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TWO STUDIES ON FUNCTIONS OF SEVERAL VARIABLES

by

ZUOWEI SHEN

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To My Wife, Lu

and

To My Parents

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ABSTRACT

An important aspect in the study of multivariate box splines is the fact that polynomials contained in the linear span of lattice translates of a box spline are precisely the common kernel of a family of linear partial operators. Dahmen and Micchelli were the first to compute the dimension of this kernel and to realize that the dimension theory extends to general linear operators indexed by a finite set X with matroid structure. Dahmen and Micchelli related the dimension problem to solvability of certain systems of linear operator equations.

The first study of this thesis is related to the study of Dahmen and Micchelli. Let G be a semigroup of linear operators on a linear space into itself with the operation of composition. A subset of G is indexed by a finite set X with matroid structure. The kernel space is induced by the matroid structure on X and the dimension can be given as sum of the dimensions of simpler kernels if and only if G has s- dimensional additivity. The method also allows us to discuss some other structures on the index set. As applications, we show that constant coefficient partial differential equations and difference equations corresponding to polynomials in s variables have the s-dimensional additivity property. Another general problem related to this is the solvability of systems of linear operator equations. We also discuss conditions under which simple necessary conditions, the compatibility conditions, are sufficient for the system of linear operator equations to have solutions. In particular, we apply these results to systems of partial differential and difference equations.

The second part of this thesis is wavelet decompositons. We discuss orthogonal decompositions of multiresolution approximation generated by box splines in dimensions $s = 1, 2, 3$.

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CHAPTER 1

INTRODUCTION

The first basic question addressed in the first four chapters of this thesis is one of expressing the dimension of the intersection of kernels of linear operators that arise naturally in multivariate approximation theory in terms of the more easily computable dimensions of some basic building blocks. The theory, as it has progressed, connects concepts arising in multivariate approximation theory with ideas from general algebra and algebraic geometry. We shall first describe the present setting for the problem and then describe its development and our motivation from the point of view of approximation theory.

In the first four chapters of thesis, G will denote a semigroup of commuting linear operators on a linear space S over a field k with the group operation taken as composition of linear operators. An important property for our study is the **s-dimensional additivity** of such a semigroup G. This concept can be described as follows: From the two subsets of linear operators

$$F_1 = \{\ell_1, \dots, \ell_j, \dots, \ell_s\} \quad \text{and} \quad F_2 = \{\ell_1, \dots, \tilde{\ell}_j, \dots, \ell_s\}$$

in G, a new subset F is formed by

$$F := \{\ell_1, \dots, \tilde{\ell}_j \ell_j, \dots, \ell_s\}.$$

We say that G has **s-dimensional additivity** if for arbitrary F_1, F_2 and F as above:

$$\dim K(F) = \dim K(F_1) + \dim K(F_2),$$

where, for any subset F^* of s operators from G, $K(F^*)$ is the intersection of kernel spaces

$$K(F^*) := \bigcap_{\ell \in F^*} \ker \ell, \qquad \ker \ell := \{f \in S : \ell f = 0\}. \tag{1.1}$$

The kernels of interest to us arise when an external structure is imposed on a subset of G through the structure of a finite index set. Let X be an index set with cardinality $|X| < \infty$. A matroid structure is imposed on X by a collection of "independent" subsets, $\mathcal{T}$, satisfying: i) The empty set is in $\mathcal{T}$. ii) If $V \in \mathcal{T}$, then any subset of V is in $\mathcal{T}$. iii) For arbitrary $U, V \in \mathcal{T}$ with $|U| = |V| + 1$, there exists $x \in U \backslash V$ such that $V \cup \{x\} \in \mathcal{T}$. (See, for example, the book of Welsh [W] for a detailed account of matroids.) Of course, there can be many matroid structures on a given X, but we shall simply say the matroid X to refer to X with some fixed matroid structure and only specify the structure if it has some importance. One exception is that for an arbitrary subset Y of a matroid X, we will always assume that the submatroid structure is imposed.

For any matroid there is a rank function, $\varrho : 2^X \to \mathbb{Z}_+$, defined on subsets $V \subseteq X$ by

$$\varrho(V) := \max\{|Y| : Y \subseteq V, \quad Y \in \mathcal{T}\}.$$

A maximal independent subset of X is called a base for the matroid X. Every base of X has the same cardinality, $\varrho(X)$, which is called the rank of X. We deal mainly with matroids of rank s (this number is connected with the s-dimensional additivity of G).

The collection $\mathcal{B}(X)$ of all bases for the matroid X is described as

$$\mathcal{B}(X) := \{B \subseteq X : |B| = \varrho(B) = \varrho(X)\}.$$

For a subcollection $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, we define $\mathcal{A}(X, \mathcal{B}_1^{(X)})$ to be all the subsets of X which intersect all bases in $\mathcal{B}_1^{(X)}$; i.e.,

$$\mathcal{A}(X, \mathcal{B}_1^{(X)}) := \{V \subseteq X : V \cap B \neq \emptyset \quad \forall B \in \mathcal{B}_1^{(X)}\}.$$

We are now in a position to describe the kernel spaces of interest: For a matroid X with rank $\varrho(X) = s$, and the commutative semigroup G of linear operators on S, we take L_X to be the image of some mapping $X \to G$ that associates a linear operator $\ell_x \in G$ to each $x \in X$. For any subset $V \subseteq X$, we set

$$L_V = \{\ell_x : \quad x \in V\},$$

and define

$$\ell_V = \prod_{x \in V} \ell_x,$$

to be the composition (in any order) of the operators from L_V.

The main problem is to describe the dimension of the kernel spaces

$$K(L_X, \mathcal{B}_1^{(X)}) := \{f \in S : \ell_V f = 0, \quad \forall V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\}, \tag{1.2}$$

in terms of the dimensions of the kernels $K(L_B)$ given in (1.1) for the s linear operators L_B, $B \in \mathcal{B}_1^{(X)}$.

How did such a question arise from approximation theory and why would its answer be interesting? Kernels of the type (1.2) appear in de Boor and Höllig's paper [BH], the first one dealing extensively with the properties of box splines. Without going into details, for a given set of nonzero vectors X that span $\mathbb{R}^s$ (with the natural matroid structure on X), the box spline is a compactly supported piecewise polynomial function with the polynomial pieces from $D(X)$ which is the kernel in (1.2) with $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ and the operators $\ell_x = D_x$, the directional derivative in the direction of x, $x \in X$. The dimension formula

$$\dim D(X) = |\mathcal{B}(X)| \tag{1.3}$$

was first shown by Dahmen and Micchelli [DM_2]. Its importance is derived from the fact that when the directions X are in $\mathbb{Z}^s$, the polynomials in the linear space spanned by the integer translates of the box spline are precisely the functions in $D(X)$, and this plays an essential role in both de Boor and Höllig's and Dahmen and Micchelli's studies of the algebraic and approximation properties of such spaces ([BH] & [$DM_{1,2,3}$]).

There is a natural relation between the directional derivatives D_x and the difference operators $\nabla_x : f \mapsto f - f(\cdot - x)$, $x \in \mathbb{R}^s$.

The corresponding kernel space $\nabla(X)$ given as in (1.2) with $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ and $\ell_x = \nabla_x$, and the formula ([DM_3]) for its dimension

$$\dim \nabla(X) = \sum_{B \in \mathcal{B}(X)} |\det B|, \tag{1.4}$$

were also crucial in Dahmen and Micchelli's studies of the algebraic properties of box splines.

As a natural extension of the box spline, Ron [Ro_1] introduced the exponential box splines for a set of directions X and complex numbers $c_X = \{c_x\}_{x \in X}$. The kernels $D_{c_X}(X)$ and $\nabla_{c_X}(X)$ defined as in (1.2) for the differential operators $D_{x,c_x} := D_x - c_x$, $x \in X$, and the difference operators $\nabla_{x,c_x} : f \mapsto f - \exp(c_x) f(\cdot - x)$, $x \in X$, played the same type of essential role in the studies of exponential box splines by Dahmen and Micchelli [$DM_{4,5}$], Ron [R], and Ben-Artzi and Ron [BeR]. In particular, the dimension formulas for $D_{c_X}(X)$ and $\nabla_{c_X}(X)$ are the same as given in (1.3) and (1.4) respectively. This (nontrivial) extension precipitated two separate but related lines of study into the deeper algebraic ideas behind the box spline theory.

Dahmen and Micchelli ([DM_5]) realized in their investigations of exponential box splines that the problem could be formulated for $\mathcal{B}(X)$ in terms of matroids and linear operators. Their results were announced previously in [DM_4]. When $|X| = s$, the sums in (1.3) and (1.4) reduce to one summand and the definitions (1.1) and (1.2) agree. Therefore, both (1.3) and (1.4) are expressible as

$$\dim K(L_X, \mathcal{B}(X)) = \sum_{B \in \mathcal{B}(X)} \dim K(L_B). \tag{1.5}$$

Dahmen and Micchelli proved the following theorm in [DM_5].

(1.6)Theorem. *[DM$_5$] For any matriod X with $\varrho(X) = s$, and any $L_X \subset G$ associated with X.*

$$\dim K(L_X, \mathcal{B}(X)) \leq \sum_{B \in \mathcal{B}(X)} \dim K(L_B). \tag{1.7}$$

Furthermore, their studies also led to an important theorem which gave a sufficient condition [DM$_5$, Theorem 3.3] for equality based on the solvability of certain systems of operator equations [DM$_5$, Theorem 3.2].

(1.8)Theorem. *[DM$_5$] Suppose that L_X is associated with a matriod X of rank s and $\dim K(L_B) < \infty$. If the system of equations*

$$\ell_y f = \phi_y \qquad y \in Y,$$

has, for each $Y \in \mathcal{B}(_r(X))$, the independent subsets of cardinality r in X, $1 \leq r \leq s$, a solution provided that the compatibity conditions

$$\ell_{y'} \phi_y = \ell_y \phi_{y'}$$

hold for each $y, y \in Y$, then equality holds in Theorem (1.6).

In another paper [DM$_6$], Dahmen and Micchelli investigated the dimensions of certain spline spaces and the relationship of these questions to syzygies and the kernels of systems of differential equations. In particular, they proved that if the linear operators in G are the partial differential operators given by homogeneous polynomials on $\mathbb{R}^s$, then (1.5) holds for $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$ in the cases $s = 2$ and for general s but with restricted X (a fixed basis of $\mathbb{R}^s$ with each of its elements having arbitrary multipicity). They conjectured that it would always hold and this conjecture initiated our the study in the first part of this thesis.

Along other lines, de Boor, Dyn, and Ron ([DR$_{1,2}$], [BR$_{1,2,3}$], and [BDR]) were taking advantage of polynomial ideals, their varieties and codimensions to gain new insights and results for various problems in multivariate interpolation, approximation and spline theory. They also encountered dimension problems of the type considered here, usually in the context of partial differential operators given by polynomials, but sometimes free from any matroid structure. For example, in the case when G consists of differential operators given by affine polynomials, de Boor and Ron [BR$_2$, Theorem 6.6] give the lower bound

$$\dim K(L_X, \mathcal{B}_1^{(X)}) \geq |\mathcal{B}_1^{(X)}|$$

for arbitrary $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, in contrast to the upper bound in (1.7). They also prove that equality holds for these special operators if $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$ [BR$_2$, Theorem 6.9].

(1.9)Theorem. *[BR] For* $L_X = \{p_{x,\mu_x}(D) = D_x + \mu_x : x \in X\}$

$$\dim K(L_X, \mathcal{B}_1^{(X)}) \geq |\mathcal{B}_1^{(X)}|.$$

Morever, if $\mathcal{B}_1^{(X)}$ *is an order closed subset of* $\mathcal{B}(X)$*, then the equality holds.*

We have used these two approaches in our research. As Dahmen and Micchelli pointed out, the dimension of kernels of linear operators is related to the solvability of the certain systems of linear operator equations. In Chapter 2, we closely investigate solvability problems of systems of linear operator equations and extend Theorem (1.9) to the case of general linear operators from some semigroup G. It turns out that s-dimensional additivity is the key to this problem. In Chapter 3 and 4, we use polynomial ideals, their varieties and codimensions together with the results obtained in Chapter 2 to discuss the s-dimensional additivity and corresponding dimension problems for partial differential and difference operators given by polynomials of s varibles. In particular, we confirm the conjecture in Chapter 3. We also establish some solvability criteria for systems of differential and difference equations. The research began with [Sh] where the Dahmen and Micchelli conjecture, Theorem (3.7), was solved as a consequence of Theorem (2.16) and a version of Theorem (2.10). In joint research with Professors Jia and Riemenschneider [JRS], these ideas were extended to the order closed case and the s-dimensional additivity of constant coefficient partial differential and difference operators were established. This research led naturally to the consideration of solvability of systems of linear operator equations and its connection to s-dimensional additivity (see [SJR]). The results in chapter 2-4 are based on the papers [Sh], [JRS], and [SJR].

The second topic of this thesis is wavelet decompositions. This problem has been studied by Meyer and Mallat for the univariate case (see [Me] [M]). Daubechies [D] gives a construction of compactly supported wavelets, along with a comprehensive overview of the subject. Dahmen and Micchelli [DM_7] using stationary subdivision techniques provided some improvement of [D] and gave an alternative derivation of Daubechies' theorem. The cardinal spline approach to wavelet decompositions for the univariate case was considered by Chui and Wang in [CW]. In Chapter 5, we will give a constructive way to obtain orthogonal decompositions of multiresolution approximation generated by box splines in dimensions $s = 1, 2, 3$. This construction was part of a joint work with Professor Riemenschneider (see [RS]).

