REPRESENTATIONS OF TWISTED YANGIANS OF TYPES B, C AND D

by

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Abstract

In the first part of this dissertation, we prove a generalization of a theorem of Drinfeld's [Dri85] which allows one to rebuild the Yangian of an arbitrary simple Lie algebra starting from any of its finite-dimensional modules satisfying a non-triviality condition. This is achieved using the so-called R-matrix formalism, and the resulting realizing of the Yangian is called its R-matrix presentation. When the underlying module is assumed to be irreducible, our result coincides with Drinfeld's and, in particular, makes available a proof of his theorem – which has never appeared in the literature.

In addition, we provide a detailed study of the algebraic structure of the extended Yangian and prove several generalizations of results which are known to hold in the special case where the underlying module is the vector representation of a classical Lie algebra.

In the second part of this dissertation, we address the problem of classifying the finite-dimensional irreducible representations for twisted Yangians associated to orthogonal and symplectic symmetric pairs of Lie algebras. We lay the foundation needed to solve this problem by developing a highest weight theory and proving that the highest weight of a finite-dimensional irreducible module necessarily satisfies a set of relations involving a distinguished complex scalar and a tuple of polynomials whose set of roots are invariant under certain reflections.

Our main results on this topic provide a complete classification of finite-dimensional irreducible modules for twisted Yangians associated to a large family of orthogonal and symplectic symmetric pairs.

Preface

The content of this thesis is largely based on three publications, one of which is the sole work of the author and two of which are are joint work with Nicolas Guay and Vidas Regelskis. Chapter 2 contains the article

[Wen18] C. Wendlandt, *The R-matrix presentation for the Yangian of a simple Lie algebra*, Commun. Math. Phys. **363** (2018), no. 1, 289–332.

Chapters 4 and 5 are based on the two publications

[GRW17] N. Guay, V. Regelskis and C. Wendlandt, *Representations of twisted Yangians of types B, C, D: I*, Selecta Math. (N.S.) **23** (2017), no. 3, 2071–2156.

[GRW19b] N. Guay, V. Regelskis and C. Wendlandt, *Representations of twisted Yangians of types B, C, D: II*, Transform. Groups (2019), doi:10.1007/s00031-019-09514-x.

I am the main author of these articles, and was responsible for the majority of the manuscript composition and proof writing.

Dedicated to *Eszter*

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Contents

Chapter 1

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbb{C}$, and let $\mathfrak{g}[z]$ denote the polynomial current algebra associated to $\mathfrak{g}$. That is, $\mathfrak{g}[z]$ is the Lie algebra of polynomial maps $f : \mathbb{C} \to \mathfrak{g}$, with Lie bracket given pointwise.

The Yangian $Y(\mathfrak{g})$ associated to $\mathfrak{g}$ is a quantum group of affine type which provides the canonical filtered Hopf algebra deformation of the enveloping algebra $U(\mathfrak{g}[z])$. In particular, it is a filtered Hopf algebra over $\mathbb{C}$, with associated graded algebra $\mathrm{gr}Y(\mathfrak{g})$ isomorphic to $U(\mathfrak{g}[z])$ as a graded Hopf algebra:

$$U(\mathfrak{g}[z]) \cong \mathrm{gr}Y(\mathfrak{g}).$$

Yangians originally appeared under the guise of their representations in the work of mathematical physicists studying the quantum inverse scattering method (see [KS82a, KS82b], for instance). They were later formally introduced by V. Drinfeld [Dri85], who laid the rigorous foundation for what has grown into a captivating theory. Since Drinfeld's pioneering work, this theory has become entangled with many different areas and topics in mathematics and mathematical physics. For instance, various applications of Yangians to the theories of classical Lie algebras [Naz91, NT94, Naz98, Mol06, Mol07], finite W-algebras [RS99, Rag01, BR01, BK06, BK08, Bro09, Bro11], classical W-algebras and affine vertex algebras [MM14, MM17, Mol13] have been discovered. In addition, the representation theory of Yangians and affine quantum groups has now manifested itself in many geometric settings [Var00, Nak13, KWWY14, KTW$^+$15, SV17, SV18, YZ18a, YZ18b, FKP$^+$18, MO19, Li19]. Many of these developments are in some way related to the fundamental fact that representations of Yangians produce rational R-matrices – rational solutions of the quantum Yang-Baxter equation (see (2.2.9)).

In this dissertation, we consider two different topics in the representation theory of Yangians related to R-matrices.

In its first part, we consider the topic of the R-matrix formalism for the Yangian of an arbitrary complex simple Lie algebra. Our main goal, which is realized in Chapter 2, is to prove a generalization of a theorem of Drinfeld's [Dri85] which allows one to rebuild the Yangian from any finite-dimensional, non-trivial, representation. This goal, and our approach to realizing it, is described in detail in §1.1 below.

In its second part, we focus on the representation theory of orthogonal and symplectic twisted Yangians. Twisted Yangians are coideal subalgebras of Yangians associated to symmetric pairs of Lie algebras and are intimately linked to an elegant relation called the reflection equation. The orthogonal and symplectic Yangians which we consider were introduced by N. Guay and V. Regelskis in [GR16], and are built inside a special instance of the R-matrix presentation of the Yangian constructed in Chapter 2. Our goal, which is realized in Chapters 4 and 5, is to lay the groundwork needed to solve the problem of classifying the finite-dimensional irreducible representation of these twisted Yangians. In fact, we solve this classification problem completely for a large family of twisted Yangians. This is described in §1.2.

1.1 The R-matrix presentation of the Yangian

1.1.1 Background and motivation

Yangians admit at least three important presentations: Drinfeld's original (or "J") presentation, the R-matrix (or "RTT") realization, and Drinfeld's new (or "current") presentation [Dri85, Dri88, FRT90]. Here, we will be primarily concerned with the R-matrix presentation.

The general construction of the R-matrix presentation of the Yangian and its equivalence with Drinfeld's original (or J) presentation was succinctly explained in [Dri85, Theorem 6].

Drinfeld's construction begins with a fixed finite-dimensional irreducible representation V of $Y(\mathfrak{g})$, where $\mathfrak{g}$ is any simple Lie algebra and the Yangian $Y(\mathfrak{g})$ is in Drinfeld's original presentation (see §2.2). Starting from this data, one may define a Hopf algebra $X(\mathfrak{g})$ – the extended Yangian – whose generators are organized into a matrix

$$T(u) \in \mathrm{End}V \otimes X(\mathfrak{g})[\![u^{-1}]\!],$$

with defining relations encoded in the ternary matrix relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v),$$

called the RTT-relation. Here $R(u)$ is a solution of the quantum Yang-Baxter equation associated to V, which comes from evaluating a formal series

$$\mathcal{R}(u) \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[\![u^{-1}]\!],$$

called the *universal R-matrix* of the Yangian, on $V \otimes V$. After being translated to fit our informal setup, Theorem 6 of [Dri85] can be expressed as follows[1].

Theorem A. *There is an epimorphism of Hopf algebras*

$$\widetilde{\Phi} : X(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$$

whose kernel is generated by central elements $\{c_r\}_{r\in\mathbb{N}}$ which satisfy

$$\Delta(c(u)) = c(u) \otimes c(u),$$
$$\text{where} \quad c(u) = 1 + \sum_{r\geq 1} c_r u^{-r} \in X(\mathfrak{g})[\![u^{-1}]\!]$$

and Δ is the coproduct for $X(\mathfrak{g})$.

The R-matrix presentation of the Yangian associated to V is then defined to be the quotient

$$Y_R(\mathfrak{g}) = X(\mathfrak{g})/(c(u)-1),$$

where $(c(u)-1)$ denotes the ideal generated by $\{c_r\}_{r\in\mathbb{N}}$.

1) This is stated more precisely in Theorem 2.7.2.

There are, however, many important aspects of this construction which remain mysterious:

(a) $Y_R(\mathfrak{g})$ has only been explicitly studied in the special cases where $\mathfrak{g}$ is a classical Lie algebra (i.e. $\mathfrak{g} = \mathfrak{sl}_N, \mathfrak{so}_N$ or $\mathfrak{sp}_N$) and V is the vector representation $\mathbb{C}^N$.

(b) A proof of Theorem A has never appeared in the literature in full generality.

(c) No general expression for the central series $c(u)$ has been given, nor has any alternate procedure for describing $\mathrm{Ker}(\widetilde{\Phi})$.

(d) It is not clear how well Theorem A generalizes when the irreducibility assumption on V is dropped.

This brings us to the first main goal behind our work in Chapter 2: To address the points (b)-(d) in detail and, in particular, to show that Theorem A admits a generalization where the irreducibility assumption on V is removed.

Our second goal concerns understanding the algebraic structure of the extended Yangian $X(\mathfrak{g})$. To provide some context, let us temporarily narrow our focus to the setting of (a). In these cases, $X(\mathfrak{g})$ has the following properties:

(e) The series $c(u)$ may be chosen so that it's coefficients $\{c_r\}_{r\in\mathbb{N}}$ are algebraically independent and generate the center $ZX(\mathfrak{g})$ of $X(\mathfrak{g})$. In particular,

$$ZX(\mathfrak{g}) \cong \mathbb{C}[c_r]_{r\in\mathbb{N}} = \mathbb{C}[c_1, c_2, \ldots].$$

(f) $X(\mathfrak{g})$ admits the tensor product decomposition

$$X(\mathfrak{g}) \cong ZX(\mathfrak{g}) \otimes Y_R(\mathfrak{g}) \cong \mathbb{C}[c_r]_{r\in\mathbb{N}} \otimes Y_R(\mathfrak{g}).$$

(g) There is an isomorphism of graded Hopf algebras

$$U(\mathfrak{g}[z] \oplus \mathbb{C}[z]) \cong \mathrm{gr} X(\mathfrak{g}).$$

(h) There is a family of automorphisms $\{m_f\} \subset \mathrm{Aut}(X(\mathfrak{g}))$, indexed by

$$f = f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!],$$

such that $Y_R(\mathfrak{g})$ is equal to the subalgebra of $X(\mathfrak{g})$ fixed by all m_f:

$$Y_R(\mathfrak{g}) = \left\{ Y \in X(\mathfrak{g}) \, : \, m_f(Y) = Y \quad \forall\, f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!] \right\}.$$

When $\mathfrak{g} = \mathfrak{sl}_N$, such a $c(u)$ is provided by the *quantum determinant* $\mathrm{qdet}T(u)$ (see (2.7.7)), while if $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$, one may take $c(u)$ to instead by the series $\mathsf{z}(u)$ defined by (2.7.17). Proofs of the above assertions may be found in [Mol07, Chapter 1] for $\mathfrak{g}$ equal to $\mathfrak{sl}_N$ and [AMR06, AAC+03] for $\mathfrak{g}$ equal to $\mathfrak{so}_N$ or $\mathfrak{sp}_N$.

Many known applications of Yangians are specific to the case where $\mathfrak{g} = \mathfrak{sl}_N$ and make extensive use of $X(\mathfrak{sl}_N)$, which is often denoted $Y(\mathfrak{gl}_N)$ and provides a filtered deformation of $U(\mathfrak{gl}_N[z])$. In particular, the fact that $X(\mathfrak{sl}_N)$ has a large center governed by $\mathrm{qdet}T(u)$ has led to many interesting results: see [Mol07, Chapter 7].

Our second goal in Chapter 2 is to prove that (e)–(h) admit a generalization which holds for any complex simple Lie algebra $\mathfrak{g}$, with the underlying module V chosen to be any finite-dimensional $Y(\mathfrak{g})$-module satisfying a non-triviality condition.

1.1.2 Main Results

We now provide a description of the of main results of Chapter 2, which realize both goals laid out in §1.1.1. Let $\mathfrak{g}$ be an arbitrary simple Lie algebra over $\mathbb{C}$, and fix V to be any finite-dimensional $Y(\mathfrak{g})$-module which has a non-trivial irreducible summand when viewed as a module over $\mathfrak{g} \subset Y(\mathfrak{g})$.

The extended Yangian associated to V, which we will denote by $X_{\mathcal{I}}(\mathfrak{g})$, can be constructed as in §1.1.1 (see Definition 2.4.1) and has a natural filtered Hopf algebra structure. Here $\mathcal{I}$ is an indexing set satisfying

$$|\mathcal{I}| = \dim \mathrm{End}_{Y(\mathfrak{g})} V$$

and is omitted as a subscript of $X_{\mathcal{I}}(\mathfrak{g})$ when V is irreducible. We then define the Yangian $Y_R(\mathfrak{g})$ as the quotient

$$Y_R(\mathfrak{g}) = X_{\mathcal{I}}(\mathfrak{g})/(\mathcal{Z}(u) - 1),$$
$$\text{where} \quad \mathcal{Z}(u) = S_{\mathcal{I}}^2(T(u))T(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1} \in \mathrm{End}V \otimes X_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!].$$

Here $S_{\mathcal{I}}$ is the antipode of $X_{\mathcal{I}}(\mathfrak{g})$ (see (2.4.2)), $c_{\mathfrak{g}}$ is the eigenvalue of the Casimir element of $\mathfrak{g}$ in the adjoint representation, and $(\mathcal{Z}(u)-1)$ is the ideal generated by the coefficients of $\mathcal{Z}(u)$.

Our first collection of results, summarized in the following theorem, provide strong justification for this definition.

Theorem B. *The Yangian $Y_R(\mathfrak{g})$ has the following properties.*

(1) There is an epimorphism of filtered Hopf algebras

$$\widetilde{\Phi} : X_{\mathcal{I}}(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$$

with kernel $\mathrm{Ker}(\widetilde{\Phi}) = (\mathcal{Z}(u)-1)$. *In particular, $\widetilde{\Phi}$ induces an isomorphism*

$$\Phi : Y_R(\mathfrak{g}) \xrightarrow{\sim} Y(\mathfrak{g}).$$

(2) There is an isomorphism of graded Hopf algebras $U(\mathfrak{g}[z]) \cong \mathrm{gr} Y_R(\mathfrak{g})$.

(3) The center $ZY_R(\mathfrak{g})$ of $Y_R(\mathfrak{g})$ is trivial: $ZY_R(\mathfrak{g}) = \mathbb{C}\cdot 1$.

In §2.5, this theorem is broken into three distinct parts: see Theorem 2.5.2, Theorem 2.5.5 and Corollary 2.5.6. We note that Part (1) does not immediately imply Theorem A, as it does not state that $\mathcal{Z}(u)$ is a central, grouplike series multiple of the identity matrix when V is assumed irreducible.

The next theorem realizes our second goal described in §1.1.1 by providing generalizations of (e), (f), (g) and (h).

Theorem C. *Let $ZX_{\mathcal{I}}(\mathfrak{g})$ denote the center of $X_{\mathcal{I}}(\mathfrak{g})$. Then:*

(1) There is an isomorphism of filtered Hopf algebras

$$X_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[y_\lambda^{(r)}]_{\lambda\in\mathcal{I}, r\in\mathbb{N}} \otimes Y_R(\mathfrak{g}).$$

(2) There is an isomorphism of graded Hopf algebras

$$U(\mathfrak{g}[z] \oplus \mathfrak{z}_{\mathcal{I}}[z]) \cong \mathrm{gr} X_{\mathcal{I}}(\mathfrak{g}),$$

where $\mathfrak{z}_{\mathcal{I}}$ is a commutative Lie algebra of dimension $|\mathcal{I}| = \dim \mathrm{End}_{Y(\mathfrak{g})} V$.

(3) The matrix $\mathcal{Z}(u)$ belongs to

$$\mathrm{End}_{Y(\mathfrak{g})}V \otimes ZX_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!],$$

and its coefficients generate a polynomial algebra $\mathbb{C}[z_\lambda^{(r)}]_{\lambda\in\mathcal{I},r\geq 2}$ satisfying

$$\mathbb{C}[z_\lambda^{(r)}]_{\lambda\in\mathcal{I},r\geq 2} \cong ZX_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[y_\lambda^{(r)}]_{\lambda\in\mathcal{I},r\in\mathbb{N}}.$$

(4) There is a family of automorphisms $\{m_{\mathbf{f}}\} \subset \mathrm{Aut}(X_{\mathcal{I}}(\mathfrak{g}))$ which are indexed by

$$\mathbf{f} = \mathbf{f}(u) \in I + \mathrm{End}_{Y(\mathfrak{g})}V \otimes u^{-1}\mathbb{C}[\![u^{-1}]\!],$$

such that $Y_R(\mathfrak{g})$ is equal to the subalgebra of $X_{\mathcal{I}}(\mathfrak{g})$ fixed by all $m_{\mathbf{f}}$:

$$Y_R(\mathfrak{g}) = \left\{Y \in X_{\mathcal{I}}(\mathfrak{g}) \,:\, m_{\mathbf{f}}(Y) = Y \quad \forall\, \mathbf{f}(u) \in I + \mathrm{End}_{Y(\mathfrak{g})}V \otimes u^{-1}\mathbb{C}[\![u^{-1}]\!]\right\}.$$

This theorem is a stripped down combination of Theorem 2.6.3, Proposition 2.6.6, Theorem 2.6.7, Proposition 2.6.9 and Theorem 2.6.11.

The relation between the elements $\{z_\lambda^{(r)}\}_{\lambda\in\mathcal{I},r\geq 2}$ of Part (3) and $\mathcal{Z}(u)$ is given precisely by

$$\mathcal{Z}(u) = I + \sum_{\lambda\in\mathcal{I}} X_\lambda^\bullet \otimes z_\lambda(u), \quad \text{where} \quad z_\lambda(u) = \sum_{r\geq 2} z_\lambda^{(r)} u^{-r}$$

and $\{X_\lambda^\bullet\}_{\lambda\in\mathcal{I}}$ is a fixed basis of $\mathrm{End}_{Y(\mathfrak{g})}V$ containing the identity operator I.

Similarly, the central elements $\{y_\lambda^{(r)}\}_{\lambda\in\mathcal{I},r\in\mathbb{N}}$ from Parts (1) and (3) are encoded as the coefficients of a matrix $\mathcal{Y}(u)$. This matrix is realized explicitly as the unique solution of the formal difference equation

$$\mathcal{Y}(u + \tfrac{1}{2}c_{\mathfrak{g}})\mathcal{Z}(u) = \mathcal{Y}(u) \quad \text{in} \quad I + \mathrm{End}_{Y(\mathfrak{g})}V \otimes u^{-1}X_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!].$$

The image of $\mathcal{Y}(u)$ and $\mathcal{Z}(u)$ under the coproduct, counit and antipode of $X_{\mathcal{I}}(\mathfrak{g})$ is explicitly computed in Lemma 2.6.8 and Corollary 2.6.10. In particular, one has

$$\Delta_{\mathcal{I}}(\mathcal{Y}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Y}_{[2]}(u) \;\text{ and }\; \Delta_{\mathcal{I}}(\mathcal{Z}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Z}_{[2]}(u)\mathcal{Y}_{[1]}(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1}, \quad (1.1.1)$$

where the notation is as in §2.1 and $\Delta_{\mathcal{I}}$ is the coproduct for $X_{\mathcal{I}}(\mathfrak{g})$.

Let us now explain how the above results imply Theorem A. Suppose that V is irreducible. Then $\mathrm{End}_{Y(\mathfrak{g})}V = \mathbb{C}I$ and the matrices $\mathcal{Z}(u)$ and $\mathcal{Y}(u)$ necessarily take the form

$$\mathcal{Z}(u) = z(u) \cdot I \quad \text{and} \quad \mathcal{Y}(u) = y(u) \cdot I,$$
$$\text{where} \quad z(u) \in 1 + u^{-1}X(\mathfrak{g})[\![u^{-1}]\!] \quad \text{and} \quad y(u) \in 1 + u^{-1}X(\mathfrak{g})[\![u^{-1}]\!]$$

In addition, the formulas (1.1.1) collapse to

$$\Delta(z(u)) = z(u) \otimes z(u) \quad \text{and} \quad \Delta(y(u)) = y(u) \otimes y(u).$$

Combining these observations with Part (1) of Theorem B gives the following theorem, which is a summary of the first two parts of Theorem 2.7.2.

Theorem D. *Theorem A holds with $c(u)$ taken to be either $z(u)$ or $y(u)$.*

More generally, Part (1) of Theorem B, combined with Part (3) of Theorem C and (1.1.1) should be viewed as a generalization of Theorem A. This is explained further in Remark 2.7.3.

Our proofs of Theorems B, C and D are based on the construction of "matrix" presentations for the Lie algebra $\mathfrak{g}$ and for an auxiliary Lie algebra $\mathfrak{g}_{\mathcal{I}}$ satisfying

$$\mathfrak{g}_{\mathcal{I}} \cong \mathfrak{g} \oplus \mathfrak{z}_{\mathcal{I}}.$$

They are obtained in §2.3 by studying $\mathrm{End}V$ as a $\mathfrak{g}$-module equipped with an adjoint action, and provide the $\mathfrak{g}$-analogues of the R-matrix presentation of the Yangian and of the extended Yangian. In particular, Proposition 2.3.4 gives a procedure for rebuilding $\mathfrak{g}$ from the evaluation of its Casimir two tensor $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ on the tensor square $V \otimes V$ of any fixed finite-dimensional, non-trivial, $\mathfrak{g}$-module V.

These realizations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathcal{I}}$ naturally lead to the so-called r-matrix presentations of the current algebras $\mathfrak{g}[z]$ and $\mathfrak{g}_{\mathcal{I}}[z]$: see Propositions 2.3.9 and 2.3.16. Our treatment of these topics in §2.3 appears to be novel, and leads to interesting results even for the simple Lie algebra $\mathfrak{g}$.

1.2 Representations of twisted Yangians of types B, C and D

1.2.1 Background and motivation

The goal of the second part of this dissertation, which is contained in Chapters 3–5, is to address the problem of classifying all finite-dimensional irreducible representations for twisted Yangians associated to symmetric pairs of types B, C and D.

Twisted Yangians provide one of the main examples of quantum symmetric pairs of affine type and can be defined starting from any symmetric pair structure $(\mathfrak{g}, \mathfrak{g}^{\vartheta})$ on a simple Lie algebra $\mathfrak{g}$, where $\mathfrak{g}^{\vartheta}$ denotes the Lie subalgebra of $\mathfrak{g}$ consisting of elements fixed by a given involution $\vartheta \in \mathrm{Aut}(\mathfrak{g})$. A twisted Yangian $Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}$ associated to such a pair is a certain left coideal subalgebra of $Y(\mathfrak{g})$ which provides a filtered deformation of the enveloping algebra $U(\mathfrak{g}[z]^{\check{\vartheta}})$, where $\check{\vartheta}$ is the non-trivial extension of ϑ to an involution of $\mathfrak{g}[z]$ given by

$$\check{\vartheta}(f)(z) = \vartheta(f(-z)) \quad \forall \quad z \in \mathbb{C}$$

and $\mathfrak{g}[z]^{\check{\vartheta}}$ is the subalgebra of $\mathfrak{g}[z]$ consisting of elements fixed by $\check{\vartheta}$. The left coideal property means that the restriction of the coproduct Δ of $Y(\mathfrak{g})$ to $Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}$ satisfies

$$\Delta(Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}) \subset Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}$$

and the filtered deformation property means that there is an isomorphism of graded algebras

$$U(\mathfrak{g}[z]^{\check{\vartheta}}) \cong \mathrm{gr} Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}$$

compatible with the isomorphism of Hopf algebras $U(\mathfrak{g}[z]) \cong \mathrm{gr} Y(\mathfrak{g})$.

For a general definition of $Y(\mathfrak{g}, \mathfrak{g}^{\vartheta})^{tw}$ given in terms of generators and relations which is compatible with Drinfeld's original presentation of the Yangian, we refer the reader to the recent work of S. Belliard and V. Regelskis [BR17]. Our approach to studying twisted Yangians of orthogonal and symplectic type will instead follow [GR16], where they are constructed within the R-matrix presentation of the Yangians $Y(\mathfrak{so}_N)$ and $Y(\mathfrak{sp}_N)$ associated to the vector representation $\mathbb{C}^N$ (as in (a) of §1.1.1).

A complete list of symmetric pairs of types B, C and D is given by

$$\begin{gathered}\mathrm{B0}:(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n+1}),\quad \mathrm{D0}:(\mathfrak{so}_{2n},\mathfrak{so}_{2n}),\quad \mathrm{C0}:(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n}),\\ \mathrm{BI}:(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n+1-q}\oplus\mathfrak{so}_q),\quad \mathrm{DI}:(\mathfrak{so}_{2n},\mathfrak{so}_{2n-q}\oplus\mathfrak{so}_q),\quad \mathrm{CII}:(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n-q}\oplus\mathfrak{sp}_q),\\ \mathrm{CI}:(\mathfrak{sp}_{2n},\mathfrak{gl}_n)\quad\text{and}\quad \mathrm{DIII}:(\mathfrak{so}_{2n},\mathfrak{gl}_n),\end{gathered}$$

where q is necessarily even for the type CII pairs and the labeling comes from Cartan's classification of symmetric spaces: see [Hel01, Chapter X]. Our focus will be entirely on twisted Yangians associated to symmetric pairs which arise from inner automorphisms. By Table II and Theorem 5.16 of [Hel01, Chapter X], this includes all symmetric pairs from the above list except for those of the form $(\mathfrak{so}_{2n},\mathfrak{so}_{2n-q}\oplus\mathfrak{so}_q)$ with q taking odd values. The pairs $(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})$ which we do consider are therefore given by the list

$$(\mathfrak{g}_{2n},\mathfrak{gl}_n)\quad\text{and}\quad(\mathfrak{g}_N,\mathfrak{g}_{N-q}\oplus\mathfrak{g}_q),\quad\text{where}\quad 0\le q<N\quad\text{and}\quad q\in 2\mathbb{Z}.$$

Here (and henceforth) $\mathfrak{g}_N$ always denotes either $\mathfrak{so}_N$ or $\mathfrak{sp}_N$, where $N\geq 2$ if $\mathfrak{g}_N=\mathfrak{sp}_N$ and $N\geq 3$ if $\mathfrak{g}_N=\mathfrak{so}_N$. In particular, we include the non-simple case $\mathfrak{so}_4\cong\mathfrak{sl}_2\oplus\mathfrak{sl}_2$.

The problem of classifying finite-dimensional irreducible representations of Yangians and twisted Yangians has a history dating back to Drinfeld's work in the 1980's. In [Dri88, Theorem 2], it was established that the finite-dimensional irreducible representations of $Y(\mathfrak{g})$ are classified up to isomorphism by tuples of monic polynomials

$$(P_i(u))_{i=1}^{\mathrm{rank}(\mathfrak{g})}\in\mathbb{C}[u]^{\mathrm{rank}(\mathfrak{g})},$$

which have since been called *Drinfeld polynomials*. Drinfeld's classification result was reproved for symplectic and orthogonal Yangians in their R-matrix presentations in [AMR06]: see §4.1.

For twisted Yangians of type A (see §3.4), the analogous classification problem was solved by A. Molev for symmetric pairs of type AI and AII [Mol92, Mol98, Mol07] and by A. Molev and E. Ragoucy for symmetric pairs of type AIII [MR02]. The resulting classifications are again given in terms of monic polynomials, but these polynomials are now subject to additional conditions and, when $\mathfrak{sl}_N^{\vartheta}$ has a center, there is a complex parameter which also plays a role.

A common theme in the above results is that the polynomials at the heart of each classification manifest themselves in certain relations satisfied by the highest weight of a finite-dimensional irreducible module. Therefore, it should not come as a surprise that our approach to addressing the problem of classifying the finite-dimensional irreducible representations of $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ will involve first developing a highest weight theory. Moreover, following Molev and Ragoucy, we shall work almost exclusively with the extended twisted Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, which is a coideal subalgebra of the extended Yangian $X(\mathfrak{g}_N)$ whose relation to $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ mirrors $X(\mathfrak{g}_N)$'s relation with $Y(\mathfrak{g}_N)$.

1.2.2 Main Results

In Chapter 3 we will give a detailed survey of Guay and Regelskis' construction of the twisted Yangian $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and of the extended Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$. For the sake of the reader, we will include several proofs to illustrate how many results can be obtained as an application of the general theory of Chapter 2.

In Chapter 4, we develop a highest weight theory for $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and demonstrate that it provides the right tool for distinguishing finite-dimensional irreducible modules. The definitions which form the basis of this theory are given in §4.2.2. For the moment, it will be sufficient to know that highest weights take the form of tuples

$$\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_N^+} \in \prod_{i\in\mathcal{I}_N^+} (g_{ii} + u^{-1}\mathbb{C}[\![u^{-1}]\!]),$$
$$\text{where} \quad \mathcal{I}_N^+ = \{2n - N + 1, \dots, n\} \ \text{ for } \ n = \lfloor N/2 \rfloor$$

and the g_{ii} take values in $\{\pm 1\}$ determined by the underlying involution ϑ.

One may define a Verma module $M(\mu(u))$ associated to any such $\mu(u)$ and, provided it is non-trivial, it has a unique irreducible quotient $V(\mu(u))$. Our main results of Chapter 4 include a characterization of precisely when $M(\mu(u))$ is non-trivial, and a proof that every finite-dimensional irreducible module is of highest weight type:

Theorem E. *The Verma module $M(\mu(u))$ is non-trivial if and only if the relations*

$$\begin{gathered} \widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i) \\ u\,\mathcal{g}(u)\widetilde{\mu}_0(\kappa-u) = (\kappa-u)\,\mathcal{g}(\kappa-u)\widetilde{\mu}_0(u) \end{gathered} \tag{1.2.1}$$

hold for all $i \in \mathcal{I}_N^+ \setminus \{n\}$*, where* $\kappa = c_{\mathfrak{g}_N}/4$*. Moreover, every finite-dimensional irreducible module* V *satisfies*

$$V \cong V(\mu(u))$$

for a unique tuple $\mu(u)$ *solving the relations* (1.2.1).

The auxiliary tuple $\widetilde{\mu}(u) = (\widetilde{\mu}_i(u))_{i\in\mathcal{I}_N^+}$ which appears in (1.2.1) is defined by

$$\widetilde{\mu}_i(u) = (2u - n + i)\mu_i(u) + \sum_{\ell=i+1}^{n} \mu_\ell(u) \quad \forall \quad i \in \mathcal{I}_N^+,$$

and $\mathcal{g}(u)$ is a rational function of u determined by ϑ: see (3.3.3). Theorem E will be a consequence of Theorem 4.2.6, Proposition 4.2.9 and Theorem 4.4.4.

The second assertion of Theorem E reduces the problem of classifying the finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ up to isomorphism to the problem of determining an explicit description of the set

$$\left\{ \mu(u) \in \prod_{i\in\mathcal{I}_N^+} (g_{ii} + u^{-1}\mathbb{C}[\![u^{-1}]\!]) \; : \; \dim V(\mu(u)) < \infty \right\}.$$

Our first main result of Chapter 5 provides a list of conditions on $\mu(u)$ which are necessarily satisfied whenever it belongs to the above set. Set $\delta = \delta_{\mathfrak{g}_N, \mathfrak{sp}_N}$.

Theorem F. *Suppose the* $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$*-module* $V(\mu(u))$ *is finite-dimensional. Then there exists monic polynomials* $P_1(u), \dots, P_n(u)$ *in* u*, with*

$$\begin{aligned} P_1(u) &= P_1(-u + \kappa + 2^\delta), \\ P_i(u) &= P_i(-u + n - i + 2) \quad \forall \quad 2 \le i \le n, \end{aligned} \tag{1.2.2}$$

together with a scalar $\alpha \in \mathbb{C} \setminus Z(P_{\mathscr{k}(\mathcal{G})+1}(u))$[2] *such that*

$$\begin{aligned} \frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} &= \frac{P_i(u+1)}{P_i(u)} \left(\frac{\alpha - u}{\alpha + u - \ell}\right)^{\delta_{i,\mathscr{k}(\mathcal{G})+1}} \quad \forall \quad 2 \le i \le n, \\ \frac{u}{\kappa - u} \cdot \frac{\widetilde{\mu}_{\mathsf{a}}(\kappa - u)}{\widetilde{\mu}_{\mathsf{b}}(u)} &= \frac{\mathcal{g}(\kappa - u)}{\mathcal{g}(u)} \cdot \frac{P_1(u + \mathsf{d})}{P_1(u)} \left(\frac{\alpha - u}{\alpha + u - \kappa + \mathsf{d} - 2^\delta}\right)^{\delta_{\mathscr{k}(\mathcal{G}),0}}. \end{aligned}$$

2) $Z(P(u))$ denotes the set of roots of a fixed polynomial $P(u)$: see Definition 5.2.3.

Moreover, if ϑ is non-trivial and $\mathfrak{g}_N^\vartheta$ is semisimple, then α satisfies

$$2^{1-\delta}\left(\alpha - \frac{N}{4}\right) \in \mathbb{Z}.$$

The symbols $\mathsf{a}, \mathsf{b}, \mathsf{d}$ and $\mathscr{k}(\mathcal{G})$ all take non-negative integer values and are defined at the beginning of §5.2.2. The statement of the above theorem will be proven in Propositions 5.2.5 and 5.2.13.

The freeness of the scalar α which appears in Theorem F is controlled by the existence of non-trivial one-dimensional representations for $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. This observation motivates our second main result of Chapter 5, which provides a classification of all one-dimensional representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. The below theorem summarizes the $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ version of this result.

Theorem G. *In what follows, $V(\mathcal{G})$ denotes the one-dimensional representation of $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ obtained by restricting the counit of $Y(\mathfrak{g}_N)$.*

(1) If $\mathfrak{g}_N^\vartheta$ is semisimple, then every one-dimensional representation of $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ is isomorphic to $V(\mathcal{G})$.

(2) If $\mathfrak{g}_N^\vartheta$ has a one-dimensional center, then $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ has a one-parameter family of representations

$$\{V(\alpha)\}_{\alpha\in\mathbb{C}} \quad \text{with} \quad \dim V(\alpha) = 1 \quad \forall \quad \alpha \in \mathbb{C}.$$

Additionally, a $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module V is one-dimensional if and only if there is $\alpha \in \mathbb{C}$ such that

$$V \cong V(\alpha).$$

Note that every symmetric pair $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ under consideration satisfies the hypothesis of either Part (1) or Part (2) except if $\mathfrak{g}_N^\vartheta = \mathfrak{so}_2 \oplus \mathfrak{so}_2$, in which case $\mathfrak{g}_N$ is the non-simple Lie algebra $\mathfrak{so}_4$. This exceptional case is discussed in Remark 5.3.7. Part (1) of Theorem G is stated as Corollary 5.3.2 in §5.3, while Part (2) is a combination of Corollaries 5.3.6 and 5.3.11 from the same section.

The final set of main results of Chapter 5 are also the most significant results contained in the second part of this dissertation. They provide a complete classification of all finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, up to isomorphism, for all symmetric pairs $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ of the form

$$(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}),$$
$$(\mathfrak{g}_N, \mathfrak{g}_N), \quad (\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2),$$

For those pairs appearing on the second line above, the necessary conditions of Theorem F are also sufficient conditions and give rise to the desired classifications. These results are proven in §5.4. Here, we will only present those results which pertain to pairs $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})$, where $n \geq 2$.

Given $\alpha, \beta \in \mathbb{C}$ such that $\alpha - \beta \in \mathbb{Z}$, define the string $S(\alpha, \beta) \subset \mathbb{C}$ by

$$S(\alpha, \beta) = \{\beta + k \,:\, k \in \mathbb{Z} \text{ and } 0 \leq k \leq \alpha - \beta - 1\}.$$

The next theorem provides a complete description of when the $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module $V(\mu(u))$ is finite-dimensional.

Theorem H. *Let $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_{2n+1}^+}$ satisfy (1.2.1). Then the $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $P_1(u), \ldots, P_n(u)$ in u satisfying (1.2.2), together with $\alpha \in \mathbb{C} \setminus Z(P_1(u))$ such that*

$$\alpha - \tfrac{N}{4} \in \tfrac{1}{2}\mathbb{Z},$$
$$S(\alpha, \tfrac{N}{2} - \alpha) \cup S(\alpha + \tfrac{1}{2}, \tfrac{N}{2} - \alpha + \tfrac{1}{2}) \subset Z(P_2(u)),$$
$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u + 1 - \frac{\delta_{i1}}{2})}{P_i(u)} \left(\frac{\alpha - u}{\alpha + u - n}\right)^{\delta_{i,1}} \quad \forall \quad 1 \leq i \leq n.$$

This theorem will be proven in §5.5 as Theorem 5.5.7. By applying it in conjunction with certain structural properties of $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$, we will be able to prove the classification results stated in the next, and final, theorem of this section.

Theorem I. *Let $(1 + u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$ denote the subset of $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ consisting of series invariant under the transformation $u \mapsto \kappa - u$. Then*

(1) The isomorphism classes of finite-dimensional irreducible representations of

$X(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$ are parameterized by tuples

$$(g(u);(\alpha,(P_i(u))_{i=1}^n))\in(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}\times\mathbb{C}\times\mathbb{C}[u]^n$$

which satisfy the following set of conditions:

$$\begin{gathered}\textit{each }\ P_i(u)\ \textit{ is monic,}\\ \alpha\in\mathbb{C}\setminus Z(P_1(u)),\quad \alpha-\tfrac{N}{4}\in\tfrac{1}{2}\mathbb{Z},\\ S(\alpha,\tfrac{N}{2}-\alpha)\cup S(\alpha+\tfrac{1}{2},\tfrac{N}{2}-\alpha+\tfrac{1}{2})\subset Z(P_2(u)),\\ P_1(u)=P_1(-u+\kappa+1)\quad \textit{and}\quad P_i(u)=P_i(-u+n-i+2)\ \forall\, i\geq 2.\end{gathered}\tag{1.2.3}$$

(2) The isomorphism classes of finite-dimensional irreducible representations of $Y(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$ are parameterized by tuples

$$(\alpha,(P_i(u))_{i=1}^n)\in\mathbb{C}\times\mathbb{C}[u]^n$$

which satisfy the conditions (1.2.3).

Parts (1) and (2) of this theorem will appear as Proposition 5.5.8 and Corollary 5.5.9, respectively, in §5.5.2.

In Chapter 6, we will provide an indication of how well the above results generalize to the twisted Yangians associated to the remaining symmetric pairs of types B, C and D. We will also briefly comment on some natural problems which are motivated by both parts of this dissertation.

Chapter 2

The R-matrix Presentation of the Yangian

In this chapter, we develop the R-matrix formalism for the Yangian associated to an arbitrary simple Lie algebra $\mathfrak{g}$ and prove the results outlined in §1.1.2. The various pieces of Theorems B, C and D will be proven in §2.5, §2.6 and §2.7, respectively.

The rest of this chapter will proceed as follows. Section 2.1 serves as a preliminary section where relevant notation and basic facts are gathered. That section will also serve, in part, as a preliminary section for the rest of this dissertation. In §2.2, we recall the definition of $Y(\mathfrak{g})$ in Drinfeld's original presentation and survey some of its main properties, following [Dri85]. In §2.3, we construct the r-matrix presentations of the current algebras $\mathfrak{g}[z]$ and $\mathfrak{g}_{\mathcal{I}}[z]$ by first developing a matrix formalism for the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{\mathcal{I}}$. The results of this section should be viewed as classical analogues of Theorems B, C and D.

In §2.4, we introduce the extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$ and the R-matrix presentation of the Yangian $Y_R(\mathfrak{g})$ associated to a fixed finite-dimensional $Y(\mathfrak{g})$-module V. In addition, we establish some of their basic algebraic properties. After proving our main results in §2.5, §2.6 and §2.7.1, we conclude Chapter 2 in §2.7.2 by explaining in more detail how the results of §2.6 generalize results which are known to hold in the special case where $\mathfrak{g} = \mathfrak{sl}_N$, $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and the underlying $Y(\mathfrak{g})$-module V is the vector representation $\mathbb{C}^N$.

2.1 Preliminaries

2.1.1 Simple Lie algebras and their polynomial current algebras

Throughout this chapter we assume that $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra with symmetric non-degenerate invariant bilinear form $(\cdot,\cdot)$. Following the notation of [Dri85], we fix an orthonormal basis $\{X_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{g}$ with respect to this form, where Λ is an indexing set of size $\dim\mathfrak{g}$. Let $\{\alpha_{\lambda\nu}^{\gamma}\}_{\lambda,\nu,\gamma\in\Lambda}$ be the structure constants with respect to this basis:

$$[X_\lambda, X_\nu] = \sum_{\gamma\in\Lambda} \alpha_{\lambda\nu}^{\gamma} X_\gamma.$$

In particular, $\alpha_{\lambda\nu}^{\gamma} = -\alpha_{\nu\lambda}^{\gamma}$ and $\alpha_{\lambda\nu}^{\gamma} = -\alpha_{\lambda\gamma}^{\nu}$ for all $\lambda,\nu,\gamma\in\Lambda$, the second of these equalities being a consequence of the invariance of the bilinear form $(\cdot,\cdot)$.

Let Ω and ω denote the Casimir elements

$$\Omega = \sum_{\lambda\in\Lambda} X_\lambda\otimes X_\lambda \in \mathfrak{g}\otimes\mathfrak{g} \quad \text{and} \quad \omega = \sum_{\lambda\in\Lambda} X_\lambda^2 \in U(\mathfrak{g}),$$

and let $c_{\mathfrak{g}}$ denote the eigenvalue of ω in the adjoint representation. Here $U(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$. More generally, the notation $U(\mathfrak{a})$ will be used to denote the enveloping algebra of an arbitrary complex Lie algebra $\mathfrak{a}$, and Δ will denote the standard coproduct on $U(\mathfrak{a})$.

The polynomial current algebra of a complex Lie algebra $\mathfrak{a}$ is the Lie algebra which is equal to $\mathfrak{a}[z]=\mathfrak{a}\otimes\mathbb{C}[z]$ as a vector space, with Lie bracket given by

$$[X\otimes f(z), Y\otimes g(z)] = [X,Y]_{\mathfrak{g}}\otimes f(z)g(z) \quad \forall\, X,Y\in\mathfrak{a} \;\text{ and }\; f(z),g(z)\in\mathbb{C}[z].$$

Equivalently, $\mathfrak{a}[z]$ is the space of polynomial maps $\mathbb{C}\to\mathfrak{g}$ with Lie bracket given pointwise. If $\mathfrak{a}=\mathfrak{g}$ is a complex simple Lie algebra, then the enveloping algebra $U(\mathfrak{g}[z])$ is isomorphic to the unital associative algebra generated by elements $\{X_\lambda z^r :$

$\lambda \in \Lambda$, $r \geq 0\}$ subject to the defining relations

$$[X_\lambda z^r, X_\mu z^s] = \sum_{\gamma \in \Lambda} \alpha_{\lambda\mu}^{\gamma} X_\gamma z^{r+s} \quad \forall\, \lambda, \mu \in \Lambda \;\text{ and }\; r, s \geq 0. \tag{2.1.1}$$

The Lie algebra $\mathfrak{a}[z]$ is graded: we have $\mathfrak{a}[z] = \bigoplus_{k \geq 0} \mathfrak{a} z^k$, with $\mathfrak{a} z^k = \mathfrak{a} \otimes \mathbb{C} z^k$. If $\mathfrak{a} = \mathfrak{g}$ is simple, then $\mathfrak{g}[z]$ is generated as a Lie algebra by $\mathfrak{g}$ and $\mathfrak{g} z$.

In addition to having the structure of a Lie algebra, $\mathfrak{g}[z]$ admits the structure of a Lie bialgebra determined by the classical r-matrix

$$r_{\mathfrak{g}} = - \sum_{\lambda \in \Lambda, k \geq 0} X_\lambda v^k \otimes X_\lambda u^{-k-1} \in \mathfrak{g}[v] \widehat{\otimes} \mathfrak{g}(\!(u^{-1})\!).$$

That is, its Lie bialgebra cocommutator $\delta : \mathfrak{g}[z] \to \mathfrak{g}[z] \otimes \mathfrak{g}[z] \cong (\mathfrak{g} \otimes \mathfrak{g})[v, u]$ is given by

$$\delta(f(z))(v, u) = [f(v) \otimes 1 + 1 \otimes f(u), r_{\mathfrak{g}}] \quad \forall \quad f(z) \in \mathfrak{g}[z].$$

That the right-hand side of the above expression indeed belongs to $(\mathfrak{g} \otimes \mathfrak{g})[v, u]$ follows from the observation that $r_{\mathfrak{g}}$ may be identified with the element

$$-\frac{\Omega}{u - v} = - \sum_{k \geq 0} \Omega v^k u^{-k-1} \in (\mathfrak{g} \otimes \mathfrak{g}) \otimes (\mathbb{C}[v])[\![u^{-1}]\!],$$

together with the fact that $[\Delta(X), \Omega] = 0$ for all $X \in \mathfrak{g}$. The statement that $r_{\mathfrak{g}}$ is an r-matrix is meant to indicate that it is a solution of the classical Yang-Baxter equation with spectral parameter: see [ES02, §6.3.2], as well as §6.2 of *loc. cit.* for a more complete description of the Lie bialgebra structure on $\mathfrak{g}[z]$.

A deep understanding of the bialgebra $(\mathfrak{g}[z], \delta)$ will not be needed here, although the r-matrix $\frac{\Omega}{u-v}$ will play a significant role. We, however, adapt the viewpoint that this element be treated as a rational function in $u - v$ which can be expanded as a formal series in $(\mathfrak{g} \otimes \mathfrak{g}) \otimes \mathbb{C}[\![v^{\pm 1}, u^{\pm 1}]\!]$ in various ways: see Remark 2.3.10.

2.1.2 Matrix, formal series, and miscellaneous notation

In what follows, all vector spaces and algebras are assumed to be over the complex numbers $\mathbb{C}$, and we will maintain this assumption for the remainder of this thesis.

Suppose that W is an arbitrary vector space and that V is a finite-dimensional vector space of dimension N with a fixed basis $\{e_1, \ldots, e_N\}$, and let $\{E_{ij}\}_{1\leq i,j\leq N}$ denote the elementary matrices of $\mathrm{End}V$ with respect to this basis. We will often be working with spaces of the form $(\mathrm{End}V)^{\otimes m} \otimes W$, with $m \geq 1$. Given

$$A = \sum_{i,j=1}^{N} E_{ij} \otimes a_{ij} \in \mathrm{End}V \otimes W$$

and $1 \leq k \leq m$, we set

$$A_k = \sum_{i,j=1}^{N} 1^{\otimes(k-1)} \otimes E_{ij} \otimes 1^{\otimes(m-k)} \otimes a_{ij} \in (\mathrm{End}V)^{\otimes m} \otimes W.$$

If W is a formal power series ring or if more generally $A = A(u)$ depends on a formal parameter u, we will indicate this by writing $A_a(u)$ in place of A_a (and rather than $A(u)_a$).

Similarly, if $\mathscr{A}$ is a unital algebra and $B = \sum_{i=1}^{r} a_i \otimes b_i \in \mathscr{A} \otimes \mathscr{A}$ with $1 \leq k < l \leq m$ and $m \geq 2$, then we will denote by B_{kl} the element

$$B_{kl} = \sum_{i=1}^{r} 1^{\otimes(k-1)} \otimes a_i \otimes 1^{\otimes(l-k-1)} \otimes b_i \otimes 1^{\otimes(m-l)} \in \mathscr{A}^{\otimes m}.$$

We instead write $B_{kl}(u)$ if $B = B(u)$ depends on a formal parameter u.

Throughout this thesis, we will consider embeddings of elements $A(u) \in \mathrm{End}V \otimes \mathscr{A}[\![u^{-1}]\!]$ into $\mathrm{End}V \otimes (\mathscr{A} \otimes \mathscr{A})[\![u^{-1}]\!]$. With this in mind, given

$$A(u) = \sum_{i,j=1}^{N} E_{ij} \otimes a_{ij}(u) \in \mathrm{End}V \otimes \mathscr{A}[\![u^{-1}]\!]$$

and $1 \leq k \leq 2$, we define

$$A_{[k]}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes 1^{\otimes(k-1)} \otimes a_{ij}(u) \otimes 1^{\otimes(2-k)} \in \mathrm{End}V \otimes (\mathscr{A} \otimes \mathscr{A})[\![u^{-1}]\!].$$

Now suppose that W_1 and W_2 are arbitrary vector spaces, and let $\phi : W_1 \to W_2$ be a linear map. Then, given

$$a(u) = \sum_{r\geq 0} a_r u^{-r} \in W_1[\![u^{-1}]\!] \quad \text{and} \quad b(u) = \sum_{r\geq 0} b_r u^{-r} \in W_2[\![u^{-1}]\!],$$

we will write $\phi(a(u)) = b(u)$ to indicate that $\phi(a_r) = b_r$ for all $r \geq 0$. Conversely, we will use expressions of the form $\phi(a(u)) = b(u)$ (understood in the same way) to define linear maps, algebra homomorphisms and anti-homomorphisms. Similarly, expressions of the form

$$\phi(A(u)) = (\mathrm{id} \otimes \phi)A(u) = B(u)$$

with $A(u) \in \mathrm{End}V \otimes W_1[\![u^{-1}]\!]$ and $B(u) \in \mathrm{End}V \otimes W_2[\![u^{-1}]\!]$ will be used to define and interpret transformations $\phi : W_1 \to W_2$.

For any two vector spaces W_1 and W_2, let

$$\sigma_{W_1,W_2} : W_1 \otimes W_2 \to W_2 \otimes W_1$$

be the canonical permutation operator, which is defined on simple tensors by

$$\sigma_{W_1,W_2}(w_1 \otimes w_2) = w_2 \otimes w_1 \quad \forall\, w_1 \in W_1 \ \text{ and } \ w_2 \in W_2.$$

In practice, we will drop the subscripts and simply write $\sigma = \sigma_{W_1,W_2}$; the underlying vector spaces will always be clear from context. We will also write

$$R_{21} = \sigma(R) \in W_2 \otimes W_1 \quad \forall\, R \in W_1 \otimes W_2.$$

Finally, for any unital associative algebra $\mathscr{A}$ we denote by $\mathrm{Lie}(\mathscr{A})$ the Lie algebra which is equal to $\mathscr{A}$ as a vector space and has Lie bracket equal to the commutator bracket:

$$[a_1, a_2] = a_1 a_2 - a_2 a_1 \quad \forall\, a_1, a_1 \in \mathscr{A}.$$

2.2 The Yangian of a simple Lie algebra

In this section we recall the definition for the Yangian of $\mathfrak{g}$ in its J-presentation, as well as some of its properties which will play a role in §2.5 and §2.6. Aside from Proposition 2.2.2 and a few brief remarks, all of the contents of this section appeared in Drinfeld's seminal paper [Dri85].

Definition 2.2.1 ([Dri85])**.** The Yangian $Y(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by the set of elements $\{X, J(X) : X \in \mathfrak{g}\}$ subject to the defining relations

$$XY - YX = [X,Y]_{\mathfrak{g}}, \quad J([X,Y]) = [J(X),Y], \tag{2.2.1}$$

$$J(cX + dY) = cJ(X) + dJ(Y), \tag{2.2.2}$$

$$\begin{aligned}&[J(X),[J(Y),Z]] - [X,[J(Y),J(Z)]]\\ &\qquad = \sum_{\lambda,\mu,\nu\in\Lambda} ([X,X_\lambda],[[Y,X_\mu],[Z,X_\nu]])\{X_\lambda,X_\mu,X_\nu\},\end{aligned} \tag{2.2.3}$$

$$\begin{aligned}&[[J(X),J(Y)],[Z,J(W)]] + [[J(Z),J(W)],[X,J(Y)]]\\ &\qquad = \sum_{\lambda,\mu,\nu\in\Lambda} \Big(([X,X_\lambda],[[Y,X_\mu],[[Z,W],X_\nu]])\\ &\qquad\qquad + ([Z,X_\lambda],[[W,X_\mu],[[X,Y],X_\nu]])\Big)\{X_\lambda,X_\mu,J(X_\nu)\},\end{aligned} \tag{2.2.4}$$

for all $X,Y,Z,W \in \mathfrak{g}$ and $c,d \in \mathbb{C}$, where

$$\{x_1,x_2,x_3\} = \frac{1}{24}\sum_{\pi\in\mathfrak{S}_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \quad \forall\, x_1,x_2,x_3 \in Y(\mathfrak{g}).$$

The algebra $Y(\mathfrak{g})$ is equipped with an ascending filtration $\mathbf{F}^J$ defined by

$$\deg X = 0 \quad \text{and} \quad \deg J(X) = 1 \quad \forall\, X \in \mathfrak{g}.$$

For each $k \geq 0$, let $\mathbf{F}_k^J$ denote the subspace of $Y(\mathfrak{g})$ spanned by elements of degree less than or equal to k and denote by $\bar{X}$ and $\overline{J(X)}$ the images of X and $J(X)$, respectively, in $\mathbf{F}_0^J$ and $\mathbf{F}_1^J/\mathbf{F}_0^J$, respectively. By convention, we also set $\mathbf{F}_{-1}^J = \{0\}$. A proof of the following well-known result, dating back to [Dri85], was made available recently in [GRW19a].

Proposition 2.2.2 ([GRW19a, Proposition 2.2]). *The associated graded algebra*

$$\mathrm{gr} Y(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathbf{F}_k^J/\mathbf{F}_{k-1}^J$$

is isomorphic to $U(\mathfrak{g}[z])$. *An isomorphism* $\varphi_J : U(\mathfrak{g}[z]) \xrightarrow{\sim} \mathrm{gr}\, Y(\mathfrak{g})$ *is provided by the assignment*

$$X_\lambda z \mapsto \overline{J(X_\lambda)}, \quad X_\lambda \mapsto \overline{X}_\lambda \quad \forall\, \lambda \in \Lambda.$$

We pause momentarily to comment on the relations (2.2.3) and (2.2.4). It was pointed out in [Dri85] that

(*a*) when $\mathfrak{g} \cong \mathfrak{sl}_2$ the relation (2.2.3) follows from (2.2.1) together with (2.2.2), and

(*b*) when $\mathfrak{g} \ncong \mathfrak{sl}_2$ the relation (2.2.4) follows from the relations (2.2.1)-(2.2.3).

One way of seeing this is to appeal to the proof of [GRW19a, Theorem 2.6]. A careful reading of that proof together with [GNW18, 3(ii)] shows that if $\mathfrak{g} \ncong \mathfrak{sl}_2$ then the relation (2.2.4) can be omitted and the relation (2.2.3) can even be replaced with the relation

$$[J(h), J(h')] = \frac{1}{4} \sum_{\alpha,\beta\in\Delta_+} \alpha(h)\beta(h')[x_\alpha^- x_\alpha^+, x_\beta^- x_\beta^+] \quad \forall\, h, h' \in \mathfrak{h},$$

where $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$, Δ_+ denotes the set of positive roots of $\mathfrak{g}$, and for each $\alpha \in \Delta_+$ $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ are such that $(x_\alpha^+, x_\alpha^-) = 1$. If instead $\mathfrak{g} \cong \mathfrak{sl}_2$, then the proof of [GRW19a, Theorem 2.6] found in Appendix A of *loc. cit.* shows that the relation (2.2.3) can be omitted and (2.2.4) can be replaced with

$$[[J(e), J(f)], J(h)] = (fJ(e) - J(f)e)h,$$

where $\{e, f, h\}$ is the standard $\mathfrak{sl}_2$-triple and $(\cdot,\cdot)$ has been normalized to equal the trace form.

By [Dri85, Theorem 2], $Y(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S given by

$$\begin{gathered}\Delta(X) = X \otimes 1 + 1 \otimes X, \\ \Delta(J(X)) = J(X) \otimes 1 + 1 \otimes J(X) + \tfrac{1}{2}[X \otimes 1, \Omega], \\ \epsilon(X) = \epsilon(J(X)) = 0, \\ S(X) = -X, \quad S(J(x)) = -J(X) + \tfrac{1}{4}c_{\mathfrak{g}}X,\end{gathered} \tag{2.2.5}$$

where X is an arbitrary element of $\mathfrak{g}$. A proof that Δ is an algebra homomorphism may be found in [GNW18].

The enveloping algebra $U(\mathfrak{g}[z])$ has a one parameter family of Hopf algebra automorphisms $\overline{\tau}_c$, indexed by $c \in \mathbb{C}$, which are determined by

$$\overline{\tau}_c : Xz^r \to X(z+c)^r \quad \forall\, r \geq 0 \text{ and } X \in \mathfrak{g}.$$

The Yangian $Y(\mathfrak{g})$ also possesses such a family of Hopf algebra automorphisms which can be viewed as quantizations of these shift automorphisms. Explicitly, for each $c \in \mathbb{C}$, there is a Hopf algebra automorphism τ_c of $Y(\mathfrak{g})$ given by the assignment

$$X \mapsto X, \quad J(X) \mapsto J(X) + cX \quad \forall\, X \in \mathfrak{g}. \tag{2.2.6}$$

By replacing $c \in \mathbb{C}$ with a formal variable u, we obtain an automorphism τ_u of the polynomial algebra $Y(\mathfrak{g})[u]$ or even of the formal power series algebra $Y(\mathfrak{g})(\!(u^{-1})\!)$. Given complex numbers $c, d \in \mathbb{C}$ and formal variables u, v, we will write $\tau_{c,d} = \tau_c \otimes \tau_d$ and $\tau_{u,v} = \tau_u \otimes \tau_v$. We will also denote by Δ^{op} the opposite coproduct of $Y(\mathfrak{g})$; that is,

$$\Delta^{\mathrm{op}} = \sigma \circ \Delta, \quad \text{where} \quad \sigma = \sigma_{Y(\mathfrak{g}),Y(\mathfrak{g})}.$$

The next corollary follows immediately from the definition of the antipode S given in (2.2.5).

Corollary 2.2.3. *The square of the antipode S is given by $S^2 = \tau_{-\frac{1}{2}c_{\mathfrak{g}}}$.*

We are now prepared to introduce the universal R-matrix of $Y(\mathfrak{g})$.

Theorem 2.2.4 ([Dri85, Theorem 3])**.** *There is a unique formal series*

$$\mathcal{R}(u) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k u^{-k} \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[\![u^{-1}]\!]$$

satisfying the relations

$$(\mathrm{id}\otimes\Delta)\mathcal{R}(u)=\mathcal{R}_{12}(u)\mathcal{R}_{13}(u), \tag{2.2.7}$$

$$\tau_{0,u}\Delta^{\mathrm{op}}(Y)=\mathcal{R}(u)^{-1}(\tau_{0,u}\Delta(Y))\mathcal{R}(u)\quad \forall\, Y\in Y(\mathfrak{g}). \tag{2.2.8}$$

The series $\mathcal{R}(u)$ is called the universal R-matrix of $Y(\mathfrak{g})$ and it also satisfies the quantum Yang-Baxter equation

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v)=\mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v), \tag{2.2.9}$$

as well as the relations

$$\mathcal{R}_{12}(u)\mathcal{R}_{21}(-u)=1,\quad \tau_{c,d}\mathcal{R}(u)=\mathcal{R}(u+d-c), \tag{2.2.10}$$

$$\begin{aligned}\mathcal{R}(u)=1+\Omega u^{-1}+\sum_{\lambda\in\Lambda}(J(X_\lambda)\otimes X_\lambda-X_\lambda\otimes J(X_\lambda))u^{-2}+\tfrac{1}{2}\Omega^2u^{-2}\\ +O(u^{-3}).\end{aligned} \tag{2.2.11}$$

Note that (2.2.8) should be viewed as a relation in $(Y(\mathfrak{g})\otimes Y(\mathfrak{g}))(\!(u^{-1})\!)$ and the quantum Yang-Baxter equation (2.2.9) can be interpreted as an equality in the space $(Y(\mathfrak{g})\otimes Y(\mathfrak{g})\otimes Y(\mathfrak{g}))[\![v^{\pm1},u^{\pm1}]\!]$.

In addition to those properties of $\mathcal{R}(u)$ listed in the above theorem, standard arguments show that

$$(\mathrm{id}\otimes S)\mathcal{R}(u)=\mathcal{R}(u)^{-1}\quad \text{and}\quad (\mathrm{id}\otimes\epsilon)(\mathcal{R}(u))=1. \tag{2.2.12}$$

We end this section by recalling a result which concerns the uniqueness and rationality of $\mathcal{R}(u)$ when evaluated on any two finite-dimensional irreducible representations. Let ρ_V and ρ_W be finite-dimensional irreducible representations of $Y(\mathfrak{g})$ on the spaces V and W, respectively, and set

$$\mathcal{R}_{V,W}(u)=(\rho_V\otimes\rho_W)\mathcal{R}(-u).$$

Theorem 2.2.5 ([Dri85, Theorem 4] and [GRW19a, Theorem 3.10])**.** *Up to multiplication by elements of $\mathbb{C}[\![u^{-1}]\!]$, $\mathcal{R}_{V,W}(u)$ is the unique solution $R(u)\in\mathrm{End}(V\otimes$*

$W)[\![u^{-1}]\!]$ *of the equation*

$$(\rho_V \otimes \rho_W)(\tau_{u,v}\Delta(J(X)))R(u-v) \\ = R(u-v)(\rho_V \otimes \rho_W)(\tau_{u,v}\Delta^{\mathrm{op}}(J(X))) \quad \forall\, X \in \mathfrak{g}. \tag{2.2.13}$$

Additionally, there exists a formal series $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ *such that*

$$f(u)\mathcal{R}_{V,W}(u) \in \mathrm{End}(V \otimes W) \otimes \mathbb{C}(u).$$

The negative sign which appears in the definition of $\mathcal{R}_{V,W}(u)$ does not play an important role in this result and has been included so that, up to multiplication by a formal series, $\mathcal{R}_{\mathbb{C}^N,\mathbb{C}^N}(u)$ coincides with the R-matrix $R(u)$ given by (2.7.4) if $\mathfrak{g} = \mathfrak{sl}_N$ and (2.7.15) if $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$: see [GRW19a, Proposition 3.13].

2.3 The r-matrix presentation of the current algebra $\mathfrak{g}[z]$

An important ingredient needed to prove the isomorphism between the Drinfeld Yangian $Y(\mathfrak{g})$ and the RTT-Yangian $Y_R(\mathfrak{g})$ (see §2.4) is a presentation of the polynomial current algebra $\mathfrak{g}[z]$ which is determined by the image of the Casimir element Ω, or more precisely the classical r-matrix of $\mathfrak{g}[z]$, under a fixed representation of the Lie algebra $\mathfrak{g}$. In this section we obtain such a realization of $\mathfrak{g}[z]$ (see Corollary 2.3.7 and Proposition 2.3.9), and also for the current algebra $(\mathfrak{g} \oplus \mathfrak{z}_{\mathcal{I}})[z]$ of a certain trivial central extension $\mathfrak{g} \oplus \mathfrak{z}_{\mathcal{I}}$ of $\mathfrak{g}$ (see Proposition 2.3.16). The polynomial current algebra $(\mathfrak{g} \oplus \mathfrak{z}_{\mathcal{I}})[z]$ will play an analogous role to $\mathfrak{g}[z]$ in the study of the extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$.

2.3.1 Setup

Let V be a finite-dimensional $\mathfrak{g}$-module with associated homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

Set $N = \dim V$, and assume that V is not isomorphic to a direct sum of N copies of the trivial representation. The following setup will be used throughout this chapter, with the exception that from §2.3.3 onwards V will be assumed to be a finite-dimensional $Y(\mathfrak{g})$-module.

As in the preliminary section, we fix a basis $\{e_i\}_{1\leq i\leq N}$ of V and let $\{E_{ij}\}_{1\leq i,j\leq N}$ denote the usual elementary matrices with respect to this basis. Let Ω_ρ denote the image of Ω under $\rho\otimes\rho$:

$$\Omega_\rho = (\rho\otimes\rho)(\Omega).$$

Since $\mathfrak{g}$ is simple and $\mathrm{Ker}(\rho) \subsetneq \mathfrak{g}$, the homomorphism ρ is injective, and hence

$$\{X_\lambda^\bullet = \rho(X_\lambda)\}_{\lambda\in\Lambda}$$

is a linearly independent set in $\mathfrak{gl}(V)$ which spans a Lie subalgebra $\rho(\mathfrak{g})$ isomorphic to $\mathfrak{g}$. The Lie algebra $\mathfrak{gl}(V)$ acts on itself via the adjoint action, and we may restrict this action to $\mathfrak{g}\cong\rho(\mathfrak{g})$ to obtain a finite-dimensional representation of $\mathfrak{g}$. We denote the resulting $\mathfrak{g}$-module by $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$, and we let ϱ denote the corresponding Lie algebra homomorphism:

$$\varrho : \mathfrak{g} \to \mathrm{End}(\mathfrak{gl}(V)).$$

We use the same notation when $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$ is viewed as a $U(\mathfrak{g})$-module.

The space $\mathrm{span}\{X_\lambda^\bullet\}_{\lambda\in\Lambda}$ forms a submodule of $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$ isomorphic to the adjoint representation of $\mathfrak{g}$. Accordingly, we will write

$$\mathrm{ad}(\mathfrak{g}) = \mathrm{span}\{X_\lambda^\bullet\}_{\lambda\in\Lambda}$$

when the space on the right-hand side is viewed as a $\mathfrak{g}$-submodule of $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$.

We will extend the basis $\{X_\lambda^\bullet\}_{\lambda\in\Lambda}$ of $\mathrm{ad}(\mathfrak{g})$ to a basis $\{X_\lambda^\bullet\}_{\lambda\in\Lambda^\bullet}$ of $\mathrm{End}V$ which respects the decomposition of $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$ into irreducible submodules. Consider the subspace of intertwiners $\mathcal{E}_\mathfrak{g}$ defined by

$$\mathcal{E}_\mathfrak{g} = \mathrm{End}_\mathfrak{g}V.$$

This is a submodule of $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$ isomorphic to a direct sum of copies of the trivial representation $\mathbb{C}_\mathfrak{g}$ of $\mathfrak{g}$. As $\mathcal{E}_\mathfrak{g}$ intersects with $\mathrm{ad}(\mathfrak{g})$ trivially, the direct sum $\mathrm{ad}(\mathfrak{g})\oplus\mathcal{E}_\mathfrak{g}$

is also a submodule of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$. By complete reducibility, there is a submodule W' of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$ complimentary to $\mathrm{ad}(\mathfrak{g}) \oplus \mathcal{E}_{\mathfrak{g}}$. Let

$$W' = W_1 \oplus \cdots \oplus W_m \tag{2.3.1}$$

be its decomposition into a direct sum of irreducible $\mathfrak{g}$-submodules of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$, and set $W = \mathcal{E}_{\mathfrak{g}} \oplus W'$. In summary, we have the $\mathfrak{g}$-module decomposition

$$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V)) = \mathrm{ad}(\mathfrak{g}) \oplus W = \mathrm{ad}(\mathfrak{g}) \oplus \mathcal{E}_{\mathfrak{g}} \oplus W' = \mathrm{ad}(\mathfrak{g}) \oplus \mathcal{E}_{\mathfrak{g}} \oplus W_1 \oplus \cdots \oplus W_m.$$

Note that, by definition, every trivial subrepresentation of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$ consists of endomorphisms which commute with $\rho(\mathfrak{g})$, and hence is contained in $\mathcal{E}_{\mathfrak{g}}$. In particular, this implies that $W_i \ncong \mathbb{C}_{\mathfrak{g}}$ for any $1 \leq i \leq m$. Let $\mathcal{J}$ and Λ_i, for each $1 \leq i \leq m$, be indexing sets such that $\{X_\lambda^\bullet\}_{\lambda \in \mathcal{J}}$ is a basis for $\mathcal{E}_{\mathfrak{g}}$, and $\{X_\lambda^\bullet\}_{\lambda \in \Lambda_i}$ is a basis for W_i for each fixed $1 \leq i \leq m$. We then set

$$\Lambda^c = \mathcal{J} \sqcup \Lambda_1 \sqcup \cdots \sqcup \Lambda_m \quad \text{and} \quad \Lambda^\bullet = \Lambda \cup \Lambda^c.$$

Finally, we define a family of complex scalars $\{c_{ij}^\lambda, a_{ij}^\lambda : \lambda \in \Lambda^\bullet, 1 \leq i, j \leq N\}$ by

$$X_\lambda^\bullet = \sum_{i,j=1}^{N} c_{ij}^\lambda E_{ij} \quad \text{and} \quad E_{ij} = \sum_{\lambda \in \Lambda^\bullet} a_{ij}^\lambda X_\lambda^\bullet. \tag{2.3.2}$$

2.3.2 The Lie algebras $\mathfrak{g}_{\mathcal{J}}$, $\mathfrak{g}_\rho$ and their polynomial current algebras

We now turn to giving a presentation for the enveloping algebra of $\mathfrak{g}$ which is governed by Ω_ρ. This naturally leads to the desired presentation of the polynomial current algebra $\mathfrak{g}[z]$: see Corollary 2.3.7 and Proposition 2.3.9.

2.3.2.1 $U_\rho(\mathfrak{g})$ and the extended enveloping algebra $U_{\mathcal{J}}(\mathfrak{g})$

We begin by defining an algebra $U_{\mathcal{J}}(\mathfrak{g})$ which can be viewed as an extension of $U(\mathfrak{g})$. It will be proven in Proposition 2.3.6 that this algebra is isomorphic to the enveloping algebra of the Lie algebra $\mathfrak{g} \oplus \mathfrak{z}_{\mathcal{J}}$, where $\mathfrak{z}_{\mathcal{J}}$ is a commutative Lie algebra of dimension

$\dim \mathrm{End}_{\mathfrak{g}} V$.

Definition 2.3.1. The extended enveloping algebra $U_{\mathcal{J}}(\mathfrak{g})$ is defined to be the unital associative $\mathbb{C}$-algebra generated by elements $\{F_{ij}^{\mathcal{J}}\}_{1\leq i,j\leq N}$ subject to the defining relation

$$[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = [\Omega_\rho, F_2^{\mathcal{J}}] \quad \text{in} \quad (\mathrm{End}V)^{\otimes 2} \otimes U_{\mathcal{J}}(\mathfrak{g}), \tag{2.3.3}$$

where $F^{\mathcal{J}} = \sum_{i,j=1}^N E_{ij} \otimes F_{ij}^{\mathcal{J}} \in \mathrm{End}V \otimes U_{\mathcal{J}}(\mathfrak{g})$ and Ω_ρ has been identified with $\Omega_\rho \otimes 1$.

For each $\lambda \in \Lambda^\bullet$, set $X_\lambda^{\mathcal{J}} = \sum_{i,j=1}^N a_{ij}^\lambda F_{ij}^{\mathcal{J}}$ (see (2.3.2)) so that

$$F^{\mathcal{J}} = \sum_{\lambda\in\Lambda^\bullet} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{J}},$$

and let $K = \sum_{i,j=1}^N E_{ij} \otimes k_{ij}$ be the element of $\mathrm{End}V \otimes U_{\mathcal{J}}(\mathfrak{g})$ defined by

$$K = \sum_{i,j=1}^N E_{ij} \otimes k_{ij} = \sum_{\lambda\in\Lambda^c} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{J}}. \tag{2.3.4}$$

Given an arbitrary vector space $\mathbf{U}$ and $A = \sum_{i,j=1}^N E_{ij} \otimes \mathbf{u}_{ij} \in \mathrm{End}V \otimes \mathbf{U}$, define

$$\omega(A) = \sum_{i,j=1}^N \omega(E_{ij}) \otimes \mathbf{u}_{ij} \in \mathrm{End}V \otimes \mathbf{U}, \quad \text{where} \quad \omega(E_{ij}) = \varrho(\omega)(E_{ij}),$$

and let $\nabla : \mathrm{End}V \otimes \mathrm{End}V \to \mathrm{End}V$ denote the multiplication (or composition) map.

Lemma 2.3.2. *K satisfies the following properties:*

(1) The coefficients k_{ij} of K are central,

(2) $[\Omega_\rho, K_2] = 0 = [\Omega_\rho, K_1]$ and $\omega(K) = 0$,

(3) $X_\lambda^{\mathcal{J}} = 0$ for all $\lambda \in \Lambda^c \setminus \mathcal{J}$. In particular, $K = \sum_{\lambda\in\mathcal{J}} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{J}}$.

Proof. Consider first (1). After setting $\mathsf{F} = F^{\mathcal{J}} - K \in \mathrm{ad}(\mathfrak{g}) \otimes U_{\mathcal{J}}(\mathfrak{g})$, (2.3.3) gives

$$[K_1, F_2^{\mathcal{J}}] = [\Omega_\rho, F_2^{\mathcal{J}}] - [\mathsf{F}_1, F_2^{\mathcal{J}}] \in \mathrm{ad}(\mathfrak{g}) \otimes \mathrm{End}V \otimes U_{\mathcal{J}}(\mathfrak{g}). \tag{2.3.5}$$

Since $[K_1, F_2^{\mathcal{J}}] \in W \otimes \mathrm{End}V \otimes U_{\mathcal{J}}(\mathfrak{g})$, both sides of this equality must vanish, which proves (1).

Proof of (2). By Part (1) and (2.3.5), we have

$$[\Omega_\rho, K_2] = [\mathsf{F}_2, \Omega_\rho] + [\mathsf{F}_1, \mathsf{F}_2] \in \mathrm{ad}(\mathfrak{g}) \otimes \mathrm{ad}(\mathfrak{g}) \otimes U_{\mathcal{J}}(\mathfrak{g}). \tag{2.3.6}$$

As W is a submodule of $\mathrm{ad}_\mathfrak{g}(\mathfrak{gl}(V))$, $[\Omega_\rho, K_2] \in \mathrm{ad}(\mathfrak{g}) \otimes W \otimes U_{\mathcal{J}}(\mathfrak{g})$. Therefore $[\Omega_\rho, K_2] = 0$, and applying the permutation operator $\sigma \otimes 1$ to both sides of this equality gives $[\Omega_\rho, K_1] = 0$. These two relations also imply that

$$0 = (\nabla \otimes 1)([\Omega_\rho, K_2 - K_1]) = \sum_{\lambda \in \Lambda, \mu \in \Lambda^c} \left[X_\lambda^\bullet, [X_\lambda^\bullet, X_\mu^\bullet]\right] \otimes X_\mu^{\mathcal{J}} = \omega(K).$$

Proof of (3). On each irreducible component W_i of W' (see (2.3.1)), ω operates as multiplication by a scalar c_i. Hence, from the equality $\omega(K) = 0$ and the fact that $\omega(X_\mu^\bullet) = 0$ for all $\mu \in \mathcal{J}$, we obtain

$$0 = \sum_{i=1}^m c_i \left(\sum_{\mu \in \Lambda_i} X_\mu^\bullet \otimes X_\mu^{\mathcal{J}} \right) = \sum_{\mu \in \Lambda^c \setminus \mathcal{J}} X_\mu^\bullet \otimes c_\mu X_\mu^{\mathcal{J}}, \tag{2.3.7}$$

where in the second equality we have defined c_μ, for each $\mu \in \Lambda^c \setminus \mathcal{J}$, to be equal to c_i for the unique $i \in \{1, \ldots, m\}$ such that $\mu \in \Lambda_i$. It is well known result from the classical theory of simple Lie algebras over $\mathbb{C}$ that the Casimir element operates as a nonzero scalar in every non-trivial finite-dimensional irreducible module. Therefore, $c_i \neq 0$ for all $1 \leq i \leq m$ and (2.3.7) implies that $X_\mu^{\mathcal{J}} = 0$ for all $\mu \in \Lambda^c \setminus \mathcal{J}$. □

The next lemma gives two equivalent definitions of K and proves that there is a morphism $U(\mathfrak{g}) \to U_{\mathcal{J}}(\mathfrak{g})$.

Lemma 2.3.3. *The matrices $F^{\mathcal{J}}$ and K satisfy the identities*

$$[\Omega_\rho, F_2^{\mathcal{J}}] = [F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = [F_1^{\mathcal{J}}, \Omega_\rho], \tag{2.3.8}$$

$$F^{\mathcal{J}} - 2c_\mathfrak{g}^{-1}(\nabla \otimes 1)[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = K = F^{\mathcal{J}} - c_\mathfrak{g}^{-1}\omega(F^{\mathcal{J}}). \tag{2.3.9}$$

Moreover, the assignment $X_\lambda \mapsto -X_\lambda^{\mathcal{J}}$ for all $\lambda \in \Lambda$ extends to a homomorphism

$$\iota_{\mathcal{J}} : U(\mathfrak{g}) \to U_{\mathcal{J}}(\mathfrak{g}).$$

Proof. Applying the permutation operator $\sigma \otimes 1$ to $[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = [\Omega_\rho, F_2^{\mathcal{J}}]$ gives

$$-[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = [\Omega_\rho, F_1^{\mathcal{J}}],$$

which implies (2.3.8).

By Part (2) of Lemma 2.3.2, $F^{\mathcal{J}} - c_{\mathfrak{g}}^{-1}\omega(F^{\mathcal{J}}) = K$. Since

$$(\nabla \otimes 1)[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = (\nabla \otimes 1)[\Omega_\rho, F_2^{\mathcal{J}}],$$

the relation (2.3.8) yields

$$(\nabla \otimes 1)[F_1^{\mathcal{J}}, F_2^{\mathcal{J}}] = \tfrac{1}{2}(\nabla \otimes 1)([\Omega_\rho, F_2^{\mathcal{J}}] - [\Omega_\rho, F_1^{\mathcal{J}}]) = \tfrac{1}{2}\omega(F^{\mathcal{J}}),$$

which proves (2.3.9).

As for the second part of the lemma, we obtain from Part (2) of Lemma 2.3.2 and (2.3.6) that $[\mathsf{F}_1, \mathsf{F}_2] = [\Omega_\rho, \mathsf{F}_2]$, where $\mathsf{F} = F^{\mathcal{J}} - K$. Expanding in terms of the basis $\{X_\lambda^\bullet \otimes X_\mu^\bullet\}_{\lambda,\mu\in\Lambda}$ of $\mathrm{ad}(\mathfrak{g}) \otimes \mathrm{ad}(\mathfrak{g})$ gives

$$[X_\lambda^{\mathcal{J}}, X_\mu^{\mathcal{J}}] = \sum_{\gamma\in\Lambda} \alpha_{\lambda\gamma}^{\mu} X_\gamma^{\mathcal{J}} = -\sum_{\gamma\in\Lambda} \alpha_{\lambda\mu}^{\gamma} X_\gamma^{\mathcal{J}} \quad \forall\, \lambda, \mu \in \Lambda.$$

Thus, the assignment $X_\lambda \mapsto -X_\lambda^{\mathcal{J}}$, for all $\lambda \in \Lambda$, extends to a homomorphism $\iota_{\mathcal{J}} : U(\mathfrak{g}) \to U_{\mathcal{J}}(\mathfrak{g})$. □

We now simultaneously define the algebra $U_\rho(\mathfrak{g})$ as a quotient of $U_{\mathcal{J}}(\mathfrak{g})$ and prove that it is isomorphic to the enveloping algebra $U(\mathfrak{g})$.

Proposition 2.3.4. *Let $U_\rho(\mathfrak{g})$ be the quotient of $U_{\mathcal{J}}(\mathfrak{g})$ by the two-sided ideal generated by the coefficients of the central matrix K. Equivalently, $U_\rho(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by elements $\{F_{ij}\}_{1\le i,j\le N}$ subject to the defining relations*

$$[F_1, F_2] = [\Omega_\rho, F_2], \tag{2.3.10}$$
$$F = c_{\mathfrak{g}}^{-1}\omega(F), \tag{2.3.11}$$

where $F = \sum_{i,j=1}^N E_{ij} \otimes F_{ij} \in \mathrm{End}V \otimes U_\rho(\mathfrak{g})$.

Then $U_\rho(\mathfrak{g})$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$. An isomorphism ϕ_ρ is given by

$$\phi_\rho : U_\rho(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}), \quad F \mapsto -(\rho \otimes 1)\Omega. \tag{2.3.12}$$

Proof. Set $\mathcal{F} = \sum_{i,j=1}^N E_{ij} \otimes \mathcal{F}_{ij} = -(\rho \otimes 1)\Omega$. By (2.3.2), the element $\mathcal{F}_{ij} = \phi_\rho(F_{ij})$ is equal to $-\sum_{\lambda\in\Lambda} c_{ij}^\lambda X_\lambda$.

Step 1: ϕ_ρ is a homomorphism of algebras.

Recall that $[\Omega, \Delta(X)] = 0$ for all $X \in \mathfrak{g}$. This implies that, in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, we have the identity $[\Omega_{13}, \Omega_{23}] = -[\Omega_{12}, \Omega_{23}]$. Applying the homomorphism $\rho\otimes\rho\otimes 1$ to both sides of this identity, we obtain the relation

$$[\mathcal{F}_1, \mathcal{F}_2] = [\Omega_\rho, \mathcal{F}_2] \quad \text{in} \quad (\mathrm{End} V)^{\otimes 2} \otimes \mathfrak{g}.$$

Hence, the assignment (2.3.12) preserves the relation (2.3.10).

Since we also have $\mathcal{F} = -\sum_{\lambda\in\Lambda} X_\lambda^\bullet \otimes X_\lambda \in \mathrm{ad}(\mathfrak{g}) \otimes \mathfrak{g}$, and ω acts on $\mathrm{ad}(\mathfrak{g})$ as multiplication by the scalar $c_\mathfrak{g}$, the relation $\mathcal{F} = c_\mathfrak{g}^{-1}\omega(\mathcal{F})$ is satisfied, and thus ϕ_ρ is a homomorphism.

Step 2: ϕ_ρ is an isomorphism.

For each $\lambda \in \Lambda^\bullet$, define X_λ^ρ to be the image of $X_\lambda^\mathcal{J}$ under the natural quotient map $q : U_\mathcal{J}(\mathfrak{g}) \twoheadrightarrow U_\rho(\mathfrak{g})$.

Since $q(K) = 0$, $X_\lambda^\rho = 0$ for all $\lambda \in \Lambda^c$. Let $\psi = q \circ \iota_\mathcal{J} : U(\mathfrak{g}) \to U_\rho(\mathfrak{g})$, where $\iota_\mathcal{J} : U(\mathfrak{g}) \to U_\mathcal{J}(\mathfrak{g})$ is the morphism from Lemma 2.3.3. Then $\phi_\rho \circ \psi = \mathrm{id}_{U(\mathfrak{g})}$, and to see that $\psi \circ \phi_\rho = \mathrm{id}_{U_\rho(\mathfrak{g})}$ it suffices to note that $\{X_\lambda^\rho\}_{\lambda\in\Lambda}$ generates $U_\rho(\mathfrak{g})$, which is immediate since it is the image of the generating set $\{X_\lambda^\mathcal{J}\}_{\lambda\in\Lambda^\bullet}$ of $U_\mathcal{J}(\mathfrak{g})$. This proves that ϕ_ρ is an isomorphism with inverse ψ. □

Remark 2.3.5. After expanding $\Omega_\rho = \sum_{i,j,k,l=1}^N \mathsf{c}_{ij}^{kl} E_{ij} \otimes E_{kl}$, we may rewrite the relations (2.3.10) and (2.3.11) of $U_\rho(\mathfrak{g})$ more explicitly in terms of the generators F_{ij}. They are

$$\begin{aligned} [F_{ij}, F_{kl}] &= \sum_{a=1}^N \left(\mathsf{c}_{ij}^{ka} F_{al} - \mathsf{c}_{ij}^{al} F_{ka}\right) \\ F_{ij} &= 2c_\mathfrak{g}^{-1} \sum_{a=1}^N [F_{ia}, F_{aj}] \end{aligned} \tag{2.3.13}$$

for all $1 \le i, j, k, l \le N$, where to obtain the second relation we have employed that, by (2.3.9), $c_{\mathfrak{g}}^{-1}\omega(F) = 2c_{\mathfrak{g}}^{-1}(\nabla \otimes 1)[F_1, F_2]$.

Let $\mathfrak{g}_\rho$ be the Lie subalgebra of $\mathrm{Lie}(U_\rho(\mathfrak{g}))$ generated by $\{X_\lambda^\rho\}_{\lambda\in\Lambda}$, or equivalently by $\{F_{ij}\}_{1\leq i,j\leq N}$. Then Proposition 2.3.4 implies that (2.3.10) and (2.3.11) are defining relations for $\mathfrak{g}_\rho$ and that $\phi_\rho|_{\mathfrak{g}_\rho}$ is an isomorphism of Lie algebras $\mathfrak{g}_\rho \xrightarrow{\sim} \mathfrak{g}$. Consequently $U(\mathfrak{g}_\rho) \cong U_\rho(\mathfrak{g})$, and we will henceforth exploit this fact and denote $U_\rho(\mathfrak{g})$ instead by $U(\mathfrak{g}_\rho)$.

We now return to the study of the algebra $U_{\mathcal{J}}(\mathfrak{g})$. Define $\mathfrak{z}_{\mathcal{J}}$ to be the commutative Lie algebra with basis $\{\mathcal{K}_\lambda^{\mathcal{J}}\}_{\lambda\in\mathcal{J}}$, and identify the enveloping algebra $U(\mathfrak{z}_{\mathcal{J}})$ with

$$\mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}].$$

We will denote the matrix $\sum_{\lambda\in\mathcal{J}} X_\lambda^\bullet \otimes \mathcal{K}_\lambda^{\mathcal{J}} \in \mathrm{End}V \otimes \mathfrak{z}_{\mathcal{J}}$ by $\mathscr{K}^{\mathcal{J}}$.

Proposition 2.3.6. *The assignment $F^{\mathcal{J}} \mapsto F + \mathscr{K}^{\mathcal{J}}$ extends to an isomorphism of algebras*

$$\phi_{\mathcal{J}} : U_{\mathcal{J}}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}] \otimes U(\mathfrak{g}_\rho). \tag{2.3.14}$$

Proof. Since $\mathscr{K}^{\mathcal{J}} \in \mathcal{E}_{\mathfrak{g}} \otimes \mathfrak{z}_{\mathcal{J}}$, we have $[\Omega_\rho, \mathscr{K}_2^{\mathcal{J}}] = 0$. As the coefficients of $\mathscr{K}^{\mathcal{J}}$ are also central and F satisfies (2.3.10), $F + \mathscr{K}^{\mathcal{J}}$ satisfies the defining relation (2.3.3) of $U_{\mathcal{J}}(\mathfrak{g})$. Thus the assignment $F^{\mathcal{J}} \mapsto F + \mathscr{K}^{\mathcal{J}}$ extends to a homomorphism

$$\phi_{\mathcal{J}} : U_{\mathcal{J}}(\mathfrak{g}) \to \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}] \otimes U(\mathfrak{g}_\rho).$$

Since the coefficients of K are central, we deduce that there is an algebra homomorphism $\psi_{\mathfrak{z}_{\mathcal{J}}} : \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}] \to U_{\mathcal{J}}(\mathfrak{g})$ given by $\mathscr{K}^{\mathcal{J}} \mapsto K$. Let $\iota = \iota_{\mathcal{J}} \circ \phi_\rho : U(\mathfrak{g}_\rho) \to U_{\mathcal{J}}(\mathfrak{g})$. Since $[\iota(X), \psi_{\mathfrak{z}_{\mathcal{J}}}(Y)] = 0$ for all $X \in U(\mathfrak{g}_\rho)$ and $Y \in \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}]$, there is a unique homomorphism

$$\psi_{\mathcal{J}} = \psi_{\mathfrak{z}_{\mathcal{J}}} \otimes \iota : \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} \,:\, \lambda \in \mathcal{J}] \otimes U(\mathfrak{g}_\rho) \to U_{\mathcal{J}}(\mathfrak{g})$$

satisfying $\psi_{\mathcal{J}}(\mathscr{K}^{\mathcal{J}}) = K$ and $\psi_{\mathcal{J}}(F) = \mathsf{F}$, where we recall that $\mathsf{F} = F^{\mathcal{J}} - K$. As $\phi_{\mathcal{J}}$ is completely determined by $\phi_{\mathcal{J}}(K) = \mathscr{K}^{\mathcal{J}}$ and $\phi_{\mathcal{J}}(\mathsf{F}) = F$, it follows immediately that $\psi_{\mathcal{J}} = \phi_{\mathcal{J}}^{-1}$. □

Define $\mathfrak{g}_{\mathcal{J}}$ to be the Lie subalgebra of $\mathrm{Lie}(U_{\mathcal{J}}(\mathfrak{g}))$ generated by $\{F_{ij}^{\mathcal{J}}\}_{1\leq i,j\leq N}$. Then the restriction $\phi_{\mathcal{J}}|_{\mathfrak{g}_{\mathcal{J}}}$ (see (2.3.14)) and its composition with $\mathrm{id} \otimes \phi_\rho$ (see (2.3.12)) produce isomorphisms

$$\mathfrak{g}_{\mathcal{J}} \xrightarrow{\sim} \mathfrak{g}_\rho \oplus \mathfrak{z}_{\mathcal{J}} \xrightarrow{\sim} \mathfrak{g} \oplus \mathfrak{z}_{\mathcal{J}}, \tag{2.3.15}$$

and we have $U(\mathfrak{g}_{\mathcal{J}}) \cong U_{\mathcal{J}}(\mathfrak{g})$. Accordingly, we will henceforth denote $U_{\mathcal{J}}(\mathfrak{g})$ by $U(\mathfrak{g}_{\mathcal{J}})$.

2.3.2.2 The polynomial current algebras $\mathfrak{g}_\rho[z]$ and $\mathfrak{g}_{\mathcal{J}}[z]$

As a consequence of Proposition 2.3.4 and the comments following Remark 2.3.5, the current algebras $\mathfrak{g}_\rho[z]$ and $\mathfrak{g}[z]$ are isomorphic. Similarly, Proposition 2.3.6 and the isomorphism (2.3.15) imply that $\mathfrak{g}_{\mathcal{J}}[z] \cong (\mathfrak{g} \oplus \mathfrak{z}_{\mathcal{J}})[z]$. The former identification leads to the so called r-matrix realization of $\mathfrak{g}[z]$, as we will illustrate in this subsection.

Corollary 2.3.7. *An isomorphism $\phi_\rho^z : \mathfrak{g}_\rho[z] \to \mathfrak{g}[z]$ is provided by the assignment*

$$\phi_\rho^z : F^{(r)} \mapsto -(\rho \otimes 1)(\Omega z^r) \quad \forall\, r \geq 0, \quad \textit{where}$$

$$F^{(r)} = \sum_{i,j=1}^{N} E_{ij} \otimes F_{ij} z^r \in \mathrm{End} V \otimes \mathfrak{g}_\rho[z] \quad \textit{and} \quad \Omega z^r = \sum_{\lambda \in \Lambda} X_\lambda \otimes X_\lambda z^r \in \mathfrak{g} \otimes \mathfrak{g}[z].$$

In particular, $U(\mathfrak{g}[z])$ is isomorphic to the unital associative $\mathbb{C}$-algebra generated by the family of elements $\{F_{ij}^{(r)} = F_{ij} z^r \,:\, 1 \leq i,j \leq N,\, r \in \mathbb{Z}_{\geq 0}\}$ subject to the defining relations

$$[F_1^{(r)}, F_2^{(s)}] = [\Omega_\rho, F_2^{(r+s)}] \quad \forall\, r,s \geq 0, \tag{2.3.16}$$

$$F^{(r)} = c_{\mathfrak{g}}^{-1} \omega(F^{(r)}) \quad \forall\, r \geq 0. \tag{2.3.17}$$

Proof. The corollary follows from Proposition 2.3.4, the three sentences following Remark 2.3.5, and the definition of the current algebra $\mathfrak{g}[z]$ (see (2.1.1)). □

Remark 2.3.8. The relations (2.3.16) and (2.3.17) are, of course, just the defining relations of $U(\mathfrak{g}_\rho[z])$. Omitting the relation (2.3.17) gives the definition of $U(\mathfrak{g}_{\mathcal{J}}[z])$.

Introduce the generating matrix

$$\begin{aligned} F(u) &= \sum_{i,j=1}^{N} E_{ij} \otimes F_{ij}(u) \in \mathrm{End} V \otimes (\mathfrak{g}_\rho[z])[\![u^{-1}]\!], \quad \text{where} \\ F_{ij}(u) &= \sum_{r \geq 0} F_{ij}^{(r)} u^{-r-1} \in (\mathfrak{g}_\rho[z])[\![u^{-1}]\!]. \end{aligned} \tag{2.3.18}$$

Using this notation, we can express the defining relations of $\mathfrak{g}[z]$ (or more precisely those of $\mathfrak{g}_\rho[z]$) using the classical r-matrix $\frac{\Omega}{u-v}$ associated to its standard Lie bialgebra structure.

Proposition 2.3.9. *The defining relations* (2.3.16) *and* (2.3.17) *are equivalent to the relations*

$$[F_1(u), F_2(v)] = \left[\frac{\Omega_\rho}{u-v}, F_1(u) + F_2(v)\right], \tag{2.3.19}$$

$$F(u) = c_{\mathfrak{g}}^{-1}\omega(F(u)). \tag{2.3.20}$$

The relation (2.3.19) *independently serves as the defining relation of* $U(\mathfrak{g}_{\mathcal{J}}[z])$.

Proof. It is clear that the relation (2.3.20) is equivalent to (2.3.17). To prove the equivalence of (2.3.19) with (2.3.16), we will expand

$$(u-v)^{-1} = \sum_{p\geq 0} v^p u^{-p-1} \in (\mathbb{C}[v])[\![u^{-1}]\!], \tag{2.3.21}$$

view (2.3.19) as an equality in the space $(\mathrm{End}V)^{\otimes 2} \otimes U(\mathfrak{g}_\rho[z])[\![v^{\pm 1}, u^{-1}]\!]$, and compare the coefficient of $v^s u^{-r}$ on each side for $s \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$. Note that (2.3.21) is not the unique expansion of $(u-v)^{-1}$ in $\mathbb{C}[\![v^{\pm 1}, u^{\pm 1}]\!]$, and thus there are other equivalent ways of viewing (2.3.19): see Remark 2.3.10.

Expanding (2.3.19) using (2.3.21), we obtain

$$\begin{aligned}\sum_{r,s\geq 0} &[F_1^{(r)}, F_2^{(s)}] v^{-s-1} u^{-r-1} \\ &= \sum_{p,a,b\geq 0} \left([\Omega_\rho, F_1^{(a)}] v^p u^{-p-a-2} + [\Omega_\rho, F_2^{(b)}] v^{p-b-1} u^{-p-1}\right)\end{aligned} \tag{2.3.22}$$

Comparing the coefficient of $u^{-r-1}v^{-s-1}$ in both sides, for $r, s \in \mathbb{Z}_{\geq 0}$, we obtain (2.3.17):

$$[F_1^{(r)}, F_2^{(s)}] = [\Omega_\rho, F_2^{(r+s)}] \quad \forall\, r, s \geq 0.$$

We must also compare the coefficient of $v^s u^{-r}$ (for $r, s \in \mathbb{Z}_{\geq 0}$) in both sides of (2.3.22) to guarantee that this relation does not imply any additional relations which are not satisfied in $U(\mathfrak{g}_\rho[z])$. If $0 \leq r < 2$ or $s > r-2$, the coefficient of $u^{-r}v^s$ on both sides of (2.3.19) is zero. Otherwise, we obtain

$$0 = [\Omega_\rho, F_1^{(r-s-2)}] + [\Omega_\rho, F_2^{(r-s-2)}],$$

which is also a consequence of (2.3.16): this can be deduced from (2.3.16) in the same way that the relation (2.3.8) of Lemma 2.3.3 was deduced from (2.3.3). □

Remark 2.3.10. In the proof of Proposition 2.3.9 we have expanded the rational expression $(u-v)^{-1}$ as an element of $(\mathbb{C}[v])[\![u^{-1}]\!]$ and then interpreted (2.3.19) as an equality in $(\mathrm{End}V)^{\otimes 2} \otimes U(\mathfrak{g}_\rho[z])[\![v^{\pm 1}, u^{-1}]\!]$. As mentioned in the proof of the proposition, this is not the only way we could have proceeded. Working in a more general framework, (2.3.19) should be viewed as an equality in $(\mathrm{End}V)^{\otimes 2} \otimes U(\mathfrak{g}_\rho[z])[\![v^{\pm 1}, u^{\pm 1}]\!]$. In particular, $(u-v)^{-1}$ can be expanded as the formal series $-\sum_{p\geq 0} u^p v^{-p-1}$ in $(\mathbb{C}[u])[\![v^{-1}]\!]$, leading to an equivalent set of defining relations.

An alternative expansion involves multiplying both sides of (2.3.19) by the polynomial $u-v$ and then expanding both sides as elements of $(\mathrm{End}V)^{\otimes 2} \otimes U(\mathfrak{g}_\rho[z])[\![u^{-1}, v^{-1}]\!]$: see for instance §1.1 of [Mol07].

2.3.3 The extended Lie algebra $\mathfrak{g}_\mathcal{I}$ and its polynomial current algebra

In this subsection we consider an algebra $U_\mathcal{I}(\mathfrak{g})$ which is constructed from a fixed finite-dimensional $Y(\mathfrak{g})$-module. Like $U(\mathfrak{g}_\mathcal{J}) = U_\mathcal{J}(\mathfrak{g})$, it is an extension of the enveloping algebra $U(\mathfrak{g}_\rho)$, but the role played by $\mathrm{End}_\mathfrak{g} V$ is instead played by $\mathrm{End}_{Y(\mathfrak{g})} V$. Consequently, $U_\mathcal{I}(\mathfrak{g})$ encodes certain information about the underlying $Y(\mathfrak{g})$-module structure which $U(\mathfrak{g}_\mathcal{J})$ does not.

Henceforth, we assume that V is a finite-dimensional $Y(\mathfrak{g})$-module with corresponding homomorphism $\rho : Y(\mathfrak{g}) \to \mathrm{End}V$. We also assume that V contains a non-trivial irreducible submodule. This hypothesis guarantees that V has at least one non-trivial irreducible component when viewed as a $\mathfrak{g}$-module (via restriction), and hence that we are in the situation of §2.3.1. In particular, all the definitions and results of the previous subsection apply.

Going forward, we will need to specialize our basis $\{X_\lambda^\bullet\}_{\lambda \in \mathcal{J}}$ of $\mathcal{E}_\mathfrak{g} = \mathrm{End}_\mathfrak{g}(V)$. Let $\mathcal{E} \subset \mathcal{E}_\mathfrak{g}$ denote the subspace of $Y(\mathfrak{g})$-module endomorphisms, and let $\mathcal{E}_c$ be a subspace of $\mathcal{E}_{\mathfrak{g}}$ complimentary to $\mathcal{E}$:

$$\mathcal{E} = \mathrm{End}_{Y(\mathfrak{g})}V \subset \mathcal{E}_{\mathfrak{g}}, \quad \mathcal{E}_{\mathfrak{g}} = \mathcal{E} \oplus \mathcal{E}_c.$$

We may then partition $\mathcal{J} = \mathcal{I} \sqcup \mathcal{I}_c$ and choose $\{X_\lambda^\bullet\}_{\lambda \in \mathcal{J}}$ in such a way that $\{X_\lambda^\bullet\}_{\lambda \in \mathcal{I}}$ is a basis of $\mathcal{E}$ and $\{X_\lambda^\bullet\}_{\lambda \in \mathcal{I}_c}$ is a basis of $\mathcal{E}_c$.

2.3.3.1 The extended enveloping algebra $U_{\mathcal{I}}(\mathfrak{g})$

Following our convention of labeling $X^\bullet = \rho(X)$ for each $X \in \mathfrak{g}$, we will write $J(X^\bullet)$ for the image of $J(X)$ in $\mathrm{End}V$ under ρ. In addition, we define a module homomorphism

$$J : \mathrm{ad}(\mathfrak{g}) \to \mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V)), \quad X^\bullet \mapsto J(X^\bullet) \quad \forall\, X \in \mathfrak{g}.$$

Definition 2.3.11. Define $U_{\mathcal{I}}(\mathfrak{g})$ to be the quotient of $U(\mathfrak{g}_{\mathcal{J}})$ by the two-sided ideal generated by the relation $[K_2, (1 \otimes J)(\Omega_\rho)] = [K_1, (J \otimes 1)(\Omega_\rho)]$. That is, $U_{\mathcal{I}}(\mathfrak{g})$ is the associative unital $\mathbb{C}$-algebra generated by elements $\{F_{ij}^{\mathcal{I}}\}_{1 \le i,j \le N}$ subject to the defining relations

$$[F_1^{\mathcal{I}}, F_2^{\mathcal{I}}] = [\Omega_\rho, F_2^{\mathcal{I}}], \tag{2.3.23}$$

$$[K_2^{\mathcal{I}}, (1 \otimes J)(\Omega_\rho)] = [K_1^{\mathcal{I}}, (J \otimes 1)(\Omega_\rho)] \tag{2.3.24}$$

where $F^{\mathcal{I}} = \sum_{i,j=1}^N E_{ij} \otimes F_{ij}^{\mathcal{I}} \in \mathrm{End}V \otimes U_{\mathcal{I}}(\mathfrak{g})$ and $K^{\mathcal{I}} = F^{\mathcal{I}} - c_{\mathfrak{g}}^{-1}\omega(F^{\mathcal{I}})$.

We now work towards establishing an analogue of Proposition 2.3.6. For each $\lambda \in \Lambda^\bullet$, let $X_\lambda^{\mathcal{I}}$ denote the image of $X_\lambda^{\mathcal{J}}$ in $U_{\mathcal{I}}(\mathfrak{g})$. Explicitly, $X_\lambda^{\mathcal{I}} = \sum_{i,j=1}^N a_{ij}^\lambda F_{ij}^{\mathcal{I}}$ (see (2.3.2)) and we have $F^{\mathcal{I}} = \sum_{\lambda \in \Lambda^\bullet} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{I}}$. In fact, by Part (3) of Lemma 2.3.2, we have

$$K^{\mathcal{I}} = \sum_{\lambda \in \mathcal{J}} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{I}} \quad \text{and} \quad F^{\mathcal{I}} = \sum_{\lambda \in \Lambda \cup \mathcal{J}} X_\lambda^\bullet \otimes X_\lambda^{\mathcal{I}}.$$

As a first step, we construct for each $x \in \mathcal{E}_{\mathfrak{g}}$ a $\mathfrak{g}$-module $W(x)$ which is either zero or isomorphic to $\mathrm{ad}(\mathfrak{g})$, but which cannot have a nonzero intersection with $\mathrm{ad}(\mathfrak{g})$. Fix $x \in \mathcal{E}_{\mathfrak{g}}$ and let

$$W(x) = \mathrm{span}\{[x, J(X_\lambda^\bullet)]\}_{\lambda \in \Lambda}.$$

Note that $W(x)$ is a submodule of the $\mathfrak{g}$-module $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$, and that there is a module homomorphism

$$\varphi_x : \mathrm{ad}(\mathfrak{g}) \to W(x),\ X_\lambda^\bullet \mapsto [x, J(X_\lambda^\bullet)] \quad \forall\, \lambda \in \Lambda.$$

This homomorphism is surjective and, by Schur's lemma, it is either an isomorphism or the zero morphism. We also have $\mathcal{E} = \{x \in \mathcal{E}_{\mathfrak{g}} : \varphi_x = 0\} = \{x \in \mathcal{E}_{\mathfrak{g}} : W(x) = 0\}$.

