac{2u_{n-1} + W_{n-1} - X}{X} + \frac{X - W_{n-1} - 2u_{n-1} + y_n - y_{n-1}}{X + y_{n-1} + y_n + Z_{n-1}}\right)\right] = 0.$$

We separate the expectation into three terms,

$$E\left[\left(\frac{1}{X} - \frac{1}{X + y_{n-1} + y_n + Z_{n-1}}\right)^2 (2u_{n-1} + W_{n-1} - X)\right]$$
$$+ E\left[\frac{y_n}{X + y_{n-1} + y_n + Z_{n-1}}\left(\frac{1}{X} - \frac{1}{X + y_{n-1} + y_n + Z_{n-1}}\right)\right]$$
$$- E\left[\frac{y_{n-1}}{X + y_{n-1} + y_n + Z_{n-1}}\left(\frac{1}{X} - \frac{1}{X + y_{n-1} + y_n + Z_{n-1}}\right)\right] = 0.$$

It is important to recall that y_{n-1} and y_n are i.i.d. Since the second and third terms have the same structure, the difference must be zero, which leads to

$$E\left[\left(\frac{1}{X} - \frac{1}{X + y_{n-1} + y_n + Z_{n-1}}\right)^2 (2u_{n-1} + W_{n-1} - X)\right] = 0.$$

It is obvious that $\left(\frac{1}{X} - \frac{1}{X+y_{n-1}+y_n+Z_{n-1}}\right)^2$ is always positive. Using that u_{n-1}, W_{n-1}, and X are deterministic, the above equation is true if and only if

$$2u_{n-1} + W_{n-1} - X = 0.$$

This yields the optimal strategy in period $n-1$, namely

$$u^*_{n-1} = \frac{X - W_{n-1}}{2}.$$

By putting the optimal strategy back into the value function $V\left(n-1, W_{n-1}, Z_{n-1}\right)$, we obtain

$$V\left(n-1,W_{n-1},Z_{n-1}\right)=\frac{1}{2}E\left[\left(\frac{X-W_{n-1}}{X}-\frac{X-W_{n-1}+2y_{n-1}}{X+y_{n-1}+y_n+Z_{n-1}}\right)^2\right].$$

Observing the structure of the value function and optimal strategy in period $n-1$, we find some interesting patterns. For the optimal strategy, we just need to split our remaining shares evenly. For the value function, $\frac{1}{2}$ is the inverse of 2, which is the number of the remaining trading periods. $\frac{X-W_{n-1}}{X}$ is the portion of our remaining volume. $\frac{X-W_{n-1}+2y_{n-1}}{X+y_{n-1}+y_n+Z_{n-1}}$ is the portion of the remaining market volume. Based on these observations, we make the following induction hypotheses on the value function and optimal strategy in any period $n-i, 0<i<n-1$,

$$u^*_{n-i}=\frac{X-W_{n-i}}{i+1}, \tag{3.1}$$

$$V\left(n-i,W_{n-i},Z_{n-i}\right)=\frac{1}{i+1}E\left[\left(\frac{X-W_{n-i}}{X}-\frac{X-W_{n-i}+(i+1)y_{n-i}}{X+\sum_{j=n-i}^{n}y_j+Z_{n-i}}\right)^2\right].$$

Applying the Bellman equation, we get the value function at the beginning of period $n-i-1$,

$$\begin{aligned}
&V\left(n-i-1,W_{n-i-1},Z_{n-i-1}\right)\\
&=\min_{u_{n-i-1}}E\left[\left(\frac{u_{n-i-1}}{X}-\frac{u_{n-i-1}+y_{n-i-1}}{X+\sum_{j=n-i-1}^{n}y_j+Z_{n-i-1}}\right)^2\right.\\
&\left.+\frac{1}{i+1}\left(\frac{X-W_{n-i-1}-u_{n-i-1}}{X}-\frac{X-W_{n-i-1}-u_{n-i-1}+(i+1)y_{n-i}}{X+\sum_{j=n-i-1}^{n}y_j+Z_{n-i-1}}\right)^2\right].
\end{aligned}$$

By using the same method, we can get

$$
\begin{aligned}
&u^*_{n-i-1} = \frac{X - W_{n-i-1}}{i+2}, \\
&V\left(n-i-1, W_{n-i-1}, Z_{n-i-1}\right) \\
&= \frac{1}{i+2} E\left[\left(\frac{X - W_{n-i-1}}{X} - \frac{X - W_{n-i-1} + (i+2)y_{n-i-1}}{X + \sum_{j=n-i-1}^{n} y_j + Z_{n-i-1}}\right)^2\right].
\end{aligned}
$$

The solution is consistent with our induction hypotheses regarding the optimal strategy and value function. By backward induction, we conclude that these properties hold for every trading period $i, 0 < i < n-1$.

For the first period with $W_1 = 0$ and $Z_1 = 0$, we get the optimal strategy in period 1, u^*_1 and the value function, $V(1, 0, 0)$,

$$
\begin{aligned}
u^*_1 &= \frac{X}{n}, \\
V\left(1, 0, 0\right) &= \frac{1}{n} E\left[\left(1 - \frac{X + ny_1}{X + \sum_{j=1}^{n} y_j}\right)^2\right].
\end{aligned}
$$

By substituting, we get all the optimal strategies,

$$
u^*_1 = u^*_2 = \cdots = u^*_n = \frac{X}{n}.
$$

Therefore, in the n-period i.i.d. case, our optimal strategy is to buy $\frac{X}{n}$ shares in each period, which is exactly the TWAP strategy. The trading strategies in each period are exactly the same, and there is no adjustment on our strategy based on the market information, although we collect more information from the market as time goes by. This is reasonable because past market information is independent of future market volumes by assumption. This information is not helpful in inferring future market volumes, which are i.i.d. Therefore, we make no adjustment based on the past market information, which leads to continuing splitting the remaining order evenly.

3.2 Solution for general independent volumes

In the previous section, we analyzed i.i.d. volumes. However, when we observe the intraday volumes, we find some U-shape patterns. That is, the volumes at the beginning and end of the day are higher than those in the middle of the day. In this case, i.i.d. volumes may not be close to the reality. In this section, we consider general independent volumes. We still do not specify the distribution type and parameters. It makes the optimal strategy more complicated, but also more interesting. As expected, we cannot get an explicit formula for the optimal strategy, but we can implement it through Monte-Carlo simulation. The structure of this section is the same as in the i.i.d. case, we first consider the two-period case, then the n-period case.

3.2.1 Two-period case

As in Section 3.1.1, we aim to minimize

$$f(u_1) = E\left[\left(\frac{u_1}{X} - \frac{u_1 + y_1}{X + y_1 + y_2}\right)^2 + \left(\frac{X - u_1}{X} - \frac{X - u_1 + y_2}{X + y_1 + y_2}\right)^2\right].$$

Unlike the i.i.d. case, we cannot replace y_2 in the numerator of $\frac{X-u_1+y_2}{X+y_1+y_2}$ by y_1. Take the derivative and set it equal to zero,

$$f'(u_1) = E\left[\frac{y_1 + y_2}{X^2\left(X + y_1 + y_2\right)^2}\left(u_1 y_1 + u_1 y_2 - X y_1\right)\right] = 0.$$

Then, we can calculate our optimal strategy

$$u_1^* = \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}X, \quad u_2^* = \left(1 - \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}\right)X. \tag{3.2}$$

Hence, the optimal strategy is to buy $\frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}X$ shares in period one, and the remaining shares in period two. It seems that the optimal strategy is complicated, in particular, if we want to generalize it to the n-period case. Nevertheless, we can use the historical trading data to estimate the distributions of y_1 and y_2. Moreover, our trading volume X is fixed. We can calculate the optimal strategy numerically. Observing the structure of u_1^*, we find that the numerator and denominator look very similar. We sperate the numerator into two parts

$$u_1^* = \frac{E\left[\frac{y_1}{y_1+y_2}\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}X.$$

If $\frac{y_1}{y_1+y_2}$ is independent of $y_1 + y_2$, we are able to simplify it to

$$u_1^* = E\left[\frac{y_1}{y_1+y_2}\right]X, \quad u_2^* = E\left[\frac{y_2}{y_1+y_2}\right]X.$$

If we can get this strategy, it substantially reduces the time to calculate it. It also makes sense to have a strategy proportional to the expectation of the relative market volumes. We are asking ourselves if we can find a distribution type which achieves it and fits the data well. We discuss this in Section 3.3.

3.2.2 n-period case

As in Section 3.1.2, it is a dynamic programming problem. We can no longer get an explicit formula for the value function, but we show in Appendix A.1 that the optimal strategy in the n-period case with $W_1 = 0$ and $Z_1 = 0$ is given by

$$u_i^*(Z_i) = \frac{E\left[(Z_i + (n-i+1)y_i)\frac{Z_i + \sum_{j=i}^n y_j}{(X+Z_i+\sum_{j=i}^n y_j)^2}\right]}{(n-i+1)E\left[\frac{(Z_i+\sum_{j=i}^n y_j)^2}{(X+Z_i+\sum_{j=i}^n y_j)^2}\right]}X - \frac{\sum_{j=1}^{i-1} u_j^*}{n-i+1}, \tag{3.3}$$

for any i, $1 \le i \le n-1$, where u_i^* is a function of Z_i, the cumulative market volume up to period $i-1$. Because the y_j are independent, we have

$$u_i^* = \frac{E\left[\left(\frac{1}{n-i+1}\sum_{\ell=1}^{i-1} y_\ell + y_i\right)\frac{\sum_{j=1}^n y_j}{\left(X+\sum_{j=1}^n y_j\right)^2}\middle|\mathcal{F}_{i-1}\right]}{E\left[\frac{\left(\sum_{j=1}^n y_j\right)^2}{\left(X+\sum_{j=1}^n y_j\right)^2}\middle|\mathcal{F}_{i-1}\right]}X - \frac{\sum_{j=1}^{i-1} u_j^*}{n-i+1},$$

where $\mathcal{F}_{i-1}$ is the σ-algebra generated by $y_1, \ldots, y_{i-1}$.

Remarks. 1) As mentioned in Section 3.1.2, it is sensible that the optimal strategy should be positive. In Section 4.3, we use real data to calculate the optimal strategies. In all these examples, the optimal strategies are strictly positive. Generally, with reasonable parameter choices, the optimal strategy generated by (3.3) is positive. However, in some extreme case, the optimal strategy may be negative. For example, consider three trading periods. The expectation of y_1 is large and that of y_2 is almost zero. However, the actual y_1 is very small (almost zero). In this case, u_1^* is still very big. For u_2^*, the first term is almost zero, but the second term is negative, which gives us a negative strategy in period two.