CHAPTER 2

SOLVABILITY OF SYSTEMS LINEAR OPERATOR EQUATIONS AND DIMENSION FORMULAE

We wish to determine some conditions on $\mathcal{B}_1^{(X)}$ under which the dimension formula (1.5) holds. To this purpose, we first discuss solvability of systems of linear operator equations.

Let G be a semigroup of commuting linear operators on a linear space S with the group operation of composition. The solvability of the system of equations

$$\ell_i f = \phi_i, \qquad i = 1, \ldots, r, \tag{2.1}$$

where $\ell_i \in G$ and $\phi_i \in S$, was considered by Dahmen and Micchelli in their studies of the dimension of the kernel space of certain linear operators (see [DM$_5$]). It is clear that the compatibility conditions

$$\ell_j \phi_i = \ell_i \phi_j \quad (i \neq j) \tag{2.2}$$

are necessary for the system (2.1) to have a solution in S. However, in general, the compatibility conditions do not provide sufficient conditions for the system (2.1) to be solvable in S. In this Chapter, we shall discuss what kind of conditions on operators will make the conditions (2.2) sufficient for the system (2.1) to be solvable in S.

First, we observe the easy case in which one of the operators is invertible.

(2.3)Theorem. *Let G be a commutative semigroup of linear operators on S, and let $\ell_1, \ldots, \ell_r$ be elements of G. Assume that one of them, say ℓ_1, is invertible on S. Then for given $\phi_1, \ldots, \phi_r$ in S, the system of equations*

$$\ell_j f = \phi_j, \quad j = 1, \ldots, r,$$

has a solution in S if and only if the compatibility conditions

$$\ell_j \phi_k = \ell_k \phi_j, \quad 1 \le j < k \le r,$$

hold.

Proof. We claim that ℓ_1^{-1} commutes with each ℓ_j, $j = 1, \ldots, r$. Indeed, it follows from $\ell_1 \ell_j = \ell_j \ell_1$ that

$$\ell_1^{-1} \ell_j = \ell_1^{-1}(\ell_j \ell_1)\ell_1^{-1} = \ell_1^{-1}(\ell_1 \ell_j)\ell_1^{-1} = \ell_j \ell_1^{-1}.$$

Let $f = \ell_1^{-1}\phi_1$. Then f is a solution to the system. ♠

Let M be a subspace of S. An element $\ell \in G$ is called **nilpotent** on M if for any $\phi \in M$, there exists a positive integer m such that $\ell^m \phi = 0$ (m may depend on ϕ). We say that M is **compatible** with G if the following two conditions are satisfied:

(i) M is invariant under G, i.e., for any $\ell \in G$, $\ell(M) \subseteq M$;

(ii) For any $\ell \in G$, $\ell|_M$ is either invertible or nilpotent.

(2.4)Theorem. *Let G be a commutative semigroup of linear operators on S which possesses s-dimensional additivity. Suppose that S is a direct sum of two subspaces M and N which are invariant under G. Moreover, assume that $\ell_1, \ldots, \ell_s \in G$ are nilpotent on M and have the property* $\dim(\ker(\ell_1, \ldots, \ell_s)) < \infty$. *Let $r \in \{1, \ldots, s\}$. Then for given $\phi_1, \ldots, \phi_r \in M$, the system of operator equations*

$$\ell_j f = \phi_j, \qquad j = 1, \ldots, r, \tag{2.5}$$

has a solution in M if and only if the compatibility conditions

$$\ell_j \phi_k = \ell_k \phi_j, \quad 1 \le j < k \le r, \tag{2.6}$$

hold.

Proof. Obviously, the compatibility conditions (2.6) are necessary for the system (2.5) to have a solution.

For the sufficiency part, we first consider the special case where $r = s$ and $\phi_2 = \cdots = \phi_s = 0$. Since $\ell_1|_M$ is nilpotent, there exists a positive integer m such that $\ell_1^m \phi_1 = 0$. Let

$$F := \ker(\ell_1^{m+1}, \ell_2, \ldots, \ell_s),$$

$$F' := \ker(\ell_1^m, \ell_2, \ldots, \ell_s),$$

$$F'' := \ker(\ell_1, \ell_2, \ldots, \ell_s).$$

By the compatibility conditions (2.6), $\ell_j \phi_1 = 0$ for $j = 2, \ldots, s$. Hence $\phi_1 \in F'$. Observe that ℓ_1 is a linear mapping from F to F' with F'' being its kernel. Hence

$$\dim(F) = \dim(F'') + \dim(\ell_1(F)).$$

On the other hand, since G possesses s-dimensional additivity, we have

$$\dim(F) = \dim(F'') + \dim(F').$$

Comparing these two equations gives $\ell_1(F) = F'$. Since $\phi_1 \in F'$, it follows that there exists $f \in F$ such that $\ell_1 f = \phi_1$. This f also satisfies $\ell_j f = 0$, $j = 2, \ldots, s$. Since S is a direct sum of M and N, f has a unique decomposition $f = f_1 + f_2$, where $f_1 \in M$ and $f_2 \in N$. But both M and N are invariant under G, hence the element f_1 satisfies $\ell_1 f_1 = \phi_1$ and $\ell_j f_1 = 0$, $j = 2, \ldots, s$.

The general case will be proved by induction on r. For the case $r = 1$, the compatibility conditions (2.6) are satisfied since there is only one equation. The solvability of equation $\ell_1 = \phi_1$ can be proved in the same way as the following induction step. Suppose that the theorem holds for $r - 1$. Given $\phi_1, \ldots, \phi_r \in M$, by the induction hypothesis, we can find an $f_1 \in M$ such that

$$\ell_i f_1 = \phi_i, \quad i = 1, \ldots, r - 1.$$

Choose f_1 to be 0 in the case $r = 1$. Let $g = \ell_r f_1$. Then $g \in M$ and

$$\ell_i g = \ell_i(\ell_r f_1) = \ell_r(\ell_i f_1) = \ell_r \phi_i, \quad i = 1, \ldots, r-1.$$

This together with the compatibility conditions implies

$$\ell_i(\phi_r - g) = \ell_i \phi_r - \ell_r \phi_i = 0, \quad i = 1, \ldots, r-1. \tag{2.7}$$

Moreover, since $\ell_j|_M$ $(j = r+1, \ldots, s)$ are nilpotent, there exists a positive integer m such that

$$\ell_j^m(\phi_r - g) = 0, \quad j = r+1, \ldots, s. \tag{2.8}$$

Consider the following system of operator equations for h:

$$\begin{aligned} &\ell_i h = 0, \quad i = 1, \ldots, r-1, \\ &\ell_r h = \phi_r - g \\ &\ell_j^m h = 0, \quad j = r+1, \ldots, s. \end{aligned} \tag{2.9}$$

By (2.7) and (2.8), the compatibility conditions corresponding to the system (2.9) are satisfied. This is just the special case we discussed before. Therefore the system (2.9) has a solution h in M. Let $f = f_1 + h$. Then

$$\ell_i f = \ell_i f_1 + \ell_i h = \phi_i, \quad i = 1, \ldots, r-1,$$

and

$$\ell_r f = \ell_r f_1 + \ell_r h = g + (\phi_r - g) = \phi_r.$$

This shows that f is a solution to the system (2.5). ♠

To carry our investigations further, we need the following concepts. For a matroid X, we say that a set of linear operators $L_X \subseteq G$ is **well associated** with $(X, \mathcal{B}_1^{(X)})$, if $\dim K(L_B) < \infty$ for any $B \in \mathcal{B}_1^{(X)}$. If, for any $L_X \subseteq G$, the formula

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} \dim K(L_B)$$

holds, then G is said to be **excellently associated** with $(X, \mathcal{B}_1^{(X)})$.

Let us recall that $\mathcal{B}(X)$ is the collection of all bases for the matroid X. For a subcollection $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$, we define $\mathcal{A}(X, \mathcal{B}_1^{(X)})$ to be all the minimal subsets of X which intersect all bases in $\mathcal{B}_1^{(X)}$; i.e.,

$$\mathcal{A}(X, \mathcal{B}_1^{(X)}) := \{V \subseteq X : V \cap B \neq \emptyset \quad \forall B \in \mathcal{B}_1^{(X)} \quad \text{and} \quad \forall y \in V \quad \exists \quad B \in \mathcal{B}_1^{(X)} \quad \text{such that } (V \backslash \{y\}) \cap B = \emptyset\}.$$

We also set

$$\mathcal{M}(X, \mathcal{B}_1^{(X)}) := \{X \backslash V : V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\}.$$

Equivalently, $\mathcal{M}(X, \mathcal{B}_1^{(X)})$ is the set of all maximal subsets of X such that $X \backslash M$ intersect all $B \in \mathcal{B}_1^{(X)}$, or, the maximal subsets of X that do not contain any elements of $\mathcal{B}_1^{(X)}$. When $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$, $\mathcal{M}(X, \mathcal{B}_1^{(X)}) =: \mathcal{H}(X)$ is the set of "hyperplanes" for the matroid structure.

Note that we have changed the definition of $\mathcal{A}(X, \mathcal{B}_1^{(X)})$, but this does not affect the definition of the space $K(L_X, \mathcal{B}_1^{(X)})$.

We give the solvability of a special system of linear operator equations in the next theorem. The usefulness of the solvability of the system in investigation of the dimension of the kernel spaces was recognized by Dahmen and Micchelli in [DM_5] (see (1.8) Theorem).

(2.10)Theorem. *Let $X \cup \zeta$ be a matroid with rank $\varrho(X \cup \zeta) = s$ and let G be a semi-group of linear operators with s-dimensional additivity and $\mathcal{B}_1^{(X \cup \zeta)} \subseteq \mathcal{B}(X \cup \zeta)$. Suppose that $L_{X \cup \zeta} \subseteq G$ and $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$ are given and satisfy the following conditions:*

i) L_X and $L_{M \cup \zeta}$ are well associated to $(X, \mathcal{B}_1^{(X)})$ and $(M \cup \zeta, \mathcal{B}_1^{(M \cup \zeta)})$ respectively;

ii) G is excellently associated to $(M \cup \zeta, \mathcal{B}_1^{(M \cup \zeta)})$;

iii) For any $B \in \mathcal{B}_1^{(M\cup\zeta)}$ *and* $y \in X\backslash M$, $(B\backslash\{\zeta\}) \cup \{y\} \in \mathcal{B}_1^{(X)}$.

Then the system

$$
(2.11) \qquad \begin{aligned} \ell_{X\backslash M} f &= \varphi \\ \ell_V f &= 0 \qquad \forall V \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\backslash\{X\backslash M\} \end{aligned}
$$

is solvable for any $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$.

Proof. To each $x \in M \cup \zeta$ we associate two linear operators $\widetilde{\ell}_x$ and $\bar{\ell}_x$ as follows:

$$
\widetilde{\ell}_x = \begin{cases} \ell_\zeta \ell_{X\backslash M}, & \text{if } x = \zeta \\ \ell_x, & \text{otherwise,} \end{cases}
$$

and

$$
\bar{\ell}_x = \begin{cases} \ell_{X\backslash M}, & \text{if } x = \zeta \\ \ell_x, & \text{otherwise.} \end{cases}
$$

Then we set

$$
\widetilde{L}_{M\cup\zeta} := \{\, \widetilde{\ell}_x : \quad x \in M \cup \zeta \,\}
$$

and

$$
\bar{L}_{M\cup\zeta} := \{\, \bar{\ell}_x : \quad x \in M \cup \zeta \,\}.
$$

Both $\widetilde{L}_{M\cup\zeta}$ and $\bar{L}_{M\cup\zeta}$ are well associated with $(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$. Thus, for any $V \subseteq M \cup \zeta$, we have

$$
\widetilde{\ell}_V = \begin{cases} \ell_V, & \text{if } \zeta \notin V; \\ \ell_{X\backslash M} \ell_V, & \text{if } \zeta \in V. \end{cases}
$$

Hence, $\ell_{X\backslash M}$ maps $K(\widetilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ into $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ with $K(\bar{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ as its kernel. Moreover, since G has s-dimensional additivity and is excellently associated to $M \cup \zeta$, we have

$$
\begin{aligned}
\dim K(\widetilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) &= \sum_{B \in \mathcal{B}_1^{(M\cup\zeta)}} \dim K(\widetilde{L}_B) \\
&= \sum_{B \in \mathcal{B}_1^{(M\cup\zeta)}} \dim K(\bar{L}_B) + \sum_{B \in \mathcal{B}_1^{(M\cup\zeta)}} \dim K(L_B) \\
&= \dim K(\bar{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) + \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}).
\end{aligned}
$$

Hence $\ell_{X\backslash M}$ is surjective, since the dimension of $K(\widetilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ is finite.

