Singular Loci of Rationally Smooth Orbit Closures in Flag Varieties

By

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Abstract

Let G be a semi-simple algebraic group over $\mathbb{C}$, B a Borel subgroup, and T a maximal torus contained in B. In the first part of this thesis, we examine the singular loci of rationally smooth T-orbit closures $X = \overline{T \cdot x}$ in the flag variety G/B in types A and D. In type A, we prove that a T-orbit closure X in G/B is smooth if and only if it is rationally smooth. In the type D case, where this statement is known to be false, we investigate how the method used to prove the type A case fails. In particular, for $y \in X$, $S := T_y$ the stabilizer of y (assumed to be connected), and $Y = \overline{S \cdot x}$, we give a description of the S-weights of the tangent space $T_y(\Sigma)$, where $\Sigma \subseteq Y$ is any irreducible S-stable surface containing y.

In the second part of this thesis, we examine the maximal singularities of affine Schubert varieties $X(w)$ in the affine flag variety $\mathcal{G}/\mathcal{B}$ in type $A^{(1)}$, which are equipped with the action of a particular torus $\widehat{T}$. Let $E^{-}(X(w), u)$ be the set consisting of the $\widehat{T}$-curves C in $X(w)$ which contain a $\widehat{T}$-fixed point u, but whose $\widehat{T}$-fixed point set $C^{\widehat{T}} = \{u, v\}$ satisfies $u < v \leq w$. We obtain a partial characterization of the set $E^{-}(X(w), u)$, where the $\widehat{T}$-fixed point u is a maximal singularity of $X(w)$. Furthermore, we provide a necessary condition for a $\widehat{T}$-fixed point of a *rationally smooth* affine Schubert variety to be a maximal singularity. Finally, we prove that the affine permutation w

corresponding to any rationally smooth, but singular, Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ contains the pattern 3412. Using this result, we provide a proof of a conjecture by Billey-Crites that states that a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ is smooth if and only if it is indexed by an affine permutation w that avoids the patterns 3412 and 4231.

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Contents

Chapter 1

Introduction

In this thesis, we investigate the smoothness of certain rationally smooth subvarieties of flag varieties in two different contexts.

Our first context is the classical algebraic group theory setting: given a semisimple algebraic group G (eg. $\mathrm{SL}_n(\mathbb{C})$) and a Borel subgroup B (eg. for $G = \mathrm{SL}_n(\mathbb{C})$, the subgroup of upper triangular matrices) of G, we form the flag variety G/B. The group G, and hence any subgroup of G, acts as a group of transformations on G/B (via left translation). One subgroup action of particular interest is that of a maximal torus T (eg. for $G = \mathrm{SL}_n(\mathbb{C})$, the subgroup of diagonal matrices) contained in B. The irreducible T-stable subvarieties of G/B enjoy the structure of what we call in this thesis a *T-variety* (See Definition 2.6.1 below). Many authors have examined the homogeneous space G/B. Important examples of irreducible T-stable subvarieties of G/B that often arise in these explorations are T-orbit closures and Schubert varieties $X(w)$, i.e. closures of B-orbits of T-fixed points w of G/B. Properties of T-orbit closures have been examined by Morand in [33] and by Carrell and Kurth in [14]. A good resource for information on Schubert varieties is the book [3] by Billey-Lakshmibai.

A great deal of attention has been paid to characterizing the smooth loci of T-stable subvarieties of G/B (as well as other partial flag varieties of G), but there is also a substantial amount of interest in determining the loci of *rationally smooth* points of such varieties and in relating the two properties (See Definition 2.13.1 below). In the case of Schubert varieties, significant work on these topics has been done by Kazhdan-Lusztig (see [23],[24]), Lakshmibai-Seshadri (see [30]), Carrell-Peterson (see [12], [13]), Kumar (see [27]), Arabia (see [1]), Boe-Graham (see [7]), and Carrell-Kuttler (see [15]), amongst others. A considerable amount of the notable research on the singular loci of Schubert varieties and rational smoothness of Schubert varieties has been included the aforementioned book [3]. In the more general context of arbitrary varieties with torus actions, rational smoothness has been studied by Arabia (see [1]) and Brion (see [9], [10]).

Carrell-Peterson showed that a Schubert variety $X(w)$ in G/B is rationally smooth at a T-fixed point x if and only if the number of irreducible T-stable curves (T-curves) containing y is equal to the dimension of $X(w)$ for all $x \leq y \leq w$,[1] that is, the Bruhat graph of $X(w)$ is $(\dim X(w))$-regular at x and all vertices above x (See Theorem E in [13]). More generally, Brion (see [9]) showed that a T-variety X is rationally smooth at an attractive[2] T-fixed point x if it is rationally smooth in a punctured neighbourhood of x and the number of T-curves in X containing x is equal to the dimension of X.

Dale Peterson showed that, when working over the field $\mathbb{C}$ of complex numbers, if G is simply laced (i.e. of type A, D, or E), then all rationally smooth Schubert varieties in G/B are nonsingular (more generally, Peterson showed that the smooth and rationally smooth loci coincide for such Schubert varieties; see [15]). This was originally proven for type A by Deodhar (see [18]). From the research of Carrell-Kuttler, it is known that Peterson's theorem does not extend to arbitrary T-varieties in G/B, when G is simply laced. Indeed, Carrell-Kuttler have produced an example in the type D case of a T-orbit closure X in G/B containing a T-fixed point at which X is rationally smooth, but singular (see Example 7.1 in [15]).

In this thesis, based on the work of Carrell-Kuttler in [15], we considered this problem for T-orbit closures in G/B in the type A case. In [15], Carrell-Kuttler obtained a sufficient condition for an attractive T-fixed point of a T-variety to be nonsingular (see Theorem 1.4 in [15] or Theorem 2.11.5 below). This condition involves so-called *Peterson translates* along curves. Peterson translates have figured prominently in our work throughout this thesis. Using

[1] where $\leq$ denotes the Bruhat-Chevalley order on the Weyl group W.

[2] See Definition 2.9.1 below. Note, any T-fixed point of a T-variety in G/B is attractive.

Carrell-Kuttler's condition, we have proven that, in type A, a T-orbit closure in G/B is rationally smooth if and only if it is smooth (see Theorem 3.9.5 below). We remark that some preliminary investigations on this problem were carried out by Yasmin Omanovic as part of an undergraduate summer research project supervised by Jochen Kuttler, but our work is independent of these investigations.

The second context appearing in this thesis is an infinite dimensional analogue to the situation above which is referred to as the Kac-Moody setting. The flag variety under consideration is $\mathcal{G}/\mathcal{B}$, where $\mathcal{G} = \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{B} = ev^{-1}(B)$, where $ev : \mathcal{P} := \mathrm{SL}_n(\mathbb{C}[[x]]) \to \mathrm{SL}_n(\mathbb{C})$ is entry-wise evaluation at $x = 0$ and B is the Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting of upper triangular matrices. The quotient $\mathcal{G}/\mathcal{B}$ is an ind-variety, that is, a direct limit of finite dimensional projective varieties over $\mathbb{C}$. In this case, $\mathcal{G}$, $\mathcal{B}$, and hence $\mathcal{G}/\mathcal{B}$ are equipped with the action of the product torus $\widehat{T} = T \times S$, where T is the subset of $\mathcal{G}$ consisting of the diagonal matrices with constant entries, which acts by translation on $\mathcal{G}$, and $S = \mathbb{C}^*$ which acts on $\mathcal{G}$ through its obvious action on $\mathbb{C}((x))$ (i.e. $(s \cdot f)(x) = f(s \cdot x)$). This torus plays a role analogous to the maximal torus in the classical G/B framework.

In this context, there is a natural generalization of the concept of a Schubert variety known as an affine Schubert variety. Once again, there is interest in describing the singular and rationally smooth loci of affine Schubert varieties and, in particular, knowing to what extent the results in the classical setting carry over to the affine context. One particularly influential such result from the classical backdrop is due to Lakshmibai-Sandhya. In [29], they introduced the concept of pattern avoidance for permutations in S_n and proved that a classical Schubert variety $X(w)$ in the type A case is smooth if and only if its associated permutation w avoids the patterns 3412 and 4231 (see Theorem 1 in [29] or Theorem 6.1.1 below). This result was generalized by Billey-Postnikov to all types (see [4]). Also in [29], Lakshmibai-Sandhya formulated a conjecture that would identify the maximal singularities of $X(w)$ with those permutations having a specific combinatorial relationship to either the 3412 or 4231 pattern in w (by a maximal singularity of $X(w)$ we mean a singular fixed point for the action of the maximal torus which is maximal with respect to the Bruhat-Chevalley order; note that these points determine the singular locus of $X(w)$). This conjecture was later proven concurrently and independently by Billey-Warrington (see [5]), Cortez (see [16]), Kassel-Lascoux-Reutenauer (see [22]), and Manivel (see [32]).

Returning to the affine setting, in [2] Billey-Crites defined the concept of pattern avoidance for affine permutations and then used it to give a full characterization of rationally smooth Schubert varieties in $\mathcal{G}/\mathcal{B}$ for $n \geq 3$ (see Theorem 1.1 in [2] or Theorem 6.1.3 below). From this theorem, they proved that a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ is singular if its corresponding affine permutation w contains either of the patterns 3412 or 4231 (see Corollary 1.2 in [2] or Corollary 6.1.5 below). Furthermore, they conjectured that converse is also true, which would mean that a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ is smooth if and only if it is indexed by an affine permutation w that avoids the pattern 3412 or 4231 (see Conjecture 1 in [2] or Conjecture 6.1.6 below). They have verified this up to $n = 5$.

In this thesis, we provide a proof of the Billey-Crites conjecture (See Theorem 6.5.2 below). This conjecture was independently proven by Richmond-Slofstra in [35] using a different method from the one used in this document. Our work on this project began with some initial discussions with Andrew Crites, one of the coauthors of [2], during which a general strategy was developed for finding the proof. However, the proof, as presented in this thesis, is more involved and provides a more detailed description of the maximal singularities of singular rationally smooth Schubert varieties in $\mathcal{G}/\mathcal{B}$ than was initially discussed. In order to prove this conjecture, it sufficed to show that the affine permutation w corresponding to any rationally smooth, but singular, Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ contains the pattern 3412. Our approach to solving this problem involved an examination of the maximal singularities of singular rationally smooth Schubert varieties in $\mathcal{G}/\mathcal{B}$.

Our work on maximal singularities is both geometric and combinatorial in nature, involving $\widehat{T}$- weights of certain tangent spaces, irreducible $\widehat{T}$-stable curves and surfaces, and the Bruhat-Chevalley order on the affine Weyl group $\widehat{W}$. While Peterson translates are crucial to the proof of our main result in the classical G/B setting, they figure far more prominently in our work in the affine setting. In addition, our results on irreducible $\widehat{T}$-stable surfaces in $\mathcal{G}/\mathcal{P}$ as presented in [11] are central to our research here. In Theorem 5.9.5, we provide a necessary condition for a $\widehat{T}$-fixed point of a rationally smooth

Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ to be a maximal singularity. Specifically, we showed that the Bruhut graph of $X(w)$ contains one of two "kite patterns" (see Definitions 5.8.1 and 5.8.2 below) going through each maximal singularity. Now, for every maximal singularity of $X(w)$, the maximal $\widehat{T}$-fixed point (with respect to the Bruhat-Chevalley order on $\widehat{W}$) appearing in its kite pattern contains the pattern 3412 (see Theorem 6.4.1 below). Using this, we are able to show that w contains the pattern 3412 (See Theorem 6.5.1 below).

Regarding the structure of thesis, in Chapter 2, we provide some general background material which applies to our work in both the classical and affine settings. Specifically, we recall some standard facts about linear algebraic groups and varieties, including facts concerning torus actions and T-orbits and their closures. It is in this chapter that we introduce and review some well-know results about those objects which are central to our work, namely, T-varieties, T-curves, T-surfaces, attractive T-fixed points, and Peterson translates. None of the material presented in this chapter is original to this thesis.

Our main result in Chapter 3 is that, in the (classical) type A case, a T-orbit closure X in G/B is smooth if and only if it is rationally smooth (see Theorem 3.9.5). The structure of this chapter is as follows: after providing some well-known background information on the general G/B context, we introduce the notion of *combinatorial regularity* (see Definition 3.4.1 below) for points of T-varieties. We work with this notion since it is equivalent to the concept of rational smoothness for T-orbit closures in G/B under the assumption that all stabilizers in T are connected (which we show using a result of Brion, see Theorem 3.8.1), but is more user-friendly for the methods deployed in this chapter. Then, to each point y in a T-orbit closure X we associate a subvariety Y of X which has the structure of an S-variety, where S is the stabilizer of y in T. Using well-established techniques, we relate the (combinatorially) regular loci of X and Y. We then use an inductive argument, which reduces to applying the methods of Carrell-Kutter to Y (see Theorem 1.4 in [15] or Theorem 2.11.5 below), to prove that X is regular whenever it is combinatorially regular (see Theorem 3.9.4). Our main result follows. The statement of Theorem 3.9.5 is known to be false in general in the type D case owing to an example of Carrell-Kuttler (see Example 7.1 in [15]). As such, in Section 3.10, we analyze the failure of the method we used for type A in the type D context. Specifically, in the type A case, all S-surfaces in Y

through y are nonsingular, whereas in the type D case this is no longer true (see Examples 3.10.7 and 3.10.8). For an S-surface Σ which is singular at y, the tangent space $T_y(\Sigma)$ has at least three S-weights. In Lemma 3.10.5, we give a description of the possible S-weights of $T_y(\Sigma)$.

Chapter 4 is primarily dedicated to describing the $\mathcal{G}/\mathcal{B}$ context and presenting the background material required for our work in Chapter 5 on maximal singularities of Schubert varieties in $\mathcal{G}/\mathcal{B}$. Although most of the chapter contains previously-known results there is some original material presented there, including Example 4.8.11, Lemma 4.9.4, and the material in Sections 4.12 and 4.13.

Chapter 5 is where we present our work on maximal singularities of Schubert varieties $X(w)$ in $\mathcal{G}/\mathcal{B}$. Much of our effort is focused on describing the set $E^{-}(X(w),u)$ consisting of the $\widehat{T}$-curves in $X(w)$ which contain a $\widehat{T}$-fixed point u, but whose $\widehat{T}$-fixed point set $C^{\widehat{T}} = \{u,v\}$ satisfies $u < v \leq w$. Describing $E^{-}(X(w),u)$ in turn gives descriptions of weights of $T_u\big(X(w)\big)$, the tangent space of $X(w)$ at u. Indeed, almost all characterizations of the elements of $E^{-}(X(w),u)$ are accompanied by the corresponding characterizations of the roots associated with the $\widehat{T}$-curves. In Section 5.2 we define four types of root/$\widehat{T}$-curve pairs: Type I (strong, weak), Type II, and pseudo Type II. These root/$\widehat{T}$-curve pair types stem from our work in [11] on $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{P}$. Using these pair types, we obtain a partial characterization of the set $E^{-}(X(w),u)$, where the $\widehat{T}$-fixed point u is a maximal singularity of any singular Schubert variety in $\mathcal{G}/\mathcal{B}$ (see Theorem 5.6.1 and Theorem 5.6.3). In the case that $X(w)$ is rationally smooth, we prove that $E^{-}(X(w),u)$ contains a strong Type I pair of $\widehat{T}$-curves, again when the $\widehat{T}$-fixed point u is a maximal singularity (see Theorem 5.7.2). From this, we obtain our main result of this chapter: a maximal singularity $u \in X(w)^{\widehat{T}}$ of a rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ satisfies one of two "kite properties", that is, the Bruhut graph of $X(w)$ contains one of the two aforementioned kite patterns passing through u (see Definitions 5.8.1 and 5.8.2 and Theorem 5.9.5).

In Chapter 6, we first review the work of Billey-Crites on pattern avoidance and (rationally) smooth Schubert varieties in $\mathcal{G}/\mathcal{B}$ as presented in [2]. We then define the notion of a *wide* affine permutation (see Definition 6.3.1 below) and develop some theory regarding wide affine permutations related to the Bruhat-Chevalley order on the affine Weyl group $\widehat{W}$ and to our work on maximal singularities (and kite properties) in Chapter 5, specifically Theorem 5.9.5. Wide permutations are significant since they all contain the pattern 3412. Using the theory of wide affine permutations we prove that the affine permutation w corresponding to a singular rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ contains the pattern 3412 (See Theorem 6.5.1). We conclude the chapter by providing a proof of Conjecture 6.1.6 (See Theorem 6.5.2 below).

Before we go any further, we will establish some notation and state our universal assumptions used in this thesis. We will always work over $\mathbb{C}$. We will view varieties as sets of closed points and assume all algebraic groups are affine. The coordinate ring of an affine variety X will be denoted by $\mathbb{C}[X]$ and, if X is irreducible, its quotient field will be denoted by $\mathbb{C}(X)$. The tangent space of a variety X at a point $x \in X$ will be denoted by $T_x(X)$. When we specify that an algebraic group acts on a variety, we assume it acts morphically. For a variety X with the action of an algebraic group T, we will write X^T for the set of T-fixed points of X. For a point $x \in X$, we will use T_x and $T \cdot x$ to denote the stabilizer of x and the orbit of x, respectively. Given a finite-dimensional complex vector space V, we will often identify V with the associated affine space $\mathbb{A}(V)$. In particular, this will enable us to talk about subvarieties of V. Finally, if S is a set, then $|S|$ denotes the cardinality of the set.

Chapter 2

Preliminaries

In this chapter, we will provide the general background information which underlies both of our research projects. None of the material presented in this chapter is original to this thesis. We will review some basic facts regarding linear algebraic groups and varieties, many of which can be found in the books by Borel (see [8]), Humphreys (see [20]), and Hartshorne (see [19]). In [11], we stated some well-known results in these areas and provided a few of the proofs as a convenience to the reader. As many of those results are also relevant to this thesis, we will state them again here, but this time without proofs. In some instances, we will refer the reader back to [11] as an example of where certain proofs can be found.

2.1 Tori

Definition 2.1.1. A *torus* T is an algebraic group isomorphic to G_m^n, for some $n \in \mathbb{N}$. A *diagonalizable* group is an algebraic group which is isomorphic to a closed subgroup of a torus.

For example, the set of all $n \times n$ diagonal matrices in the general linear group $\mathrm{GL}_n(\mathbb{C})$ is a torus. As a closed subgroup of G_m, the group μ_n of n^{th} roots of unity is a diagonalizable group.

Every connected diagonalizable group over $\mathbb{C}$ is a torus. Homomorphic images of tori are again tori. Notably, any connected subgroup of a torus and any quotient of a torus by a torus is again a torus.

2.2 Characters and One-Parameter Subgroups

A *character* of an algebraic group G is a homomorphisms $\chi : G \to \mathrm{G}_m$. The set of all such homomorphisms, denoted $X(G)$, has the natural structure of an abelian group (we use additive notation $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$, for all $g \in G$). The set $X(G)$ can be identified with the group-like elements of the Hopf algebra $\mathbb{C}[G]$.

If $\phi : G \mapsto H$ is a homomorphism of algebraic groups, then there is an induced group homomorphism $\phi^* : X(H) \to X(G)$ given by $\phi^*(\chi) = \chi \circ \phi$ (alternatively, restrict the $\mathbb{C}$-algebra homomorphism $\phi^* : \mathbb{C}[H] \to \mathbb{C}[G]$ to the group-like elements). Thus, we have a contravariant functor from the category of algebraic groups to the category of abelian groups sending G to $X(G)$.

Restricting this functor to the subcategory of diagonalizable groups yields an anti-equivalence between the latter category and the category of finitely generated abelian groups, under which the n-dimensional tori correspond to free abelian groups of rank n. In particular, $X(\mathrm{G}_m^n)$ is the free abelian group on the n projections

$$(c_1, c_2, \ldots, c_n) \mapsto c_i.$$

In the other direction, a homomorphism $\lambda : \mathrm{G}_m \to G$ is called a *one-parameter subgroups* (or cocharacter) of G. If G is a diagonalizable group, then the set $Y(G)$ of all one-parameter subgroups also has a natural structure of a free abelian group of finite rank ($(\lambda_1 + \lambda_2)(c) = \lambda_1(c)\lambda_2(c)$, for all $c \in \mathrm{G}_m$). In this case, there is a pairing

$$X(G) \times Y(G) \to X(\mathrm{G}_m) \simeq \mathbb{Z}$$

given by

$$(\chi, \lambda) \mapsto \langle \chi, \lambda \rangle,$$

where $\langle \chi, \lambda \rangle$ is the integer such that $(\chi \circ \lambda)(c) = c^{\langle \chi, \lambda \rangle}$. If G is a torus T, then the pairing is non-degenerate, that is, it identifies $X(T)$ with $Y(T)^* = \mathrm{Hom}_{\mathbb{Z}}(Y(T), \mathbb{Z})$. In particular, if T is an n-dimensional torus, then $Y(T)$ is a free abelian group of rank n. Concretely, $Y(\mathrm{G}_m)$ is the free abelian group on the n cocharacters

$$c \mapsto (1, 1, \ldots, 1, c, 1, \ldots, 1).$$

2.3 Representations of Algebraic Groups

Finite-dimensional complex vector spaces with torus actions play a prominent role in our work as we will often be able to reduce questions involving varieties with T-actions to problems concerning vector spaces.

Let V be a $\mathbb{C}$-vector space. A (rational) representation $\rho : G \to \mathrm{GL}(V)$ induces a linear action of G (as an abstract group) on V given by $g \cdot v = \rho(g)(v)$. In this case, we say that the vector space V is a *G-module* and that the algebraic group G *acts on* V. Of particular interest to us, is the case in which G is a torus T.

We will now recall some well-known results concerning T-modules. We begin with a fundamental fact, which we will use frequently. Let V be a T-module. Then V has a *weight space decomposition*:

$$V = \bigoplus_{\alpha \in X(T)} V_\alpha,$$

where

$$V_\alpha = \{v \in V \mid t \cdot v = \alpha(t)v, \text{ for all } t \in T\}.$$

The α for which $V_\alpha \neq 0$ are called the *weights* of T in V and V_α is called a *weight space*. We denote the set of weights by $\Omega_T(V)$ or simply $\Omega(V)$ if the torus involved is clear.

Remark 2.3.1. More generally, it suffices in the above discussion that T be a diagonalizable group.

If $v = \sum v_\alpha \in V$, where $v_\alpha \in V_\alpha$, then the set

$$s(v) := \{\alpha \mid v_\alpha \neq 0\}$$

in $X(T)$ is called the *support* of v. Also, a T-module V is called *multiplicity free* if $\dim V_\alpha = 1$, for all $\alpha \in \Omega(V)$.

Any G_m-module V has an induced $\mathbb{Z}$-grading defined by $V_d := V_\alpha$, where $\alpha(c) = c^d$. For an arbitrary torus T, we obtain a G_m-module structure on a T-module V by fixing a $\lambda \in Y(T)$ and defining $c \cdot v := \lambda(c) \cdot v$. This in turn induces a $\mathbb{Z}$-grading on a V by setting

$$V_d := \bigoplus_{\substack{\alpha \in X(T) \\ \langle \alpha, \lambda \rangle = d}} V_\alpha.$$

Using this $\mathbb{Z}$-grading, we define

$$V_- := \bigoplus_{d<0} V_d, \qquad V_0 := \bigoplus_{d=0} V_d, \qquad \text{and} \qquad V_+ := \bigoplus_{d>0} V_d.$$

If W is a T-stable subspace of a T-module V, then both W and V/W inherit T-modules structures from V. They are endowed with the weight space decompositions

$$W = \bigoplus_{\alpha \in X(T)} W_\alpha, \qquad \text{where} \quad W_\alpha = W \cap V_\alpha,$$

and

$$V/B = \bigoplus_{\alpha \in X(T)} (V/W)_\alpha, \qquad \text{where} \quad (V/W)_\alpha = V_\alpha / W_\alpha.$$

In particular,

$$\Omega(W),\ \Omega\big(V/W\big) \subseteq \Omega(V).$$

In the event that V is multiplicity free, these decompositions become

$$W = \bigoplus_{\alpha \in \Gamma} V_\alpha$$

and

$$V/W = \bigoplus_{\alpha \in \Omega(V) \setminus \Gamma} V_\alpha,$$

for some $\Gamma \subseteq \Omega(V)$.

In addition, the dual space V^* inherits a T-module structure from V via the action $(t \cdot f)(v) = f(t^{-1} \cdot v)$, for all $t \in T$, $f \in V^*$. It is clear that V^* decomposes under this action as

$$V^* = \bigoplus_{\alpha \in \Omega(V)} (V_\alpha)^*,$$

where the weight of $(V_\alpha)^*$ is $-\alpha$, and hence

$$\Omega(V^*) = -\Omega(V).$$

2.4 Algebraic Group Actions on Varieties

Let G be an algebraic group and let X be a variety. By a (left) action of G on X, we mean a morphism $G \times X \to X$ that equips the underlying set X with a left action of the abstract group G. We will write $g \cdot x$ to denote the image of (g, x) under such a morphism.

A map $f : X \to Y$ of varieties with G-actions is called G-*equivariant* if

$$g \cdot (f(x)) = f(g \cdot x)$$

for all $x \in X$, $g \in G$.

Example 2.4.1. To give an action of an algebraic group G on a finite-dimensional $\mathbb{C}$-vector space V is equivalent to giving an action of G on the affine space $\mathbb{A}(V)$.

Example 2.4.2. If V is a G-module, then the action of G on V induces an action of G on the projective space $\mathbb{P}(V)$, where G acts on each line $[v]$ by the rule $g \cdot [v] = [g \cdot v]$.

We will use X^G, $G \cdot x$, and G_x to denote the set of fixed points of G, the orbit of x, and the stabilizer of x, respectively. The following basic facts can be found in chapters 7 and 8 of [20]:

Lemma 2.4.3. *Let G be any algebraic group and let X be variety with a G-action.*

1) *G-orbits are open in their closures.*

2) *G-orbits are irreducible, if G is connected.*

3) *G-orbits are smooth.*

4) *G-orbit closures are G-stable.*

5) *Every G-orbit closure contains a closed orbit.*

6) *X^G is closed in X.*

7) *G_x is a closed subgroup of G, for all $x \in X$.*

From Property 1) of Lemma 2.4.3, we observe that $G \cdot x$ can be equipped with the structure of a locally closed subvariety of X. The Orbit-Stabilizer Theorem is then valid, i.e the natural map

$$G/G_x \to G \cdot x$$

is a G-equivariant isomorphism of varieties.

In addition to G-actions on varieties, we also want to consider G-actions on structures associated with varieties.

Let X be a variety with G-action and let $x \in X^G$. There is an induced action of G on the tangent space $T_x(X)$: let $g \in G$ and also use g to denote the map $g : X \to X$ defined by $y \mapsto g \cdot y$, with differential $d_x g : T_x(X) \to T_x(X)$, then the induced action of G on $T_x(X)$ is given by $g \cdot \delta = d_x g(\delta)$. If Y is any G-stable subvariety of X containing x, then $T_x(Y)$ is a G-stable subspace of $T_x(X)$.

In the case that X is an affine variety with G-action, there is an induced G-action on the coordinate ring $\mathbb{C}[X]$ given by

$$(g \cdot f)(x) = f(t^{-1} \cdot x),$$

for all $g \in G$, $f \in \mathbb{C}[X]$, and $x \in X$. This, in turn, induces a G-action on the field of rational functions $\mathbb{C}(X)$ in the obvious way. Moreover, if $x \in X^G$, then the maximum ideal $\mathfrak{m}_x$ in $\mathbb{C}[X]$ of functions which vanish at x is also G-stable.

The action of G on $\mathbb{C}[X]$ is locally finite, that is, for each $f \in \mathbb{C}[X]$, there exists a finite dimensional subspace of $\mathbb{C}[X]$ containing $G \cdot f$. In particular, this yields that $\mathbb{C}[X]$ is the union of finite dimensional G-stable subspaces, upon each of which G acts rationally. In the case that G is a torus T, we obtain a weight space decomposition

$$\mathbb{C}[X] = \bigoplus_{\alpha \in X(T)} \mathbb{C}[X]_\alpha.$$

If $f, g \in \mathbb{C}[X]$ are elements of weight α and β, respectively, then $f/g \in \mathbb{C}(X)$ has weight $\alpha - \beta$. Furthermore, in light of the given G actions on $\mathbb{C}[X]$ and $T_x(X)$, the canonical isomorphism

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_x(X)^*,$$

where the coset $f + \mathfrak{m}_x^2$ is identified with $d_x f$, is G-equivariant when x is fixed by G. In particular, taking G to be a torus T yields that

$$\Omega\big(T_x(X)^*\big) = \Omega\big(\mathfrak{m}_x/\mathfrak{m}_x^2\big) \subset \Omega\big(\mathbb{C}[x]\big).$$

This is significant as it enables us to apply results about the weights of $\mathbb{C}[X]$ to the weights of $T_x(X)^*$ and, subsequently, to the weights of $T_x(X)$, when X is an affine variety with T-action.

2.5 Torus Orbits and Torus Orbit Closures

In this section, we will review some useful facts concerning T-orbits and their closures, including some lemmas which will be utilized in our work on torus orbit closures in Chapter 3.

In addition to satisfying the properties listed in Lemma 2.4.3, T-orbits are also affine, which follows from the Orbit-Stabilizer Theorem and the fact that T is commutative.

In the context of T-modules, we have a convenient tool for computing dimensions of T-orbits:

Lemma 2.5.1. *If v is an element of a T-module V with support $s(v)$, then $\dim T \cdot v = \operatorname{rank} M$, where M is the $\mathbb{Z}$-module generated by $s(v)$.*

Proof. See for example Lemma 2.9 in [11]. □

As a consequence, a T-orbit $T \cdot v$ is one dimensional if and only if the elements of $s(v)$ are proportional over $\mathbb{Q}$ and at least one is nonzero.

One of the benefits of working with affine T-orbit closures X is the connection between the geometry of the T-action on X and the representation of T on $\mathbb{C}[X]$ described in the following well-known lemma.

Lemma 2.5.2. *Let X be an affine variety with T-action. The following are equivalent:*

1) *X has finitely many orbits.*

2) *X has an open dense orbit.*

3) *$\mathbb{C}[X]$ is multiplicity free.*

Proof. For example, see Lemma 2.15 in [11]. □

More can be said in the case of affine T-orbit closures, as indicted in the next lemma:

Lemma 2.5.3. *If X is an affine variety with T-action containing an open dense orbit $T \cdot x$, then $X \setminus (T \cdot x)$ is pure of codimension one, that is, every irreducible component of $X \setminus (T \cdot x)$ has codimension one.*

Proof. More generally, it is well known that if U is a dense open affine subvariety of a variety X, then $X \setminus U$ is pure of codimension one. In the case where X is normal, this may be deduced from Proposition 6.3 A in [19]. The general case then follows by passing to the normalization of X. □

Consequently, in light of Lemma 2.5.2, the dimension of T-orbits in an affine T-orbit closure X go down by one, in other words, if $\dim X = n$, then there is a T-orbit of dimension i, for all $0 \leq i \leq n$.

Now, if T is a torus with closed subgroup S and X is a variety with T-action, then

$$s \cdot (t, x) := (ts^{-1}, s \cdot x)$$

defines an action of S on the product $T \times X$. Taking the quotient of $T \times X$ by this S-action we obtain $T \times^S X$, the contracted product of T and X, whose points are identified with the S-orbits in $T \times X$. The variety $T \times^S X$ is equipped with an action of T by setting

$$t' \cdot \overline{(t, x)} := \overline{(t't, x)}$$

Furthermore, the product $(T/S) \times X$ is equipped with a T-action by defining

$$t' \cdot (\bar{t}, x) = (\overline{t't}, t' \cdot x)$$

A routine calculation shows the following:

Lemma 2.5.4. *Let T act on $T \times^S X$ and $(T/S) \times X$ as above. Then the map*

$$\begin{aligned} \psi : T \times^S X &\to (T/S) \times X \\ \overline{(t, x)} &\mapsto (\bar{t}, t \cdot x) \end{aligned}$$

is a well-defined T-equivariant isomorphism.

Now, for any S-stable closed subvariety Z of X, we can again form the contracted product $T \times^S Z$. Equipped with a T-action on the left factor, $T \times^S Z$

is a T-stable closed subvariety of $T \times^S X$. Thus, if we fix an S-stable closed subvariety Y of X, we obtain an inclusion preserving map

$$\theta : \left\{ \begin{array}{c} S\text{-stable closed} \\ \text{subvarieties of } Y \end{array} \right\} \to \left\{ \begin{array}{c} T\text{-stable closed} \\ \text{subvarieties of } T \times^S Y \end{array} \right\}$$
$$Z \mapsto T \times^S Z$$

Given a T-stable closed subvariety of W of $T \times^S Y$, let Z denote its preimage under the natural map

$$Y \to T \times^S Y$$
$$y \mapsto \overline{(1, y)}$$

A straightforward calculation then shows that the map sending W to Z is an inverse of θ, so that:

Lemma 2.5.5. *θ is a bijection.*

We now consider a special case of this situation. Suppose that we are given a T-equivariant morphism $f : X \to T/S$ and let $u \in T/S$. Then S acts on the fibre $Y := f^{-1}(u)$.

Lemma 2.5.6. *The map*

$$\eta : T \times^S Y \to X$$
$$\overline{(t, y)} \mapsto t \cdot y$$

is a well-defined T-equivariant isomorphism.

Proof. The map

$$T \times Y \to X$$
$$(t, y) \mapsto t \cdot y$$

is constant on S-orbits since it send $s \cdot (t, y) = (ts^{-1}, s \cdot y)$ to $t \cdot y$. As such, η is well-defined. Furthermore, it is clear that η is T-equivariant. Now consider the diagram

$$\begin{tikzcd}
T \times^S Y \arrow[r, hook] \arrow[dr, "\eta"'] & T \times^S X \arrow[r, "\psi"] & (T/S) \times X \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

The first horizontal map on the left is the inclusion of $T \times^S Y$ as a closed subvariety of $T \times^S X$, and the map π is the projection

$$\begin{aligned}
\pi : (T/S) \times X &\to X \\
(\bar{t}, x) &\mapsto x
\end{aligned}$$

It follows from the definitions of η and ψ that this diagram commutes. Now let $\Gamma \subseteq (T/S) \times X$ be the image of $T \times^S Y$ in $(T/S) \times X$ under the closed embedding formed by the horizontal composition. By the commutativity of this diagram, η is an isomorphism if the projection π induces an isomorphism $\Gamma \to X$. According the definition of ψ, Γ consists of pairs $(\bar{t}, t \cdot y)$ such that $t \in T$ and $y \in Y$. Now, let $x \in X$. Then $x = t \cdot y$ for some $t \in T$ and some $y \in Y$. Indeed, setting $f(x) := w \in T/S$, we know that there exists a $t \in T$ such that $\overline{t^{-1}}w = u$ and hence $f(t^{-1} \cdot x) = \overline{t^{-1}}f(x) = \overline{t^{-1}}w = u$. Therefore, $x = t \cdot (t^{-1} \cdot x)$, where $t \in T$ and $t^{-1} \cdot x \in Y$. Consequently, since

$$f(x) = f(t \cdot y) = \bar{t}f(y) = \bar{t}u,$$

Γ is the graph of the morphism

$$\begin{aligned}
X &\to T/S \\
x &\mapsto f(x)u^{-1}
\end{aligned}$$

As such, the projection π yields an isomorphism from Γ to X, as required. $\square$

The subsequent lemma provides us with a nice relationship between T-orbit closures and S-orbit closures of elements of the fibre $Y := f^{-1}(u)$.

Lemma 2.5.7. *Let T be a torus with subgroup S, let X be a variety with T-action, and let $f : X \to T/S$ be a T-equivariant map. If $x, y \in f^{-1}(u)$, for some $u \in T/S$, then $x \in \overline{T \cdot y}$ if and only if $x \in \overline{S \cdot y}$.*

Proof. One direction is immediate since $\overline{S \cdot y} \subset \overline{T \cdot y}$.

Now consider an element $a = \overline{(1,y)} \in T \times^S Y$. The T-orbit closure $\overline{T \cdot a}$ in $T \times^S Y$ is the smallest T-stable closed subvariety of $T \times^S Y$ containing a. As a result of Lemma 2.5.5, this is equal to $T \times^S Z$, where Z is the smallest S-stable closed subvariety of Y containing y. However, $\overline{S \cdot y}$ is the smallest S-stable closed subvariety of Y containing y. Ergo, $\overline{T \cdot a} = T \times^S \overline{S \cdot y}$. Furthermore, by Lemma 2.5.6, the map

$$\begin{aligned} \eta : T \times^S Y &\to X \\ \overline{(t, y')} &\mapsto t \cdot y' \end{aligned}$$

identifies $\overline{T \cdot a}$ with $\overline{T \cdot y}$. Hence, if $x \in \overline{T \cdot y}$, then $\overline{(1,x)} \in \overline{T \cdot a} = T \times^S \overline{S \cdot y}$. Consequently, $\overline{(1,x)} = \overline{(t,w)}$, for some $t \in T$ and some $w \in \overline{S \cdot y}$. This means that $s \cdot (1,x) = (s^{-1}, s \cdot x) = (t,w)$ for some $s \in S$. In particular, we have that $s \cdot x = w$ and so $x = s^{-1} \cdot w$. Finally, since $\overline{S \cdot y}$ is S-stable, we obtain that $x \in \overline{S \cdot y}$, as required. □

Before we state the last fact of this section, we will first define some notation which appears in the lemma. Given a morphism $f : G_m \to X$ of varieties, the expression

$$\lim_{g \to 0} f(g) = y$$

is used to mean that we extend f to a morphism $\tilde{f} : \mathbb{A}^1 \to X$, by defining $\tilde{f}(0) = y$. This gives us the usual notion of a limit when working in a vector space.

Lemma 2.5.8. *Let X be an irreducible affine or projective variety with T-action that contains an open T-orbit $T \cdot x$. Then for any $y \in X$, there exists a $\lambda \in Y(T)$ such that $\lim_{g \to 0} \lambda(g) \cdot x \in T \cdot y$.*

Proof. See [25] and Lemma 2.1 in [14]. □

This lemma, which is a version of the Hilbert-Mumford criterion, is significant as it gives us a method in concrete examples of determining which T-orbits are contained in a given T-orbit closure.

2.6 T-Varieties

We now consider an important type of variety with T-action:

Definition 2.6.1. A *T-variety* X is an irreducible variety with a T-action such that X^T is finite and for all $x \in X^T$ there is an open T-stable affine neighborhood of x, which we will denote by X_x.

Remark 2.6.2. If X is a T-variety, then any closed irreducible T-stable subvariety Y of X is also a T-variety by taking $Y_y := Y \cap X_y$ for all y in the finite set $Y^T = Y \cap X^T$.

A key example for our purposes is:

Example 2.6.3. If V is a multiplicity free T-module of dimension $n+1$, then any closed irreducible T-stable subvariety of the projective n-space $\mathbb{P}(V)$ is a T-variety. (See Lemma 2.13 in [11], for example.)

This can be generalized as follows:

Example 2.6.4. As a corollary of Sumihiro's Theorem (see [36], [37]), if Y is a normal variety with T-action, then any closed irreducible T-stable subvariety X with finite T-fixed point set X^T is a T-variety.

2.7 T-Curves

Definition 2.7.1. Let X be a variety with T-action. A curve in X which is the closure of a one-dimensional T-orbit is called a *T-curve*.

Example 2.7.2. If C is a closed irreducible curve with T-action for which $C \neq C^T$, then C is a T-orbit closure and hence a T-curve. In particular, any closed T-stable irreducible curve in a T-variety X is a T-curve.

A specific type of T-curve of interest to us is:

Definition 2.7.3. A T-curve in a variety X with T-action is said to be *good* if $C = \overline{T \cdot z}$, where X is nonsingular at z.

One useful fact pertaining to T-curves is given in the following lemma.

Lemma 2.7.4. *If V is a multiplicity free T-module such that no two weights are proportional, then the only T-curves contained in V are the weight spaces V_α.*

Proof. See, for example, Lemma 2.20 in [11]. □

Let X be a variety with T-action. Denote by $E_T(X)$ the set of all closed irreducible T-stable curves in X and denote by $E_T(X,x)$ the set of all closed irreducible T-stable curves in X containing a T-fixed point x. If the torus involved is clear, we will denote these sets simply by $E(X)$ and $E(X,x)$, respectively. Moreover, if X is a T-variety, then both $E(X)$ and $E(X,x)$ consist entirely of T-curves.

From Lemma 2.7.4, we know that if V is a multiplicity free T-module where no two weights are proportional, then $E(V,0)$ is a finite set.

In the event that $E(X)$ is a finite set such that $|C^T| = 2$ for all $C \in E(X)$, then X has an affiliated graph $\Gamma(X)$, called a *Bruhat graph*, whose vertex set is X^T and edge set is $E(X)$. More precisely, two T-fixed points x and y are connected by an edge if and only if there is a T-curve C in $E(X)$ such that $C^T = \{x, y\}$.

When working with T-varieties of dimension at least 1, we are guaranteed by the following lemma, that there is at least one T-curve passing through each T-fixed point.

Lemma 2.7.5. *If $x \in X^T$, where X is a T-variety, then $|E(X,x)| \geq \dim X$.*

Proof. See Lemma 2 in [13] or Lemma 2.4 in [15]. □

In particular, this means that the Bruhat graph of X has at least $\dim X$ edges attached to every vertex. It was proven in Lemma 2.3 of [15] that the Bruhat graph of a projective T-variety is connected.

Furthermore, in the T-variety setting, there is also a useful relationship between the T-curves in $E(X,x)$ and the T-stable lines in $T_x(X)$ as explained in the next lemma.

Lemma 2.7.6. *Let X be a T-variety, let $x \in X^T$, and let L be any T-stable line in $T_x(X)$. If X is nonsingular at x, then there exists a $C \in E(X,x)$ such that $T_x(C) = L$.*

Proof. Let L_1 be a T-stable line in $T_x(X)$. Therefore $L_1 \subseteq T_x(X)_\alpha$, for some $\alpha \in \Omega\big(T_x(X)\big)$. Let $f_1 \in \mathbb{C}[X_x]_{-\alpha}$ such that $d_x f_1$ spans L_1^*. Now choose T-eigenvectors $f_2, \ldots, f_n \in \mathfrak{m}_x$ such that $d_x f_1, d_x f_2, \ldots, d_x f_n$ is a basis of $T_x(X)^*$. Let $L_i := \operatorname{Span}(f_i)$, then

$$T_x(X) = \bigoplus_{i=1}^{n} L_i$$

Let $C_x := Z(f_2, \ldots f_n) \subseteq X_x$. Thus,

$$\dim C_x \geq \dim X - (n-1) = n - (n-1) = 1$$

since X is nonsingular at x. Furthermore, since

$$T_x(C_x) = \bigcap_{i=2}^{n} \ker(d_x f_i) = L_1,$$

we have that $\dim C_x \leq 1$ and hence $\dim C_x = 1$. Clearly, $x \in C_x$ and since the f_i are T-eigenvectors, C_x is T-stable. Now, since x is a nonsingular T-fixed point of C_x, the irreducible component of C_x containing x is a T-stable curve. Thus, replacing C_x with this irreducible component if necessary, and taking $C = \overline{C_x} \subset X$, we have $C \in E(X,x)$ such that $T_x(C) = L_1$ as required. □

2.8 T-Fixed Points

Thus far and in much of what follows, a great deal of emphasis has and will be placed on the set of T-fixed points of a variety with T-action. Indeed, T-fixed points are a useful diagnostic tool, in particular, when working with projective varieties. Borel's fixed point theorem guarantees that:

Lemma 2.8.1. *Every nonempty closed T-stable subset of a projective variety X with T-action contains a T-fixed point. In particular, $X^T \neq \emptyset$.*

Proof. See Theorem 10.4 in [8]. □

Hence, whenever the set of all points with a specific property forms a closed T-stable subset of X, if there is a point with that property, then there is a T-fixed point with that property. Thus the T-fixed points are indicators of whether or not a projective variety possesses certain properties.

Of particular interest to us is identifying the presence of singular points. The set $\operatorname{Sing} X$ of singular points of a variety X is a proper closed subset of X (see Corollary 17.2 in [8]). If X is endowed with a T-action, then $\operatorname{Sing} X$ is clearly T-stable. Hence,

Lemma 2.8.2. *For any projective variety X with T-action, the set $\operatorname{Sing} X$ of singular points of X is a proper closed T-stable subset of X which contains a T-fixed point, if nonempty.*

Thus, determining if a projective variety with T-action is singular or nonsingular can be accomplished by checking if any of the T-fixed points are singular.

2.9 Attractive T-Fixed Points

Much of our work relies heavily on the following aptly named object:

Definition 2.9.1. Let X be a T-variety. A T-fixed point x and X_x are called *attractive* if there is a $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega\big(T_x(X)\big)$.

Remark 2.9.2. Let X be a T-variety with attractive T-fixed point x and let $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega(T_x(X))$. If Y is a closed irreducible T-stable subvariety of X which contains x, then x and $Y_x := Y \cap X_x$ are attractive in Y since $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega\big(T_x(Y)\big) \subseteq \Omega\big(T_x(X)\big)$.

The justification for the term "attractive" is apparent from statement 3) of Lemma 2.9.6 below. However, attractive points/neighbourhoods are also deserving of their names based upon their usefulness, as evidenced by the following lemma:

Lemma 2.9.3. *If X is an affine variety with T-action, then there is a T-equivariant embedding $X \hookrightarrow V$, for some T-module V. In the case that x is an attractive fixed point of some T-variety X, then X_x embeds into $T_x(X)$.*

Proof. For example, see Lemma 2.23 in [11]. □

Remark 2.9.4. If x is a nonsingular attractive T-fixed point of a T-variety X, then

$$X_x \simeq T_x(X)$$

(T-equivariantly).