2) We can regain (3.1) from (3.3) if $(y_j)_{j=1,\dots,n}$ are identically distributed. Indeed, we then have that

$$E\left[(Z_i + (n-i+1)\, y_\ell)\, \frac{Z_i + \sum_{j=i}^{n} y_j}{\left(X + Z_i + \sum_{j=i}^{n} y_j\right)^2}\right]$$

takes the same value for all $\ell = i, \dots, n$. Therefore, we obtain

$$E\left[(Z_i + (n-i+1)\, y_i)\, \frac{Z_i + \sum_{j=i}^{n} y_j}{\left(X + Z_i + \sum_{j=i}^{n} y_j\right)^2}\right]$$
$$= E\left[\frac{\left(Z_i + \sum_{j=i}^{n} y_j\right)^2}{\left(X + Z_i + \sum_{j=i}^{n} y_j\right)^2}\right]$$

and then it follows from (3.3) that

$$u_i^* = \frac{X - \sum_{j=1}^{i-1} u_j^*}{n-i+1},$$

which is equivalent to (3.1).

Observing the structure of the optimal strategy, we still find some interesting patterns. The first term mainly depends on the market information. It includes at period i, the past cumulative market volume $\sum_{\ell=1}^{i-1} y_\ell$, the estimation of the future market volumes through $\{y_i, ..., y_n\}$, and our total trading volume X. Although it is not an explicit solution, we can calculate it numerically. The second term depends on the past information regarding our own trading. The numerator is the sum of the volumes we have already traded before our decision, and the denominator is the number of the remaining trading periods.

The first term can be seen as a strategy in period i if we have no trading before that time. However, we have already executed $\sum_{j=1}^{i-1} u_j^*$ before period i. We have to make adjustments to take our previous trading into consideration. It is reasonable to have an optimal strategy which depends on all the information that we need to make our decision.

3.3 Solution for independent Gamma distributed volumes

As mentioned in Section 3.2.1, we are asking ourselves if we can find a distribution type which fits the data well and makes the optimal strategy proportional to the expected relative market volumes. In this section, we show that the Gamma distribution with same scale parameter works. There are three reasons why we choose Gamma distribution with same scale parameter. Firstly, it is reasonable from mathematical and financial perspectives. Since trading volumes are positive, the Gamma distribution meets this first requirement. More importantly, the Gamma distribution is widely used in actuarial science to model the accumulation of losses, which is comparable with the accumulation of trading volume. Secondly, in Chapter 4, we show that it fits the data well through statistical tests. Thirdly, it leads to an explicit solution. In the general case, the solution looks very complicated. Our goal is to find a distribution type to simplify the solution. The Lukacs' proportion-sum independence theorem and the relation between the Gamma distribution and the Beta distribution can help us realize our goal.

Before we start this special case, we recall this useful theorem from probability theory.

Theorem 3.1: ***Lukacs' proportion-sum independence theorem*** *[15]*

If y_1 and y_2 are non-degenerate, independent random variables, then the random variables $y_1 + y_2$ and $\frac{y_1}{y_1+y_2}$ are independently distributed if and only if both y_1 and y_2 have Gamma distributions with same scale parameter.

3.3.1 Two-period case

We assume the morning market volume y_1 and afternoon market volume y_2 satisfy

$$y_1 \sim \Gamma(k_1, \theta), \quad y_2 \sim \Gamma(k_2, \theta).$$

They have the same scale parameter. Since it is a special case of general volumes, we apply the result obtained from Section 3.2.1, namely,

$$u_1^* = \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]} X.$$

The numerator is equal to $E\left[\frac{y_1}{y_1+y_2}\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]$. The first term $\frac{y_1}{y_1+y_2}$ is the fraction, and the second term $\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}$ is a function of the sum $y_1 + y_2$. According to Lukacs' proportion-sum independence theorem, these two terms are independent. Then, we calculate the optimal strategy

$$u_1^* = \frac{E\left[\frac{y_1}{y_1+y_2}\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]} X = E\left[\frac{y_1}{y_1+y_2}\right] X,$$

which is proportional to the relative market volumes. It is exactly what we expected. We also find another nice property of the Gamma distribution with same scale parameter. If $y_1 \sim \Gamma(k_1, \theta), y_2 \sim \Gamma(k_2, \theta)$, then $\frac{y_1}{y_1+y_2} \sim \beta(k_1, k_2)$.

The property of the Beta distribution gives us

$$u_1^* = \frac{k_1}{k_1 + k_2} X.$$

Indeed, it is also proportional to the expected market volume since

$$u_1^* = \frac{k_1}{k_1 + k_2} X = \frac{k_1 \theta}{k_1 \theta + k_2 \theta} X = \frac{E[y_1]}{E[y_1] + E[y_2]} X. \tag{3.4}$$

In period one, we should buy $\frac{k_1}{k_1+k_2} X$ shares, and $\frac{k_2}{k_1+k_2} X$ shares in period two. With this optimal strategy, in practice, we just need to estimate two parameters and directly get the result, which increases the speed.

In the financial industry, one of the most popular and efficient VWAP strategies is the empirical strategy, which means executing the order proportionally to the average of the historical relative market volumes. Since our Gamma strategy is proportional to the expectation of relative market volumes by (3.4), it corresponds to the empirical strategy. However, we will see in the next section that this does not hold for the n-period case with $n > 2$.

3.3.2 n-period case

Since it is a special case of independent general volumes, we directly use the result in Section 3.2.2. By (3.3), the optimal strategy in period $i, 1 \le i \le n-1$, is

$$u_i^* = \frac{E\left[(Z_i + (n-i+1)y_i) \frac{Z_i + \sum_{j=i}^n y_j}{(X + Z_i + \sum_{j=i}^n y_j)^2}\right]}{(n-i+1) E\left[\frac{(Z_i + \sum_{j=i}^n y_j)^2}{(X + Z_i + \sum_{j=i}^n y_j)^2}\right]} X - \frac{\sum_{j=1}^{i-1} u_j^*}{n-i+1}. \tag{3.5}$$

In this n-period case, we also wish to get a sequence of optimal strategies which are proportional to the expected relative market volumes, just like the two-period case. Unfortunately, we are unable to obtain it. As in the two-period case, we rewrite the numerator of the first term as

$$E\left[\frac{Z_i+(n-i+1)y_i}{Z_i+\sum_{j=i}^n y_j}\frac{(Z_i+\sum_{j=i}^n y_j)^2}{(X+Z_i+\sum_{j=i}^n y_j)^2}\right].$$

Although the second term is a function of $\sum_{j=i}^n y_j$, the first term cannot be treated as the fraction of Gamma distributed random variables, due to the existence of market information Z_i in the denominator. That is, we cannot apply Lukacs' theorem in the n-period case. A solution in closed form is not available in this situation. However, we can use MLE (maximum likelihood estimation) to approximate the parameters of the Gamma distribution, then use Monte-Carlo simulation to get the strategy numerically (see Appendix B.1). We will introduce the MLE for this case.

We want to estimate the parameters for the market volumes in all periods. There are n random variables $(y_i)_{i=1,...,n}$ corresponding to periods $i = 1, ..., n$. Since we assume that these variables are independent and Gamma distributed with same scale parameter, we have $n+1$ parameters to approximate, namely, the shape parameters k_i of y_i for $i = 1, ..., n$ and the scale parameter θ of all the random variables.

We use 60 successive days to estimate the parameters for the trading strategy on the next day. For each random variable y_i, there are 60 observations. Let $y_i^{(j)}$ denote the jth observation of y_i. Based on the independence assumption, we get the log-likelihood function

$$\begin{aligned}\ell(k_1,\cdot\cdot\cdot,k_n,\theta) = &\sum_{i=1}^{n}(k_i-1)\sum_{j=1}^{60}\ln\left(y_i^{(j)}\right) - \sum_{i=1}^{n}\sum_{j=1}^{60}\frac{y_i^{(j)}}{\theta} \\ &- \sum_{i=1}^{n}\left(60k_i\ln(\theta) + 60\ln\left(\Gamma(k_i)\right)\right)\end{aligned}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ is the Gamma function. Take the derivative of the log-likelihood function with respect to θ and set it equal to zero

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n}\sum_{j=1}^{60}\frac{y_i^{(j)}}{\theta^2} - \sum_{i=1}^{n}\frac{60k_i}{\theta} = 0,$$

which gives us the estimation of θ, namely,

$$\hat{\theta} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{60} y_i^{(j)}}{60\sum_{i=1}^{n} k_i}.$$

Then we can substitute it back into the log-likelihood function, which yields

$$\begin{aligned}\ell(k_1,\ldots,k_n) = &\sum_{i=1}^{n}(k_i-1)\sum_{j=1}^{60}\ln\left(y_i^{(j)}\right) - 60\sum_{i=i}^{n}k_i \\ &- 60\left(\sum_{i=1}^{n}k_i\right)\ln\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{60} y_i^{(j)}}{60\sum_{i=1}^{n} k_i}\right) - 60\sum_{i=1}^{n}\ln\left(\Gamma(k_i)\right).\end{aligned}$$

To approximate the values of $k_1,\ldots,k_n$, we can take the partial derivatives of $\ell(k_1,\ldots,k_n)$ with respect to $k_1,\ldots,k_n$. It gives us a system of n equations with n variables. It is nonlinear system, but we can solve it numerically. The ith equation is

$$\frac{1}{60}\sum_{j=1}^{60}\ln\left(y_i^{(j)}\right) - \ln\left(\frac{1}{60}\sum_{i=1}^{n}\sum_{j=1}^{60}y_i^{(j)}\right) + \ln\left(\sum_{i=1}^{n}k_i\right) - \psi(k_i) = 0,$$

where $\psi(k_i) = \frac{\Gamma'(k_i)}{\Gamma(k_i)}$ is the digamma function.

After we have estimated these n shape parameters and the scale parameter, we are able to generate samples of the random variables $(y_i)_{i=1,\ldots,n}$. This helps us to calculate numerically the expectations $E\left[\frac{(Z_i+\sum_{j=i}^n y_j)^2}{(X+Z_i+\sum_{j=i}^n y_j)^2}\right]$ and $E\left[(Z_i+(n-i+1)y_i)\frac{Z_i+\sum_{j=i}^n y_j}{(X+Z_i+\sum_{j=i}^n y_j)^2}\right]$ appearing in denominator and numerator of (3.5).

The previous Gamma strategy is dynamic and quite complicated. The cost of collecting market information and doing the calculations on time may not be ignored. We next analyze the gain in performance compared to a static and deterministic strategy. This comparison is useful to decide whether the gain in performance outweighs the costs due to a higher complexity. A static Gamma strategy means that we make all the decisions on $(u_i)_{i=1,\ldots,n}$ at the beginning of the trading day, under the same Gamma assumptions. That is, $(u_i)_{i=1,\ldots,n}$ has to be deterministic. Recall the n period model, where we aim to minimize

$$E\left[\sum_{i=1}^{n}\left(\frac{u_i}{X} - \frac{u_i+y_i}{X+\sum_{j=1}^n y_j}\right)^2\right]$$

subject to $\sum_{i=1}^n u_i = X$.