Now that the image of the mapping $\ell_{X\backslash M}$ is $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$, for any given $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$, we can find a function $f \in K(\widetilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ such that

$$\ell_{X\backslash M} f = \varphi.$$

We claim that this f also satisfies

$$\ell_W f = 0 \qquad \forall W \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\backslash\{X\backslash M\}.$$

Let $W \in \mathcal{A}(X, \mathcal{B}_1^{(X)})\backslash\{X\backslash M\}$. Then there is $y \in X\backslash M$, such that $y \notin W$. Otherwise, $W = X\backslash M$ by the minimal property of W. We want to show that W intersects any base in $\mathcal{B}_1^{(M\cup\zeta)}$. For this purpose, we pick $B \in \mathcal{B}_1^{(M\cup\zeta)}$. Then $\zeta \in B$, and $\widetilde{B} := (B\backslash\{\zeta\}) \cup \{y\} \in \mathcal{B}_1^{(X)}$ by iii). Therefore,

$$W \cap B = W \cap (B\backslash\{\zeta\}) = W \cap \widetilde{B} \neq \emptyset.$$

This shows that W intersects any base in $\mathcal{B}_1^{(M\cup\zeta)}$. Therefore, by the very definition of $\mathcal{A}(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$, W must contain some $V \in \mathcal{A}(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$. Since $\zeta \notin W$, we have $\zeta \notin V$, hence $\ell_V = \widetilde{\ell}_V$. Now, $f \in K(\widetilde{L}_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$ and $V \in \mathcal{A}(M \cup \zeta, \mathcal{B}_1^{(M\cup\zeta)})$ imply

$$\ell_V f = \widetilde{\ell}_V f = 0.$$

It follows that

$$\ell_W f = \ell_{(\ldots)} \ell_V f = 0.$$

Therefore, for the given φ, the system (2.11) is solvable. ♠

Next, we use the last theorem to prove the main result about the dimension of kernel spaces of linear operators. The conditions imposed on $\mathcal{B}_1^{(X)}$ under which the dimension formula (1.5) will hold is that $\mathcal{B}_1^{(X)}$ be an order closed subset of

$\mathcal{B}(X)$. This concept was first introduced by de Boor and Ron in [BR$_2$]. In fact, they proved the dimension formula (1.5) for the case that the linear operators are the differential operators induced by affine polynomials.

Suppose that a total order on X is given. This order induces a partial order on $\mathcal{B}(X)$;

$$B = (x_1, \ldots, x_s) \leq \widetilde{B} = (\widetilde{x}_1, \ldots, \widetilde{x}_s) \quad \Longleftrightarrow \quad x_j \leq \widetilde{x}_j, \quad j = 1, \ldots, s,$$

where the elements of each sequence are arranged in an increasing order. We say that $\mathcal{B}_1^{(X)} \subseteq \mathcal{B}(X)$ is an **order closed subset** of $\mathcal{B}(X)$, if

$$B_1 \in \mathcal{B}_1^{(X)}, \quad B_2 \in \mathcal{B}(X), \quad \text{and} \quad B_2 \leq B_1 \implies B_2 \in \mathcal{B}_1^{(X)}.$$

(2.12)Theorem. *Let X be an arbitrary matroid with rank $\varrho(X) = s$ and $\mathcal{B}_1^{(X)}$ be an arbitrary order closed subset of $\mathcal{B}(X)$. If G is a semigroup of linear operators on S, then, for arbitrary $L_X \subseteq G$,*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} \dim K(L_B),$$

if and only if G has the s-dimensonally additive property.

Proof. "$\Longrightarrow$" If L_X is not well associated with $(X, \mathcal{B}_1^{(X)})$, then the equality holds simply because there exists a $B \in \mathcal{B}_1^{(X)}$ such that $\dim K(L_B) = \infty$ and $K(L_B) \subseteq K(L_X, \mathcal{B}_1^{(X)})$.

For the case that L_X is well associated with $(X, \mathcal{B}_1^{(X)})$, we will prove the equality by induction on $|X|$. When $\varrho(X) = |X| = s$, the theorem is obviously true. Suppose now that the theorem holds for all X with $s \leq |X| \leq n$ and $\varrho(X) = s$. We want to establish it for $(Y, \mathcal{B}_1^{(Y)})$ with $|Y| = n + 1$, where $\mathcal{B}_1^{(Y)}$ is an order closed subset of $\mathcal{B}(Y)$ and $\varrho(Y) = s$. Let

$$\zeta := \sup\{\, y \in Y : \varrho(Y \backslash \{y\}) = s \,\}.$$

We write X for $Y\backslash\zeta$. Then $Y = X \cup \zeta$.

Let $\mathcal{P}$ be the mapping given by

$$\mathcal{P}f = (\ell_{X\backslash M} f)_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})}, \qquad f \in K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}).$$

Since the mapping $\ell_{X\backslash M}$ maps $K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)})$ to $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$, the mapping $\mathcal{P}$ maps $K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)})$ to the Cartesian product

$$\prod_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}).$$

Note that $K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) = 0$ if $\mathcal{B}_1^{(M\cup\zeta)} = \emptyset$.

Observe that $\ker(\mathcal{P}) = K(L_X, \mathcal{B}_1^{(X)})$. Therefore,

(2.13)

$$\dim K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}) \le \dim K(L_X, \mathcal{B}_1^{(X)}) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}).$$

Equality will hold in (2.13) if the mapping $\mathcal{P}$ is surjective. Before showing this, we pick out the nontrivial components in the image of $\mathcal{P}$: We claim

$$M \in \mathcal{M}(X, \mathcal{B}_1^{(X)}) \text{ and } \mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset \quad \Longrightarrow \quad M \in \mathcal{H}(X).$$

For this purpose, we shall show $\rho(M) < s$. Suppose to the contrary that $\rho(M) = s$. Since $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$, there exists a base $B \in \mathcal{B}_1^{(M\cup\zeta)}$; it follows that $\zeta \in B$ and $\rho(B\backslash\zeta) = s - 1$. But $\rho(M) = s$; hence there exists some $y \in M$ such that $\rho((B\backslash\zeta) \cup y) = s$. By the very definition of ζ, we have $y < \zeta$. Therefore, $B_1 := (B\backslash\zeta) \cup y \in \mathcal{B}_1^{(X\cup\zeta)}$, because $\mathcal{B}_1^{(X\cup\zeta)}$ is order closed. Thus, M would contain a base B_1 in $\mathcal{B}_1^{(X)}$. This contradicts the choice of M. Hence, $\varrho(M) < s$, so $M \subseteq H$ for some $H \in \mathcal{H}(X)$. But $(X\backslash H) \cap B \neq \emptyset$, for any $B \in \mathcal{B}(X)$; it follows that $H = M$

by the maximality of M. In particular, this implies that the following union is a disjoint union:

$$\mathcal{B}_1^{(X\cup\zeta)} = \mathcal{B}_1^{(X)} \cup \left(\cup_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \mathcal{B}_1^{(M\cup\zeta)} \right).$$

This fact will be used below.

The mapping $\mathcal{P}$ is surjective if and only if for each $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$ with $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$, the system (2.11) is solvable for any $\varphi \in K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)})$. Thus, it suffices to prove that Theorem 2.10 can be applied here. Condition i) of Theorem 2.10 holds since we have already restricted ourselves to the well associated case. Condition ii) holds by the induction hypothesis since $|M \cup \zeta| \leq |X|$. Condition iii) holds because of the following reason. Suppose that $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$. Let $B \in \mathcal{B}_1^{(M\cup\zeta)}$ and $y \in X\backslash M$. Note that $\mathcal{B}_1^{(M\cup\zeta)} \neq \emptyset$ implies $\varrho(M \cup \zeta) = s$. Thus, if $y \in X\backslash M$, then $\rho(X \cup \zeta\backslash y) \geq \rho(M \cup \zeta) = s$. Hence, by the choice of ζ, we have $y < \zeta$. Since $\mathcal{B}_1^{(X\cup\zeta)}$ is order closed, for any $B \in \mathcal{B}_1^{(M\cup\zeta)}$ and $y \in X\backslash M$, we have $\widetilde{B} := (B\backslash\zeta)\cup y \in \mathcal{B}_1^{(X)}$. Therefore, Theorem 2.10 can be applied and equality holds in (2.13):

(2.14)

$$\dim K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}) = \dim K(L_X, \mathcal{B}_1^{(X)}) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}).$$

Finally, applying the induction hypothesis to X and to each $M \cup \zeta$, $M \in \mathcal{M}(X, \mathcal{B}_1^{(X)})$, we obtain

(2.15)

$$\begin{aligned}
\dim K(L_X, \mathcal{B}_1^{(X)}) &+ \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \dim K(L_{M\cup\zeta}, \mathcal{B}_1^{(M\cup\zeta)}) \\
&= \sum_{B\in\mathcal{B}_1^{(X)}} \dim K(L_B) + \sum_{M\in\mathcal{M}(X,\mathcal{B}_1^{(X)})} \sum_{B\in\mathcal{B}_1^{(M\cup\zeta)}} \dim K(L_B) \\
&= \sum_{B\in\mathcal{B}_1^{(X\cup\zeta)}} \dim K(L_B).
\end{aligned}$$

Substituting this into (2.14) gives the desired result:

$$\dim K(L_{X\cup\zeta}, \mathcal{B}_1^{(X\cup\zeta)}) = \sum_{B\in\mathcal{B}_1^{(X\cup\zeta)}} \dim K(L_B).$$

"$\Longleftarrow$" Suppose F_1, F_2 and F are the sets of linear operators given as above. Let $X = \{e_1, \ldots, e_j, \ldots, e_s, \tilde{e}_j\}$ where $e_1, \ldots, e_s$ are linearly independent vectors in $\mathbb{R}^s$, and $\tilde{e}_j = e_j$. Then X has a natural matroid structure with $\varrho(X) = s$. Moreover, $\mathcal{B}(X) = \{B_1, B_2\}$ with,

$$B_1 = \{e_1, \ldots, e_j, \ldots, e_s\}, \quad B_2 = \{e_1, \ldots, \tilde{e}_j, \ldots, e_s\}$$

The set of linear operators $L_X := \{\ell_1, \ldots, \ell_j, \ldots, \ell_s, \tilde{\ell}_j\}$ is associated to X by the correspondence $e_j \leftrightarrow \ell_j$, $j = 1, \ldots, s$ and $\tilde{e}_j \leftrightarrow \tilde{\ell}_j$. We observe that $K(F) = K(L_X)$. Therefore, since G is excellently associated with X, we have

$$\begin{aligned}\dim K(F) = \dim K(L_X) &= \dim K(L_{B_1}) + \dim K(L_{B_2}) \\ &= \dim K(F_1) + \dim K(F_2).\end{aligned}$$

Thus, G is s-dimensionally additive. ♠

For the special case that $\mathcal{B}_1^{(X)} = \mathcal{B}(X)$, we have following theorem.

(2.16)Theorem. *The semigroup G is s-dimensionally additive if and only if G is excellently associated with any matroid X of rank $\varrho(X) = s$, i.e. for any $L_X \subset G$ associated with X,*

$$\dim K(L_X) = \sum_{B\in\mathcal{B}(X)} \dim K(L_B).$$

CHAPTER 3

APPLICATION TO CONSTANT COEFFICIENT PARTIAL DIFFERENTIAL OPERATORS

The first part of this chapter consists of background material from algebraic geometry. It is meant to give a concise account of the material that is required for our purposes with appropriate references for the details.

Let k be an algebraically closed field, and k^s the s-dimensional affine space over k. Denote the ring of polynomials in s indeterminates over the field k by $k[Z] = k[Z_1, \ldots, Z_s]$. The ideal generated by $p_1, \ldots, p_m \in k[Z]$ will be denoted by $(p_1, \ldots, p_m)$. The codimension of an ideal I, denoted by $\operatorname{codim}(I)$, is the dimension of the quotient linear space $k[Z]/I$ over k.

For a multi-index $\alpha \in \mathbb{N}^s$, the formal differential operator D^α on $k[Z]$ is defined by

$$D^\alpha Z^\beta := \frac{\beta!}{(\beta-\alpha)!} Z^{\beta-\alpha}.$$

Here we make the convention that $Z^{\beta-\alpha} = 0$, if $\alpha \not\leq \beta$. For a polynomial $p(Z) = \sum_\alpha a_\alpha Z^\alpha$, the corresponding partial differential operator $p(D)$ is defined by $p(D) := \sum_\alpha a_\alpha D^\alpha$.

An ideal I of $k[Z]$ determines its variety

$$\operatorname{Var}(I) := \{ a \in k^s : p(a) = 0 \quad \forall p \in I \}.$$

Such a variety is a finite irredundant union of irreducible varieties (e.g. see [K, p.122]).

Given an algebraic variety V, we denote by $I(V)$ the ideal of all polynomials which vanish on V. The ring

$$k[V] := k[Z]/I(V)$$

is called the coordinate ring of V. If V is irreducible, then $k[V]$ is an integral domain. In this case the quotient field of $k[V]$ is called the field of rational functions on V, and is written $k(V)$.

Suppose that $\theta = (\theta_1, \ldots, \theta_s)$ is an isolated zero of I; that is, $\{\theta\}$ is one of the irreducible components of $\mathrm{Var}\,(I)$. Let

$$I = \bigcap_{j=1}^{n} I_j$$

be a reduced primary decomposition, where I_j is P_j-primary for $j = 1, \ldots, n$, and the prime ideals $P_1, \ldots, P_n$ are all different. To the component $\{\theta\}$ of $\mathrm{Var}\,(I)$ there corresponds one prime ideal, say P_1, such that $P_1 = (Z_1 - \theta_1, \ldots, Z_s - \theta_s)$. Then we have

$$\theta \notin \mathrm{Var}\,(P_j) = \mathrm{Var}\,(I_j), \qquad j = 2, \ldots, n,$$

for otherwise $\{\theta\}$ would not be a component of $\mathrm{Var}\,(I)$. In what follows we write I_θ for I_1.