Remark 2.9.5. If Y is a closed irreducible T-stable subvariety of X containing the attractive T-fixed point x, then one can choose T-equivariant embeddings $Y_x \hookrightarrow T_x(Y)$ and $X_x \hookrightarrow T_x(X)$ compatible with the natural embedding $T_x(Y) \hookrightarrow T_x(X)$, that is, such that the diagram below commutes:

$$\begin{array}{ccc} Y_x = Y \cap X_x & \hookrightarrow & X_x \\ \downarrow & & \downarrow \\ T_x(Y) & \hookrightarrow & T_x(X) \end{array}$$

We will state two equivalent notions of attractiveness. In order to do this, we first make a clarifying remark: a subset S of $\Omega(V)$ is said to lie on one side of a hyperplane in $X(T) \otimes \mathbb{Q} \simeq \mathbb{Q}^n$ if there is a linear function on $\mathbb{Q}^n$ that is strictly positive on S.

Lemma 2.9.6. *Let X be a T-variety and let $x \in X^T$. The following are equivalent:*

1) *x is attractive.*

2) *$\Omega(T_x(X))$ lies on one side of a hyperplane in $X(T) \otimes \mathbb{Q}$.*

3) *There exists a $\lambda \in Y(T)$ such that $\lim\limits_{g \to 0} \lambda(g) \cdot y = x$, for all $y \in X_x$.*

Proof. For example, see Lemma 2.25 in [11]. □

Remark 2.9.7. The equivalence of 1) and 3) can be made more precise as follows: let $\lambda \in Y(T)$. Then $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega(T_x(X))$ if and only if $\lim\limits_{g \to 0} \lambda(g) \cdot y = x$, for all $y \in X_x$.

There are two immediate consequences of the equivalence of 1) and 3) worth noting.

Corollary 2.9.8. *Let X be a T-variety and let $x \in X^T$ be attractive. The open neighbourhood X_x is unique.*

Proof. See, for example, [11]. □

Corollary 2.9.9. *Let X be a T-variety and let $x \in X^T$ be attractive. The only T-fixed point in X_x is x.*

Proof. Clear. □

Furthermore, keeping in mind that X_x embeds into $T_x(X)$, an attractive T-fixed point x is equal to 0 when viewed as element of $T_x(X)$. Indeed, write $x = \sum v_{\alpha_i} \in X_x$, for some $\alpha_i \in \Omega(T_x(X))$, and let λ be an element of $Y(T)$ which makes x is attractive. Therefore,

$$x = \lim_{c \to 0} \lambda(c) \cdot x = \lim_{c \to 0} \sum \alpha_i(\lambda(c)) v_{\alpha_i} = \lim_{c \to 0} \sum c^{d_i} v_{\alpha_i} = 0,$$

since $d_i := \langle \alpha_i, \lambda \rangle > 0$.

Additionally, in subsequent chapters we will require the following result pertaining to attractive fixed points. This is Lemma 2.1 in [15].

Lemma 2.9.10. *Given a map $f : X \to Y$ of affine T-varieties, where X contains an attractive T-fixed point x, then f is a finite morphism if and only if the fibre $f^{-1}(f(x))$ is a finite set.*

2.10 T-Orbit Closures with Attractive T-Fixed Points

Now let X be a T-variety which contains a dense T-orbit $T \cdot y$ and suppose that $x \in X^T$ is attractive. By Lemma 2.9.3, there is a T-equivariant embedding of the attractive affine neighbourhood X_x into $T_x(X)$. Clearly, $T \cdot y \subseteq X_x$, since $X_x \cap (T \cdot y)$ is nonempty and T-stable. Therefore, we also have $X_x = \overline{T \cdot y}$ in $T_x(X)$. It now follows from Lemma 2.5.2 that $\mathbb{C}[X_x]$ is multiplicity free. Recall that if $\mathfrak{m}_x$ is the ideal in $\mathbb{C}[X_x]$ of all functions vanishing at x, then $T_x(X)^* \simeq \mathfrak{m}_x/\mathfrak{m}_x^2$ (T-equivariantly) under the assignment $d_x f \mapsto \bar{f}$. As a consequence, $T_x(X)$ is multiplicity free as well. Specifically, this means that $T_x(X)$ has a weight space decomposition

$$T_x(X) = \bigoplus_{\alpha \in \Gamma} T_x(X)_\alpha,$$

where $\Gamma = \Omega\big(T_x(X)\big)$ and each $T_x(X)_\alpha$ has dimension 1. Certainly, if we know Γ and $\dim X$, we can determine whether or not a T-fixed point $x \in X$ is singular. (For example, if we know the support $s(y)$ of y, viewed as an element of $T_x(X)$, we can compute $\dim X_x = \dim X$ using Lemma 2.5.1 and then compare $\dim X$ with $|\Gamma| = \dim T_x(X)$.)

More can be said about Γ if we take into account the fact that

$$-\Gamma = \Omega(T_x(X)^*) \subseteq \Omega(\mathbb{C}[X_x]).$$

Before we elaborate further, we will gives some clarifying remarks.

Let $h \in T_x(X)^*$, then since X_x embeds in $T_x(X)$, we can restrict h to X_x. Computing the differential of $h|_{X_x}$ at x, we obtain that $d_x(h|_{X_x}) = h$. In light of the above assignment, we also have $h|_{X_x} \in \mathfrak{m}_x$. In addition, if $h|_{X_x} = 0$, then $d_x(h|_{X_x}) = h = 0$ or equivalently, if $h \neq 0$, then $h|_{X_x} \neq 0$.

Returning to our previous discussion, there is a convenient relationship between the weights Γ of $T_x(X)$ and the restrictions to X_x of the variables on the weight spaces $T_x(X)_\alpha$.

Lemma 2.10.1. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq \Omega(T_x(X))$ and let $z_{\alpha_i} \in \mathbb{C}[X_x]$ be the restriction to X_x of a variable on $T_x(X)_{\alpha_i}$, for each i. If*

$$\sum_{i=1}^{m} a_i\alpha_i = \sum_{i=1}^{m} b_i\alpha_i,$$

where $a_i, b_i \in \mathbb{N}_0$, then

$$\prod_{i=1}^{m} z_{\alpha_i}^{a_i} = \prod_{i=1}^{m} z_{\alpha_i}^{b_i},$$

up to scalars.

Proof. We first note that the weight of z_{α_i} is $-\alpha_i$. Let

$$d = -\sum_{i=1}^{m} a_i\alpha_i.$$

Then,

$$\prod_{i=1}^{m} z_{\alpha_i}^{a_i}, \prod_{i=1}^{m} z_{\alpha_i}^{b_i} \in \mathbb{C}[X_x]_d.$$

Since X_x is a T-orbit closure, by Lemma 2.5.2, the integral domain $\mathbb{C}[X_x]$ is a multiplicity free T-module. Therefore,

$$\prod_{i=1}^{m} z_{\alpha_i}^{a_i} = c\prod_{i=1}^{m} z_{\alpha_i}^{b_i},$$

for some $c \in \mathbb{C}$ (which we assume to be 1 by an appropriate choice of variables). $\square$

From the relationship specified in the previous lemma, we obtain another well-known fact about the weights Γ of $T_x(X)$, which is a useful tool when attempting to identify the elements of Γ:

Lemma 2.10.2. *Let X be a T-variety containing a dense open T-orbit and let $x \in X^T$ be attractive. Any linear combination $\sum_{i=1}^{m} a_i\alpha_i$ of weights of $T_x(X)$, with $m, a_i \in \mathbb{N}$ and $\sum_{i=1}^{m} a_i > 1$ is not a weight of $T_x(X)$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be weights of $T_x(X)$ and let $z_{\alpha_i} \in \mathbb{C}[X_x]$ be the restriction to X_x of the variable on $T_x(X)_{\alpha_i}$, for each i. Now, suppose that $\omega = \sum a_i\alpha_i$ is a weight of $T_x(X)$, such that $\sum a_i > 1$, and let $z_\omega \in \mathbb{C}[X_x]$ be the restriction to X_x of a variable on $T_x(X)_\omega$. Therefore, $z_\omega = c\prod z_{\alpha_i}^{a_i}$, for some $c \in \mathbb{C}$, by Lemma 2.10.1. As was discussed above, $z_{\alpha_i} \in \mathfrak{m}_x$, for each i. Thus, since $\sum a_i > 1$, we obtain that $\prod z_{\alpha_i}^{a_i} \in \mathfrak{m}_x^2$ and consequently $d_x z_\omega = 0$ (in $\mathfrak{m}_x/\mathfrak{m}_x^2$), which is a contradiction. $\square$

2.11 Peterson Translates

One particularly useful method for determining the regular locus of T-variety is presented by Carrell and Kuttler in [15]. The key construction in this method is an object known as a Peterson translate: let X be a T-variety, let x be an attractive T-fixed point of X, whose attractive neighbourhood will be denoted X_x, and let $C \in E(X, x)$, which we will assume is smooth. As a T-curve, C is a T-orbit closure, say $C = \overline{T \cdot z}$, for some $z \in C$. The set $C_x := X_x \cap C = T \cdot z \cup \{x\}$ is an open affine T-stable attractive neighbourhood of x in C (see Remark 2.9.2).

Now, for any $t \in T$, the map $t : X \to X$ given by $z \mapsto t \cdot z$ is an isomorphism and hence the differential $d_c t : T_c(X) \to T_{t\cdot c}(X)$ is an isomorphism for any $c \in X$, but in particular for any $c \in C \setminus C^T$. Thus, the tangent spaces $T_c(X)$ along the open orbit $C \setminus C^T$ have a common dimension, which we will denote as d.

Furthermore, since the attractive neighbourhood X_x embeds into $T_x(X)$, the differential of this embedding at any $c \in C \setminus C^T$ enables us to view $T_c(X)$ as a subspace of $T_x(X)$, for all $c \in C \setminus C^T$.

Let $V := T_x(X)$ and set $G(d, V)$ to be the Grassmannian of d-planes in V. Then the map $\varphi\colon C \setminus C^T \to G(d, V)$, given by $c \mapsto T_c(X)$, extends uniquely to a map $\tilde{\varphi} : C_x \to G(d, v)$. The point $\tilde{\varphi}(x)$ of $G(d, V)$ is the following object:

Definition 2.11.1. The *Peterson translate* of X along C is the limit

$$\tau_C(X,x) := \lim_{c \to x} T_c(X),$$

where $c \in C \setminus C^T$.

Now, for all $t \in T$, set $t : V \to V$ to be the map given by $v \mapsto t \cdot v$ (which can be restricted to X_x). In this setting we have

$$\begin{array}{ccc} X_x & \hookrightarrow & V \\ t\downarrow & & \downarrow t \\ X_x & \hookrightarrow & V \end{array} \qquad\qquad \begin{array}{ccc} T_c(X) & \hookrightarrow & V \\ d_c t\downarrow & & \downarrow t \\ T_{t\cdot c}(X) & \hookrightarrow & V \end{array}$$

where the diagram on the right is obtained from the one on the left by taking differentials at the appropriate points. From the commutativity of the right diagram it follows that φ is T-equivariant, which implies that $\varphi(C \setminus C^T)$ is T-stable.

Let $\Gamma_\varphi := \{(c, \varphi(c))\}$ denote the graph of φ in $C \setminus C^T \times G(d,V)$. Since φ is T-equivariant, Γ_φ is stable under the diagonal action of T on $C \setminus C^T \times G(d,V)$. Its closure $\overline{\Gamma_\varphi}$ in $C_x \times G(d,V)$ is the graph of the unique extension $\tilde{\varphi}$ of φ to C_x and is therefore equal to $\Gamma_\varphi \cup \{(x, \tau_C(X,x))\}$. In particular, $\overline{\Gamma_\varphi}$ is T-stable under the diagonal T-action on $C_x \times G(d,V)$. Since Γ_φ is also T-stable, it follows that $(x, \tau_C(X,x))$ is a T-fixed point. Therefore, the Peterson translate $\tau_C(X,x)$ is a T-stable subspace of $V = T_x(X)$.

From the construction of the Peterson translate of X along C, it is clear that

$$\dim X \leq \dim \tau_C(X,x) \leq \dim T_x(X).$$

Hence, if X is nonsingular at x, then

$$\tau_C(X,x) = T_x(X)$$

for all $C \in E(X,x)$.

Furthermore, if C is a good T-curve in X, that is $C = \overline{T \cdot z}$, where z is a nonsingular point of X, then

$$\dim \tau_C(X, x) = \dim X.$$

In addition to constructing Peterson translates for X, we can also construct them for any closed irreducible T-stable subvariety Y of X. If X and Y both contain an attractive point x and a smooth curve C through x, then there is a convenient relationship between the two Peterson translates $\tau_C(Y, x)$ and $\tau_C(X, x)$. In order to describe this relationship, we first identify $T_x(Y)$ with a subspace of $T_x(X)$ in the usual way and fix a commutative diagram

$$\begin{array}{ccc} Y_x = Y \cap X_x & \hookrightarrow & X_x \\ \downarrow & & \downarrow \\ T_x(Y) & \hookrightarrow & T_x(X) \end{array}$$

of T-equivariant embeddings as in Remark 2.9.5.

Lemma 2.11.2. *In the situation above, we have that $\tau_C(Y, x) \subseteq \tau_C(X, x)$ as T-modules.*

Proof. Let d be the common dimension of the tangent spaces $T_c(X)$ for all $c \in C \backslash C^T$ and let m be the common dimension of the tangent spaces $T_c(Y)$ for all $c \in C \backslash C^T$. Let $F_{m,d}$ be the closed subvariety of $G\big(m, T_x(Y)\big) \times G\big(d, T_x(X)\big)$ consisting of points (A, B) such that $A \subseteq B$, that is,

$$F_{m,d} = \mathrm{Flag}(m, d, T_x(X)) \cap \Big(G\big(m, T_x(Y)\big) \times G\big(d, T_x(X)\big)\Big),$$

where $\mathrm{Flag}(m, d, T_x(X))$ is the variety of (m, d)-flags in $T_x(X)$. (Recall that we are identifying $T_x(Y)$ with a subspace of $T_x(X)$.) Since C is smooth and $F_{m,d}$ is projective, the map $\rho : C \setminus C^T \to F_{m,d}$ given by $c \mapsto \big(T_c(Y), T_c(X)\big)$ has a unique extension $\tilde{\rho} : C_x \to F_{m,d}$.

Now let $\pi_m : F_{m,d} \twoheadrightarrow G\big(m, T_x(Y)\big)$ and $\pi_d : F_{m,d} \twoheadrightarrow G\big(d, T_x(X)\big)$ be the projection maps. The composition map

$$\varphi_m := \pi_m \circ \rho : C \setminus C^T \to G\big(m, T_x(Y)\big)$$

has a unique extension

$$\tilde{\varphi}_m : C_x \to G\big(m, T_x(Y)\big).$$

By definition, $\tilde{\varphi}_m(x) = \tau_C(Y, x)$. Likewise, the map

$$\varphi_d := \pi_d \circ \rho : C \setminus C^T \to G\big(d, T_x(X)\big)$$

has unique extension

$$\tilde{\varphi}_d : C_x \to G\big(d, T_x(X)\big).$$

Again, by definition, $\tilde{\varphi}_d(x) = \tau_C(X, x)$. By the uniqueness of the maps involved, we obtain that the following diagrams commute:

$$\begin{tikzcd}
C \setminus C^T \arrow[r, "\rho"] \arrow[d, hook] & F_{m,d} \arrow[d, "\pi_m"] \\
C_x \arrow[ur, "\tilde{\rho}"] \arrow[r, "\tilde{\varphi}_m"'] & G\big(m, T_x(Y)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
C \setminus C^T \arrow[r, "\rho"] \arrow[d, hook] & F_{m,d} \arrow[d, "\pi_d"] \\
C_x \arrow[ur, "\tilde{\rho}"] \arrow[r, "\tilde{\varphi}_d"'] & G\big(d, T_x(X)\big)
\end{tikzcd}$$

As a result,

$$\tilde{\rho}(x) = \big(\tilde{\varphi}_m(x), \tilde{\varphi}_d(x)\big) = \big(\tau_C(Y, x), \tau_C(X, x)\big).$$

Since the image of $\tilde{\rho}$ lies in $\mathrm{Flag}(m, d, T_x(X))$, it follows that

$$\tau_C(Y, x) \subseteq \tau_C(X, x).$$

$\square$

Corollary 2.11.3. *The tangent space $T_x(C)$ is a T-stable subspace of $\tau_C(X, x)$.*

Proof. The fact that C is smooth, in conjunction with Lemma 2.11.2, yields

$$T_x(C) = \tau_C(C, x) \subseteq \tau_C(X, x).$$

$\square$

In addition to Peterson translates, we will also use the following structure:

Definition 2.11.4. Let X be any T-variety with T-fixed point x. The *tangent space* to $E(X, x)$ at x is the subspace of $T_x(X)$ given by

$$TE(X, x) := \sum_{C \in E(X,x)} T_x(C).$$

In the case that x is a nonsingular point of X, Lemma 2.7.6 gives that every T-stable line in $T_x(X)$ is tangent to some T-curve in $E(X, x)$ and therefore

$$TE(X, x) = T_x(X).$$

Since $\tau_C(X, x)$ and $TE(X, x)$ are both subspaces of $T_x(X)$, it is natural to ask if there is any connection between them. If x is in the regular locus of X, then the two coincide for all $C \in E(X, x)$. Conversely, one is lead to ask: are there circumstances under which $\tau_C(X, x) = TE(X, x)$ implies that X is nonsingular at x? One particular set of such conditions is given in the following theorem. (This is Theorem 1.4 in [15].)

Theorem 2.11.5. *Let X be a T-variety with an attractive T-fixed point x. Assume that $E(X, x)$ contains only smooth curves and that for any distinct $C, D \in E(X, x)$, the T-weights of $T_x(C)$ and $T_x(D)$ as T-modules are not equal. If either*

1) *$TE(X, x) = \tau_C(X, x)$ for at least two good T-curves $C \in E(X, x)$, or*

2) *X is Cohen-Macaulay at x and $TE(X, x) = \tau_C(X, x)$ for at least one good $C \in E(X, x)$,*

holds, then X is nonsingular at x.

2.12 T-Surfaces

In regards to Theorem 2.11.5, matters can be simplified further by considering surfaces. Let X be a variety with T-action and let C be a closed irreducible

T-stable curve in X. We define $\Sigma_T(X,C)$ (or simply $\Sigma(X,C)$ if the torus involved is clear) be the set of all closed irreducible T-stable surfaces in X which contain C. If X is a T-variety, then whenever C is a good T-curve, the Peterson translate $\tau_C(X,x)$ of X depends on the Peterson translates $\tau_C(\Sigma,x)$, for $\Sigma \in \Sigma(X,C)$ as outlined below (cf. Lemma 5.1 in [15]):

Lemma 2.12.1. *Let X be a T-variety, let $x \in X$ be an attractive T-fixed point, and let $C \in E(X,x)$ be good. Then*

$$\tau_C(X,x) = \sum_{\Sigma \in \Sigma(X,C)} \tau_C(\Sigma,x).$$

This is a powerful result as it enables us to reduce our considerations to the T-stable surfaces contained in X.

Due to a result by Brion, one of the benefits of working with surfaces with T-actions is that they often contain exactly two closed irreducible T-stable curves.

Lemma 2.12.2. *Suppose Σ is an irreducible surface with T-action which contains an attractive point x such that $|E(\Sigma,x)|$ is finite. Then $|E(\Sigma,x)| = 2$.*

Proof. See Corollary 1 and Corollary 2 in Section 1.4 of [9]. □

As is the case for curves, we are particularly interested in those surfaces which contain a dense open T-orbit.

Definition 2.12.3. Let X be a variety with T-action. A surface in X which is the closure of a two-dimensional T-orbit is called a *T-surface.*

For example, any irreducible T-stable surface in a variety X with T-action for which both X^T and $E(X)$ are finite is a T-surface.

Now let $\Sigma = \overline{T \cdot y}$ be a T-variety and suppose that Σ contains an attractive T-fixed point x. We will also assume that $E(\Sigma,x)$ is a finite set which contains only smooth T-curves. By Lemma 2.12.2, we know that $|E(\Sigma,x)| = 2$, so let C, D be these two T-curves. Finally, we will also assume that the T-weights of $T_x(C)$ and $T_x(D)$ as T-modules, say α and β, respectively, are not equal.

By Lemma 2.9.3, there is a T-equivariant embedding of Σ_x into $T_x(\Sigma)$ and since C and D are smooth, by Remark 2.9.4, $C_x \simeq T_x(C)$ and $D_x \simeq T_x(D)$. We have

$$C_x, D_x \hookrightarrow \Sigma_x \hookrightarrow T_x(\Sigma) = \bigoplus_{\alpha \in \Gamma} T_x(\Sigma)_\alpha,$$

where $\Gamma = \Omega\big(T_x(\Sigma)\big)$. As in Section 2.10, we have that $\Sigma_x = \overline{T \cdot x}$ when viewed in $T_x(\Sigma)$ and hence that $T_x(\Sigma)$ is multiplicity free. Thus, $T_x(C) = T_x(\Sigma)_\alpha$ and $T_x(D) = T_x(\Sigma)_\beta$.

For simplicity, we will write $V_\alpha := T_x(\Sigma)_\alpha$ and $V_\beta := T_x(\Sigma)_\beta$

Let $z_\alpha, z_\beta \in \mathbb{C}[\Sigma_x]$ be the restrictions to Σ_x of the variables on V_α and V_β in $T_x(\Sigma)$, respectively. As such, z_α and z_β have weight $-\alpha$ and $-\beta$, respectively, in the T-representation on $\mathbb{C}[\Sigma_x]$. The following lemma is a version of a fact that is presented in the proof of Proposition 5.2 in [15]. This lemma will, once again, enable us to use a result regarding coordinate rings to understand weights of tangent spaces.

Lemma 2.12.4. *Let $-\omega \in \Omega(\mathbb{C}[\Sigma_x])$. Then there exists an $N \in \mathbb{N}$ such that $N(-\omega) = a(-\alpha) + b(-\beta)$, for some $a, b \in \mathbb{N}_0$.*

Proof. Let $\rho : \Sigma_x \to V_\alpha \oplus V_\beta$ be the restriction to Σ_x of the unique T-equivariant projection $\tilde{\rho} : T_x(\Sigma) \twoheadrightarrow V_\alpha \oplus V_\beta$, so that $d_x\rho = \tilde{\rho}$. Since x is attractive, $x = 0$ in $T_x(\Sigma)$, and hence $\rho(x) = 0$. If the fibre over 0 is infinite, then its dimension is at least 1. Moreover, by attractiveness, every irreducible component contains x. Therefore, by Lemma 2.7.5, it contains at least one T-curve passing through x, which we may assume is C_x. Thus $\rho(C_x) = 0$, which yields that $d_x\rho(T_x(C_x)) = 0$, which in turn implies that

$$\tilde{\rho}(V_\alpha) = \tilde{\rho}(T_x(C_x)) = 0.$$

As this is clearly a contradiction, the fibre over $\rho(x)$ is finite and thus by Lemma 2.9.10, ρ is a finite morphism. As such, since $V_\alpha \oplus V_\beta$ is affine,by definition the $\mathbb{C}[z_\alpha, z_\beta]$-algebra $\mathbb{C}[\Sigma_x]$ is a finitely generated $\mathbb{C}[z_\alpha, z_\beta]$-module, that is, $\rho^* : \mathbb{C}[z_\alpha, z_\beta] \to \mathbb{C}[\Sigma_x]$ is a finite ring homomorphism. In particular, ρ^* is integral and so $\mathbb{C}[\Sigma_x]$ is integral over $\mathbb{C}[z_\alpha, z_\beta]$.

Let $f \neq 0 \in \mathbb{C}[\Sigma_x]_{-\omega}$. There is an $N \in \mathbb{N}$ such that

$$f^N = h_{N-1} f^{N-1} + \cdots + h_1 f + h_0,$$

where $h_{N-i} \in \mathbb{C}[z_\alpha, z_\beta]$, for $1 \leq i \leq N$, and $h_0 \neq 0$. Since $\mathbb{C}[z_\alpha, z_\beta]$ has a weight space decomposition with respect to T,

$$h_{N-i} = \sum_{\mu \in X(T)} h_{N-i,\mu},$$

where $h_{N-i,\mu} \in \mathbb{C}[z_\alpha, z_\beta]_\mu$. Also, as $f \in \mathbb{C}[\Sigma_x]_{-\omega}$, we know that f^N has weight $N(-\omega)$ and hence

$$f^N = \sum_{i=1}^{N} h_{N-i,i(-\omega)} f^{N-i},$$

where $h_{0,N(-\omega)} \neq 0$ and each summand has weight $N(-\omega)$. Therefore, $N(-\omega)$ is a weight of $\mathbb{C}[z_\alpha, z_\beta]$. □

Consequently, we have:

Lemma 2.12.5. *Let $\omega \in \Omega\big(T_x(\Sigma)\big)$. Then there exists an $N \in \mathbb{N}$ such that $N\omega = a\alpha + b\beta$, for some $a, b \in \mathbb{N}_0$.*

2.13 Rationally Smooth Varieties

Definition 2.13.1. Let X be an equidimensional variety. A point $x \in X$ is said to be *rationally smooth* if there is an open neighbourhood U_x of x in the analytic topology on X such that for every $y \in U_x$, the relative singular cohomology group $H^j(X, X \setminus \{y\}, \mathbb{Q})$ is 0 when $j \neq 2\dim X$ and isomorphic to $\mathbb{Q}$ otherwise. We say that X is *rationally smooth* if it is rationally smooth at each of its points.

The locus of rationally smooth points on a variety X is open in the Zariski topology on X and contains the smooth locus of X (see [26, p471]). The notion of rational smoothness turns out to be useful for studying singularities of Schubert varieties and their torus orbit closures. Later we will give some equivalent conditions for rational smoothness in the latter context.

Chapter 3

In the Context of G/B

In this chapter we consider the singular locus of a rationally smooth T-orbit closure X in G/B in types A and D. If every point of X has a connected stabilizer in T, then it follows from the work of Brion that rational smoothness is equivalent to a more concrete notion which is referred to as combinatorial regularity (see Definition 3.4.1 below). Using this, we show that in type A, a T-orbit closure in G/B is rationally smooth if and only if it is smooth (see Theorem 3.9.5 below). In type D, Carrell and Kuttler have shown in [15] that there are rationally smooth T-orbit closures in G/B that are singular (see Example 7.1 in [15]). Nonetheless, we have studied the case for type D, in particular, how the techniques applied in the type A case fail for type D.

3.1 Notation and Terminology

Let G be a connected semi-simple algebraic group, B a Borel subgroup, and $T \subset B$ a maximal torus. Let $W := N_G(T)/T$ denote the Weyl group of (G, T). It is well-known that W is a finite group. The group W acts homomorphically on T by conjugation, which induces (left) actions of W on the character group $X(T)$ and the cocharacter group $Y(T)$ in the obvious way.

The homogeneous space G/B, known as the *flag variety* of G, is an irreducible smooth projective variety with a T-action defined in the obvious way. As a projective variety, we know from Borel's fixed point theorem that

$$(G/B)^T \neq \emptyset.$$

Moreover, $(G/B)^T$ is a finite set. Indeed, there is a well known one-to-one correspondence between the Weyl group W and $(G/B)^T$ under which w corresponds to $\dot{w}B$, where $\dot{w}$ is a representative of w in $N_G(T)$. Henceforth, we will identify the T-fixed points of G/B with the corresponding element of W by simply writing w for $\dot{w}B$. (For example, we will write e for the coset represented by the identity element of G.) Moreover, the natural action of G on G/B determines a transitive action of W on $(G/B)^T$. With the identification above, this action corresponds to the transitive action of W on itself by left multiplication.

Since G/B is normal (even smooth), it follows from Example 2.6.4 that G/B and any of its closed irreducible T-stable subvarieties are T-varieties in the sense of Definition 2.6.1. Concretely, for any $w \in (G/B)^T$,

$$(G/B)_w := (\dot{w}U^-) \cdot e$$

is a T-stable open affine neighborhood, where $\dot{w} \in N_G(T)$ is a representative of w and U^- is the unipotent radical of B^-, the Borel opposite B. For convenience, we will use the notation U_w to denote $(G/B)_w$.

Furthermore, for any closed irreducible T-stable subvariety X of G/B and any $w \in X^T$, the set

$$X_w := U_w \cap X$$

is an open affine T-stable neighborhood of w in X.

3.2 Weight Space Decompositions

Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$ denote the Lie algebras of G, B, and T, respectively. Since G is semi-simple and we are working over the characteristic 0 field $\mathbb{C}$, $\mathfrak{g}$ is also semi-simple. By restricting the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ to T, we obtain an action of T on $\mathfrak{g}$ which yields the *root space* decomposition:

$$\mathfrak{g} = (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha) \oplus \mathfrak{h},$$

where $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid t \cdot g = \alpha(t)g, \text{ for all } t \in T\}$. The elements of $\Phi := \Omega(\mathfrak{g}) \backslash \{0\}$ are called the *roots* of G with respect to T. It is well-known that $\dim \mathfrak{g}_\alpha = 1$, for all $\alpha \in \Phi$.

The natural action of the Weyl group W on $X(T)$ restricts to an action of W on Φ. For any $w \in W$ and $\alpha \in \Phi$, we have

$$\mathfrak{g}_{w \cdot \alpha} = \dot{w} \mathfrak{g}_\alpha$$

(here $\dot{w}$ is any representative of w in $N_G(T)$, acting on $\mathfrak{g}$ through the adjoint action of G).

Let $\alpha \in \Phi$. We will denote by s_α the unique reflection automorphism in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ which sends α to $-\alpha$. The natural map

$$W \to \mathrm{Aut}_{\mathbb{R}}\big(X(T) \otimes_{\mathbb{Z}} \mathbb{R}\big)$$

is injective and identifies W with the subgroup of $\mathrm{Aut}_{\mathbb{R}}\big(X(T) \otimes_{\mathbb{Z}} \mathbb{R}\big)$ generated by the s_α. (See Theorem 27.1 in [20].)

There is a notion of positivity defined on Φ which arises from the root space decomposition of $\mathfrak{b}$: the set of positive roots, denoted Φ^+, is

$$\Phi^+ := \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subset \mathfrak{b}\}.$$

This is equivalent to setting

$$\Phi^+ := \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle > 0\},$$

where $\lambda \in Y(T)$ that depends on the choice of Borel subgroup B. The set of negative roots will be denoted by Φ^-. The notation $\alpha > 0$ indicates that $\alpha \in \Phi^+$ and $\alpha < 0$ indicates that $\alpha \in \Phi^-$. Therefore, since we also have $\mathfrak{h} \subset \mathfrak{b}$, we obtain that

$$\mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha.$$

Tangent spaces are crucial in the study of singularities and in the G/B context they have useful descriptions. For the T-fixed point $e \in G/B$:

$$T_e(G/B) \simeq \mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha.$$

For an arbitrary $w \in (G/B)^T$ we have:

$$T_w(G/B) \simeq \mathfrak{g}/w\mathfrak{b}w^{-1} \simeq \bigoplus_{w^{-1}(\alpha)<0} \mathfrak{g}_\alpha.$$

Also, Φ is a reduced root system and, as such, the roots in Φ^-, or more generally in $w(\Phi^-)$, for any $w \in W$, are nonproportional. In particular, this means that the T-curves in $T_w(G/B)$ are the weight spaces $\mathfrak{g}_\alpha$, by Lemma 2.7.4.

From the characterizations above, an important fact about G/B can be obtained:

Lemma 3.2.1. *Every element of $(G/B)^T$ is attractive.*

Proof. From our notion of positivity on Φ, we know that there is a $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle < 0$ for all $\alpha < 0$ and hence $\langle \alpha, -\lambda \rangle > 0$ for all $\alpha < 0$. Thus, the T-fixed point e is attractive since $\Omega\big(T_e(G/B)\big) = \Phi^-$. An arbitrary T-fixed point w is also attractive, since for any $\alpha \in \Omega\big(T_w(G/B)\big) = w\big(\Phi^-\big)$, we have $w^{-1}(\alpha) < 0$, so that $\langle \alpha, w(-\lambda) \rangle = \langle w^{-1}(\alpha), -\lambda \rangle > 0$. $\square$

As a consequence of Lemma 3.2.1, in conjunction with Remark 2.9.4, we obtain that for every $w \in (G/B)^T$,

$$U_w \simeq T_w(G/B).$$

Moreover, in view of Lemma 2.8.1, we observe that G/B can be covered by the attractive T-stable open affine neighbourhoods U_w, for $w \in (G/B)^T$.

3.3 T-Curves in G/B

As G/B is a T-variety, we know that every irreducible T-stable curve in G/B is a T-curve, that is, the closure of a one dimensional T-orbit. The T-curves in G/B are very well understood objects. Recall that, for any T-stable subvariety X of G/B, the set $E(X)$ consists of all T-curves in X.

Lemma 3.3.1. *Let X be a nonempty T-stable subvariety of G/B. Then the following hold:*

1) *$E(X)$ is finite.*
2) *Every element of $E(X)$ is smooth.*
3) *Every element of $E(X)$ contains exactly two T-fixed points.*
4) *If $C \neq D$ are elements of $E(X)$ with nonempty intersection, then $C \cap D = \{x\}$, for some $x \in X^T$.*

Proof. See Theorems D and F in [13]. $\square$

As a result of Lemma 3.3.1, we obtain a nice description of certain surfaces in G/B: since $(G/B)^T$ and $E(G/B)$ are both finite sets, the T-stable irreducible surfaces in G/B are T-surfaces, that is, they are closures of two-dimensional T-orbits.

A more precise characterization of each T-curve in G/B can be given in terms of its T-fixed points. Recall that $E(G/B, w)$ is the set of T-curves in G/B which pass through the T-fixed point w. Also, let U_α be the unique connected T-stable subgroup of G with Lie algebra $\mathfrak{g}_\alpha$.

Lemma 3.3.2. *Let $w \in (G/B)^T$ and let $C \in E(G/B, w)$. Then $C = \overline{U_\alpha \cdot w}$, for some $\alpha \in \Phi$, and $C^T = \{w, s_\alpha w\}$, where s_α is the reflection in the Weyl group W corresponding to α. Moreover, $T_w(C) = \mathfrak{g}_\alpha$ and $T_{s_\alpha w}(C) = \mathfrak{g}_{-\alpha}$.*

Proof. This also follows from Theorems D and F in [13]. $\square$

Taking into account that all T-fixed points in G/B are attractive, for any $C = \overline{U_\alpha \cdot w} \in E(G/B, w)$, we in fact have

$$C_w \simeq T_w(C) = \mathfrak{g}_\alpha.$$

Lemma 3.3.3. *There are precisely d T-curves through $w \in (G/B)^T$, where $d = \dim G/B$.*

Proof. The assignment $C \mapsto C_w$ gives a one-to-one correspondence between T-curves in G/B passing through w and T-curves in the open subset U_w passing through w. Since U_w is T-equivariantly isomorphic to $T_w(G/B)$ (with w being mapped to 0) the claim follows from Lemma 2.7.4. $\square$

3.4 Combinatorially Regular Points

In order to introduce the notion of combinatorial regularity, we will first establish some notation. For a T-variety X, we specify a partial order $\leq_X$ on the elements of X by setting

$$z \leq_X w \qquad \text{if and only if} \qquad \overline{T \cdot z} \subseteq \overline{T \cdot w}.$$

We shall use the notation

$$z \approx_X w$$

to denote when two points of X are equal under this partial order. Note that

$$z \approx_X w \qquad \text{if and only if} \qquad z \in T \cdot w$$

Let $w \in X$ with stabilizer T_w. Define

$$X_{T,w} := \{q \in X |\ w \in \overline{T_w \cdot q}\ \}.$$

It is immediate that $X_{T,w}$ is a T_w-stable subset of X which contains only one T_w-fixed point, namely w. Now, $X_{T,w}$ is a well-known object, in particular, it is known that $X_{T,w}$ is locally closed. We provide a verification of this for T-orbit closures in G/B below. The argument for a general T-variety is similar.

Definition 3.4.1. A T-variety X with a dense T-orbit is said to be *combinatorially regular* at z if, for all $w \geq_X z$, $X_{T,w}$ is irreducible and

$$|E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}.$$

X is *combinatorially regular* if it is combinatorially regular at all $z \in X$.

Let $x \in G/B$ and set $X := \overline{T \cdot x}$. Fix $y \in X$, let $S = T_y$ be the stabilizer of y, and set $Y = X_{T,y} := \{q \in X |\ y \in \overline{S \cdot q}\ \}$. Suppose S is connected.

The significance of the set Y goes beyond its appearance in the definition above. We will show in this chapter that for any point $z \in Y$, X is combinatorially regular at z if and only if Y is combinatorially regular at z. Furthermore, z is a nonsingular point of X if and only if z is a nonsingular point of Y. Using these facts, the problem of determining if a combinatorially regular X is nonsingular at z has been reduced to determining if the combinatorially regular Y is nonsingular at z. The benefit of this is that Y possesses properties that simplify the process of identifying nonsingular points. We would like to note that the results and techniques appearing in Sections 3.5 through 3.8, including the two facts mentioned above, are refinements of well-established ideas to our context. We begin by examining some of the properties of Y.

3.5 Properties of Y

In this section, we will continue to use the notation as defined in the previous section. As mentioned above, Y is an S-stable subset of X containing y such that $Y^S = \{y\}$.

As $y \in G/B$, it is contained in an attractive T-stable neighbourhood U_p of some T-fixed point $p \in G/B$. Now, $T \cdot x \subseteq X \cap U_p$. It is straightforward to show that $Y \subseteq X \cap U_p$ and that, for every $z \in Y$, $y \in \overline{S \cdot z}$, where the closure is taken in U_p. Thus, in what follows, we may assume that X is affine by replacing X with the affine T-variety $X \cap U_p = \overline{T \cdot x}$, where the closure is taken in U_p. Consequently,

$$Y \subseteq X \subseteq U_p \simeq T_p(U_p) \simeq V := T_p(G/B) = \bigoplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha,$$

for some subset $\widetilde{\Phi} \subseteq \Phi$.

By Lemma 2.5.8, there exists a $\lambda \in Y(T)$ such that

$$\lim_{g \to 0} \lambda(g)x \in T \cdot y$$

and hence there is a $t \in T$ such that

$$\lim_{g \to 0} \lambda(g)(t \cdot x) = y.$$

Replacing x with $t^{-1} \cdot x$, we have

$$\lim_{g \to 0} \lambda(g)x = y.$$

We use λ to obtain a G_m-action on V and hence a $\mathbb{Z}$-grading on V. Thus we may write:

$$V = V_- \oplus V_0 \oplus V_+ \;\; = \bigoplus_{\substack{\alpha \in \widetilde{\Phi} \\ \langle \alpha, \lambda \rangle < 0}} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\alpha \in \widetilde{\Phi} \\ \langle \alpha, \lambda \rangle = 0}} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\alpha \in \widetilde{\Phi} \\ \langle \alpha, \lambda \rangle > 0}} \mathfrak{g}_\alpha$$

As an element of V, $x = x_- + x_0 + x_+$, where $x_- \in V_-$, $x_0 \in V_0$, and $x_+ \in V_+$. Since the above limit exists, $x_- = 0$, which yields that $x \in V_0 \oplus V_+$ and consequently

$$X = \overline{T \cdot x} \subseteq V_0 \oplus V_+.$$

Furthermore,

$$\lim_{g \to 0} \lambda(g)x = x_0,$$

but since it is also equal to y, we have $x_0 = y$. Thus

$$x = y + x_+$$

and, moreover, $y \in V_0$, i.e. y is a fixed point under the G_m-action induced by λ. Hence $\lambda(\mathrm{G}_m) \subseteq S$ and therefore

$$\lambda \in Y(S).$$

Now, for any $v = v_0 + v_+ \in V_0 \oplus V_+$, $\lim_{g \to 0} \lambda(g)v$ exists. Specifically,

$$\lim_{g \to 0} \lambda(g)v = v_0.$$

Consequently, viewed as a map on $V_0 \oplus V_+$, $\lim_{g \to 0} \lambda(g)$ is the T-equivariant linear projection

$$f : V_0 \oplus V_+ \to V_0$$

In particular, $f(x) = y$ and $f(y) = y$. Subsequently,

$$f(X) = f(\overline{T \cdot x}) \subseteq \overline{f(T \cdot x)} = \overline{T \cdot f(x)} = \overline{T \cdot y}$$

and thus we obtain $f(X) \subseteq \overline{T \cdot y}$. In particular, this gives us that the points of $f(X)$ are S-fixed points.

Let

$$\begin{aligned} X_{\lambda,y} :=& \{q \in X | \lim_{g \to 0} \lambda(g)q = y\} \\ =& \{q \in X | f(q) = y\} \\ =& f^{-1}(y) \cap X. \end{aligned}$$

Note that $x \in X_{\lambda,y}$. As λ has image in S,

$$X_{\lambda,y} \subseteq Y := \{q \in X | \ y \in \overline{S \cdot q}\ \}.$$

Let $z = z_0 + z_+ \in Y$. Then $y \in \overline{S \cdot z}$. Since $z_0 = f(z) \in f(X)$ is an S-fixed point we have

$$f(\overline{S \cdot z}) \subseteq \overline{S \cdot f(z)} = \overline{S \cdot z_0} = \{z_0\}.$$

In particular, $f(y) = z_0$, but as $f(y) = y$, we obtain $f(z) = y$. Thus, $Y \subseteq X_{\lambda,y}$ and so we have

$$Y = X_{\lambda,y}$$

Therefore, Y is a closed subset of X containing x (and so a locally closed subset of the original X in G/B).

Remark 3.5.1. An arbitrary T-variety X comes equipped with a covering by T-stable affine open subsets. For any $w \in X$, w lies in some T-stable affine neighbourhood, say U, which then also contains $X_{T,w}$. As U is an affine T-variety, there is a T-equivariant embedding of U into some T-module V_w (Lemma 2.9.3). Using an argument identical to the one given directly above with U in place of $X \cap U_p$ and V_w in place of $V := T_p(G/B)$, we obtain that $X_{T,w}$ is a closed subset of U and hence a locally closed subset of X.

Returning to the nature of Y, it is also evident that $Y \cap T \cdot y = \{y\}$. As Y is S-stable, $\overline{S \cdot x} \subseteq Y$. Since for any $z \in Y \subseteq X = \overline{T \cdot x}$ we have that $f(z) = y = f(x)$, by Lemma 2.5.7 applied to the projection

$$T \cdot y \times V_+ \to T \cdot y \simeq T/S,$$

which is the restriction of f to the irreducible affine variety $T \cdot y \times V_+$, we obtain that $z \in \overline{S \cdot x}$ and hence $Y \subseteq \overline{S \cdot x}$. As a result,

$$Y = \overline{S \cdot x}$$

and thus we may regard Y as an irreducible affine subvariety of X. In particular, Y is an S-variety with sole S-fixed point y.

In summary,

Proposition 3.5.2.

$$Y := X_{T,y} = X_{\lambda,y} = f^{-1}(y) \cap X = \overline{S \cdot x}$$

As a result,

Corollary 3.5.3.

1. *y is an attractive fixed point of the S-variety Y.*
2. *$\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega_S(T_y(Y))$.*
3. *The elements of $\Omega_S(T_y(Y))$ are all non-zero.*
4. *There is an S-equivariant embedding $Y \hookrightarrow T_y(Y)$.*
5. *$y = 0$ as an element of $T_y(Y)$.*
6. *$T_y(Y)$ is multiplicity free.*

Proof. Since $\lim\limits_{g \to 0} \lambda(g)q = y$ for all q in the affine S-variety Y, the first assertion is immediate from Lemma 2.9.6. The second statement is the definition of what it means for y to be attractive. The third observation follows immediately from point number two. Statement four is a consequence of Lemma 2.9.3 since y is attractive and $Y_y = Y$. Point five is clear given item four. Finally, as Y has an open dense S-orbit, by Lemma 2.5.2, $\mathbb{C}[Y]$ is multiplicity free and subsequently $T_y(Y)$ is as well. □

Remark 3.5.4. During the realization of Y as $\overline{S \cdot x}$, we replaced x with $t^{-1} \cdot x$. In general, given $w \in X = \overline{T \cdot x}$, as above we have that there exists a $\widetilde{\lambda} \in Y(T_w)$ and a $\tilde{t} \in T$ such that

$$\lim_{g \to 0} \widetilde{\lambda}(g)(\tilde{t} \cdot x) = w$$

Without relabeling, it follows that

$$X_{T,w} = \overline{T_w \cdot (\tilde{t} \cdot x)}$$

We shall use this remark freely throughout the next section.

For the remainder of this section, we will elaborate on the structure of $T_y(Y)$.

We have the following situation:

$$Y \subseteq X \subseteq V = \bigoplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha$$

for some subset $\widetilde{\Phi} \subseteq \Phi$. However, as Y is an S-variety, we would like to have a decomposition in terms of the set of weights $\Omega_S(\mathfrak{g})$ of $\mathfrak{g}$ with respect to S, where S acts via the restriction of the T-action on $\mathfrak{g}$ (which is the same as restricting the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ to S). It is clear that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\alpha|_S}$. As a result we have an embedding,

$$Y \hookrightarrow \bigoplus_{\alpha|_S \in \Gamma} \mathfrak{g}_{\alpha|_S} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$$

for some subset Γ of $\Omega_S(\mathfrak{g})$.

Taking the differential at y of this embedding, we obtain

$$T_y(Y) \hookrightarrow \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha.$$

By Corollary 3.5.3, we know that $\langle \alpha, \lambda \rangle > 0$, for all $\alpha \in \Omega_S(T_y(Y))$. Consequently,

Lemma 3.5.5. $\Omega_S(T_y(Y)) \subseteq \{\alpha \in \Omega_S(\mathfrak{g}) | \langle \alpha, \lambda \rangle > 0\}$

Furthermore, since $T_y(Y)$ is multiplicity free by Corollary 3.5.3, we also have

Lemma 3.5.6.

$$T_y(Y) = \bigoplus_{\alpha \in \Omega_S(T_y(Y))} L_\alpha,$$

where each L_α is a line in $\mathfrak{g}_\alpha$, for some $\alpha \in \Omega_S(\mathfrak{g})$.