Although we have n decision variables, we only make $n-1$ decisions at the same time due to the total volume constraint. The last strategy u_n is automatically determined by the other decisions. Since all the decision variables are deterministic, we do not need dynamic programming. To enable the objective function to achieve the minimum, we first replace u_n by $X - \sum_{i=1}^{n-1} u_i$ and then take the partial derivative of the objective function with respect to u_i, for $i = 1, \ldots, n-1$. We set each partial derivative equal to zero, which gives us $n-1$ equations with $n-1$ decision variables. The ith equation is

$$E\left[2\left(\frac{u_i}{X}-\frac{u_i+y_i}{X+\sum_{j=1}^{n}y_j}\right)\left(\frac{1}{X}-\frac{1}{X+\sum_{j=1}^{n}y_j}\right)\right.$$
$$\left.+\,2\left(\frac{X-\sum_{j=1}^{n-1}u_j}{X}-\frac{X-\sum_{j=1}^{n-1}u_j+y_n}{X+\sum_{j=1}^{n}y_j}\right)\left(-\frac{1}{X}+\frac{1}{X+\sum_{j=1}^{n}y_j}\right)\right]=0.$$

By simplifying, we obtain

$$\left(\sum_{j=1}^{n-1}u_j+u_i-X\right)E\left[\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}\right]$$
$$=E\left[\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}\frac{y_i-y_n}{\sum_{j=1}^{n}y_j}\right]X.$$

The right-hand side looks very similar to the two-period Gamma case. Unlike the dynamic Gamma strategy, there is no Z_j in the expectation, which allows us to apply the Lukacs' theorem. The first term in the expectation, $\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}$ is the function of the sum $\sum_{j=1}^{n}y_j$. The second term is the difference of two fractions, $\frac{y_i}{\sum_{j=1}^{n}y_j}$ and $\frac{y_n}{\sum_{j=1}^{n}y_j}$, both of which are independent of the sum. Therefore, these two terms are independent, which yields

$$\left(\sum_{j=1}^{n-1}u_j+u_i-X\right)E\left[\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}\right]$$
$$=E\left[\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}\right]E\left[\frac{y_i-y_n}{\sum_{j=1}^{n}y_j}\right]X.$$

The term $E\left[\frac{(\sum_{j=1}^{n}y_j)^2}{X^2(X+\sum_{j=1}^{n}y_j)^2}\right]$ on both sides can be canceled. The remaining of the right-hand side can be calculated by the property of the Beta distribution, namely,

$$E\left[\frac{y_1 - y_n}{\sum_{j=1}^n y_j}\right] X = E\left[\frac{y_1}{\sum_{j=1}^n y_j}\right] X - E\left[\frac{y_n}{\sum_{j=1}^n y_j}\right] X = \frac{k_i - k_n}{\sum_{j=1}^n k_j} X.$$

The system of equations is

$$\begin{cases} 2u_1 + u_2 + \cdots + u_{n-1} - X = \dfrac{k_1 - k_n}{k_1 + \cdots + k_n} X, \\ \vdots \\ u_1 + \cdots + 2u_i + \cdots + u_{n-1} - X = \dfrac{k_i - k_n}{k_1 + \cdots + k_n} X, \\ \vdots \\ u_1 + \cdots + u_{n-2} + 2u_{n-1} - X = \dfrac{k_{n-1} - k_n}{k_1 + \cdots + k_n} X. \end{cases}$$

This system gives us our optimal strategy in period i, namely

$$u_i = \frac{k_i}{\sum_{j=1}^n k_j} = \frac{k_i \theta}{\sum_{j=1}^n k_j \theta} = \frac{E[y_i]}{\sum_{j=1}^n E[y_j]}.$$

The static Gamma strategy is proportional to the relative market volumes, also to the expected market volumes, which exactly meets our expectation. In other words, the static Gamma strategy corresponds to the n-period empirical strategy, like the Gamma strategy in two periods as presented in Section 3.3.1. Therefore, we do not analyze them separately. We will compare the performance of the dynamic Gamma strategy with that of the empirical strategy in Section 4.3. The result shows that the dynamic Gamma strategy may improve the performance, but is computationally more costly.

Chapter 4

Data fit and analysis

In this chapter, we fit the two-period Gamma based model to real data. Then we introduce a multivariate Gamma distribution to describe the correlation between intraday volumes in the two-period case. We also present how to estimate the parameters of the multivariate Gamma distribution through the method of moments. Finally, we compare the performance of the strategies that we obtain with that of the empirical strategy.

4.1 Statistical test

In this section, we show that the two-period Gamma based model fits the data well. We divide a trading day into two parts of equal length: morning session and afternoon session. We analyze the morning and afternoon market volumes. In the Gamma based model, we assume the morning and afternoon market volumes are independent and Gamma distributed with same scale parameter. Under this assumption, the relative market volumes are Beta distributed. To reduce the influence of seasonal fluctuations on market volumes, we investigate the relative market volumes rather than the absolute volumes.

We analyze data from 120 trading days, starting from April 24, 2012[1]. We consolidate the data to obtain the morning and afternoon volumes. To reflect generality, we choose five exemplary representative companies from three major markets: North America, Europe and Asia. The companies are MSFT (Microsoft) from US, VOD (Vodafone) from UK, PC (Petro China), ICBC (Industrial and Commercial Bank of China) and SINOPEC (China Petroleum and Chemical Corporation) from China. We choose more companies from China for three reasons: firstly, there is a lunch break in the Chinese market, which naturally divides a trading day into two two-hour trading sessions. The situation is exactly as in our two-period model. Secondly, the Chinese market is booming. The total trading volume at the Shanghai Stock Exchange ranks the third all over the world, following NYSE and NASDAQ. We are interested in such an emerging market. Thirdly, algorithmic trading is not very popular in China right now. However, its popularity is expected to increase quickly.

In the two-period case, the relative market volume in the afternoon is equal to one minus the relative market volume in the morning. Therefore, we only test the relative market volume in the morning, since the result for the afternoon is exactly the same. Maximum likelihood estimation is applied to approximate the two shape parameters of the Beta distribution in the first 60 trading days. The second 60 trading days are the test data set. We use a Kolmogorov-Smirnov (K-S) test to obtain p-values (see Appendix B.2). The null hypothesis of the K-S test is that the relative market volumes are Beta distributed. The result of the K-S test is shown in Table 4.1.

From the table, we see that, except for MSFT, the p-values are bigger than

[1]Data is used with the permission of Bloomberg L.P.

	p-value
VOD	0.1510
MSFT	0.0145
PC	0.3493
ICBC	0.6513
SINOPEC	0.3802

Table 4.1: p-values of K-S test

15%. This means the K-S test does not reject our null hypothesis. Of course, one can never prove the null hypothesis by means of a statistical test, and real trading volume is not Gamma distributed. However, the p-values indicate that our assumptions of Gamma distributed volumes is reasonable, in particular, for our prime example of the Chinese market.

We also completed the statistical test in the n-period case by dividing a trading day into 10-minute trading periods. However, the p-values were not as good as in the two-period case. The reason is that, in the two-period case, we test the Beta distribution, which reduces the influence of the fluctuations in the total market volume, which is highly volatile. However, in the n-period case, we test the Gamma distribution, using absolute and not relative volumes. It is to be expected that we cannot obtain a good statistical result in the n-period case. However, in Section 4.3, we also show that our dynamic Gamma strategy works better than the empirical strategy in the chosen stocks, except for MSFT. This means our dynamic Gamma strategy is still meaningful although the underlying model assumption may not fit well in the n-period case.

4.2 Correlated market volumes

In the previous parts, we mainly discussed independent situations. Intuitively, there should be some correlations among the intraday volumes. We are seeking for a correlation structure suitable for the intraday volumes. Based on our previous discussion, a Gamma distribution with correlation structure seems natural. We also notice that in actuarial science, multivariate Gamma distributions are widely used to decompose risk capital. In this section, we introduce the multivariate Gamma distribution to our setting in the two-period case by decomposing market volume into a common part and a unique part. The n-period correlated case is much more difficult. We cannot use the same technique as in Chapter 3 because its application hinged on the independence assumption, which allowed us to calculate the value function by taking the derivative with respect to the control variable.

Let v_0, v_1, v_2 be random variables which are mutually independent and Gamma distributed with shape parameters k_i and scale parameters θ_i. Set

$$y_1 = \frac{\theta_1}{\theta_0} v_0 + v_1, \quad y_2 = \frac{\theta_2}{\theta_0} v_0 + v_2.$$

Applying the property of Gamma distributions, we deduce

$$y_1 \sim \Gamma(k_0 + k_1, \theta_1), \quad y_2 \sim \Gamma(k_0 + k_2, \theta_2).$$

In the two-period case, we only make one decision at the beginning of the trading day. When we make the decision, there is no available updated market information. Both u_1 and u_2 are deterministic. In this situation, even though the morning and afternoon market volumes are correlated, the case is the same as in Section 3.2.1. Recall from (3.2) that the optimal strategy is given by,

$$u_1^* = \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}X, \quad u_2^* = \left(1 - \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}\right)X.$$

Because y_1 and y_2 are not independent, we need to estimate y_1 and y_2 to calculate our strategy numerically. Since y_1 and y_2 are generated by v_0, v_1 and v_2, we should estimate six parameters of these three random variables. However, y_1 and y_2 depend only on five parameters and there is no dependence on θ_0. This means that the value of θ_0 makes no difference, as long as it is positive. In the simulation, we set it equal to one.

We do not have the data of v_0, v_1 and v_2. Instead, we can use the data of market volumes to estimate the parameters. Denote the morning volume on day i by y_1^i and the afternoon volume on day i by y_2^i. Then, we can calculate the first and second moments and the covariance of these two variables

$$\begin{aligned}
m_1^1 &= \frac{y_1^1 + y_1^2 + \cdots + y_1^n}{n}, & m_1^2 &= \frac{(y_1^1)^2 + (y_1^2)^2 + \cdots + (y_1^n)^2}{n}, \\
m_2^1 &= \frac{y_2^1 + y_2^2 + \cdots + y_2^n}{n}, & m_2^2 &= \frac{(y_2^1)^2 + (y_2^2)^2 + \cdots + (y_2^n)^2}{n}, \\
\mathrm{Cov}(y_1, y_2) &= \frac{\sum_{i=1}^n (y_1^i - m_1^1)(y_2^i - m_2^1)}{n}.
\end{aligned}$$

Applying the property of the Gamma distribution, we can link the moments and covariance to the five parameters that we need to estimate by five equations

$$
\begin{aligned}
m_1^1 =& (\hat{k_0}+\hat{k_1})\hat{\theta}_1, & m_1^2 &= (\hat{k_0}+\hat{k_1}+1)(\hat{k_0}+\hat{k_1})(\hat{\theta}_1)^2,\\
m_2^1 =& (\hat{k_0}+\hat{k_2})\hat{\theta}_2, & m_2^2 &= (\hat{k_0}+\hat{k_2}+1)(\hat{k_0}+\hat{k_2})(\hat{\theta}_2)^2,\\
\mathrm{Cov}(y_1,y_2) =& E[y_1y_2]-E[y_1]E[y_2]=\hat{k_0}\hat{\theta}_1\hat{\theta}_2. & &
\end{aligned}
$$

Solving the system gives

$$
\begin{aligned}
\hat{k_0} =& \frac{\mathrm{Cov}(y_1,y_2)}{\hat{\theta}_1\hat{\theta}_2}, & \hat{\theta}_1 &= \frac{(m_1^2-(m_1^1)^2)}{m_1^1},\\
\hat{k_1} =& \frac{(m_1^1)^2}{(m_1^2-(m_1^1)^2)}-\hat{k_0}, & \hat{\theta}_2 &= \frac{(m_2^2-(m_2^1)^2)}{m_2^1},\\
\hat{k_2} =& \frac{(m_2^1)^2}{(m_2^2-(m_2^1)^2)}-\hat{k_0}. & &
\end{aligned}
$$

With these parameters, we can generate samples of the random variables v_0, v_1, and v_2, which indirectly generate samples for the random variables y_1 and y_2. Then, we can calculate our multivariate Gamma strategy numerically. In Section 4.3, we analyze the performance of the multivariate Gamma strategy in the two-period case.