The set

$$S_\theta := \{\, g \in k[Z] : g(\theta) \neq 0 \,\}$$

is a multiplicative set of $k[Z] =: R$. Let $\mathcal{O}_\theta := S_\theta^{-1} R$ be the quotient ring of R by S_θ (the **localization** of R at S_θ); i.e.,

$$\mathcal{O}_\theta = \{f/g : f, g \in k[Z], \quad g(\theta) \neq 0\}.$$

Thus, $\mathcal{O}_\theta$ is the local ring of the point θ (e.g. see [Sf, Chapter 2]). If I is an ideal of R, then $S_\theta^{-1} I$ is an ideal of $\mathcal{O}_\theta$.

(3.1)Lemma. $S_\theta^{-1}I = S_\theta^{-1}I_\theta$.

Proof. Since $\theta \notin \mathrm{Var}(I_j)$ for $j > 1$, we can find a polynomial $f_j \in k[Z]$ such that f_j vanishes on $\mathrm{Var}(I_j)$ but $f_j(\theta) \neq 0$. By Hilbert's Nullstellensatz, $f_j^\ell \in I_j$ for some positive integer ℓ. Thus, we have

$$f_j^\ell \in S_\theta \cap I_j.$$

It follows that

$$1 \in S_\theta^{-1}I_j, \qquad j = 2, \ldots, n.$$

This gives

$$S_\theta^{-1}I = \cap_{j=1}^{n}(S_\theta^{-1}I_j) = S_\theta^{-1}I_\theta,$$

as desired. ♠

Let I be an ideal of $k[Z_1, \ldots, Z_s]$. If θ is an isolated zero of I, we define the **intersection index** of I at θ as follows:

$$\mathrm{ind}_\theta(I) := \dim(\mathcal{O}_\theta/(S_\theta^{-1}I)).$$

If θ is an isolated zero of I, then we also can talk about the multiplicity of I at θ. Let

$$P_{I,\theta} := \{p \in k[Z] : p(D)f(\theta) = 0, \quad \forall f \in I\}.$$

The space $P_{I,\theta}$ is finite dimensional and is called the multiplicity space of I at θ. The dimension of $P_{I,\theta}$ is called the **multiplicity** of I at θ. It is known that (e.g. see [BR$_2$, (3.12)Proposition])

$$I_\theta = \{\, f \in k[Z] : p(D)f(\theta) = 0, \quad \forall p \in P_{I,\theta}\,\}.$$

The following theorem shows that the multiplicity of I at θ is just the intersection index of I at θ.

(3.2)Theorem. *Let I be an ideal of $k[Z_1, \ldots, Z_s]$. If θ is an isolated zero of I, then*

$$\dim(\mathcal{O}_\theta/(S_\theta^{-1}I)) = \dim P_{I,\theta}.$$

Proof. For $g \in \mathcal{O}_\theta$, the residue class of g in $\mathcal{O}_\theta/(S_\theta^{-1}I)$ will be denoted by $\bar{g}$. Consider the following bilinear function for the pair (p, g) where $p \in P_{I,\theta}$ and $g \in \mathcal{O}_\theta/(S_\theta^{-1}I)$:

$$\langle p, \bar{g} \rangle := p(D)g(\theta).$$

This is well defined on $\mathcal{O}_\theta/(S_\theta^{-1}I)$, because for any $p \in P_{I,\theta}$ and $f \in S_\theta^{-1}I$, $p(D)f(\theta) = 0$. Furthermore, the bilinear function has actually a scalar product type property. To this end, suppose that $\langle p, \bar{g} \rangle = 0$ for all $\bar{g} \in \mathcal{O}_\theta/(S_\theta^{-1}I)$. Then for any $f \in k[Z]$,

$$p(D)f(\theta) = \langle p, \bar{f} \rangle = 0.$$

It follows that $p = 0$.

On the other hand, suppose that $\langle p, \bar{g} \rangle = 0$ for all $p \in P_{I,\theta}$. Let $g = f/h$, with $f, h \in k[Z]$ and $h(\theta) \neq 0$. Since $P_{I,\theta}$ is D–invariant, by Leibnitz' formula, we have

$$p(D)f(\theta) = p(D)(gh)(\theta) = 0, \quad \forall p \in P_{I,\theta}.$$

Hence, $f \in P_{I,\theta}$. This gives

$$g = \frac{f}{h} \in S_\theta^{-1}I_\theta = S_\theta^{-1}I;$$

i.e., $\bar{g} = 0$.

Define $T : P_{I,\theta} \to \mathcal{O}_\theta/(S_\theta^{-1}I)$ by

$$T(p) := \sum_{i=1}^{n} \langle p, \bar{g}_i \rangle \bar{g}_i$$

for a given fixed basis $\bar{g}_1, \ldots, \bar{g}_n$ of $\mathcal{O}_\theta/(S_\theta^{-1}I)$. Then, T is an a $1-1$ map between $P_{I,\theta}$ and $\mathcal{O}_\theta/(S_\theta^{-1}I)$.

Similarly, we define $T' : \mathcal{O}_\theta/(S_\theta^{-1}I) \to P_{I,\theta}$ by

$$T'(\bar{g}) := \sum_{i=1}^{m} \langle p_i, \bar{g} \rangle p_i$$

for a fixed basis $p_1 \dots p_m$. The map T' is a $1-1$ map between $\mathcal{O}_\theta/(S_\theta^{-1}I)$ and $P_{I,\theta}$. Therefore, these two spaces must have the same dimension. ♠

The following additivity theorem plays an essential role in our study of kernels of differential and difference operators.

(3.3)Theorem (Additivity). *If θ is an isolated common zero of $f_1, \dots, f_s$, and if f_s is the product of two polynomials, $f_s = f_s' f_s''$, then*

$$\mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s) = \mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s') + \mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s'').$$

Proof. This theorem can be proved as follows (see [Sf, Chapter IV §1.3]). We denote the ring $\mathcal{O}_\theta/(S_\theta^{-1}(f_1, \dots, f_{s-1}))$ by $\bar{\mathcal{O}}$, and the images of f_s' and f_s'' in $\bar{\mathcal{O}}$ under the canonical homomorphism by f' and f''. Then

$$\mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s) = \dim(\bar{\mathcal{O}}/(f'f'')),$$

$$\mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s') = \dim(\bar{\mathcal{O}}/(f')),$$

$$\mathrm{ind}_\theta(f_1, \dots, f_{s-1}, f_s'') = \dim(\bar{\mathcal{O}}/(f'')).$$

Since the sequence

$$0 \longrightarrow (f'')/(f'f'') \longrightarrow \bar{\mathcal{O}}/(f'f'') \longrightarrow \bar{\mathcal{O}}/(f'') \longrightarrow 0$$

is exact, we have

$$\dim(\bar{\mathcal{O}}/(f'f'')) = \dim(\bar{\mathcal{O}}/(f'')) + \dim((f'')/(f'f'')).$$

It can be shown that f'' is not a zero divisor in $\bar{\mathcal{O}}$ (see [Sf, Chapter IV §1.3, Lemma 1 and Lemma 2]). Then the homomorphism from $\bar{\mathcal{O}}$ to $(f'')/(f'f'')$ given by

$f \to ff'' + (f'f'')$ is surjective and has (f') as its kernel. Hence, $\dim((f'')/(f'f'')) = \dim(\bar{\mathcal{O}}/(f'))$. ♠

The result above combined with those of Chapter 2 can be used to gain information about the kernels of linear partial differential operators defined on the ring of formal power series in s indeterminates, $k[[Z_1, \ldots, Z_s]] =: k[[Z]]$. We take the commutative semigroup G of linear operators to be the partial differential operators

$$G_{k[Z]}(D) := \{p(D) : p \in k[Z_1, \ldots, Z_s]\}$$

defined in the usual way. For polynomials $p_1, \ldots, p_m$, there is a relationship between the dimension of the kernel space,

$$K_{(p_1,\ldots,p_m)}(D) := \{f \in k[[Z]] : p_1(D)f = 0, \ldots, p_m(D)f = 0\},$$

and the cardinality of the variety $\mathrm{Var}\,(p_1, \ldots, p_m)$; namely,

$$\dim K_{(p_1,\ldots,p_m)}(D) < \infty \Longleftrightarrow |\mathrm{Var}\,(p_1, \ldots, p_m)| < \infty,$$

see [DM_6, Proposition 2.1] and [BR_2, Corollary 3.21]. In fact, [BR_2] gives an explicit formula

$$\text{(3.4)} \quad \dim K_{(p_1,\ldots,p_m)}(D) = \mathrm{codim}\,(p_1, \ldots, p_m) = \sum_{\theta \in \mathrm{Var}\,(p_1,\ldots,p_m)} \mathrm{ind}_\theta(p_1, \ldots, p_m).$$

These results were proved for $\mathbf{C}$, but hold equally well for any algebraically closed field k (the exponential function is defined by its formal power series).

As an immediate consequence of Theorem 3.3 and (3.4), we have

(3.5) Corollary. *If $p_s = p'_s p''_s$, then the kernel spaces for the ideals*

$$I = (p_1, \ldots, p_{s-1}, p_s), \quad I' = (p_1, \ldots, p_{s-1}, p'_s), \quad \text{and} \quad I'' = (p_1, \ldots, p_{s-1}, p''_s),$$

satisfy the relation

$$\dim(K_I(D)) = \dim(K_{I'}(D)) + \dim(K_{I''}(D)).$$

In particular, $G_{k[Z]}(D)$ is a semigroup of linear operators with s-dimensional additivity.

For any matroid X and a collection of bases $\mathcal{B}_1^{(X)}$, we can consider the kernel spaces (1.2) for arbitrary operators, $L_X \subset G_{k[Z]}(D)$, associated with X. By Corollary 3.5 and Theorem 2.12, we have

(3.6) Theorem. *If the matroid X has rank $\varrho(X) = s$ and $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$, then for $L_X \subset G_{k[Z]}(D)$,*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} \dim K(L_B).$$

We now give a formula for the intersection index for some special polynomial ideals. Let $(p_1, \ldots, p_s)$ be the ideal generated by the homogeneous polynomials $p_1, \ldots, p_s \in k[Z_1, \ldots, Z_s]$. If zero is the only common zero of $p_1, \ldots, p_s$, then the intersection index of $p_1, \ldots, p_s$ at 0 is just $\operatorname{codim}(p_1, \ldots, p_s)$. Moreover,

$$\operatorname{codim}(p_1, \ldots, p_s) = \prod_{i=1}^{s} \deg p_i.$$

When $k = \mathbf{C}$, Stiller [St] established a formula for $\operatorname{codim}(p_1, \ldots, p_s)$, while Dahmen and Micchelli [DM_6] pointed out that this formula could be written in the above form.

Taking the semi-group $G_{Hom}(D)$ to be the differential operators generated by s- variable homogeneous polynomials, we have the following result which confirms a conjecture of Dahmen and Micchelli [DM_6].

(3.7)Theorem. *If the matroid X has rank $\varrho(X) = s$, then for $L_X \subset G_{Hom}(D)$,*

$$\dim K(L_X, \mathcal{B}(X)) = \sum_{B \in \mathcal{B}(X)} \dim K(L_B).$$

Moreover, if $\dim K(L_B) < \infty$ for all $B \in \mathcal{B}(X)$, then

$$\dim K(L_X, \mathcal{B}(X)) = \sum_{B \in \mathcal{B}(X)} \prod_{b \in B} \deg q_b.$$

Another case in which an explicit formula can be given is discussed in the following example.

(3.8)Example. *Consider the semigroup $G_\Pi(D) \subset G_{k[Z]}(D)$ defined by products of linear polynomials in $k[Z]$; i.e.,*

$$G_\Pi(D) := \{p(D) : p(Z) = \prod_{j=1}^{m} (\lambda_j \cdot Z - c_j),\ \lambda_j \in k^s,\ c_j \in k,\ j = 1, \ldots, m,\ m = m(p)\},$$

where $\lambda \cdot Z := \lambda(1)Z_1 + \ldots + \lambda(s)Z_s$ for $\lambda = (\lambda(1), \ldots, \lambda(s)) \in k^s$ and $Z = (Z_1, \ldots, Z_s)$. Let X be a matroid and $L_X \subset G_\Pi(D)$ be the associated operators. For each polynomial

$$p_b(Z) = \prod_{j=1}^{m_b} (\lambda_{b,j} \cdot Z - c_{b,j}), \qquad b \in B \in \mathcal{B}(X),$$

there is a corresponding set of elements from k^s given by

$$\Lambda_b := \{\lambda_{b,j} : j = 1, \ldots, m_b\}.$$

For any $B \in \mathcal{B}(X)$, we consider Λ_B to be the set of all possible matrices in $k^{s \times s}$ with columns indexed by $b \in B$ and with the bth column chosen from Λ_b. Let Ω_B be the set of all matrices in Λ_B of rank $< s$.

(3.9)Corollary. *If the set $L_X \subset G_\Pi(D)$ is well associated to the matroid X, $\varrho(X) = s$, and $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$, then*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B \in \mathcal{B}_1^{(X)}} \Big[\prod_{b \in B} \deg p_b - |\Omega_B|\Big].$$

Proof. By Theorem 3.6, we only have to show that

$$\dim K(L_B) = \Big[\prod_{b \in B} \deg p_b - |\Omega_B|\Big], \qquad \forall B \in \mathcal{B}_1^{(X)}.$$

From given $B \in \mathcal{B}(X)$ and the associated L_B, we construct a matroid Y_B and a corresponding set of operators $\widetilde{L}_{Y_B}$. The matroid Y_B consists of the elements b taken $\deg p_b$ times with the notion of independence inherited from B. For the operators $\widetilde{L}_{Y_B}$, we take any one-to-one correspondence between $\{y \in Y_B : y = b\}$ and the linear factors of p_b. Thus, a basis $W \in \mathcal{B}(Y_B)$ corresponds to a selection of linear factors from p_b, $b \in B$, i.e. to a matrix $\Lambda(W) = [\lambda_{b,j(W)}]_{b \in B} \in \Lambda_B$, and the operators $\widetilde{L}_W$ are just the linear partial differential operators $\{(\lambda_{b,j(W)} \cdot D - c_{b,j(W)})\}$.