3.6 Combinatorially Regular Points: Relating X and Y

Continuing with the notation established in the previous sections, in this section we shall provide a proof of the fact that for any $z \in Y$, X is combinatorially regular at z if and only if Y is combinatorially regular at z. To that end, we begin with the following lemma.

Lemma 3.6.1. *If $z, w \in X$ such that $z \approx_X w$, then*

$$X_{T,z} \ \ \textit{is irreducible} \quad \textit{and} \quad |E_{T_z}(X_{T,z}, z)| = \dim X_{T,z}$$

$$\textit{if and only if}$$

$$X_{T,w} \ \ \textit{is irreducible} \quad \textit{and} \quad |E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}.$$

Proof. Since $z \approx_X w$, we have that $z \in T \cdot w$, i.e. $z = t \cdot w$ for some $t \in T$. It is obvious that $T_z = T_w$. In addition, if

$$\lim_{g \to 0} \widetilde{\lambda}(g)(\tilde{t} \cdot x) = w$$

for some $\widetilde{\lambda} \in Y(T_w)$ and some $\tilde{t} \in T$, then

$$\lim_{g \to 0} \widetilde{\lambda}(g)((t \cdot \tilde{t}) \cdot x) = t \cdot w = z.$$

Consequently,

$$X_{T,w} = \overline{T_w \cdot (\tilde{t} \cdot x)} \ \text{ and } \ X_{T,z} = \overline{T_w \cdot ((t \cdot \tilde{t}) \cdot x)}$$

From these descriptions it is clear that $X_{T,w}$ is irreducible if and only if $X_{T,z}$ is irreducible and $\dim X_{T,w} = \dim X_{T,z}$. Furthermore, the assignment

$$\begin{gathered} E_{T_w}(X_{T,w}, w) \to E_{T_z}(X_{T,z}, z) \\ C \mapsto t \cdot C \end{gathered}$$

is a bijection, which yields

$$|E_{T_w}(X_{T,w}, w)| = |E_{T_z}(X_{T,z}, z)|$$

The result follows. $\square$

A useful consequence of the above Lemma is the following:

Corollary 3.6.2. *Let $X = \overline{T \cdot x}$ be affine. Let $y_1, y_2, \ldots, y_r \in X$ such that $X = \bigcup_{i=1}^{r}(T \cdot y_i)$. Then X is combinatorially regular if and only if X_{T,y_i} is irreducible and*

$$|E_{T_{y_i}}(X_{T,y_i}, y_i)| = \dim X_{T,y_i}$$

for all i.

Proof. Note that by Lemma 2.5.2, the affine T-orbit closure X has finitely many orbits.

One direction is clear. For the other, assume that X_{T,y_i} is irreducible and $|E_{T_{y_i}}(X_{T,y_i}, y_i)| = \dim X_{T,y_i}$, for all i. Let $z \in X$. We will verify that X is combinatorially regular at z. Let $w \in X$ such that $w \geq_X z$. Since $w \in X = \bigcup_{i=1}^{r}(T \cdot y_i)$, we obtain that $w \in T \cdot y_i$, for some i and hence $w \approx_X y_i$. Thus, as established in Lemma 3.6.1, $X_{T,w}$ is irreducible and $|E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}$. Consequently, X is combinatorially regular at z, for all $z \in X$, and as a result X is combinatorially regular. $\square$

We will make use of Corollary 3.6.2 in Section 3.10 when generating local examples of combinatorially regular T-orbit closures.

Returning to the work in progress, let $w \in Y$. By Proposition 3.5.2, $f(w) = y$ and so for any $t \in T_w$,

$$t \cdot y = t \cdot f(w) = f(t \cdot w) = f(w) = y$$

and thus $t \in S_w$. Hence $T_w \subseteq S_w$. As $S \subseteq T$, it is clear that $S_w \subseteq T_w$, so that

$$S_w = T_w. \tag{3.1}$$

By applying Lemma 2.5.8 to the S-variety $Y = \overline{S \cdot x}$, we obtain that there exists a $\widetilde{\lambda} \in Y(S)$ and an $s \in S$ such that

$$\lim_{g \to 0} \widetilde{\lambda}(g)(s \cdot x) = w$$

and as a result

$$\begin{aligned} Y_{S,w} &:= \{q \in Y | w \in \overline{S_w \cdot q}\} \\ &= \overline{S_w \cdot (s \cdot x)} \end{aligned}$$

However, as $s \in S \subseteq T$ and $\widetilde{\lambda} \in Y(S) \subseteq Y(T)$, we also have

$$\begin{aligned} X_{T,w} &:= \{q \in X |\ w \in \overline{T_w \cdot q}\ \} \\ &= \overline{T_w \cdot (s \cdot x)} \\ &= \overline{S_w \cdot (s \cdot x)} \end{aligned}$$

Therefore, for all $w \in Y$,

$$X_{T,w} = Y_{S,w}. \tag{3.2}$$

With this description in hand, we are now in a position to prove the following:

Lemma 3.6.3. *Let $z \in Y$. Then X is combinatorially regular at z if and only if Y is combinatorially regular at z.*

Proof. Let $z \in Y$. Suppose that X is combinatorially regular at z and take w to be any element of Y such that $z \leq_Y w$. Therefore, $\overline{S \cdot z} \subseteq \overline{S \cdot w}$, so that, in particular, $z \in \overline{S \cdot w} \subseteq \overline{T \cdot w}$. Hence $\overline{T \cdot z} \subseteq \overline{T \cdot w}$ and thus $z \leq_X w$. Since X is combinatorially regular at z, $X_{T,w}$ is irreducible and

$$|E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}.$$

However, as $T_w = S_w$ and $X_{T,w} = Y_{S,w}$ (by Equations (3.1) and (3.2) above), we in fact have that $Y_{S,w}$ is irreducible and

$$|E_{S_w}(Y_{S,w}, w)| = \dim Y_{S,w}$$

and as a result, Y is combinatorially regular at z.

For the opposite direction, assume that Y is combinatorially regular at z and let $w \in X$ such that $z \leq_X w$. Accordingly, $\overline{T \cdot z} \subseteq \overline{T \cdot w}$. Since $z \in Y$, by definition $y \in \overline{S \cdot z} \subseteq \overline{T \cdot z}$. Let $W := \overline{T \cdot w}$ in X. Thus $y \in W$. By Lemma 2.5.8 applied to W we find that there is a $\widetilde{\lambda} \in Y(T)$ and a $t \in T$ such that

$$\lim_{g \to 0} \widetilde{\lambda}(g)(t \cdot w) = y$$

and hence

$$W_{T,y} = \overline{S \cdot (t \cdot w)}$$

Thus,

$$t \cdot w \in W_{T,y} \subseteq X_{T,y} = Y.$$

Seeing that $z, t \cdot w \in Y$, we have $f(z) = y = f(t \cdot w)$. From this and the fact that $z \in \overline{T \cdot w} = \overline{T \cdot (t \cdot w)}$, it follows that $z \in \overline{S \cdot (t \cdot w)}$, by Lemma 2.5.7. Therefore, $z \leq_Y t \cdot w$. As Y is combinatorially regular at z, we know that $Y_{S,t \cdot w}$ is irreducible and

$$|E_{S_{t \cdot w}}(Y_{S,t \cdot w}, t \cdot w)| = \dim Y_{S,t \cdot w}$$

and subsequently from Equation 3.2, $X_{T,t \cdot w}$ is irreducible and

$$|E_{T_{t \cdot w}}(X_{T,t \cdot w}, t \cdot w)| = \dim X_{T,t \cdot w}.$$

Finally, since $t \cdot w \approx_X w$, from Lemma 3.6.1, we obtain that $X_{T,w}$ is irreducible and

$$|E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}.$$

Consequently, X is combinatorially regular at z. $\square$

3.7 Regular Points: Relating X and Y

This section is devoted to showing that for any $z \in Y$, z is a nonsingular point of X if and only if z is a nonsingular point of Y. The key to this is the following set:

Let

$$U := f^{-1}(T \cdot y) \cap X,$$

then U is an open T-stable subset of X which contains y and hence contains the orbit $T \cdot y$. Let $u \in U$, then $T \cdot u \subseteq U$ and we form $\overline{T \cdot u}$, where the closure is taken in U. As $f(u) \in T \cdot y$, there is a $t \in T$ such that

$$\lim_{g \to 0} \lambda(g)(t \cdot u) = f(t \cdot u) = y.$$

Therefore, $y \in U$ is a limit point of $\overline{T \cdot u}$ and is thus contained in $\overline{T \cdot u}$. Subsequently, $\overline{T \cdot y} \subseteq \overline{T \cdot u}$ and thus

$$\dim T \cdot y \leq \dim T \cdot u.$$

Also, if $z \in Y \setminus \{y\}$, then $z \notin T \cdot y$ as $T \cdot y \cap Y = \{y\}$. It follows that

$$T \cdot y \subset \overline{T \cdot z} \setminus T \cdot z,$$

and so

$$\dim T \cdot y < \dim T \cdot z.$$

Therefore, we have:

Lemma 3.7.1. *Let $U = f^{-1}(T \cdot y) \cap X$, then $T \cdot y$ is the minimal orbit in U. In particular, if $z \in Y \setminus \{y\}$, then $\dim T \cdot y < \dim T \cdot z$.*

The significance of U is the fact that

$$U \simeq T \cdot y \times Y$$

as T-varieties.

To prove this, we will use the following description of T.

Lemma 3.7.2. *$T \simeq S \times (T/S)$.*

Proof. The exact sequence of tori

$$1 \to S \hookrightarrow T \twoheadrightarrow T/S \to 1$$

yields the exact sequence

$$0 \to X(T/S) \hookrightarrow X(T) \twoheadrightarrow X(S) \to 0.$$

As $X(S)$ is a torsion-free finitely generated abelian group, it is free and hence a projective $\mathbb{Z}$-module. As a result, the latter sequence splits, and for that reason $X(T) \simeq X(S) \oplus X(T/S)$. Accordingly, the projection $X(T) \to X(T/S)$ yields a section $T/S \to T$ of the quotient map $T \to T/S$. Consequently, the exact sequence of tori splits and $T \simeq S \times (T/S)$. $\square$

Lemma 3.7.3. *Let $U := f^{-1}(T \cdot y) \cap X$, then $U \simeq T \cdot y \times Y$, as T-varieties.*

Proof. Since $T \cdot y$ is a locally closed subset of V_0, $f^{-1}(T \cdot y)$ is locally closed in $V_0 \oplus V_+$ and hence carries a unique structure as a subvariety, namely

$$f^{-1}(T \cdot y) = T \cdot y \times V_+.$$

We obtain an S-action on V_+ by restricting the T-action. Hence we have:

$$\begin{array}{ccccccc} T \times^S V_+ & \simeq & T/S \times V_+ & \simeq & T \cdot y \times V_+ & = & f^{-1}(T \cdot y) \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ S \times^S V_+ & \simeq & \{eS\} \times V_+ & \simeq & \{y\} \times V_+ & = & f^{-1}(y) \end{array}$$

(Here $A \times^S B$ denotes the contracted product, i.e. the quotient of $A \times B$ by the S-action $s \cdot (a, b) = (as^{-1}, s \cdot b)$. See Section 2.5, in particular, Lemma 2.5.4.) Now, $f^{-1}(T \cdot y) \cap X = U$ is a closed subvariety of $f^{-1}(T \cdot y)$ and hence also of $T \times^S V_+$.

With this identification, U has the form $T \times^S F$, where

$$F = U \cap V_+ = (f^{-1}(T \cdot y) \cap X) \cap f^{-1}(y) = f^{-1}(y) \cap X = Y$$

in $f^{-1}(T \cdot y)$. By Lemma 3.7.2, $T \simeq S \times (T/S)$. Thus we may use the projection $T \to S$ to obtain an action of T on Y. As a result,

$$U \simeq T \times^S Y \simeq T/S \times Y \simeq T \cdot y \times Y.$$

$\square$

We are now in a position to prove the following lemma:

Lemma 3.7.4.

If $z \in Y$, then X is regular at z if and only if Y is regular at z.

Proof. We have the following situation:

$$\begin{array}{ccc} U & \subseteq & X \\ \wr\| & & \cup\| \\ T \cdot y \times Y & \rightarrow & Y \end{array}$$

where $T \cdot y \times Y \rightarrow Y$ is the S-equivariant projection map.

As U is open in X, X is nonsingular at z if and only if U is nonsingular at z. Since $T \cdot y$ is nonsingular, the map $T \cdot y \times Y \rightarrow Y$ is smooth. As a result, U is nonsingular at z if and only if Y is nonsingular at z. $\square$

There are two important consequences of this set up that we would like to note. The first is that

$$\dim Y = \dim X - \dim T \cdot y. \tag{3.3}$$

The second is given in the following remark:

Remark 3.7.5. For any $z \in Y$, X is rationally smooth at z if and only if Y is rationally smooth at z. Indeed, X is rationally smooth at z if and only if U is rationally smooth at z. Furthermore, since $T \cdot y \times Y \rightarrow Y$ is smooth, U is rationally smooth at z if and only if Y is rationally smooth at z (see Proposition A1 in [9]).

3.8 Combinatorial Regularity and Rational Smoothness

As previously mentioned, our interest in the notion of combinatorial regularity comes from its equivalence to the concept of rational smoothness for T-orbit closures in G/B under the assumption that all stabilizers in T are connected. This fact follows from the work of Brion in [9].

Theorem 3.8.1. *Let $x \in G/B$, set $X := \overline{T \cdot x}$, and let $y \in X$. Suppose that the stabilizers T_z are connected for all $z \in X$. Then X is rationally smooth at y if and only if X is combinatorially regular at y. In particular, X is rationally smooth if and only if X is combinatorially regular.*

Proof. Let $y \in X$ and form Y and S as above. Suppose that X is rationally smooth at y. Then by Remark 3.7.5, Y is rationally smooth at y. Now let $w \neq y \in Y$. According to the definition of Y, $y \in \overline{S \cdot w}$ and hence the rationally smooth locus of Y meets $\overline{S \cdot w}$. Furthermore, since $S \cdot w$ is dense in $\overline{S \cdot w}$ and the rationally smooth locus of Y is open, the rationally smooth locus of Y meets $S \cdot w$. Moreover, since it is also S-stable, it contains w. Therefore, Y is rationally smooth.

We know that Y is irreducible and, since Y is rationally smooth at y, from [9, Cor. 2] we obtain that

$$|E_S(Y, y)| = \dim Y.$$

By Equations (3.1) and (3.2) above, we know that $T_w = S_w$ and $X_{T,w} = Y_{S,w}$, respectively. By assumption, T_w is connected and so, just as was done for Y, we determine that $X_{T,w}$ is irreducible. Furthermore, Remark 3.7.5 yields that X is rationally smooth at w. Since Remark 3.7.5 equally applies to X and $X_{T,w}$, we obtain that $X_{T,w}$ is rationally smooth at w. A second application of [9, Cor. 2] yields that

$$|E_{T_w}(X_{T,w}, w)| = \dim X_{T,w}$$

and hence

$$|E_{S_w}(Y_{S,w}, w)| = \dim Y_{S,w}.$$

Therefore, Y is combinatorially regular at y and hence X is combinatorially regular at y, as a result of Lemma 3.6.3.

Conversely, suppose that X is combinatorially regular at y. Thus, for all $z \geq_X y$, $X_{T,z}$ is irreducible and

$$|E_{T_z}(X_{T,z}, z)| = \dim X_{T,z}.$$

As such, X is also combinatorially regular at all $z \in X$ such that $z \geq_X y$. In addition, this means that

$$|E_S(Y, y)| = \dim Y.$$

We will prove by induction on the codimension of $T \cdot y$ in X (i.e. the dimension of Y) that X is rationally smooth at y. If $\operatorname{codim} T \cdot y = 0$, so that $\dim Y = 0$, then $y \in T \cdot x$. Since the rationally smooth locus of X is open and T-stable, and since $T \cdot x$ is dense in X, it follows that X is rationally smooth at y.

Now suppose that $\operatorname{codim} T \cdot y \geq 1$ and assume that any combinatorially regular point of X whose T-orbit has codimension less that $\operatorname{codim} T \cdot y$ is rationally smooth. Let $w \in Y \setminus \{y\}$. Since $y \in \overline{S \cdot w}$, we have that $\overline{T \cdot y} \subseteq \overline{T \cdot w}$, hence $w \geq_X y$, and therefore X is combinatorially regular at w. Furthermore, since $w \in Y \setminus \{y\} \subseteq U$, Lemma 3.7.1 yields that $\dim T \cdot y < \dim T \cdot w$. Thus, by our induction assumption, X is rationally smooth at w and subsequently Y is rationally smooth at w, by Remark 3.7.5. Hence Y is rationally smooth on $Y \setminus \{y\}$. Thus, Y is rationally smooth in a punctured neighbourhood of the attractive S-fixed point y and $|E_S(Y, y)| = \dim Y$. It now follows from [9, Cor. 2] that Y is rationally smooth at y and hence, by Remark 3.7.5, X is rationally smooth at y. □

3.9 Combinatorially Regular Orbit Closures in G/B for $G = \mathrm{PGL}_{n+1}$

In this section, continuing with the notation adopted in Sections 3.4 to 3.8, we will prove Theorem 3.9.4 which states that in type A, a combinatorially regular T-orbit closure X is in fact regular. Our preliminary results leading up to Theorem 3.9.4 use techniques which are based on well-established ideas. Following Theorem 3.9.4, we will prove our main result in the type A case, namely Theorem 3.9.5. To that end, let $G = \mathrm{PGL}_{n+1}$. Since some of the constructions in the previous sections depend on the fact that S is connected, we begin by verifying this.

Consider the quotient map of reductive groups:

$$\mathrm{GL}_{n+1} \twoheadrightarrow \mathrm{PGL}_{n+1}.$$

The preimage $\widetilde{T}$ of T is a maximal torus and hence is conjugate to the torus consisting of invertible diagonal matrices D, i.e. $g\widetilde{T}g^{-1} = D$. The preimage $\widetilde{S}$ of S is the stabilizer $\widetilde{T}_y$ and $g\widetilde{T}_y g^{-1} = D_{gy}$. Since it is the common kernel of some roots of GL_{n+1} with respect to D, D_{gy} consists of all invertible diagonal matrices with particular sets of entries identical and is thus a torus. Consequently, $\widetilde{S} = \widetilde{T}_y$ is a torus and so its image S is as well.

As we are interested in the regular points of Y, we return to the subject of the weights of the tangent space $T_y(Y)$.

Since G is an algebraic group of adjoint type and S is connected,

$$S = \ker(\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_n),$$

for some simple system $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi$. Let

$$\Delta(S) = \{\alpha_1|_S, \alpha_2|_S, \ldots, \alpha_r|_S\}.$$

To simplify notation, we will denote $\alpha_i|_S$ by α_i for each i. As every element of $X(S)$ can be obtained from an element of $X(T)$ by restriction, every $\beta \in \Omega_S(\mathfrak{g})$ has the form

$$\beta = \sum_{\alpha_i \in \Delta(S)} \beta_i \alpha_i,$$

where all $\beta_i \in \{0, 1\}$ or all $\beta_i \in \{0, -1\}$.

Lemma 3.9.1. *No two weights of the S-module $T_y(Y)$ are proportional.*

Proof. Let

$$\beta = \sum_{\alpha_i \in \Delta(S)} \beta_i \alpha_i \quad \text{and} \quad \gamma = \sum_{\alpha_i \in \Delta(S)} \gamma_i \alpha_i$$

be any elements of $\Omega_S(T_y(Y))$ and suppose that $\gamma = r\beta$, for some $r \in \mathbb{Q}$. Since the α_i are linearly independent, for each i we obtain $\gamma_i = r\beta_i$, where $\gamma_i, \beta_i \in \{0, \pm 1\}$. By Corollary 3.5.3, $\gamma \neq 0$. Thus there exists a $\gamma_i \neq 0$, so that $\gamma_i, \beta_i \in \{\pm 1\}$ and hence $r = \pm 1$. Consequently $\gamma = \pm\beta$. However, as $\langle \gamma, \lambda \rangle, \langle \beta, \lambda \rangle > 0$ (Corollary 3.5.3), $\gamma = \beta$. $\square$

Since the weights of $T_y(Y)$ are non-proportional, by Lemma 2.7.4 the S-curves in $T_y(Y)$ are weight spaces. Thus for any S-curve C in Y,

$$C \hookrightarrow Y \hookrightarrow T_y(Y) = \bigoplus L_\alpha$$

and hence $C = L_\alpha$, for some $\alpha \in \Omega_S(T_y(Y))$. Thus

$$T_y(C) = T_y(L_\alpha) \simeq L_\alpha$$

and hence C is nonsingular at y. Since $C = \overline{S \cdot z}$ for some $z \in Y \setminus \{y\}$, C is regular along $S \cdot z$, and $C^S = \{y\}$, we obtain that C is regular. Consequently we have:

Lemma 3.9.2. *If C is an S-curve in Y containing y, then C is regular. In particular, if $\dim Y = 1$, then Y is regular.*

Furthermore, if C and D are distinct S-curves in $T_y(Y)$, then their tangents spaces at y, $T_y(C)$ and $T_y(D)$, are distinct. Indeed, suppose not and let $T_y(C) = L_\alpha = T_y(D)$ and

$$T_y(C \cup D) = \bigoplus_{\beta \in \Phi' \subset \Omega_S(T_y(Y))} L_\beta.$$

As the tangent space $T_y(C \cup D)$ of the singular variety $C \cup D$ has dimension at least 2, there are at least 2 weights in Φ'. Now, the connected component $(\ker \alpha)^\circ$ of the kernel of α acts trivially on $T_y(C) \simeq C$ and $T_y(D) \simeq D$, so also on $C \cup D$ and consequently on $T_y(C \cup D)$. Therefore, the weights in Φ' are proportional, which contradicts the fact that $T_y(Y)$ has non-proportional weights.

Consequently, in this setting,

$$TE(Y, y) = \bigoplus_{C \in E_S(Y,y)} T_y(C)$$

with dimension $|E_S(Y, y)|$.

The other vital ingredient in the proof of Theorem 3.9.4 is the Peterson Translate $\tau_C(Y, y)$. In order to apply Lemma 2.12.1, we will use the following lemma regarding S-surfaces in Y.

Lemma 3.9.3. *Assume* $\dim Y \geq 2$ *and let* $\Sigma \in \Sigma_S(Y, y)$*, then* Σ *is nonsingular at* y*.*

Proof. If $\dim Y = 2$, then $\Sigma = Y$ in what follows. By Lemma 2.12.2, there are two S-curves C and D contained in Σ. Since $\Sigma_y = \Sigma$, we have

$$C, D \subseteq \Sigma \hookrightarrow T_y(\Sigma) \hookrightarrow T_y(Y) = \bigoplus L_\alpha,$$

so that $C = L_\beta$ and $D = L_\gamma$, for some $\beta, \gamma \in \Omega_S(T_y(Y))$, such that $\beta \neq \gamma$. Furthermore, $\beta \neq -\gamma$ since $\langle \beta, \lambda \rangle, \langle \gamma, \lambda \rangle > 0$ (see Corollary 3.5.3).

To show that Σ is nonsingular at y, we will show that β and γ are the only two weights of $T_y(\Sigma)$. To that end, let $\omega \in \Omega_S(T_y(\Sigma))$. We write $\beta = \sum \beta_i \alpha_i$, $\gamma = \sum \gamma_i \alpha_i$, and $\omega = \sum \omega_i \alpha_i$, where the α_i are those elements of $\Delta(S)$ for which the coefficients β_i, γ_i, and ω_i are not all 0. In addition, all $\beta_i \in \{0, 1\}$ or all $\beta_i \in \{0, -1\}$, all $\gamma_i \in \{0, 1\}$ or all $\gamma_i \in \{0, -1\}$, and all $\omega_i \in \{0, 1\}$ or all $\omega_i \in \{0, -1\}$.

By Lemma 2.12.5 there exists an $N \in \mathbb{N}$ and $g, b \in \mathbb{Z}_{\geq 0}$ such that

$$N\omega = b\beta + g\gamma.$$

We may assume that $\gcd(N, b, g) = 1$.

We will prove that either $b = 0$ or $g = 0$ (exclusively since $N \neq 0$ and $\omega \neq 0$ (Corollary 3.5.3)). Suppose, for the sake of contradiction, that $b, g > 0$. From the linearly independence of the roots α_i we obtain

$$N\omega_i = b\beta_i + g\gamma_i$$

for all i. Now, there exists an i for which either $\beta_i = 0$ or $\gamma_i = 0$ (exclusively since $N \neq 0$ and by assumption, β_i, γ_i, and ω_i are not all 0). Indeed, if not, then $\beta_i = \beta_j = \pm 1$ and $\gamma_i = \gamma_j = \pm 1$ for all i and j, which implies that $\beta = \pm\gamma$.

Without loss of generality, we may assume that $\beta_i = 0$ (so that $\gamma_i \neq 0$). As a result, we obtain that $N\omega_i = g\gamma_i$, where $\gamma_i = \omega_i = \pm 1$, since $N, g > 0$. Thus $N = g$. As $\beta \neq 0$ (Corollary 3.5.3), there is a j such that $\beta_j \neq 0$ (so $\beta_j = \pm 1$). Accordingly,

$$N\omega_j = b\beta_j + N\gamma_j$$

which yields

$$N(\omega_j - \gamma_j) = b\beta_j.$$

Consequently,

$$N | b\beta_j$$

and since $b\beta_j = \pm b$, we obtain

$$N | \pm b.$$

However, $\gcd(N, b) = \gcd(N, b, g) = 1$, from which we obtain that $N = 1$. Thus

$$\omega = b\beta + \gamma$$

where $b > 0$. By Lemma 2.10.2 this implies that ω is not a weight of $T_y(\Sigma)$, hence providing us with the required contradiction.

We thus conclude that exactly one of b and g is 0. Thus, ω is proportional to either β or γ, which by Lemma 3.9.1 yields that $\omega = \beta$ or $\omega = \gamma$. As a result, β and γ are the only weights of $T_y(\Sigma)$ and hence Σ is nonsingular at y. $\square$

Theorem 3.9.4. *Let $G = \mathrm{PGL}_{n+1}$, B a Borel subgroup, and $T \subseteq B$ a maximal torus. Let $x \in G/B$ and $X := \overline{T \cdot x}$. If X is combinatorially regular, then X is regular.*

Proof. Assume X is combinatorially regular. As X is regular along the open orbit $T \cdot x$, it remains to show that X is nonsingular at all points in $X \setminus T \cdot x$. Let $y \in X \setminus T \cdot x$ and form Y as above. We will prove by induction on the codimension of $T \cdot y$, i.e. the dimension of Y, that X is nonsingular at all points of Y. To that end, Lemma 3.7.1 shows that for any $z \in Y \setminus \{y\} \subseteq U$, $\operatorname{codim} T \cdot y > \operatorname{codim} T \cdot z$.

If $\operatorname{codim} T \cdot y = 1$, then $\dim Y = 1$. By Lemma 3.9.2, Y is regular and hence, by Lemma 3.7.4, X is nonsingular along Y.

Now suppose that $\operatorname{codim} T \cdot y = d$, for $d \geq 2$, and assume that any T-orbit with codimension strictly less than d is contained in the regular locus of X. Thus, as $\operatorname{codim} T \cdot y > \operatorname{codim} T \cdot z$ for all $z \in Y \setminus \{y\}$, X is regular along $Y \setminus \{y\}$. It remains to show that Y is nonsingular at y.

To accomplish this, let $C \in E_S(Y, y)$, then $C = \overline{S \cdot z}$ for some $z \in Y \setminus \{y\}$ and hence C is good. Let $D \in E_S(Y, y)$ other than C and set $T_y(C) = L_\gamma$ and $T_y(D) = L_\beta$. Let $\rho : Y \twoheadrightarrow TE(Y, y)$ be the restriction to Y of the S-equivariant projection $\widetilde{\rho} : T_y(Y) \twoheadrightarrow TE(Y, y)$, so that $d_y\rho = \widetilde{\rho}$. Then $\rho^{-1}(0)$ is finite, as otherwise $\rho^{-1}(0)$ would contain an S-curve through y whose tangent space would be an element of $TE(Y, y)$ that vanishes under $\widetilde{\rho}$. Therefore, by Lemma 2.9.10, ρ is a finite surjective morphism and thus, by a dimensional argument, one of the irreducible component of $\rho^{-1}(L_\gamma \oplus L_\beta)$ is a surface Σ containing C and D. Fortunately, by Lemma 3.9.3, Σ is nonsingular at y. From this and Lemma 2.12.1 we have

$$T_y(D) \hookrightarrow T_y(\Sigma) = \tau_C(\Sigma, y) \subseteq \tau_C(Y, y).$$

Consequently, $TE(Y, y) \subseteq \tau_C(Y, y)$. Since X is combinatorially regular at y, by Lemma 3.6.3, Y is combinatorially regular at y and hence

$$\dim Y = |E_S(Y, y)| = \dim TE(Y, y).$$

As C is good, $\dim \tau_C(Y, y) = \dim Y$ and subsequently

$$TE(Y, y) = \tau_C(Y, y).$$

As this is true for at least two good curves, it follows from Theorem 2.11.5 that Y is nonsingular at y. Finally, by Lemma 3.7.4, X is nonsingular at y. □

Theorem 3.9.5. *Let $G = \mathrm{PGL}_{n+1}$, B a Borel subgroup, and $T \subseteq B$ a maximal torus. Let $x \in G/B$ and $X := \overline{T \cdot x}$. Then X is rationally smooth if and only if X is smooth.*

Proof. One direction is clear. Now assume that X is rationally smooth. Thus, by Theorem 3.8.1 X is combinatorially regular and hence, by Theorem 3.9.4 X is smooth. □

3.10 Combinatorially Regular Orbit Closures in G/B for $G = \mathrm{PSO}_{2n}$

When considering type D, we use the constructions from Sections 3.5 through 3.7. Throughout this section, we shall assume that all stabilizers S are connected. We then attempt to employ the method used to investigate type A in Section 3.9. It is still the case in type D that no two weights of $T_y(Y)$ are proportional and hence that the S-curves in Y containing y are regular. However, this approach breaks down when examining the possible relations between the weights of the tangent space $T_y(\Sigma)$ of an S-surface in Y containing y. If $\Sigma \in \Sigma_S(Y,y)$, then Σ contains two S-curves, say $C = L_\beta$ and $D = L_\gamma$, from which it follows that β and γ are both weights of $T_y(\Sigma)$. Now, if $w \in \Omega_S(T_y(Y))$, then there exists an $N \in \mathbb{N}$ and $b, g \in \mathbb{N}$ such that $N\omega = b\beta + g\gamma$. In type A, we were able to use this equation to verify that β and γ are the only two weights of $T_y(\Sigma)$, from which it followed that the S-surfaces in Y are nonsingular. As a consequence of this, we obtained that combinatorially regular T-orbit closures in G/B are regular. In type D, however, there are two possible bad relations resulting from $N\omega = b\beta + g\gamma$:

$$2\omega = \beta + \gamma$$
$$2\omega = 2\beta + \gamma$$

Furthermore, we know from the work of Carrell and Kuttler that there are rationally smooth T-orbit closures in G/B that are singular (see Example 7.1 in [15]).

In this section, we will verify that no two weights of $T_y(Y)$ are proportional and then prove that these two equations are the only two possible bad relations involving weights of $T_y(\Sigma)$. We will also provide a local example of a combinatorially regular orbit closure which is singular (see Example 3.10.6).

Once again, as G is an algebraic group of adjoint type, we have that

$$S = \bigcap_{i \in I} \ker(\alpha_i),$$

for some simple system $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi$ and some indexing set $I \subseteq \{1, 2, \ldots, n\}$.

Remark 3.10.1. For type D considerations, we will make explicit the a particular ordering of the roots α_i. Consider the following algebraic group:

$$SO(B) := \{g \in \mathrm{SL}_n(\mathbb{C}) | g^T B g = B\},$$

where B is the following $2n \times 2n$ matrix:

$$B = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$$

In SO(B), there is a maximal torus consisting of diagonal matrices

$$\mathrm{diag}(g_1, g_2, \ldots, g_n, g_n^{-1}, g_{n-1}^{-1}, \ldots, g_1^{-1}).$$

Let e_i be the projection onto the i^{th} diagonal entry, for $1 \leq i \leq n$. The set $\Delta' = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$ is a system of simple roots for a root system of type D_n. As such, there is an element w of the Weyl group $N_G(T)/T$ such that $w\Delta' = \Delta$. We now relabel the elements of Δ, so that

$$\begin{aligned} \alpha_i &= w(e_i - e_{i+1}), \quad \text{for} \quad 1 \leq i \leq n-1 \\ \alpha_n &= w(e_{n-1} + e_n) \end{aligned}$$

and change the set I accordingly.

Let

$$\Delta(S) = \{\alpha_1|_S, \alpha_2|_S, \ldots, \alpha_n|_S\} \setminus \{\alpha_i|_S \mid i \in I\}.$$

To simplify notation, we will denote $\alpha_i|_S$ by α_i for each i.

Thus, every $\beta \in \Omega_S(\mathfrak{g})$ has the form

$$\beta = \sum_{\alpha_i \in \Delta(S)} \beta_i \alpha_i,$$

where all $\beta_i \in \{0, 1, 2\}$ or all $\beta_i \in \{0, -1, -2\}$.

Using this descriptions, we will verify the following:

Lemma 3.10.2. *If S is connected, then no two weights of the S-module $T_y(Y)$ are proportional.*

Proof. Let $\beta, \gamma \in \Omega_S(T_y(Y))$. By Corollary 3.5.3, $\beta, \gamma \neq 0$. Hence,

$$\beta = \sum \beta_i \alpha_i \text{ and } \gamma = \sum \gamma_i \alpha_i,$$

where $\beta_i, \gamma_i \in \{0, \pm 1, \pm 2\}$ and the α_i appearing in the sums are those elements of $\Delta(S)$ for which the coefficients β_i and γ_i are not both 0.

Suppose that $b\beta = g\gamma$, for some non-zero $b, g \in \mathbb{Z}$. By the linear independence of the α_i, for all i we obtain

$$b\beta_i = g\gamma_i$$

Since β_i and γ_i are not both 0 and $b, g \neq 0$, it follows that $\beta_i, \gamma_i \in \{\pm 1, \pm 2\}$. This results in six possibilities:

$$b = \pm g$$
$$b = \pm 2g$$
$$g = \pm 2b.$$

These yield:

$$\beta = \pm\gamma$$
$$\beta = \pm 2\gamma$$
$$\gamma = \pm 2\beta$$

However, since $\langle \beta, \lambda \rangle, \langle \gamma, \lambda \rangle > 0$ by Corollary 3.5.3, we eliminate $\beta = -\gamma$, $\beta = -2\gamma$, and $\gamma = -2\beta$ as possibilities. Furthermore, we can eliminate the linear combinations $\beta = 2\gamma$ and $\gamma = 2\beta$ using Lemma 2.10.2. Therefore, $\beta = \gamma$. $\square$

Consequently, as in type A we have

Lemma 3.10.3. *If C is an S-curve in Y containing y, where S is connected, then C is regular.*

To verify that there are only two possible bad relationships between the weights of $T_y(\Sigma)$, we will use a property of the coefficients of a root of type D_n using the simple system indicated above.

Remark 3.10.4 (2-Section Condition)**.** Let $\beta = \sum_{i=1}^{n} \beta_i \alpha_i$ be a root of type D_n, where $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi$ is the simple system described above. A coefficient of ± 2 may only appear on $\alpha_2, \alpha_3, \ldots, \alpha_{n-2}$. The sum $\sum_{i=2}^{n-2} \beta_i \alpha_i$ is what we shall refer to as the 2-*section*. If $\beta_i = \pm 2$ for some i, then $\beta_j = \pm 2$ for all j such that $i \leq j \leq n-2$.

Restricting to S deletes some of the simple roots appearing in β, but doesn't change the coefficients on the simple roots that remain. Thus it is still the case that, for elements of $\Omega_S(\mathfrak{g})$, once a coefficient of ± 2 appears in the 2-section, all subsequent coefficients on the surviving simple roots in the 2-section are ± 2.

In order to prove Lemma 3.10.5, we will consider a number of cases involving possible combinations of values of coefficients. The 2-section condition will be used in the proof of Lemma 3.10.5 to eliminate cases with following conditions:

Suppose $\beta = \sum \beta_i \alpha_i$ and $\gamma = \sum \gamma_i \alpha_i$ are elements of $\Omega_S(\mathfrak{g})$, such that

$$\begin{array}{ll} \beta_i = \pm 2 & \beta_j = 0, \pm 1 \\ \gamma_i = 0, \pm 1 & \gamma_j = \pm 2 \end{array}$$

for some $i \neq j$. If $i < j$, then β violates the 2-section condition and if $i > j$, then γ breaks the 2-section condition. Consequently, this situation cannot occur.

Lemma 3.10.5. *Assume S is connected. Let $\Sigma \in \Sigma_S(Y, y)$ and suppose that Σ contains the two S-curves $C = L_\beta$ and $D = L_\gamma$, so that β and γ are distinct weights of $T_y(\Sigma)$. Assume that ω is a third distinct element of $\Omega_S(T_y(\Sigma))$. Then one of the following holds:*

$$2\omega = \beta + \gamma$$
$$2\omega = 2\beta + \gamma$$

Proof. Let

$$\omega = \sum \omega_i \alpha_i, \quad \beta = \sum \beta_i \alpha_i, \quad \text{and} \quad \gamma = \sum \gamma_i \alpha_i,$$

where $\omega_i, \beta_i, \gamma_i \in \{0, \pm 1, \pm 2\}$ such that

$$\begin{array}{lccc} & \text{all } \omega_i \geq 0 & \text{or} & \text{all } \omega_i \leq 0, \\ & \text{all } \beta_i \geq 0 & \text{or} & \text{all } \beta_i \leq 0, \\ \text{and} & \text{all } \gamma_i \geq 0 & \text{or} & \text{all } \gamma_i \leq 0. \end{array} \tag{3.4}$$

Furthermore, the α_i appearing in the sums are those elements of $\Delta(S)$ for which the coefficients ω_i, β_i, and γ_i are not all 0 ($\omega, \beta, \gamma \neq 0$ by Corollary 3.5.3). From Lemma 2.12.5 we obtain that there exist an $N \in \mathbb{N}$ and $b, g \in \mathbb{Z}_{\geq 0}$ such that

$$N\omega = b\beta + g\gamma. \tag{3.5}$$

We may assume that $\gcd(N, b, g) = 1$. As an immediate consequence of equation 3.5, since $\Delta(S)$ is a linearly independent set, we have that for each i

$$N\omega_i = b\beta_i + g\gamma_i. \tag{3.6}$$

As the weights of $T_y(Y)$ are non-proportional (Lemma 3.10.2), the weights of $T_y(\Sigma)$ are also non-proportional. From this, equation 3.5, and the fact that $N \neq 0$ and $\omega \neq 0$, it follows that neither b nor g is 0. Moreover, if $N = 1$, then $\omega = b\beta + g\gamma$, where $b + g > 1$, which by Lemma 2.10.2 yields that ω is not a weight of $T_y(\Sigma)$. Consequently, as ω is a weight of $T_y(\Sigma)$, $N \geq 2$.

We will now show that there is an i such that ω_i, β_i, or γ_i is 0. To that end, assume that $\omega_i, \beta_i, \gamma_i \in \{\pm 1, \pm 2\}$, for all i. According to Corollary 3.5.3, for the distinct weights β and γ we have that $\langle \beta, \lambda \rangle, \langle \gamma, \lambda \rangle > 0$ and subsequently $\beta \neq \pm\gamma$. Thus, in view of 3.4 above, there exists an i such that $\beta_i \neq \pm\gamma_i$. As $\omega_i, \beta_i, \gamma_i \in \{\pm 1, \pm 2\}$, $\omega_i = \pm\beta_i$ or $\omega_i = \pm\gamma_i$.

Without loss of generality, we may assume that $\omega_i = \pm\beta_i$. As it is also the case that $\langle \omega, \lambda \rangle > 0$ (Corollary 3.5.3), $\omega \neq \pm\beta$. Therefore, there exists a $j \neq i$ for which $\omega_j \neq \pm\beta_j$. By 3.6 above we have a system of equations

$$\begin{aligned} N\omega_i &= b\beta_i + g\gamma_i \\ N\omega_j &= b\beta_j + g\gamma_j \end{aligned}$$

which yields

$$N(\omega_i - \omega_j) = b(\beta_i - \beta_j) + g(\gamma_i - \gamma_j) \tag{3.7}$$

Taking into account that $\omega_i, \beta_i \in \{\pm 1, \pm 2\}$, $\omega_i = \pm\beta_i$, $\omega_j \neq \pm\beta_j$, and 3.4 above, we have that either

$$\begin{array}{ccc} \omega_i = \omega_j & \quad\text{or}\quad & \omega_i \neq \omega_j. \\ \beta_i \neq \beta_j & & \beta_i = \beta_j \end{array}$$

In the first case, equation 3.7 becomes $0 = b(\beta_i - \beta_j) + g(\gamma_i - \gamma_j)$. As g, b, and $\beta_i - \beta_j$ are all non-zero, we also have $\gamma_i - \gamma_j \neq 0$ and hence $\gamma_i \neq \gamma_j$. Therefore, we have that $\beta_i \neq \beta_j$, $\gamma_i \neq \gamma_j$, and $\beta_i \neq \pm\gamma_i$, with $\beta_i, \beta_j, \gamma_i, \gamma_j \in \{\pm 1, \pm 2\}$ such that 3.4 holds. Thus, we have either

$$\begin{array}{ccccc} \beta_i = \pm 1 & \beta_j = \pm 2 & & \beta_i = \pm 2 & \beta_j = \pm 1 \\ & & \text{or} & & \\ \gamma_i = \pm 2 & \gamma_j = \pm 1 & & \gamma_i = \pm 1 & \gamma_j = \pm 2 \end{array}$$

In both situations we obtain a contradiction with the 2-section condition (see Remark 3.10.4).

In the second case, equation 3.7 becomes $N(\omega_i - \omega_j) = g(\gamma_i - \gamma_j)$. We obtain a contradiction in this instance as we did in case one, but using the coefficients of ω in place of the coefficients of β.

Thus ω_i, β_i, or γ_i is 0, for some i. Hence, from 3.6 above we obtain the following three possibilities:

$$0 = b\beta_i + g\gamma_i \qquad N\omega_i = g\gamma_i \qquad N\omega_i = b\beta_i$$

It follows from these equations that exactly one of ω_i, β_i, and γ_i is 0, since not all three are 0 by assumption and $N, b, g \neq 0$.

Case 1: $0 = b\beta_i + g\gamma_i$

In this case, $\omega_i = 0$. Since $b, g > 0$, we also have either $\beta_i \in \{1, 2\}$ with $\gamma_i \in \{-1, -2\}$ or $\beta_i \in \{-1, -2\}$ with $\gamma_i \in \{1, 2\}$. This leads to three subcases:

$$b = g \qquad b = 2g \qquad g = 2b$$

As $\gcd(N, b, g) = 1$, $\gcd(N, b) = 1$ and $\gcd(N, g) = 1$ in all three subcases.

As $\omega \neq 0$, there exists a j such that $\omega_j \neq 0$. By 3.6 we know that

$$N\omega_j = b\beta_j + g\gamma_j,$$

which produces:

$$N\omega_j = b(\beta_j + \gamma_j) \qquad N\omega_j = g(2\beta_j + \gamma_j) \qquad N\omega_j = b(\beta_j + 2\gamma_j) \tag{3.8}$$

Noting that $N, b, g, \omega_j \neq 0$, we know:

$$\beta_j + \gamma_j \neq 0 \qquad 2\beta_j + \gamma_j \neq 0 \qquad \beta_j + 2\gamma_j \neq 0$$

Consequently, since $\gcd(N, b) = 1$ and $\gcd(N, g) = 1$, we obtain:

$$b|\omega_j \qquad g|\omega_j \qquad b|\omega_j$$

Taking into account that $\omega_j \in \{\pm 1, \pm 2\}$, our three subcases result in:

$$\begin{array}{cccccc} \multicolumn{2}{c}{\text{Subcase 1}} & \multicolumn{2}{c}{\text{Subcase 2}} & \multicolumn{2}{c}{\text{Subcase 3}} \\ b = 1 & b = 2 & b = 2 & b = 4 & b = 1 & b = 2 \\ g = 1 & g = 2 & g = 1 & g = 2 & g = 2 & g = 4 \\ \omega_j = \pm 1 & \omega_j = \pm 2 & \omega_j = \pm 1 & \omega_j = \pm 2 & \omega_j = \pm 1 & \omega_j = \pm 2 \end{array} \tag{3.9}$$

Using these values, after simplification 3.8 becomes:

$$\pm N = \beta_j + \gamma_j \qquad \pm N = 2\beta_j + \gamma_j \qquad \pm N = \beta_j + 2\gamma_j \tag{3.10}$$

In view of 3.4, since β_i and γ_i have opposite signs, we have one of β_j and γ_j is an element of $\{0, -1, -2\}$ and the other is an element of $\{0, 1, 2\}$. In addition, since $N \neq 0$, from 3.10 we know

$$\beta_j \neq -\gamma_j \qquad 2\beta_j \neq -\gamma_j \qquad \beta_j \neq -2\gamma_j$$

Based on this, we compute:

$$\beta_j + \gamma_j = \pm 1, \pm 2 \qquad 2\beta_j + \gamma_j = \pm 1, \pm 2, \pm 3, \pm 4 \qquad \beta_j + 2\gamma_j = \pm 1, \pm 2, \pm 3, \pm 4$$

Subcase 2 is that $b = 2g$, which resulted from $\beta_i = \pm 1$, $\gamma_i = \mp 2$. We obtained $2\beta_j + \gamma_j = \pm 3$ from $\beta_j = \pm 2$ and $\gamma_j = \mp 1$, whereas $2\beta_j + \gamma_j = \pm 4$ arose from $\beta_j = \pm 2$ and $\gamma_j = 0$. Hence, $2\beta_j + \gamma_j \neq \pm 3, \pm 4$ since the 2-section condition fails:

$$\beta_i = \pm 1 \qquad \beta_j = \pm 2$$

$$\gamma_i = \mp 2 \qquad \gamma_j = 0, \mp 1$$

By symmetry, we determine that $\beta_j + 2\gamma_j \neq \pm 3, \pm 4$ for Subcase 3 as well.