Lemma 2.3.12. *There does not exist $x \in \mathcal{E}_{\mathfrak{g}}$ such that $W(x) \cap \mathrm{ad}(\mathfrak{g}) \neq \{0\}$.*

Proof. Suppose that $x \in \mathcal{E}_{\mathfrak{g}}$ satisfies $W(x) \cap \mathrm{ad}(\mathfrak{g}) \neq \{0\}$. Then $W(x)$ is irreducible and, since the same is true for $\mathrm{ad}(\mathfrak{g})$, we have $W(x) = \mathrm{ad}(\mathfrak{g})$. In particular, φ_x must be an isomorphism, and by Schur's lemma, every module homomorphism $W(x) \to \mathrm{ad}(\mathfrak{g})$ is a scalar multiple of $\varphi_x^{-1} : [x, J(X_\lambda^\bullet)] \mapsto X_\lambda^\bullet$. As the identity map provides such a homomorphism, there exists $c \in \mathbb{C}^\times$ such that $[x, J(X_\lambda^\bullet)] = cX_\lambda^\bullet$ for all $\lambda \in \Lambda$. After re-normalizing x if necessary, we can assume that $c = 1$. Consider the linear map

$$\mathrm{ad}_x : \mathrm{End}V \to \mathrm{End}V, \quad X \mapsto [x, X] \quad \forall\, X \in \mathrm{End}V.$$

Since $\mathrm{ad}_x(J(X_\lambda^\bullet)) = X_\lambda^\bullet$ and $\mathrm{ad}_x(X_\lambda^\bullet) = 0$ for all $\lambda \in \Lambda$, we deduce from the fact that ad_x is a derivation that it restricts to a linear map

$$\mathrm{ad}_{\rho,x} : \rho(Y(\mathfrak{g})) \to \rho(Y(\mathfrak{g})).$$

Given a monomial X in the variables $\{J(X_\lambda^\bullet), X_\gamma^\bullet\}_{\lambda,\gamma\in\Lambda}$, we denote by $\ell(X)$ the degree of this monomial with respect to the assignment $\deg X_\gamma^\bullet = 0$ and $\deg J(X_\lambda^\bullet) = 1$. For each $k \geq 0$, let $\mathbf{H}_k$ denote the subspace of $\rho(Y(\mathfrak{g}))$ which is spanned by monomials X such that $\ell(X) \leq k$, i.e. $\mathbf{H}_k = \rho(\mathbf{F}_k^J)$, where $\mathbf{F}^J = \{\mathbf{F}_k^J\}_{k\geq 0}$ is the filtration defined below Definition 2.2.1. We then have $\mathrm{ad}_{\rho,x}(\mathbf{H}_0) = 0$ and $\mathrm{ad}_{\rho,x}(\mathbf{H}_k) \subset \mathbf{H}_{k-1}$ for all $k \geq 1$. This follows from the facts that $\mathrm{ad}_{\rho,x}(J(X_\lambda^\bullet)) = X_\lambda^\bullet$ for all $\lambda \in \Lambda$, $\mathrm{ad}_{\rho,x}(X_\gamma^\bullet) = 0$ for all $\gamma \in \Lambda$, and that $\mathrm{ad}_{\rho,x}$ is a derivation. We will break the remainder of our proof into two steps:

Step 1: There exists $k \geq 1$ such that $\mathrm{ad}_{\rho,x}^k = 0$.

Note that $\mathbf{H}_{k-1} \subset \mathrm{Ker}(\mathrm{ad}_{\rho,x}^k)$ for all $k \geq 1$. Indeed, since $\mathrm{ad}_{\rho,x}(\mathbf{H}_{k-1}) \subset \mathbf{H}_{k-2}$ for all $k \geq 1$ (here $\mathbf{H}_a = \{0\}$ for all $a < 0$), we obtain inductively that $\mathrm{ad}_{\rho,x}^k(\mathbf{H}_{k-1}) \subset \mathbf{H}_{-1} = \{0\}$. Since $\rho(Y(\mathfrak{g})) \subset \mathrm{End}V$ is finite-dimensional, it has a finite basis

$\{B_1, \dots, B_{\dim \rho(Y(\mathfrak{g}))}\}$ consisting of monomials B_i in the variables $\{J(X_\lambda^\bullet), X_\gamma^\bullet\}_{\lambda,\gamma\in\Lambda}$. Let ℓ denote the finite integer $\max\{\ell(B_i) : 1 \leq i \leq \dim \rho(Y(\mathfrak{g}))\}$. Then each B_i belongs to $\mathbf{H}_\ell$ and hence so does all of $\rho(Y(\mathfrak{g}))$. Since $\mathrm{ad}_{\rho,x}^{\ell+1}(\mathbf{H}_\ell) = 0$, $\mathrm{ad}_{\rho,x}^{\ell+1}$ is identically zero.

Step 2: The image of $\mathrm{ad}_{\rho,x}^k$ contains $\rho(\mathfrak{g}) \cong \mathfrak{g}$ for every $k \geq 1$.

For each $k \geq 1$ and k-tuple $\alpha_1, \dots, \alpha_k \in \Lambda$, set

$$A_{\alpha_1,\dots,\alpha_k} = [J(X_{\alpha_1}^\bullet), [J(X_{\alpha_2}^\bullet), \cdots, [J(X_{\alpha_{k-1}}^\bullet), J(X_{\alpha_k}^\bullet)] \cdots]],$$
$$Y_{\alpha_1,\dots,\alpha_k} = [X_{\alpha_1}^\bullet, [X_{\alpha_2}^\bullet, \cdots, [X_{\alpha_{k-1}}^\bullet, X_{\alpha_k}^\bullet] \cdots]].$$

If $k = 1$, then it is understood that $A_\alpha = J(X_\alpha^\bullet)$ and $Y_\alpha = X_\alpha^\bullet$.

Claim: $\mathrm{ad}_{\rho,x}^k(A_{\alpha_1,\dots,\alpha_k}) = k! Y_{\alpha_1,\dots,\alpha_k}$ for all $k \geq 1$.

We will prove the claim by induction on k. If $k = 1$ then it is just the statement that $\mathrm{ad}_{\rho,x}(J(X_\alpha^\bullet)) = X_\alpha^\bullet$. Suppose inductively that the claim holds whenever $k = l$, and consider $\mathrm{ad}_{\rho,x}^{l+1}(A_{\alpha_1,\dots,\alpha_{l+1}})$. We have

$$\mathrm{ad}_{\rho,x}^{l+1}(A_{\alpha_1,\dots,\alpha_{l+1}}) = \sum_{j=0}^{l+1} \binom{l+1}{j} \left[\mathrm{ad}_{\rho,x}^j(J(X_{\alpha_1}^\bullet)), \mathrm{ad}_{\rho,x}^{l+1-j}(A_{\alpha_2,\dots,\alpha_{l+1}})\right].$$

Since $\mathrm{ad}_{\rho,x}^2(J(X_{\alpha_1}^\bullet)) = 0$ and $\mathrm{ad}_{\rho,x}^{l+1}(A_{\alpha_2,\dots,\alpha_{l+1}}) = 0$ (since $A_{\alpha_2,\dots,\alpha_{l+1}} \in \mathbf{H}_l$), the only term of the sum on the right-hand side which does not necessarily vanish corresponds to $j = 1$. As $\mathrm{ad}_{\rho,x}(J(X_{\alpha_1}^\bullet)) = X_{\alpha_1}^\bullet$ and, by induction, $\mathrm{ad}_{\rho,x}^l(A_{\alpha_2,\dots,\alpha_{l+1}}) = l! Y_{\alpha_2,\dots,\alpha_{l+1}}$, we have

$$\mathrm{ad}_{\rho,x}^{l+1}(A_{\alpha_1,\dots,\alpha_{l+1}}) = (l+1)l! \left[X_{\alpha_1}^\bullet, Y_{\alpha_2,\dots,\alpha_{l+1}}\right] = (l+1)! Y_{\alpha_1,\dots,\alpha_{l+1}}.$$

This completes the proof of the claim.

To complete the proof of Step 2, it remains to note that, since $\rho(\mathfrak{g})$ is a simple Lie algebra, it is perfect and thus spanned by the collection of elements $\{Y_{\alpha_1,\dots,\alpha_k}\}_{\alpha_i\in\Lambda}$ for any fixed $k \geq 1$.

We can now finish the proof of the lemma. By Step 1, there exists $k \geq 1$ such that $\mathrm{ad}_{\rho,x}^k = 0$. By Step 2, $\rho(\mathfrak{g}) \subset \mathrm{ad}_{\rho,x}^k(\rho(Y(\mathfrak{g}))) = \{0\}$, which is a contradiction. Therefore there cannot exist $x \in \mathcal{E}_\mathfrak{g}$ such that $W(x) \cap \mathrm{ad}(\mathfrak{g}) \neq \{0\}$. □

This leads us to the following analogue of Part (3) of Lemma 2.3.2.

Lemma 2.3.13. *We have $X^{\mathcal{I}}_\mu = 0$ for all $\mu \in \mathcal{I}_c$. In particular, $K^{\mathcal{I}} = \sum_{\lambda \in \mathcal{I}} X^\bullet_\lambda \otimes X^{\mathcal{I}}_\lambda$.*

Proof. Since $[X^\bullet_\mu, J(X^\bullet_\lambda)] = 0$ for all $\mu \in \mathcal{I}$ and $\lambda \in \Lambda$, (2.3.24) is equivalent to

$$\sum_{\lambda \in \Lambda, \mu \in \mathcal{I}_c} X^\bullet_\lambda \otimes [X^\bullet_\mu, J(X^\bullet_\lambda)] \otimes X^{\mathcal{I}}_\mu = \sum_{\lambda \in \Lambda, \mu \in \mathcal{I}_c} [X^\bullet_\mu, J(X^\bullet_\lambda)] \otimes X^\bullet_\lambda \otimes X^{\mathcal{I}}_\mu. \tag{2.3.25}$$

Let's first show that for any fixed $\lambda \in \Lambda$, $\{[X^\bullet_\mu, J(X^\bullet_\lambda)]\}_{\mu \in \mathcal{I}_c}$ is a linearly independent set. Suppose that

$$\sum_{\mu \in \mathcal{I}_c} a_\mu [X^\bullet_\mu, J(X^\bullet_\lambda)] = 0 \ \text{ for some } \ \{a_\mu\}_{\mu \in \mathcal{I}_c} \subset \mathbb{C}.$$

Then $x = \sum_{\mu \in \mathcal{I}_c} a_\mu X^\bullet_\mu$ must belong to $\mathcal{E}$, because φ_x cannot be an isomorphism as its kernel contains $X^\bullet_\lambda$. Since x also belongs to $\mathcal{E}_c$, we must have $x = 0$. The assertion then follows from the linear independence of the set $\{X^\bullet_\mu\}_{\mu \in \mathcal{I}_c}$.

Next, we deduce that, for any fixed $\lambda \in \Lambda$, the set $\{X^\bullet_\gamma, [X^\bullet_\mu, J(X^\bullet_\lambda)]\}_{\gamma \in \Lambda, \mu \in \mathcal{I}_c}$ must also be linearly independent. Indeed, if $0 \neq \sum_{\mu \in \mathcal{I}_c} a_\mu [X^\bullet_\mu, J(X^\bullet_\lambda)] \in \mathrm{ad}(\mathfrak{g})$, then $x = \sum_{\mu \in \mathcal{I}_c} a_\mu X^\bullet_\mu$ is such that $W(x) \cap \mathrm{ad}(\mathfrak{g}) \neq \{0\}$. By Lemma 2.3.12, no such x can exist, and hence we have shown that $\mathrm{span}_{\mu \in \mathcal{I}_c}\{[X^\bullet_\mu, J(X^\bullet_\lambda)]\}$ intersects trivially with $\mathrm{ad}(\mathfrak{g})$, from which the linear independence of $\{X^\bullet_\gamma, [X^\bullet_\mu, J(X^\bullet_\lambda)]\}_{\gamma \in \Lambda, \mu \in \mathcal{I}_c}$ follows automatically from the previous assertion and the linear independence of $\{X^\bullet_\gamma\}_{\gamma \in \Lambda}$.

Let $\{f_\mu\}_{\mu \in \Lambda^\bullet} \subset (\mathrm{End} V)^*$ denote the dual basis to $\{X^\bullet_\lambda\}_{\lambda \in \Lambda^\bullet} \subset \mathrm{End} V$. By the linear independence of $\{X^\bullet_\gamma, [X^\bullet_\mu, J(X^\bullet_\lambda)]\}_{\gamma \in \Lambda, \mu \in \mathcal{I}_c}$, applying $f_\lambda \otimes \mathrm{id} \otimes \mathrm{id}$ to both sides of (2.3.25) for a fixed $\lambda \in \Lambda$ yields

$$\sum_{\mu \in \mathcal{I}_c} [X^\bullet_\mu, J(X^\bullet_\lambda)] \otimes X^{\mathcal{I}}_\mu = 0.$$

The linear independence of $\{[X^\bullet_\mu, J(X^\bullet_\lambda)]\}_{\mu \in \mathcal{I}_c}$ then implies $X^{\mathcal{I}}_\mu = 0$ for all $\mu \in \mathcal{I}_c$. □

We define $\mathfrak{z}_{\mathcal{I}}$ similarly to $\mathfrak{z}_{\mathcal{J}}$: it is the commutative Lie algebra with basis $\{\mathcal{K}^{\mathcal{I}}_\lambda\}_{\lambda \in \mathcal{I}}$. We identify its enveloping algebra with the polynomial ring $\mathbb{C}[\mathcal{K}^{\mathcal{I}}_\lambda : \lambda \in \mathcal{I}]$, and set $\mathscr{K}^{\mathcal{I}} = \sum_{\lambda \in \mathcal{I}} X^\bullet_\lambda \otimes \mathcal{K}^{\mathcal{I}}_\lambda \in \mathrm{End} V \otimes \mathfrak{z}_{\mathcal{I}}$. We are now prepared to state the analogue of Proposition 2.3.6.

Proposition 2.3.14. *The assignment $F^{\mathcal{I}} \mapsto F + \mathscr{K}^{\mathcal{I}}$ extends to an isomorphism of algebras*

$$\phi_{\mathcal{I}} : U_{\mathcal{I}}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{I}} : \lambda \in \mathcal{I}] \otimes U(\mathfrak{g}_\rho). \tag{2.3.26}$$

Proof. Let $\pi : \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} : \lambda \in \mathcal{J}] \twoheadrightarrow \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{I}} : \lambda \in \mathcal{I}]$ be the surjection given by

$$\pi(\mathcal{K}_\lambda^{\mathcal{J}}) = \begin{cases} \mathcal{K}_\lambda^{\mathcal{I}} & \text{if } \lambda \in \mathcal{I}, \\ 0 & \text{if } \lambda \in \mathcal{I}_c. \end{cases}$$

Consider the tensor product $\pi \otimes \mathrm{id} : \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{J}} : \lambda \in \mathcal{J}] \otimes U(\mathfrak{g}_\rho) \twoheadrightarrow \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{I}} : \lambda \in \mathcal{I}] \otimes U(\mathfrak{g}_\rho)$. Its kernel is precisely the ideal generated by $\{\mathcal{K}_\lambda^{\mathcal{J}}\}_{\lambda \in \mathcal{I}_c}$, which is the image of the ideal generated by $\{X_\mu^{\mathcal{J}}\}_{\mu \in \mathcal{I}_c}$ under the isomorphism $\phi_{\mathcal{J}}$ of Proposition 2.3.6. By Lemma 2.3.13 and the definition of $U_{\mathcal{I}}(\mathfrak{g})$, this ideal is contained in the two-sided ideal $\mathscr{I}$ of $U_{\mathcal{J}}(\mathfrak{g})$ generated by the relation $[K_2, (1 \otimes J)(\Omega_\rho)] = [K_1, (J \otimes 1)(\Omega_\rho)]$, hence $\mathrm{Ker}((\pi \otimes \mathrm{id}) \circ \phi_{\mathcal{J}}) \subset \mathscr{I}$. Since $[\mathscr{K}_2^{\mathcal{I}}, (1 \otimes J)(\Omega_\rho)] = [\mathscr{K}_1^{\mathcal{I}}, (J \otimes 1)(\Omega_\rho)]$ trivially holds in $\mathbb{C}[\mathcal{K}_\lambda^{\mathcal{I}} : \lambda \in \mathcal{I}] \otimes U(\mathfrak{g}_\rho)$, we indeed have the equality $\mathrm{Ker}((\pi \otimes \mathrm{id}) \circ \phi_{\mathcal{J}}) = \mathscr{I}$. Thus $(\pi \otimes \mathrm{id}) \circ \phi_{\mathcal{J}}$ induces an isomorphism $\phi_{\mathcal{I}} : U_{\mathcal{I}}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathcal{K}_\lambda^{\mathcal{I}} : \lambda \in \mathcal{I}] \otimes U(\mathfrak{g}_\rho)$ which is given by $F^{\mathcal{I}} \mapsto F + \mathscr{K}^{\mathcal{I}}$. □

We conclude our discussion of $U_{\mathcal{I}}(\mathfrak{g})$ by emphasizing that Proposition 2.3.14 can be naturally interpreted at the level of Lie algebras. Letting $\mathfrak{g}_{\mathcal{I}}$ denote the Lie subalgebra of $\mathrm{Lie}(U_{\mathcal{I}}(\mathfrak{g}))$ generated by $\{F_{ij}^{\mathcal{I}}\}_{1 \le i,j \le N}$, we find that $\phi_{\mathcal{I}}|_{\mathfrak{g}_{\mathcal{I}}}$ and its composition with $\mathrm{id} \otimes \phi_\rho$ induce isomorphisms

$$\mathfrak{g}_{\mathcal{I}} \xrightarrow{\sim} \mathfrak{g}_\rho \oplus \mathfrak{z}_{\mathcal{I}} \xrightarrow{\sim} \mathfrak{g} \oplus \mathfrak{z}_{\mathcal{I}}, \tag{2.3.27}$$

and moreover that $U(\mathfrak{g}_{\mathcal{I}}) \cong U_{\mathcal{I}}(\mathfrak{g})$. With this in mind, $U_{\mathcal{I}}(\mathfrak{g})$ will be denoted $U(\mathfrak{g}_{\mathcal{I}})$ from this point on.

2.3.3.2 The extended polynomial current algebra $\mathfrak{g}_{\mathcal{I}}[z]$

By (2.3.23), (2.3.24) and (2.3.27), the enveloping algebra $U(\mathfrak{g}_{\mathcal{I}}[z])$ is isomorphic to the unital associative $\mathbb{C}$-algebra generated by elements $\{\mathbb{F}_{ij}^{(r)} = F_{ij}^{\mathcal{I}} z^r : 1 \le i, j \le$

N, $r \in \mathbb{Z}_{\geq 0}\}$ subject to the defining relations

$$[\mathbb{F}_1^{(r)}, \mathbb{F}_2^{(s)}] = [\Omega_\rho, \mathbb{F}_2^{(r+s)}] \quad \forall\, r, s \geq 0, \tag{2.3.28}$$

$$[\mathbb{K}_2^{(r)}, (1 \otimes J)(\Omega_\rho)] = [\mathbb{K}_1^{(r)}, (J \otimes 1)(\Omega_\rho)] \quad \forall\, r \geq 0, \tag{2.3.29}$$

where $\mathbb{F}^{(a)} = \sum_{i,j=1}^N E_{ij} \otimes \mathbb{F}_{ij}^{(a)} \in \mathrm{End} V \otimes U(\mathfrak{g}_{\mathcal{I}}[z])$ and $\mathbb{K}^{(a)} = \mathbb{F}^{(a)} - c_{\mathfrak{g}}^{-1}\omega(\mathbb{F}^{(a)})$ for all $a \geq 0$.

Following (2.3.18), let us define

$$\begin{aligned} \mathbb{F}(u) &= \sum_{i,j=1}^N E_{ij} \otimes \mathbb{F}_{ij}(u) \in \mathrm{End} V \otimes (\mathfrak{g}_{\mathcal{I}}[z])[\![u^{-1}]\!], \qquad \text{where} \\ \mathbb{F}_{ij}(u) &= \sum_{r \geq 0} \mathbb{F}_{ij}^{(r)} u^{-r-1} \in (\mathfrak{g}_{\mathcal{I}}[z])[\![u^{-1}]\!]. \end{aligned} \tag{2.3.30}$$

Recall that, for each $\lambda \in \Lambda^\bullet$, $X_\lambda^{\mathcal{I}} = \sum_{i,j} a_{ij}^\lambda F_{ij}^{\mathcal{I}} \in \mathfrak{g}_{\mathcal{I}}$, where the family of scalars $\{a_{ij}^\lambda\}$ is defined in (2.3.2). To every $\lambda \in \Lambda^\bullet$ we associate the series $\mathbb{X}_\lambda(u) = \sum_{r \geq 0} \mathbb{X}_\lambda^{(r)} u^{-r-1} \in (\mathfrak{g}_{\mathcal{I}}[z])[\![u^{-1}]\!]$, where $\mathbb{X}_\lambda^{(r)} = X_\lambda^{\mathcal{I}} z^r$.

Finally, we set $\mathcal{K}_\lambda^{(r)} = \mathcal{K}_\lambda^{\mathcal{I}} z^{r-1}$, so that $U(\mathfrak{z}_{\mathcal{I}}[z]) \cong \mathbb{C}[\mathcal{K}_\lambda^{(r)} : \lambda \in \mathcal{I},\, r \geq 1]$, and define

$$\mathscr{K}(u) = \sum_{\lambda \in \mathcal{I}} X_\lambda^\bullet \otimes \mathcal{K}_\lambda(u), \qquad \text{where} \qquad \mathcal{K}_\lambda(u) = \sum_{r \geq 1} \mathcal{K}_\lambda^{(r)} u^{-r}.$$

We can now state the polynomial current algebra version of Proposition 2.3.14:

Proposition 2.3.15. *The assignment $\mathbb{F}(u) \mapsto F(u) + \mathscr{K}(u)$ extends to an isomorphism of algebras*

$$\phi_{\mathcal{I}}^z : U(\mathfrak{g}_{\mathcal{I}}[z]) \xrightarrow{\sim} \mathbb{C}[\mathcal{K}_\lambda^{(r)} : \lambda \in \mathcal{I},\, r \geq 1] \otimes U(\mathfrak{g}_\rho[z]). \tag{2.3.31}$$

Proof. The isomorphism $\mathfrak{g}_{\mathcal{I}} \xrightarrow{\sim} \mathfrak{g}_\rho \oplus \mathfrak{z}_{\mathcal{I}}$ furnished by Proposition 2.3.14 (see (2.3.27)) extends to an isomorphism $\mathfrak{g}_{\mathcal{I}}[z] \xrightarrow{\sim} (\mathfrak{g}_\rho \oplus \mathfrak{z}_{\mathcal{I}})[z] \cong \mathfrak{g}_\rho[z] \oplus \mathfrak{z}_{\mathcal{I}}[z]$, which induces the desired isomorphism $\phi_{\mathcal{I}}^z$ between the corresponding enveloping algebras. □

Setting $\mathbb{K}(u) = \sum_{r \geq 0} \mathbb{K}^{(r)} u^{-r-1}$, we have $\phi_{\mathcal{I}}^z(\mathbb{K}(u)) = \mathscr{K}(u)$ and $\mathbb{K}(u) = \mathbb{F}(u) - c_{\mathfrak{g}}^{-1}\omega(\mathbb{F}(u))$. By Lemma 2.3.13, $\mathbb{K}(u)$ can be equivalently defined by $\mathbb{K}(u) = \sum_{\lambda \in \mathcal{I}} X_\lambda^\bullet \otimes \mathbb{X}_\lambda(u)$.

We will end this section by rewriting the defining relations of $U(\mathfrak{g}_{\mathcal{I}}[z])$ using the classical r-matrix formalism, which is achieved with the use of Proposition 2.3.9.

Proposition 2.3.16. *The defining relations* (2.3.28) *and* (2.3.29) *are equivalent to the relations*

$$[\mathbb{F}_1(u), \mathbb{F}_2(v)] = \left[\frac{\Omega_\rho}{u-v}, \mathbb{F}_1(u) + \mathbb{F}_2(v)\right], \tag{2.3.32}$$

$$[\mathbb{K}_2(u), (1 \otimes J)(\Omega_\rho)] = [\mathbb{K}_1(u), (J \otimes 1)(\Omega_\rho)], \tag{2.3.33}$$

where $\mathbb{K}(u) = \mathbb{F}(u) - c_{\mathfrak{g}}^{-1}\omega(\mathbb{F}(u))$.

2.4 The R-matrix presentation of the Yangian $Y(\mathfrak{g})$

We have now reached the second and main part of this chapter, where we will focus on establishing the Yangian version of the results of §2.3 and studying them in more detail. In this section specifically, we define the extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$, the *RTT*-Yangian $Y_R(\mathfrak{g})$, and we then study some of their basic properties.

We continue to assume that V is a fixed finite-dimensional $Y(\mathfrak{g})$-module with corresponding homomorphism ρ, and that V has a non-trivial (not necessarily proper) irreducible submodule. We let $R(u)$ denote the image of the universal R-matrix $\mathcal{R}(-u)$ (see Theorem 2.2.4) under $\rho \otimes \rho$:

$$R(u) = (\rho \otimes \rho)\mathcal{R}(-u) \in \mathrm{End}(V \otimes V)[\![u^{-1}]\!].$$

We adapt all of the notation from §2.3. In particular, we fix a basis $\{e_1, \ldots, e_N\}$ of V and we let $\{E_{ij}\}_{1 \le i,j \le N}$ denote the usual elementary matrices with respect to this basis.

2.4.1 The extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$

In this subsection we define and study a Hopf algebra $X_{\mathcal{I}}(\mathfrak{g})$ larger than $Y(\mathfrak{g})$ which we will eventually prove (in §2.6) is a filtered deformation of $U(\mathfrak{g}_{\mathcal{I}}[z])$.

2.4.1.1 Definition of the extended Yangian

We begin with the definition of $X_{\mathcal{I}}(\mathfrak{g})$ as an algebra.

Definition 2.4.1. The extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$ is the unital associative $\mathbb{C}$-algebra generated by elements $\{t_{ij}^{(r)} : 1 \le i,j \le N, r \ge 1\}$ subject to the defining *RTT*-relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \\ \text{in} \quad (\mathrm{End}V)^{\otimes 2} \otimes X_{\mathcal{I}}(\mathfrak{g})[\![v^{\pm 1}, u^{\pm 1}]\!], \tag{2.4.1}$$

where $T(u) = \sum_{i,j=1}^{N} E_{ij} \otimes t_{ij}(u)$ with $t_{ij}(u) = \delta_{ij} + \sum_{r\ge 1} t_{ij}^{(r)} u^{-r}$ for all $1 \le i,j \le N$, and $R(u-v)$ has been identified with $R(u-v) \otimes 1$.

Remark 2.4.2. An equivalent definition is obtained by replacing $R(u)$ by $f(u)R(u)$ for any fixed $f(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$. In particular, if V is irreducible then, by Theorem 2.2.5, $R(u)$ can be replaced with a rational R-matrix.

Since no explicit description of the coefficients $\mathcal{R}_k$ of $\mathcal{R}(u)$ is known, $R(u)$ cannot be computed directly by evaluating $\mathcal{R}(-u)$. In practice, $R(u)$ is obtained by instead solving the equation (2.2.13). By Theorem 2.2.5, this determines $R(u)$ up to multiplication by elements of $\mathbb{C}[\![u^{-1}]\!]$, provided V is irreducible. See for example [GRW19a, Proposition 3.13].

Note that $X_{\mathcal{I}}(\mathfrak{g})$ comes equipped with a natural action on the underlying $Y(\mathfrak{g})$-module V. Namely, there is an algebra homomorphism

$$X_{\mathcal{I}}(\mathfrak{g}) \to \mathrm{End}V, \quad T(u) \mapsto R(u).$$

A standard argument (see [Mol07, Theorem 1.5.1] and [FRT90]) shows that $X_{\mathcal{I}}(\mathfrak{g})$ is a Hopf algebra with coproduct $\Delta_{\mathcal{I}}$, antipode $S_{\mathcal{I}}$, and counit $\epsilon_{\mathcal{I}}$ given by

$$\Delta_{\mathcal{I}}(T(u)) = T_{[1]}(u)T_{[2]}(u), \quad S_{\mathcal{I}}(T(u)) = T(u)^{-1}, \quad \epsilon_{\mathcal{I}}(T(u)) = I, \tag{2.4.2}$$

respectively. Expressing $\Delta_{\mathcal{I}}$ in terms of the generating series $t_{ij}(u)$ and the generators $t_{ij}^{(r)}$, we have

$$\Delta_{\mathcal{I}}(t_{ij}(u)) = \sum_{a=1}^{N} t_{ia}(u) \otimes t_{aj}(u) \quad \text{and} \quad \Delta_{\mathcal{I}}(t_{ij}^{(r)}) = \sum_{a=1}^{N}\sum_{b=0}^{r} t_{ia}^{(b)} \otimes t_{aj}^{(r-b)},$$

where $t_{kl}^{(0)} = \delta_{kl}$ for all $1 \leq k, l \leq N$.

2.4.1.2 Automorphisms of $X_{\mathcal{I}}(\mathfrak{g})$

The extended Yangian $X_{\mathcal{I}}(\mathfrak{g})$ has at least three important families of automorphisms. The first family we will discuss turns out to be closely tied to the Yangian $Y_R(\mathfrak{g})$, as we will make precise in §2.6.2.

Recall that $\mathcal{E} = \mathrm{End}_{Y(\mathfrak{g})}V \subset \mathrm{End}V$, and consider the tensor product $\mathcal{E}\otimes u^{-1}\mathbb{C}[\![u^{-1}]\!]$. This space can be identified with $\prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$, i.e. the collection of all tuples $(f_\lambda(u))_{\lambda\in\mathcal{I}} \subset u^{-1}\mathbb{C}[\![u^{-1}]\!]$, the identification being given by

$$\begin{aligned}(f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}&(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda \\ &\mapsto \mathbf{f}^\circ(u) = \sum_{\lambda\in\mathcal{I}} X_\lambda^\bullet \otimes f_\lambda(u) \in \mathcal{E}\otimes u^{-1}\mathbb{C}[\![u^{-1}]\!].\end{aligned} \tag{2.4.3}$$

Here $(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$ just denotes a copy of $u^{-1}\mathbb{C}[\![u^{-1}]\!]$ associated to λ. The following lemma demonstrates that $X_{\mathcal{I}}(\mathfrak{g})$ admits a family of automorphisms indexed by $\prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$.

Lemma 2.4.3. *Let $(f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$ and set $\mathbf{f}(u) = I + \mathbf{f}^\circ(u)$. Then the assignment*

$$m_{\mathbf{f}} : T(u) \mapsto \mathbf{f}(u)T(u) \tag{2.4.4}$$

extends to an automorphism $m_{\mathbf{f}}$ of $X_{\mathcal{I}}(\mathfrak{g})$.

Proof. Using that $\mathbf{f}(u) \in \mathcal{E}\otimes\mathbb{C}[\![u^{-1}]\!]$ and $R(u) \in (\rho(Y(\mathfrak{g}))\otimes\rho(Y(\mathfrak{g})))[\![u^{-1}]\!]$, we can conclude that $\mathbf{f}(u)$ satisfies the defining RTT-relation of $X_{\mathcal{I}}(\mathfrak{g})$. Indeed, by definition $\mathcal{E}$ is the centralizer of $\rho(Y(\mathfrak{g}))$ in $\mathrm{End}V$, which implies $R(u-v)\mathbf{f}_a(u) = \mathbf{f}_a(u)R(u-v)$ for $a \in \{1,2\}$. Moreover, $[\mathbf{f}_1(u), \mathbf{f}_2(v)] = 0$, from which the assertion follows easily.

Applying this observation in conjunction with $[\mathbf{f}_1(u), T_2(v)] = 0 = [\mathbf{f}_2(v), T_1(u)]$, we deduce that $m_{\mathbf{f}}$ extends to an algebra endomorphism of $X_{\mathcal{I}}(\mathfrak{g})$. The invertibility of $m_{\mathbf{f}}$ follows from the invertibility of $\mathbf{f}(u)$ as an element $\mathcal{E}[\![u^{-1}]\!]$. □

The second family of automorphisms is indexed by the complex numbers. For each $c \in \mathbb{C}$, the assignment

$$T(u) \mapsto T(u-c) \tag{2.4.5}$$

extends to an automorphism of $X_{\mathcal{I}}(\mathfrak{g})$. These automorphisms are closely related to the automorphisms τ_c of $Y(\mathfrak{g})$ defined in (2.2.6). The third family of automorphisms will be introduced in Chapter 3: see Lemma 3.2.4.

2.4.1.3 The associated graded algebra $\mathrm{gr}X_{\mathcal{I}}(\mathfrak{g})$

By (2.2.11), the R-matrix $R(u)$ admits an expansion

$$\begin{aligned} R(u) &= I + \sum_{k\geq 1} R^{(k)} u^{-k} \\ &= I - \Omega_\rho u^{-1} + \left((J\otimes 1 - 1\otimes J)(\Omega_\rho) + \tfrac{1}{2}\Omega_\rho^2\right) u^{-2} + \sum_{k\geq 3} R^{(k)} u^{-r} \end{aligned} \tag{2.4.6}$$

with $R^{(k)} = (-1)^k(\rho\otimes\rho)(\mathcal{R}_k)$ for each $k\geq 1$. Setting $T^\circ(u) = T(u) - I$, the defining relation (2.4.1) can be rewritten as

$$\begin{aligned} &[T_1^\circ(u), T_2^\circ(v)] \\ &\quad = \frac{1}{u-v}\Big([\Omega_\rho, T_1^\circ(u)] + [\Omega_\rho, T_2^\circ(v)] \\ &\qquad\qquad + \Omega_\rho T_1^\circ(u) T_2^\circ(v) - T_2^\circ(v) T_1^\circ(u)\Omega_\rho\Big) \\ &\qquad + \sum_{k\geq 2}\frac{1}{(u-v)^k}\Big([T_2^\circ(v), R^{(k)}] + [T_1^\circ(u), R^{(k)}] \\ &\qquad\qquad + T_2^\circ(v)T_1^\circ(u)R^{(k)} - R^{(k)}T_1^\circ(u)T_2^\circ(v)\Big), \end{aligned} \tag{2.4.7}$$

where Ω_ρ and $R^{(k)}$ have been identified with $\Omega_\rho\otimes 1$ and $R^{(k)}\otimes 1$, respectively.

The degree assignment

$$\deg t_{ij}^{(r)} = r-1 \quad \forall\, 1\leq i,j\leq N \quad \text{and} \quad r\geq 1 \tag{2.4.8}$$

equips $X_{\mathcal{I}}(\mathfrak{g})$ with the structure of a filtered algebra. Let $\mathbf{F}_k(X_{\mathcal{I}}(\mathfrak{g}))$ (or $\mathbf{F}_k^{\mathcal{I}}$ for brevity) denote the subspace spanned by elements of degree less than or equal to k, and set $\bar{t}_{ij}^{(r)}$ to be the image of $t_{ij}^{(r)}$ in $\mathbf{F}_{r-1}^{\mathcal{I}}/\mathbf{F}_{r-2}^{\mathcal{I}}\subset \mathrm{gr}X_{\mathcal{I}}(\mathfrak{g})$.

Proposition 2.4.4. *The assignment*

$$\varphi_{\mathcal{I}} : \mathbb{F}_{ij}^{(r-1)} \mapsto \bar{t}_{ij}^{(r)} \quad \forall\, 1\leq i,j\leq N,\ r\geq 1 \tag{2.4.9}$$

extends to a surjective morphism of algebras $\varphi_{\mathcal{I}}: U(\mathfrak{g}_{\mathcal{I}}[z]) \twoheadrightarrow \mathrm{gr} X_{\mathcal{I}}(\mathfrak{g})$.

Proof. Let $\mathbb{T}(u) = \sum_{k\geq 1} \mathbb{T}^{(k)} u^{-k}$, where $\mathbb{T}^{(k)} = \sum_{i,j=1}^{N} E_{ij} \otimes \bar{t}_{ij}^{(k)}$.

Step 1: The relation $[\mathbb{T}_1(u), \mathbb{T}_2(v)] = \left[\frac{\Omega_\rho}{u-v}, \mathbb{T}_1(u) + \mathbb{T}_2(v)\right]$ is satisfied.

For each $k > 0$, we expand $(u-v)^{-k}$ as an element of $(\mathbb{C}[v])[\![u^{-1}]\!]$:

$$(u-v)^{-k} = \sum_{s\geq 0} \binom{k+s-1}{s} v^s u^{-s-k}. \tag{2.4.10}$$

Note the following simple fact: if

$$\mathbb{A}(u,v) = \sum_{a,b\geq 1} \mathbb{A}_{a,b} u^{-a} v^{-b} \quad \text{with } \mathbb{A}_{a,b} \in (\mathrm{End} V)^{\otimes 2} \otimes \mathbf{F}^{\mathcal{I}}_{a+b-c},$$

then we have

$$\begin{gathered} \frac{1}{(u-v)^k} \mathbb{A}(u,v) = \sum_{a\in\mathbb{Z}_{\geq k+1}, b\in\mathbb{Z}} \mathbb{B}_{a,b} u^{-a} v^{-b}, \quad \text{where} \\ \mathbb{B}_{a,b} \in (\mathrm{End} V)^{\otimes 2} \otimes \mathbf{F}^{\mathcal{I}}_{a+b-c-k} \quad \forall\, a,b \geq 0 \end{gathered} \tag{2.4.11}$$

and $\mathbf{F}^{\mathcal{I}}_{-l} = \{0\}$ for all $l \in \mathbb{N}$. Here c is assumed to be a fixed positive integer depending on $\mathbb{A}(u,v)$.

For each $l \geq 0$, set

$$\mathbf{F}_l(u,v) = (\mathrm{End} V)^{\otimes 2} \otimes \prod_{a\in\mathbb{Z}_{\geq 0}, b\in\mathbb{Z}} \mathbf{F}^{\mathcal{I}}_{a+b-l} u^{-a} v^{-b} \subset (\mathrm{End} V)^{\otimes 2} \otimes X_{\mathcal{I}}(\mathfrak{g})[\![v^{\pm 1}, u^{-1}]\!],$$

and note that $\mathbf{F}_l(u,v)/\mathbf{F}_{l+1}(u,v)$ can be naturally identified with

$$(\mathrm{End} V)^{\otimes 2} \otimes \prod_{a\in\mathbb{Z}_{\geq 0}, b\in\mathbb{Z}} (\mathrm{gr}_{a+b-l} X_{\mathcal{I}}(\mathfrak{g})) u^{-a} v^{-b} \subset (\mathrm{End} V)^{\otimes 2} \otimes (\mathrm{gr} X_{\mathcal{I}}(\mathfrak{g}))[\![v^{\pm 1}, u^{-1}]\!],$$

where $\mathrm{gr}_k X_{\mathcal{I}}(\mathfrak{g})$ denotes the k-th graded component of $\mathrm{gr} X_{\mathcal{I}}(\mathfrak{g})$, which is understood to equal zero if $k < 0$.

We will simultaneously show both sides of (2.4.7) belong to $\mathbf{F}_2(u,v)$ and compute their images in the quotient $\mathbf{F}_2(u,v)/\mathbf{F}_3(u,v)$. By the above observation this yields an identity in $(\mathrm{End} V)^{\otimes 2} \otimes (\mathrm{gr} X_{\mathcal{I}}(\mathfrak{g}))[\![v^{\pm 1}, u^{-1}]\!]$.

If $\mathbb{A}(u,v) = \Omega_\rho T_1^\circ(u)T_2^\circ(v)$ or $\mathbb{A}(u,v) = T_2^\circ(v)T_1^\circ(u)\Omega_\rho$, then the integer c (see (2.4.11)) is equal to 2 and hence $(u-v)^{-1}\mathbb{A}(u,v) \equiv 0 \mod \mathbf{F}_3(u,v)$.

If instead $\mathbb{A}(u,v)$ is equal to one of the terms that appears within the parentheses on the second line of the right-hand side of (2.4.7) (i.e. a term involving $R^{(k)}$ with $k \geq 2$), then $c = 1$ or 2 but $k \geq 2$. Therefore the observation (2.4.11) yields that $(u-v)^{-k}\mathbb{A}(u,v) \equiv 0 \mod \mathbf{F}_3(u,v)$.

Since $[T_1^\circ(u), T_2^\circ(v)]$ and $[\frac{\Omega_\rho}{u-v}, T_1^\circ(u) + T_2^\circ(v)]$ belong to $\mathbf{F}_2(u,v)$ with images

$$[\mathbb{T}_1(u), \mathbb{T}_2(v)] \quad \text{and} \quad \left[\frac{\Omega_\rho}{u-v}, \mathbb{T}_1(u) + \mathbb{T}_2(v)\right]$$

in $\mathbf{F}_2(u,v)/\mathbf{F}_3(u,v)$, respectively, we obtain the relation

$$[\mathbb{T}_1(u), \mathbb{T}_2(v)] = \left[\frac{\Omega_\rho}{u-v}, \mathbb{T}_1(u) + \mathbb{T}_2(v)\right].$$

Note that Step 1 implies that there is a surjective algebra homomorphism

$$U(\mathfrak{g}_{\mathcal{J}}[z]) \twoheadrightarrow \mathrm{gr} X_{\mathcal{I}}(\mathfrak{g}).$$

To verify that it factors through $U(\mathfrak{g}_{\mathcal{I}}[z])$, we must show that the assignment $\varphi_{\mathcal{I}}$ preserves the relation (2.3.33). In order to state this more precisely we define, following (2.3.2),

$$\bar{t}_\lambda^{(k)} = \sum_{i,j=1}^{N} a_{ij}^\lambda \bar{t}_{ij}^{(k)} \quad \forall\, \lambda \in \Lambda^\bullet \quad \text{and} \quad k \geq 1.$$

Then the statement that $\varphi_{\mathcal{I}}$ preserves (2.3.33) is equivalent to the statement that, for each $k \geq 1$, $\mathbb{D}^{(k)} = \sum_{\lambda \in \mathcal{J}} X_\lambda^\bullet \otimes \bar{t}_\lambda^{(k)}$ satisfies

$$[\mathbb{D}_2^{(k)}, (1 \otimes J)(\Omega_\rho)] = [\mathbb{D}_1^{(k)}, (J \otimes 1)(\Omega_\rho)]. \tag{2.4.12}$$

Step 2: the relation (2.4.12) is satisfied for every $k \geq 1$.

We will divide this step of the proof into a few smaller steps.

Step 2.1: The relation

$$[\mathbb{T}_2^{(k)}, (J \otimes 1 - 1 \otimes J)(\Omega_\rho)] = -[\mathbb{T}_1^{(k)}, (J \otimes 1 - 1 \otimes J)(\Omega_\rho)] \tag{2.4.13}$$

holds in $\mathrm{gr}X_{\mathcal{I}}(\mathfrak{g})$ for all $k \geq 1$.

We will prove (2.4.13) by expanding (2.4.7) in two different ways. First, we compute for each $k \geq 1$ the v^0u^{-k-2} coefficient of both sides of (2.4.7) modulo $\mathbf{F}^{\mathcal{I}}_{k-2}$, using the expansion (2.4.10). Using (2.4.11), it is not difficult to deduce that no term on the right-hand side of (2.4.7) involving $R^{(k)}$ with $k \geq 3$ makes a contribution, and the same is true for the terms $T_2^\circ(v)T_1^\circ(u)R^{(2)}$ and $R^{(2)}T_1^\circ(u)T_2^\circ(v)$. As the coefficient of v^0u^{-k-2} in $[T_1^\circ(u), T_2^\circ(v)]$ is zero, we are left with the equivalence

$$\begin{aligned}0 \equiv &[\Omega_\rho, T_1^{(k+1)}] + [\Omega_\rho, T_2^{(k+1)}] \\ &+ \sum_{a=1}^{k}(\Omega_\rho T_1^{(k+1-a)}T_2^{(a)} - T_2^{(a)}T_1^{(k+1-a)}\Omega_\rho) \\ &+ [T_1^{(k)}, R^{(2)}] + (k+1)[T_2^{(k)}, R^{(2)}] \mod \mathbf{F}^{\mathcal{I}}_{k-2}.\end{aligned} \tag{2.4.14}$$

Next, we compute the u^0v^{-k-2} coefficient of both sides of (2.4.7) modulo $\mathbf{F}^{\mathcal{I}}_{k-2}$ after expanding $(u-v)^{-r}$ as an element of $(\mathbb{C}[u])[\![v^{-1}]\!]$ and viewing (2.4.7) as an equality in $(\mathrm{End}V)^{\otimes 2} \otimes X_{\mathcal{I}}(\mathfrak{g})[\![u^{\pm 1}, v^{-1}]\!]$. Using the symmetry and skew-symmetry between u and v in the relation (2.4.7), we deduce from (2.4.14) the equivalence

$$\begin{aligned}0 \equiv &-[\Omega_\rho, T_1^{(k+1)}] - [\Omega_\rho, T_2^{(k+1)}] \\ &- \sum_{b=1}^{k}(\Omega_\rho T_1^{(b)}T_2^{(k+1-b)} - T_2^{(k+1-b)}T_1^{(b)}\Omega_\rho) \\ &+ (k+1)[T_1^{(k)}, R^{(2)}] + [T_2^{(k)}, R^{(2)}] \mod \mathbf{F}^{\mathcal{I}}_{k-2}.\end{aligned} \tag{2.4.15}$$

Adding (2.4.14) and (2.4.15) and dividing by $k+2$, we obtain

$$\begin{aligned}&[T_2^{(k)}, R^{(2)}] \equiv -[T_1^{(k)}, R^{(2)}] \mod \mathbf{F}^{\mathcal{I}}_{k-2} \\ &\qquad\qquad \implies [\mathbb{T}_2^{(k)}, R^{(2)}] = -[\mathbb{T}_1^{(k)}, R^{(2)}] \text{ in } \mathrm{gr}X_{\mathcal{I}}(\mathfrak{g}).\end{aligned}$$

Recall from (2.4.6) that $R^{(2)} = (J \otimes 1 - 1 \otimes J)(\Omega_\rho) + \frac{1}{2}\Omega_\rho^2$. Substituting this into the above equality gives

$$\begin{aligned}&[\mathbb{T}_2^{(k)}, (J \otimes 1 - 1 \otimes J)(\Omega_\rho)] + \tfrac{1}{2}[\mathbb{T}_2^{(k)}, \Omega_\rho^2] \\ &\qquad = -[\mathbb{T}_1^{(k)}, (J \otimes 1 - 1 \otimes J)(\Omega_\rho)] - \tfrac{1}{2}[\mathbb{T}_1^{(k)}, \Omega_\rho^2].\end{aligned} \tag{2.4.16}$$

Since $\mathbb{T}^{(k)}$ is a homomorphic image of $F^{\mathcal{J}}z^{k-1} \in U(\mathfrak{g}_{\mathcal{J}}[z])$, Lemma 2.3.3 yields

$$[\mathbb{T}_2^{(k)}, \Omega_\rho] = -[\mathbb{T}_1^{(k)}, \Omega_\rho],$$

from which the identity

$$\tfrac{1}{2}[\mathbb{T}_2^{(k)}, \Omega_\rho^2] = -\tfrac{1}{2}[\mathbb{T}_1^{(k)}, \Omega_\rho^2]$$

follows directly. Therefore the relation (2.4.16) implies the relation (2.4.13).

Step 2.2: We have

$$[\mathbb{D}_2^{(k)}, (J\otimes 1 - 1\otimes J)(\Omega_\rho)] = -[\mathbb{D}_1^{(k)}, (J\otimes 1 - 1\otimes J)(\Omega_\rho)]. \tag{2.4.17}$$

for each $k \geq 1$.

For each $k \geq 1$, set $\mathbb{L}^{(k)} = \mathbb{T}^{(k)} - \mathbb{D}^{(k)}$, so that $\mathbb{L}^{(k)} = \sum_{\lambda\in\Lambda} X_\lambda^\bullet \otimes \bar{t}_\lambda^{(k)}$. Using that $J : \mathrm{ad}(\mathfrak{g}) \to \mathrm{ad}_{\mathfrak{g}}(\mathfrak{gl}(V))$ is a morphism of $\mathfrak{g}$-modules, it is straightforward to derive from the relation $[\Omega_\rho, \mathbb{L}_2^{(k)}] = -[\Omega_\rho, \mathbb{L}_1^{(k)}]$ that

$$[\mathbb{L}_2^{(k)}, (J\otimes 1 - 1\otimes J)(\Omega_\rho)] = -[\mathbb{L}_1^{(k)}, (J\otimes 1 - 1\otimes J)(\Omega_\rho)]. \tag{2.4.18}$$

Subtracting (2.4.18) from (2.4.13) yields (2.4.17).

Since $\mathbb{D}^{(k)}$ is a homomorphic image of $\sum_{i,j=1}^{N} E_{ij}\otimes K_{ij}z^{k-1} \in \mathrm{End}V \otimes U(\mathfrak{g}_{\mathcal{J}}[z])$, Lemma 2.3.2 implies that $[\Omega_\rho, \mathbb{D}_2^{(k)}] = 0 = [\Omega_\rho, \mathbb{D}_1^{(k)}]$. We thus also have

$$[\mathbb{D}_2^{(k)}, (J\otimes 1)(\Omega_\rho)] = 0 = [\mathbb{D}_1^{(k)}, (1\otimes J)(\Omega_\rho)].$$

Subtracting this identity from (2.4.17) leaves us with the equality (2.4.12).

By Step 1, Step 2 and Proposition 2.3.16, the assignment (2.4.9) extends to an epimorphism $\varphi_{\mathcal{I}} : U(\mathfrak{g}_{\mathcal{I}}[z]) \twoheadrightarrow \mathrm{gr}X_{\mathcal{I}}(\mathfrak{g})$. □

We will prove that $\varphi_{\mathcal{I}}$ is in fact an isomorphism, but this will be delayed until §2.6.1.

2.4.2 The RTT-Yangian $Y_R(\mathfrak{g})$

Our present goal is to give an exposition of $Y_R(\mathfrak{g})$ analogous to that given for $X_\mathcal{I}(\mathfrak{g})$ in the previous subsection.

2.4.2.1 Definition of the RTT-Yangian

Let us begin with the definition of the Yangian $Y_R(\mathfrak{g})$:

Definition 2.4.5. The RTT-Yangian $Y_R(\mathfrak{g})$ is the quotient of $X_\mathcal{I}(\mathfrak{g})$ by the two-sided ideal generated by the elements $z_{ij}^{(r)}$, for $1 \leq i, j \leq N$ and $r \geq 1$, defined by

$$\mathcal{Z}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes z_{ij}(u) = S_\mathcal{I}^2(T(u))T(u + \tfrac{1}{2}c_\mathfrak{g})^{-1}, \tag{2.4.19}$$

where $z_{ij}(u) = \delta_{ij} + \sum_{r\geq 1} z_{ij}^{(r)} u^{-r}$ for each pair of indices $1 \leq i, j \leq N$.

The ideal of $X_\mathcal{I}(\mathfrak{g})$ generated by $\{z_{ij}^{(r)} : 1 \leq i, j \leq N,\ r \geq 1\}$ will be conveniently denoted by $(\mathcal{Z}(u) - I)$. Note that it is not obvious that this ideal is a Hopf ideal, and hence that $Y_R(\mathfrak{g})$ inherits the structure of a Hopf algebra from $X_\mathcal{I}(\mathfrak{g})$. This will, however, be a consequence of Lemma 2.5.1 and Theorem 2.5.2, which will be proven in the next section.

We will denote the images of $t_{ij}^{(r)}, t_{ij}(u)$ and $T(u)$ in $Y_R(\mathfrak{g})$ by $\tau_{ij}^{(r)}, \tau_{ij}(u)$ and $\mathcal{T}(u)$, respectively.

For each $c \in \mathbb{C}$ the automorphism (2.4.5) factors through the Yangian $Y_R(\mathfrak{g})$ yielding an automorphism given by the assignment $\mathcal{T}(u) \mapsto \mathcal{T}(u - c)$. We will prove in §2.6.2 that each automorphism $m_\mathfrak{f}$ of $X_\mathcal{I}(\mathfrak{g})$ (see Lemma 2.4.3) also induces an automorphism of $Y_R(\mathfrak{g})$, but that these turn out to all be equal to the identity map. This fact will be used to give an equivalent characterization of $Y_R(\mathfrak{g})$.

2.4.2.2 The associated graded algebra $\mathrm{gr}Y_R(\mathfrak{g})$

The RTT-Yangian $Y_R(\mathfrak{g})$ inherits an algebra filtration from $X_\mathcal{I}(\mathfrak{g})$ via the quotient filtration; this is equivalent to assigning $\deg \tau_{ij}^{(r)} = r - 1$. Let $\bar{\tau}_{ij}^{(r)}$ denote the image of $\tau_{ij}^{(r)}$ in $\mathbf{F}_{r-1}(Y_R(\mathfrak{g}))/\mathbf{F}_{r-2}(Y_R(\mathfrak{g})) = \mathrm{gr}_{r-1} Y_R(\mathfrak{g})$.

Proposition 2.4.6. *The assignment*

$$\varphi : F_{ij}^{(r-1)} \mapsto \bar{\tau}_{ij}^{(r)} \quad \forall\, 1 \leq i, j \leq N,\ r \geq 1$$

extends to a surjective algebra morphism $\varphi : U(\mathfrak{g}_\rho[z]) \twoheadrightarrow \mathrm{gr} Y_R(\mathfrak{g})$.

Proof. We will take a slightly more explicit route than taken in the proof of Proposition 2.4.4 and work directly with the generators $\bar{\tau}_{ij}^{(r)}$ of $\mathrm{gr}Y_R(\mathfrak{g})$. By (2.3.13), Corollary 2.3.7 and Proposition 2.4.4 it suffices to show that

$$\tfrac{1}{2}\bar{\tau}_{ij}^{(r)} = c_{\mathfrak{g}}^{-1} \sum_{a=1}^{N} [\bar{\tau}_{ia}^{(r)}, \bar{\tau}_{aj}^{(1)}] \quad \forall\, r \geq 1. \tag{2.4.20}$$

In $Y_R(\mathfrak{g})$ we have, by (2.4.19), the relation $\mathcal{T}(u + \frac{1}{2}c_{\mathfrak{g}}) = S_{\mathcal{I}}^2(\mathcal{T}(u))$ where $S_{\mathcal{I}}^2(\mathcal{T}(u))$ is understood to equal the image of $S_{\mathcal{I}}^2(T(u))$ in $Y_R(\mathfrak{g})$ under the natural quotient map. Since

$$\left(u + \tfrac{1}{2}c_{\mathfrak{g}}\right)^{-k} = \sum_{s \geq 0} \binom{k+s-1}{s} \left(-\tfrac{1}{2}c_{\mathfrak{g}}\right)^{s} u^{-s-k} \quad \forall\, k \geq 1,$$

the u^{-r-1} coefficient of $\tau_{ij}(u + \frac{1}{2}c_{\mathfrak{g}})$ is equal to

$$\tau_{ij}^{(r+1)} - \tfrac{r}{2} c_{\mathfrak{g}} \tau_{ij}^{(r)} \mod \mathbf{F}_{r-2}(Y_R(\mathfrak{g})). \tag{2.4.21}$$

Let $\widehat{T}^{(r)} = \sum_{i,j=1}^{N} E_{ij} \otimes \hat{t}_{ij}^{(r)}$ denote the u^{-r} coefficient of $T(u)^{-1}$. In particular, $\widehat{T}^{(r)}$ can be determined inductively from the relation

$$\widehat{T}^{(r)} = -\sum_{b=1}^{r} T^{(b)} \widehat{T}^{(r-b)} = -\sum_{b=1}^{r} \sum_{i,j=1}^{N} E_{ij} \otimes \left(\sum_{a=1}^{N} t_{ia}^{(b)} \hat{t}_{aj}^{(r-b)} \right).$$

By definition of the antipode $S_{\mathcal{I}}$, we thus have

$$S_{\mathcal{I}}^2(t_{ij}^{(r+1)}) = -S_{\mathcal{I}}(\sum_{b=1}^{r+1} \sum_{a=1}^{N} t_{ia}^{(b)} \hat{t}_{aj}^{(r+1-b)}) = -\sum_{b=1}^{r+1} \sum_{a=1}^{N} S_{\mathcal{I}}(\hat{t}_{aj}^{(r+1-b)}) \hat{t}_{ia}^{(b)}.$$

Expanding the right-hand side and using that $S_{\mathcal{I}}$ is a filtration preserving map with

$$S_{\mathcal{I}}(t_{kl}^{(s)}) = \hat{t}_{kl}^{(s)} \equiv -t_{kl}^{(s)} \mod \mathbf{F}_{s-2}^{\mathcal{I}} \quad \forall\, s \geq 1,$$

we obtain

$$\begin{aligned}
S_{\mathcal{I}}^2(t_{ij}^{(r+1)}) &= \sum_{b=1}^{r+1}\sum_{a=1}^{N} S_{\mathcal{I}}(\hat{t}_{aj}^{(r+1-b)})(t_{ia}^{(b)} + \sum_{d=1}^{b-1}\sum_{c=1}^{N} t_{ic}^{(d)}\hat{t}_{ca}^{(b-d)})\\
&\equiv t_{ij}^{(r+1)} - \sum_{b=1}^{r}\sum_{a=1}^{N}\hat{t}_{aj}^{(r+1-b)}t_{ia}^{(b)} + \sum_{d=1}^{r}\sum_{c=1}^{N} t_{ic}^{(d)}\hat{t}_{cj}^{(r+1-d)} \mod \mathbf{F}_{r-2}^{\mathcal{I}}\\
&\equiv t_{ij}^{(r+1)} + \sum_{b=1}^{r}\sum_{a=1}^{N}[t_{aj}^{(r+1-b)}, t_{ia}^{(b)}] \mod \mathbf{F}_{r-2}^{\mathcal{I}}.
\end{aligned}$$

Combining this with the relation $[\mathbb{T}_1^{(r)}, \mathbb{T}_2^{(s)}] = [\Omega_\rho, \mathbb{T}_2^{(r+s)}]$ of $\mathrm{gr}X_{\mathcal{I}}(\mathfrak{g})$ (which holds by Proposition 2.4.4), we arrive at the relation

$$S_{\mathcal{I}}^2(t_{ij}^{(r+1)}) \equiv t_{ij}^{(r+1)} + r\sum_{a=1}^{N}[t_{aj}^{(1)}, t_{ia}^{(r)}] \mod \mathbf{F}_{r-2}^{\mathcal{I}}.$$

As the same relation must hold in $Y_R(\mathfrak{g})/\mathbf{F}_{r-2}(Y_R(\mathfrak{g}))$ with each generator $t_{kl}^{(s)}$ replaced by $\tau_{kl}^{(s)}$, equating the resulting expression with (2.4.21) and subtracting $\tau_{ij}^{(r+1)}$ from both sides gives (2.4.20). □

We conclude this section by noting a simple, but rather useful, corollary of Proposition 2.4.6.

Corollary 2.4.7. *The algebra $Y_R(\mathfrak{g})$ is generated by the elements $\tau_{ij}^{(r)}$ with $1 \leq i, j \leq N$ and $1 \leq r \leq 2$.*

Proof. Since $U(\mathfrak{g}_\rho[z])$ is generated by $\{F_{ij}^{(0)}, F_{ij}^{(1)}\}_{1\leq i,j\leq N}$ and $\varphi : U(\mathfrak{g}_\rho[z]) \to \mathrm{gr}Y_R(\mathfrak{g})$ is surjective, $\mathrm{gr}Y_R(\mathfrak{g})$ is generated by $\{\bar{\tau}_{ij}^{(1)}, \bar{\tau}_{ij}^{(2)}\}_{1\leq i,j\leq N}$. If $r > 2$, then we may write $\bar{\tau}_{ij}^{(r)}$ as a homogeneous polynomial Q in the variables $\{\bar{\tau}_{kl}^{(1)}, \bar{\tau}_{kl}^{(2)}\}_{1\leq i,j\leq N}$ of degree $r-1$. Let P be the polynomial in $\{\tau_{kl}^{(1)}, \tau_{kl}^{(2)}\}_{1\leq i,j\leq N}$ obtained from Q by replacing $\bar{\tau}_{kl}^{(s)}$ with $\tau_{kl}^{(s)}$ for $s = 1, 2$ and $1 \leq k, l \leq N$. Then $P \in \mathbf{F}_{r-1}(Y_R(\mathfrak{g}))$ and

$$\tau_{ij}^{(r)} - P \in \mathbf{F}_{r-2}(Y_R(\mathfrak{g})).$$

The result thus follows by a straightforward induction on $r \geq 1$. □

2.5 Equivalence of the two definitions of the Yangian

In this section we prove that, irrespective of the choice of V, we always have $Y_R(\mathfrak{g}) \cong Y(\mathfrak{g})$. In the process we prove that the surjection $\varphi : U(\mathfrak{g}_\rho[z]) \twoheadrightarrow \mathrm{gr}Y_R(\mathfrak{g})$ from Proposition 2.4.6 is an isomorphism, yielding a Poincaré-Birkhoff-Witt theorem for $Y_R(\mathfrak{g})$: see Theorem 2.5.5. This in turn implies that the center of $Y_R(\mathfrak{g})$ is trivial, as will be explained in Corollary 2.5.6.

The first step in proving the equivalence of the two Yangians is the construction of a surjective Hopf algebra homomorphism $X_\mathcal{I}(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$, and this is the content of the next lemma.

Lemma 2.5.1. *The assignment*

$$\widetilde{\Phi} : T(u) \to (\rho \otimes 1)(\mathcal{R}(-u)) \tag{2.5.1}$$

extends to a surjective homomorphism of Hopf algebras $\widetilde{\Phi} : X_\mathcal{I}(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$.

Proof. The lemma follows from the same kind of arguments as used to prove [GRW19a, Theorem 3.16]. By (2.2.9), $\mathcal{R}(u)$ satisfies

$$\mathcal{R}_{12}(v-u)\mathcal{R}_{13}(-u)\mathcal{R}_{23}(-v) = \mathcal{R}_{23}(-v)\mathcal{R}_{13}(-u)\mathcal{R}_{12}(v-u).$$

Applying the homomorphism $\rho \otimes \rho \otimes 1$ to both sides of this relation we obtain that $\widetilde{\Phi}(T(u))$ satisfies the defining RTT-relation (2.4.1). Therefore, $\widetilde{\Phi}$ extends to a homomorphism $\widetilde{\Phi} : X_\mathcal{I}(\mathfrak{g}) \to Y(\mathfrak{g})$. By (2.2.11),

$$\begin{aligned}\mathcal{R}(-u) =& 1 - \Omega u^{-1} + \sum_{\lambda\in\Lambda}(J(X_\lambda)\otimes X_\lambda - X_\lambda \otimes J(X_\lambda))u^{-2} \\ &+ \tfrac{1}{2}\Omega^2 u^{-2} + O(u^{-3}).\end{aligned} \tag{2.5.2}$$

After applying $\rho \otimes 1$ to both sides, we obtain that

$$\widetilde{\Phi}(t_{ij}^{(1)}) = \mathcal{F}_{ij} \quad \text{and} \quad \widetilde{\Phi}(t_{ij}^{(2)}) \equiv J(\mathcal{F}_{ij}) \mod \mathbf{F}_0^J \quad \forall\, 1 \le i,j \le N, \tag{2.5.3}$$

where we recall that the elements $\mathcal{F}_{ij} \in \mathfrak{g}$, which were defined in the proof of Proposition 2.3.4, are determined by $\sum_{i,j=1}^{N} E_{ij} \otimes \mathcal{F}_{ij} = -(\rho \otimes 1)\Omega$.

Since $\mathbf{F}_0^J = U(\mathfrak{g})$ is generated by $\{\mathcal{F}_{ij}\}_{1\leq i,j\leq N}$, this shows that $\widetilde{\Phi}$ is surjective. The proof that $\widetilde{\Phi}$ is a coalgebra morphism commuting with the antipodes of $X_{\mathcal{I}}(\mathfrak{g})$ and $Y(\mathfrak{g})$ follows from the relations (2.2.7) and (2.2.12): see the proof of [GRW19a, Theorem 3.16]. □

We are now prepared to prove that $Y_R(\mathfrak{g})$ and $Y(\mathfrak{g})$ are isomorphic.

Theorem 2.5.2. *The homomorphism $\widetilde{\Phi}$ factors through the quotient algebra*

$$Y_R(\mathfrak{g}) = X_{\mathcal{I}}(\mathfrak{g})/(\mathcal{Z}(u) - I)$$

to yield an isomorphism of algebras

$$\Phi : Y_R(\mathfrak{g}) \xrightarrow{\sim} Y(\mathfrak{g}), \quad \mathcal{T}(u) \mapsto (\rho \otimes 1)(\mathcal{R}(-u)).$$

Proof. By Corollary 2.2.3 the relation $S^2 = \tau_{-\frac{1}{2}c_{\mathfrak{g}}}$ is satisfied in $Y(\mathfrak{g})$ and by the second identity of (2.2.10) we have $(1 \otimes \tau_{-\frac{1}{2}c_{\mathfrak{g}}})(\mathcal{R}(-u)) = \mathcal{R}(-u - \frac{1}{2}c_{\mathfrak{g}})$. This shows that

$$(1 \otimes S^2)(\mathcal{R}(-u))\mathcal{R}(-u - \tfrac{1}{2}c_{\mathfrak{g}})^{-1} = 1.$$

Applying $\rho \otimes 1$ to both sides of this equality and using that $\widetilde{\Phi}$ is a morphism of Hopf algebras, we arrive at the relation

$$\widetilde{\Phi}(\mathcal{Z}(u)) = \widetilde{\Phi}(S_{\mathcal{I}}^2(T(u)))\widetilde{\Phi}(T(u + \tfrac{1}{2}c_{\mathfrak{g}}))^{-1} = I.$$

This proves that $(\mathcal{Z}(u) - I) \subset \mathrm{Ker}(\widetilde{\Phi})$ and hence that $\widetilde{\Phi}$ factors through $Y_R(\mathfrak{g})$ to yield an algebra epimorphism $\Phi : Y_R(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$ determined by $\mathcal{T}(u) \mapsto (\rho \otimes 1)(\mathcal{R}(-u))$.

By (2.5.3),

$$\Phi(\tau_{ij}^{(1)}) = \mathcal{F}_{ij} \quad \text{and} \quad \Phi(\tau_{ij}^{(2)}) \equiv J(\mathcal{F}_{ij}) \mod U(\mathfrak{g}) \quad \forall\, 1 \leq i, j \leq N$$

Since, by Corollary 2.4.7, $Y_R(\mathfrak{g})$ is generated by $\{\tau_{ij}^{(1)}, \tau_{ij}^{(2)}\}_{1\leq i,j\leq N}$, this shows that Φ is a filtered homomorphism. To conclude that Φ is an isomorphism, it is enough to show that the associated graded morphism $\mathrm{gr}(\Phi) : \mathrm{gr}Y_R(\mathfrak{g}) \to \mathrm{gr}Y(\mathfrak{g})$ is an isomorphism.

Set

$$\varphi_{\bullet} = (\phi_{\rho}^{z})^{-1} \circ \varphi_J^{-1} \circ \mathrm{gr}(\Phi) : \mathrm{gr}Y_R(\mathfrak{g}) \to U(\mathfrak{g}_{\rho}[z]),$$

where φ_J is the isomorphism $U(\mathfrak{g}[z]) \xrightarrow{\sim} \mathrm{gr}Y(\mathfrak{g})$ of Proposition 2.2.2 and ϕ_{ρ}^{z} : $U(\mathfrak{g}_{\rho}[z]) \xrightarrow{\sim} U(\mathfrak{g}[z])$ is the isomorphism of Corollary 2.3.7. This morphism sends $\bar{\tau}_{ij}^{(r)}$ to $F_{ij}^{(r-1)} = F_{ij}z^{r-1}$ for all $1 \leq r \leq 2$ and $1 \leq i, j \leq N$. Consider the composition $\varphi_{\bullet} \circ \varphi$ where $\varphi : U(\mathfrak{g}_{\rho}[z]) \twoheadrightarrow \mathrm{gr}Y_R(\mathfrak{g})$ is the epimorphism of Proposition 2.4.6. This composition sends $F_{ij}^{(r)}$ to $F_{ij}^{(r)}$ for $r = 0, 1$ and hence is equal to the identity morphism. Therefore $\mathrm{gr}(\Phi)$, and thus Φ, is an isomorphism. □

In particular, we have shown that the ideal $(\mathcal{Z}(u) - I)$ is the kernel of the Hopf algebra morphism $\widetilde{\Phi}$, and hence is a Hopf ideal. The Yangian $Y_R(\mathfrak{g})$ thus inherits from $X_{\mathcal{I}}(\mathfrak{g})$ the unique Hopf algebra structure such that Φ becomes an isomorphism of Hopf algebras. Explicitly, it has coproduct Δ_R, antipode S_R, and counit ϵ_R given by

$$\Delta_R(\mathcal{T}(u)) = \mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u), \quad S_R(\mathcal{T}(u)) = \mathcal{T}(u)^{-1}, \quad \epsilon_R(\mathcal{T}(u)) = I. \tag{2.5.4}$$

As was noted in Remark 2.4.2, the coefficients $\mathcal{R}_k$ of the universal R-matrix $\mathcal{R}(u)$ have not been explicitly written down, and consequently the elements $\Phi(\tau_{ij}^{(r)})$ do not in general admit an explicit description. Nonetheless, such a description does exist for the images of the elements $\{\tau_{ij}^{(1)}, \tau_{ij}^{(2)}\}_{1\leq i,j\leq N}$ which, by Corollary 2.4.7, do generate $Y_R(\mathfrak{g})$. Since the J-presentation $Y(\mathfrak{g})$ of the Yangian is defined only in terms of degree one and degree zero generators, it is perhaps more natural to rephrase this observation by stating that Φ^{-1} can be concretely described, which is the purpose of the next corollary.

Corollary 2.5.3. *For each $1 \leq i, j \leq N$ let $\{b_{kl}^{(ij)}\}_{1\leq k,l\leq N} \subset \mathbb{C}$ be defined by*

$$(J \otimes 1)(\mathcal{F}) = \sum_{i,j=1}^{N} E_{ij} \otimes \left(\sum_{k,l=1}^{N} b_{kl}^{(ij)} \mathcal{F}_{kl} \right).$$

Then Φ^{-1} is determined on the generators $\{\mathcal{F}_{ij}, J(\mathcal{F}_{ij})\}_{1\leq i,j\leq N}$ of $Y(\mathfrak{g})$ by

$$\begin{aligned}\mathcal{F}_{ij} &\mapsto \tau_{ij}^{(1)},\\ J(\mathcal{F}_{ij}) &\mapsto \tau_{ij}^{(2)} - \tfrac{1}{2}\sum_{a=1}^{N}\tau_{ia}^{(1)}\tau_{aj}^{(1)} + \sum_{k,l=1}^{N} b_{kl}^{(ij)}\tau_{kl}^{(1)}\end{aligned} \tag{2.5.5}$$

for all $1 \leq i, j \leq N$.

Proof. For each $r \geq 1$, set $\mathcal{T}^{(r)} = \sum_{i,j=1}^{N} E_{ij} \otimes \tau_{ij}^{(r)}$ and define

$$J(\mathcal{F}) = \sum_{i,j=1}^{N} E_{ij} \otimes J(\mathcal{F}_{ij}) \in \mathrm{End}V \otimes Y(\mathfrak{g}).$$

Then, using the expansion (2.5.2) we find that $\Phi(\mathcal{T}^{(1)}) = \mathcal{F}$ and $\Phi(\mathcal{T}^{(2)}) = J(\mathcal{F}) - (J \otimes 1)(\mathcal{F}) + \frac{1}{2}\mathcal{F}^2$. Thus,

$$\Phi^{-1}(J(\mathcal{F})) = \mathcal{T}^{(2)} - \tfrac{1}{2}(\mathcal{T}^{(1)})^2 + (J \otimes 1)(\mathcal{T}^{(1)}),$$

which implies (2.5.5). □

Remark 2.5.4. When $\mathfrak{g}$ is a symplectic or orthogonal Lie algebra and V is its vector representation, it was proven in Proposition 3.19 of [GRW19a] directly that the assignment (2.5.5) extends to an isomorphism $Y(\mathfrak{g}) \xrightarrow{\sim} Y_R(\mathfrak{g})$. In that case, and more generally in any case where $\rho(J(X)) = 0$ for all $X \in \mathfrak{g}$, the term involving the coefficients $b_{kl}^{(ij)}$ in (2.5.5) vanishes and we have

$$J(\mathcal{F}_{ij}) \mapsto \tau_{ij}^{(2)} - \tfrac{1}{2}\sum_{a=1}^{N}\tau_{ia}^{(1)}\tau_{aj}^{(1)}.$$

In the process of proving Theorem 2.5.2 we have also shown that the homomorphism φ of Proposition 2.4.6 is injective. We thus obtain the following Poincaré-Birkhoff-Witt type theorem for $Y_R(\mathfrak{g})$:

Theorem 2.5.5. *The surjective homomorphism $\varphi : U(\mathfrak{g}_\rho[z]) \twoheadrightarrow \mathrm{gr}Y_R(\mathfrak{g})$ of Proposition 2.4.6, which is given by $F_{ij}^{(r-1)} \to \bar{\tau}_{ij}^{(r)}$, is an isomorphism of algebras. Consequently, the assignment*

$$F_{ij} \mapsto \tau_{ij}^{(1)} \quad \forall\, 1 \leq i, j \leq N$$

defines an embedding $U(\mathfrak{g}_\rho) \hookrightarrow Y_R(\mathfrak{g})$.

The above theorem can be employed to obtain a complete description of the center of $Y_R(\mathfrak{g})$:

Corollary 2.5.6. *The center of* $Y_R(\mathfrak{g})$ *is equal to* $\mathbb{C} \cdot 1$.

Proof. The center of the universal enveloping algebra $U(\mathfrak{g}_\rho[z]) \cong U(\mathfrak{g}[z])$ is known to be trivial: see for instance [Mol07, Lemma 1.7.4]. As a consequence of Theorem 2.5.5, the same must be true for the associated graded algebra $\mathrm{gr}Y_R(\mathfrak{g})$, and thus the Yangian $Y_R(\mathfrak{g})$. See also [Ols92, Theorem 1.12], [Mol07, Theorem 1.7.5], and [AMR06, Corollary 3.9] for the version of this result corresponding to the case where $\mathfrak{g}$ is equal to $\mathfrak{sl}_N, \mathfrak{so}_N$, or $\mathfrak{sp}_N$ and $V = \mathbb{C}^N$, which is proven in the exact same way. □

2.6 Structure of the extended Yangian

Using the results of the previous section one can extract a fair amount of information about the extended Yangian $X_\mathcal{I}(\mathfrak{g})$, and in fact prove several results which are known to hold when $\mathfrak{g}$ is a classical Lie algebra and V is its vector representation. Making this explicit is the main goal of the current section.

2.6.1 The tensor product decomposition, the center, and the PBW theorem

In this subsection we will prove that $X_\mathcal{I}(\mathfrak{g})$ is isomorphic to the tensor product of a polynomial algebra in countably many variables with the Yangian $Y_R(\mathfrak{g})$. This will allow us to deduce a Poincaré-Birkhoff-Witt type theorem for $X_\mathcal{I}(\mathfrak{g})$ and also to obtain a complete description of its center.

For brevity, we shall write

$$\mathbb{C}[\mathcal{y}_\lambda^{(r)}]_{\lambda,r} = \mathbb{C}[\mathcal{y}_\lambda^{(r)} \,:\, \lambda \in \mathcal{I},\, r \geq 1].$$

Definition 2.6.1. We define the auxiliary algebra $\mathfrak{X}_\mathcal{I}(\mathfrak{g})$ to be the tensor product of $\mathbb{C}[\mathcal{y}_\lambda^{(r)}]_{\lambda,r}$ with $Y_R(\mathfrak{g})$:

$$\mathfrak{X}_\mathcal{I}(\mathfrak{g}) = \mathbb{C}[\mathcal{y}_\lambda^{(r)}]_{\lambda,r} \otimes Y_R(\mathfrak{g}).$$

Our present goal is to prove the deformed version of Proposition 2.3.15. Namely, we will prove that $X_{\mathcal{I}}(\mathfrak{g})$ and $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ are isomorphic algebras. Define $\mathscr{Y}(u) \in \mathcal{E} \otimes (\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!]$ by

$$\mathscr{Y}(u) = I + \sum_{\lambda \in \mathcal{I}} X_\lambda^\bullet \otimes \mathscr{y}_\lambda(u), \qquad \text{where} \qquad \mathscr{y}_\lambda(u) = \sum_{r \geq 1} \mathscr{y}_\lambda^{(r)} u^{-r}.$$

It will also be useful to expand

$$\mathscr{Y}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes y_{ij}(u), \qquad \text{where} \qquad \mathscr{y}_{ij}(u) = \delta_{ij} + \sum_{\lambda \in \mathcal{I}} c_{ij}^\lambda \mathscr{y}_\lambda(u)$$

and the c_{ij}^λ are as in (2.3.2).

Set

$$\mathscr{T}(u) = \mathscr{Y}(u)\mathcal{T}(u) \in \mathrm{End}V \otimes \mathfrak{X}_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!],$$

and denote by $\mathscr{t}_{ij}(u) = \delta_{ij} + \sum_{r \geq 1} \mathscr{t}_{ij}^{(r)} u^{-r}$ the (i,j)-th entry of $\mathscr{T}(u)$ (that is, $\mathscr{T}(u) = \sum_{i,j=1}^N E_{ij} \otimes \mathscr{t}_{ij}(u)$). We then have

$$\mathscr{t}_{ij}^{(r)} = \tau_{ij}^{(r)} + \mathscr{y}_{ij}^{(r)} + \sum_{a=1}^{N} \sum_{c=1}^{r-1} y_{ia}^{(c)} \tau_{aj}^{(r-c)} \quad \forall\, 1 \leq i,j \leq N,\, r \geq 1. \tag{2.6.1}$$

The degree assignment

$$\deg \mathscr{y}_\lambda^{(r)} = r - 1 \quad \forall\, \lambda \in \mathcal{I} \quad \text{and} \quad r \geq 1$$

defines a grading on the polynomial algebra $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}$. Let $\mathbf{G}_k$ denote the subspace spanned by monomials of degree equal to k and denote the direct sum $\oplus_{i=0}^k \mathbf{G}_k$ by $\mathbf{H}_k$. In particular, we have $\mathscr{y}_{ij}^{(r)} \in \mathbf{G}_{r-1}$ for all $r \geq 1$ and $1 \leq i,j \leq N$. After equipping $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ with the tensor product filtration defined by

$$\mathbf{F}_r(\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})) = \sum_{k+l=r} \mathbf{H}_k \otimes \mathbf{F}_l(Y_R(\mathfrak{g})) = \bigoplus_{a=0}^{r} \mathbf{G}_a \otimes \mathbf{F}_{r-a}(Y_R(\mathfrak{g})),$$

it becomes a filtered algebra with $\mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \otimes \mathrm{gr}Y_R(\mathfrak{g})$. It is immediate from (2.6.1) that the following relations are satisfied in $\mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$:

$$\bar{\mathscr{t}}_{ij}^{(r)} = \bar{\tau}_{ij}^{(r)} + \bar{\mathscr{y}}_{ij}^{(r)}, \quad \forall\, 1 \leq i,j \leq N,\, r \geq 1.$$

Here $\bar{t}_{ij}^{(r)}$ and $\bar{y}_{ij}^{(r)}$ denote the images of $t_{ij}^{(r)}$ and $y_{ij}^{(r)}$, respectively, in

$$\mathbf{F}_{r-1}(\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}))/\mathbf{F}_{r-2}(\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})) \subset \mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}).$$

It follows from Theorem 2.5.5 that the assignment

$$(\bar{y}_{ij}^{(r)}, \bar{\tau}_{ij}^{(r)}) \mapsto (y_{ij}^{(r)}, F_{ij}^{(r-1)}) \quad \forall\, r \geq 1 \quad \text{and} \quad 1 \leq i, j \leq N$$

extends to an isomorphism $\mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[y_{\lambda}^{(r)}]_{\lambda,r} \otimes U(\mathfrak{g}_{\rho}[z])$. Composing with the inverse of the isomorphism $\phi_{\mathcal{I}}^{z}$ of Proposition 2.3.15 (after identifying $\mathcal{K}_{\lambda}^{(r)}$ with $y_{\lambda}^{(r)}$) yields an isomorphism

$$\varphi_{\mathfrak{X}} : \mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}_{\mathcal{I}}[z]), \quad \bar{t}_{ij}^{(r)} \mapsto \mathbb{F}_{ij}^{(r-1)} = F_{ij}^{\mathcal{I}} z^{r-1}. \tag{2.6.2}$$

Remark 2.6.2. In Step 2 of the proof of Proposition 2.4.4, it was useful to expand $T(u)$ with respect to the basis $\{X_{\lambda}^{\bullet}\}_{\lambda\in\Lambda^{\bullet}}$ of $\mathrm{End}V$. It is also sometimes more natural to expand $\mathcal{T}(u)$ and $\mathcal{T}(u)$ in this way. Setting $t_{\lambda}^{(r)} = \sum_{i,j=1}^{N} a_{ij}^{\lambda} t_{ij}^{(r)}$ and $\tau_{\lambda}^{(r)} = \sum_{i,j=1}^{N} a_{ij}^{\lambda} \tau_{ij}^{(r)}$ for each $\lambda \in \Lambda^{\bullet}$ and $r \geq 1$, we obtain

$$\mathcal{T}(u) = I + \sum_{\lambda\in\Lambda^{\bullet}} X_{\lambda}^{\bullet} \otimes t_{\lambda}(u) \quad \text{and} \quad \mathcal{T}(u) = I + \sum_{\lambda\in\Lambda^{\bullet}} X_{\lambda}^{\bullet} \otimes \tau_{\lambda}(u),$$

where $(t_{\lambda}(u), \tau_{\lambda}(u)) = (\sum_{r\geq 1} t_{\lambda}^{(r)} u^{-r}, \sum_{r\geq 1} \tau_{\lambda}^{(r)} u^{-r})$ for all $\lambda \in \Lambda^{\bullet}$. We then have

$$\bar{t}_{\lambda}^{(r)} = \begin{cases} \bar{\tau}_{\lambda}^{(r)} & \text{if } \lambda \in \Lambda, \\ \bar{y}_{\lambda}^{(r)} & \text{if } \lambda \in \mathcal{I}, \\ 0 & \text{otherwise,} \end{cases}$$

in $\mathrm{gr}\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$, and the isomorphism $\varphi_{\mathfrak{X}}$ from (2.6.2) is also determined by $\bar{\tau}_{\lambda}^{(r)} \mapsto \mathbb{X}_{\lambda}^{(r-1)}$ for all $\lambda \in \Lambda$ and $\bar{y}_{\lambda}^{(r)} \mapsto \mathbb{X}_{\lambda}^{(r-1)}$ for all $\lambda \in \mathcal{I}$: see §2.3.3.2.

The next theorem is the first main result of this section, and, as previously suggested, it may be viewed as the Yangian analogue of Proposition 2.3.15.

Theorem 2.6.3. *The assignment $T(u) \mapsto \mathcal{T}(u)$ extends uniquely to yield an algebra isomorphism*

$$\Phi_{\mathcal{I}} : X_{\mathcal{I}}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) = \mathbb{C}[y_{\lambda}^{(r)}]_{\lambda,r} \otimes Y_{R}(\mathfrak{g}).$$

Proof. Since $\mathscr{Y}(u) \in \mathcal{E} \otimes (\mathbb{C}[\boldsymbol{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!]$ and $\mathcal{T}(u)$ satisfies the RTT-relation (2.4.1), the same argument as used to prove Lemma 2.4.3 shows that $\mathscr{T}(u) = \mathscr{Y}(u)\mathcal{T}(u)$ also satisfies (2.4.1). Therefore, $\Phi_\mathcal{I} : T(u) \mapsto \mathscr{T}(u)$ extends uniquely to an algebra homomorphism $X_\mathcal{I}(\mathfrak{g}) \to \mathfrak{X}_\mathcal{I}(\mathfrak{g})$. By (2.6.1), $\Phi_\mathcal{I}$ is filtration preserving. To prove that $\Phi_\mathcal{I}$ is an isomorphism, we will follow the same argument as employed to prove Theorem 2.5.2 and show that the associated graded morphism $\mathrm{gr}(\Phi_\mathcal{I})$ is an isomorphism.

The composition $\mathrm{gr}(\Phi_\mathcal{I}) \circ \varphi_\mathcal{I}$, where $\varphi_\mathcal{I} : U(\mathfrak{g}_\mathcal{I}[z]) \twoheadrightarrow \mathrm{gr}X_\mathcal{I}(\mathfrak{g})$ is the epimorphism of Proposition 2.4.4, sends $\mathbb{F}_{ij}^{(r-1)}$ to $\bar{\boldsymbol{t}}_{ij}^{(r)}$ for all $r \geq 1$ and $1 \leq i, j \leq N$. Composing with the isomorphism $\varphi_\mathfrak{X} : \mathrm{gr}\mathfrak{X}_\mathcal{I}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}_\mathcal{I}[z])$ defined in (2.6.2) therefore gives the identity map $\mathrm{id}_{U(\mathfrak{g}_\mathcal{I}[z])}$. This implies that $\mathrm{gr}(\Phi_\mathcal{I})$ is indeed an isomorphism, and the same must be true of $\Phi_\mathcal{I}$. □

Our next goal is to use Theorem 2.6.3 to obtain a complete description of the center of $X_\mathcal{I}(\mathfrak{g})$, and to prove a Poincaré-Birkhoff-Witt theorem for $X_\mathcal{I}(\mathfrak{g})$. We will need a few preliminary lemmas, the first being a consequence of Theorem 2.5.2.

Lemma 2.6.4. *We have*

$$\mathcal{T}(u) \in \rho(Y(\mathfrak{g})) \otimes Y_R(\mathfrak{g})[\![u^{-1}]\!] \subset \mathrm{End}V \otimes Y_R(\mathfrak{g})[\![u^{-1}]\!].$$

Consequently, the commutation relations

$$\mathscr{Y}(u)\mathcal{T}(u) = \mathcal{T}(u)\mathscr{Y}(u) \quad \textit{and} \quad \mathscr{Y}(u)\mathscr{T}(u) = \mathscr{T}(u)\mathscr{Y}(u)$$

are satisfied in $\mathrm{End}V \otimes \mathfrak{X}_\mathcal{I}(\mathfrak{g})$.

Proof. Since $\mathcal{R}(u) \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[\![u^{-1}]\!]$, Theorem 2.5.2 implies the first part of the Lemma. As $\mathscr{Y}(u) \in \mathcal{E} \otimes (\mathbb{C}[\boldsymbol{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!]$ and $\mathcal{E}$ is the centralizer of $\rho(Y(\mathfrak{g}))$ in $\mathrm{End}V$, $[\mathscr{Y}(u), \mathcal{T}(u)] = 0 = [\mathscr{Y}(u), \mathscr{T}(u)]$. □

Next, define $\mathcal{Y}(u)$ to be the preimage of $\mathscr{Y}(u)$ under $\Phi_\mathcal{I}$:

$$\mathcal{Y}(u) = I + \sum_{\lambda \in \mathcal{I}} X_\lambda^\bullet \otimes y_\lambda(u) = \Phi_\mathcal{I}^{-1}(\mathscr{Y}(u)) \in \mathcal{E} \otimes X_\mathcal{I}(\mathfrak{g})[\![u^{-1}]\!],$$

and write $y_\lambda(u) = \sum_{r \geq 1} y_\lambda^{(r)} u^{-r}$. As was the case for $\mathscr{Y}(u)$, we shall also make use of the expansion of $\mathcal{Y}(u)$ with respect to the basis of elementary matrices $\{E_{ij}\}_{1 \leq i,j \leq N}$.

That is, we may write

$$\mathcal{Y}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes y_{ij}(u) \quad \text{with} \quad y_{ij}(u) = \delta_{ij} + \sum_{\lambda \in \mathcal{I}} c_{ij}^{\lambda} y_\lambda(u).$$

For each $\lambda \in \mathcal{I}$ (resp. $1 \leq i,j \leq N$) and $r \geq 1$, the element $y_\lambda^{(r)}$ (resp. $y_{ij}^{(r)}$) belongs to $\mathbf{F}_{r-1}^{\mathcal{I}}$, and we will denote by $\bar{y}_\lambda^{(r)}$ (resp. $\bar{y}_{ij}^{(r)}$) its image in the quotient $\mathbf{F}_{r-1}^{\mathcal{I}}/\mathbf{F}_{r-2}^{\mathcal{I}} = \mathrm{gr}_{r-1} X_{\mathcal{I}}(\mathfrak{g})$.

Lemma 2.6.5. *The following statements hold:*

(1) $\mathcal{Z}(u) = \mathcal{Y}(u)\mathcal{Y}(u + \frac{1}{2}c_{\mathfrak{g}})^{-1} \in \mathcal{E} \otimes X_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!]$,

(2) $z_{ij}^{(r+1)} \in \mathbf{F}_{r-1}^{\mathcal{I}}$ *for all* $1 \leq i,j \leq N$ *and* $r \geq 0$ *(where* $\mathbf{F}_{-1}^{\mathcal{I}} = \{0\}$*),*

(3) $\bar{z}_{ij}^{(r+1)} = \frac{r}{2} c_{\mathfrak{g}} \bar{y}_{ij}^{(r)} \quad \forall\, 1 \leq i,j \leq N$ *and* $r \geq 0$*, where* $\bar{z}_{ij}^{(r+1)}$ *denotes the image of* $z_{ij}^{(r+1)}$ *in* $\mathrm{gr}_{r-1} X_{\mathcal{I}}(\mathfrak{g})$.

Proof. Consider (1). Since $\mathscr{Y}(u)$ is an invertible element of

$$(\mathcal{E} \otimes \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!] \cong \mathcal{E} \otimes (\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!],$$

we obtain an automorphism $S_{\mathscr{Y}}$ of $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}$ which is determined by $\mathscr{Y}(u) \mapsto \mathscr{Y}(u)^{-1}$. Consider the tensor product $S_{\mathfrak{X}} = S_{\mathscr{Y}} \otimes S_R$, where we recall from (2.5.4) that S_R is the antipode of $Y_R(\mathfrak{g})$, and it is given by $\mathcal{T}(u) \mapsto \mathcal{T}(u)^{-1}$. Then $S_{\mathfrak{X}}$ is the anti-automorphism of the algebra $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) = \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \otimes Y_R(\mathfrak{g})$ completely determined by

$$S_{\mathfrak{X}}(\mathscr{T}(u)) = \mathscr{Y}(u)^{-1}\mathcal{T}(u)^{-1} = \mathcal{T}(u)^{-1}\mathscr{Y}(u)^{-1} = \mathscr{T}(u)^{-1},$$

where in the second equality we have appealed to Lemma 2.6.4. Consequently,

$$S_{\mathfrak{X}} \circ \Phi_{\mathcal{I}} = \Phi_{\mathcal{I}} \circ S_{\mathcal{I}}.$$

Since $\Phi : Y_R(\mathfrak{g}) \to Y(\mathfrak{g})$ is a Hopf algebra morphism and $(1 \otimes S^2)\mathcal{R}(-u) = \mathcal{R}(-u - \frac{1}{2}c_{\mathfrak{g}})$, we have $S_R^2(\mathcal{T}(u)) = \mathcal{T}(u + \frac{1}{2}c_{\mathfrak{g}})$. Therefore,

$$\begin{aligned}\Phi_{\mathcal{I}}(\mathcal{Z}(u)) &= \Phi_{\mathcal{I}}(S_{\mathcal{I}}^2(T(u))T(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1}) \\ &= S_{\mathfrak{X}}^2(\mathscr{T}(u))\mathscr{T}(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1} = S_{\mathfrak{X}}^2(\mathscr{Y}(u))\mathscr{Y}(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1}.\end{aligned}$$

Since $S_{\mathfrak{X}}$ restricts to an automorphism of $\mathbb{C}[\mathscr{Y}_\lambda^{(r)}]_{\lambda,r}$ (namely $S_{\mathscr{Y}}$) and $S_{\mathfrak{X}}(\mathscr{Y}(u)) = \mathscr{Y}(u)^{-1}$, we have $S_{\mathfrak{X}}^2(\mathscr{Y}(u)) = \mathscr{Y}(u)$, and we may thus conclude that

$$\Phi_{\mathcal{I}}(\mathcal{Z}(u)) = \mathscr{Y}(u)\mathscr{Y}(u + \tfrac{1}{2}c_{\mathfrak{g}})^{-1},$$

and hence that $\mathcal{Z}(u) = \mathcal{Y}(u)\mathcal{Y}(u + \frac{1}{2}c_{\mathfrak{g}})^{-1}$. Since $\mathcal{E} = \mathrm{End}_{Y(\mathfrak{g})}V$ is an algebra, $\mathcal{Z}(u)$ also belongs to $\mathcal{E} \otimes X_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!]$. This observation concludes the proof of (1).

Proof of (2). The (i,j)-th entry of the u^{-r-1} coefficient of $\mathcal{Y}(u + \frac{1}{2}c_{\mathfrak{g}})^{-1}$ is equal to

$$-y_{ij}^{(r+1)} \quad \mathrm{mod}\ \mathbf{F}_{r-1}^{\mathcal{I}}.$$

It is a straightforward consequence of this fact that the u^{-r-1} coefficient of the (i,j)-th entry of $\mathcal{Y}(u)\mathcal{Y}(u + \frac{1}{2}c_{\mathfrak{g}})^{-1}$, which is equal to $z_{ij}^{(r+1)}$, is contained in $\mathbf{F}_{r-1}^{\mathcal{I}}$.

Proof of (3). The argument we give is similar to the proof of Proposition 2.4.6. By (1), we have

$$\mathcal{Z}(u)\mathcal{Y}(u + \tfrac{1}{2}c_{\mathfrak{g}}) = \mathcal{Y}(u).$$

Taking the (i,j)-th coefficient of both sides yields

$$\sum_{a=1}^{N} z_{ia}(u) y_{aj}(u + \tfrac{1}{2}c_{\mathfrak{g}}) = y_{ij}(u).$$

Writing $y_{aj}(u + \frac{1}{2}c_{\mathfrak{g}}) = \sum_{r\geq 0} y_{aj}^{\circ(r)} u^{-r}$, we have $y_{aj}^{\circ(r)} \in \mathbf{F}_{r-1}^{\mathcal{I}}$ for each $r \geq 0$ and

$$\begin{aligned} y_{ij}(u) &= \sum_{a=1}^{N} z_{ia}(u) y_{aj}(u + \tfrac{1}{2}c_{\mathfrak{g}}) \\ &= y_{ij}(u + \tfrac{1}{2}c_{\mathfrak{g}}) + z_{ij}(u) + \sum_{a=1}^{N} \sum_{k,s\geq 1} z_{ia}^{(k)} y_{aj}^{\circ(s)} u^{-k-s} \end{aligned} \tag{2.6.3}$$

By (2), $z_{ia}^{(k)} y_{aj}^{\circ(s)} \in \mathbf{F}_{k+s-3}^{\mathcal{I}}$. Thus, the coefficient of u^{-r-1} in the summation on the right-hand side of the above equality is contained in $\mathbf{F}_{r-2}^{\mathcal{I}}$. On the other hand, the same argument as used to establish (2.4.21) allows us to deduce that the u^{-r-1} coefficient of $y_{ij}(u + \frac{1}{2}c_{\mathfrak{g}})$ is equivalent to $y_{ij}^{(r+1)} - \frac{r}{2}c_{\mathfrak{g}} y_{ij}^{(r)}$ modulo $\mathbf{F}_{r-2}^{\mathcal{I}}$. Thus, (2.6.3) implies that

$$y_{ij}^{(r+1)} \equiv y_{ij}^{(r+1)} + z_{ij}^{(r+1)} - \tfrac{r}{2}c_{\mathfrak{g}} y_{ij}^{(r)} \quad \mathrm{mod}\ \mathbf{F}_{r-2}^{\mathcal{I}},$$

and hence that $\bar{z}_{ij}^{(r+1)} = \frac{r}{2} c_{\mathfrak{g}} \bar{y}_{ij}^{(r)}$ for all $1 \leq i, j \leq N$, $r \geq 0$. $\square$

For each $\lambda \in \Lambda^{\bullet}$, set $z_\lambda(u) = \sum_{r \geq 1} z_\lambda^{(r)} u^{-r}$ with $z_\lambda^{(r)} = \sum_{i,j=1}^N a_{ij}^\lambda z_{ij}^{(r)}$. Then, by Part (1) of Lemma 2.6.5,

$$\mathcal{Z}(u) = I + \sum_{\lambda \in \Lambda^{\bullet}} X_\lambda^{\bullet} \otimes z_\lambda(u) = I + \sum_{\lambda \in \mathcal{I}} X_\lambda^{\bullet} \otimes z_\lambda(u).$$

The following Proposition gives a complete description of the center of $X_{\mathcal{I}}(\mathfrak{g})$ in terms of the coefficients $z_\lambda^{(r)}$ of $\mathcal{Z}(u)$.

Proposition 2.6.6. *Let $ZX_{\mathcal{I}}(\mathfrak{g})$ denote the center of $X_{\mathcal{I}}(\mathfrak{g})$. The set of elements $\{y_\lambda^{(r)}\}_{\lambda \in \mathcal{I}, r \geq 1}$ is algebraically independent and generates $ZX_{\mathcal{I}}(\mathfrak{g})$, and the same is true of the set $\{z_\lambda^{(r)}\}_{\lambda \in \mathcal{I}, r \geq 2}$. Consequently,*

$$\mathbb{C}[y_\lambda^{(r)} : \lambda \in \mathcal{I}, r \geq 1] \cong ZX_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[z_\lambda^{(r)} : \lambda \in \mathcal{I}, r \geq 2].$$

Proof. By Corollary 2.5.6, the center of $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ is equal to the polynomial algebra $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda, r}$. Since the isomorphism $\Phi_{\mathcal{I}}$ of Theorem 2.6.3 satisfies

$$\Phi_{\mathcal{I}}(y_\lambda^{(r)}) = \mathscr{y}_\lambda^{(r)} \quad \forall \lambda \in \mathcal{I} \quad \text{and} \quad r \geq 1,$$

the set $\{y_\lambda^{(r)}\}_{\lambda \in \mathcal{I}, r \geq 1}$ must be an algebraically independent set which generates the center of $X_{\mathcal{I}}(\mathfrak{g})$. In particular, $ZX_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[y_\lambda^{(r)} : \lambda \in \mathcal{I}, r \geq 1]$.

Since the coefficients $\{z_\lambda^{(r)}\}_{\lambda \in \mathcal{I}, r \geq 2}$ are central, the assignment $y_\lambda^{(r)} \mapsto z_\lambda^{(r+1)}$, for all $\lambda \in \mathcal{I}$ and $r \geq 1$, extends to an algebra endomorphism

$$\varphi_{y,z} : ZX_{\mathcal{I}}(\mathfrak{g}) \cong \mathbb{C}[y_\lambda^{(r)} : \lambda \in \mathcal{I}, r \geq 1] \to ZX_{\mathcal{I}}(\mathfrak{g}).$$

By Part (2) of Lemma 2.6.5, $\varphi_{y,z}$ is a filtered morphism, and by Part (3) of Lemma 2.6.5 the associated graded morphism $\mathrm{gr}(\varphi_{y,z})$ is just the rescaling automorphism of $\mathbb{C}[y_\lambda^{(r)} : \lambda \in \mathcal{I}, r \geq 1]$ determined by

$$y_\lambda^{(r)} \mapsto 2(rc_{\mathfrak{g}})^{-1} y_\lambda^{(r)} \quad \forall \lambda \in \mathcal{I} \quad \text{and} \quad r \geq 1.$$

Thus $\varphi_{y,z}$ is an automorphism of $ZX_{\mathcal{I}}(\mathfrak{g})$ and hence $\{z_\lambda^{(r)}\}_{\lambda \in \mathcal{I}, r \geq 2}$ is an algebraically independent set which generates the center $ZX_{\mathcal{I}}(\mathfrak{g})$ of $X_{\mathcal{I}}(\mathfrak{g})$. $\square$

With the help of Theorem 2.6.3 or, more accurately, its proof, we obtain the following Poincaré-Birkhoff-Witt theorem for $X_{\mathcal{I}}(\mathfrak{g})$:

Theorem 2.6.7. *The surjective homomorphism $\varphi_{\mathcal{I}} : U(\mathfrak{g}_{\mathcal{I}}[z]) \to \mathrm{gr} X_{\mathcal{I}}(\mathfrak{g})$ of Proposition 2.4.4, which is given by $\mathbb{F}_{ij}^{r-1} \mapsto \bar{t}_{ij}^{(r)}$, is an isomorphism of algebras. Consequently, the assignment*

$$F_{ij}^{\mathcal{I}} \mapsto t_{ij}^{(1)} \quad \forall\, 1 \leq i, j \leq N \tag{2.6.4}$$

defines an embedding $U(\mathfrak{g}_{\mathcal{I}}) \hookrightarrow X_{\mathcal{I}}(\mathfrak{g})$, while the assignment

$$F_{ij} \mapsto t_{ij}^{(1)} - 2c_{\mathfrak{g}}^{-1} z_{ij}^{(2)} \quad \forall\, 1 \leq i, j \leq N \tag{2.6.5}$$

defines an embedding $U(\mathfrak{g}_{\rho}) \hookrightarrow X_{\mathcal{I}}(\mathfrak{g})$.

Proof. The injectivity of $\varphi_{\mathcal{I}}$ was proven in the course of the proof of Theorem 2.6.3, and that (2.6.4) defines an embedding follows immediately.

As for the last statement of the theorem, consider the embedding

$$\iota_R : Y_R(\mathfrak{g}) \hookrightarrow X_{\mathcal{I}}(\mathfrak{g}), \quad \mathcal{T}(u) \mapsto \mathcal{Y}(u)^{-1} T(u).$$

It sends $\tau_{ij}^{(1)}$ to $t_{ij}^{(1)} - y_{ij}^{(1)}$ for all $1 \leq i, j \leq N$. Composing with the embedding

$$U(\mathfrak{g}_{\rho}) \hookrightarrow Y_R(\mathfrak{g}), \quad F_{ij} \mapsto \tau_{ij}^{(1)} \quad \forall\, 1 \leq i, j \leq N$$

of Theorem 2.5.5, we obtain an injection

$$U(\mathfrak{g}_{\rho}) \hookrightarrow X_{\mathcal{I}}(\mathfrak{g}), \quad F_{ij} \mapsto t_{ij}^{(1)} - y_{ij}^{(1)} \quad \forall\, 1 \leq i, j \leq N.$$

The proof that this coincides with (2.6.5) is completed by noting that, by Part (3) of Lemma 2.6.5, we have $y_{ij}^{(1)} = 2c_{\mathfrak{g}}^{-1} z_{ij}^{(2)}$ for all $1 \leq i, j \leq N$. □

2.6.2 The Yangian as a Hopf subalgebra of the extended Yangian

By Theorem 2.6.3, $Y_R(\mathfrak{g})$ may also be identified as a subalgebra of $X_\mathcal{I}(\mathfrak{g})$ via the embedding

$$\iota_R : Y_R(\mathfrak{g}) \hookrightarrow X_\mathcal{I}(\mathfrak{g}), \quad \mathcal{T}(u) \mapsto \mathcal{Y}(u)^{-1}T(u), \tag{2.6.6}$$

which played a role in the proof of Theorem 2.6.7. In this subsection we study $Y_R(\mathfrak{g})$ from this viewpoint, our main goals being to show that ι_R is a Hopf algebra morphism, to study the behaviour of the center under the coproduct $\Delta_\mathcal{I}$, and to show that $Y_R(\mathfrak{g})$ can in fact be realized as a fixed point subalgebra of $X_\mathcal{I}(\mathfrak{g})$.