With the above construction, $K(L_B) = K(\widetilde{L}_{Y_B})$ and

$$\dim K(\widetilde{L}_W) = \begin{cases} 1, & \text{if } \Lambda(W) \in \Lambda_B \backslash \Omega_B; \\ 0, & \text{if } \Lambda(W) \in \Omega_B. \end{cases}$$

Therefore, by Theorem 3.6 once again,

$$\dim K(L_B) = \dim K(\widetilde{L}_{Y_B}) = \sum_{W \in \mathcal{B}(Y_B)} \dim K(\widetilde{L}_W)$$
$$= \prod_{b \in B} \deg p_b - |\Omega_B|. \qquad \spadesuit$$

When $k = \mathbf{C}$, $X = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{C}^{s \times n}$ has nonzero columns, and $L_X = \{D_\lambda - c_\lambda, \lambda \in X\}$, then Corollary 3.9 contains the result of de Boor and Ron [BR$_2$, Theorem 6.9].

Corollary (3.5) can also be used to discuss solvability of systems of differential equations. It turns out the s-dimensional addvitivity plays an important role here.

Recently, Dahmen and Micchelli [DM_6], among other things, investigated the solvability problem when G is the set of all partial differential operators, $p(D)$, induced by homogeneous polynomials in s variables and $S := \pi(\mathbb{R}^s)$, the linear space of polynomials in s variables. Their studies led to the following theorem.

(3.10)Theorem. *[DM_6] Let $p_1, \ldots, p_n$ be homogeneous polynomials on $\mathbb{R}^s$. Consider the system of differential equations*

$$p_j(D)f = \phi_j, \quad j = 1, \ldots, n,$$

where ϕ_j and $f \in \pi(\mathbb{R}^s)$ and there is some integer N such that

$$\deg \phi_j + \deg p_j = N, \quad j = 1, \ldots, n.$$

Then the above system of differential equations has a solution in $\pi(\mathbb{R}^s)$ if and only if whenever $q_1, \ldots, q_n \in \pi(\mathbb{R}^s)$ satisfy

$$\sum_{j=1}^{n} q_j p_j = 0, \quad \deg q_j = N - \deg p_j$$

it follows that

$$\sum_{j=1}^{n} q_j(D)\phi_j = 0.$$

When $s = 2$, and p_1, p_2 have no common factors, they obtained the following

(3.11)Corollary. *[DM_6] Let p_1 and p_2 be any homogeneous polynomials on $\mathbb{R}^2$ with no common nontrivial zeros. Then, the system of equations*

$$p_1(D)f = \phi_1, \quad p_2(D) = \phi_2; \qquad \phi_1, \phi_2 \in \pi(\mathbb{R}^2)$$

has a solution if and only if

$$p_2(D)\phi_1 = p_1(D)\phi_2.$$

We observe that the conditions in Corollary (3.11) are easier to check than the conditions in Theorem (3.10). However, it was not clear at that time whether the above corollary can be extended to the case $s > 2$. Our next goal is to extend the above corollary to the case $s > 2$ and to the case that $p(D)$ are the differential operators induced by arbitrary polynomials in s variables. To this end, we need some further basic notions and results of algebraic geometry.

(3.12)Definition. *The dimension of an irreducible variety V,* $\dim(V)$, *is the transcendence degree of $k(V)$ over k. The dimension of a variety is the maximum of the dimensions of its irreducible components.*

It is obvious that a single point has dimension 0. Let $V \subseteq k^s$ be an algebraic variety. For a polynomial $f \in k[Z]$, we define the variety $V_f := \{a \in V : f(a) = 0\}$.

The following theorem is useful for proving some of the results in this section. The proof of this theorem can be found in [L], Chap.II, Theorem 11 and [Sh], Chap.1 §6, Theorem 4.

(3.13)Theorem. *(Dimension Theorem) Let V be an irreducible variety of dimension $n \geq 1$ in k^s. If a polynomial $f \in k[Z]$ does not vanish on all of V and $V_f \neq \emptyset$, then each irreducible component of V_f has dimension $n - 1$.*

(3.14)Theorem. *Suppose that every irreducible component of a variety U in k^s has dimension n ($n \geq 1$). Let $\theta = (\theta_1, \ldots, \theta_s) \in U$. Then there exists a vector $v = (v_1, \ldots, v_s) \in k^s$ such that*

$$F_v := (Z_1 - \theta_1)v_1 + \cdots + (Z_s - \theta_s)v_s \in k[Z]$$

does not vanish on any component of U. Consequently, every irreducible component of $U \cap V(F_v)$ has dimension $n - 1$.

Proof. Let $U = U_1 \cup \cdots \cup U_m$ be the decomposition of U into irreducible components with $U_i \not\subseteq U_j$ for all $i \neq j$. Let

$$L_j := \{ v = (v_1, \ldots, v_s) \in k^s : F_v \text{ vanishes on all of } U_j \}, \quad j = 1, \ldots, m.$$

Clearly, L_j is a linear subspace of k^s for each j. The dimension of L_j is at most $s - n$, for otherwise we would have $\dim(U_j) < n$. Since $n \geq 1$, the set $k^s \setminus \cup_{j=1}^m L_j$ is nonempty. For any $v \in k^s \setminus \cup_{j=1}^m L_j$, F_v does not vanish on any of U_j for each $j = 1, \ldots, m$. ♠

(3.15)Corollary. *Let $U \subseteq k^s$ be an algebraic variety with all its irreducible components having the same dimension n. For any $\theta \in U$ there exist n polynomials $p_1, \ldots, p_n$ of degree 1 such that they vanish at θ and that $U \cap V(p_1, \ldots, p_n)$ is a finite set.*

Finally, we have following theorem which extends Corollary (3.11).

(3.16)Theorem. *Let $p_1, \ldots, p_r$ ($r \leq s$) be polynomials on k^s. Assume that the variety $V(p_1, \ldots, p_r)$ is either empty or each of its irreducible components has dimension $s - r$. Then for given polynomials (resp. exponential polynomials) $\phi_1, \ldots, \phi_r$, the system of differential equations*

$$p_j(D)f = \phi_j, \quad j = 1, \ldots, r, \tag{3.17}$$

has a polynomial (resp. exponential polynomial) solution f if and only if the compatibility conditions

$$p_j(D)\phi_k = p_k(D)\phi_j, \quad 1 \leq j < k \leq r,$$

hold.

Proof. Let S be the linear space of all exponential polynomials on k^s and $G := G_{\pi(k^s)}(D)$. It is clear that we only need to consider $\phi_j \in e_\theta \pi(k^s)$ for some $\theta = (\theta_1, \dots, \theta_s) \in k^s$ since $e_\theta \pi(k^s)$ is G-invariant. Let $M = e_\theta \pi(k^s)$. Given $\phi_1, \dots, \phi_r \in M$, we consider the solvability for $f \in M$ of the system (3.17). If $p_j(\theta) \neq 0$ for some j, then $p_j(D)$ is invertible on M. Hence, the theorem follows from Theorem (2.3).

If $p_j(\theta) = 0$ for all $j = 1, \dots, r$, then $\theta \in V(p_1, \dots, p_r)$. By hypothesis, every irreducible component of $V(p_1, \dots, p_r)$ has dimension $s - r$. Applying Corollary (3.15) to $V(p_1, \dots, p_r)$, we see that there exist polynomials $p_{r+1}, \dots, p_s$ of degree one such that they vanish at θ and the set

$$V(p_1, \dots, p_r, p_{r+1}, \dots, p_s)$$

is finite. Thus, an application of Theorem (2.4) gives the desired result. ♠

(3.18)Corollary. *Let $p_1, \dots, p_s$ be polynomials on k^s such that $V(p_1, \cdots, p_s)$ is a finite set. The system of differential equations*

$$p_j(D)f = \phi_j, \qquad j = 1, \dots, s, \tag{3.19}$$

where ϕ_j's are exponential polynomials, has a solution in the the space of exponential polynomials if and only if the compatibility conditions

$$p_j(D)\phi_k = p_k(D)\phi_j, \qquad 1 \leq j < k \leq s,$$

hold. ♠

CHAPTER 4

APPLICATION TO LINEAR DIFFERENCE OPERATORS

Let $\mathbb{Z}$ be the set of integers and s be a positive integer. As before k is an algebraically closed field. A mapping from $\mathbb{Z}^s$ to k is called an s-variate k-sequence, and we denote the linear space of all s-variate k-sequences by A. We wish to consider translation operators on A. These can be best described using the primitive translation operators τ_j given by

$$\tau_j f = f(\cdot + e_j), \qquad j = 1, \ldots, s, \quad \text{for} \quad f \in A,$$

where

$$e_j \;=\; (0, \;\ldots\; , \underset{j\text{th}}{1}, \;\ldots\; , 0),$$

$j = 1, \ldots, s$, are the canonical unit vectors in $\mathbb{Z}^s$. For a multiindex $\alpha \in \mathbb{N}^s$, we define

$$\tau^\alpha := \tau_1^{\alpha_1} \cdots \tau_s^{\alpha_s}.$$

For a polynomial $p \in k[Z_1, \ldots, Z_s]$, $p(Z) = \sum a_\alpha Z^\alpha$, there is a corresponding translation operator

$$p(\tau) := \sum a_\alpha \tau^\alpha.$$

Similarly, for $\theta \in k^s$, we define

$$p(\theta\tau) := \sum a_\alpha \theta^\alpha \tau^\alpha.$$

If $q \in k[Z_1, \ldots, Z_s]$, then the sequence given by $\beta \mapsto q(\beta)$, $\beta \in \mathbb{Z}^s$ will also be simply denoted by q. Similarly, if Q is a subspace of $k[Z_1, \ldots, Z_s]$, then the sequence space $\{\beta \mapsto q(\beta) : q \in Q\}$ will again be denoted by Q. For any pair of polynomials $p, q \in k[Z]$, the notation $p(\tau)q$ means the sequence obtained by applying the difference operator $p(\tau)$ to the sequence $q : \beta \mapsto q(\beta)$, $\beta \in \mathbb{Z}^s$.

Given an ideal I of $k[Z]$, the kernel space $K_I(\tau)$ of all the difference operators $p(\tau)$, $p \in I$, is defined by

$$K_I(\tau) := \{f \in A : p(\tau)f = 0 \quad \forall p \in I\}.$$

We wish to single out some special elements in $K_I(\tau)$. Let

$$(k\backslash\{0\})^s = \{(a_1, \ldots, a_s) \in k^s : a_1 \neq 0, \ldots, a_s \neq 0\}.$$

For any $\theta \in (k\backslash\{0\})^s$, we denote by $\theta^{(\ )}$ the sequence given by $\beta \mapsto \theta^\beta$, $\beta \in \mathbb{Z}^s$. It follows from the definition of $K_I(\tau)$ that

$$\theta^{(\ )} \in K_I(\tau) \iff \theta \in \mathrm{Var}\,(I).$$

(4.1)Theorem. *The dimension of $K_I(\tau)$ is finite if and only if the set $\mathrm{Var}\,(I) \cap (k\backslash\{0\})^s$ is finite. Moreover, in this case, to each $\theta \in \mathrm{Var}\,(I) \cap (k\backslash\{0\})^s$, there corresponds a translation invariant space $Q_{I,\theta}$ of polynomials such that*

$$K_I(\tau) = \bigoplus_{\theta \in \mathrm{Var}\,(I) \cap (k\backslash\{0\})^s} \theta^{(\ )} Q_{I,\theta}.$$

Proof. When $k = \mathbf{C}$, the proof of this theorem can be found in [L], [DJM], and [BR$_3$]. Their proofs can be easily carried over for the case when k is an algebraically closed field. Let us find the spaces $Q_{I,\theta}$ explicitly. Observe that for $\alpha \in \mathbb{N}^s$ and $q \in A$, we have

$$\tau^\alpha(\theta^{(\ )}q) = \theta^{(\ )}(\theta^\alpha \tau^\alpha q).$$

It follows that for any $f \in I$,

$$f(\tau)(\theta^{(\ )}q) = \theta^{(\ )}(f(\theta\tau)q).$$

Thus,

$$\theta^{(\ )}q \in K_I(\tau) \iff f(\theta\tau)q = 0, \quad \forall f \in I.$$

This shows that

$$(4.2) \qquad Q_{I,\theta} = \{q \in k[Z] : f(\theta\tau)q = 0, \quad \forall f \in I\}. \qquad \spadesuit$$

Recall that if θ is an isolated zero of I, then the multiplicity space, $P_{I,\theta}$, of I at θ is defined as

$$P_{I,\theta} := \{p \in k[Z] : p(D)f(\theta) = 0, \quad \forall f \in I\}.$$

The following theorem shows that the spaces $Q_{I,\theta}$ and $P_{I,\theta}$ have the same dimension.

(4.3)Theorem. $\dim(Q_{I,\theta}) = \dim(P_{I,\theta})$.