Thus we have:

$$\beta_j + \gamma_j = \pm 1, \pm 2 \qquad 2\beta_j + \gamma_j = \pm 1, \pm 2 \qquad \beta_j + 2\gamma_j = \pm 1, \pm 2$$

As $N \geq 2$, we see that $N = 2$ in all three subcases. Furthermore, as $\gcd(N, b, g) = 1$, the possibilities presented in 3.9 reduce to:

Subcase 1	Subcase 2	Subcase 3
$N = 2$	$N = 2$	$N = 2$
$b = 1$	$g = 1$	$b = 1$
$g = 1$	$b = 2$	$g = 2$

The resulting relations are:

$$2\omega = \beta + \gamma \qquad\qquad 2\omega = 2\beta + \gamma \qquad\qquad 2\omega = \beta + 2\gamma$$

Case 2: $N\omega_i = g\gamma_i$

In this case, $\beta_i = 0$ and either $\omega_i, \gamma_i \in \{1, 2\}$ or $\omega_i, \gamma_i \in \{-1, -2\}$. We will write γ as follows:

$$\gamma = \sum \gamma_k \alpha_k = \sum -\gamma'_k \alpha_k,$$

where $\gamma'_k = -\gamma_k$. It is still that the case that $\gamma'_k \in \{0, 1, 2\}$ for all k or $\gamma'_k \in \{0, -1, -2\}$ for all k. Henceforth, we will use $-\gamma'_k$ in place of γ_k for all k. Accordingly, $N\omega_i = g\gamma_i$ becomes

$$N\omega_i + g\gamma'_i = 0$$

where either $\omega_i \in \{1, 2\}$ and $\gamma'_i \in \{-1, -2\}$ or $\omega_i \in \{-1, -2\}$ and $\gamma'_i \in \{1, 2\}$. We may now argue as we did in Case 1, but with a few minor modifications, resulting from the fact that N and b have interchanged roles, but have different restrictions: $N \geq 2$, whereas $b \geq 1$. As before, we have three subcases:

$$N = g \qquad\qquad N = 2g \qquad\qquad g = 2N$$

Following the procedure in Case 1, since $N \geq 2$, we see that 3.9 above becomes

Subcase 1	Subcase 2		Subcase 3
$N = 2$	$N = 2$	$N = 4$	$N = 2$
$g = 2$	$g = 1$	$g = 2$	$g = 4$
$\beta_j = \pm 2$	$\beta_j = \pm 1$	$\beta_j = \pm 2$	$\beta_j = \pm 2$

for some $j \neq i$, and 3.10 becomes:

$$\pm b = \omega_j + \gamma'_j \qquad \pm b = 2\omega_j + \gamma'_j \qquad \pm b = \omega_j + 2\gamma'_j$$

We then determine that:

$$\omega_j + \gamma_j' = \pm 1, \pm 2 \qquad 2\omega_j + \gamma_j' = \pm 1, \pm 2 \qquad \omega_j + 2\gamma_j' = \pm 1, \pm 2$$

So $b = 1$ or 2 in all subcases, however, as $\gcd(N, b, g) = 1$, we reduce to:

Subcase 1	Subcase 2		Subcase 3
$b = 1$	$b = 1$ or 2	$b = 1$	$b = 1$
$N = 2$	$N = 2$	$N = 4$	$N = 2$
$g = 2$	$g = 1$	$g = 2$	$g = 4$

We can further reduce this list by once again using the 2-section condition. For Case 2, we assumed that $\beta_i = 0$. For Subcase 2, we used $\gamma_i = \pm 2$ and the instance in Subcase 2 in which $b = 1$, $N = 4$, and $g = 2$ was obtained by taking $\beta_j = \pm 2$ and $\gamma_j = \pm 1$. Thus

$$\beta_i = 0 \qquad \beta_j = \pm 2$$

$$\gamma_i = \pm 2 \qquad \gamma_j = \pm 1$$

violates the 2-section condition. Similarly, we exclude Subcase 3 (using the coefficients of ω in place of those of γ).

Thus Case 2 yields:

Subcase 1	Subcase 2
$b = 1$	$b = 1$ or 2
$N = 2$	$N = 2$
$g = 2$	$g = 1$

The resulting relations are:

$$2\omega = \beta + 2\gamma \qquad 2\omega = \beta + \gamma \qquad 2\omega = 2\beta + \gamma$$

<u>Case 3</u>: $N\omega_i = b\beta_i$

By symmetry, we obtain the same three relations as in Case 2.

Thus, all three case yield the relations:

$$2\omega = 2\beta + \gamma \qquad\qquad 2\omega = \beta + \gamma \qquad\qquad 2\omega = \beta + 2\gamma$$

By symmetry, we reduce to the two equations:

$$2\omega = \beta + \gamma \qquad\qquad 2\omega = 2\beta + \gamma$$

$\square$

As mentioned above, we have produced a local example of a combinatorially regular T-orbit closure that is singular. Before providing this example, we outline the procedure used to examine such local examples.

We construct $X = \overline{T \cdot x}$ in the tangent space of some T-fixed point $p \in G/B$,

$$T_p(G/B) = \bigoplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha,$$

for some subset $\widetilde{\Phi} \subseteq \Phi$. As this tangent space is open in G/B, the closure of X in G/B is a T-orbit closure.

Suppose

$$x = \sum x_\alpha \in \bigoplus_{\alpha \in \widetilde{\Phi}} \mathfrak{g}_\alpha,$$

According to Lemma 2.5.8, for every point $z \in X = \overline{T \cdot x}$, there exists a $\lambda \in Y(T)$ such that

$$\lim_{g \to 0} \lambda(g)x \in T \cdot z$$

Given any $\lambda \in Y(T)$ and any $g \in \mathrm{G}_m$, we have

$$\lambda(g)x = \sum \lambda(g)x_\alpha = \sum (\alpha \circ \lambda)(g)x_\alpha = \sum g^{\langle \alpha, \lambda \rangle} x_\alpha$$

Hence, for

$$\lim_{g \to 0} \lambda(g)x = \lim_{g \to 0} \sum g^{\langle \alpha, \lambda \rangle} x_\alpha$$

to exist, we must have $\langle \alpha, \lambda \rangle \geq 0$, for all $\alpha \in s(x)$, and the result of taking this limit is a partial sum of x:

$$y := \sum_{\substack{\alpha \in s(x) \\ \langle \alpha, \lambda \rangle = 0}} x_\alpha.$$

As we will see in Example 3.10.6, it is not necessarily the case that all partial sums of x can be obtained in this way. Certain partial sums of x may be unattainable as no such λ may exist.

In summary, every T-orbit in X contains exactly one partial sum of x realized in the above manner. In particular, by determining all such partial sums, we essentially determine all T-orbits of X.

As established in Lemma 3.6.2, to verify that X is combinatorially regular, it suffices to show that $X_{T,y}$ is irreducible and

$$|E_{T_y}(X_{T,y}, y)| = \dim X_{T,y}$$

holds for one point y in each T-orbit in X. Thus, for each attainable partial sum y we construct $Y := X_{T,y}$, verify that the above equation holds, and confirm that T_y is connected, which by Proposition 3.5.2 yields that Y is irreducible. To compute Y, we use the fact that

$$Y \subseteq X \subseteq \operatorname{Span}\big(\ x_\alpha \mid \alpha \in s(x)\ \big) \simeq \mathbb{A}^{|s(x)|}.$$

Hence any $w \in Y$ can be written in the form $w = \sum_{\alpha \in s(x)} w_\alpha x_\alpha$, where the w_α are elements of $\mathbb{C}$ which satisfy the defining equations of X, which by Lemma 2.10.1 result from integral relations amongst roots in the support $s(x)$ of x.

As stated in Proposition 3.5.2,

$$Y = X_{\lambda,y} := \{w \in X | \lim_{g \to 0} \lambda(g)w = y\}.$$

Since

$$\lim_{g \to 0} \lambda(g)w = \lim_{g \to 0} \sum \lambda(g) w_\alpha x_\alpha = \lim_{g \to 0} \sum g^{\langle \alpha, \lambda \rangle} w_\alpha x_\alpha = y,$$

$w_\alpha = 1$ for all α in the support $s(y)$ of y. Therefore Y consists of all elements of X of the form

$$y + \sum_{\alpha \in s(x) \setminus s(y)} w_\alpha x_\alpha.$$

Note, that the λ used to compute Y for a given partial sum y is not unique. In fact, as demonstrated in Section 3.5, $Y = X_{\lambda,y}$ for any λ where $\lim_{g \to 0} \lambda(g)x = y$.

We will conclude by commenting on how we will indicate the form to be used to state an appropriate one-parameter subgroup λ for each partial sum y. Let $T' = T/T_x$. As $X = \overline{T \cdot x} = \overline{T' \cdot x}$, we may replace T with T' in the above process. Doing so will enable us to specify an isomorphism between T and $(\mathbb{C}^*)^n$ using a basis of the $\mathbb{Z}$-module generated by the support $s(x)$ of x, since the character group $X(T')$ is generated by the support $s(x)$. The choice of basis may depend on the y under consideration. In Examples 3.10.6 and 3.10.8, we use this isomorphism to verify that T_y is connected. When we specify a particular one-parameter subgroup, we will state its image in $(\mathbb{C}^*)^n$ under this isomorphism, from which we may readily compute the value of $\langle \alpha, \lambda \rangle$ for each $\alpha \in s(x)$.

For our example, we take $G = \mathrm{PSO}_{\mathbf{8}}$. Let $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_3 + e_4$ the simple system of roots in D_4 as described in Remark 3.10.1

Example 3.10.6. Let $x = x_\alpha + x_\beta + x_\gamma + x_\delta + x_\epsilon$, where

$$\begin{aligned}
\alpha &= \alpha_1 \\
\beta &= \alpha_3 \\
\gamma &= \alpha_4 \\
\delta &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \\
\epsilon &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4
\end{aligned}$$

These roots satisfy

$$2\epsilon = \alpha + \beta + \gamma + \delta$$

and so the support of x, $s(x)$, generates a $\mathbb{Z}$-module of rank 4.

As such, in view of Lemma 2.10.1, $X = \overline{T \cdot x}$ is isomorphic to an affine variety of dimension 4 given by the equation $z_\epsilon^2 = z_\alpha z_\beta z_\gamma z_\delta$, where $z_\alpha, z_\beta, z_\gamma, z_\delta, z_\epsilon \in \mathbb{C}[X]$ are variables of weights $-\alpha, -\beta, -\gamma, -\delta$, and $-\epsilon$, respectively. It is clear that X is singular at 0. To compute the orbits contained in X, we determine which of the partial sums of x can be obtained from $\lim_{g \to 0} \lambda(g)x$, for some $\lambda \in Y(T)$.

For

$$\lim_{g \to 0} \lambda(g)x = \lim_{g \to 0}(g^{\langle \alpha, \lambda \rangle} x_\alpha + g^{\langle \beta, \lambda \rangle} x_\beta + g^{\langle \gamma, \lambda \rangle} x_\gamma + g^{\langle \delta, \lambda \rangle} x_\delta + g^{\langle \epsilon, \lambda \rangle} x_\epsilon)$$

to exist we require

$$\langle \alpha, \lambda \rangle,\ \langle \beta, \lambda \rangle,\ \langle \gamma, \lambda \rangle,\ \langle \delta, \lambda \rangle,\ \langle \epsilon, \lambda \rangle \geq 0,$$

in which case

$$\lim_{g \to 0} \lambda(g)x = \sum_{\substack{\rho \in s(x) \\ \langle \rho, \lambda \rangle = 0}} x_\rho.$$

Since $2\epsilon = \alpha + \beta + \gamma + \delta$,

$$2\langle \epsilon, \lambda \rangle = \langle \alpha, \lambda \rangle + \langle \beta, \lambda \rangle + \langle \gamma, \lambda \rangle + \langle \delta, \lambda \rangle \tag{3.11}$$

Thus $\langle \epsilon, \lambda \rangle = 0$ if and only if $\langle \alpha, \lambda \rangle = \langle \beta, \lambda \rangle = \langle \gamma, \lambda \rangle = \langle \delta, \lambda \rangle = 0$. From this we ascertain that the orbits

$$T \cdot x_\epsilon$$

$$T \cdot (x_\alpha + x_\epsilon),\ T \cdot (x_\beta + x_\epsilon),\ T \cdot (x_\gamma + x_\epsilon),\ T \cdot (x_\delta + x_\epsilon),$$

$$T \cdot (x_\alpha + x_\beta + x_\epsilon),\ T \cdot (x_\alpha + x_\gamma + x_\epsilon),\ T \cdot (x_\alpha + x_\delta + x_\epsilon),$$

$$T \cdot (x_\beta + x_\gamma + x_\epsilon),\ T \cdot (x_\beta + x_\delta + x_\epsilon),\ T \cdot (x_\gamma + x_\delta + x_\epsilon)$$

$$T \cdot (x_\alpha + x_\beta + x_\gamma + x_\epsilon),\ T \cdot (x_\alpha + x_\beta + x_\delta + x_\epsilon),$$

$$T \cdot (x_\alpha + x_\gamma + x_\delta + x_\epsilon),\ T \cdot (x_\beta + x_\gamma + x_\delta + x_\epsilon),$$

$$T \cdot (x_\alpha + x_\beta + x_\gamma + x_\delta)$$

are not contained in X.

As an example, for there to exist a $\lambda \in Y(T)$ such that $\lim_{g\to 0} \lambda(g)x = x_\gamma + x_\delta + x_\epsilon$, we would require

$$\langle \gamma, \lambda\rangle = \langle \delta, \lambda\rangle = \langle \epsilon, \lambda\rangle = 0$$

whilst

$$\langle \alpha, \lambda\rangle, \langle \beta, \lambda\rangle > 0.$$

However, this would violate Equation 3.11. Subsequently, no such λ exits and so X does not contain the orbit $T \cdot (x_\gamma + x_\delta + x_\epsilon)$.

An alternate argument that these orbits are not in X is as follows: since the defining equation of X is $z_\epsilon^2 = z_\alpha z_\beta z_\gamma z_\delta$, a point in X has a nonzero coefficient on x_ϵ if and only if the point has nonzero coefficients on x_α, x_β, x_γ, and x_δ.

The orbits contained in X are

$$\{0\},$$

$$T \cdot x_\alpha,\ T \cdot x_\beta,\ T \cdot x_\gamma,\ T \cdot x_\delta,$$

$$T\cdot(x_\alpha+x_\beta),\ T\cdot(x_\alpha+x_\gamma),\ T\cdot(x_\alpha+x_\delta),\ T\cdot(x_\beta+x_\gamma),\ T\cdot(x_\beta+x_\delta),\ T\cdot(x_\gamma+x_\delta),$$

$$T \cdot (x_\alpha + x_\beta + x_\gamma),\ T \cdot (x_\alpha + x_\beta + x_\delta),\ T \cdot (x_\alpha + x_\gamma + x_\delta),\ T \cdot (x_\beta + x_\gamma + x_\delta),$$

$$T \cdot x.$$

In what follows, we consider each of these orbits individually: we will specify a point y in that orbit, provide an explicit isomorphism between T and $(\mathbb{C}^*)^4$ using a basis for the support $s(x)$ of x, verify that the stabilizer T_y is connected, state Y and the T-curves contained in Y, and provide a λ for which $\lim_{g\to 0} \lambda(g)x = y$ (which verifies that the orbit in question is actually contained in X) and that can be used to determine Y. To define λ, we will indicate the image of $\lambda(s)$ in $(\mathbb{C}^*)^4$. In each case, Y is both singular and combinatorially regular at y (identified with 0 when viewed as an element of $T_y(Y)$, see Corollary 3.5.3).

For $y = 0$, $T_y = T$ and hence is connected. We have $Y = \overline{T \cdot x} = X$, so $\dim Y = 4$ and contains the four T-curves $\overline{T \cdot x_\alpha}$, $\overline{T \cdot x_\beta}$, $\overline{T \cdot x_\gamma}$, and $\overline{T \cdot x_\delta}$.

For $y = x_\alpha$, we identify T with $(\mathbb{C}^*)^4$ using $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$. Then $T_y = \ker(\alpha) = \{(1, \beta(t), \gamma(t), \epsilon(t))\} \simeq (\mathbb{C}^*)^3$ and so is connected. We see that $Y = \{x_\alpha + bx_\beta + cx_\gamma + dx_\delta + ex_\epsilon | bcd = e^2\}$, which can be obtained by taking $\lambda(s) = (1, s, s, s^2)$. Hence Y is a 3-dimensional affine variety given by the equation $bcd = e^2$ containing the three T_y-curves $b = c = e = 0$, $b = d = e = 0$, and $c = d = e = 0$.

For $y = x_\beta$, we use $T \simeq (\mathbb{C}^*)^4$ via $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$. In this case, $T_y = \ker(\beta) = \{(\alpha(t), 1, \gamma(t), \epsilon(t))\} \simeq (\mathbb{C}^*)^3$, which again is connected. We find that $Y = \{ax_\alpha + x_\beta + cx_\gamma + dx_\delta + ex_\epsilon | acd = e^2\}$. We can derive Y using $\lambda(s) = (s, 1, s, s^2)$. Consequently, Y is an affine variety of dimension 3 given by the equation $acd = e^2$ and so contains three T_y-curves: $a = c = e = 0$, $a = d = e = 0$, and $c = d = e = 0$.

For $y = x_\gamma$, we also use $T \simeq (\mathbb{C}^*)^4$ given by $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$. Thus, $T_y = \ker(\gamma) = \{(\alpha(t), \beta(t), 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^3$ and is therefore connected. We have $Y = \{ax_\alpha + bx_\beta + x_\gamma + dx_\delta + ex_\epsilon | abd = e^2\}$ which can be computed using $\lambda(s) = (s, s, 1, s^2)$. Subsequently, Y is a 3 dimensional affine variety given by $abd = e^2$. Hence, $a = b = e = 0$, $a = d = e = 0$, and $b = d = e = 0$ are the three T_y-curves in Y.

For $y = x_\delta$, we instead use $T \simeq (\mathbb{C}^*)^4$ via $t \mapsto (\beta(t), \gamma(t), \delta(t), \epsilon(t))$, so that $T_y = \ker(\delta) = \{(\beta(t), \gamma(t), 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^3$. Hence T_y is connected. We compute $Y = \{ax_\alpha + bx_\beta + cx_\gamma + x_\delta + ex_\epsilon | abc = e^2\}$ and this can be done using $\lambda(s) = (s, s, 1, s^2)$. Therefore, we have that Y is an affine variety of dimension 3 defined by $abc = e^2$. There are three T_y-curves in Y, namely, $a = b = e = 0$, $a = c = e = 0$, and $b = c = e = 0$.

For $y = x_\alpha + x_\beta$, we view T as $(\mathbb{C}^*)^4$ using the map $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$. This yields $T_y = \ker(\alpha) \cap \ker(\beta) = \{(1, 1, \gamma(t), \delta(t))\} \simeq (\mathbb{C}^*)^2$ and hence T_y is connected. Now, $Y = \{x_\alpha + x_\beta + cx_\gamma + dx_\delta + ex_\epsilon | cd = e^2\}$ which we can obtain via $\lambda(s) = (1, 1, s, s)$. Thus, Y is a surface given by the equation $cd = e^2$ containing the two T_y-curves $c = e = 0$ and $d = e = 0$. As Y is a singular surface, we know from Lemma 3.10.5 that the weights of its tangent space at y (identified with 0, see Corollary 3.5.3) satisfy one of two equations. In this case, the weights of $T_y(Y)$, namely $\gamma|_{T_y}$, $\delta|_{T_y}$, and $\epsilon|_{T_y}$, satisfy the relation $2\epsilon|_{T_y} = \gamma|_{T_y} + \delta|_{T_y}$ (see Example 3.10.7).

For $y = x_\alpha + x_\gamma$, we identify T with $(\mathbb{C}^*)^4$ by sending $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$. As a result, $T_y = \ker(\alpha) \cap \ker(\gamma) = \{(1, \beta(t), 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^2$ and thus is connected. We can use $\lambda(s) = (1, s, 1, s)$ to determine that, in this situation, $Y = \{x_\alpha + bx_\beta + x_\gamma + dx_\delta + ex_\epsilon | bd = e^2\}$. Hence, Y is a surface defined by $bd = e^2$ which contains the two T_y-curves $b = e = 0$ and $d = e = 0$. The weights of $T_y(Y)$ are $\beta|_{T_y}$, $\delta|_{T_y}$, and $\epsilon|_{T_y}$ and they satisfy the equation $2\epsilon|_{T_y} = \beta|_{T_y} + \delta|_{T_y}$.

For $y = x_\alpha + x_\delta$, we use $t \mapsto (\alpha(t), \gamma(t), \delta(t), \epsilon(t))$ to obtain $T \simeq (\mathbb{C}^*)^4$. We have that T_y is connected since $T_y = \ker(\alpha) \cap \ker(\delta) = \{(1, \gamma(t), 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^2$. To see that $Y = \{x_\alpha + bx_\beta + cx_\gamma + x_\delta + ex_\epsilon | bc = e^2\}$, we choose $\lambda(s) = (1, s, 1, s)$. Consequently, Y is a surface given by $bc = e^2$ and the two T_y-curves contained in Y are $b = e = 0$ and $c = e = 0$. For this surface, we have that the weights of $T_y(Y)$ are $\beta|_{T_y}$, $\gamma|_{T_y}$, and $\epsilon|_{T_y}$ and that $2\epsilon|_{T_y} = \beta|_{T_y} + \gamma|_{T_y}$.

For $y = x_\beta + x_\gamma$, the map $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$ allows us to identify T with $(\mathbb{C}^*)^4$. Subsequently, the stabilizer T_y is connected as $T_y = \ker(\beta) \cap \ker(\gamma) = \{(\alpha(t), 1, 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^2$. Here, $Y = \{ax_\alpha + x_\beta + x_\gamma + dx_\delta + ex_\epsilon | ad = e^2\}$. To find this, we may take $\lambda(s) = (s, 1, 1, s)$. Thus, Y is a surface defined by the equation $ad = e^2$. There are two T_y-curves contained in Y, specifically, $a = e = 0$ and $d = e = 0$. In this case, the weights of $T_y(Y)$ are $\alpha|_{T_y}$, $\delta|_{T_y}$, and $\epsilon|_{T_y}$ and they satisfy $2\epsilon|_{T_y} = \alpha|_{T_y} + \delta|_{T_y}$.

For $y = x_\beta + x_\delta$, we have $T \simeq (\mathbb{C}^*)^4$ from the map $t \mapsto (\beta(t), \gamma(t), \delta(t), \epsilon(t))$. Therefore, $T_y = \ker(\beta) \cap \ker(\delta) = \{(1, \gamma(t), 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^2$ and hence is connected. In this case, $Y = \{ax_\alpha + x_\beta + cx_\gamma + x_\delta + ex_\epsilon | ac = e^2\}$ and thus is a surface given by $ac = e^2$. To determine Y, we can use $\lambda(s) = (1, s, 1, s)$. The two T_y-curves in Y are $a = e = 0$ and $c = e = 0$. The weights of $T_y(Y)$ are $\alpha|_{T_y}$, $\gamma|_{T_y}$, and $\epsilon|_{T_y}$ which satisfy $2\epsilon|_{T_y} = \alpha|_{T_y} + \gamma|_{T_y}$.

For $y = x_\gamma + x_\delta$, we again use $t \mapsto (\beta(t), \gamma(t), \delta(t), \epsilon(t))$ to view T as $(\mathbb{C}^*)^4$. This gives us that $T_y = \ker(\gamma) \cap \ker(\delta) = \{(\beta(t), 1, 1, \epsilon(t))\} \simeq (\mathbb{C}^*)^2$ and so T_y is connected. We can choose $\lambda(s) = (s, 1, 1, s)$ in order to determine that $Y = \{ax_\alpha + bx_\beta + x_\gamma + x_\delta + ex_\epsilon | ab = e^2\}$ and is therefore a surface defined by $ab = e^2$. The two T_y-curves contained in Y are $a = e = 0$ and $b = e = 0$. We have that $2\epsilon|_{T_y} = \alpha|_{T_y} + \beta|_{T_y}$, where $\alpha|_{T_y}$, $\beta|_{T_y}$, and $\epsilon|_{T_y}$ are the weights of $T_y(Y)$.

For $y = x_\alpha + x_\beta + x_\gamma$, identify T with $(\mathbb{C}^*)^4$ using $t \mapsto (\alpha(t), \beta(t), \gamma(t), \epsilon(t))$, in which case $T_y = \ker(\alpha) \cap \ker(\beta) \cap \ker(\gamma) = \{(1, 1, 1, \epsilon(t))\} \simeq \mathbb{C}^*$ and hence is connected. We have that $Y = \{x_\alpha + x_\beta + x_\gamma + dx_\delta + ex_\epsilon | d = e^2\}$ which can be obtained using $\lambda(s) = (1, 1, 1, s)$. Consequently Y is a T_y-curve given by $d = e^2$.

For $y = x_\alpha + x_\beta + x_\delta$, we view T as $(\mathbb{C}^*)^4$ via $t \mapsto (\alpha(t), \beta(t), \delta(t), \epsilon(t))$. As such, T_y is connected since $T_y = \ker(\alpha) \cap \ker(\beta) \cap \ker(\delta) = \{(1, 1, 1, \epsilon(t))\} \simeq \mathbb{C}^*$. We can take $\lambda(s) = (1, 1, 1, s)$ to see that $Y = \{\, x_\alpha + x_\beta + cx_\gamma + x_\delta + ex_\epsilon \,|\, c = e^2 \,\}$ and is hence a T_y-curve defined by the equation $c = e^2$.

For $y = x_\alpha + x_\gamma + x_\delta$, we obtain $T \simeq (\mathbb{C}^*)^4$ from $t \mapsto (\alpha(t), \gamma(t), \delta(t), \epsilon(t))$. In this situation, $T_y = \ker(\alpha) \cap \ker(\gamma) \cap \ker(\delta) = \{(1, 1, 1, \epsilon(t))\} \simeq \mathbb{C}^*$. Therefore, T_y is connected. We find that $Y = \{\, x_\alpha + bx_\beta + x_\gamma + x_\delta + ex_\epsilon \,|\, b = e^2 \,\}$, which can be done by choosing $\lambda(s) = (1, 1, 1, s)$. Thus Y is a T_y-curve with defining equation $b = e^2$.

For $y = x_\beta + x_\gamma + x_\delta$, we have that $T \simeq (\mathbb{C}^*)^4$ using $t \mapsto (\beta(t), \gamma(t), \delta(t), \epsilon(t))$. Thus T_y is connected since $T_y = \ker(\beta) \cap \ker(\gamma) \cap \ker(\delta) = \{(1, 1, 1, \epsilon(t))\} \simeq \mathbb{C}^*$. We see that $Y = \{\, ax_\alpha + x_\beta + x_\gamma + x_\delta + ex_\epsilon \,|\, a = e^2 \,\}$, which can be determined using $\lambda(s) = (1, 1, 1, s)$. Consequently, Y is a T_y-curve given by the equation $a = e^2$.

In the proof of Lemma 3.10.5, we used the process of elimination determine that $2\omega = \beta + \gamma$ and $2\omega = 2\beta + \gamma$ are the only two possible relationships amongst the weights of $T_y(\Sigma)$. We did not, however, show that these relations actually occur. We will now provide examples to verify that these bad relations do in fact occur.

Example 3.10.7. In Example 3.10.6 above, we explicitly construct a T-orbit closure X and produce several different sets Y. There are six cases in which Y itself is a surface and in all six cases, the weights of $T_y(\Sigma)$ all satisfy an equation of the form $2\omega = \beta + \gamma$.

Example 3.10.8. Let $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4 - e_5, \alpha_5 = e_5 - e_6, \alpha_6 = e_5 + e_6$ be the simple system of roots in D_6 as indicated in Remark 3.10.1.

Let $x = x_\alpha + x_\beta + x_\gamma + x_\delta + x_\epsilon + x_\zeta$, where

$$\begin{aligned}
\alpha &= \alpha_1 \\
\beta &= \alpha_5 \\
\gamma &= \alpha_6 \\
\delta &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\
\epsilon &= \alpha_3 + \alpha_4 \\
\zeta &= \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6
\end{aligned}$$

These roots satisfy

$$2\zeta + \alpha = \beta + \gamma + \delta + 2\epsilon$$

and so the support of x, $s(x)$, generates a $\mathbb{Z}$-module of rank 5.

As such, $X = \overline{T \cdot x}$ is isomorphic to an affine variety of dimension 5 given by the equation $z_\zeta^2 z_\alpha = z_\beta z_\gamma z_\delta z_\epsilon^2$, where $z_\alpha, z_\beta, z_\gamma, z_\delta, z_\epsilon, z_\zeta \in \mathbb{C}[X]$ are variables of weights $-\alpha, -\beta, -\gamma, -\delta, -\epsilon$, and $-\zeta$ respectively. It is clear that X is singular at 0.

We identify T with $(\mathbb{C}^*)^6$ using $t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t))$. Now, $T_y = \ker(\alpha_1) \cap \ker(\alpha_5) \cap \ker(\alpha_6) = \{1, \alpha_2(t), \alpha_3(t), \alpha_4(t), 1, 1)\} \simeq (C^*)^3$ and hence is connected. Using $\lambda(s) = (1, s, s, s, 1, 1)$, we see that $y = x_\alpha + x_\beta + x_\gamma$ is a point of X and that $Y = \{x_\alpha + x_\beta + x_\gamma + dx_\delta + ex_\epsilon + fx_\zeta | f^2 = de^2\}$. Hence, Y is a singular surface defined by $f^2 = de^2$ which contains the two T_y-curves $f = d = 0$ and $f = e = 0$. The weights of $T_y(Y)$ are

$$\begin{aligned}
\delta|_{T_y} &= 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\
\epsilon|_{T_y} &= \alpha_3 + \alpha_4 \\
\zeta|_{T_y} &= \alpha_2 + 2\alpha_3 + 2\alpha_4
\end{aligned}$$

and they satisfy the equation $2\zeta|_{T_y} = \delta|_{T_y} + 2\epsilon|_{T_y}$.

Chapter 4

In the Context of $\mathcal{G}/\mathcal{B}$

In this chapter, we will provide an overview of some of the objects and concepts in the $\mathcal{G}/\mathcal{B}$ setting that we require for our work. Much of this information can be found in the book by Kumar ([26]). The interested reader is directed there for a more detailed examination of this material.

4.1 Notation and Terminology

Let $\mathcal{G} = \mathrm{SL}_n(\mathbb{C}((x)))$ and $\mathcal{P} = \mathrm{SL}_n(\mathbb{C}[[x]])$, for some $n \in \mathbb{N}$. Also, let $\mathcal{B} = ev^{-1}(B)$, where $ev : \mathcal{P} \to \mathrm{SL}_n(\mathbb{C})$ is entry-wise evaluation at $x = 0$ and B is the Borel subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting of upper triangular matrices. Thus, $\mathcal{B}$ is the subgroup of $\mathcal{P}$ composed of matrices whose entries below the diagonal have no constant term.

The quotients $\mathcal{G}/\mathcal{P}$ and $\mathcal{G}/\mathcal{B}$ are projective ind-varieties and hence can be expressed in the form

$$\lim_{\to} X_i,$$

where each X_i is an irreducible normal finite-dimensional projective variety. We refer to $\mathcal{G}/\mathcal{P}$ as the *affine Grassmannian.*

The n-dimensional torus of interest in this setting is $\widehat{T} = T \times S$, where $T \subset \mathcal{B}$ is the maximal torus comprised of diagonal matrices in $SL_n(\mathbb{C})$, which acts on $\mathcal{G}$ by conjugation, and $S = \mathbb{C}^*$ is the element of the automorphism group of $\mathcal{G}$

which acts on each $g \in \mathcal{G}$ by acting on each entry of g according to the rule

$$s \cdot \left(\sum_{i=\ell}^{\infty} x^i \right) = \sum_{i=\ell}^{\infty} s^i x^i,$$

for all $s \in S$. These actions commute and hence $\widehat{T}$ acts on $\mathcal{G}$. As $\mathcal{B}$ is also $\widehat{T}$-stable, the $\widehat{T}$ action descends to the quotient $\mathcal{G}/\mathcal{B}$.

Let

$$\widehat{W} := N_{\mathcal{G} \rtimes S}(\widehat{T})/\widehat{T} \simeq N_{\mathcal{G}}(\widehat{T})/T$$

denote the *affine Weyl group* of $\mathcal{G}$. The identity element of $\widehat{W}$ will be denoted as e and we will define $w^0 = e$, for all $w \in \widehat{W}$. As in the classical setting, the $\widehat{T}$-fixed points of $\mathcal{G}/\mathcal{B}$ are in a one-to-one correspondence with the points of the set $\widehat{W}$, and moreover, $\widehat{W}$ acts transitively on $(\mathcal{G}/\mathcal{B})^{\widehat{T}}$. Henceforth, we will identify the elements of $(\mathcal{G}/\mathcal{B})^{\widehat{T}}$ with the elements of $\widehat{W}$. In addition, the affine Weyl group $\widehat{W}$ acts on $\widehat{T}$ by conjugation, which induces actions of $\widehat{W}$ on the character group $X(\widehat{T})$ and the cocharacter group $Y(\widehat{T})$ in the obvious way.

4.2 Weight Space Decompositions

Let $\mathfrak{g}$ denote the Lie algebra of $SL_n(\mathbb{C})$, let $\mathfrak{b}$ is the Lie algebra of B, and let $\mathfrak{h}$ denote the Lie algebra of T. The set of roots of $SL_n(\mathbb{C})$ with respect to T form a root system of type A_{n-1} and will be denoted as Φ. We will use the notation (ij) to represent the root $e_i - e_j$, since the notation e_i will be used to denote a different object later on.

Let $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[x, x^{-1}]$ and let $\hat{\mathfrak{b}} := \mathfrak{b} \oplus \big(\mathfrak{g} \otimes x\mathbb{C}[x]\big)$.

The torus $\widehat{T}$ acts on $\hat{\mathfrak{g}}$ by the rule

$$(t, s) \cdot (g \otimes x^i) = tgt^{-1} \otimes s^i x^i.$$

In particular, for $g \in \mathfrak{g}_\alpha$, the action is

$$(t, s) \cdot (g \otimes x^i) = \alpha(t) g \otimes s^i x^i = \alpha(t) s^i (g \otimes x^i).$$

Accordingly, the roots of $\widehat{T}$ in $\hat{\mathfrak{g}}$ are

$$\widehat{\Phi} := \{\alpha + h\delta \mid \alpha \in \Phi \text{ and } h \in \mathbb{Z}\} \cup \{h\delta \mid h \in \mathbb{Z} \setminus \{0\}\},$$

where we define $(\alpha + h\delta)(t, s) := \alpha(t)s^h$. For $\hat{\alpha} = \alpha + h\delta \in \widehat{\Phi}$, we define $\mathrm{Re}(\hat{\alpha}) := \alpha$.

The induced action of the affine Weyl group $\widehat{W}$ on the character group $X(\widehat{T})$ restricts to an action on $\widehat{\Phi}$. This action satisfies

$$w(\alpha + h\delta) = w(\alpha) + h\delta,$$

and, in particular, we have

$$w(h\delta) = h\delta$$

for all $h \in \mathbb{Z} \setminus \{0\}$.

The *imaginary* roots are the element of $\{h\delta \mid h \in \mathbb{Z} \setminus \{0\}\}$. All other roots are said to be *real*. We will denote the set of imaginary roots by $\mathrm{Im}(\widehat{\Phi})$.

There is a notion of positivity on $\widehat{\Phi}$ which arises from the weight space decomposition of $\hat{\mathfrak{b}}$, that is,

$$\widehat{\Phi}^+ := \{\alpha + h\delta \mid h > 0 \text{ or } h = 0 \text{ and } \alpha > 0\}.$$

We can also describe $\widehat{\Phi}^+$ in terms of one-parameter subgroups: we first recall that

$$\Phi^+ = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle > 0\},$$

for some $\lambda \in Y(T)$. Furthermore, as Φ is finite there exists a $k \in \mathbb{N}$ such that $|\langle \alpha, \lambda \rangle| < k$ for all $\alpha \in \Phi$. Now, using $\hat{\lambda} := \lambda + k\delta$ to represent the element $(\lambda, k) \in Y(\widehat{T}) \simeq Y(T) \oplus \mathbb{Z}$, we know that

$$\langle \alpha + h\delta, \lambda + k\delta \rangle = m + kh,$$

where $|m| < k$, and hence

$$\begin{aligned}\widehat{\Phi}^+ &= \{\alpha + h\delta \mid \langle \alpha + h\delta, \lambda + k\delta \rangle > 0\} \\ &= \{\hat{\alpha} \in \widehat{\Phi} \mid \langle \hat{\alpha}, \hat{\lambda} \rangle > 0\}.\end{aligned}$$

Thus we also have,

$$\begin{aligned}\widehat{\Phi}^- &= \{\alpha + h\delta \mid h < 0 \text{ or } h = 0 \text{ and } \alpha < 0\}\\ &= \{\hat{\alpha} \in \widehat{\Phi} \mid \langle \hat{\alpha}, -\hat{\lambda}\rangle > 0\}.\end{aligned}$$

We will use the notation $\hat{\alpha} > 0$ to indicate that $\hat{\alpha} \in \widehat{\Phi}^+$ and $\hat{\alpha} < 0$ to indicate that $\hat{\alpha} \in \widehat{\Phi}^-$. The positive imaginary roots will be denoted by $\mathrm{Im}(\widehat{\Phi})^+$ and the negative imaginary roots will be denoted by $\mathrm{Im}(\widehat{\Phi})^-$.

Regarding the weight spaces $\hat{\mathfrak{g}}_{\hat{\alpha}}$, for a real root $\hat{\alpha} = \alpha + h\delta$, we have

$$\hat{\mathfrak{g}}_{\hat{\alpha}} = \mathfrak{g}_\alpha \otimes (\mathbb{C}[x, x^{-1}])_h.$$

Since $\mathfrak{g}_\alpha$ and $(\mathbb{C}[x, x^{-1}])_h$ are both 1-dimensional, $\dim \hat{\mathfrak{g}}_{\hat{\alpha}} = 1$. If $\hat{\alpha} = h\delta$ is imaginary, we have

$$\hat{\mathfrak{g}}_{\hat{\alpha}} = \mathfrak{h} \otimes (\mathbb{C}[x, x^{-1}])_h.$$

The quotient space $\hat{\mathfrak{g}}/\hat{\mathfrak{b}}$ decomposes as

$$\hat{\mathfrak{g}}/\hat{\mathfrak{b}} = \bigoplus_{\hat{\alpha}<0} \hat{\mathfrak{g}}_{\hat{\alpha}}.$$

Relating this to tangent spaces, for $e \in \widehat{W}$ we have

$$T_e(\mathcal{G}/\mathcal{B}) = \hat{\mathfrak{g}}/\hat{\mathfrak{b}}$$

and for arbitrary $w \in \widehat{W}$, we have

$$T_w(\mathcal{G}/\mathcal{B}) = \hat{\mathfrak{g}}/w\hat{\mathfrak{b}}w^{-1} = \bigoplus_{w^{-1}(\hat{\alpha})<0} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

Specifically,

$$\Omega\big(T_w(\mathcal{G}/\mathcal{B})\big) = w\Big(\widehat{\Phi}^-\Big) = \{\hat{\alpha} \in \widehat{\Phi} \mid w^{-1}(\hat{\alpha}) < 0\}.$$

Since $w^{-1}(\hat{\alpha}) \in \widehat{\Phi}^-$, we know that $\langle w^{-1}(\hat{\alpha}), -\hat{\lambda}\rangle > 0$. Accordingly, we also have

$$\Omega\big(T_w(\mathcal{G}/\mathcal{B})\big) = \{\hat{\alpha} \in \widehat{\Phi} \mid \langle \hat{\alpha}, w(-\hat{\lambda}) \rangle > 0\}.$$

Due to the fact that $w^{-1}(-\hat{\alpha}) = -w^{-1}(\hat{\alpha})$, we know that for any $\hat{\alpha} \in \widehat{\Phi}$, exactly one of $\hat{\alpha}$ or $-\hat{\alpha}$ appears as a weight of $T_w(\mathcal{G}/\mathcal{B})$. Furthermore, for $h > 0$,

$$w^{-1}(h\delta) = h\delta > 0.$$

As such, for any $w \in \widehat{W}$, the set $\Omega\big(T_w(\mathcal{G}/\mathcal{B})\big)$ does not contain any positive imaginary roots.

4.3 The Affine Weyl Group $\widehat{W}$

In this section, we will further discuss the affine Weyl group $\widehat{W}$, in particular, its realization as a Coxeter group. Information on Coxeter groups, including some general theory presented in this section, is available in [21] and [6]. We begin by reviewing two ways to view $\widehat{W}$ and its elements.

Analogous to the classical case, the affine Weyl group can be realized as a group generated by reflections $s_{\hat{\alpha}}$ associated to elements $\hat{\alpha} \in \widehat{\Phi}$, where in this case $\hat{\alpha}$ is required to be real. In what follows, we will outline this identification as presented in [28].

Let $W = S_n$ be the Weyl group of $SL_n(\mathbb{C})$, that is, the symmetric group on the set $\{1, 2, \ldots, n\}$, and let S_∞ be the group of bijections of $\mathbb{Z}$. Let $r + nq \in \mathbb{Z}$, where q is any integer and r is an integer such that $1 \leq r \leq n$. The set S_n can be viewed as a subgroup of S_∞ by defining for each $\sigma \in S_n$

$$\sigma(r + nq) = \sigma(r) + nq.$$

Moreover, we embed $\mathbb{Z}^n$ in S_∞ by assigning $z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n$ to the map σ_z defined by

$$\sigma_z(r + nq) = r + n(q + z_r).$$

Let

$$\widetilde{W} = \{\sigma \in S_\infty \mid \sigma(q+n) = \sigma(q) + n, \; \forall q \in \mathbb{Z}\},$$

then $\widetilde{W} = W \ltimes \mathbb{Z}^n$ and the affine Weyl group $\widehat{W}$ is the subgroup of $\widetilde{W}$ consisting of those (w, z) for which

$$\sum_{i=1}^{n} z_i = 0.$$

Suppose that $\hat{\alpha} = \alpha + h\delta \in \widehat{\Phi}$ is a real root, where $\alpha = (ij) \in \Phi^+$ with $1 \leq i < j \leq n$. Let s_α be the reflection in W associated to α and let $\{e_k\}$ represent the standard basis of $\mathbb{Z}^n$.

The reflection $s_{\hat{\alpha}}$ associated to $\hat{\alpha}$ is the element $\big(s_\alpha, h(e_i - e_j)\big) \in \widehat{W}$. Thus $s_{\hat{\alpha}}$ acts on $\mathbb{Z}$ as follows:

$$s_{\hat{\alpha}}(r+nq) := \begin{cases} r+nq & \text{if } \; r \neq i, j \\ j + n(q+h) & \text{if } \; r = i \\ i + n(q-h) & \text{if } \; r = j \end{cases}$$

Moreover, we define $s_{-\hat{\alpha}} := s_{\hat{\alpha}}$.

Example 4.3.1. Let $n = 5$, let $\hat{\alpha} = (14) + 2\delta$, let and $\hat{\beta} = -(25) + \delta$, where $(14), (25) \in \Phi^+$. Note that $-\hat{\beta} = (25) - \delta$.

$$\begin{aligned}
s_{\hat{\alpha}}(6) &= s_{\hat{\alpha}}(1 + 5(1)) = 4 + 5(1+2) = 19 \\
s_{\hat{\alpha}}(-3) &= s_{\hat{\alpha}}(2 + 5(-1)) = 2 + 5(-1) = -3 \\
s_{\hat{\alpha}}(8) &= s_{\hat{\alpha}}(3 + 5(1)) = 3 + 5(1) = 8 \\
s_{\hat{\alpha}}(4) &= s_{\hat{\alpha}}(4 + 5(0)) = 1 + 5(0-2) = -9 \\
s_{\hat{\alpha}}(0) &= s_{\hat{\alpha}}(5 + 5(-1) = 5 + 5(-1) = 0
\end{aligned}$$

$$\begin{aligned}
s_{\hat{\beta}}(6) &= s_{-\hat{\beta}}(6) = s_{-\hat{\beta}}(1 + 5(1)) = 1 + 5(1) = 6 \\
s_{\hat{\beta}}(-3) &= s_{-\hat{\beta}}(-3) = s_{-\hat{\beta}}(2 + 5(-1)) = 5 + 5(-1 + (-1)) = -5 \\
s_{\hat{\beta}}(8) &= s_{-\hat{\beta}}(8) = s_{-\hat{\beta}}(3 + 5(1)) = 3 + 5(1) = 8 \\
s_{\hat{\beta}}(4) &= s_{-\hat{\beta}}(4) = s_{-\hat{\beta}}(4 + 5(0))) = 4 + 5(0) = 4 \\
s_{\hat{\beta}}(0) &= s_{-\hat{\beta}}(0) = s_{-\hat{\beta}}(5 + 5(-1)) = 2 + 5(-1 - (-1)) = 2
\end{aligned}$$

Alternatively, $\widehat{W}$ is the *affine symmetric group* $\widehat{S}_n$, which is defined (See [31], Section 3.6) to be the set of all permutations w of $\mathbb{Z}$ satisfying:

$$w(z+n) = w(z) + n \quad \text{and} \quad \sum_{z=1}^{n} w(z) = \frac{n(n+1)}{2}.$$

The elements of $\widehat{S}_n$ are referred to as *affine permutations* and can be expressed in one-line notation: if $w \in \widehat{W}$ and we let $w_z = w(z)$, for all $z \in \mathbb{Z}$, then w can be written as an infinite string

$$\cdots w_{-3}w_{-2}w_{-1}w_0w_1w_2w_3 \cdots$$

As w commutes with shifting by n, any $w \in \widehat{S}_n$ can be defined by specifying w_1, w_2, ..., w_n (or the image of any n consecutive integers). As such, unless we require additional entries in the string, we will typically write w in standard window form:

$$w = [w_1, w_2, \ldots, w_n].$$

For example, in $\widehat{S}_5$, we have the element

$$w = [-3, 6, 0, 8, 4].$$

This description of the elements of $\widehat{W} = \widehat{S}_n$ is the one presented in [2] when introducing the notion of pattern avoidance. We will discuss pattern avoidance in Chapter 6.