4.3 Performance

So far, we have obtained three strategies in the two-period case: TWAP, Gamma and multivariate Gamma strategies. In practice, the empirical strategy is often used and works very well. The empirical strategy is obtained as follows. When we make the decision on one day, we collect the trading data in the past several days. Then, we take the average of relative market volumes in each period. The trading strategy in each period is just proportional to the average historical relative market volumes. We showed in Section 3.3.1 that the Gamma strategy corresponds to the empirical strategy. In this part, we compare the performance of TWAP, multivariate Gamma strategy and empirical strategy by rolling parameter estimations in a two-period model (see Appendix B.3). Recall the two-period model with optimization criterion

$$\text{Performance} = E\left[\left(\frac{u_1}{X} - \frac{u_1 + y_1}{X + y_1 + y_2}\right)^2 + \left(\frac{X - u_1}{X} - \frac{X - u_1 + y_2}{X + y_1 + y_2}\right)^2\right].$$

We compare the average performance of each strategy. Here, we use the 120 days data to compute the performance of each strategy from day 61 to day 120. Then we take the average of performance in these 60 days.

To calculate estimations of the parameters for the multivariate Gamma and empirical strategies, we use a rolling method. For multivariate Gamma strategy, we use 60 successive days to estimate the distribution parameters. For the empirical strategy, we take the average of relative market volumes, in the 60 successive days. These parameters and average are used to calculate strategies on day 61. Then, we put the strategies into the model with the real morning and afternoon market volumes on day 61, and get the performance on day 61 for each strategy. Then, moving one day forward, we repeat the same work to get the performance on day 62. We repeat this from day 61 to day 120. Finally, we take the average of the performance over the 60 trading days.

For the TWAP strategy, it is easier. We do not need to estimate the parameters or take average. Since the strategy is fixed, we just execute one half in the morning and the other in the afternoon. We can directly use the strategy with real data from day 61 to day 120 to get the performance, then take the average over the 60 trading days.

Since the trader's total volume X has an effect on the performance, we compare the average performance for different values of X. We consider 20 cases. The minimum value of X is 5% of the average total market volume. We increase it by 5% until 100% of the average total market volume.

In Figures 4.1–4.5, we plot five graphs for the five companies. Each graph includes three curves, which represent the average performance of these three strategies. We find several interesting patterns in the graphs:

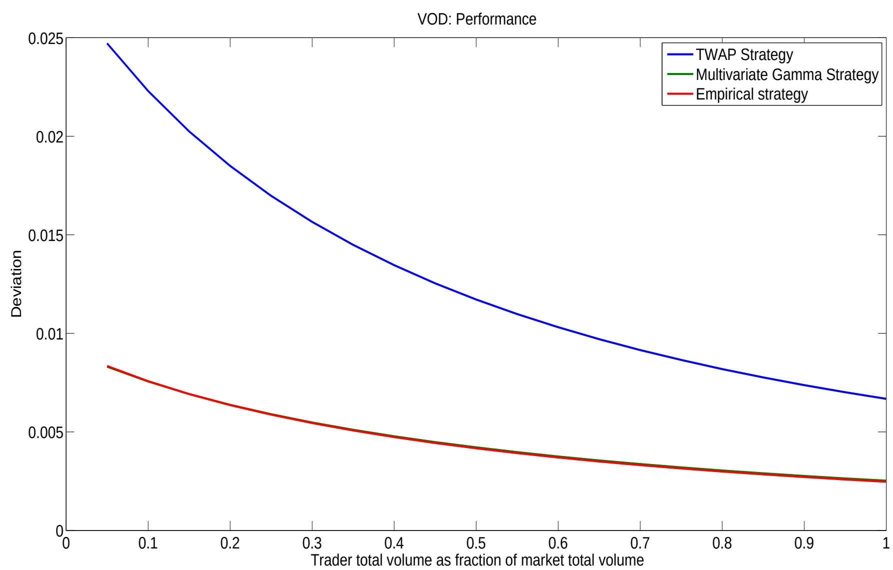

Figure 4.1: VOD: performance in the two-period model

1. The multivariate Gamma strategy performs nearly the same as the empirical strategy, which results from assuming independent and Gamma distributed volume. The reason is that the two-period case is static. Therefore, information about the correlation cannot be used beneficially because we are not allowed to update the strategy during the day. Both

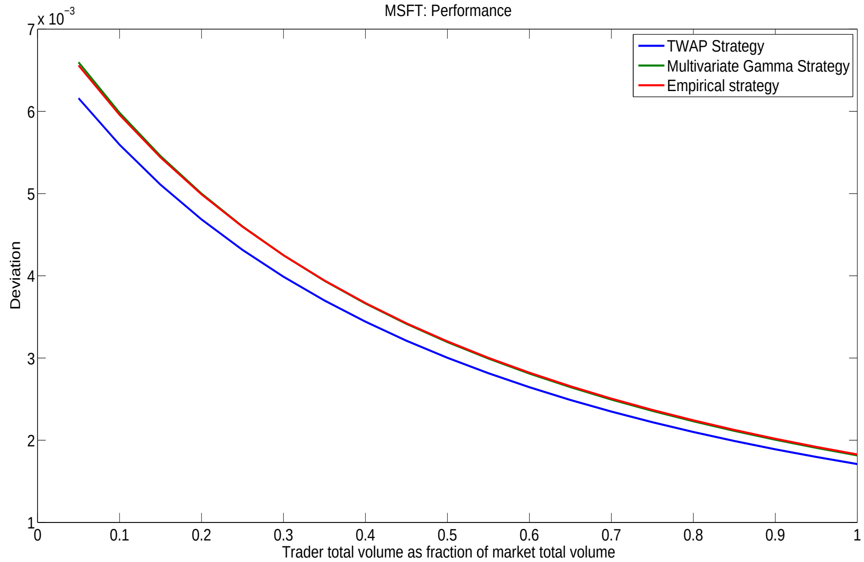

Figure 4.2: MSFT: performance in the two-period model

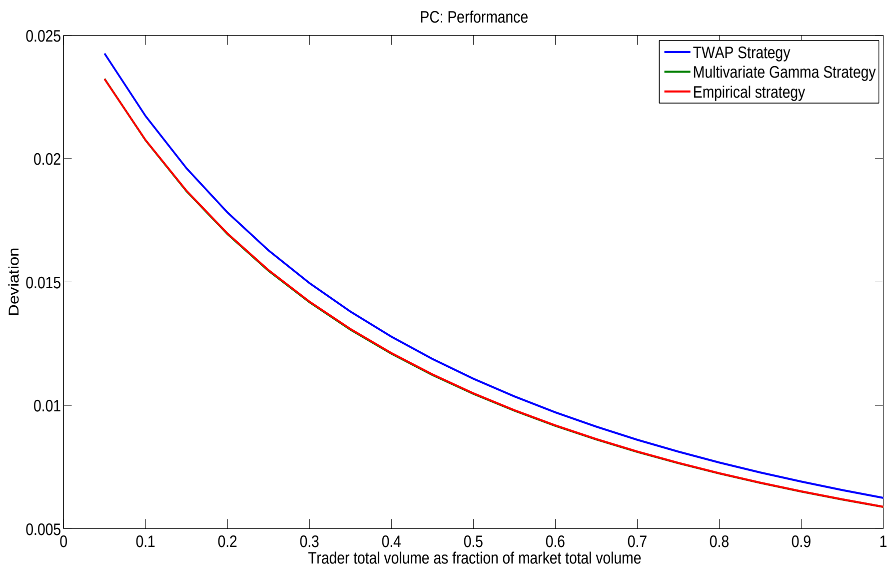

Figure 4.3: PC: performance in the two-period model

strategies perform better than the TWAP strategy, except for MSFT.