Proof. To prove that $\dim(Q_{I,\theta}) = \dim(P_{I,\theta})$, it suffices to establish a linear isomorphism between the spaces $P_{I,\theta}$ and $Q_{I,\theta}$. For this purpose, we introduce

$$[Z]^\beta := [Z_1]^{\beta_1} \cdots [Z_s]^{\beta_s}, \qquad \beta \in \mathbb{N}^s,$$

with

$$[Z_j]^{\beta_j} := Z_j(Z_j - 1)\cdots(Z_j - \beta_j + 1).$$

Let Δ_j be the jth forward difference operator:

$$\Delta_j q := q(\cdot + e_j) - q, \qquad q \in A.$$

It is easy to verify that

$$\Delta^\alpha[\ ]^\beta = \frac{\beta!}{(\beta-\alpha)!}[\ ]^{\beta-\alpha}.$$

Here we make the convention that $[\ ]^{\beta-\alpha} = 0$ if $\alpha \not\leq \beta$.

To each $p \in k[Z]$, $p(Z) = \sum_\beta b_\beta Z^\beta$, we associate $q(Z) = \sum_\beta b_\beta \theta^{-\beta}[Z]^\beta$. The mapping $\sigma : p \mapsto q$ is a linear automorphism on $k[Z]$. We want to show that σ is an isomorphism from $P_{I,\theta}$ to $Q_{I,\theta}$. For this purpose, we compute $p(D)f(\theta)$ and $f(\theta\tau)q$ as follows. Suppose $f(Z) = \sum_\alpha a_\alpha (Z-\theta)^\alpha$. Then

$$p(D)f(\theta) = \sum_\alpha a_\alpha b_\alpha \alpha!,$$

and

$$\begin{aligned} f(\theta\tau)q(Z) &= \Big(\sum_\alpha a_\alpha(\theta\Delta)^\alpha\Big)\Big(\sum_\beta b_\beta \theta^{-\beta}[Z]^\beta\Big) \\ &= \sum_{\alpha,\beta} a_\alpha b_\beta \frac{\beta!}{(\beta-\alpha)!}\theta^{-(\beta-\alpha)}[Z]^{\beta-\alpha} \\ &= \sum_\gamma \big(\sum_\alpha a_\alpha b_{\alpha+\gamma}(\alpha+\gamma)!\big)\frac{\theta^{-\gamma}}{\gamma!}[Z]^\gamma. \end{aligned}$$

Let $q \in Q_{I,\theta}$ and $p = \sigma^{-1}(q)$. Then for any $f \in I$, $f(\theta\tau)q = 0$; hence, $\sum_\alpha a_\alpha b_\alpha \alpha! = 0$ from the above formula. It follows that $p(D)f(\theta) = 0$, $\forall f \in I$; i.e., $p \in P_{I,\theta}$. Conversely, suppose $p \in P_{I,\theta}$ and $q = \sigma(p)$. Then $D^\gamma p \in P_{I,\theta}$ for any $\gamma \in \mathbb{N}^s$, since $P_{I,\theta}$ is D-invariant. We have

$$D^\gamma p = \sum_\beta b_\beta \frac{\beta!}{(\beta-\gamma)!}Z^{\beta-\gamma} = \sum_\beta b_{\beta+\gamma}\frac{(\beta+\gamma)!}{\beta!}Z^\beta.$$

Therefore,

$$0 = (D^\gamma p)(D)f(\theta) = \sum_\alpha a_\alpha \frac{(\gamma+\alpha)!}{\alpha!} b_{\alpha+\gamma}\alpha! = \sum_\alpha a_\alpha b_{\alpha+\gamma}(\alpha+\gamma)!.$$

This shows that $f(\theta\tau)q = 0$ for all $f \in I$; i.e., $q \in Q_{I,\theta}$. We conclude that σ is a linear isomorphism from $P_{I,\theta}$ to $Q_{I,\theta}$. ♠

Suppose now that I is generated by s polynomials $f_1, \ldots, f_s$ from $k[Z]$, and that $\mathrm{Var}(I) \cap (k\backslash\{0\})^s$ is a finite set. Then $\dim(K_I(\tau))$ is finite, so by Theorem 4.1

and Theorem 4.3, we have

$$\begin{aligned}\dim(K_I(\tau)) &= \sum_{\theta \in \mathrm{Var}\,(I) \cap (k\backslash\{0\})^s} \dim(Q_{I,\theta}) \\ &= \sum_{\theta \in \mathrm{Var}\,(I) \cap (k\backslash\{0\})^s} \dim(P_{I,\theta}) \\ &= \sum_{\theta \in \mathrm{Var}\,(I) \cap (k\backslash\{0\})^s} \mathrm{ind}_\theta(f_1, \ldots, f_s).\end{aligned}$$

(4.4)Theorem. *Suppose $f_1, \ldots, f_{s-1}, f'_s, f''_s \in k[Z]$ and $f_s = f'_s f''_s$. For the ideals*

$$I = (f_1, \ldots, f_{s-1}, f_s), \quad I' = (f_1, \ldots, f_{s-1}, f'_s), \quad \text{and} \quad I'' = (f_1, \ldots, f_{s-1}, f''_s),$$

we have the relation

$$\dim(K_I(\tau)) = \dim(K_{I'}(\tau)) + \dim(K_{I''}(\tau)).$$

Proof. If one of the dimensions of $K_{I'}(\tau)$ or $K_{I''}(\tau)$ is infinite, then the dimension of $K_I(\tau)$ is also infinite, since $K_I(\tau)$ contains both $K_{I'}(\tau)$ and $K_{I''}(\tau)$. Suppose that both $K_{I'}(\tau)$ and $K_{I''}(\tau)$ are finite dimensional. Then $\mathrm{Var}\,(I') \cap (k\backslash\{0\})^s$ and $\mathrm{Var}\,(I'') \cap (k\backslash\{0\})^s$ are finite sets, hence $\mathrm{Var}\,(I) \cap (k\backslash\{0\})^s$ is finite as well. Thus, by Theorem 3.3 and the above results, we obtain

$$\begin{aligned}\dim(K_I(\tau)) &= \sum_{\theta \in \mathrm{Var}\, \cap (k\backslash\{0\})^s} \mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f_s) \\ &= \sum_{\theta \in \mathrm{Var}\, \cap (k\backslash\{0\})^s} \Big(\mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f'_s) + \mathrm{ind}_\theta(f_1, \ldots, f_{s-1}, f''_s)\Big) \\ &= \dim(K_{I'}(\tau)) + \dim(K_{I''}(\tau)). \quad \spadesuit\end{aligned}$$

Again we can combine this theorem with the results in Chapter 2 to obtain information about the kernels of partial difference equations on the sequence space A. The semigroup of commutative operators in this case will be

$$G_{k[Z]}(\tau) := \{p(\tau) : p \in k[Z_1, \ldots, Z_s]\}.$$

The statement of Theorem 4.4 is simply the s-dimensional additivity of $G_{k[Z]}(\tau)$.

(4.5)Theorem. *The semigroup $G_{k[Z]}(\tau)$ of difference operators has s-dimensional additivity. In particular, Theorem 2.12 holds for $G_{k[Z]}(\tau)$.*

A special case of this theorem has arisen previously in the study of the algebraic properties of box splines and exponential box splines. In that case an explicit formula for $\dim K(L_B)$, $B \in \mathcal{B}_1^{(X)}$, can be given. Our next example extends these explicit formulae to a wider class of polynomials.

(4.6)Example. Let $k = \mathbf{C}$ be the field of complex numbers. The polynomials in $\mathbf{C}[Z]$ to be associated with the matroid X in this example will be taken from the subset

$$\mathbb{P} := \{\prod_{j=1}^{m}(Z^{\lambda_j} - \mu_j) : \lambda_j \in \mathbb{N}^s \quad \text{and} \quad \mu_j \in \mathbf{C}\backslash\{0\}, \qquad j = 1, \ldots, m\}.$$

The translation operators, $\mathbb{P}(\tau) := \{p(\tau) : p \in \mathbb{P}\}$, correspond to products of difference operators (with $\mu = \exp(c)$)

$$\begin{aligned} \Delta_{\lambda,c} f :=& f(\cdot + \lambda) - \exp(c) f(\cdot) \\ =:& \nabla_{\lambda,c} f(\cdot + \lambda) = \tau^{\lambda} \nabla_{\lambda,c} f. \end{aligned} \tag{4.7}$$

The difference operators $\nabla_{\lambda,c}$ arise quite naturally in the study of exponential box splines. Here, we following the notion and setup of [DM$_5$] and briefly recall the exponential box splines case. Their relation to our situation is quite clear from (4.7): If $p_\ell(Z) = Z^{\lambda_\ell} - \exp(c_\ell)$, $\ell = 1, \ldots, s$, then

$$\begin{aligned} K_{(p_1,\ldots,p_s)}(\tau) =& \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) \\ :=& \{f : \nabla_{\lambda_\ell, c_\ell} f = 0, \quad \ell = 1, \ldots, s\}. \end{aligned} \tag{4.8}$$

When $\dim \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) < \infty$, an explicit formula for the dimension can be given:

$$\dim \nabla_{c_1,\ldots,c_s}([\lambda_1, \ldots, \lambda_s]) = |\det[\lambda_1, \ldots, \lambda_s]|. \tag{4.9}$$

If $\lambda_1, \ldots, \lambda_s$ are linearly dependent, then $\dim K_{(p_1,\ldots,p_s)}$ is either 0 or ∞ depending on whether $p_1, \ldots, p_s$ have no common zeros in $(\mathbb{C}\backslash\{0\})^s$ or (necessarily) infinitely many common zeros in $(\mathbb{C}\backslash\{0\})^s$.

The formula (4.9) is a special case of the following setup: Let $X = \Lambda$ be a matrix of rank s in $\mathbb{Z}^{s\times n}$ with nonzero columns. To each $\lambda \in \Lambda$ we choose a $c_\lambda \in \mathbb{C}$ and associate the difference operator given by

$$\nabla_{\lambda,c_\lambda} f = f - \exp(c_\lambda) f(\cdot - \lambda).$$

For $V \subseteq \Lambda$, let $\nabla_{V,c_V} := \prod_{v\in V} \nabla_{v,c_v}$. Then the kernel space

$$\nabla_{c_\Lambda}(\Lambda) := \{f : \nabla_{H,c_H} f = 0, \quad \forall H \in \mathcal{H}(\Lambda)\},$$

has dimension given by

$$\dim(\nabla_{c_\Lambda}(\Lambda)) = \sum_{B\in\mathcal{B}(\Lambda)} |\det(B)|$$

As was done in Example 3.8, we wish to separate the matroid structure from the associated linear difference operators, using the matroid only as an index set for L_X (as opposed to using it to give the directions of the translations), and at the same time for each index we consider a product of difference operators (induced by the polynomials from $\mathbb{P}$). In order to state the result, let Λ_B be the set of all possible matrices with columns indexed by $b \in B$ and with the bth column chosen from the exponents in p_b, namely, $\Lambda_b := \{\lambda_{b,j} : j = 1, \ldots, m_b\}$.

(4.10)Corollary. *Let the set $L_X \subset \mathbb{P}(\tau)$ be associated with a matroid X, $\varrho(X) = s$, and suppose $\mathcal{B}_1^{(X)}$ is an order closed subset of $\mathcal{B}(X)$. If for every $B \in \mathcal{B}_1^{(X)}$, the polynomials $\{p_b\}_{b\in B}$ have only finitely many common zeros in $(\mathbb{C}\backslash\{0\})^s$, then*

$$\dim K(L_X, \mathcal{B}_1^{(X)}) = \sum_{B\in\mathcal{B}_1^{(X)}} \sum_{W\in\Lambda_B} |\det W|.$$

Proof. By Theorem 4.5, we only need to show that

$$\dim K(L_B) = \sum_{W \in \Lambda_B} |\det W|.$$

This can be done using the techniques in the proof of Corollary 3.9 together with (4.8) and (4.9) above. ♠

Finally, as we did before, we briefly discuss the solvability for the system of difference equations.

Let $V \subseteq k^s$ be an algebraic variety. A subset $U \subseteq V$ is said to be closed, if U itself is an algebraic variety. A subset $O \subseteq V$ is said to be open, if $V \backslash O$ is closed in V. Let O be a nonempty open subset of an irreducible algebraic variety V. If a polynomial $f \in k[Z]$ vanishes on O, then it must vanish on V, for otherwise $V = (V \backslash O) \cup V_f$ gives a decomposition of V with $V \backslash O \neq V$ and $V_f \neq V$. This contradicts the irreducibility of V. Thus, $I(O) = I(V)$. Hence, the coordinate ring $k[O]$ is the same as the coordinate ring $k[V]$. Consequently, the quotient field $k(O)$ of $k[O]$ is the same as $k(V)$. In particular, this shows that $\dim(O) = \dim(V)$ for any nonempty open subset O of an irreducible variety V. Moreover, if V is irreducible, then $k[O] = k[V]$ is an integral domain, hence O is also irreducible.

Let V be an algebraic variety and let

$$V = U_1 \cup \cdots \cup U_m$$

be the decomposition of V into its irreducible components. For an open subset $O \subseteq V$, $O \cap U_j$ is open in U_j for each $j = 1, \ldots, m$. If $O \cap U_j$ is nonempty, then $O \cap U_j$ is an irreducible component of O. Thus, after discarding some possible empty sets,

$$O = (O \cap U_1) \cup \cdots \cup (O \cap U_m)$$

gives a decomposition of O into its irreducible components.

The results of Theorem (3.14) and its corollary can be extended to the case where U is an open subset of an algebraic variety.