Let $i < j$ be integers such that $1 \leq i \leq n$ and $i \not\equiv j \mod n$. An *affine transposition*, denoted $t_{i,j}$, is the permutation that interchanges $i + nq$ and $j+nq$, for all $q \in \mathbb{Z}$, and fixes any integer k which is not congruent modulo n to either i or j. The reflections in $\widehat{S}_n$ are the affine transformations and the simple reflections, which generate $\widehat{S}_n$ as a Coxeter group, are the transformations

$$s_0 := t_{n,n+1} \qquad \text{and}$$
$$s_i := t_{i,i+1}, \quad \text{for } 1 \leq i \leq n-1.$$

Relating our two notations for elements of $\widehat{W} = \widehat{S}_n$, we have $s_0 = s_{\hat{\alpha}}$, where $\hat{\alpha} = (1n) - \delta$, and $s_i = s_{\hat{\alpha}}$, where $\hat{\alpha}$ is the root $(i\, i{+}1) \in \Phi^+$, for $1 \leq i \leq n-1$.

For $w = [w_1, w_2, \ldots, w_n]$, we compute $s_{\hat{\alpha}} w$ by the rule

$$s_{\hat{\alpha}} w = [s_{\hat{\alpha}}(w_1), s_{\hat{\alpha}}(w_2), \ldots, s_{\hat{\alpha}}(w_n)],$$

where $s_{\hat{\alpha}}$ is the reflection associated to some $\hat{\alpha} \in \widehat{\Phi}$.

Example 4.3.2. Let $n = 5$, let $\hat{\alpha} = (14) + 2\delta$, let $\hat{\beta} = -(25) + \delta$, and let $w = [-3, 6, 0, 8, 4]$.

Using our computations in Example 4.3.1, we calculate

$$s_{\hat{\alpha}} w = [s_{\hat{\alpha}}(-3), s_{\hat{\alpha}}(6), s_{\hat{\alpha}}(0), s_{\hat{\alpha}}(8), s_{\hat{\alpha}}(4)] = [-3, 19, 0, 8, -9]$$

and

$$s_{\hat{\beta}} w = [s_{\hat{\beta}}(-3), s_{\hat{\beta}}(6), s_{\hat{\beta}}(0), s_{\hat{\beta}}(8), s_{\hat{\beta}}(4)] = [-5, 6, 2, 8, 4].$$

As a Coxeter group, $\widehat{W}$ comes equipped with a particular length function ℓ. We set

$$\ell(e) = 0.$$

Now, for any $w \in \widehat{W}$, we can write w in infinitely many ways as a product of (not necessarily distinct) simple reflections

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}.$$

If $w \neq e$, such an expression is said to be *reduced* if k is minimal. For $w \neq e$, we set $\ell(w)$ to be the value of k for which $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w.

As in the classical S_n case, the length of an element of $\widehat{S}_n$ can be computed using the notion of an inversion. Let $w = [w_1, w_2, \ldots, w_n]$. An *affine inversion* in w is a pair of indices (i, j), where $1 \leq i \leq n$, for which $i < j$, but $w_i > w_j$. We will denote the set of all affine inversions in w as $\mathrm{Inv}_{\widehat{S}_n}(w)$ and use $\left|\mathrm{Inv}_{\widehat{S}_n}(w)\right|$ to denote the number of elements in $\mathrm{Inv}_{\widehat{S}_n}(w)$. According to Proposition 8.3.1 in [6]

$$\ell(w) = \left|\mathrm{Inv}_{\widehat{S}_n}(w)\right|.$$

For example, in $\widehat{S}_5$, for

$$w = [-3, 6, 0, 8, 4]2, 11, 5, 13, 9, 7, \dots$$

we compute that

$$\mathrm{Inv}_{\widehat{S}_n}(w) = \{(2,3), (2,5), (2,6), (2,8), (4,5), (4,6), (4,8), (4,11), (5,6)\}$$

and thus $\ell(w) = 9$.

The length function ℓ on $\widehat{W}$ can be used to define a partial order on $\widehat{W}$ referred to as the *Bruhat-Chevalley ordering* on $\widehat{W}$. First, we write $u \to w$ if $w = s_{\hat{\alpha}}u$ for some reflection $s_{\hat{\alpha}} \in \widehat{W}$ and $\ell(u) < \ell(w)$. Many authors will use $w = us_{\hat{\alpha}}$ instead of $s_{\hat{\alpha}}u$. The two approaches are related as follows: if $w = us_{\hat{\alpha}}$, then $w = s_{\hat{\beta}}u$, where $\hat{\beta} = w(\hat{\alpha})$.

Now set $u < w$, if there exits $u_1, u_2, \dots, u_k \in \widehat{W}$ such that

$$u \to u_1 \to u_2 \to \dots \to u_k \to w.$$

If $u \leq w$, then the interval from u to w is defined to be

$$[u, w] := \{y \in \widehat{W}\} | u \leq y \leq w\}.$$

For $u < w$, we also want to consider the set of roots defined as follows:

$$R(u, w) := \{\hat{\alpha} \in \widehat{\Phi}^+ \,|\, \hat{\alpha} \text{ is real and } u < s_{\hat{\alpha}}u \leq w\}.$$

The theory of Coxeter groups gives us another useful fact regarding $\widehat{W}$: let $\hat{\alpha} \in \Phi^+$ be real, then

$$w < s_{\hat{\alpha}}w \quad \text{if and only if} \quad w^{-1}(\hat{\alpha}) > 0$$

(See Lemma 5.7 and Proposition 5.7 in [21]).

Relating this to our description of the weights of $T_w(\mathcal{G}/\mathcal{B})$, we obtain:

Lemma 4.3.3. *Let $w \in \widehat{W}$ and let $\hat{\alpha} \in \Phi^+$ be real.*

The following are equivalent:

1) $\hat{\alpha} \in \Omega\big(T_w(\mathcal{G}/\mathcal{B})\big)$

2) $w^{-1}(\hat{\alpha}) < 0$

3) $s_{\hat{\alpha}} w < w$

The following are equivalent:

1) $-\hat{\alpha} \in \Omega\big(T_w(\mathcal{G}/\mathcal{B})\big)$

2) $w^{-1}(-\hat{\alpha}) < 0$

3) $w < s_{\hat{\alpha}} w$

4.4 Schubert Varieties in $\mathcal{G}/\mathcal{B}$

For each $w \in \widehat{W}$, the object
$$X(w) := \overline{\mathcal{B}w}$$
is an irreducible finite-dimensional ($\dim X(w) = \ell(w)$) normal projective $\widehat{T}$-stable variety known as an *affine Schubert variety.* It can also be described as a union of $\mathcal{B}$-orbits, each containing a unique $\widehat{T}$-fixed point:

$$X(w) = \bigcup_{y \in [e,w]} \mathcal{B}y.$$

As such, the Bruhat-Chevalley order on $\widehat{W}$ can also be formulated in terms of Schubert varieties by setting

$$u \leq w \quad \Longleftrightarrow \quad X(u) \subseteq X(w) \quad \Longleftrightarrow \quad u \in X(w).$$

In terms of the ind-variety structure of $\mathcal{G}/\mathcal{B}$, for each X_i, one may choose $X_i = X(w_i)$, for some $w_i \in \widehat{W}$ so that

$$\mathcal{G}/\mathcal{B} = \lim_{\rightarrow} X(w_i).$$

The set $(\mathcal{G}/\mathcal{B})^{\widehat{T}}$ of $\widehat{T}$-fixed points of $\mathcal{G}/\mathcal{B}$ is discrete and thus

$$X(w)^{\widehat{T}} = X(w) \cap (\mathcal{G}/\mathcal{B})^{\widehat{T}} = [e, w]$$

is finite. Accordingly, since $X(w)$ is normal and irreducible, Example 2.6.4 establishes that affine Schubert varieties and any of their closed irreducible $\widehat{T}$-stable subvarieties are $\widehat{T}$-varieties. Furthermore, Schubert varieties in $\mathcal{G}/\mathcal{B}$ are Cohen-Macaulay (see Theorem 8.2.2 in [26] or see [34] for the classical G/P setting).

Moreover, for any $u \in X(w)^{\widehat{T}}$, we observe that

$$T_u\big(X(w)\big) \subset T_u(\mathcal{G}/\mathcal{B}) = \hat{\mathfrak{g}}/u\hat{\mathfrak{b}}u^{-1} = \bigoplus_{u^{-1}(\hat{\alpha})<0} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

as a $\widehat{T}$-stable subspace and hence

$$\begin{aligned} \Omega\big(T_u(X(w)\big) \subset \Omega\big(T_u(\mathcal{G}/\mathcal{B})\big) &= \{\hat{\alpha} \in \widehat{\Phi} \mid u^{-1}(\hat{\alpha}) < 0\} \\ &= \{\hat{\alpha} \in \widehat{\Phi} \mid \langle \hat{\alpha}, u(-\hat{\lambda})\rangle > 0\}, \end{aligned}$$

where $\hat{\lambda}$ is the element of $Y(\widehat{T})$ specified in Section 4.2. There are several consequences of this worth mentioning.

Remark 4.4.1.

1) Every element of $X(w)^{\widehat{T}}$ is attractive.

2) Every weight $\hat{\alpha}$ of $T_u\big(X(w)\big)$ satisfies $u^{-1}(\hat{\alpha}) < 0$.

3) The set $\Omega\Big(T_u\big(X(w)\big)\Big)$ does not contain any elements of $\mathrm{Im}(\widehat{\Phi})^+$.

In addition, as $X(w)$ is $\mathcal{B}$-stable, the stabilizer $\mathcal{B}_u$ acts on $T_u\big(X(w)\big)$.[1] Taking the differential, we obtain an action of its Lie algebra $\hat{\mathfrak{b}} \cap u\hat{\mathfrak{b}}u^{-1}$ on $T_u\big(X(w)\big)$ which is induced by restricting the usual adjoint action of $u\hat{\mathfrak{b}}u^{-1}$ on $\hat{\mathfrak{g}}/u\hat{\mathfrak{b}}u^{-1}$.

Lemma 4.4.2. *Let $\hat{\beta} \in \Omega\Big(T_u\big(X(w)\big)\Big)$ and let $\hat{\alpha} \in \widehat{\Phi}^+$ such that $u^{-1}(\hat{\alpha}) > 0$. If $\hat{\alpha} + \hat{\beta} \in \widehat{\Phi}$ and $u^{-1}(\hat{\alpha} + \hat{\beta}) < 0$, then $\hat{\alpha} + \hat{\beta}$ is a weight of $T_u\big(X(w)\big)$.*

Proof. Recall that $[\hat{\mathfrak{g}}_{\hat{\alpha}}, \hat{\mathfrak{g}}_{\hat{\beta}}] \subseteq \hat{\mathfrak{g}}_{\hat{\alpha}+\hat{\beta}}$ for all $\hat{\alpha}, \hat{\beta} \in \widehat{\Phi}$ (see [26, page 9]). Since the weight space $\hat{\mathfrak{g}}_{\hat{\alpha}}$ lies in $\hat{\mathfrak{b}} \cap u\hat{\mathfrak{b}}u^{-1}$ if and only if $\hat{\alpha} \in \widehat{\Phi}^+$ and $u^{-1}(\hat{\alpha}) > 0$, the claim follows from the remarks preceding the statement of the lemma. $\square$

[1] $\mathcal{B}_u$ is not to be confused with the set of unipotent elements of $\mathcal{B}$.

4.5 $\widehat{T}$-Curves in $X(w)$

Since a Schubert variety $X(w)$ is a $\widehat{T}$-variety, every closed irreducible T-stable curve in $X(w)$ is a $\widehat{T}$-curve. The $\widehat{T}$-curves in $X(w)$ are well understood. We will now review some information about these curves. (See Section 12.1 in [26], in particular, Proposition 12.1.7. For the classical G/B setting, see [13], in particular, Theorems D and F.)

Denote by $\mathcal{U}_{\hat{\alpha}}$ the unique subgroup of $\mathcal{G}$ normalized by $\widehat{T}$ which has Lie algebra $\hat{\mathfrak{g}}_{\hat{\alpha}}$ and denote by $\mathcal{G}_{\hat{\alpha}}$ the copy of $\mathrm{SL}_2(\mathbb{C})$ in $\mathcal{G}$ which is generated by $\mathcal{U}_{\hat{\alpha}}$ and $\mathcal{U}_{-\hat{\alpha}}$. Using this notation, the $\widehat{T}$-curves in $X(w)$ through $u \in X(w)^{\widehat{T}}$ can be described as follows:

$$\begin{aligned} E(X(w),u) = \ & \{\mathcal{G}_{\hat{\alpha}}u \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } s_{\hat{\alpha}}u \leq w\} \\ = \ & \{\overline{\mathcal{U}_{\hat{\alpha}}u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } s_{\hat{\alpha}}u < u \leq w\} \\ & \cup \ \{\overline{\mathcal{U}_{-\hat{\alpha}}u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } u < s_{\hat{\alpha}}u \leq w\} \end{aligned}$$

The $\widehat{T}$-curves in $E(X(w),u)$ are smooth and distinct. For $C \neq D \in E(X(w),u)$, we have that $C \cap D = \{u\}$. Since every element of $X(w)^{\widehat{T}}$ is attractive in $X(w)$, by Remark 2.9.2, every element of $C^{\widehat{T}}$ is attractive in C. Since C is smooth and u is attractive, by Remark 2.9.4, there is a T-equivariant isomorphism

$$C_u \simeq T_u(C).$$

We will let

$$E^+(X(w),u) := \{\overline{\mathcal{U}_{\hat{\alpha}}u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } s_{\hat{\alpha}}u < u \leq w\}$$

and

$$E^-(X(w),u) := \{\overline{\mathcal{U}_{-\hat{\alpha}}u} \mid \hat{\alpha} \in \hat{\Phi}^+ \text{ is real and } u < s_{\hat{\alpha}}u \leq w\}.$$

The $\widehat{T}$-curve $C := \overline{\mathcal{U}_{-\hat{\alpha}}u}$, where $\hat{\alpha} \in \widehat{\Phi}^+$, has $\widehat{T}$-fixed point set $C^{\widehat{T}} = \{u, s_{\hat{\alpha}}u\}$, with $u < s_{\hat{\alpha}}u$. The $\widehat{T}$-fixed point u of C has attractive neighbourhood

$$C_u = X(w)_u \cap C = C \setminus \{s_{\hat{\alpha}}u\} = \mathcal{U}_{-\hat{\alpha}}u.$$

The tangent space of C at u is $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}} \subset T_u\big(X(w)\big)$, which in particular indicates that $u^{-1}(-\hat{\alpha}) < 0$.

Likewise, the $\widehat{T}$-curve $D := \overline{\mathcal{U}_{\hat{\alpha}}u}$, where $\hat{\alpha} \in \widehat{\Phi}^+$, has $D^{\widehat{T}} = \{u, s_{\hat{\alpha}}u\}$, with $s_{\hat{\alpha}}u < u$. The $\widehat{T}$-fixed point u of D has attractive neighbourhood

$$D_u = X(w)_u \cap D = D \setminus \{s_{\hat{\alpha}}u\} = \mathcal{U}_{\hat{\alpha}}u.$$

The tangent space of D at u is $T_u(D) = \hat{\mathfrak{g}}_{\hat{\alpha}} \subset T_u\big(X(w)\big)$, which specifies that $u^{-1}(\hat{\alpha}) < 0$.

Since $X(w)^{\widehat{T}}$ is finite and there is a unique $\widehat{T}$-curve containing any pair of $\widehat{T}$-fixed points, it is clear that the set $E\big(X(w)\big)$ of all $\widehat{T}$-curves in $X(w)$ is finite. As such, $X(w)$ has an affiliated Bruhat graph $\Gamma\big(X(w)\big)$.

For $C := \overline{\mathcal{U}_{-\hat{\alpha}}u} \in E^-(X(w), u)$ (with $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$), we also have that

$$C = \overline{\mathcal{U}_{\hat{\alpha}}(s_{\hat{\alpha}}u)} \in E^+(X(w), s_{\hat{\alpha}}u)$$

(with $T_{s_{\hat{\alpha}}u}(C) = \hat{\mathfrak{g}}_{\hat{\alpha}}$). We convey this information in the piece of the Bruhat graph $\Gamma\big(X(w)\big)$ which corresponds to C as follows:

$$\begin{tikzcd} s_{\hat{\alpha}}u \arrow[d, dash, "\hat{\alpha}" near start, "-\hat{\alpha}" near end] \\ u \end{tikzcd}$$

Also, as indicated in Lemma 2.7.5,

$$|E(X(w), u)| \geq \dim X(w).$$

Specifically, this means that there are at least $\dim X(w)$ edges attached to every vertex in $\Gamma\big(X(w)\big)$.

Recall from Section 4.3 that $R(u, w) = \{\hat{\alpha} \in \widehat{\Phi}^+ \mid \hat{\alpha} \text{ is real and } u < s_{\hat{\alpha}} u \leq w\}$. Deodhar's Inequality states that

$$|R(u, w)| \geq \ell(w) - \ell(u)$$

(See Corollary 11.1.20 in [26]) from which it follows that

$$|E^-(X(w), u)| \geq \dim X(w) - \dim X(u).$$

Regarding the tangent space $T_u\big(X(w)\big)$, the elements of $\Omega\Big(T_u\big(X(w)\big)\Big)$ which are weights of the tangent spaces at u of the $\widehat{T}$-curves in $X(w)$ are significant in much of what is to come. For ease of reference, we make the following definition.

Definition 4.5.1. Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$, and let $C \in E(X(w), u)$. The elements of

$$\Omega(w, u) := \bigcup_{C \in E(X(w), u)} \Omega(T_u(C))$$

are called the *curve weights* of $X(w)$ at u.

Remark 4.5.2. All elements of $\Omega(w, u)$ are real.

For any $C \in E(X(w), u)$, since u is attractive, by Lemma 2.9.3 we have

$$C_u \hookrightarrow X(w)_u \hookrightarrow T_u\big(X(w)\big).$$

Let $C_u = \overline{\widehat{T} \cdot v}$, for some $v \in T_u\big(X(w)\big)$. Since $\dim C_u = 1$, we know from Lemma 2.5.1 that the rank of the $\mathbb{Z}$-submodule M of $X(\widehat{T})$ generated by the support $s(v)$ of v is 1. Accordingly, $M \simeq \mathbb{Z}$ and hence the elements of $s(v)$ are proportional. The connected component $(\widehat{T}_v)^\circ$ of the stabilizer

$$\widehat{T}_v = \bigcap_{\hat{\chi} \in M} \ker \hat{\chi}$$

is a codimension 1 torus which acts trivially on $\widehat{T} \cdot v$, thus on C_u, and therefore on $T_u(C)$. Now let $T_u(C) = \hat{\mathfrak{g}}_{\hat{\alpha}}$, for some real root $\hat{\alpha} \in u(\widehat{\Phi}^-)$. Hence,

$(\widehat{T}_v)^\circ \subseteq \ker \hat{\alpha}$. Let $N = \ker(X(\widehat{T}) \twoheadrightarrow X((\widehat{T}_v)^\circ))$ (restriction), so $N \simeq \mathbb{Z}$ and contains M. Consequently, $M = zN$ for some $z \in \mathbb{Z}$, which gives us that $z\hat{\alpha} \in M$. As such, the elements of $s(v)$ are proportional to $\hat{\alpha}$ and since none of the other roots in $u(\widehat{\Phi}^-)$ is proportional to the real root $\hat{\alpha}$, it must be the case that $v \in \hat{\mathfrak{g}}_{\hat{\alpha}}$. Consequently, $C_u = \hat{\mathfrak{g}}_{\hat{\alpha}}$. Hence we have proven:

Lemma 4.5.3. *For any $C \in E(X(w), u)$, the image of C_u under the $\widehat{T}$-equivariant embedding $X(w)_u \hookrightarrow T_u\big(X(w)\big)$ is $T_u(C)$.*

We conclude this section by making the following remark:

Remark 4.5.4. The tangent space to $E(X(w), u)$ at u, described in Definition 2.11.4, in this setting has the form

$$TE(X(w), u) = \bigoplus_{C \in E(X(w),u)} T_u(C) \quad = \bigoplus_{\hat{\alpha} \in \Omega(w,u)} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

In particular, this means the dimension of $TE(X(w), u)$ is $|E(X(w), u)|$.

4.6 Rationally Smooth Schubert Varieties in $\mathcal{G}/\mathcal{B}$.

The concept of rational smoothness (see Definition 2.13.1) has a concrete reformulation for Schubert varieties in $\mathcal{G}/\mathcal{B}$ (or in classical G/B).

Theorem 4.6.1. *Let $w \in \widehat{W}$. The following are equivalent:*

1) $X(w)$ is rationally smooth.

2) $|R(u, w)| = \ell(w) - \ell(u)$, for all $u \leq w$.

3) $|E(X(w), u)| = \ell(w)$, for all $u \leq w$.

Proof. See Theorems 12.2.8 and 12.2.14 in [26]. See [13], in particular Theorems B and E, for the classical G/B setting. □

Remark 4.6.2. In particular, this means that for a rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$, for all $u \leq w$ we have

$$|E^{-}(X(w), u)| = \dim X(w) - \dim X(u)$$

and

$$|E(X(w), u)| = \dim X(w).$$

It follows from the latter equation that there are exactly $\dim X(w)$ edges attached to every vertex in $\Gamma\big(X(w)\big)$.

Example 4.6.3. For $n = 2$, $X(w)$ is rationally smooth for all $w \in \widehat{W}$ (see [26, p476] or [2, p108]).

We conclude this section with a fact which is important in the study of rationally smooth Schubert varieties.

Remark 4.6.4. As mentioned in Section 2.13, the smooth locus of a Schubert variety is contained in the rationally smooth locus. In particular, this means that any smooth Schubert variety is rationally smooth.

4.7 Singular Points of $X(w)$

According to Lemma 2.8.2, the singular locus $\mathrm{Sing}\big(X(w)\big)$ of a Schubert variety $X(w)$ is a proper closed $\widehat{T}$-stable subset of $X(w)$ which contains a $\widehat{T}$-fixed point, if $\mathrm{Sing}\big(X(w)\big) \neq \emptyset$. Hence to study the singular locus of $X(w)$ we study its $\widehat{T}$-fixed point set

$$X(w)^{\widehat{T}} = [e, w] \subset \widehat{W}.$$

Much of our work in this thesis focuses on the detection of maximal singularities of Schubert varieties. By *maximal singularity* of $X(w)$, we mean a point

$$u \in \mathrm{Sing}\big(X(w)\big) \cap [e, w]$$

such that for any $v \in \mathrm{Sing}\big(X(w)\big) \cap [e, w]$,

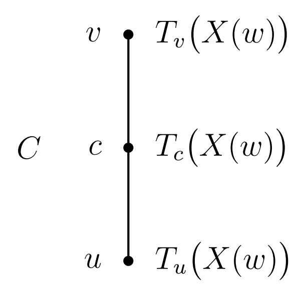
$$u \not< v,$$

with respect to the Bruhat-Chevalley ordering on $\widehat{W}$.

To investigate maximal singularities of $X(w)$, we will make use of several well-known facts about $\operatorname{Sing}\bigl(X(w)\bigr)$. First, it is clear that $X(w) = \overline{\mathcal{B}w}$ is nonsingular at w. Furthermore, due to a result by Chevalley (recalled in Proposition 12.1.1 in [26]), we know that affine Schubert varieties in $\mathcal{G}/\mathcal{B}$ are nonsingular in codimension 1.

As indicated in Section 12.1 in [26], the set $\operatorname{Sing}\bigl(X(w)\bigr)$ is also a $\mathcal{B}$-stable closed subvariety of $X(w)$ and as such, if nonempty, $\operatorname{Sing}\bigl(X(w)\bigr)$ can be expressed as a union of specific Schubert varieties $X(u)$, for some $u < w$. Some of the information given below, such as Lemma 4.7.3, follows easily from this fact. However, as $\widehat{T}$-curves are essential to our work, we will focus on how they can be used in this setting.

Let $v \in X(w)^{\widehat{T}}$ and let $C := \overline{\mathcal{U}_{\hat{\alpha}} v} \in E^{+}(X(w), v)$, with $C^{\widehat{T}} = \{u, v\}$, where $u < v$. Consider the tangent spaces of $X(w)$ along C:



Since $X(w)$ is $\mathcal{B}$-stable and since $\mathcal{U}_{\hat{\alpha}} \subset \mathcal{B}$, we know that $X(w)$ is also $\mathcal{U}_{\hat{\alpha}}$-stable.

Remark 4.7.1. It follows that

$$\dim T_v(X(w)) = \dim T_c(X(w))$$

for all points c in the orbit $\mathcal{U}_{\hat{\alpha}} v = C \setminus \{u\}$.

Thus $X(w)$ is nonsingular at v if and only if it is nonsingular along $\mathcal{U}_{\hat{\alpha}} v$. In particular, this means that C is good (see Definition 2.7.3) if and only if $X(w)$ is nonsingular at v.

Now, C is also equal to the curve $\overline{\mathcal{U}_{-\hat{\alpha}}u} \in E^-(X(w), u)$. Since the attractive neighbourhood $X(w)_u$ embeds into $T_u\big(X(w)\big)$, for any c in the open $\widehat{T}$-orbit $C \setminus C^{\widehat{T}} \subseteq X(w)_u$ we have

$$T_c\big(X(w)\big) \hookrightarrow T_u\big(X(w)\big).$$

by taking the differential of the embedding at c.

Remark 4.7.2. Thus,

$$\dim X(w) \leq \dim T_c\big(X(w)\big) \leq \dim T_u\big(X(w)\big)$$

for all $c \in C$.

As such, if $X(w)$ is nonsingular at u, then it is also nonsingular at v, or equivalently, if $X(w)$ is singular at v, then it is also singular at u.

More generally, we have the following:

Lemma 4.7.3. *Let $u, v \in X(w)^{\widehat{T}}$ such that $u < v$. If v is a singular point of $X(w)$, then u is a singular point of $X(w)$.*

Proof. If $v = s_{\hat{\alpha}}u$, for some reflection $s_{\hat{\alpha}} \in \widehat{W}$, where $\hat{\alpha} \in \widehat{\Phi}^+$, then $C = \overline{\mathcal{U}_{-\hat{\alpha}}u}$ is a $\widehat{T}$-curve in $E^-(X(w), u)$ with $C^{\widehat{T}} = \{u, v\}$. According to the argument given directly above, if v is a singular point of $X(w)$, so is u.

For an arbitrary v such that $u < v$, inductively, if v is a singular point of $X(w)$, then u is also a singular point since

$$u < s_{\hat{\alpha}_1}u < s_{\hat{\alpha}_2}s_{\hat{\alpha}_1}u < \cdots < s_{\hat{\alpha}_k}s_{\hat{\alpha}_{k-1}} \cdots s_{\hat{\alpha}_2}s_{\hat{\alpha}_1}u = v$$

for some reflections $s_{\hat{\alpha}_1}, s_{\hat{\alpha}_2}, \ldots, s_{\hat{\alpha}_k} \in \widehat{W}$, for some $k \geq 1$. □

There is one particular consequences of this discussion preceding Lemma 4.7.3 that we would like to highlight:

Remark 4.7.4. Let $u \in X(w)^{\widehat{T}}$ and let $C \in E^-(X(w), u)$ with $C^{\widehat{T}} = \{u, v\}$, so that $u < v$. If $X(w)$ is nonsingular at u or v, then C is good. In particular, if u is a maximal singularity of $X(w)$, then C is good.

The $\widehat{T}$-curves in $X(w)$ also yield another useful tool for detecting singularities in $X(w)$. Recall from Section 2.11 that, if $X(w)$ is nonsingular at a $\widehat{T}$-fixed point u, then $TE(X(w),u) = T_u\big(X(w)\big)$. Thus, in view of Remark 4.5.4, for a nonsingular $\widehat{T}$-fixed point u of $X(w)$ we have

$$T_u\big(X(w)\big) = \bigoplus_{C\in E(X(w),u)} T_u(C) \quad = \bigoplus_{\hat{\alpha}\in\Omega(w,u)} \hat{\mathfrak{g}}_{\hat{\alpha}}$$

As a consequences, we obtain:

Lemma 4.7.5. *Let $u \in X(w)^{\widehat{T}}$. If $\Omega\Big(T_u\big(X(w)\big)\Big)$ contains an element of* $\mathrm{Im}(\widehat{\Phi}^-)$*, then u is a singular point of $X(w)$.*

Proof. If $X(w)$ is nonsingular at u, then $\Omega\Big(T_u\big(X(w)\big)\Big) = \Omega(w,u)$. However, all elements of $\Omega(w,u)$ are real (see Remark 4.5.2). Thus, if $\Omega\Big(T_u\big(X(w)\big)\Big)$ contains an imaginary weight, then u is a singular point of $X(w)$. □

Note: this result appears as Lemma 4.6 in [28], in the context of Schubert varieties in $\mathcal{G}/\mathcal{P}$.

4.8 Peterson Translates of $X(w)$

In this section we will review some well-known results regarding Peterson translates of Schubert Varieties in $\mathcal{G}/\mathcal{B}$, including facts that are based upon material presented in [15], such as Proposition 3.4. We have also drawn heavily upon the ideas presented in Section 8 of [15] (which we have translated to the affine setting). Of the material in this section, only Example 4.8.11 is original to this thesis.

Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$, and let $C \in E(X(w),u)$. Recall from Section 2.11 that the Peterson translate $\tau_C\big(X(w),u\big)$ is a $\widehat{T}$-stable subspace of $T_u\big(X(w)\big)$ which satisfies

$$\dim X(w) \le \dim \tau_C\big(X(w),u\big) \le \dim T_u\big(X(w)\big),$$

where the dimension of $\tau_C\big(X(w),u\big)$ is equal to the common dimension of the tangent spaces $T_c(X)$ along the orbit $C \setminus C^{\widehat{T}}$. In the case that C is good, this yields that

$$\dim X(w) = \dim \tau_C\big(X(w), u\big).$$

Also recall that if $X(w)$ is nonsingular at u, then

$$\tau_C\big(X(w), u\big) = TE(X(w), u) = T_u\big(X(w)\big),$$

where

$$TE(X(w), u) = \bigoplus_{\hat{\alpha} \in \Omega(w,u)} \hat{\mathfrak{g}}_{\hat{\alpha}}.$$

Remark 4.8.1. In particular, this means that $\Omega\Big(\tau_C\big(X(w), u\big)\Big) = \Omega(w, u)$ for all $C \in E(X(w), u)$, whenever $X(w)$ is nonsingular at u.

We note that the requirements of Theorem 2.11.5 are fulfilled by any Schubert variety in $\mathcal{G}/\mathcal{B}$. In fact, since Schubert varieties are Cohen-Macaulay, by condition 2) of Theorem 2.11.5, we only require $TE(X(w), u) = \tau_C\big(X(w), u\big)$ for one good $\widehat{T}$-curve C to conclude that $X(w)$ is nonsingular at u.

Remark 4.8.2. If u is a singular $\widehat{T}$-fixed point of $X(w)$, then

$$TE\big(X(w), u\big) \neq \tau_C\big(X(w), u\big),$$

for all good $\widehat{T}$-curves $C \in E(X(w), u)$.

Since $\tau_C\big(X(w), u\big)$ is a $\widehat{T}$-stable subspace of $T_u\big(X(w)\big)$, it has a weight space decomposition as a $\widehat{T}$-module and

$$\Omega\Big(\tau_C\big(X(w), u\big)\Big) \subseteq \Omega\Big(T_u\big(X(w)\big)\Big).$$

Consequently, we have the following:

Lemma 4.8.3.

1) $u^{-1}(\hat{\alpha}) < 0$ *for all* $\hat{\alpha} \in \Omega\Big(\tau_C\big(X(w), u\big)\Big)$.
2) $\tau_C\big(X(w), u\big)$ *has no positive imaginary weights.*
3) *If* $\Omega\Big(\tau_C\big(X(w), u\big)\Big)$ *contains an element of* $\mathrm{Im}(\widehat{\Phi}^-)$*, then* u *is a singular point of* $X(w)$.

Proof. Statements 1) and 2) follow from Remark 4.4.1 and statement 3) is a consequence of Lemma 4.7.5. □

From Corollary 2.11.3 we know that the tangent space $T_u(C)$ is a $\widehat{T}$-stable subspace of $\tau_C\big(X(w), u\big)$.

Remark 4.8.4. As such, the weight of $T_u(C)$ as a $\widehat{T}$-module is a weight of $\tau_C\big(X(w), u\big)$.

In addition to $T_u(C)$, the Peterson translate $\tau_C\big(X(w), u\big)$ also contains the tangent spaces at u of all $\widehat{T}$-curves in $E^+(X(w), u)$:

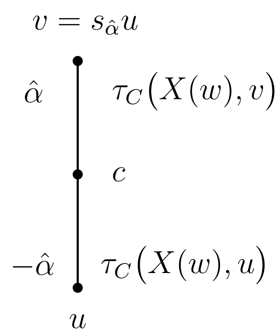
Lemma 4.8.5. *Let* $X(w)$ *be a Schubert variety in* $\mathcal{G}/\mathcal{B}$*, let* $u \in X(w)^{\widehat{T}}$*, and let* $C \in E(X(w), u)$*. If* $D \in E^+(X(w), u)$*, then* $T_u(D) \subseteq \tau_C\big(X(w), u\big)$*, as a* $\widehat{T}$*-stable subspace.*

Proof. Let $D \in E^+(X(w), u)$. If $D = C \in E^+(X(w), u)$, then we know that $T_u(D) = T_u(C) \subseteq \tau_C\big(X(w), u\big)$. See the proof of Theorem 5.9 in [11] for the case in which $C \neq D$. □

Remark 4.8.6. Accordingly, the weight of $T_u(D)$ as a $\widehat{T}$-module is a weight of $\tau_C\big(X(w), u\big)$, for all $\widehat{T}$-curves $D \in E^+(X(w), u)$.

Therefore, in terms of the set $E(X(w), u)$, the question remains: for which $D \in E^-(X(w), u)$ is $T_u(D) \subseteq \tau_C\big(X(w), u\big)$?

Now consider a $\widehat{T}$-curve $C \in E^-(X(w), u)$. We know that $C = \overline{\mathcal{U}_{-\hat{\alpha}} u}$, for some $\hat{\alpha} \in \widehat{\Phi}^+$, and that $C^{\widehat{T}} = \{u, s_\alpha u\}$, where $u < s_{\hat{\alpha}} u \leq w$. Let $v := s_{\hat{\alpha}} u$. There are two Peterson translates related to this C: $\tau_C\big(X(w), u\big)$ and $\tau_C\big(X(w), v\big)$.



We know from Remark 4.7.1 and by the construction of $\tau_C\big(X(w), v\big)$ that

$$\dim T_v(X(w)) = \dim T_c(X(w)) = \dim \tau_C\big(X(w), v\big)$$

for all points $c \in C \setminus C^{\widehat{T}}$. Therefore, since $\tau_C\big(X(w), v\big)$ is a subspace of $T_v\big(X(w)\big)$, it is clear that

$$\tau_C\big(X(w), v\big) = T_v\big(X(w)\big).$$

In terms of the other $\widehat{T}$-fixed point of C, the Peterson translate $\tau_C\big(X(w), u\big)$ constructed at u has a useful property for determining its weights:

Lemma 4.8.7. *Let $C \in E^-(X(w), u)$, so that $C = \overline{\mathcal{U}_{-\hat\alpha} u}$, for some $\hat\alpha \in \widehat{\Phi}^+$, and let $\hat\beta$ be any weight of $\tau_C\big(X(w), u\big)$. If $\hat\alpha + \hat\beta \in \widehat{\Phi}$ and $u^{-1}(\hat\alpha + \hat\beta) < 0$, then $\hat\alpha + \hat\beta$ is a weight of $\tau_C\big(X(w), u\big)$.*

Proof. Since $\mathcal{U}_{\hat\alpha} \subset \mathcal{B}$, $\mathcal{U}_{\hat\alpha}$ acts on $X(w)$. This action restricts to an action on the curve

$$C = \overline{\mathcal{U}_{-\hat\alpha} u} = \overline{\mathcal{U}_{\hat\alpha}(s_{\hat\alpha} u)} = \mathcal{U}_{\hat\alpha}(s_{\hat\alpha} u) \cup \{u\},$$

for which u is a fixed point. This induces an action of $\mathcal{U}_{\hat\alpha}$ on $T_u\big(X(w)\big)$ and hence on the Grassmannian of d-planes in $T_u\big(X(w)\big)$, where $d = \dim T_c(X(w))$ for all $c \in C \setminus C^T$. The map

$$\begin{aligned} \varphi\colon C \setminus C^T &\to G\Big(d, T_u\big(X(w)\big)\Big) \\ c &\mapsto T_c(X(w)) \end{aligned}$$

is $\mathcal{U}_{\hat\alpha}$-equivariant and a similar argument to the one given in the discussion following Definition 2.11.1 shows that the Peterson translate $\tau_C\big(X(w), u\big)$ is a $\mathcal{U}_{\hat\alpha}$-stable subspace of $T_u\big(X(w)\big)$.

Let $\hat\beta \in \Omega\Big(\tau_C\big(X(w),u\big)\Big) \subset \Omega\Big(T_u\big(X(w)\big)\Big)$. If $\hat\alpha+\hat\beta \in \widehat\Phi$ and $u^{-1}(\hat\alpha+\hat\beta) < 0$, then $\hat\alpha+\hat\beta \in \Omega\Big(T_u\big(X(w)\big)\Big)$, by Lemma 4.4.2. However, since $\tau_C\big(X(w),u\big)$ is $\mathcal{U}_{\hat\alpha}$-stable and since the Lie algebra of $\mathcal{U}_{\hat\alpha}$ is $\hat{\mathfrak{g}}_{\hat\alpha}$, we have

$$[\hat{\mathfrak{g}}_{\hat\alpha}, \hat{\mathfrak{g}}_{\hat\beta}] \subseteq \hat{\mathfrak{g}}_{\hat\alpha+\hat\beta} \subset \tau_C\big(X(w),u\big).$$

Therefore, $\hat\alpha+\hat\beta$ is a weight of $\tau_C\big(X(w),u\big)$. □

For both of the Peterson translates of $X(w)$ along $C = \overline{\mathcal{U}_{-\hat\alpha}u}$, there is a nice description of their respective weights in terms of $\hat\alpha$.

Lemma 4.8.8. *Let $C \in E^-(X(w),u)$, so that $C = \overline{\mathcal{U}_{-\hat\alpha}u}$, for some $\hat\alpha \in \widehat\Phi^+$. Let $v = s_{\hat\alpha}u$. Let $c \in C \setminus C^{\widehat T}$ and let $\widehat S = \widehat T_c$, the stabilizer of c. Then:*

1) $\widehat S = \ker\hat\alpha$.

2) $V := \tau_C\big(X(w),v\big)$ *or* $\tau_C\big(X(w),u\big)$ *is an $\widehat S$-module such that the weight space for each $\hat\omega \in \Omega_{\widehat S}(V) \subseteq X(\widehat S)$ is a $\widehat T$-module which decomposes as*

$$V_{\hat\omega} = \bigoplus_{\substack{\hat\beta \in X(\widehat T)\\ \hat\beta|_{\widehat S} = \hat\omega}} V_{\hat\beta}$$

Furthermore, if $\hat\beta \in \Omega_{\widehat T}(V_{\hat\omega})$, then all $\widehat T$-weights of $V_{\hat\omega}$ are elements of $\hat\beta + \mathbb{Z}\hat\alpha$ and hence are elements of the $\hat\alpha$-string through $\hat\beta$.

Proof. To prove statement 1), we first fix an isomorphism $\varphi : \mathrm{G}_a \to \mathcal{U}_{\hat\alpha}$ (See [26, pages 175, 177, 189, 455]). Since $C = \overline{\mathcal{U}_{\hat\alpha}v} = \mathcal{U}_{\hat\alpha}v \cup \{u\}$, $c = \varphi(b)v$ for some $b \in \mathrm{G}_a$. Furthermore, $t\cdot c = t\varphi(b)t^{-1}v = \varphi(\hat\alpha(t)b)v$.

Therefore,

$$\begin{aligned} t \in \widehat S &\iff c = t\cdot c \\ &\iff \varphi(b)v = \varphi(\hat\alpha(t)b)v \\ &\iff \varphi(b) = \varphi(\hat\alpha(t)b) \\ &\iff b = \hat\alpha(t)b \\ &\iff \hat\alpha(t) = 1 \\ &\iff t \in \ker\hat\alpha \end{aligned}$$

Thus, $\widehat{S} = \ker \hat{\alpha}$, as required.

Now consider statement 2). The stabilizer $\widehat{S}$ is a codimension 1 diagonalizable subgroup of $\widehat{T}$ which acts on the Peterson translate $V := \tau_C\big(X(w), v\big)$ or $\tau_C\big(X(w), u\big)$ by restricting the action of $\widehat{T}$ on V. As such, V also has a weight space decomposition as an $\widehat{S}$-module:

$$V = \bigoplus_{\hat{\omega} \in X(\widehat{S})} V_{\hat{\omega}}$$

Since the actions of $\widehat{S}$ and $\widehat{T}$ on V commute, it follows that each weight space $V_{\hat{\omega}}$ in this decomposition is also $\widehat{T}$-stable.

Moreover, from the exact sequence of diagonalizable groups

$$1 \to \widehat{S} \hookrightarrow \widehat{T} \xrightarrow{\hat{\alpha}} \mathrm{G}_m \to 1,$$

we obtain the exact sequence

$$0 \to \mathbb{Z} \hookrightarrow X(\widehat{T}) \xrightarrow[\text{(restriction)}]{\rho} X(\widehat{S}) \to 0,$$

where $\ker \rho = \mathbb{Z}\hat{\alpha}$. Consequently, for each weight $\hat{\omega} \in X(\widehat{S})$, we have

$$V_{\hat{\omega}} = \bigoplus_{\substack{\hat{\beta} \in X(\widehat{T}) \\ \hat{\beta}|_{\widehat{S}} = \hat{\omega}}} V_{\hat{\beta}}$$

and for any $\hat{\beta}_1, \hat{\beta}_2 \in \Omega_{\widehat{T}}(V_{\hat{\omega}})$, we have that $(\hat{\beta}_1 - \hat{\beta}_2)\Big|_{\widehat{S}} = 0$, and therefore $\hat{\beta}_1 - \hat{\beta}_2 \in \mathbb{Z}\hat{\alpha}$. Thus, if we fix some $\hat{\beta} \in \Omega_{\widehat{T}}(V_{\hat{\omega}})$, then any other weight $\hat{\gamma} \in \Omega_{\widehat{T}}(V_{\hat{\omega}})$ is an element of $\hat{\beta} + \mathbb{Z}\hat{\alpha}$.

□

Remark 4.8.9. The number of elements of $\hat{\beta} + \mathbb{Z}\hat{\alpha}$ which appear as weights of a $\widehat{T}$-module V is referred to as the *length* of the $\hat{\alpha}$-string through $\hat{\beta}$ in V.

As $\tau_C\big(X(w), u\big)$ and $\tau_C\big(X(w), v\big)$ are constructed using the same $\widehat{T}$-orbit, it is natural to ask if there are any connections between them. One particularly useful relationship is given in the following lemma:

Lemma 4.8.10. *Let $C \in E^{-}(X(w),u)$, so that $C = \overline{\mathcal{U}_{-\hat{\alpha}}u}$, for some $\hat{\alpha} \in \widehat{\Phi}^{+}$. Let $v = s_{\hat{\alpha}}u$. Let $c \in C \setminus C^{\widehat{T}}$ and let $\widehat{S} = \widehat{T}_c$, the stabilizer of c. Then:*

1) *The Peterson translates $\tau_C\big(X(w),u\big)$ and $\tau_C\big(X(w),v\big)$ are isomorphic as $\widehat{S}$-modules.*

2) *If $\hat{\beta}$ is a $\widehat{T}$-weight of $\tau_C\big(X(w),u\big)$, then there exists a (not necessarily unique) $z \in \mathbb{Z}$ such that $\hat{\beta} + z\hat{\alpha}$ is a $\widehat{T}$-weight of $\tau_C\big(X(w),v\big)$.*

3) *If $\hat{\beta}$ is a $\widehat{T}$-weight of $\tau_C\big(X(w),v\big)$, then there exists a (not necessarily unique) $z \in \mathbb{Z}$ such that $\hat{\beta} + z\hat{\alpha}$ is a $\widehat{T}$-weight of $\tau_C\big(X(w),u\big)$.*

4) *If v is a nonsingular point of $X(w)$ and $\hat{\beta}$ is a $\widehat{T}$-weight of either $\tau_C\big(X(w),u\big)$ or $\tau_C\big(X(w),v\big)$, then the length of the $\hat{\alpha}$-string through $\hat{\beta}$ in $\tau_C\big(X(w),v\big)$ is greater than or equal to the length of the $\hat{\alpha}$-string through $\hat{\beta}$ in $\tau_C\big(X(w),u\big)$.*

Proof. Statement 1) holds by Proposition 3.4 in [15].

Now let $V := \tau_C\big(X(w),u\big)$ and $W := \tau_C\big(X(w),v\big)$.