2. As X is getting bigger, the three average performances are getting lower

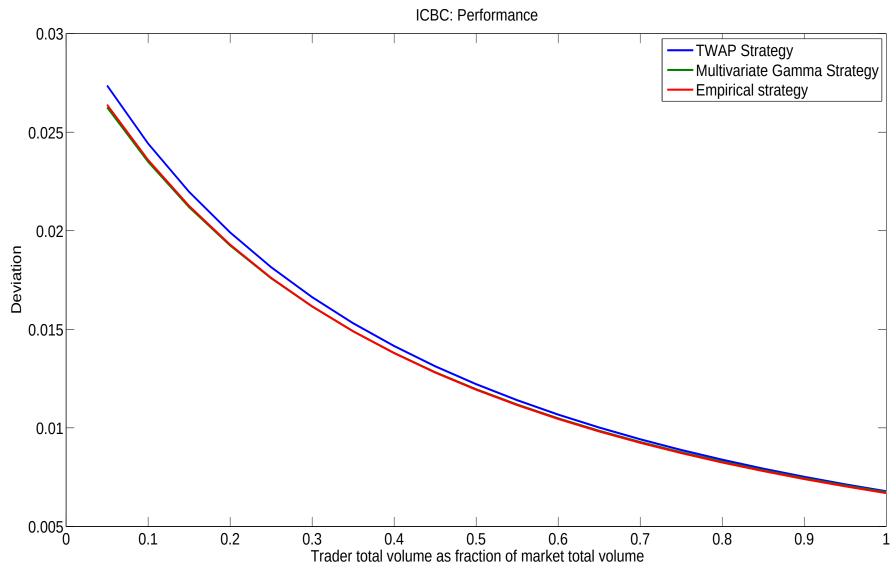

Figure 4.4: ICBC: performance in the two-period model

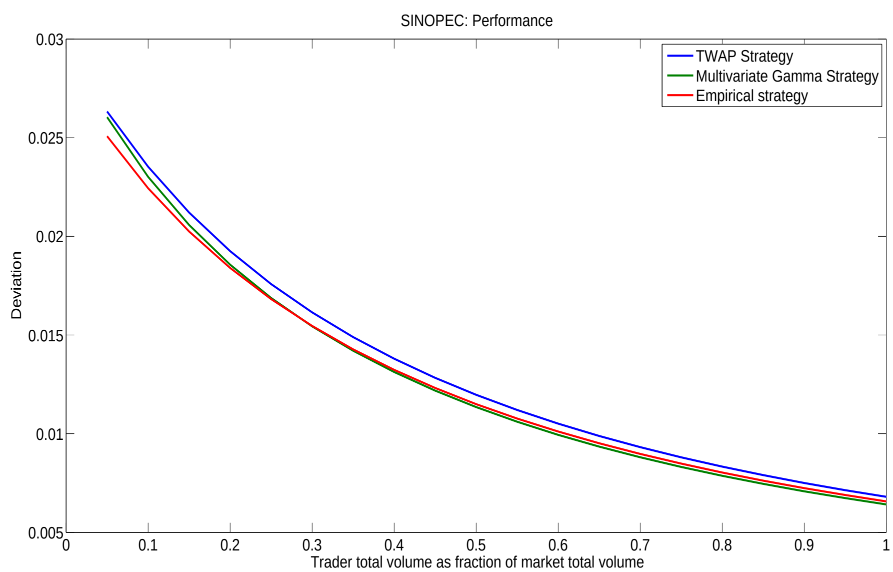

Figure 4.5: SINOPEC: performance in the two-period model

and closer. Observe that the performance here is only a criterion for the deviation of our relative volume to that of the market, and it does not say anything about the price impact. The bigger our order, the easier to follow the market pattern, since our own volume counts a large parts of the market.

3. For VOD, the multivariate Gamma strategy and the empirical strategy perform much better than the TWAP strategy. This is due to lower morning market volume in the London market. In the afternoon, the traders from the US also get into the market, which leads to higher market volume in the afternoon.

4. For MSFT, TWAP is the best. The relative market volume of MSFT in the morning is approximately the same as that in the afternoon. In this case, splitting our order evenly is reasonable. Moreover, in the statistical part, the p-value for MSFT is very low, which means that using historical data to estimate future market volumes may not work well.

5. For SINOPEC, at the left part, the empirical strategy beats the multivariate Gamma strategy. Intuitively, the multivariate Gamma strategy should be better. Here, the reason may be that there is more fluctuation in the total market volume of SINOPEC. The empirical strategy uses the average of the historical market relative volumes, which reduces the influence of fluctuations in the total market volumes. On the contrary, the absolute market volume is used to estimate the parameters of the multivariate Gamma strategy. The estimation error may negatively affect its performance.

After investigating the two-period case, we compare the performance of the dynamic Gamma strategy with the empirical strategy, which corresponds to

	Number of days	Improvement
VOD	33/60	1.51%
MSFT	23/60	-16.98%
PC	39/60	3.18%
ICBC	46/60	2.88%
SINOPEC	44/60	4.20%

Table 4.2: Performance of dynamic Gamma strategy and empirical strategy in the n-period case

the static Gamma strategy in the n-period case (see Appendix B.4). Here, we divide a trading day into 10-minute trading periods, and fix the trader's total volume by setting it to 20% of the total market volume. We compare two criteria: the number of days when the dynamic Gamma strategy performs better and the improvement[2] in the average performance.

As shown in Table 4.2, except for MSFT, the dynamic Gamma strategy performs better, compared with the empirical strategy. Especially, in the exemplary stocks from the China market, the dynamic Gamma strategy performs better in approximately 75% of the 60 days. It gives an improvement between 1.5% and 4.2% in the average performance. Although the improvement is not that big, it can be important considering the large order size. This indicates that, although the dynamic Gamma strategy is more complex and time consuming to implement, it may be worth it due to the better performance. However, this is just a snapshot based on a few stocks and over a relatively short trading period. More research would need to be done to confirm or revise these findings; see Chapter 6.

[2]Improvement=$1 - \frac{\text{Average performance of dynamic Gamma strategy}}{\text{Average performance of empirical strategy}}$

Chapter 5

Incorporating linear price impact

So far, we have considered the minimization of expected deviations between our and the market relative volumes in any period. Finding such a strategy whose relative volume is close to that of the market automatically reduces the price impact of our order. Therefore, it is not really necessary to incorporate a price impact separately in our problem formulation. Still, we introduce in this section an additional linear (temporary) price impact in the optimization criterion. This allows us to study how such a price impact affects the optimal strategies.

We consider a buy order. Sell orders can be treated analogously by changing the signs. Price impact eventually increases the cost of our order. Assume the stock prices without price impact denoted by $(p_i)_{i=1,\ldots,n}$ is a martingale. We denote by the coefficient of the linear (temporary) price impact. The actual price per share we pay in period i is given by $p_i + \kappa u_i$. The total costs of our order are

$$E\left[\sum_{i=1}^{n}(p_i+\kappa u_i)u_i\right]=E\left[\sum_{i=1}^{n}p_iu_i\right]+E\left[\sum_{i=1}^{n}\kappa u_i^2\right],$$

which consists of two parts. The first part is the cost without price impact, and the second one is the cost originated from the price impact. Applying the martingale property of $(p_i)_{i=1,\dots,n}$ and using that $(u_i)_{i=1,\dots,n}$ is predictable, we can deduce that

$$E\left[\sum_{i=1}^{n}p_iu_i\right]=E\left[\sum_{i=1}^{n-1}(p_i-p_n)u_i+p_nX\right]=p_0X,$$

which is a constant independent of our trading strategy. When we compare different strategies, we can just consider the second part, $E\left[\sum_{i=1}^{n}\kappa u_i^2\right]$.

Now, we want to add the price impact term to our original model. However, the value we get from the original model is a deviation of relative volumes, but the additional term represents costs. We cannot directly combine them. We first set up a linear mapping from the volume deviation to cost. Then, we have a new model with price impact. We aim to minimize

$$E\left[\lambda\sum_{i=1}^{n}\left(\frac{u_i}{X}-\frac{u_i+y_i}{X+\sum_{j=1}^{n}y_j}\right)^2+\sum_{i=1}^{n}\kappa u_i^2\right]$$

over $(u_i)_{i=1,\dots,n}$, subject to $\sum_{i=1}^{n}u_i=X$, where $\lambda>0$ is the coefficient of the linear mapping.

To simplify the model, we divide both parts by λ. The objective function is equivalent to

$$E\left[\sum_{i=1}^{n}\left(\frac{u_i}{X}-\frac{u_i+y_i}{X+\sum_{j=1}^{n}y_j}\right)^2+\sum_{i=1}^{n}\tilde{\kappa}u_i^2\right],$$

where $\tilde{\kappa}=\frac{\kappa}{\lambda}$.

The value of $\tilde{\kappa}$ depends on our optimization preferences through λ and varies for different stocks through κ. Intuitively, if $\tilde{\kappa}$ is large and the second term dominates the first, we will split our order evenly over the time. As in Chapter 3, we solve this problem in three cases.

5.1 Solution for general i.i.d. volumes

In the case of i.i.d. volume, the optimal strategy is still the TWAP strategy with

$$u_1^*=u_2^*=\cdots=u_n^*=\frac{X}{n}. \tag{5.1}$$

To see this, we note that

$$\min_{u_i} E\left[\sum_{i=1}^{n}\left(\frac{u_i}{X}-\frac{u_i+y_i}{X+\sum_{j=1}^{n}y_j}\right)^2+\sum_{i=1}^{n}\tilde{\kappa}u_i^2\right] \tag{5.2}$$

$$\geq \min_{u_i} E\left[\sum_{i=1}^{n}\left(\frac{u_i}{X}-\frac{u_i+y_i}{X+\sum_{j=1}^{n}y_j}\right)^2\right]+\tilde{\kappa}\min_{u_i} E\left[\sum_{i=1}^{n}u_i^2\right], \tag{5.3}$$

where the minima are subject to $\sum_{i=1}^{n}u_i=X$.

Jensen's inequality yields

$$\frac{1}{n}\sum_{i=1}^{n} u_i^2 \geq \frac{1}{n^2}\left(\sum_{i=1}^{n} u_i\right)^2 = \frac{1}{n^2}X^2$$

with equality for $(u_i^*)_{i=1,\dots,n}$ given in (5.1). By Section 3.1.2, the two optimization problems in (5.3) have the same minimizer, which thus is also the minimization of the original problem. In the case of i.i.d. volumes, we can minimize simultaneously the price impact and the volume deviation by slitting our order evenly.

5.2 Solution for general independent volumes

We start this section with the two-period case. The goal is to minimize

$$E\left[\left(\frac{u_1}{X} - \frac{u_1 + y_1}{X + y_1 + y_2}\right)^2 + \tilde{\kappa}u_1^2 + \left(\frac{X - u_1}{X} - \frac{X - u_1 + y_2}{X + y_1 + y_2}\right)^2 + \tilde{\kappa}(X - u_1)^2\right].$$

Taking the derivative and setting it equal to zero, we obtain

$$E\left[\left(\left(\frac{1}{X} - \frac{1}{X + y_1 + y_2}\right)^2 + \tilde{\kappa}\right)(2u_1 - X)\right] + E\left[\frac{y_2^2 - y_1^2}{X(X + y_1 + y_2)^2}\right] = 0.$$

Here, y_1 and y_2 are not i.i.d. The second term is not equal to zero. We just keep it and solve for the optimal strategy

$$u_1^* = \frac{X}{2} + \frac{E\left[\frac{y_1^2 - y_2^2}{X^2(X+y_1+y_2)^2}\right]}{2E\left[\frac{(y_1+y_2)^2}{X^2(X+y_1+y_2)^2} + \tilde{\kappa}\right]}X.$$

The optimal strategy can be decomposed into two parts. The first term is a TWAP strategy. The second term is an adjustment, which mainly depends on the deviation between y_1 and y_2, the coefficient of linear price impact, and our total volume.

If $\tilde{\kappa} \to 0$, we have

$$u_1^* \to \frac{X}{2} + \frac{E\left[\frac{y_1^2 - y_2^2}{X^2(X+y_1+y_2)^2}\right]}{2E\left[\frac{(y_1+y_2)^2}{X^2(X+y_1+y_2)^2}\right]}X = \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]}X.$$

It is exactly the same as the solution in Section 3.2.1, which is the special case $\tilde{\kappa} = 0$.

If $\tilde{\kappa} \to \infty$, the second term vanishes, which leads to

$$u_1^* \to \frac{X}{2}. \tag{5.4}$$

This means, if we have a huge linear price impact on the market, our optimal strategy is a TWAP strategy, which helps us to reduce the price impact.