In order to distinguish between the identifications of $Y_R(\mathfrak{g})$ as a quotient and as a subalgebra of $X_\mathcal{I}(\mathfrak{g})$, we shall denote by $\widetilde{Y}_R(\mathfrak{g}) \subset X_\mathcal{I}(\mathfrak{g})$ the isomorphic copy of $Y_R(\mathfrak{g})$ obtained from the embedding ι_R. We also set

$$\widetilde{\mathcal{T}}(u) = \mathcal{Y}(u)^{-1}T(u) = \sum_{i,j} E_{ij} \otimes \widetilde{\tau}_{ij}(u).$$

The first and main step in showing that ι_R is a morphism of Hopf algebras is to study the behaviour of $\mathcal{Y}(u)$ under the coproduct, counit, and antipode of $X_\mathcal{I}(\mathfrak{g})$. This is the purpose of the next lemma.

Lemma 2.6.8. *The central matrix $\mathcal{Y}(u)$ satisfies*

$$\Delta_\mathcal{I}(\mathcal{Y}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Y}_{[2]}(u), \quad S_\mathcal{I}(\mathcal{Y}(u)) = \mathcal{Y}(u)^{-1}, \quad \epsilon_\mathcal{I}(\mathcal{Y}(u)) = I.$$

Proof. We have already demonstrated in the course of the proof of Lemma 2.6.5 that $S_\mathcal{I}(\mathcal{Y}(u)) = \mathcal{Y}(u)^{-1}$. More precisely, we showed that $S_\mathfrak{X}(\mathscr{Y}(u)) = \mathscr{Y}(u)^{-1}$, where $S_\mathfrak{X}$ is the antiautomorphism of $\mathfrak{X}_\mathcal{I}(\mathfrak{g})$ determined by $S_\mathfrak{X}(\mathscr{T}(u)) = \mathscr{T}(u)^{-1}$. Since

$$S_\mathfrak{X} = \Phi_\mathcal{I} \circ S_\mathcal{I} \circ \Phi_\mathcal{I}^{-1},$$

this implies that $S_\mathcal{I}(\mathcal{Y}(u)) = \mathcal{Y}(u)^{-1}$.

The Hopf algebra axioms dictate that $\epsilon_\mathcal{I} \circ S_\mathcal{I} = \epsilon_\mathcal{I}$, and hence $\epsilon_\mathcal{I}(\mathcal{Y}(u)) = \epsilon_\mathcal{I}(\mathcal{Y}(u))^{-1}$. The equality $\epsilon_\mathcal{I}(\mathcal{Y}(u)\mathcal{Y}(u)^{-1}) = I$ then implies that $\epsilon_\mathcal{I}(\mathcal{Y}(u))^2 = I$. Since the identity matrix I is the unique square root of itself belong to $I + u^{-1}(\mathrm{End}V)[\![u^{-1}]\!]$, we can conclude that $\epsilon_{\mathcal{I}}(\mathcal{Y}(u)) = I$.

It remains to see that $\Delta_{\mathcal{I}}(\mathcal{Y}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Y}_{[2]}(u)$. Let $\Delta_{\mathscr{Y}}$ be the algebra morphism $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \to \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \otimes \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}$ determined by

$$\Delta_{\mathscr{Y}}(\mathscr{Y}(u)) = \mathscr{Y}_{[1]}(u)\mathscr{Y}_{[2]}(u) \in \mathcal{E} \otimes (\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \otimes \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r})[\![u^{-1}]\!].$$

We then obtain an algebra morphism

$$\Delta_{\mathfrak{X}} = \sigma_{23} \circ (\Delta_{\mathscr{Y}} \otimes \Delta_R) : \mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \to \mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \otimes \mathfrak{X}_{\mathcal{I}}(\mathfrak{g}),$$

where $\sigma_{23} = \mathrm{id}_{\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}} \otimes \sigma \otimes \mathrm{id}_{Y_R(\mathfrak{g})}$ and $\sigma : Y_R(\mathfrak{g}) \otimes \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \to \mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \otimes Y_R(\mathfrak{g})$ is the flip map. By definition,

$$\Delta_{\mathfrak{X}}(\mathscr{T}(u)) = \mathscr{Y}_{[1]}(u)\mathscr{Y}_{[2]}(u)\mathcal{T}_{[1]}(u)\mathcal{T}_{[2]}(u) \in \mathrm{End}V \otimes (\mathfrak{X}_{\mathcal{I}}(\mathfrak{g}) \otimes \mathfrak{X}_{\mathcal{I}}(\mathfrak{g}))[\![u^{-1}]\!].$$

Since $\mathscr{Y}_{[2]}(u)$ commutes with $\mathcal{T}_{[1]}(u)$, we can rewrite this as

$$\Delta_{\mathfrak{X}}(\mathscr{T}(u)) = \mathscr{Y}_{[1]}(u)\mathcal{T}_{[1]}(u)\mathscr{Y}_{[2]}(u)\mathcal{T}_{[2]}(u) = \mathscr{T}_{[1]}(u)\mathscr{T}_{[2]}(u),$$

and hence $(\Phi_{\mathcal{I}} \otimes \Phi_{\mathcal{I}}) \circ \Delta_{\mathcal{I}} = \Delta_{\mathfrak{X}} \circ \Phi_{\mathcal{I}}$. This implies that $\Delta_{\mathcal{I}} = (\Phi_{\mathcal{I}}^{-1} \otimes \Phi_{\mathcal{I}}^{-1}) \circ \Delta_{\mathfrak{X}} \circ \Phi_{\mathcal{I}}$, and consequently

$$\Delta_{\mathcal{I}}(\mathcal{Y}(u)) = (\Phi_{\mathcal{I}}^{-1} \otimes \Phi_{\mathcal{I}}^{-1})(\mathscr{Y}_{[1]}(u)\mathscr{Y}_{[2]}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Y}_{[2]}(u). \qquad \square$$

The above lemma leads us to the first main result of this subsection. Let $\epsilon_{\mathscr{Y}}$ be the homomorphism $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r} \to \mathbb{C}$, $\mathscr{Y}(u) \mapsto I$, and recall that $\Phi_{\mathcal{I}} : X_{\mathcal{I}}(\mathfrak{g}) \to \mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ is the algebra isomorphism of Theorem 2.6.3.

Proposition 2.6.9. *$\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}$ is a Hopf algebra with coproduct $\Delta_{\mathscr{Y}}$, counit $\epsilon_{\mathscr{Y}}$ and antipode $S_{\mathscr{Y}}$, and if $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ is equipped with the standard tensor product of Hopf algebras structure, $\Phi_{\mathcal{I}} : X_{\mathcal{I}}(\mathfrak{g}) \to \mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ becomes an isomorphism of Hopf algebras. In particular, The embedding $\iota_R : Y_R(\mathfrak{g}) \hookrightarrow X_{\mathcal{I}}(\mathfrak{g})$ is a morphism of Hopf algebras.*

Proof. $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ becomes a Hopf algebra, and $\Phi_{\mathcal{I}}$ a Hopf algebra isomorphism, after being given the coproduct $(\Phi_{\mathcal{I}} \otimes \Phi_{\mathcal{I}}) \circ \Delta_{\mathcal{I}} \circ \Phi_{\mathcal{I}}^{-1}$ (which, by Lemma 2.6.8, is $\Delta_{\mathfrak{X}}$), counit $\epsilon_{\mathcal{I}} \circ \Phi_{\mathcal{I}}^{-1}$ (which, by Lemma 2.6.8, is $\epsilon_{\mathfrak{X}}$), and antipode $\Phi_{\mathcal{I}} \circ S_{\mathcal{I}} \circ \Phi_{\mathcal{I}}^{-1}$ (which, by Lemma 2.6.8, is $S_{\mathfrak{X}}$). Since the tuple $(\Delta_{\mathcal{Y}}, \epsilon_{\mathcal{Y}}, S_{\mathcal{Y}})$ coincides with

$$(\Delta_{\mathfrak{X}}|_{\mathbb{C}[y_\lambda^{(r)}]_{\lambda,r}}, \epsilon_{\mathfrak{X}}|_{\mathbb{C}[y_\lambda^{(r)}]_{\lambda,r}}, S_{\mathfrak{X}}|_{\mathbb{C}[y_\lambda^{(r)}]_{\lambda,r}}),$$

it endows $\mathbb{C}[y_\lambda^{(r)}]_{\lambda,r}$ with the structure of a Hopf algebra.

Since $\Delta_{\mathfrak{X}} = \sigma_{23}\circ(\Delta_{\mathcal{Y}}\otimes\Delta_R)$, $\epsilon_{\mathfrak{X}} = \eta\circ(\epsilon_{\mathcal{Y}}\otimes\epsilon_R)$ (where $\eta : \mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$ is the natural isomorphism), and $S_{\mathfrak{X}} = S_{\mathcal{Y}} \otimes S_R$, the Hopf algebra structure on $\mathfrak{X}_{\mathcal{I}}(\mathfrak{g})$ induced from $X_{\mathcal{I}}(\mathfrak{g})$ via $\Phi_{\mathcal{I}}$ coincides with the Hopf algebra structure obtained via the standard tensor product of Hopf algebras construction. □

Before moving onto the last main result of this subsection, we note the following corollary of Lemma 2.6.8.

Corollary 2.6.10. *The central matrix $\mathcal{Z}(u)$ satisfies*

$$\Delta_{\mathcal{I}}(\mathcal{Z}(u)) = \mathcal{Y}_{[1]}(u)\mathcal{Z}_{[2]}(u)\mathcal{Y}_{[1]}(u+\tfrac{1}{2}c_{\mathfrak{g}})^{-1},$$
$$S_{\mathcal{I}}(\mathcal{Z}(u)) = \mathcal{Y}(u)^{-1}\mathcal{Y}(u+\tfrac{1}{2}c_{\mathfrak{g}}) \quad \textit{and} \quad \epsilon_{\mathcal{I}}(\mathcal{Z}(u)) = I.$$

Proof. By Lemma 2.6.5, $\mathcal{Z}(u) = \mathcal{Y}(u)\mathcal{Y}(u+\frac{1}{2}c_{\mathfrak{g}})^{-1}$. Therefore, by Lemma 2.6.8, we have

$$\begin{aligned}\Delta_{\mathcal{I}}(\mathcal{Z}(u)) &= \mathcal{Y}_{[1]}(u)\mathcal{Y}_{[2]}(u)\mathcal{Y}_{[2]}(u+\tfrac{1}{2}c_{\mathfrak{g}})^{-1}Y_{[1]}(u+\tfrac{1}{2}c_{\mathfrak{g}})^{-1}\\ &= \mathcal{Y}_{[1]}(u)\mathcal{Z}_{[2]}(u)\mathcal{Y}_{[1]}(u+\tfrac{1}{2}c_{\mathfrak{g}})^{-1}.\end{aligned}$$

Similarly, $\epsilon_{\mathcal{I}}(\mathcal{Z}(u)) = \epsilon(\mathcal{Y}(u))\epsilon(\mathcal{Y}(u+\frac{1}{2}c_{\mathfrak{g}}))^{-1} = I$. Lastly, since the restriction of $S_{\mathcal{I}}$ to the center $ZX_{\mathcal{I}}(\mathfrak{g})$ is an automorphism,

$$S_{\mathcal{I}}(\mathcal{Z}(u)) = S_{\mathcal{I}}(\mathcal{Y}(u))S_{\mathcal{I}}(\mathcal{Y}(u+\tfrac{1}{2}c_{\mathfrak{g}}))^{-1} = \mathcal{Y}(u)^{-1}\mathcal{Y}(u+\tfrac{1}{2}c_{\mathfrak{g}}).$$ □

Recall that, by Lemma 2.4.3, for each $(f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$ there is an automorphsim $m_{\mathfrak{f}}$ of $X_{\mathcal{I}}(\mathfrak{g})$ determined by the assignment (2.4.4). The next theorem proves that $\widetilde{Y}_R(\mathfrak{g})$ can be realized as a fixed point subalgebra of $X_{\mathcal{I}}(\mathfrak{g})$.

Theorem 2.6.11. *The Yangian $\widetilde{Y}_R(\mathfrak{g})$ is equal to the subalgebra of $X_{\mathcal{I}}(\mathfrak{g})$ fixed by all automorphisms $m_{\mathfrak{f}}$:*

$$\widetilde{Y}_R(\mathfrak{g}) = \left\{ Y \in X_{\mathcal{I}}(\mathfrak{g}) \,:\, m_{\mathfrak{f}}(Y) = Y \quad \forall\, (f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda \right\}. \tag{2.6.7}$$

Proof. Recall from (2.4.3) that $(f_\lambda(u))_{\lambda\in\mathcal{I}}$ is identified with the matrix $\mathbf{f}^\circ(u) = \sum_{\lambda\in\mathcal{I}} X_\lambda^\bullet \otimes f_\lambda(u)$, and that, by (2.4.4), $m_{\mathbf{f}}(T(u)) = \mathbf{f}(u)T(u)$, where $\mathbf{f}(u) = I + \mathbf{f}^\circ(u)$. Let us denote the right-hand side of (2.6.7) by $X_\mathcal{I}(\mathfrak{g})^{m_{\mathbf{f}}}$.

For each $(f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$, the assignment

$$\mathscr{Y}(u) \mapsto \mathbf{f}(u)\mathscr{Y}(u)$$

extends to an automorphism $m_{\mathbf{f}}^{\mathscr{Y}}$ of $\mathbb{C}[\mathscr{y}_\lambda^{(r)}]_{\lambda,r}$. Consider the automorphism $m_{\mathbf{f}}^{\mathfrak{X}} = m_{\mathbf{f}}^{\mathscr{Y}} \otimes \mathrm{id}$ of $\mathfrak{X}_\mathcal{I}(\mathfrak{g})$. It satisfies

$$m_{\mathbf{f}}^{\mathfrak{X}}(\mathscr{T}(u)) = m_{\mathbf{f}}^{\mathfrak{X}}(\mathscr{Y}(u))m_{\mathbf{f}}^{\mathfrak{X}}(\mathcal{T}(u)) = \mathbf{f}(u)\mathscr{T}(u),$$

and thus $m_{\mathbf{f}}^{\mathfrak{X}} \circ \Phi_\mathcal{I} = \Phi_\mathcal{I} \circ m_{\mathbf{f}}$. It follows that

$$m_{\mathbf{f}}(\mathcal{Y}(u)) = \mathbf{f}(u)\mathcal{Y}(u) \tag{2.6.8}$$

for every tuple $(f_\lambda(u))_{\lambda\in\mathcal{I}}$. Therefore, for each $(f_\lambda(u))_{\lambda\in\mathcal{I}} \in \prod_{\lambda\in\mathcal{I}}(u^{-1}\mathbb{C}[\![u^{-1}]\!])_\lambda$,

$$\begin{aligned} m_{\mathbf{f}}(\widetilde{\mathcal{T}}(u)) &= m_{\mathbf{f}}(\mathcal{Y}(u))^{-1}m_{\mathbf{f}}(T(u)) \\ &= \mathcal{Y}(u)^{-1}\mathbf{f}(u)^{-1}\mathbf{f}(u)T(u) = \mathcal{Y}(u)^{-1}T(u) = \widetilde{\mathcal{T}}(u). \end{aligned}$$

This proves that $\widetilde{Y}_R(\mathfrak{g}) \subset X_\mathcal{I}(\mathfrak{g})^{m_{\mathbf{f}}}$.

To obtain the reverse inclusion, we employ similar techniques as used to prove [AMR06, Theorem 3.1]. Suppose towards a contradiction that there is $X \in X_\mathcal{I}(\mathfrak{g})^{m_{\mathbf{f}}} \setminus \widetilde{Y}_R(\mathfrak{g})$. By Theorem 2.6.3 we may write X as a polynomial in the variables $\{y_\lambda^{(r)}\}_{\lambda\in\mathcal{I},r\geq 1}$ with coefficients in $\widetilde{Y}_R(\mathfrak{g})$. This polynomial is non-constant by assumption. Only finitely many variables can appear in this polynomial, so there is $m \geq 1$ such that X depends only on the variables $\{y_\lambda^{(r)}\}_{\lambda\in\mathcal{I},r=1,\ldots,m}$. We take m to be minimal with this property, and we fix $\mu \in \mathcal{I}$ such that X depends on $y_\mu^{(m)}$.

Let $X = \sum_{a\geq 0} X_a (y_\mu^{(m)})^a$ be the expansion of X as a polynomial in the single variable $y_\mu^{(m)}$ and set $P(y_\mu^{(m)}) = \sum_{a\geq 1} X_a(y_\mu^{(m)})^a$. The polynomial $P(y_\mu^{(m)})$ has degree at least 1, as otherwise X would not depend on $y_\mu^{(m)}$. For each $w \in \mathbb{C}$, define

$\mathbf{f}_w^\circ(u) = (f_\lambda(u))_{\lambda\in\mathcal{I}}$ by

$$f_\lambda(u) = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ wu^{-m} & \text{if } \lambda = \mu. \end{cases}$$

As a matrix in $\mathcal{E} \otimes u^{-1}\mathbb{C}[\![u^{-1}]\!]$, $\mathbf{f}_w^\circ(u) = X_\mu^\bullet \otimes wu^{-m}$. Note that

$$\mathbf{f}_w(u)\mathcal{Y}(u) = (I + X_\mu^\bullet \otimes wu^{-m})\mathcal{Y}(u) = \mathcal{Y}(u) + X_\mu^\bullet \otimes wu^{-m} + (X_\mu^\bullet \otimes wu^{-m})\mathcal{Y}^\circ(u),$$

where $\mathcal{Y}^\circ(u) = \mathcal{Y}(u) - I$. This implies that, for $1 \leq r \leq m$ and $\lambda \in \mathcal{I}$, the image of $y_\lambda^{(r)}$ under $m_{\mathbf{f}_w}$ is given by

$$m_{\mathbf{f}_w}(y_\lambda^{(r)}) = \begin{cases} y_\lambda^{(r)} & \text{if } (\lambda, r) \neq (\mu, m), \\ y_\mu^{(m)} + w & \text{if } (\lambda, r) = (\mu, m). \end{cases}$$

Consequently,

$$X = m_{\mathbf{f}_w}(X) = X_0 + m_{\mathbf{f}_w}(P(y_\mu^{(m)})) = X_0 + P(y_\mu^{(m)} + w) \quad \forall\, w \in \mathbb{C}.$$

Here $P(y_\mu^{(m)} + w)$ is the polynomial obtained from $P(y_\mu^{(m)})$ by substituting $y_\mu^{(m)} \mapsto y_\mu^{(m)} + w$. This allows us to deduce that

$$P(y_\mu^{(m)}) = P(y_\mu^{(m)} + w) \quad \forall\, w \in \mathbb{C}. \tag{2.6.9}$$

For each $w \in \mathbb{C}$, let ev_w be the algebra endomorphism of $\mathbb{C}[y_\lambda^{(r)}]_{\lambda,r}$ given by $y_\lambda^{(r)} \mapsto y_\lambda^{(r)}$ for all $(\lambda, r) \neq (\mu, m)$ and $y_\mu^{(m)} \mapsto -w$. Note that $\cap_{w\in\mathbb{C}}\mathrm{Ker}(\mathrm{ev}_w) = \{0\}$. We can extend ev_w to obtain an endomorphism $\mathrm{ev}_w^{\mathfrak{X}}$ of $\mathfrak{X}_\mathcal{I}(\mathfrak{g})$ by setting $\mathrm{ev}_w^{\mathfrak{X}} = \mathrm{ev}_w \otimes \mathrm{id}$. We then have $\mathrm{Ker}(\mathrm{ev}_w^{\mathfrak{X}}) = \mathrm{Ker}(\mathrm{ev}_w) \otimes Y_R(\mathfrak{g})$ and $\cap_{w\in\mathbb{C}}\mathrm{Ker}(\mathrm{ev}_w^{\mathfrak{X}}) = \{0\}$.

The equality (2.6.9) implies that $\mathrm{ev}_w^{\mathfrak{X}}(\Phi_\mathcal{I}(P(y_\mu^{(m)}))) = 0$ for all $w \in \mathbb{C}$. This shows that $\Phi_\mathcal{I}(P(y_\mu^{(m)})) = 0$, and thus that $P(y_\mu^{(m)}) = 0$. This contradicts the fact that $P(y_\mu^{(m)})$ is a non-constant polynomial of degree at least 1. Thus no such X can exist, and we may conclude that $\widetilde{Y}_R(\mathfrak{g}) = X_\mathcal{I}(\mathfrak{g})^{m_\mathbf{f}}$. □

2.7 Drinfeld's theorem and classical Lie algebras

When V is assumed to be irreducible, one can recover from the results of §2.4, §2.5 and §2.6 a proof of [Dri85, Theorem 6]. Our first task is to formalize this statement: this will be accomplished in §2.7.1. We will conclude in §2.7.2 by explaining how many of the results of this chapter reduce to, and have been motivated by, results which are known to hold when V is the vector representation of a classical Lie algebra $\mathfrak{g}$.

2.7.1 Drinfeld's theorem and the irreducibility assumption

We now restrict our attention to the setting where the underlying $Y(\mathfrak{g})$-module V is irreducible. As has been explained in Remark 2.4.2, this situation has additional practical value, since, at least in principle, $R(u)$ can be computed by solving the equation (2.2.13) and, after a suitable re-normalization, is equal to a rational R-matrix.

Since V is irreducible, Schur's lemma implies that

$$\mathcal{E} = \mathrm{End}_{Y(\mathfrak{g})} V = \mathbb{C} \cdot I.$$

In particular, the indexing set $\mathcal{I}$ contains a single element, say ς, and the basis element $X_\varsigma^\bullet$ of $\mathcal{E}$ can be chosen to equal the identity matrix I. With this in mind, we shall henceforth denote $X_\mathcal{I}(\mathfrak{g})$ simply by $X(\mathfrak{g})$ whenever V is assumed to be irreducible.

Set

$$z(u) = 1 + \sum_{r \geq 2} z_r u^{-r} = 1 + z_\varsigma(u) \quad \text{and} \quad y(u) = 1 + \sum_{r \geq 1} y_r u^{-r} = 1 + y_\varsigma(u). \tag{2.7.1}$$

The observation made in the previous paragraph implies the first part of the following result.

Corollary 2.7.1. *The matrices $\mathcal{Z}(u)$ and $\mathcal{Y}(u)$ are equal to $z(u) \cdot I$ and $y(u) \cdot I$, respectively. In particular, $z(u)$ is uniquely determined by the relation*

$$S_\mathcal{I}^2(T(u))T(u + \tfrac{1}{2}c_\mathfrak{g})^{-1} = z(u) \cdot I = T(u + \tfrac{1}{2}c_\mathfrak{g})^{-1} S_\mathcal{I}^2(T(u)).$$

Proof. The relation $z(u)\cdot I = S_{\mathcal{I}}^2(T(u))T(u+\frac{1}{2}c_{\mathfrak{g}})^{-1}$ is immediate from (2.4.19). This relation, together with the centrality of $z(u)$, implies that

$$z(u)T(u+\tfrac{1}{2}c_{\mathfrak{g}}) = T(u+\tfrac{1}{2}c_{\mathfrak{g}})z(u) = S_{\mathcal{I}}^2(T(u)),$$

and hence that $z(u)\cdot I = T(u+\frac{1}{2}c_{\mathfrak{g}})^{-1}S_{\mathcal{I}}^2(T(u))$. □

These simplifications allow us to write down a proof of the following theorem, whose first two parts are precisely the statement of [Dri85, Theorem 6].

Theorem 2.7.2 ([Dri85, Theorem 6])**.** *The following three statements are satisfied:*

(1) There is an epimorphism of Hopf algebras $\widetilde{\Phi}: X(\mathfrak{g}) \twoheadrightarrow Y(\mathfrak{g})$ such that

$$\widetilde{\Phi}(T(u)) = (\rho\otimes 1)(\mathcal{R}(-u)).$$

(2) There is a series $c(u) = 1+\sum_{r\geq 1} c_r u^{-r}$, whose coefficients $\{c_r\}_{r\geq 1}$ are central and generate $\mathrm{Ker}(\widetilde{\Phi})$ *as an ideal, which satisfies*

$$\Delta_{\mathcal{I}}(c(u)) = c(u)\otimes c(u). \tag{2.7.2}$$

(3) The coefficients of $c(u)$ generate the center of $X(\mathfrak{g})$, which is a polynomial algebra in countably many variables.

Proof. The first statement is precisely Lemma 2.5.1, which we have seen holds even when V is not irreducible. Let us turn to (2). There are two natural candidates for the series $c(u)$, the first being $z(u)$ and the second being $y(u)$, and both satisfy the desired properties. If $c(u) = z(u)$, then by Corollaries 2.6.10 and 2.7.1 we have

$$\Delta_{\mathcal{I}}(z(u)) = (y(u)\otimes 1)(1\otimes z(u))(y(u+\tfrac{1}{2}c_{\mathfrak{g}})^{-1}\otimes 1) = z(u)\otimes z(u),$$

while Theorem 2.5.2 gives $\mathrm{Ker}(\widetilde{\Phi}) = (z(u)-1)$. If instead $c(u) = y(u)$, then it is immediate from Lemma 2.6.8 and Corollary 2.7.1 that $c(u)$ satisfies the grouplike property (2.7.2). As the ideal $(y(u)-1)$ generated by the coefficients $\{y_r\}_{r\geq 1}$ is equal to $(z(u)-1)$, we also have $\mathrm{Ker}(\widetilde{\Phi}) = (y(u)-1)$.

As for part (3), Proposition 2.6.6 and Corollary 2.7.1 guarantee that both $\{z_r\}_{r\geq 2}$ and $\{y_r\}_{r\geq 1}$ are algebraically independent sets which generate $ZX(\mathfrak{g})$. □

Remark 2.7.3. More generally, when V is not assumed to be irreducible, we have shown that $C(u) = \mathcal{Y}(u) \in I + \mathcal{E} \otimes u^{-1}X_{\mathcal{I}}(\mathfrak{g})[\![u^{-1}]\!]$ has central coefficients which generate the ideal $(\mathcal{Z}(u) - I) = \mathrm{Ker}(\widetilde{\Phi})$, and moreover that $C(u)$ satisfies $\Delta(C(u)) = C_{[1]}(u)C_{[2]}(u)$. This should be viewed as a generalization of (2), and the statement that the coefficients $y_\lambda^{(r)}$ of $C(u)$ are algebraically independent generators of $ZX_{\mathcal{I}}(\mathfrak{g})$ (see Proposition 2.6.6) should be viewed as a generalization of (3).

Remark 2.7.4. In the special case where $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$ and $V = \mathbb{C}^N$, Theorem 2.7.2 follows from [GRW19a, Theorem 3.16] and the results of [AMR06].

In the proof of Theorem 2.7.2 we have observed that the series $z(u)$ is grouplike. It is thus also the case that $S_{\mathcal{I}}(z(u)) = z(u)^{-1}$ (as can also be seen from Corollary 2.6.10). The next corollary summarizes these results.

Corollary 2.7.5. *When V is irreducible the formulas of Corollary 2.6.10 reduce to*

$$\Delta_{\mathcal{I}}(z(u)) = z(u) \otimes z(u), \quad S_{\mathcal{I}}(z(u)) = z(u)^{-1}, \quad \epsilon_{\mathcal{I}}(z(u)) = 1.$$

We conclude this subsection by noting that, since $\mathcal{E} = \mathbb{C} \cdot I$, every automorphism $m_{\mathbf{f}}$ (see (2.4.4)) takes the form

$$T(u) \mapsto f(u)T(u) \tag{2.7.3}$$

for a series $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ uniquely determined by $\mathbf{f}(u) = I \otimes f(u)$. With this in mind, we will denote $m_{\mathbf{f}}$ by m_f whenever V is assumed to be irreducible, which will henceforth be the case.

2.7.2 The vector representation of the Yangian of a classical Lie algebra

We now narrow our focus to the case where $\mathfrak{g}$ is a Lie algebra of classical type and V is specialized to its vector representation, our goal being to briefly highlight results in the literature which have motivated some of the results of this paper, with emphasis on the results of §2.6.

We remark that these specializations fall into the slightly more general framework in which the representation V of $Y(\mathfrak{g})$ is irreducible as a $\mathfrak{g}$-module. Considering only such modules leads to fairly significant simplifications. For instance, $\mathfrak{g}_{\mathcal{I}}$ always coincides with $\mathfrak{g}_{\mathcal{J}}$ and hence §2.3.3 and Step 2 of the proof of Proposition 2.4.4 are no longer needed. There are, however, examples where $R(u)$ has been computed when V is not irreducible as a $\mathfrak{g}$-module: see [CP91].

2.7.2.1 The special linear Lie algebra $\mathfrak{sl}_N$

Fix $N \geq 2$, let $\{e_1, \ldots, e_N\}$ denote the standard basis of $\mathbb{C}^N$, and view $\mathfrak{g} = \mathfrak{sl}_N$ as the space of traceless $N \times N$ matrices. Fixing the invariant form $(\cdot,\cdot)$ to be the trace form, we have

$$\Omega_\rho = P - \frac{1}{N}I \quad \text{and} \quad c_{\mathfrak{g}} = 2N,$$

where $P = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}$ is the permutation operator σ on $\mathbb{C}^N \otimes \mathbb{C}^N$. Additionally, we have $\mathfrak{g}_{\mathcal{J}} = \mathfrak{g}_{\mathcal{I}} \cong \mathfrak{gl}_N$.

It is well known that the $\mathfrak{sl}_N$-module $\mathbb{C}^N$ admits a $Y(\mathfrak{sl}_N)$-module structure defined by allowing $J(X)$, for each $X \in \mathfrak{sl}_N$, to act as the zero operator: see for instance Example 1 of [Dri85]. In this case $(\rho \otimes \rho)(\mathcal{R}(-u))$ is, up to multiplication by a formal series in u^{-1}, equal to Yang's R-matrix

$$R(u) = I - \frac{P}{u}, \tag{2.7.4}$$

as can be deduced by directly solving the equation (2.2.13) with $V = W = \mathbb{C}^N$. The associated extended Yangian $X(\mathfrak{sl}_N)$ is usually denoted $Y(\mathfrak{gl}_N)$ in the literature, and has been studied extensively. Using the above rational form of $R(u)$, one deduces that the defining RTT-relation (2.4.1) can be rewritten in terms of the series $\{t_{ij}(u)\}_{1\leq i,j\leq N}$ as

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}\Big(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)\Big). \tag{2.7.5}$$

In what follows we do not attempt to provide a full account of the history behind each result, but instead refer the reader to the appropriate results in the monograph [Mol07] where a detailed bibliography is given.

The central series $y(u)$ and $z(u)$ (adapting the notation from (2.7.1)) both admit rather concrete descriptions. The series $y(u)$ is equal to the series $\widetilde{d}(u)$ which has appeared in the proof of [Mol07, Theorem 1.8.2]: it is the unique central series in

$1 + u^{-1} ZX(\mathfrak{sl}_N)[\![u^{-1}]\!]$ such that

$$\widetilde{d}(u)\widetilde{d}(u-1)\cdots\widetilde{d}(u-N+1) = \mathrm{qdet}T(u), \tag{2.7.6}$$

where $\mathrm{qdet}T(u)$ is the quantum determinant of the generating matrix $T(u)$: see Definition 1.6.5 of [Mol07]. By [Mol07, Proposition 1.6.6], it is given by

$$\mathrm{qdet}T(u) = \sum_{\pi \in \mathfrak{S}_N} \mathrm{sign}(\pi) \cdot t_{\pi(1),1}(u) \cdots t_{\pi(N),N}(u-N+1). \tag{2.7.7}$$

The series $z(u)$ is related to the series

$$\mathsf{z}(u) = \frac{\mathrm{qdet}T(u-1)}{\mathrm{qdet}T(u)},$$

which was defined in [Mol07, (1.68)], by $z(u) = \mathsf{z}(u+N)$, as can be seen using [Mol07, Theorem 1.9.9]. The relation $z(u) = 1$ is equivalent to $\mathrm{qdet}T(u) = 1$, as was pointed out in the original statement of [Dri85, Theorem 6].

Theorem 2.6.3 reduces to the statements of Theorems 1.7.5 and 1.8.2 of [Mol07], and Proposition 2.6.6 follows from these same results together with [Mol07, Corollary 1.9.7]. The Poincaré-Birkhoff-Witt theorem for $X(\mathfrak{sl}_N)$ (Theorem 2.6.7 with $(\mathfrak{g}, V) = (\mathfrak{sl}_N, \mathbb{C}^N)$) is given in [Mol07, Theorem 1.4.1].

The description of $Y_R(\mathfrak{sl}_N)$ as the subalgebra of $X(\mathfrak{sl}_N)$ consisting of all elements stable under all automorphisms of the form m_f, which is provided by Theorem 2.6.11, was actually taken as the definition of $Y_R(\mathfrak{sl}_N)$ in [Mol07]. It was then proven in Corollary 1.8.3 of [Mol07] that $Y_R(\mathfrak{sl}_N)$ could be equivalently characterized as in Definition 2.4.1. According to [Mol07, Bibliographical notes 1.8], the description of $Y_R(\mathfrak{sl}_N)$ using the automorphisms m_f is originally due to Drinfeld, as is the more general fact that $Y_R(\mathfrak{sl}_N)$ can be realized as a subalgebra of $X(\mathfrak{sl}_N)$: see Theorem 1.13 of [Ols92].

2.7.2.2 The orthogonal and symplectic Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_{2n}$

Still assuming $N \geq 2$, let $n \in \mathbb{N}$ be defined by $N = 2n$ (if N is even) and $N = 2n+1$ (if N is odd). We now assume that $\mathfrak{g}_N = \mathfrak{g}$ is either equal to $\mathfrak{so}_N$ or $\mathfrak{sp}_N$, where N is necessarily even in the latter case. It is convenient to relabel the standard basis of

$\mathbb{C}^N$ using the indexing set

$$\mathcal{I}_N = \{-n, \ldots, -1, (0), 1, \ldots, n\}, \tag{2.7.8}$$

where $(0) = 0$ if $N = 2n+1$ and should be omitted otherwise. That is, we denote the standard basis of $\mathbb{C}^N$ by $\{e_{-n}, \ldots, e_{-1}, (e_0), e_1, \ldots, e_N\}$. Let $t : \mathrm{End}\mathbb{C}^N \to \mathrm{End}\mathbb{C}^N$ denote the transposition determined by

$$(E_{ij})^t = \theta_{ij} E_{-j,-i} \quad \text{where} \quad \theta_{ij} = \begin{cases} 1 & \text{if } \mathfrak{g}_N = \mathfrak{so}_N, \\ \mathrm{sign}(i)\mathrm{sign}(j) & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N. \end{cases} \tag{2.7.9}$$

The Lie algebra $\mathfrak{g}_N$ can then be realized as the Lie subalgebra $\mathfrak{sl}_N^\theta$ of $\mathfrak{sl}_N$ consisting of elements fixed by the involution $\theta \in \mathrm{Aut}(\mathfrak{sl}_N)$ defined by

$$\theta(X) = -X^t \quad \forall\, X \in \mathfrak{sl}_N. \tag{2.7.10}$$

We will see below that this realization of $\mathfrak{g}_N$ is consistent with that given by Proposition 2.3.4.

Letting $(\cdot,\cdot)$ be equal to one half of the trace form, we have $\Omega_\rho = P - Q$ and $c_\mathfrak{g} = 4\kappa$, where

$$P = \sum_{i,j \in \mathcal{I}_N} E_{ij} \otimes E_{ji}, \quad Q = P^{t_2} = \sum_{i,j \in \mathcal{I}_N} \theta_{ij} E_{ij} \otimes E_{-i,-j}$$
$$\text{and} \quad \kappa = \begin{cases} N/2 - 1 & \text{if } \mathfrak{g}_N = \mathfrak{so}_N, \\ n + 1 & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N. \end{cases} \tag{2.7.11}$$

We will also write $\kappa = N/2 \mp 1$, where $\mp = -(\pm)$ and

$$\pm = \begin{cases} + & \text{if } \mathfrak{g}_N = \mathfrak{so}_N, \\ - & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N. \end{cases} \tag{2.7.12}$$

The presentation of $\mathfrak{g}_N$ furnished by Proposition 2.3.4 takes the following explicit form: Firstly, the above data implies that the relation (2.3.10) is equivalent to

$$[F_{ij}, F_{kl}] = \delta_{jk} F_{il} - \delta_{il} F_{kj} + \delta_{j,-l} \theta_{ij} F_{k,-i} - \delta_{i,-k} \theta_{ij} F_{-j,l}.$$

As for the relation (2.3.11), using (2.3.13) and the above relation, we deduce that it is equivalent to the set of relations

$$\pm F_{ij} = \delta_{ij}\mathrm{tr}(F) \mp \theta_{ij}F_{-j,-i} \quad \forall\, i,j \in \mathcal{I}_N.$$

Taking $i = j$ and summing over $i \in \mathcal{I}_N$ gives $\mathrm{tr}(F) = 0$, and hence (2.3.11) implies that

$$F_{ij} + \theta_{ij}F_{-j,-i} = 0 \quad \forall\, i,j \in \mathcal{I}_N.$$

Since this relation clearly implies that $\mathrm{tr}(F) = 0$, it is equivalent to (2.3.11). In summary, $\mathfrak{g}_N$ is isomorphic to the Lie algebra generated by $\{F_{ij}\}_{i,j\in\mathcal{I}_N}$, subject only to the relations

$$\begin{gathered}[F_{ij}, F_{kl}] = \delta_{jk}F_{il} - \delta_{il}F_{kj} + \delta_{j,-l}\theta_{ij}F_{k,-i} - \delta_{i,-k}\theta_{ij}F_{-j,l} \\ F_{ij} + \theta_{ij}F_{-j,-i} = 0\end{gathered} \tag{2.7.13}$$

for all $i, j, k, l \in \mathcal{I}_N$. As hinted at earlier, this recovers the realization of $\mathfrak{g}_N$ as the fixed point subalgebra $\mathfrak{sl}_N^\theta$, with θ as in (2.7.10). Indeed, the identification is given by

$$F_{ij} = E_{ij} - \theta_{ij}E_{-j,-i} = E_{ij} + \theta(E_{ij}) \quad \forall\, i,j \in \mathcal{I}_N. \tag{2.7.14}$$

As in the $\mathfrak{g} = \mathfrak{sl}_N$ case, it is well known that the vector representation $\mathbb{C}^N$ of $\mathfrak{g}_N$ extends to a representation of $Y(\mathfrak{g}_N)$ by setting $\rho(J(X)) = 0$ for all $X \in \mathfrak{g}_N$. For an explicit proof see [GRW19a, Proposition 3.1]. The R-matrix $(\rho\otimes\rho)(\mathcal{R}(-u))$ can be computed from (2.2.13) and is equal to

$$R(u) = I - \frac{P}{u} + \frac{Q}{u-\kappa}, \tag{2.7.15}$$

up to multiplication by an invertible element of $\mathbb{C}[\![u^{-1}]\!]$. This has certainly been known for a long time (see [KS82b] and [Dri85, Example 2]), but for a complete proof we refer the reader to Proposition 3.13 of the recent paper [GRW19a]. The RTT-Yangian $Y_R(\mathfrak{g}_N)$ and the extended Yangian $X(\mathfrak{g}_N)$ have not been studied to the same extent as their $\mathfrak{sl}_N$ analogues, although there has been an increase in efforts over the last fifteen years [AAC+03, AMR06, MM14, MM17, GRW19a, JLM18]. In [GRW19a], a proof of Theorem A was given in [GRW19a] using the algebraic theory developed in [AMR06]. Using (2.7.15), the defining relation (2.4.1) of $X(\mathfrak{g}_N)$ can be rewritten in terms of the generating series $\{t_{ij}(u)\}_{i,j\in\mathcal{I}_N}$ as

$$\begin{aligned}[t_{ij}(u),t_{kl}(v)] =& \frac{1}{u-v}\Big(t_{kj}(u)t_{il}(v)-t_{kj}(v)t_{il}(u)\Big)\\ &-\frac{1}{u-v-\kappa}\sum_{a\in\mathcal{I}_N}\Big(\delta_{k,-i}\theta_{ia}t_{aj}(u)t_{-a,l}(v)-\delta_{l,-j}\theta_{ja}t_{k,-a}(v)t_{ia}(u)\Big).\end{aligned} \tag{2.7.16}$$

It was proven in [AAC+03] (see also [AMR06, (2.26)]) that there is a central series $\mathsf{z}(u)\in 1+u^{-1}ZX(\mathfrak{g}_N)[\![u^{-1}]\!]$ determined by

$$\begin{aligned}\mathsf{z}(u)\cdot I = T^t(u+\kappa)T(u) = T(u)T^t(u+\kappa),\\ \text{where}\quad T^t(u)=\sum_{i,j\in\mathcal{I}_N}(E_{ij})^t\otimes t_{ij}(u).\end{aligned} \tag{2.7.17}$$

By comparing (2.31) of [AMR06] with the relation $S_{\mathcal{I}}^2(T(u)) = z(u)T(u+2\kappa)$ of Corollary 2.7.1, we can conclude that

$$z(u)=\frac{\mathsf{z}(u)}{\mathsf{z}(u+\kappa)}.$$

Conversely $y(u)$ is equal to the central series $\mathsf{y}(u)$ defined in [AMR06, Theorem 3.1]: it is uniquely determined by

$$y(u)y(u+\kappa)=\mathsf{z}(u). \tag{2.7.18}$$

It was also noted in the statement of [Dri85, Theorem 6] that, when $(\mathfrak{g},V)=(\mathfrak{so}_N,\mathbb{C}^N)$, the coefficients of $\mathsf{z}(u)-1$ generate the kernel of the epimorphism $\widetilde{\Phi}$ from Lemma 2.5.1 as an ideal.

Theorem 2.6.3 with $(\mathfrak{g},V)=(\mathfrak{g}_N,\mathbb{C}^N)$ is precisely Theorem 3.1 of [AMR06], while Corollary 2.6.6 is deduced from that same theorem of [AMR06] together with [AMR06, Corollary 3.9]. The Poincaré-Birkhoff-Witt theorem for $X(\mathfrak{g}_N)$ when $V=\mathbb{C}^N$ was stated and proven in Corollary 3.10 of [AMR06]: see also [AMR06, Theorem 3.6], which is exactly Theorem 2.5.5 in the particular case being discussed.

Just as was the case for $\mathfrak{g}=\mathfrak{sl}_N$ with $V=\mathbb{C}^N$, the authors of [AMR06] first defined $Y_R(\mathfrak{g}_N)$ as the fixed point subalgebra of $X(\mathfrak{g}_N)$ under all automorphisms m_f, and then in [AMR06, Corollary 3.2] proved that it could be equivalently defined as a quotient of $X(\mathfrak{g}_N)$.

To conclude this subsection, we would like to mention that there is an alternative route to proving that $Y_R(\mathfrak{g})$ and $Y(\mathfrak{g})$ are isomorphic when $\mathfrak{g} = \mathfrak{sl}_N$, $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and $V = \mathbb{C}^N$. This alternative uses the Gauss decomposition of the generating matrix $T(u)$ of $X(\mathfrak{g})$ to construct an isomorphism

$$\Phi_D : Y_R(\mathfrak{g}) \xrightarrow{\sim} Y_D(\mathfrak{g}),$$

where $Y_D(\mathfrak{g})$ denotes the current (or "Drinfeld's new") realization of the Yangian. For $\mathfrak{g} = \mathfrak{sl}_N$, such an isomorphism was constructed by J. Brundan and A. Kleshchev in [BK05] (see also [Mol07, §3.1]). For $\mathfrak{g} = \mathfrak{so}_N$ and $\mathfrak{g} = \mathfrak{sp}_N$ this has been achieved in the recent paper [JLM18] of N. Jing, M. Liu and A. Molev.

One may then compose Φ_D with the inverse of the isomorphism $Y(\mathfrak{g}) \xrightarrow{\sim} Y_D(\mathfrak{g})$ from Theorem 1 of [Dri88] to obtain an isomorphism

$$Y_R(\mathfrak{g}) \xrightarrow{\sim} Y(\mathfrak{g}).$$

Although a proof of [Dri88, Theorem 1] did not appear in Drinfeld's original paper, one has recently been made available in [GRW19a, Theorem 2.6], where $Y_D(\mathfrak{g})$ was denoted $Y^{\mathrm{cr}}(\mathfrak{g})$.

Chapter 3

Twisted Yangians of Classical Type

We have now reached the second part of this thesis, where our attention will deviate from the study of the Yangian $Y(\mathfrak{g})$ itself and instead focus on the theory of twisted Yangians – certain coideal subalgebras of Yangians associated to symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\vartheta})$ of Lie algebras.

In this chapter, we provide a introduction to the theory of twisted Yangians associated to symmetric pairs of classical type, with emphasis on those of type B, C, D of the form

$$(\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{g}_N, \mathfrak{g}_{N-q} \oplus \mathfrak{g}_q) \quad \text{with} \quad 0 \leq q < N, \quad q \in 2\mathbb{Z}. \tag{3.0.1}$$

Here $\mathfrak{g}_N$ always denotes the symplectic Lie algebra $\mathfrak{sp}_N$ or the orthogonal Lie algebra $\mathfrak{so}_N$, as in §2.7.2.2. These twisted Yangians, introduced by Guay and Regelskis in [GR16], are built inside the special instance of the R-matrix presentation of the Yangian considered in §2.7.2.2. Our exposition will mostly follow [GR16], though we will present the theory in a manner more consistent with the general theory of Chapter 2 where possible.

One novel feature of our exposition is §3.2, where a general construction of symmetric pairs (coming from an adjoint action) is given which is consistent with the R-matrix formalism of Chapter 2. This construction should form the basis of a more general construction of twisted Yangians inside the R-matrix presentation of the Yangian, which does not yet exist.

The rest of this chapter will unfold as follows. In §3.1.1, we gather notation which is specific to the setting of §2.7.2– this is the setting in which we will be working for the remainder of this thesis (excluding §3.2.1). In §3.2.2, we construct explicit realizations of the pairs (3.0.1). The twisted Yangians $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ associated to symmetric pairs $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ of the form (3.0.1) are then defined in §3.3, which is the main section of this chapter. That section gives a detailed survey of the algebraic theory of twisted Yangians of types B, C and D, and should remind the reader of many of the results we have encountered in Chapter 2. In §3.4, the last section of this Chapter, we give a very brief introduction to twisted Yangians of type A, which play an important role in Chapters 4 and 5.

3.1 Notation

As indicated above, the remainder of this thesis will take place within the framework of §2.7.2, and almost exclusively in the special case where $\mathfrak{g}$ is of orthogonal or symplectic type. In this section, we introduce and recall notation which reflects our narrowed focus. We will break from this notation only in §3.2.1, where a general construction for symmetric pairs is given which is compatible with the R-matrix formalism of Chapter 2.

In addition to this specialized notation, we shall also frequently use the following standard terminology. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are unital associative $\mathbb{C}$-algebras, V is any $\mathscr{B}$-module, and Ψ is an algebra homomorphism

$$\Psi : \mathscr{A} \to \mathscr{B}.$$

Then we denote by $\Psi^*(V)$ the $\mathscr{A}$-module which is equal to V as a vector space and has module structure given by

$$X \cdot v = \Psi(X)v \quad \forall \quad X \in \mathscr{A} \text{ and } v \in V.$$

In other words, Ψ^* is the usual restriction of scalars functor associated to Ψ.

3.1.1 The Lie algebras $\mathfrak{g}_N$ and $\mathring{\mathfrak{g}}_N$

As in §2.7.2.2, we fix $N \geq 2$ and define $n \in \mathbb{N}$ by $N = 2n+1$ or $N = 2n$, depending on the parity of N. Except for in §3.4.2, we will always label the standard basis of $\mathbb{C}^N$ using the indexing set

$$\mathcal{I}_N = \{-n, \ldots, -1, (0), 1, \ldots, n\}$$

defined in (2.7.8). In addition, we introduce the set of non-negative integers

$$\mathcal{I}_N^+ = \mathcal{I}_N \cap \mathbb{Z}_{\geq 0}.$$

We will continue to write $\mathfrak{g}_N$ for either $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and we will always assume that these Lie algebras are in the presentations given by Proposition 2.3.4 with ρ the vector representation. As demonstrated in (2.7.13), this means that $\mathfrak{g}_N$ is the complex Lie algebra generated by $\{F_{ij}\}_{i,j\in\mathcal{I}_N}$, subject to the defining relations

$$[F_{ij}, F_{kl}] = \delta_{jk}F_{il} - \delta_{il}F_{kj} + \delta_{j,-l}\theta_{ij}F_{k,-i} - \delta_{i,-k}\theta_{ij}F_{-j,l},$$
$$F_{ij} + \theta_{ij}F_{-j,-i} = 0.$$

In this presentation, a natural choice of Cartan subalgebra is given by

$$\mathfrak{h}_N = \mathrm{span}_{\mathbb{C}}\{F_{ii} \,:\, 1 \leq i \leq n\}. \tag{3.1.1}$$

The theory developed in Chapter 2 is also valid for complex semisimple Lie algebras which are not necessarily simple, provided the underlying representation V is faithful. In particular, our fixed realization of $\mathfrak{g}_N$ holds for $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. It also holds for the one-dimensional Lie algebra $\mathfrak{so}_2$, though we will assume that $n \geq 2$ when working with Yangians and twisted Yangians associated to $\mathfrak{so}_{2n}$.

By (2.7.14), $\mathfrak{g}_N$ can be viewed as the fixed point subalgebra $\mathfrak{sl}_N^\theta$ (see (2.7.10)) by identifying

$$F_{ij} = E_{ij} - E_{ij}^t = E_{ij} - \theta_{ij}E_{-j-i} \quad \forall \quad i,j \in \mathcal{I}_N,$$

where the transpose t is defined in (2.7.9). We will also continue to employ the notation κ, $\pm$ and $\mp$: see (2.7.11) and (2.7.12).

Consider now the Lie algebras $(\mathfrak{g}_N)_\mathcal{J}$ and $(\mathfrak{g}_N)_\mathcal{I}$ from §2.3. As $\mathbb{C}^N$ is irreducible as a $\mathfrak{g}_N$-module, both of these Lie algebras coincide and are trivial one-dimensional central extensions of $\mathfrak{g}_N$. We denote both of these Lie algebras by $\mathring{\mathfrak{g}}_N$ and set

$$\mathring{F} = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \mathring{F}_{ij}, \qquad \text{where} \qquad \mathring{F}_{ij} = F_{ij}^{\mathcal{J}}$$

and $\{F_{ij}^{\mathcal{J}}\}_{i,j\in\mathcal{I}_N}$ are as in Definition 2.3.1. With this notation, $\mathring{\mathfrak{g}}_N$ is the complex Lie algebra generated by $\{\mathring{F}_{ij}\}_{i,j\in\mathcal{I}_N}$, subject to the defining relations

$$[\mathring{F}_{ij}, \mathring{F}_{kl}] = \delta_{jk}\mathring{F}_{il} - \delta_{il}\mathring{F}_{kj} + \delta_{j,-l}\theta_{ij}\mathring{F}_{k,-i} - \delta_{i,-k}\theta_{ij}\mathring{F}_{-j,l}.$$

The matrix K (see (2.3.4)) takes the form $K = \mathcal{K} \cdot I$ for a central element $\mathcal{K} \in \mathring{\mathfrak{g}}_N$ and, by Proposition 2.3.6, we have the Lie algebra decomposition

$$\mathring{\mathfrak{g}}_N = \mathfrak{g}_N \oplus \mathbb{C}\mathcal{K},$$

where the generators F_{ij} of $\mathfrak{g}_N$ are related to $\mathring{F}_{ij}$ and $\mathcal{K}$ by

$$\mathring{F}_{ij} = F_{ij} + \delta_{ij}\mathcal{K} \quad \forall \quad i, j \in \mathcal{I}_N.$$

Moreover, the second relation of (2.7.13) implies that $\mathcal{K}$ is determined by

$$\mathring{F}_{ij} + \theta_{ij}\mathring{F}_{-j,-i} = 2\delta_{ij}\mathcal{K}.$$

Taking the sum over $i = j \in \mathcal{I}_N$ gives the equivalent expression

$$\mathcal{K} = \frac{1}{N}\mathrm{tr}(\mathring{F}).$$

For the generators of the current algebra $\mathring{\mathfrak{g}}_N[z]$, we will adopt the notation used for $(\mathfrak{g}_N)_\mathcal{I}[z]$ in (2.3.30). That is, we set $\mathbb{F}_{ij}^{(r)} = \mathring{F}_{ij}z^r$ and then define

$$\mathbb{F}(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \mathbb{F}_{ij}(u), \qquad \text{where} \qquad \mathbb{F}_{ij}(u) = \sum_{r\geq 0} \mathbb{F}_{ij}^{(r)} u^{-r-1}.$$

In addition, we will write $\mathcal{K}_r = \mathcal{K}z^r$ and $\mathcal{K}(u) = \sum_{r\geq 0} \mathcal{K}_r u^{-r-1}$, so that in the notation of §2.3.3.2 we have $\mathbb{K}(u) = \mathcal{K}(u) \cdot I$.

To conclude this subsection, we record vector space bases of $\mathfrak{g}_N$ and $\mathring{\mathfrak{g}}_N$. Introduce $\mathcal{B}_N \subset \mathcal{I}_N \times \mathcal{I}_N$ by

$$\mathcal{B}_N = \{(i,j) \in \mathcal{I}_N \times \mathcal{I}_N \,:\, i+j \geq \delta_{\mathfrak{g}_N, \mathfrak{so}_N}\}. \tag{3.1.2}$$

As $\mathring{\mathfrak{g}}_N = \mathfrak{g}_N \oplus \mathbb{C}\mathcal{K}$ with $\mathring{F} = F + \mathcal{K} \cdot I$, it follows from (2.7.14) that $\mathfrak{g}_N$ and $\mathring{\mathfrak{g}}_N$ have bases

$$\{F_{ij}\}_{(i,j)\in\mathcal{B}_N} \quad \text{and} \quad \{\mathring{F}_{ij}\}_{(i,j)\in\mathcal{B}_N} \cup \{\mathcal{K}\}, \tag{3.1.3}$$

respectively.

3.1.2 The Yangians $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$

Henceforth, $X(\mathfrak{g}_N)$ shall be as in §2.7.2.2 of Chapter 2. It is generated by $\{t_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N}$, subject to the defining RTT-relation (2.4.1) with $\{1,\ldots,N\}$ replaced by $\mathcal{I}_N$ and $R(u)$ taken to be the rational R-matrix (2.7.15). That is,

$$R(u) = I - \frac{P}{u} + \frac{Q}{u-\kappa},$$

with P, Q and κ as in (2.7.11). The defining relation (2.4.1) can be equivalently expressed in terms of the generating series $\{t_{ij}(u)\}_{i,j\in\mathcal{I}_N}$ as in (2.7.16).

It will be useful to note that the operators P and Q satisfy

$$\begin{gathered} P^2 = I, \quad Q^2 = NQ, \quad PQ = \pm Q = QP, \\ PA_1 = A_2P, \quad QA_1 = QA_2^t \quad \text{and} \quad QA_1Q = \operatorname{tr}(A)Q, \end{gathered} \tag{3.1.4}$$

where $A \in \operatorname{End}(\mathbb{C}^N)$. In particular, $\frac{1}{N}Q$ is a projection operator whose image is the one-dimensional subspace $\mathbb{C}v_Q$ of $\mathbb{C}^N \otimes \mathbb{C}^N$, where

$$v_Q = \sum_{i\in\mathcal{I}_N} \theta_i e_i \otimes e_{-i} \quad \text{with} \quad \theta_i = \begin{cases} \operatorname{sign}(i) & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N, \\ 1 & \text{if } \mathfrak{g}_N = \mathfrak{so}_N. \end{cases} \tag{3.1.5}$$

As a consequence of the first three relations in (3.1.4), the R-matrix $R(u)$ satisfies

$$R(u)R(-u) = \left(1 - \frac{1}{u^2}\right) I \quad \text{and} \quad R(u) = R(\kappa - u)^t, \tag{3.1.6}$$

where the transpose t is as in (2.7.9) and is applied in either the first or second tensor factor.

As we will be exclusively working in the R-matrix realization of $Y(\mathfrak{g}_N)$ associated to $\mathbb{C}^N$ (as in §2.7.2.2), we shall henceforth write

$$Y(\mathfrak{g}_N) = Y_R(\mathfrak{g}_N),$$

where the underlying R-matrix $R(u)$ is understood to be given by (2.7.15), as above. As the J-presentation of $Y(\mathfrak{g}_N)$ introduced in §2.2 will no longer appear, this notation should not cause any confusion. In addition, we will usually drop the subscripts "$\mathcal{I}$" and "R" which featured prominently in Chapter 2. In particular, the coproduct, antipode and counit for both $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$ will be denoted

$$\Delta, \quad S \quad \text{and} \quad \epsilon,$$

respectively. These are defined explicitly in (2.4.2) and (2.5.4).

We will also follow the same notational conventions for the Yangian and extended Yangian associated to $\mathfrak{sl}_N$, with the role of §2.7.2.2 played instead by §2.7.2.1. In particular, in this case $R(u)$ is given by Yang's R-matrix

$$R(u) = I - \frac{P}{u},$$

and the defining RTT-relation of $X(\mathfrak{sl}_N) = Y(\mathfrak{gl}_N)$ is given in terms of $\{t_{ij}(u)\}_{i,j\in\mathcal{I}_N}$ in (2.7.5).

3.2 Symmetric pairs and twisted current algebras

Our first order of business is to introduce a presentation for the symmetric pairs (3.0.1) which is compatible with the R-matrix formulation of the Yangian studied in Chapter 2. In fact, we take a more general route applicable to a wide range of symmetric pairs.

3.2.1 Adjoint action and symmetric pairs

Let us for a moment return to the general setting of §2.3.2, where $\mathfrak{g}$ is an arbitrary simple Lie algebra and V is a finite-dimensional, faithful, $\mathfrak{g}$-module of dimension N, with $\mathfrak{g}$-action given by $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Let $\mathrm{G}(\Omega)$ be the subgroup of $\mathrm{GL}(V)$ defined by

$$\mathrm{G}(\Omega) = \{\mathcal{G} \in \mathrm{GL}(V) \,:\, \mathrm{Ad}(\mathcal{G}_1\mathcal{G}_2)(\Omega_\rho) = \Omega_\rho\},$$

where $\mathrm{Ad}(G)(X) = GXG^{-1}$ for all $G \in \mathrm{GL}(V \otimes V)$ and $X \in \mathfrak{gl}(V \otimes V)$.

Let $\exp(\mathfrak{g})$ denote the image of $\rho(\mathfrak{g}) \cong \mathfrak{g}$ under the matrix exponential map

$$\exp : \mathfrak{gl}(V) \to \mathrm{GL}(V)$$

and define $\mathrm{G} \subset \mathrm{GL}(V)$ to be the subgroup generated by $\exp(\mathfrak{g})$:

$$\mathrm{G} = \langle \exp(\mathfrak{g}) \rangle \subset \mathrm{GL}(V).$$

Recall that $U(\mathfrak{g}_\mathcal{J}) = U_\mathcal{J}(\mathfrak{g})$ is the extension of $U(\mathfrak{g})$ introduced in Definition 2.3.1, and $U(\mathfrak{g}_\rho) = U_\rho(\mathfrak{g})$ is the presentation of $U(\mathfrak{g})$ obtained in Proposition 2.3.4. The following simple lemma will give us a familiar source of automorphisms of $U(\mathfrak{g})$.

Lemma 3.2.1. *The group* G *is a subgroup of* $\mathrm{G}(\Omega)$*. Moreover:*

(1) For any $\mathcal{G} \in \mathrm{G}(\Omega)$*, the assignment*

$$\mathrm{Ad}_\mathcal{J}(\mathcal{G}) : F^\mathcal{J} \mapsto \mathcal{G}F^\mathcal{J}\mathcal{G}^{-1}$$

extends uniquely to an automorphism $\mathrm{Ad}_\mathcal{J}(\mathcal{G})$ *of* $U(\mathfrak{g}_\mathcal{J})$*.*

(2) If in addition $\mathcal{G} \in \lambda\mathrm{G}$ *for some* $\lambda \in \mathbb{C}^\times$*, then the assignment*

$$\mathrm{Ad}_\rho(\mathcal{G}) : F \mapsto \mathcal{G}F\mathcal{G}^{-1}$$

extends uniquely to an automorphism $\mathrm{Ad}_\rho(\mathcal{G})$ *of* $U(\mathfrak{g}_\rho)$*.*

Proof. Suppose that $\mathcal{G} = \exp(\rho(X))$ for some $X \in \mathfrak{g}$. Then

$$\begin{aligned}
\mathrm{Ad}(\mathcal{G}_1\mathcal{G}_2)(\Omega_\rho) &= \mathrm{Ad}(\exp(\rho(X)\otimes 1 + 1\otimes\rho(X)))(\Omega_\rho)\\
&= \exp\left((\rho\otimes\rho)\mathrm{ad}(\Delta(X))\right)(\Omega_\rho)\\
&= \Omega_\rho,
\end{aligned}$$

where the last equality is a consequence of the relation $[\Delta(X),\Omega]=0$. It follows that $\mathrm{G}\subset\mathrm{G}(\Omega)$.

Suppose now that $\mathcal{G}\in\mathrm{G}(\Omega)$. We must show that the assignment $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ preserves the defining relation (2.3.3) of $U(\mathfrak{g}_{\mathcal{J}})$. We have

$$[\mathcal{G}_1F_1^{\mathcal{J}}\mathcal{G}_1^{-1},\mathcal{G}_2F_2^{\mathcal{J}}\mathcal{G}_2^{-1}] = \mathcal{G}_1\mathcal{G}_2[F_1^{\mathcal{J}},F_2^{\mathcal{J}}]\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} = \mathcal{G}_1\mathcal{G}_2[\Omega_\rho,F_2^{\mathcal{J}}]\mathcal{G}_1^{-1}\mathcal{G}_2^{-1}.$$

Since $\mathcal{G},\mathcal{G}^{-1}\in\mathrm{G}(\Omega)$, we have

$$\mathcal{G}_1\mathcal{G}_2[\Omega_\rho,F_2^{\mathcal{J}}]\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} = \mathcal{G}_1\mathcal{G}_2(\Omega_\rho F_2^{\mathcal{J}} - F_2^{\mathcal{J}}\Omega_\rho)\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} = [\Omega_\rho,\mathcal{G}_2F_2^{\mathcal{J}}\mathcal{G}_2^{-1}].$$

Thus, the assignment $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ extends to an algebra endomorphism of $U(\mathfrak{g}_{\mathcal{J}})$. As it is invertible with inverse $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G}^{-1})$, Part (1) holds.

To prove Part (2), it suffices to assume $\mathcal{G}\in\mathrm{G}$. We claim that the automorphism $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ fixes the matrix K from (2.3.4). Indeed, by Lemma 2.3.2, $K\in\mathcal{E}_{\mathfrak{g}}\otimes U(\mathfrak{g}_{\mathcal{J}})$, where $\mathcal{E}_{\mathfrak{g}} = \mathrm{End}_{\mathfrak{g}}V$. Hence, we have

$$[\exp(\rho(X)),K]=0 \quad \forall\, X\in\mathfrak{g},$$

which implies that $\mathcal{G}K\mathcal{G}^{-1}=K$. The relation (2.3.9) of Lemma 2.3.3, together with the decomposition $F^{\mathcal{J}} = F + K$ (see Proposition 2.3.6), then gives

$$\begin{aligned}
\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})(K) &= \mathrm{Ad}_{\mathcal{J}}(\mathcal{G})(F^{\mathcal{J}} - c_{\mathfrak{g}}^{-1}\omega(F^{\mathcal{J}}))\\
&= \mathcal{G}F\mathcal{G}^{-1} + K - c_{\mathfrak{g}}^{-1}\omega(\mathcal{G}F\mathcal{G}^{-1}+K) = K,
\end{aligned}$$

where the last equality follows from

$$\omega(K)=0, \quad \mathrm{Ad}(\mathcal{G})(\rho(\mathfrak{g}))\subset\rho(\mathfrak{g}) \quad \text{and} \quad \mathcal{G}F\mathcal{G}^{-1}\in\mathrm{ad}(\mathfrak{g})\otimes U_{\mathcal{J}}(\mathfrak{g}),$$

the third fact being a consequence of the second fact and that $F \in \mathrm{ad}(\mathfrak{g}) \otimes U(\mathfrak{g}_{\mathcal{J}})$.

As $U(\mathfrak{g}_\rho)$ is isomorphic to the quotient of $U(\mathfrak{g}_{\mathcal{J}})$ by the ideal generated by the coefficients of K, we can conclude that $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ induces an automorphism of $U(\mathfrak{g}_\rho)$ as in Part (2). □

Remark 3.2.2. It should be emphasized that $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ and $\mathrm{Ad}_\rho(\mathcal{G})$ are given by conjugating $F^{\mathcal{J}}$ and F, respectively, in the first tensor factor, and not the second (which would have no meaning in general).

Suppose now that V is also a $Y(\mathfrak{g})$-module, as in §2.3.3. Since the automorphism $\mathrm{Ad}_{\mathcal{J}}(\mathcal{G})$ of $U(\mathfrak{g}_{\mathcal{J}})$ fixes K for any $\mathcal{G} \in \lambda\mathrm{G}$, it also preserves the relation (2.3.24) of $U(\mathfrak{g}_{\mathcal{I}}) = U_{\mathcal{I}}(\mathfrak{g})$. We thus obtain the following corollary.

Corollary 3.2.3. *Suppose that $\mathcal{G} \in \lambda\mathrm{G}$ for some $\lambda \in \mathbb{C}^\times$. Then the assignment*

$$\mathrm{Ad}_{\mathcal{I}}(\mathcal{G}) : F^{\mathcal{I}} \mapsto \mathcal{G} F^{\mathcal{I}} \mathcal{G}^{-1}$$

extends uniquely to an automorphism $\mathrm{Ad}_{\mathcal{I}}(\mathcal{G})$ of $U(\mathfrak{g}_{\mathcal{I}})$.

It will be useful for us to note that Lemma 3.2.1 also admits a Yangian analogue. Let $R(u)$ be as in §2.4, and define the group

$$\mathrm{G}(R) = \{\mathcal{G} \in \mathrm{GL}(V) \,:\, \mathrm{Ad}(\mathcal{G}_1\mathcal{G}_2)(R(u)) = R(u)\}$$

We then have the following result.

Lemma 3.2.4. *The group G is a subgroup of $\mathrm{G}(R)$. Moreover:*

(1) For any $\mathcal{G} \in \mathrm{G}(R)$, the assignment

$$\mathrm{Ad}_R(\mathcal{G}) : T(u) \mapsto \mathcal{G} T(u) \mathcal{G}^{-1}$$

extends uniquely to an automorphism $\mathrm{Ad}_R(\mathcal{G})$ of $X_{\mathcal{I}}(\mathfrak{g})$.

(2) If in addition $\mathcal{G} \in \lambda\mathrm{G}$ for some $\lambda \in \mathbb{C}^\times$, then the assignment

$$\mathrm{Ad}_R(\mathcal{G}) : \mathcal{T}(u) \mapsto \mathcal{G}\mathcal{T}(u)\mathcal{G}^{-1}$$

extends uniquely to an automorphism $\mathrm{Ad}_R(\mathcal{G})$ of $Y_R(\mathfrak{g})$.

Proof. The relation (2.2.8) implies that $[\mathcal{R}(u), \Delta(X)] = 0$ for all $X \in \mathfrak{g}$. Hence, the same argument as used in the proof of Lemma 3.2.1 to show $\mathrm{G} \subset \mathrm{G}(\Omega)$ can be used to show $\mathrm{G} \subset \mathrm{G}(R)$.

Consider now Part (1). For any $\mathcal{G} \in \mathrm{G}(R)$, we have

$$\begin{aligned} R(u-v)\mathcal{G}_1 T_1(u)\mathcal{G}_1^{-1}\mathcal{G}_2 T_2(v)\mathcal{G}_2^{-1} &= R(u-v)\mathcal{G}_1\mathcal{G}_2 T_1(u)T_2(v)\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} \\ &= \mathcal{G}_1\mathcal{G}_2 R(u-v)T_1(u)T_2(v)\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} \\ &= \mathcal{G}_1\mathcal{G}_2 T_2(v)T_1(u)R(u-v)\mathcal{G}_1^{-1}\mathcal{G}_2^{-1} \\ &= \mathcal{G}_2 T_2(v)\mathcal{G}_2^{-1}\mathcal{G}_1 T_1(u)\mathcal{G}_1^{-1}R(u-v). \end{aligned}$$

Therefore, the assignment $\mathrm{Ad}_R(\mathcal{G})$ uniquely extends to an algebra endomorphism of $X_\mathcal{I}(\mathfrak{g})$, which is invertible with inverse $\mathrm{Ad}_R(\mathcal{G}^{-1})$.

To prove Part (2), it suffices to show that $\mathrm{Ad}_R(\mathcal{G}) \in \mathrm{Aut}(X_\mathcal{I}(\mathfrak{g}))$ from Part (1) satisfies

$$\mathrm{Ad}_R(\mathcal{G})(\mathcal{Z}(u)) = \mathcal{Z}(u) \quad \forall \quad \mathcal{G} \in \mathrm{G}, \tag{3.2.1}$$

where $\mathcal{Z}(u)$ is as in (2.4.19). Since the antipode $S_\mathcal{I}$ commutes with $\mathrm{Ad}_R(\mathcal{G})$, we have

$$\mathrm{Ad}_R(\mathcal{G})(\mathcal{Z}(u)) = \mathcal{G}S_\mathcal{I}^2(T(u))\mathcal{G}^{-1}\mathcal{G}T(u+\tfrac{1}{2}c_\mathfrak{g})^{-1}\mathcal{G}^{-1} = \mathcal{G}\mathcal{Z}(u)\mathcal{G}^{-1}.$$

By Lemma 2.6.5, $\mathcal{Z}(u) \in \mathcal{E}_\mathfrak{g} \otimes X_\mathcal{I}(\mathfrak{g})[\![u^{-1}]\!]$, where $\mathcal{E}_\mathfrak{g} = \mathrm{End}_\mathfrak{g} V$. As $\mathcal{G}$ is in the group $\mathrm{G} = \langle \exp(\mathfrak{g}) \rangle$, it commutes with all $\mathfrak{g}$-intertwiners. Therefore, $\mathcal{G}\mathcal{Z}(u)\mathcal{G}^{-1} = \mathcal{Z}(u)$, and we may conclude that (3.2.1) holds. □

Let us now shift our focus to symmetric pairs of Lie algebras. We continue to assume V is a finite-dimensional $Y(\mathfrak{g})$-module which is faithful as a $\mathfrak{g}$-module. Let us fix an element $\mathcal{G} \in \mathrm{G}(\Omega)$ which belongs to $\lambda\mathrm{G}$ for some $\lambda \in \mathbb{C}^\times$ and satisfies $\mathcal{G}^2 = I$. Let ϑ be the involution

$$\vartheta = \mathrm{Ad}_\rho(\mathcal{G})|_{\mathfrak{g}_\rho} \in \mathrm{Aut}(\mathfrak{g}_\rho). \tag{3.2.2}$$

From any ϑ as above, we obtain a symmetric pair of Lie algebras

$$(\mathfrak{g}_\rho, \mathfrak{g}_\rho^\vartheta), \qquad \text{where} \qquad \mathfrak{g}_\rho^\vartheta = \{X \in \mathfrak{g}_\rho \, : \, \vartheta(X) = X\}.$$

Such a pair is always associated with an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_\rho^\vartheta \oplus \mathfrak{p} \quad \text{where} \quad \mathfrak{p} = \{X \in \mathfrak{g}_\rho \,:\, \vartheta(X) = -X\}.$$

Each symmetric pair of type B, C and D (i.e. when $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$) can be realized in the form $(\mathfrak{g}_\rho, \mathfrak{g}_\rho^\vartheta)$, with (ρ, V) the vector representation of $Y(\mathfrak{g}_N)$ on $\mathbb{C}^N$ as in §2.7.2. We shall spell this out explicitly in §3.2.2 for all pairs of the form (3.0.1). For the moment, we shift our attention to twisted polynomial current algebras, which are the classical analogues of twisted Yangians.

Let $\nu : \mathbb{C}[z] \to \mathbb{C}[z]$ be involution given by

$$\nu : f(z) \mapsto f(-z) \quad \forall \quad z \in \mathbb{C}[z].$$

The tensor product $\check{\vartheta} = \vartheta \otimes \nu$ is then an involution of $\mathfrak{g}_\rho[z] = \mathfrak{g}_\rho \otimes \mathbb{C}[z]$, which is given by

$$\check{\vartheta}(F(u)) = -\mathcal{G}F(-u)\mathcal{G}^{-1}, \tag{3.2.3}$$

where $F(u) \in \mathrm{End}V \otimes (\mathfrak{g}_\rho[z])[\![u^{-1}]\!]$ is as in (2.3.18).

The *twisted* polynomial current algebra associated to this data is

$$\begin{aligned} \mathfrak{g}_\rho[z]^{\check{\vartheta}} &= \{f(z) \in \mathfrak{g}_\rho[z] \,:\, \check{\vartheta}(f(z)) = f(z)\} \\ &= \bigoplus_{n\geq 0}(\mathfrak{g}_\rho^\vartheta \otimes \mathbb{C}z^{2n}) \oplus \bigoplus_{n\geq 0}(\mathfrak{p} \otimes \mathbb{C}z^{2r+1}) \\ &= \mathfrak{g}_\rho^\vartheta[z^2] \oplus z\mathfrak{p}[z^2]. \end{aligned} \tag{3.2.4}$$

There are two important extensions of $\check{\vartheta}$ to an involution of $\mathfrak{g}_\mathcal{I}[z]$ which occur in the theory of twisted Yangians:

(1) The extension obtained by replacing ϑ by $\mathrm{Ad}_\mathcal{I}(\mathcal{G})|_{\mathfrak{g}_\mathcal{I}}$ in the definition $\check{\vartheta}$.

(2) The extension obtained by replacing ϑ with

$$\vartheta_\mathcal{I} = \sigma_K \circ \mathrm{Ad}_\mathcal{I}(\mathcal{G})|_{\mathfrak{g}_\mathcal{I}}$$

in the definition of $\check{\vartheta}$, where σ_K is the involution of $\mathfrak{g}_\mathcal{I} \cong \mathfrak{g}_\rho \oplus \mathfrak{z}_\mathcal{I}$ which fixes $\mathfrak{g}_\rho$

and satisfies $\sigma_K(K^{\mathcal{I}}) = -K^{\mathcal{I}}$ (see §2.3.3). Equivalently, $\vartheta_{\mathcal{I}}$ is given by

$$\vartheta_{\mathcal{I}}(F^{\mathcal{I}}) = \mathcal{G}F\mathcal{G}^{-1} - K^{\mathcal{I}} = \mathcal{G}F^{\mathcal{I}}\mathcal{G}^{-1} - 2K^{\mathcal{I}}.$$

The extension (1) is relevant to the theory of twisted Yangians of type AIII considered in §3.4. However, it is (2) which occurs in the theory of orthogonal and symplectic twisted Yangians introduced in [GR16], and we will therefore only consider it in our current discussion.

Observe that with the above definition of $\vartheta_{\mathcal{I}}$, the commutative subalgebra $\mathfrak{z}_{\mathcal{I}}$ belongs to eigenspace for the eigenvalue -1, which is

$$\mathfrak{p} \oplus \mathfrak{z}_{\mathcal{I}}.$$

Let us set $\check{\vartheta} = \vartheta_{\mathcal{I}} \otimes \nu \in \mathrm{Aut}(\mathfrak{g}_{\mathcal{I}}[z])$, as suggested by (2) above. If

$$\mathbb{F}(u), \mathbb{K}(u) \in \mathrm{End}V \otimes (\mathfrak{g}_{\mathcal{I}}[z])[\![u^{-1}]\!]$$

are as in (2.3.30), then we have

$$\check{\vartheta}(\mathbb{F}(u)) = 2\mathbb{K}(-u) - \mathcal{G}\mathbb{F}(-u)\mathcal{G}^{-1},$$
$$\check{\vartheta}(F(u)) = -\mathcal{G}F(-u)\mathcal{G}^{-1}, \quad \check{\vartheta}(\mathbb{K}(u)) = \mathbb{K}(-u).$$

In particular, $\check{\vartheta} \in \mathrm{Aut}(\mathfrak{g}_{\mathcal{I}}[z])$ does indeed restrict to the automorphism $\check{\vartheta} \in \mathrm{Aut}(\mathfrak{g}_\rho[z])$ (see (3.2.3)), which justifies our choice of notation. In addition, we find that

$$\mathfrak{g}_{\mathcal{I}}[z]^{\check{\vartheta}} = \mathfrak{g}_\rho^{\vartheta}[z^2] \oplus z\mathfrak{p}[z^2] \oplus z\mathfrak{z}_{\mathcal{I}}[z^2]. \tag{3.2.5}$$

The next lemma provides a useful spanning set for $\mathfrak{p}$, $\mathfrak{g}_\rho^{\vartheta}$, $\mathfrak{g}_\rho[z]^{\check{\vartheta}}$ and $\mathfrak{g}_{\mathcal{I}}[z]^{\check{\vartheta}}$.

Lemma 3.2.5. *Define*

$$F^{\mathfrak{p}} = \sum_{i,j=1}^{N} E_{ij} \otimes F_{ij}^{\mathfrak{p}} = F\mathcal{G} - \mathcal{G}F, \quad F^{\vartheta} = \sum_{i,j=1}^{N} E_{ij} \otimes F_{ij}^{\vartheta} = F\mathcal{G} + \mathcal{G}F,$$
$$F(u)^{\check{\vartheta}} = \sum_{i,j=1}^{N} E_{ij} \otimes F_{ij}(u)^{\check{\vartheta}} = F(u)\mathcal{G} - \mathcal{G}F(-u),$$

$$\mathbb{F}(u)^{\check{\vartheta}} = \sum_{i,j=1}^{N} E_{ij} \otimes \mathbb{F}_{ij}(u)^{\check{\vartheta}} = \mathbb{F}(u)\mathcal{G} - \mathcal{G}\mathbb{F}(-u) + 2\mathbb{K}(-u)\mathcal{G},$$

Then $\mathfrak{p}$, $\mathfrak{g}_\rho^\vartheta$, $\mathfrak{g}_\rho[z]^{\check{\vartheta}}$ and $\mathfrak{g}_{\mathcal{I}}[z]^{\check{\vartheta}}$ are linearly spanned by the coefficients of $F^{\mathfrak{p}}$, F^ϑ, $F(u)^{\check{\vartheta}}$ and $\mathbb{F}(u)^{\check{\vartheta}}$, respectively.

Proof. By (2.3.10), $\mathfrak{g}_\rho$ is spanned by the coefficients F_{ij} of F. As $\mathcal{G}$ is invertible, $\mathfrak{g}_\rho$ is also spanned by the coefficients of $F\mathcal{G}$. As

$$\vartheta(F^\vartheta) = F^\vartheta, \quad \vartheta(F^{\mathfrak{p}}) = -F^{\mathfrak{p}},$$
$$\text{and} \quad 2F\mathcal{G} = F^\vartheta + F^{\mathfrak{p}},$$

it follows from the decomposition $\mathfrak{g}_\rho = \mathfrak{g}_\rho^\vartheta \oplus \mathfrak{p}$ that $\mathfrak{g}_\rho^\vartheta$ (resp. $\mathfrak{p}$) is the linear span of the coefficients of F^ϑ (resp. $F^{\mathfrak{p}}$). The argument is similar for $F(u)^{\check{\vartheta}}$ and $\mathbb{F}(u)^{\check{\vartheta}}$. □

3.2.2 Symmetric pairs of type B, C and D

Let us now return to the special case where $\mathfrak{g} = \mathfrak{g}_N$ is $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and (ρ, V) is the vector representation of the Yangian $Y(\mathfrak{g}_N)$ in the space $\mathbb{C}^N$. We henceforth assume all notation is as in §3.1; in particular, the indexing set $\{1, \dots, N\}$ is replaced with $\mathcal{I}_N$ (see (2.7.8)).

Define

$$\mathrm{G}_N(\mathbb{C}) = \{A \in \mathrm{SL}_N(\mathbb{C}) \,:\, AA^t = I\}, \tag{3.2.6}$$

so that $\mathrm{G}_N(\mathbb{C}) \cong \mathrm{SO}_N(\mathbb{C})$ if $\mathfrak{g}_N = \mathfrak{so}_N$ and $\mathrm{G}_N(\mathbb{C}) \cong \mathrm{SP}_N(\mathbb{C})$ if $\mathfrak{g}_N = \mathfrak{sp}_N$. By (2.7.14),

$$\mathfrak{g}_N = \{X \in \mathfrak{sl}_N \,:\, X = -X^t\},$$

and hence the exponential map $\exp : \mathfrak{g}_N \to \mathrm{G}$ has image contained in $\mathrm{G}_N(\mathbb{C})$. As $\mathrm{G}_N(\mathbb{C})$ is a connected Lie group, we have

$$\mathrm{G} = \langle \exp(\mathfrak{g}) \rangle = \mathrm{G}_N(\mathbb{C}).$$

We shall be particularly interested in those involutions ϑ as in (3.2.2) for which the underlying matrix $\mathcal{G}$ is diagonal. When this is the case, we will write $\mathrm{Ad}(\mathcal{G})$ in place of $\mathrm{Ad}_\rho(\mathcal{G})|_{\mathfrak{g}_N}$, so that

$$\vartheta = \mathrm{Ad}(\mathcal{G}).$$

The following lemma provides justification for this notation.

Lemma 3.2.6. *Let $\mathcal{G} = \sum_{i,j\in\mathcal{I}} g_{ij}E_{ij} \in \lambda\mathrm{G}_N(\mathbb{C})$ for some $\lambda \in \mathbb{C}^\times$. Then, under the identification*

$$F_{ij} = E_{ij} - \theta_{ij}E_{-j,-i} \in \mathfrak{sl}_N,$$

we have

$$\mathrm{Ad}_\rho(\mathcal{G})(F_{ij}) = \mathcal{G}'F_{ij}(\mathcal{G}')^{-1}, \quad \textit{where} \quad \mathcal{G}' = \sum_{i,j\in\mathcal{I}_N} g_{ji}E_{ij}.$$

In particular, if $\mathcal{G}$ satisfies $\mathcal{G} = \mathcal{G}'$, then

$$\mathrm{Ad}_\rho(\mathcal{G})(F_{ij}) = \mathcal{G}F_{ij}\mathcal{G}^{-1} \quad \forall \quad i,j \in \mathcal{I}_N.$$

Proof. Assume that $\mathcal{G} \in \mathrm{GL}_N(\mathbb{C})$ is arbitrary and write $\mathcal{G}^{-1} = \sum_{i,j\in\mathcal{I}_N} g^*_{ij}E_{ij}$. Then

$$\mathcal{G}E_{ij}\mathcal{G}^{-1} = \sum_{a,b\in\mathcal{I}_N} g_{ai}g^*_{jb}E_{ab}. \tag{3.2.7}$$

If in addition $\mathcal{G} \in \lambda\mathrm{G}_N(\mathbb{C})$, then by (3.2.6) we have $\mathcal{G}^t = \lambda^2\mathcal{G}^{-1}$. It follows that

$$\mathcal{G}F_{ij}\mathcal{G}^{-1} = \sum_{a,b\in\mathcal{I}_N} g_{ai}g^*_{jb}E_{ab} - \theta_{ij}\sum_{a,b\in\mathcal{I}_N} g_{-b,-j}g^*_{-i,-a}E_{-b,-a} = \sum_{a,b\in\mathcal{I}_N} g_{ai}g^*_{jb}F_{ab}.$$

On the other hand, (3.2.7) implies that

$$\mathrm{Ad}_\rho(\mathcal{G})(F) = \sum_{i,j\in\mathcal{I}_N}\left(\sum_{a,b\in\mathcal{I}_N} g_{ai}g^*_{jb}E_{ab}\right)\otimes F_{ij} = \sum_{i,j\in\mathcal{I}_N} E_{ij}\otimes\left(\sum_{a,b\in\mathcal{I}_N} g_{ia}g^*_{bj}F_{ab}\right).$$

Therefore, $\mathrm{Ad}_\rho(\mathcal{G})(F_{ij}) = \sum_{a,b\in\mathcal{I}_N} g_{ia}g^*_{bj}F_{ab} = \mathcal{G}'F_{ij}(\mathcal{G}')^{-1}$. □

Another benefit to working under the hypothesis that $\mathcal{G}$ is diagonal is that, by Lemma 3.2.5, the Lie algebras $\mathfrak{g}_N^\vartheta$, $\mathfrak{g}_N[z]^{\check{\vartheta}}$ and $\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}$ all admit simple descriptions. In particular, it is not difficult to write down bases of these Lie algebras using (3.1.3).

Given $\lambda \in \mathbb{C}^\times$ and $\mathcal{G} \in \lambda\mathrm{G}_N(\mathbb{C})$ such that $\mathcal{G}^2 = I$, we define $\mathcal{B}^+_\mathcal{G}, \mathcal{B}^-_\mathcal{G} \subset \mathcal{B}_N$ by

$$\mathcal{B}^+_\mathcal{G} = \{(i,j) \in \mathcal{B}_N \,:\, g_{ii} = g_{jj}\} \quad \text{and} \quad \mathcal{B}^-_\mathcal{G} = \{(i,j) \in \mathcal{B}_N \,:\, g_{ii} = -g_{jj}\},$$

where $\mathcal{G} = \sum_{i\in\mathcal{I}_N} g_{ii}E_{ii}$. As $\mathcal{G}$ is diagonal, it has entries in $\{\pm 1\}$ and (3.1.3) together with Lemma 3.2.5 imply that $\mathfrak{g}_N^{\vartheta}$ and $\mathfrak{p}$ have bases

$$\{F_{ij}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+} \quad \text{and} \quad \{F_{ij}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-}, \tag{3.2.8}$$

respectively. Let $\{F_{ij}(u)^{\check{\vartheta}}\}_{i,j\in\mathcal{I}_N}$ and $\{\mathbb{F}_{ij}(u)^{\check{\vartheta}}\}_{i,j\in\mathcal{I}_N}$ be as in Lemma 3.2.5. We will expand these series as

$$F_{ij}(u)^{\check{\vartheta}} = \sum_{r\geq 0} \check{F}_{ij}^{(r)} u^{-r-1} \quad \text{and} \quad \mathbb{F}_{ij}(u)^{\check{\vartheta}} = \sum_{r\geq 0} \check{\mathbb{F}}_{ij}^{(r)} u^{-r-1}.$$

Corollary 3.2.7. *The Lie algebra* $\mathfrak{g}_N[z]^{\check{\vartheta}}$ *has basis*

$$\{\check{F}_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\geq 0} \cup \{\check{F}_{ij}^{(2r+1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\geq 0}, \tag{3.2.9}$$

while $\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}$ *has basis given by*

$$\{\check{\mathbb{F}}_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\geq 0} \cup \{\check{\mathbb{F}}_{ij}^{(2r+1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\geq 0} \cup \{\mathcal{K}_{2r+1}\}_{r\geq 0}. \tag{3.2.10}$$

Proof. Consider first $\mathfrak{g}_N[z]^{\check{\vartheta}}$. By (3.2.4) and (3.2.8), it has basis given by

$$\{F_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\geq 0} \cup \{F_{ij}^{(2r+1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\geq 0},$$

On the other hand, Lemma 3.2.5 gives

$$\check{F}_{ij}^{(r)} = (g_{jj} + (-1)^r g_{ii}) F_{ij}^{(r)} \quad \forall \quad i,j\in\mathcal{I}_N \quad \text{and} \quad r\geq 0,$$

which implies that (3.2.9) is indeed a basis.

As for $\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}$, (3.2.5) and (3.2.8) imply that it has basis

$$\{F_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\geq 0} \cup \{F_{ij}^{(2r+1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\geq 0} \cup \{\mathcal{K}_{2r+1}\}_{r\geq 0}.$$

That (3.2.10) also provides a basis is then a consequence of the relations

$$\begin{aligned}\check{\mathbb{F}}_{ij}^{(2r)} &= (g_{ii} + g_{jj})\mathbb{F}_{ij}^{(2r)} - 2g_{jj}\delta_{ij}\mathcal{K}_{2r}\\ &= (g_{ii} + g_{jj})F_{ij}^{(2r)},\end{aligned}$$

$$\begin{aligned}\check{\mathbb{F}}_{ij}^{(2r+1)} &= (g_{jj} - g_{ii})\mathbb{F}_{ij}^{(2r)} + 2g_{jj}\delta_{ij}\mathcal{K}_{2r+1} \\ &= (g_{jj} - g_{ii})F_{ij}^{(2r)} + 2g_{jj}\delta_{ij}\mathcal{K}_{2r+1}.\end{aligned}$$ □

We will now fix particular choices of $\mathcal{G}$ which will lead us to an explicit realization for the symmetric pairs (3.0.1).

Given $q \in 2\mathbb{Z}$ satisfying $0 \leq q < N$, we define two distinct matrices $\mathcal{G}$ by

$$\mathcal{G} = \sum_{i=1}^{n}(E_{ii} - E_{-i,-i}) \in \mathrm{GL}_{2n}(\mathbb{C}), \tag{3.2.11}$$

$$\mathcal{G} = I - 2\sum_{a=1}^{q/2}(E_{N+1-a,N+1-a} + E_{a-N-1,a-N-1}) \in \mathrm{GL}_N(\mathbb{C}). \tag{3.2.12}$$

In both cases, we will continue to write

$$\mathcal{G} = \sum_{i,j\in\mathcal{I}_N} g_{ij}E_{ij} = \sum_{i\in\mathcal{I}_N} g_{ii}E_{ii}.$$

Proposition 3.2.8. *Let $\vartheta \in \mathrm{Aut}(\mathfrak{g}_N)$ be as in (3.2.2) and set $p = N - q$. Then:*

(1) If $\mathcal{G}$ is as in (3.2.11), then $\sqrt{-1}\mathcal{G} \in \mathrm{G}_N(\mathbb{C})$ and

$$\begin{aligned}\mathfrak{gl}_n \cong \mathfrak{g}_{2n}^{\vartheta} &= \mathrm{span}_{\mathbb{C}}\{F_{ij}^{\vartheta} = (g_{ii} + g_{jj})F_{ij} \,:\, i,j \in \mathcal{I}_{2n}\} \\ &= \mathrm{span}_{\mathbb{C}}\{F_{ij} \,:\, 1 \leq i,j \leq n\}.\end{aligned}$$

(2) If $\mathcal{G}$ is as in (3.2.12), then $\mathcal{G} \in \mathrm{G}_N(\mathbb{C})$ and

$$\mathfrak{g}_p \oplus \mathfrak{g}_q \cong \mathfrak{g}_N^{\vartheta} = \mathrm{span}_{\mathbb{C}}\{F_{ij}^{\vartheta} = (g_{ii} + g_{jj})F_{ij} \,:\, i,j \in \mathcal{I}_N\},$$

where

$$\mathfrak{g}_p \cong \mathrm{span}_{\mathbb{C}}\{F_{ij} : i,j \in \mathcal{I}_p\} \quad \text{and} \quad \mathfrak{g}_q \cong \mathrm{span}_{\mathbb{C}}\{F_{ij} : |i|,|j| > n - q/2\}.$$

Proof. Consider first Part (1). Setting

$$X = \sum_{i=1}^{n} \frac{\pi\sqrt{-1}}{2}(E_{ii} - E_{-i,-i}) \in \rho(\mathfrak{g}_{2n}),$$

we have $\sqrt{-1}\mathcal{G} = \exp(X) \in \mathrm{G}_N(\mathbb{C})$. Alternatively, by (3.2.6), $\sqrt{-1}\mathcal{G} \in \mathrm{G}_N(\mathbb{C})$

follows from $\mathcal{G} = -\mathcal{G}^t$ and $\det(\mathcal{G}) = (-1)^n$. By Lemma 3.2.5, we have

$$\mathfrak{g}_{2n}^{\vartheta} = \mathrm{span}_{\mathbb{C}}\{F_{ij}^{\vartheta} = (g_{ii} + g_{jj})F_{ij} \,:\, i, j \in \mathcal{I}_{2n}\} = \mathrm{span}_{\mathbb{C}}\{F_{ij} \,:\, 1 \le i, j \le n\}.$$

The defining relations (2.7.13) imply that

$$[F_{ij}, F_{kl}] = \delta_{jk}F_{il} - \delta_{il}F_{kj} \quad \forall \quad 1 \le i, j, k, l \le n.$$

Hence, there is a Lie algebra homomorphism $\mathfrak{gl}_n \to \mathfrak{g}_{2n}^{\vartheta}$ given by $E_{ij} \mapsto F_{ij}$ for all $1 \le i, j \le n$. Since $\{F_{ij}\}_{i,j=1}^{n}$ is a linearly independent set, it is an isomorphism.

Consider now Part (2). Setting

$$X = \sum_{a=1}^{q/2} \pi\sqrt{-1}(E_{N+1-a,N+1-a} - E_{a-N-1,a-N-1}) \in \rho(\mathfrak{g}_N),$$

we have $\mathcal{G} = \exp(X) \in \mathrm{G}_N(\mathbb{C})$. Alternatively, $\mathcal{G} = \mathcal{G}^t$ and $\det(\mathcal{G}) = 1$ imply that $\mathcal{G} \in \mathrm{G}_N(\mathbb{C})$. Next, by Lemma 3.2.5, we have

$$\begin{aligned}\mathfrak{g}_N^{\vartheta} &= \mathrm{span}_{\mathbb{C}}\{F_{ij}^{\vartheta} = (g_{ii} + g_{jj})F_{ij} \,:\, i, j \in \mathcal{I}_N\} \\ &= \mathrm{span}_{\mathbb{C}}\{F_{ij} \,:\, i, j \in \mathcal{I}_p\} + \mathrm{span}_{\mathbb{C}}\{F_{ij} : |i|, |j| > n - q/2\}\end{aligned}$$

The defining relations (2.7.13) imply that both summands above are Lie subalgebras of $\mathfrak{g}_N$ and, moreover, that the sum is in fact a direct sum of Lie algebras. It is clear that

$$\mathfrak{g}_p \cong \mathrm{span}_{\mathbb{C}}\{F_{ij} \,:\, i, j \in \mathcal{I}_p\} \subset \mathfrak{g}_N^{\vartheta}.$$

An isomorphism of Lie algebras $\mathfrak{g}_q \xrightarrow{\sim} \mathrm{span}_{\mathbb{C}}\{F_{ij} : |i|, |j| > n - q/2\}$ is given by

$$F_{ij} \mapsto F_{i+\mathrm{sign}(i)(n-q/2),j+\mathrm{sign}(j)(n-q/2)} \quad \forall \quad i, j \in \mathcal{I}_q. \qquad \square$$

Henceforth, we will assume that the symmetric pairs

$$(\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \quad \text{where} \quad 0 \le q < N \ \text{ and } \ q \in 2\mathbb{Z},$$

are of the form $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})$ with $\vartheta = \mathrm{Ad}(\mathcal{G})$ as in (3.2.2), and $\mathcal{G}$ as in (3.2.11) or (3.2.12). These choices of $\mathcal{G}$ are not unique, and we refer the reader to [GR16, §3.1] for a discussion of this point. They are chosen due to the fact that they are diagonal, which leads to a very simple description of $\mathfrak{g}_N^\vartheta$ and also simplifies the treatment of the representation theory of twisted Yangians given in Chapters 4 and 5.

3.3 Twisted Yangians of type B, C and D

In this section, we recall the definitions and main properties of the orthogonal and symplectic twisted Yangians associated to the pairs $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ of §3.2.2.

We introduce the following notation: For any $\mathcal{G} \in \mathrm{GL}_N(\mathbb{C})$ for which $\sqrt{-1}\mathcal{G}$ or $\mathcal{G}$ belongs to $\mathrm{G}_N(\mathbb{C})$ (see (3.2.6)), we define $(\pm)_\mathcal{G} \in \{\pm\}$ by

$$\mathcal{G}\mathcal{G}^t = (\pm)_\mathcal{G} I.$$

In all cases we consider, $\mathcal{G}$ will be of the form (3.2.11) or (3.2.12) and we will write $(\pm) = (\pm)_\mathcal{G}$. Explicitly, we have

$$(\pm) = \begin{cases} + & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \\ - & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n). \end{cases}$$

3.3.1 The matrix $\mathcal{G}(u)$

The starting point for defining a twisted Yangian using the R-matrix formalism is a matrix

$$\mathcal{G}(u) \in \mathrm{End}(\mathbb{C}^N)[\![u^{-1}]\!]$$

which is the expansion at $u = \infty$ of a $\mathrm{End}(\mathbb{C}^N)$-valued rational function of u, and is a solution of the so-called *reflection equation.* For us, this equation takes the form

$$R(u-v)\mathcal{G}_1(u)R(u+v)\mathcal{G}_2(v) = \mathcal{G}_2(v)R(u+v)\mathcal{G}_1(u)R(u-v), \tag{3.3.1}$$

where $R(u)$ is given by (2.7.15). The matrix $\mathcal{G}(u)$ will equip the corresponding twisted Yangian with a trivial representation, and will also provide the necessary ingredient for rebuilding it using the reflection algebra formalism: see §3.3.5.

The desired source of solutions to (3.3.1) is provided by Lemma 4.1 of [GR16].

Given $\mathcal{G}$ as in (3.2.11) or (3.2.12), we define

$$\mathcal{G}(u) = \sum_{i,j\in\mathcal{I}_N} g_{ij}(u)E_{ij} = \frac{\mathrm{tr}(\mathcal{G})I - 4u\mathcal{G}}{\mathrm{tr}(\mathcal{G}) - 4u}. \tag{3.3.2}$$

Lemma 3.3.1 ([GR16, Lemma 4.1])**.** *The matrix $\mathcal{G}(u)$ given by (3.3.2) is a solution of (3.3.1). Moreover, $\mathcal{G}(u)$ satisfies the unitary relation*

$$\mathcal{G}(u)\mathcal{G}(-u) = I.$$

Hereafter, we will assume that $\mathcal{G}(u)$ is always given by (3.3.2). We also define

$$\mathscr{g}(u) = \begin{cases} \dfrac{N-4u}{\mathrm{tr}(\mathcal{G})-4u} & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \\ u^{-1} & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n). \end{cases} \tag{3.3.3}$$

In particular, if $\mathrm{tr}(\mathcal{G}) \neq 0$, then $\mathscr{g}(u)$ is uniquely determined by

$$\mathrm{tr}(\mathcal{G}(u)) = \mathscr{g}(u)\mathrm{tr}(\mathcal{G}). \tag{3.3.4}$$

The rational function $\mathscr{g}(u)$ will play an important role throughout this thesis, as will the series $p_{\mathcal{G}}(u)$ defined by the following lemma.

Lemma 3.3.2 ([GR16, Lemma 5.1])**.** *There is a unique series $p_{\mathcal{G}}(u) \in \mathbb{C}[\![u^{-1}]\!]$ satisfying the relation*

$$Q\mathcal{G}_1(u)R(2u-\kappa)\mathcal{G}_2^{-1}(\kappa-u) = \mathcal{G}_2^{-1}(\kappa-u)R(2u-\kappa)\mathcal{G}_1(u)Q = p_{\mathcal{G}}(u)Q. \tag{3.3.5}$$

Proof. Multiplying both sides of (3.3.1) by $\mathcal{G}_2(v)^{-1}$ yields

$$\mathcal{G}_2(v)^{-1}R(u-v)\mathcal{G}_1(u)R(u+v) = R(u+v)\mathcal{G}_1(u)R(u-v)\mathcal{G}_2(v)^{-1}.$$

Multiplying both sides by $u+v-\kappa$ and taking the limit as $v \mapsto \kappa - u$ yields

$$\mathcal{G}_2(\kappa-u)^{-1}R(2u-\kappa)\mathcal{G}_1(u)Q = Q\mathcal{G}_1(u)R(2u-\kappa)\mathcal{G}_2(\kappa-u)^{-1}.$$

As Q projects $\mathbb{C}^N \otimes \mathbb{C}^N$ onto a one dimensional subspace, the above equality implies that there is a unique series $p_{\mathcal{G}}(u)$ satisfying (3.3.5). □

In the special case where $\mathcal{G} = I$, the relation (3.3.5) reduces to

$$QR(2u - \kappa) = R(2u - \kappa)Q = p_I(u)Q. \tag{3.3.6}$$

Since $PQ = \pm Q = QP$ and $Q^2 = N$, we find that

$$p_I(u) = 1 \mp \frac{1}{2u - \kappa} + \frac{N}{2u - 2\kappa}.$$

Equivalently,

$$p_I(u) = \frac{u}{\kappa - u} \cdot \frac{\kappa - 2u \mp 1}{2u - \kappa}. \tag{3.3.7}$$

Using this observation, we can compute $p_{\mathcal{G}}(u)$ in general.

Proposition 3.3.3. *$p_{\mathcal{G}}(u)$ satisfies*

$$p_I(u)p_{\mathcal{G}}(u)^{-1} = g(\kappa - u)g(u)^{-1}. \tag{3.3.8}$$

and is given explicitly by

$$p_{\mathcal{G}}(u) = (\pm)1 \mp \frac{1}{2u - \kappa} + \frac{\mathrm{tr}(\mathcal{G}(u))}{2u - 2\kappa}. \tag{3.3.9}$$

Proof. Suppose first that $\mathrm{tr}(\mathcal{G}) = 0$. In this case, $\mathcal{G}(u) = \mathcal{G}$ and Lemma 3.3.2 gives

$$\mathcal{G}_2 R(2u - \kappa)\mathcal{G}_1 Q = p_{\mathcal{G}}(u)Q.$$

By (3.1.4) and $\mathcal{G}^2 = I$, the left-hand side is

$$\mathcal{G}_2\mathcal{G}_1 Q \mp \frac{Q}{2u - \kappa} + \frac{\mathrm{tr}(\mathcal{G})Q}{2u - 2\kappa} = (\pm)Q \mp \frac{Q}{2u - \kappa}.$$

and hence we obtain

$$p_{\mathcal{G}}(u) = (\pm)1 \mp \frac{1}{2u - \kappa}.$$

This proves (3.3.9) when $\mathrm{tr}(\mathcal{G}) = 0$, and the expression (3.3.8) is easily obtained using (3.3.7) and (3.3.3) in these cases.

Suppose now that $\mathrm{tr}(\mathcal{G}) \neq 0$. Left multiplying (3.3.5) by $QG_2(\kappa - u)$, we obtain

$$QR(2u - \kappa)\mathcal{G}_1(u)Q = p_{\mathcal{G}}(u)Q\mathcal{G}_2(\kappa - u)Q.$$

Since $\mathcal{G}(u)^t = \mathcal{G}(u)$, (3.3.6) and $Q\mathcal{G}_2(u) = Q\mathcal{G}_1(u)^t$ yield the identity

$$p_I(u)Q\mathcal{G}_1(u)Q = p_{\mathcal{G}}(u)Q\mathcal{G}_1(\kappa - u)Q,$$

which by (3.1.4) is equivalent to

$$p_I(u)\mathrm{tr}(\mathcal{G}(u))Q = p_{\mathcal{G}}(u)\mathrm{tr}(\mathcal{G}(\kappa - u))Q.$$

Using the relation (3.3.4), we can conclude that (3.3.8) holds.