To prove statement 2), let $\hat{\beta} \in \Omega_{\widehat{T}}(V)$. Thus, by Lemma 4.8.8, $\hat{\beta} \in \Omega_{\widehat{T}}(V_{\hat{\omega}})$, where $\hat{\omega} \in X(\widehat{S})$ such that $\hat{\beta}\Big|_{\widehat{S}} = \hat{\omega}$. By part 1), $\hat{\omega}$ is also an $\widehat{S}$-weight of W and

$$1 \leq \dim V_{\hat{\omega}} = \dim W_{\hat{\omega}}.$$

Hence, by Lemma 4.8.8, there exits a $\widehat{T}$-weight $\hat{\gamma} \in \Omega_{\widehat{T}}(W)$ which satisfies $\hat{\gamma}|_{\widehat{S}} = \hat{\omega}$. Therefore, $\hat{\gamma}, \hat{\beta} \in X(\widehat{T})$ such that $(\hat{\gamma} - \hat{\beta})\Big|_{\widehat{S}} = 0$. By Lemma 4.8.8, we know that $\widehat{S} = \ker\hat{\alpha}$ and hence we have $\hat{\gamma} - \hat{\beta} \in \mathbb{Z}\hat{\alpha}$, so that $\hat{\gamma} = \hat{\beta} + z\hat{\alpha}$, for some $z \in \mathbb{Z}$. Thus, statement 2) holds. The proof of statement 3) is similar.

Now, to prove statement 4), suppose that v is a nonsingular $\widehat{T}$-fixed point of $X(w)$. Therefore, by Lemma 4.8.3, the set $\Omega_{\widehat{T}}(W)$ contains only real roots and consequently $\dim W_{\hat{\gamma}} = 1$, for all $\hat{\gamma} \in \Omega_{\widehat{T}}(W)$. Let $\hat{\beta} \in \Omega_{\widehat{T}}(V)$ and let $\hat{\omega} \in X(\widehat{S})$ such that $\hat{\beta}\Big|_{\widehat{S}} = \hat{\omega}$. By Lemma 4.8.8, we have

$$V_{\hat{\omega}} = \bigoplus_{\substack{z\in\mathbb{Z} \\ \hat{\beta}+z\hat{\alpha}\in\Omega_{\widehat{T}}(V)}} V_{\hat{\beta}+z\hat{\alpha}}$$

and hence the length of the $\hat{\alpha}$-string through $\hat{\beta}$ in V is less than or equal to $\dim V_{\hat{\omega}}$.

It now follows from part 2), Lemma 4.8.8, and the fact that v is a nonsingular point of $X(w)$ that

$$W_{\hat{\omega}} = \bigoplus_{\substack{z' \in \mathbb{Z} \\ \hat{\beta}+z'\hat{\alpha} \in \Omega_{\widehat{T}}(W)}} W_{\hat{\beta}+z'\hat{\alpha}}$$

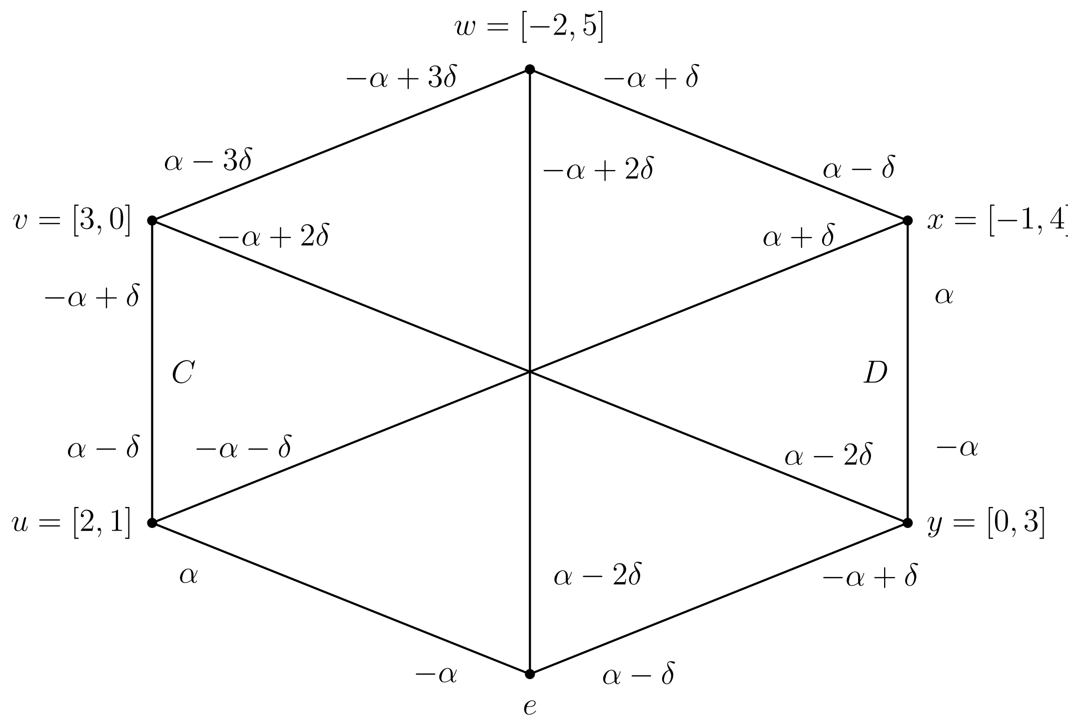
where each $W_{\hat{\beta}+z'\hat{\alpha}}$ has dimension 1. In particular, this mean that the length of the $\hat{\alpha}$-string through $\hat{\beta}$ in W is equal to $\dim W_{\hat{\omega}}$. Since $\dim V_{\hat{\omega}} = \dim W_{\hat{\omega}}$, by part 1), we obtain that statement 4) holds for $\hat{\beta} \in \Omega_{\widehat{T}}(V)$. The case for $\hat{\beta} \in \Omega_{\widehat{T}}(W)$ is similar. $\square$

At this point, we will give an example in which we compute some of the Peterson translates of a particular Schubert variety.

Example 4.8.11. Let us consider the case when $n = 2$.

Let $w = [-2, 5] \in \widehat{W} = \widehat{S}_2$ and form the Schubert variety $X(w) = \overline{\mathcal{B}w} \subseteq \mathcal{G}/\mathcal{B}$. From Example 4.6.3, we know that $X(w)$ is rationally smooth and hence from Theorem 4.6.1 we know that $|E(X(w), w')| = \ell(w) = 3$, for all $w' \in [e, w]$.

Let $u = [2, 1]$, $v = [3, 0]$, $y = [0, 3]$, and $x = [-1, 4]$. Let $\alpha = (12) \in \Phi$ and set $C = \overline{\mathcal{U}_{\alpha-\delta}u}$ and $D = \overline{\mathcal{U}_{-\alpha}y}$. The Bruhat graph $\Gamma(X(w))$ is



As Schubert varieties are nonsingular in codimension 1, we know that $X(w)$ is nonsingular at x and v.

Since $X(w)$ is nonsingular at x, we know

$$\tau_D\big(X(w), x\big) = T_x\big(X(w)\big) = TE\big(X(w), x\big) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\delta} \oplus \mathfrak{g}_{\alpha-\delta}$$

Thus, $\tau_D\big(X(w), x\big)$ has three weights, each weight on an α-string of length 1. Consequently, by Lemma 4.8.10, we know that $\tau_D\big(X(w), y\big)$ has three weights, each of which is on an α-string of length 1.

Since $D = \overline{\mathcal{U}_{-\alpha} y}$, we know that $T_y(D) = \mathfrak{g}_{-\alpha}$ and hence $-\alpha$ is a weight of $\tau_D\big(X(w), y\big)$, by Remark 4.8.4. Furthermore, by Remark 4.8.6, we determine that $-\alpha + \delta$ is a weight of $\tau_D\big(X(w), y\big)$. In fact, $-\alpha + \delta$ is the weight of $\tau_D\big(X(w), y\big)$ which is obtained, in the sense of Lemma 4.8.10, from the weight $\alpha + \delta$ of $\tau_D\big(X(w), x\big)$. Indeed, $\alpha + \delta$ corresponds to a weight of $\tau_D\big(X(w), y\big)$ which must be one of the following roots

$$\alpha + \delta \qquad \delta \qquad -\alpha + \delta$$

However, we know from Lemma 4.8.3 that $\tau_D\big(X(w), y\big)$ has no positive imaginary weight, so it cannot be δ. Also, since $-\alpha$ is a weight of $\tau_D\big(X(w), y\big)$, we obtain from Lemma 4.8.3 that $y^{-1}(-\alpha) < 0$ and hence $y^{-1}(\alpha) > 0$. Thus,

$$y^{-1}(\alpha + \delta) = y^{-1}(\alpha) + \delta > 0$$

and therefore $\alpha + \delta$ is not a weight of $\tau_D\big(X(w), y\big)$, by Lemma 4.8.3. This leaves $-\alpha + \delta$ as the only possibility.

All that remains is to determine the third weight of $\tau_D\big(X(w), y\big)$. By Lemma 4.8.10, the weight $\alpha-\delta$ of $\tau_D\big(X(w), x\big)$ corresponds to a weight $\hat{\alpha}$ of $\tau_D\big(X(w), y\big)$ which is on the α-string through $\alpha - \delta$. The candidates for $\hat{\alpha}$ are:

$$\alpha - \delta \qquad\qquad - \delta \qquad\qquad - \alpha - \delta$$

To identify $\hat{\alpha}$, we first note that since $-\alpha + \delta$ is a weight of $\tau_D\big(X(w), y\big)$, we know $y^{-1}(-\alpha + \delta) < 0$ from Lemma 4.8.3. Hence,

$$y^{-1}(\alpha - \delta) = -y^{-1}(-\alpha + \delta) > 0,$$

which implies that $\alpha - \delta$ is not a weight of $\tau_D\big(X(w), y\big)$, by Lemma 4.8.3. Subsequently $\hat{\alpha} \neq \alpha - \delta$. Furthermore, if $-\alpha - \delta$ is a weight of $\tau_D\big(X(w), y\big)$, then from Lemma 4.8.7, we obtain that

$$\alpha + (-\alpha - \delta) = -\delta \in \Omega\Big(\tau_D\big(X(w), y\big)\Big),$$

since $y^{-1}(-\delta) = -\delta < 0$. However, this would mean that the α-string through $\alpha - \delta$ in $\Omega\Big(\tau_D\big(X(w), y\big)\Big)$ has length at least 2, which is impossible since it has length 1. As a result, we deduce that $\hat{\alpha} \neq -\alpha - \delta$. Therefore, by the process of elimination, we obtain that $\hat{\alpha} = -\delta$. Hence,

$$\tau_D\big(X(w), y\big) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\delta} \oplus \mathfrak{g}_{-\alpha+\delta}$$

We note that since $-\delta$ is a weight of $\tau_D\big(X(w), y\big)$, the Schubert variety $X(w)$ is singular at y by Lemma 4.8.3.

So now let's consider the $\widehat{T}$-curve C. Again, since $X(w)$ is nonsingular at the u, we know

$$\tau_C\big(X(w), v\big) = T_v\big(X(w)\big) = TE\big(X(w), v\big) = \mathfrak{g}_{-\alpha+\delta} \oplus \mathfrak{g}_{-\alpha+2\delta} \oplus \mathfrak{g}_{\alpha-3\delta}$$

Thus, $\tau_C\big(X(w), u\big)$ has three weights, each weight on a $(-\alpha + \delta)$-string of length 1. Accordingly, by Lemma 4.8.10, the Peterson translate $\tau_C\big(X(w), u\big)$ also has three weights, each of which is on an $(-\alpha + \delta)$-string of length 1. Using a similar argument as in the previous case, we know that $\alpha - \delta$ and α are weights of $\tau_C\big(X(w), u\big)$, where α corresponds to the weight $-\alpha + 2\delta$ of $\tau_C\big(X(w), v\big)$. So now, by Lemma 4.8.10, the weight $\alpha - 3\delta$ of $\tau_C\big(X(w), v\big)$ corresponds to a weight $\hat{\alpha}$ of $\tau_C\big(X(w), u\big)$ which is on the $(-\alpha + \delta)$-string through $\alpha - 3\delta$. The options for $\hat{\alpha}$ are:

$$\alpha - 3\delta \qquad\quad -2\delta \qquad\quad -\alpha - \delta$$

However, using Lemma 4.8.7, if $\alpha - 3\delta$ is a weight of $\tau_C\big(X(w), u\big)$, then

$$-2\delta = (-\alpha + \delta) + (a - 3\delta)$$

and

$$-\alpha - \delta = (-\alpha + \delta) + (-2\delta)$$

are also weights of $\tau_C\big(X(w), u\big)$. Likewise, if -2δ is a weight of $\tau_C\big(X(w), u\big)$, then $-\alpha - \delta$ is as well. Thus, since the $(-\alpha + \delta)$-string through $\alpha - 3\delta$ in $\Omega\Big(\tau_C\big(X(w), u\big)\Big)$ has length 1, $\hat{\alpha} = -\alpha - \delta$. Consequently,

$$\tau_C\big(X(w), u\big) = \mathfrak{g}_{\alpha-\delta} \oplus \mathfrak{g}_{-\alpha-\delta} \oplus \mathfrak{g}_{\alpha}$$

Furthermore, since $X(w)$ is nonsingular at v, the $\widehat{T}$-curve C is good (see Remark 4.7.4). Therefore since

$$\tau_C\big(X(w), u\big) = TE\big(X(w), u\big)$$

we obtain from Theorem 2.11.5 that $X(w)$ is nonsingular at u.

We conclude this section by summarizing some useful equivalences discussed throughout the past few sections. For the sake of convenience, we have assembled these statement into two lemmas. We first present a list for positive real roots $\hat{\alpha}$:

Lemma 4.8.12. *Let $\hat{\alpha} \in \widehat{\Phi}^+$ be real. Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$, and let $C \in E(X(w), u)$. The following are equivalent:*

1) $u^{-1}(\hat{\alpha}) < 0$

2) $s_{\hat{\alpha}} u < u$

3) $\overline{\mathcal{U}_{\hat{\alpha}} u} \in E^+(X(w), u)$

4) $T_u\left(\overline{\mathcal{U}_{\hat{\alpha}} u}\right) \subseteq \tau_C\big(X(w), u\big)$ *as a $\widehat{T}$-stable subspace*

5) $\hat{\alpha} \in \Omega\Big(\tau_C\big(X(w), u\big)\Big)$

6) $\hat{\alpha} \in \Omega\Big(T_u\big(X(w)\big)\Big)$

7) $\hat{\alpha} \in \Omega\big(T_u(\mathcal{G}/\mathcal{B})\big)$

Proof. Statement 1) implies statement 2) by Lemma 4.3.3. Statement 2) implies statement 3) by the definition of the $E^+(X(w), u)$ (see Section 4.5). From Lemma 4.8.5, we determine that statement 3) implies statement 4). Since $T_u\left(\overline{\mathcal{U}_{\hat{\alpha}} u}\right) = \hat{\mathfrak{g}}_{\hat{\alpha}}$, statement 5) follows from statement 4). We can obtain that statement 5) implies statement 6) and statement 6) implies statement 7) from the fact that

$$\tau_C\big(X(w), u\big) \subset T_u\big(X(w)\big) \subset T_u(\mathcal{G}/\mathcal{B})$$

as $\widehat{T}$-stable subspaces. Finally, statement 7) implies statement 1) by Lemma 4.3.3. $\square$

Unfortunately, for negative real roots $-\hat{\alpha}$, we do not have such an extensive list of equivalences unless, we include an additional condition.

Lemma 4.8.13. *Let $\hat{\alpha} \in \widehat{\Phi}^+$ be real. Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$ be a nonsingular point of $X(w)$, and let $C \in E(X(w), u)$. The following are equivalent:*

1) $u < s_{\hat{\alpha}} u \leq w$

2) $\overline{\mathcal{U}_{-\hat{\alpha}} u} \in E^-(X(w), u)$

3) $T_u\Big(\overline{\mathcal{U}_{-\hat{\alpha}} u}\Big) \subseteq \tau_C\big(X(w), u\big)$ *as a $\widehat{T}$-stable subspace*

4) $-\hat{\alpha} \in \Omega\Big(\tau_C\big(X(w), u\big)\Big)$

5) $-\hat{\alpha} \in \Omega\Big(T_u\big(X(w)\big)\Big)$

Proof. Statement 1) is equivalent to statement 2) by the definition of the $E^-(X(w), u)$ (see Section 4.5). Since $X(w)$ is nonsingular at u, we have that

$$\tau_C\big(X(w), u\big) = TE(X(w), u) = T_u\big(X(w)\big)$$

Furthermore, we have that $T_u\Big(\overline{\mathcal{U}_{-\hat{\alpha}} u}\Big) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$. The equivalence of statements 2), 3), 4), and 5) now follows. $\square$

4.9 $\widehat{T}$- Surfaces in $\mathcal{G}/\mathcal{B}$

Let Σ be a closed irreducible $\widehat{T}$-stable surface in $\mathcal{G}/\mathcal{B}$. As such, Σ is a closed irreducible subvariety of some Schubert variety $X(w)$. Since $X(w)$ is a $\widehat{T}$-variety and $E\big(X(w)\big)$ is finite, Σ is a $\widehat{T}$-variety which also a $\widehat{T}$-surface, that is, the closure of a two-dimensional $\widehat{T}$-orbit. Furthermore, Lemma 2.12.2 establishes that $|E(\Sigma, u)| = 2$, for any $u \in \Sigma^{\widehat{T}}$. In the case that $X(w)$ is a rationally smooth Schubert variety, it is well known that the converse also holds, that is, any two $\widehat{T}$-curves passing through a $\widehat{T}$-fixed point u of $X(w)$ are contained in some $\widehat{T}$-surface:

Lemma 4.9.1. *Let $X(w)$ be a rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$ and let u be a $\widehat{T}$-fixed point of $X(w)$. If $C, D \in E(X(w), u)$, then there exists a $\widehat{T}$-surface $\Sigma \in \Sigma\big(X(w), u\big)$ which contains C and D.*

Proof. Let $\pi : X(w)_u \to TE(X(w),u)$ be the restriction to $X(w)_u$ of the $\widehat{T}$-equivariant projection $\tilde{\pi} : T_u\big(X(w)\big) \twoheadrightarrow TE(X(w),u)$. Using a similar argument as given in the proof of Lemma 2.12.4, we obtain that π is a finite morphism. Since $X(w)$ is rationally smooth, we know from Theorem 4.6.1 that $\dim X(w) = |E(X(w),u)|$. Therefore,

$$\dim X(w)_u = \dim TE(X(w),u),$$

which in turn implies that the finite morphism π is surjective.

Let $\Sigma' = T_u(C) \oplus T_u(D)$. For dimension reasons, some irreducible component Σ of $\pi^{-1}(\Sigma')$ is a $\widehat{T}$- surface, which by Lemma 2.12.2 contains two $\widehat{T}$-curves, say C'_u and D'_u, for some $C', D' \in E(X(w),u)$. However, by Lemma 4.5.3, $\pi(C'_u) = T_u(C')$ and $\pi(D'_u) = T_u(D')$. It follows that $\{C', D'\} = \{C, D\}$ and hence C, D are contained in the $\widehat{T}$-surface Σ. $\square$

In Section 4.3 in [11], we obtained some results regarding $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{P}$, but in Remark 4.13 of [11], we ascertain that all of our work, including material presented here in Theorem 4.9.2 and Lemma 4.9.3, also holds for $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{B}$. In the $\mathcal{G}/\mathcal{P}$ setting, the two $\widehat{T}$-curves in $E(\Sigma,u)$, for $u = e$, have the form $C = \overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $D = \overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = \beta + h_\beta\delta$ are elements of $\widehat{\Phi}^+$ with $\alpha, \beta \neq 0$ and $h_\alpha, h_\beta > 0$. The only difference in the $\mathcal{G}/\mathcal{B}$ context is that $h_\alpha, h_\beta \geq 0$. However, as indicated in Remark 4.13, the deductions involved in the $\mathcal{G}/\mathcal{P}$ case never used the assumption that $h_\alpha, h_\beta \neq 0$. We have incorporated some of these results into the following theorem:

Theorem 4.9.2. *Let Σ be a $\widehat{T}$-surface in $\mathcal{G}/\mathcal{B}$, let $u \in \Sigma^{\widehat{T}}$, and let C and D be the two $\widehat{T}$-curves in $E(\Sigma,u)$. Let $\hat{\alpha} := \alpha + h_\alpha\delta$ and $\hat{\beta} := \beta + h_\beta\delta$ be elements of $u\Big(\widehat{\Phi}^+\Big)$ such that $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$ and $T_u(D) = \hat{\mathfrak{g}}_{-\hat{\beta}}$. If*

1) $\beta \neq \pm\alpha$,

2) $\beta = \alpha$ *and* $|h_\beta - h_\alpha| = 1$, *or*

3) $\beta = -\alpha$ *and* $h_\beta + h_\alpha = 1$,

then Σ is nonsingular at u.

Proof. For $u = e$, see Section 4.3 in [11]. Refer to Theorem 4.3 in [11] for the case in which $\beta \neq \pm\alpha$. Theorem 4.5 in [11] addresses the case where $\beta = \alpha$. The case for which $\beta = -\alpha$ is handled in Theorem 4.14 in [11].

For $u \neq e$, consider the $\widehat{T}$-surface $u^{-1}\Sigma$, which contains e. The two $\widehat{T}$-curves in $E(u^{-1}\Sigma, e)$ are $u^{-1}C$ and $u^{-1}D$, which have $T_e(u^{-1}C) = \hat{\mathfrak{g}}_{-u^{-1}(\hat{\alpha})}$ and $T_e(u^{-1}D) = \hat{\mathfrak{g}}_{-u^{-1}(\hat{\beta})}$.

Let $u^{-1}(\alpha) := \gamma + h_\gamma\delta$ and $u^{-1}(\beta) := \omega + h_\omega\delta$. Accordingly, we have

$$u^{-1}(\hat{\alpha}) = u^{-1}(\alpha) + h_\alpha\delta = \gamma + (h_\gamma + h_\alpha)\delta$$

and

$$u^{-1}(\hat{\beta}) = u^{-1}(\beta) + h_\beta\delta = \omega + (h_\omega + h_\beta)\delta$$

Suppose that $\beta \neq \pm\alpha$. Since $u^{-1}(\alpha) = \gamma + h_\gamma\delta$ and $u^{-1}(\beta) = \omega + h_\omega\delta$, we also have

$$u(\gamma) = \alpha - h_\gamma\delta \quad \text{and} \quad u(\omega) = \beta - h_\omega\delta.$$

Since $\beta \neq \pm\alpha$, we obtain that $u(\omega) \neq \pm u(\gamma)$ and consequently $\omega \neq \pm\gamma$. Therefore, $u^{-1}\Sigma$ is nonsingular at e and hence Σ is nonsingular at u.

Now, suppose that $\beta = \alpha$ and $|h_\beta - h_\alpha| = 1$. Thus, $u^{-1}(\beta) = u^{-1}(\alpha)$, so that $\omega = \gamma$ and $h_\omega = h_\gamma$. Thus,

$$|(h_\omega + h_\beta) - (h_\gamma + h_\alpha)| = |h_\gamma + h_\beta - h_\gamma - h_\alpha| = |h_\beta - h_\alpha| = 1$$

Subsequently, $u^{-1}\Sigma$ is nonsingular at e and therefore Σ is nonsingular at u.

Finally, suppose that $\beta = -\alpha$ and $h_\beta + h_\alpha = 1$. Thus, $u^{-1}(\beta) = -u^{-1}(\alpha)$, so that $\omega = -\gamma$ and $h_\omega = -h_\gamma$. Thus,

$$(h_\omega + h_\beta) + (h_\gamma + h_\alpha) = -h_\gamma + h_\beta + h_\gamma + h_\alpha = h_\beta + h_\alpha = 1$$

Once again, this yields that $u^{-1}\Sigma$ is nonsingular at e and hence Σ is nonsingular at u. $\square$

Section 4.3 in [11] also provides us with another piece of useful information. For a $\widehat{T}$-surface to be singular at a $\widehat{T}$-fixed point u, we know that $\dim T_u(\Sigma) > 2$, which means that $\Omega\big(T_u(\Sigma)\big)$ must contain at least one weight in addition to $-\hat{\alpha}$ and $-\hat{\beta}$. The possible forms for such a weight are described in Section 4.3 in [11] and presented in the following lemma:

Lemma 4.9.3. *Let Σ be a $\widehat{T}$-surface in $\mathcal{G}/\mathcal{B}$ which is singular at $u \in \Sigma^{\widehat{T}}$. Let C and D be the two $\widehat{T}$-curves in $E(\Sigma, u)$. Let $\hat{\alpha} := \alpha + h_\alpha\delta$ and $\hat{\beta} := \beta + h_\beta\delta$ be elements of $u\Big(\widehat{\Phi}^+\Big)$ such that $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$ and $T_u(D) = \hat{\mathfrak{g}}_{-\hat{\beta}}$. Then any weight of $T_u(\Sigma)$, other than $-\hat{\alpha}$ and $-\hat{\beta}$, is a weight $-\hat{\gamma}$, where $\hat{\gamma} \in u\Big(\widehat{\Phi}^+\Big)$, and either*

1) *$\beta = \alpha$ and $\hat{\gamma} = \alpha + h_\gamma\delta$ where $h_\alpha < h_\gamma < h_\beta$ (assuming $h_\alpha < h_\beta$), or*

2) *$\beta = -\alpha$ and $\hat{\gamma} = \hat{\alpha} + l\delta$, $l\delta$, or $\hat{\beta} + l\delta$, for some integer $l \geq 1$.*

Proof. For the case in which $u = e$, see Lemma 4.2, Theorem 4.3, Theorem 4.5, and Theorem 4.14 in [11].

For $u \neq e$, since Σ is singular at u, the set $\Omega\big(T_u(\Sigma)\big)$ contains at least one weight in addition to $-\hat{\alpha}$ and $-\hat{\beta}$. Let $-\hat{\gamma}$, where $\hat{\gamma} := \gamma + h_\gamma\delta \in u\Big(\widehat{\Phi}^+\Big)$, be any such weight.

Now consider the $\widehat{T}$-surface $u^{-1}\Sigma$, which is singular at e. We know that $-u^{-1}(\hat{\alpha})$, $-u^{-1}(\hat{\beta}), -u^{-1}(\hat{\gamma})$ are weights of $T_e(u^{-1}\Sigma)$.

Let $u^{-1}(\alpha) := \zeta + h_\zeta\delta$ and $u^{-1}(\beta) := \omega + h_\omega\delta$. Correspondingly, we have

$$u^{-1}(\hat{\alpha}) = u^{-1}(\alpha) + h_\alpha\delta = \zeta + (h_\zeta + h_\alpha)\delta \in \widehat{\Phi}^+$$

and

$$u^{-1}(\hat{\beta}) = u^{-1}(\beta) + h_\beta\delta = \omega + (h_\omega + h_\beta)\delta \in \widehat{\Phi}^+$$

We also know that

$$u(\zeta) = \alpha - h_\zeta\delta \quad \text{and} \quad u(\omega) = \beta - h_\omega\delta$$

Since $u^{-1}\Sigma$ is singular at e, one of two situations occurs.

For the first situation, we have $\omega = \zeta$. Therefore, $u(\omega) = u(\zeta)$ and hence $\beta = \alpha$ and $h_\omega = h_\zeta$. Thus, in this case we have

$$\hat{\alpha} = \alpha + h_\alpha \delta \quad \text{and} \quad \hat{\beta} = \alpha + h_\beta \delta,$$

where $h_\alpha \neq h_\beta$, as well as

$$u^{-1}(\hat{\alpha}) = \zeta + (h_\zeta + h_\alpha)\delta \quad \text{and} \quad u^{-1}(\hat{\beta}) = \zeta + (h_\zeta + h_\beta)\delta.$$

Without loss of generality, assume $h_\alpha < h_\beta$. Consequently, for this situation, we also know that $u^{-1}(\hat{\gamma}) = \zeta + h\delta$ where $h_\zeta + h_\alpha < h < h_\zeta + h_\beta$. Subsequently,

$$\hat{\gamma} = u(\zeta + h\delta) = u(\zeta) + h\delta = (\alpha - h_\zeta \delta) + h\delta = \alpha + (h - h_\zeta)\delta,$$

where $h_\alpha < h - h_\zeta < h_\beta$, as required.

For the second situation, we have that $\omega = -\zeta$ and $u^{-1}(\hat{\gamma}) = u^{-1}(\hat{\alpha}) + l\delta$, $l\delta$, or $u^{-1}(\hat{\beta}) + l\delta$, for some integer $l \geq 1$. It follows that

$$\hat{\gamma} = u\big(u^{-1}(\hat{\alpha}) + l\delta\big) = u\big(u^{-1}(\hat{\alpha})\big) + l\delta = \hat{\alpha} + l\delta,$$

$$\hat{\gamma} = u(l\delta) = l\delta, \qquad \text{or}$$

$$\hat{\gamma} = u\Big(u^{-1}(\hat{\beta}) + l\delta\Big) = u\Big(u^{-1}(\hat{\beta})\Big) + l\delta = \hat{\beta} + l\delta,$$

for some integer $l \geq 1$. Moreover, since $\omega = -\zeta$, we have $u(\omega) = -u(\zeta)$, which implies $\beta = -\alpha$. $\square$

Based upon our work with $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{B}$, we have developed the following tool for computing some of the weights of $\tau_C\big(X(w), u\big)$:

Lemma 4.9.4. *Let $X(w)$ be a rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$, let u be a $\widehat{T}$-fixed point of $X(w)$, and let $C \in E(X(w), u)$, with $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$, where $\hat{\alpha} = \alpha + h_\alpha \delta \in u\Big(\widehat{\Phi}^+\Big)$.*

If $D \in E^+(X(w), u)$, so that $D = \overline{\mathcal{U}_{\hat{\beta}} u}$, for some $\hat{\beta} \in \widehat{\Phi}^+ \cap u(\widehat{\Phi}^-)$, then $\hat{\beta}$ is a weight of $\tau_C\big(X(w), u\big)$. If $D \in E^-(X(w), u)$, so that $D = \overline{\mathcal{U}_{-\hat{\beta}} u}$, where $\hat{\beta} = \beta + h_\beta \delta \in \widehat{\Phi}^+ \cap u(\widehat{\Phi}^+)$ such that

1) $\beta \neq \pm\alpha$,

2) $\beta = \alpha$ *and* $|h_\beta - h_\alpha| = 1$, *or*

3) $\beta = -\alpha$ *and* $h_\beta + h_\alpha = 1$,

then $-\hat{\beta}$ is a weight of $\tau_C\big(X(w), v\big)$.

Proof. If $D = \overline{\mathcal{U}_{\hat{\beta}} u} \in E^+(X(w), u)$, then Lemma 4.8.5 gives us that

$$\hat{\mathfrak{g}}_{\hat{\beta}} = T_u(D) \subseteq \tau_C\big(X(w), u\big)$$

and so $\hat{\beta}$ is a weight of $\tau_C\big(X(w), u\big)$. Suppose that $D = \overline{\mathcal{U}_{-\hat{\beta}} u} \in E^-(X(w), u)$. By Lemma 4.9.1, there is a $\widehat{T}$-surface $\Sigma \in \Sigma\big(X(w), u\big)$ containing C and D. Thus, by Theorem 4.9.2 Σ is nonsingular at u and hence $\tau_C(\Sigma, u) = T_u(\Sigma)$. Thus, we establish that

$$\hat{\mathfrak{g}}_{-\hat{\beta}} = T_u(D) \subseteq T_u(\Sigma) = \tau_C(\Sigma, u) \subseteq \tau_C\big(X(w), u\big)$$

with the last inclusion coming from Lemma 2.11.2. Therefore, $-\hat{\beta}$ is a weight of $\tau_C\big(X(w), v\big)$. $\square$

4.10 Reflection Formulas

In much of what follows, we want to express reflections $s_{\hat{\alpha}}$ associated to elements $\hat{\alpha} \in \widehat{\Phi}$ as a product of other such reflections, in particular, reflection of the form s_α and $s_{\alpha-\delta}$, for some $\alpha \in \Phi$. In this section, we will state several key reflection formulas.

We begin with a well-known formula (See Lemma 5.7 in [21]):

Remark 4.10.1.

$$s_{\hat{\alpha}} s_{\hat{\beta}} s_{\hat{\alpha}}^{-1} = s_{s_{\hat{\alpha}}(\hat{\beta})}$$

for any real roots $\hat{\alpha}, \hat{\beta} \in \widehat{\Phi}$.

The remainder of the formulas in this section are based on the descriptions of the roots involved. Let $r + nq \in \mathbb{Z}$, where $1 \leq r \leq n$ and $q \in \mathbb{Z}$. Recall that for a real root $\hat{\alpha} = \alpha + h\delta \in \widehat{\Phi}$, where $\alpha = (ij) \in \Phi^+$ with $1 \leq i < j \leq n$, the reflection $s_{\hat{\alpha}}$ acts on $\mathbb{Z}$ as follows:

$$s_{\hat{\alpha}}(r + nq) := \begin{cases} r + nq & \text{if} \quad r \neq i, j \\ j + n(q + h) & \text{if} \quad r = i \\ i + n(q - h) & \text{if} \quad r = j \end{cases}$$

In order to condense some proofs, instead of considering $r = i$ and $r = j$ separately, we will let $r = i$ and then account for the last two cases in the definition above by writing

$$s_{\hat{\alpha}}(z) = s_{\hat{\alpha}}(i + nq) = j + n(q \pm h)$$

Lemma 4.10.2. *Let $\hat{\alpha} = \alpha + h\delta \in \widehat{\Phi}$ be a real root and let $s_{\hat{\alpha}} \in \widehat{W}$ be the reflection associated to $\hat{\alpha}$. For any integer $k \geq 0$, we have*

$$s_{\hat{\alpha}+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_{\hat{\alpha}} \tag{4.1}$$

and

$$s_{\hat{\alpha}-k\delta} = (s_{\alpha-\delta} s_\alpha)^k s_{\hat{\alpha}} \tag{4.2}$$

Proof. Let $z = r + nq \in \mathbb{Z}$, where $1 \leq r \leq n$ and $q \in \mathbb{Z}$.

We begin by considering equation (4.1). If $r \neq i, j$, then it is immediate that

$$s_{\hat{\alpha}+k\delta}(z) = z = (s_\alpha s_{\alpha-\delta})^k s_{\hat{\alpha}}(z)$$

So now assume that $r = i$. For $k = 0$ we have

$$s_{\hat{\alpha}+0\delta} = s_{\hat{\alpha}} = e \cdot s_{\hat{\alpha}} = (s_\alpha s_{\alpha-\delta})^0 s_{\hat{\alpha}}$$

Inductively, we have

$$\begin{aligned}
(s_\alpha s_{\alpha-\delta})^k s_{\hat{\alpha}}(z) &= s_\alpha s_{\alpha-\delta}(s_\alpha s_{\alpha-\delta})^{k-1} s_{\hat{\alpha}}(z) \\
&= s_\alpha s_{\alpha-\delta} s_{\hat{\alpha}+(k-1)\delta}(z) \\
&= s_\alpha s_{\alpha-\delta} s_{\hat{\alpha}+(k-1)\delta}(i + nq) \\
&= s_\alpha s_{\alpha-\delta}\Big(j + n\big(q \pm (h + k - 1)\big)\Big) \\
&= s_\alpha\Big(i + n\big(q \pm (h + k - 1) \mp (-1)\big)\Big) \\
&= s_\alpha\Big(i + n\big(q \pm (h + k)\big)\Big) \\
&= j + n\big(q \pm (h + k)\big) \\
&= s_{\hat{\alpha}+k\delta}(i + nq) \\
&= s_{\hat{\alpha}+k\delta}(z)
\end{aligned}$$

and hence $s_{\hat{\alpha}+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_{\hat{\alpha}}$ for all integers $k \geq 0$. Moving on to equation (4.2), once again it is immediate that

$$s_{\hat{\alpha}-k\delta}(z) = z = (s_{\alpha-\delta} s_\alpha)^k s_{\hat{\alpha}}(z)$$

when $r \neq i, j$. Now taking $r = i$, for $k = 0$, we have

$$s_{\hat{\alpha}-0\delta} = s_{\hat{\alpha}} = e \cdot s_{\hat{\alpha}} = (s_{\alpha-\delta} s_\alpha)^0 s_{\hat{\alpha}}$$

Consequently, equation (4.2) holds for $k = 0$. Proceeding by induction on k, we obtain

$$
\begin{aligned}
(s_{\alpha-\delta}s_\alpha)^k s_{\hat{\alpha}}(z) &= s_{\alpha-\delta}s_\alpha (s_{\alpha-\delta}s_\alpha)^{k-1} s_{\hat{\alpha}}(z) \\
&= s_{\alpha-\delta}s_\alpha s_{\hat{\alpha}-(k-1)\delta}(z) \\
&= s_{\alpha-\delta}s_\alpha s_{\hat{\alpha}-(k-1)\delta}(i+nq) \\
&= s_{\alpha-\delta}s_\alpha \Big(j + n\big(q \pm (h-k+1)\big)\Big) \\
&= s_{\alpha-\delta} \Big(i + n\big(q \pm (h-k+1)\big)\Big) \\
&= j + n\big(q \pm (h-k+1) \pm (-1)\big) \\
&= j + n\big(q \pm (h-k)\big) \\
&= s_{\hat{\alpha}-k\delta}(i+nq) \\
&= s_{\hat{\alpha}-k\delta}(z)
\end{aligned}
$$

and therefore, $s_{\hat{\alpha}-k\delta} = (s_{\alpha-\delta}s_\alpha)^k s_{\hat{\alpha}}$ for all integers $k \geq 0$. □

Remark 4.10.3. We will make frequent use of a special case of Lemma 4.10.2 in which $\hat{\alpha} = \alpha \in \Phi$, specifically

$$s_{\alpha+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_\alpha,$$

where $k \geq 0$ is an integer and

$$s_{\alpha-k\delta} = (s_{\alpha-\delta}s_\alpha)^{k-1} s_{\alpha-\delta}$$

for any integer $k \geq 1$.

Finally, we will require the following equations in Section 5.8 to define the concept of a kite property.

Lemma 4.10.4. *Let $\alpha \in \Phi$. For any integer $k \geq 1$, we have*

$$s_{\alpha-\delta}s_\alpha = s_{\alpha-(k+1)\delta}s_{\alpha-k\delta}$$

and

$$s_\alpha s_{\alpha-\delta} = s_{\alpha+(k+1)\delta}s_{\alpha+k\delta}$$

Proof. These follow immediately from Remark 4.10.3. The equation

$$s_{\alpha-\delta}s_\alpha = s_{\alpha-(k+1)\delta}s_{\alpha-k\delta}$$

can be obtained from

$$s_{\alpha-(k+1)\delta} = (s_{\alpha-\delta}s_\alpha)^k s_{\alpha-\delta} = s_{\alpha-\delta}s_\alpha(s_{\alpha-\delta}s_\alpha)^{k-1}s_{\alpha-\delta} = s_{\alpha-\delta}s_\alpha s_{\alpha-k\delta}$$

and

$$s_\alpha s_{\alpha-\delta} = s_{\alpha+(k+1)\delta}s_{\alpha+k\delta}$$

comes from

$$s_{\alpha+(k+1)\delta} = (s_\alpha s_{\alpha-\delta})^{k+1}s_\alpha = s_\alpha s_{\alpha-\delta}(s_\alpha s_{\alpha-\delta})^k s_\alpha = s_\alpha s_{\alpha-\delta}s_{\alpha+k\delta}$$

□

4.11 The Bruhat-Chevalley Order on $\widehat{W}$ and Reflections

From Lemma 4.10.2, it should be apparent that alternating products of s_α and $s_{\alpha-\delta}$ will be important in what follows. Of particular interest to us are the relationships between $\widehat{T}$-fixed points of the form

$$u, \quad s_\alpha u, \quad s_{\alpha-\delta}s_\alpha u, \quad s_\alpha s_{\alpha-\delta}s_\alpha u, \quad s_{\alpha-\delta}s_\alpha s_{\alpha-\delta}s_\alpha u, \quad \ldots \tag{4.3}$$

or of the form

$$u, \quad s_{\alpha-\delta}u, \quad s_\alpha s_{\alpha-\delta}u, \quad s_{\alpha-\delta}s_\alpha s_{\alpha-\delta}u, \quad s_\alpha s_{\alpha-\delta}s_\alpha s_{\alpha-\delta}u, \quad \ldots \tag{4.4}$$

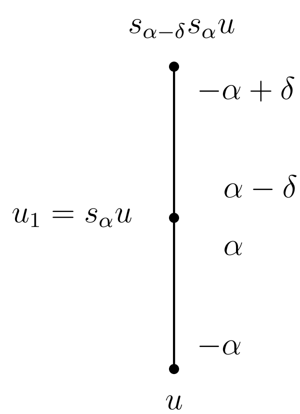
in terms of the Bruhat-Chevalley order on $\widehat{W}$. A useful tool in understanding such relationships is given in the following lemma:

Lemma 4.11.1. *Let $u \in \widehat{W}$ and let $\alpha \in \Phi^+$. If $u < s_\alpha u$, then $s_\alpha u < s_{\alpha-\delta}s_\alpha u$. If $u < s_{\alpha-\delta}u$, then $s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u$.*

Proof. Let $u_1 = s_\alpha u$, and suppose that $u < u_1$. Therefore, $u = s_\alpha^{-1}u_1 = s_\alpha u_1$ and so $s_\alpha u_1 < u_1$. Hence, by Lemma 4.3.3, $u_1^{-1}(\alpha) < 0$, from which we obtain that

$$u_1^{-1}(\alpha - \delta) = u_1^{-1}(\alpha) - \delta < 0$$

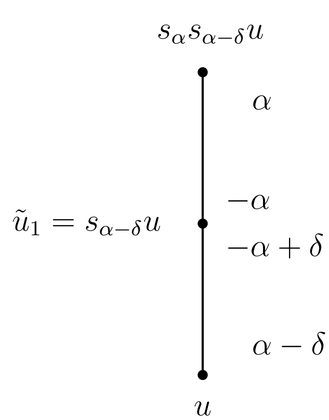
Consequently, since $\alpha - \delta < 0$, by Lemma 4.3.3, we have $u_1 < s_{a-\delta}u_1$, that is, $s_\alpha u < s_{\alpha-\delta}s_\alpha u$. The Bruhat graph for this case is:



Now let $\tilde{u}_1 = s_{\alpha-\delta}u$ and assume $u < \tilde{u}_1$. Thus, $u = s_{\alpha-\delta}^{-1}\tilde{u}_1 = s_{\alpha-\delta}\tilde{u}_1$, which yields $s_{\alpha-\delta}\tilde{u}_1 < \tilde{u}_1$. As $-\alpha + \delta > 0$, by Lemma 4.3.3,

$$(\tilde{u}_1)^{-1}(-\alpha + \delta) = (\tilde{u}_1)^{-1}(-\alpha) + \delta < 0,$$

which forces $(\tilde{u}_1)^{-1}(-\alpha) < 0$. The Bruhat graph for this situation is:



$\square$

To simplify matters, we introduce the following notation:

Let $u_0 = u$ and for $l \geq 1$ (an integer) set

$$u_l = \begin{cases} (s_\alpha s_{\alpha-\delta})^{\frac{l-1}{2}} s_\alpha u & \text{if } \; l \text{ is odd} \\ (s_{\alpha-\delta} s_\alpha)^{\frac{l}{2}} u & \text{if } \; l \text{ is even} \end{cases}$$

or recursively set

$$u_l = \begin{cases} s_\alpha u_{l-1} & \text{if } \; l \; \text{ is odd} \\ s_{\alpha-\delta} u_{l-1} & \text{if } \; l \; \text{ is even} \end{cases}$$

Similarly, let $\tilde{u}_0 = u$ and for $l \geq 1$ (an integer) set

$$\tilde{u}_l = \begin{cases} (s_{\alpha-\delta} s_\alpha)^{\frac{l-1}{2}} s_{\alpha-\delta} u & \text{if } \; l \; \text{ is odd} \\ (s_\alpha s_{\alpha-\delta})^{\frac{l}{2}} u & \text{if } \; l \; \text{ is even} \end{cases}$$

or recursively set

$$\tilde{u}_l = \begin{cases} s_{\alpha-\delta} \tilde{u}_{l-1} & \text{if } \; l \; \text{ is odd} \\ s_\alpha \tilde{u}_{l-1} & \text{if } \; l \; \text{ is even} \end{cases}$$

With this notation, Sequences (4.3) and (4.4) become

$$u_0, \;\; u_1, \;\; u_2, \;\; u_3, \;\; u_4, \;\; \ldots \tag{4.5}$$

and

$$\tilde{u}_0, \;\; \tilde{u}_1, \;\; \tilde{u}_2, \;\; \tilde{u}_3, \;\; \tilde{u}_4, \;\; \ldots \tag{4.6}$$

respectively.

Lemma 4.11.2. *Let $u \in \widehat{W}$ and let $\alpha \in \Phi^+$. Using the notation given above, if $u_i < u_{i+1}$, for some integer $i \geq 0$, then $u_l < u_k$ for all integers $i \leq l < k$, and if $\tilde{u}_j < \tilde{u}_{j+1}$, for some integer $j \geq 0$, then $\tilde{u}_l < \tilde{u}_k$ for all integers $j \leq l < k$.*

Proof. Suppose that $u_i < u_{i+1}$, for some integer $i \geq 0$. Repeated application of Lemma 4.11.1 yields the following:

$$u_i < u_{i+1} < u_{i+2} < u_{i+3} < u_{i+4} < u_{i+5} \cdots$$

In particular,

$$u_{i+a} < u_{i+b},$$

for any $0 \leq a < b$. So, if k, L are any integers such that $i \leq l < k$, then $l = i + a$,and $k = i + b$, for some $0 \leq a < b$, and hence

$$u_l < u_k,$$

as required.