After analyzing the two-period case, we go to the n-period case. By use dynamic programming as in Section 3.2.2 (see Appendix A.2), we obtain the optimal strategy in period $i, 1 \le i \le n-1$,

$$u_i^* = \frac{X - \sum_{j=1}^{i-1} u_j^*}{n+1-i} + \frac{E\left[\frac{\left(\sum_{j=i}^n y_j + Z_i\right)\left((n-i)y_i - \sum_{j=i+1}^n y_j\right)}{X^2\left(X + \sum_{j=i}^n y_j + Z_i\right)^2}\right]}{(n+1-i)E\left[\frac{\left(\sum_{j=i}^n y_j + Z_i\right)^2}{X^2\left(X + \sum_{j=i}^n y_j + Z_i\right)^2} + \tilde{\kappa}\right]}X.$$

This optimal strategy can also been decomposed into two parts. Without the second term, the first term indeed is a TWAP strategy. The second term is an adjustment based on the difference among the market volumes in different periods and the linear price impact. It is very similar to Corollary 3.4 of Frei and Westray [7], which shows that the optimal strategy in their continuous-time setting can be decomposed into two parts: TWAP and an adjustment due to the randomness and jumps of the relative market volumes.

5.3 Solution for independent Gamma distributed volumes

Since it is a special case of Section 5.2. We directly apply the result of that part. Recall the optimal strategy in the two-period case is

$$u_1^* = \frac{X}{2} + \frac{E\left[\frac{y_1^2 - y_2^2}{X^2(X+y_1+y_2)^2}\right]}{2E\left[\frac{(y_1+y_2)^2}{X^2(X+y_1+y_2)^2} + \tilde{\kappa}\right]} X.$$

We rewrite it as

$$u_1^* = \frac{E\left[\frac{y_1(y_1+y_2)}{(X+y_1+y_2)^2}\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2} X + \frac{\tilde{\kappa}X^3}{2E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + 2\tilde{\kappa}X^2}.$$

Apply Lukacs' theorem to the first term, which yields

$$\begin{aligned} u_1^* = & \frac{\frac{k_1}{k_1+k_2} X \left(E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2 \right)}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2} \\ & - \frac{\frac{k_1}{k_1+k_2}\tilde{\kappa}X^3}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2} + \frac{\tilde{\kappa}X^3}{2\left(E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2\right)}. \end{aligned}$$

After simplifying, we get the optimal strategy

$$u_1^* = \frac{k_1}{k_1+k_2} X + \frac{\frac{k_2-k_1}{k_1+k_2}\tilde{\kappa}X^2}{2\left(E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right] + \tilde{\kappa}X^2\right)} X.$$

The optimal strategy can still be decomposed into two parts. The first term is the empirical strategy. The second part is an adjustment mainly based on the deviation between y_1 and y_2, the coefficient of the linear price impact $\tilde{\kappa}$ and our total volume X.

If $\tilde{\kappa} \to 0$, we have

$$u_1^* \to \frac{k_1}{k_1 + k_2} X.$$

This is the empirical strategy, which corresponds to the optimal strategy in Section 3.3.1 for the special case $\tilde{\kappa} = 0$.

If $\tilde{\kappa} \to \infty$, $\tilde{\kappa} X^2$ dominates $E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]$. It give us, as in (5.4),

$$u_1^* \to \frac{k_1}{k_1 + k_2} X + \frac{\frac{k_2 - k_1}{k_1 + k_2} X^3}{2X^2} = \frac{X}{2}.$$

For the n-period case, we cannot apply Lukacs' theorem and thus there is no fully explicit solution. We still can calculate it numerically as in Section 3.3.2.

Chapter 6

Future work

In this thesis, we minimize the expected deviations in relative trading volumes. A better estimation of the market relative volume contributes to an improvement of the trading strategies. In a next step, we can extend the current work in several directions.

1. For the intraday volumes, we mainly discuss the independent situations, and introduce a multivariate Gamma distribution to describe correlation structure in the two-period case. We can extend the multivariate Gamma distribution to n-period case by applying copula. However, the difficulty of the dynamic programming is significantly increased. In this case, our control variable is no longer independent of the filtration, which means we cannot directly take the derivative of the value function with respect to the control variable.

2. In the data fit part, we find that our model does not work well for MSFT, even though other companies show good data fit. We can also investigate more companies from the US market to see if our model is suitable for

other companies from the US market. Moreover, we can analyze the performance deviations between the dynamic Gamma strategy and the empirical strategy for more companies and more trading days to see whether we obtain a statistically significant improvement.

3. We assume that the market volumes on different days are independent and volumes in corresponding periods are identically distribution. This enables us to estimate the future volumes by historical data. However, we also observe seasonal patterns in the market volumes. To better estimate the market volumes, we could incorporate the seasonal patterns. To decide how many days should be used to estimate the parameters is also very interesting.

4. As the seminal Black-Scholes model, which can be derived from the discrete-time model, we could also study the behavior of the current model in the limit and investigate the continuous case. However, our Gamma strategy is not in closed form.

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Appendix A

Auxiliary calculations

A.1 n-period case without price impact

In period n, we execute all the remaining volume. We start the dynamic programming at period $n-1$. The value function at the beginning of this period is

$$V(n-1, W_{n-1}, Z_{n-1}) = \min_{u_{n-1}} E\left[\left(\frac{u_{n-1}}{X} - \frac{u_{n-1}+y_{n-1}}{X+y_{n-1}+y_n+Z_{n-1}}\right)^2 + \left(\frac{X-W_{n-1}-u_{n-1}}{X} - \frac{X-W_{n-1}-u_{n-1}+y_n}{X+y_{n-1}+y_n+Z_{n-1}}\right)^2\right].$$

By using the same method as in Section 3.2.1, we can get the optimal strategy in period $n-1$,

$$\begin{aligned} u^*_{n-1} &= -\frac{W_{n-1}}{2} + \frac{E\left[(Z_{n-1}+2y_{n-1})\frac{Z_{n-1}+y_{n-1}+y_n}{(X+Z_{n-1}+y_{n-1}+y_n)^2}\right]}{2E\left[\frac{(Z_{n-1}+y_{n-1}+y_n)^2}{(X+Z_{n-1}+y_{n-1}+y_n)^2}\right]}X \\ &= -\frac{W_{n-1}}{2} + b(n-1, Z_{n-1}), \end{aligned}$$

where we use the notation

$$b(n-j,Z)=\frac{E\left[(Z+(j+1)y_{n-j})\frac{Z+\sum\limits_{\ell=n-j}^{n}y_\ell}{\left(X+Z+\sum\limits_{\ell=n-j}^{n}y_\ell\right)^2}\right]}{(j+1)E\left[\frac{\left(Z+\sum\limits_{\ell=n-j}^{n}y_\ell\right)^2}{\left(X+Z+\sum\limits_{\ell=n-j}^{n}y_\ell\right)^2}\right]}X,$$

for $Z\geq 0$ and $0\leq j<n$.

Then, we go back to period $n-2$. From the law of motion, we know

$$W_{n-1}=W_{n-2}+u_{n-2},\quad Z_{n-1}=Z_{n-2}+y_{n-2}.$$

We use u_{n-2}, W_{n-2}, Z_{n-2} and y_{n-2} to express u^*_{n-1} and u^*_n. That is,

$$\begin{aligned}u^*_{n-1}&=-\frac{W_{n-2}+u_{n-2}}{2}+b(n-1,Z_{n-2}+y_{n-2}),\\ u^*_n&=X-\frac{W_{n-2}+u_{n-2}}{2}-b(n-1,Z_{n-2}+y_{n-2}).\end{aligned}$$

The value function at the beginning of period $n-2$ equals

$$\begin{aligned}&V(n-2,W_{n-2},Z_{n-2})=\min_{u_{n-2}}f(u_{n-2},n-2,W_{n-2},Z_{n-2}),\\ &f(u_{n-2},n-2,W_{n-2},Z_{n-2})=E\left[\left(\frac{u_{n-2}}{X}-\frac{u_{n-2}+y_{n-2}}{X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n}\right)^2\right.\\ &+\left(\frac{\left(-\frac{W_{n-2}+u_{n-2}}{2}+B\right)(Z_{n-2}+y_{n-2}+y_{n-1}+y_n)-y_{n-1}X}{X(X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n)}\right)^2\\ &\left.+\left(\frac{\left(X-\frac{W_{n-2}+u_{n-2}}{2}+B\right)(Z_{n-2}+y_{n-2}+y_{n-1}+y_n)-y_nX}{X(X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n)}\right)^2\right],\end{aligned}$$

where $B = b(n-1, Z_{n-2}+y_{n-2})$. Here, u_{n-2} is the only control variable. Moreover, it is deterministic. We take the derivative of $f(u_{n-2}, n-2, W_{n-2}, Z_{n-2})$ with respect to u_{n-2} and set it equal to zero. This gives,

$$\frac{\partial f(u_{n-2}, n-2, W_{n-2}, Z_{n-2})}{\partial u_{n-2}} = E\left[\frac{Z_{n-2}+y_{n-2}+y_{n-1}+y_n}{X^2\left(X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n\right)^2}\right.$$
$$\left.\left(\Big(Z_{n-2}+y_{n-2}+y_{n-1}+y_n\Big)\Big(3u_{n-2}+W_{n-2}\Big) - XZ_{n-2} - 3Xy_{n-2}\right)\right] = 0.$$

It is surprising that $b(n-1, Z_{n-2}+y_{n-2})$ has disappeared after taking the derivative. The reason is that $b(n-1, Z_{n-2}+y_{n-2})$ is independent of u_{n-2} and some signs of $b(n-1, Z_{n-2}+y_{n-2})$ are positive, and some are negative. They are canceled in the calculation. Then, we can get the optimal strategy in period $n-2$, namely

$$u^*_{n-2} = -\frac{W_{n-2}}{3} + b(n-2, Z_{n-2}),$$

which has the same form as u^*_{n-1}. The optimal strategy is not a closed-form solution. Hence, the value function is not available in closed form. We cannot do the same backward induction as in Section 3.1.2. However, when observing u^*_{n-1} and u^*_{n-2}, we are able to find the same structure. Based on this observation, we use as induction hypothesis that the optimal strategy in period $n-i$, $0 < i < n-1$, is

$$u^*_{n-i} = -\frac{W_{n-i}}{i+1} + b(n-i, Z_{n-i}).$$

With this assumption, we consider the value function in period $n-i-1$, which contains $u_{n-i-1}, u^*_{n-i}, \ldots, u^*_n$. To minimize the value function, we try to avoid