It remains to show that (3.3.9) holds whenever $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$ with $p \neq q$. This was proven in [GR16, Lemma 5.1] directly. Below, we offer a different proof using (3.3.8).

From (3.3.7) and (3.3.3), we obtain

$$p_{\mathcal{G}}(u) = p_I(u)\frac{g(u)}{g(\kappa - u)} = \frac{u}{\kappa - u} \cdot \frac{N - 4u}{\mathrm{tr}(\mathcal{G}) - 4u} \cdot \frac{\mathrm{tr}(\mathcal{G}) - 4\kappa + 4u}{2\kappa - 4u}$$

As $\lim_{u\to\infty} p_{\mathcal{G}}(u) = 1$, $p_{\mathcal{G}}(u)$ admits a partial fraction decomposition of the form

$$p_{\mathcal{G}}(u) = 1 + \frac{z_1}{u - \kappa} + \frac{z_2}{u - \kappa/2} + \frac{z_3}{u - \mathrm{tr}(\mathcal{G})/4}, \tag{3.3.10}$$

and the coefficients z_i are readily computed and found to be

$$z_1 = \mathrm{tr}(\mathcal{G})\frac{4\kappa - N}{8\kappa - 2\mathrm{tr}(\mathcal{G})}, \quad z_2 = \mp\frac{1}{2}, \quad z_3 = \mathrm{tr}(\mathcal{G})\frac{N - \mathrm{tr}(\mathcal{G})}{8\kappa - 2\mathrm{tr}(\mathcal{G})}.$$

On the other hand, we have the partial fraction decomposition

$$\frac{\mathrm{tr}(\mathcal{G}(u))}{2u - 2\kappa} = \mathrm{tr}(\mathcal{G})\frac{N - 4u}{(\mathrm{tr}(\mathcal{G}) - 4u)(2u - 2\kappa)} = \frac{z_1}{u - \kappa} + \frac{z_3}{u - \mathrm{tr}(\mathcal{G})/4}.$$

Substituting this identity and the value $z_2 = \mp\frac{1}{2}$ into (3.3.10) returns (3.3.9). □

The last property of $p_{\mathcal{G}}(u)$ which we will need is given by the next corollary.

Corollary 3.3.4. *The series $p_{\mathcal{G}}(u)$ satisfies*

$$p_{\mathcal{G}}(u)p_{\mathcal{G}}(\kappa - u) = 1 - \frac{1}{(2u - \kappa)^2}. \tag{3.3.11}$$

Proof. By (3.3.6) and (3.1.6), $p_I(u)p_I(\kappa-u)$ is uniquely determined by

$$Np_I(u)p_I(\kappa-u)Q = QR(2u-\kappa)R(\kappa-2u)Q = N\left(1-\frac{1}{(2u-\kappa)^2}\right)Q.$$

This proves that $p_I(u)$ satisfies (3.3.11). In the general case, (3.3.8) yields

$$p_{\mathcal{G}}(u)p_{\mathcal{G}}(\kappa-u) = p_I(u)p_I(\kappa-u) = 1-\frac{1}{(2u-\kappa)^2}.$$ □

Remark 3.3.5. We will often suppress the subscript $\mathcal{G}$ of $p_{\mathcal{G}}(u)$ and simply write

$$p(u) = p_{\mathcal{G}}(u).$$

3.3.2 Definitions and first properties

We are now prepared to define the twisted Yangians associated to the symmetric pairs $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})$ of the form (3.0.1).

Definition 3.3.6. Let $\mathcal{G}(u)$ be as in (3.3.2). Then:

(1) The *extended twisted Yangian* $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is the subalgebra of $X(\mathfrak{g}_N)$ generated by the coefficients $\{s_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$ of

$$S(u) = T(u-\kappa/2)\mathcal{G}(u)T^t(-u+\kappa/2) \in \mathrm{End}(\mathbb{C}^N)\otimes X(\mathfrak{g}_N)[\![u^{-1}]\!],$$
$$\text{where } S(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij}\otimes s_{ij}(u) \quad \text{and} \quad s_{ij}(u) = g_{ij} + \sum_{r\geq 1} s_{ij}^{(r)}u^{-r}. \tag{3.3.12}$$

(2) The *twisted Yangian* $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is the subalgebra of $Y(\mathfrak{g}_N)$ generated by the coefficients $\{\sigma_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$ of

$$\mathcal{S}(u) = \mathcal{T}(u-\kappa/2)\mathcal{G}(u)\mathcal{T}^t(-u+\kappa/2) \in \mathrm{End}(\mathbb{C}^N)\otimes Y(\mathfrak{g}_N)[\![u^{-1}]\!],$$
$$\text{where } \mathcal{S}(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij}\otimes \sigma_{ij}(u) \quad \text{and} \quad \sigma_{ij}(u) = g_{ij} + \sum_{r\geq 1} \sigma_{ij}^{(r)}u^{-r}. \tag{3.3.13}$$

Let us discuss some immediate consequences of these definitions.

Since $T(u) = y(u)\mathcal{T}(u)$, the relation (2.7.17) implies that

$$S(u)S(-u) = w(u) \cdot I \quad \text{and} \quad \mathcal{S}(u)\mathcal{S}(-u) = I,$$
$$\text{where} \quad w(u) = \mathsf{z}(u - \kappa/2)\mathsf{z}(-u - \kappa/2) \in ZX(\mathfrak{g}_N)[\![u^{-2}]\!].$$

The first relation above implies that the coefficients of $w(u)$ belong to the center $ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. We shall expand $w(u)$ as

$$w(u) = 1 + \sum_{r \geq 1} w_{2r} u^{-2r} \in 1 + u^{-2} ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}[\![u^{-2}]\!]. \tag{3.3.14}$$

We may then define $q(u) = 1 + \sum_{r\geq 1} q_r u^{-r}$ to be the unique solution of the equation

$$w(u) = q(u)q(u+\kappa) \quad \text{in} \quad 1 + u^{-1} ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}[\![u^{-1}]\!]. \tag{3.3.15}$$

The uniqueness of $q(u)$ together with (2.7.18) yields

$$q(u) = q(\kappa - u) \quad \text{and} \quad q(u) = y(u - \kappa/2)y(-u + \kappa/2). \tag{3.3.16}$$

In particular, $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ is a subalgebra of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and we have

$$S(u) = q(u)\mathcal{S}(u).$$

The following lemma, which is a restatement of Proposition 3.2 and Corollary 3.2 in [GR16], shows that both $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ have a coideal structure.

Lemma 3.3.7. *The twisted Yangians $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ are left coideal subalgebras of $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$, respectively. That is, we have*

$$\Delta(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) \subset X(\mathfrak{g}_N) \otimes X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw},$$
$$\Delta(Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) \subset Y(\mathfrak{g}_N) \otimes Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}.$$

The lemma follows from (2.4.2), (2.5.4) and Lemma 3.3.1, which give

$$\Delta(S(u)) = T_{[1]}(u - \kappa/2) S_{[2]}(u) T^t_{[1]}(-u + \kappa/2),$$
$$\Delta(\mathcal{S}(u)) = \mathcal{T}_{[1]}(u - \kappa/2) \mathcal{S}_{[2]}(u) \mathcal{T}^t_{[1]}(-u + \kappa/2).$$

These formulas are expanded in terms of $s_{ij}(u)$ and $\sigma_{ij}(u)$ as

$$\begin{aligned}\Delta(s_{ij}(u)) &= \sum_{a,b\in\mathcal{I}_N} \theta_{bj} t_{ia}(u-\kappa/2) t_{-j,-b}(-u+\kappa/2)\otimes s_{ab}(u),\\ \Delta(\sigma_{ij}(u)) &= \sum_{a,b\in\mathcal{I}_N} \theta_{bj} \tau_{ia}(u-\kappa/2) \tau_{-j,-b}(-u+\kappa/2)\otimes \sigma_{ab}(u).\end{aligned} \tag{3.3.17}$$

As alluded to in §3.3.1, the restriction of the counit ϵ of $X(\mathfrak{g}_N)$ to $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ is given by the matrix $\mathcal{G}(u)$:

$$\epsilon(S(u)) = \mathcal{G}(u) \quad \text{and} \quad \epsilon(\mathcal{S}(u)) = \mathcal{G}(u).$$

This equips both $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ with a trivial representation: a one-dimensional representation with action given by ϵ. We will always denote this representation by $V(\mathcal{G})$.

In Chapter 2, an important role was played by the automorphisms m_f (see (2.4.4) and (2.7.3)) of $X(\mathfrak{g}_N)$. A similar story unfolds in the twisted Yangian setting. By definition of $S(u)$, we have

$$m_f(S(u)) = f(u-\kappa/2)f(-u+\kappa/2)S(u) \quad \forall \quad f(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!].$$

Consequently, m_f restricts to an automorphism of $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$. Note that for any $f(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ the corresponding series

$$g(u) = f(u-\kappa/2)f(-u+\kappa/2) \tag{3.3.18}$$

satisfies

$$g(u) = g(\kappa-u) \quad \text{and} \quad m_f(S(u)) = g(u)S(u).$$

Conversely, if $g(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ is an arbitrary series satisfying $g(u)=g(\kappa-u)$, then we may construct $f(u)$ such that $m_f(S(u)) = g(u)S(u)$. Indeed, we can take $f(u)$ to be the unique series in $1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying

$$g(u) = f(u)f(u-\kappa/2).$$

As $g(u)=g(\kappa-u)$, the uniqueness of $f(u)$ implies that $f(u)=f(-u+\kappa/2)$, and we thus have $m_f(S(u)) = g(u)S(u)$, as claimed.

Hence, for any $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying $g(u) = g(\kappa - u)$, the assignment

$$\nu_g : S(u) \mapsto g(u)S(u) \tag{3.3.19}$$

extends to an automorphism $\nu_g \in \mathrm{Aut}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$ with the property that

$$\nu_g = m_f|_{X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}}$$

for any $f(u)$ satisfying (3.3.18). Moreover (2.6.8), (3.3.15) and (3.3.16) imply that

$$\nu_g(q(u)) = g(u)q(u) \quad \text{ and } \quad \nu_g(w(u)) = g(u)g(u+\kappa)w(u). \tag{3.3.20}$$

As a corollary to the above discussion and Theorem 2.6.11, we obtain the following characterization of $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, which was partially given in [GR16, Corollary 3.1].

Corollary 3.3.8. *The twisted Yangian $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ is equal to the subalgebra of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ fixed by all automorphisms ν_g:*

$$Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} = \{Y \in X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \, : \, \nu_g(Y) = Y \quad \forall \quad g(u)\},$$

where $g(u)$ varies over the subset of $1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ consisting of series invariant under the transformation $u \mapsto \kappa - u$.

A second family of automorphisms for $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ is provided by Lemma 3.2.4. Suppose that $A \in \lambda \mathrm{G}_N(\mathbb{C})$ (see (3.2.6)) for some $\lambda \in \mathbb{C}^\times$. Then the automorphism $\mathrm{Ad}_R(A)$ of $X(\mathfrak{g}_N)$ satisfies

$$\begin{aligned}\mathrm{Ad}_R(A)(S(u)) &= AT(u-\kappa/2)A^{-1}\mathcal{G}(u)(A^{-1})^t T^t(-u+\kappa/2)A^t \\ &= AT(u-\kappa/2)A^{-1}\mathcal{G}(u)AT^t(-u+\kappa/2)A^{-1}.\end{aligned}$$

where we have used that $A^{-1} = \lambda^2 A^t$. If in addition $A^{-1}\mathcal{G}A = \mathcal{G}$, then

$$A^{-1}\mathcal{G}(u)A = \frac{\mathrm{tr}(\mathcal{G})I - 4uA^{-1}\mathcal{G}A}{\mathrm{tr}(\mathcal{G}) - 4u} = \mathcal{G}(u).$$

We can thus conclude that, for every A satisfying

$$A \in \mathbb{C}^\times \cdot \mathrm{G}_N(\mathbb{C}) \quad \text{ and } \quad A\mathcal{G}A^{-1} = \mathcal{G},$$

the assignment

$$\mathrm{Ad}_{\vartheta}(A) : S(u) \mapsto AS(u)A^{-1} \tag{3.3.21}$$

extends to an automorphism $\mathrm{Ad}_{\vartheta}(A) \in \mathrm{Aut}(X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})$ which satisfies

$$\mathrm{Ad}_{\vartheta}(A) = \mathrm{Ad}_R(A)|_{X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}}.$$

Moreover, the second part of Lemma 3.2.4 implies that $\mathrm{Ad}_{\vartheta}(A)$ restricts to an automorphism of $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$.

3.3.3 Poincaré-Birkhoff-Witt Theorem

Thanks to Theorems 2.5.5 and 2.6.7, it is not difficult to establish a Poincaré-Birkhoff-Witt type theorem for $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$. Recall from (2.4.8) that the filtration $\{\mathbf{F}_k(X(\mathfrak{g}_N))\}_{k\geq 0}$ on $X(\mathfrak{g}_N)$ is given the degree assignment

$$\deg t_{ij}^{(r)} = r - 1 \quad \forall \quad i, j \in \mathcal{I}_N \;\text{ and }\; r \in \mathbb{N}$$

Both $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ inherit a filtered structure from $X(\mathfrak{g}_N)$, and we denote the corresponding filtrations by

$$\{\mathbf{F}_k(X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})\}_{k\geq 0} \quad \text{and} \quad \{\mathbf{F}_k(Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})\}_{k\geq 0}.$$

It is immediate from the definitions of $S(u)$ and $\mathcal{S}(u)$ that

$$s_{ij}^{(r)} \in \mathbf{F}_{r-1}(X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}) \quad \text{and } \sigma_{ij}^{(r)} \in \mathbf{F}_{r-1}(X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})$$

for all $i, j \in \mathcal{I}_N$ and $r \in \mathbb{N}$. Let $\bar{s}_{ij}^{(r)}$ and $\bar{\sigma}_{ij}^{(r)}$ denote the images of $s_{ij}^{(r)}$ and $\sigma_{ij}^{(r)}$ in $\mathrm{gr}_{r-1}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $\mathrm{gr}_{r-1}Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, respectively. We then define

$$\begin{gathered}
\mathbb{S}(u) = \sum_{k\geq 1} \mathbb{S}^{(k)} u^{-k} \in \mathrm{End}(\mathbb{C}^N) \otimes (\mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})[\![u^{-1}]\!], \\
\boldsymbol{\Sigma}(u) = \sum_{k\geq 1} \boldsymbol{\Sigma}^{(k)} u^{-k} \in \mathrm{End}(\mathbb{C}^N) \otimes (\mathrm{gr}Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})[\![u^{-1}]\!], \\
\text{where} \quad \mathbb{S}^{(k)} = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \bar{s}_{ij}^{(k)} \quad \text{and} \quad \boldsymbol{\Sigma}^{(k)} = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \bar{\sigma}_{ij}^{(k)}.
\end{gathered}$$

The below theorem was given in [GR16, Proposition 3.3] for $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$. Let $\mathrm{Res}_u(\mathcal{G}(u))$ denote the residue of $\mathcal{G}(u)$ at $u = 0$. Explicitly,

$$\mathrm{Res}_u(\mathcal{G}(u)) = (\mathcal{G} - I)\frac{\mathrm{tr}(\mathcal{G})}{4}.$$

Theorem 3.3.9. *Let $\mathbb{F}(u)^{\check{\vartheta}}$ and $F(u)^{\check{\vartheta}}$ be as in Lemma 3.2.5. Then:*

(1) The assignment

$$\varphi_{\vartheta} : \mathbb{F}(u)^{\check{\vartheta}} \mapsto \mathbb{S}(u) - \mathrm{Res}_u(\mathcal{G}(u)) \tag{3.3.22}$$

extends to an isomorphism of graded algebras

$$\varphi_{\vartheta} : U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}) \xrightarrow{\sim} \mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}.$$

(2) φ_{ϑ} restricts to an isomorphism of graded algebras

$$U(\mathfrak{g}_N[z]^{\check{\vartheta}}) \xrightarrow{\sim} \mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw},$$

which is given by $F(u)^{\check{\vartheta}} \mapsto \mathbf{\Sigma}(u) - \mathrm{Res}_u(\mathcal{G}(u))$.

Proof. Let φ_{ϑ} denote the restriction of the isomorphism $\varphi_{\mathcal{I}} : U(\mathring{\mathfrak{g}}_N[z]) \xrightarrow{\sim} \mathrm{gr}X(\mathfrak{g}_N)$ from Theorem 2.6.7 to the subalgebra $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}) \subset U(\mathring{\mathfrak{g}}_N[z])$. This is an injection

$$\varphi_{\vartheta} : U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}) \hookrightarrow \mathrm{gr}X(\mathfrak{g}_N),$$

and hence to prove Part (1) it suffices to show that φ_{ϑ} has image equal to the subalgebra $\mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ of $\mathrm{gr}X(\mathfrak{g}_N)$, and that it is indeed given by the formula (3.3.22). As the coefficients of $\mathbb{S}(u)$ generate $\mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, it suffices to prove the latter assertion.

Note that $\mathbb{S}(u)$ is the image of

$$S(u) - \mathcal{G} \in \mathrm{End}(\mathbb{C}^N) \otimes \prod_{r \geq 0} \mathbf{F}_{r-1}(X(\mathfrak{g}_N))u^{-r}$$

in the space

$$\mathrm{End}(\mathbb{C}^N) \otimes \prod_{r \geq 0} \mathrm{gr}_{r-1}(X(\mathfrak{g}_N))u^{-r} \subset \mathrm{End}(\mathbb{C}^N) \otimes (\mathrm{gr}X(\mathfrak{g}_N))[\![u^{-1}]\!]. \tag{3.3.23}$$

On the other hand, the image of $T(u-\kappa/2)\mathcal{G}(u)T^t(-u+\kappa/2)-\mathcal{G}$ is easily seen to be

$$\mathbb{T}(u)\mathcal{G}+\mathcal{G}\mathbb{T}^t(-u)+\mathrm{Res}_u(\mathcal{G}(u)),$$

where $\mathbb{T}(u)$ is the image of $T(u)-1$ in (3.3.23), as in the proof of Proposition 2.4.4. As $\varphi_{\mathcal{I}}(\mathbb{F}(u))=\mathbb{T}(u)$, we thus have

$$\begin{aligned}\varphi_{\vartheta}^{-1}(\mathbb{S}(u)) &= \mathbb{F}(u)\mathcal{G}+\mathcal{G}\mathbb{F}^t(-u)+\mathrm{Res}_u(\mathcal{G}(u))\\ &= \mathbb{F}(u)\mathcal{G}-\mathcal{G}\mathbb{F}(-u)+2\mathbb{K}(-u)\mathcal{G}+\mathrm{Res}_u(\mathcal{G}(u)),\end{aligned} \tag{3.3.24}$$

where we have used that

$$\mathbb{F}(u)=F(u)+\mathbb{K}(u),\quad F^t(u)=-F(u)\quad \text{and}\quad \mathbb{K}^t(u)=\mathbb{K}(u).$$

The second equality above is a consequence of the relation in (2.7.13), while the third equality is due to the fact that $\mathbb{K}(u)=\mathcal{K}(u)\cdot I$ is a central series multiple of the identity matrix.

Combining Lemma 3.2.5 with (3.3.24), we obtain the desired equality

$$\varphi_{\vartheta}(\mathbb{F}(u)^{\check{\vartheta}})=\mathbb{S}(u)-\mathrm{Res}_u(\mathcal{G}(u)).$$

Consider now Part (2). It is enough to prove that

$$\varphi_{\vartheta}(F(u)^{\check{\vartheta}})=\mathbf{\Sigma}(u)-\mathrm{Res}_u(\mathcal{G}(u)).$$

It follows from the second equality in (3.3.16) that

$$q_r\equiv(1+(-1)^r)y_r \mod \mathbf{F}_{r-2}(X(\mathfrak{g}_N)). \tag{3.3.25}$$

For each $r\in\mathbb{N}$, let $\bar{y}_r$ denote the image of y_r in $\mathrm{gr}_{r-1}X(\mathfrak{g}_N)$, so that $\varphi_{\mathcal{I}}(\mathcal{K}_{r-1})=\bar{y}_r$. As $q(u)\mathcal{S}(u)=S(u)$, we obtain

$$\varphi_{\vartheta}^{-1}(\mathbb{S}(u))=\mathbb{F}(u)^{\check{\vartheta}}+\mathrm{Res}_u(\mathcal{G}(u))=\varphi_{\vartheta}^{-1}(\mathbf{\Sigma}(u))+\mathbb{K}(u)\mathcal{G}+\mathbb{K}(-u)\mathcal{G}.$$

As $\mathbb{K}(u)$ commutes with $\mathcal{G}$ and $\mathbb{F}(u) = F(u) + \mathbb{K}(u)$, this gives

$$\begin{aligned}\varphi_\vartheta^{-1}(\boldsymbol{\Sigma}(u)) &= \mathbb{F}(u)\mathcal{G} - \mathcal{G}\mathbb{F}(-u) + \mathbb{K}(-u)\mathcal{G} - \mathbb{K}(u)\mathcal{G} + \mathrm{Res}_u(\mathcal{G}(u)) \\ &= F(u)\mathcal{G} - \mathcal{G}F(-u) + \mathrm{Res}_u(\mathcal{G}(u)) \\ &= F(u)^{\check{\vartheta}} + \mathrm{Res}_u(\mathcal{G}(u)). \end{aligned}$$ □

Remark 3.3.10. The above theorem shows that $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ are filtered algebra deformations of the enveloping algebras $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$ and $U(\mathfrak{g}_N[z]^{\check{\vartheta}})$, respectively. In fact, they are filtered *coideal* deformations of these enveloping algebras.

Indeed, the Hopf algebra structure on $X(\mathfrak{g}_N)$ is filtered, and induces a Hopf structure on $\mathrm{gr}X(\mathfrak{g}_N) \cong U(\mathring{\mathfrak{g}}_N[z])$ such that the isomorphism $\varphi_\mathcal{I}$ of Theorem 2.6.7 is an isomorphism of Hopf algebras. In particular,

$$(\varphi_\mathcal{I} \otimes \varphi_\mathcal{I}) \circ \bar{\Delta} \circ \varphi_\mathcal{I}^{-1} = \mathrm{gr}(\Delta),$$

where $\bar{\Delta}$ is the standard coproduct on $U(\mathring{\mathfrak{g}}_N[z])$. The coideal structure on $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ given by Lemma 3.3.7 then induces the trivial coideal structure on $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}) \subset U(\mathring{\mathfrak{g}}_N[z])$ given by its standard Hopf subalgebra structure. That is, we have

$$(\varphi_\mathcal{I} \otimes \varphi_\vartheta) \circ \bar{\Delta}|_{U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})} \circ \varphi_\vartheta^{-1} = \mathrm{gr}(\Delta|_{X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}}).$$

Now let us briefly discuss a few consequences of Theorem 3.3.9.

As $U(\mathfrak{g}_N^\vartheta) \subset U(\mathfrak{g}_N[z]^{\check{\vartheta}}) \subset U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$ consists of elements of degree zero and $\mathbf{F}_0(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$ embeds into $\mathrm{gr}X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, Theorem 3.3.9 yields an embedding

$$U(\mathfrak{g}_N^\vartheta) \hookrightarrow Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \subset X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}.$$

In order to make this explicit, let us write

$$\bar{g}_{ij} = (g_{ij} - \delta_{ij})\frac{\mathrm{tr}(\mathcal{G})}{4},$$

so that $\mathrm{Res}_u(\mathcal{G}(u)) = \sum_{i,j\in\mathcal{I}_N} \bar{g}_{ij}E_{ij}$. In addition, we recall from Lemma 3.2.5 and Proposition 3.2.8 that

$$F_{ij}^\vartheta = (g_{ii} + g_{jj})F_{ij} \quad \forall \quad i, j \in \mathcal{I}_N.$$

Corollary 3.3.11 ([GR16, Corollary 3.3]). *The assignment*

$$F_{ij}^{\vartheta} \mapsto \sigma_{ij}^{(1)} - \bar{g}_{ij} = s_{ij}^{(1)} - \bar{g}_{ij} \quad \forall \quad i, j \in \mathcal{I}_N \tag{3.3.26}$$

extends to an injective algebra homomorphism

$$U(\mathfrak{g}_N^{\vartheta}) \hookrightarrow Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw} \subset X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}.$$

As any graded basis of the associated graded algebra $\mathrm{gr}(\mathscr{A})$ ($\mathscr{A}$ being a $\mathbb{N}$-filtered unital, associative $\mathbb{C}$-algebra) may be lifted to a basis of $\mathscr{A}$, an important consequence of Theorem 3.3.9 is that graded bases of $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$ and $U(\mathfrak{g}_N[z]^{\check{\vartheta}})$ may be lifted to bases of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, respectively.

Using Corollary 3.2.7, the formula (3.3.22) and the relation (3.3.25), we obtain the following version of Theorem 3.2 and Corollary 3.4 from [GR16].

Corollary 3.3.12. *Fix any total orderings on the sets*

$$\mathcal{B}_Y = \{\sigma_{ij}^{(2r-1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\in\mathbb{N}} \cup \{\sigma_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\in\mathbb{N}},$$
$$\mathcal{B}_X = \{s_{ij}^{(2r-1)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^+, r\in\mathbb{N}} \cup \{s_{ij}^{(2r)}\}_{(i,j)\in\mathcal{B}_{\mathcal{G}}^-, r\in\mathbb{N}} \cup \{q_{2r}\}_{r\in\mathbb{N}}.$$

Then the set of ordered monomials in the elements of $\mathcal{B}_Y$ is a basis of $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, and the set of ordered monomials in the elements of $\mathcal{B}_X$ is a basis of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$.

3.3.4 Tensor product decomposition and the center

By Proposition 2.6.6, the center $ZX(\mathfrak{g}_N)$ of $X(\mathfrak{g}_N)$ is isomorphic to the polynomial algebra $\mathbb{C}[y_r]_{r\in\mathbb{N}}$. Similarly, the subalgebra $W(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})$ of $ZX(\mathfrak{g}_N)$ generated by $\{q_r\}_{r\in\mathbb{N}}$ is isomorphic to the polynomial ring $\mathbb{C}[q_{2r}]_{r\in\mathbb{N}}$.

Lemma 3.3.13. *The algebra homomorphism*

$$\varphi_W : \mathbb{C}[x_{2r}]_{r\in\mathbb{N}} \to W(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}), \quad x_{2r} \mapsto q_{2r} \quad \forall\, r \in \mathbb{N}.$$

is an isomorphism.

Proof. The injectivity of φ_W follows from Corollary 3.3.12, though we will not make use of this here. We view $\mathbb{C}[x_{2r}]_{r\in\mathbb{N}}$ as a graded (and thus filtered) algebra by setting

$$\deg x_{2r} = 2r-1 \quad \forall \quad r \in \mathbb{N}.$$

The homomorphism φ_W is then filtered, and to prove it is an isomorphism it suffices to prove that

$$\mathrm{gr}(\varphi_W) : \mathrm{gr}\mathbb{C}[x_{2r}]_{r\in\mathbb{N}} \cong \mathbb{C}[x_{2r}]_{r\in\mathbb{N}} \to \mathrm{gr}W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$$

is an isomorphism.

By (3.3.25), the image of q_r in the associated graded algebra

$$\mathrm{gr}W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) \subset \mathrm{gr}ZX(\mathfrak{g}_N) \cong \mathbb{C}[y_r]_{r\in\mathbb{N}}$$

is $(1+(-1)^r)y_r$, and thus $\mathrm{gr}ZX(\mathfrak{g}_N) \cong \mathbb{C}[y_{2r}]_{r\in\mathbb{N}}$. Therefore, $\mathrm{gr}(\varphi_W)$ can be identified with the isomorphism

$$\mathbb{C}[x_{2r}]_{r\in\mathbb{N}} \xrightarrow{\sim} \mathbb{C}[y_{2r}]_{r\in\mathbb{N}}, \quad x_{2r} \mapsto 2y_{2r} \quad \forall\, r \in \mathbb{N}. \qquad \square$$

It follows from (3.3.15) that $W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is also generated by $\{w_{2r}\}_{r\in\mathbb{N}}$ and that

$$\mathbb{C}[q_{2r}]_{r\in\mathbb{N}} \cong W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) \cong \mathbb{C}[w_{2r}]_{r\in\mathbb{N}}.$$

Employing this identification and restricting the isomorphism

$$X(\mathfrak{g}_N) \xrightarrow{\sim} ZX(\mathfrak{g}_N) \otimes Y(\mathfrak{g}_N) \cong \mathbb{C}[y_r]_{r\in\mathbb{N}} \otimes Y(\mathfrak{g}_N),$$

of Theorem 2.6.3 to the subalgebra $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, we recover [GR16, Theorem 3.1].

Theorem 3.3.14. *The identification $S(u) = q(u)\mathcal{S}(u)$ induces an isomorphism*

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \cong \mathbb{C}[q_{2r}]_{r\in\mathbb{N}} \otimes Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \cong \mathbb{C}[w_{2r}]_{r\in\mathbb{N}} \otimes Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}.$$

One consequence of this theorem is that we may realize $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ as a quotient of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. Let

$$\epsilon_\vartheta = \epsilon|_{W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)} \otimes \mathrm{id} : X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \twoheadrightarrow Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}, \qquad (3.3.27)$$

where, as usual, ϵ denotes the counit of $X(\mathfrak{g}_N)$. As $\epsilon(q(u)) = \epsilon(w(u)) = 1$, the kernel $\mathrm{Ker}(\epsilon|_{W(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)})$ is generated as an ideal by the coefficients of $q(u)-1$ or, equivalently, by the coefficients of $w(u) - 1$.

Corollary 3.3.15. *The epimorphism ϵ_ϑ induces an isomorphism*

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}/(w(u) - 1) \xrightarrow{\sim} Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}.$$

The same assertion holds with $w(u)$ replaced by $q(u)$.

Remark 3.3.16. In fact, this is how the twisted Yangian $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ was first defined in [GR16].

We have not yet shown that $W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is the whole center $ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. By Theorem 3.3.14, this will be true if $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ has trivial center. This is proven using the argument used to prove Corollary 2.5.6 after replacing the role of [Mol07, Lemma 1.7.4] by [GR16, Proposition 3.4]. The following corollary summarizes this result.

Corollary 3.3.17 ([GR16, Corollary 3.5])**.** *$ZY(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ satisfy*

$$ZY(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} = \mathbb{C} \cdot 1,$$
$$ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} = W(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) \cong \mathbb{C}[q_{2r}]_{r\in\mathbb{N}} \cong \mathbb{C}[w_{2r}]_{r\in\mathbb{N}}.$$

In particular, Theorem 3.3.14 implies that

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \cong ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \otimes Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}.$$

3.3.5 The reflection algebra construction

Though we have been able to prove many fundamental properties of the twisted Yangians $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, we have not yet given a set of defining relations for either algebra in closed form. This was achieved in [GR16, §4–§5] using the reflection equation algebra formalism, which is intimately tied to the theory of twisted Yangians.

In this subsection, we survey this construction. Although the techniques used to develop it are very similar to those we have used in Chapter 2, our exposition will be less complete than in §3.3.2–§3.3.4, where the results consisted almost entirely of corollaries to our earlier work on (extended) Yangians.

We begin by defining the so-called *reflection algebra* associated to the symmetric pairs $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ from (3.0.1). We continue to assume that $\mathcal{G}$ is as in (3.2.11) or (3.2.12).

Definition 3.3.18. The reflection algebra $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}$ is the unital associative $\mathbb{C}$-algebra generated by $\{\mathsf{s}_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$, which are subject to the defining reflection equation

$$\begin{gathered} R(u-v)\mathsf{S}_1(u)R(u+v)\mathsf{S}_2(v) = \mathsf{S}_2(v)R(u+v)\mathsf{S}_1(u)R(u-v) \\ \text{in} \quad \mathrm{End}(\mathbb{C}^N)^{\otimes 2} \otimes \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}[\![u^{\pm 1}, v^{\pm 1}]\!], \end{gathered} \tag{3.3.28}$$

where $\mathsf{S}(u) \in \mathrm{End}(\mathbb{C}^N) \otimes \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}[\![u^{-1}]\!]$ is given by

$$\mathsf{S}(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \mathsf{s}_{ij}(u), \quad \text{with} \quad \mathsf{s}_{ij}(u) = g_{ij} + \sum_{r\geq 1} \mathsf{s}_{ij}^{(r)} u^{-r-1}.$$

By Lemma 3.3.1, $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}$ has a one dimensional representation given by the matrix $\mathcal{G}(u)$ (see (3.3.2)). Equivalently, there is an algebra homomorphism

$$\epsilon_{\mathsf{X}} : \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}} \to \mathbb{C}, \quad \mathsf{S}(u) \mapsto \mathcal{G}(u). \tag{3.3.29}$$

The reflection algebra also admits analogues of the automorphisms (3.3.19) and (3.3.21) of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. Indeed, for any $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, the assignment

$$\nu_g : \mathsf{S}(u) \mapsto g(u)\mathsf{S}(u) \tag{3.3.30}$$

extends to an automorphism $\nu_g \in \mathrm{Aut}(\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}})$. We will see in a short while that these automorphisms induce the automorphisms $\nu_g \in \mathrm{Aut}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$ defined in (3.3.19), which will provide justification for our notation.

Similarly, if $A \in \mathrm{G}(R)$ satisfies $A\mathcal{G}A^{-1} = \mathcal{G}$, then the same type of argument as used to prove Part (1) of Lemma 3.2.4 shows that the assignment

$$\mathsf{Ad}_\vartheta(A) : \mathsf{S}(u) \to A\mathsf{S}(u)A^{-1} \tag{3.3.31}$$

extends to an automorphism $\mathsf{Ad}_\vartheta(A) \in \mathrm{Aut}(\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}})$.

Remark 3.3.19. If $A\mathcal{G}A^{-1}$ is not equal to $\mathcal{G}$, then the assignment (3.3.31) does not preserve the unit. However, $A\mathsf{S}(u)A^t$ will still be a solution of the reflection equation (3.3.28).

The following theorem, which combines Proposition 5.1, Theorem 5.1 and Theorem 5.2 of [GR16], can be viewed as universal version of Lemma 3.3.2. Let $p(u) = p_{\mathcal{G}}(u)$ be as in (3.3.9).

Theorem 3.3.20. *There exists a unique central series*

$$\mathsf{c}(u) = 1 + \sum_{r\geq 1} \mathsf{c}_r u^{-r-1} \in 1 + u^{-1} Z\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}[\![u^{-1}]\!]$$

satisfying the relation

$$Q\mathsf{S}_1(u)R(2u-\kappa)\mathsf{S}_2^{-1}(\kappa-u) = \mathsf{S}_2^{-1}(\kappa-u)R(2u-\kappa)\mathsf{S}_1(u)Q = p(u)\mathsf{c}(u)Q. \quad (3.3.32)$$

Moreover, $\mathsf{c}(u)$ *is uniquely determined by*

$$p(u)\mathsf{c}(u)\mathsf{S}(\kappa-u) = \mathsf{S}^t(u) \mp \frac{\mathsf{S}(u)}{2u-\kappa} + \frac{\mathrm{tr}(\mathsf{S}(u))\cdot I}{2u-2\kappa}. \quad (3.3.33)$$

Proof. The same argument as used to prove Lemma 3.3.2 shows that there exists a unique series $\mathsf{x}(u) \in \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}[\![u^{-1}]\!]$ satisfying

$$Q\mathsf{S}_1(u)R(2u-\kappa)\mathsf{S}_2^{-1}(\kappa-u) = \mathsf{S}_2^{-1}(\kappa-u)R(2u-\kappa)\mathsf{S}_1(u)Q = \mathsf{x}(u)Q.$$

We then define $\mathsf{c}(u) = p(u)^{-1}\mathsf{x}(u)$. Applying the homomorphism ϵ_{X} from (3.3.29) to both sides of the above and appealing to Lemma 3.3.2, we find that $\epsilon_{\mathsf{X}}(\mathsf{c}(u)) = 1$, and hence $\mathsf{c}(u)$ has constant term 1.

The proof that $\mathsf{c}(u)$ has coefficients belonging to the center $Z\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}$ of $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}$ is more complicated, and we refer the reader to [GR16, Theorem 5.1] for complete details.

Consider now the relation (3.3.33). If it holds, then it uniquely determines $\mathsf{c}(u)$ as $\mathsf{S}(\kappa-u)$ is invertible. Let us prove that it indeed holds. By (3.3.32), we have

$$Q\mathsf{S}_1(u)R(2u-\kappa) = p(u)\mathsf{c}(u)Q\mathsf{S}_2(\kappa-u).$$

Using the relations from (3.1.4) to expand the left-hand side, we find that

$$Q\left(\mathsf{S}_2^t(u) \mp \frac{\mathsf{S}_2(u)}{2u-\kappa} + \frac{\mathrm{tr}(\mathsf{S}(u))\cdot I}{2u-2\kappa}\right) = p(u)\mathsf{c}(u)Q\mathsf{S}_2(\kappa-u).$$

Fixing $i, j \in \mathcal{I}_N$ and applying both sides of the above to $e_{-i}\otimes e_j$ gives

$$\theta_{-i}v_Q \otimes \left(\theta_{ij}\mathsf{s}_{-j-i}(u) \mp \frac{\mathsf{s}_{ij}(u)}{2u-\kappa} + \delta_{ij}\frac{\mathrm{tr}(\mathsf{S}(u))}{2u-2\kappa}\right) = \theta_{-i}v_Q \otimes p(u)\mathsf{c}(u)\mathsf{s}_{ij}(\kappa-u),$$

where v_Q and θ_{-i} are as in (3.1.5). As v_Q is nonzero, these relations are equivalent to (3.3.33). □

As a corollary to the above theorem, we obtain a universal version of the relation (3.3.8) from Proposition 3.3.3.

Corollary 3.3.21. *The central series* $\mathsf{c}(u) \in \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}[\![u^{-1}]\!]$ *the equivalent relations*

$$\begin{aligned} \mathsf{c}(u)p(u)\mathrm{tr}(\mathsf{S}(\kappa-u)) &= p_I(u)\mathrm{tr}(\mathsf{S}(u)),\\ \mathsf{c}(u)\mathscr{g}(u)\mathrm{tr}(\mathsf{S}(\kappa-u)) &= \mathscr{g}(\kappa-u)\mathrm{tr}(\mathsf{S}(u)). \end{aligned} \tag{3.3.34}$$

These relations uniquely determine $\mathsf{c}(u)$ *when* $\mathrm{tr}(\mathcal{G}) \neq 0$.

Proof. Taking the trace of both sides of (3.3.33) gives the first relation in (3.3.34). That this is equivalent to the second relation is due to (3.3.8). Since

$$\mathrm{tr}(\mathsf{S}(u)) \in \mathrm{tr}(\mathcal{G}) + u^{-1}\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}[\![u^{-1}]\!],$$

the series $\mathrm{tr}(\mathsf{S}(\kappa-u))$ will be a unit in $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}[\![u^{-1}]\!]$ provided $\mathrm{tr}(\mathcal{G}) \neq 0$, in which case

$$\mathsf{c}(u) = \frac{\mathscr{g}(\kappa-u)}{\mathscr{g}(u)}\cdot\frac{\mathrm{tr}(\mathsf{S}(u))}{\mathrm{tr}(\mathsf{S}(\kappa-u))}.$$ □

We are now in a position to state the main theorem of this subsection, which relates the extended Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ to the reflection algebra $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}$ and, consequently, gives a concrete set of defining relations for the former algebra.

Theorem 3.3.22 ([GR16, Theorem 4.2])**.** *The assignment* $\mathsf{S}(u) \mapsto S(u)$ *extends to* *an epimorphism of algebras*

$$\Phi_\vartheta : \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}} \twoheadrightarrow X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$$

with kernel $\mathrm{Ker}(\Phi_\vartheta) = (\mathsf{c}(u) - 1)$.

Proof. We will not give a complete proof, but we will show $\mathsf{S}(u) \mapsto S(u)$ defines an epimorphism Φ_ϑ which induces a surjection

$$\Phi_\vartheta^{\mathsf{c}} : \mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}/(\mathsf{c}(u) - 1) \twoheadrightarrow X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}. \tag{3.3.35}$$

These results have been established in Lemmas 4.2 and 4.3 of [GR16], though our arguments will differ slightly. To show that there is an epimorphism as in the statement of the theorem, we must show $S(u)$ satisfies the reflection equation (3.3.28).

By (2.7.17), we have

$$T^t(u) = \mathsf{z}(u-\kappa)T(u-\kappa)^{-1}, \tag{3.3.36}$$

and hence, by the centrality of $\mathsf{z}(u)$ and the definition of $S(u)$ (see (3.3.12)), it suffices to prove that

$$\begin{aligned} R(u-v)T_1(\mathring{u})\mathcal{G}_1(u)T_1(\mathring{\bullet u})^{-1}R(u+v)T_2(\mathring{v})\mathcal{G}_2(v)T_2(\mathring{\bullet v})^{-1} \\ = T_2(\mathring{v})\mathcal{G}_2(v)T_2(\mathring{\bullet v})^{-1}R(u+v)T_1(\mathring{u})\mathcal{G}_1(u)T_1(\mathring{\bullet u})^{-1}R(u-v), \end{aligned} \tag{3.3.37}$$

where $\mathring{u} = u - \kappa/2$ and $\mathring{\bullet u} = -u - \kappa/2$. We will make use of the relations

$$\begin{aligned} T_2(v)^{-1}R(u-v)T_1(u) &= T_1(u)R(u-v)T_2(v)^{-1}, \\ T_1(v)^{-1}R(u-v)T_2(u) &= T_2(u)R(u-v)T_1(v)^{-1}, \\ R(u-v)T_2(v)^{-1}T_1(u)^{-1} &= T_1(u)^{-1}T_2(v)^{-1}R(u-v), \\ R(u-v)T_1(v)^{-1}T_2(u)^{-1} &= T_2(u)^{-1}T_1(v)^{-1}R(u-v). \end{aligned} \tag{3.3.38}$$

The first and third relations follow immediately from (2.4.1), while the second and fourth relations follow from the first and third, respectively, after applying the permutation operator $\sigma \otimes 1$ (here $\sigma = \sigma_{\mathbb{C}^N,\mathbb{C}^N}$ in the notation of §2.1).

As $\mathring{v} - \overset{\bullet}{u} = u + v$, the second relation of (3.3.38) implies that

$$
\begin{aligned}
R(u-v)T_1(\mathring{u})\mathcal{G}_1(u)T_1(\overset{\bullet}{u})^{-1}R(u+v)T_2(\mathring{v})\mathcal{G}_2(v)T_2(\overset{\bullet}{v})^{-1}& \\
&= R(u-v)T_1(\mathring{u})T_2(\mathring{v})\mathcal{G}_1(u)R(u+v)\mathcal{G}_2(v)T_1(\overset{\bullet}{u})^{-1}T_2(\overset{\bullet}{v})^{-1} \\
&= T_2(\mathring{v})T_1(\mathring{u})R(u-v)\mathcal{G}_1(u)R(u+v)\mathcal{G}_2(v)T_1(\overset{\bullet}{u})^{-1}T_2(\overset{\bullet}{v})^{-1},
\end{aligned}
$$

where to obtain the last equality we have applied (2.4.1). By Lemma 3.3.1, $\mathcal{G}(u)$ is a solution to the reflection equation. Therefore,

$$
\begin{aligned}
R(u-v)T_1(\mathring{u})\mathcal{G}_1(u)T_1(\overset{\bullet}{u})^{-1}R(u+v)T_2(\mathring{v})\mathcal{G}_2(v)T_2(\overset{\bullet}{v})^{-1}& \\
&= T_2(\mathring{v})T_1(\mathring{u})\mathcal{G}_2(v)R(u+v)\mathcal{G}_1(u)R(u-v)T_1(\overset{\bullet}{u})^{-1}T_2(\overset{\bullet}{v})^{-1} \\
&= T_2(\mathring{v})\mathcal{G}_2(v)T_1(\mathring{u})R(u+v)T_2(\overset{\bullet}{v})^{-1}\mathcal{G}_1(u)T_1(\overset{\bullet}{u})^{-1}R(u-v),
\end{aligned}
$$

where to obtain the last equality we have employed the fourth relation of (3.3.38). A single application of the first relation in (3.3.38) then completes the proof of (3.3.37).

To prove that $\Phi_\vartheta(\mathsf{c}(u)) = 1$, it is enough to show that

$$
QS_1(u)R(2u-\kappa) = p(u)QS_2(\kappa - u). \tag{3.3.39}
$$

Indeed, by the proof of the relation (3.3.33), this will imply that (3.3.33) holds with $\mathsf{c}(u)$ replaced by 1 and $\mathsf{S}(u)$ by $S(u)$.

Since $QT_1(u) = QT_2^t(u)$ (see (3.1.4)), the left-hand side of (3.3.39) is equal to

$$
Q\mathcal{G}_1(u)T_2^t(u-\kappa/2)T_1^t(-u+\kappa/2)R(2u-\kappa) = Q\mathcal{G}_1(u)R(2u-\kappa)T_1^t(-u+\kappa/2)T_2^t(u-\kappa/2),
$$

where we have used (3.3.36) together with the last relation of (3.3.38). By (3.3.5) and (3.1.4), we thus have

$$
\begin{aligned}
QS_1(u)R(2u-\kappa) &= p(u)QT_1^t(-u+\kappa/2)\mathcal{G}_2(\kappa-u)T_2^t(u-\kappa/2) \\
&= p(u)QT_2(-u+\kappa/2)\mathcal{G}_2(\kappa-u)T_2^t(u-\kappa/2) \\
&= p(u)QS_2(\kappa-u).
\end{aligned}
$$

Therefore, $\Phi_\vartheta(\mathsf{c}(u)) = 1$, and we can conclude that Φ_ϑ induces $\Phi_\vartheta^{\mathsf{c}}$ as in (3.3.35).

Now let us provide a partial sketch of the proof that $\Phi_\vartheta^{\mathsf{c}}$ is an isomorphism, using the recurring ideas of Chapter 2. One defines a filtration on $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}$ by assigning filtration degrees

$$\deg \mathsf{s}_{ij}^{(r)} = r-1 \quad \forall \quad r \in \mathbb{N}.$$

This induces a filtration on the quotient $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}/(\mathsf{c}(u)-1)$ such that $\Phi_{\vartheta}^{\mathsf{c}}$ is filtered. It thus suffices to prove that the associated graded morphism

$$\mathrm{gr}(\Phi_{\vartheta}^{\mathsf{c}}) : \mathrm{gr}(\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}/(\mathsf{c}(u)-1)) \to \mathrm{gr} X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw} \cong U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$$

is an isomorphism. This is achieved by composing $\mathrm{gr}(\Phi_{\vartheta}^{\mathsf{c}})$ with an epimorphism

$$U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}}) \twoheadrightarrow \mathrm{gr}(\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{\mathsf{tw}}/(\mathsf{c}(u)-1))$$

and showing that the resulting map is an automorphism of $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$. To construct an epimorphism as above, one needs to identify defining relations for $U(\mathring{\mathfrak{g}}_N[z]^{\check{\vartheta}})$. This can be done in general using a variant of the construction given in §2.3.

In [GR16], this was first done for $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ rather than $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$: see the proof of [GR16, Theorem 4.1]. □

Corollary 3.3.23. *The extended twisted Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is isomorphic to the unital associative $\mathbb{C}$-algebra generated by $\{s_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$, subject only to the relations*

$$R(u-v)S_1(u)R(u+v)S_2(v) = S_2(v)R(u+v)S_1(u)R(u-v), \tag{3.3.40}$$

$$p(u)S(\kappa-u) = S^t(u) \mp \frac{S(u)}{2u-\kappa} + \frac{\mathrm{tr}(S(u))\cdot I}{2u-2\kappa}, \tag{3.3.41}$$

where $S(u) \in \mathrm{End}(\mathbb{C}^N) \otimes X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}[\![u^{-1}]\!]$ is given by

$$S(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes s_{ij}(u), \qquad \text{with} \qquad s_{ij}(u) = g_{ij} + \sum_{r\geq 1} s_{ij}^{(r)} u^{-r-1}.$$

The defining relation (3.3.41) will be called the *symmetry relation* for $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$. It is equivalent to

$$p(u)s_{ij}(\kappa-u) = \theta_{ij}s_{-j,-i}(u) \mp \frac{s_{ij}(u)}{2u-\kappa} + \delta_{ij}\frac{\mathrm{tr}(S(u))}{2u-2\kappa} \quad \forall \quad i,j \in \mathcal{I}_N,$$

with $p(u) = p_{\mathcal{G}}(u)$ given explicitly by (3.3.9).

Using the expansion (2.7.15) of $R(u)$, one can also expand the reflection equation (3.3.40) in terms of the generating series $\{s_{ij}(u)\}_{i,j\in\mathcal{I}_N}$. This yields the following relations for all $i,j,k,l\in\mathcal{I}_N$:

$$
\begin{aligned}
&[s_{ij}(u), s_{kl}(v)]\\
&= \frac{1}{u-v}\Big(s_{kj}(u)\,s_{il}(v) - s_{kj}(v)\,s_{il}(u)\Big)\\
&+ \frac{1}{u+v}\sum_{a\in\mathcal{I}_N}\Big(\delta_{kj}\,s_{ia}(u)\,s_{al}(v) - \delta_{il}\,s_{ka}(v)\,s_{aj}(u)\Big)\\
&- \frac{1}{u^2-v^2}\sum_{a\in\mathcal{I}_N}\delta_{ij}\Big(s_{ka}(u)\,s_{al}(v) - s_{ka}(v)\,s_{al}(u)\Big)\\
&- \frac{1}{u-v-\kappa}\sum_{a\in\mathcal{I}_N}\Big(\delta_{k,-i}\,\theta_{ia}\,s_{aj}(u)\,s_{-a,l}(v) - \delta_{l,-j}\,\theta_{aj}\,s_{k,-a}(v)\,s_{ia}(u)\Big)\\
&- \frac{1}{u+v-\kappa}\Big(\theta_{j,-k}\,s_{i,-k}(u)\,s_{-j,l}(v) - \theta_{i,-l}\,s_{k,-i}(v)\,s_{-l,j}(u)\Big)\\
&+ \frac{\theta_{i,-j}}{(u+v)(u-v-\kappa)}\sum_{a\in\mathcal{I}_N}\Big(\delta_{k,-i}\,s_{-j,a}(u)\,s_{al}(v) - \delta_{l,-j}\,s_{ka}(v)\,s_{a,-i}(u)\Big)\\
&+ \frac{\theta_{i,-j}}{(u-v)(u+v-\kappa)}\Big(s_{k,-i}(u)\,s_{-j,l}(v) - s_{k,-i}(v)\,s_{-j,l}(u)\Big)\\
&- \frac{\theta_{ij}}{(u-v-\kappa)(u+v-\kappa)}\sum_{a\in\mathcal{I}_N}\Big(\delta_{k,-i}\,s_{aa}(u)\,s_{-j,l}(v) - \delta_{l,-j}\,s_{k,-i}(v)\,s_{aa}(u)\Big).
\end{aligned}
\tag{3.3.42}
$$

By Corollary 3.3.15, to obtain defining relations for $Y(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$, one replaces $s_{ij}^{(r)}$, $s_{ij}(u)$, and $S(u)$ by $\sigma_{ij}^{(r)}$, $\sigma_{ij}(u)$ and $\mathcal{S}(u)$, respectively, in Corollary 3.3.23 and imposes the additional *unitary relation*

$$\mathcal{S}(u)\mathcal{S}(-u) = I.$$

As a consequence of Corollary 3.3.21, one can equivalently characterize the symmetry relation (3.3.41) in terms of the trace of $S(u)$ provided $\mathrm{tr}(\mathcal{G})\neq 0$. This is spelled out in the next result.

Corollary 3.3.24. *The symmetry relation* (3.3.41) *implies that*

$$\not{g}(u)\mathrm{tr}(S(\kappa-u)) = \not{g}(\kappa-u)\mathrm{tr}(S(u)).$$

If $\mathrm{tr}(\mathcal{G})\neq 0$*, then this relation is equivalent to* (3.3.41).

Next, as promised, we explain how the automorphism ν_g of (3.3.30) induces the automorphism (3.3.19) of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, following [GR16, Proposition 5.2]. Let $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. By (3.3.32), we have

$$\nu_g(\mathsf{c}(u)) = g(u)g(\kappa - u)^{-1}\mathsf{c}(u). \tag{3.3.43}$$

Consequently, ν_g fixes $\mathsf{c}(u)$ if and only if $g(u) = g(\kappa - u)$. When this is the case, Theorem 3.3.22 implies that ν_g induces an automorphism of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ which is necessarily determined by $S(u) \mapsto g(u)S(u)$, and hence coincides with the automorphism (3.3.19).

The reflection algebra $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}$ has many other interesting properties which we are not able to discuss in detail here. For instance, the subalgebra generated by the coefficients of $c(u)$ is a polynomial algebra $\mathbb{C}[c_{2r-1}]_{r\in\mathbb{N}}$, and there is an isomorphism

$$\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}} \xrightarrow{\sim} \mathbb{C}[c_{2r-1}]_{r\in\mathbb{N}} \otimes X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$$

which identifies $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ with a subalgebra of $\mathsf{X}(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{\mathsf{tw}}$ fixed by a subfamily of the automorphisms ν_g. The reader is referred to §5 of [GR16] for more details.

3.3.6 Equivalent presentations

We now address two potential ambiguities which arise with our definitions of the twisted Yangians $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$.

Suppose first that $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is of the form $(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$, where in addition to our usual assumption that $0 \le q < N$ is an even integer, we assume $p = N - q$ is even and strictly less that N (in particular, $N = 2n$). In this case our definition of the automorphism ϑ (see (3.2.2)) can take as input the matrix $\mathcal{G} = \mathcal{G}_q$ or $\mathcal{G} = \mathcal{G}_p$, where

$$\mathcal{G}_q = I - 2\sum_{a=1}^{q/2}(E_{N+1-a,N+1-a} + E_{a-N-1,a-N-1}),$$
$$\mathcal{G}_p = I - 2\sum_{a=1}^{p/2}(E_{N+1-a,N+1-a} + E_{a-N-1,a-N-1}).$$

Definition 3.3.6 then gives two different extended twisted Yangians associated to the pair $(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$. The first takes as input $\mathcal{G}(u) = \mathcal{G}_q(u)$ and the second is defined using $\mathcal{G}(u) = \mathcal{G}_p(u)$. We follow the convention that $\mathfrak{g}_q$ is always the right summand, so $X(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}$ corresponds to $\mathcal{G}_q(u)$ and $X(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw}$ corresponds to $\mathcal{G}_p(u)$. Such a distinction is only necessary when the underlying involution $\vartheta = \mathrm{Ad}(\mathcal{G})$ is not specified. As

$$\mathfrak{g}_p \oplus \mathfrak{g}_q \cong \mathfrak{g}_q \oplus \mathfrak{g}_p,$$

one would hope that there is an isomorphism

$$X(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw} \xrightarrow{\sim} X(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}$$

which induces an isomorphism between $Y(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}$. Our first goal of this subsection is to construct such an isomorphism explicitly.

For each $a \in \{p, q\}$, let

$$(p(u)_a, \mathscr{g}(u)_a, w(u)_a, S(u)_a)$$

be the tuple $(p_{\mathcal{G}_a}(u), \mathscr{g}(u), w(u), S(u))$ corresponding to $X(\mathfrak{g}_N, \mathfrak{g}_{N-a} \oplus \mathfrak{g}_a)^{tw}$. Set

$$A = \sum_{i=1}^{n} (E_{i,n-i+1} + E_{-i,-n+i-1}).$$

The matrix A belongs to $\mathrm{G}_N(\mathbb{C})$ (see (3.2.6)) and satisfies

$$A\mathcal{G}_q A^{-1} = A\mathcal{G}_q A^t = -\mathcal{G}_p \quad \text{and} \quad A^2 = I.$$

Proposition 3.3.25. *The assignment*

$$\Theta_{p,q} : S(u)_p \mapsto \left(\frac{\mathrm{tr}(\mathcal{G}_q) - 4u}{\mathrm{tr}(\mathcal{G}_q) + 4u}\right) AS(u)_q A^t \tag{3.3.44}$$

extends to an isomorphism

$$\Theta_{p,q} : X(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw} \xrightarrow{\sim} X(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}.$$

Moreover, $\Theta_{p,q}$ induces an isomorphism between $Y(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}$.

Proof. First note that

$$\mathcal{G}_p(u) = \frac{\mathrm{tr}(\mathcal{G}_p)I - 4u\mathcal{G}_p}{\mathrm{tr}(\mathcal{G}_p) - 4u} = -\frac{\mathrm{tr}(\mathcal{G}_q)AA^t - 4uA\mathcal{G}_qA^t}{-\mathrm{tr}(\mathcal{G}_q) - 4u}$$
$$= \left(\frac{\mathrm{tr}(\mathcal{G}_q) - 4u}{\mathrm{tr}(\mathcal{G}_q) + 4u}\right) A\mathcal{G}_q(u)A^t.$$

In particular, the assignment $\Theta_{p,q}$ defined by (3.3.44) preserves the unit.

Since $A \in \mathrm{G}_N(\mathbb{C}) \subset \mathrm{G}(R)$, (3.3.31) and Remark 3.3.19 imply that $AS(u)_qA^{-1}$, and thus $\Theta_{p,q}(S(u)_p)$, satisfies the reflection equation (3.3.40).

Next, note that

$$\frac{\mathrm{tr}(\mathcal{G}_q) - 4u}{\mathrm{tr}(\mathcal{G}_q) + 4u} = -\frac{\mathscr{g}(u)_p}{\mathscr{g}(u)_q}$$

and consider the symmetry relation (3.3.41). We have

$$p(u)_qAS(\kappa - u)_qA^t = AS^t(u)_qA^t \mp \frac{AS(u)_qA^t}{2u-\kappa} + \frac{\mathrm{tr}(AS(u)_qA^t)\cdot I}{2u - 2\kappa}$$

It follows that $\Theta_{p,q}(S(u)_p)$ will satisfy (3.3.41) provided

$$p(u)_q\mathscr{g}(u)_p\mathscr{g}(u)_q^{-1} = p(u)_p\mathscr{g}(\kappa - u)_p\mathscr{g}(\kappa - u)_q^{-1}.$$

This relation is a consequence of the relation (3.3.8) of Proposition 3.3.3. Therefore, (3.3.44) defines a homomorphism $\Theta_{p,q}$. It is easily seen to be an isomorphism: in fact, it satisfies

$$\Theta_{q,p} = \Theta_{p,q}^{-1}.$$

Since $\Theta_{p,q}$ also satisfies

$$\Theta_{p,q}(w(u)_pI) = \Theta_{p,q}(S(u)_pS(-u)_p) = AS(u)S(-u)A^t = w(u)_qI,$$

Corollary 3.3.15 implies that it induces an isomorphism between $Y(\mathfrak{g}_N, \mathfrak{g}_q \oplus \mathfrak{g}_p)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)^{tw}$. □

Remark 3.3.26. Proposition 3.3.25 provides justification for the restriction $p \geq q$ which was imposed in [GR16, GRW17, GRW19b]. We will, however, not impose this condition until Chapter 5, as it will not have any bearing on our results until that point. One reason for working in this generality is that the restriction of scalars functor $\Theta^*_{p,q}$ does not commute with the notion of highest weight module to be defined in Chapter 4, but instead interchanges the role of highest and lowest weight theory.

Let us now consider instead those symmetric pairs which are of type BI. That is,

$$(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q),$$

where $p = N - q = 2n + 1 - q$ is necessarily odd. In [GRW17, GRW19b], the type BI classification was refined into

$$\begin{aligned} \mathrm{BI}(a) &: (\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q) \ \text{ with } \ p > q, \\ \mathrm{BI}(b) &: (\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q) \ \text{ with } \ p < q. \end{aligned}$$

In the latter case, the underlying matrix $\mathcal{G}$ used to define the pair was replaced with $-\mathcal{G}$, where $\mathcal{G}$ is as in (3.2.12). That is,

$$\mathcal{G}_{\mathrm{GRW}} = -\mathcal{G} = I - 2 \sum_{a \in \mathcal{I}_p} E_{aa}.$$

Such a refinement is in fact not necessary, and the definition given in this thesis leads to a more uniform theory. However, we should still show that the definition given here is consistent with that from [GRW17, GRW19b]. In other words, we should establish an isomorphism

$$X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw} \cong X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw}_{\mathrm{GRW}} \tag{3.3.45}$$

whenever $p < q$, where $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw}_{\mathrm{GRW}}$ is defined by Definition 3.3.6 with $\mathcal{G}(u)$ replaced by

$$\mathcal{G}(u)_{\mathrm{GRW}} = \frac{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}})I - 4u\mathcal{G}_{\mathrm{GRW}}}{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}}) - 4u} = \left(\frac{\mathrm{tr}(\mathcal{G}) - 4u}{\mathrm{tr}(\mathcal{G}) + 4u}\right) \mathcal{G}(u).$$

Let $S(u)_{\mathrm{GRW}}$ be the generating matrix $S(u)$ corresponding to $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw}_{\mathrm{GRW}}$. We then have the following proposition.

Proposition 3.3.27. *Suppose $p < q$. Then the assignment*

$$\Theta_q^{\mathrm{BI}(b)} : S(u) \mapsto \left(\frac{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}}) - 4u}{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}}) + 4u}\right) S(u)_{\mathrm{GRW}}$$

extends to an isomorphism

$$\Theta_q^{\mathrm{BI}(b)} : X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw} \xrightarrow{\sim} X(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw}_{\mathrm{GRW}}.$$

In addition, $\Theta_q^{\mathrm{BI}(b)}$ induces an isomorphism

$$Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw} \cong Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_p \oplus \mathfrak{so}_q)^{tw}_{\mathrm{GRW}}.$$

Proof. The proposition is proven in the same way as Proposition 3.3.25, verbatim. □

3.4 Twisted Yangians of type A

We conclude this chapter by providing an unfairly brief introduction to twisted Yangians associated to symmetric pairs of type A. These pairs take the form

$$\begin{gathered} \mathrm{A0} : (\mathfrak{sl}_N, \mathfrak{sl}_N), \\ \mathrm{AIII} : (\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q) \cong (\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathbb{C} \oplus \mathfrak{sl}_q) \cong (\mathfrak{sl}_N, \mathfrak{gl}_p \oplus \mathfrak{sl}_q), \\ \mathrm{AI} : (\mathfrak{sl}_N, \mathfrak{so}_N) \quad \text{and} \quad \mathrm{AII} : (\mathfrak{sl}_N, \mathfrak{sp}_N), \end{gathered}$$

where $0 < q < N$ and $p = N - q$. It will be convenient for us to enlarge the AIII family as

$$\mathrm{AIII} : (\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q) \quad \text{with} \quad 0 \leq q < N,$$

so that the special case $q = 0$ encodes the trivial pairs $(\mathfrak{sl}_N, \mathfrak{sl}_N)$ of type A0.

The pairs $(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)$ can be explicitly realized using the construction of §3.2: one takes $\vartheta = \mathrm{Ad}(\mathcal{G}) = \mathrm{Ad}_\rho(\mathcal{G})|_{\mathfrak{sl}_N}$ (as in (3.2.2)) with

$$\mathcal{G} = 2\sum_{a=1}^{p} E_{aa} - I = I - 2\sum_{a=1}^{q} E_{N-a+1,N-a+1}, \tag{3.4.1}$$

where in this case we replace $\mathcal{I}_N$ by $\{1, \dots, N\}$.

We have already seen the symmetric pairs $(\mathfrak{sl}_N, \mathfrak{g}_N)$ of AI and AII realized using the involution θ introduced in (2.7.10). Indeed, this is precisely our fixed realization of $\mathfrak{g}_N$: see §3.1.1. We note that θ is not an inner automorphism, and thus not equal to an involution of the form (3.2.2).

In what follows, we will write $Y(\mathfrak{gl}_N)$ for $X(\mathfrak{gl}_N)$, as is the convention in the literature: see §2.7.2.1 and §3.1.1. In this section alone, $R(u)$ will always denote the rational R-matrix (2.7.4).

3.4.1 Twisted Yangians associated to $(\mathfrak{sl}_N, \mathfrak{g}_N)$

Fix $N \geq 2$ if $\mathfrak{g}_N = \mathfrak{sp}_N$ and $N \geq 3$ if $\mathfrak{g} = \mathfrak{so}_N$. The definitions and results given below are borrowed from [Mol07], though we shall proceed in an order consistent with the previous sections.

Definition 3.4.1. Let t be the transpose (2.7.9). Then:

(1) The *extended twisted Yangian* $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ is the subalgebra of $Y(\mathfrak{gl}_N)$ generated by the coefficients $\{s_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$ of

$$S(u) = T(u)T^t(-u) \in \mathrm{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[\![u^{-1}]\!],$$
$$\text{where} \quad S(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes s_{ij}(u) \quad \text{and} \quad s_{ij}(u) = \delta_{ij} + \sum_{r\geq 1} s_{ij}^{(r)} u^{-r}.$$

(2) The *twisted Yangian* $Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ is the subalgebra of $Y(\mathfrak{sl}_N)$ generated by the coefficients $\{\sigma_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$ of

$$\mathcal{S}(u) = \mathcal{T}(u)\mathcal{T}^t(-u) \in \mathrm{End}(\mathbb{C}^N) \otimes Y(\mathfrak{sl}_N)[\![u^{-1}]\!],$$
$$\text{where} \quad \mathcal{S}(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \sigma_{ij}(u) \quad \text{and} \quad \sigma_{ij}(u) = \delta_{ij} + \sum_{r\geq 1} \sigma_{ij}^{(r)} u^{-r}.$$

The center of the twisted Yangian $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ can be described using the *Sklyanin determinant* $\mathrm{sdet} S(u)$, which serves as a twisted Yangian analogue of the quantum determinant $\mathrm{qdet} T(u)$ given by (2.7.7). To define it, we introduce the scalar series

$\alpha_N(u)$ by

$$\alpha_N(u) = \begin{cases} 1 & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N, \\ \dfrac{2u+1}{2u-N+1} & \text{if } \mathfrak{g}_N = \mathfrak{so}_N. \end{cases}$$

The series $\mathrm{sdet} S(u)$ is then given by

$$\mathrm{sdet} S(u) = \alpha_N(u)\mathrm{qdet} T(u)\mathrm{qdet} T(-u+N-1) \in ZY(\mathfrak{gl}_N)[\![u^{-1}]\!] \tag{3.4.2}$$

By Proposition 2.5.1 and Theorem 2.5.3 of [Mol07], the coefficients of $\mathrm{sdet} S(u)$ belong to $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$, and thus to its center $ZX(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$.

Now recall from §2.7.2.1 that the central series $y(u)$ from (2.7.1) is equal to the series $\widetilde{d}(u)$ uniquely determined by (2.7.6). Set

$$\mathring{q}(u) = y(u)y(-u) = 1 + \sum_{r \geq 1} \mathring{q}_{2r} u^{-2r} \in 1 + u^{-1} ZY(\mathfrak{gl}_N)[\![u^{-1}]\!].$$

Then $\mathring{q}(u)$ is the unique solution of

$$\alpha_N(u)^{-1}\mathrm{sdet} S(u) = \mathring{q}(u)\mathring{q}(u-1)\cdots\mathring{q}(u-N+1)$$

in $1 + u^{-1}ZY(\mathfrak{gl}_N)[\![u^{-1}]\!]$, and it thus has coefficients in $ZX(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$. As

$$S(u) = \mathring{q}(u)\mathcal{S}(u),$$

the natural embedding $Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \hookrightarrow Y(\mathfrak{gl}_N)$ does in fact have image contained in $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$. Analogously to Theorem 3.3.14, restricting the isomorphism of Theorem 2.6.3 leads to a tensor decomposition

$$X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \cong ZX(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \otimes Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \cong \mathbb{C}[\mathring{q}_{2r}]_{r\in\mathbb{N}} \otimes Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw},$$

which in turn implies that

$$Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \cong X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}/(\mathring{q}(u)-1) \cong X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}/(\alpha_N(u)^{-1}\mathrm{sdet} S(u) - 1).$$

The twisted Yangian $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ may also be reconstructed using the reflection algebra formalism; however, one must alter the underlying reflection equation. The following theorem, which is a consequence of [Mol07, Theorem 2.4.3] and [Mol07, Proposition 2.15.1], provides the desired presentation.

Theorem 3.4.2. *The extended twisted Yangian $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ is isomorphic to the unital associative $\mathbb{C}$-algebra generated by $\{s_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$, subject only to the relations*

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v), \tag{3.4.3}$$

$$S^t(u) = S(-u) \pm \frac{S(u)-S(-u)}{2u}, \tag{3.4.4}$$

where $S(u) \in \mathrm{End}(\mathbb{C}^N) \otimes X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}[\![u^{-1}]\!]$ is given by

$$S(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes s_{ij}(u), \quad \text{with} \quad s_{ij}(u) = \delta_{ij} + \sum_{r\geq 1} s_{ij}^{(r)} u^{-r-1}.$$

The relation (3.4.4) is called the *symmetry relation* for $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$. The defining reflection equation (3.4.3) may be expanded explicitly as

$$\begin{aligned}
[\, s_{ij}(u), s_{kl}(v)] = {} & \frac{1}{u-v}\Big(s_{kj}(u)\, s_{il}(v) - s_{kj}(v)\, s_{il}(u)\Big) \\
& - \frac{1}{u+v}\Big(\theta_{k,-j}\, s_{i,-k}(u)\, s_{-j,l}(v) - \theta_{i,-l}\, s_{k,-i}(v)\, s_{-l,j}(u)\Big) \\
& + \frac{1}{u^2-v^2}\,\theta_{i,-j}\Big(s_{k,-i}(u)\, s_{-j,l}(v) - s_{k,-i}(v)\, s_{-j,l}(u)\Big),
\end{aligned} \tag{3.4.5}$$

where i, j, k, l take values in $\mathcal{I}_N$.

A truly remarkable property of the twisted Yangians $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ is that they admit *evaluation homomorphisms* onto the enveloping algebra $U(\mathfrak{g}_N)$. These are defined in the following proposition.

Proposition 3.4.3 ([Mol07, Proposition 2.1.2])**.** *The assignment*

$$\mathrm{ev} : s_{ij}(u) \mapsto \delta_{ij} + F_{ij}\left(u \pm \frac{1}{2}\right)^{-1} \quad \forall \quad i,j \in \mathcal{I}_N$$

extends to an algebra epimorphism $\mathrm{ev} : X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw} \twoheadrightarrow U(\mathfrak{g}_N)$*, called the evaluation homomorphism. Additionally, the assignment*

$$F_{ij} \mapsto s_{ij}^{(1)} \quad \forall \quad i,j \in \mathcal{I}_N$$

extends to an algebra embedding $\iota : U(\mathfrak{g}_N) \hookrightarrow X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ *which satisfies*

$$\mathrm{ev} \circ \iota = \mathrm{id}_{U(\mathfrak{g}_N)}.$$

For the above formulation of [Mol07, Proposition 2.1.2], we refer the reader to [Mol07, (2.106)].

The algebras $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ and $Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ were introduced by G. Olshanski in the 1991 paper [Ols92], where they were denoted $Y^{\pm}(N)$ and $[Y^{\pm}(N)]$, respectively. In [Mol07], they are denoted $Y(\mathfrak{g}_N)$ and $SY(\mathfrak{g}_N)$. It was in [Ols92] where the terminology *twisted Yangian* was first used. In this subsection, we have barely even scratched the surface of Olshanski's original work. Since then, there have been extensive advances – including several interesting applications of the evaluation homomorphisms of Proposition 3.4.3. We refer to the reader to [Mol07] for a more satisfying exposition.

3.4.2 Twisted Yangians associated to $(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)$

We now fix $N \geq 2$ and replace $\mathcal{I}_N$ by the indexing set $\{1, \dots, N\}$. In addition, we fix an integer $0 \leq q \leq N$ and set $p = N - q$.

Definition 3.4.4. Let $\mathcal{G} = \sum_{i,j=1}^{N} g_{ij} E_{ij}$ be given by (3.4.1). Then:

(1) The *extended twisted Yangian* $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ is the subalgebra of $Y(\mathfrak{gl}_N)$ generated by the coefficients $\{b_{ij}^{(r)}\}_{1 \leq i,j \leq N, r \in \mathbb{N}}$ of

$$B(u) = T(u)\mathcal{G}T(-u)^{-1} \in \mathrm{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[\![u^{-1}]\!],$$
$$\text{where } \; B(u) = \sum_{i,j=1}^{N} E_{ij} \otimes b_{ij}(u) \quad \text{and} \quad b_{ij}(u) = g_{ij} + \sum_{r \geq 1} b_{ij}^{(r)} u^{-r}.$$

(2) The *twisted Yangian* $Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ is the subalgebra of $Y(\mathfrak{sl}_N)$ generated by the coefficients $\{\sigma_{ij}^{(r)}\}_{1 \leq i,j \leq N, r \in \mathbb{N}}$ of

$$\mathcal{B}(u) = \mathcal{T}(u)\mathcal{G}\mathcal{T}(-u)^{-1} \in \mathrm{End}(\mathbb{C}^N) \otimes Y(\mathfrak{sl}_N)[\![u^{-1}]\!],$$
$$\text{where } \; \mathcal{B}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes \sigma_{ij}(u) \quad \text{and} \quad \sigma_{ij}(u) = g_{ij} + \sum_{r \geq 1} \sigma_{ij}^{(r)} u^{-r}.$$

Remark 3.4.5. We have included the value $q = N$ in order to simplify our lives in §4.4. However, in this case our notation becomes a bit misleading as the underlying symmetric pair is of course not actually $(\mathfrak{sl}_N, \mathfrak{gl}_N)$, but rather $(\mathfrak{sl}_N, \mathfrak{sl}_N)$. This will not cause any confusion; after all, $X(\mathfrak{sl}_N, \mathfrak{sl}_N)^{tw}$ and $X(\mathfrak{sl}_N, \mathfrak{gl}_N)^{tw}$ as defined above are equal as subalgebras of $Y(\mathfrak{gl}_N)$, with the generating matrix $B(u)$ of both equal up to a negative sign. The same assertion holds for $Y(\mathfrak{sl}_N, \mathfrak{sl}_N)^{tw}$ and $Y(\mathfrak{sl}_N, \mathfrak{gl}_N)^{tw}$.

Like in the $(\mathfrak{sl}_N, \mathfrak{g}_N)$ case, there is a Sklyanin determinant $\mathrm{sdet}B(u)$ which may be used to describe the center $ZX(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ of $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$. We will not take this approach here, and instead refer the reader to [MR02, Roz10].

Recall that $y(u) = \widetilde{d}(u)$, where $\widetilde{d}(u)$ is as in (2.7.6), and define

$$\mathring{q}(u) = y(u)y(-u)^{-1} = 1 + \sum_{r \geq 1} \mathring{q}_r u^{-r} \in 1 + u^{-1} ZY(\mathfrak{gl}_N)[\![u^{-1}]\!].$$

It is not immediate from this definition that the coefficients of $\mathring{q}(u)$ actually belong to $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$. However, they do generate a subalgebra of

$$ZY(\mathfrak{gl}_N) \cong \mathbb{C}[y_r]_{r \in \mathbb{N}}$$

isomorphic to the polynomial ring $\mathbb{C}[\mathring{q}_{2r-1}]_{r \in \mathbb{N}}$ in the odd coefficients of $q(u)$. Indeed, this is just a consequence of the fact that

$$\mathring{q}_r \equiv y_r - (-1)^r y_r \mod \mathbf{F}_{r-2}(Y(\mathfrak{gl}_N)).$$

Moreover, $\mathring{q}(u)$ satisfies the relation

$$B(u) = \mathring{q}(u)\mathcal{B}(u).$$

Restricting the isomorphism of Theorem 2.6.3 to $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ thus gives an injection

$$X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw} \hookrightarrow \mathbb{C}[\mathring{q}_{2r-1}]_{r \in \mathbb{N}} \otimes Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw},$$

which is easily seen to be surjective by considering the associated graded map. In particular, the coefficients of $\mathring{q}(u)$ do belong to (the center of) $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ and the proof of Corollary 3.3.15 shows that

$$Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw} \cong X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}/(\mathring{q}(u) - 1).$$

It was argued in the proof of [MR02, Theorem 3.4] that $Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ has trivial center, and hence we also have

$$X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw} \cong ZY(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw} \otimes Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw},$$
$$\text{and} \quad ZY(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw} \cong \mathbb{C}[\mathring{q}_{2r-1}]_{r\in\mathbb{N}}.$$

Now let us turn to the reflection algebra formalism for $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$.

Definition 3.4.6. The reflection algebra $\mathsf{X}(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{\mathsf{tw}}$ is the unital associative $\mathbb{C}$-algebra generated by $\{\mathsf{b}_{ij}^{(r)}\}_{i,j\in\mathcal{I}_N, r\in\mathbb{N}}$, which are subject to the defining reflection equation

$$\begin{gathered} R(u-v)\mathsf{B}_1(u)R(u+v)\mathsf{B}_2(v) = \mathsf{B}_2(v)R(u+v)\mathsf{B}_1(u)R(u-v) \\ \text{in} \quad \mathrm{End}(\mathbb{C}^N)^{\otimes 2} \otimes \mathsf{X}(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{\mathsf{tw}}[\![u^{\pm 1}, v^{\pm 1}]\!], \end{gathered} \tag{3.4.6}$$

where $\mathsf{B}(u) \in \mathrm{End}(\mathbb{C}^N) \otimes \mathsf{X}(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{\mathsf{tw}}[\![u^{-1}]\!]$ is given by

$$\mathsf{B}(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes \mathsf{b}_{ij}(u), \quad \text{with} \quad \mathsf{b}_{ij}(u) = g_{ij} + \sum_{r\geq 1} \mathsf{b}_{ij}^{(r)} u^{-r-1}.$$

By expanding $R(u)$ in (3.4.6), we find that it is equivalent to

$$\begin{aligned} [\mathsf{b}_{ij}(u), \mathsf{b}_{kl}(v)] \equiv & \frac{1}{u-v}\Big(\mathsf{b}_{kj}(u)\mathsf{b}_{i\ell}(v) - \mathsf{b}_{kj}(v)\mathsf{b}_{il}(u)\Big) \\ & + \frac{1}{u+v}\sum_{a=1}^{n}\Big(\delta_{kj}\mathsf{b}_{ia}(u)\mathsf{b}_{al}(v) - \delta_{il}\mathsf{b}_{ka}(v)\mathsf{b}_{aj}(u)\Big) \\ & - \frac{1}{u^2-v^2}\sum_{a=1}^{n}\delta_{ij}\Big(\mathsf{b}_{ka}(u)\mathsf{b}_{al}(v) - \mathsf{b}_{ka}(v)\mathsf{b}_{al}(u)\Big), \end{aligned} \tag{3.4.7}$$

where i, j, k, l take values in $\{1, \ldots, N\}$. In Proposition 2.1 of [MR02], it was shown that there exists an even central series

$$\mathsf{f}(u) \in 1 + u^{-1}\mathsf{X}(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{\mathsf{tw}}[\![u^{-1}]\!]$$

uniquely determined by

$$\mathsf{B}(u)\mathsf{B}(-u) = \mathsf{f}(u)I.$$

One then has the following analogue of Theorem 3.3.22 and Corollary 3.3.23, which is equivalent to [MR02, Theorem 3.1].

Theorem 3.4.7. *The assignment* $\mathsf{B}(u) \mapsto B(u)$ *extends to an epimorphism of algebras*

$$\mathsf{X}(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{\mathsf{tw}} \twoheadrightarrow X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$$

with kernel equal to $(\mathsf{f}(u) - 1)$. *Consequently,* $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ *isomorphic to the unital associative* $\mathbb{C}$*-algebra generated by* $\{b_{ij}^{(r)}\}_{1\leq i,j\leq N, r\in\mathbb{N}}$*, subject only to the relations*

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v),$$
$$B(u)B(-u) = I,$$

where $B(u) \in \mathrm{End}(\mathbb{C}^N) \otimes X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}[\![u^{-1}]\!]$ *is given by*

$$B(u) = \sum_{i,j\in\mathcal{I}_N} E_{ij} \otimes b_{ij}(u), \qquad \textit{with} \qquad b_{ij}(u) = g_{ij} + \sum_{r\geq 1} b_{ij}^{(r)} u^{-r-1}.$$

Reflection algebras of type AIII were introduced by E. Sklyanin in the 1988 paper [Skl88], while the twisted Yangians $X(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ and $Y(\mathfrak{sl}_N, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)^{tw}$ were introduced by A. Molev and E. Ragoucy in [MR02], where they were denoted $\mathcal{B}(N,q)$ and $\mathcal{SB}(N,q)$, respectively. This notation was also used in [GRW16, GRW17, GRW19b], where $\mathcal{B}(N,q)$ was called the *Molev-Ragoucy reflection algebra*. They have been less studied then their type AI and AII counterparts, but will play a very important role in Chapters 4 and 5 below.

Chapter 4

Highest Weight Theory for Twisted Yangians

We now turn to studying the representation theory for the twisted Yangians of types B, C and D introduced in §3.3, with a emphasis on the finite-dimensional irreducible representations of the extended twisted Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$[1]. Our goal in this chapter is to lay the foundations needed to classify the finite-dimensional irreducible representations of both $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ – a classification which will be partially achieved in Chapter 5.

The first ingredient needed is a highest weight theory for $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ compatible with its finite-dimensional irreducible modules. Such a theory is developed in §4.2 using a familiar approach. In §4.2.2, we give the relevant definitions and first results, including a proof that every finite-dimensional irreducible module is of highest weight type: see Theorem 4.2.6. In §4.2.3, we introduce Verma modules and study the compatibility of our notion of highest weight module with the coideal structure of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$: see Proposition 4.2.11 and Corollary 4.2.12.

In §4.3 and §4.4, we construct two other tools instrumental to the study of finite-dimensional irreducible $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules. The first of these tools, developed in §4.3, addresses the problem that there is no obvious embedding

$$X(\mathfrak{g}_{N-2}, \mathfrak{g}_{N-2}^{\vartheta(2)})^{tw} \hookrightarrow X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}.$$

1) The emphasis on $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, rather than $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, will be explained in §4.2.1.

(Here our notation is as in §4.3.) Our solution to this problem, which is inspired by a result of [MR02], is to show that one can still build a $X(\mathfrak{g}_{N-2},\mathfrak{g}_{N-2}^{\vartheta(2)})^{tw}$-module structure on a subspace of any $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$-module V, in a manner which is compatible with highest weight theory: see Proposition 4.3.4 and Corollary 4.3.7.

The tool constructed in §4.4 is similar, but the role of $X(\mathfrak{g}_{N-2},\mathfrak{g}_{N-2}^{\vartheta(2)})^{tw}$ is played instead by a twisted Yangian $X(\mathfrak{sl}_n,\mathfrak{sl}_k\oplus\mathfrak{gl}_\ell)^{tw}$ of type AIII: see Proposition 4.4.1. We use this tool to prove Theorem 4.4.4, which characterizes exactly when a $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ Verma module is non-trivial. In addition, it will allow us to associate a $(n-1)$-tuple of polynomials $(P_i(u))_{i=2}^n$, together with a scalar α, to any finite-dimensional irreducible module: see Proposition 4.4.5.

Our approach to the representation theory of $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ depends, in many ways, on the representation theory of the extended Yangian $X(\mathfrak{g}_N)$, which has been studied by D. Arnaudon, A. Molev and E. Ragoucy in [AMR06]. For this reason, we have included an exposition to their results in the first section of this chapter.

4.1 Representations of $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$

Let us begin by recalling some of the results developed in [AMR06] for the extended Yangians $X(\mathfrak{g}_N)$ and the Yangian $Y(\mathfrak{g}_N)$.

4.1.1 Representations of $X(\mathfrak{g}_N)$

A representation V of $X(\mathfrak{g}_N)$ is a *highest weight representation* if there exists a nonzero vector $\xi\in V$ such that $V=X(\mathfrak{g}_N)\,\xi$ and

$$\begin{aligned} t_{ij}(u)\xi &= 0 \qquad \forall \quad i<j\in\mathcal{I}_N,\\ t_{ii}(u)\xi &= \lambda_i(u)\xi \qquad \forall \quad i\in\mathcal{I}_N, \end{aligned}$$

where, for each $i\in\mathcal{I}_N$, $\lambda_i(u)$ is a formal power series in $\mathbb{C}[\![u^{-1}]\!]$ of the form

$$\lambda_i(u)=1+\sum_{r=1}^{\infty}\lambda_i^{(r)}u^{-r},\quad \lambda_i^{(r)}\in\mathbb{C}.$$

The vector ξ is called the *highest weight vector* of V and the N-tuple $\lambda(u) = (\lambda_i(u))_{i\in\mathcal{I}_N}$ is called the *highest weight* of V.

Theorem 4.1.1 ([AMR06, Theorem 5.1]). *Every finite-dimensional irreducible representation V of $X(\mathfrak{g}_N)$ is a highest weight representation. Moreover, V contains a unique highest weight vector, up to multiplication by a nonzero scalar.*

Given an N-tuple $\lambda(u)$, the Verma module $M(\lambda(u))$ is defined as the quotient of $X(\mathfrak{g}_N)$ by the left ideal generated by all the coefficients of the series

$$\begin{aligned} t_{ij}(u) \quad &\text{with} \quad i < j \in \mathcal{I}_N, \\ t_{ii}(u) - \lambda_i(u) \quad &\text{with} \quad i \in \mathcal{I}_N. \end{aligned}$$

It is not the case that $M(\lambda(u))$ is always non-trivial. However, there is a precise classification of when this occurs due to Arnaudon, Molev and Ragoucy.

Proposition 4.1.2 ([AMR06, Proposition 5.14]). *$M(\lambda(u))$ is non-trivial if and only if the components of the highest weight satisfy*

$$\frac{\lambda_{-i}(u)}{\lambda_{-i-1}(u)} = \frac{\lambda_{i+1}(u-\kappa+n-i)}{\lambda_i(u-\kappa+n-i)} \quad \textit{for} \quad i \in \mathcal{I}_N^+ \setminus \{n\}. \tag{4.1.1}$$

The following simple lemma from [GRW17] will play an important role in establishing the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-analogue of the above result in §4.4.

Lemma 4.1.3. *Fix a tuple $(\lambda_i(u))_{i\in\mathcal{I}_N^+}$ with $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ for each $i \in \mathcal{I}_N^+$, and let $\nu(u)$ be any series of the same form. Then*

(1) If $N = 2n+1$, then there is a unique N-tuple $\lambda(u)$ extending $(\lambda_i(u))_{i\in\mathcal{I}_N^+}$ with the property that the $X(\mathfrak{g}_N)$-module $M(\lambda(u))$ is non-trivial.

(2) If $N = 2n$, then for each fixed $k \in \mathcal{I}_N^+$ there exists a unique N-tuple $\lambda(u)$ extending $(\lambda_i(u))_{i\in\mathcal{I}_N^+}$ with the property that $\lambda_{-k}(u) = \nu(u)$ and the $X(\mathfrak{g}_N)$-module $M(\lambda(u))$ is non-trivial.

Proof. Suppose first that $N = 2n+1$. The condition (4.1.1) forces us to define

$$\lambda_{-1}(u) = \frac{\lambda_0(u-\kappa+n)}{\lambda_1(u-\kappa+n)}\lambda_0(u)$$

and recursively

$$\lambda_{-i-1}(u) = \frac{\lambda_i(u-\kappa+n-i)}{\lambda_{i+1}(u-\kappa+n-i)}\lambda_{-i}(u) \quad \forall \quad 1 \le i \le n-1.$$

In this way we can associate a unique N-tuple $\lambda(u)$ to $(\lambda(u))_{i\in\mathcal{I}_N^+}$ satisfying the claimed properties.

If instead $N = 2n$, then the condition (4.1.1) alone no longer uniquely determines an N-tuple $\lambda(u)$ from $(\lambda(u))_{i\in\mathcal{I}_N^+}$. However, fixing $k \in \mathcal{I}_N^+$ and setting $\lambda_{-k}(u) = \nu(u)$, a simple modification of the argument used in the $N = 2n+1$ case shows that the condition (4.1.1) does produce a unique $2n$-tuple with the desired properties. □

If $M(\lambda(u))$ is non-trivial, then it has a unique irreducible (non-zero) quotient $L(\lambda(u))$, and any irreducible highest weight $X(\mathfrak{g}_N)$-module with the highest weight $\lambda(u)$ is isomorphic to $L(\lambda(u))$. In particular, by Theorem 4.1.1 above, every finite-dimensional irreducible module is isomorphic to a module of this form.

Theorem 4.1.4 ([AMR06, Theorem 5.16]). *Let $\lambda(u)$ satisfy (4.1.1), so the Verma module $M(\lambda(u))$ is non-trivial. Then the irreducible $X(\mathfrak{g}_N)$-module $L(\lambda(u))$ is finite-dimensional if and only if there exist monic polynomials $P_1(u), \dots, P_n(u)$ in u such that*

$$\frac{\lambda_{i-1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)} \quad \textit{for all} \quad 2 \le i \le n,$$

and in addition

$$\begin{aligned} \frac{\lambda_0(u)}{\lambda_1(u)} &= \frac{P_1(u+1/2)}{P_1(u)} \quad \textit{if} \quad \mathfrak{g}_N = \mathfrak{so}_{2n+1}, \\ \frac{\lambda_{-1}(u)}{\lambda_1(u)} &= \frac{P_1(u+2)}{P_1(u)} \quad \textit{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}, \\ \frac{\lambda_{-1}(u)}{\lambda_2(u)} &= \frac{P_1(u+1)}{P_1(u)} \quad \textit{if} \quad \mathfrak{g}_N = \mathfrak{so}_{2n}. \end{aligned}$$

The polynomials $P_1(u), \dots, P_n(u)$ are called the *Drinfeld polynomials* associated to $L(\lambda(u))$: they are uniquely determined by the highest weight $\lambda(u)$.

Theorem 4.1.4 gives rise to an elegant parameterization of the set of isomorphism classes of finite-dimensional irreducible $X(\mathfrak{g}_N)$-modules. To make this precise, let us introduce the notation

$$\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N)) \quad \text{and} \quad \mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N))$$

for the set of isomorphism classes of finite-dimensional irreducible $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$ modules, respectively. In general, we shall write $[V]$ to denote the isomorphism class of a module V. However, we will drop the brackets when making use of the natural identification

$$\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N)) = \{L(\lambda(u)) \,:\, \dim L(\lambda(u)) < \infty\}$$ [2]

in order to emphasize $L(\lambda(u)) \ncong L(\lambda^\sharp(u))$ for $\lambda(u) \neq \lambda^\sharp(u)$.

We may now define a function

$$\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N)) \to \{(P_i(u))_{i=1}^n \in \mathbb{C}[u]^n \,:\, P_i(u) \ \text{monic}\} \tag{4.1.2}$$

which assigns to $L(\lambda(u))$ the Drinfeld polynomials $(P_i(u))_{i=1}^n$ furnished by Theorem 4.1.4.

It is not difficult to show (using Lemma 4.1.3, for instance) that this is a surjective function. It is not, however, injective: Two finite-dimensional irreducible modules $L(\lambda(u))$ and $L(\lambda^\sharp(u))$ share the same n-tuple of Drinfeld polynomials if and only if there is $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$L(\lambda^\sharp(u)) = m_f^*(L(\lambda(u))),$$

where m_f is the automorphism (2.7.3) (see also (2.4.4)).

The next proposition explains how to modify (4.1.2) to account for this observation. Recall that $y(u) \in ZX(\mathfrak{g}_N)[\![u^{-1}]\!]$ is the central series defined in (2.7.1) (see also (2.7.18)).

Proposition 4.1.5. *The isomorphism classes of finite-dimensional irreducible representations of $X(\mathfrak{g}_N)$ are parameterized by tuples*

$$(f(u); (P_i(u))_{i=1}^n) \in (1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]) \times \mathbb{C}[u]^n,$$

2) On the right-hand side it is implictly assumed that $L(\lambda(u))$ exists, and thus that $\lambda(u)$ satisfies (4.1.1).

where each $P_i(u)$ *is monic.*

The underlying correspondence $\Gamma_{X(\mathfrak{g}_N)}$ *is given*

$$\Gamma_{X(\mathfrak{g}_N)}(L(\lambda(u))) = (f(u); (P_i(u))_{i=1}^n), \quad \textit{where} \tag{4.1.3}$$

(a) $f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ *is the unique scalar series such that*

$$y(u)|_{m_f^*(L(\lambda(u)))} = \mathrm{id}_{m_f^*(L(\lambda(u)))}.$$

(b) $(P_i(u))_{i=1}^n$ *is the n-tuple of Drinfeld polynomials associated to* $L(\lambda(u))$*.*

Proof. Assume that $L(\lambda(u))$ is finite-dimensional. As $y(u)$ is a central series, it operates in $L(\lambda(u))$ as multiplication by a scalar series $f(u)_\lambda \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$. By (2.6.8), we have

$$m_f(y(u)) = f(u)y(u) \quad \forall\, f(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!], \tag{4.1.4}$$

and hence $f(u) = f(u)_\lambda^{-1}$ is the unique series such that $y(u)$ operates as the identity operator in $m_f^*(L(\lambda(u)))$. This justifies the existence of $f(u)$ as in (a), and thus the existence of $\Gamma_{X(\mathfrak{g}_N)}$ as in (4.1.3).

We now establish the bijectivity of this correspondence. By (4.1.4), we have

$$m_h^*(L(\lambda(u))) \mapsto (h(u)^{-1}f(u); (P_i(u))_{i=1}^n).$$

Note that for any fixed $\lambda(u)$ (and thus fixed $(P_i(u))_{i=1}^n$), $h(u)^{-1}f(u)$ can be made to take arbitrary values in $1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ by varying $h(u)$ appropriately. The surjectivity of $\Gamma_{X(\mathfrak{g}_N)}$ thus follows from the surjectivity of (4.1.2).

As $h(u)^{-1}f(u) = f(u)$ if and only if $h(u) = 1$, we can also conclude from the discussion preceding the statement of the proposition that $\Gamma_{X(\mathfrak{g}_N)}$ is injective. □

The following result, which in fact plays a crucial role in the proof of Theorem 4.1.4 (see [AMR06, Theorem 5.16]), illustrates that $\Gamma_{X(\mathfrak{g}_N)}$ translates tensor products of modules to multiplication of polynomials.

Proposition 4.1.6 ([AMR06, Lemma 5.17])**.** *Let* $\xi \in L(\lambda(u))$ *and* $\xi^\sharp \in L(\lambda^\sharp(u))$ *be* *highest weight vectors. Then the $X(\mathfrak{g}_N)$-module*

$$X(\mathfrak{g}_N)(\xi \otimes \xi^\sharp) \subset L(\lambda(u)) \otimes L(\lambda^\sharp(u))$$

is a highest weight module with the highest weight vector $(\lambda_i(u)\lambda_i^\sharp(u))_{i\in\mathcal{I}_N}$. In particular, if $L(\lambda(u))$ and $L(\lambda^\sharp(u))$ are finite-dimensional with

$$\Gamma_{X(\mathfrak{g}_N)}(L(\lambda(u)) = (f(u), (P_i(u))_{i=1}^n) \quad \text{and} \quad \Gamma_{X(\mathfrak{g}_N)}(L(\lambda^\sharp(u)) = (f^\sharp(u), (P_i^\sharp(u))_{i=1}^n),$$

then the irreducible quotient V of $X(\mathfrak{g}_N)(\xi \otimes \xi^\sharp)$ satisfies

$$\Gamma_{X(\mathfrak{g}_N)}([V]) = (f(u)f^\sharp(u), (P_i(u)P_i^\sharp(u))_{i=1}^n).$$

4.1.2 Representations of $Y(\mathfrak{g}_N)$

Let us now focus on the finite-dimensional irreducible representations of the Yangian $Y(\mathfrak{g}_N)$. Due to Theorem 2.6.3, one can obtain a classification of such modules from the results of §4.1.1 with little effort.

Let

$$\epsilon_y : X(\mathfrak{g}_N) \twoheadrightarrow Y(\mathfrak{g}_N), \quad T(u) \mapsto \mathcal{T}(u)$$

be the natural quotient map. Here the notation comes from the observation that under the identification

$$X(\mathfrak{g}_N) \cong \mathbb{C}[y_r \,:\, r \geq 1] \otimes Y(\mathfrak{g}_N) \cong ZX(\mathfrak{g}_N) \otimes Y(\mathfrak{g}_N)$$

given by Theorem 2.6.3 and Proposition 2.6.6, we have $\epsilon_y = \epsilon_{\mathcal{Y}} \otimes \mathrm{id}$ with $\epsilon_{\mathcal{Y}}$ equal to the restriction of the counit ϵ of $X(\mathfrak{g}_N)$ to $ZX(\mathfrak{g}_N)$ (see above Proposition 2.6.9).

Corollary 4.1.7 ([AMR06, Corollary 5.19])**.**

(1) The function

$$\Gamma : \mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N)) \to \mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N)), \quad [V] \mapsto [\epsilon_y^*(V)]$$

is injective with image equal to

$$\{L(\lambda(u)) \in \mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N)) \, : \, y(u)|_{L(\lambda(u))} = \mathrm{id}_{L(\lambda(u))}\} \tag{4.1.5}$$

(2) The isomorphism classes of finite-dimensional irreducible $Y(\mathfrak{g}_N)$-modules are parameterized by tuples of monic polynomials

$$(P_i(u))_{i=1}^n \in \mathbb{C}[u]^n.$$

The parameterization is given by $\Gamma_{Y(\mathfrak{g}_N)} = \Gamma_{X(\mathfrak{g}_N)} \circ \Gamma$.

Proof. It is clear that Γ is injective with image contained in (4.1.5). Suppose that $L(\lambda(u))$ is finite-dimensional and that $y(u)|_{L(\lambda(u))}$ is the identity operator. Consider the $Y(\mathfrak{g}_N)$-module $\iota^*(L(\lambda(u)))$, where $\iota : Y(\mathfrak{g}_N) \hookrightarrow X(\mathfrak{g}_N)$ is the embedding (2.6.6) (as usual, we drop the subscript R).

As $\mathcal{T}(u)$ operates as $T(u)$ in $\iota^*(L(\lambda(u)))$, it is an irreducible $Y(\mathfrak{g}_N)$-module which satisfies

$$\epsilon_y^*(\iota^*(L(\lambda(u)))) = L(\lambda(u)).$$

Hence $\Gamma_{Y(\mathfrak{g}_N)}([\iota^*(L(\lambda(u)))]) = L(\lambda(u))$, and we may conclude that Part (1) of the corollary holds.

As for Part (2), observe that (4.1.5) is mapped bijectively onto the set of tuples $(1; (P_i(u))_{i=1}^n)$ under $\Gamma_{X(\mathfrak{g}_N)}$, which can naturally be identified with the set of n-tuples of monic polynomials in u. Hence, the desired conclusion follows from Part (1). □

We conclude this subsection with a brief discussion of some of the most elementary finite-dimensional irreducible $Y(\mathfrak{g}_N)$-modules: the *fundamental representations*.

Definition 4.1.8. Fix $\alpha \in \mathbb{C}$ and $1 \leq i \leq n$. The fundamental representation $L(i : \alpha)$ is the unique, up to isomorphism, finite-dimensional irreducible representation of $Y(\mathfrak{g}_N)$ satisfying

$$\Gamma_{Y(\mathfrak{g}_N)}([L(i : \alpha)]) = (P_j(u))_{j=1}^n, \quad \text{where } \; P_j(u) = (u - \alpha)^{\delta_{ij}}.$$

These representations play an important role in the representation theory of the Yangian, where they serve as a basic building block. This is illustrated in part by the following simple, but deep consequence of Proposition 4.1.6 and Corollary 4.1.7.

Corollary 4.1.9 ([AMR06, Corollary 5.20]). *Suppose that V is a finite-dimensional irreducible $Y(\mathfrak{g}_N)$-module. Then there is $m \geq 0$, $1 \leq i_1, \dots, i_m \leq n$ and $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ such that V is isomorphic to the unique irreducible quotient of*

$$Y(\mathfrak{g}_N)(\xi_1 \otimes \cdots \otimes \xi_m) \subset L(i_1 : \alpha_1) \otimes \cdots \otimes L(i_m : \alpha_m),$$

where, for each $1 \leq k \leq n$, $\xi_k \subset L(i : \alpha)$ is a highest weight vector and both sides are identified with the trivial representation if $m = 0$.

For an explicit description of the fundamental representations $L(i : \alpha)$ compatible with the R-matrix presentation of the Yangian, the reader is referred to [AMR06, §5.4].

4.1.3 From Drinfeld polynomials to dominant integral $\mathfrak{g}_N$-weights

Let us now recall some aspects of the representation theory of $\mathfrak{g}_N$. Following §4.2 of [Mol07], for any n-tuple

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$$

we denote by $V(\lambda)$ the irreducible $\mathfrak{g}_N$-module with the highest weight λ. That is, $V(\lambda)$ is the irreducible module generated by a nonzero vector ξ such that

$$F_{ij}\xi = 0 \quad \forall \quad i < j \in \mathcal{I}_N,$$
$$F_{kk}\xi = \lambda_k \xi \quad \forall \quad 1 \leq k \leq n.$$

The module $V(\lambda)$ is finite-dimensional if and only if

$$\begin{aligned} \lambda_{i-1} - \lambda_i \in \mathbb{Z}_{\geq 0} \quad \forall \quad 2 \leq i \leq n \;\text{ and} \\ -\lambda_1 \in \mathbb{Z}_{\geq 0} \quad \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_N, \\ -2\lambda_1 \in \mathbb{Z}_{\geq 0} \quad \text{if} \quad \mathfrak{g}_N = \mathfrak{so}_{2n+1}, \\ -\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0} \quad \text{if} \quad \mathfrak{g}_N = \mathfrak{so}_{2n}. \end{aligned} \tag{4.1.6}$$

We can express this in terms of a more standard Chevalley-Serre type Cartan basis as follows. Following [GRW19a, §3.1][3], we set

$$(d_i, h_i) = (1, F_{i-1,i-1} - F_{ii}) \quad \forall \quad 2 \leq i \leq n \quad \text{and}$$
$$(d_1, h_1) = \begin{cases} (2, -2F_{11}) & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N, \\ (1/2, -F_{11}) & \text{if } \mathfrak{g}_N = \mathfrak{so}_{2n+1}, \\ (1, -F_{11} - F_{22}) & \text{if } \mathfrak{g}_N = \mathfrak{so}_{2n}. \end{cases} \tag{4.1.7}$$

Then the conditions (4.1.6) are equivalent to the requirement that

$$d_i^{-1} h_i(\xi) \in \mathbb{Z}_{\geq 0}\xi \quad \forall \quad 1 \leq i \leq n. \tag{4.1.8}$$

By Theorems 2.5.5 and 2.6.7 and the relations (2.7.17) and (2.7.18), the assignment

$$F_{ij} \mapsto \tau_{ij}^{(1)} = t_{ij}^{(1)} - \delta_{ij} y_1 = \frac{1}{2}(t_{ij} - \theta_{ij} t_{-j,-i}) \quad \forall\, i, j \in \mathcal{I}_N$$

extends to an embedding $U(\mathfrak{g}_N) \hookrightarrow Y(\mathfrak{g}_N) \subset X(\mathfrak{g}_N)$, and consequently we may regard any $X(\mathfrak{g}_N)$ or $Y(\mathfrak{g}_N)$ module as a $\mathfrak{g}_N$-module. Using this embedding and the relations of Theorem 4.1.4, we deduce the following corollary.