Likewise, if $\tilde{u}_j < \tilde{u}_{j+1}$, for some integer $j \geq 0$, then repeated application of Lemma 4.11.1 gives us that

$$\tilde{u}_j < \tilde{u}_{j+1} < \tilde{u}_{j+2} < \tilde{u}_{j+3} < \tilde{u}_{j+4} < \tilde{u}_{j+5} < \cdots$$

Specifically, we have

$$\tilde{u}_{j+a} < \tilde{u}_{j+b}$$

for any $0 \leq a < b$. Once again, if k, L are any integers such that $j \leq l < k$, then $l = j + a$,and $k = j + b$, for some $0 \leq a < b$, and consequently

$$\tilde{u}_l < \tilde{u}_k,$$

as required. □

Sequences (4.5) and (4.6) *might* begin with a strictly decreasing portion, which must terminate since the Bruhat-Chevalley order on $\widehat{W}$ has the unique minimal element e. Thus we have:

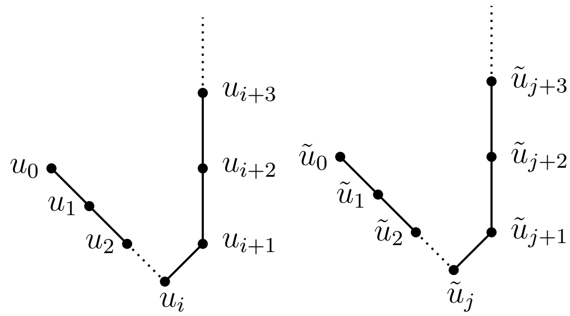
$$u_0 > u_1 > u_2 > \cdots > u_i \quad \text{and} \quad \tilde{u}_0 > \tilde{u}_1 > \tilde{u}_2 > \cdots > \tilde{u}_j$$

for some $i, j \geq 0$.

Thereafter, as indicated in Lemma 4.11.2, the sequences become strictly increasing:

$$u_i < u_{i+1} < u_{i+2} < \cdots \quad \text{and} \quad \tilde{u}_j < \tilde{u}_{j+1} < \tilde{u}_{j+2} < \cdots$$

Bruhat graphs depicting these two sequences are:



These descriptions were obtained by considering these sequences in isolation, however, we can determine even more about their behaviour if we consider their influence on each other. Indeed, we observe that

$$u^{-1}(\alpha) < 0 \implies u^{-1}(-\alpha) > 0 \implies u^{-1}(-\alpha+\delta) = u^{-1}(-\alpha) + \delta > 0$$

and

$$u^{-1}(-\alpha+\delta) = u^{-1}(-\alpha) + \delta < 0 \implies u^{-1}(-\alpha) < 0 \implies u^{-1}(\alpha) > 0.$$

Thus, by Lemma 4.3.3, we can not have both

$$u > s_\alpha u \quad \text{and} \quad u > s_{\alpha-\delta} u,$$

and hence we have:

Remark 4.11.3. Either

$$u < s_\alpha u \quad \text{or} \quad u < s_{\alpha-\delta} u,$$

inclusively.

Therefore, for a fixed u, at least one of (4.5) or (4.6) is strictly increasing.

4.12 More on the Bruhat-Chevalley Order on $\widehat{W}$ and Reflections

In this section, we closely examine relationships amongst the terms of Sequence (4.3) (or Sequence (4.5)) and relationships amongst the terms of Sequence (4.4) (or Sequence (4.6)) with respect to the Bruhat-Chevalley order on $\widehat{W}$. We present our results in several lemmas. These lemmas are of a technical nature and we created them to address specific situations that we encountered in our work.

Before we begin, we will make a comment regarding our notation. From Remark 4.10.3, we know that

$$s_{\alpha+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_\alpha \quad \text{and} \quad s_{\alpha-k\delta} = (s_{\alpha-\delta} s_\alpha)^{k-1} s_{\alpha-\delta}$$

Thus, in terms of Sequences (4.5) and (4.6), $s_{\alpha+k\delta}u = u_{2k+1}$ and $s_{\alpha-k\delta}u = \tilde{u}_{2k-1}$

Lemma 4.12.1. *Let $u \in \widehat{W}$ and let $\alpha \in \Phi^+$. Using the notation defined in Section 4.11, if $u < s_{\alpha+k\delta}u$, where $k \geq 0$ is an integer, then $u_l < s_{\alpha+k\delta}u$, for all integers l such that $0 \leq l \leq 2k$. If $u < s_{\alpha-k\delta}u$, where $k \geq 1$ is an integer, then $\tilde{u}_l < s_{\alpha-k\delta}u$, for all integers l where $0 \leq l \leq 2k-2$.*

Proof. Suppose that $u < s_{\alpha+k\delta}u$, for some integer $k \geq 0$. By Remark 4.10.3 we know that

$$s_{\alpha+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_\alpha.$$

Thus, for Sequence (4.5), we have:

$$u_0 = u, \;\; u_1 = s_\alpha u, \;\; u_2 = s_{\alpha-\delta} s_\alpha u, \; \ldots u_{2k} = (s_{\alpha-\delta} s_\alpha)^k u, \;\; u_{2k+1} = (s_\alpha s_{\alpha-\delta})^k s_\alpha u$$
$$= s_{\alpha+k\delta} u$$

As mentioned above, this sequence might begin with a strictly decreasing portion, but since

$$u_0 = u < s_{\alpha+k\delta}u = u_{2k+1},$$

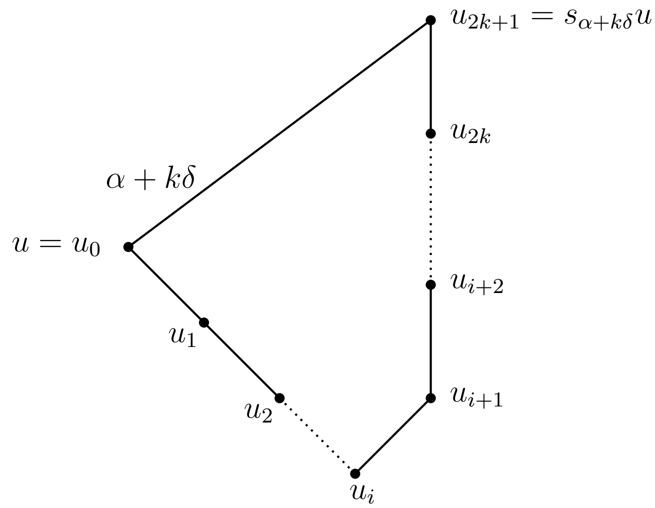
there is an index i, where $0 \leq i \leq 2k$, for which $u_i < u_{i+1}$ (take i to be the smallest such value). As a consequence of Lemma 4.11.2, the remainder of the sequence is strictly increasing. Thus, we have

$$u_0 > u_1 > u_2 > \cdots > u_i$$

(which consists only of u_0, if $i = 0$) followed by

$$u_i < u_{i+1} < \cdots < u_{2k} < u_{2k+1}$$

We see immediately that $u_l < u_{2k+1} = s_{\alpha+k\delta}u$, for all l with $i \leq l \leq 2k$. Furthermore, since $u < s_{\alpha+k\delta}u$, for any l such that $0 \leq l \leq i$, we also have $u_l < u_0 = u < s_{\alpha+k\delta}u$. Therefore, $u_l < s_{\alpha+k\delta}u$, for all l where $0 \leq l \leq 2k$. A Bruhat graph representing this situation is:



So now assume that $u < s_{\alpha-k\delta}u$, for some integer $k \geq 1$. This time, we obtain from Remark 4.10.3 that

$$s_{\alpha-k\delta} = (s_{\alpha-\delta}s_\alpha)^{k-1}s_{\alpha-\delta},$$

so that, for Sequence (4.6) we have:

$$\tilde{u}_0 = u, \quad \tilde{u}_1 = s_{\alpha-\delta}u, \quad \tilde{u}_2 = s_\alpha s_{\alpha-\delta}u, \quad \ldots \quad \tilde{u}_{2k-1} = (s_{\alpha-\delta}s_\alpha)^{k-1}s_{\alpha-\delta}u$$
$$= s_{\alpha-k\delta}u$$

Again, since

$$\tilde{u}_0 = u < s_{\alpha-k\delta}u = \tilde{u}_{2k-1},$$

there exists a smallest index j, satisfying $0 \leq j \leq 2k-2$, for which $\tilde{u}_j < \tilde{u}_{j+1}$.

Once more, we have

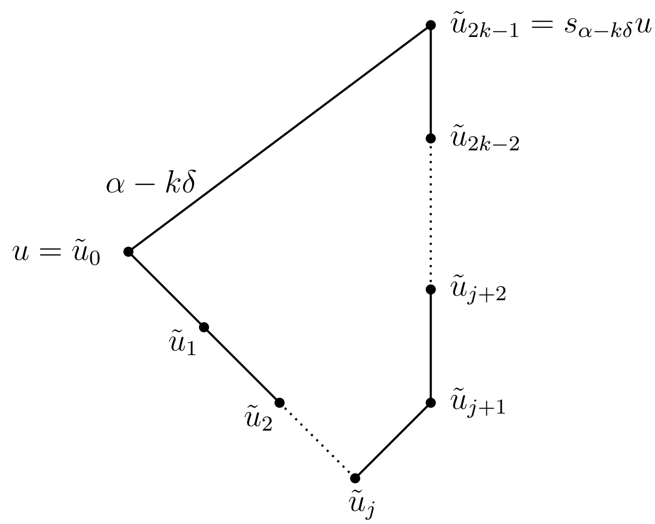
$$\tilde{u}_0 > \tilde{u}_1 > \tilde{u}_2 > \cdots > \tilde{u}_j$$

and

$$\tilde{u}_j < \tilde{u}_{j+1} < \cdots < \tilde{u}_{2k-2} < \tilde{u}_{2k-1}$$

At this point, by repeating the argument in the previous case, we obtain that $\tilde{u}_l < s_{\alpha-k\delta}u$, for all integers l where $0 \leq l \leq 2k-2$, as required.

A Bruhat graph depicting this scenario is:



□

Lemma 4.12.2. *Let $u \in \widehat{W}$ and let $\alpha \in \Phi^+$. If $u < s_{\alpha+k\delta}u$, where $k \geq 0$ is an integer, then $s_{\alpha+l\delta}u < s_{\alpha+k\delta}u$, for all integers l where $0 \leq l < k$, and $s_{\alpha+l\delta}s_{\alpha+k\delta}u < s_{\alpha+k\delta}u$, for all integers l where $0 \leq l \leq k$. If $u < s_{\alpha-k\delta}u$, for some integer $k \geq 1$, then $s_{\alpha-l\delta}u < s_{\alpha-k\delta}u$, for all integers l such that $1 \leq l < k$, and $s_{\alpha-l\delta}s_{\alpha-k\delta}u < s_{\alpha-k\delta}u$, for all integers l such that $1 \leq l \leq k$.*

Proof. Assume that $u < s_{\alpha+k\delta}u$, where $k \geq 0$ is an integer. Since, for $k = 0$, the first part of the statement for this case holds vacuously and the second part clearly holds, we will assume that $k \geq 1$.

Let l be an integer. If $l = k$, then

$$s_{\alpha+l\delta}s_{\alpha+k\delta}u = s_{\alpha+k\delta}s_{\alpha+k\delta}u = u < s_{\alpha+k\delta}u,$$

as required.

Now let $0 \leq l < k$. By Remark 4.10.3, we have

$$s_{\alpha+k\delta} = (s_\alpha s_{\alpha-\delta})^k s_\alpha \quad \text{and} \quad s_{\alpha+l\delta} = (s_\alpha s_{\alpha-\delta})^l s_\alpha$$

from which we can compute that

$$s_{\alpha+k\delta} = (s_\alpha s_{\alpha-\delta})^l (s_\alpha s_{\alpha-\delta})^{k-l} s_\alpha = (s_\alpha s_{\alpha-\delta})^l s_\alpha (s_{\alpha-\delta} s_\alpha)^{k-l} = s_{\alpha+l\delta}(s_{\alpha-\delta}s_\alpha)^{k-l}.$$

In particular, this means that

$$s_{\alpha+l\delta}s_{\alpha+k\delta}u = (s_{\alpha-\delta}s_\alpha)^{k-l}u.$$

Therefore, in regards to Sequence (4.5), we obtain that

$$s_{\alpha+k\delta}u = u_{2k+1},$$

$$s_{\alpha+l\delta}u = u_{2l+1},$$

and

$$s_{\alpha+l\delta}s_{\alpha+k\delta}u = u_{2(k-l)}.$$

Since $0 \leq l < k$, it is clear that $0 < 2l + 1, 2(k - l) < 2k + 1$. Thus, by applying Lemma 4.12.1, we determine that

$$u_{2l+1} < u_{2k+1} \quad \text{and} \quad u_{2(k-l)} < u_{2k+1},$$

so that,

$$s_{\alpha+l\delta}u < s_{\alpha+k\delta}u \quad \text{and} \quad s_{\alpha+l\delta}s_{\alpha+k\delta}u < s_{\alpha+k\delta}u.$$

Now suppose that $u < s_{\alpha-k\delta}u$, where $k \geq 1$ is an integer. Again, for $k = 1$, the first part of the statement for this case vacuously holds and the second part is obviously true. Hence, we will assume that $k \geq 2$.

This time, if $l = k$, then

$$s_{\alpha-l\delta}s_{\alpha-k\delta}u = s_{\alpha-k\delta}s_{\alpha-k\delta}u = u < s_{\alpha-k\delta}u,$$

as claimed.

Moreover, for any integer l where $1 \leq l < k$, Remark 4.10.3 gives us that

$$s_{\alpha-k\delta} = (s_{\alpha-\delta}s_{\alpha})^{k-1}s_{\alpha-\delta},$$

$$s_{\alpha-l\delta} = (s_{\alpha-\delta}s_{\alpha})^{l-1}s_{\alpha-\delta},$$

and

$$\begin{aligned} s_{\alpha-k\delta} = (s_{\alpha-\delta}s_{\alpha})^{l-1}(s_{\alpha-\delta}s_{\alpha})^{k-l}s_{\alpha-\delta} &= (s_{\alpha-\delta}s_{\alpha})^{l-1}s_{\alpha-\delta}(s_{\alpha}s_{\alpha-\delta})^{k-l} \\ &= s_{\alpha-l\delta}(s_{\alpha}s_{\alpha-\delta})^{k-l} \end{aligned}$$

From this, we see that $s_{\alpha-l\delta}s_{\alpha-k\delta}u = (s_{\alpha}s_{\alpha-\delta})^{k-l}u$. Hence, referring to Sequence (4.6), we determine that

$$s_{\alpha-k\delta}u = \tilde{u}_{2k-1},$$

$$s_{\alpha-l\delta}u = \tilde{u}_{2l-1},$$

and

$$s_{\alpha-l\delta}s_{\alpha-k\delta}u = \tilde{u}_{2(k-l)}.$$

Again, as $1 \leq l < k$, we know that $1 \leq 2l-1, 2(k-l) < 2k-1$. It now follows from Lemma 4.12.1 that

$$\tilde{u}_{2l-1} < \tilde{u}_{2k-1} \quad \text{and} \quad \tilde{u}_{2(k-l)} < \tilde{u}_{2k-1}$$

and hence

$$s_{\alpha-l\delta}u < s_{\alpha-k\delta}u \quad \text{and} \quad s_{\alpha-l\delta}s_{\alpha-k\delta}u < s_{\alpha-k\delta}u.$$

$\square$

Lemma 4.12.3. *If $u < s_{\alpha+i\delta}u$ for some integer $i \geq 0$, then $s_{\alpha+i\delta}u < s_{\alpha+k\delta}u$, for all integers $k > i$ and thus, in particular, $u < s_{\alpha+k\delta}u$, for all integers $k \geq i$. If $u < s_{\alpha-j\delta}u$ for some integer $j \geq 1$, then $s_{\alpha-j\delta}u < s_{\alpha-k\delta}u$ for all integers $k > j$, and thus, in particular, $u < s_{\alpha-k\delta}u$ for all integers $k \geq j$.*

Proof. Suppose that $u < s_{\alpha+i\delta}u$, for some integer $i \geq 0$, and let k be any integer such that $k \geq i$. The statement clearly holds for $k = i$, so assume $k > i$.

Using Lemma 4.12.2, we compute that

$$s_\alpha(s_{\alpha+i\delta}u) < s_{\alpha+i\delta}u.$$

Also, by Remark 4.10.3, we obtain that

$$s_{\alpha+i\delta} = (s_\alpha s_{\alpha-\delta})^i s_\alpha$$

and

$$s_\alpha s_{\alpha+i\delta} = s_\alpha(s_\alpha s_{\alpha-\delta})^i s_\alpha = (s_{\alpha-\delta}s_\alpha)^i.$$

Thus, regarded as terms of Sequence (4.5),

$$s_\alpha(s_{\alpha+i\delta}u) = u_{2i} \quad \text{and} \quad s_{\alpha+i\delta}u = u_{2i+1}.$$

Using this notation in our work above, we have that

$$u_{2i} < u_{2i+1}.$$

Thus, since $2i < 2i + 1 < 2k + 1$, according Lemma 4.11.2

$$u_{2i+1} < u_{2k+1}.$$

Consequently, since

$$u_{2k+1} = s_{\alpha+k\delta}u,$$

we in fact have

$$s_{\alpha+i\delta}u < s_{\alpha+k\delta}u,$$

which in turn implies that

$$u < s_{\alpha+k\delta}u.$$

Now instead suppose that $u < s_{\alpha-j\delta}u$, for some integer $j \geq 1$, and let k be any integer such that $k \geq j$. Again, as the statement clearly holds for $k = j$, assume $k > j$.

Lemma 4.12.2

$$s_{\alpha-\delta}\big(s_{\alpha-j\delta}u\big) < s_{\alpha-j\delta}u.$$

In addition, from Remark 4.10.3, we determine that

$$s_{\alpha-j\delta} = (s_{\alpha-\delta}s_{\alpha})^{j-1}s_{\alpha-\delta}$$

and

$$s_{\alpha-\delta}s_{\alpha-j\delta} = s_{\alpha-\delta}(s_{\alpha-\delta}s_{\alpha})^{j-1}s_{\alpha-\delta} = (s_{\alpha}s_{\alpha-\delta})^{j-1}.$$

Thus, referring to Sequence (4.6), we have

$$s_{\alpha-\delta}\big(s_{\alpha-j\delta}u\big) = u_{2j-2} \quad \text{and} \quad s_{\alpha-j\delta}u = u_{2j-1}.$$

Using this notation, our work above gives us that

$$u_{2j-2} < u_{2j-1}.$$

Hence, as $2j - 2 < 2j - 1 < 2k - 1$, Lemma 4.11.2 yields

$$u_{2j-1} < u_{2k-1}.$$

Thus,

$$s_{\alpha-j\delta}u < s_{\alpha-k\delta}u,$$

since

$$u_{2k-1} = s_{\alpha-k\delta}u,$$

and therefore, as required,

$$u < s_{\alpha-k\delta}u.$$

$\square$

Lemma 4.12.4. *Let $u \in \widehat{W}$ and let $\alpha \in \Phi^+$. If $u < s_{\alpha+k\delta}u$, where $k \geq 0$ is an integer, then $s_{\alpha+k\delta}u < s_{\alpha-l\delta}(s_{\alpha+k\delta}u)$ for all integers $l \geq 1$. If $u < s_{\alpha-k\delta}u$, where $k \geq 1$ is an integer, then $s_{\alpha-k\delta}u < s_{\alpha+l\delta}(s_{\alpha-k\delta}u)$ for all integers $l \geq 0$.*

Proof. Suppose that $u < s_{\alpha+k\delta}u$, where $k \geq 0$ is an integer. Hence, Lemma 4.12.2 gives us that

$$s_\alpha(s_{\alpha+k\delta}u) < s_{\alpha+k\delta}u.$$

This can be restated as

$$s_\alpha(s_{\alpha+k\delta}u) < s_\alpha\big(s_\alpha(s_{\alpha+k\delta}u)\big).$$

Applying Lemma 4.11.1 to this yields

$$s_\alpha\big(s_\alpha(s_{\alpha+k\delta}u)\big) < s_{\alpha-\delta}\Big(s_\alpha\big(s_\alpha(s_{\alpha+k\delta}u)\big)\Big)$$

which simplifies to

$$s_{\alpha+k\delta}u < s_{\alpha-\delta}(s_{\alpha+k\delta}u).$$

Hence, as a consequence of Lemma 4.12.3, we obtain that

$$s_{\alpha+k\delta}u < s_{\alpha-l\delta}(s_{\alpha+k\delta}u),$$

for all $l \geq 1$.

Now instead suppose that $u < s_{\alpha-k\delta}u$, where $k \geq 1$ is an integer. Thus, we can use Lemma 4.12.2 to compute that

$$s_{\alpha-\delta}(s_{\alpha-k\delta}u) < s_{\alpha-k\delta}u$$

which is the same as

$$s_{\alpha-\delta}(s_{\alpha-k\delta}u) < s_{\alpha-\delta}\big(s_{\alpha-\delta}(s_{\alpha-k\delta}u)\big).$$

From this, we are able to calculate, using Lemma 4.11.1, that

$$s_{\alpha-\delta}\big(s_{\alpha-\delta}(s_{\alpha-k\delta}u)\big) < s_\alpha\Big(s_{\alpha-\delta}\big(s_{\alpha-\delta}(s_{\alpha-k\delta}u)\big)\Big)$$

and so

$$s_{\alpha-k\delta}u < s_{\alpha}(s_{\alpha-k\delta}u).$$

To this, we apply Lemma 4.12.3 to obtain that

$$s_{\alpha-k\delta}u < s_{\alpha+l\delta}(s_{\alpha-k\delta}u),$$

for all $l \geq 0$. $\square$

4.13 The Bruhat-Chevalley Order on $\widehat{W}$ and $E^{-}(X(w), u)$

Based upon our results in the previous section, we have developed a tool for identifying elements of $E^{-}(X(w), u)$.

Lemma 4.13.1. *Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$, and let $\alpha \in \Phi^{+}$. If $\overline{\mathcal{U}_{\alpha-k\delta}u} \in E^{-}(X(w), u)$, for some integer $k \geq 2$, and $u < s_{\alpha-l\delta}u$, for some integer l with $1 \leq l < k$, then $\overline{\mathcal{U}_{\alpha-l\delta}u} \in E^{-}(X(w), u)$. If $\overline{\mathcal{U}_{-\alpha-k\delta}u} \in E^{-}(X(w), u)$, for some integer $k \geq 1$, and $u < s_{\alpha+l\delta}u$, for some integer l with $0 \leq l < k$, then $\overline{\mathcal{U}_{-\alpha-l\delta}u} \in E^{-}(X(w), u)$.*

Proof. Suppose that $\overline{\mathcal{U}_{\alpha-k\delta}u} \in E^{-}(X(w), u)$, for some integer $k \geq 2$, which, in particular, means that

$$u < s_{\alpha-k\delta}u \leq w.$$

Also suppose that $u < s_{\alpha-l\delta}u$, for some integer l with $1 \leq l < k$. Thus, by Lemma 4.12.2

$$s_{\alpha-l\delta}u < s_{\alpha-k\delta}u,$$

and hence

$$u < s_{\alpha-l\delta}u < w.$$

Therefore $\overline{\mathcal{U}_{\alpha-l\delta}u} \in E^{-}(X(w), u)$.

Now instead suppose that $\overline{\mathcal{U}_{-\alpha-k\delta}u} \in E^{-}(X(w), u)$, for some $k \geq 1$. Thus,

$$u < s_{\alpha+k\delta}u \leq w.$$

As well, suppose that $u < s_{\alpha+l\delta}u$, for some integer l with $0 \leq l < k$. This time, Lemma 4.12.2 yields that

$$s_{\alpha+l\delta}u < s_{\alpha+k\delta}u.$$

Consequently,

$$u < s_{\alpha+l\delta}u < w$$

and so $\overline{\mathcal{U}_{-\alpha-l\delta}u} \in E^-(X(w), u)$. $\square$

Chapter 5

Maximal Singularities of Schubert Varieties in $\mathcal{G}/\mathcal{B}$

In this chapter, we will present our work on maximal singularities of Schubert varieties $X(w)$ in $\mathcal{G}/\mathcal{B}$. Much of our work focuses on describing the set $E^{-}(X(w),u)$ for $u \in X(w)^{\widehat{T}}$ with the goal of understanding the nature of this set in the case that u is a maximal singularity.

In [11], we examined the two $\widehat{T}$-curves in the set $E(\Sigma,u)$ and their corresponding weights in the dual tangent space $T_u(\Sigma)^*$ for a $\widehat{T}$-surface Σ in $\mathcal{G}/\mathcal{P}$ and $u \in \Sigma^{\widehat{T}}$ (where u was assumed to be e). These results carry over to $\widehat{T}$-surfaces in $\mathcal{G}/\mathcal{B}$. In particular, we showed that the roots corresponding to the pair of $\widehat{T}$-curves in $E(\Sigma,u)$ satisfied one of two relationships when Σ is singular at u.

As a result of Lemma 5.1 in [15] (see Lemma 2.12.1 above), questions involving singularities of a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ can often be reduced to considering the $\widehat{T}$-surfaces contained in $X(w)$. Thus, the aforementioned relationships we found in [11] are relevant to the work in this thesis and, as such, we have introduced terminology to identify these relationships.

We refer to the pairs of roots which satisfy these relationships as either a Type I pair of roots or a Type II pair of roots and the corresponding $\widehat{T}$-curves are then either a Type I or a Type II pair of curves. (See Section 5.2 below).

Our work on this project began with some initial discussions with Andrew Crites regarding how to prove that the affine permutation w indexing a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$ contains the pattern 3412. The strategy discussed involved considering the presence of either a Type I or Type II pair of $\widehat{T}$-curves passing through (possibly non-maximal) $\widehat{T}$-fixed points. So when we commenced working on obtaining a description of the maximal singularities of a rationally smooth Schubert variety, we first considered the presence of either a Type I or Type II pair of $\widehat{T}$-curves passing through the maximal singularity. We soon realized that a specific case of a Type I pair of $\widehat{T}$-curves, which we have called a strong Type I pair, was key in obtaining a characterization of maximal singularities.

In this chapter, we will provide a partial characterization of the set $E^{-}(X(w), u)$, where $X(w)$ is an arbitrary singular Schubert variety in $\mathcal{G}/\mathcal{B}$ and the $\widehat{T}$-fixed point u is a maximal singularity of $X(w)$ (see Theorem 5.6.1 and Theorem 5.6.3). Furthermore, we will also provide necessary conditions for a $\widehat{T}$-fixed point of a rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ to be a maximal singularity (see Theorem 5.9.5 and Lemma 5.10.1).

5.1 Chain Properties

In Chapter 4 we spent a significant amount of time examining the behaviour of the Sequences (4.3)/(4.5) and (4.4)/(4.6). It should come as no surprise that they will play a vital role in this chapter. One of the reasons that these sequences are useful is that they can be used to detect singular $\widehat{T}$-fixed points in Schubert varieties. In order to make this relationship explicit, we begin with the following definitions:

Definition 5.1.1. Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$ and let $u \in X(w)^{\widehat{T}}$. Then u is said to satisfy the *$(-\alpha)$–chain property* in $X(w)$ if there is a root $\alpha \in \Phi^+$ such that

$$s_{\alpha-\delta}u < u < s_\alpha u < s_{\alpha-\delta}s_\alpha u \leq w.$$

Definition 5.1.2. Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$ and let $u \in X(w)^{\widehat{T}}$. Then u is said to satisfy the *$(\alpha - \delta)$-chain property* in $X(w)$ if there is a root $\alpha \in \Phi^+$ such that

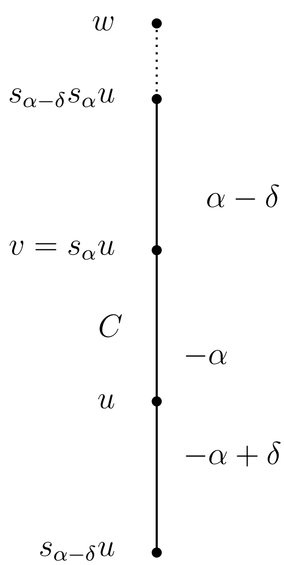
$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u \leq w.$$

Lemma 5.1.3. *Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$ and let $u \in X(w)^{\widehat{T}}$. If u satisfies either the $(-\alpha)$–chain property or the $(\alpha-\delta)$–chain property in $X(w)$, then u is a singular point of $X(w)$.*

Proof. Suppose that u satisfies the $(-\alpha)$–chain property in $X(w)$, for some $\alpha \in \Phi^+$, that is,

$$s_{\alpha-\delta}u < u < s_\alpha u < s_{\alpha-\delta}s_\alpha u \leq w.$$

Let $v := s_\alpha u$ and let $C := \overline{\mathcal{U}_{-\alpha}u} \in E^-(X(w), u)$. A Bruhat graph summarizing this case is:



If v is a singular point of $X(w)$, then by Lemma 4.7.3, u is also a singular point of $X(w)$.

So now assume that $X(w)$ is nonsingular at v. Thus, we know from Lemma 4.8.13 that $\alpha - \delta$ is a weight of $\tau_C\big(X(w), v\big)$. As such, by Lemma 4.8.10, $\Omega\Big(\tau_C\big(X(w), u\big)\Big)$ has at least one element which is in the α-string through $\alpha - \delta$. The candidates for this element are:

$$\alpha - \delta \qquad -\delta \qquad -\alpha - \delta$$

Since $s_{\alpha-\delta}u < u$, Lemma 4.8.12 specifies that

$$u^{-1}(-\alpha+\delta) < 0$$

As a result,

$$u^{-1}(\alpha-\delta) > 0$$

from which it follows that $\alpha-\delta$ is not a weight of $\tau_C\big(X(w),u\big)$, by Lemma 4.8.3. Thus, either $-\delta$ or $-\alpha-\delta$ is a weight of $\tau_C\big(X(w),u\big)$. However, if $-\alpha-\delta$ is a weight of $\tau_C\big(X(w),u\big)$, then by Lemma 4.8.7,

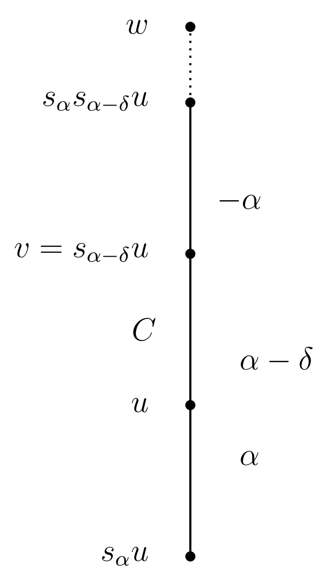
$$\alpha+(-\alpha-\delta) = -\delta \;\in\; \Omega\Big(\tau_C\big(X(w),u\big)\Big)$$

since $u^{-1}(-\delta) = -\delta < 0$. Consequently, $-\delta$ is a weight of $\tau_C\big(X(w),u\big)$. Thus, as a result of Lemma 4.8.3, we obtain that u is a singular point of $X(w)$.

Next suppose that u satisfies the $(\alpha-\delta)$–chain property in $X(w)$, so that

$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u \leq w$$

Relabel $v = s_{\alpha-\delta}u$ and $C := \overline{\mathcal{U}_{\alpha-\delta}u} \in E^-(X(w),u)$. A Bruhat graph representing this case is:



As before, if v is a singular point of $X(w)$, then u is also a singular point of $X(w)$ due to Lemma 4.7.3. So assume that $X(w)$ is nonsingular at v. Consequently, it follows from Lemma 4.8.13 that $-\alpha$ is a weight of $\tau_C\big(X(w),v\big)$. As a result, Lemma 4.8.10 guarantees that at least one member of the $(-\alpha+\delta)$-string through $-\alpha$ is a weight of $\tau_C\big(X(w),u\big)$. The possibilities for this weight are:

$$-\alpha \qquad -\delta \qquad \alpha - 2\delta$$

However, $s_\alpha u < u$, so that by Lemma 4.8.12

$$u^{-1}(\alpha) < 0$$

which means,

$$u^{-1}(-\alpha) > 0.$$

Therefore, $-\alpha$ is not in $\Omega\Big(\tau_C\big(X(w),u\big)\Big)$, by Lemma 4.8.3. Thus it remains to consider $-\delta$ and $\alpha-2\delta$. If $\alpha-2\delta$ is a weight of $\tau_C\big(X(w),u\big)$, then by Lemma 4.8.7,

$$(-\alpha+\delta)+(\alpha-2\delta) = -\delta \ \in \ \Omega\Big(\tau_C\big(X(w),u\big)\Big),$$

since $u^{-1}(-\delta) = -\delta < 0$. Thus, we must have $-\delta \in \Omega\Big(\tau_C\big(X(w),u\big)\Big)$ and once again, Lemma 4.8.3 gives us that u is a singular point of $X(w)$, as required. $\square$

5.2 Root / $\widehat{T}$-Curve Pair Types

Motivated by our work on $\widehat{T}$-surfaces and the pair of $\widehat{T}$-curves they contain through each $\widehat{T}$-fixed point (see [11] for the $\mathcal{G}/\mathcal{P}$ case and Section 4.9 for the $\mathcal{G}/\mathcal{B}$ case), we have identified four key types of pairs of $\widehat{T}$-curves which have proven useful in studying singularities of Schubert varieties.

The definition of each $\widehat{T}$-curve pair type involves first defining a type of pair of roots. That being the case, for the remainder of this section, we will let $\hat{\alpha} := \alpha + h_\alpha\delta$ and $\hat{\beta} := \beta + h_\beta\delta$ be elements of $\widehat{\Phi}^+$. We will also let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$ and $u \in X(w)^{\widehat{T}}$.

Definition 5.2.1. The pair $\hat{\alpha}, \hat{\beta} \in \Phi^+$ is said to be of *Type I* if $\beta = -\alpha$ and $h_\beta + h_\alpha \geq 2$. Thus,

$$\hat{\beta} = \beta + h_\beta\delta = -\alpha + h_\beta\delta = -\alpha - h_\alpha\delta + (h_\beta + h_\alpha)\delta = -\hat{\alpha} + (h_\beta + h_\alpha)\delta$$

Setting $k = h_\beta + h_\alpha$, we have that a Type I pair is one which satisfies

$$\hat{\beta} = -\hat{\alpha} + k\delta,$$

for some integer $k \geq 2$.

Furthermore, $\widehat{T}$-curves $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$ are said to be a *Type I pair* if $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha}$ and $\hat{\beta}$ form a Type I pair of roots.

Remark 5.2.2. When working with a Type I pair of roots, we will assume, without loss of generality, that $\alpha \in \Phi$ is positive. Therefore, in order for $\hat{\beta} = -\alpha + h_\beta\delta \in \widehat{\Phi}^+$, we require $h_\beta \geq 1$.

Definition 5.2.3. The pair $\hat{\alpha}, \hat{\beta} \in \Phi^+$ is said to be of *strong Type I* if it is a Type I pair, i.e. $\beta = -\alpha$ (where $\alpha > 0$) and $h_\beta + h_\alpha \geq 2$, with the added condition that $h_\alpha = 0$ or $h_\beta = 1$, exclusively. In other words, a strong Type I pair has either the form

$$\hat{\alpha} = \alpha \quad \text{and} \quad \hat{\beta} = -\alpha + k\delta, \quad \text{where} \quad k = h_\beta \geq 2$$

or

$$\hat{\alpha} = \alpha + k\delta \quad \text{and} \quad \hat{\beta} = -\alpha + \delta, \quad \text{where} \quad k = h_\alpha \geq 1.$$

The pair $\hat{\alpha}, \hat{\beta} \in \Phi^+$ is said to be of *weak Type I* if it is a Type I pair which is not strong. Explicitly, a weak Type I pair has the form $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = \beta + h_\beta\delta$, where $h_\alpha \geq 1$ and $h_\beta \geq 2$.

Moreover, $\widehat{T}$-curves $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$ are said to be a *strong Type I pair* (respectively, *weak Type I pair*) if $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha}$ and $\hat{\beta}$ form a strong Type I (respectively, weak Type I) pair of roots.

Definition 5.2.4. The pair $\hat{\alpha}, \hat{\beta} \in \Phi^+$ is said to be of *Type II* if $\beta = \alpha$ and $|h_\beta - h_\alpha| \geq 2$. Assuming that $h_\beta > h_\alpha$, we have

$$\hat{\beta} = \hat{\alpha} + k\delta,$$

for some integer $k \geq 2$. A pair of $\widehat{T}$-curves $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$ is said to be a *Type II pair* if $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}} u}$, where $\hat{\alpha}$ and $\hat{\beta}$ form a Type II pair of roots.

Definition 5.2.5. The pair $\hat{\alpha}, \hat{\beta} \in \Phi^+$ is said to be of *pseudo Type II* if $\beta = \alpha$ and $|h_\beta - h_\alpha| = 1$. Again, assuming that $h_\beta > h_\alpha$, a pseudo Type II pair $\hat{\alpha}, \hat{\beta}$ is one in which

$$\hat{\beta} = \hat{\alpha} + \delta.$$

A pair of $\widehat{T}$-curves $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$ is said to be a *pseudo Type II pair* if $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}} u}$, where $\hat{\alpha}$ and $\hat{\beta}$ form a pseudo Type II pair of roots.

Note that, unlike for a Type I pair of roots where we may assume that $\alpha > 0$, in the definitions of Type II and pseudo Type II root pairs, α may be positive or negative.

5.3 Pseudo Type II Pairs of Roots / $\widehat{T}$-Curves

Although the focus of this section is $\widehat{T}$-fixed points $u \in X(w)$ which have a pseudo Type II pair of $\widehat{T}$-curves in $E^-(X(w), u)$, we begin with an observation that applies to $\widehat{T}$-fixed points u for which $E^-(X(w), u)$ contains either a Type II or pseudo Type II pair.

Remark 5.3.1. If $E^-(X(w), u)$ contains a Type II or pseudo Type II pair of $\widehat{T}$-curves, then $X(w)$ is singular at u. Indeed, if $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$ are a Type II or pseudo Type II pair of $\widehat{T}$-curves, then $\hat{\alpha}, \hat{\beta} \in \Phi^+$ form a Type II or pseudo Type II pair of roots, that is, $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = \alpha + h_\beta\delta$, where $\alpha \in \Phi$ and $h_\beta - h_\alpha \geq 1$. As $-\hat{\alpha}, -\hat{\beta} \in \Omega\Big(T_u\big(X(w)\big)\Big)$, from Lemma 4.4.2, we obtain

$$\hat{\alpha} - \hat{\beta} = (h_\alpha - h_\beta)\delta = -(h_\beta - h_\alpha)\delta \in \Omega\Big(T_u\big(X(w)\big)\Big)$$

since $u^{-1}\big(-(h_\beta - h_\alpha)\delta\big) = -(h_\beta - h_\alpha)\delta < 0$. Subsequently, since $\Omega\Big(T_u\big(X(w)\big)\Big)$ contains an element of $\mathrm{Im}(\widehat{\Phi}^-)$, we obtain from Lemma 4.7.5 that $X(w)$ is singular at u.

Returning to the case that $E^-(X(w), u)$ contains a pseudo Type II pair of $\widehat{T}$-curves, even though u is a singular point of $X(w)$, such a u can not be a maximal singularity, as indicated in the following lemma. Note that in the proof of this lemma 5.3.2, we give an alternative argument showing that u is a singular point of $X(w)$.

Lemma 5.3.2. *If $u \in X(w)^{\widehat{T}}$ such that $E^-(X(w), u)$ contains a pseudo Type II pair of $\widehat{T}$-curves, then u is a singular point of $X(w)$ which is not a maximal singularity.*

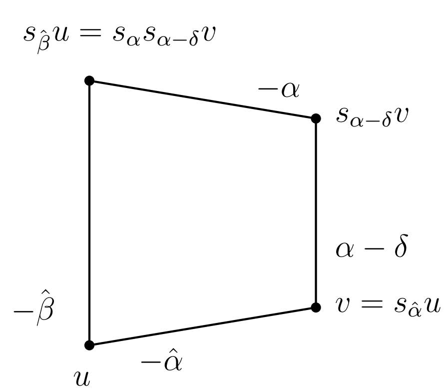
Proof. Let $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}} u}$ be a pseudo Type II pair of $\widehat{T}$-curves in $E^-(X(w), u)$. Thus $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = \hat{\alpha} + \delta$, where $\alpha \in \Phi$. We also know that

$$u < s_{\hat{\alpha}}u \leq w \quad \text{and} \quad u < s_{\hat{\beta}}u \leq w.$$

Let $v = s_{\hat{\alpha}}u$. Furthermore, Lemma 4.10.2 indicates that

$$s_{\hat{\beta}}u = s_\alpha s_{\alpha-\delta} s_{\hat{\alpha}} u = s_\alpha s_{\alpha-\delta} v.$$

Suppose that $\alpha \in \Phi^+$. Thus, $h_\alpha \geq 0$. A Bruhat graph representing this case is:



Since $u < v = s_{\alpha+h_\alpha\delta}u$, where $\alpha \in \Phi^+$ and $h_\alpha \geq 0$, we learn from Lemma 4.12.2 that

$$s_\alpha v = s_\alpha(s_{\alpha+h_\alpha\delta}u) < s_{\alpha+h_\alpha\delta}u = v$$

and we obtain from Lemma 4.12.4 that

$$v = s_{\alpha+h_\alpha\delta}u < s_{\alpha-\delta}(s_{\alpha+h_\alpha\delta}u) = s_{\alpha-\delta}v.$$

This in turn implies that

$$s_{\alpha-\delta}v < s_\alpha(s_{\alpha-\delta}v)$$

by Lemma 4.11.1. Collectively, we have determine that

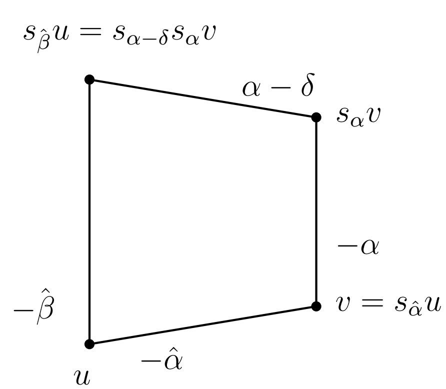
$$s_\alpha v < v < s_{\alpha-\delta}v < s_\alpha s_{\alpha-\delta}v \leq w.$$

Hence, v satisfies the $(\alpha-\delta)$–chain property in $X(w)$ and therefore, by Lemma 5.1.3, v is a singular point of $X(w)$. Thus, since $u < v$, we simultaneously deduce that u is a singular point of $X(w)$ (see Lemma 4.7.3) which is not a maximal singularity.

Now instead, suppose that $\alpha \in \Phi^-$, so that $h_\alpha \geq 1$. Relabel the roots, so that $\hat{\alpha} = -\alpha + h_\alpha\delta$ and $\hat{\beta} = \hat{\alpha} + \delta$, where $\alpha \in \Phi^+$. This time, with our relabeling, Lemma 4.10.2 and Remark 4.10.3 specify that

$$s_{\hat{\beta}}u = s_{-\alpha}s_{-\alpha-\delta}s_{\hat{\alpha}}u = s_\alpha s_{\alpha+\delta}s_{\hat{\alpha}}u = s_\alpha(s_\alpha s_{\alpha-\delta}s_\alpha)s_{\hat{\alpha}}u = s_{\alpha-\delta}s_\alpha s_{\hat{\alpha}}u = s_{\alpha-\delta}s_\alpha v.$$

A Bruhat graph illustrating this case is:



Since $u < v = s_{-\alpha+h_\alpha\delta}u = s_{\alpha-h_\alpha\delta}u$, where $\alpha \in \Phi^+$ and $h_\alpha \geq 1$, Lemma 4.12.2 tells us that

$$s_{\alpha-\delta}v = s_{\alpha-\delta}(s_{\alpha-h_\alpha\delta}u) < s_{\alpha-h_\alpha\delta}u = v$$

and Lemma 4.12.4 gives us that

$$v = s_{\alpha-h_\alpha\delta}u < s_\alpha(s_{\alpha-h_\alpha\delta}u) = s_\alpha v.$$

Applying Lemma 4.11.1 to this yields

$$s_\alpha v < s_{\alpha-\delta}(s_\alpha v).$$

In summary, we have obtained that

$$s_{\alpha-\delta}v < v < s_\alpha v < s_{\alpha-\delta}s_\alpha v \leq w.$$

Thus we have shown that v satisfies the $(-\alpha)$–chain property in $X(w)$ and hence, by Lemma 5.1.3, we know that v is a singular point of $X(w)$. Therefore, since $u < v$, we determine not only that u is a singular point of $X(w)$ (via Lemma 4.7.3), but also that u is not a maximal singularity. $\square$

Corollary 5.3.3. *If $u \in X(w)^{\widehat{T}}$ for which $E^-(X(w), u)$ contains a pseudo Type II pair of $\widehat{T}$-curves, then the codimension of $X(u)$ in $X(w)$ is at least 3.*

Proof. From the proof of Lemma 5.3.2, we have either

$$u < v < s_{\alpha-\delta}v < s_\alpha s_{\alpha-\delta}v \leq w$$

or

$$u < v < s_\alpha v < s_{\alpha-\delta}s_\alpha v \leq w.$$

Subsequently, $\ell(w) - \ell(u) \geq 3$ and so the codimension of $X(u)$ in $X(w)$ is at least 3. $\square$

5.4 Type II Pairs of Roots / $\widehat{T}$-Curves

We now consider the case in which the set $E^{-}(X(w), u)$ contains a Type II pair of $\widehat{T}$-curves. We showed in Remark 5.3.1 that such a u is a singular point of $X(w)$. As with the pseudo Type II case, a $\widehat{T}$-fixed point u for which $E^{-}(X(w), u)$ contains a Type II pair of $\widehat{T}$-curves can not be a maximal singularity.