$u^*_{n-i}, \dots, u^*_n$. Therefore, we use u_{n-i-1} to express them,

$$
\begin{aligned}
u^*_{n-i} =& -\frac{u_{n-i-1} + W_{n-i-1}}{i+1} + b(n-i, Z_{n-i-1} + y_{n-i-1}) \\
&\vdots \\
u^*_{n-i+j} =& -\frac{u_{n-i-1} + W_{n-i-1}}{i+1} + b\left(n-i+j, Z_{n-i-1} + \sum_{\ell=n-i-1}^{n-i-1+j} y_\ell\right) \\
& - \sum_{k=1}^{j} \frac{1}{i+1-k} b\left(n-i-1+k, Z_{n-i-1} + \sum_{\ell=n-i-1}^{n-i-2+k} y_\ell\right) \\
&\vdots \\
u^*_n =& X - \frac{u_{n-i-1} + W_{n-i-1}}{i+1} \\
& - \sum_{k=1}^{i} \frac{1}{i+1-k} b\left(n-i-1+k, Z_{n-i-1} + \sum_{\ell=n-i-1}^{n-i-2+k} y_\ell\right)
\end{aligned}
$$

Although it looks complicated, $b\left(n-i-1+k, Z_{n-i-1} + \sum_{\ell=n-i-1}^{n-i-2+k} y_\ell\right)$ does not depend on u_{n-i-1}. When we take the derivative of $f(u_{n-i-1}, n-i-1, W_{n-i-1}, Z_{n-i-1})$ with respect to u_{n-i-1}, $b\left(n-i-1+k, Z_{n-i-1} + \sum_{\ell=n-i-1}^{n-i-2+k} y_\ell\right)$ makes no difference. It gives

$$
\begin{aligned}
E &\left[\frac{\left(Z_\ell + \sum_{j=\ell}^{n} y_j\right)((i+2)u_\ell + W_\ell - X) - X\left((i+1)y_\ell - \sum_{j=\ell+1}^{n} y_j\right)}{X\left(X + Z_\ell + \sum_{j=\ell}^{n} y_j\right)} \right. \\
& \left. \times \frac{Z_\ell + \sum_{j=\ell}^{n} y_j}{X\left(X + Z_\ell + \sum_{j=\ell}^{n} y_j\right)} \right] = 0,
\end{aligned}
$$

where $\ell = n-i-1$. By solving this equation, we get the optimal strategy in period $n-i-1$

$$u^*_{n-i-1} = -\frac{W_{n-i-1}}{i+2} + b(n-i-1, Z_{n-i-1}).$$

It confirms our induction hypothesis. Then going back to period one with $W_1 = 0$ and $Z_1 = 0$, we can calculate

$$u^*_1 = \frac{E\left[\frac{\sum_{j=1}^n y_j}{(X+\sum_{j=1}^n y_j)^2} y_1\right]}{E\left[\frac{(\sum_{j=1}^n y_j)^2}{(X+\sum_{j=1}^n y_j)^2}\right]} X.$$

If $n = 2$ (two-period case), we get $u^*_1 = \frac{E\left[\frac{y_1+y_2}{(X+y_1+y_2)^2} y_1\right]}{E\left[\frac{(y_1+y_2)^2}{(X+y_1+y_2)^2}\right]} X$, which is consistent with the result that we obtain in Section 3.2.1. By substituting, we obtain (3.3).

A.2 n-period case with price impact

We start the dynamic programming in period $n-1$. The value function is,

$$V(n-1, W_{n-1}, Z_{n-1}) = \min_{u_{n-1}} \left[\left(\frac{u_{n-1}}{X} - \frac{u_{n-1} + y_{n-1}}{X + y_{n-1} + y_n + Z_{n-1}} \right)^2 + \tilde{\kappa} u^2_{n-1} \right.$$
$$\left. + \left(\frac{X - W_{n-1} - u_{n-1}}{X} - \frac{X - W_{n-1} - u_{n-1} + y_n}{X + y_{n-1} + y_n + Z_{n-1}} \right)^2 + \tilde{\kappa}(X - W_{n-1} - u_{n-1})^2 \right].$$

Here, the situation is similar to the two-period case, except for the existence of updated information W_{n-1} and Z_{n-1}. By using the same method as in Section 5.2, we obtain the optimal strategy in period $n-1$

$$u^*_{n-1} = \frac{X - W_{n-1}}{2} + \frac{E\left[\frac{(y_{n-1}+y_n+Z_{n-1})(y_{n-1}-y_n)}{X^2(X+y_{n-1}+y_n+Z_{n-1})^2}\right]}{2E\left[\frac{(y_{n-1}+y_n+Z_{n-1})^2}{X^2(X+y_{n-1}+y_n+Z_{n-1})^2} + \tilde{\kappa}\right]} X.$$

The first term is a TWAP strategy, and second term is an adjustment based on the deviation between future volumes and the coefficient of the linear price impact. To continue the dynamic programming, we rewrite the optimal strategy, which helps us simplify the process

$$u_{n-1}^* = -\frac{W_{n-1}}{2} + \frac{E\left[\frac{(y_{n-1}+y_n+Z_{n-1})(2y_{n-1}+Z_{n-1})}{X^2(X+y_{n-1}+y_n+Z_{n-1})^2} + \tilde{\kappa}\right]}{2E\left[\frac{(y_{n-1}+y_n+Z_{n-1})^2}{X^2(X+y_{n-1}+y_n+Z_{n-1})^2} + \tilde{\kappa}\right]} X.$$

The second term is very complicated. However, it is not dependent on our decision variables. When we take the derivative of the value function with respect to the decision variable, it keeps the same. We use the abbreviation

$$b(n-j, Z) = \frac{E\left[(Z+(j+1)y_{n-j})\frac{Z+\sum\limits_{\ell=n-j}^{n} y_\ell}{X^2\left(X+Z+\sum\limits_{\ell=n-j}^{n} y_\ell\right)^2} + \tilde{\kappa}\right]}{(j+1)E\left[\frac{\left(Z+\sum\limits_{\ell=n-j}^{n} y_\ell\right)^2}{X^2\left(X+Z+\sum\limits_{\ell=n-j}^{n} y_\ell\right)^2} + \tilde{\kappa}\right]} X,$$

for $Z \geq 0$ and $0 \leq j < n$. The optimal strategy in period $n-1$ and period n can be rewritten as

$$u_{n-1}^* = -\frac{W_{n-1}}{2} + b(n-1, Z_{n-1}), \quad u_n^* = X - \frac{W_{n-1}}{2} - b(n-1, Z_{n-1}).$$

Then, we go back to period $n-2$. We use $u_{n-2}, W_{n-2}, y_{n-2}$ and Z_{n-2} to write

$$u_{n-1}^* = -\frac{W_{n-2}+u_{n-2}}{2} + b(n-1, Z_{n-2}+y_{n-2}),$$

$$u_n^* = X - \frac{W_{n-2}+u_{n-2}}{2} - b(n-1, Z_{n-2}+y_{n-2}).$$

By plugging in, we get the value function in period $n-2$,

$$V(n-2, W_{n-2}, Z_{n-2}) = \min_{u_{n-2}} f(u_{n-2}, n-2, W_{n-2}, Z_{n-2}),$$

$$f(u_{n-2}, n-2, W_{n-2}, Z_{n-2}) = E\Bigg[\left(\frac{u_{n-2}}{X} - \frac{u_{n-2}+y_{n-2}}{X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n}\right)^2$$

$$+\left(\frac{\left(-\frac{W_{n-2}+u_{n-2}}{2}+B\right)(Z_{n-2}+y_{n-2}+y_{n-1}+y_n) - y_{n-1}X}{X(X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n)}\right)^2$$

$$+\left(\frac{\left(X-\frac{W_{n-2}+u_{n-2}}{2}+B\right)(Z_{n-2}+y_{n-2}+y_{n-1}+y_n) - y_nX}{X(X+Z_{n-2}+y_{n-2}+y_{n-1}+y_n)}\right)^2$$

$$+\tilde{\kappa}u_{n-2}^2 + \tilde{\kappa}\left(-\frac{W_{n-2}+u_{n-2}}{2}+B\right)^2$$

$$+\tilde{\kappa}\left(X-\frac{W_{n-2}+u_{n-2}}{2}-B\right)^2\Bigg],$$

where $B = b(n-1, Z_{n-2}+y_{n-2})$. We take the derivative of $f(u_{n-2}, n-2, W_{n-2}, Z_{n-2})$ with respect to u_{n-2} and set it equal to zero

$$E\Bigg[(3u_{n-2}+W_{n-2}-X)\left(\frac{(y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}{X^2(X+y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}+\tilde{\kappa}\right)$$
$$-\frac{(y_{n-2}+y_{n-1}+y_n+Z_{n-2})(2y_{n-2}-y_{n-1}-y_n)}{X(X+y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}\Bigg]=0,$$

which gives us the optimal strategy in period $n-2$, namely

$$u_{n-2}^* = \frac{X-W_{n-2}}{3} + \frac{E\left[\frac{(y_{n-2}+y_{n-1}+y_n+Z_{n-2})(2y_{n-2}-y_{n-1}-y_n)}{X^2(X+y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}\right]}{3E\left[\frac{(y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}{X^2(X+y_{n-2}+y_{n-1}+y_n+Z_{n-2})^2}+\tilde{\kappa}\right]}X.$$

As the strategy in period $n-1$, we rewrite the strategy to continue the backward induction,

$$u_{n-2}^* = -\frac{W_{n-2}}{3} + b(n-2, Z_{n-2})$$

Since the optimal strategy is not in closed form, even though we put it back to the value function, we are unable to obtain a value function in closed form. However, we still can continue our backward induction for the form of the optimal strategy. Based on our observation that the optimal strategy in period $n-2$ and that in period $n-1$ have the same structure, we consider the induction hypothesis that, for any $i, 0 < i < n-1$, the optimal strategy in period $n-i$ is

$$u^*_{n-i} = -\frac{W_{n-i}}{i+1} + b(n-i, Z_{n-i}).$$

Now, we consider the optimal strategy in period $n-i-1$. We use u_{n-i-1} to represent the optimal strategy $(u^*_{n-i+j})_{j=0,\ldots,i}$ through the induction hypothesis,

$$\begin{aligned}
u^*_{n-i} =& -\frac{u_{n-i-1} + W_{n-i-1}}{i+1} + b(n-i, Z_{n-i-1} + y_{n-i-1}) \\
&\vdots \\
u^*_{n-i+j} =& -\frac{u_{n-i-1} + W_{n-i-1}}{i+1} + b\left(n-i+j, Z_{n-i-1} + \sum_{l=n-i-1}^{n-i-1+j} y_l\right) \\
& - \sum_{k=1}^{j} \frac{1}{i+1-k} b\left(n-i-1+k, Z_{n-i-1} + \sum_{l=n-i-1}^{n-i-2+k} y_l\right) \\
&\vdots \\
u^*_n =& X - \frac{u_{n-i-1} + W_{n-i-1}}{i+1} \\
& - \sum_{k=1}^{i} \frac{1}{i+1-k} b\left(n-i-1+k, Z_{n-i-1} + \sum_{l=n-i-1}^{n-i-2+k} y_l\right)
\end{aligned}$$

We plug these strategies into the value function in period $n-i-1$. Then, we take the derivative of $f(u_{n-i-1}, n-i-1, Z_{n-i-1}, W_{n-i-1})$ with respect to

u_{n-i-1} and set it equal to zero, which gives us

$$E\left[\frac{\left(Z_\ell+\sum_{j=\ell}^{n} y_j\right)((i+2)u_\ell+W_\ell-X)-X\left((i+1)y_\ell-\sum_{j=\ell}^{n} y_j\right)}{X\left(X+Z_\ell+\sum_{j=\ell}^{n} y_j\right)}\right.$$

$$\left.\times\frac{Z_\ell+\sum_{j=\ell}^{n} y_j}{X\left(X+Z_\ell+\sum_{j=\ell}^{n} y_j\right)}+\tilde{\kappa}\left((i+2)u_\ell+W_\ell-X\right)\right]=0,$$

Where $\ell = n-i-1$. By solving this equation, we obtain the optimal strategy in period $n-i-1$

$$u^*_{n-i-1}=\frac{X-W_{n-i-1}}{i+2}+\frac{E\left[\frac{\left(\sum_{j=n-i-1}^{n} y_j+Z_{n-i-1}\right)\left((i+1)y_{n-i-1}-\sum_{j=n-i}^{n} y_j\right)}{X^2\left(X+\sum_{j=n-i-1}^{n} y_j+Z_{n-i-1}\right)^2}\right]}{(i+2)E\left[\frac{\left(\sum_{j=n-i-1}^{n} y_j+Z_{n-i-1}\right)^2}{X^2\left(X+\sum_{j=n-i-1}^{n} y_j+Z_{n-i-1}\right)^2}+\tilde{\kappa}\right]}X$$

It has exactly the same structure as that of u^*_{n-i}, which confirms the induction hypothesis.