Corollary 4.1.10. *Suppose that $L(\lambda(u))$ is finite-dimensional with highest weight vector ξ and Drinfeld tuple $\mathbf{P} = (P_i(u))_{i=1}^n$. Then the $\mathfrak{g}_N$-module $U(\mathfrak{g}_N)\xi$ is a highest weight module with highest weight $\lambda_{\mathbf{P}} = (\lambda_{\mathbf{P},i})_{i=1}^n$ whose components are given by*

$$\lambda_{\mathbf{P},i} = \lambda_{\mathbf{P}} - \sum_{a=2}^{i} \deg P_a(u) \quad \forall \quad 1 \leq i \leq n, \quad \textit{where}$$
$$\lambda_{\mathbf{P}} = \begin{cases} -\deg P_1(u) & \textit{if } \mathfrak{g}_N = \mathfrak{sp}_{2n}, \\ -\frac{1}{2}\deg P_1(u) & \textit{if } \mathfrak{g}_N = \mathfrak{so}_{2n+1}, \\ \frac{1}{2}(\deg P_2(u) - \deg P_1(u)) & \textit{if } \mathfrak{g}_N = \mathfrak{so}_{2n}. \end{cases} \tag{4.1.9}$$

Equivalently, we have

$$d_i^{-1} h_i(\xi) = \deg P_i(u)\xi \in \mathbb{Z}_{\geq 0}\xi \quad \forall \quad 1 \leq i \leq n.$$

3) To obtain the formulation given in [GRW19a], (d_i, h_i) should be replaced with (d_{i-1}, h_{i-1}).

4.2 Highest weight theory for twisted Yangians

In this section, we develop a highest weight theory for representations of the extended twisted Yangians $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ of type B, C and D considered in Chapter 3.

4.2.1 From twisted Yangians to extended twisted Yangians

Before getting into the heart of the matter, it is worth taking a moment to explain the emphasis on the extended twisted Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ rather than the twisted Yangian $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ itself. Let

$$\epsilon_\vartheta : X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \twoheadrightarrow Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}, \quad S(u) \mapsto \mathcal{S}(u)$$

be the natural quotient homomorphism, as in (3.3.27). Recall that, under the identification

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \cong ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \otimes Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$$

provided by Theorem 3.3.14, $\epsilon_\vartheta = \epsilon|_{ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}} \otimes \mathrm{id}$, where ϵ is the counit of $X(\mathfrak{g}_N)$. Equivalently, $\epsilon_\vartheta = \epsilon_y|_{X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}}$, where ϵ_y is as in §4.1.2.

Following §4.1.2, we will use the notation

$$\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) \quad \text{and} \quad \mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$$

to denote the set of isomorphism classes of all finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, respectively. As in §4.1, we will generally write $[V]$ for the isomorphism class of a module V.

Lemma 4.2.1. *The function*

$$\Gamma^\vartheta : \mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) \to \mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}), \quad [V] \mapsto [\epsilon_\vartheta^*(V)]$$

is injective with image equal to

$$\{[V] \in \mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) \, : \, q(u)|_V = \mathrm{id}_V\}.$$

Moreover, for each $[V] \in \mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw})$*, there exists a unique series*

$$g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!] \quad \textit{with} \quad g(u) = g(\kappa - u)$$

such that $[\nu_g^*(V)] \in \mathrm{Im}(\Gamma^{\vartheta})$*.*

Proof. The first part of the lemma is proven using the same argument as used to establish Part (1) of Corollary 4.1.7. It thus suffices to show that, if V is a finite-dimensional irreducible $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module, then there is a unique element $g(u)$ of $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ with $g(u) = g(\kappa - u)$ and

$$q(u)|_{\nu_g^*(V)} = \mathrm{id}_{\nu_g^*(V)}. \tag{4.2.1}$$

As $q(u)$ is a central series, it operates in V as multiplication by a scalar series $q_V(u)$. Since $q(u)$ satisfies $q(u) = q(\kappa - u)$, we also have $q_V(u) = q_V(\kappa - u)$. By (3.3.20), $q(u)$ satisfies

$$\nu_g(q(u)) = g(u)q(u) \quad \forall \quad g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!] \ \text{ with } \ g(u) = g(\kappa - u).$$

Therefore, $g(u) = q_V(u)^{-1}$ is the unique series such that (4.2.1) holds. □

Remark 4.2.2. By Corollary 3.3.8, $\nu_g^*(V)$ and V are always identical as $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules. A right inverse to Γ^{ϑ} is thus given by

$$[V] \mapsto [\iota_{\vartheta}^*(V)],$$

where ι_{ϑ} is the natural embedding of $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ into $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$.

Our goal is to develop a highest weight theory for $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ which will allow us to simultaneously classify the finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ using the above lemma, in the same way that Part (1) of Corollary 4.1.7 allowed us to pass from the parameterization of $\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N))$ given by Proposition 4.1.5 to the parameterization of $\mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N))$ by Drinfeld polynomials given in Part (2) of Corollary 4.1.7.

4.2.2 Definitions and first results

Definition 4.2.3. A representation V of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that $V = X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}\xi$ and

$$\begin{aligned} s_{ij}(u)\xi &= 0 \qquad \forall \quad i < j \in \mathcal{I}_N, \\ s_{ii}(u)\xi &= \mu_i(u)\xi \qquad \forall \quad i \in \mathcal{I}_N^+, \end{aligned}$$

where each $\mu_i(u)$ is a formal power series in $\mathbb{C}[\![u^{-1}]\!]$ of the form

$$\mu_i(u) = g_{ii} + \sum_{r=1}^{\infty} \mu_i^{(r)} u^{-r}, \quad \mu_i^{(r)} \in \mathbb{C}.$$

The vector ξ is called the *highest weight vector* and $(N-n)$-tuple $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_N^+}$ is called the *highest weight.*

Given a highest weight representation V with highest weight vector ξ, a natural question to ask is whether or not ξ is a simultaneous eigenvector for the diagonal elements $s_{-i,-i}(u)$ with $1 \leq i \leq n$. This is indeed the case. By the symmetry relation (3.3.41) we have

$$\begin{aligned} s_{-i,-i}(u) + &\frac{1}{2u-2\kappa} \sum_{\ell=1}^{n} s_{-\ell,-\ell}(u) \\ &= p(u) s_{ii}(\kappa - u) \pm \frac{s_{ii}(u)}{2u-\kappa} - \frac{1}{2u-2\kappa} \sum_{\ell \in \mathcal{I}_N^+} s_{\ell\ell}(u). \end{aligned} \tag{4.2.2}$$

Summing over $1 \leq i \leq n$ yields

$$\begin{aligned} &\left(\frac{2u-2\kappa+n}{2u-2\kappa}\right) \sum_{\ell=1}^{n} s_{-\ell,-\ell}(u) \\ &\quad = \sum_{\ell=1}^{n} \left(p(u) s_{\ell\ell}(\kappa-u) \pm \frac{s_{\ell\ell}(u)}{2u-\kappa} \right) - \frac{n}{2u-2\kappa} \sum_{\ell \in \mathcal{I}_N^+} s_{\ell\ell}(u). \end{aligned} \tag{4.2.3}$$

Substituting this equation back into (4.2.2) leads to the following result.

Proposition 4.2.4. *Let V be a highest weight representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ with the highest weight vector ξ and the highest weight $\mu(u)$. Then ξ is an eigenvector for the action of $s_{-i,-i}(u)$ for all $1 \leq i \leq n$. More explicitly, for each $1 \leq i \leq n$ we have* *the relation:*

$$\begin{aligned}(2\kappa - 2u - n)&s_{-i,-i}(u)\xi \\ &= \sum_{\ell=1}^{n} \beta_{i,\ell}(u)\left(p(u)\mu_\ell(\kappa-u) \pm \frac{\mu_\ell(u)}{2u-\kappa}\right)\xi + \sum_{\ell\in\mathcal{I}_N^+}\mu_\ell(u)\xi,\end{aligned} \tag{4.2.4}$$

where $\beta_{i,\ell}(u) = 1$ *if* $\ell \neq i$ *and* $\beta_{i,\ell}(u) = (2\kappa - 2u - n + 1)$ *otherwise.*

Given a highest weight $\mu(u)$, we shall frequently make use of the corresponding tuple $\widetilde{\mu}(u) = (\widetilde{\mu}_i(u))_{i\in\mathcal{I}_N^+}$ whose components are given by

$$\widetilde{\mu}_i(u) = (2u - n + i)\mu_i(u) + \sum_{\ell=i+1}^{n} \mu_\ell(u) \quad \forall \quad i \in \mathcal{I}_N^+. \tag{4.2.5}$$

The following proposition imposes one important restriction on $\widetilde{\mu}_0(u)$.

Proposition 4.2.5. *Suppose that* $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$ *and let* V *be a highest weight representation of* $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ *with the highest weight* $\mu(u)$*. Then the series* $\widetilde{\mu}_0(u)$ *satisfies*

$$u\,\mathcal{g}(u)\widetilde{\mu}_0(\kappa-u) = (\kappa-u)\,\mathcal{g}(\kappa-u)\widetilde{\mu}_0(u), \tag{4.2.6}$$

where $\mathcal{g}(u)$ *is the rational function of* u *defined in* (3.3.3)*.*

Proof. Let $\xi \in V$ be a highest weight vector. By the symmetry relation (3.3.41) we have

$$s_{00}(u) = p(u)s_{00}(\kappa-u) + \frac{s_{00}(u)}{2u-\kappa} - \frac{\mathrm{tr}(S(u))}{2u-2\kappa},$$

which can be rearranged to

$$\begin{aligned}\left(1 - \frac{1}{2u-\kappa} + \frac{1}{2u-2\kappa}\right)&s_{00}(u) + \frac{1}{2u-2\kappa}\sum_{\ell=1}^{n} s_{\ell\ell}(u) \\ &= p(u)s_{00}(\kappa-u) - \frac{1}{2u-2\kappa}\sum_{\ell=1}^{n} s_{-\ell,-\ell}(u).\end{aligned} \tag{4.2.7}$$

Multiplying both sides of this relation by $(2\kappa - 2u - n)$ and substituting in relation (4.2.3), the right-hand side becomes:

$$p(u)\left((2\kappa - 2u - n)s_{00}(\kappa-u) + \sum_{\ell=1}^{n} s_{\ell\ell}(\kappa-u)\right)$$

$$+\left(\frac{1}{2u-\kappa}-\frac{n}{2u-2\kappa}\right)\sum_{\ell=1}^{n}s_{\ell\ell}(u)-\frac{n}{2u-2\kappa}s_{00}(u).$$

Therefore, on $\mathbb{C}\xi$, (4.2.7) can be expressed as

$$\begin{aligned}&p(u)\widetilde{\mu}_0(\kappa-u)\\&\quad=(2u-n)\left(-1-\frac{1}{2u-\kappa}\right)\mu_0(u)+\left(-1-\frac{1}{2u-\kappa}\right)\sum_{\ell=1}^{n}\mu_\ell(u).\end{aligned}$$

Using Proposition 3.3.3, we deduce that the relation (4.2.6) follows from

$$-1-\frac{1}{2u-\kappa}=p_I(u)\frac{\kappa-u}{u},\tag{4.2.8}$$

which is readily verified using (3.3.7), as in the type CI and DIII instances of Proposition 3.3.3. □

Recall from Lemma 3.2.5 that the family of generators $\{F^{\vartheta}_{ij}\}_{i,j\in\mathcal{I}_N}\subset U(\mathfrak{g}^{\vartheta}_N)$ are defined by

$$F^{\vartheta}_{ij}=(g_{ii}+g_{jj})F_{ij}\quad\forall\quad i,j\in\mathcal{I}_N.$$

Using the embedding of Corollary 3.3.11, we may restrict the adjoint action of $\mathrm{Lie}(X(\mathfrak{g}_N,\mathfrak{g}^{\vartheta}_N)^{tw})$ to $\mathfrak{g}^{\vartheta}_N$. Appealing to the explicit form of the reflection equation (3.3.42), we find that the resulting $\mathfrak{g}^{\vartheta}_N$-module structure on $X(\mathfrak{g}_N,\mathfrak{g}^{\vartheta}_N)^{tw}$ is given by

$$\begin{aligned}&[F^{\vartheta}_{ij},s_{k\ell}(v)]\\&\quad=(g_{ii}+g_{jj})\left(\delta_{kj}s_{i\ell}(v)-\delta_{i\ell}s_{kj}(v)-\delta_{k,-i}\theta_{ij}s_{-j,\ell}(v)+\delta_{\ell,-j}\theta_{ij}s_{k,-i}(v)\right).\end{aligned}\tag{4.2.9}$$

Recall from (3.1.1) that our fixed choice of Cartan subalgebra for $\mathfrak{g}_N$ is

$$\mathfrak{h}_N=\mathrm{span}_{\mathbb{C}}\{F_{ii}\,:\,1\le i\le n\}.$$

As the involution ϑ satisfies

$$\vartheta|_{\mathfrak{h}_N}=\mathrm{id}_{\mathfrak{h}_N},$$

$\mathfrak{h}_N$ is also a Cartan subalgebra for $\mathfrak{g}^{\vartheta}_N$. For each $1\le i\le n$, define $\epsilon_i\in\mathfrak{h}_N^*$ by

$$\epsilon_i(F_{kk})=\delta_{ik}\quad\forall\,1\le k\le n.$$

In addition, define auxiliary elements $\alpha_{k,\ell} \in \mathfrak{h}_N^*$ by

$$\alpha_{k,\ell} = \mathrm{sign}(k)\epsilon_{|k|} - \mathrm{sign}(\ell)\epsilon_{|\ell|} \quad \forall\, k,\ell \in \mathcal{I}_N,$$

where ϵ_0 is the zero functional. Then from equation (4.2.9) we obtain

$$[F_{ii}^{\vartheta}, s_{k\ell}(v)] = 2g_{ii}\left(\delta_{ik} - \delta_{i\ell} - \delta_{i,-k} + \delta_{i,-\ell}\right) s_{k\ell}(v) = \alpha_{k,\ell}(F_{ii}^{\vartheta}) s_{k\ell}(v) \tag{4.2.10}$$

for all $1 \leq i \leq n$ and $k,\ell \in \mathcal{I}_N$.

Let Δ_+ be the standard set of positive roots of $\mathfrak{g}_N$ for our choice of $\mathfrak{h}_N$ (as in [AMR06]), so

$$\Delta_+ = \{\alpha_{k,\ell} \,:\, k < \ell \in \mathcal{I}_N \text{ and } (k,\ell) \in \mathcal{B}_N\},$$

where $\mathcal{B}_N$ is defined in (3.1.2). Let $\preceq$ be the corresponding partial ordering on $\mathfrak{h}_N^*$. That is, $\preceq$ is defined by

$$\mu \preceq \lambda \iff \lambda - \mu \in Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha.$$

We are now in a position to prove the first theorem of this section.

Theorem 4.2.6. *Every finite-dimensional irreducible representation V of the twisted Yangian $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is a highest weight representation. Additionally, V contains a unique highest weight vector ξ up to scalar multiplication.*

Proof. Define the subspace V^0 of V by

$$V^0 = \{\xi \in V : s_{ij}(u)\xi = 0 \quad \forall\, i < j \in \mathcal{I}_N\}.$$

Step 1: V^0 is nonzero.

Via the embedding $\mathfrak{g}_N^{\vartheta} \hookrightarrow X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, we may view V as a $\mathfrak{g}_N^{\vartheta}$-module. Since V is finite-dimensional, the F_{ii}^{ϑ} have a mutual weight vector ξ. Let L be the set of all weights of the $\mathfrak{g}_N^{\vartheta}$-module V, so L is a nonempty finite set. Therefore, there exists $\mu \in L$ such that $\mu + \alpha_{k,\ell}$ is not a weight for any $k < \ell \in \mathcal{I}_N$. Then the μ-weight vector ξ must belong to V^0. Indeed, suppose there exists $k < \ell \in \mathcal{I}_N$ such that $s_{k\ell}(u)\xi \neq 0$.

Then from (4.2.10) we obtain:

$$F_{ii}^{\vartheta}\,(s_{k\ell}(v)\xi) = (\alpha_{k,\ell} + \mu)\,(F_{ii}^{\vartheta})s_{k\ell}(v)\xi$$

for all $1 \le i \le n$. This contradicts the maximality of μ, and so we must have $\xi \in V^0$. Therefore V^0 is nonzero.

Step 2: The subspace V^0 is preserved by the operators $s_{ii}(u)$ for all $i \in \mathcal{I}_N^+$.

We will consider separately the cases when N is even and when N is odd.

Step 2.1: $N = 2n$.

By definition of V^0, we must show that $s_{k\ell}(u)s_{ii}(v) \equiv 0$ for all $k < \ell$ and $1 \le i \le n$, where $\equiv$ denotes equality of operators on V^0.

Claim: It suffices to show that

$$s_{k\ell}(u)s_{ii}(v) \equiv 0, \quad s_{-k,\ell}(u)s_{ii}(v) \equiv 0 \quad \text{and} \quad s_{-j,j}(u)s_{ii}(v) \equiv 0$$

for all $0 < k < \ell$ and $i, j > 0$.

This claim follows from the symmetry relation (3.3.41).

Step 2.1.1: $s_{k\ell}(u)s_{ii}(v) \equiv 0$ for all $0 < k < \ell$ and $i > 0$.

Assume first that $k < i$. Then it is immediate from (3.3.42) and the relation

$$s_{k\ell}(u)s_{ii}(v) \equiv [s_{k\ell}(u), s_{ii}(v)]$$

that $s_{k\ell}(u)s_{ii}(v) \equiv 0$ unless $i = \ell$. If $i = \ell$, we obtain

$$s_{k\ell}(u)s_{\ell\ell}(v) \equiv \frac{1}{u+v}\sum_{a=\ell}^{n} s_{ka}(u)s_{a\ell}(v),$$

and thus for $a \ge \ell > k$, (3.3.42) yields

$$s_{ka}(u)s_{a\ell}(v) \equiv [s_{ka}(u), s_{a\ell}(v)] \equiv \frac{1}{u+v}\sum_{b=\ell}^{n} s_{kb}(u)s_{b\ell}(v) \equiv s_{k\ell}(u)s_{\ell\ell}(v).$$

Therefore we obtain
$$\left(1 - \frac{n-\ell+1}{u+v}\right) s_{k\ell}(u)s_{\ell\ell}(v) \equiv 0,$$
and so we must have $s_{k\ell}(u)s_{\ell\ell}(v) \equiv 0$, as desired.

If instead $k \geq i$, then we write $[s_{k\ell}(u), s_{ii}(v)] = -[s_{ii}(v), s_{k\ell}(u)]$. Relation (3.3.42) then gives

$$\begin{aligned}
[s_{ii}(v), s_{k\ell}(u)] &\equiv \frac{\delta_{ik}}{v+u}\sum_{a=\ell}^{n} s_{ia}(v)s_{a\ell}(u) - \frac{1}{v^2-u^2}\sum_{a=\ell}^{n}(s_{ka}(v)s_{a\ell}(u) - s_{ka}(u)s_{a\ell}(v)) \\
&\equiv \left(\frac{\delta_{ik}}{v+u} - \frac{1}{v^2-u^2}\right)\sum_{a=\ell}^{n} s_{ka}(v)s_{a\ell}(u) + \frac{1}{v^2-u^2}\sum_{a=\ell}^{n} s_{ka}(u)s_{a\ell}(v).
\end{aligned}$$

From the above proof that $s_{k\ell}(u)s_{\ell\ell}(v) \equiv 0$, we see that the right-hand side of the previous line is $\equiv 0$. This completes the proof that $s_{k\ell}(u)s_{ii}(v) \equiv 0$ for all $0 < k < \ell$ and $i > 0$.

Step 2.1.2: $s_{-k,\ell}(u)s_{ii}(v) \equiv 0$ for all $i > 0$ and $\ell > k > 0$.

This is an immediate consequence of relation (3.3.42) unless $i = \ell$ or $i = k$. The case $i = \ell$ is similar to Step 2.1.1, so we concentrate on the case $i = k$. By (3.3.42) we have

$$\begin{aligned}
s_{-k,\ell}(u)s_{kk}(v) \equiv &- \frac{1}{u-v-\kappa}\sum_{a=k}^{n} s_{-a,\ell}(u)s_{ak}(v) \\
&+ \frac{1}{(u+v)(u-v-\kappa)}\sum_{a=k}^{n} s_{-\ell,a}(u)s_{ak}(v).
\end{aligned} \tag{4.2.11}$$

Since $s_{-a,\ell}(u)s_{ak}(v) \equiv [s_{-a,\ell}(u), s_{ak}(v)]$, for $a \geq k$ we have

$$\begin{aligned}
s_{-a,\ell}(u)s_{ak}(v) &\equiv \frac{\delta_{a\ell}}{u+v}\sum_{b=k}^{n} s_{-a,b}(u)s_{bk}(v) \\
&\quad - \frac{1}{u-v-\kappa}\left(\sum_{b=k}^{n} s_{-b,\ell}(u)s_{bk}(v) - \frac{1}{u+v}\sum_{b=k}^{n} s_{-\ell,b}(u)s_{bk}(v)\right) \\
&\equiv \frac{\delta_{a\ell}}{u+v}\sum_{b=k}^{n} s_{-\ell,b}(u)s_{bk}(v) + s_{-k,\ell}(u)s_{kk}(v).
\end{aligned}$$

Substituting this result back into (4.2.11), we obtain

$$\left(1+\frac{n-k+1}{u-v-\kappa}\right)s_{-k,\ell}(u)s_{kk}(v)\equiv 0$$

and so we must have $s_{-k,\ell}(u)s_{kk}(v)\equiv 0$ whenever $0<k<\ell$.

Step 2.1.3: $s_{-j,j}(u)s_{ii}(v)\equiv 0$ for all $i,j>0$.

To begin, it is an immediate consequence of (3.3.42) that $s_{-j,j}(u)s_{ii}(v)\equiv 0$ unless $i=j$, so without loss of generality we may assume $i=j$. We have

$$\begin{aligned}[s_{-i,i}(u),s_{ii}(v)]\equiv\left(\frac{1}{u+v}+\frac{1}{(u+v)(u-v-\kappa)}\right)\sum_{a=i}^{n}s_{-i,a}(u)s_{ai}(v)\\ -\frac{1}{u-v-\kappa}\sum_{a=i}^{n}s_{-a,i}(u)s_{ai}(v).\end{aligned}\tag{4.2.12}$$

Let us compute $s_{-a,i}(u)s_{ai}(v)$ for $a>i$. From (3.3.42) we see that

$$\begin{aligned}&s_{-a,i}(u)s_{ai}(v)\\ &\quad\equiv-\frac{1}{u-v-\kappa}\sum_{b=i}^{n}s_{-b,i}(u)s_{bi}(v)+\frac{1}{(u+v)(u-v-\kappa)}\sum_{b=i}^{n}s_{-i,b}(u)s_{bi}(v)\\ &\quad\equiv[s_{-i,i}(u),s_{ii}(v)]-\frac{1}{u+v}\sum_{a=i}^{n}s_{-i,a}(u)s_{ai}(v),\end{aligned}$$

where the second equivalence follows from (4.2.12). Substituting this result back into relation (4.2.12), we get

$$\begin{aligned}&\left(1+\frac{n-i+1}{u-v-\kappa}\right)[s_{-i,i}(u),s_{ii}(v)]\\ &\qquad\equiv\left(\frac{1}{u+v}+\frac{n-i+1}{(u-v-\kappa)(u+v)}\right)\sum_{a=i}^{n}s_{-i,a}(u)s_{ai}(v),\end{aligned}$$

from which we obtain

$$[s_{-i,i}(u),s_{ii}(v)]\equiv\frac{1}{u+v}\sum_{a=i}^{n}s_{-i,a}(u)s_{ai}(v).\tag{4.2.13}$$

By (3.3.42), we have that for all $a > i$

$$s_{-i,a}(u)s_{ai}(v) = \frac{1}{u+v}\sum_{b=i}^{n} s_{-i,b}(u)s_{bi}(v).$$

Substituting this into (4.2.13) leads to $[s_{-i,i}(u), s_{ii}(v)] \equiv 0$. This completes the proof of Step 2 when $N = 2n$.

Step 2.2: $N = 2n+1$.

The argument is essentially the same in this case. By the symmetry relation (3.3.41), it suffices to show that

$$s_{k\ell}(u)s_{ii}(v), \quad s_{-k,\ell}(u)s_{ii}(v), \quad s_{-j,j}(u)s_{ii}(v) \quad \text{and} \quad s_{0j}(u)s_{ii}(v)$$

all vanish on V^0 whenever $0 < k < \ell$, $j > 0$ and $i \geq 0$.

Step 2.2.1: $s_{k\ell}(u)s_{ii}(v)$, $s_{-k,\ell}(u)s_{ii}(v)$ and $s_{-j,j}(u)s_{ii}(v)$ operate as zero on V^0 whenever $0 < k < \ell$, $j > 0$ and $i \geq 0$.

The same arguments as those given for the $N = 2n$ case show that

$$s_{k\ell}(u)s_{ii}(v) \equiv s_{-k,\ell}(u)s_{ii}(v) \equiv s_{-j,j}(u)s_{ii}(v) \equiv 0$$

whenever $i, j > 0$ and $\ell > k > 0$. Moreover, given the same restrictions on j, ℓ and k, the reflection equation (3.3.42) immediately yields

$$s_{-k,\ell}(u)s_{00}(v) \equiv s_{-j,j}(u)s_{00}(v) \equiv 0.$$

Additionally, if $0 < k < \ell$ then $s_{k\ell}(u)s_{00}(v) \equiv -[s_{00}(v), s_{k\ell}(u)]$ and (3.3.42) gives

$$[s_{00}(v), s_{k\ell}(u)] \equiv -\frac{1}{v^2-u^2}\sum_{a=\ell}^{n}(s_{ka}(v)s_{a\ell}(u) - s_{ka}(u)s_{a\ell}(v)).$$

By the same argument as in Step 2.1.1, the right-hand side vanishes.

Step 2.2.2: $s_{0j}(u)s_{ii}(v) \equiv 0$ for all $j > 0$ and $i \geq 0$.

Assume first $i > 0$. Then (3.3.42) implies that $s_{0j}(u)s_{ii}(v) \equiv 0$ unless $i = j$. Moreover, the proof that $s_{0j}(u)s_{jj}(v) \equiv 0$ proceeds identically to the proof that

$s_{-k,\ell}(u)s_{\ell\ell}(v) \equiv 0$ for all $\ell > k > 0$.

To prove that $s_{0j}(u)s_{00}(v) \equiv 0$ for all $j > 0$, note first that

$$s_{0j}(u)s_{00}(v) \equiv -[s_{00}(v), s_{0j}(u)]$$

and hence, by (3.3.42), we have

$$\begin{aligned}[s_{00}(v), s_{0j}(u)] \equiv &\left(1 - \frac{1}{v-u} + \frac{1}{v-u-\kappa}\right) B(v,u) \\ &+ \frac{1}{v-u} B(u,v) - \frac{1}{v-u-\kappa} \sum_{a=j}^{n} s_{-a,0}(v) s_{aj}(u),\end{aligned} \tag{4.2.14}$$

where $B(u,v) = \frac{1}{u+v}\sum_{a=j}^{n} s_{0a}(u)s_{aj}(v)$. However, since $s_{0j}(u)s_{jj}(v) \equiv 0$ by the previous step, (3.3.42) yields

$$0 \equiv s_{0j}(u)s_{jj}(v) \equiv \frac{1}{u+v}\sum_{a=j}^{n} s_{0a}(u)s_{aj}(v) = B(u,v),$$

and the symmetry relation (3.3.41) gives

$$\sum_{a=j}^{n} s_{-a,0}(v)s_{aj}(u) \equiv p(v)(\kappa - v + u)B(\kappa - v, u) \pm \frac{v+u}{2v-\kappa} B(v,u) \equiv 0.$$

Therefore, by (4.2.14) we have $s_{0j}(u)s_{jj}(v) \equiv 0$ for all $j > 0$.

Step 3: Viewed as operators on V^0, $s_{ii}(u)$ and $s_{jj}(v)$ commute for all $i, j \in \mathcal{I}_N^+$.

Again, we will treat the cases $N = 2n$ and $N = 2n + 1$ separately.

Step 3.1: $N = 2n$.

Let us define the operator $A_{ij}(u,v)$ on V^0 by

$$A_{ij}(u,v) = s_{ij}(u)s_{ji}(v) - s_{ij}(v)s_{ji}(u). \tag{4.2.15}$$

As a consequence of (3.3.42) we have:

$$A_{ii}(u,v) \equiv \frac{1}{u+v}\sum_{a=i}^{n} A_{ia}(u,v). \tag{4.2.16}$$

On the other hand, for $0 < i < j$ we have $s_{ji}(v)s_{ij}(u) \equiv s_{ji}(u)s_{ij}(v) \equiv 0$, and hence we can rewrite $A_{ij}(u,v)$ as

$$A_{ij}(u,v) \equiv [s_{ij}(u), s_{ji}(v)] + [s_{ji}(u), s_{ij}(v)].$$

Using (3.3.42) to compute $[s_{ij}(u), s_{ji}(v)]$ and $[s_{ji}(u), s_{ij}(v)]$, we obtain

$$\begin{aligned} A_{ij}(u,v) \equiv & \frac{1}{u-v}\left([s_{ii}(u), s_{jj}(v)] + [s_{jj}(u), s_{ii}(v)]\right) \\ & + \frac{1}{u+v}\left(\sum_{a=i}^{n} A_{ia}(u,v) + \sum_{a=j}^{n} A_{ja}(u,v)\right). \end{aligned} \tag{4.2.17}$$

We apply (3.3.42) again to compute

$$[s_{ii}(u), s_{jj}(v)] \equiv -\frac{1}{u^2-v^2}\sum_{a=j}^{n} A_{ja}(u,v), \tag{4.2.18}$$

from which it follows that $[s_{ii}(u), s_{jj}(v)] + [s_{jj}(u), s_{ii}(v)] \equiv 0$. Combining this with (4.2.16), equation (4.2.17) can be rewritten as

$$A_{ij}(u,v) \equiv A_{ii}(u,v) + A_{jj}(u,v). \tag{4.2.19}$$

Taking the sum as j goes from $i+1$ to n and adding $A_{ii}(u,v)$ to both sides we arrive at the relation

$$\sum_{j=i}^{n} A_{ij}(u,v) \equiv (n-i+1)A_{ii}(u,v) + \sum_{j=i+1}^{n} A_{jj}(u,v).$$

However, by (4.2.16), the left hand side is equivalent to $(u+v)A_{ii}(u,v)$, so we may rewrite the above as

$$(u+v-n+i-1)A_{ii}(u,v) \equiv \sum_{j=i+1}^{n} A_{jj}(u,v).$$

A simple downward induction on i then proves that $A_{ii}(u,v) \equiv 0$ for all $i \in \mathcal{I}_N^+$.

Since $A_{ii}(u,v) = [s_{ii}(u), s_{ii}(v)]$, this proves that $s_{ii}(u)$ and $s_{ii}(v)$ commute for all

$i \in \mathcal{I}_N^+$. Moreover, combining equations (4.2.18) and (4.2.16), we have that

$$[s_{ii}(u), s_{jj}(v)] \equiv -\frac{1}{u^2 - v^2} \sum_{a=j}^{n} A_{ja}(u,v) \equiv -\frac{1}{u-v} A_{jj}(u,v) \equiv 0 \tag{4.2.20}$$

for all $j > i > 0$.

Step 3.2: $N = 2n + 1$.

The arguments from Step 3.1 show that $[s_{ii}(u), s_{jj}(v)] \equiv 0$ whenever $1 \leq i, j \leq n$, so it suffices to show that $[s_{00}(u), s_{jj}(v)] \equiv 0$ for all $j \geq 0$. Suppose first that $j > 0$. Then by (3.3.42) and (4.2.20), we have

$$[s_{00}(u), s_{jj}(v)] \equiv -\frac{1}{u^2 - v^2} \sum_{a=j}^{n} A_{ja}(u,v) \equiv -\frac{1}{u-v} A_{jj}(u,v) \equiv 0. \tag{4.2.21}$$

Hence, it remains to see $[s_{00}(u), s_{00}(v)] \equiv 0$. The same calculations as those used to obtain (4.2.17) give

$$A_{0j}(u,v) \equiv \frac{1}{u+v} \sum_{a=0}^{n} A_{0a}(u,v) \quad \forall\, j > 0. \tag{4.2.22}$$

Summing this expression over $1 \leq j \leq n$ and adding $A_{00}(u,v)$ to both sides yields

$$A_{00}(u,v) \equiv (u + v - n) A_{0j}(u,v) \quad \forall\, j > 0. \tag{4.2.23}$$

It follows from (3.3.42) that

$$\begin{aligned}
&\left(1 - \frac{1}{u-v} + \frac{1}{u+v-\kappa} - \frac{1}{(u-v)(u+v-\kappa)}\right) A_{00}(u,v) \\
&\qquad \equiv \left(1 - \frac{1}{u-v} + \frac{1}{u-v-\kappa}\right) \frac{1}{u+v-n} A_{00}(u,v) \\
&\qquad\quad - \frac{1}{u-v-\kappa} \sum_{a=0}^{n} (s_{-a,0}(u) s_{a0}(v) - s_{0a}(v) s_{0,-a}(u)) \\
&\qquad\quad - \frac{1}{(u-v-\kappa)(u+v-\kappa)} \sum_{a=-n}^{0} [s_{aa}(u), s_{00}(v)].
\end{aligned} \tag{4.2.24}$$

Since $[s_{aa}(u), s_{00}(v)] \equiv 0$ for any $a > 0$, the symmetry relation (3.3.41) implies that

$$[s_{-a,-a}(u), s_{00}(v)] \equiv -\frac{1}{2u-2\kappa}\sum_{b=0}^{n}[s_{-b,-b}(u), s_{00}(v)]. \tag{4.2.25}$$

Taking the sum of both sides as a goes from 1 to n and adding $A_{00}(u,v)$, we obtain the two identities

$$\begin{aligned}\left(1+\frac{n}{2u-2\kappa}\right)\sum_{b=0}^{n}[s_{-b,-b}(u), s_{00}(v)] &\equiv A_{00}(u,v),\\ (2\kappa-2u-n)[s_{-a,-a}(u), s_{00}(v)] &\equiv A_{00}(u,v),\end{aligned} \tag{4.2.26}$$

for any $a > 0$.

On the other hand, the explicit form of the defining reflection equation (3.3.42) implies that

$$\begin{aligned}[s_{-a,-a}(u), s_{00}(v)] \equiv & -\frac{1}{u^2-v^2}\sum_{b=0}^{n}A_{0b}(u,v) + \frac{1}{(u-v)(u+v-\kappa)}A_{0a}(u,v)\\ & -\frac{1}{u+v-\kappa}(s_{-a,0}(u)s_{a0}(v) - s_{0a}(v)s_{0,-a}(u)).\end{aligned}$$

Multiplying both sides by $(u+v-n)$ and appealing to (4.2.22), (4.2.23) and (4.2.26) we obtain

$$\begin{aligned}&\left(\frac{u+v-n}{2\kappa-2u-n}+\frac{1}{u-v}-\frac{1}{(u-v)(u+v-\kappa)}\right)A_{00}(u,v)\\ &\qquad\equiv -\left(\frac{u+v-n}{u+v-\kappa}\right)(s_{-a,0}(u)s_{a0}(v) - s_{0a}(v)s_{0,-a}(u))\end{aligned}$$

for any $a > 0$. Taking the sum of both sides as a goes from 1 to n, adding $-\frac{u+v-n}{u+v-\kappa}A_{00}(u,v)$, and then multiplying both sides by $\frac{u+v-\kappa}{u+v-n}$ we get

$$\begin{aligned}&\left(\frac{n(u+v-\kappa)}{2\kappa-2u-n}+\frac{n(u+v-\kappa)}{(u-v)(u+v-n)}-\frac{n}{(u-v)(u+v-n)}-1\right)A_{00}(u,v)\\ &\qquad\equiv -\sum_{a=0}^{n}(s_{-a,0}(u)s_{a0}(v) - s_{0a}(v)s_{0,-a}(u)).\end{aligned} \tag{4.2.27}$$

Substituting (4.2.27) and (4.2.26) into (4.2.24), we obtain a relation of the form

$$f(u,v)A_{00}(u,v) \equiv 0 \quad \text{with} \;\; f(u,v) \in 1 + u^{-1}\mathbb{C}[v][\![u^{-1}]\!].$$

This implies that we must have $A_{00}(u,v) \equiv 0$.

Step 4: V is a highest weight representation.

By Step 2, we may view $\{s_{ii}^{(r)}\}_{i\in\mathcal{I}_N^+, r\in\mathbb{N}}$ as a family of linear endomorphisms of V^0. By Step 3, this is a commutative family, and hence has a common eigenvector $\xi \in V^0$. That is, there is $\{\mu_i^{(r)}\}_{i\in\mathcal{I}_N^+, r\in\mathbb{N}} \subset \mathbb{C}$ such that

$$s_{ii}(u)\xi = \mu_i(u)\xi, \quad \text{where} \quad \mu_i(u) = g_{ii} + \sum_{r=1}^{\infty} \mu_i^{(r)} u^{-r} \quad \forall\, i \in \mathcal{I}_N^+.$$

Therefore, the submodule $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}\xi$ is a highest weight representation with highest weight vector ξ and highest weight $(\mu_i(u))_{i\in\mathcal{I}_N^+}$. As V was assumed to be irreducible, we must have $V = X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}\xi$. This proves that every finite-dimensional irreducible representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is a highest weight representation.

Step 5: Uniqueness of the highest weight vector.

Let $\mu \in \mathfrak{h}_N^*$ be the weight of the $\mathfrak{g}_N^{\vartheta}$-module V corresponding to ξ, so

$$\mu(F_{ii}^{\vartheta}) = \mu_i^{(1)} - \bar{g}_{ii} \quad \forall \quad 1 \le i \le n.$$

As the central series $q(u)$ must act as a scalar series multiple of id_V, Corollary 3.3.12 implies that V is spanned by elements of the form

$$s_{j_1,i_1}^{(r_1)} \cdots s_{j_m,i_m}^{(r_m)} \xi \tag{4.2.28}$$

with $j_a > i_a$, $(j_a, i_a) \in \mathcal{B}_N$ (see (3.1.2)), $r_a \ge 1$ for all $1 \le a \le m$, and $m \ge 0$. It follows by (4.2.10) that $v \in V$ can only belong to the μ-weight space V_μ if $v \in \mathbb{C}\cdot\xi$, thus V_μ is one dimensional, and moreover any other weight λ of V satisfies $\lambda \prec \mu$. □

We now determine how the coefficients of the distinguished central series $w(u)$ (see (3.3.14)) act on any highest weight representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$.

Proposition 4.2.7. *Let V be a highest weight representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ with the highest weight $\mu(u)$. Then*

$$w(u)|_V = \mu_n(-u)\mu_n(u)\mathrm{id}_V.$$

Proof. Let $\xi \in V$ be the highest weight vector. Since the coefficients of $w(u)$ belong to the center $ZX(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and V is spanned by elements of the form given in (4.2.28), the action of the $2r$-th coefficient w_{2r} of $w(u)$ on V is completely determined by its action on ξ. By (3.3.14) we have the relation $S(u)S(-u) = w(u) \cdot I$. Applying the $(n,n)^{th}$ entry of both sides to the highest weight vector ξ we obtain

$$w(u)\xi = \sum_{\ell \in \mathcal{I}_N} s_{n\ell}(u)s_{\ell n}(-u)\xi = s_{nn}(u)s_{nn}(-u)\xi = \mu_n(-u)\mu_n(u)\xi. \qquad \square$$

4.2.3 Verma modules and tensor products

We now introduce a universal highest weight module associated to any highest weight. As usual, these are called *Verma modules.*

Definition 4.2.8. Let $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_N^+}$ with $\mu_i(u)$ in $g_{ii} + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ for each $i \in \mathcal{I}_N^+$. We define the Verma module $M(\mu(u))$ over $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ as the quotient

$$M(\mu(u)) = X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}/\mathbf{J}$$

where $\mathbf{J}$ is left ideal of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ generated by the coefficients of

$$s_{ij}(u) \quad \forall\, i < j \in \mathcal{I}_N,$$
$$s_{kk}(u) - \mu_k(u) \quad \forall\, k \in \mathcal{I}_N^+.$$

We will see in Theorem 4.4.4 below that, similarly to the Verma modules for $X(\mathfrak{g}_N)$ (see Proposition 4.1.2), some choices of $\mu(u)$ may result in $M(\mu(u))$ being trivial. In fact, by Proposition 4.2.5, we already know that in the type B case $\widetilde{\mu}_0(u)$ necessarily satisfies the invariance relation (4.2.6) whenever $M(\mu(u))$ is non-trivial.

If $M(\mu(u))$ is non-trivial, then it is a highest weight module with the highest weight $\mu(u)$ and the highest weight vector $1_{\mu(u)}$ equal to the image of the identity element $1 \in X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ under the natural quotient map $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \twoheadrightarrow M(\mu(u))$.

As a consequence of Corollary 3.3.12, $M(\mu(u))$ is spanned by elements of the form

$$s_{j_1,i_1}^{(r_1)} \cdots s_{j_m,i_m}^{(r_m)} 1_{\mu(u)}$$

with $j_a > i_a$, $(j_a, i_a) \in \mathcal{B}_N$ (see (3.1.2)), $r_a \geq 1$ for all $1 \leq a \leq m$, and $m \geq 0$. Using this fact, together with the commutator relation (4.2.10), one can prove the following standard result.

Proposition 4.2.9. *Suppose $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_N^+}$ is such that the Verma module $M(\mu(u))$ is non-trivial. Then*

(1) If K is a submodule of $M(\mu(u))$, then $K = \bigoplus_\lambda K_\lambda$ where

$$K_\lambda = \{v \in K \,:\, F_{ii}^{\vartheta} v = \lambda_i v \;\; \forall \;\; 1 \leq i \leq n\} = M(\mu(u))_\lambda \cap K.$$

(2) If K is a proper submodule of $M(\mu(u))$, then $K \subseteq \bigoplus_{\lambda \neq \mu} M(\mu(u))_\lambda$, where

$$\mu = (\mu_i^{(1)} - \bar{g}_{ii})_{i \in \mathcal{I}_N^+}.$$

(3) $M(\mu(u))$ admits a unique irreducible quotient $V(\mu(u))$.

(4) Any irreducible highest weight $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module with the highest weight $\mu(u)$ is isomorphic to $V(\mu(u))$.

Recall from the proof of Theorem 4.2.6 that, given an $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module V, V^0 is the subspace

$$V^0 = \{\xi \in V \,:\, s_{ij}(u)\xi = 0 \quad \forall\, i < j \in \mathcal{I}_N\}.$$

The next corollary follows from a modification of the proof of the uniqueness of the highest weight vector in Theorem 4.2.6 and is analogous to Corollaries 3.2.8 and 4.2.7 of [Mol07].

Corollary 4.2.10. *Assume that the Verma module $M(\mu(u))$ is non-trivial and let $\xi \in V(\mu(u))$ be a highest weight vector. Then $V(\mu(u))^0 = \mathbb{C}\xi$.*

Since $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ is a left coideal subalgebra of $X(\mathfrak{g}_N)$, the tensor product of an $X(\mathfrak{g}_N)$-module L and an $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module V inherits the structure of an

$X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module via the coproduct Δ (see Lemma 3.3.7). More explicitly, for all $x \in X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, the action on $L \otimes V$ is given by

$$x \cdot w \otimes v = \Delta(x)(w \otimes v) \quad \forall\, w \in L \text{ and } v \in V.$$

In particular, we may take $L = L(\lambda(u))$ for some N-tuple $\lambda(u) = (\lambda_i(u))_{i \in \mathcal{I}_N}$ satisfying (4.1.1), and $V = V(\mu(u))$, where $\mu(u)$ is such that the Verma module $M(\mu(u))$ is non-trivial. If $L(\lambda(u))$ has the highest weight vector ξ and $V(\mu(u))$ has the highest weight vector η, then we may consider the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module

$$X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}(\xi \otimes \eta).$$

Our present goal is to show that this module is of highest weight type, and to compute its highest weight explicitly.

Set $t'_{ij}(u) = \theta_{ij} t_{-j,-i}(u)$ for all $i, j \in \mathcal{I}_N$. After rewriting

$$[t_{ij}(u), t'_{kl}(v)] = -[t'_{kl}(v), t_{ij}(u)],$$

the defining relation (2.7.16) of $X(\mathfrak{g}_N)$ takes the form

$$\begin{aligned}[t_{ij}(u), t'_{kl}(v)] =& \frac{1}{v-u}\Big(\theta_{j,-k} t_{i,-k}(u) t'_{-j,l}(v) - \theta_{i,-l} t'_{k,-i}(v) t_{-l,j}(u)\Big) \\ &- \frac{1}{v-u-\kappa} \sum_{a \in \mathcal{I}_N} \Big(\delta_{jk} t_{ia}(u) t'_{al}(v) - \delta_{il} t'_{ka}(v) t_{aj}(u)\Big).\end{aligned} \tag{4.2.29}$$

Recall that, given a tuple of series $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_N^+}$, $\widetilde{\mu}(u) = (\widetilde{\mu}_i(u))_{i \in \mathcal{I}_N^+}$ is the corresponding tuple whose components have been defined in (4.2.5).

Proposition 4.2.11. *Let $\xi \in L(\lambda(u))$ and $\eta \in V(\mu(u))$ be highest weight vectors. Then $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}(\xi \otimes \eta)$ is a highest weight $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module with the highest weight vector $\xi \otimes \eta$, and the highest weight $\gamma(u) = (\gamma_i(u))_{i \in \mathcal{I}_N^+}$ whose components are determined by the relations*

$$\widetilde{\gamma}_i(u) = \widetilde{\mu}_i(u) \lambda_i(u - \kappa/2) \lambda_{-i}(-u + \kappa/2) \quad \forall \quad i \in \mathcal{I}_N^+. \tag{4.2.30}$$

Proof. We will use the symbol $\equiv$ to denote equality of operators on the spaces $\mathbb{C}(\xi \otimes \eta)$

or $\mathbb{C}\xi$. We begin by showing that

$$s_{ij}(u) \cdot (\xi \otimes \eta) = 0 \quad \forall\, i < j.$$

By the symmetry relation (3.3.41), it is enough to consider the cases where $i < 0 < j$ or $0 \leq i < j$. By (3.3.17), we have

$$\Delta(s_{ij}(u)) \equiv \sum_{b \leq a} t_{ia}(u - \kappa/2) t'_{bj}(-u + \kappa/2) \otimes s_{ab}(u).$$

Moreover, we have $t'_{bj}(v)\xi = 0$ whenever $b < j$, so we can assume $b \geq j$. Since $i < j \leq b \leq a$, we have $t_{ia}(u)\xi = 0$ and also $a, b > 0$ since $j > 0$. By (4.2.29), we have

$$t_{ia}(u) t'_{bj}(v) \equiv \frac{\delta_{ab}}{u - v + \kappa} \sum_{c=j}^{n} t_{ic}(u) t'_{cj}(v). \tag{4.2.31}$$

Taking $a = b$ in the above and summing over $a \geq j$ gives

$$\sum_{a=j}^{n} t_{ia}(u) t'_{aj}(v) \equiv \frac{n - j + 1}{u - v + \kappa} \sum_{a=j}^{n} t_{ia}(u) t'_{aj}(v).$$

It follows from this that the right-hand side of (4.2.31) vanishes. This completes the proof that $\Delta(s_{ij}(u))(\xi \otimes \eta) = 0$ for all $i < j$.

Next, we compute $\Delta(s_{ii}(u))(\xi \otimes \eta)$ for all $i \in \mathcal{I}_N^+$. Using computations similar to those above, one can argue that $t_{ia}(u) t'_{bi}(v)\xi = 0$ whenever $a > b$. Thus,

$$\begin{aligned} &\Delta(s_{ii}(u))(\xi \otimes \eta) \\ &\quad = \sum_{a=i}^{n} t_{ia}(u - \kappa/2) t'_{ai}(-u + \kappa/2)\xi \otimes s_{aa}(u)\eta = (\hat{s}_{ii}(u)\xi) \otimes \eta, \end{aligned} \tag{4.2.32}$$

where $\hat{s}_{ii}(u)$ is the operator defined by the formula

$$\hat{s}_{ii}(u) = \sum_{a=i}^{n} \mu_a(u) t_{ia}(u - \kappa/2) t'_{ai}(-u + \kappa/2).$$

As a consequence of our work so far, it remains only to determine the eigenvalue $\gamma_i(u)$ of the operator $\hat{s}_{ii}(u)$ corresponding to the vector ξ. Define the operator $A_i(u)$ by the formula

$$A_i(u) = \sum_{a=i}^{n} t_{ia}(u-\kappa/2)t'_{ai}(-u+\kappa/2). \tag{4.2.33}$$

We first show that $A_i(u)\xi = \mu_i^\bullet(u)\xi$ for some scalar series $\mu_i^\bullet(u)$. By (4.2.29), for all $a > i \geq 0$, we have

$$\begin{aligned} & t_{ia}(u-\kappa/2)t'_{ai}(-u+\kappa/2) \\ & \quad \equiv \frac{1}{2u}\left(\sum_{b=i}^{n} t_{ib}(u-\kappa/2)t'_{bi}(-u+\kappa/2) - \sum_{b=a}^{n} t'_{ab}(-u+\kappa/2)t_{ba}(u-\kappa/2)\right). \end{aligned} \tag{4.2.34}$$

This implies that

$$A_i(u) \equiv t_{ii}\,(u-\kappa/2)\,t'_{ii}\,(-u+\kappa/2) + \frac{n-i}{2u}A_i(u) - \frac{1}{2u}\sum_{a=i+1}^{n} B_a(u),$$

where

$$B_a(u) = \sum_{b=a}^{n} t'_{ab}\,(-u+\kappa/2)\,t_{ba}\,(u-\kappa/2)\,.$$

Consequently,

$$\frac{2u-n+i}{2u}A_i(u) \equiv t_{ii}\,(u-\kappa/2)\,t'_{ii}\,(-u+\kappa/2) - \frac{1}{2u}\sum_{a=i+1}^{n} B_a(u). \tag{4.2.35}$$

Using the same method, one shows using (4.2.29) that

$$\frac{2u-n+i}{2u}B_i(u) \equiv t_{ii}\,(u-\kappa/2)\,t'_{ii}\,(-u+\kappa/2) - \frac{1}{2u}\sum_{a=i+1}^{n} A_a(u) \tag{4.2.36}$$

for all $i \in \mathcal{I}_N^+$. An easy downward induction then shows

$$B_i(u) \equiv A_i(u) \quad \forall\, i \in \mathcal{I}_N^+.$$

Substituting this result back into (4.2.35), we obtain

$$\frac{2u-n+i}{2u}A_i(u) \equiv t_{ii}\,(u-\kappa/2)\,t'_{ii}(-u+\kappa/2) - \frac{1}{2u}\sum_{a=i+1}^{n} A_a(u). \tag{4.2.37}$$

It follows by downward induction on $i \in \mathcal{I}_N^+$ that there is a tuple of formal series $\mu^\bullet(u) = (\mu_i^\bullet(u))_{i\in\mathcal{I}_N^+}$ such that $A_i(u)\xi = \mu_i^\bullet(u)\xi$ for all $i \in \mathcal{I}_N^+$. Moreover, the components of $\mu^\bullet(u)$ are determined by the relations

$$\widetilde{\mu}_i^\bullet(u) = 2u\lambda_i(u-\kappa/2)\lambda_{-i}(-u+\kappa/2) \quad \forall \quad i \in \mathcal{I}_N^+.$$

As $B_i(u) \equiv A_i(u)$ for all $i \in \mathcal{I}_N^+$, we may express (4.2.34) as

$$t_{ia}(u-\kappa/2)t'_{ai}(-u+\kappa/2) \equiv \frac{1}{2u}\left(A_i(u) - A_a(u)\right) \quad \forall\, a > i.$$

This gives $\hat{s}_{ii}(u)\xi = \gamma_i(u)\xi$ with

$$\gamma_i(u) = \mu_i(u)\lambda_i(u-\kappa/2)\lambda_{-i}(-u+\kappa/2) + \frac{1}{2u}\sum_{a=i+1}^{n} \mu_a(u)\left(\mu_i^\bullet(u) - \mu_a^\bullet(u)\right). \quad (4.2.38)$$

We now want to obtain the formula (4.2.30). Since

$$\mu_i^\bullet(u) = \frac{1}{2u-n+i}\left(\widetilde{\mu}_i^\bullet(u) - \sum_{a\geq i+1} \mu_a^\bullet(u)\right),$$

equation (4.2.38) implies that

$$\begin{aligned}(2u-n+i)\gamma_i(u) = {}& \frac{2u-n+i}{2u}\mu_i(u)\widetilde{\mu}_i^\bullet(u) + \frac{1}{2u}\sum_{j\geq i+1} \mu_j(u)\widetilde{\mu}_i^\bullet(u) \\ & - \frac{1}{2u}\sum_{a,j\geq i+1} \mu_j(u)\mu_a^\bullet(u) - \frac{2u-n+i}{2u}\sum_{j\geq i+1}\mu_j(u)\mu_j^\bullet(u).\end{aligned} \quad (4.2.39)$$

A straightforward downward induction on $i \in \mathcal{I}_n$ then shows that

$$\sum_{j\geq i+1}\gamma_j(u) = \frac{1}{2u}\sum_{a,j\geq i+1}\mu_j(u)\mu_a^\bullet(u) + \frac{2u-n+i}{2u}\sum_{j\geq i+1}\mu_j(u)\mu_j^\bullet(u).$$

Combining this with (4.2.39) proves that (4.2.30) holds for all $i \in \mathcal{I}_N^+$. □

Define the non-negative integers ℓ and $\hbar$ by

$$\ell = \begin{cases} 0 & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^\rho) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n), \\ q/2 & \text{otherwise,}\end{cases} \quad \text{and} \quad \hbar = n - \ell. \quad (4.2.40)$$

An important instance of Proposition 4.2.11 occurs when $\mu(u) = (g_{ii}(u))_{i\in\mathcal{I}_N^+}$, in which case $V(\mu(u))$ coincides with the trivial representation $V(\mathcal{G})$ with action given by the counit ϵ.

As the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules $L(\lambda(u))$ and $L(\lambda(u))\otimes V(\mathcal{G})$ are naturally isomorphic, (4.2.30) provides formulas for the highest weight of the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module

$$X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}\xi \subset L(\lambda(u)).$$

Corollary 4.2.12. *Let $\xi \in L(\lambda(u))$ be a highest weight vector. Then $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}\xi$ is a highest weight module with the highest weight $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_N^+}$ whose components are determined by*

$$\widetilde{\mu}_i(u) = \begin{cases} 2u\lambda_i(u-\kappa/2)\lambda_{-i}(-u+\kappa/2) & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n) \\ 2u\bar{g}_i(u)\lambda_i(u-\kappa/2)\lambda_{-i}(-u+\kappa/2) & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q) \end{cases} \tag{4.2.41}$$

where $\bar{g}_i(u)$ is the rational function of u

$$\bar{g}_i(u) = \begin{cases} g(u) & \text{if } i \leq k, \\ \left(\dfrac{\operatorname{tr}(\mathcal{G})+4u}{\operatorname{tr}(\mathcal{G})-4u}\right) & \text{if } i \geq k+1. \end{cases}$$

Proof. If $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n)$, then $\widetilde{g}_{ii}(u) = 2u$ for all $i \in \mathcal{I}_N^+$, and Proposition 4.2.11 gives

$$\widetilde{\mu}_i(u) = 2u\lambda_i(u-\kappa/2)\lambda_{-i}(-u+\kappa/2) \quad \forall\, i \in \mathcal{I}_N^+.$$

Suppose instead that $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})$ is of the form $(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$. Then

$$g_{ii}(u) = 1 \quad \forall \quad i \leq k \qquad \text{and} \qquad g_{ii}(u) = \frac{\operatorname{tr}(\mathcal{G})+4u}{\operatorname{tr}(\mathcal{G})-4u} \quad \forall \quad i > k.$$

It follows that $\widetilde{g}_{ii}(u) = \bar{g}_i(u)$ for all $i > k$, and

$$\begin{aligned} \widetilde{g}_{ii}(u) &= (2u-\ell)\left(\frac{\operatorname{tr}(\mathcal{G})-4u}{\operatorname{tr}(\mathcal{G})-4u}\right) + \ell\left(\frac{\operatorname{tr}(\mathcal{G})+4u}{\operatorname{tr}(\mathcal{G})-4u}\right) \\ &= 2u\left(\frac{\operatorname{tr}(\mathcal{G})+4\ell-4u}{\operatorname{tr}(\mathcal{G})-4u}\right) = 2ug(u) \end{aligned}$$

for all $i \leq k$. Hence, (4.2.41) follows from (4.2.30). □

4.3 Lowering the rank

Given a positive integer $m < n - \delta_{\mathfrak{g}_N,\mathfrak{so}_{2n}}$, consider the natural embedding

$$\iota_m : \mathfrak{g}_{N-2m} \hookrightarrow \mathfrak{g}_N, \quad F_{ij} \mapsto F_{ij} \quad \forall\, i,j \in \mathcal{I}_{N-2m}.$$

This injection satisfies $\iota_m(\mathfrak{g}_{N-2m}^{\vartheta(m)}) \subset \mathfrak{g}_N^{\vartheta}$, where

$$\vartheta(m) = \mathrm{Ad}(\mathcal{G}_m) \quad \text{and} \quad \mathcal{G}_m = \sum_{i,j\in\mathcal{I}_{N-2m}} g_{ij}E_{ij}.$$

In particular, ι_m can be viewed as an embedding of symmetric pairs

$$\iota_m : (\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)}) \hookrightarrow (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}).$$

Any $\mathfrak{g}_N^{\vartheta}$-module V can be studied as a $\mathfrak{g}_{N-2m}^{\vartheta(m)}$-module via the functor ι_m^* which sends V to $\iota_m^*(V)$, the $\mathfrak{g}_{N-2m}^{\vartheta(m)}$-module equal to V as a vector space, and in which

$$F_{ij}(v) = \iota_m(F_{ij})v \quad \forall\, i,j \in \mathcal{I}_{N-2m} \ \text{ and } \ v \in V.$$

Restriction functors of this type play an important role in the finite-dimensional representation theory of semisimple Lie algebras as they preserve information about highest weight theory and can be used for inductive arguments.

Unfortunately, one cannot mimic the above in the twisted Yangian setting as the assignment $s_{ij}(u) \mapsto s_{ij}(u)$ for $i,j \in \mathcal{I}_{N-2m}$ does not extend to a homomorphism

$$X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw} \to X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}.$$

In this section, we show that, despite this fact, there is a functor[4] from the category of finite-dimensional $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules to the category of finite-dimensional $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-modules which plays a role entirely analogous to ι_m^*.

4) We will not make any explicit reference to a functor below, but this interpretation should be implicitly understood.

4.3.1 From $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ to $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$

Let $1 \leq m < n - \delta_{\mathfrak{g}_N, \mathfrak{so}_{2n}}$. In what follows we will consider

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw} \quad \text{and} \quad X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$$

simultaneously, and for this reason it will be convenient to employ the notation $p_m(u)$ for the rational function $p_{\mathcal{G}_m}(u)$ defined in (3.3.9). That is,

$$p_m(u) = (\pm)1 \mp \frac{1}{2u-\kappa} + \frac{\mathrm{tr}(\mathcal{G}_m(u))}{2u-2\kappa}, \qquad \text{where} \quad \mathcal{G}_m(u) = \frac{\mathrm{tr}(\mathcal{G}_m)I - 4u\mathcal{G}_m}{\mathrm{tr}(\mathcal{G}_m) - 4u}.$$

For each $i, j \in \mathcal{I}_{N-2m}$, define $s_{ij}^{\circ m}(u) \in g_{ij} + u^{-1}X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}[\![u^{-1}]\!]$ by

$$s_{ij}^{\circ m}(u) = s_{ij}(u + \tfrac{m}{2}) + \frac{\delta_{ij}}{2u} \sum_{a=n-m+1}^{n} s_{aa}(u + \tfrac{m}{2}). \tag{4.3.1}$$

Given an arbitrary representation V of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, let $V_{(+,m)} \subset V$ be the subspace

$$V_{(+,m)} = \{v \in V \,:\, s_{ij}(u)v = 0 \quad \forall\, i < j \text{ with } n-m+1 \leq j \leq n\}. \tag{4.3.2}$$

If V is finite-dimensional and nonzero, then it contains an irreducible submodule and hence, by Theorem 4.2.6, a highest weight vector $\xi \in V_{(+,m)}$. In particular, $V_{(+,m)}$ is nonzero.

Our goal is to show that $V_{(+,m)}$ can be given the structure of a $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module using the series (4.3.1). This will be realized in Proposition 4.3.4 after treating the $m = 1$ case in Lemmas 4.3.1 and 4.3.3 below. In this setting, we will write

$$V_+ = V_{(+,1)} \quad \text{and} \quad s_{ij}^\circ(u) = s_{ij}^{\circ 1}(u) \quad \forall\, i, j \in \mathcal{I}_{N-2}.$$

Lemma 4.3.1. *Let V be an $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module. Then*

(1) V_+ is stable under the action of $\{s_{ij}^\circ(u)\}_{i,j \in \mathcal{I}_{N-2}}$. That is, we may view

$$s_{ij}^\circ(u) \in \mathrm{End}(V_+)[\![u^{-1}]\!] \quad \forall\, i, j \in \mathcal{I}_{N-2}.$$

(2) Setting

$$\mathsf{s}_{ij}(u)\cdot v = s^{\circ}_{ij}(u)v \quad \forall\, v\in V_+ \;\; \textit{and} \;\; i,j\in\mathcal{I}_{N-2} \tag{4.3.3}$$

equips V_+ *with the structure of an* $\mathsf{X}(\mathfrak{g}_{N-2},\mathfrak{g}_{N-2}^{\vartheta(1)})^{\mathsf{tw}}$*-module.*

Proof. The proof of the Lemma is straightforward, but very technical.

Step 1: $s^{\circ}_{ij}(u)\in \mathrm{End}(V_+)[\![u^{-1}]\!]$ for all $i,j\in\mathcal{I}_{N-2}$.

We will use $\equiv$ to denote equality of operators on V_+. We first show that

$$s_{kn}(u)s_{ij}(v)\equiv 0 \quad \forall\, k<n \;\; \text{and} \;\; i,j\in\mathcal{I}_{N-2}.$$

Since $s_{kn}(u)s_{ij}(v)\equiv -[s_{ij}(v),s_{kn}(u)]$, it is enough to show $[s_{ij}(v),s_{kn}(u)]\equiv 0$. By (3.3.42),

$$\begin{aligned}
&[s_{ij}(v),s_{kn}(u)]\\
&\quad\equiv \frac{\delta_{kj}}{v+u}s_{in}(v)s_{nn}(u)-\frac{\delta_{ij}}{v^2-u^2}\left(s_{kn}(v)s_{nn}(u)-s_{kn}(u)s_{nn}(v)\right)\\
&\qquad -\frac{\delta_{k,-i}}{v-u-\kappa}\theta_{i,-n}s_{-n,j}(v)s_{nn}(u)+\frac{\delta_{k,-i}}{(v+u)(v-u-\kappa)}\theta_{i,-j}s_{-j,n}(v)s_{nn}(u).
\end{aligned}$$

By the symmetry relation (3.3.41), it remains only to see $s_{\ell n}(v)s_{nn}(u)\equiv 0$ for any $\ell\in\mathcal{I}_{N-2}$. Using the expansion (3.3.42), we compute

$$s_{\ell n}(v)s_{nn}(u)\equiv \frac{1}{v+u}s_{\ell n}(v)s_{nn}(u). \tag{4.3.4}$$

Therefore, $s_{\ell n}(v)s_{nn}(u)\equiv 0$ for all $\ell\in\mathcal{I}_{N-2}$. This completes the proof that V_+ is stable under the action of all $s_{ij}(u)$ with $i,j\in\mathcal{I}_{N-2}$. Moreover, it shows that V_+ is stable under the action of the operator $s_{nn}(u)$. Thus, by definition V_+ is also stable under the action of all operators $s^{\circ}_{ij}(u)$.

Step 2: The action (4.3.3) defines an algebra homomorphism

$$\mathsf{X}(\mathfrak{g}_{N-2},\mathfrak{g}_{N-2}^{\vartheta(1)})^{\mathsf{tw}}\to \mathrm{End}(V_+).$$

First observe that

$$[s_{-n,-n}(u), s_{nn}(v)] \equiv 0, \quad [s_{nn}(u), s_{nn}(v)] \equiv 0 \quad \text{and} \quad [s_{nn}(u), s_{ij}(v)] \equiv 0 \tag{4.3.5}$$

for all $i, j \in \mathcal{I}_{N-2}$. To see this, note that by (3.3.42) we have

$$[s_{nn}(u), s_{nn}(v)] \equiv \left(\frac{1}{u-v} + \frac{1}{u+v} - \frac{1}{u^2-v^2}\right)[s_{nn}(u), s_{nn}(v)],$$

which implies $[s_{nn}(u), s_{nn}(v)] \equiv 0$. Furthermore,

$$[s_{ij}(u), s_{nn}(v)] \equiv -\frac{\delta_{ij}}{u^2-v^2}[s_{nn}(u), s_{nn}(v)] \equiv 0 \quad \forall\, i, j \in \mathcal{I}_{N-2}.$$

The symmetry relation (3.3.41) together with the second and third equivalences of (4.3.5) then give

$$[s_{-n,-n}(u), s_{nn}(v)] \equiv -\frac{1}{2u-2\kappa}\sum_{a=-n}^{n}[s_{aa}(u), s_{nn}(v)] \equiv -\frac{1}{2u-2\kappa}[s_{-n,-n}(u), s_{nn}(v)].$$

Hence, $[s_{-n,-n}(u), s_{nn}(v)] \equiv 0$.

Now let $i, j, k, \ell \in \mathcal{I}_{N-2}$. As a consequence of the second and third equivalences in (4.3.5), we have

$$[s^{\circ}_{ij}(u), s^{\circ}_{k\ell}(v)] \equiv [s_{ij}(\mathring{u}), s_{k\ell}(\mathring{v})],$$

where

$$\mathring{u} = u + 1/2 \quad \text{and} \quad \mathring{v} = v + 1/2.$$

Thus, appealing to (3.3.42), we obtain the expression

$$\begin{aligned}
&[s^{\circ}_{ij}(u), s^{\circ}_{k\ell}(v)] \\
&\quad \equiv \frac{1}{u-v}\left(s_{kj}(\mathring{u})s_{i\ell}(\mathring{v}) - s_{kj}(\mathring{v})s_{i\ell}(\mathring{u})\right) \\
&\qquad + \frac{1}{\mathring{u}+\mathring{v}}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{kj}s_{ia}(\mathring{u})s_{a\ell}(\mathring{v}) - \delta_{i\ell}s_{ka}(\mathring{v})s_{aj}(\mathring{u})\right) \\
&\qquad - \frac{\delta_{ij}}{\mathring{u}^2-\mathring{v}^2}\sum_{a\in\mathcal{I}_{N-2}}\left(s_{ka}(\mathring{u})s_{a\ell}(\mathring{v}) - s_{ka}(\mathring{v})s_{a\ell}(\mathring{u})\right) \\
&\qquad - \frac{1}{\mathring{u}-\mathring{v}-\kappa}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{k,-i}\theta_{ia}s_{aj}(\mathring{u})s_{-a,\ell}(\mathring{v}) - \delta_{l,-j}\theta_{aj}s_{k,-a}(\mathring{v})s_{ia}(\mathring{u})\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\mathring{u}+\mathring{v}-\kappa}\left(\theta_{j,-k} s_{i,-k}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \theta_{i,-\ell} s_{k,-i}(\mathring{v}) s_{-\ell,j}(\mathring{u})\right) \\
&+ \frac{\theta_{i,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)} \sum_{a\in\mathcal{I}_{N-2}} \left(\delta_{k,-i} s_{-j,a}(\mathring{u}) s_{a\ell}(\mathring{v}) - \delta_{\ell,-j} s_{ka}(\mathring{v}) s_{a,-i}(\mathring{u})\right) \\
&+ \frac{\theta_{i,-j}}{(\mathring{u}-\mathring{v})(\mathring{u}+\mathring{v}-\kappa)} \left(s_{k,-i}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - s_{k,-i}(\mathring{v}) s_{-j,\ell}(\mathring{u})\right) \\
&- \frac{\theta_{ij}}{(\mathring{u}-\mathring{v}-\kappa)(\mathring{u}+\mathring{v}-\kappa)} \sum_{a\in\mathcal{I}_{N-2}} \left(\delta_{k,-i} s_{aa}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{k,-i}(\mathring{v}) s_{aa}(\mathring{u})\right) \\
&+ \frac{1}{\mathring{u}+\mathring{v}}\left(\delta_{kj} s_{in}(\mathring{u}) s_{n\ell}(\mathring{v}) - \delta_{i\ell} s_{kn}(\mathring{v}) s_{nj}(\mathring{u})\right) \qquad (4.3.6) \\
&- \frac{\delta_{ij}}{\mathring{u}^2-\mathring{v}^2}\left(s_{kn}(\mathring{u}) s_{n\ell}(\mathring{v}) - s_{kn}(\mathring{v}) s_{n\ell}(\mathring{u})\right) \qquad (4.3.7) \\
&- \frac{1}{\mathring{u}-\mathring{v}-\kappa}\left(\delta_{k,-i}\theta_{i,-n} s_{-n,j}(\mathring{u}) s_{n\ell}(\mathring{v}) - \delta_{l,-j}\theta_{-n,j} s_{kn}(\mathring{v}) s_{i,-n}(\mathring{u})\right) \qquad (4.3.8) \\
&+ \frac{\theta_{i,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)}\left(\delta_{k,-i} s_{-j,n}(\mathring{u}) s_{n\ell}(\mathring{v}) - \delta_{\ell,-j} s_{kn}(\mathring{v}) s_{n,-i}(\mathring{u})\right) \qquad (4.3.9) \\
&- \frac{\theta_{ij}}{(\mathring{u}-\mathring{v}-\kappa)(\mathring{u}+\mathring{v}-\kappa)}\left(\delta_{k,-i} s_{-n,-n}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{k,-i}(\mathring{v}) s_{-n,-n}(\mathring{u})\right) \\
&- \frac{\theta_{ij}}{(\mathring{u}-\mathring{v}-\kappa)(\mathring{u}+\mathring{v}-\kappa)}\left(\delta_{k,-i} s_{nn}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{k,-i}(\mathring{v}) s_{nn}(\mathring{u})\right).
\end{aligned}
$$

We now need to rewrite (4.3.6)–(4.3.9) in a way that will enable us to compare the right-hand side above with the right-hand side of the reflection equation (3.3.42) for $\mathsf{X}(\mathfrak{g}_{N-2}, \mathfrak{g}_{N-2}^{\vartheta(1)})^{\mathsf{tw}}$ with $\{\mathsf{s}_{ij}^1(u)\}_{i,j\in\mathcal{I}_{N-2}}$ replaced by $\{s_{ij}^\circ(u)\}_{i,j\in\mathcal{I}_{N-2}}$.

Step 2.1: Re-expressing (4.3.6).

Using the reflection equation (3.3.42) for $[s_{in}(\mathring{u}), s_{n\ell}(\mathring{v})]$ with $i \in \mathcal{I}_{N-2}$ and rearranging the terms yields

$$
\begin{aligned}
\frac{1}{\mathring{u}+\mathring{v}} s_{in}(\mathring{u}) s_{n\ell}(\mathring{v}) \equiv\ & \frac{1}{(\mathring{u}-\mathring{v})(\mathring{u}+\mathring{v}-1)}\left(s_{nn}(\mathring{u}) s_{i\ell}(\mathring{v}) - s_{nn}(\mathring{v}) s_{i\ell}(\mathring{u})\right) \\
&+ \frac{1}{(\mathring{u}+\mathring{v})(\mathring{u}+\mathring{v}-1)} \sum_{a\in\mathcal{I}_{N-2}} s_{ia}(\mathring{u}) s_{a\ell}(\mathring{v}) \qquad (4.3.10) \\
&- \frac{\delta_{i\ell}}{(\mathring{u}+\mathring{v})(\mathring{u}+\mathring{v}-1)} s_{nn}(\mathring{v}) s_{nn}(\mathring{u}).
\end{aligned}
$$

This computation, together with (4.3.5), implies that the expression (4.3.6) can be rewritten as

$$\begin{aligned}
&\frac{1}{\mathring{u}+\mathring{v}}\left(\delta_{kj}s_{in}(\mathring{u})s_{n\ell}(\mathring{v})-\delta_{i\ell}s_{kn}(\mathring{v})s_{nj}(\mathring{u})\right)\\
&\quad\equiv\frac{1}{(\mathring{u}-\mathring{v})(\mathring{u}+\mathring{v}-1)}\Big(\delta_{kj}\left(s_{nn}(\mathring{u})s_{i\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{i\ell}(\mathring{u})\right)\\
&\qquad\qquad\qquad+\delta_{i\ell}\left(s_{nn}(\mathring{v})s_{kj}(\mathring{u})-s_{nn}(\mathring{u})s_{kj}(\mathring{v})\right)\Big)\\
&\qquad+\frac{1}{(\mathring{u}+\mathring{v})(\mathring{u}+\mathring{v}-1)}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{kj}s_{ia}(\mathring{u})s_{a\ell}(\mathring{v})-\delta_{i\ell}s_{ka}(\mathring{v})s_{aj}(\mathring{u})\right).
\end{aligned}\tag{4.3.11}$$

Step 2.2: Re-expressing (4.3.7).

Similarly, (4.3.10) and (4.3.5) imply that (4.3.7) can be expressed as:

$$\begin{aligned}
&-\frac{\delta_{ij}}{\mathring{u}^2-\mathring{v}^2}\left(s_{kn}(\mathring{u})s_{n\ell}(\mathring{v})-s_{kn}(\mathring{v})s_{n\ell}(\mathring{u})\right)\\
&\qquad\equiv-\frac{\delta_{ij}}{(\mathring{u}^2-\mathring{v}^2)(\mathring{u}+\mathring{v}-1)}\sum_{a\in\mathcal{I}_{N-2}}\left(s_{ka}(\mathring{u})s_{a\ell}(\mathring{v})-s_{ka}(\mathring{v})s_{a\ell}(\mathring{u})\right).
\end{aligned}\tag{4.3.12}$$

Step 2.3: Re-expressing (4.3.8).

An analogous but more lengthy computation to that used in obtaining (4.3.10) gives the relation

$$\begin{aligned}
&\frac{\mathring{u}-\mathring{v}-\kappa+1}{\mathring{u}-\mathring{v}-\kappa}s_{-n,j}(\mathring{u})s_{n\ell}(\mathring{v})\\
&\equiv-\frac{1}{\mathring{u}-\mathring{v}-\kappa}\sum_{a\in\mathcal{I}_{N-2}}\theta_{-n,a}s_{aj}(\mathring{u})s_{-a,\ell}(\mathring{v})+\frac{\delta_{\ell,-j}}{\mathring{u}-\mathring{v}-\kappa}\theta_{-n,j}s_{nn}(\mathring{v})s_{-n,-n}(\mathring{u})\\
&\quad-\frac{1}{\mathring{u}+\mathring{v}-\kappa}\left(\theta_{j,-n}s_{-n,-n}(\mathring{u})s_{-j,\ell}(\mathring{v})-\theta_{-n,-\ell}s_{nn}(\mathring{v})s_{-\ell,j}(\mathring{u})\right)\\
&\quad+\frac{\theta_{-n,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)}\sum_{a\in\mathcal{I}_{N-2}}s_{-j,a}(\mathring{u})s_{a\ell}(\mathring{v})\\
&\quad+\frac{\theta_{-n,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)}s_{-j,n}(\mathring{u})s_{n\ell}(\mathring{v})-\frac{\theta_{-n,-j}\delta_{\ell,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)}s_{nn}(\mathring{v})s_{nn}(\mathring{u})\\
&\quad+\frac{\theta_{-n,-j}}{(\mathring{u}-\mathring{v})(\mathring{u}+\mathring{v}-\kappa)}\left(s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{-j,\ell}(\mathring{u})\right)\\
&\quad-\frac{\theta_{-n,j}}{(\mathring{u}-\mathring{v}-\kappa)(\mathring{u}+\mathring{v}-\kappa)}\sum_{a\in\mathcal{I}_{N}}\left(s_{aa}(\mathring{u})s_{-j,\ell}(\mathring{v})-\delta_{\ell,-j}s_{nn}(\mathring{v})s_{aa}(\mathring{u})\right).
\end{aligned}\tag{4.3.13}$$

Similarly, since $s_{kn}(\mathring{v})s_{i,-n}(\mathring{u}) \equiv -[s_{i,-n}(\mathring{u}), s_{kn}(\mathring{v})]$, we have

$$
\begin{aligned}
&\frac{\mathring{u}-\mathring{v}-\kappa+1}{\mathring{u}-\mathring{v}-\kappa} s_{kn}(\mathring{v})s_{i,-n}(\mathring{u}) \\
&\equiv -\frac{1}{\mathring{u}-\mathring{v}-\kappa} \sum_{a\in\mathcal{I}_{N-2}} \theta_{a,-n} s_{k,-a}(\mathring{v}) s_{ia}(\mathring{u}) + \frac{\delta_{k,-i}}{\mathring{u}-\mathring{v}-\kappa} \theta_{i,-n} s_{-n,-n}(\mathring{u}) s_{nn}(\mathring{v}) \\
&\quad + \frac{1}{\mathring{u}+\mathring{v}-\kappa} \left(\theta_{-n,-k} s_{i,-k}(\mathring{u}) s_{nn}(\mathring{v}) - \theta_{i,-n} s_{k,-i}(\mathring{v}) s_{-n,-n}(\mathring{u})\right) \\
&\quad + \frac{\theta_{in}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)} \sum_{a\in\mathcal{I}_{N-2}} s_{ka}(\mathring{v}) s_{a,-i}(\mathring{u}) \\
&\quad + \frac{\theta_{in}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)} s_{kn}(\mathring{v}) s_{n,-i}(\mathring{u}) - \frac{\theta_{in}\delta_{k,-i}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)} s_{nn}(\mathring{u}) s_{nn}(\mathring{v}) \\
&\quad - \frac{\theta_{in}}{(\mathring{u}-\mathring{v})(\mathring{u}+\mathring{v}-\kappa)} \left(s_{k,-i}(\mathring{u}) s_{nn}(\mathring{v}) - s_{k,-i}(\mathring{v}) s_{nn}(\mathring{u})\right) \\
&\quad + \frac{\theta_{i,-n}}{(\mathring{u}-\mathring{v}-\kappa)(\mathring{u}+\mathring{v}-\kappa)} \sum_{a\in\mathcal{I}_{N}} \left(\delta_{k,-i} s_{aa}(\mathring{u}) s_{nn}(\mathring{v}) - s_{k,-i}(\mathring{v}) s_{aa}(\mathring{u})\right).
\end{aligned}
\tag{4.3.14}
$$

Using the two equivalences (4.3.13) and (4.3.14), together with (4.3.5), we obtain the following expression for (4.3.8):

$$
\begin{aligned}
&\frac{-1}{\mathring{u}-\mathring{v}-\kappa} \left(\delta_{k,-i}\theta_{i,-n} s_{-n,j}(\mathring{u}) s_{n\ell}(\mathring{v}) - \delta_{l,-j}\theta_{-n,j} s_{kn}(\mathring{v}) s_{i,-n}(\mathring{u})\right) \\
&= f_-(u,v) \sum_{a\in\mathcal{I}_{N-2}} \left(\delta_{k,-i}\theta_{ia} s_{aj}(\mathring{u}) s_{-a,\ell}(\mathring{v}) - \delta_{l,-j}\theta_{aj} s_{k,-a}(\mathring{v}) s_{ia}(\mathring{u})\right) \\
&\quad - \theta_{i,-j} \frac{f_-(u,v)}{\mathring{u}+\mathring{v}} \sum_{a\in\mathcal{I}_{N-2}} \left(\delta_{k,-i} s_{-j,a}(\mathring{u}) s_{a\ell}(\mathring{v}) - \delta_{\ell,-j} s_{ka}(\mathring{v}) s_{a,-i}(\mathring{u})\right) \\
&\quad + \theta_{ij} \frac{f_-(u,v)}{\mathring{u}+\mathring{v}-\kappa} \sum_{a\in\mathcal{I}_{N}} \left(\delta_{k,-i} s_{aa}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{k,-i}(\mathring{v}) s_{aa}(\mathring{u})\right) \\
&\quad + f_+(u,v)\delta_{k,-i} \left(\theta_{ij} s_{-n,-n}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - \theta_{i,-\ell} s_{nn}(\mathring{v}) s_{-\ell,j}(\mathring{u})\right) \\
&\quad + f_+(u,v)\delta_{\ell,-j} \left(\theta_{j,-k} s_{i,-k}(\mathring{u}) s_{nn}(\mathring{v}) - \theta_{ij} s_{k,-i}(\mathring{v}) s_{-n,-n}(\mathring{u})\right) \\
&\quad - \theta_{i,-j}\delta_{k,-i} \frac{f_+(u,v)}{\mathring{u}-\mathring{v}} \left(s_{nn}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - s_{nn}(\mathring{v}) s_{-j,\ell}(\mathring{u})\right) \\
&\quad - \theta_{i,-j}\delta_{\ell,-j} \frac{f_+(u,v)}{\mathring{u}-\mathring{v}} \left(s_{k,-i}(\mathring{u}) s_{nn}(\mathring{v}) - s_{k,-i}(\mathring{v}) s_{nn}(\mathring{u})\right) \\
&\quad - \theta_{i,-j} \frac{f_-(u,v)}{\mathring{u}+\mathring{v}} \left(\delta_{k,-i} s_{-j,n}(\mathring{u}) s_{n,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{kn}(\mathring{v}) s_{n,-i}(\mathring{u})\right),
\end{aligned}
\tag{4.3.15}
$$

where $f_\pm(u,v)$ is given by

$$f_\pm(u,v) = \frac{1}{(\mathring{u} \pm \mathring{v} - \kappa)(\mathring{u} - \mathring{v} - \kappa + 1)}.$$

Step 2.4: Re-expressing (4.3.9).

If we add the last line of (4.3.15) to (4.3.9), we obtain

$$\frac{\theta_{i,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa+1)} \left(\delta_{k,-i} s_{-j,n}(\mathring{u}) s_{n,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{kn}(\mathring{v}) s_{n,-i}(\mathring{u})\right).$$

We can also re-express this using (4.3.10) and (4.3.5). This yields

$$\begin{aligned}
&\frac{\theta_{i,-j}}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa+1)} \left(\delta_{k,-i} s_{-j,n}(\mathring{u}) s_{n,\ell}(\mathring{v}) - \delta_{\ell,-j} s_{kn}(\mathring{v}) s_{n,-i}(\mathring{u})\right) \\
&= \theta_{i,-j}\delta_{k,-i} \frac{f(u,v)}{\mathring{u}-\mathring{v}} \left(s_{nn}(\mathring{u}) s_{-j,\ell}(\mathring{v}) - s_{nn}(\mathring{v}) s_{-j,\ell}(\mathring{u})\right) \\
&+ \theta_{i,-j}\delta_{\ell,-j} \frac{f(u,v)}{\mathring{u}-\mathring{v}} \left(s_{nn}(\mathring{v}) s_{k,-i}(\mathring{u}) - s_{nn}(\mathring{u}) s_{k,-i}(\mathring{v})\right) \\
&+ \theta_{i,-j} \frac{f(u,v)}{\mathring{u}+\mathring{v}} \sum_{a\in\mathcal{I}_{N-2}} \left(\delta_{k,-i} s_{-j,a}(\mathring{u}) s_{a\ell}(\mathring{v}) - \delta_{l,-j} s_{ka}(\mathring{v}) s_{a,-i}(\mathring{u})\right),
\end{aligned} \tag{4.3.16}$$

where $f(u,v)$ is given by

$$f(u,v) = \frac{1}{(\mathring{u}+\mathring{v}-1)(\mathring{u}-\mathring{v}-\kappa+1)}.$$

Step 2.5: The reflection equation (3.3.42) is preserved.

Next, observe that the following identities hold:

$$\begin{aligned}
\frac{1}{\mathring{u}+\mathring{v}} + \frac{1}{(\mathring{u}+\mathring{v})(\mathring{u}+\mathring{v}-1)} &= \frac{1}{u+v}, \\
f_-(u,v) - \frac{1}{\mathring{u}-\mathring{v}-\kappa} &= -\frac{1}{u-v-\mathring{\kappa}}, \\
\frac{1}{(\mathring{u}+\mathring{v})(\mathring{u}-\mathring{v}-\kappa)} - \frac{f_-(u,v)}{\mathring{u}+\mathring{v}} + \frac{f(u,v)}{\mathring{u}+\mathring{v}} &= \frac{1}{(u-v-\mathring{\kappa})(u+v)},
\end{aligned}$$

where $\mathring{\kappa} = \kappa - 1$ is $\frac{1}{4} c_{\mathfrak{g}_{N-2}}$. Therefore, combining the new expressions (4.3.11), (4.3.12), (4.3.15) and (4.3.16) and substituting them back into (4.3.6)–(4.3.9) gives:

$$
\begin{aligned}
&[s^{\circ}_{ij}(u), s^{\circ}_{k\ell}(v)] \\
&\equiv \frac{1}{u-v}\left(s_{kj}(\mathring{u})s_{i\ell}(\mathring{v}) - s_{kj}(\mathring{v})s_{i\ell}(\mathring{u})\right) \\
&+ \frac{1}{u+v}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{kj}s_{ia}(\mathring{u})s_{a\ell}(\mathring{v}) - \delta_{i\ell}s_{ka}(\mathring{v})s_{aj}(\mathring{u})\right) \\
&- \frac{\delta_{ij}}{u^2-v^2}\sum_{a\in\mathcal{I}_{N-2}}\left(s_{ka}(\mathring{u})s_{a\ell}(\mathring{v}) - s_{ka}(\mathring{v})s_{a\ell}(\mathring{u})\right) \\
&- \frac{1}{u-v-\mathring{\kappa}}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{k,-i}\theta_{ia}s_{aj}(\mathring{u})s_{-a,\ell}(\mathring{v}) - \delta_{l,-j}\theta_{aj}s_{k,-a}(\mathring{v})s_{ia}(\mathring{u})\right) \\
&- \frac{1}{u+v-\mathring{\kappa}}\left(\theta_{j,-k}s_{i,-k}(\mathring{u})s_{-j,\ell}(\mathring{v}) - \theta_{i,-\ell}s_{k,-i}(\mathring{v})s_{-\ell,j}(\mathring{u})\right) \\
&+ \frac{\theta_{i,-j}}{(u+v)(u-v-\mathring{\kappa})}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{k,-i}s_{-j,a}(\mathring{u})s_{a\ell}(\mathring{v}) - \delta_{\ell,-j}s_{ka}(\mathring{v})s_{a,-i}(\mathring{u})\right) \\
&+ \frac{\theta_{i,-j}}{(u-v)(u+v-\mathring{\kappa})}\left(s_{k,-i}(\mathring{u})s_{-j,\ell}(\mathring{v}) - s_{k,-i}(\mathring{v})s_{-j,\ell}(\mathring{u})\right) \\
&- \frac{\theta_{ij}}{(u-v-\mathring{\kappa})(u+v-\mathring{\kappa})}\sum_{a\in\mathcal{I}_{N-2}}\left(\delta_{k,-i}s_{aa}(\mathring{u})s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j}s_{k,-i}(\mathring{v})s_{aa}(\mathring{u})\right) \\
&+ \mathcal{B}(u,v),
\end{aligned}
$$

where $\mathcal{B}(u,v)$ is defined as the operator

$$
\begin{aligned}
&\mathcal{B}(u,v) \\
&= \frac{\delta_{kj}}{u^2-v^2}\left(s_{nn}(\mathring{u})s_{i\ell}(\mathring{v}) - s_{nn}(\mathring{v})s_{i\ell}(\mathring{u})\right) \\
&+ \frac{\delta_{i\ell}}{u^2-v^2}\left(s_{nn}(\mathring{v})s_{kj}(\mathring{u}) - s_{nn}(\mathring{u})s_{kj}(\mathring{v})\right) \\
&+ \frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{k,-i}\left(\theta_{ij}s_{-n,-n}(\mathring{u})s_{-j,\ell}(\mathring{v}) - \theta_{i,-\ell}s_{nn}(\mathring{v})s_{-\ell,j}(\mathring{u})\right) \\
&+ \frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{\ell,-j}\left(\theta_{j,-k}s_{i,-k}(\mathring{u})s_{nn}(\mathring{v}) - \theta_{ij}s_{k,-i}(\mathring{v})s_{-n,-n}(\mathring{u})\right) \\
&- \frac{\theta_{i,-j}\delta_{k,-i}}{(u-v)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v}) - s_{nn}(\mathring{v})s_{-j,\ell}(\mathring{u})\right) \\
&- \frac{\theta_{i,-j}\delta_{\ell,-j}}{(u-v)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{k,-i}(\mathring{u})s_{nn}(\mathring{v}) - s_{k,-i}(\mathring{v})s_{nn}(\mathring{u})\right)
\end{aligned}
$$

$$+\frac{\theta_{i,-j}\delta_{k,-i}}{(u-v)(u+v)(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{-j,\ell}(\mathring{u})\right)$$
$$+\frac{\theta_{i,-j}\delta_{\ell,-j}}{(u-v)(u+v)(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{v})s_{k,-i}(\mathring{u})-s_{nn}(\mathring{u})s_{k,-i}(\mathring{v})\right)$$
$$-\frac{\theta_{ij}}{(u-v-\mathring{\kappa})(u+v-\mathring{\kappa})}\left(\delta_{k,-i}s_{-n,-n}(\mathring{u})s_{-j,\ell}(\mathring{v})-\delta_{\ell,-j}s_{k,-i}(\mathring{v})s_{-n,-n}(\mathring{u})\right)$$
$$-\frac{\theta_{ij}}{(u-v-\mathring{\kappa})(u+v-\mathring{\kappa})}\left(\delta_{k,-i}s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})-\delta_{\ell,-j}s_{k,-i}(\mathring{v})s_{nn}(\mathring{u})\right).$$

Adding terms together, and applying (4.3.5) where necessary, we obtain the equivalence of operators

$$\begin{aligned}
&\mathcal{B}(u,v)\\
&\equiv\frac{\delta_{kj}}{u^2-v^2}\left(s_{nn}(\mathring{u})s_{i\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{i\ell}(\mathring{u})\right)\\
&+\frac{\delta_{i\ell}}{u^2-v^2}\left(s_{nn}(\mathring{v})s_{kj}(\mathring{u})-s_{nn}(\mathring{u})s_{kj}(\mathring{v})\right)\\
&+\frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{\ell,-j}\left(\theta_{j,-k}s_{i,-k}(\mathring{u})s_{nn}(\mathring{v})+\theta_{ij}s_{k,-i}(\mathring{v})s_{nn}(\mathring{u})\right)\\
&-\frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{k,-i}\left(\theta_{i,-\ell}s_{nn}(\mathring{v})s_{-\ell,j}(\mathring{u})+\theta_{ij}s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})\right)\\
&-\frac{\mathring{\kappa}\delta_{k,-i}\theta_{i,-j}}{(u^2-v^2)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{-j,\ell}(\mathring{u})\right)\\
&-\frac{\mathring{\kappa}\delta_{\ell,-j}\theta_{i,-j}}{(u^2-v^2)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{v})s_{k,-i}(\mathring{u})-s_{nn}(\mathring{u})s_{k,-i}(\mathring{v})\right).
\end{aligned}\tag{4.3.17}$$

Let $\mathcal{D}(u,v)$ be the expression on the right-hand side of the reflection equation (3.3.42) for $\mathsf{X}(\mathfrak{g}_{N-2},\mathfrak{g}_{N-2}^{\vartheta(1)})^{\mathsf{tw}}$ with $\{\mathsf{s}_{ij}^{1}(u)\}_{i,j\in\mathcal{I}_{N-2}}$ replaced by $\{s_{ij}^{\circ}(u)\}_{i,j\in\mathcal{I}_{N-2}}$. A lengthy computation using the definition of the elements $s_{ij}^{\circ}(u)$ and again appealing to (4.3.5) where necessary yields the equivalence

$$\mathcal{D}(u,v)\equiv[s_{ij}^{\circ}(u),s_{k\ell}^{\circ}(v)]-\mathcal{B}(u,v)+\mathcal{A}(u,v),$$

where $\mathcal{A}(u,v)$ is the operator defined by:

$$\begin{aligned}
&\mathcal{A}(u,v)\\
&\equiv\frac{\delta_{kj}}{u^2-v^2}\left(s_{nn}(\mathring{u})s_{i\ell}(\mathring{v})-s_{nn}(\mathring{v})s_{i\ell}(\mathring{u})\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\delta_{i\ell}}{u^2-v^2}\left(s_{nn}(\mathring{v})s_{kj}(\mathring{u}) - s_{nn}(\mathring{u})s_{kj}(\mathring{v})\right)\\
&+ \frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{\ell,-j}\left(\theta_{j,-k}s_{i,-k}(\mathring{u})s_{nn}(\mathring{v}) + \theta_{ij}s_{k,-i}(\mathring{v})s_{nn}(\mathring{u})\right)\\
&- \frac{1}{(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\delta_{k,-i}\left(\theta_{i,-\ell}s_{nn}(\mathring{v})s_{-\ell,j}(\mathring{u}) + \theta_{ij}s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v})\right)\\
&+ \frac{\mathring{\kappa}\theta_{ij}+\theta_{i,-j}}{u(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(\delta_{k,-i}s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j}s_{nn}(\mathring{u})s_{k,-i}(\mathring{v})\right)\\
&- \frac{\mathring{\kappa}\delta_{k,-i}\theta_{i,-j}}{(u^2-v^2)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v}) - s_{nn}(\mathring{v})s_{-j,\ell}(\mathring{u})\right)\\
&- \frac{\mathring{\kappa}\delta_{\ell,-j}\theta_{i,-j}}{(u^2-v^2)(u+v-\mathring{\kappa})(u-v-\mathring{\kappa})}\left(s_{nn}(\mathring{v})s_{k,-i}(\mathring{u}) - s_{nn}(\mathring{u})s_{k,-i}(\mathring{v})\right)\\
&- \frac{\theta_{ij}(N-2)}{2u(u-v-\mathring{\kappa})(u+v-\mathring{\kappa})}\left(\delta_{k,-i}s_{nn}(\mathring{u})s_{-j,\ell}(\mathring{v}) - \delta_{\ell,-j}s_{k,-i}(\mathring{v})s_{nn}(\mathring{u})\right).
\end{aligned}
$$

Therefore, to complete the proof of the lemma it remains only to see $\mathcal{A}(u,v) \equiv \mathcal{B}(u,v)$. Comparing the above expression for $\mathcal{A}(u,v)$ with (4.3.17), we see that it is enough to show

$$\mathring{\kappa}\theta_{ij} + \theta_{i,-j} - \theta_{ij}(\tfrac{N}{2} - 1) = 0$$

This follows from the identities

$$\kappa = \frac{N}{2} \mp 1, \quad \frac{N}{2} - 1 = \kappa - 1 \pm 1 = \mathring{\kappa} \pm 1 \quad \text{and} \quad \theta_{i,-j} = \pm\theta_{i,j}. \qquad \square$$

Remark 4.3.2. The definition of the operators $\{s^\circ_{ij}(u)\}_{i,j\in\mathcal{I}_{N-2}}$ are motivated by the proof of [MR02, Theorem 4.6], where a similar result to Lemma 4.3.1 played an integral role.

One could postulate that the action of the reflection algebra $\mathsf{X}(\mathfrak{g}_{N-2}, \mathfrak{g}_{N-2}^{\vartheta(1)})^{\mathsf{tw}}$ on V_+ given by Lemma 4.3.1 factors through $X(\mathfrak{g}_{N-2}, \mathfrak{g}_{N-2}^{\vartheta(1)})^{tw}$. We will soon see that if $(\mathfrak{g}_N, \mathfrak{g}_N^\rho) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n)$, then this is indeed the case. However, if the pair $(\mathfrak{g}_N, \mathfrak{g}_N^\rho)$ is of the form $(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$, then the operators (4.3.1) fail to satisfy the defining symmetry relation (3.3.41) of the algebra $X(\mathfrak{g}_{N-2}, \mathfrak{g}_{N-2}^{\vartheta(1)})^{tw}$.

The next Lemma shows that this issue can be avoided by replacing $s^\circ_{ij}(u)$ with $h(u)s^\circ_{ij}(u)$ for a suitable formal series $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$.

Lemma 4.3.3. *Let V be an $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module such that V_+ is nonzero, and fix*

$h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. Then

$$s_{ij}(u) \cdot v = h(u) s^{\circ}_{ij}(u) v \quad \forall\, v \in V_+ \;\; \textit{and} \;\; i, j \in \mathcal{I}_{N-2} \tag{4.3.18}$$

defines an $X(\mathfrak{g}_{N-2}, \mathfrak{g}^{\vartheta(1)}_{N-2})^{tw}$-module structure on V_+ if and only if

$$h(u) h(\kappa - 1 - u)^{-1} = p_1(u) p(u + \tfrac{1}{2})^{-1}. \tag{4.3.19}$$

Moreover, a solution $h(u)$ of (4.3.19) exists in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$.

Proof. Let us equip V_+ with the $\mathsf{X}(\mathfrak{g}_{N-2}, \mathfrak{g}^{\vartheta(1)}_{N-2})^{\mathsf{tw}}$-module structure given by Lemma 4.3.1. By Theorem 3.3.22,

$$X(\mathfrak{g}_{N-2}, \mathfrak{g}^{\vartheta(1)}_{N-2})^{tw} \cong \mathsf{X}(\mathfrak{g}_{N-2}, \mathfrak{g}^{\vartheta(1)}_{N-2})^{\mathsf{tw}} / (\mathsf{c}(u) - 1)$$

and thus the first assertion of the lemma is equivalent to the statement that $\mathsf{c}(u)$ operates as the identity operator in the twisted module $\nu_h^*(V_+)$ if and only if $h(u)$ satisfies (4.3.19). By (3.3.43), the automorphsim ν_h satisfies

$$\nu_h(\mathsf{c}(u)) = h(\mathring{\kappa} - u)^{-1} h(u) \mathsf{c}(u), \qquad \text{where} \;\; \mathring{\kappa} = \kappa - 1,$$

and it therefore suffices to prove that

$$\mathsf{c}(u)|_{V_+} = p(u + \tfrac{1}{2}) p_1(u)^{-1} \mathrm{id}_{V_+}. \tag{4.3.20}$$

Fix $i \in \mathcal{I}_{N-2}$. Since $s^{\circ}_{ii}(u) \in g_{ii} + u^{-1}\mathrm{End}(V_+)[\![u^{-1}]\!]$ is invertible, it follows from (3.3.33) that the action of $\mathsf{c}(u)$ on $V_{(+,m)}$ is completely determined by

$$p_1(u) \mathsf{c}(u) s^{\circ}_{ii}(\mathring{\kappa} - u) = s^{\circ}_{-i,-i}(u) \mp \frac{s^{\circ}_{ii}(u)}{2u - \mathring{\kappa}} + \frac{\sum_{k \in \mathcal{I}_{N-2}} s^{\circ}_{kk}(u)}{2u - 2\mathring{\kappa}}.$$

The identity (4.3.20) will therefore hold if

$$p(\mathring{u}) s^{\circ}_{ii}(\mathring{\kappa} - u) = s^{\circ}_{-i,-i}(u) \mp \frac{s^{\circ}_{ii}(u)}{2u - \mathring{\kappa}} + \frac{\sum_{k \in \mathcal{I}_{N-2}} s^{\circ}_{kk}(u)}{2u - 2\mathring{\kappa}}, \tag{4.3.21}$$

where $\mathring{u} = u + \frac{1}{2}$. By (4.3.1), we have

$$\begin{aligned} s^{\circ}_{-i,-i}(u) &\mp \frac{s^{\circ}_{ii}(u)}{2u - \mathring{\kappa}} + \frac{\sum_{k\in\mathcal{I}_{N-2}} s^{\circ}_{kk}(u)}{2u - 2\mathring{\kappa}} \\ &= s_{-i,-i}(\mathring{u}) \mp \frac{s_{ii}(\mathring{u})}{2u - \mathring{\kappa}} + \frac{\sum_{k\in\mathcal{I}_{N-2}} s_{kk}(\mathring{u})}{2u - 2\mathring{\kappa}} + p_I(u)\frac{s_{nn}(\mathring{u})}{2u}, \end{aligned} \tag{4.3.22}$$

where $p_I(u) = p_{I_{N-2}}(u)$. By the symmetry relation in $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$,

$$p(\mathring{u})s_{jj}(\kappa - \mathring{u}) = s_{-j,-j}(\mathring{u}) \mp \frac{s_{jj}(\mathring{u})}{2u - \mathring{\kappa}} + \frac{\mathrm{tr}(S(\mathring{u}))\cdot I}{2u - 2\mathring{\kappa} - 1} \quad \forall\, j \in \mathcal{I}_N.$$

This implies that

$$\begin{aligned} &p(\mathring{u})s^{\circ}_{ii}(\mathring{\kappa} - u) \\ &\quad= p(\mathring{u})s_{ii}(\kappa - \mathring{u}) + \frac{p(\mathring{u})}{2\mathring{\kappa} - 2u}\, s_{nn}(\kappa - \mathring{u}) \\ &\quad= s_{-i,-i}(\mathring{u}) \mp \frac{s_{ii}(\mathring{u})}{2u - \mathring{\kappa}} + \frac{\sum_{k\in\mathcal{I}_{N-2}} s_{kk}(\mathring{u})}{2u - 2\mathring{\kappa}} \\ &\quad+ \left(\frac{1}{2u - 2\mathring{\kappa} - 1} + \frac{1}{2\mathring{\kappa} - 2u} + \frac{1}{(2\mathring{\kappa} - 2u)(2u - 2\mathring{\kappa} - 1)}\right) s_{-n,-n}(\mathring{u}) \\ &\quad+ \left(\frac{1}{2u - 2\mathring{\kappa} - 1} \mp \frac{1}{(2u - \mathring{\kappa})(2\mathring{\kappa} - 2u)} + \frac{1}{(2u - 2\mathring{\kappa} - 1)(2\mathring{\kappa} - 2u)}\right) s_{nn}(\mathring{u}) \\ &\quad= s_{-i,-i}(\mathring{u}) \mp \frac{s_{ii}(\mathring{u})}{2u - \mathring{\kappa}} + \frac{\sum_{k\in\mathcal{I}_{N-2}} s_{kk}(\mathring{u})}{2u - 2\mathring{\kappa}} + p_I(u)\frac{s_{nn}(\mathring{u})}{2u}, \end{aligned}$$

where in the last line we have used (3.3.7). Comparing with (4.3.22), we deduce that (4.3.21) holds.