(4.11)Theorem. *Let $O \subseteq k^s$ be an open subset of an algebraic variety with all its irreducible components having the same dimension n. Then for any $\theta \in O$ there exist n polynomials $p_1, \ldots, p_n$ of degree 1 such that they vanish at θ and that $O \cap V(p_1, \ldots, p_n)$ is a finite set.*

Finally we establish the following theorem about difference equations.

(4.12)Theorem. *Let $p_1, \ldots, p_r$ $(r \leq s)$ be polynomials on k^s. Assume that the intersection of the variety $V(p_1, \ldots, p_r)$ with $(k\backslash\{0\})^s$ is either empty or each of its irreducible components has dimension $s - r$. Then for given polynomial sequences (resp. exponential polynomial sequences) $\phi_1, \ldots, \phi_r$, the system of difference equations*

$$p_j(\tau)f = \phi_j, \quad j = 1, \ldots, r, \tag{4.13}$$

has a polynomial sequence (resp. exponential polynomial sequence) solution f if and only if the compatibility conditions

$$p_j(\tau)\phi_k = p_k(\tau)\phi_j, \quad 1 \leq j < k \leq r,$$

hold.

Proof. Let S be the linear space of all exponential polynomial k-sequences on $\mathbb{Z}^s$. Without loss of generality, we may assume that $\phi_1, \ldots, \phi_r \in \theta^{(\ )}\pi(k^s) =: M$ for some $\theta = (\theta_1, \ldots, \theta_s) \in (k\backslash\{0\})^s$.

If $p_j(\theta) \neq 0$ for some j, then by Theorem (2.3), the system (3.28) has a solution in M provided the compatibility conditions hold.

Suppose that $p_j(\theta) = 0$ for all $j = 1, \ldots, r$. Then $\theta \in V(p_1, \ldots, p_r)$, and hence each irreducible component of $V(p_1, \ldots, p_r) \cap (k\backslash\{0\})^s$ has dimension $s - r$ by the hypothesis. Again we can find polynomials $p_{r+1}, \dots, p_s$ of degree 1 such that they vanish at θ and the set

$$V(p_1, \dots, p_r, p_{r+1}, \dots, p_s) \cap (k \backslash \{0\})^s$$

is finite. Thus, an application of Theorem (2.4) gives the desired result. ♠

CHAPTER 5

WAVELET DECOMPOSITIONS

In this chapter, we discuss the orthogonal decompositions from multiresolution approximations to $L_2(\mathbb{R}^s)$ generated by lower dimension box splines. To this end, we start with the definition of box spline.

The ordinary **box spline**, M_Ξ, in $\mathbb{R}^s$ associated with an $s \times n$ integer matrix $\Xi \in \mathbb{Z}^{s\times n}$ is defined by

$$\langle M_\Xi, \psi \rangle := \int_{[0,1]^n} \psi(\Xi x)\, dx, \qquad \psi \in C(\mathbb{R}^s). \tag{5.1}$$

Its Fourier transform is

$$\widehat{M_\Xi}(y) = \prod_{\xi\in\Xi} \frac{1-\exp(-iy\xi)}{iy\xi}.$$

Similarly, a **centered box spline** is defined by

$$\langle M_\Xi^c, \psi \rangle := \int_{[-1/2,1/2]^n} \psi(\Xi x)\, dx, \qquad \psi \in C(\mathbb{R}^s). \tag{5.2}$$

Its Fourier transform

$$\widehat{M_\Xi^c}(y) = \prod_{\xi\in\Xi} \frac{\sin(y\xi/2)}{y\xi/2} \tag{5.3}$$

plays an essential role in our discussions. (Our notational conventions are that for all set relations involving a matrix Ξ, the matrix is considered as a set of its columns, and that the product $y\xi$ of two vectors in $\mathbb{R}^s$ is the usual inner product. The same column ξ can appear several times in Ξ; the number of times being its **multiplicity**.)

Clearly, $M_\Xi^c = M_\Xi(\cdot + c_\Xi)$, where

$$c_\Xi = \sum_{\xi\in\Xi} \xi/2$$

is the **center** of the box spline.

When $\operatorname{rank}\Xi = s$, the box spline is a piecewise polynomial of total degree at most $n-s$, symmetric about the center, and with compact support.

The space associated with M_Ξ^c is generated by its integer translates:

$$(5.4) \qquad \mathcal{S}(M_\Xi^c) := \{M_\Xi^c *' a := \sum_{j\in\mathbb{Z}^s} a(j) M_\Xi^c(\cdot - j) : \ a : \mathbb{Z}^s \to \mathbf{C}\}.$$

When the sequences, a, are restricted to a particular space, we so label $\mathcal{S}$; e.g., $\mathcal{S}(M_\Xi^c, \ell_2)$ is derived by restricting the sequences to $\ell_2(\mathbb{Z}^s)$. Of course, it is important to have the functions in (5.4) uniquely representable by $M_\Xi^c *' a$; i.e.,$M_\Xi^c *' a = M_\Xi^c *' b$ implies $a = b$. This is the (global) linear independence of the integer translates of M_Ξ^c which is characterized by the statement

$$(5.5) \qquad \forall u \in \mathbf{C}^s \ \exists\, j \in \mathbb{Z}^s \quad \text{such that} \quad \widehat{M_\Xi^c}(2\pi j - u) \neq 0.$$

(see [L], [DM_1], and [Ro_2]). In our case of an integer matrix, the linear independence (and therefore (5.5)) is equivalent to the matrix Ξ being unimodular,

$$(5.6) \qquad |\det Z| = 1 \qquad \forall \quad \text{column bases } Z \subset \Xi$$

(de Boor and Höllig [BH], Dahmen and Micchelli [DM_2], Jia [J_1-J_2]).

Another concept is the **characteristic polynomial** of M_Ξ^c defined by

$$(5.7) \qquad P := P_\Xi := \sum_{j\in\mathbb{Z}^s} M_\Xi^c(j)\exp(ij\cdot).$$

From the Poisson summation formula and the symmetry about the origin of M_Ξ^c, we have

$$(5.8) \qquad P_\Xi(y) = \sum_{j\in\mathbb{Z}^s} \widehat{M_\Xi^c}(2\pi j - y) = \sum_{j\in\mathbb{Z}^s} \widehat{M_\Xi^c}(y + 2\pi j).$$

It is well known that if Ξ is a unimodular integer matrix of rank s with columns of only even multiplicities, $P_\Xi > 0$. In particular,

$$P_{\Xi \cup \Xi}(y) := P_2 = \sum_{j \in \mathbb{Z}^s} |\widehat{M_\Xi^c}(y + 2\pi j)|^2 > 0.$$

Let

$$\widehat{K} := \widehat{M_\Xi}/(P_2)^{1/2}, \tag{5.9}$$

then, the function K satisfies

$$\sum_{j \in \mathbb{Z}^s} |\widehat{K}(y + 2\pi j)|^2 = \sum_{j \in \mathbb{Z}^s} \widehat{M_2^c}(y + 2\pi j)/P_2(y) = 1, \tag{5.10}$$

since $|\widehat{M_\Xi}| = |\widehat{M_\Xi^c}|$. Notice that in (5.9), the polynomial in the denominator is defined using the Fourier transform of the centered box spline while the ordinary box spline appears in the numerator. The function K has the representation

$$K = \sum_{j \in \mathbb{Z}^s} a_K(j) M_\Xi(\cdot - j) \tag{5.11}$$

where the coefficient sequence a_K is given by

$$a_K(j) := \frac{1}{(2\pi)^s} \int_{[-\pi,\pi]^s} \frac{\exp(-ijy)}{(P_2(y))^{1/2}}\, dy, \qquad j \in \mathbb{Z}^s.$$

Furthermore, we have the following property of function K.

(5.12)Theorem. *If M_Ξ is the box spline associated with a unimodular matrix Ξ of rank s, then the translates $\{K(\cdot - j)\}_{j \in \mathbb{Z}^s}$ of the function K given in (5.9) form an orthonormal basis for the space $S(M_\Xi, \ell_2)$. Moreover,*

$$|a_K(j)| \le \text{const} \exp(-|j|/\text{const}) \ \ \textit{and}\ \ |K(x)| \le \text{const} \exp(-|x|/\text{const}), \tag{5.13}$$

for some positive constants and, for any $p \in [1, \infty]$,

$$\|K *' a\|_p \leq \text{const}\|a\|_p. \tag{5.14}$$

Proof. The fact that $\{K(\cdot - j)\}_{j \in \mathbb{Z}^s}$ is an orthonormal system is a consequence of (5.10). The first of the inequalities (5.13) follows from the fact that $1/(P_2)^{1/2}$ is analytic in a neighborhood of $[-\pi, \pi]^s$ in $\mathbb{C}^s$, while the second follows from the first and the compact support of M_Ξ.

For (5.14), it is enough to realize that $|K(x)| \leq \text{const}\exp(-|k|/\text{const})$ for all $x \in k + [-1/2, 1/2]^s$ and that convolution with an exponentially decaying sequence is a bounded operator on $\ell_p(\mathbb{Z}^s)$.

It only remains to show that

$$\mathcal{S}(M_\Xi, \ell_2) = V_0 =: \{\sum_{j \in \mathbb{Z}^s} a(j)K(\cdot - j), \quad a \in \ell_2(\mathbb{Z}^s)\}. \tag{5.15}$$

For this we observe that

$$\widehat{M_\Xi} = \widehat{K}(P_2)^{1/2} \Longrightarrow M_\Xi = \sum_{j \in \mathbb{Z}^s} a_M(j)K(\cdot - j),$$

where the coefficients a_M, being the Fourier coefficients of $(P_2)^{1/2}$, have exponential decay. The equivalence (5.15) now follows since for any sequence b with exponential decay, $\|a*b\|_2 \leq \text{const}\|a\|_2$, and $K *' a = M_\Xi *'(a_K * a)$, $M_\Xi *' a = K *'(a_M * a)$. ♠

The last useful notion is multiresolution approximation of $L_2(\mathbb{R}^s)$. A scale of spaces $\{V_\nu\}_{\nu \in \mathbb{Z}}$ is a **multiresolution approximation** of $L_2(\mathbb{R}^s)$ if the following hold:

(i) $V_\nu \subset V_{\nu+1}$, $\nu \in \mathbb{Z}$.

(ii) $\cup_{\nu \in \mathbb{Z}} V_\nu$ is dense in $L_2(\mathbb{R}^s)$ and $\cap_{\nu \in \mathbb{Z}} V_\nu = \{0\}$.

(iii) $f \in V_\nu \Longleftrightarrow f(2\cdot) \in V_{\nu+1}$, $\forall \nu \in \mathbb{Z}$.

(iv) $f \in V_\nu \Longrightarrow f(\cdot - 2^{-\nu}j) \in V_\nu$, $\forall \nu \in \mathbb{Z}$, and $\forall j \in \mathbb{Z}^s$.

(v) There is an isomorphism from V_0 onto $\ell_2(\mathbb{Z}^s)$ which commutes with the action of $\mathbb{Z}^s$.

The spaces V_ν in the present case are

$$\begin{aligned} V_\nu :=& \{ \sum_{j\in\mathbb{Z}^s} a(j)K(2^\nu \cdot - j) : a \in \ell_2(\mathbb{Z}^s)\} \\ =& \{ \sum_{j\in\mathbb{Z}^s} a(j)M_\Xi(2^\nu \cdot - j) : a \in \ell_2(\mathbb{Z}^s)\}. \end{aligned}$$

When do these form a multiresolution approximation for $L_2(\mathbb{R}^s)$? For item (i), it suffices to show that the generator, $K(\cdot/2)/2^s$, of V_{-1} belongs to V_0. Now, $K(\cdot/2)/2^s \in V_0$ is equivalent to $(K(\cdot/2)/2^s)\widehat{\ }(y) = \widehat{K}(2y) = H(y)\widehat{K}(y)$ for some 2π-periodic function $H \in L_2([0, 2\pi]^s)$, and

$$\begin{aligned} H(y) := \frac{\widehat{K}(2y)}{\widehat{K}(y)} &= \frac{\widehat{M_\Xi}(2y)(P_2(y))^{1/2}}{\widehat{M_\Xi}(y)(P_2(2y))^{1/2}} \\ &= 2^{-\#\Xi}\frac{(P_2(y))^{1/2}}{(P_2(2y))^{1/2}} \prod_{\xi\in\Xi}(1 + \exp(-i\xi y)). \end{aligned} \tag{5.16}$$

Since P_2 is positive and 2π-periodic, H is also 2π-periodic. Therefore,

$$K(\cdot/2)/2^s = \sum_{j\in\mathbb{Z}^s} a_H(j)K(\cdot - j),$$

where the coefficients a_H are real-valued and have exponential decay. Item (ii) holds by the well-known approximation properties of box spline spaces and the exponential decay of the orthogonal basis functions $K(\cdot - j)$. Items (iii) and (iv) are obvious. Finally, the mapping $a \mapsto K{*}a$ is an isomorphism since $\{K(\cdot - j)\}$ is an orthogonal base for V_0.