Appendix B

MATLAB code

B.1 MLE for the dynamic Gamma strategy

```
function [k,theta]=mlegamma(A)
%The function mlegamma returns shape parameters and scale ...
   parameter through maximum likelihood estimation. A is an ...
   m*n matrix, which includes the market volumes. n is the ...
   number of trading periods. k is an n-dimensional vector, ...
   which contains n shape parameters, and theta is the scale ...
   parameter.
[m,n]=size(A);
s=sum(sum(A));
l=sum(log(A));
d=log(s/m);
function F=myfun(x);
F=l/m-d+log(sum(x))-psi(x);
end
x=ones(1,n);
```

```
k=fsolve(@myfun,x);
theta=s/(m*sum(k));
end
```

B.2 K-S test for the two-period Gamma based model

```
function p = betatest(M)
%The betatest function returns the p-value of the K-S test. M ...
    is the market relative volume in the morning. It contains ...
    120 data. The first 60 data is used to estimate the ...
    parameters of the Beta distribution. The second 60 data is ...
    the test dataset for the K-S test.
A=M(1:60);
[a,¬]=mle('beta',A);
pd = ProbDistUnivParam('beta',[a(1),a(2)]);
[¬,p] = kstest(M(61:120),pd);
```

B.3 Performance comparison in the two-period case

```
function compareX(M,A,P,n)
%Compare the average performance of TWAP strategy, ...
    multivariate Gamma strategy and empirical strategy by ...
```

```
    rolling parameter estimations. M is the morning volume, A ...
    is the afternoon volume, and n is the length of days that ...
    are used to estimate parameters. Our volume relative to ...
    the market is chosen variably as 0.05, 0.1,..., 1
average=mean(M)+mean(A);
a=zeros(20,1);
c=zeros(20,1);
d=zeros(20,1);
for i=1:20
    X=i*5/100*average;
    [¬,a(i,1)]=TTPer(M,A,n,X);
    [¬,c(i,1)]=TMGPer(M,A,n,X);
    [¬,d(i,1)]=TEPer(M,A,n,X);
end
X=0.05:0.05:1;
figure, plot(X, a, X, c, X, d, 'LineWidth', 2),
set(gca, 'FontSize', 16),
title('SINOPEC: Performance'),
legend('TWAP Strategy', 'Multivariate Gamma Strategy',
'Empirical strategy'),
xlabel('Trader total volume as fraction of market total volume');
ylabel('Deviation');
```

B.3.1 Performance of the TWAP strategy

```
function [u1,TTPer]=TTPer(M,A,n,X)
%The function TTPer returns the average performance of TWAP ...
    strategy in two-period case. M is the morning volume, A is ...
    the afternoon volume, n is the length of days that are ...
```

```
    used to estimate parameters, and X is our total volume ...
    relative to that of the market.
l=length(M);
c=l−n;
per=zeros(c,1);
u1=zeros(c,1);
for i=1:c
    u1(i,1)=X/2;
    y1=M(i+n,1);
    y2=A(i+n,1);
    per(i,1)=(u1(i,1)/X−(u1(i,1)+y1)/(X+y1+y2))^2+((X−u1(i,1))/X
    −(X−u1(i,1)+y2)/(X+y1+y2))^2;
end
TTPer=mean(per);
```

B.3.2 Performance of the multivariate Gamma strategy

```
function [u1,TMGPer]=TMGPer(M,A,n,X)
%The function TMGPer returns the average performance of ...
    multivariate Gamma strategy in two−period case. M is the ...
    morning volume, A is the afternoon volume, n is the length ...
    of days that are used to estimate parameters, and X is our ...
    total volume relative to that of the market.
rng('default');
l=length(M);
c=l−n;
Per=zeros(c,1);
u1=zeros(c,1);
for i=1:c
```

```
    d=i+n-1;
    one=sum(M(i:d));
    two=sum(M(i:d).^2);
    m1M=one/n;   %First moment of morning volumes
    m2M=two/n;   %Second moment of morning volumes
    one1=sum(A(i:d));
    two1=sum(A(i:d).^2);
    m1A=one1/n;   %First moment of afternoon volumes
    m2A=two1/n;   %Second moment of afternoon volumes
    theta1=(m2M-m1M*m1M)/m1M;
    theta2=(m2A-m1A*m1A)/m1A;
    meanm=mean(M(i:d));
    meana=mean(A(i:d));
    a=sum((M(i:d)-meanm).*(A(i:d)-meana));
    covMA=a/n;   %Cov between morning and afternoon volumes
    k0=covMA/theta1/theta2;
    k1=m1M*m1M/(m2M-m1M*m1M)-k0;
    k2=m1A*m1A/(m2A-m1A*m1A)-k0;
    x0=gamrnd(k0,1,10000,1);
    x1=gamrnd(k1,theta1,10000,1);
    x2=gamrnd(k2,theta2,10000,1);
    y1=theta1*x0+x1;
    y2=theta2*x0+x2;
    top=y1.*(y1+y2)./(X+y1+y2).^2;
    bottom=(y1+y2).^2./(X+y1+y2).^2;
    u1(i,1)=mean(top)/mean(bottom)*X;
    mv=M(i+n,1);
    av=A(i+n,1);
    Per(i,1)=(u1(i,1)/X-(u1(i,1)+mv)/(X+mv+av))^2+((X-u1(i,1))/X
    -(X-u1(i,1)+av)/(X+mv+av))^2;
end
```

```
TMGPer=mean(Per);
```

B.3.3 Performance of the empirical strategy

```
function [u1,TEPer]=TEPer(M,A,n,X)
%The function TEPer returns the average performance of ...
    empirical strategy in two-period case. M is the market ...
    volumes in the morning, A is the market volumes in the ...
    afternoon, n is the length of days that are used to ...
    estimate parameters, and X is our total volume relative to ...
    that of the market.
P=M./(M+A);
l=length(P);
c=l-n;
per=zeros(c,1);
u1=zeros(c,1);
for i=1:c
    d=i+n-1;
    avepor=mean(P(i:d,1));
    u1(i,1)=avepor*X;
    y1=M(i+n,1);
    y2=A(i+n,1);
    per(i,1)=(u1(i,1)/X-(u1(i,1)+y1)/(X+y1+y2))^2+((X-u1(i,1))/X
    -(X-u1(i,1)+y2)/(X+y1+y2))^2;
end
TEPer=mean(per);
```

B.4 Performance of dynamic Gamma strategy and empirical strategy in the n-period case

```
function [dynerr,emperr]=degamma(A)
% Compare the performance of dynamic Gamma strategy (DGS) and ...
    empirical strategy (ES). A is an m*n matrix, which stores ...
    the market volumes. m is the number of trading days. n is ...
    the number of trading periods on each day. A(i,j) means ...
    the market volume in period j on day i.
rng('default');
X=0.2*sum(A(1,:));   %X is our total volume on one day. We ...
    assume it is 20% of the market total volume on day 1.
[m,n]=size(A);  %m is the number of trading days, n is the ...
    number of periods
c=m-60;
dynerr=zeros(c,1);   %Performance of DGS.
emperr=zeros(c,1);   %Performance of ES.
ave=zeros(c,n);   %ave(i,j): average market volume in period j ...
    from day i to day i+59.
Z=zeros(c,n); %Z(i,j): total market volume before period j on ...
    day i+60.
du=zeros(c,n); %du(i,j): DGS in period j on day i+60.
eu=zeros(c,n); %eu(i,j): ES in period j on day i+60.
As=repmat(sum(A,2),1,n);
prop=A./As;
for i=1:1:c   %Compute the performance from day 1+60 to day c+60.
    d=i+59;
```

```
    B=A(i:d,1:n);
    [k,theta]=mlegamma(B);
    sumy=sum(A(60+i,1:n));
    y=gamrnd(repmat(k,1000,1),theta);
    Z(i,2:n)=cumsum(A(i+60,1:n-1));
    top=zeros(1000,1);
    bottom=zeros(1000,1);
    for q=1:1000
        top(q,1)=(Z(i,1)+n*y(q,1))*(Z(i,1)+sum(y(q,1:n)))/(X+Z(i,1)
        +sum(y(q,1:n)))^2;
        bottom(q,1)=(Z(i,1)+sum(y(q,1:n)))^2/(X+Z(i,1)
        +sum(y(q,1:n)))^2;
    end
    du(i,1)=mean(top)/mean(bottom)/n*X;
    for j=2:n
           top=(Z(i,j)+(n-j+1)*y(:,j)).*(Z(i,j)+sum(y(:,j:n),2))
           ./(X+Z(i,j)+sum(y(:,j:n),2)).^2;
           bottom=(Z(i,j)+sum(y(:,j:n),2)).^2./(X+Z(i,j)
           +sum(y(:,j:n),2)).^2;
       du(i,j)=mean(top)/(n-j+1)*X/mean(bottom)-sum(du(i,1:(j-1)))
       /(n-j+1);
    end
        dynerr(i,1)=sum((du(i,:)/X-(du(i,:)+A(60+i,:))
        /(X+sumy)).^2);
      %Compute the performance of ES
        ave(i,:)=mean(prop(i:i+59,:));
        eu(i,:)=ave(i,:)*X;
        emperr(i,1)=sum((eu(i,:)/X-(eu(i,:)+A(60+i,:))
        /(X+sumy)).^2);
end
```