We now turn to establishing the existence of $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying (4.3.19). Set

$$c(u) = p(u + \tfrac{1}{2})p_1(u)^{-1} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!].$$

Let $h(u)$ be the unique solution of $h(u)^2 = c(u)^{-1}$ in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. It follows from (3.3.11) that $c(u)$ satisfies

$$c(\mathring{\kappa} - u) = p_1(u)p(u + \tfrac{1}{2})^{-1} = c(u)^{-1}.$$

Hence we have

$$1 = c(u)^{-1}c(\mathring{\kappa} - u)^{-1} = (h(u)h(\mathring{\kappa} - u))^2.$$

As 1 is the unique square root of itself in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, we can conclude that $h(u)h(\mathring{\kappa} - u) = 1$, and thus that

$$h(u)h(\mathring{\kappa} - u)^{-1} = h(u)^2 = c(u)^{-1} = p_1(u)p(u + \tfrac{1}{2})^{-1}. \qquad \square$$

We now once again assume that $1 \leq m < n - \delta_{\mathfrak{g}_N, \mathfrak{so}_{2n}}$. The next proposition generalizes Lemmas 4.3.1 and 4.3.3.

Proposition 4.3.4. *Let V be an $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module. Then*

(1) $V_{(+,m)}$ is stable under the action of $\{s_{ij}^{\circ m}(u)\}_{i,j \in \mathcal{I}_{N-2m}}$. That is, we may view

$$s_{ij}^{\circ m}(u) \in \mathrm{End}(V_{(+,m)})[\![u^{-1}]\!] \quad \forall\, i, j \in \mathcal{I}_{N-2m}.$$

(2) For any $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, setting

$$\mathsf{s}_{ij}(u) \cdot v = h(u)s_{ij}^{\circ m}(u)v \quad \forall\, v \in V_{(+,m)} \;\; \text{and} \;\; i, j \in \mathcal{I}_{N-2m} \tag{4.3.23}$$

equips $V_{(+,m)}$ with the structure of an $\mathsf{X}(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{\mathsf{tw}}$-module.

(3) Provided $V_{(+,m)}$ is nonzero, (4.3.23) descends to an $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-action if and only if

$$h(u)h(\kappa - m - u)^{-1} = p_m(u)p(u + \tfrac{m}{2})^{-1}. \tag{4.3.24}$$

Moreover, a solution $h(u)$ of (4.3.24) exists in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$.

Proof. First note that if $k+1 < n - \delta_{\mathfrak{g}_N,\mathfrak{so}_{2n}}$ then, for each $i,j \in \mathcal{I}_{N-2(k+1)}$, we have

$$\begin{aligned}
& s_{ij}^{\circ k}(u+\tfrac{1}{2}) + \frac{\delta_{ij}}{2u} s_{n-k,n-k}^{\circ k}(u+\tfrac{1}{2}) \\
=& s_{ij}(u+\tfrac{k+1}{2}) + \frac{\delta_{ij}}{2u+1}\sum_{a=n-k+1}^{n} s_{aa}^{\circ k}(u+\tfrac{k+1}{2}) \\
& + \frac{1}{2u}\left(s_{n-k,n-k}(u+\tfrac{k+1}{2}) + \frac{1}{2u+1}\sum_{a=n-k+1}^{n} s_{aa}^{\circ k}(u+\tfrac{k+1}{2})\right) \\
& = s_{ij}(u+\tfrac{k+1}{2}) + \frac{\delta_{ij}}{2u}\sum_{a=n-k}^{n} s_{aa}^{\circ k}(u+\tfrac{k+1}{2}) \\
& = s_{ij}^{\circ(k+1)}(u).
\end{aligned} \tag{4.3.25}$$

By the existence statement for $h(u)$ in Lemma 4.3.3, we may choose

$$\{h_a(u)\}_{a=1}^m \subset 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$$

satisfying the set of relations

$$h_a(u)h_a(\kappa - a - u)^{-1} = p_a(u)p_{a-1}(u+\tfrac{1}{2})^{-1} \quad \forall\, 1 \le a \le m. \tag{4.3.26}$$

Combining Lemmas 4.3.1 and 4.3.3 with the relation (4.3.25) and the definition of $V_{(+,m)}$, we deduce using a simple induction argument that

$$s_{ij}(u)\cdot v = d(u)s_{ij}^{\circ m}(u)v \quad \forall\, v \in V_{(+,m)} \text{ and } i,j \in \mathcal{I}_{N-2m} \tag{4.3.27}$$

defines an $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module structure on $V_{(+,m)}$, where

$$d(u) = h_m(u)h_{m-1}(u+\tfrac{1}{2})\cdots h_1(u+\tfrac{m-1}{2}).$$

Since $d(u)$ is invertible, this implicitly implies that Part (1) holds.

Consider now Part (2). Replacing $\{s_{ij}(u)\}_{i,j\in\mathcal{I}_{N-2m}}$ by $\{\mathsf{s}_{ij}(u)\}_{i,j\in\mathcal{I}_{N-2m}}$ in (4.3.27) equips $V_{(+,m)}$ with an $\mathsf{X}(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{\mathsf{tw}}$-module structure in which $\mathsf{c}(u)$ operates as the identity.

Given a fixed series $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, set $g(u) = h(u)d(u)^{-1}$. Then the action of $\mathsf{X}(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{\mathsf{tw}}$ on the module $\nu_g^*(V_{(+,m)})$ is given by (4.3.23), and thus Part (2) holds.

Now let us turn to proving Part (3). By (3.3.43), we have

$$\nu_g(\mathsf{c}(u)) = g(\kappa - m - u)g(u)^{-1}\mathsf{c}(u).$$

It follows that the central series $\mathsf{c}(u)$ operates as

$$g(\kappa - m - u)g(u)^{-1}\mathrm{id}_{V_{(+,m)}} = h(\kappa - m - u)h(u)^{-1}d(\kappa - m - u)^{-1}d(u)\mathrm{id}_{V_{(+,m)}}$$

in $\nu_g^*(V_{(+,m)})$. Thus, the action (4.3.23) descends to an $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-action if and only if

$$h(\kappa - m - u)h(u)^{-1} = d(\kappa - m - u)d(u)^{-1}.$$

By (4.3.26) and the definition of $d(u)$, the right-hand side is

$$\begin{aligned}\prod_{a=0}^{m-1} & h_{m-a}(\kappa - m - u + \tfrac{a}{2})h_{m-a}(u + \tfrac{a}{2})^{-1} \\ &= \prod_{a=0}^{m-1} p_{m-a}(u + \tfrac{a}{2})p_{m-a-1}(u + \tfrac{a+1}{2})^{-1} \\ &= p_m(u)p(u + \tfrac{m}{2})^{-1}.\end{aligned}$$

Since $h(u) = d(u)$ is a solution to (4.3.24), we may conclude that Part (3) of the Proposition holds. □

4.3.2 Highest weight properties

In this subsection we show that in the special case where $V = V(\mu(u))$, additional information is encoded in the $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module $V_{(+,m)}$ of Proposition 4.3.4. First, we prove that there is a particular series $h(u)$ which can be regarded as the most natural solution of (4.3.24).

Let $\mathscr{g}_m(u)$ be the rational function from (3.3.3) associated to $(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})$. That is,

$$\mathscr{g}_m(u) = \begin{cases} \dfrac{N - 2m - 4u}{\mathrm{tr}(\mathcal{G}_m) - 4u} & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \\ u^{-1} & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n). \end{cases}$$

Proposition 4.3.5. *Fix $1 \leq m < n - \delta_{\mathfrak{g}_N,\mathfrak{so}_{2n}}$. Then the series*

$$h(u) = \frac{u}{u+\frac{m}{2}} \cdot \mathcal{g}_m(u)\mathcal{g}(u+\tfrac{m}{2})^{-1} \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!] \tag{4.3.28}$$

satisfies (4.3.24). If $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$, this is the unique choice of $h(u)$ with the property that

$$V(\mathcal{G}_m) \cong V(\mathcal{G})_{(+,m)},$$

where the $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module structure on $V(\mathcal{G})_{(+,m)}$ is given by (4.3.23).

Proof. Let us first show that $h(u)$, as defined in (4.3.28), satisfies the equation (4.3.24). By Proposition 3.3.3 and (3.3.7), we have

$$\begin{aligned} h(\mathring{\kappa} - u) &= \frac{\mathring{\kappa} - u}{\mathring{\kappa} - u + \frac{m}{2}} \cdot \frac{p_{I_{N-2m}}(u)}{p_{I_N}(u+\frac{m}{2})} \cdot \frac{p(u+\frac{m}{2})}{p_m(u)} \cdot \frac{\mathcal{g}_m(u)}{\mathcal{g}(u+\frac{m}{2})} \\ &= \frac{p(u+\frac{m}{2})}{p_m(u)} h(u), \end{aligned}$$

where $\mathring{\kappa} = \kappa - m$. Thus, $h(u)$ satisfies (4.3.24).

Suppose now that $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$ and let $h(u)$ be an arbitrary solution of (4.3.24). Since $V(\mathcal{G})_{(+,m)} = V(\mathcal{G})$ as vector spaces, $V(\mathcal{G})$ is a $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module with action given by (4.3.23). Equivalently,

$$\begin{aligned} s_{ij}(u) \cdot \xi &= h(u) g_{ij}^{\circ m}(u) \xi \quad \forall\, i,j \in \mathcal{I}_{N-2m}, \\ \text{where} \quad g_{ij}^{\circ m}(u) &= g_{ij}(u+\tfrac{m}{2}) + \frac{\delta_{ij}}{2u} \sum_{a=n-m+1}^{n} g_{aa}(u+\tfrac{m}{2}) \end{aligned}$$

and ξ is any nonzero vector in the one-dimensional space $V(\mathcal{G})$. In particular,

$$V(\mathcal{G})_{(+,m)} \cong V\Big((h(u) g_{ii}^{\circ m}(u))_{i \in \mathcal{I}_{N-2m}}\Big).$$

Since any highest weight $\mu(u)$ is completely determined by the corresponding series $\widetilde{\mu}(u)$ defined in (4.2.5), $V(\mathcal{G})_{(+,m)}$ is isomorphic to $V(\mathcal{G}_m)$ if and only if

$$h(u)\widetilde{g_{ii}^{\circ m}}(u) = \widetilde{g_{ii}^{m}}(u) \quad \forall\, i \in \mathcal{I}_{N-2m}^{+}, \tag{4.3.29}$$

where $\mathcal{G}_m(u) = \sum_{i,j \in \mathcal{I}_{N-2m}} g_{ij}^{m}(u) E_{ij}$. The equality (4.3.31) proven in Corollary 4.2.10

below gives $\widetilde{g_{ii}^{\circ m}}(u) = \widetilde{g_{ii}}(u + \frac{m}{2})$ for all $i \in \mathcal{I}_{N-2m}^+$. Since there is at most one $h(u)$ satisfying the above equations, we are left to show that $h(u)$ given by (4.3.28) satisfies

$$h(u) = \widetilde{g_{ii}^m}(u)\widetilde{g_{ii}}(u + \tfrac{m}{2})^{-1} \quad \forall\, i \in \mathcal{I}_{N-2m}^+.$$

If $i \leq \hbar$, then the proof of Corollary 4.2.12 shows that the right-hand side is

$$\frac{u}{u + \frac{m}{2}} \cdot \mathcal{g}_m(u)\mathcal{g}(u + \tfrac{m}{2})^{-1},$$

as desired. If instead $i \geq \hbar + 1$, then we also have $\hbar \leq n - m$, and the proof of Corollary 4.2.12 gives

$$\begin{aligned}
\widetilde{g_{ii}^m}(u)\widetilde{g_{ii}}(u + \tfrac{m}{2})^{-1} &= \frac{u}{u + \frac{m}{2}} \left(\frac{\mathrm{tr}(\mathcal{G}_m) + 4u}{\mathrm{tr}(\mathcal{G}_m) - 4u}\right)\left(\frac{\mathrm{tr}(\mathcal{G}) - 2m - 4u}{\mathrm{tr}(\mathcal{G}) + 2m + 4u}\right) \\
&= \frac{u}{u + \frac{m}{2}} \left(\frac{\mathrm{tr}(\mathcal{G}) - 2m - 4u}{\mathrm{tr}(\mathcal{G}) + 2m - 4u}\right) \\
&= \frac{u}{u + \frac{m}{2}} \cdot \mathcal{g}_m(u)\mathcal{g}(u + \tfrac{m}{2})^{-1}.
\end{aligned}$$

□

Remark 4.3.6. When $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_{2n}, \mathfrak{gl}_{2n})$, the expression (4.3.28) collapses to $h(u) = 1$. Though this is certainly the most natural solution of (4.3.24), it does not preserve the trivial representation in the sense described in the second assertion of Proposition 4.3.5. In fact, no solution to (4.3.24) does: solving the system (4.3.29) yields

$$h(u) = \frac{u}{u + \frac{m}{2}},$$

for which (4.3.24) does not hold.

Henceforth, given a $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module V and fixed positive integer $m < n - \delta_{\mathfrak{g}_N, \mathfrak{so}_{2n}}$, we will write $V_{(m)}$ for the $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module which is equal to

$$V_{(+,m)} = \{v \in V \,:\, s_{ij}(u)v = 0 \quad \forall\, i < j \text{ with } n - m + 1 \leq j \leq n\}$$

as a vector space, and has module structure given by

$$s_{ij}(u) \cdot v = h(u) s_{ij}^{\circ m}(u) v \quad \forall\, v \in V_{(+,m)} \text{ and } i,j \in \mathcal{I}_{N-2m},$$

where

$$h(u) = \frac{u}{u+\frac{m}{2}} \cdot g_m(u) g(u+\tfrac{m}{2})^{-1}, \tag{4.3.30}$$

$$s_{ij}^{\circ m}(u) = s_{ij}(u+\tfrac{m}{2}) + \frac{\delta_{ij}}{2u} \sum_{a=n-m+1}^{n} s_{aa}(u+\tfrac{m}{2}) \quad \forall\, i,j \in \mathcal{I}_{N-2m}.$$

If in addition V is isomorphic to a module of the form $V(\mu(u))$, we will write V_m for the cyclic submodule

$$V_m = X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw} \xi \subset V_{(m)},$$

where $\xi \in V(\mu(u))$ is any highest weight vector.

Corollary 4.3.7. *Fix a positive integer $m < n - \delta_{\mathfrak{g}_N, \mathfrak{so}_{2n}}$ and let $\xi \in V(\mu(u))$ be a highest weight vector. Then*

(1) $V(\mu(u))_m$ is a highest weight module with the highest weight

$$h(u)\mu^{\circ m}(u) = (h(u)\mu_i^{\circ m}(u))_{i \in \mathcal{I}_{N-2m}^+},$$

where the components of $\mu^{\circ m}(u)$ are uniquely determined by

$$\widetilde{\mu_i^{\circ m}}(u) = \widetilde{\mu}_i(u+\tfrac{m}{2}) \quad \forall\, i \in \mathcal{I}_{N-2m}^+. \tag{4.3.31}$$

(2) $V(\mu(u))_m$ is the unique highest weight submodule of $V(\mu(u))_{(m)}$. In particular, if $V(\mu(u))$ is finite-dimensional then

$$V(\mu(u))_m \cong V(h(u)\mu^{\circ m}(u)).$$

Proof. It is clear that $V(\mu(u))_m$ is a highest weight module with the highest weight $(h(u)\mu_i^{\circ m}(u))_{i \in \mathcal{I}_{N-2m}^+}$, where

$$\mu_i^{\circ m}(u) = \mu_i(u+\tfrac{m}{2}) + \frac{1}{2u} \sum_{a=n-m+1}^{n} \mu_a(u+\tfrac{m}{2}) \quad \forall\, i \in \mathcal{I}_{N-2m}^+.$$

To prove Part (1), it thus suffices to show that, for each $i \in \mathcal{I}_{N-2m}^+$, the corresponding series $\widetilde{\mu_i^{\circ m}}(u)$ is indeed given by (4.3.31). By definition,

$$\begin{aligned}
\widetilde{\mu_i^{\circ m}}(u) &= (2u-n+m+i)\mu_i^{\circ m}(u) + \sum_{k=i+1}^{n-m} \mu_k^{\circ m}(u) \\
&= (2u-n+m+i)\mu_i(u+\tfrac{m}{2}) + \sum_{k=i+1}^{n-m} \mu_k(u+\tfrac{m}{2}) \\
&\quad + \left(\frac{2u-n+m+i}{2u} + \frac{n-m-i}{2u}\right) \sum_{a=n-m+1}^{n} \mu_a(u+\tfrac{m}{2}) \\
&= (2(u+\tfrac{m}{2})-n+i)\mu_i(u+\tfrac{m}{2}) + \sum_{a=i+1}^{n} \mu_a(u+\tfrac{m}{2}) \\
&= \widetilde{\mu}_i(u+\tfrac{m}{2}).
\end{aligned}$$

Now let us turn to proving Part (2). Suppose that K is any highest weight submodule of $V(\mu(u))_{(m)}$, and let $\xi_K \in K$ be a highest weight vector. Since $V(\mu(u))_{(m)}$ is equal to $V(\mu(u))_{(+,m)}$ as a vector space,

$$s_{ij}(u)\xi_K = 0 = s_{-j,-i}(u)\xi_K \quad \forall\, i<j \ \text{ with } \ n-m+1 \le j \le n.$$

Since ξ_K is a highest weight vector of K, we must also have

$$s_{ij}(u+\tfrac{m}{2})\xi_K = 0 \implies s_{ij}(u)\xi_K = 0 \quad \forall\, i<j \in \mathcal{I}_{N-2m},$$

Combining these two facts gives $s_{ij}(u)\xi_K = 0$ for all $i < j \in \mathcal{I}_N$, and thus $\xi \in V(\mu(u))^0$. By Corollary 4.2.10, $V(\mu(u))^0 = \mathbb{C}\xi$ and therefore ξ_K is a nonzero scalar multiple of ξ. As ξ generates $V(\mu(u))_m$, this implies $K = V(\mu(u))_m$.

The second assertion of Part (2) follows from the first assertion and the fact that, if $V(\mu(u))_m$ is finite-dimensional, every proper nonzero submodule $K \subset V(\mu(u))_m$ must contain an irreducible submodule, and hence a highest weight vector. □

4.4 Reduction to type AIII

Consider now the embedding

$$\iota_{\mathfrak{gl}} : \mathfrak{gl}_n \hookrightarrow \mathfrak{g}_N, \quad E_{ij} \mapsto F_{ij} \quad \forall\, 1 \leq i, j \leq n.$$

Let $\vartheta_{\mathfrak{gl}}$ be the involution of $\mathfrak{gl}_n$ defined by $\vartheta_{\mathfrak{gl}} = \mathrm{Ad}(\mathcal{G}_{\mathfrak{gl}})$, where

$$\mathcal{G}_{\mathfrak{gl}} = \sum_{i,j=1}^{n} g_{ij} E_{ij} = \sum_{i=1}^{\hbar} E_{ii} - \sum_{i=\hbar+1}^{n} E_{ii}$$

and $\hbar$ is the non-negative integer defined in (4.2.40). In particular,

$$\mathfrak{gl}_n^{\vartheta_{\mathfrak{gl}}} \cong \mathfrak{gl}_{\hbar} \oplus \mathfrak{gl}_{\ell}$$

and $\iota_{\mathfrak{gl}}$ may be viewed as an embedding of symmetric pairs

$$\iota_{\mathfrak{gl}} : (\mathfrak{gl}_n, \mathfrak{gl}_{\hbar} \oplus \mathfrak{gl}_{\ell}) \hookrightarrow (\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}).$$

Our goal in this section is construct a twisted Yangian analogue of the functor $\iota_{\mathfrak{gl}}^*$ and apply it to study highest weight $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules. Analogously to the last section, this will involve constructing an $X(\mathfrak{sl}_n, \mathfrak{sl}_{\hbar} \oplus \mathfrak{gl}_{\ell})^{tw}$-module structure on a (generally proper) subspace of any $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module, where $X(\mathfrak{sl}_n, \mathfrak{sl}_{\hbar} \oplus \mathfrak{gl}_{\ell})^{tw}$ is the extended twisted Yangian of type AIII from §3.4.

4.4.1 From $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ to $X(\mathfrak{sl}_n, \mathfrak{sl}_{\hbar} \oplus \mathfrak{gl}_{\ell})^{tw}$

Let V be an arbitrary representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$, and define J be the left ideal in $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ generated by the coefficients of the series $s_{-i,j}(u)$ with $i \in \mathcal{I}_N^+$ and $1 \leq j \leq n$. We define V^J to be the subspace of V annihilated by J:

$$V^J = \{v \in V \,:\, s_{-i,j}(u)v = 0 \quad \forall\, i \in \mathcal{I}_N^+ \text{ and } 1 \leq j \leq n\}. \tag{4.4.1}$$

Note that if V is finite-dimensional or a highest weight module, then V^J always contains a highest weight vector and hence is nonzero.

Proposition 4.4.1. *Let V be an $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module. Then*

(1) V^J is stable under the action of $\{s_{ij}(u)\}_{1\leq i,j\leq n}$. That is, we may view

$$s_{ij}(u) \in \mathrm{End}(V^J)[\![u^{-1}]\!] \quad \forall\, 1 \leq i,j \leq n.$$

(2) Setting

$$\mathsf{b}_{ij}(u)\cdot v = s_{ij}(u)v \quad \forall\, v \in V^J \;\; \text{and} \;\; 1 \leq i,j \leq n \tag{4.4.2}$$

equips V^J with the structure of an $\mathsf{X}(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_\ell)^{\mathsf{tw}}$-module.

(3) Setting

$$b_{ij}(u)\cdot v = \sigma_{ij}(u)v \quad \forall\, v \in V^J \;\; \text{and} \;\; 1 \leq i,j \leq n \tag{4.4.3}$$

equips V^J with the structure of an $X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_\ell)^{tw}$-module.

Proof. We begin with Part (1). We must show that $s_{-i,j}(u)s_{k\ell}(v) = 0 \mod J$ for all $i \in \mathcal{I}_N^+$ and $1 \leq j,k,\ell \leq n$, or equivalently

$$[s_{-i,j}(u), s_{k\ell}(v)] \equiv 0 \quad \forall\, i \in \mathcal{I}_N^+ \;\text{ and }\; 1 \leq j,k,\ell \leq n, \tag{4.4.4}$$

where $\equiv$ is used to denote equality of operators on V^J. Let us first show the above equivalence holds assuming $1 \leq i \leq n$. This is immediate if $k \neq i,j$ by (3.3.42). Consider the case where $i = j = k$. As a consequence of the relation (3.3.42), we have

$$\begin{aligned}[s_{-i,i}(u), s_{i\ell}(v)] \equiv &\left(\frac{1}{u+v} + \frac{1}{(u+v)(u-v-\kappa)}\right)\sum_{a=1}^{n} s_{-i,a}(u)s_{a\ell}(v) \\ &- \frac{1}{u-v-\kappa}\sum_{a=1}^{n} s_{-a,i}(u)s_{a\ell}(v).\end{aligned} \tag{4.4.5}$$

Computing $s_{-a,i}(u)s_{a\ell}(v)$ for $a \neq i$, we obtain

$$\begin{aligned}[s_{-a,i}(u), s_{a\ell}(v)] \equiv &-\frac{1}{u-v-\kappa}\sum_{b=1}^{n} s_{-b,i}(u)s_{b\ell}(v) \\ &+ \frac{1}{(u+v)(u-v-\kappa)}\sum_{b=1}^{n} s_{-i,b}(u)s_{b\ell}(v) \\ \equiv &\, [s_{-i,i}(u), s_{i\ell}(v)] - \frac{1}{u+v}\sum_{b=1}^{n} s_{-i,b}(u)s_{b\ell}(v),\end{aligned}$$

where the last equivalence is a direct consequence of equation (4.4.5). Substituting the above result back into (4.4.5), we get

$$\begin{aligned}
&[s_{-i,i}(u), s_{i\ell}(v)] \\
&\equiv \left(\frac{1}{u+v} + \frac{1}{(u+v)(u-v-\kappa)}\right)\sum_{a=1}^{n} s_{-i,a}(u)s_{a\ell}(v) \\
&- \frac{n}{u-v-\kappa}[s_{-i,i}(u), s_{i\ell}(v)] + \frac{n-1}{(u+v)(u-v-\kappa)}\sum_{b=1}^{n} s_{-i,b}(u)s_{b\ell}(v) \\
&\equiv \left(1 + \frac{n}{u-v-\kappa}\right)\frac{1}{u+v}\sum_{a=1}^{n} s_{-i,a}(u)s_{a\ell}(v) - \frac{n}{u-v-\kappa}[s_{-i,i}(u), s_{i\ell}(v)],
\end{aligned}$$

which implies that

$$[s_{-i,i}(u), s_{i\ell}(v)] \equiv \frac{1}{u+v}\sum_{a=1}^{n} s_{-i,a}(u)s_{a\ell}(v). \tag{4.4.6}$$

By (3.3.42), for all $a \neq i$ and $a \geq 1$, we have the relation

$$s_{-i,a}(u)s_{a\ell}(v) \equiv \frac{1}{u+v}\sum_{b=1}^{n} s_{-i,b}(u)s_{b\ell}(v) \equiv [s_{-i,i}(u), s_{i\ell}(v)].$$

Substituting this into (4.4.6), we arrive at

$$[s_{-i,i}(u), s_{i\ell}(v)] \equiv \frac{n}{u+v}[s_{-i,i}(u), s_{i\ell}(v)],$$

which allows us to conclude that $[s_{-i,i}(u), s_{i\ell}(v)] \equiv 0$ for all $1 \leq i \leq n$.

Now, let us consider the case $i \neq j$. As a consequence of relation (3.3.41), it is enough to consider the case where $j = k$. By (3.3.42) we have:

$$[s_{-i,j}(u), s_{j\ell}(v)] \equiv \frac{1}{u+v}\sum_{a=1}^{n} s_{-i,a}(u)s_{a\ell}(v).$$

However, by (4.4.6), the right-hand side of the above is equivalent to 0. Thus, we have shown that (4.4.4) holds for $1 \leq i \leq n$.

If $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, then we must also show

$$[s_{0j}(u), s_{k\ell}(v)] \equiv 0 \quad \forall\, 1 \leq j, k, \ell \leq n.$$

This is immediate from (3.3.42) unless $j=k$, and in this case we obtain

$$[s_{0j}(u), s_{j\ell}(v)] \equiv \frac{1}{u+v}\sum_{a=1}^{n} s_{0a}(u)s_{a\ell}(v). \tag{4.4.7}$$

However, the same computation shows that

$$s_{0a}(u)s_{a\ell}(v) \equiv \frac{1}{u+v}\sum_{b=1}^{n} s_{0b}(u)s_{b\ell}(v) \equiv [s_{0j}(u), s_{j\ell}(v)],$$

and so (4.4.7) yields

$$\left(1-\frac{n}{u+v}\right)[s_{0j}(u), s_{j\ell}(v)] \equiv 0.$$

This completes the proof of Part (1).

Consider now Part (2). By (3.3.42), we have the following equivalence of operators for all $1 \le i,j,k,l \le n$:

$$\begin{aligned}
[s_{ij}(u), s_{kl}(v)] \equiv{}& \frac{1}{u-v}\left(s_{kj}(u)s_{i\ell}(v) - s_{kj}(v)s_{il}(u)\right)\\
&+\frac{1}{u+v}\sum_{a=1}^{n}\left(\delta_{kj}s_{ia}(u)s_{al}(v) - \delta_{il}s_{ka}(v)s_{aj}(u)\right)\\
&-\frac{1}{u^2-v^2}\sum_{a=1}^{n}\delta_{ij}\left(s_{ka}(u)s_{al}(v) - s_{ka}(v)s_{al}(u)\right).
\end{aligned}$$

This is precisely the explicit form (3.4.7) of the defining reflection equation for the algebra $\mathsf{X}(\mathfrak{sl}_n, \mathfrak{sl}_{\mathscr{k}} \oplus \mathfrak{gl}_\ell)^{\mathsf{tw}}$. Hence, Part (2) of the Proposition holds.

As for Part (3), since the generating series $\sigma_{ij}(u) = q(u)^{-1}s_{ij}(u)$ for the subalgebra $Y(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ of $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ satisfy the defining relations (3.3.40) and (3.3.41), Part (2) implies that setting

$$\mathsf{b}_{ij}(u)\cdot v = \sigma_{ij}(u)v \quad \forall\, v \in V^J \text{ and } 1 \le i,j \le n$$

equips V^J with a $\mathsf{X}(\mathfrak{sl}_n, \mathfrak{sl}_{\mathscr{k}} \oplus \mathfrak{gl}_\ell)^{\mathsf{tw}}$-module structure. By Theorem 3.4.7, we have

$$\begin{gathered}
X(\mathfrak{sl}_n, \mathfrak{sl}_{\mathscr{k}} \oplus \mathfrak{gl}_\ell)^{tw} \cong \mathsf{X}(\mathfrak{sl}_n, \mathfrak{sl}_{\mathscr{k}} \oplus \mathfrak{gl}_\ell)^{\mathsf{tw}}/(\mathsf{f}(u)-1),\\
\text{where} \quad \mathsf{f}(u)I = \mathsf{B}(u)\mathsf{B}(-u).
\end{gathered}$$

Hence, it suffices to show that $\sum_{a=1}^{n}\sigma_{ia}(u)\sigma_{aj}(-u) \equiv \delta_{ij}$ for all $1 \le i,j \le n$. This is a consequence of the relation $\sum_{a\in\mathcal{I}_N}\sigma_{ia}(u)\sigma_{aj}(-u)=\delta_{ij}$ in $Y(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$. □

Remark 4.4.2. The above proposition has been inspired by the proof of Proposition 4.2.8 in [Mol07], where a similar result was established with $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$ replaced by $X(\mathfrak{sl}_N,\mathfrak{g}_N)^{tw}$ and $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$ replaced by $Y(\mathfrak{gl}_n)$.

In order to apply Proposition 4.4.1 to study highest weight representations of the twisted Yangian $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$, we first must recall some of the representation theoretic results for $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$ developed in [MR02].

4.4.2 Representations of $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$

A representation V of $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$ is a *highest weight representation* if there exists a nonzero vector $\xi\in V$ such that $V=X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}\xi$ and

$$b_{ij}(u)\xi=0 \qquad \forall \quad 1\le i<j\le n,$$
$$b_{ii}(u)\xi=\mu_i(u)\xi \qquad \forall \quad 1\le i\le n,$$

where, for each $1\le i\le n$, $\mu_i(u)$ is a formal power series in $\mathbb{C}[\![u^{-1}]\!]$ of the form

$$\mu_i(u)=g_{ii}+\sum_{r=1}^{\infty}\mu_i^{(r)}u^{-r},\quad \mu_i^{(r)}\in\mathbb{C}.$$

As usual, we call $\mu(u)=(\mu_i(u))_{i=1}^n$ the highest weight of V, and the vector ξ the highest weight vector. By [MR02, Theorem 4.1], every finite-dimensional irreducible module V is a highest weight representation.

Given an n-tuple $\mu(u)=(\mu_i(u))_{i=1}^n$, the $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$ Verma module $M(\mu(u))$ is defined as the quotient of $X(\mathfrak{sl}_n,\mathfrak{sl}_{\mathscr{k}}\oplus\mathfrak{gl}_{\ell})^{tw}$ by the left ideal generated by all the coefficients of the series

$$b_{ij}(u)\quad\text{with}\quad 1\le i<j\le n,$$
$$b_{ii}(u)-\mu_i(u)\quad\text{with}\quad 1\le i\le n.$$

When it is non-trivial, $M(\mu(u))$ is a highest weight module with the highest weight $\mu(u)$ and highest weight vector equal to the image of the unit 1.

A classification of non-trivial Verma modules was obtained by Molev and Ragoucy in [MR02]: By [MR02, Theorem 4.2], $M(\mu(u))$ is non-trivial if and only if the components of the highest weight $\mu(u)$ satisfy the relations

$$\mu_n(u)\mu_n(-u) = 1, \tag{4.4.8}$$
$$\widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i). \tag{4.4.9}$$

for all $1 \leq i < n$, where the components of $\widetilde{\mu}(u)$ are defined in (4.2.5).

For each n-tuple $\mu(u) = (\mu_i(u))_{i=1}^n$ whose components satisfy (4.4.8) and (4.4.9), the Verma module $M(\mu(u))$ admits a unique irreducible quotient $V(\mu(u))$. Up to isomorphism, it is the unique irreducible highest weight module with the highest weight $\mu(u)$. In particular, every finite-dimensional irreducible module is isomorphic to a module of this form.

One of the main results of [MR02] is a complete classification of finite-dimensional irreducible representations of $X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw}$. By [MR02, Theorem 4.6], $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $P_2(u), \dots, P_n(u)$ in u, with

$$P_i(u) = P_i(-u+n-i+2) \quad \forall \quad 2 \leq i \leq n, \tag{4.4.10}$$

together with a scalar $\alpha \in \mathbb{C}$ such that $P_{\Bbbk+1}(\alpha) \neq 0$ and

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u+1)}{P_i(u)} \left(\frac{\alpha - u}{\alpha + u - \ell}\right)^{\delta_{i,\Bbbk+1}} \quad \forall \quad 2 \leq i \leq n. \tag{4.4.11}$$

Remark 4.4.3.

(1) When $\Bbbk = 0$ or $\Bbbk = n$, we have $\Bbbk + 1 \notin \{2, \dots, n\}$ and α plays no role in the above classification. In particular, (4.4.11) becomes

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u+1)}{P_i(u)} \quad \forall \quad 2 \leq i \leq n.$$

(2) In [MR02], the condition $\ell \leq n/2$ was assumed. However, this condition can be removed and the proof of Theorem 4.6 in [MR02] goes through after only a small modification.

4.4.3 Applications to $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$

Using Proposition 4.4.1 and [MR02, Theorem 4.2], we may now make precise the sufficient and necessary conditions on the tuple $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_N^+}$ which results in a non-trivial $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ Verma module $M(\mu(u))$.

Theorem 4.4.4. *The $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ Verma module $M(\mu(u))$ is non-trivial if and only if the relations*

$$\begin{gathered}\widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i)\\ u\,\mathcal{g}(u)\widetilde{\mu}_0(\kappa-u) = (\kappa-u)\,\mathcal{g}(\kappa-u)\widetilde{\mu}_0(u)\end{gathered} \tag{4.4.12}$$

hold for all $i \in \mathcal{I}_N^+ \setminus \{n\}$, where the second equation is omitted if $\mathfrak{g}_N \neq \mathfrak{so}_{2n+1}$.

Proof. Suppose that $V = M(\mu(u))$ is non-trivial and recall that $1_{\mu(u)} \in M(\mu(u))$ is a highest weight vector associated to the highest weight $\mu(u)$. If $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, then the second relation of (4.4.12) follows immediately from Proposition 4.2.5. Thus, to show that the relations of (4.4.12) are necessarily satisfied by $\mu(u)$, it suffices to show that

$$\widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i) \quad \forall \quad i \in \mathcal{I}_N^+ \setminus \{n\}.$$

Since $1_{\mu(u)}$ belongs to the subspace V^J of V (see (4.4.1)) V^J is nonzero and by Proposition 4.4.1 admits the structure of a $X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw}$-module. Consider the submodule

$$W = X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw} 1_{\mu(u)} \subset V^J.$$

As the central series $q(u)$ is uniquely determined by

$$w(u) = q(u)q(u+\kappa),$$

Proposition 4.2.7 implies that

$$q(u)|_V = q_{\mu}(u)\mathrm{id}_V, \quad \text{ where } \quad q_{\mu}(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!].$$

As $\sigma_{ij}(u) = q(u)^{-1}s_{ij}(u)$, it follows that W is a highest weight representation of $W = X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw}$ with the highest weight

$$\mu^{\sharp}(u) = (q_{\mu}(u)^{-1}\mu_i(u))_{i=1}^n.$$

Therefore, the $X(\mathfrak{sl}_n, \mathfrak{sl}_k \oplus \mathfrak{gl}_\ell)^{tw}$ Verma module $M(\mu^\sharp(u))$ is non-trivial, and by (4.4.9), we have

$$\widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i)$$

for all $1 \le i \le n-1$. If $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, then we must also show that

$$\widetilde{\mu}_0(u)\widetilde{\mu}_0(-u+n) = \widetilde{\mu}_1(u)\widetilde{\mu}_1(-u+n). \tag{4.4.13}$$

In fact, the same argument as used in [MR02] to establish (4.4.9) can be applied to show that (4.4.13) holds. Let us recall the main steps of this argument. Define

$$\beta_i(u,v) = \sum_{a=i}^{n} s_{ia}(u)s_{ai}(v) \quad \forall \quad i \in \mathcal{I}_N^+$$

Using $\equiv$ to denote equality of operators on $\mathbb{C}1_{\mu(u)}$, we have

$$\beta_i(u,v) - \beta_i(v,u) = \sum_{a=i}^{n} A_{ia}(u,v) \equiv 0,$$

where the definition of $A_{ij}(u,v)$ has been given in (4.2.15), and the second equivalence has been proven in Step 3.1 of the proof of Theorem 4.2.6 for $i>0$, and in Step 3.2 of the same proof for $i=0$. As a consequence, we have $\beta_i(u,v) \equiv \beta_i(v,u)$ for all $i \ge 0$. From (3.3.42) we obtain

$$\begin{aligned}\beta_i(u,v) \equiv s_{ii}(u)s_{ii}(v) &+ \frac{1}{u-v}\sum_{a=i+1}^{n}\left(s_{aa}(u)s_{ii}(v) - s_{aa}(v)s_{ii}(u)\right) \\ &+ \frac{1}{u+v}\sum_{a=i+1}^{n}\left(\beta_i(u,v) - \beta_a(v,u)\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left(\frac{u+v-n+i}{u+v}\right)\beta_i(u,v) \equiv s_{ii}(u)s_{ii}(v) &- \frac{1}{u+v}\sum_{a=i+1}^{n}\beta_a(v,u) \\ &+ \frac{1}{u-v}\sum_{a=i+1}^{n}\left(s_{aa}(u)s_{ii}(v) - s_{aa}(v)s_{ii}(u)\right).\end{aligned} \tag{4.4.14}$$

Subtracting (4.4.14) with $i=1$ from (4.4.14) with $i=0$ and rearranging, we obtain

$$\begin{aligned}\frac{u+v-n}{u+v}&\left(\beta_0(u,v)-\beta_1(u,v)\right)\\ &\equiv s_{00}(u)s_{00}(v)+\frac{1}{u-v}\sum_{a=1}^{n}(s_{aa}(u)s_{00}(v)-s_{aa}(v)s_{00}(u))\\ &-s_{11}(u)s_{11}(v)-\frac{1}{u-v}\sum_{a=2}^{n}(s_{aa}(u)s_{11}(v)-s_{aa}(v)s_{11}(u)).\end{aligned}$$

Substituting $v\mapsto n-u$, the left-hand side becomes the zero operator and, after applying both sides to $1_{\mu(u)}$, we arrive at the relation

$$\begin{aligned}\mu_0(u)\mu_0(v)&+\frac{1}{2u-n}\sum_{a=1}^{n}(\mu_a(u)\mu_0(v)-\mu_a(v)\mu_0(u))\\ &=\mu_1(u)\mu_1(v)+\frac{1}{2u-n}\sum_{a=2}^{n}(\mu_a(u)\mu_1(v)-\mu_a(v)\mu_1(u)).\end{aligned}$$

By expanding equation (4.4.13) (using the definition of $\widetilde{\mu}_i(u)$), we see that it is equivalent to the above relation. Therefore, we may conclude that the relations (4.4.12) are necessarily satisfied when $M(\mu(u))$ is non-trivial.

Conversely, suppose that the components of $\mu(u)=(\mu_i(u))_{i\in\mathcal{I}_N^+}$ satisfy (4.4.12). Let $h(u)$ be the rational function in u defined by

$$h(u)=\frac{\mathrm{tr}(\mathcal{G})+4\ell-4u}{\mathrm{tr}(\mathcal{G})+4u}. \tag{4.4.15}$$

Note that $h(u)$ satisfies the relation $h(u)h(\ell-u)=1$. Define

$$f_i(u)=(h(u))^{-\delta_{i\mathfrak{k}}}\frac{\widetilde{\mu}_i(u)}{\widetilde{\mu}_{i+1}(u)}\quad\forall\quad i\in\mathcal{I}_N^+\setminus\{n\}.$$

Then, by the first relation of (4.4.12), we have

$$f_i(u)f_i(n-u-i)=1\quad\forall\quad i\in\mathcal{I}_N\setminus\{n\}.$$

Thus, for each $i\in\mathcal{I}_N\setminus\{n\}$, there exists $g_i(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$f_i(u)=g_i(u)g_i(n-u-i)^{-1}.$$

We will use these series to construct a non-trivial $X(\mathfrak{g}_N)$ Verma module $M(\lambda(u))$ containing an $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ highest weight module with the highest weight $\mu(u)$.

If $N = 2n$, fix $\lambda_n(u), \lambda_{-n}(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying the relation

$$\mu_n(u) = \left(\frac{\mathrm{tr}(\mathcal{G}) - 4ug_{nn}}{\mathrm{tr}(\mathcal{G}) - 4u}\right)\lambda_n(u - \kappa/2)\lambda_{-n}(-u + \kappa/2) \tag{4.4.16}$$

For each $i \in \mathcal{I}_N^+ \setminus \{n\}$ define $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ recursively in terms of $\lambda_{i+1}(u)$ by

$$\lambda_i(u - \kappa/2) = g_i(u)\lambda_{i+1}(-u - \kappa/2 + n - i)^{-1}.$$

By Lemma 4.1.3, there is a unique N-tuple $\lambda(u)$ extending $(\lambda_{-n}(u), \lambda_1(u), \ldots, \lambda_n(u))$ with the property that the $X(\mathfrak{g}_N)$ Verma module $M(\lambda(u))$ is non-trivial.

If instead $N = 2n + 1$ then, by the second relation of (4.4.12), there exists $\lambda_0(u)$ in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$\widetilde{\mu}_0(u) = 2u\,\mathcal{g}(u)\lambda_0(u - \kappa/2)\lambda_0(-u + \kappa/2). \tag{4.4.17}$$

For each $i \in \mathcal{I}_N^+ \setminus \{n\}$, define $\lambda_{i+1}(u)$ in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ recursively in terms of $\lambda_i(u)$ by

$$\lambda_{i+1}(-u - \kappa/2 + n - i) = g_i(u)\lambda_i(u - \kappa/2)^{-1}.$$

Then, by Lemma 4.1.3, there is a unique N-tuple $\lambda(u)$ extending $(\lambda_i(u))_{i \in \mathcal{I}_N^+}$ such that the $X(\mathfrak{g}_N)$ Verma module $M(\lambda(u))$ is non-trivial.

In either case, we have produced a nontrivial $X(\mathfrak{g}_N)$ Verma module $M(\lambda(u))$ with the highest weight $\lambda(u)$ whose components satisfy the relations

$$\begin{aligned}\frac{\widetilde{\mu}_i(u)}{\widetilde{\mu}_{i+1}(u)} &= h(u)^{\delta_{i\mathfrak{k}}}\frac{\lambda_i(u - \kappa/2)\lambda_{i+1}(-u - \kappa/2 + n - i)}{\lambda_{i+1}(u - \kappa/2)\lambda_i(-u - \kappa/2 + n - i)} \\ &= h(u)^{\delta_{i\mathfrak{k}}}\frac{\lambda_i(u - \kappa/2)\lambda_{-i}(-u + \kappa/2)}{\lambda_{i+1}(u - \kappa/2)\lambda_{-i-1}(-u + \kappa/2)}\end{aligned}$$

for all $i \in \mathcal{I}_N^+ \setminus \{n\}$, in addition to the relation (4.4.16) if $N = 2n$ and (4.4.17) if $N = 2n + 1$.

By Corollary 4.2.12, the module $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}1_{\lambda(u)} \subset M(\lambda(u))$ is a non-trivial highest weight module with highest weight is equal to $\mu(u)$. It follows that the

$X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$ Verma module $M(\mu(u))$ is non-trivial. □

We conclude this chapter with a second application of Proposition 4.4.1, which gives a first hint at what a classification of finite-dimensional $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules will look like.

Proposition 4.4.5. *Suppose the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module $V(\mu(u))$ is finite-dimensional. Then there exists monic polynomials $P_2(u), \ldots, P_n(u)$ in u, with*

$$P_i(u) = P_i(-u + n - i + 2) \quad \forall \quad 2 \leq i \leq n, \tag{4.4.18}$$

together with a scalar $\alpha \in \mathbb{C}$ such that $P_{\Bbbk+1}(\alpha) \neq 0$ and

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u+1)}{P_i(u)} \left(\frac{\alpha - u}{\alpha + u - \ell} \right)^{\delta_{i,\Bbbk+1}} \quad \forall \quad 2 \leq i \leq n. \tag{4.4.19}$$

Proof. Let $\xi \in V(\mu(u))$ be a highest weight vector. As in the proof of Theorem 4.4.4, one uses Proposition 4.4.1 to construct a $X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw}$ highest weight module

$$X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw} \xi \subset V(\mu(u))^J$$

with the highest weight $\mu^{\sharp}(u) = (q_{\mu}(u)^{-1}\mu_i(u))_{i=1}^n$, where $q_{\mu}(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. As $V(\mu(u))$ is finite-dimensional, the same must be true of the $X(\mathfrak{sl}_n, \mathfrak{sl}_{\Bbbk} \oplus \mathfrak{gl}_{\ell})^{tw}$-module $V(\mu^{\sharp}(u))$. Therefore, the proposition follows from [MR02, Theorem 4.6]; see (4.4.10) and (4.4.11). □

Chapter 5

Finite-Dimensional Irreducible Modules

Our work so far reduces the problem of classifying finite-dimensional irreducible $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules to the problem of determining precisely for which highest weights

$$\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_N^+} \in \prod_{i \in \mathcal{I}_N^+} (g_{ii} + u^{-1}\mathbb{C}[\![u^{-1}]\!])$$

the irreducible highest weight module $V(\mu(u))$ is finite-dimensional. By Theorem 4.4.4, the set of all such $\mu(u)$ is contained in the subset of $\prod_{i \in \mathcal{I}_N^+}(g_{ii} + u^{-1}\mathbb{C}[\![u^{-1}]\!])$ consisting of $\mu(u)$ satisfying

$$\widetilde{\mu}_i(u)\widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\widetilde{\mu}_{i+1}(-u+n-i),$$
$$u\,\mathcal{g}(u)\widetilde{\mu}_0(\kappa-u) = (\kappa-u)\,\mathcal{g}(\kappa-u)\widetilde{\mu}_0(u),$$

where $\widetilde{\mu}(u) = (\widetilde{\mu}_i(u))_{i \in \mathcal{I}_N^+}$ is defined by (4.2.5). Indeed, if these conditions are not satisfied, then $V(\mu(u))$ does not even exist. In addition, Proposition 4.4.5 tells us that, if $V(\mu(u))$ is finite-dimensional, there is a tuple

$$(\alpha, (P_i(u))_{i=2}^n) \subset \mathbb{C} \times \mathbb{C}[u]^n$$

satisfying the relations (4.4.18) and (4.4.19).

In this chapter, we strengthen these necessary conditions significantly (in §5.2), classify all one-dimensional representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ (in §5.3) and, for pairs $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ of the form

$$(\mathfrak{g}_{2n}, \mathfrak{gl}_n), \quad (\mathfrak{g}_N, \mathfrak{g}_N), \quad (\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2) \quad \text{and} \quad (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}),$$

we obtain a complete classification of all finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and of $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ (in §5.4 and §5.5).

Our proofs of these results will, in part, rely on a detailed study of twisted Yangians associated to the low rank symmetric pairs

$$(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta), \quad (\mathfrak{so}_3, \mathfrak{so}_3^\vartheta) \quad \text{and} \quad (\mathfrak{so}_4, \mathfrak{so}_4^\vartheta),$$

which is carried out in §5.1. Such a study is possible due to the existence of isomorphisms, constructed in [GRW16], between twisted Yangians of the above pairs and twisted Yangians of type A associated to the pairs $(\mathfrak{sl}_2, \mathfrak{sp}_2)$ and $(\mathfrak{sl}_2, \mathfrak{so}_2)$.

Henceforth, we will assume that for symmetric pairs

$$(\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q) \quad \text{with} \quad N, p, q \in 2\mathbb{Z},$$

we have $q \leq p$. At the level of the symmetric pair this assumption changes nothing, but it fixes a unique choice of twisted Yangian corresponding to each pair: see §3.3.6 and, in particular, Proposition 3.3.25 and Remark 3.3.26. We note, however, that the arguments we give in this chapter do apply in the $q < p$ case after making only very small modifications.

5.1 Low rank twisted Yangians

The isomorphisms of low rank classical Lie algebras

$$\mathfrak{gl}_1 \cong \mathbb{C} \cong \mathfrak{so}_2,$$

$$\mathfrak{sl}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sp}_2 \quad \text{and} \quad \mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$$

induce isomorphisms of symmetric pairs

$$(\mathfrak{sp}_2, \mathfrak{gl}_1) \cong (\mathfrak{sl}_2, \mathfrak{so}_2) \cong (\mathfrak{so}_3, \mathfrak{so}_2),$$
$$(\mathfrak{sp}_2, \mathfrak{sp}_2) \cong (\mathfrak{sl}_2, \mathfrak{sp}_2) \cong (\mathfrak{so}_3, \mathfrak{so}_3),$$
$$(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2) \cong (\mathfrak{sl}_2, \mathfrak{so}_2) \oplus (\mathfrak{sl}_2, \mathfrak{so}_2),$$
$$(\mathfrak{so}_4, \mathfrak{gl}_2) \cong (\mathfrak{sl}_2, \mathfrak{sp}_2) \oplus (\mathfrak{sl}_2, \mathfrak{so}_2) \quad \text{and} \quad (\mathfrak{so}_4, \mathfrak{so}_4) \cong (\mathfrak{sl}_2, \mathfrak{sp}_2) \oplus (\mathfrak{sl}_2, \mathfrak{sp}_2).$$

In [GRW16], we constructed the twisted Yangian analogues of these isomorphisms. In this section, we use the isomorphisms of [GRW16], together with the classification results for finite-dimensional irreducible modules of the twisted Yangians associated to $(\mathfrak{sl}_2, \mathfrak{sp}_2)$ and $(\mathfrak{sl}_2, \mathfrak{so}_2)$ [Mol92, Mol98, Mol07], to study the representation theory of extended twisted Yangians of type B, C and D when the rank of $\mathfrak{g}_N$ is one or two.

Throughout this section, we will make use of the following terminology: A $\mathfrak{g}_N^\vartheta$-module V is said to be a highest weight module with the highest weight $(\mu_i)_{i=1}^n \subset \mathbb{C}^n$ if it is generated by a nonzero vector ξ such that

$$F_{ij}^\vartheta \xi = 0 \quad \forall \quad i < j \in \mathcal{I}_N,$$
$$F_{ii}\xi = \mu_i \xi \quad \forall \quad 1 \le i \le n.$$

5.1.1 Low rank twisted Yangians of type AI and AII

The definitions and main properties of the (extended) twisted Yangians $X(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ and $Y(\mathfrak{sl}_N, \mathfrak{g}_N)^{tw}$ were briefly surveyed in §3.4. In this section, we will only be concerned with the $N = 2$ case. For the sake of clarity, we will denote the generators of $X(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$ (resp. $Y(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$) by $\{s_{ij}^{\circ(r)}\}_{i,j \in \mathcal{I}_2, r \in \mathbb{N}}$ (resp. $\{\sigma_{ij}^{\circ(r)}\}_{i,j \in \mathcal{I}_2, r \in \mathbb{N}}$). Similarly, we will write

$$S^\circ(u) = \sum_{i,j \in \mathcal{I}_2} E_{ij} \otimes s_{ij}^\circ(u) \quad \text{and} \quad \mathcal{S}^\circ(u) = \sum_{i,j \in \mathcal{I}_2} E_{ij} \otimes \sigma_{ij}^\circ(u), \quad \text{where}$$
$$s_{ij}^\circ(u) = \delta_{ij} + \sum_{r \ge 1} s_{ij}^{\circ(r)} u^{-r} \quad \text{and} \quad \sigma_{ij}^\circ(u) = \delta_{ij} + \sum_{r \ge 1} \sigma_{ij}^{\circ(r)} u^{-r} \quad \forall\, i, j \in \mathcal{I}_2.$$

In addition, we will exploit the following explicit formulas for the Sklyanin determinant $\mathrm{sdet} S^\circ(u)$ (see (3.4.2)) which can be found in [MNO96, §4]:

$$\begin{aligned}\mathrm{sdet} S^\circ(u) &= \frac{2u+1}{2u\pm 1}\Big(s^\circ_{-1,-1}(u-1)s^\circ_{-1,-1}(-u) \mp s^\circ_{-1,1}(u-1)s^\circ_{1,-1}(-u)\Big)\\ &= \frac{2u+1}{2u\pm 1}\Big(s^\circ_{11}(-u)s^\circ_{11}(u-1) \mp s^\circ_{1,-1}(-u)s^\circ_{-1,1}(u-1)\Big). \end{aligned} \tag{5.1.1}$$

We now give a brief overview of those results from the representation theory of $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$ which will be applied in §5.1.2–§5.1.5. For a more complete survey, we refer the reader to Chapter 4 of the monograph [Mol07].

A representation V of $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that $V = X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}\xi$,

$$s^\circ_{-1,1}(u)\xi = 0 \quad \text{and} \quad s^\circ_{11}(u)\xi = \mu(u)\xi$$

for a scalar series $\mu(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. As usual, we call $\mu(u)$ the highest weight of V and the vector ξ a highest weight vector.

Given an arbitrary series $\mu(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$, the $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$ Verma module $M(\mu(u))$ is defined to be the quotient of $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$ by the left ideal generated by the coefficients of the series

$$s^\circ_{-1,1}(u) \quad \text{and} \quad s^\circ_{11}(u) - \mu(u).$$

Contrary to the $X(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)^{tw}$ case, the $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$ Verma module $M(\mu(u))$ is always non-trivial. It admits a unique irreducible quotient $V(\mu(u))$, and any irreducible highest weight module with the highest weight $\mu(u)$ is isomorphic to $V(\mu(u))$. In particular, every finite-dimensional irreducible $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$-module V is isomorphic to $V(\mu(u))$ for some $\mu(u)$.

The following theorem is a restatement of Theorems 4.4 and 5.4 of [Mol92] (see also Theorems 4.3.3 and 4.4.3 of [Mol07]).

Theorem 5.1.1. *The $X(\mathfrak{sl}_2,\mathfrak{g}_2)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ together with a scalar $\gamma \in \mathbb{C}$ such that*

$$P(\gamma) \neq 0,$$

$$
\begin{aligned}
P(u) &= P(-u+1), \\
\frac{\mu(-u)}{\mu(u)} &= \frac{2u+1}{2u\mp 1}\cdot\frac{P(u+1)}{P(u)}\cdot\frac{u-\gamma}{u\pm\gamma}.
\end{aligned}
\tag{5.1.2}
$$

Remark 5.1.2. If $\mathfrak{g}_2 = \mathfrak{sp}_2$, then the symbols $\pm$ and $\mp$ take their upper values and (5.1.2) reduces to

$$
\frac{\mu(-u)}{\mu(u)} = \frac{P(u+1)}{P(u)}.
$$

In this case, the scalar γ and the condition $P(\gamma) \neq 0$ play no role in the above theorem and should be omitted. Additionally, the polynomial $P(u)$ is always uniquely determined by $\mu(u)$ and, if $\mathfrak{g}_2 = \mathfrak{so}_2$, then the pair $(\gamma, P(u))$ is unique.

Consider now the twisted Yangian $Y(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$. The highest weight module $V(\mu(u))$ remains irreducible when restricted to $Y(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$. Moreover, in complete analogy with Lemma 4.2.1, the isomorphism class of any finite-dimensional irreducible $Y(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$-module has a unique representative of the form $V(\mu(u))$ with

$$
\mathrm{sdet} S^{\circ}(u)|_{V(\mu(u))} = \mathrm{id}_{V(\mu(u))}.
$$

By (5.1.1), the above condition means precisely that

$$
\frac{2u+1}{2u\pm 1}\mu(-u)\mu(u-1) = 1. \tag{5.1.3}
$$

By [Mol07, Corollary 4.3.4], Theorem 5.1.1 implies that the isomorphism classes of finite-dimensional irreducible $Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$-modules are parameterized by monic polynomials

$$
P(u) \in \mathbb{C}[u] \quad \text{with} \quad P(u) = P(-u+1).
$$

Similarly, by [Mol07, Corollary 4.4.5], the isomorphism classes of finite-dimensional irreducible $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-modules are parameterized by pairs

$$
(\gamma, P(u)) \in \mathbb{C} \times \mathbb{C}[u],
$$
$$
\text{where } P(u) \text{ is monic }, \quad P(u) = P(-u+1) \text{ and } P(\gamma) \neq 0.
$$

5.1.2 Low rank twisted Yangians of type C

We now begin our analysis of twisted Yangians associated to symmetric pairs of type B, C and D in the low rank setting. In this subsection, we consider the symplectic pairs $(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)$. That is, $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ takes the form

$$(\mathfrak{sp}_2, \mathfrak{gl}_1) \quad \text{and} \quad (\mathfrak{sp}_2, \mathfrak{sp}_2).$$

Let $K = E_{11} - E_{-1,-1} \in \mathrm{End}(\mathbb{C}^2)$. By [GRW16, Corollary 4.2], there are isomorphisms of algebras

$$\varphi_{\mathrm{C0}} : X(\mathfrak{sp}_2, \mathfrak{sp}_2)^{tw} \xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \quad S(u) \mapsto S^\circ(u/2 - 1/2), \tag{5.1.4}$$

$$\varphi_{\mathrm{CI}} : X(\mathfrak{sp}_2, \mathfrak{gl}_1)^{tw} \xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}, \quad S(u) \mapsto S^\circ(u/2 - 1/2)K. \tag{5.1.5}$$

Moreover, these isomorphisms induce algebra isomorphisms

$$Y(\mathfrak{sp}_2, \mathfrak{sp}_2)^{tw} \cong Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \quad \text{and} \quad Y(\mathfrak{sp}_2, \mathfrak{gl}_1)^{tw} \cong Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}.$$

With the help of these isomorphisms and Theorem 5.1.1, we can now prove the following classification result for finite-dimensional irreducible representations of the extended twisted Yangian $X(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)^{tw}$.

Proposition 5.1.3. *Let $\mu(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$. Then the irreducible $X(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u, with*

$$P(u) = P(-u + 4),$$

in addition to a scalar $\alpha \in \mathbb{C}$ with $P(\alpha) \neq 0$ if $\mathfrak{sp}_2^\vartheta = \mathfrak{gl}_1$, such that

$$\frac{\widetilde{\mu}(2-u)}{\widetilde{\mu}(u)} = \frac{P(u+2)}{P(u)} \cdot \frac{2-u}{u} \qquad \text{if} \quad \mathfrak{sp}_2^\vartheta = \mathfrak{sp}_2, \tag{5.1.6}$$

$$\frac{\widetilde{\mu}(2-u)}{\widetilde{\mu}(u)} = \frac{P(u+2)}{P(u)} \cdot \frac{\alpha - u}{\alpha + u - 2} \quad \text{if} \quad \mathfrak{sp}_2^\vartheta = \mathfrak{gl}_1. \tag{5.1.7}$$

Moreover, when they exist, the polynomial $P(u)$ and the scalar α are uniquely determined.

Proof. We will only include a detailed proof for the case $\mathfrak{sp}_2^\vartheta = \mathfrak{gl}_1$; the $\mathfrak{sp}_2^\vartheta = \mathfrak{sp}_2$

case is very similar.

The isomorphism φ_{CI} from (5.1.5) defines an equivalence between the highest weight representations of $X(\mathfrak{sp}_2,\mathfrak{gl}_1)^{tw}$ and those of $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$. To see this, given a series $\mu^\circ(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$, let $V(\mu^\circ(u))$ denote the irreducible $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module with the highest weight $\mu^\circ(u)$, as in §5.1.1.

Then, viewed as a $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module via φ_{CI}, the irreducible $X(\mathfrak{sp}_2,\mathfrak{gl}_1)^{tw}$-module $V(\mu(u))$ is isomorphic to $V(\mu^\circ(u))$ with

$$\mu^\circ(u)=\mu(2u+1).$$

Indeed, if $\xi\in V(\mu(u))$ is the highest weight vector, then we have

$$s^\circ_{-1,1}(u)\cdot\xi=\varphi_{\text{CI}}^{-1}(s^\circ_{-1,1}(u))\xi=s_{-1,1}(2u+1)\xi=0,$$
$$s^\circ_{11}(u)\cdot\xi=\varphi_{\text{CI}}^{-1}(s^\circ_{11}(u))\xi=s_{11}(2u+1)\xi=\mu(2u+1)\xi.$$

By Theorem 5.1.1, the $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module $V(\mu^\circ(u))$ is finite-dimensional if and only there exists a pair $(Q(u),\gamma)$, where $Q(u)$ is a monic polynomial in u with $Q(u)=Q(-u+1)$, $\gamma\in\mathbb{C}$ is such that $Q(\gamma)\neq 0$, and

$$\frac{\mu^\circ(-u)}{\mu^\circ(u)}=\frac{2u+1}{2u-1}\cdot\frac{Q(u+1)}{Q(u)}\cdot\frac{u-\gamma}{u+\gamma}$$

Rewriting this condition using $\mu(u)$ and substituting $u\mapsto\frac{u-1}{2}$, we obtain the expression

$$\frac{(2-u)\,\mu(2-u)}{u\mu(u)}=\frac{Q(\frac{u+1}{2})}{Q(\frac{u-1}{2})}\cdot\frac{2\gamma-(u-1)}{(u-1)+2\gamma}. \tag{5.1.8}$$

Set $P(u)=2^{\deg Q(u)}Q\left(\frac{u-1}{2}\right)$ and $\alpha=2\gamma+1$, so that $P(u)$ is a monic polynomial with $P(u)=P(-u+4)$ (since $Q(u)=Q(-u+1)$) and $P(\alpha)=2^{\deg Q(u)}Q(\gamma)\neq 0$. Then, by (5.1.8), we have shown that $V(\mu(u))$ is finite-dimensional if and only if there exists a pair $(P(u),\alpha)$ as in the statement of the proposition, satisfying

$$\frac{\widetilde{\mu}(2-u)}{\widetilde{\mu}(u)}=\frac{P(u+2)}{P(u)}\cdot\frac{\alpha-u}{\alpha+u-2}.$$

The uniqueness of the pair $(P(u),\alpha)$ follows immediately from the uniqueness of $(Q(u),\alpha)$ (alternatively, it follows from Lemma 5.2.1 below). □

We conclude our brief analysis of the twisted Yangians associated to $(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)$ with two applications of Proposition 3.4.3. Recall that

$$F_{ij}^\vartheta = (g_{ii} + g_{jj})F_{ij} \in U(\mathfrak{sp}_2^\vartheta) \quad \forall\, i, j \in \mathcal{I}_2.$$

Composing the isomorphisms (5.1.4) and (5.1.5) with the evaluation morphism ev from Proposition 3.4.3, we obtain the following.

Proposition 5.1.4. *The assignments*

$$\begin{aligned}
\mathrm{ev}_{\mathrm{C0}} &: s_{ij}(u) \mapsto \delta_{ij} + F_{ij}^\vartheta (u-2)^{-1}, \\
\mathrm{ev}_{\mathrm{CI}} &: s_{ij}(u) \mapsto g_{ij} + F_{ij}^\vartheta u^{-1},
\end{aligned}$$

for all $i, j \in \mathcal{I}_2$, extend to algebra epimorphisms

$$\mathrm{ev}_{\mathrm{C0}} : X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \twoheadrightarrow U(\mathfrak{sp}_2) \quad \text{and} \quad \mathrm{ev}_{\mathrm{CI}} : X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2).$$

For any $\gamma \in \mathbb{C}$, let $V(\gamma)$ denote the irreducible highest weight representation of $\mathfrak{sp}_2^\vartheta$ with the highest weight γ. That is, $V(\gamma)$ is the irreducible module generated by a nonzero vector ξ such that

$$F_{-1,1}^\vartheta \xi = 0 \quad \text{and} \quad F_{11}\xi = \gamma\xi.$$

If $\mathfrak{sp}_2^\vartheta = \mathfrak{gl}_1$, then $U(\mathfrak{so}_2)$ is a polynomial algebra in one variable and $V(\gamma)$ is always one-dimensional.

We may view $V(\gamma)$ as an $X(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)^{tw}$-module by pulling back via the evaluation homomorphisms $\mathrm{ev}_{\mathrm{C0}}$ and $\mathrm{ev}_{\mathrm{CI}}$.

Corollary 5.1.5. *Given $\mu \in \mathbb{C}$, $V(\gamma)$ is isomorphic to the irreducible $X(\mathfrak{sp}_2, \mathfrak{sp}_2^\vartheta)^{tw}$-module $V(\mu(u))$ with*

$$\mu(u) = \begin{cases} 1 + 2\mu(u-2)^{-1} & \text{if } \mathfrak{sp}_2^\vartheta = \mathfrak{sp}_2, \\ 1 + (2\mu)u^{-1} & \text{if } \mathfrak{sp}_2^\vartheta = \mathfrak{gl}_1. \end{cases} \tag{5.1.9}$$

We will see in Corollary 5.3.11 that the family of one-dimensional representations $\{V(\gamma)\}_{\gamma \in \mathbb{C}}$ yield a complete list of one-dimensional $Y(\mathfrak{sp}_2, \mathfrak{gl}_1)^{tw}$-modules, up to isomorphism (in the notation of §5.3.3, $V(\gamma)$ is $V(\alpha)$ with $\alpha = 2\gamma$).

5.1.3 Low rank twisted Yangians of type D0 and DIII

We now consider those symmetric pairs $(\mathfrak{so}_4, \mathfrak{so}_4^\vartheta)$ of the form

$$(\mathfrak{so}_4, \mathfrak{so}_4) \quad \text{and} \quad (\mathfrak{so}_4, \mathfrak{gl}_2).$$

The isomorphisms constructed in [GRW16] which concern these pairs involve the tensor products

$$Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \quad \text{and} \quad Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}.$$

To distinguish between the two tensor factors, we will write $\sigma_{ij}^\circ(u) = \sigma_{ij}^\circ(u) \otimes 1$ and $s_{ij}^\bullet(u) = 1 \otimes s_{ij}^\circ(u)$. Similarly, we will denote

$$\mathcal{S}^\circ(u) = \mathcal{S}^\circ(u) \otimes 1 \quad \text{and} \quad S^\bullet(u) = 1 \otimes S^\circ(u).$$

Let $V = \mathbb{C}^2 \otimes \mathbb{C}^2$ with ordered basis given by

$$v_{-2} = e_{-1} \otimes e_{-1}, \quad v_{-1} = e_{-1} \otimes e_1, \quad v_1 = e_1 \otimes e_{-1} \quad \text{and} \quad v_2 = -e_1 \otimes e_1.$$

By identifying V with $\mathbb{C}^4$ equipped with canonical basis $\{v_i\}_{i \in \mathcal{I}_4}$, we can consider $S(u)$ as an element of $\operatorname{End} V \otimes X(\mathfrak{so}_4, \mathfrak{so}_4^\vartheta)^{tw}[\![u^{-1}]\!]$, where $\mathfrak{so}_4^\vartheta$ is either $\mathfrak{gl}_2$ or $\mathfrak{so}_4$. By Corollaries 4.9 and 4.13 of [GRW16], the assignments

$$\begin{aligned} \varphi_{\mathrm{D0}} &: S(u) \mapsto \mathcal{S}^\circ(u - 1/2) S^\bullet(u - 1/2), \\ \varphi_{\mathrm{DIII}} &: S(u) \mapsto \mathcal{S}^\circ(u - 1/2) K_1 S^\bullet(u - 1/2), \end{aligned}$$

where we recall that $K = E_{11} - E_{-1,-1}$, extend to algebra isomorphisms

$$\varphi_{\mathrm{D0}} : X(\mathfrak{so}_4, \mathfrak{so}_4)^{tw} \xrightarrow{\sim} Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \tag{5.1.10}$$

$$\varphi_{\mathrm{DIII}} : X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw} \xrightarrow{\sim} Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}. \tag{5.1.11}$$

These isomorphisms can be recovered from embeddings

$$\begin{aligned}
\widetilde{\varphi}_{\mathrm{D0}} : X(\mathfrak{so}_4, \mathfrak{so}_4)^{tw} &\hookrightarrow X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \\
& S(u) \mapsto S^{\circ}(u-1/2)S^{\bullet}(u-1/2), \\
\widetilde{\varphi}_{\mathrm{DIII}} : X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw} &\xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \\
& S(u) \mapsto S^{\circ}(u-1/2)K_1 S^{\bullet}(u-1/2),
\end{aligned} \tag{5.1.12}$$

by composing with the natural quotient map $X(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw} \twoheadrightarrow Y(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$ in the first tensor factor. Here $S^{\circ}(u)$ is identified with $S^{\circ}(u) \otimes 1$.

By Corollaries 4.10 and 4.14 of [GRW16], the restrictions of φ_{D0} and φ_{DIII} to the twisted Yangians $Y(\mathfrak{so}_4, \mathfrak{so}_4^{\vartheta})^{tw}$ yield isomorphisms

$$\begin{aligned}
Y(\mathfrak{so}_4, \mathfrak{so}_4)^{tw} &\cong Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \otimes Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \\
Y(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw} &\cong Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}.
\end{aligned}$$

Using the above isomorphisms, we obtain the following classification result.

Proposition 5.1.6. *The irreducible representation $V(\mu(u))$ of $X(\mathfrak{so}_4, \mathfrak{so}_4^{\vartheta})^{tw}$ is finite-dimensional if and only if there exist monic polynomials $P(u)$ and $Q(u)$ in u, with*

$$P(u) = P(-u+2) \quad \textit{and} \quad Q(u) = Q(-u+2),$$

in addition to a scalar $\alpha \in \mathbb{C}$ with $Q(\alpha) \neq 0$ if $\mathfrak{so}_4^{\vartheta} = \mathfrak{gl}_2$, such that

$$\frac{\widetilde{\mu}_1(u)}{\widetilde{\mu}_2(u)} = \frac{P(u+1)}{P(u)}, \tag{5.1.13}$$

$$\frac{\widetilde{\mu}_1(1-u)}{\widetilde{\mu}_2(u)} = \frac{Q(u+1)}{Q(u)} \cdot \frac{1-u}{u} \qquad \textit{if} \quad \mathfrak{so}_4^{\vartheta} = \mathfrak{so}_4, \tag{5.1.14}$$

$$\frac{\widetilde{\mu}_1(1-u)}{\widetilde{\mu}_2(u)} = \frac{Q(u+1)}{Q(u)} \cdot \frac{\alpha-u}{\alpha+u-1} \qquad \textit{if} \quad \mathfrak{so}_4^{\vartheta} = \mathfrak{gl}_2. \tag{5.1.15}$$

Moreover, when they exist, the pair $(Q(u), P(u))$ and the scalar α are uniquely determined.

Proof. We will give details of the proof only for the case $\mathfrak{so}_4^{\vartheta} = \mathfrak{gl}_2$.

It is a general fact that any simple finite-dimensional module over a tensor product $A \otimes B$ of two associative unital $\mathbb{C}$-algebras A and B is of the form $M_A \otimes M_B$, where

M_A (resp. M_B) is a simple, finite-dimensional module over A (resp. over B): see Theorem 3.10.2 in [EGH+11]. We show more precisely that the $X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw}$-module $V(\mu(u))$, viewed as a $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$-module via the isomorphism φ_{DIII}, is isomorphic to

$$V(\lambda^\circ(u)) \otimes V(\lambda^\bullet(u)),$$

where the $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module $V(\lambda^\circ(u))$ is viewed as a $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ by restriction, and the series $\lambda^\circ(u)$ and $\lambda^\bullet(u)$ are completely determined by the two relations

$$\begin{aligned} \lambda^\bullet(u)\lambda^\bullet(u-1) - \tfrac{1}{2u}\,(\lambda^\bullet(u) - \lambda^\bullet(-u))\lambda^\bullet(u-1) \\ = \mu_1(-u+1/2)\mu_2(u-1/2), \end{aligned} \tag{5.1.16}$$

$$\lambda^\circ(u) = \mu_2(u+1/2)\lambda^\bullet(u)^{-1}. \tag{5.1.17}$$

Equivalently, $\lambda^\circ(u)$ and $\lambda^\bullet(u)$ are completely determined by

$$\begin{aligned} \lambda^\circ(-u)\lambda^\circ(u-1) &= 1, \\ \widetilde{\mu}_2(u) &= 2u\,\lambda^\circ(u-1/2)\lambda^\bullet(u-1/2), \\ \widetilde{\mu}_1(u) &= 2u\,\lambda^\circ(u-1/2)\lambda^\bullet(-u+1/2). \end{aligned} \tag{5.1.18}$$

We will need the following explicit formulas for the images of the generators $s_{ij}(u)$ under the isomorphism φ_{DIII} from (5.1.11):

$$\begin{aligned}
s_{-2,-2}(u) &\mapsto -\sigma^\circ_{-1,-1}(\widetilde{u})s^\bullet_{-1,-1}(\widetilde{u}), & s_{1,-2}(u) &\mapsto -\sigma^\circ_{1,-1}(\widetilde{u})s^\bullet_{-1,-1}(\widetilde{u}),\\
s_{-2,-1}(u) &\mapsto -\sigma^\circ_{-1,-1}(\widetilde{u})s^\bullet_{-1,1}(\widetilde{u}), & s_{1,-1}(u) &\mapsto -\sigma^\circ_{1,-1}(\widetilde{u})s^\bullet_{-1,1}(\widetilde{u}),\\
s_{-2,1}(u) &\mapsto \sigma^\circ_{-1,1}(\widetilde{u})s^\bullet_{-1,-1}(\widetilde{u}), & s_{11}(u) &\mapsto \sigma^\circ_{11}(\widetilde{u})s^\bullet_{-1,-1}(\widetilde{u}),\\
s_{-2,2}(u) &\mapsto -\sigma^\circ_{-1,1}(\widetilde{u})s^\bullet_{-1,1}(\widetilde{u}), & s_{12}(u) &\mapsto -\sigma^\circ_{11}(\widetilde{u})s^\bullet_{-1,1}(\widetilde{u}),\\
s_{-1,-2}(u) &\mapsto -\sigma^\circ_{-1,-1}(\widetilde{u})s^\bullet_{1,-1}(\widetilde{u}), & s_{2,-2}(u) &\mapsto \sigma^\circ_{1,-1}(\widetilde{u})s^\bullet_{1,-1}(\widetilde{u}),\\
s_{-1,-1}(u) &\mapsto -\sigma^\circ_{-1,-1}(\widetilde{u})s^\bullet_{11}(\widetilde{u}), & s_{2,-1}(u) &\mapsto \sigma^\circ_{1,-1}(\widetilde{u})s^\bullet_{11}(\widetilde{u}),\\
s_{-1,1}(u) &\mapsto \sigma^\circ_{-1,1}(\widetilde{u})s^\bullet_{1,-1}(\widetilde{u}), & s_{21}(u) &\mapsto -\sigma^\circ_{11}(\widetilde{u})s^\bullet_{1,-1}(\widetilde{u}),\\
s_{-1,2}(u) &\mapsto -\sigma^\circ_{-1,1}(\widetilde{u})s^\bullet_{11}(\widetilde{u}), & s_{22}(u) &\mapsto \sigma^\circ_{11}(\widetilde{u})s^\bullet_{11}(\widetilde{u}),
\end{aligned} \tag{5.1.19}$$

where $\widetilde{u} = u - 1/2$. It is explained how to obtain these formulas from the definition of φ_{DIII} (see (5.1.11)) at the end of the proof of Proposition 4.8 in [GRW16]. These formulas together with the expression (5.1.1) and the fact that $\mathrm{sdet}\mathcal{S}^\circ(u) = 1$ give

$$
\begin{aligned}
&\varphi_{\mathrm{DIII}}\left(s_{11}(-\widetilde{u})s_{22}(\widetilde{u}) - s_{1,-2}(-\widetilde{u})s_{-1,2}(\widetilde{u})\right)\\
&\quad= \left(\sigma^\circ_{11}(-u)\sigma^\circ_{11}(u-1) - \sigma^\circ_{1,-1}(-u)\sigma^\circ_{-1,1}(u-1)\right) s^\bullet_{-1,-1}(-u)s^\bullet_{11}(u-1) \qquad (5.1.20)\\
&\quad= s^\bullet_{-1,-1}(-u)s^\bullet_{11}(u-1).
\end{aligned}
$$

Letting $\xi \in V(\mu(u))$ denote the highest weight vector, this gives

$$
s^\bullet_{-1,-1}(-u)s^\bullet_{11}(u-1)\xi = \mu_1(-\widetilde{u})\mu_2(\widetilde{u})\xi.
$$

Employing the defining symmetry relation (3.4.4) of $X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$, we can rewrite this as

$$
\left(s^\bullet_{11}(u) - \tfrac{1}{2u}\left(s^\bullet_{11}(u) - s^\bullet_{11}(-u)\right)\right) s^\bullet_{11}(u-1)\xi = \mu_1(-\widetilde{u})\mu_2(\widetilde{u})\xi.
$$

By induction on the coefficients $s^{\bullet(r)}_{11}$ of $s^\bullet_{11}(u)$, this implies that there exists

$$
\lambda^\bullet(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]
$$

such that $s^\bullet_{11}(u)\xi = \lambda^\bullet(u)\xi$, and $\lambda^\bullet(u)$ is determined by (5.1.16). Again appealing to the formulas (5.1.19), we have $\varphi_{\mathrm{DIII}}(s_{22}(u)) = \sigma^\circ_{11}(\widetilde{u})s^\bullet_{11}(\widetilde{u})$, which implies that ξ is an eigenvector for the action of $\sigma^\circ_{11}(u)$ with weight $\lambda^\circ(u)$ determined by the relation (5.1.17). Notice that it now follows immediately from the explicit formulas (5.1.19) that

$$
\sigma^\circ_{-1,1}(u)\xi = s^\bullet_{-1,1}(u)\xi = 0.
$$

Conversely, any vector η with the property that $\sigma^\circ_{-1,1}(u)\eta = s^\bullet_{-1,1}(u)\eta = 0$ and which is a weight vector for $s^\bullet_{ii}(u)$ must be a highest weight vector of the $X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw}$-module $V(\mu(u))$ by (5.1.19), hence a scalar multiple of ξ. Thus, by the irreducibility of $V(\mu(u))$ we can conclude that

$$
V(\mu(u)) \cong V(\lambda^\circ(u)) \otimes V(\lambda^\bullet(u)). \qquad (5.1.21)
$$

To see that (5.1.16) and (5.1.17) are equivalent to the relations given in equation (5.1.18), we observe first that relation (5.1.17) is clearly equivalent to

$$
\widetilde{\mu}_2(u) = 2u\,\lambda^\circ(u-1/2)\lambda^\bullet(u-1/2).
$$

Notice also that we may rewrite (5.1.16) as

$$\begin{aligned}\lambda^\bullet(u)\lambda^\bullet(u-1) - \tfrac{1}{2u}\left(\lambda^\bullet(u) - \lambda^\bullet(-u)\right)\lambda^\bullet(u-1)\\ = \mu_1(-u+1/2)\lambda^\circ(u-1)\lambda^\bullet(u-1).\end{aligned} \tag{5.1.22}$$

Since $\mathrm{sdet}\mathcal{S}^\circ(u) = 1$, by (5.1.3) we have $\lambda^\circ(-u)^{-1} = \lambda^\circ(u-1)$. Using this, we may rewrite (5.1.22) as

$$\begin{aligned}&\lambda^\bullet(-u+1/2)\lambda^\circ(u-1/2)\\ &\quad - \tfrac{1}{1-2u}\left(\lambda^\bullet(-u+1/2)\lambda^\circ(u-1/2) - \lambda^\bullet(u-1/2)\lambda^\circ(u-1/2)\right) = \mu_1(u),\end{aligned}$$

which is equivalent to

$$2u\,\lambda^\bullet(-u+1/2)\lambda^\circ(u-1/2) = (2u-1)\mu_1(u) + \mu_2(u) = \widetilde{\mu}_1(u).$$

As a consequence of the isomorphism of $Y(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2,\mathfrak{sp}_2)^{tw}$-modules (5.1.21), we can deduce exactly when $V(\mu(u))$ is finite-dimensional. Indeed, by Theorem 5.1.1, $V(\lambda^\circ(u)) \otimes V(\lambda^\bullet(u))$ is finite-dimensional if and only if there exists $\gamma \in \mathbb{C}$ together with monic polynomials $P^\circ(u)$, $P^\bullet(u)$ such that

$$P^\circ(\gamma) \neq 0, \quad P^\circ(u) = P^\circ(-u+1), \quad P^\bullet(u) = P^\bullet(-u+1) \tag{5.1.23}$$

and the following equalities hold:

$$\begin{aligned}\frac{\lambda^\circ(-u)}{\lambda^\circ(u)} &= \frac{2u+1}{2u-1}\cdot\frac{P^\circ(u+1)}{P^\circ(u)}\cdot\frac{u-\gamma}{u+\gamma},\\ \frac{\lambda^\bullet(-u)}{\lambda^\bullet(u)} &= \frac{P^\bullet(u+1)}{P^\bullet(u)}.\end{aligned} \tag{5.1.24}$$

In this case, the triple $(\gamma, P^\circ(u), P^\bullet(u))$ is unique. Since

$$\frac{\widetilde{\mu}_1(u)}{\widetilde{\mu}_2(u)} = \frac{2u\cdot\lambda^\circ(u-1/2)\lambda^\bullet(-u+1/2)}{2u\cdot\lambda^\circ(u-1/2)\lambda^\bullet(u-1/2)} = \frac{\lambda^\bullet(-u+1/2)}{\lambda^\bullet(u-1/2)},$$

the second equation in (5.1.24) is equivalent to

$$\frac{\widetilde{\mu}_1(u)}{\widetilde{\mu}_2(u)} = \frac{P^\bullet(u+1/2)}{P^\bullet(u-1/2)} = \frac{P(u+1)}{P(u)}, \qquad \text{where } \ P(u) = P^\bullet(u-1/2),$$

which is precisely (5.1.13). Similarly, since

$$\frac{\widetilde{\mu}_1(1-u)}{\widetilde{\mu}_2(u)} = \frac{2(1-u)\lambda^\circ(-u+1/2)\lambda^\bullet(u-1/2)}{2u\lambda^\circ(u-1/2)\lambda^\bullet(u-1/2)} = \frac{1-u}{u}\cdot\frac{\lambda^\circ(-u+1/2)}{\lambda^\circ(u-1/2)},$$

we may rewrite the first equation in (5.1.24) as

$$\begin{aligned}\frac{\widetilde{\mu}_1(1-u)}{\widetilde{\mu}_2(u)} &= \frac{P^\circ(u-1/2+1)}{P^\circ(u-1/2)}\cdot\frac{\gamma+1/2-u}{u-1/2+\gamma}\\ &= \frac{Q(u+1)}{Q(u)}\cdot\frac{\alpha-u}{\alpha+u-1},\end{aligned}$$

where we have set $\alpha = \gamma + 1/2$ and $Q(u) = P^\circ(u-1/2)$. This is precisely (5.1.15). Moreover, by (5.1.23), we have $Q(\alpha) \neq 0$, $P(u) = P(-u+1)$ and $Q(u) = Q(-u+1)$.

Finally, we note that the uniqueness of the triple $(\alpha, Q(u), P(u))$ is immediate from the uniqueness of $(\gamma, P^\circ(u), P^\bullet(u))$. □

We now turn to the construction of evaluation morphisms

$$X(\mathfrak{so}_4,\mathfrak{so}_4)^{tw} \twoheadrightarrow U(\mathfrak{so}_4) \quad \text{and} \quad X(\mathfrak{so}_4,\mathfrak{gl}_2)^{tw} \twoheadrightarrow U(\mathfrak{gl}_2).$$

Let Ω_ϑ be the Casimir element of $U(\mathfrak{so}_4)$ if $\mathfrak{so}_4^\vartheta = \mathfrak{so}_4$, or of $U(\mathfrak{sl}_2) \subset U(\mathfrak{gl}_2)$ if $\mathfrak{so}_4^\vartheta = \mathfrak{gl}_2$, defined by

$$\Omega_\vartheta = \begin{cases} F_{11}^2 + F_{22}^2 - 2F_{22} + 2F_{21}F_{12} + 2F_{2,-1}F_{-1,2} & \text{if } \mathfrak{so}_4^\vartheta = \mathfrak{so}_4,\\ \frac{1}{2}(F_{22}-F_{11})^2 + F_{12}F_{21} + F_{21}F_{12} & \text{if } \mathfrak{so}_4^\vartheta = \mathfrak{gl}_2.\end{cases}$$

Here we recall that $U(\mathfrak{so}_4^\vartheta) \subset U(\mathfrak{so}_4)$ is generated by $F_{ij}^\vartheta = (g_{ii}+g_{jj})F_{ij}$ for all $i,j \in \mathcal{I}_4$. If $\mathfrak{so}_4^\vartheta = \mathfrak{gl}_2$, define the auxiliary central element $z \in U(\mathfrak{gl}_2)$ by

$$z = F_{11}^2 + F_{22}^2 + F_{12}F_{21} + F_{21}F_{12} = \Omega_\vartheta + \tfrac{1}{2}(F_{11}+F_{22})^2.$$

In the following proposition it will be convenient to denote the Casimir element Ω_ϑ corresponding to $\mathfrak{so}_4^\vartheta = \mathfrak{so}_4$ simply by Ω.

Proposition 5.1.7. *The assignments*

$$\mathrm{ev}_{\mathrm{D0}} : S(u) \mapsto I + \frac{F^{\vartheta}}{u-1} + \frac{(F^{\vartheta})^2 - 2F^{\vartheta} - 2\Omega \cdot I}{2(u-1)^2}, \tag{5.1.25}$$

$$\mathrm{ev}_{\mathrm{DIII}} : S(u) \mapsto \mathcal{G} + \frac{F^{\vartheta}}{u} + \mathcal{G}\frac{(F^{\vartheta})^2 - 2z \cdot I}{2u(u-1)}, \tag{5.1.26}$$

extend to algebra epimorphisms

$$\mathrm{ev}_{\mathrm{D0}} : X(\mathfrak{so}_4, \mathfrak{so}_4)^{tw} \twoheadrightarrow U(\mathfrak{so}_4) \quad \textit{and} \quad \mathrm{ev}_{\mathrm{DIII}} : X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw} \twoheadrightarrow U(\mathfrak{gl}_2).$$

Proof. Suppose first that $\mathfrak{so}_4^{\vartheta} = \mathfrak{gl}_2$. Consider the Lie algebra $\mathfrak{so}_2 \oplus \mathfrak{sp}_2$. Denote the generators of $\mathfrak{so}_2$ in this direct sum by F_{ij}°, and those of $\mathfrak{sp}_2$ by $F_{ij}^{\bullet}$, where $i, j \in \mathcal{I}_2$. The Lie algebra $\mathfrak{so}_2$ is one-dimensional with basis F_{11}°, while $\mathfrak{sp}_2$ is three-dimensional with basis $\{F_{1,1}^{\bullet}, F_{-1,1}^{\bullet}, F_{1,-1}^{\bullet}\}$. Let Φ be the isomorphism $\mathfrak{so}_2 \oplus \mathfrak{sp}_2 \xrightarrow{\sim} \mathfrak{gl}_2$ given by

$$F_{11}^{\circ} \mapsto F_{11} + F_{22}, \quad F_{11}^{\bullet} \mapsto F_{22} - F_{11}, \quad F_{-1,1}^{\bullet} \mapsto -2F_{12}, \quad F_{1,-1}^{\bullet} \mapsto -2F_{21}.$$

It induces an isomorphism $\widehat{\Phi} : U(\mathfrak{so}_2) \otimes U(\mathfrak{sp}_2) \xrightarrow{\sim} U(\mathfrak{gl}_2)$, and we thus obtain an algebra epimorphism

$$\widehat{\Phi} \circ (\mathrm{ev} \otimes \mathrm{ev}) : X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \twoheadrightarrow U(\mathfrak{gl}_2),$$

where $\mathrm{ev} : X(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw} \twoheadrightarrow U(\mathfrak{g}_2)$ is as in Proposition 3.4.3. Writing this map explicitly, we have $s_{ij}^{\circ}(u) \mapsto 0$ for $i \neq j$, and

$$s_{-1,-1}^{\circ}(u) \mapsto 1 - \frac{F_{11} + F_{22}}{u + 1/2}, \quad s_{-1,-1}^{\bullet}(u) \mapsto 1 + \frac{F_{11} - F_{22}}{u - 1/2}, \quad s_{-1,1}^{\bullet}(u) \mapsto -\frac{2F_{12}}{u - 1/2},$$

$$s_{11}^{\circ}(u) \mapsto 1 + \frac{F_{11} + F_{22}}{u + 1/2}, \quad s_{11}^{\bullet}(u) \mapsto 1 + \frac{F_{22} - F_{11}}{u - 1/2}, \quad s_{1,-1}^{\bullet}(u) \mapsto -\frac{2F_{21}}{u - 1/2},$$

where $s_{ij}^{\circ}(u)$ is identified with $s_{ij}^{\circ}(u) \otimes 1$.

The proof is now completed as follows: Composing $\widehat{\Phi} \circ (\mathrm{ev} \otimes \mathrm{ev})$ with the embedding $\widetilde{\varphi}_{\mathrm{DIII}}$ from (5.1.12) gives a homomorphism $\mathrm{ev}_{\mathrm{DIII}} : X(\mathfrak{so}_4, \mathfrak{gl}_2)^{tw} \to U(\mathfrak{gl}_2)$. It remains to see that it is given by the assignment (5.1.26) and that it is surjective. However, if it is indeed given by (5.1.26) then it must be surjective, so it remains only to check the former claim. This can be shown by a direct calculation using the formulas (5.1.19).

For instance, since

$$\widetilde{\varphi}_{\mathrm{DIII}}(s_{-2,-2}(u)) = -s^{\circ}_{-1,-1}(u-1/2)s^{\bullet}_{-1,-1}(u-1/2),$$

we have

$$\begin{aligned}\mathrm{ev}_{\mathrm{DIII}}(s_{-2,-2}(u)) &= -\left(1-\frac{F_{11}+F_{22}}{u}\right)\left(1+\frac{F_{11}-F_{22}}{u-1}\right)\\ &= -1+\frac{2F_{22}}{u}+\frac{F_{11}^2-F_{22}^2-F_{11}+F_{22}}{u(u-1)}.\end{aligned} \tag{5.1.27}$$

On the other hand, since $F_{ij}^{\vartheta} = (g_{ii}+g_{jj})F_{ij}$ and $F_{ij} = -F_{-j,-i}$ in $\mathfrak{so}_4$, the coefficient of $E_{-2,-2}$ in the right-hand side of (5.1.26) is given by

$$\begin{aligned}-1+\frac{2F_{22}}{u}&+\frac{-2F_{22}^2-2F_{12}F_{21}+F_{11}^2+F_{22}^2+F_{12}F_{21}+F_{21}F_{12}}{u(u-1)}\\ &= -1+\frac{2F_{22}}{u}+\frac{F_{11}^2-F_{22}^2+F_{22}-F_{11}}{u(u-1)},\end{aligned}$$

which coincides with (5.1.27). The images of the remaining generators can be verified similarly.

If instead $\mathfrak{so}_4^{\vartheta} = \mathfrak{so}_4$, the argument is similar. Denote the generators corresponding to the first copy of $\mathfrak{sp}_2$ in the direct sum $\mathfrak{sp}_2 \oplus \mathfrak{sp}_2$ by F°_{ij}, and those corresponding to the second copy of $\mathfrak{sp}_2$ by $F^{\bullet}_{ij}$, where in both cases $i, j \in \mathcal{I}_2$. A basis for $\mathfrak{sp}_2 \oplus \mathfrak{sp}_2$ is then given by the union of $\{F^{\circ}_{1,1}, F^{\circ}_{-1,1}, F^{\circ}_{1,-1}\}$ and $\{F^{\bullet}_{1,1}, F^{\bullet}_{-1,1}, F^{\bullet}_{1,-1}\}$. Let Φ be the isomorphism $\mathfrak{sp}_2 \oplus \mathfrak{sp}_2 \xrightarrow{\sim} \mathfrak{so}_4$ given by

$$\begin{aligned}&F^{\circ}_{11} \mapsto F_{11}+F_{22}, \quad F^{\circ}_{-1,1} \mapsto 2F_{-2,1}, \quad F^{\circ}_{1,-1} \mapsto 2F_{1,-2}\\ &F^{\bullet}_{11} \mapsto F_{22}-F_{11}, \quad F^{\bullet}_{-1,1} \mapsto -2F_{12}, \quad F^{\bullet}_{1,-1} \mapsto -2F_{21}.\end{aligned}$$

Φ induces an isomorphism $\widehat{\Phi} : U(\mathfrak{sp}_2) \otimes U(\mathfrak{sp}_2) \xrightarrow{\sim} U(\mathfrak{so}_4)$, and so the composition $\widehat{\Phi}\circ(\mathrm{ev}\otimes\mathrm{ev})$ is a surjective homomorphism $X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_4)$. The composition of this map with the embedding $\widetilde{\varphi}_{\mathrm{D0}}$ from (5.1.12) gives a homomorphism $\mathrm{ev}_{\mathrm{D0}} : X(\mathfrak{so}_4, \mathfrak{so}_4)^{tw} \to U(\mathfrak{so}_4)$. If it is indeed given by the assignment (5.1.25) then it is surjective, so we need only verify that this is the case. This can be shown directly by first computing the explicit images $\widetilde{\varphi}_{\mathrm{D0}}(s_{ij}(u))$ (as in (5.1.19)), and then performing computations similar to those carried out in the $\mathfrak{so}_4^{\vartheta} = \mathfrak{gl}_2$ case. □

Given $\mu_1, \mu_2 \in \mathbb{C}$, let $V(\mu_1, \mu_2)$ denote the irreducible $\mathfrak{so}_4^\vartheta$-module with the highest weight (μ_1, μ_2). The pull-back of $V(\mu_1, \mu_2)$ via the appropriate epimorphism from Proposition 5.1.7 is an irreducible $X(\mathfrak{so}_4, \mathfrak{so}_4^\vartheta)^{tw}$-module, which we call an evaluation module.

Corollary 5.1.8. *Given $\mu_1, \mu_2 \in \mathbb{C}$, the evaluation module $V(\mu_1, \mu_2)$ is isomorphic to the $X(\mathfrak{so}_4, \mathfrak{so}_4^\vartheta)^{tw}$-module $V(\mu_1(u), \mu_2(u))$ where*

$$\left.\begin{aligned} \mu_1(u) &= 1 + \frac{2\mu_1}{u} + \frac{\mu_1^2 - \mu_2^2 + \mu_1 - \mu_2}{u(u-1)} \\ \mu_2(u) &= 1 + \frac{2\mu_2}{u} + \frac{\mu_2^2 - \mu_1^2 + \mu_2 - \mu_1}{u(u-1)} \end{aligned}\right. \qquad \text{if } \mathfrak{so}_4^\vartheta = \mathfrak{gl}_2,$$

$$\left.\begin{aligned} \mu_1(u) &= 1 + \frac{2\mu_1}{u-1} + \frac{\mu_1^2 - \mu_2^2}{(u-1)^2} \\ \mu_2(u) &= 1 + \frac{2\mu_2}{u-1} + \frac{\mu_2^2 - \mu_1^2}{(u-1)^2} \end{aligned}\right. \qquad \text{if } \mathfrak{so}_4^\vartheta = \mathfrak{so}_4.$$

Proof. Consider first the case where $\mathfrak{so}_4^\vartheta = \mathfrak{gl}_2$. We first show that z acts on $V(\mu_1, \mu_2)$ as the scalar $\mu_1^2 + \mu_2^2 + \mu_1 - \mu_2$. Since z belongs to the center of $U(\mathfrak{gl}_2)$ and $V(\mu_1, \mu_2)$ is a highest weight module, z acts by scalar multiplication. Therefore, it suffices to determine how z operates on the highest weight vector ξ. We have

$$\begin{aligned} (F_{11}^2 + F_{22}^2 + F_{12}F_{21} + F_{21}F_{12})\xi &= (F_{11}^2 + F_{22}^2 + F_{11} - F_{22})\xi \\ &= (\mu_1^2 + \mu_2^2 + \mu_1 - \mu_2)\xi, \end{aligned}$$

as desired. The formula(5.1.26) now shows ξ is a highest weight vector for $X(\mathfrak{so}_4, \mathfrak{sl}_2)^{tw}$ with the highest weight $(\mu_1(u), \mu_2(u))$ as in the statement of the corollary.

If instead $\mathfrak{so}_4^\vartheta = \mathfrak{so}_4$, observe that the Casimir element Ω operates on $V(\mu_1, \mu_2)$ as multiplication by the scalar $\mu_1^2 + \mu_2^2 - 2\mu_2$. The corollary now follows from the formula (5.1.25). □

5.1.4 Low rank twisted Yangians of type DI

In this subsection, we consider the extended twisted Yangian associated to the symmetric pair

$$(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2).$$

The story unfolds similarly to §5.1.3; the only difference is that the fixed point subalgebra $\mathfrak{g}_N^{\vartheta}$ is commutative of dimension 2, which leads to an extra parameter in the classification given by Proposition 5.1.9 below.

By [GRW16, Corollary 4.17], there is an isomorphism of algebras

$$\begin{aligned} \varphi_{\mathrm{DI}} : X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw} &\xrightarrow{\sim} Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}, \\ S(u) &\mapsto -\mathcal{S}_1^{\circ}(u - 1/2) K_1 S_2^{\bullet}(u - 1/2) K_2. \end{aligned} \tag{5.1.28}$$

This isomorphism can be obtained from the embedding

$$\begin{aligned} \widetilde{\varphi}_{\mathrm{DI}} : X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw} &\hookrightarrow X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}, \\ S(u) &\mapsto -S_1^{\circ}(u - 1/2) K_1 S_2^{\bullet}(u - 1/2) K_2. \end{aligned} \tag{5.1.29}$$

by applying the natural quotient map $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \twoheadrightarrow Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ in the first tensor factor. Here it understood that all notation is as in §5.1.3. The sign difference between (5.1.28) and (4.59) of [GRW16] is due to the fact that the matrix $\mathcal{G}$ that we use equals the matrix $-\mathcal{G}'$ used in [GRW16].

Proposition 5.1.9. *The $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $Q(u)$ and $P(u)$ together with scalars $\alpha, \beta \in \mathbb{C}$, such that*

$$\begin{gathered} P(\alpha) \neq 0 \neq Q(\beta), \\ P(u) = P(-u+2), \quad Q(u) = Q(-u+2), \\ \frac{\widetilde{\mu}_1(u)}{\widetilde{\mu}_2(u)} = \frac{P(u+1)}{P(u)} \cdot \frac{\alpha - u}{\alpha + u - 1}, \\ \frac{\widetilde{\mu}_1(1-u)}{\widetilde{\mu}_2(u)} = \frac{u}{1-u} \cdot \frac{Q(u+1)}{Q(u)} \cdot \frac{\beta - u}{\beta + u - 1}. \end{gathered}$$

Moreover, when it exists, the tuple $(\alpha, \beta, Q(u), P(u))$ is uniquely determined by $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}_4^+}$.

Proof. The proof of this proposition is very similar to that of Proposition 5.1.6. We begin by showing that the $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$-module $V(\mu(u))$, viewed as a $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module via the isomorphism φ_{DI}, is isomorphic to

$$V(\mu^{\circ}(u)) \otimes V(\mu^{\bullet}(u)),$$

where the pair $(\mu^\circ(u), \mu^\bullet(u))$ is completely determined by the relations

$$\begin{aligned} \mu^\circ(-u)\mu^\circ(1-u) &= 1, \\ \widetilde{\mu}_2(u) &= -2u\,\mu^\circ(u-1/2)\,\mu^\bullet(u-1/2), \\ \widetilde{\mu}_1(u) &= (2u-2)\,\mu^\circ(u-1/2)\,\mu^\bullet(-u+1/2). \end{aligned} \tag{5.1.30}$$

Writing the map (5.1.28) explicitly we have that

$$\varphi_{\mathrm{DI}}\ :\ \begin{cases} s_{11}(u) \mapsto \sigma^\circ_{11}(\widetilde{u})\, s^\bullet_{-1,-1}(\widetilde{u}), & s_{-1,2}(u) \mapsto \sigma^\circ_{-1,1}(\widetilde{u})\, s^\bullet_{11}(\widetilde{u}), \\ s_{22}(u) \mapsto -\sigma^\circ_{11}(\widetilde{u})\, s^\bullet_{11}(\widetilde{u}), & s_{1,-2}(u) \mapsto -\sigma^\circ_{1,-1}(\widetilde{u})\, s^\bullet_{-1,-1}(\widetilde{u}), \end{cases} \tag{5.1.31}$$

where $\widetilde{u} = u - 1/2$. Moreover, a computation analogous to (5.1.20) shows that

$$\varphi_{\mathrm{DI}}\ :\ s_{1,-2}(-\widetilde{u})\, s_{-1,2}(\widetilde{u}) - s_{11}(-\widetilde{u})\, s_{22}(\widetilde{u}) \mapsto s^\bullet_{-1,-1}(-u)\, s^\bullet_{11}(u-1).$$

Letting $\xi \in V(\mu(u))$ denote the highest weight vector, this in turn implies that

$$s^\bullet_{-1,-1}(-u) s^\bullet_{11}(u-1)\,\xi = -\mu_1(-\widetilde{u})\,\mu_2(\widetilde{u})\,\xi.$$

Using the symmetry relation (3.4.4) for $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$, we can rewrite the equality above as

$$\left(s^\bullet_{11}(u) + \frac{s^\bullet_{11}(u) - s^\bullet_{11}(-u)}{2u} \right) s^\bullet_{11}(u-1)\xi = -\mu_1(-\widetilde{u})\,\mu_2(\widetilde{u})\xi.$$

By induction on the coefficients $s^{\bullet(r)}_{11}$ of $s^\bullet_{11}(u)$, this implies that there exists $\mu^\bullet(u)$ in $1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying

$$s^\bullet_{11}(u)\xi = \mu^\bullet(u)\xi.$$

Moreover, $\mu^\bullet(u)$ is uniquely determined by the relation

$$\left(\mu^\bullet(u) + \frac{\mu^\bullet(u) - \mu^\bullet(-u)}{2u} \right) \mu^\bullet(u-1) = -\mu_1(-\widetilde{u})\,\mu_2(\widetilde{u}). \tag{5.1.32}$$

Moreover, (5.1.31) implies that ξ is also an eigenvector for the action of $\sigma^\circ_{11}(u)$ with weight $\mu^\circ(u)$ defined by

$$\mu_2(u) = -\mu^\circ(\widetilde{u})\mu^\bullet(\widetilde{u}). \tag{5.1.33}$$

The relations (5.1.32) and (5.1.33) are in fact equivalent to (5.1.30). Indeed, since $\operatorname{sdet} \mathcal{S}^\circ(u) = 1$ we have $\mu^\circ(-u)^{-1} = \mu^\circ(u-1)$, and using this relation we can rewrite (5.1.32) as

$$\mu_1(u) = \mu^\circ(\tilde{u}) \left(\mu^\bullet(-\tilde{u}) + \frac{\mu^\bullet(\tilde{u}) - \mu^\bullet(-\tilde{u})}{2\tilde{u}} \right).$$

This relation, together with $\mu^\circ(-u)^{-1} = \mu^\circ(u-1)$ and (5.1.33), yields (5.1.30). On the other hand, it is easily seen that (5.1.30) has a unique solution; hence the claimed equivalence holds.