In order to find a dense orthogonal system in $L_2(\mathbb{R}^s)$, we begin by looking for the orthogonal complement, $V_{\nu-1}^\perp$, of $V_{\nu-1}$ in V_ν. More precisely, we search for an orthogonal decomposition of V_0 into 2^s spaces where each space is generated by the (orthogonal) translates of a single function and hope that V_{-1} is one of these spaces. If this were the case, then there is an orthonormal set of 2^s functions $K_1, \cdots, K_{2^s}$, such that each K_μ has orthogonal translates, $K_\mu(\cdot/2)/2^s \in V_0$, and for any $a \in \ell_2(\mathbb{Z}^s)$, there exist sequences $b_\mu \in \ell_2(\mathbb{Z}^s)$ for which

$$\sum_{j\in\mathbb{Z}^s} a(j)K(\cdot - j) = \sum_{\mu=1}^{2^s} \sum_{j\in\mathbb{Z}^s} b_\mu(j)(K_\mu(\cdot/2 - j)/2^s) \tag{5.17}$$

is an orthonormal decomposition. The requirement that $K_\mu(\cdot/2)/2^s \in V_0$ is equivalent to

$$(K_\mu(\cdot/2)/2^s)\widehat{\ }(y) =: H_\mu(y)\widehat{K}(y), \quad \mu = 1, \cdots, 2^s, \tag{5.18}$$

with the functions $H_\mu \in L_2([0, 2\pi]^s)$.

Taking the Fourier transform of (5.17) and using (5.18), we find

$$A(y)\widehat{K}(y) = \sum_{\mu=1}^{2^s} B_\mu(y)H_\mu(y)\widehat{K}(y) \quad \text{or} \quad A(y) = \sum_{\mu=1}^{2^s} B_\mu(y)H_\mu(y), \tag{5.19}$$

where

$$A(y) = \sum_{j\in\mathbb{Z}^s} a(j)\exp(-ijy) \quad \text{and} \quad B_\mu(y) = \sum_{j\in\mathbb{Z}^s} b_\mu(j)\exp(-i2jy),$$

with the B_μ π-periodic in each variable.

By the orthonormality of the translates of K and Parseval's theorem, the orthogonality of the decomposition (5.17) is equivalent to the relation

$$\|A\|^2_{L_2([0,2\pi]^s)} = \sum_{\mu=1}^{2^s} \|B_\mu\|^2_{L_2([0,\pi]^s)}. \tag{5.20}$$

This relation will be compared to the one obtained by computing norms on both sides of the second equation in (5.19). To describe the latter, we let $\mathcal{A} = 2\mathbb{Z}^s$ and $\mathcal{G} = \mathcal{G}_s := \mathbb{Z}^s/\mathcal{A}$ be its factor group identified with the integer points in

$[0,2)^s$. Then computing the norms on both sides of that equation by using the inner product, we obtain

$$(5.21)\qquad \begin{aligned}\|A\|^2_{L_2([0,2\pi]^s)} &= \sum_{\mu=1}^{2^s}\sum_{\tau=1}^{2^s}\int_{[0,2\pi]^s} B_\mu(y)H_\mu(y)\overline{B_\tau(y)H_\tau(y)}\,dy\\ &= \sum_{\mu=1}^{2^s}\sum_{\tau=1}^{2^s}\int_{[0,\pi]^s} B_\mu(y)\overline{B_\tau(y)}\sum_{k\in\mathcal{G}} H_\mu(y+\pi k)\overline{H_\tau(y+\pi k)}\,dy.\end{aligned}$$

The equations (5.20) and (5.21) would be the same provided the functions H_μ satisfy the relations

$$(5.22)\qquad \begin{aligned}&\sum_{k\in\mathcal{G}}|H_\mu(\cdot+\pi k)|^2 = 1, \qquad \mu=1,\cdots,2^s, \quad \text{and}\\ &\sum_{k\in\mathcal{G}} H_\mu(\cdot+\pi k)\overline{H_\tau(\cdot+\pi k)} = 0, \qquad \mu\neq\tau.\end{aligned}$$

Therefore, the problem is reduced to the construction of functions H_μ with the desired properties. This is aided by the fact that the function H in (5.16) satisfies the first of the relations (5.22) since

$$(5.23)\qquad \begin{aligned}1 = \sum_{j\in\mathbb{Z}^s}|\widehat{K}(2y+2\pi j)|^2 &= \sum_{j\in\mathbb{Z}^s}|H(y+\pi j)|^2|\widehat{K}(y+\pi j)|^2\\ &= \sum_{k\in\mathcal{G}}\sum_{j\in\mathbb{Z}^s}|H(y+\pi k+2\pi j)|^2|\widehat{K}(y+\pi k+2\pi j)|^2\\ &= \sum_{k\in\mathcal{G}}|H(y+\pi k)|^2.\end{aligned}$$

We find it convenient to index the functions H_μ by the 2^s elements in $\mathcal{G}$ and to choose these functions to be of the form

$$(5.24)\qquad H_k(y) = \exp(iy\eta(k))\begin{cases} H(y+\pi k), & \text{if } 2c_{\Xi}k \text{ is even;}\\ \overline{H(y+\pi k)}, & \text{if } 2c_{\Xi}k \text{ is odd;}\end{cases} \qquad k\in\mathcal{G},$$

with $\eta(k)\in\mathcal{G}$. Then the first of the requirements (5.22) is met (by applying (5.23)) and it remains to choose $\eta(k)$ in such a way that the second condition is also met.

For $k_1 \neq k_2$, consider the terms in the second equation of (5.22) with $k = k^*$, say, and again with $k = k_2 + k_1 + k^*$ (they are distinct since $k_1 + k_2 \neq 0$). To help distinquish the cases, set $G_k(y + \pi k) := H_k(y)\exp(-iy\eta(k))$; i.e.,$G_k$ is either H or $\overline{H}$. The term for $k = k^*$ is

$$H_{k_1}(\cdot + \pi k^*)\overline{H_{k_2}(\cdot + \pi k^*)} = (-1)^{(\eta(k_1)-\eta(k_2))k^*}\exp(i(\eta(k_1) - \eta(k_2))\cdot)$$
$$\times G_{k_1}(\cdot + \pi(k_1 + k^*))\overline{G_{k_2}(\cdot + \pi(k_2 + k^*))},$$

while since G_k is 2π-periodic, the term for $k = k^* + k_1 + k_2$ is

$$(-1)^{(\eta(k_2)-\eta(k_2))(k^*+k_1+k_2)}\exp(i(\eta(k_1) - \eta(k_2))\cdot)$$
$$\times G_{k_1}(\cdot + \pi(k_2 + k^*))\overline{G_{k_2}(\cdot + \pi(k_1 + k^*))}.$$

The exponential factors of these two terms will be of opposite sign if and only if

(5.25) $\qquad (\eta(k_1) - \eta(k_2))(k_1 + k_2)$ is odd for all $k_1 \neq k_2$.

Therefore, the terms themselves will be equal and of opposite sign once we have shown that

(5.26)
$$G_{k_1}(\cdot + \pi(k_1 + k^*))\overline{G_{k_2}(\cdot + \pi(k_2 + k^*))} = G_{k_1}(\cdot + \pi(k_2 + k^*))\overline{G_{k_2}(\cdot + \pi(k_1 + k^*))}.$$

If one of $2c_\Xi k_1$ and $2c_\Xi k_2$ is even and the other is odd, then G_{k_1} and $\overline{G}_{k_2}$ are either both H or both $\overline{H}$ and (5.26) is immediate. In case the parities of $2c_\Xi k_1$ and $2c_\Xi k_2$ are the same, one of G_{k_1} and $\overline{G}_{k_2}$ is H and the other is $\overline{H}$, and we use the observation obtained from (5.16) that

$$\overline{H(y + \pi k)} = \exp(i2c_\Xi y)(-1)^{2c_\Xi k}H(y + \pi k),$$

since $\sum_{\xi\in\Xi}\xi = 2c_\Xi$. Then, if say both $2c_\Xi k_1$ and $2c_\Xi k_2$ are odd, we have

$$G_{k_1}(\cdot + \pi(k_1 + k^*))\overline{G_{k_2}(\cdot + \pi(k_2 + k^*))}$$
$$= \exp(i2c_\Xi\cdot)(-1)^{2c_\Xi(k^*+k_1)}H(\cdot + \pi(k^* + k_1))H(\cdot + \pi(k^* + k_2))$$
$$= \exp(i2c_\Xi\cdot)(-1)^{2c_\Xi(k^*+k_2)}H(\cdot + \pi(k^* + k_2))H(\cdot + \pi(k^* + k_1))$$
$$= G_{k_1}(\cdot + \pi(k_2 + k^*))\overline{G_{k_2}(\cdot + \pi(k_1 + k^*))}.$$

Therefore, it is sufficient to choose η as a 1-1 mapping on $\mathcal{G}$ so that (5.25) holds. To ensure that $H_k = H$ for $k = 0 \in \mathcal{G}$, we need $\eta(0) = 0$.

Finally, when H_k are chosen as in (5.24), their Fourier coefficients exhibit exponential decay. Therefore, any function of the form

$$\sum_{j\in\mathbb{Z}^s} b_k(j)K_k(\cdot/2 - j)/2^s, \qquad b_k \in \ell_2(\mathbb{Z}^s),$$

belongs to V_0.

Assume for the moment that a mapping η can be found with the properties

(5.27) $$\eta(0) = 0, \quad \text{and} \quad (\eta(k_1) + \eta(k_2))(k_1 + k_2) \text{ is odd if } k_1 \neq k_2.$$

Let $O_{-1,k}$ denote the spaces generated by the $\mathbb{Z}^s$ translates of $K_k(\cdot/2)/2^s$, $k \in \mathcal{G}$; then $O_{-1,0} = V_{-1}$. The question remains as to whether we have captured all of V_0 in this way; i.e., whether

$$V_{-1}^{\perp} = \bigoplus_{\substack{k\in\mathcal{G}\\k\neq 0}} O_{-1,k}.$$

This is equivalent to whether given arbitrary $A \in L_2([0, 2\pi]^s)$, the functions B_μ in (5.19) can be found. The π-periodicity of B_μ in each variable allows the expansion of (5.19) into 2^s equations in the unknowns B_μ: (For convenience, we again change the index to the set $\mathcal{G}$.)

(5.28) $$A(y + \pi k) = \sum_{k^*\in\mathcal{G}} B_{k^*}(y)H_{k^*}(y + \pi k), \quad \forall k \in \mathcal{G}.$$

Therefore, the decomposition

(5.29) $$V_0 = V_{-1} \oplus V_{-1}^{\perp} = V_{-1} \oplus \bigoplus_{\substack{k\in\mathcal{G}\\k\neq 0}} O_{-1,k}$$

exists if the system of equations (5.28) is solvable. But this system is solvable provided the mapping η exists. Indeed, if the rows of the matrix W of the system (5.28) are indexed by $k \in \mathcal{G}$ and the columns by $k^* \in \mathcal{G}$, then

$$W = [H_{k^*}(\cdot + \pi k)]_{k\in\mathcal{G},k^*\in\mathcal{G}}\,.$$

The conditions (5.22) imply that

$$\det(W^T\overline{W}) = \prod_{k_1\in\mathcal{G}}\left(\sum_{k\in\mathcal{G}}|H_{k_1}(\cdot+\pi k)|^2\right) = 1.$$

The spaces $O_{\nu,k}$ generated by the $\mathbb{Z}^s$ translates of $K_k(2^\nu\cdot)$, $k\in\mathcal{G}\backslash\{0\}$,$\nu\in\mathbb{Z}$, give an orthogonal decomposition of $L_2(\mathbb{R}^s)$.

(5.30)Theorem. *If M_Ξ is the box spline associated with a unimodular matrix $\Xi\in\mathbb{Z}^{s\times n}$, then the decomposition (5.29) exists for $s=1,2,3$. In particular, there are spline functions K_k, $k\in\mathcal{G}$, for which the set of dilations and translations,*

$$\left\{K_k(2^\nu\cdot-j): j\in\mathbb{Z}^s,\ \nu\in\mathbb{Z},\ k\in\mathcal{G}\backslash\{0\}\right\},$$

form a dense orthogonal set in $L_2(\mathbb{R}^s)$.

Proof. From the above discussion, we must find a 1-1 mapping $\eta:\mathcal{G}_s\to\mathcal{G}_s$ with the properties (5.27).

The mapping η with $\eta(0)=0$ and $\eta(1)=1$ is easy to construct if $s=1$.

When $s=2$, one choice of the mapping η is

$$\begin{matrix}(0,0)\mapsto(0,0) & (0,1)\mapsto(0,1)\\ (1,0)\mapsto(1,1) & (1,1)\mapsto(1,0).\end{matrix}\tag{5.31}$$

Finally, a suitable mapping η in the case of $s=3$ is

$$(0,0,0)\mapsto(0,0,0),\ (1,0,0)\mapsto(1,1,0),\ (0,1,0)\mapsto(0,1,1),$$

$$(1,1,0)\mapsto(1,0,0),\ (0,0,1)\mapsto(1,0,1),\ (1,0,1)\mapsto(0,0,1),$$

$$(0,1,1)\mapsto(0,1,0),\ (1,1,1)\mapsto(1,1,1).$$ ♠

It should be noted that it is impossible to find the mapping η for $s\geq 4$; *i.e.*, the choice (5.24) for the functions H_k cannot work in general. When $c_\Xi\in\mathbb{Z}^s$, then $G_k(y)=H(y)=\exp(-iyc_\Xi)G^*(y)$ for every $k\in\mathcal{G}$ where G^* is a real valued function. In this case, the complex exponentials could be factored out of the matrix W to obtain a $2^s \times 2^s$ matrix W_1 with entries $\pm x_k$ for 2^s indeterminants x_k such that

$$W_1^T W_1 = (\sum_{k \in G} x_k^2)^{2^s} I.$$

Such matrices exist only for $s = 0, 1, 2, 3$ (see Taussky [T]). Of course it would be interesting to determine whether there are choices of the functions H_k that would lead to a solvable system (5.28).

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