Finally, since $[\sigma^\circ_{ij}(u), s^\bullet_{kl}(v)] = 0$, it follows immediately from the definition of ξ and the two formulas

$$\varphi_{\mathrm{DI}}(s_{-1,2}(u)) = \sigma^\circ_{-1,1}(\tilde{u})\, s^\bullet_{11}(\tilde{u}) \quad \text{and} \quad \varphi_{\mathrm{DI}}(s_{12}(u)) = \sigma^\circ_{11}(\tilde{u}) s^\bullet_{-1,1}(\tilde{u})$$

that $\sigma^\circ_{-1,1}(u)\,\xi = s^\bullet_{-1,1}(u)\,\xi = 0$. Thus, by the irreducibility of $V(\mu(u))$ we can conclude that

$$V(\mu(u)) \cong V(\mu^\circ(u)) \otimes V(\mu^\bullet(u)).$$

with $(\mu^\circ(u), \mu^\bullet(u))$ the unique solution of (5.1.30).

We can now use the isomorphism above to determine exactly when $V(\mu(u))$ is finite-dimensional. By Theorem 5.1.1, the module $V(\mu^\circ(u)) \otimes V(\mu^\bullet(u))$ is finite-dimensional if and only if there exists a tuple $(\gamma^\circ, \gamma^\bullet, P^\circ(u), P^\bullet(u))$, where $\gamma^\circ, \gamma^\bullet \in \mathbb{C}$ and $P^\circ(u), Q^\bullet(u)$ are monic polynomials in u such that

$$P^\circ(\gamma^\circ) \neq 0 \neq P^\bullet(\gamma^\bullet),$$
$$P^\circ(u) = P^\circ(-u+1), \quad P^\bullet(u) = P^\bullet(-u+1),$$

and the following equations hold:

$$\begin{aligned} \frac{\mu^\circ(-u)}{\mu^\circ(u)} &= \frac{2u+1}{2u-1} \cdot \frac{P^\circ(u+1)}{P^\circ(u)} \cdot \frac{u-\gamma^\circ}{u+\gamma^\circ}, \\ \frac{\mu^\bullet(-u)}{\mu^\bullet(u)} &= \frac{2u+1}{2u-1} \cdot \frac{P^\bullet(u+1)}{P^\bullet(u)} \cdot \frac{u-\gamma^\bullet}{u+\gamma^\bullet}. \end{aligned} \tag{5.1.34}$$

Set $P(u) = P^\bullet(\tilde{u})$, $Q(u) = P^\circ(\tilde{u})$, $\alpha = \gamma^\bullet + \frac{1}{2}$ and $\beta = \gamma^\circ + \frac{1}{2}$. Substituting $u \mapsto \tilde{u}$, the above relations become

$$\frac{\mu^\circ(-\tilde{u})}{\mu^\circ(\tilde{u})} = \frac{2u}{2u-2}\cdot\frac{P(u+1)}{P(u)}\cdot\frac{u-\alpha}{u+\alpha-1},$$
$$\frac{\mu^\bullet(-\tilde{u})}{\mu^\bullet(\tilde{u})} = \frac{2u}{2u-2}\cdot\frac{Q(u+1)}{Q(u)}\cdot\frac{u-\beta}{u+\beta-1}.$$

By (5.1.30),

$$\frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)} = \frac{2-2u}{2u}\cdot\frac{\mu^\bullet(-\tilde{u})}{\mu^\bullet(\tilde{u})} \quad \text{and} \quad \frac{\tilde{\mu}_1(1-u)}{\tilde{\mu}_2(u)} = \frac{\mu^\circ(-\tilde{u})}{\mu^\circ(\tilde{u})}.$$

Therefore, (5.1.34) is equivalent to

$$\frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)} = \frac{P(u+1)}{P(u)}\cdot\frac{\alpha-u}{u+\alpha-1} \quad \text{and} \quad \frac{\tilde{\mu}_1(1-u)}{\tilde{\mu}_2(u)} = \frac{u}{1-u}\cdot\frac{Q(u+1)}{Q(u)}\cdot\frac{\beta-u}{u+\beta-1}.$$

Moreover, $(\alpha, P(u))$ satisfies

$$P(u) = P^\bullet(u-1/2) = P^\bullet(-u+1/2+1) = P(-u+2),$$
$$P(\alpha) = P^\bullet(\alpha-1/2) = P^\bullet(\gamma^\bullet) \neq 0.$$

and the same is true of $(\beta, Q(u))$. Finally, the uniqueness of $(\alpha, \beta, Q(u), P(u))$ follows either from the uniqueness of $(\gamma^\circ, \gamma^\bullet, P^\circ(u), P^\bullet(u))$ or Lemma 5.2.1 below. □

We now construct an evaluation morphism $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2 \oplus \mathfrak{so}_2)$. Note that the fixed point subalgebra $U(\mathfrak{so}_2 \oplus \mathfrak{so}_2) \subset U(\mathfrak{so}_4)$ is generated by the elements F_{11} and F_{22}.

Proposition 5.1.10. *The assignment*

$$\mathrm{ev}_{\mathrm{DI}} : s_{ij}(u) \mapsto g_{ij} + 2g_{ij}F_{ij}u^{-1} + \delta_{ij}(F_{11}^2 - F_{22}^2)u^{-2} \tag{5.1.35}$$

defines a surjective algebra homomorphism

$$\mathrm{ev}_{\mathrm{DI}} : X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2 \oplus \mathfrak{so}_2).$$

Proof. The Lie algebra $\mathfrak{so}_2$ is one-dimensional and hence $\mathfrak{so}_2 \oplus \mathfrak{so}_2 = \mathbb{C}F_{11}^\circ \oplus \mathbb{C}F_{11}^\bullet$, where F_{11}° and $F_{11}^\bullet$ denote the generator F_{11} for the first and second copy of $\mathfrak{so}_2$, respectively.

Let Φ be the isomorphism

$$\Phi : \mathfrak{so}_2 \oplus \mathfrak{so}_2 \xrightarrow{\sim} \mathfrak{so}_4^{\vartheta},$$
$$F_{11}^{\circ} \mapsto F_{11} + F_{22}, \quad F_{11}^{\bullet} \mapsto F_{22} - F_{11}.$$

Then Φ induces an isomorphism

$$\widehat{\Phi} : U(\mathfrak{so}_2) \otimes U(\mathfrak{so}_2) \xrightarrow{\sim} U(\mathfrak{so}_4^{\vartheta}) = U(\mathfrak{so}_2 \oplus \mathfrak{so}_2),$$

and so the composition $\widehat{\Phi} \circ (\mathrm{ev} \otimes \mathrm{ev})$ yields a surjective homomorphism

$$\widehat{\Phi} \circ (\mathrm{ev} \otimes \mathrm{ev}) : X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2 \oplus \mathfrak{so}_2),$$

where $\mathrm{ev} : X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2)$ is given by Proposition 3.4.3. Composing $\widehat{\Phi} \circ (\mathrm{ev} \otimes \mathrm{ev})$ with the embedding $\widetilde{\varphi}_{\mathrm{DI}}$ from (5.1.29), we obtain $\mathrm{ev}_{\mathrm{DI}}$ as in (5.1.35). □

The morphism $\mathrm{ev}_{\mathrm{DI}}$ allows us to equip any $\mathfrak{so}_2 \oplus \mathfrak{so}_2$-module with the structure of a $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$-module. As usual, modules obtained this way are called evaluation modules. Let $V(\mu_1, \mu_2)$ denote the irreducible $\mathfrak{so}_4^{\vartheta} = \mathfrak{so}_2 \oplus \mathfrak{so}_2$ module with the highest weight (μ_1, μ_2). This is the one-dimensional representation of $\mathfrak{so}_4^{\vartheta}$ in which F_{ii} acts as multiplication by $\mu_i \in \mathbb{C}$. The corollary below follows directly from the formula (5.1.35).

Corollary 5.1.11. *For any $\mu_1, \mu_2 \in \mathbb{C}$, the module $V(\mu_1, \mu_2)$ is isomorphic to the $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$-module $V(\mu(u))$ with*

$$\mu_i(u) = g_{ii} + 2g_{ii}\mu_i u^{-1} + (\mu_1^2 - \mu_2^2)u^{-2} \quad \textit{for} \quad 1 \leq i \leq 2.$$

The collection $\{V(\mu_1, \mu_2)\}_{\mu_1, \mu_2 \in \mathbb{C}}$ provides a family of one-dimensional representations of $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$ indexed by $\mathbb{C} \times \mathbb{C}$. Note that the trivial representation $V(\mathcal{G})$ may be recovered in the special case where $(\mu_1, \mu_2) = (0, 0)$. In Remark 5.3.7, it will be explained that these are essentially all of the one-dimensional representations of $X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$.

5.1.5 Low rank twisted Yangians of type B

Our discussion of low rank twisted Yangians concludes in this subsection with an analysis of the finite-dimensional irreducible representations for the extended twisted Yangians associated to the pairs $(\mathfrak{so}_3, \mathfrak{so}_3^{\vartheta})$. That is, $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})$ takes the form

$$(\mathfrak{so}_3, \mathfrak{so}_3) \quad \text{and} \quad (\mathfrak{so}_3, \mathfrak{so}_2).$$

The latter pairs correspond to $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n-1} \oplus \mathfrak{so}_2)$ with $n = 1$, as $\mathfrak{so}_1 = \{0\}$. We begin by recalling the relevant isomorphisms from [GRW16].

Let V be the three-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ with basis $\{v_i\}_{i \in \mathcal{I}_3}$ given by

$$v_{-1} = e_{-1} \otimes e_{-1}, \quad v_0 = \tfrac{1}{\sqrt{2}}(e_{-1} \otimes e_1 + e_1 \otimes e_{-1}) \quad \text{and} \quad v_1 = -e_1 \otimes e_1.$$

Upon identifying V with $\mathbb{C}^3$ we may view

$$S(u) = \sum_{i,j \in \mathcal{I}_3} E_{ij}^V \otimes s_{ij}(u) \in \mathrm{End}(V) \otimes X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}[\![u^{-1}]\!],$$
$$\text{where} \quad E_{ij}^V v_k = \delta_{jk} v_i \quad \forall \quad i, j, k \in \mathcal{I}_3.$$

Let $R^\circ(u) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ denote the rational R-matrix (2.7.4) corresponding to $N = 2$. That is,

$$R^\circ(u) = I - P^\circ u, \quad \text{where} \quad P^\circ = \sum_{i,j \in \mathcal{I}_2} E_{ij} \otimes E_{ji}$$

and $\mathcal{I}_2 = \{\pm 1\}$. The operator

$$Q_V = \tfrac{1}{2} R^\circ(-1) = \tfrac{1}{2}(I + P^\circ) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

is then a projector of $\mathbb{C}^2 \otimes \mathbb{C}^2$ onto the subspace V. By Proposition 4.3 of [GRW16], there are isomorphisms of algebras

$$\begin{aligned}
\varphi_{\mathrm{B0}} : X(\mathfrak{so}_3, \mathfrak{so}_3)^{tw} &\xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}, \\
S(u) &\mapsto Q_V S_1^\circ(2u-1) R^\circ(-4u+1)^t S_2^\circ(2u), \\
\varphi_{\mathrm{BI}} : X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw} &\xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}, \\
S(u) &\mapsto \frac{1-4u}{1+4u} Q_V S_1^\circ(2u-1) R^\circ(-4u+1)^t S_2^\circ(2u) K_1 K_2
\end{aligned} \tag{5.1.36}$$

where the transpose t is that corresponding $\mathfrak{g}_2$ in $X(\mathfrak{sl}_2, \mathfrak{g}_2)^{tw}$, and it is applied to either the first or second tensor factor of $R_{12}^\circ(-4u+1)$. In addition $K = E_{11} - E_{-1,-1}$, as in the previous subsections.

Remark 5.1.12. The discrepancy between the definition of φ_{BI} and that of the isomorphism from [GRW16, (4.34)] and [GRW19b, (3.11)] is due to the factor

$$\frac{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}}) - 4u}{\mathrm{tr}(\mathcal{G}_{\mathrm{GRW}}) + 4u} = \frac{1-4u}{1+4u}$$

which appears in Proposition 3.3.27.

By [GRW16, Proposition 4.3], the isomorphisms φ_{B0} and φ_{BI} induce isomorphisms

$$Y(\mathfrak{so}_3, \mathfrak{so}_3)^{tw} \cong Y(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw} \quad \text{and} \quad Y(\mathfrak{so}_3, \mathfrak{so}_2)^{tw} \cong Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw},$$

though these will not play an important role in this subsection.

Before applying φ_{B0} and φ_{BI} to study the representation theory of $X(\mathfrak{so}_3, \mathfrak{so}_3^\vartheta)^{tw}$, we pause to explain how the definitions (5.1.36) should be interpreted. Let's first consider φ_{B0}. Setting $\widetilde{u} = u - 1/2$, we have

$$\begin{aligned}
s_{-1,-1}(u) \mapsto & s^\circ_{-1,-1}(2\widetilde{u}) s^\circ_{-1,-1}(2u) - \tfrac{1}{4u-1} s^\circ_{-1,1}(2\widetilde{u}) s^\circ_{1,-1}(2u), \\
s_{-1,0}(u) \mapsto & \tfrac{1}{\sqrt{2}} s^\circ_{-1,-1}(2\widetilde{u}) s^\circ_{-1,1}(2u) \\
& + \tfrac{1}{\sqrt{2}(4u-1)} \left(4u\, s^\circ_{-1,1}(2\widetilde{u}) s^\circ_{-1,-1}(2u) - s^\circ_{-1,1}(2\widetilde{u}) s^\circ_{11}(2u) \right), \\
s_{-1,1}(u) \mapsto & -\tfrac{4u}{4u-1} s^\circ_{-1,1}(2\widetilde{u}) s^\circ_{-1,1}(2u), \\
s_{0,-1}(u) \mapsto & \tfrac{1}{\sqrt{2}} s^\circ_{1,-1}(2\widetilde{u}) s^\circ_{-1,-1}(2u) \\
& + \tfrac{1}{\sqrt{2}(4u-1)} \left(4u\, s^\circ_{-1,-1}(2\widetilde{u}) s^\circ_{1,-1}(2u) - s^\circ_{11}(2\widetilde{u}) s^\circ_{1,-1}(2u) \right), \\
s_{00}(u) \mapsto & \tfrac{1}{8u-2} (4u\, s^\circ_{-1,-1}(2\widetilde{u}) - s^\circ_{11}(2\widetilde{u})) s^\circ_{11}(2u)
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{1}{8u-2}(4u\, s^{\circ}_{11}(2\widetilde{u}) - s^{\circ}_{-1,-1}(2\widetilde{u}))s^{\circ}_{-1,-1}(2u) \\
&+ \tfrac{1}{2}\left(s^{\circ}_{1,-1}(2\widetilde{u})s^{\circ}_{-1,1}(2u) + s^{\circ}_{-1,1}(2\widetilde{u})s^{\circ}_{1,-1}(2u)\right), \\
s_{01}(u) \mapsto &- \tfrac{1}{\sqrt{2}}s^{\circ}_{-1,1}(2\widetilde{u})s^{\circ}_{11}(2u) \\
&- \tfrac{1}{\sqrt{2}(4u-1)}\left(4u\, s^{\circ}_{11}(2\widetilde{u})s^{\circ}_{-1,1}(2u) - s^{\circ}_{-1,-1}(2\widetilde{u})s^{\circ}_{-1,1}(2u)\right), \\
s_{1,-1}(u) \mapsto &- \tfrac{4u}{4u-1}s^{\circ}_{1,-1}(2\widetilde{u})s^{\circ}_{1,-1}(2u), \\
s_{10}(u) \mapsto &- \tfrac{1}{\sqrt{2}}s^{\circ}_{11}(2\widetilde{u})s^{\circ}_{1,-1}(2u) \\
&- \tfrac{1}{\sqrt{2}(4u-1)}\left(4u\, s^{\circ}_{1,-1}(2\widetilde{u})s^{\circ}_{11}(2u) - s^{\circ}_{1,-1}(2\widetilde{u})s^{\circ}_{-1,-1}(2u)\right), \\
s_{11}(u) \mapsto & s^{\circ}_{11}(2\widetilde{u})s^{\circ}_{11}(2u) - \tfrac{1}{4u-1}s^{\circ}_{1,-1}(2\widetilde{u})s^{\circ}_{-1,1}(2u).
\end{aligned}
$$

To obtain the above from the assignment given in (5.1.36), one must expand $S(u)v_k$ and $\varphi_{\mathrm{B0}}(S(u))v_k$, for each $k \in \mathcal{I}_3 = \{0, \pm 1\}$, as linear combinations of the basis $\{v_i\}_{i\in\mathcal{I}_3}$ of V and then compare coefficients.

As an example, we consider the case where $k = 1$. Since

$$
S(u)v_1 = \sum_{i\in\mathcal{I}_3} v_i \otimes s_{i1}(u),
$$

this computation will allow us to compute the images of $s_{-1,1}(u), s_{01}(u)$ and $s_{11}(u)$. Since $v_1 = -e_1 \otimes e_1$, a straightforward computation shows that

$$
\begin{aligned}
\varphi_{\mathrm{B0}}(S(u))v_1 = &\; - Q_V S^{\circ}_1(2u-1)R^{\circ}(-4u+1)^t S^{\circ}_2(2u)(e_1 \otimes e_1) \\
= &\; \tfrac{1}{8u-2}\sum_{i,k\in\mathcal{I}_2} (e_k \otimes e_1 + e_1 \otimes e_k) \otimes s^{\circ}_{ki}(2u-1)s^{\circ}_{i1}(2u) \\
&- \tfrac{2u}{4u-1}\sum_{i,k\in\mathcal{I}_2} (e_k \otimes e_i + e_i \otimes e_k) \otimes s^{\circ}_{k1}(2u-1)s^{\circ}_{i1}(2u). \qquad (5.1.37)
\end{aligned}
$$

From this expression we can easily compute the $X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$-coefficients of $v_{-1} = e_{-1} \otimes e_{-1}$ and $v_1 = -e_1 \otimes e_1$, which must coincide with the images of $s_{-1,1}(u)$ and $s_{11}(u)$, respectively. We obtain

$$
\begin{aligned}
\varphi_{\mathrm{B0}}(s_{-1,1}(u)) &= -\tfrac{4u}{4u-1}s^{\circ}_{-1,1}(2u-1)s^{\circ}_{-1,1}(2u), \\
\varphi_{\mathrm{B0}}(s_{11}(u)) &= s^{\circ}_{11}(2u-1)s^{\circ}_{11}(2u) - \tfrac{1}{4u-1}s^{\circ}_{1,-1}(2u-1)s^{\circ}_{-1,1}(2u).
\end{aligned}
$$

Similarly, the coefficients of $e_{-1} \otimes e_1$ and $e_1 \otimes e_{-1}$ in (5.1.37) are both equal to

$$\tfrac{1}{8u-2}(s^\circ_{-1,-1}(2u-1)s^\circ_{-1,1}(2u)+s^\circ_{-1,1}(2u-1)s^\circ_{11}(2u)) \\ - \tfrac{4u}{8u-2}(s^\circ_{-1,1}(2u-1)s^\circ_{11}(2u) + s^\circ_{11}(2u-1)s^\circ_{-1,1}(2u)).$$

Since $v_0 = \frac{1}{\sqrt{2}}(e_{-1} \otimes e_1 + e_1 \otimes e_{-1})$, the image of $s_{01}(u)$ must coincide with the above expression multiplied by $\sqrt{2}$. After rearranging, this gives

$$\begin{aligned} \varphi_{\mathrm{B0}}(s_{01}(u)) = &- \tfrac{1}{\sqrt{2}} s^\circ_{-1,1}(2u-1)s^\circ_{11}(2u) \\ &- \tfrac{1}{\sqrt{2}(4u-1)}(4us^\circ_{11}(2u-1)s^\circ_{-1,1}(2u) - s^\circ_{-1-1}(2u-1)s^\circ_{-1,1}(2u)). \end{aligned}$$

The remaining images can all be computed by repeating this procedure with $k = 0$ and $k = 1$.

Similar calculations show that the images of the generators $s_{ij}(u)$ of $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ under φ_{BI} are given by the following formulas.

$$\begin{aligned}
s_{-1,-1}(u) &\mapsto \tfrac{1-4u}{1+4u} s^\circ_{-1,-1}(2\widetilde{u})s^\circ_{-1,-1}(2u) - \tfrac{1}{4u+1} s^\circ_{-1,1}(2\widetilde{u})s^\circ_{1,-1}(2u), \\
s_{-1,0}(u) &\mapsto \tfrac{1}{\sqrt{2}}\left(\tfrac{4u-1}{4u+1}\right) s^\circ_{-1,-1}(2\widetilde{u})s^\circ_{-1,1}(2u) \\
&\quad + \tfrac{1}{\sqrt{2}(4u+1)}\Big(s^\circ_{-1,1}(2\widetilde{u})s^\circ_{11}(2u) + 4us^\circ_{-1,1}(2\widetilde{u})s^\circ_{-1,-1}(2u)\Big), \\
s_{-1,1}(u) &\mapsto \tfrac{4u}{4u+1}\, s^\circ_{-1,1}(2\widetilde{u})s^\circ_{-1,1}(2u), \\
s_{0,-1}(u) &\mapsto \tfrac{1}{\sqrt{2}}\left(\tfrac{1-4u}{4u+1}\right) s^\circ_{1,-1}(2\widetilde{u})s^\circ_{-1,-1}(2u) \\
&\quad - \tfrac{1}{\sqrt{2}(4u+1)}\Big(s^\circ_{11}(2\widetilde{u})s^\circ_{1,-1}(2u) + 4u\, s^\circ_{-1,-1}(2\widetilde{u})s^\circ_{1,-1}(2u)\Big), \\
s_{00}(u) &\mapsto \tfrac{4u-1}{8u+2}\Big(s^\circ_{-1,1}(2\widetilde{u})s^\circ_{1,-1}(2u) + s^\circ_{1,-1}(2\widetilde{u})s^\circ_{-1,1}(2u)\Big) \\
&\quad + \tfrac{1}{8u+2}(s^\circ_{-1,-1}(2\widetilde{u}) + 4u\, s^\circ_{11}(2\widetilde{u}))s^\circ_{-1,-1}(2u) \\
&\quad + \tfrac{1}{8u+2}(4u\, s^\circ_{-1,-1}(2\widetilde{u}) + s^\circ_{11}(2\widetilde{u}))s^\circ_{11}(2u), \\
s_{01}(u) &\mapsto \tfrac{1}{\sqrt{2}(4u+1)} s^\circ_{-1,-1}(2\widetilde{u})s^\circ_{-1,1}(2u) \\
&\quad + \tfrac{1}{\sqrt{2}}\left(\tfrac{4u-1}{4u+1}\right) s^\circ_{-1,1}(2\widetilde{u})s^\circ_{11}(2u) + \tfrac{4u}{\sqrt{2}(4u+1)} s^\circ_{11}(2\widetilde{u})s^\circ_{-1,1}(2u), \\
s_{1,-1}(u) &\mapsto \tfrac{4u}{4u+1}\, s^\circ_{1,-1}(2\widetilde{u})s^\circ_{1,-1}(2u), \\
s_{10}(u) &\mapsto \tfrac{1}{\sqrt{2}}\left(\tfrac{1-4u}{4u+1}\right) s^\circ_{11}(2\widetilde{u})s^\circ_{1,-1}(2u) \\
&\quad - \tfrac{1}{\sqrt{2}(4u+1)}\Big(4u\, s^\circ_{1,-1}(2\widetilde{u})\, s^\circ_{11}(2u) + s^\circ_{1,-1}(2\widetilde{u})\, s^\circ_{-1,-1}(2u)\Big), \\
s_{11}(u) &\mapsto \tfrac{1-4u}{4u+1} s^\circ_{11}(2\widetilde{u})s^\circ_{11}(2u) - \tfrac{1}{4u+1} s^\circ_{1,-1}(2\widetilde{u})s^\circ_{-1,1}(2u).
\end{aligned} \tag{5.1.38}$$

The last ingredient we will need is the following lemma.

Lemma 5.1.13. *Suppose that* $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_3^+}$ *satisfies* (4.4.12). *That is,*

$$\widetilde{\mu}_0(u)\widetilde{\mu}_0(1-u) = \widetilde{\mu}_1(u)\widetilde{\mu}_1(1-u)$$
$$u\,\mathscr{g}(u)\widetilde{\mu}_0(1/2-u) = (1/2-u)\,\mathscr{g}(1/2-u)\widetilde{\mu}_0(u).$$

Then there exists a unique series $\mu^\circ(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ *satisfying*

$$\begin{aligned} \widetilde{\mu}_1(u) &= 2u\,\bar{\mathscr{g}}(u)\mu^\circ(2u)\mu^\circ(2u-1), \\ \widetilde{\mu}_0(u) &= 2u\,\mathscr{g}(u)\mu^\circ(2u)\mu^\circ(1-2u), \\ \textit{where} \quad \bar{\mathscr{g}}(u) &= \begin{cases} \dfrac{1-4u}{1+4u} & \textit{if } \mathfrak{so}_3^\vartheta = \mathfrak{so}_2, \\ 1 & \textit{if } \mathfrak{so}_3^\vartheta = \mathfrak{so}_3, \end{cases} \end{aligned} \tag{5.1.39}$$

Proof. It suffices to prove the lemma in the case where $\mathfrak{so}_3^\vartheta = \mathfrak{so}_3$. Indeed, if instead $\mathfrak{so}_3^\vartheta = \mathfrak{so}_2$, then we may define the $X(\mathfrak{so}_3, \mathfrak{so}_3)^{tw}$ highest weight $\mu^\sharp(u)$ from the $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ highest weight $\mu(u)$ by

$$\widetilde{\mu}_0^\sharp(u) = \mathscr{g}(u)^{-1}\widetilde{\mu}_0(u) \quad \text{and} \quad \widetilde{\mu}_1^\sharp(u) = \bar{\mathscr{g}}(u)^{-1}\widetilde{\mu}_1(u).$$

Since $\mathscr{g}(u)\mathscr{g}(1-u) = \bar{\mathscr{g}}(u)\bar{\mathscr{g}}(1-u)$, $\mu^\sharp(u)$ satisfies the assumptions of the lemma for $\mathfrak{so}_3^\vartheta = \mathfrak{so}_3$, and a unique solution $\mu^\circ(u)$ of (5.1.39) for $\mu^\sharp(u)$ gives a unique solution of (5.1.39) for $\mu(u)$.

Next, note that if $h(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ and $\gamma \in \mathbb{C}$, then there exists a unique series $k(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$h(u) = k(u)k(u+\gamma).$$

This is readily observed by expanding both sides.

Now let $\mu(u)$ be a $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ highest weight satisfying the assumptions of the lemma and set

$$f(u) = \frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)}.$$

By the above observation, there is a unique series $\lambda(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$\mu_1(u) = \lambda(u)\lambda(u-1/2)$. Set $\mu^\circ(u) = \lambda(u/2)$. Then

$$\widetilde{\mu}_1(u) = 2u\,\mu^\circ(2u)\mu^\circ(2u-1).$$

The uniqueness of $\lambda(u)$ guarantees that $\mu^\circ(u)$ will be the unique solution of (5.1.39) provided it satisfies

$$\widetilde{\mu}_0(u) = 2u\,\mu^\circ(2u)\mu^\circ(1-2u).$$

Since $f(u)f(1-u) = 1$, there exists a series $d(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$f(u) = d(1-2u)d(2u-1)^{-1}.$$

For instance, $d(u) = \mathring{f}\left(\frac{1-u}{2}\right)$, where $\mathring{f}(u)$ is the unique solution of the equation

$$\mathring{f}(u)^2 = f(u) \quad \text{in } \ 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!],$$

has this property. Set $g(u) = d(2u-1)\mu^\circ(2u-1)^{-1}$. Then

$$f(u) = \frac{\mu^\circ(1-2u)g(1-u)}{\mu^\circ(2u-1)g(u)} = \frac{2u\mu^\circ(2u)\mu^\circ(1-2u)g(1-u)g(u)^{-1}}{\widetilde{\mu}_1(u)},$$

which implies that $\widetilde{\mu}_0(u) = 2u\,\mu^\circ(2u)\mu^\circ(1-2u)g(1-u)g(u)^{-1}$. Since $\widetilde{\mu}_0(u)$ satisfies $u\,\widetilde{\mu}_0(1/2-u) = (1/2-u)\,\widetilde{\mu}_0(u)$, we obtain

$$g(1-u)g(u)^{-1} = g(u+1/2)g(1/2-u)^{-1}.$$

This implies that $k(u) = g(1-u)$ is the unique solution of

$$k(u+1/2)k(u) = g(u+1/2)g(u) \quad \text{in} \quad 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!].$$

Therefore, we must have $g(u) = k(u) = g(1-u)$. Hence

$$\begin{aligned}\widetilde{\mu}_0(u) &= 2u\,\mu^\circ(2u)\mu^\circ(1-2u)g(1-u)g(u)^{-1}\\ &= 2u\,\mu^\circ(2u)\mu^\circ(1-2u).\end{aligned}$$

□

Proposition 5.1.14. *Suppose that $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_3^+}$ satisfies the condition (4.4.12). Then the $X(\mathfrak{so}_3, \mathfrak{so}_3^\vartheta)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists* *a monic polynomial $P(u)$ in u, with*

$$P(u) = P(-u + 3/2),$$

in addition to a scalar $\alpha \in \mathbb{C}$ with $P(\alpha) \neq 0$ if $\mathfrak{so}_3^\vartheta = \mathfrak{so}_2$, such that

$$\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)} = \frac{P(u+1/2)}{P(u)} \qquad \text{if} \quad \mathfrak{so}_3^\vartheta = \mathfrak{so}_3, \tag{5.1.40}$$

$$\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)} = \frac{P(u+1/2)}{P(u)} \cdot \frac{\alpha - u}{\alpha + u - 1} \qquad \text{if} \quad \mathfrak{so}_3^\vartheta = \mathfrak{so}_2. \tag{5.1.41}$$

Moreover, when they exist, the polynomial $P(u)$ and scalar α are unique.

Proof. We will include a detailed proof the $\mathfrak{so}_3^\vartheta = \mathfrak{so}_2$; the proof in the $\mathfrak{so}_3^\vartheta = \mathfrak{so}_3$ case is nearly identical and is spelled out explicitly in [GRW17, Proposition 5.8].

By Lemma 5.1.13, there is a unique series $\mu^\circ(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ satisfying

$$\begin{aligned} \widetilde{\mu}_1(u) &= 2u\left(\frac{1-4u}{1+4u}\right)\mu^\circ(2u)\mu^\circ(2u-1), \\ \widetilde{\mu}_0(u) &= 2u\left(\frac{4u-3}{4u+1}\right)\mu^\circ(2u)\mu^\circ(1-2u). \end{aligned} \tag{5.1.42}$$

Since $\widetilde{\mu}_0(u) = (2u-1)\,\mu_0(u) + \mu_1(u)$ and $\widetilde{\mu}_1(u) = 2u\,\mu_1(u)$, these relations are equivalent to

$$\begin{aligned} \mu_1(u) &= \left(\frac{1-4u}{1+4u}\right)\mu^\circ(2u)\mu^\circ(2u-1), \\ \mu_0(u) &= \frac{1}{2\widetilde{u}}\left(\frac{4u-3}{4u+1}\cdot 2u\mu^\circ(-2\widetilde{u}) + \frac{4u-1}{4u+1}\cdot\mu^\circ(2\widetilde{u})\right)\mu^\circ(2u), \end{aligned} \tag{5.1.43}$$

where we recall that $\widetilde{u} = u - 1/2$.

Let $V(\mu^\circ(u))$ denote the irreducible highest weight $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module with the highest weight $\mu^\circ(u)$ and let ξ denote its highest weight vector. We may view $V(\mu^\circ(u))$ as a $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-module via the isomorphism φ_{BI} from (5.1.36). It is immediate from (5.1.38) that $s_{ij}(u)\xi = 0$ for all $i < j$ and

$$s_{11}(u)\xi = \left(\frac{1-4u}{1+4u}\right)\mu^\circ(2\widetilde{u})\mu^\circ(2u)\xi = \mu_1(u)\xi. \tag{5.1.44}$$

To compute $s_{00}(u)\xi$ we need to use the following formula, which follows from (3.4.5):

$$\begin{aligned}[s^\circ_{-1,1}(2\tilde{u}), s^\circ_{1,-1}(2u)] = \tfrac{1}{4u-1}\left(s^\circ_{11}(2u)\, s^\circ_{11}(2\tilde{u}) - s^\circ_{-1,-1}(2\tilde{u})\, s^\circ_{-1,-1}(2u)\right) \\ + \tfrac{4u}{4u-1}\left(s^\circ_{11}(2u)\, s^\circ_{-1,-1}(2\tilde{u}) - s^\circ_{11}(2\tilde{u})\, s^\circ_{-1,-1}(2u)\right).\end{aligned}$$

Combining this formula with the equality $[s^\circ_{ii}(u), s^\circ_{jj}(v)]\xi = 0$ for all $i, j \in \{\pm 1\}$, we obtain

$$s_{00}(u)\xi = \frac{1}{4u+1}\left(4u\, s^\circ_{-1,-1}(2\tilde{u}) + s^\circ_{11}(2\tilde{u})\right) s^\circ_{11}(2u)\xi. \tag{5.1.45}$$

By the defining symmetry relation (3.4.4) of $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$, we have

$$s^\circ_{-1,-1}(2\tilde{u}) = \frac{1}{4\tilde{u}}\left((4u-3)s^\circ_{11}(-2\tilde{u}) + s^\circ_{11}(2\tilde{u})\right).$$

Substituting this into (5.1.45) and appealing to (5.1.43), we obtain

$$s_{00}(u)\xi = \frac{1}{2\tilde{u}}\left(\frac{4u-3}{4u+1}\cdot 2u\mu^\circ(-2\tilde{u}) + \frac{4u-1}{4u+1}\cdot \mu^\circ(2\tilde{u})\right)\mu^\circ(2u)\xi = \mu_0(u)\xi. \tag{5.1.46}$$

Equalities (5.1.44) and (5.1.46) show that, as an $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-module, $V(\mu^\circ(u))$ is isomorphic to $V(\mu(u))$.

We can now use this isomorphism to determine exactly when $V(\mu(u))$ is finite-dimensional. By Theorem 5.1.1, this occurs precisely when there exists a monic polynomial $Q(u)$ together with $\gamma \in \mathbb{C}$ such that $Q(\gamma) \neq 0$ and

$$\begin{gathered} Q(u) = Q(-u+1), \\ \frac{\mu^\circ(-u)}{\mu^\circ(u)} = \frac{2u+1}{2u-1}\cdot\frac{Q(u+1)}{Q(u)}\cdot\frac{u-\gamma}{u+\gamma}. \end{gathered}$$

Using (5.1.42), the second relation above can be rewritten as

$$\begin{aligned}\frac{\tilde{\mu}_0(u)}{\tilde{\mu}_1(u)} &= \frac{4u-3}{1-4u}\cdot\frac{\mu^\circ(1-2u)}{\mu^\circ(2u-1)} \\ &= \frac{Q(2u)}{Q(2u-1)}\cdot\frac{\gamma-2u+1}{\gamma+2u-1} = \frac{P(u+1/2)}{P(u)}\cdot\frac{\alpha-u}{\alpha+u-1},\end{aligned}$$

where

$$P(u) = 2^{-\deg Q}Q(2u-1) \quad \text{and} \quad \alpha = \frac{\gamma+1}{2}. \tag{5.1.47}$$

With this definition of $(\alpha, P(u))$ we have $P(\alpha) \neq 0$ and $P(u) = P(-u+3/2)$ as desired. Finally, the uniqueness of $(\alpha, P(u))$ is guaranteed by the uniqueness of $(\gamma, Q(u))$. □

As in §5.1.2–§5.1.4, we conclude this subsection with a brief discussion of evaluation homomorphisms and evaluation modules. Let Ω denote the Casimir element

$$\Omega = F_{11}^2 - F_{11} + 2F_{10}F_{01}$$

of the Lie algebra $\mathfrak{so}_3$, and set

$$\Omega(u) = \left(\frac{4u+1}{4u}\right)\Omega.$$

Recall that $F^\vartheta = \sum_{i,j\in\mathcal{I}_3} E_{ij} \otimes F_{ij}^\vartheta \in \mathrm{End}(\mathbb{C}^3) \otimes U(\mathfrak{so}_3^\vartheta)$.

Proposition 5.1.15. *The assignments*

$$\mathrm{ev}_{\mathrm{B0}} : S(u) \mapsto I + \frac{u}{u-3/4}\left(\frac{F^\vartheta}{u-1/4} + \frac{(F^\vartheta)^2 - 2F^\vartheta - 2\Omega(u)\cdot I}{2(u-1/4)^2}\right), \tag{5.1.48}$$

$$\mathrm{ev}_{\mathrm{BI}} : S(u) \mapsto \mathcal{G}(u) - \frac{4}{(4u+1)^2}\left(4u\, F^\vartheta + (F_{11}^\vartheta)^2 \cdot I\right), \tag{5.1.49}$$

extend to algebra epimorphisms

$$\mathrm{ev}_{\mathrm{B0}} : X(\mathfrak{so}_3, \mathfrak{so}_3)^{tw} \twoheadrightarrow U(\mathfrak{so}_3) \quad \textit{and} \quad \mathrm{ev}_{\mathrm{BI}} : X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2).$$

Proof. Consider first the $\mathfrak{so}_3^\vartheta = \mathfrak{so}_2$ case. We identify $\mathfrak{so}_2 = \mathbb{C}F_{11}^\circ$ in its standard realization with $\mathfrak{so}_3^\vartheta$ via the isomorphism $F_{11}^\circ \mapsto F_{11}^\vartheta$. Composing the evaluation morphism ev from Proposition 3.4.3 (where $\mathfrak{g}_N = \mathfrak{so}_2$) with φ_{BI} from (5.1.36) and using the aforementioned identification, we obtain an epimorphism

$$\mathrm{ev} \circ \varphi_{\mathrm{BI}} : X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw} \twoheadrightarrow U(\mathfrak{so}_2)$$

which satisfies $s_{ij}(u) \mapsto 0$ if $i \neq j$ and

$$s_{-1,-1}(u) \mapsto \frac{1-4u}{1+4u} - \frac{4}{(4u+1)^2}\left((F_{-1,-1}^\vartheta)^2 + 4uF_{-1,-1}^\vartheta\right),$$

$$s_{00}(u) \mapsto 1 - \frac{8u}{(1-4u)(1+4u)}\left(F_{-1,-1}^\vartheta + F_{11}^\vartheta\right)$$

$$
\begin{aligned}
&\quad - \frac{2}{(1+4u)^2(1-4u)}\Big((F^\vartheta_{11} + 4u\, F^\vartheta_{-1,-1})\, F^\vartheta_{11} + (F^\vartheta_{-1,-1} + 4u\, F^\vartheta_{11})\, F^\vartheta_{-1,-1} \Big) \\
&= 1 - \frac{4(F^\vartheta_{11})^2}{(4u+1)^2}, \\
s_{11}(u) &\mapsto \frac{1-4u}{1+4u} - \frac{4}{(4u+1)^2}\left((F^\vartheta_{11})^2 + 4uF^\vartheta_{11} \right).
\end{aligned}
$$

Hence, $\mathrm{ev} \circ \varphi_{\mathrm{BI}}$ gives $\mathrm{ev}_{\mathrm{BI}}$ as in (5.1.49).

Suppose instead that $\mathfrak{so}_3^\vartheta = \mathfrak{so}_3$. Let $\Phi : U(\mathfrak{sp}_2) \xrightarrow{\sim} U(\mathfrak{so}_3)$ be the isomorphism given by

$$
F^\circ_{11} \mapsto 2F_{11}, \quad F^\circ_{-1,1} \mapsto 2\sqrt{2}F_{-1,0}, \quad F^\circ_{1,-1} \mapsto 2\sqrt{2}F_{0,-1},
$$

where the usual generators of $\mathfrak{g}_2 = \mathfrak{sp}_2$ are denoted $\{F^\circ_{ij}\}_{i,j\in\mathcal{I}_2}$. We thus obtain an epimorphism

$$
\mathrm{ev}_{\mathrm{B0}} = \Phi \circ \mathrm{ev} \circ \varphi_{\mathrm{B0}} : X(\mathfrak{so}_3, \mathfrak{so}_3)^{tw} \twoheadrightarrow U(\mathfrak{so}_3)
$$

To complete the proof of the proposition, it remains only to see that this map agrees with $\mathrm{ev}_{\mathrm{B0}}$ as given in (5.1.48). This can be checked directly using the explicit formulas for $\varphi_{\mathrm{B0}}(s_{ij}(u))$ (see above (5.1.37)) and that ev is given by

$$
s^\circ_{ij}(u) \mapsto \delta_{ij} + F^\circ_{ij}\left(u - \frac{1}{2}\right)^{-1}.
$$

For example,

$$
\begin{aligned}
s_{11}(u) &\mapsto \left(1 + \frac{F_{11}}{u-3/4}\right)\left(1 + \frac{F_{11}}{u-1/4}\right) - \frac{1}{2u-1/2}\left(\frac{F_{0,-1}}{u-3/4}\right)\left(\frac{F_{-1,0}}{u-1/4}\right) \\
&= 1 + \frac{u}{u-3/4}\left(\frac{2F_{11}}{u-1/4} + \frac{F_{11}^2 - F_{11} - \frac{1}{4u}(F_{11}^2 - F_{11} + 2F_{0,-1}F_{-1,0})}{(u-1/4)^2} \right).
\end{aligned}
$$

Conversely, since $F^\vartheta_{ij} = 2F_{ij}$ for all $i,j \in \mathcal{I}_2$, the coefficient of E_{11} on the right-hand side of (5.1.48) is

$$
\begin{aligned}
1 + &\frac{u}{u-3/4}\left(\frac{2F_{11}}{u-1/4} + \frac{2F_{10}F_{01} + 2F_{11}^2 - 2F_{11} - \frac{4u+1}{4u}(F_{11}^2 - F_{11} + 2F_{10}F_{01})}{(u-1/4)^2} \right) \\
&= 1 + \frac{u}{u-3/4}\left(\frac{2F_{11}}{u-1/4} + \frac{F_{11}^2 - F_{11} - \frac{1}{4u}(F_{11}^2 - F_{11} + 2F_{10}F_{01})}{(u-1/4)^2} \right).
\end{aligned}
$$

As $F_{10}F_{01} = F_{0,-1}F_{-1,0}$, this shows that $\mathrm{ev}_{\mathrm{B0}}(s_{11}(u))$ is indeed given as claimed in

(5.1.48). The images of the other generators can be checked similarly, or one can use (4.2.9) and the fact that $X(\mathfrak{so}_3, \mathfrak{so}_3)^{tw}$ is generated by the coefficients of $s_{11}(u)$ and the elements $F_{ij} \in \mathfrak{so}_3$. □

Let $V(\mu)$ denote the irreducible $\mathfrak{so}_3^\vartheta$-module with the highest weight $\mu \in \mathbb{C}$. As a consequence of Proposition 5.1.15, we can, and will, view $V(\mu)$ as an irreducible module over $X(\mathfrak{so}_3, \mathfrak{so}_3^\vartheta)^{tw}$.

Corollary 5.1.16. *Let $\mu \in \mathbb{C}$. Then, as an $X(\mathfrak{so}_3, \mathfrak{so}_3^\vartheta)^{tw}$-module, $V(\mu)$ is isomorphic to $V(\mu(u))$ with*

$$\left.\begin{aligned} \mu_0(u) &= \frac{(4u+1)^2-(4\mu)^2}{(4u+1)^2} \\ \mu_1(u) &= -\frac{(4u+4\mu)^2-1}{(4u+1)^2} \end{aligned}\right\} \quad \text{if } \mathfrak{so}_3^\vartheta = \mathfrak{so}_2,$$

$$\left.\begin{aligned} \mu_0(u) &= 1-16\mu\frac{\mu(4u+1)+4u-1}{(4u-3)(4u-1)^2} \\ \mu_1(u) &= 1+16\mu\frac{\mu+2u-1}{(4u-3)(4u-1)} \end{aligned}\right\} \quad \text{if } \mathfrak{so}_3^\vartheta = \mathfrak{so}_3.$$

Proof. This follows from (5.1.48) and (5.1.49) after observing that Ω operates on $V(\mu)$ as scalar multiplication by $\mu^2 - \mu$. □

When $\mathfrak{so}_3^\vartheta = \mathfrak{so}_2$, $V(\mu)$ is always one dimension with $F_{11} \in \mathfrak{so}_3^\vartheta$ operating as $\mu \cdot \mathrm{id}$. In analogy to the two-parameter family $\{V(\mu_1, \mu_2)\}_{\mu_1,\mu_2\in\mathbb{C}}$ of the previous subsection, the one-parameter family $\{V(\mu)\}_{\mu\in\mathbb{C}}$ of $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-modules, which includes $V(\mathcal{G}) = V(0)$, contains every one-dimensional representation of $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ up to twisting by automorphisms of the form ν_g. Indeed, this will be shown in Proposition 5.3.5 below, where $V(\mu)$ corresponds to $V(\alpha)$ with $\alpha = -\mu - 1/4$.

5.2 Necessary conditions in the general setting

Henceforth, we will assume that $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is any of the orthogonal or symplectic pairs introduced in Chapter 3 which is not equal to $(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)$. That is, $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ belongs to one of the families

$$(\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \quad \text{where} \quad (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q) \neq (\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2),$$

and $0 \leq q < N$ takes even values and satisfies $q \leq p$ if $p = N - q$ is also even.

In this section we use the machinery developed in Chapter 4, together with the results of §5.1, to obtain a set of conditions on the highest weight $\mu(u)$ of the $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-module $V(\mu(u))$, which are satisfied whenever this module is finite-dimensional: see Propositions 5.2.5 and 5.2.13. The extended twisted Yangian associated to the pair $(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)$ has been studied in §5.1.4, and is exceptional due to its two-parameter family of one-dimensional representations, which exist as a consequence of the fact that $\mathfrak{so}_4^{\vartheta} = \mathfrak{so}_2 \oplus \mathfrak{so}_2$ is a two-dimensional commutative Lie algebra.

As a prerequisite to our study, we begin by proving two elementary results pertaining to polynomials satisfying certain symmetry relations.

5.2.1 Preliminaries on polynomials

The two technical lemmas presented below are inspired by similar results which appeared in Chapters 3 and 4 of [Mol07]. They will play a small, but important, role in the next chapters.

Lemma 5.2.1. *Let $\alpha, \beta \in \mathbb{C}$, $l \in \mathbb{Z}$ and $m \in \mathbb{Q}$. Suppose that $P(u)$ and $Q(u)$ are both monic polynomials satisfying*

$$P(u) = P(-u + l), \quad Q(u) = Q(-u + l),$$
$$P(\alpha) \neq 0 \neq Q(\beta),$$
$$\frac{P(u+m)}{P(u)} \cdot \frac{\alpha - u}{\alpha + u - l + m} = \frac{Q(u+m)}{Q(u)} \cdot \frac{\beta - u}{\beta + u - l + m}. \tag{5.2.1}$$

Then $P(u) = Q(u)$ and $\alpha = \beta$.

Proof. This is a generalization of a result proven as part of the proof of [Mol07, Theorem 4.4.3]. There the statement of the lemma was proven in the special case where $l = m = 1$. The same argument works in the general case, and we repeat it here for the sake of the reader.

If $\alpha = \beta$, then

$$\frac{P(u+m)}{P(u)} = \frac{Q(u+m)}{Q(u)},$$

which implies that the rational function $f(u) = Q(u)/P(u)$ is periodic. This is impossible unless $f(u)$ is constant. Since $P(u)$ and $Q(u)$ are monic, we can conclude that $f(u) = 1$, and hence $P(u) = Q(u)$.

Therefore it suffices to show that an equality of the form (5.2.1) is impossible unless $\alpha = \beta$. We prove this by induction on k, where $k = \frac{1}{2}(\deg P(u) + \deg Q(u))$.

If $k = 0$, then this follows from the fact that (5.2.1) collapses to

$$\frac{\alpha - u}{\alpha + u - l + m} = \frac{\beta - u}{\beta + u - l + m}.$$

Suppose inductively that (5.2.1) is impossible whenever $\alpha \neq \beta$ and $k < M$ for some $M \in \mathbb{N}$. Assume now $k = M$. By symmetry, we may assume without loss of generality that $\deg P(u) \geq 2$. Additionally, without loss of generality we may assume that $P(u)$ and $Q(u)$ have no common roots. Let u_0 be a root of $P(u)$ such that $u_0 + m$ is not a root. Then (5.2.1) implies $u_0 = l - m - \beta$.

Write $P(u) = P^\sharp(u)(u - u_0)(u + u_0 - l)$ and set $\beta^\sharp = l - u_0$. Then we have

$$\frac{P^\sharp(u+m)}{P^\sharp(u)} \cdot \frac{\alpha - u}{\alpha + u - l + m} = \frac{Q(u+m)}{Q(u)} \cdot \frac{\beta^\sharp - u}{\beta^\sharp + u - l + m},$$

and $\alpha \neq \beta^\sharp$ due to the fact that $P(\beta^\sharp) = 0$. By the induction hypothesis, this is impossible. □

Lemma 5.2.2. *Let $\alpha \in \mathbb{C}$, $l \in \mathbb{Z}$ and $m \in \mathbb{Q}$. Suppose that $P(u)$ is a monic polynomial such that $P(u) = P(-u + l)$. Then there exists a pair $(\ell_\alpha^m, P_\alpha^m(u))$, where $\ell_\alpha^m \in \mathbb{Z}_{\geq 0}$ and $P_\alpha^m(u)$ is a monic polynomial, satisfying*

$$P_\alpha^m(u) = P_\alpha^m(-u + l), \quad P_\alpha^m(\alpha - m\,\ell_\alpha^m) \neq 0,$$
$$\frac{P(u+m)}{P(u)} \cdot \frac{\alpha - u}{\alpha + u - l + m} = \frac{P_\alpha^m(u+m)}{P_\alpha^m(u)} \cdot \frac{(\alpha - m\,\ell_\alpha^m) - u}{(\alpha - m\,\ell_\alpha^m) + u - l + m}. \tag{5.2.2}$$

Moreover, the pair $(\ell_\alpha^m, P_\alpha^m(u))$ is unique with $P_\alpha^m(u)$ equal to $P(u)$ divided by

$$Q(u) = \prod_{k=0}^{\ell_\alpha^m - 1} (u - \alpha + km)(u - l + \alpha - km). \tag{5.2.3}$$

Proof. We first define the pair $(\ell_\alpha^m, P_\alpha^m(u))$ and show that it satisfies the desired properties. For each $a \geq 0$, set

$$P^{(a)}(u) = \frac{P(u)}{\prod_{k=0}^{a-1}(u-\alpha+km)(u-l+\alpha-km)} \in \mathbb{C}(u), \tag{5.2.4}$$

where $P^{(0)}(u) = P(u)$. Note that $P^{(a)}(u)$ will be a monic polynomial in u satisfying $P^{(a)}(u) = P^{(a)}(-u+l)$ whenever $P(u)$ is divisible by $\prod_{k=0}^{a-1}(u-\alpha+km)(u-l+\alpha-km)$. Define

$$\ell = \begin{cases} 0 \ \text{ if } \ P(\alpha) \neq 0, \\ \min_{k\geq 1}\{P^{(k-1)}(\alpha-(k-1)m) = 0,\ P^{(k)}(\alpha-km) \neq 0\} \ \text{ otherwise.} \end{cases}$$

It a straightforward consequence of the above definitions that $P^{(\ell)}(u)$ is a monic polynomial in u satisfying $P^{(\ell)}(u) = P^{(\ell)}(-u+l)$. We may now set

$$\ell_\alpha^m = \ell \quad \text{and} \quad P_\alpha^m(u) = P^{(\ell)}(u).$$

By definition of ℓ_α^m, we have $P_\alpha^m(\alpha - m\,\ell_\alpha^m) \neq 0$. Moreover,

$$\begin{aligned} \frac{P(u+m)}{P(u)} &= \frac{P_\alpha^m(u+m)}{P_\alpha^m(u)} \cdot \frac{\prod_{k=0}^{\ell_\alpha^m-1}(u-\alpha+(k+1)m)(u-l+\alpha-(k-1)m)}{\prod_{k=0}^{\ell_\alpha^m-1}(u-\alpha+km)(u-l+\alpha-km)} \\ &= \frac{P_\alpha^m(u+m)}{P_\alpha^m(u)} \cdot \frac{(\alpha-\ell_\alpha^m m)-u}{(\alpha-\ell_\alpha^m m)+u-l+m} \cdot \frac{\alpha+u-l+m}{\alpha-u}, \end{aligned}$$

which implies that (5.2.2) holds.

Finally, note that the uniqueness of $(\ell_\alpha^m, P_\alpha^m(u))$ is an immediate corollary of Lemma 5.2.1. □

The statement of Lemma 5.2.2 did not explicitly appear in [Mol07], but similar ideas were needed in the proof of Theorem 4.4.14 therein. We note that the assumption that $m \in \mathbb{Q}$ is not necessary in either of the above two lemmas. However, we will only be concerned with this case.

We conclude this brief subsection by introducing some convenient notation related to polynomials.

Definition 5.2.3. Let $P(u) \in \mathbb{C}[u]$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha - \beta \in \mathbb{Z}$.

(1) We denote the zero set (or zero locus) of $P(u)$ by $Z(P(u))$:

$$Z(P(u)) = \{z \in \mathbb{C} \,:\, P(z) = 0\}.$$

(2) We define the string $S(\alpha, \beta) \subset \mathbb{C}$ to be the set

$$S(\alpha,\beta) = \begin{cases} \{\beta, \beta+1, \dots, \alpha-1\} & \text{if } \alpha - \beta \in \mathbb{Z}_{>0}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.2.5}$$

5.2.2 Associating polynomials to $V(\mu(u))$

Let us now define auxiliary parameters $\mathsf{a}, \mathsf{b} \in \mathcal{I}_N^+$ and $\mathsf{d} \in \mathbb{Q}$ by

$$(\mathsf{a}, \mathsf{b}, \mathsf{d}) = \begin{cases} (1,1,2) \;\text{ if }\; \mathfrak{g}_N = \mathfrak{sp}_N, \\ (0,1,1/2) \;\text{ if }\; \mathfrak{g}_N = \mathfrak{so}_{2n+1}, \\ (1,2,1) \;\text{ if }\; \mathfrak{g}_N = \mathfrak{so}_{2n}. \end{cases} \tag{5.2.6}$$

These parameters are chosen so that, in the notation of (4.1.7), we have

$$h_1 = -F_{\mathsf{aa}} - F_{\mathsf{bb}} \quad \text{ and } \quad \mathsf{d} = d_1.$$

Moreover, with this notation the second set of relations in Theorem 4.1.4 may be expressed uniformly as

$$\frac{\lambda_{-\mathsf{a}}(u)}{\lambda_{\mathsf{b}}(u)} = \frac{P_1(u+\mathsf{d})}{P_1(u)}. \tag{5.2.7}$$

We will also set

$$\hbar(\mathcal{G}) = \hbar\delta_{\mathcal{G},\mathcal{G}^t} \quad \text{ and } \quad \delta = \delta_{\mathfrak{g}_N, \mathfrak{sp}_N}.$$

For the next lemma, we briefly return to the low rank setting of §5.1.

Lemma 5.2.4. *Let $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ be a symmetric pair from §5.1 not equal to $(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)$. Then, if $V(\mu(u))$ is finite-dimensional, there exists a monic polynomial $Q(u)$ in u together with a scalar $\alpha \in \mathbb{C} \setminus Z(Q(u))$ such that*

$$Q(u) = Q(-u + \kappa + 2^\delta),$$

$$\frac{u}{\kappa-u}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa-u)}{\widetilde{\mu}_{\mathsf{b}}(u)}=\frac{g(\kappa-u)}{g(u)}\cdot\frac{Q(u+\mathsf{d})}{Q(u)}\left(\frac{\alpha-u}{\alpha+u-\kappa+\mathsf{d}-2^{\delta}}\right)^{\delta_{\hbar(\mathcal{G}),0}}.$$

Proof. If $\mathfrak{g}_N=\mathfrak{sp}_2$ or $\mathfrak{g}_N=\mathfrak{so}_4$, then this follows from Propositions 5.1.3 and 5.1.6 together with the definitions of κ, d, δ and $g(u)$. If instead $\mathfrak{g}_N=\mathfrak{so}_3$, then by Proposition 4.4.4, we have

$$\frac{u}{\kappa-u}\cdot\frac{\widetilde{\mu}_0(\kappa-u)}{\widetilde{\mu}_1(u)}\frac{g(u)}{g(\kappa-u)}=\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)},$$

and the lemma follows from Proposition 5.1.14 together with the facts that $\kappa=1/2=\mathsf{d}$ and $2^{\delta}=1$. □

Using this lemma, we can improve upon Proposition 4.4.5 of Chapter 4.

Proposition 5.2.5. *Suppose the $X(\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})^{tw}$-module $V(\mu(u))$ is finite-dimensional. Then there exists monic polynomials $P_1(u),\ldots,P_n(u)$ in u, with*

$$\begin{aligned}P_1(u)&=P_1(-u+\kappa+2^{\delta}),\\ P_i(u)&=P_i(-u+n-i+2)\quad\forall\quad 2\le i\le n,\end{aligned}\tag{5.2.8}$$

together with a scalar $\alpha\in\mathbb{C}\setminus Z(P_{\hbar(\mathcal{G})+1}(u))$ such that

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)}=\frac{P_i(u+1)}{P_i(u)}\left(\frac{\alpha-u}{\alpha+u-\ell}\right)^{\delta_{i,\hbar+1}}\quad\forall\quad 2\le i\le n,\tag{5.2.9}$$

$$\frac{u}{\kappa-u}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa-u)}{\widetilde{\mu}_{\mathsf{b}}(u)}=\frac{g(\kappa-u)}{g(u)}\cdot\frac{P_1(u+\mathsf{d})}{P_1(u)}\left(\frac{\alpha-u}{\alpha+u-\kappa+\mathsf{d}-2^{\delta}}\right)^{\delta_{\hbar(\mathcal{G}),0}}.\tag{5.2.10}$$

Proof. The existence of $P_2(u),\ldots,P_n(u)$ along with $\alpha\in\mathbb{C}\setminus Z(P_{\hbar+1}(u))$ (provided $\hbar(\mathcal{G})\neq 0$) satisfying (5.2.8) and (5.2.9) was established in Proposition 4.4.5. Therefore, it suffices to show that there exists $P_1(u)$ (together with $\alpha\in\mathbb{C}\setminus Z(P_1(u))$ if $\hbar(\mathcal{G})=0$) satisfying $P_1(u)=P_1(-u+\kappa+2^{\delta})$ as well as the relation (5.2.10).

For this, we will make use of the results of §4.3. Define

$$m=n-2\ \text{ if }\ \mathfrak{g}_N=\mathfrak{so}_{2n}\qquad\text{and}\qquad m=n-1\ \text{ if }\ \mathfrak{g}_N=\mathfrak{sp}_{2n}\ \text{ or }\ \mathfrak{so}_{2n+1}.$$

The symmetric pair $(\mathfrak{g}_{N-2m},\mathfrak{g}_{N-2m}^{\vartheta(m)})$ (see §4.3) is then given by

$$(\mathfrak{g}_{N-2m},\mathfrak{g}_{N-2m}^{\vartheta(m)})=\begin{cases}(\mathfrak{g}_{N-2m},\mathfrak{gl}_{n-m}) & \text{if } (\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})=(\mathfrak{g}_{2n},\mathfrak{gl}_n),\\ (\mathfrak{g}_{N-2m},\mathfrak{g}_{N-2m}) & \text{if } (\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})=(\mathfrak{g}_N,\mathfrak{g}_p\oplus\mathfrak{g}_q),\ p>1,\\ (\mathfrak{so}_3,\mathfrak{so}_2) & \text{if } (\mathfrak{g}_N,\mathfrak{g}_N^{\vartheta})=(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n}).\end{cases}$$

Here we note that $(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})=(\mathfrak{so}_{2n+1},\mathfrak{so}_1\oplus\mathfrak{so}_{2n})$ and that the symmetric pair $(\mathfrak{so}_4,\mathfrak{so}_2\oplus\mathfrak{so}_2)$ does not appear in the above list.

As $V(\mu(u))$ is finite-dimensional, the $X(\mathfrak{g}_{N-2m},\mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module $V(\mu(u))_m$ from Corollary 4.3.7 is finite-dimensional. Moreover, it is isomorphic to $V(\mu^\sharp(u))$ with

$$\mu^\sharp(u)=(h(u)\mu_i^{\circ m}(u))_{i\in\mathcal{I}_{N-2m}},$$

where $h(u)$ is given by (4.3.28) and $(\mu_i^{\circ m}(u))_{i\in\mathcal{I}_{N-2m}}$ is determined by (4.3.31). By Lemma 5.2.4, there exists a monic polynomial $Q(u)$ in u, with

$$Q(u)=Q(-u+\kappa-m+2^{\delta}), \tag{5.2.11}$$

together with a scalar $\mathring{\alpha}\in\mathbb{C}\setminus Z(Q(u))$ such that

$$\frac{u}{\mathring{\kappa}-u}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}^\sharp(\mathring{\kappa}-u)}{\widetilde{\mu}_{\mathsf{b}}^\sharp(u)}=\frac{\mathcal{g}_m(\mathring{\kappa}-u)}{\mathcal{g}_m(u)}\cdot\frac{Q(u+\mathsf{d})}{Q(u)}\left(\frac{\mathring{\alpha}-u}{\mathring{\alpha}+u-\mathring{\kappa}+\mathsf{d}-2^{\delta}}\right)^{\delta_{\mathcal{k}(\mathcal{G}_m),0}}, \tag{5.2.12}$$

where $\mathring{\kappa}=\kappa-m$. By definition of $\mu^\sharp(u)$, the left-hand side is equal to

$$\frac{u+\frac{m}{2}}{\kappa-u-\frac{m}{2}}\cdot\frac{\mathcal{g}_m(\mathring{\kappa}-u)}{\mathcal{g}_m(u)}\cdot\frac{\mathcal{g}(u+\frac{m}{2})}{\mathcal{g}(\kappa-u-\frac{m}{2})}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa-u-\frac{m}{2})}{\widetilde{\mu}_{\mathsf{b}}(u+\frac{m}{2})}.$$

Substituting this back into (5.2.12), shifting $u\mapsto u-\frac{m}{2}$, and using $\delta_{\mathcal{k}(\mathcal{G}),0}=\delta_{\mathcal{k}(\mathcal{G}_m),0}$, we obtain

$$\frac{u}{\kappa-u}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa-u)}{\widetilde{\mu}_{\mathsf{b}}(u)}=\frac{\mathcal{g}(\kappa-u)}{\mathcal{g}(u)}\cdot\frac{Q(u-\frac{m}{2}+\mathsf{d})}{Q(u-\frac{m}{2})}\left(\frac{\mathring{\alpha}+\frac{m}{2}-u}{\mathring{\alpha}+\frac{m}{2}+u-\kappa+\mathsf{d}-2^{\delta}}\right)^{\delta_{\mathcal{k}(\mathcal{G}),0}}$$

Setting $\alpha=\mathring{\alpha}+\frac{m}{2}$ (if $\mathcal{k}(\mathcal{G})=0$) and $P_1(u)=Q(u-\frac{m}{2})$, the relation (5.2.10) holds, we have $\alpha\in\mathbb{C}\setminus Z(P_1(u))$ (provided $\mathcal{k}(\mathcal{G})=0$), and by (5.2.11) the first identity in (5.2.8) holds. □

At this point, it is clear that finite-dimensional irreducible $X(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta})^{tw}$-modules are intimately connected to tuples

$$(\alpha, (P_i(u))_{i=1}^n) \subset \mathbb{C} \times \mathbb{C}[u]^n$$

which satisfy certain conditions. In light of this, it is desirable to have terminology which enables us to quickly translate between the language of highest weights and that of finite sequences $(\alpha, (P_i(u))_{i=1}^n)$ of the form described in Proposition 5.2.5.

Definition 5.2.6. Suppose that $(P_i(u))_{i=1}^n \subset \mathbb{C}[u]^n$ is an n-tuple of monic polynomials in u such that

$$P_1(u) = P_1(-u + \kappa + 2^{\delta}),$$
$$P_i(u) = P_i(-u + n - i + 2) \quad \forall \quad 2 \leq i \leq n,$$

Then, given $\alpha \in \mathbb{C} \setminus Z(P_{\mathfrak{k}(\mathcal{G})+1}(u))$, we say that $\mu(u)$ is *associated* to the tuple $(\alpha, (P_i(u))_{i=1}^n)$ if the relations (5.2.9) and (5.2.10) of Proposition 5.2.5 are satisfied and additionally (4.2.6) holds when $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$.

Remark 5.2.7. If $(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_N, \mathfrak{g}_N)$, then the parameter α plays no role in Proposition 5.2.5 or in the above definition, and should be removed.

Note that it follows from the relations (5.2.8), (5.2.9), (5.2.10) and (4.2.6) that (4.4.12) holds, and hence that if $\mu(u)$ is associated to a tuple $(\alpha, (P_i(u))_{i=1}^n)$ then the irreducible module $V(\mu(u))$ exists: see Theorem 4.4.4.

In the case where $V(\mu(u))$ is also finite-dimensional, we follow the convention in the literature and call $(\alpha, (P_i(u))_{i=1}^n)$ the *Drinfeld tuple* associated to $V(\mu(u))$ and the polynomials $P_1(u), \ldots, P_n(u)$ the *Drinfeld polynomials.*

Lemma 5.2.8. *Suppose that $\mu(u)$ is associated to $(\alpha, (P_i(u))_{i=1}^n)$. Then this is the unique tuple associated to $\mu(u)$. Moreover, if $\mu^\sharp(u)$ is also associated to $(\alpha, (P_i(u))_{i=1}^n)$ then there exists $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $g(u) = g(\kappa - u)$ and*

$$V(\mu^\sharp(u)) \cong \nu_g^*(V(\mu(u))).$$

Proof. The uniqueness of $(\alpha, (P_i(u))_{i=1}^n)$ is an immediate consequence of Lemma 5.2.1. Let's turn to the second statement of the lemma. Suppose that $\mu^\sharp(u)$ is also associated to $(\alpha, (P_i(u))_{i=1}^n)$. We need to show there is $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that

$$g(u) = g(\kappa - u) \quad \text{and} \quad \widetilde{\mu}_i^\sharp(u) = g(u)\widetilde{\mu}_i(u) \quad \forall\, i \in \mathcal{I}_N^+.$$

If this is true, then $V(\mu^\sharp(u))$ and $\nu_g^*(V(\mu(u)))$ will have the same highest weight and hence be isomorphic. From (5.2.9) and (5.2.10) we obtain the equalities

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{\widetilde{\mu}_{i-1}^\sharp(u)}{\widetilde{\mu}_i^\sharp(u)} \quad \forall\, 2 \le i \le n \quad \text{and} \quad \frac{\widetilde{\mu}_{\mathsf{a}}(\kappa - u)}{\widetilde{\mu}_{\mathsf{b}}(u)} = \frac{\widetilde{\mu}_{\mathsf{a}}^\sharp(\kappa - u)}{\widetilde{\mu}_{\mathsf{b}}^\sharp(u)}, \tag{5.2.13}$$

Setting $g(u) = \mu_n^\sharp(u)\mu_n(u)^{-1}$, we obtain from the first relations in (5.2.13) that

$$\widetilde{\mu}_i^\sharp(u) = g(u)\widetilde{\mu}_i(u) \quad \forall\, i \in \mathcal{I}_N^+ \setminus \{0\}.$$

If $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, then the relation (4.2.6) together with the second relation of (5.2.13) gives

$$\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)} = \frac{\widetilde{\mu}_0^\sharp(u)}{\widetilde{\mu}_1^\sharp(u)}$$

and hence $\widetilde{\mu}_0^\sharp(u) = g(u)\widetilde{\mu}_0(u)$ also holds.

To complete the proof, note that the second relation in (5.2.13) now implies that $g(u) = g(\kappa - u)$, as desired. □

We conclude this subsection by providing a general description of the relationship between a tuple $(\alpha, (P_i(u))_{i=1}^n)$ associated to a $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ highest weight $\mu(u)$, and the corresponding tuple $(\alpha^\sharp, (P_i^\sharp(u))_{i=1}^{n-m})$ associated to the highest weight $\mu^\sharp(u)$ of the $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module $V(\mu(u))_m$ from Corollary 4.3.7.

Corollary 5.2.9. *Suppose that $\mu(u)$ is associated to $(\alpha, (P_i(u))_{i=1}^n)$, and let $1 \le m < n - \delta_{\mathfrak{g}_N, \mathfrak{so}_{2n}}$. Then the highest weight $\mu^\sharp(u)$ of the $X(\mathfrak{g}_{N-2m}, \mathfrak{g}_{N-2m}^{\vartheta(m)})^{tw}$-module $V(\mu(u))_m$ is associated to the tuple*

$$(\alpha^\sharp, (P_i^\sharp(u))_{i=1}^{n-m}) = (\alpha - \tfrac{m}{2}, (P_i(u + \tfrac{m}{2})_{i=1}^{n-m})),$$

where $\alpha^\sharp = \alpha - \frac{m}{2}$ is omitted if $\mathfrak{g}_N^{\vartheta(m)} = \mathfrak{g}_{N-2m}$.

Proof. It is clear that $(P_i^\sharp(u))_{i=1}^{n-m}$ satisfies (5.2.8) and that $P_{\Bbbk(\mathcal{G}_m)+1}^\sharp(\alpha^\sharp) \neq 0$. Hence, it suffices to show that the relations (5.2.9) and (5.2.10) hold for

$$\mu^\sharp(u) \quad \text{and} \quad (\alpha^\sharp, (P_i^\sharp(u))_{i=1}^{n-m}).$$

By Corollary 4.3.7, $\mu^\sharp(u) = h(u)\mu^{\circ m}(u)$ with $h(u)$ given by (4.3.28) and $\mu^{\circ m}(u)$ determined by

$$\widetilde{\mu_i^{\circ m}}(u) = \widetilde{\mu}_i(u + \tfrac{m}{2}) \quad \forall i \in \mathcal{I}_{N-2m}^+.$$

The relation (5.2.9) then implies that, for each $2 \le i \le n-m$, we have

$$\begin{aligned}\frac{\widetilde{\mu}_{i-1}^\sharp(u)}{\widetilde{\mu}_i^\sharp(u)} = \frac{\widetilde{\mu}_{i-1}(u+\frac{m}{2})}{\widetilde{\mu}_i(u+\frac{m}{2})} &= \frac{P_i(u+\frac{m}{2}+1)}{P_i(u+\frac{m}{2})}\left(\frac{\alpha-\frac{m}{2}-u}{\alpha+\frac{m}{2}+u-\ell}\right)^{\delta_{i,\hbar+1}} \\ &= \frac{P_i^\sharp(u+1)}{P_i^\sharp(u)}\left(\frac{\alpha^\sharp-u}{\alpha^\sharp+u-(\ell-m)}\right)^{\delta_{i,\hbar+1}}.\end{aligned}$$

It remains to show (5.2.10) holds when $(\widetilde{\mu}_{\mathsf{a}}(u), \widetilde{\mu}_{\mathsf{b}}(u), g(u), P_1(u), \alpha, \kappa)$ is replaced by

$$(\widetilde{\mu}_{\mathsf{a}}^\sharp(u), \widetilde{\mu}_{\mathsf{b}}^\sharp(u), g_m(u), P_1(u+\tfrac{m}{2}), \alpha - \tfrac{m}{2}, \kappa - m).$$

This is shown by repeating the computation done in the proof of Proposition 5.2.5. □

5.2.3 Conditions on α

Our aim for the rest of this section is to show that if $(\alpha, (P_i(u))_{i=1}^n)$ is a Drinfeld tuple then $\alpha - N/4$ must take half integer values whenever $\mathfrak{g}_N^\vartheta$ is semisimple. This will be made precise in Proposition 5.2.13. First, we proceed with two lemmas.

Lemma 5.2.10. *Suppose that the $X(\mathfrak{g}_N)$-module $L(\lambda(u))$ is finite-dimensional with Drinfeld polynomials $(Q_i(u))_{i=1}^n$ and that $\mu(u)$ is associated to a tuple $(\alpha, (P_i(u))_{i=1}^n)$ such that $\alpha, \alpha^t \notin Z(Q_{\hbar(\mathcal{G})+1}(u-\kappa/2))$, where*

$$\alpha^t = \begin{cases} -\alpha + \ell + 1 & \text{if } \hbar(\mathcal{G}) \ge 1, \\ -\alpha + \kappa + 2^\delta & \text{if } \hbar(\mathcal{G}) = 0. \end{cases}$$

Let $\xi \in L(\lambda(u))$ and $\eta \in V(\mu(u))$ be highest weight vectors. Then the highest weight

$\gamma(u)$ *of the* $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$*-module*

$$X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}(\xi \otimes \eta) \subset L(\lambda(u)) \otimes V(\mu(u))$$

is associated to the tuple $(\alpha, ((Q_i \odot P_i)(u))_{i=1}^n)$, *where*

$$(Q_i \odot P_i)(u) = \begin{cases} (-1)^{\deg Q_i} Q_i(u-\kappa/2)Q_i(-u+n-i+2-\kappa/2)\, P_i(u) & \text{if } i > 1 \\ (-1)^{\deg Q_1} Q_1(u-\kappa/2)Q_1(-u+\kappa/2+2^\delta)\, P_i(u) & \text{if } i = 1. \end{cases}$$

Proof. By Propositions 4.1.2 and 4.2.11, we have

$$\begin{aligned} \frac{\widetilde{\gamma}_{i-1}(u)}{\widetilde{\gamma}_i(u)} &= \frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} \cdot \frac{\lambda_{i-1}(u-\kappa/2)\lambda_i(-u-\kappa/2+n-i+1)}{\lambda_i(u-\kappa/2)\lambda_{i-1}(-u-\kappa/2+n-i+1)} \\ &= \frac{Q_i(u-\kappa/2+1)Q_i(-u+n-i+1-\kappa/2)}{Q_i(u-\kappa/2)Q_i(-u+n-i+2-\kappa/2)} \cdot \frac{P_i(u+1)}{P_i(u)} \left(\frac{\alpha-u}{\alpha+u-\ell}\right)^{\delta_{i,\Bbbk+1}} \end{aligned}$$

for all $i > 1$. Similarly,

$$\begin{aligned} &\frac{\kappa-u}{u} \cdot \frac{g(u)}{g(\kappa-u)} \cdot \frac{\widetilde{\gamma}_{\mathsf{a}}(\kappa-u)}{\widetilde{\gamma}_{\mathsf{b}}(u)} \\ &= \frac{\lambda_{\mathsf{a}}(\kappa/2-u)}{\lambda_{-\mathsf{b}}(\kappa/2-u)} \cdot \frac{Q_1(u-\kappa/2+1)}{Q_1(u-\kappa/2)} \cdot \frac{P_1(u+\mathsf{d})}{P_1(u)} \left(\frac{\alpha-u}{\alpha+u-\kappa+\mathsf{d}-2^\delta}\right)^{\delta_{\Bbbk(\mathcal{G}),0}}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{\lambda_{\mathsf{a}}(\kappa/2-u)}{\lambda_{-\mathsf{b}}(\kappa/2-u)} = \frac{Q_1(-u+\kappa/2+2^\delta-\mathsf{d})}{Q_1(-u+\kappa/2+2^\delta)}. \tag{5.2.14}$$

If $\mathfrak{g}_N = \mathfrak{sp}_N$, then $\mathsf{a} = \mathsf{b}$ and the above holds by (5.2.7). If instead $\mathfrak{g}_N = \mathfrak{so}_N$, then $\mathsf{b} = \mathsf{a}+1$ and Proposition 4.1.2 gives

$$\frac{\lambda_{\mathsf{a}}(u-\kappa+n-\mathsf{a})}{\lambda_{-\mathsf{b}}(u)} = \frac{\lambda_{\mathsf{b}}(u-\kappa+n-\mathsf{a})}{\lambda_{-\mathsf{a}}(u)}.$$

If $\mathfrak{g}_N = \mathfrak{so}_{2n}$, then substituting $u \mapsto -u + \frac{\kappa}{2}$ and using (5.2.7) yields (5.2.14). If $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, then $\mathsf{a} = 0$ and after shifting $u \mapsto -u + \frac{\kappa}{2}$, we can rewrite the above as

$$\frac{\lambda_0(\kappa/2-u)}{\lambda_{-\mathsf{b}}(\kappa/2-u)} = \frac{\lambda_{\mathsf{b}}(-u+\kappa/2+1/2)}{\lambda_0(-u+\kappa/2+1/2)}.$$

By (5.2.7), this coincides with the right-hand side of (5.2.14). □

Remark 5.2.11. If α or α^t is a root of $Q_{\Bbbk(\mathcal{G})+1}(u-\kappa/2)$, then $(\alpha, (Q_i \odot P_i)(u))_{i=1}^n)$ will still satisfy the relations of Proposition 5.2.5, except that the condition

$$\alpha \notin Z((Q_{\Bbbk(\mathcal{G})+1} \odot P_{\Bbbk(\mathcal{G})+1})(u))$$

will fail to hold and one must apply Lemma 5.2.2 to obtain the correct tuple associated to $\gamma(u)$.

To state the next lemma, we recall from §5.1 that a $\mathfrak{g}_N^\vartheta$-module V is said to be a highest weight module with the highest weight $(\mu_i)_{i=1}^n \subset \mathbb{C}^n$ if it is generated by a nonzero vector ξ such that

$$F_{ij}^\vartheta \xi = 0 \quad \forall \quad i < j \in \mathcal{I}_N,$$
$$F_{ii}\xi = \mu_i \xi \quad \forall \quad 1 \leq i \leq n.$$

Lemma 5.2.12. *Suppose that $\mu(u)$ is associated to $(\alpha, \mathbf{P} = (P_i(u))_{i=1}^n)$, and let ξ be a highest weight vector in $V(\mu(u))$. Then the $\mathfrak{g}_N^\vartheta$-module $U(\mathfrak{g}_N^\vartheta)\xi$ is a highest weight module with highest weight $\mu = (\mu_i)_{i=1}^n$ given by*

$$\mu_i = \frac{1}{2}\lambda_{\mathbf{P},i} + \mu_\alpha^\vartheta(i) \quad \forall \quad 1 \leq i \leq n, \tag{5.2.15}$$

where $\lambda_{\mathbf{P},i}$ is as in (4.1.9) and $\mu_\alpha^\vartheta(i) \in \mathbb{C}$ is given by

$$\mu_\alpha^\vartheta(i) = \begin{cases} \delta_{i>\Bbbk}\left(\alpha - \dfrac{N}{4}\right) & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q), \\ \dfrac{\alpha - \kappa}{2} & \text{if } (\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n). \end{cases}$$

Proof. It follows from Corollary 3.3.11 that $U(\mathfrak{g}_N^\vartheta)\xi$ is a highest weight module. It thus suffices to show that, for each $1 \leq i \leq n$, the F_{ii}-weight of ξ is given by (5.2.15).

Assume first that $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q)$. Let $\mu_\alpha(u) = (\mu_{\alpha,i}(u))_{i \in \mathcal{I}_N^+}$ be the $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-highest weight determined by

$$\widetilde{\mu}_{\alpha,i}(u) = \begin{cases} 2u\, \mathcal{g}(u) & \text{if } i \leq \Bbbk, \\ 2u\, \mathcal{g}(u)\left(\dfrac{\ell - \alpha - u}{u - \alpha}\right) & \text{if } i \geq \Bbbk + 1. \end{cases} \tag{5.2.16}$$

As $\mu_\alpha(u)$ satisfies the relations of Theorem 4.4.4, the irreducible module $V(\mu_\alpha(u))$ exists.

Since $P_i(u) = P_i(-u+n-i+2)$ for all $i \geq 2$, there exists monic polynomials $Q_2(u), \ldots, Q_n(u)$ such that

$$P_i(u) = (-1)^{\deg Q_i(u)} Q_i(u-\kappa/2) Q_i(-u+n-i+2-\kappa/2).$$

Similarly, since $P_1(u) = P_1(-u+\kappa+2^\delta)$, there is a monic polynomial $Q_1(u)$ such that

$$P_1(u) = (-1)^{\deg Q_1(u)} Q_1(u-\kappa/2) Q_1(-u+\kappa/2+2^\delta).$$

Let $L(\lambda(u))$ be a finite-dimensional irreducible $X(\mathfrak{g}_N)$-module with Drinfeld polynomials $(Q_j(u))_{j=1}^n$, and suppose $\xi^\sharp \in L(\lambda(u))$ and $\eta \in V(\mu_\alpha(u))$ are highest weight vectors. Then, as a consequence of Lemmas 5.2.8 and 5.2.10, there is $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $g(u) = g(\kappa - u)$ and $\nu_g^*(V(\mu(u)))$ is isomorphic to the irreducible quotient of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}(\xi^\sharp \otimes \eta)$. As $\mathfrak{g}_N^\vartheta$ acts identically in $\nu_g^*(V(\mu(u)))$ and $V(\mu(u))$, we can assume without loss of generality that $g(u) = 1$.

Observe that

$$\begin{aligned}
\mathrm{Res}_u\Big(\frac{1}{2u}\widetilde{\mu}_{\alpha,i}(u)\Big) &= \begin{cases} \mu_{\alpha,i}^{(1)} - \ell & \text{if } \ i \leq \hbar, \\ \mu_{\alpha,i}^{(1)} & \text{if } \ i \geq \hbar+1, \end{cases} \\
\mathrm{Res}_u(g(u)) &= \frac{\mathrm{tr}(\mathcal{G}) - N}{4} = -\ell, \\
\mathrm{Res}_u\left(\frac{\ell-\alpha-u}{u-\alpha}\right) &= \ell - 2\alpha,
\end{aligned}$$

where Res_u denotes the formal residue operator at $u = 0$. We thus obtain from (5.2.16) the relations

$$\mu_{\alpha,i}^{(1)} = \begin{cases} 0 & \text{if } \ i \leq \hbar, \\ 2\ell - 2\alpha & \text{if } \ i \geq \hbar + 1. \end{cases} \tag{5.2.17}$$

Since $F_{ii}^\vartheta = 2g_{ii}F_{ii}$, Corollary 3.3.11 yields

$$2g_{ii}\mu_{\alpha,i} + (g_{ii}-1)\frac{\mathrm{tr}(\mathcal{G})}{4} = \mu_{\alpha,i}^{(1)} \quad \forall \quad i \in \mathcal{I}_N^+,$$

where $\mu_{\alpha,i}$ denotes the F_{ii}-weight of η. Combining this with (5.2.17), we obtain

$$\mu_{\alpha,i} = \begin{cases} 0 & \text{if } \ i \leq \Bbbk, \\ \alpha - \frac{N}{4} & \text{if } \ i \geq \Bbbk + 1. \end{cases}$$

Since

$$F_{ii}(\xi^\sharp \otimes \eta) = F_{ii}\xi^\sharp \otimes \eta + \mu_{\alpha,i}(\xi^\sharp \otimes \eta) \quad \forall \quad i \in \mathcal{I}_N^+,$$

The relation (5.2.15) now follows immediately from the formulas (4.1.9) of Corollary 4.1.10 and the fact that $\deg P_i(u) = 2 \deg Q_i(u)$ for each i.

If instead $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n)$, then one should replace (5.2.16) by

$$\mu_{\alpha,i}(u) = 1 + \frac{\alpha - \kappa}{u} \quad \forall \quad 1 \leq i \leq n.$$

The F_{ii}-weight of $\xi^\sharp \otimes \eta$ is then given by

$$F_{ii}(\xi^\sharp \otimes \eta) = F_{ii}\xi^\sharp \otimes \eta + \frac{\alpha - \kappa}{2}(\xi^\sharp \otimes \eta) \quad \forall \quad i \in \mathcal{I}_N^+$$

and hence (5.2.15) again follows from Corollary 4.1.10. □

Now let us restrict our attention to the setting where $\mathfrak{g}_N^\vartheta$ is semisimple, but not the whole Lie algebra $\mathfrak{g}_N$. This means that

$$(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q) \quad \text{with} \quad \mathfrak{g}_q \ncong \mathfrak{so}_2 \ \text{ and } \ q \neq 0. \tag{5.2.18}$$

where we recall that we always assume $q \in 2\mathbb{Z}$ and that $p \geq q$ whenever $p \in 2\mathbb{Z}$. Let $\{d_{q,i}\}_{i=1}^{\ell} \subset \mathbb{Q}$ and $\{h_{q,i}\}_{i=1}^{\ell} \subset \mathfrak{g}_{2\ell} \cap \mathfrak{h}$ be given by

$$(d_{q,i}, h_{q,i}) = (1, F_{\Bbbk+i-1,\Bbbk+i-1} - F_{\Bbbk+i,\Bbbk+i}) \quad \forall \quad 2 \leq i \leq \ell,$$

$$(d_{q,1}, h_{q,1}) = \begin{cases} (2, -2F_{\Bbbk+1,\Bbbk+1}) & \text{if } \ \mathfrak{g}_N = \mathfrak{sp}_N, \\ (1, -F_{\Bbbk+1,\Bbbk+1} - F_{\Bbbk+2,\Bbbk+2}) & \text{if } \ \mathfrak{g}_N = \mathfrak{so}_N. \end{cases}$$

Let $\mu(u)$ and ξ be as in Lemma 5.2.12. This lemma implies that $U(\mathfrak{g}_{2\ell})\xi \subset V(\mu(u))$ is itself a highest weight module with highest weight determined by

$$d_{q,i}^{-1} h_{q,i}(\xi) = \frac{1}{2} \deg P_{\Bbbk+i}(u)\xi \quad \forall \quad 2 \leq i \leq \ell,$$

$$d_{q,1}^{-1}h_{q,1}(\xi) = \begin{cases} \left(\frac{N}{4} - \alpha - \frac{1}{2}\lambda_{\mathbf{P},\hbar+1}\right)\xi & \text{if } \mathfrak{g}_N = \mathfrak{sp}_N, \\ \left(\frac{N}{2} - 2\alpha - \frac{1}{2}\lambda_{\mathbf{P},\hbar+1} - \frac{1}{2}\lambda_{\mathbf{P},\hbar+2}\right)\xi & \text{if } \mathfrak{g}_N = \mathfrak{so}_N. \end{cases}$$

If in addition $V(\mu(u))$ is assumed to be finite-dimensional then, by (4.1.8), the coefficients of ξ on the right-hand sides of the above equations must take non-negative integer values. As $\deg P_i(u) \in 2\mathbb{Z}$ for each $1 \leq i \leq n$, this observation together with (4.1.9) yields the following proposition.

Proposition 5.2.13. *Suppose that $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is as in (5.2.18) and let $\mu(u)$ and $(\alpha, \mathbf{P})$ be as in Lemma 5.2.12 with $V(\mu(u))$ finite-dimensional. Then α satisfies*

$$2^{1-\delta}\left(\alpha - \frac{N}{4}\right) \in \mathbb{Z}$$

in addition to the relation

$$4\alpha - N \leq 2\sum_{a=1}^{\hbar+1} \deg P_a(u) \quad \text{if } \mathfrak{g}_N = \mathfrak{sp}_N,$$

$$4\alpha - N \leq \deg(P_1(u)P_{\hbar+2}(u)) + 2\sum_{a=2}^{\hbar+1} \deg P_a(u) \quad \text{if } \mathfrak{g}_N = \mathfrak{so}_{2n+1},$$

$$4\alpha - N \leq \deg(P_1(u)P_2(u)P_{\hbar+2}(u)) + 2\sum_{a=3}^{\hbar+1} \deg P_a(u) \quad \text{if } \mathfrak{g}_N = \mathfrak{so}_{2n},$$

where we recall that $\delta = \delta_{\mathfrak{g}_N, \mathfrak{sp}_{2n}}$.

5.3 One-dimensional representations

In this section we classify the one-dimensional representations of all twisted Yangians of type B, C and D considered in this thesis. In particular, we will prove that $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ (and thus $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$) admits non-trivial one-dimensional representations if and only if $\mathfrak{g}_N^\vartheta$ has a non-trivial center. This occurs precisely when

$$(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2).$$

In these cases, $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ admits a family $\{V(\alpha)\}_{\alpha \in \mathbb{C}}$ of one-dimensional representations, which we will construct explicitly in §5.3.2 and §5.3.3. These one-parameter families will play a crucial role in the classification results for finite-dimensional irreducible representations of $X(\mathfrak{g}_{2n}, \mathfrak{gl}_n)^{tw}$ and $X(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$ to be presented in §5.4.2 and §5.4.3, respectively.

In what follows, we continue to assume that $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is not equal to $(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)$.

5.3.1 The semisimple setting

We begin by classifying the one-dimensional representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ under the assumption that $\mathfrak{g}_N^\vartheta$ is a complex semisimple Lie algebra. Under this hypothesis, we have

$$(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_p \oplus \mathfrak{g}_q) \quad \text{with} \quad \mathfrak{g}_q \ncong \mathfrak{so}_2.$$

Proposition 5.3.1. *Assume that $\mathfrak{g}_N^\vartheta$ is a complex, semisimple Lie algebra. Then, a representation V of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ is one-dimensional if and only if*

$$V \cong \nu_g^*(V(\mathcal{G}))$$

for some $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ with $g(u) = g(\kappa - u)$.

Proof. Suppose that V is a one-dimensional representation of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$. By Proposition 5.2.5, V can be assigned a tuple $(\alpha, (P_i(u))_{i=1}^n)$, where the scalar α should be omitted if $\mathfrak{g}_N^\vartheta = \mathfrak{g}_N$. As $\mathfrak{g}_N^\vartheta$ is semisimple, it admits no nontrivial one-dimensional representations. Consequently, when viewed as a $\mathfrak{g}_N^\vartheta$-module, V is isomorphic to the trivial representation. Therefore, relation (5.2.15) of Lemma 5.2.12 becomes equivalent to

$$-\lambda_{\mathbf{P},i} = \delta_{i>\Bbbk}\left(2\alpha - \frac{N}{2}\right) \quad \forall \quad 1 \leq i \leq n,$$

from which it can be deduced that

$$\deg P_a(u) = \begin{cases} 2\alpha - \frac{N}{2} & \text{if } \; a = \Bbbk + 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathfrak{g}_N^\vartheta = \mathfrak{g}_N$, then $(P_i(u))_{i=1}^n$ is equal to $\mathbf{1} = (1, \dots, 1)$, the Drinfeld tuple corresponding to the trivial representation $V(\mathcal{G})$. In this case, the desired conclusion then follows from Lemma 5.2.8.

Therefore, we may assume without loss of generality that $\mathfrak{g}_N^\vartheta = \mathfrak{g}_p \oplus \mathfrak{g}_q$ with $q \geq 4$. It suffices to show that

$$\deg P_{\hbar+1}(u) = 0,$$

as this will imply $(\alpha, (P_i(u))_{i=1}^n) = (\frac{N}{4}, \mathbf{1})$. Since $V(\mathcal{G})$ is also associated to this tuple, the desired conclusion will follow from Lemma 5.2.8.

Case 1: $\hbar > 0$.

Suppose first that $\hbar > 0$, and set $M = N - 2\ell + 2$. By Corollary 4.3.7, we may regard $V = V_{\ell-1}$ as a one-dimensional representation of $X(\mathfrak{g}_M, \mathfrak{g}_{M-2} \oplus \mathfrak{g}_2)^{tw}$. By Corollary 5.2.9, the highest weight of this module is associated to

$$(\gamma, (Q_i(u))_{i=1}^{\hbar+1}), \qquad \text{where } \gamma = \alpha - \frac{\ell - 1}{2}$$

$$\text{and} \qquad Q_a(u) = \begin{cases} P_{\hbar+1}(u + \frac{\ell-1}{2}) & \text{if } i = \hbar + 1, \\ 1 & \text{otherwise.} \end{cases}$$

By Proposition 4.4.1, we may then also regard $V = (V_{\ell-1})^J$ as a one-dimensional representation of the twisted Yangian $X(\mathfrak{sl}_{\hbar+1}, \mathfrak{sl}_\hbar \oplus \mathfrak{gl}_1)^{tw}$. This representation corresponds to the tuple $(\gamma, (Q_i(u))_{i=2}^{\hbar+1})$ in the classification given by [MR02, Theorem 4.6] (see (4.4.10) and (4.4.11)).

Since the $X(\mathfrak{sl}_{\hbar+1}, \mathfrak{sl}_\hbar \oplus \mathfrak{gl}_1)^{tw}$-module V is one-dimensional, it inherits the structure of a $X(\mathfrak{sl}_2, \mathfrak{gl}_1)^{tw} = X(\mathfrak{sl}_2, \mathfrak{sl}_1 \oplus \mathfrak{gl}_1)^{tw}$-module by setting

$$b_{ij}(u) \cdot v = b_{i+\hbar-1, j+\hbar-1}(u)v \quad \forall\, v \in V \text{ and } 1 \leq i, j \leq 2.$$

This can be verified directly, but it also follows from a more general result observed in the first part of the proof of [MR02, Theorem 4.6]. The resulting $X(\mathfrak{sl}_2, \mathfrak{gl}_1)^{tw}$-module has Drinfeld tuple $(\gamma, Q_{\hbar+1}(u))$.

The twisted Yangians $X(\mathfrak{sl}_2, \mathfrak{gl}_1)^{tw}$ and $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ are isomorphic (see [MR02, Proposition 4.3] as well as the remarks concluding [MR02, §4.2]), hence V can also be viewed as a one-dimensional representation of $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$. The arguments used to prove [MR02, Proposition 4.4] show that this irreducible $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module corresponds to the pair $(\gamma - \frac{1}{2}, Q_{\hbar+1}(u + \frac{1}{2}))$ under the correspondence of Theorem 5.1.1. On the other hand, Corollary 4.4.5 of [Mol07] implies that $V \cong V(\gamma - 1)$ as a

$Y(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module, where $V(\gamma-1)$ is the one-dimensional representation obtained by restricting the irreducible $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module with the highest weight

$$\gamma(u)=\frac{1+(\gamma-1/2)u^{-1}}{1+1/2u^{-1}}$$

(see equation [Mol07, (4.21)]). This module corresponds to the pair $(\gamma-\frac{1}{2},1)$, so we obtain

$$\deg P_{\Bbbk+1}(u)=\deg Q_{\Bbbk+1}(u)=0.$$

Case 2: $\Bbbk=0$.

In this case, $V=V_{\ell-1}$ is a one-dimensional representation of $X(\mathfrak{so}_3,\mathfrak{so}_2)^{tw}$ which, by Corollary 5.2.9, is associated to the pair $(\gamma,Q_1(u))=(\alpha-\frac{\ell-1}{2},P_1(u+\frac{\ell-1}{2}))$. Moreover, it can be made into a $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module via the isomorphism (5.1.36), and the proof of Proposition 5.1.14 shows that, as a $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-module, V corresponds to the pair $(2\gamma-1,2^{\deg Q_1}Q_1(\frac{u+1}{2}))$: see (5.1.47). Repeating the last part of the argument of Case 1, we are able to conclude that $\deg P_1(u)=0$, which completes the proof of the Proposition. □

The below corollary of Proposition 5.3.1 now follows immediately from Lemma 4.2.1.

Corollary 5.3.2. *Assume that $\mathfrak{g}_N^\vartheta$ is a complex semisimple Lie algebra. Then, up to isomorphism, $V(\mathcal{G})$ is the unique one-dimensional representation of the twisted Yangian $Y(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)^{tw}$.*

5.3.2 Twisted Yangians associated to $(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)$

We now turn to the twisted Yangians associated to pairs of the form

$$(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)=(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)\cong(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathbb{C})\quad\text{with}\quad N\neq 4.$$

Given $\alpha \in \mathbb{C}$, define

$$K(u;\alpha) = k(u)\left(I - \frac{2u}{u-\alpha}E_{-n,-n} - \frac{2u}{u+\alpha-2d}E_{nn}\right), \qquad \text{where} \quad k(u) = \frac{(u-\alpha)(u+\alpha-2d)}{(u-d)^2} \tag{5.3.1}$$

and $d = \frac{\mathrm{tr}(\mathcal{G})}{4} = N/4 - 1$.

Lemma 5.3.3. *For each $\alpha \in \mathbb{C}$, the assignment*

$$S(u) \mapsto K(u;\alpha) \in \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}[\![u^{-1}]\!]$$

defines a one-dimensional $V(\alpha)$ of $X(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$.

Proof. The proof is straightforward, but highly technical. We begin by showing that $K(u;\alpha)$ satisfies the reflection equation

$$R(u-v)K_1(u;\alpha)R(u+v)K_2(v;\alpha) = K_2(v;\alpha)R(u+v)K_1(u;\alpha)R(u-v).$$

Notice that we only need to show this for $k(u)^{-1}K(u;\alpha)$. Denote

$$G(u) = -\frac{2u}{u-\alpha}E_{-n,-n} - \frac{2u}{u+\alpha-2d}E_{nn}, \tag{5.3.2}$$

so that $K(u;\alpha) = k(u)(I + G(u))$. Since I is a solution to (3.3.40), our task is to show that

$$\begin{aligned} R(u-v)\,G_1(u)\,R(u+v) + R(u-v)\,R(u+v)\,G_2(v) \\ + R(u-v)\,G_1(u)\,R(u+v)\,G_2(v) \\ = G_2(v)\,R(u+v)\,R(u-v) + R(u+v)\,G_1(u)\,R(u-v) \\ + G_2(v)\,R(u+v)\,G_1(u)\,R(u-v). \end{aligned} \tag{5.3.3}$$

We first show that

$$\begin{aligned}
&\left(1-\frac{P}{u-v}\right)\left(G_1(u)\left(1-\frac{P}{u+v}\right)+\left(1-\frac{P}{u+v}\right)G_2(v)\right.\\
&\qquad\left.+G_1(u)\left(1-\frac{P}{u+v}\right)G_2(v)\right)+H(u,v)\\
&\quad=\left(G_2(v)\left(1-\frac{P}{u+v}\right)+\left(1-\frac{P}{u+v}\right)G_1(u)\right.\\
&\qquad\left.+G_2(v)\left(1-\frac{P}{u+v}\right)G_1(u)\right)\left(1-\frac{P}{u-v}\right),
\end{aligned}\tag{5.3.4}$$

where $H(u,v)$ is given by

$$\frac{8uv(\alpha-d)}{(u-\alpha)(v-\alpha)(u+\alpha-2d)(v+\alpha-2d)}\left(E_{-n,n}\otimes E_{n,-n}-E_{n,-n}\otimes E_{-n,n}\right).\tag{5.3.5}$$

The equality (5.3.4) reduces to

$$[P,2v\,G_2(u)-2u\,G_2(v)-(u-v)\,G_2(u)\,G_2(v)]=0.$$

This follows from the following computations:

$$\begin{aligned}
&2v\,G(u)-2u\,G(v)-(u-v)\,G(u)\,G(v)\\
&\quad=-\left(\frac{4uv}{u-\alpha}-\frac{4uv}{v-\alpha}\right)E_{-n,-n}-\left(\frac{4uv}{u+\alpha-2d}-\frac{4uv}{v+\alpha-2d}\right)E_{nn}\\
&\qquad-\frac{4uv\,(u-v)}{(u-\alpha)(v-\alpha)}E_{-n,-n}-\frac{4uv\,(u-v)}{(u+\alpha-2d)(v+\alpha-2d)}E_{nn}=0,
\end{aligned}$$

thus implying (5.3.4). Next we use

$$\begin{gathered}
Q^2=NQ,\quad (1-u^{-1}P)Q=(1-u^{-1})Q,\\
QG_1(u)Q=QG_2(u)Q=g(u)\,Q,\\
\text{where}\quad g(u)=\operatorname{tr}(G(u))=-\frac{2u}{u-\alpha}-\frac{2u}{u+\alpha-2d}
\end{gathered}$$

and subtract (5.3.4) from (5.3.3). Then (5.3.3) holds if and only if the following expression equals $H(u,v)$:

$$\frac{G_1(u)\,Q+Q\,G_2(v)+G_1(u)\,Q\,G_2(v)}{u+v-\kappa}-\frac{G_2(v)\,Q+Q\,G_1(u)+G_2(v)\,Q\,G_1(u)}{u+v-\kappa}$$

$$
\begin{aligned}
&- \frac{G_2(u)\,Q + Q\,G_2(v) + G_2(u)\,Q\,G_2(v)}{(u-v)(u+v-\kappa)} + \frac{G_2(v)\,Q + Q\,G_2(u) + G_2(v)\,Q\,G_2(u)}{(u-v)(u+v-\kappa)} \\
&+ \frac{Q\,(G_1(u) + G_2(v) + G_1(u)\,G_2(v))}{u-v-\kappa} - \frac{(G_2(v) + G_1(u) + G_2(v)\,G_1(u))\,Q}{u-v-\kappa} \\
&- \frac{Q\,(G_2(u) + G_2(v) + G_2(u)\,G_2(v))}{(u+v)(u-v-\kappa)} + \frac{(G_2(v) + G_2(u) + G_2(v)\,G_2(u))\,Q}{(u+v)(u-v-\kappa)} \\
&+ \frac{g(u)\,Q + N\,Q\,G_2(v) + g(u)\,Q\,G_2(v)}{(u-v-\kappa)(u+v-\kappa)} - \frac{N\,G_2(v)\,Q + g(u)\,Q + g(u)\,G_2(v)\,Q}{(u-v-\kappa)(u+v-\kappa)}.
\end{aligned}
$$

Since $[G(u), G(v)] = 0$, that the previous long expression is equal to $H(u,v)$ is equivalent to

$$
\begin{aligned}
&H(u,v) \\
&\;= \frac{[G_1(u) - G_2(v), Q] + G_1(u)\,Q\,G_2(v) - G_2(v)\,Q\,G_1(u)}{u+v-\kappa} \\
&\quad + \frac{[Q, G_1(u) + G_2(v) + G_1(u)\,G_2(v)]}{u-v-\kappa} \\
&\quad + \frac{[Q, G_2(u) - G_2(v)] + G_2(v)\,Q\,G_2(u) - G_2(u)\,Q\,G_2(v)}{(u-v)(u+v-\kappa)} \\
&\quad - \frac{[Q, G_2(u) + G_2(v) + G_2(u)\,G_2(v)]}{(u+v)(u-v-\kappa)} + \frac{(N + g(u))\,[Q, G_2(v)]}{(u-v-\kappa)(u+v-\kappa)}.
\end{aligned}
\tag{5.3.6}
$$

Now observe that $G^t(u) = -u\,(2d-u)^{-1} G(2d-u)$, which implies that

$$
G_1(u)Q = -\frac{u}{2d-u} G_2(2d-u)Q, \qquad QG_1(u) = -\frac{u}{2d-u} QG_2(2d-u).
$$

Moreover,

$$
\begin{aligned}
G_2(u)QG_2(v) =& \frac{4uv}{(u-\alpha)(v-\alpha)} E_{nn} \otimes E_{-n,-n} \\
&+ \frac{4uv}{(u-\alpha)(v+\alpha-2d)} E_{n,-n} \otimes E_{-n,n} \\
&+ \frac{4uv}{(u+\alpha-2d)(v-\alpha)} E_{-n,n} \otimes E_{n,-n} \\
&+ \frac{4uv}{(u+\alpha-2d)(v+\alpha-2d)} E_{-n,-n} \otimes E_{nn}.
\end{aligned}
\tag{5.3.7}
$$

Using (5.3.5), (5.3.7) and $\kappa = 2d+1$ we compute the following identity

$$
\begin{aligned}
&\frac{G_1(u)QG_2(v) - G_2(v)QG_1(u)}{u+v-\kappa} + \frac{G_2(v)QG_2(u) - G_2(u)QG_2(v)}{(u-v)(u+v-\kappa)} \\
&= \frac{4uv}{u+v-\kappa}\left(\frac{1}{(u-\alpha)(v-\alpha)} - \frac{1}{(u+\alpha-2d)(v+\alpha-2d)}\right) \\
&\qquad \cdot (E_{-n,n} \otimes E_{n,-n} - E_{n,-n} \otimes E_{-n,n}) \\
&+ \frac{4uv}{(u-v)(u+v-\kappa)}\left(\frac{1}{(u-\alpha)(v+\alpha-2d)} - \frac{1}{(v-\alpha)(u+\alpha-2d)}\right) \\
&\qquad \cdot (E_{-n,n} \otimes E_{n,-n} - E_{n,-n} \otimes E_{-n,n}) \\
&= \frac{8uv(\alpha-d)}{(u-\alpha)(v-\alpha)(u+\alpha-2d)(v+\alpha-2d)} (E_{-n,n} \otimes E_{n,-n} - E_{n,-n} \otimes E_{-n,n}) \\
&= H(u,v).
\end{aligned}
$$

By combining the identities above and denoting $\widetilde{u} = u\,(2d-u)^{-1}$, we can rewrite (5.3.6) as

$$
\begin{aligned}
\Bigg[Q, &\frac{\widetilde{u}\,G_2(2d-u) + G_2(v)}{u+v-\kappa} + \frac{G_2(u) - G_2(v)}{(u-v)(u+v-\kappa)} \\
&+ \frac{(N+g(u))G_2(v)}{(u-v-\kappa)(u+v-\kappa)} - \frac{\widetilde{u}\,G_2(2d-u) - G_2(v) + \widetilde{u}\,G_2(2d-u)G_2(v)}{u-v-\kappa} \\
&\quad - \frac{G_2(u) + G_2(v) + G_2(u)G_2(v)}{(u+v)(u-v-\kappa)}\Bigg] = 0.
\end{aligned}
$$

Denoting the commutator above by $[Q, 1 \otimes F(u,v)]$ we only need to verify that $F(u,v) = 0$, which follows by a direct computation using (5.3.2), the explicit form of $g(u)$ and $\kappa = 2d+1 = N/2-1$, as we now illustrate. After reorganizing the various terms and multiplying by $(u-v-\kappa)(u+v-\kappa)$, we obtain, with $F'(u,v) = (u-v-\kappa)(u+v-\kappa)F(u,v)$:

$$
\begin{aligned}
&F'(u,v) \\
&\quad = -\frac{2\kappa v\,G(u)}{u^2-v^2} - \frac{2uv\,G(2d-u)}{2d-u} \\
&\qquad + 2u\left(1 - \frac{1}{u-\alpha} - \frac{1}{u+\alpha-2d} + \frac{\kappa}{u^2-v^2}\right)G(v) \\
&\qquad\quad - (u+v-\kappa)\left(\frac{u\,G(2d-u)G(v)}{2d-u} + \frac{G(u)G(v)}{u+v}\right)
\end{aligned}
$$

$$= 4uv\Bigg(- \left(\frac{1}{u+\alpha-2d} - \frac{\kappa}{(u-\alpha)(u^2-v^2)}\right)E_{-n,-n}$$
$$- \left(\frac{1}{u-\alpha} - \frac{\kappa}{(u+\alpha-2d)(u^2-v^2)}\right)E_{nn}$$
$$- \left(1 - \frac{1}{u-\alpha} - \frac{1}{u+\alpha-2d} + \frac{\kappa}{u^2-v^2}\right)\left(\frac{1}{v-\alpha}E_{-n,-n} + \frac{1}{v+\alpha-2d}E_{nn}\right)$$
$$- \frac{u+v-\kappa}{(u-\alpha)(u+\alpha-2d)(u+v)}\Bigg(\frac{(u+\alpha-2d)-(u-\alpha)(u+v)}{v-\alpha}E_{-n,-n}$$
$$+ \frac{(u-\alpha)-(u+\alpha-2d)(u+v)}{v+\alpha-2d}E_{nn}\Bigg)\Bigg)$$
$$= \frac{2d+1-\kappa}{(v-\alpha)(u+\alpha-2d)}E_{-n,-n} + \frac{2d+1-\kappa}{(u-\alpha)(v+\alpha-2d)}E_{nn} = 0.$$

This completes the proof that $K(u;\alpha)$ is a solution of the reflection equation (3.3.40).

Our work thus far shows that there is an algebra homomorphism

$$\phi_a : \mathsf{X}(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{\mathsf{tw}} \to \mathbb{C}, \quad \mathsf{S}(u) \mapsto K(u;\alpha).$$

By Corollary 3.3.24, to complete the proof of the proposition it suffices to show

$$\mathscr{g}(u)\mathrm{tr}(K(\kappa-u;\alpha)) = \mathscr{g}(\kappa-u)\mathrm{tr}(K(u;\alpha)).$$

Since $\mathrm{tr}(\mathcal{G}) = N-4$ and $\kappa = N/2-1$, we have

$$\frac{\mathscr{g}(\kappa-u)}{\mathscr{g}(u)} = \frac{N-4\kappa+4u}{\mathrm{tr}(\mathcal{G})-4\kappa+4u}\cdot\frac{\mathrm{tr}(\mathcal{G})-4u}{N-4u} = \left(\frac{u+1-N/4}{u-N/4}\right)^2. \tag{5.3.8}$$

By definition of $K(u;\alpha)$,

$$\mathrm{tr}(K(u;a)) = k(u)\left(N - \frac{2u}{u-\alpha} - \frac{2u}{u+\alpha-2d}\right)$$
$$= \frac{N(u-\alpha)(u+\alpha-2d)-2u(2u-2d)}{(u-d)^2}.$$

Let $P(u)$ be the numerator of the right-hand side. Using that $N = 2\kappa+2$ and $2d = \kappa-1$ we find that

$$P(u) = N(u-\alpha)(u+\alpha-\kappa+1) - 2u(2u-\kappa+1)$$

$$= N(u-\alpha)(u+\alpha-\kappa) - 2u(2u-2\kappa) - N\alpha,$$

and hence $P(u)$ is invariant under the substitution $u \mapsto \kappa - u$. This implies that

$$\frac{\mathrm{tr}(K(u;\alpha))}{\mathrm{tr}(K(\kappa-u;\alpha))} = \left(\frac{\kappa-u-d}{u-d}\right)^2 = \left(\frac{u-N/4}{u+1-N/4}\right)^2,$$

which, by (5.3.8), yields the desired result. □

The importance of the family $\{V(\alpha)\}_{\alpha\in\mathbb{C}}$ given by Lemma 5.3.3 is encoded in the following corollary.

Corollary 5.3.4. *Let $\alpha \in \mathbb{C}$. Then $V(\alpha) \cong V(\gamma^\alpha(u))$ with $\gamma^\alpha(u) = (\gamma_i^\alpha(u))_{i\in\mathcal{I}_N^+}$ defined by*

$$\gamma_i^\alpha(u) = \frac{(u-\alpha)((-1)^{\delta_{in}}u+\alpha-2d)}{(d-u)^2} \quad \forall \quad i \in \mathcal{I}_N^+, \tag{5.3.9}$$

where $d = \frac{\mathrm{tr}(\mathcal{G})}{4} = N/4 - 1$. Moreover, $V(\alpha)$ is associated to the Drinfeld tuple

$$(\gamma, P_1(u), \dots, P_n(u)) = (\kappa-\alpha, 1, \dots, 1). \tag{5.3.10}$$

Proof. The first part of the Corollary follows immediately from Lemma5.3.3 and the definition of $K(u,\alpha)$: see (5.3.1). The Drinfeld tuple associated to $\gamma^\alpha(u)$ is then readily computed using (5.3.8) and that

$$\widetilde{\gamma}_i^\alpha(u) = 2u \cdot \frac{(u-\alpha)((-1)^{\delta_{in}}u+\alpha-\kappa+\delta_{in})}{(d-u)^2} \quad \forall \quad i \in \mathcal{I}_N^+.$$ □

The last two results of this subsection provide a complete classification of one-dimensional representations for $X(\mathfrak{so}_N, \mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)^{tw}$ and of $Y(\mathfrak{so}_N, \mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)^{tw}$ when $N \neq 4$.

Proposition 5.3.5. *Let $N \neq 4$. Then a representation V of $X(\mathfrak{so}_N, \mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)^{tw}$ is one-dimensional if and only if*

$$V \cong \nu_g^*(V(\alpha))$$

for some $\alpha \in \mathbb{C}$ and $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ with $g(u) = g(\kappa - u)$.

Proof. Let V be a one-dimensional representation with highest weight associated to $(\alpha, (P_i(u))_{i=1}^n)$. It suffices to prove that

$$\deg P_i(u) = 0 \quad \forall \quad 1 \leq i \leq n.$$

Indeed, if this is the case then, by Corollary 5.3.4, V and $V(\kappa - \alpha)$ share the same Drinfeld tuple and hence $V \cong \nu_g^*(V(\kappa - \alpha))$ by Lemma 5.2.8.

If $N = 3$, this is a consequence of the $\hbar = 0$ case in the proof of Proposition 5.3.1.

Suppose instead that $N \geq 5$. In this case, $\mathfrak{so}_{N-2}$ is semisimple and thus V is equal to the trivial representation when viewed as a $\mathfrak{so}_{N-2}$-module. Since $\mathfrak{so}_2$ is one-dimensional, F_{nn} operates in V as multiplication by a scalar γ. In particular, the highest weight of V as a $(\mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)$-module is $(\mu_i)_{i=1}^n = (0, \ldots, 0, \gamma)$. The relation (5.2.15) of Lemma 5.2.12 therefore implies that

$$\deg P_i(u) = 0 \quad \forall\, 1 \leq i < n \quad \text{ and } \quad \deg P_n(u) = 2\alpha - \frac{N}{2} - 2\gamma.$$

To complete the proof, we need to see that $\deg P_n(u) = 0$. This can shown using the same argument as given in the $\hbar > 0$ case of the proof of Proposition 5.3.1. □

Corollary 5.3.6. *Let $N \neq 4$. Then a representation V of $Y(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$ is one-dimensional if and only if there is $\alpha \in \mathbb{C}$ such that $V \cong V(\alpha)$.*

Remark 5.3.7. If V is a one-dimensional representation of

$$X(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)^{tw}$$

then there exists $(\mu_1, \mu_2) \in \mathbb{C}^2$ and $g(u) \in 1 + u^{-1}\mathbb{C}[\![u^{-1}]\!]$ such that $g(u) = g(\kappa - u)$ and

$$V \cong \nu_g^*(V(\mu_1, \mu_2)),$$

where $V(\mu_1, \mu_2)$ is as in Corollary 5.1.11. This follows from the isomorphism (5.1.28) together with the fact that the family $\{V(\gamma)\}_{\gamma \in \mathbb{C}}$, which appeared in the proof of Proposition 5.3.1, provides a complete list of the isomorphisms classes of one-dimensional representations of $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$. This fact follows from Corollary 4.4.5 of [Mol07].

5.3.3 Twisted Yangians associated to $(\mathfrak{g}_{2n}, \mathfrak{gl}_n)$

We conclude this section with a study of the one-dimensional representations for the twisted Yangians associated to symmetric pairs of the form

$$(\mathfrak{sp}_{2n}, \mathfrak{gl}_n) \text{ with } n \geq 1 \quad \text{and} \quad (\mathfrak{so}_{2n}, \mathfrak{gl}_n) \text{ with } n \geq 2.$$

We begin with the analogue of Lemma 5.3.3. For each $\alpha \in \mathbb{C}$, set

$$K(u;\alpha) = \mathcal{G} + \frac{\alpha}{u} I = \frac{\alpha\, I + u\mathcal{G}}{u}. \tag{5.3.11}$$

Lemma 5.3.8. *For each $\alpha \in \mathbb{C}$, the assignment*

$$S(u) \mapsto K(u;\alpha) \in \mathrm{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}[\![u^{-1}]\!]$$

defines a one-dimensional representation $V(\alpha)$ of $X(\mathfrak{g}_{2n}, \mathfrak{gl}_n)^{tw}$.

Proof. We begin by showing that $K(u,\alpha)$ satisfies the reflection equation (3.3.40):

$$\begin{aligned} &R_{12}(u-v)(\mathcal{G}_1 + a\, u^{-1} I_1) R_{12}(u+v)(\mathcal{G}_2 + a\, v^{-1} I_2) \\ &\qquad = (\mathcal{G}_2 + a\, v^{-1} I_2) R_{12}(u+v)(\mathcal{G}_1 + a\, u^{-1} I_1) R_{12}(u-v). \end{aligned}$$

As the matrices $\mathcal{G}$ and I are themselves solutions of (3.3.40), it suffices to show that

$$\begin{aligned} &v^{-1} R_{12}(u-v)\mathcal{G}_1 R_{12}(u+v) + u^{-1} R_{12}(u-v) R_{12}(u+v)\mathcal{G}_2 \\ &\qquad = u^{-1}\mathcal{G}_2 R_{12}(u+v) R_{12}(u-v) + v^{-1} R_{12}(u+v)\mathcal{G}_1 R_{12}(u-v). \end{aligned} \tag{5.3.12}$$

Notice first that

$$\begin{aligned} &\left(1 - \frac{P}{u-v}\right)\mathcal{G}_1\left(1 - \frac{P}{u+v}\right)v^{-1} + \left(1 - \frac{P}{u-v}\right)\left(1 - \frac{P}{u+v}\right)\mathcal{G}_2 u^{-1} \\ &= \mathcal{G}_2\left(1 - \frac{P}{u+v}\right)\left(1 - \frac{P}{u-v}\right)u^{-1} + \left(1 - \frac{P}{u+v}\right)\mathcal{G}_1\left(1 - \frac{P}{u-v}\right)v^{-1}. \end{aligned} \tag{5.3.13}$$

This is verified by expanding both sides. After subtracting (5.3.13), the left-hand side of (5.3.12) becomes

$$\frac{1}{u+v-\kappa}\left(\frac{\mathcal{G}_1 Q}{v}-\frac{P\mathcal{G}_1 Q}{(u-v)v}+\frac{Q\mathcal{G}_2}{u}-\frac{PQ\mathcal{G}_2}{(u-v)u}\right)$$
$$+\frac{1}{u-v-\kappa}\left(\frac{Q\mathcal{G}_1}{v}-\frac{Q\mathcal{G}_1 P}{v(u+v)}+\frac{Q\mathcal{G}_1 Q}{(u+v-\kappa)v}\right)$$
$$+\frac{1}{u-v-\kappa}\left(\frac{Q\mathcal{G}_2}{u}-\frac{QP\mathcal{G}_2}{(u+v)u}+\frac{Q^2\mathcal{G}_2}{(u+v-\kappa)u}\right),$$

while the right-hand side becomes

$$\frac{1}{u-v-\kappa}\left(\frac{\mathcal{G}_2 Q}{u}-\frac{\mathcal{G}_2 PQ}{u(u+v)}+\frac{\mathcal{G}_1 Q}{v}-\frac{P\mathcal{G}_1 Q}{(u+v)v}\right)$$
$$+\frac{1}{u+v-\kappa}\left(\frac{\mathcal{G}_2 Q}{u}-\frac{\mathcal{G}_2 QP}{u(u-v)}+\frac{\mathcal{G}_2 Q^2}{u(u-v-\kappa)}\right)$$
$$+\frac{1}{u+v-\kappa}\left(\frac{Q\mathcal{G}_1}{v}-\frac{Q\mathcal{G}_1 P}{(u-v)v}+\frac{Q\mathcal{G}_1 Q}{v(u-v-\kappa)}\right).$$

Multiplying both sides by $(u+v-\kappa)(u-v-\kappa)(u^2-v^2)uv$ and equating the coefficients of $u^i v^j$, we see that it suffices to establish the following relations:

$$2\kappa Q\mathcal{G}_2+QP\mathcal{G}_2-Q^2\mathcal{G}_2+PQ\mathcal{G}_2=2\kappa\mathcal{G}_2 Q+\mathcal{G}_2 QP-\mathcal{G}_2 Q^2+\mathcal{G}_2 PQ, \tag{5.3.14}$$
$$Q\mathcal{G}_1+Q\mathcal{G}_2-\mathcal{G}_1 Q+Q\mathcal{G}_2=\mathcal{G}_2 Q-Q\mathcal{G}_1+\mathcal{G}_2 Q+\mathcal{G}_1 Q, \tag{5.3.15}$$
$$-Q\mathcal{G}_1 P+QP\mathcal{G}_2+P\mathcal{G}_1 Q+PQ\mathcal{G}_2=\mathcal{G}_2 QP+Q\mathcal{G}_1 P+\mathcal{G}_2 PQ-P\mathcal{G}_1 Q. \tag{5.3.16}$$

Since $Q^2=NQ$ and $PQ=QP=\pm Q$, (5.3.14) is equivalent to

$$(2\kappa-N)(Q\mathcal{G}_2-\mathcal{G}_2 Q)=\mp 2(Q\mathcal{G}_2-\mathcal{G}_2 Q),$$

and since $\kappa=N/2\mp 1$, this equality is indeed satisfied. As $P\mathcal{G}_1=\mathcal{G}_2 P$ and $\mathcal{G}^t=-\mathcal{G}$, we have

$$-\mathcal{G}_1 Q=(P\mathcal{G}_1)^{t_1}=(\mathcal{G}_2 P)^{t_1}=\mathcal{G}_2 Q.$$

Similarly, $Q\mathcal{G}_1=-Q\mathcal{G}_2$. The relation (5.3.15) follows from these identities.

Finally, relation (5.3.16) holds since

$$P\mathcal{G}_2=\mathcal{G}_1 P, \quad P\mathcal{G}_1=\mathcal{G}_2 P \quad \text{and} \quad PQ=QP.$$

This completes the proof that $K(u;\alpha)=\mathcal{G}+\alpha\, u^{-1}I$ is a solution of (3.3.40).

We are left to show that

$$p(u)K(\kappa-u;\alpha)=K(u;\alpha)^t\mp\frac{K(u;\alpha)}{2u-\kappa}+\frac{\operatorname{tr}(K(u;\alpha))\cdot I}{2u-2\kappa},$$
$$\text{where}\quad p(u)=-1\mp\frac{1}{2u-\kappa}.$$

Since this identity holds if $K(u;\alpha)$ is replaced with $\mathcal{G}$, it suffices to show that it also holds if $K(u;\alpha)$ is replaced with $u^{-1}I$. This reduces to the identity

$$u\,p(u)=(\kappa-u)\,p_I(u)$$

which was proven in Proposition 3.3.3. □

As an immediate consequence of Lemma 5.3.8 and the definition of $K(u;\alpha)$ (see (5.3.11)), we obtain the following corollary.

Corollary 5.3.9. *Let $\alpha\in\mathbb{C}$. Then $V(\alpha)\cong V(\gamma_i^\alpha(u))$ with $\gamma^\alpha(u)=(\gamma_i^\alpha(u))_{i\in\mathcal{I}_N^+}$ defined by*

$$\gamma_i^\alpha(u)=\frac{u+\alpha}{u}\quad\forall\quad i\in\mathcal{I}_N^+.$$

In particular, $V(\alpha)$ is associated to the Drinfeld tuple

$$(\gamma,P_1(u),\dots,P_n(u))=(\kappa+\alpha,1,\dots,1).$$

We now turn to establishing the $(\mathfrak{g}_{2n},\mathfrak{gl}_n)$ versions of Proposition 5.3.5 and Corollary 5.3.6.

Proposition 5.3.10. *A representation V of $X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$ is one-dimensional if and only if*

$$V\cong\nu_g^*(V(\alpha))$$

for some $\alpha\in\mathbb{C}$ and $g(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ with $g(u)=g(\kappa-u)$.

Proof. As in the proof of Proposition 5.3.5, it suffices to show that the Drinfeld tuple associated to any one-dimensional $X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$-module V is of the form

$$(\alpha,P_1(u),\dots,P_n(u))=(\alpha,1,\dots,1).$$

Let V be such a representation. As a $\mathfrak{sl}_{2n} \subset \mathfrak{gl}_{2n}$ module, V must be isomorphic to the trivial representation. Thus

$$\mu_i - \mu_{i+1} = 0 \quad \forall \quad 1 \leq i < n,$$

where $(\mu_i)_{i=1}^n$ is the highest weight of V as a $\mathfrak{gl}_{2n}$-module. By Lemma 5.2.12, this implies that

$$\lambda_{\mathbf{P},1} = \lambda_{\mathbf{P},2} = \ldots = \lambda_{\mathbf{P},n}.$$

Using (4.1.9), we find that $\deg P_i(u) = 0$ for all $i \geq 2$. It remains to see that $\deg P_1(u) = 0$.

Set $m = n - 2^{1-\delta}$, so that

$$(\mathfrak{g}_{2(n-m)}, \mathfrak{gl}_{n-m}) = \begin{cases} (\mathfrak{sp}_2, \mathfrak{gl}_1) & \text{if } \; \mathfrak{g}_{2n} = \mathfrak{sp}_{2n}, \\ (\mathfrak{so}_4, \mathfrak{gl}_2) & \text{if } \; \mathfrak{g}_{2n} = \mathfrak{so}_{2n}. \end{cases}$$

By Corollaries 4.3.7 and 5.2.9, we may regard $V = V_m$ as a one-dimensional representation of $X(\mathfrak{g}_{2(n-m)}, \mathfrak{gl}_{n-m})^{tw}$ which is necessarily associated to

$$(\alpha^\sharp, (P_i^\sharp(u))_{i=1}^{n-m}) = (\alpha - \tfrac{m}{2}, (P_i(u + \tfrac{m}{2}))_{i=1}^{n-m}).$$

If $\mathfrak{g}_{2n} = \mathfrak{sp}_{2n}$, then we view V as a $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module via the isomorphism φ_{CI} of (5.1.5). The proof of Proposition 5.1.3 shows that, as a $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module, V corresponds to the pair

$$\left(\frac{\alpha^\sharp - 1}{2}, \frac{P_1^\sharp(2u+1)}{2^{\deg P_1^\sharp(u)}} \right).$$

On the other hand, as indicated in the proof of Proposition 5.3.1, a one-dimensional representation of $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ must have Drinfeld polynomials all equal to 1. Hence,

$$\deg P_1^\sharp(u) = \deg P_1(u + \tfrac{n-1}{2}) = \deg P_1(u) = 0.$$

If $\mathfrak{g}_{2n} = \mathfrak{so}_{2n}$, then we instead view V as a representation of

$$Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$$

via the isomorphism φ_{DIII} of (5.1.11). By [Mol07, Proposition 4.3.2], $X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$ has no non-trivial one-dimensional representations. It follows that

$$V \cong V^\sharp \otimes \mathbb{C}$$

as a representation of $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw} \otimes X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$, where $\mathbb{C}$ is the trivial representation of $X(\mathfrak{sl}_2, \mathfrak{sp}_2)^{tw}$ and $V^\sharp$ is a one-dimensional representation of $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$. The proof of Proposition 5.1.6 shows that the $Y(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$-module $V^\sharp$ corresponds to the pair

$$\left(\alpha^\sharp - \frac{1}{2}, P_1^\sharp\left(u + \frac{1}{2}\right)\right).$$

Therefore, we may conclude as in the $\mathfrak{g}_{2n} = \mathfrak{sp}_{2n}$ case that $\deg P_1(u) = 0$. □

Corollary 5.3.11. *A representation V of $Y(\mathfrak{g}_{2n}, \mathfrak{gl}_n)^{tw}$ is one-dimensional if and only if there is $\alpha \in \mathbb{C}$ such that $V \cong V(\alpha)$.*

5.4 Classification results: I

In this section, we will obtain a complete classification of all finite-dimensional irreducible representations for twisted Yangians associated to the symmetric pairs

$$(\mathfrak{g}_N, \mathfrak{g}_N), \quad (\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2),$$

where, as usual, we omit $(\mathfrak{so}_4, \mathfrak{so}_2 \oplus \mathfrak{so}_2)$. Of all the twisted Yangians of type B, C and D we have considered, those associated to the above three families of pairs are exceptional in that only for them are the necessary conditions established in Proposition 5.2.5 in fact sufficient conditions.

For the remainder of this chapter, it will be useful to employ notation introduced in §4.2.1. We recall that $\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$ and $\mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw})$ denote the set of isomorphism classes of all finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ and $Y(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$, respectively. As in §4.2.1, we will write $[V]$ for the isomorphism class of a module V, but we will drop the brackets when making use of the identification

$$\mathrm{Irrep}_{\mathrm{fd}}(X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}) = \{V(\mu(u)) \,:\, \dim V(\mu(u)) < \infty\}$$

given by Part (4) of Proposition 4.2.9.

5.4.1 Twisted Yangians associated to $(\mathfrak{g}_N, \mathfrak{g}_N)$

We begin by focusing on the case where $\vartheta = \mathrm{id}_{\mathfrak{g}_N}$, and thus $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{g}_N, \mathfrak{g}_N)$. Only in this setting is there no complex parameter α present in the conditions of Proposition 5.2.5 (see Remark 5.2.7), and this is reflected in the following theorem.

Theorem 5.4.1. *Let $\mu(u) = (\mu_i(u))_{i\in\mathcal{I}_N^+}$ satisfy (4.4.12). Then the $X(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $P_1(u), \dots, P_n(u)$ in u satisfying (5.2.8), in addition to the relations*

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u+1)}{P_i(u)} \quad \forall \quad 2 \le i \le n, \tag{5.4.1}$$

$$\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa - u)}{\widetilde{\mu}_{\mathsf{b}}(u)} = \frac{P_1(u+\mathsf{d})}{P_1(u)} \cdot \frac{\kappa - u}{u}, \tag{5.4.2}$$

where a, b and d are given by (5.2.6).

Proof. First note that, by (3.3.3) and (4.2.40), $\hbar(\mathcal{G}) = \hbar = n$ and $g(u) = 1$. Hence, the relations (5.2.9) and (5.2.10) collapse to (5.4.1) and (5.4.2), respectively. The assertion of the theorem is thus equivalent to the statement that $V(\mu(u))$ is finite-dimensional if and only if $\mu(u)$ can be associated to a tuple $(P_i(u))_{i=1}^n$ (see Definition 5.2.6).

If $V(\mu(u))$ is finite-dimensional, then Proposition 5.2.5 implies that $\mu(u)$ is associated to a tuple $(P_i(u))_{i=1}^n$.

Conversely, assume that $\mu(u)$ is associated to $(P_i(u))_{i=1}^n$. We will argue that $V(\mu(u))$ is necessarily finite-dimensional. As a consequence of (5.2.8), there are monic polynomials $Q_1(u), \dots, Q_n(u)$ satisfying

$$\begin{aligned} P_1(u) &= (-1)^{\deg Q_1} Q_1(u - \kappa/2) Q_1(-u + \kappa/2 + 2^\delta), \\ P_i(u) &= (-1)^{\deg Q_i} Q_i(u - \kappa/2) Q_i(-u + n - i + 2 - \kappa/2) \quad \forall \quad 2 \le i \le n. \end{aligned} \tag{5.4.3}$$

Fix $\lambda(u) = (\lambda_i(u))_{i\in\mathcal{I}_N}$ to be a $X(\mathfrak{g}_N)$ highest weight with the property that $L(\lambda(u))$ is finite-dimensional with Drinfeld polynomials $(Q_i(u))_{i=1}^n$. Such a $\lambda(u)$ exists by Proposition 4.1.5.

Let $\xi \in L(\lambda(u))$ be a highest weight vector. Then, by Lemma 5.2.10 applied with $V(\mu(u)) = V(\mathcal{G})$, the highest weight $\mu^\sharp(u)$ of the $X(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}$-module

$$X(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}\xi \subset L(\lambda(u))$$

is associated to $(P_i(u))_{i=1}^n$. By Lemma 5.2.8, we thus have

$$V(\mu(u)) = \nu_g^*(V(\mu^\sharp(u))$$

for some $g(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ with $g(u) = g(\kappa-u)$. As $L(\lambda(u))$ is finite-dimensional, we can conclude that $V(\mu(u))$ is finite-dimensional. □

Remark 5.4.2. When $\mathfrak{g}_N = \mathfrak{so}_{2n+1}$, the second relation of (4.4.12) implies that (5.4.2) is equivalent to

$$\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)} = \frac{P_1(u+\frac{1}{2})}{P_1(u)}.$$

We would now like to translate Theorem 5.4.1 into a parameterization of the form given by Proposition 4.1.5 for the extended Yangian $X(\mathfrak{g}_N)$. Consider the automorphism ref_κ of $\mathbb{C}[\![u^{-1}]\!]$ which sends each $f(u) \in \mathbb{C}[\![u^{-1}]\!]$ to $f(\kappa - u)$. Let us denote

$$(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa} = \{g(u) \in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!] \,:\, \mathrm{ref}_\kappa(g(u)) = g(u)\}.$$

Of course, this is precisely $1+(u-\kappa/2)^{-2}\mathbb{C}[\![(u-\kappa/2)^{-2}]\!]$, but we shall favor the above notation which emphasizes the fixed point nature of the underlying set.

Proposition 5.4.3. *The isomorphism classes of finite-dimensional irreducible representations of $X(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}$ are parameterized by tuples*

$$(g(u); (P_i(u))_{i=1}^n) \in (1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa} \times \mathbb{C}[u]^n,$$
$$\textit{where each } \; P_i(u) \; \textit{ is monic,}$$
$$P_1(u) = P_1(-u+\kappa+2^\delta) \quad \textit{and} \quad P_i(u) = P_i(-u+n-i+2) \quad \forall\, i \geq 2.$$

The underlying correspondence Γ_X^ϑ is given

$$\Gamma_X^\vartheta(V(\mu(u))) = (g(u); (P_i(u))_{i=1}^n), \quad \textit{where} \tag{5.4.4}$$

(a) $g(u) \in (1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$ *is the unique scalar series such that*

$$q(u)|_{\nu_g^*(V(\mu(u)))} = \mathrm{id}_{\nu_g^*(V(\mu(u)))}.$$

(b) $(P_i(u))_{i=1}^n$ *is the tuple of Drinfeld polynomials associated to* $V(\mu(u))$.

Proof. Assume that $V(\mu(u))$ is finite-dimensional. By Lemma 4.2.1, there does indeed indeed exist $g(u)$ as in (a). Moreover, by Lemma 5.2.8, the tuple $(P_i(u))_{i=1}^n$ associated to $\mu(u)$ is unique. From these two observations, we can conclude that there is a well-defined correspondence Γ_X^ϑ as in (5.4.4).

We now prove that Γ_X^ϑ is a bijection. Fix $(g(u); (P_i(u))_{i=1}^n)$ as in the statement of the proposition. The proof of Theorem 5.4.1 shows that there is $\mu^\sharp(u)$ such that $V(\mu^\sharp(u))$ is finite-dimensional and

$$\Gamma_X^\vartheta(V(\mu^\sharp(u))) = (f(u); (P_i(u))_{i=1}^n)$$

For some $f(u) \in (1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$. Let $h(u) = f(u)^{-1}g(u)$ and $\mu(u) = h(u)\mu^\sharp(u)$. Then $V(\mu(u)) = \nu_h^*(V(\mu^\sharp(u)))$ is finite-dimensional and

$$\Gamma_X^\vartheta(V(\mu(u))) = (g(u); (P_i(u))_{i=1}^n).$$

Hence, Γ_X^ϑ is surjective. The injectivity of Γ_X^ϑ follows from Lemma 5.2.8. □

Using the above parameterization and Lemma 4.2.1, we obtain the following classification of finite-dimensional irreducible representations for $Y(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}$.

Corollary 5.4.4. *The isomorphism classes of finite-dimensional irreducible representations of* $Y(\mathfrak{g}_N, \mathfrak{g}_N)^{tw}$ *are parameterized by tuples*

$$(P_i(u))_{i=1}^n \in \mathbb{C}[u]^n,$$
$$\textit{where each } \; P_i(u) \; \textit{ is monic,}$$
$$P_1(u) = P_1(-u+\kappa+2^\delta) \quad \textit{and} \quad P_i(u) = P_i(-u+n-i+2) \quad \forall\, i \geq 2.$$

Proof. By Lemma 4.2.1 and Proposition 5.4.3, the composition

$$\Gamma_Y^\vartheta = \Gamma_X^\vartheta \circ \Gamma^\vartheta$$

provides a bijection between $\mathrm{Irrep}_{\mathrm{fd}}(Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw})$ and the set of tuples $(1;(P_i(u))_{i=1}^n)$ with $(P_i(u))_{i=1}^n$ as in the statement of the corollary. This yields the desired parameterization. □

We end this subsection by proving a $Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}$-analogue of Corollary 4.1.9. Recall from Definition 4.1.8 that, for each $1 \leq i \leq n$ and $\alpha \in \mathbb{C}$, the fundamental representation $L(i:\alpha)$ is the unique, up to isomorphism, finite-dimensional irreducible $Y(\mathfrak{g}_N)$-module with Drinfeld tuple $(P_j(u))_{j=1}^n$, where

$$P_j(u) = (u-\alpha)^{\delta_{ji}}.$$

Corollary 5.4.5. *Let V be a finite-dimensional irreducible $Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}$-module. Then there is $m \geq 0$, $1 \leq i_1,\ldots,i_m \leq n$ and $\alpha_1,\ldots,\alpha_m \in \mathbb{C}$ such that V is isomorphic to the unique irreducible quotient of*

$$Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}(\xi_1 \otimes \cdots \otimes \xi_m) \subset L(i_1:\alpha_1)\otimes\cdots\otimes L(i_m:\alpha_m),$$

where, for each $1 \leq k \leq m$, $\xi_k \subset L(i:\alpha)$ is a highest weight vector and both sides are identified with the trivial representation if $m=0$.

Proof. Let $(P_i(u))_{i=1}^n$ be the unique Drinfeld tuple corresponding to V under the parameterization of Corollary 5.4.4. Let $L(\mathbf{Q})$ denote the unique, up to isomorphism, finite-dimensional irreducible $Y(\mathfrak{g}_N)$-module with Drinfeld tuple $\mathbf{Q} = (Q_i(u))_{i=1}^n$, where $\mathbf{Q}$ is any solution of the equations (5.4.3) (see Corollary 4.1.7).

By Corollary 4.1.9, there is m, $(i_k)_{k=1}^m$, $(\alpha_k)_{k=1}^m$ and $(\xi_k)_{k=1}^m$ such that $L(\mathbf{Q})$ is isomorphic to the unique irreducible quotient of

$$Y(\mathfrak{g}_N)(\xi_1\otimes\cdots\otimes\xi_m) \subset L(i_1:\alpha_1)\otimes\cdots\otimes L(i_m:\alpha_m).$$

Let ξ denote the image of $\xi_1\otimes\cdots\otimes\xi_m$ in $L(\mathbf{Q})$ under the natural quotient map. The proof of Theorem 5.4.1 then shows that the irreducible quotient of the $Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}$-module

$$Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}\xi \subset L(\mathbf{Q})$$

has Drinfeld tuple $(P_i(u))_{i=1}^n$, and hence is isomorphic to V. As this irreducible quotient is also isomorphic to the irreducible quotient of $Y(\mathfrak{g}_N,\mathfrak{g}_N)^{tw}(\xi_1\otimes\cdots\otimes\xi_m)$, we are done. ☐

5.4.2 Twisted Yangians associated to $(\mathfrak{g}_{2n}, \mathfrak{gl}_n)$

We now shift our attention to the twisted Yangians associated to symmetric pairs of type CI and DIII. That is, we assume

$$(\mathfrak{g}_N, \mathfrak{g}_N^{\vartheta}) = (\mathfrak{g}_{2n}, \mathfrak{gl}_n),$$

where $n \geq 1$ if $\mathfrak{g}_{2n} = \mathfrak{sp}_{2n}$ and $n \geq 2$ if $\mathfrak{g}_{2n} = \mathfrak{so}_{2n}$.

We note that the proofs of the results stated in both this subsection and in §5.4.3 are similar to those given in §5.4.1, and thus, to avoid redundancies, we shall omit details where possible.

Theorem 5.4.6. *The $X(\mathfrak{g}_{2n}, \mathfrak{gl}_n)^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $P_1(u), \dots, P_n(u)$ in u satisfying (5.2.8), together with a scalar $\alpha \in \mathbb{C} \setminus Z(P_1(u))$ such that*

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u+1)}{P_i(u)} \quad \forall \quad 2 \leq i \leq n, \tag{5.4.5}$$
$$\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa - u)}{\widetilde{\mu}_{\mathsf{b}}(u)} = \frac{P_1(u+\mathsf{d})}{P_1(u)} \cdot \frac{\alpha - u}{\alpha + u - \kappa}, \tag{5.4.6}$$

where a, b and d are given by (5.2.6).

Proof. Note that, as $\mathscr{k}(\mathcal{G}) = 0$ and $\mathscr{g}(u) = u^{-1}$, the relations (5.4.5) and (5.4.6) are equivalent to (5.2.9) and (5.2.10), respectively. Therefore, if $V(\mu(u))$ is finite-dimensional then Proposition 5.2.5 implies that there is $(\alpha, (P_i(u))_{i=1}^n)$ as in the statement of the theorem.

Conversely, assume that $\mu(u)$ is associated to a tuple $(\alpha, (P_i(u))_{i=1}^n)$. We will argue that $V(\mu(u))$ is finite-dimensional by modifying the argument used to prove Theorem 5.4.1.

By Proposition 4.1.5, there is $\lambda(u) = (\lambda_i(u))_{i \in \mathcal{I}_{2n}}$ such that the $X(\mathfrak{g}_{2n})$-module $L(\lambda(u))$ is finite-dimensional with Drinfeld tuple $\mathbf{Q} = (Q_i(u))_{i=1}^n$, where $\mathbf{Q}$ is a fixed solution to (5.4.3) (such a solution exists by (5.2.8)). Let $\xi \in L(\lambda(u))$ be a highest weight vector, and let η be any nonzero vector in the one dimensional representation $V(\alpha-\kappa)$ given by Lemma 5.3.8.

By Lemma 5.2.10 and Corollary 5.3.9, the highest weight $\mu^\sharp(u)$ of the $X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$-module

$$X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}(\xi\otimes\eta)\subset L(\lambda(u))\otimes V(\alpha-\kappa)$$

is associated to the tuple $(\alpha,(P_i(u))_{i=1}^n)$. The desired conclusion thus follows from Lemma 5.2.8 and the fact that $V(\mu^\sharp(u))$ is finite-dimensional: see the proof of Theorem 5.4.1. □

The above theorem can be translated to give a parameterization of all finite-dimensional irreducible $X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$-modules and $Y(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$-modules. This formulation is given, without proof, in Proposition 5.4.7 and Corollary 5.4.8 below. For the proofs of these results, we refer the reader to their counterparts in §5.4.1: see Proposition 5.4.3 and Corollary 5.4.4.

Proposition 5.4.7. *The isomorphism classes of finite-dimensional irreducible representations of $X(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$ are parameterized by tuples*

$$\begin{gathered}(g(u);(\alpha,(P_i(u))_{i=1}^n))\in(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}\times\mathbb{C}\times\mathbb{C}[u]^n,\\ \text{where each } P_i(u) \text{ is monic},\\ \alpha\in\mathbb{C}\setminus Z(P_1(u)),\\ P_1(u)=P_1(-u+\kappa+2^\delta)\quad\text{and}\quad P_i(u)=P_i(-u+n-i+2)\quad\forall\, i\geq 2.\end{gathered}$$

The underlying correspondence Γ_X^ϑ is given

$$\Gamma_X^\vartheta(V(\mu(u)))=(g(u);(\alpha,(P_i(u))_{i=1}^n)),\qquad\text{where}$$

(a) *$g(u)\in(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$ is the unique scalar series such that*

$$q(u)|_{\nu_g^*(V(\mu(u)))}=\mathrm{id}_{\nu_g^*(V(\mu(u)))}.$$

(b) *$(\alpha,(P_i(u))_{i=1}^n)$ is the Drinfeld tuple associated to $\mu(u)$.*

Corollary 5.4.8. *The isomorphism classes of finite-dimensional irreducible repre-*

sentations of $Y(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$ *are parameterized by tuples*

$$(\alpha,(P_i(u))_{i=1}^n)\in\mathbb{C}\times\mathbb{C}[u]^n,$$
$$\textit{where each } \; P_i(u) \; \textit{ is monic},$$
$$\alpha\in\mathbb{C}\setminus Z(P_1(u)),$$
$$P_1(u)=P_1(-u+\kappa+2^\delta) \quad \textit{and} \quad P_i(u)=P_i(-u+n-i+2) \quad \forall\, i\geq 2.$$

We conclude our analysis of the twisted Yangians of type CI and DIII with the following analogue of Corollary 5.4.5, which can be proven in the same way after taking into account the use of the one dimensional representations $V(\alpha)$ in the proof of Theorem 5.4.6.

Corollary 5.4.9. *Suppose that* V *is a finite-dimensional irreducible representation of* $Y(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}$*. Then there is* $m\geq 0$, $1\leq i_1,\ldots,i_m\leq n$ *and* $\alpha,\alpha_1,\ldots,\alpha_m\in\mathbb{C}$ *such that* V *is isomorphic to the unique irreducible quotient of*

$$Y(\mathfrak{g}_{2n},\mathfrak{gl}_n)^{tw}(\xi_1\otimes\cdots\otimes\xi_m\otimes\eta)\subset L(i_1:\alpha_1)\otimes\cdots\otimes L(i_m:\alpha_m)\otimes V(\alpha),$$

where, for each $1\leq k\leq m$, $\xi_k\subset L(i:\alpha)$ *is a highest weight vector and* $\eta\in V(\alpha)$ *is any nonzero vector.*

5.4.3 Twisted Yangians associated to $(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)$

For the remainder of this section, we assume that

$$(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)=(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathfrak{so}_2) \quad \text{with} \quad N\neq 4 \;\text{ and }\; N\geq 3.$$

Theorem 5.4.10. *Let* $\mu(u)=(\mu_i(u))_{i\in\mathcal{I}_N^+}$ *satisfy* (4.4.12)*. Then the irreducible* $X(\mathfrak{so}_N,\mathfrak{so}_{N-2}\oplus\mathfrak{so}_2)^{tw}$*-module* $V(\mu(u))$ *is finite-dimensional if and only if there exists monic polynomials* $P_1(u),\ldots,P_n(u)$ *in* u *satisfying* (5.2.8)*, together with a scalar* $\alpha\in\mathbb{C}\setminus Z(P_n(u))$ *such that*

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)}=\frac{P_i(u+1)}{P_i(u)}\left(\frac{\alpha-u}{\alpha+u-1}\right)^{\delta_{i,n}} \quad \forall \quad 2\leq i\leq n, \tag{5.4.7}$$

$$\frac{u}{\kappa-u}\cdot\frac{\widetilde{\mu}_{\mathsf{a}}(\kappa-u)}{\widetilde{\mu}_{\mathsf{b}}(u)}=\left(\frac{u+1-\frac{N}{4}}{u-\frac{N}{4}}\right)^2\frac{P_1(u+\mathsf{d})}{P_1(u)}\left(\frac{\alpha-u}{\alpha+u-1}\right)^{\delta_{1,n}}, \tag{5.4.8}$$

where a, b *and* d *are given by* (5.2.6).

Proof. If $V(\mu(u))$ is finite-dimensional, then by (5.3.8) and Proposition 5.2.5, there is $(\alpha, (P_i(u))_{i=1}^n)$ as in the statement of the theorem.

To prove that $V(\mu(u))$ is finite-dimensional whenever $\mu(u)$ is associated to a tuple $(\alpha, (P_i(u))_{i=1}^n)$, one employs the argument given in the proof of Theorem 5.4.6. The only modification required is that one must replace the one-dimensional $X(\mathfrak{g}_{2n}, \mathfrak{gl}_n)^{tw}$-module $V(\alpha - \kappa)$ by the one-dimensional $X(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$-module $V(\kappa - \alpha)$: see Lemma 5.3.3 and Corollary 5.3.4. $\square$

Remark 5.4.11. When N is odd the relation (5.4.8) is equivalent to

$$\frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)} = \frac{P_1(u+\frac{1}{2})}{P_1(u)} \left(\frac{\alpha - u}{\alpha + u - 1}\right)^{\delta_{1,n}}.$$

If N is even, then the existence of $P_1(u)$ satisfying $P_1(u) = P_1(-u+n)$ and condition (5.4.8) can be replaced by the equivalent requirement that there exists a monic polynomial $Q_1(u)$ such that $Q_1(u) = Q_1(-u+n)$, $n/2 \in Z(Q_1(u))$ and

$$\frac{\widetilde{\mu}_1(\kappa - u)}{\widetilde{\mu}_2(u)} = \frac{Q_1(u+1)}{Q_1(u)} \cdot \frac{\kappa - u}{u}.$$

We conclude this section by presenting the analogues of Proposition 5.4.7, Corollary 5.4.8 and Corollary 5.4.9. For more details, we refer the reader to the proofs given in §5.4.1.

Proposition 5.4.12. *The isomorphism classes of finite-dimensional irreducible representations of* $X(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$ *are parameterized by tuples*

$$(g(u); (\alpha, (P_i(u))_{i=1}^n)) \in (1 + u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa} \times \mathbb{C} \times \mathbb{C}[u]^n,$$
$$\textit{where each } \; P_i(u) \; \textit{ is monic,}$$
$$\alpha \in \mathbb{C} \setminus Z(P_n(u)),$$
$$P_1(u) = P_1(-u + \kappa + 1) \quad \textit{and} \quad P_i(u) = P_i(-u + n - i + 2) \quad \forall\, i \geq 2.$$

The underlying correspondence Γ_X^{ϑ} *is given*

$$\Gamma_X^{\vartheta}(V(\mu(u))) = (g(u); (\alpha, (P_i(u))_{i=1}^n)), \qquad \textit{where}$$

(a) $g(u) \in (1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$ *is the unique scalar series such that*

$$q(u)|_{\nu_g^*(V(\mu(u)))} = \mathrm{id}_{\nu_g^*(V(\mu(u)))}.$$

(b) $(\alpha, (P_i(u))_{i=1}^n)$ *is the Drinfeld tuple associated to* $\mu(u)$.

Corollary 5.4.13. *The isomorphism classes of finite-dimensional irreducible representations of* $Y(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$ *are parameterized by tuples*

$$\begin{gathered}(\alpha, (P_i(u))_{i=1}^n) \in \mathbb{C} \times \mathbb{C}[u]^n, \\ \textit{where each } \; P_i(u) \; \textit{ is monic}, \\ \alpha \in \mathbb{C} \setminus Z(P_n(u)), \\ P_1(u) = P_1(-u+\kappa+1) \quad \textit{and} \quad P_i(u) = P_i(-u+n-i+2) \quad \forall\, i \geq 2.\end{gathered}$$

Recall that, for each $\alpha \in \mathbb{C}$, $V(\alpha)$ denotes the one-dimensional representation given by Lemma 5.3.3 (see also Corollary 5.3.4).

Corollary 5.4.14. *Suppose that* V *is a finite-dimensional irreducible representation of* $Y(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}$. *Then there is* $m \geq 0$, $1 \leq i_1, \ldots, i_m \leq n$ *and* $\alpha, \alpha_1, \ldots, \alpha_m \in \mathbb{C}$ *such that* V *is isomorphic to the unique irreducible quotient of*

$$Y(\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2)^{tw}(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta) \subset L(i_1 : \alpha_1) \otimes \cdots \otimes L(i_m : \alpha_m) \otimes V(\alpha),$$

where, for each $1 \leq k \leq m$, $\xi_k \subset L(i : \alpha)$ *is a highest weight vector and* $\eta \in V(\alpha)$ *is any nonzero vector.*

5.5 Classification results: II

When $\mathfrak{g}_N^\vartheta$ is a complex semisimple Lie algebra which is a proper Lie subalgebra of $\mathfrak{g}_N$, the necessary conditions of §5.2 are no longer sufficient for determining exactly when the irreducible $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module $V(\mu(u))$ is finite-dimensional.

In this section, we illustrate this explicitly by focusing our attention on the twisted Yangians associated to pairs

$$(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}) \quad \text{with} \quad n \geq 2.$$

The culmination of our effort is a classification of all finite-dimensional irreducible modules for both $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$ and $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$: see Theorem 5.5.7, Proposition 5.5.8 and Corollary 5.5.9. Our first step towards proving these results is to study how a certain automorphism ψ_σ interacts with highest weight modules of $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$. This will play a critical role in the rest of this section, one that is similar to the role played in the analogous classification for the twisted Yangian of the symmetric pair $(\mathfrak{sl}_{2n}, \mathfrak{so}_{2n})$ by the automorphism (4.69) of [Mol07].

5.5.1 Permuting finite-dimensional modules

For any $n \geq 1$, let $\mathfrak{S}_{2n+1}$ denote the symmetric group on the set $\mathcal{I}_{2n+1}$ and let $\sigma \in \mathfrak{S}_{2n+1}$ be the transposition $(1, -1)$. That is

$$\sigma(i) = \begin{cases} -i & \text{if } i \in \{\pm 1\}, \\ i & \text{otherwise} \end{cases} \qquad \forall \quad i \in \mathcal{I}_{2n+1}.$$

Define the involutory permutation matrix A_σ by

$$A_\sigma = \sum_{i \in \mathcal{I}_{2n+1}} E_{i,\sigma(i)} \in \mathrm{GL}_{2n+1}(\mathbb{C}).$$

Since $A_\sigma^t = A_\sigma = A_\sigma^{-1}$ and $\det(A_\sigma) = -1$, $-A_\sigma \in \mathrm{G}_{2n+1}(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C})$ (see (3.2.6)). As we also have $A_\sigma \mathcal{G} A_\sigma^t = \mathcal{G}$, (3.3.21) implies that the assignment

$$\psi_\sigma : S(u) \mapsto A_\sigma S(u) A_\sigma^t$$

extends uniquely to an automorphism ψ_σ of $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$. We will write

$$\psi_\sigma^n = \psi_\sigma \tag{5.5.1}$$

when it is necessary to emphasize the rank n of $\mathfrak{so}_{2n+1}$. This is not to be confused with the n-th power $(\psi_\sigma)^n$ of ψ_σ.

Our present goal is to determine the highest weight of the twisted module

$$V(\mu(u))^{\psi_\sigma} = \psi_\sigma^*(V(\mu(u)))$$

under the assumption that $V(\mu(u))$ is finite-dimensional. We will first address this problem for the low rank pair $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}) = (\mathfrak{so}_3, \mathfrak{so}_2)$.

Lemma 5.5.1. *Suppose that the $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-module $V(\mu(u))$ is finite-dimensional with Drinfeld tuple $(\alpha, P(u))$, as in Proposition 5.1.14. Then*

$$V(\mu(u))^{\psi_\sigma} \cong V(\mu^\sharp(u)),$$

where the components of $\mu^\sharp(u)$ are uniquely determined by the relations

$$\begin{aligned} \widetilde{\mu}_0^\sharp(u) &= \widetilde{\mu}_0(u) \cdot \frac{3-2u-2\alpha}{2\alpha-2u} \cdot \frac{2u-2\alpha+2}{2u+2\alpha-1}, \\ \widetilde{\mu}_1^\sharp(u) &= \widetilde{\mu}_1(u) \cdot \frac{2u-2\alpha+1}{2u+2\alpha-2} \cdot \frac{2u-2\alpha+2}{2u+2\alpha-1}. \end{aligned} \tag{5.5.2}$$

In particular, $V(\mu(u))^{\psi_\sigma}$ has the Drinfeld tuple $(\frac{3}{2}-\alpha, P(u))$.

Proof. We appeal to the isomorphism $\varphi_{\mathrm{BI}} : X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw} \xrightarrow{\sim} X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ of (5.1.36), which we recall is given by

$$\begin{aligned} S(u) &\mapsto \frac{1-4u}{1+4u} Q_V S_1^\circ(2u-1) R^\circ(-4u+1)^t S_2^\circ(2u) K_1 K_2, \\ &\text{where} \quad Q_V = \frac{1}{2} R^\circ(-1), \quad K = E_{11} - E_{-1,-1}. \end{aligned} \tag{5.5.3}$$

Let $A = E_{1,-1} - E_{-1,1}$, and let β_A be the automorphism of $X(\mathfrak{sl}_2, \mathfrak{so}_2)^{tw}$ given by

$$\beta_A : S^\circ(u) \mapsto -A S^\circ(u) A^t.$$

More explicitly, β_A is defined on generators by the assignment

$$s_{ij}^\circ(u) \mapsto (-1)^{\delta_{i,-1}+\delta_{j,-1}} s_{-i,-j}^\circ(u) \quad \forall \quad i,j \in \mathcal{I}_2.$$

We claim that β_A and ψ_σ are related by

$$\varphi_{\mathrm{BI}}^{-1} \circ \beta_A \circ \varphi_{\mathrm{BI}} = \psi_\sigma. \tag{5.5.4}$$

Applying β_A to the right hand side of (5.5.3) and performing a few straightforward manipulations, we obtain

$$\frac{1-4u}{1+4u}Q_V(A_1A_2)S_1^\circ(2u-1)R_{12}^\circ(-4u+1)^tS_2^\circ(2u)K_1K_2(A_1A_2).$$

It follows that

$$(\varphi_{\mathrm{BI}}^{-1}\circ\beta_A\circ\varphi_{\mathrm{BI}})(S(u)) = A_VS(u)A_V,$$
$$\text{where}\quad A_V = Q_VA_1A_2Q_V\in\mathrm{End}(V)\cong\mathrm{End}(\mathbb{C}^3).$$

We have $A_Vv_i = -v_{-i}$ for all $i\in\mathcal{I}_3$, so

$$A_V = -\sum_{i\in\mathcal{I}_3}E_{i,\sigma(i)} = -A_\sigma.$$

Since $A_VS(u)A_V = A_\sigma S(u)A_\sigma$, the relation (5.5.4) does indeed hold.

Now let $V(\mu(u))$ be as in the statement of the lemma. The proof of Proposition 5.1.14 shows that there is an isomorphism of $X(\mathfrak{sl}_2,\mathfrak{so}_2)^{tw}$-modules

$$(\varphi_{\mathrm{BI}}^{-1})^*(V(\mu(u)))\cong V(\mu^\circ(u)),$$

where $\mu^\circ(u)\in 1+u^{-1}\mathbb{C}[\![u^{-1}]\!]$ is uniquely determined from the two relations

$$\widetilde{\mu}_1(u) = 2u\left(\frac{1-4u}{1+4u}\right)\mu^\circ(2u)\mu^\circ(2u-1),$$
$$\widetilde{\mu}_0(u) = 2u\left(\frac{4u-3}{4u+1}\right)\mu^\circ(2u)\mu^\circ(1-2u).$$

By Lemma 4.4.13 of [Mol07], the twisted module $\beta_A{}^*(V(\mu^\circ(u)))$ is isomorphic to $V(\mu^\bullet(u))$, where $\mu^\bullet(u)$ is given by

$$\mu^\bullet(u) = \mu^\circ(u)\cdot\frac{u-2\alpha+2}{u+2\alpha-1}.$$

As a $X(\mathfrak{so}_3,\mathfrak{so}_2)^{tw}$-module, $V(\mu^\bullet(u))$ is isomorphic to $V(\mu^\sharp(u))$, where

$$\begin{aligned}\widetilde{\mu}_1^\sharp(u) &= 2u\left(\frac{1-4u}{1+4u}\right)\mu^\bullet(2u)\mu^\bullet(2u-1)\\ &= \widetilde{\mu}_1(u)\cdot\frac{2u-2\alpha+1}{2u+2\alpha-2}\cdot\frac{2u-2\alpha+2}{2u+2\alpha-1},\end{aligned}$$

$$\begin{aligned}\widetilde{\mu}_0^\sharp(u) &= 2u\left(\frac{4u-3}{4u+1}\right)\mu^\bullet(2u)\mu^\bullet(1-2u)\\ &= \widetilde{\mu}_0(u)\cdot\frac{3-2u-2\alpha}{2\alpha-2u}\cdot\frac{2u-2\alpha+2}{2u+2\alpha-1}.\end{aligned}$$

As $\varphi_{\mathrm{BI}}^{-1}\circ\beta_A\circ\varphi_{\mathrm{BI}}=\psi_\sigma$, we can conclude that $V(\mu(u))^{\psi_\sigma}$ is isomorphic to $V(\mu^\sharp(u))$ with $\mu^\sharp(u)$ as in (5.5.2).

Finally, since $\widetilde{\mu}_0^\sharp(u)$ satisfies (4.2.6), $P(\frac{3}{2}-\alpha)\neq 0$, $P(u)=P(-u+\frac{3}{2})$ and

$$\begin{aligned}\frac{\widetilde{\mu}_0^\sharp(u)}{\widetilde{\mu}_1^\sharp(u)} &= \frac{\widetilde{\mu}_0(u)}{\widetilde{\mu}_1(u)}\cdot\frac{(\frac{3}{2}-\alpha)-u}{\alpha-u}\cdot\frac{\alpha+u-1}{(\frac{3}{2}-\alpha)+u-1}\\ &= \frac{P(u+\frac{1}{2})}{P(u)}\cdot\frac{(\frac{3}{2}-\alpha)-u}{(\frac{3}{2}-\alpha)+u-1},\end{aligned}$$

the module $V(\mu^\sharp(u))\cong V(\mu(u))^{\psi_\sigma}$ is associated to $(\frac{3}{2}-\alpha,P(u))$. □

We now generalize Lemma 5.5.1 to pairs $(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})$ for all $n\in\mathbb{N}$.

Proposition 5.5.2. *Suppose that the irreducible $X(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$-module $V(\mu(u))$ is finite-dimensional with Drinfeld tuple $(\alpha,(P_i(u))_{i=1}^n)$. Then*

$$V(\mu(u))^{\psi_\sigma}\cong V(\mu^\sharp(u)),$$

where the components of $\mu^\sharp(u)=(\mu_i(u))_{i\in\mathcal{I}_{2n+1}^+}$ are determined by

$$\mu_i^\sharp(u)=\mu_i(u)\quad\forall\quad 2\leq i\leq n, \tag{5.5.5}$$

$$\widetilde{\mu}_1^\sharp(u)=\widetilde{\mu}_1(u)\cdot\frac{2u-2\alpha+1}{2u+2\alpha-N+1}\cdot\frac{2u-2\alpha+2}{2u+2\alpha-N+2}, \tag{5.5.6}$$

$$\widetilde{\mu}_0^\sharp(u)=\widetilde{\mu}_0(u)\cdot\frac{N-2u-2\alpha}{2\alpha-2u}\cdot\frac{2u-2\alpha+2}{2u+2\alpha-N+2}. \tag{5.5.7}$$

Proof. Since $V(\mu(u))^{\psi_\sigma}$ has finite dimension and is irreducible, Theorem 4.2.6 and Proposition 4.2.9 imply that it is isomorphic to $V(\mu^\sharp(u))$ for some $\mu^\sharp(u)$. Throughout the proof, we fix highest weight vectors $\xi\in V(\mu(u))$ and $\xi_\sigma\in V(\mu(u))^{\psi_\sigma}$.

Set $m=n-1$. Since

$$\psi_\sigma(s_{ij}(u))=s_{\sigma(i)\sigma(j)}(u)\quad\forall\quad i,j\in\mathcal{I}_{2n+1},$$

the two subspaces $V(\mu(u))_{(+,m)}$ and $(V(\mu(u))^{\psi_\sigma})_{(+,m)}$ of $V(\mu(u))$ (see (4.3.2)) are identical. Consequently, the identity map provides a $\mathbb{C}$-linear isomorphism

$$\mathrm{id} : (V(\mu(u))_{(m)})^{\psi^1_\sigma} \to (V(\mu(u))^{\psi^n_\sigma})_{(m)}, \tag{5.5.8}$$

where we have used the notation (5.5.1), and the $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-modules $V(\mu(u))_{(m)}$ and $(V(\mu(u))^{\psi^n_\sigma})_{(m)}$ are defined in §4.3: see (4.3.30).

The first step of our proof is to show that this is also a $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-module isomorphism.

Step 1: The identity map (5.5.8) is an isomorphism of $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-modules.

To prove that this is the case it suffices to show that the generating series

$$\{s_{ij}(u)\}_{i,j\in\mathcal{I}_3} \subset X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}[\![u^{-1}]\!]$$

operate identically in $(V(\mu(u))_{(m)})^{\psi^1_\sigma}$ and $(V(\mu(u))^{\psi^n_\sigma})_{(m)}$ (which coincide as subspaces of $V(\mu(u))$).

Since $s_{ij}(u)$ acts in $V(\mu(u))_{(m)}$ as the operator $h(u)(s^{\circ m}_{ij}(u))$ (see (4.3.30)), it operates in $(V(\mu(u))_{(m)})^{\psi^1_\sigma}$ as the operator

$$h(u)\left(s_{\sigma(i)\sigma(j)}(u+\tfrac{m}{2}) + \frac{\delta_{ij}}{2u}\sum_{a=2}^{n} s_{aa}(u+\tfrac{m}{2})\right).$$

As $\sigma(a) = a$ for all $a \geq 2$, this is also equal to $h(u)\psi^n_\sigma(s^{\circ m}_{ij}(u))$ which is precisely the operator by which $s_{ij}(u)$ acts in $(V(\mu(u))^{\psi^n_\sigma})_{(m)}$. This completes the proof of Step 1.

Next, by Corollary 4.3.7, we can form the finite-dimensional irreducible modules

$$V(\mu(u))_m = X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}\xi \subset V(\mu(u))_{(m)},$$
$$(V(\mu(u))^{\psi^n_\sigma})_m = X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}\xi_\sigma \subset (V(\mu(u))^{\psi^n_\sigma})_{(m)}.$$

Step 2: ξ_σ is contained in $V(\mu(u))_m$ and $(V(\mu(u))_m)^{\psi^1_\sigma} \cong (V(\mu(u))^{\psi^n_\sigma})_m$.

Since the $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-module $(V(\mu(u))_m)^{\psi^1_\sigma}$ is finite-dimensional and irreducible, it is generated by a highest weight vector $\xi^\sharp_\sigma \in (V(\mu(u))_m)^{\psi^1_\sigma}$.

Since $(V(\mu(u))_m)^{\psi^1_\sigma}$ is a submodule of $(V(\mu(u))_{(m)})^{\psi^1_\sigma}$, Step 1 shows that $\xi^\sharp_\sigma$ is also contained in $(V(\mu(u))^{\psi_\sigma^n})_{(m)}$ and generates a highest weight submodule. By Part (2) of Corollary 4.3.7, this submodule must be equal to $(V(\mu(u))^{\psi_\sigma^n})_m$ and thus $\xi_\sigma^\sharp$ must be proportional to ξ_σ. This implies that ξ_σ, being a scalar multiple of $\xi_\sigma^\sharp$, is contained in $V(\mu(u))_m$.

Next, Let W be the image of the module $(V(\mu(u))_m)^{\psi_\sigma^1}$ under the isomorphism (5.5.8). As W is the irreducible submodule of $(V(\mu(u))^{\psi_\sigma^n})_{(m)}$ generated by ξ_σ, it is equal to $(V(\mu(u))^{\psi_\sigma^n})_m$.

Step 3: $\mu_i^\sharp(u) = \mu_i(u)$ for all $2 \leq i \leq n$.

Let us temporarily denote the standard family of generators of $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ by

$$\{\mathring{s}_{ij}^{(r)}\}_{i,j\in\mathcal{I}_3, r\in\mathbb{N}} \subset X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}.$$

This will distinguish them from the generators $\{s_{ij}^{(r)}\}_{i,j\in\mathcal{I}_{2n+1}, r\in\mathbb{N}} \subset X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$.

Since $V(\mu(u))_m$ is a $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$ highest weight module, it is generated by monomials of the form

$$(\mathring{s}_{i_1j_1}^{(r_1)} \cdots \mathring{s}_{i_cj_c}^{(r_c)}) \cdot \xi, \quad \text{where } i_a > j_a \in \mathcal{I}_3 \text{ and } r_a \in \mathbb{N} \quad \forall\, 1 \leq a \leq c,$$

and c takes non-negative integer values. By definition of the $X(\mathfrak{so}_3, \mathfrak{so}_2)^{tw}$-action, this implies $V(\mu(u))_m$ is also spanned by monomials of the form

$$s_{i_1j_1}^{(r_1)} \cdots s_{i_cj_c}^{(r_c)} \xi \tag{5.5.9}$$

with the same restrictions on the indices. In particular, by Step 2 the highest weight vector ξ_σ must be a linear combination of such monomials.

Let J_m be the left ideal of $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$ generated by the coefficients of

$$s_{ij}(u) \quad \forall \quad i < j \text{ with } 2 \leq j \leq n.$$

In particular, $V(\mu(u))_{(m)}$ is, as a vector space, equal to the subspace of $V(\mu(u))$ annihilated by J_m. For every $2 \leq k \leq n$ and pair $i, j \in \mathcal{I}_3$ with $j < i$, the defining relation (3.3.42) implies that

$$[s_{kk}(u), s_{ij}(v)] = 0 \mod J_m,$$

and hence the action of $s_{kk}(u)$ on a monomial of the form (5.5.9) is given by

$$s_{kk}(u)(s^{(r_1)}_{i_1j_1}\cdots s^{(r_c)}_{i_cj_c}\xi) = s^{(r_1)}_{i_1j_1}\cdots s^{(r_c)}_{i_cj_c}(s_{kk}(u)\xi) = \mu_k(u)(s^{(r_1)}_{i_1j_1}\cdots s^{(r_c)}_{i_cj_c}\xi).$$

Since $s_{\sigma(k)\sigma(k)}(u) = s_{kk}(u)$ for all $2 \le k \le n$, the above observation yields that

$$\psi_\sigma(s_{kk}(u))\xi_\sigma = \mu_k(u)\xi_\sigma \quad \forall \quad 2 \le k \le n.$$

Hence, $\mu_k(u) = \mu^\sharp_k(u)$ for all $2 \le k \le n$.

Step 4: The formulas (5.5.7) and (5.5.6) hold.

To compute $(\mu^\sharp_0(u), \mu^\sharp_1(u))$, we use Step 2 in conjunction with Lemma 5.5.1. By Corollary 4.3.7, we have

$$V(\mu(u))_m \cong V(h(u)\mu^\circ(u)) \quad \text{and} \quad V(\mu^\sharp(u))_m \cong V(h(u)\mu^\bullet(u)),$$

where $h(u)$ is given by (4.3.28), and $\mu^\circ(u) = (\mu^\circ_i(u))_{i\in\mathcal{I}_3^+}$, $\mu^\bullet(u) = (\mu^\bullet_i(u))_{i\in\mathcal{I}_3^+}$ are uniquely determined by

$$\widetilde{\mu}^\circ_i(u) = \widetilde{\mu}_i\Big(u + \tfrac{n-1}{2}\Big) \quad \text{and} \quad \widetilde{\mu}^\bullet_i(u) = \widetilde{\mu}^\sharp_i\Big(u + \tfrac{n-1}{2}\Big) \quad \forall \quad i \in \mathcal{I}_3^+.$$

By Corollary 5.2.9, $V(\mu(u))_m$ is associated to the tuple

$$\Big(\alpha - \tfrac{n-1}{2}, P\Big(u + \tfrac{n-1}{2}\Big)\Big),$$

and by Step 2, we have the sequence of isomorphisms

$$V(h(u)\mu^\circ(u))^{\psi^1_\sigma} \cong (V(\mu(u))_m)^{\psi^1_\sigma} \cong (V(\mu(u))^{\psi^n_\sigma})_m \cong V(\mu^\sharp(u))_m \cong V(h(u)\mu^\bullet(u)).$$

It thus follows from Lemma 5.5.1 that

$$\begin{aligned}
\widetilde{\mu}^\sharp_0(u + \tfrac{n-1}{2}) &= \widetilde{\mu}_0(u + \tfrac{n-1}{2}) \cdot \frac{3 - 2u - 2\alpha + n - 1}{2\alpha - n + 1 - 2u} \cdot \frac{2u - 2\alpha + n + 1}{2u + 2\alpha - n},\\
\widetilde{\mu}^\sharp_1(u + \tfrac{n-1}{2}) &= \widetilde{\mu}_1(u + \tfrac{n-1}{2}) \cdot \frac{2u - 2\alpha + n}{2u + 2\alpha - n - 1} \cdot \frac{2u - 2\alpha + n + 1}{2u + 2\alpha - n}.
\end{aligned}$$

Substituting $u \mapsto u - \frac{n-1}{2}$ we obtain the formulas (5.5.7) and (5.5.6). □

Henceforth, we assume that $n > 1$, so that $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})$ is not equal to $(\mathfrak{so}_3, \mathfrak{so}_2)$.

Proposition 5.5.3. *Suppose that $V(\mu(u))$ is finite-dimensional with Drinfeld tuple $(\alpha, (P_i(u))_{i=1}^n)$. Then*

$$\alpha - \tfrac{N}{4} \in \tfrac{1}{2}\mathbb{Z},$$
$$S(\alpha, \tfrac{N}{2} - \alpha) \cup S(\alpha + \tfrac{1}{2}, \tfrac{N}{2} - \alpha + \tfrac{1}{2}) \subset Z(P_2(u)).$$

Remark 5.5.4. By definition, the strings $S(\alpha, \frac{N}{2} - \alpha)$ and $S(\alpha + \frac{1}{2}, \frac{N}{2} - \alpha + \frac{1}{2})$ are empty unless $\alpha > \frac{N}{4}$. Therefore, the condition

$$S(\alpha, \tfrac{N}{2} - \alpha) \cup S(\alpha + \tfrac{1}{2}, \tfrac{N}{2} - \alpha + \tfrac{1}{2}) \subset Z(P_2(u))$$

is vacuous whenever $\alpha \leq \frac{N}{4}$.

Proof of Proposition 5.5.3. It was shown in Proposition 5.2.13 that $\alpha \in \frac{1}{2}\mathbb{Z} + \frac{N}{4}$. However, we will not assume this in our proof.

As a consequence of Proposition 5.5.2 we have $V(\mu(u))^{\psi_\sigma^n} \cong V(\mu^\sharp(u))$ where $\mu^\sharp(u)$ is as in (5.5.5)–(5.5.7). Since $\mu(u)$ is associated to $(\alpha, (P_i(u))_{i=1}^n)$, the components of $\widetilde{\mu}^\sharp(u)$ satisfy the relations

$$\frac{\widetilde{\mu}_0^\sharp(u)}{\widetilde{\mu}_1^\sharp(u)} = \frac{P_1(u + \frac{1}{2})}{P_1(u)} \cdot \frac{(\frac{N}{2} - \alpha) - u}{u + (\frac{N}{2} - \alpha) - n}, \tag{5.5.10}$$

$$\frac{\widetilde{\mu}_1^\sharp(u)}{\widetilde{\mu}_2^\sharp(u)} = \frac{P_2(u+1)}{P_2(u)} \cdot \frac{2u - 2\alpha + 1}{2u + 2\alpha - N + 1} \cdot \frac{2u - 2\alpha + 2}{2u + 2\alpha - N + 2}, \tag{5.5.11}$$

$$\frac{\widetilde{\mu}_{i-1}^\sharp(u)}{\widetilde{\mu}_i^\sharp(u)} = \frac{P_i(u+1)}{P_i(u)} \quad \forall \quad 3 \leq i \leq n. \tag{5.5.12}$$

On the other hand, since $V(\mu^\sharp(u))$ is finite-dimensional, Proposition 5.2.5 implies that $\mu^\sharp(u)$ can be associated to a Drinfeld tuple $(\alpha^\sharp, (Q_i(u))_{i=1}^n)$. Consequently, from relation (5.5.11) we obtain the equality

$$\frac{Q_2(u+1)}{Q_2(u)} \cdot \frac{(n-\alpha) - u}{u + (n - \alpha) - n + 1} = \frac{P_2(u+1)}{P_2(u)} \cdot \frac{(\alpha - \frac{1}{2}) - u}{u + (\alpha - \frac{1}{2}) - n + 1}.$$

Applying Lemma 5.2.2 to both sides (with $m = 1$ and $l = n$) we find that there exists monic polynomials $Q_2^\bullet(u)$ and $P_2^\bullet(u)$, together with non-negative integers ℓ_P and ℓ_Q

such that $P_2^\bullet(u) = P_2^\bullet(-u+n)$, $Q_2^\bullet(u) = Q_2^\bullet(-u+n)$, and

$$\frac{Q_2^\bullet(u+1)}{Q_2^\bullet(u)} \cdot \frac{(n-\alpha-\ell_Q)-u}{u+(n-\alpha-\ell_Q)-n+1} = \frac{P_2^\bullet(u+1)}{P_2^\bullet(u)} \cdot \frac{(\alpha-\frac{1}{2}-\ell_P)-u}{u+(\alpha-\frac{1}{2}-\ell_P)-n+1},$$

with $Q_2^\bullet(n-\alpha-\ell_Q) \neq 0$ and $P_2^\bullet(\alpha-\frac{1}{2}-\ell_P) \neq 0$. By Lemma 5.2.1, we must have $Q_2^\bullet(u) = P_2^\bullet(u)$ and $n-\alpha-\ell_Q = \alpha-\frac{1}{2}-\ell_P$. The latter relation implies that $2\alpha - \frac{N}{2} = \ell_P - \ell_Q \in \mathbb{Z}$, and thus that

$$\alpha - \tfrac{N}{4} \in \tfrac{1}{2}\mathbb{Z}.$$

If in addition $\alpha > \frac{N}{4}$, then

$$\ell_P \geq \ell_P - \ell_Q = 2\alpha - \tfrac{N}{2} > 0.$$

Since $(\ell_P, P_2^\bullet(u))$ is the pair $(\ell^1_{\alpha-1/2}, P^1_{\alpha-1/2}(u))$ from Lemma 5.2.2 (where $P(u) = P_2(u)$), $P_2^\bullet(u)$ is equal to $P_2(u)$ divided by the polynomial $Q(u)$ from (5.2.3) with $m = 1$ and α replaced by $\alpha - 1/2$. Therefore, $P_2(u)$ is divisible by the polynomial

$$\begin{aligned} P_\alpha(u) &= \prod_{k=0}^{2\alpha-\frac{N}{2}-1} (u-\alpha+1/2+k)(u-\tfrac{N}{2}+\alpha-k) \\ &= \prod_{k=0}^{2\alpha-\frac{N}{2}-1} (u-\alpha+1/2+k)(u-\alpha+1+k). \end{aligned} \tag{5.5.13}$$

The proof of the proposition is completed by observing that the roots of $P_\alpha(u)$ are precisely the elements of

$$S(\alpha, \tfrac{N}{2}-\alpha) \cup S(\alpha+\tfrac{1}{2}, \tfrac{N}{2}-\alpha+\tfrac{1}{2}). \qquad \square$$

Remark 5.5.5. The statement of Proposition 5.5.3 is much stronger than that of Proposition 5.2.13 (in the case $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta) = (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})$). The latter tells us that $\alpha \in \frac{1}{2}\mathbb{Z} + \frac{N}{4}$ and that $\alpha - \frac{N}{4} \leq \frac{1}{4}(\deg P_1(u) + \deg P_2(u))$ but says nothing about the roots of $P_2(u)$. In fact, since the strings $S(\alpha, \frac{N}{2}-\alpha)$ and $S(\alpha+\frac{1}{2}, \frac{N}{2}-\alpha+\frac{1}{2})$ are disjoint and both have length $2\alpha - \frac{N}{2}$, Proposition 5.5.3 implies that $\alpha - \frac{N}{4} \leq \frac{1}{4}\deg P_2(u)$.

Provided $\alpha > \frac{N}{4}$, the polynomial $P_\alpha(u)$ from (5.5.13) satisfies the relation

$$\frac{P_\alpha(u)}{P_\alpha(u+1)} = \frac{2u-2\alpha+1}{2u+2\alpha-N+1} \cdot \frac{2u-2\alpha+2}{2u+2\alpha-N+2}. \tag{5.5.14}$$

If instead $\alpha \leq \frac{N}{4}$, let $P_\alpha^-(u)$ be the polynomial

$$P_\alpha^-(u) = \prod_{k=0}^{\frac{N}{2}-2\alpha-1} (u-n+\alpha+k)(u-\alpha-k) = \prod_{k=0}^{\frac{N}{2}-2\alpha-1} (u-\alpha-\tfrac{1}{2}-k)(u-\alpha-k),$$

where the equality $P_\alpha^-(u)=1$ is understood to hold if $\alpha = \frac{N}{4}$. Then $P_\alpha^-(u)$ satisfies the relation

$$\frac{P_\alpha^-(u+1)}{P_\alpha^-(u)} = \frac{2u-2\alpha+1}{2u+2\alpha-N+1} \cdot \frac{2u-2\alpha+2}{2u+2\alpha-N+2}.$$

These observations together with the relations (5.5.10)–(5.5.12) imply the following corollary.

Corollary 5.5.6. *Suppose that the $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module $V(\mu(u))$ has finite dimension and Drinfeld tuple $(\alpha, (P_i(u))_{i=1}^n)$. Then the Drinfeld tuple of the finite-dimensional irreducible module $V(\mu(u))^{\psi_\sigma}$ is*

$$\left(\frac{N}{2}-\alpha, (P_i^\sharp(u))_{i=1}^n\right),$$

where $P_i^\sharp(u) = P_i(u)$ for all $i \neq 2$ and

$$P_2^\sharp(u) = \begin{cases} P_2(u)P_\alpha^-(u) & \text{if } \alpha \leq \frac{N}{4}, \\ P_2(u)/P_\alpha(u) & \text{if } \alpha > \frac{N}{4}. \end{cases} \tag{5.5.15}$$

Observe that these formulas together with those of Proposition 5.5.2 imply that, under the assumption that $V(\mu(u))$ is finite-dimensional with $\mu(u)$ associated to $(\alpha, (P_i(u))_{i=1}^n)$, we have

$$V(\mu(u)) \cong V(\mu(u))^{\psi_\sigma} \iff \alpha = \frac{N}{4},$$

and the same assertion remains true at the level of $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-modules.

5.5.2 Classification of finite-dimensional irreducibles

With Proposition 5.5.3 at our disposal we can now classify the finite-dimensional irreducible representations of $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$. We continue to assume that $n > 1$.

Theorem 5.5.7. *Let $\mu(u) = (\mu_i(u))_{i \in \mathcal{I}^+_{2n+1}}$ satisfy (4.4.12). The $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module $V(\mu(u))$ is finite-dimensional if and only if there exists monic polynomials $P_1(u), \ldots, P_n(u)$ in u satisfying (5.2.8), together with $\alpha \in \mathbb{C} \setminus Z(P_1(u))$ such that*

$$\begin{gathered} \alpha - \tfrac{N}{4} \in \tfrac{1}{2}\mathbb{Z}, \\ S(\alpha, \tfrac{N}{2} - \alpha) \cup S(\alpha + \tfrac{1}{2}, \tfrac{N}{2} - \alpha + \tfrac{1}{2}) \subset Z(P_2(u)), \end{gathered} \tag{5.5.16}$$

$$\frac{\widetilde{\mu}_{i-1}(u)}{\widetilde{\mu}_i(u)} = \frac{P_i(u + 1 - \frac{\delta_{i1}}{2})}{P_i(u)} \left(\frac{\alpha - u}{\alpha + u - n} \right)^{\delta_{i,1}} \quad \forall \quad 1 \leq i \leq n. \tag{5.5.17}$$

Proof. Since $\hbar = \hbar(\mathcal{G}) = 0$ and (4.2.6) holds, (5.5.17) is equivalent to the relations (5.2.9) and (5.2.10). Hence, if $V(\mu(u))$ is finite-dimensional, Propositions 5.2.5 and (5.5.3) imply that there $\mu(u)$ is associated to a tuple $(\alpha, (P(u)_{i=1}^n))$ as in the statement of the theorem.

Conversely, suppose that $\mu(u)$ is associated to a tuple $(\alpha, (P_i(u))_{i=1}^n)$ as in Definition 5.2.6, which in addition satisfies (5.5.16). We will show that $V(\mu(u))$ is finite-dimensional, splitting our proof into two cases.

Case 1: $\alpha \leq \frac{N}{4}$.

Define the auxiliary tuple $(P_i^\circ(u))_{i=1}^n \subset \mathbb{C}[u]^n$ by

$$\begin{gathered} P_i^\circ(u) = P_i(u) \quad \forall i > 1, \\ P_1^\circ(u) = P_1(u) \prod_{k=0}^{N/2 - 2\alpha - 1} (u - \tfrac{N}{4} + \tfrac{k}{2})(u - \tfrac{N}{4} - \tfrac{k}{2}). \end{gathered}$$

Note that $P_1^\circ(u)$ satisfies $P_1^\circ(u) = P_1^\circ(-u + \frac{N}{2}) = P^\circ(-u + \kappa + 1)$ and

$$\frac{P_1^\circ(u + \frac{1}{2})}{P_1^\circ(u)} \cdot \frac{\frac{N}{4} - u}{\frac{N}{4} + u - n} = \frac{P_1(u + \frac{1}{2})}{P_1(u)} \cdot \frac{\alpha - u}{\alpha + u - n}. \tag{5.5.18}$$

Therefore, by (5.2.8), there are monic polynomials $Q_1(u), \ldots, Q_n(u)$ satisfying

$$P_1^\circ(u) = (-1)^{\deg Q_1} Q_1(u-\kappa/2)Q_1(-u+\kappa/2+1),$$
$$P_i^\circ(u) = (-1)^{\deg Q_i} Q_i(u-\kappa/2)Q_i(-u+n-i+2-\kappa/2) \quad \forall \quad 2 \le i \le n.$$

By Proposition 4.1.5, there is an $X(\mathfrak{so}_{2n+1})$ highest weight $\lambda(u) = (\lambda_i(u))_{i\in\mathcal{I}_{2n+1}}$ with the property that $L(\lambda(u))$ is finite-dimensional with Drinfeld polynomials $(Q_i(u))_{i=1}^n$. Let $\xi \in L(\lambda(u))$ be a highest weight vector.

Then Lemma 5.2.10 applied with $V(\mu(u)) = V(\mathcal{G})$, along with (5.5.18) and Remark 5.2.11, imply that the finite-dimensional highest weight module

$$X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}\xi \subset L(\lambda(u))$$

has highest weight $\nu(u)$ which is also associated to $(\alpha, (P_i(u))_{i=1}^n)$. We may thus conclude that $V(\mu(u))$ is finite-dimensional using Lemma 5.2.8 and same argument as given in the proof of Theorem 5.4.1.

Case 2: $\alpha > \frac{N}{4}$.

Let $\mu^\sharp(u) = (\mu_i^\sharp(u))_{i\in\mathcal{I}_{2n+1}^+}$ be the tuple determined by (5.5.5)–(5.5.7). Since

$$S(\alpha, \tfrac{N}{2}-\alpha) \cup S(\alpha+\tfrac{1}{2}, \tfrac{N}{2}-\alpha+\tfrac{1}{2}) \subset Z(P_2(u)),$$

the polynomial $P_\alpha(u)$ from (5.5.13) divides $P_2(u)$. By (5.5.10)–(5.5.12) and (5.5.14), $\mu^\sharp(u)$ is associated to

$$(\tfrac{N}{2}-\alpha, (P_i^\sharp(u))_{i=1}^n), \quad \text{where}$$
$$P_2^\sharp(u) = P_2(u)/P_\alpha(u) \quad \text{and} \quad P_i^\sharp(u) = P_i(u) \quad \forall \quad i \neq 2.$$

Since $\frac{N}{2}-\alpha < \frac{N}{4}$, the argument of Case 1 implies that $V(\mu^\sharp(u))$ is finite-dimensional. By Proposition 4.7, $V(\mu^\sharp(u))^{\psi_\sigma}$ is isomorphic to $V(\mu(u))$, and thus $V(\mu(u))$ is also finite-dimensional. □

Using the above theorem and the arguments of §5.4.1, we obtain a parameterization of finite-dimensional irreducible representations of both $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$ and $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$. This is presented in Proposition 5.5.8 and Corollary 5.5.9 below;

all notation is as in §5.4.1.

Proposition 5.5.8. *The isomorphism classes of finite-dimensional irreducible representations of $X(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$ are parameterized by tuples*

$$(g(u);(\alpha,(P_i(u))_{i=1}^n))\in(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}\times\mathbb{C}\times\mathbb{C}[u]^n,$$
$$\textit{where each } P_i(u) \textit{ is monic},$$
$$\alpha\in\mathbb{C}\setminus Z(P_1(u)),\quad \alpha-\tfrac{N}{4}\in\tfrac{1}{2}\mathbb{Z},$$
$$S(\alpha,\tfrac{N}{2}-\alpha)\cup S(\alpha+\tfrac{1}{2},\tfrac{N}{2}-\alpha+\tfrac{1}{2})\subset Z(P_2(u)),$$
$$P_1(u)=P_1(-u+\kappa+1)\quad \textit{and}\quad P_i(u)=P_i(-u+n-i+2)\quad \forall\, i\geq 2.$$

The underlying correspondence Γ_X^ϑ is given

$$\Gamma_X^\vartheta(V(\mu(u)))=(g(u);(\alpha,(P_i(u))_{i=1}^n)),\qquad \textit{where}$$

(a) $g(u)\in(1+u^{-1}\mathbb{C}[\![u^{-1}]\!])^{\mathrm{ref}_\kappa}$ *is the unique scalar series such that*

$$q(u)|_{\nu_g^*(V(\mu(u)))}=\mathrm{id}_{\nu_g^*(V(\mu(u)))}.$$

(b) $(\alpha,(P_i(u))_{i=1}^n)$ *is the Drinfeld tuple associated to $\mu(u)$.*

Corollary 5.5.9. *The isomorphism classes of finite-dimensional irreducible representations of $Y(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$ are parameterized by tuples*

$$(\alpha,(P_i(u))_{i=1}^n)\in\mathbb{C}\times\mathbb{C}[u]^n,$$
$$\textit{where each } P_i(u) \textit{ is monic},$$
$$\alpha\in\mathbb{C}\setminus Z(P_1(u)),\quad \alpha-\tfrac{N}{4}\in\tfrac{1}{2}\mathbb{Z},$$
$$S(\alpha,\tfrac{N}{2}-\alpha)\cup S(\alpha+\tfrac{1}{2},\tfrac{N}{2}-\alpha+\tfrac{1}{2})\subset Z(P_2(u)),$$
$$P_1(u)=P_1(-u+\kappa+1)\quad \textit{and}\quad P_i(u)=P_i(-u+n-i+2)\quad \forall\, i\geq 2.$$

We now turn towards obtaining a result analogous to Corollary 5.4.5 for the twisted Yangian $Y(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$. Since every automorphism of the form (3.3.21) restricts to an automorphism of the twisted Yangian $Y(\mathfrak{g}_N,\mathfrak{g}_N^\vartheta)^{tw}$, we may view ψ_σ as an automorphism of $Y(\mathfrak{so}_{2n+1},\mathfrak{so}_{2n})^{tw}$.

Given $m \geq 0$, $1 \leq i_1, \ldots, i_m \leq n$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, we can form the tensor product of $Y(\mathfrak{so}_{2n+1})$ fundamental representations

$$L(i_1 : \alpha_1) \otimes \cdots \otimes L(i_m : \alpha_m), \tag{5.5.19}$$

where $L(i_k, \alpha_k)$ is as in Definition 4.1.8. For each $1 \leq k \leq m$, let ξ_i be a highest weight vector of $L(i_k : \alpha_{i_k})$ and consider the $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module

$$Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}(\xi_1 \otimes \cdots \otimes \xi_m) \subset L(i_1 : \alpha_{i_1}) \otimes \cdots \otimes L(i_m : \alpha_{i_m}), \tag{5.5.20}$$

where both sides are identified with the trivial representation if $m = 0$.

Corollary 5.5.10. *Let V be a finite-dimensional irreducible $Y(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$-module with Drinfeld tuple $(\alpha, (P_i(u))_{i=1}^n)$. Then*

(1) V is isomorphic to the unique irreducible quotient of a module of the form (5.5.20) if and only if $\alpha \leq \frac{N}{4}$,

(2) $\psi_\sigma^*(V)$ is isomorphic to the unique irreducible quotient of a module of the form (5.5.20) if and only if $\alpha \geq \frac{N}{4}$.

Proof. If V is isomorphic to the irreducible quotient of the module (5.5.20), Lemma 5.2.10 and Remark 5.2.11 with $V(\mu(u)) = V(\mathcal{G})$ imply that $\alpha = \frac{N}{4} - \frac{\ell_\alpha}{2}$, where ℓ_α is a non-negative integer. This proves the ($\Longrightarrow$) direction of (1).

Suppose now that the representation $\psi_\sigma^*(V)$ is isomorphic to the irreducible quotient of the module (5.5.20). By Corollary 5.5.6, $\psi_\sigma^*(V)$ is associated to the Drinfeld tuple

$$(\tfrac{N}{2} - \alpha, (P_i^\sharp(u))_{i=1}^n),$$

where $P_i^\sharp(u) = P_i(u)$ for all $i \neq 2$ and $P_2^\sharp(u)$ is given by (5.5.15). Hence, the same argument as given in the previous paragraph shows that $\frac{N}{2} - \alpha \leq \frac{N}{4}$, thus proving the ($\Longrightarrow$) direction of (2).

The ($\Longleftarrow$) direction of (1) and (2) is now proven using the same argument employed to prove Corollary 5.4.5, with the role of the proof of Theorem 5.4.1 played by the proof of Theorem 5.5.7. □

Chapter 6

Conclusion

In this thesis, we have considered two topics in the representation theory of Yangians. In Chapter 2, we constructed the R-matrix presentation of the Yangian starting from any fixed finite-dimensional $Y(\mathfrak{g})$-module which has a non-trivial irreducible $\mathfrak{g}$-submodule. Our results in this chapter provide a generalization of [Dri85, Theorem 6] and, in particular, make available a proof of that result. In addition, we have proven several structural properties for the extended Yangian which generalize results which have played an important role in the special case where the underlying module is the vector representation of a classical Lie algebra.

In Chapters 3–5, we have focused on the problem of classifying all finite-dimensional irreducible representations for twisted Yangians associated to orthogonal and symplectic symmetric pairs of Lie algebras. Our main results include a complete solution to this problem when the underlying pair is of the form

$$(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n}),$$
$$(\mathfrak{g}_N, \mathfrak{g}_N), \quad (\mathfrak{g}_{2n}, \mathfrak{gl}_n) \quad \text{and} \quad (\mathfrak{so}_N, \mathfrak{so}_{N-2} \oplus \mathfrak{so}_2).$$

To conclude, we now provide a brief discussion of some natural questions which arise from our work.

6.0.1 On the universal R-matrix of the Yangian

The construction of the R-matrix presentation of the Yangian carried out in Chapter 2 depends heavily on the existence of Drinfeld's universal R-matrix $\mathcal{R}(u)$, which is provided by [Dri85, Theorem 3]: see Theorem 2.2.4. Drinfeld's proof that such a remarkable series exists is based on deformation theory and, unfortunately, has not appeared in the literature[1]. Due to the non-constructive nature of the argument, there is no known expression for the evaluation of $\mathcal{R}(u)$ on the tensor square $V \otimes V$ of a given finite-dimensional $Y(\mathfrak{g})$-module V (see Remark 2.4.2). This discussion raises the following questions:

(Q1) Can the existence of $\mathcal{R}(u)$ be proven constructively?

(Q2) Given a fixed finite-dimensional $Y(\mathfrak{g})$-module V and a suitable R-matrix

$$R(u) \in \mathrm{End}(V \otimes V)[\![u^{-1}]\!],$$

can the results of Chapter 2 be proven with $(\rho \otimes \rho)\mathcal{R}(-u)$ replaced by $R(u)$ and without otherwise appealing to $\mathcal{R}(u)$?

(Q3) If the answer to (Q2) is positive, can $\mathcal{R}(u)$ be rebuilt from $R(u)$ using the R-matrix formalism?

It turns out that each of these questions has a positive answer. Question (Q1) is addressed in the forthcoming work [GTLW] of S. Gautam, V. Toledano Laredo, and the author. Therein, we construct explicitly a formal series $\mathcal{R}^G(u)$ by identifying each factor in its Gauss decomposition

$$\mathcal{R}^G(u) = \mathcal{R}^+(u)\mathcal{R}^0(u)\mathcal{R}^-(u) \in (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[\![u^{-1}]\!]. \tag{6.0.1}$$

The commutative factor $\mathcal{R}^0(u)$ was constructed by Gautam and Toledano Laredo in [GTL17]. The triangular factors $\mathcal{R}^{\pm}(u)$ are related by $\mathcal{R}^+(u)^{-1} = \mathcal{R}^-_{21}(-u)$ and arise as the unique solutions of two related consistent systems of partial differential equations, as is spelled out in detail in [GTLW]. We then prove that $\mathcal{R}^G(u)$ satisfies the defining properties of $\mathcal{R}(u)$, the main ingredient (supplemental to [GTL17]) being

1) However, the author has been informed that such a proof will appear in forthcoming work of P. Etingof and M. Gardini.

that $\mathcal{R}^-(u)$ intertwines the so-called deformed Drinfeld coproduct and the standard coproduct on $Y(\mathfrak{g})$. By the uniqueness assertion of Theorem 2.2.4 (a result proven in [GTLW]), it follows that $\mathcal{R}^G(u)$ coincides with Drinfeld's universal R-matrix $\mathcal{R}(u)$[2].

Consider now (Q2) and (Q3). Though these questions both have positive answers, they have not yet been afforded a proper treatment in the literature. For the sake of the reader, we provide a sketch of some of the relevant ideas below, with emphasis on (Q2).

Let V and ρ be as in §2.4 and set

$$\mathcal{R}_1 = \Omega \quad \text{and} \quad \mathcal{R}_2 = \sum_{\lambda\in\Lambda}(J(X_\lambda)\otimes X_\lambda - X_\lambda\otimes J(X_\lambda)) + \tfrac{1}{2}\Omega^2,$$

as in the expansion (2.2.11). Next, suppose we are given

$$R(u) = I + \sum_{k\geq 1} R^{(k)}u^{-k} \in I + u^{-1}\mathrm{End}(V\otimes V)[\![u^{-1}]\!]$$

satisfying

(1) $R^{(k)} = (-1)^k(\rho\otimes\rho)(\mathcal{R}_k)$ for $k=1$ and $k=2$,

(2) $R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$,

(3) $\left(\left(R(u+\tfrac{1}{2}c_{\mathfrak{g}})^t\right)^{-1}\right)^t = R(u)^{-1}$,

where t is the standard transposition $E_{ij}^t = E_{ji}$ and is applied in the first tensor factor.

Given the above data, we may define $X_{\mathcal{I}}(\mathfrak{g})$ and $Y_R(\mathfrak{g})$ exactly as before. The relation (3), called the *crossing symmetry* relation, is imposed so that the representation

$$\varrho : X_{\mathcal{I}}(\mathfrak{g}) \to \mathrm{End}V, \quad T(u)\mapsto R(u),$$

which exists by (2), descends to a representation of $Y_R(\mathfrak{g})$. Due to the condition (1) on $R(u)$, Propositions 2.4.4 and 2.4.6 still hold and give surjections

$$\varphi_{\mathcal{I}} : U(\mathfrak{g}_{\mathcal{I}}[z]) \twoheadrightarrow \mathrm{gr}X_{\mathcal{I}}(\mathfrak{g}) \quad \text{and} \quad \varphi : U(\mathfrak{g}[z]) \twoheadrightarrow \mathrm{gr}Y_R(\mathfrak{g}).$$

2) More precisely, we prove that $\mathcal{R}^G(u) = \mathcal{R}(-u)^{-1}$.

The Poincaré-Birkhoff-Witt theorem for $Y_R(\mathfrak{g})$ and the isomorphism $Y_R(\mathfrak{g}) \cong Y(\mathfrak{g})$ can still be proven simultaneously, but one must replace the arguments of §2.5, which make extensive use of $\mathcal{R}(u)$. This can be achieved by constructing filtered homomorphisms

$$\Phi_R : Y_R(\mathfrak{g}) \to \prod_{n\in\mathbb{N}} \mathrm{End}_{\mathbb{C}[u_1,\ldots,u_n]}(V^{\otimes n}[u_1, \ldots u_n]) \leftarrow Y(\mathfrak{g}) : \Phi_J$$

as follows. If P_k denotes the natural projection

$$\mathrm{P}_k : \prod_{n\in\mathbb{N}} \mathrm{End}_{\mathbb{C}[u_1,\ldots,u_n]}(V^{\otimes n}[u_1, \ldots u_n]) \to \mathrm{End}_{\mathbb{C}[u_1,\ldots,u_k]}(V^{\otimes k}[u_1, \ldots u_k]),$$

then Φ_J is uniquely defined by the requirement that

$$\mathrm{P}_k \circ \Phi_J = \rho^{\otimes k} \circ (\tau_{u_1} \otimes \cdots \otimes \tau_{u_k}) \circ \Delta^{(k-1)} : Y(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}[u_1,\ldots,u_k]}(V^{\otimes k}[u_1, \ldots u_k]),$$

where all notation is as in §2.2. The homomorphism Φ_R is defined similarly, with ρ replaced by ϱ and τ_{u_i} replaced with the formal analogue of (2.4.5).

It is not difficult to prove that the composite of $\mathrm{gr}(\Phi_R)$ with φ is injective, which implies that both φ and Φ_R are themselves injective. Similarly, Φ_J is injective. Using degree zero and one generators together with the condition (1), one then argues that Φ_R and Φ_J have the same image, which implies that the composites $\Phi_J^{-1} \circ \Phi_R$ and $\Phi_R^{-1} \circ \Phi_J$ are both isomorphisms.

The rest of the constructions of Chapter 2 can now proceed without any serious modifications. As for (Q3), the universal R-matrix $\mathcal{R}(u)$ can now be rebuilt from $R(u)$ via a fusion type procedure, with a little effort, in the presentation of $Y(\mathfrak{g})$ provided by $\Phi_R(Y_R(\mathfrak{g}))$.

Remark 6.0.1. We emphasize that the above discussion of (Q2) and (Q3) is not intended to be rigorous. A more complete, and rigorous, picture will hopefully be given in future work of the author.

6.0.2 A general R-matrix approach to twisted Yangians

Our approach to symmetric pairs in §3.2, together with our proofs of several results in §3.3, has hopefully left the impression that is possible to give a general theory of twisted Yangians using ideas from Chapter 2. Such a theory should allow one to construct a twisted Yangian $Y(\mathfrak{g},\mathfrak{g}^\vartheta)^{tw}$ from any involution $\vartheta = \mathrm{Ad}_\rho(\mathcal{G})|_{\mathfrak{g}_\rho}$ (see (3.2.2)) both as a coideal subalgebra of $Y_R(\mathfrak{g})$ and as a quotient of a reflection equation algebra (as in §3.3.5). We hope to pursue this direction in future work.

Right now, there are at least two missing ingredients needed for carrying out this construction in full generality:

(1) A $\mathfrak{g}_\rho[z]^{\check{\vartheta}}$-analogue of Proposition 2.3.9.

(2) A source of solutions

$$\mathcal{G}(u) \in \mathcal{G} + u^{-1}(\mathrm{End}V)[\![u^{-1}]\!]$$

of the reflection (or *boundary* Yang-Baxter) equation

$$R_{12}(u-v)\mathcal{G}_1(u)R_{21}(u+v)\mathcal{G}_2(v) = \mathcal{G}_2(v)R_{12}(u+v)\mathcal{G}_1(u)R_{21}(u-v). \quad (6.0.2)$$

Note that the reflection equation (6.0.2) reduces to (3.3.1) when $R(u)$ is symmetric, as is the case in Chapter 3. Both of the above ingredients are needed for describing $Y(\mathfrak{g},\mathfrak{g}^\vartheta)^{tw}$ precisely as a quotient of a reflection algebra.

The problem of proving that there exists a universal source of solutions to the reflection equation (6.0.2) is closely related to the (open) problem of establishing the existence of a universal K-matrix for $Y(\mathfrak{g},\mathfrak{g}^\vartheta)^{tw}$, which would serve as a twisted Yangian analogue of $\mathcal{R}(u)$: see [BK16, Kol17, Li19].

6.0.3 On the classification problem for twisted Yangians of types B, C and D

Finally, let us briefly comment on the current state of the classification problem which has been at the core of our work in Chapters 4 and 5.

Our main results of Chapter 5 do not provide a complete solution to this problem when the underlying symmetric pair is of the form

$$\begin{aligned} (\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-q} \oplus \mathfrak{sp}_q) &\quad \text{with} \quad 2 \leq q \leq 2n-q, \\ (\mathfrak{so}_{2n}, \mathfrak{so}_{2n-q} \oplus \mathfrak{so}_q) &\quad \text{with} \quad 3 \leq q \leq 2n-q, \\ (\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n+1-q} \oplus \mathfrak{so}_q) &\quad \text{with} \quad 3 \leq q \leq 2n-2, \end{aligned} \tag{6.0.3}$$

where in all cases q is assumed to be even (see (3.0.1)). When $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ is any of the above pairs, the necessary conditions established in §5.2 are not sufficient for determining precisely when the $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$-module $V(\mu(u))$ is finite-dimensional. However, there is a version of the machinery developed in §5.5.1 for $X(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})^{tw}$ which is applicable to $X(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)^{tw}$ for any pair $(\mathfrak{g}_N, \mathfrak{g}_N^\vartheta)$ of the above form. It has recently been discovered by the author, in joint work with N. Guay and V. Regelskis, that this type of approach leads to a complete solution to the underlying classification problem for all twisted Yangians

$$X(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-q} \oplus \mathfrak{sp}_q)^{tw} \quad \text{and} \quad Y(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-q} \oplus \mathfrak{sp}_q)^{tw},$$

where the same restrictions on q are imposed as in (6.0.3). These results will be presented in the forthcoming paper [GRW].

For the twisted Yangians associated to the symmetric pairs in (6.0.3) of orthogonal type, there are additional complications which arise. One difficulty involves showing that, in the notation of Definition 5.2.6, if $\mu(u)$ is associated to the tuple

$$(N/4 - 1/2, (P_i(u))_{i=1}^n), \quad \text{where} \quad P_i(u) = 1 \quad \forall \quad 1 \leq i \leq n,$$

then $V(\mu(u))$ is finite-dimensional. By Lemma 5.2.12, any $V(\mu(u))$ with this property is closely related to the spinor representation of $\mathfrak{so}_q$ with the highest weight $(-1/2, \ldots, -1/2)$. For the twisted Yangians associated to $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2n})$ studied in §5.5, this difficulty does not arise as such modules can be obtained by restricting the spinor representations for $Y(\mathfrak{so}_{2n+1})$ which are given by [AMR06, Lemma 5.18].

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