Lemma 5.4.1. *If $u \in X(w)^{\widehat{T}}$ such that $E^{-}(X(w), u)$ contains a Type II pair of $\widehat{T}$-curves, then u is a singular point of $X(w)$ which is not a maximal singularity. In addition, there exists $u' > u \in X(w)^{\widehat{T}}$ for which $E^{-}(X(w), u')$ contains a pseudo Type II pair of $\widehat{T}$-curves.*

Proof. Suppose that $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^{-}(X(w), u)$ which form a Type II pair. As such,

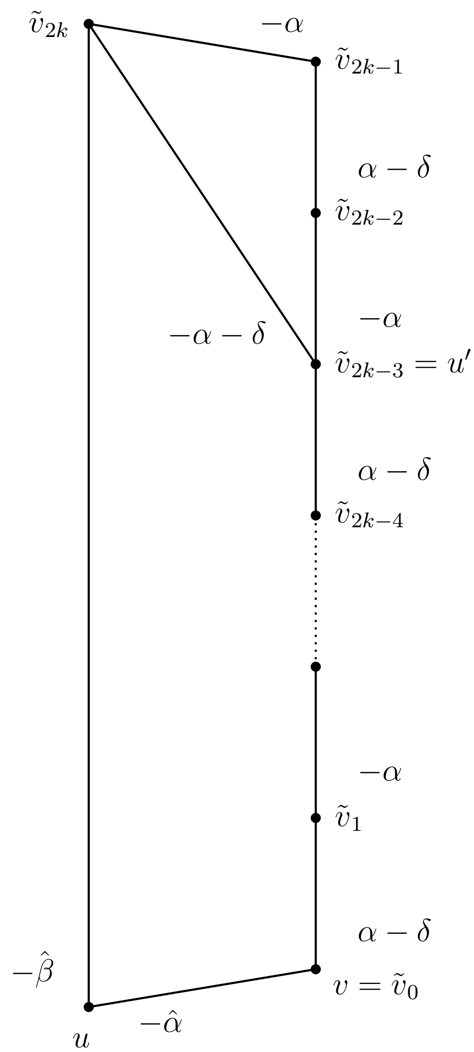
$$u < s_{\hat{\alpha}}u, s_{\hat{\beta}}u \leq w$$

and $\hat{\alpha}, \hat{\beta} \in \widehat{\Phi}^{+}$ for which $\hat{\alpha} = \alpha + h_{\alpha}\delta$ and $\hat{\beta} = \hat{\alpha} + k\delta$, for some integer $k \geq 2$.

Let $v = s_{\hat{\alpha}}u$. Using Lemma 4.10.2, we compute that

$$s_{\hat{\beta}}u = s_{\hat{\alpha}+k\delta}u = (s_{\alpha}s_{\alpha-\delta})^{k}s_{\hat{\alpha}}u = (s_{\alpha}s_{\alpha-\delta})^{k}v.$$

We will first consider the case in which $\alpha > 0$, so that $h_{\alpha} \geq 0$. In order to give a general overview of the proof, we begin with a Bruhat graph which represents this case. The technical details of the proof will follow the graph.



We will construct an instance of Sequence (4.6) starting with v: let $\tilde{v}_0 = v$, let $\tilde{v}_1 = s_{\alpha-\delta}v$ and, for $2 \leq j \leq 2k$, set

$$\tilde{v}_j = \begin{cases} (s_\alpha s_{\alpha-\delta})^{\frac{j}{2}}v & \text{if} \quad j \text{ is even} \\ (s_{\alpha-\delta}s_\alpha)^{\frac{j-1}{2}}s_{\alpha-\delta}v & \text{if} \quad j \text{ is odd} \end{cases}$$

or recursively, for $1 \leq j \leq 2k$, set

$$\tilde{v}_j = \begin{cases} s_\alpha \tilde{v}_{j-1} & \text{if} \quad j \text{ is even} \\ s_{\alpha-\delta}\tilde{v}_{j-1} & \text{if} \quad j \text{ is odd} \end{cases}$$

In particular, $\tilde{v}_{2k} = (s_\alpha s_{\alpha-\delta})^k v = (s_\alpha s_{\alpha-\delta})^k s_{\hat{\alpha}} u = s_{\hat{\beta}} u$.

Since $u < v = s_{\alpha+h_\alpha\delta}u$, where $\alpha \in \Phi^+$ and $h_\alpha \geq 0$, Lemma 4.12.4 establishes that

$$v = s_{\alpha+h_\alpha\delta}u < s_{\alpha-\delta}(s_{\alpha+h_\alpha\delta}u) = s_{\alpha-\delta}v,$$

or, in terms of our instance of Sequence (4.6),

$$\tilde{v}_0 < \tilde{v}_1.$$

Thus, as a result of Lemma 4.11.2, the entire sequence is strictly increasing and hence we have

$$u < v = \tilde{v}_0 < \tilde{v}_1 < \tilde{v}_2 < \cdots < \tilde{v}_{2k-3} < \tilde{v}_{2k-2} < \tilde{v}_{2k-1} < \tilde{v}_{2k} = s_{\hat{\beta}}u \leq w. \quad (5.1)$$

As a consequence, $\tilde{v}_j \in X(w)$, for $0 \leq j \leq 2k$ and since $k \geq 2$, we are guaranteed that $\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_4 \in X(w)$. In particular, we have that

$$v = s_{\alpha-\delta}\tilde{v}_1 < \tilde{v}_1 < s_\alpha\tilde{v}_1 = \tilde{v}_2 < s_{\alpha-\delta}s_\alpha\tilde{v}_1 = \tilde{v}_3 \leq w$$

which means that $\tilde{v}_1$ satisfies the $(-\alpha)$–chain property in $X(w)$. Thus, by Lemma 5.1.3, $\tilde{v}_1$ is a singular point of $X(w)$. Furthermore, since $u < \tilde{v}_1$, we know that u is a singular point of $X(w)$, from Lemma 4.7.3, that is not a maximal singularity.

In addition, as $k \geq 2$, so that $1 \leq 2k-3 \leq 2k$, we know $\tilde{v}_{2k-3} \in X(w)$ and $\tilde{v}_{2k-3} > u$. We claim that $E^-(X(w), \tilde{v}_{2k-3})$ contains a pseudo Type II pair. To prove this claim, we first note that, by definition,

$$\tilde{v}_{2k-2} = s_\alpha\tilde{v}_{2k-3} \quad \text{and} \quad \tilde{v}_{2k} = s_\alpha s_{\alpha-\delta}s_\alpha\tilde{v}_{2k-3},$$

but, since

$$s_{\alpha+\delta} = s_\alpha s_{\alpha-\delta}s_\alpha$$

(see Remark 4.10.3), we also have

$$\tilde{v}_{2k} = s_{\alpha+\delta}\tilde{v}_{2k-3}.$$

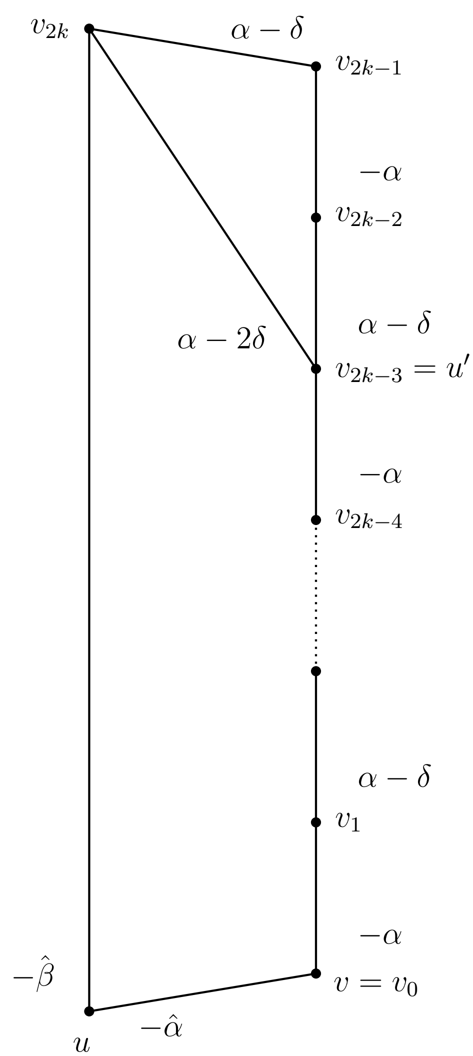
Hence, in view of (5.1), we obtain

$$\tilde{v}_{2k-3} < s_\alpha\tilde{v}_{2k-3} < s_{\alpha+\delta}\tilde{v}_{2k-3} \leq w.$$

Therefore, the $\widehat{T}$-curves $C = \overline{\mathcal{U}_{-\alpha}\tilde{v}_{2k-3}}$ and $D = \overline{\mathcal{U}_{-\alpha-\delta}\tilde{v}_{2k-3}}$ are elements of $E^{-}(X(w), \tilde{v}_{2k-3})$ which form a pseudo Type II pair (since α and $\alpha+\delta$ form a pseudo Type II pair of roots). Hence, in the case for which $\alpha > 0$, we can take u' in the statement of this Lemma to be the point $\tilde{v}_{2k-3}$.

We note that by Lemma 5.3.2, $u' = \tilde{v}_{2k-3}$ is singular point of $X(w)$ and hence, since $u < u'$, we could have used u' instead of $\tilde{v}_1$ to show that u is a non-maximal singularity of $X(w)$.

Now consider the case in which $\alpha < 0$. We first relabel so that $\alpha > 0$, making $\hat{\alpha} = -\alpha + h_\alpha\delta$ and $\hat{\beta} = \hat{\alpha} + k\delta$, where $h_\alpha \geq 1$ and $k \geq 2$. Once again, we provide a Bruhat graph depicting the entire proof:



Taking into account our relabeling, Lemma 4.10.2 and Remark 4.10.3 give us that

$$s_{\hat{\beta}}u = (s_{-\alpha}s_{-\alpha-\delta})^k s_{\hat{\alpha}}u = (s_\alpha s_{\alpha+\delta})^k s_{\hat{\alpha}}u = \big(s_\alpha(s_\alpha s_{\alpha-\delta}s_\alpha)\big)^k s_{\hat{\alpha}}u = (s_{\alpha-\delta}s_\alpha)^k s_{\hat{\alpha}}u$$
$$= (s_{\alpha-\delta}s_\alpha)^k v.$$

This time, we construct an instance of Sequence (4.5) starting with v: let $v_0 = v$, set $v_1 = s_\alpha v$ and, for $2 \leq i \leq 2k$, define

$$v_i = \begin{cases} (s_{\alpha-\delta}s_\alpha)^{\frac{j}{2}}v & \text{if } \ i \ \text{ is even} \\ (s_\alpha s_{\alpha-\delta})^{\frac{j-1}{2}}s_\alpha v & \text{if } \ i \ \text{ is odd} \end{cases}$$

or recursively, for $1 \leq i \leq 2k$, define

$$v_i = \begin{cases} s_{\alpha-\delta}v_{i-1} & \text{if } \ i \ \text{ is even} \\ s_\alpha v_{i-1} & \text{if } \ i \ \text{ is odd} \end{cases}$$

Note that $v_{2k} = (s_{\alpha-\delta}s_\alpha)^k v = (s_{\alpha-\delta}s_\alpha)^k s_{\hat{\alpha}}u = s_{\hat{\beta}}u$.

Since $u < v = s_{-\alpha+h_\alpha\delta} = s_{\alpha-h_\alpha\delta}u$, where $\alpha \in \Phi^+$ and $h_\alpha \geq 1$, we learn from Lemma 4.12.4 that

$$v = s_{\alpha-h_\alpha\delta}u < s_\alpha\Big(s_{\alpha-h_\beta\delta}u\Big) = s_\alpha v,$$

which means, in regards to our instance of Sequence (4.5), we have

$$v_0 < v_1.$$

As such, it follows from Lemma 4.11.2 that

$$u < v = v_0 < v_1 < v_2 < \cdots < v_{2k-3} < v_{2k-2} < v_{2k-1} < v_{2k} = s_{\hat{\beta}}u \leq w. \quad (5.2)$$

Thus, $v_i \in X(w)$, for $0 \leq i \leq 2k$. As $k \geq 2$, we are assured that $v_0, v_1, \ldots, v_4 \in X(w)$. Therefore, we have established that

$$v = s_\alpha v_1 < v_1 < s_{\alpha-\delta}v_1 = v_2 < s_\alpha s_{\alpha-\delta}v_1 = v_3 \leq w$$

and consequently, v_1 satisfies the $(\alpha-\delta)$–chain property in $X(w)$. Hence, v_1 is a singular point of $X(w)$, by Lemma 5.1.3. Moreover, as $u < v_1$, we deduce that u is a non-maximal singular point of $X(w)$ (see Lemma 4.7.3).

Additionally, since $k \geq 2$, so that $1 \leq 2k-3 \leq 2k$, we have $v_{2k-3} \in X(w)$ and $v_{2k-3} > u$.

Claim: $E^{-}(X(w), v_{2k-3})$ contains a pseudo Type II pair. To prove this, we first note that by definition,

$$v_{2k-2} = s_{\alpha-\delta} v_{2k-3} \quad \text{and} \quad v_{2k} = s_{\alpha-\delta} s_{\alpha} s_{\alpha-\delta} v_{2k-3},$$

and since

$$s_{\alpha-2\delta} = s_{\alpha-\delta} s_{\alpha} s_{\alpha-\delta}$$

(see Remark 4.10.3), we in fact have

$$v_{2k} = s_{\alpha-2\delta} v_{2k-3}.$$

It follows from (5.2) that

$$v_{2k-3} < s_{\alpha-\delta} v_{2k-3} < s_{\alpha-2\delta} v_{2k-3} \leq w$$

Thus, the$\widehat{T}$-curves $C = \overline{\mathcal{U}_{\alpha-\delta} v_{2k-3}}$ and $D = \overline{\mathcal{U}_{\alpha-2\delta} v_{2k-3}}$ are elements of $E^{-}(X(w), v_{2k-3})$ which form a pseudo Type II pair (since $-\alpha+\delta$ and $-\alpha+2\delta$ form a pseudo Type II pair of roots). Therefore, in the second case, we can take u' in the statement of this Lemma to be the point v_{2k-3}.

As in the first case, we know from Lemma 5.3.2 that $u' = v_{2k-3}$ is singular point of $X(w)$ and hence, since $u < u'$, we could have used u' instead of v_1 to show that u is a non-maximal singularity of $X(w)$. $\square$

Corollary 5.4.2. *If $u \in X(w)^{\widehat{T}}$ for which $E^{-}(X(w), u)$ contains a Type II pair of $\widehat{T}$-curves, then the codimension of $X(u)$ in $X(w)$ is at least 5.*

Proof. Using the notation established in the proof of Lemma 5.4.1, since $k \geq 2$, $v_{2k} \geq v_4$ and $\tilde{v}_{2k} \geq \tilde{v}_4$, and hence we have either

$$u < v_0 < v_1 < v_2 < v_3 < v_4 \leq v_{2k} \leq w$$

or

$$u < \tilde{v}_0 < \tilde{v}_1 < \tilde{v}_2 < \tilde{v}_3 < \tilde{v}_4 \leq \tilde{v}_{2k} \leq w.$$

Consequently, $\ell(w) - \ell(u) \geq 5$ and so the codimension of $X(u)$ in $X(w)$ is at least 5. $\square$

5.5 Weak Type I Pairs of Roots / $\widehat{T}$-Curves

At this point, we move on to considering the case in which $E^{-}(X(w), u)$ contains a weak Type I pair of $\widehat{T}$-curves.

Lemma 5.5.1. *If $u \in X(w)^{\widehat{T}}$ such that $E^{-}(X(w), u)$ contains a weak Type I pair of $\widehat{T}$-curves, then u is a singular point of $X(w)$, which is not a maximal singularity.*

Proof. Assume that $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}} u}$ are a weak Type I pair $\widehat{T}$-curves contained in $E^{-}(X(w), u)$. Thus, $\hat{\alpha} = \alpha + h_\alpha \delta$ and $\hat{\beta} = -\alpha + h_\beta \delta$, where $\alpha \in \Phi^{+}$, $h_\alpha \geq 1$, and $h_\beta \geq 2$. We also know that $u < s_{\hat{\alpha}} u \leq w$ and $u < s_{\hat{\beta}} u \leq w$. Furthermore, Remark 4.11.3 tells us that we must have at least one of

$$u < s_\alpha u \quad \text{or} \quad u < s_{\alpha - \delta} u.$$

If $u < s_\alpha u$, then since

$$C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}} u} = \overline{\mathcal{U}_{-\alpha - h_\alpha \delta} u} \in E^{-}(X(w), u),$$

where $h_\alpha \geq 1$, Lemma 4.13.1 indicates that

$$\overline{\mathcal{U}_{-\alpha} u} \in E^{-}(X(w), u).$$

As $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\alpha - h_\alpha \delta} u}$ and $\overline{\mathcal{U}_{-\alpha} u}$ form either a pseudo Type II or Type II pair of $\widehat{T}$-curves, we obtain from Lemmas 5.3.2 and 5.4.1 that u is a non-maximal singularity of $X(w)$.

If instead $u < s_{\alpha-\delta}u$, then using the fact that

$$C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}}u} = \overline{\mathcal{U}_{\alpha-h_\beta\delta}u} \in E^-(X(w), u),$$

where $h_\beta \geq 2$, we determine that

$$\overline{\mathcal{U}_{\alpha-\delta}u} \in E^-(X(w), u),$$

by Lemma 4.13.1. Hence, since $C_{-\hat{\beta}} = \overline{\mathcal{U}_{\alpha-h_\beta\delta}u}$ and $\overline{\mathcal{U}_{\alpha-\delta}u}$ form either a pseudo Type II or Type II pair of $\widehat{T}$-curves, Lemmas 5.3.2 and 5.4.1 once again yield that u is a non-maximal singularity of $X(w)$. □

5.6 Maximal Singularities and $E^-(X(w), u)$

In this section, we begin an analysis of the set $E^-(X(w), u)$, where a $\widehat{T}$-fixed point u is a maximal singularity of $X(w)$. In our efforts to characterize what is contained in $E^-(X(w), u)$, we have considered what is not contained in $E^-(X(w), u)$.

Theorem 5.6.1. *Let $X(w)$ be a singular Schubert variety in $\mathcal{G}/\mathcal{B}$. If the point $u \in X(w)^{\widehat{T}}$ is a maximal singularity of $X(w)$, then $E^-(X(w), u)$ does not contain a weak Type I, pseudo Type II, or Type II pair of $\widehat{T}$-curves.*

Proof. This follows immediately from Lemma 5.5.1, Lemma 5.3.2, and Lemma 5.4.1. □

Of the four types of $\widehat{T}$-curve pairs defined in Section 5.2, only strong Type I pairs of $\widehat{T}$-curves remain as candidates to appear in the set $E^-(X(w), u)$ for a maximal singularity u. We have determined that, for maximal singularities, the presence of a strong Type I pair of $\widehat{T}$-curves in $E^-(X(w), u)$ impacts what other $\widehat{T}$-curves may appear in $E^-(X(w), u)$, as specified in the following lemma:

Lemma 5.6.2. *Let $X(w)$ be a singular Schubert variety in $\mathcal{G}/\mathcal{B}$ and let the point $u \in X(w)^{\widehat{T}}$ be a maximal singularity of $X(w)$. If $E^-(X(w),u)$ contains a pair of strong Type I $\widehat{T}$-curves $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha} = \alpha + h_\alpha\delta$ and $\hat{\beta} = -\alpha + h_\beta\delta$, for some $\alpha \in \Phi^+$, then $E^-(X(w),u)$ does not contain any $\widehat{T}$-curves, other than $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$, of the form $\overline{\mathcal{U}_{-\hat{\gamma}}u}$, where $\hat{\gamma} = \pm\alpha + h_\gamma\delta \in \widehat{\Phi}^+$.*

Proof. Suppose that $E^-(X(w),u)$ contains a $\widehat{T}$-curve $C_{-\hat{\gamma}} = \overline{\mathcal{U}_{-\hat{\gamma}}u}$, where $\hat{\gamma} = \pm\alpha + h_\gamma\delta \in \widehat{\Phi}^+$, but $\hat{\gamma} \neq \hat{\alpha}, \hat{\beta}$.

Since $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ are a pair of strong Type I $\widehat{T}$-curves, there are two possibilities for $\hat{\alpha}$ and $\hat{\beta}$: $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, for some $k \geq 2$, or $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 1$.

First, assume that $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, where $k \geq 2$.

If $\hat{\gamma} = \alpha + h_\gamma\delta$, then $h_\gamma \geq 1$, since $\hat{\gamma} \neq \hat{\alpha}$. As such, $\hat{\beta}$ and $\hat{\gamma}$ form a weak Type I pair of roots and hence $E^-(X(w),u)$ contains the weak Type I pair of $\widehat{T}$-curves $C_{-\hat{\beta}}$ and $C_{-\hat{\gamma}}$. However, since u is a maximal singularity, Theorem 5.6.1 stipulates that $E^-(X(w),u)$ cannot contain a pair of weak Type I $\widehat{T}$-curves, and hence we obtain a contradiction.

Thus $\hat{\gamma} = -\alpha + h_\gamma\delta$, where $h_\gamma \geq 1$, but $h_\gamma \neq k$, since $\hat{\gamma} \neq \hat{\beta}$. Accordingly, $\hat{\beta}$ and $\hat{\gamma}$ form either a pseudo Type II or a Type II pair of roots and hence $E^-(X(w),u)$ contains the pseudo Type II or Type II pair $C_{-\hat{\beta}}$ and $C_{-\hat{\gamma}}$. This again leads to a contradiction, since the presence of a pair of pseudo Type II or Type II $\widehat{T}$-curves in $E^-(X(w),u)$ is impossible, by Theorem 5.6.1.

Now instead suppose that $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 1$.

Since $\hat{\gamma} \neq \hat{\alpha}$, if $\hat{\gamma} = \alpha + h_\gamma\delta$, then $h_\gamma \geq 0$ and $h_\gamma \neq k$. As a result, $\hat{\alpha}$ and $\hat{\gamma}$ form either a pseudo Type II or a Type II pair of roots, which means that $C_{-\hat{\alpha}}$ and $C_{-\hat{\gamma}}$ are a pair of pseudo Type II or Type II $\widehat{T}$-curves contained in $E^-(X(w),u)$. Once again, we have a contradictions, since $E^-(X(w),u)$ cannot contain a pair of weak Type I $\widehat{T}$-curves, due to Theorem 5.6.1.

Therefore, $\hat{\gamma} = -\alpha + h_\gamma\delta$, where $h_\gamma \geq 2$, since $\hat{\gamma} \neq \hat{\beta}$. That being so, $\hat{\alpha}$ and $\hat{\gamma}$ form a weak Type I pair of roots and, as a consequence, the weak Type I pair of $\widehat{T}$-curves $C_{-\hat{\alpha}}$ and $C_{-\hat{\gamma}}$ appear in $E^-(X(w),u)$. However, this is a contradiction since Theorem 5.6.1 specifies that $E^-(X(w),u)$ does not contain such a pair of $\widehat{T}$-curves. $\square$

In the following theorem, we prove that for each a $\alpha \in \Phi^+$, the set $E^-(X(w),u)$ contains at most two $\widehat{T}$-curves whose tangent spaces at a $\widehat{T}$-fixed point u have weights with real part $\pm\alpha$. Furthermore, if two such $\widehat{T}$ are present, then they form either a strong Type I pair or the weights of tangent spaces at u are $-\alpha$ and $\alpha - \delta$.

Theorem 5.6.3. *Let $X(w)$ be a singular Schubert variety in $\mathcal{G}/\mathcal{B}$ and let the point $u \in X(w)^{\widehat{T}}$ be a maximal singularity of $X(w)$. Then for each $\alpha \in \Phi^+$, the set $E^-(X(w),u)$ contains at most two $\widehat{T}$-curves of the form $\overline{\mathcal{U}_{-\hat{\alpha}}u}$, where $\hat{\alpha} = \pm\alpha + h_\alpha\delta \in \widehat{\Phi}^+$. Furthermore, if $E^-(X(w),u)$ does contain two $\widehat{T}$-curves $\overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $\overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha} = \pm\alpha + h_\alpha\delta$ and $\hat{\beta} = \pm\alpha + h_\beta\delta$ are positive roots, for some $\alpha \in \Phi^+$, then one of the following holds:*

1) *$\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, where $k \geq 1$*

2) *$\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 0$*

Proof. Let $\alpha \in \Phi^+$ and suppose that $C_{-\hat{\alpha}} := \overline{\mathcal{U}_{-\hat{\alpha}}u}$, $C_{-\hat{\beta}} := \overline{\mathcal{U}_{-\hat{\beta}}u}$, $C_{-\hat{\gamma}} := \overline{\mathcal{U}_{-\hat{\gamma}}u}$ are distinct $\widehat{T}$-curves in $E^-(X(w),u)$ such that $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma} \in \widehat{\Phi}^+$ are distinct roots with

$$\mathrm{Re}(\hat{\alpha}), \mathrm{Re}(\hat{\beta}), \mathrm{Re}(\hat{\gamma}) \in \{\pm\alpha\}.$$

We will first consider the case in which all three roots have real part α. To that end, let $\hat{\alpha} = \alpha + h_\alpha\delta$, $\hat{\beta} = \alpha + h_\beta\delta$, and $\hat{\gamma} = \alpha + h_\gamma\delta$, where $h_\alpha, h_\beta, h_\gamma \geq 0$ are three distinct integers. Without loss of generality, assume $0 \leq h_\alpha < h_\beta < h_\gamma$. Thus, $h_\gamma - h_\alpha \geq 2$ and hence $\hat{\alpha}$ and $\hat{\gamma}$ form a Type II pair. Therefore, $E^-(X(w),u)$ contains the Type II pair of $\widehat{T}$-curves $C_{-\hat{\alpha}}$ and $C_{-\hat{\gamma}}$, which is contradiction, since, by Theorem 5.6.1, the set $E^-(X(w),u)$, for a maximal singularity u, does not contain a Type II pair of $\widehat{T}$-curves. Using an identical argument, we also obtain a contradiction in the case in which $\hat{\alpha} = -\alpha + h_\alpha\delta$, $\hat{\beta} = -\alpha + h_\beta\delta$, and $\hat{\gamma} = -\alpha + h_\gamma\delta$, where $h_\alpha, h_\beta, h_\gamma \geq 0$ are three distinct integers.

We will now consider the case in which two of the roots have real part α and the other has real part $-\alpha$. Accordingly, suppose that $\hat{\alpha} = \alpha + h_\alpha\delta$,

$\hat{\beta} = \alpha + h_\beta\delta$, and $\hat{\gamma} = -\alpha + h_\gamma\delta$, where $h_\alpha, h_\beta \geq 0$ are distinct integers and $h_\gamma \geq 1$ is an integer. Without loss of generality, assume that $0 \leq h_\alpha < h_\beta$ and hence $h_\beta \geq 1$. Therefore $h_\beta + h_\gamma \geq 2$ and thus $C_{-\hat{\beta}}$ and $C_{-\hat{\gamma}}$ form a Type I pair of $\widehat{T}$-curves. If they form a weak Type I pair, then we obtain a contradiction with Theorem 5.6.1, which states that $E^-(X(w), u)$ does not contain a weak Type I pair of $\widehat{T}$-curves. If they form a strong Type I pair, then a contradiction stems from Lemma 5.6.2, which indicates that $E^-(X(w), u)$ cannot contain $C_{-\hat{\alpha}}$ as well as the strong Type I pair $C_{-\hat{\beta}}$ and $C_{-\hat{\gamma}}$.

The case in which two of the roots have real part $-\alpha$ and the other has real part α is very similar to the previous case. If $\hat{\alpha} = -\alpha + h_\alpha\delta$, $\hat{\beta} = -\alpha + h_\beta\delta$, and $\hat{\gamma} = \alpha + h_\gamma\delta$, where $1 \leq h_\alpha < h_\beta$ are distinct integers and $h_\gamma \geq 0$ is an integer, then $C_{-\hat{\beta}}$ and $C_{-\hat{\gamma}}$ still form a Type I pair, since $h_\beta \geq 2$ and so $h_\beta + h_\gamma \geq 2$. A contradiction now ensues, as above.

Therefore, $E^-(X(w), u)$ contains at most two $\widehat{T}$-curves $\overline{\mathcal{U}_{-\hat{\alpha}}u}$ and $\overline{\mathcal{U}_{-\hat{\beta}}u}$, where $\hat{\alpha} = \pm\alpha + h_\alpha\delta$ and $\hat{\beta} = \pm\alpha + h_\beta\delta$, for some $\alpha \in \Phi^+$.

So now suppose that $E^-(X(w), u)$ does contain two such $\widehat{T}$-curves. Thus, $\hat{\alpha}$ and $\hat{\beta}$ are either a weak Type I pair, a strong Type I pair, a pseudo Type II pair, a Type II pair, or the pair α and $-\alpha + \delta$. However, since $E^-(X(w), u)$ cannot contain a weak Type I pair, a pseudo Type II pair, or a Type II pair of $\widehat{T}$-curves, according to Theorem 5.6.1, $\hat{\alpha}$ and $\hat{\beta}$ must form either a strong Type I pair or be the pair α and $-\alpha + \delta$. □

5.7 Strong Type I Pairs of Roots / $\widehat{T}$-Curves

In this section, we will show that the set $E^-(X(w), u)$, for a rationally smooth Schubert variety $X(w)$ and a maximal singularity u, contains a strong Type I pair of $\widehat{T}$-curves. To do this, we first prove the following result:

Lemma 5.7.1. *Let Σ be a $\widehat{T}$-surface in $\mathcal{G}/\mathcal{B}$, let $u \in \Sigma^{\widehat{T}}$, and let C and D be the two $\widehat{T}$-curves in $E(\Sigma, u)$. Let $\hat{\alpha} := \alpha + h_\alpha\delta$ and $\hat{\beta} := \beta + h_\beta\delta$ be elements of $u\left(\widehat{\Phi}^+\right)$ such that $T_u(C) = \hat{\mathfrak{g}}_{-\hat{\alpha}}$ and $T_u(D) = \hat{\mathfrak{g}}_{-\hat{\beta}}$. If $\hat{\alpha}, \hat{\beta} \in \widehat{\Phi}^+$ and if Σ is singular at u, then $\hat{\alpha}$ and $\hat{\beta}$ form a Type I or Type II pair of roots.*

Proof. As indicated in Theorem 4.9.2, if $\beta \neq \pm\alpha$, if $\beta = \alpha$ and $|h_\beta - h_\alpha| = 1$, or if $\beta = -\alpha$ and $h_\beta + h_\alpha = 1$, then Σ is nonsingular at u. Thus, for Σ to be singular at u, either we have $|h_\beta - h_\alpha| \geq 2$, when $\beta = \alpha$, or we have $h_\alpha + h_\beta \geq 2$, when $\beta = -\alpha$. In other words, if $\hat{\alpha}, \hat{\beta} \in \widehat{\Phi}^+$, then $\hat{\alpha}$ and $\hat{\beta}$ form a Type II or Type I pair of roots, respectively. $\square$

Theorem 5.7.2. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$. If $u \in X(w)^{\widehat{T}}$ is a maximal singularity, then $E^-(X(w), u)$ contains a strong Type I pair of $\widehat{T}$-curves.*

Proof. Since $X(w)$ is nonsingular at w and in codimension 1, u must be in codimension 2 or higher. Since $X(w)$ is rationally smooth, we know from Remark 4.6.2 that $|E(X(w), u)| = \dim X(w)$ and

$$|E^-(X(w), u)| = \dim X(w) - \dim X(u) \geq 2.$$

Let $C \in E^-(X(w), u)$. Then $C = \overline{\mathcal{U}_{-\hat{\alpha}} u}$, for some real $\hat{\alpha} \in \widehat{\Phi}^+$. Since $u < s_{\hat{\alpha}} u$ and u is a maximal singularity of $X(w)$, we know from Remark 4.7.4 that C is good. Thus

$$\dim \tau_C\big(X(w), u\big) = \dim X(w) = |E(X(w), u)| = \dim TE\big(X(w), u\big).$$

Since $X(w)$ is singular at u and $X(w)$ is Cohen-Macaulay, it follows from Theorem 2.11.5 that

$$TE\big(X(w), u\big) \neq \tau_C\big(X(w), u\big)$$

and as a result

$$TE\big(X(w), u\big) \nsubseteq \tau_C\big(X(w), u\big)$$

since they have the same dimension. By Lemma 4.8.5, for any $D \in E^+(X(w), u)$,

$$T_u(D) \subseteq \tau_C\big(X(w), u\big)$$

and, as a consequence, there exists $D \in E^-(X(w), u)$ such that

$$T_u(D) \nsubseteq \tau_C\big(X(w), u\big).$$

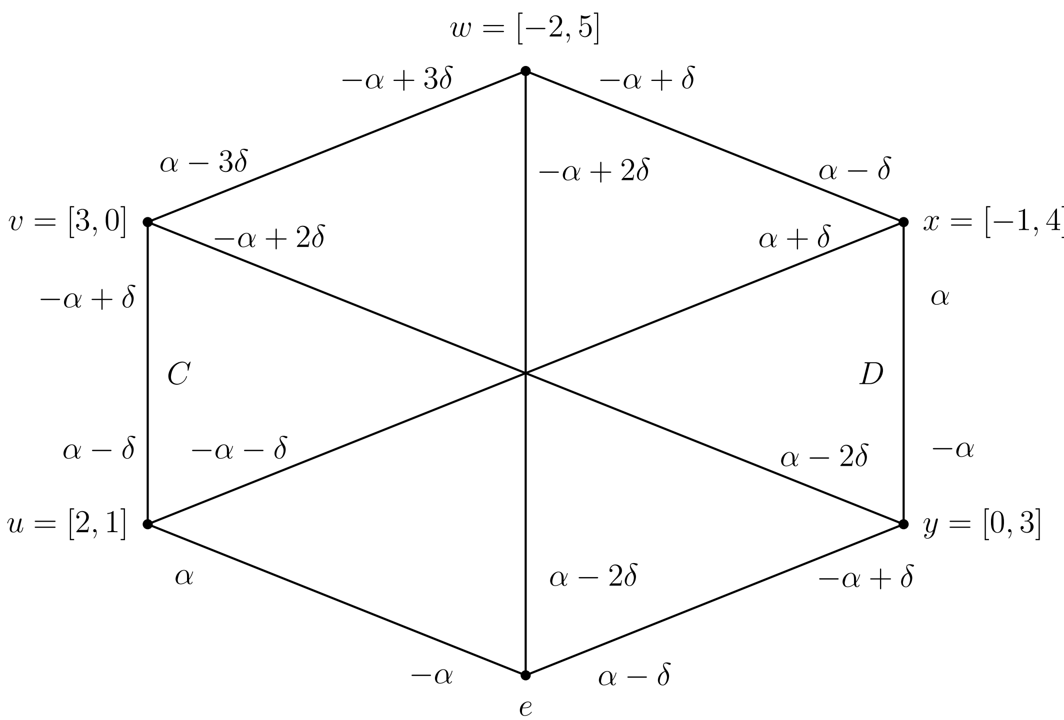
Let $D = \overline{\mathcal{U}_{-\hat{\beta}} u}$, where $\hat{\beta} \in \widehat{\Phi}^{+}$ is real. Again, as u is a maximal singularity of $X(w)$, Remark 4.7.4 indicates that the $\widehat{T}$-curve D is good. According to Lemma 4.9.1, there exists a $\widehat{T}$-surface $\Sigma \in \Sigma\big(X(w), u\big)$ which contains C and D. If Σ is nonsingular at u, then, in conjunction with Lemma 2.12.1, we obtain

$$T_u(D) \subseteq T_u(\Sigma) = \tau_C(\Sigma, u) \subseteq \tau_C\big(X(w), u\big)$$

Hence, as this contradicts the statement above, u is a singular point of Σ. It now follows from Lemma 5.7.1, that $\hat{\alpha}$ and $\hat{\beta}$ form a Type I or Type II pair of roots. However, as $C, D \in E^{-}(X(w), u)$, Theorem 5.6.1 establishes that C and D cannot be a weak Type I or a Type II pair. As a result, C and D must form a strong Type I pair, as required. $\square$

Unfortunately, for a rationally smooth Schubert variety $X(w)$, the presence of a strong Type I pair in $E^{-}(X(w), u)$ is a necessary, but not sufficient condition for a $\widehat{T}$-fixed point u to be a maximal singularity. In fact, the presence of a strong Type I pair of $\widehat{T}$-curves in $E^{-}(X(w), u)$ does not even guarantee that the point u is singular, as demonstrated in the following example:

Example 5.7.3. Consider the situation of Example 4.8.11:



The set $E^-(X(w),u)$ contains the strong Type I pair of $\widehat{T}$-curves $\overline{\mathcal{U}_{\alpha-\delta}u}$ and $\overline{\mathcal{U}_{-\alpha-\delta}u}$ and yet, as shown in Example 4.8.11, $X(w)$ is nonsingular at u.

5.8 Kite Properties

Based upon our work in previous section, it is clear that conditions in addition to the presence of a strong Type I pair of $\widehat{T}$-curves in $E^-(X(w),u)$ are required to locate maximal singularities of rationally smooth Schubert varieties. Some of these conditions come in the form of the following two definitions.

Definition 5.8.1. Let $\alpha \in \Phi^+$. A $\widehat{T}$-fixed point u in $\mathcal{G}/\mathcal{B}$ is said to satisfy the $(-\alpha)$*–kite property* if

$$s_{\alpha-\delta}u < u < s_\alpha u < s_{\alpha-\delta}s_\alpha u$$

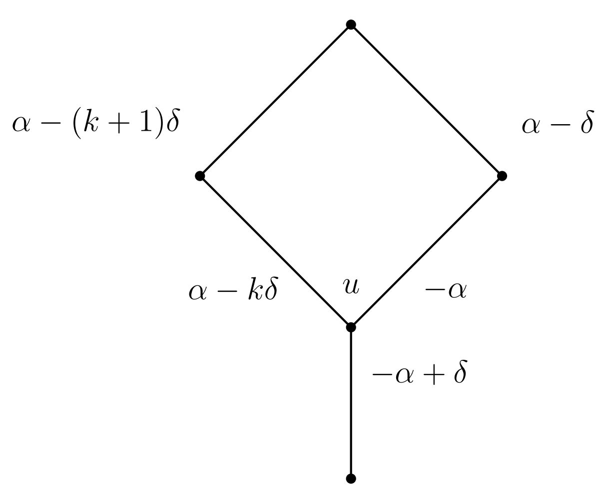
and

$$u < s_{\alpha-k\delta}u < s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u)$$

for some integer $k \geq 2$. These conditions are referred to as the $(-\alpha)$*–kite property through* u. By Lemma 4.10.4, we have that

$$s_{\alpha-\delta}s_\alpha u = s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u)$$

and hence the Bruhat graph illustrating the $(-\alpha)$–kite property through u demonstrates a kite-shaped pattern:



If $u \in X(w)$, then we say u satisfies the $(-\alpha)$*–kite property* in $X(w)$ if

$$s_{\alpha-\delta}u < u < s_{\alpha}u < s_{\alpha-\delta}s_{\alpha}u \leq w$$

and

$$u < s_{\alpha-k\delta}u < s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u) \leq w$$

for some integer $k \geq 2$.

We note that if a point u satisfies the $(-\alpha)$–kite property in $X(w)$, then the $\widehat{T}$-curves $\overline{\mathcal{U}_{-\alpha}u}$ and $\overline{\mathcal{U}_{\alpha-k\delta}u}$ appear in $E^{-}(X(w),u)$. Hence, in this situation, we are working a strong Type I pair of roots $\hat{\alpha}, \hat{\beta}$ of the form $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, where $k \geq 2$.

Definition 5.8.2. Let $\alpha \in \Phi^{+}$. A $\widehat{T}$-fixed point u in $\mathcal{G}/\mathcal{B}$ is said to satisfy the $(\alpha-\delta)$*–kite property* if

$$s_{\alpha}u < u < s_{\alpha-\delta}u < s_{\alpha}s_{\alpha-\delta}u$$

and

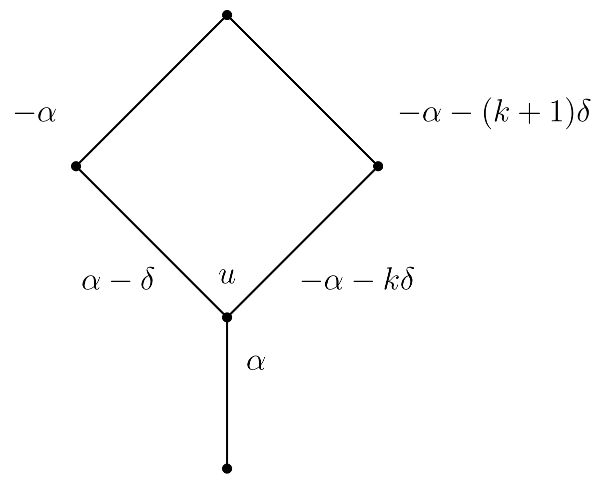
$$u < s_{\alpha+k\delta}u < s_{\alpha+(k+1)\delta}(s_{\alpha+k\delta}u)$$

for some integer $k \geq 1$.

These conditions are referred to as the $(\alpha-\delta)$*–kite property through* u. Since

$$s_{\alpha}s_{\alpha-\delta}u = s_{\alpha+(k+1)\delta}(s_{\alpha+k\delta}u)$$

(see Lemma 4.10.4), the Bruhat graph showing the $(\alpha-\delta)$–kite property through u forms a kite-shaped pattern:



If $u \in X(w)$, then we say u satisfies the $(\alpha-\delta)$–*kite property* in $X(w)$ if

$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u \leq w$$

and

$$u < s_{\alpha+k\delta}u < s_{\alpha+(k+1)\delta}(s_{\alpha+k\delta}u) \leq w$$

for some integer $k \geq 1$.

It is worth noting that if a $\widehat{T}$-fixed point u satisfies the $(\alpha-\delta)$–kite property in $X(w)$, then the $\widehat{T}$-curves $\overline{\mathcal{U}_{\alpha-\delta}u}$ and $\overline{\mathcal{U}_{\alpha+k\delta}u}$, where $k \geq 1$, are contained in $E^-(X(w),u)$ and thus, in this scenario, we are dealing with a strong Type I pair $\hat{\alpha}, \hat{\beta}$ in which $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha+\delta$.

Remark 5.8.3. As is evident from these definitions, if $u \in X(w)^{\widehat{T}}$ satisfies one of the two kite properties in $X(w)$, then it also satisfies one of the chain properties. The $(-\alpha)$–kite property through u incorporates the $(-\alpha)$–chain property:

$$s_{\alpha-\delta}u < u < s_\alpha u < s_{\alpha-\delta}s_\alpha u \leq w$$

and the $(\alpha-\delta)$–kite property through u includes the $(\alpha-\delta)$–chain property:

$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u \leq w.$$

As a consequence, we know from Lemma 5.1.3 that any $u \in X(w)$ which fulfills the conditions of either kite property is a singular point of $X(w)$. Hence we have the following result:

Lemma 5.8.4. *If $u \in X(w)^{\widehat{T}}$ which satisfies either the $(-\alpha)$–kite property in $X(w)$ or $(\alpha - \delta)$– kite property in $X(w)$, for some $\alpha \in \Phi^+$, then u is a singular point of $X(w)$.*

5.9 Kite Properties at Maximal Singularities

In this section, we will prove that every maximal singularity of a singular rationally smooth Schubert variety $X(w)$ satisfies either the $(-\alpha)$–kite property in $X(w)$ or $(\alpha - \delta)$– kite property in $X(w)$, for some $\alpha \in \Phi^+$. To that end, we start with the following lemma.

Lemma 5.9.1. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$ and let $u \in X(w)^{\widehat{T}}$. Suppose that u is a maximal singularity such that $E^-(X(w), u)$ contains a pair of strong Type I $\widehat{T}$-curves $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$. Then, in addition to $-\hat{\alpha}$, the Peterson translate $\tau_{C_{-\hat{\alpha}}}\big(X(w), u\big)$ has a weight $-\hat{\gamma} = -\hat{\alpha} - l\delta$, $-l\delta$, or $-\hat{\beta} - l\delta$, for some integer $l \geq 1$, but does not have $-\hat{\beta}$ as a weight. Likewise, in addition to $-\hat{\beta}$, the Peterson translate $\tau_{C_{-\hat{\beta}}}\big(X(w), u\big)$ has a weight $-\hat{\gamma} = -\hat{\alpha} - l\delta$, $-l\delta$, or $-\hat{\beta} - l\delta$, for some integer $l \geq 1$, but does not have $-\hat{\alpha}$ as a weight.*

Proof. Since $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ form a strong Type I pair of $\widehat{T}$-curves, we have $\hat{\alpha} = \alpha + h_\alpha \delta$ and $\hat{\beta} = -\alpha + h_\beta \delta$ are elements of $\widehat{\Phi}^+$ where $\alpha \in \Phi^+$ and $h_\beta + h_\alpha \geq 2$ such that either $h_\alpha = 0$ or $h_\beta = 1$, exclusively. To simplify notation, let $C = C_{-\hat{\alpha}}$ and $D = C_{-\hat{\beta}}$. Since u is a maximal singularity, both C and D are good according to Remark 4.7.4. In particular, this means that

$$\dim \tau_C\big(X(w), u\big) = \dim X(w).$$

Furthermore, as $X(w)$ is rationally smooth,

$$\dim X(w) = |E(X(w), u)| = \dim TE\big(X(w), u\big)$$

and hence we have established that

$$\dim \tau_C\bigl(X(w), u\bigr) = \dim TE\bigl(X(w), u\bigr).$$

However, it follows from Theorem 2.11.5 that

$$TE\bigl(X(w), u\bigr) \neq \tau_C\bigl(X(w), u\bigr),$$

which in turn indicates that

$$TE\bigl(X(w), u\bigr) \nsubseteq \tau_C\bigl(X(w), u\bigr)$$

for dimension reasons. Therefore, there exists $\widetilde{C} \in E(X(w), u)$ such that

$$T_u(\widetilde{C}) \nsubseteq \tau_C\bigl(X(w), u\bigr)$$

Clearly, $\widetilde{C} \neq C$. Moreover, from Lemma 4.8.5 we determine that $\widehat{C} \in E^-(X(w), u)$. Let $-\hat{\nu} \in \widehat{\Phi}^-$ be the weight of $T_u(\widetilde{C})$. According to Lemma 4.9.4, if $\hat{\nu} = \nu + h_\nu \delta$, where $\nu \neq \pm\alpha$, then $-\hat{\nu}$ is a weight of $\tau_C\bigl(X(w), u\bigr)$. As a result, $\hat{\nu}$ must have the form $\hat{\nu} = \pm\alpha + h_\nu \delta$. It now follows from Lemma 5.6.2 that $\widetilde{C} = D$ and hence $-\hat{\beta}$ is not a weight of $\tau_C\bigl(X(w), u\bigr)$, required.

Also, we know from Lemma 4.9.1 that there exists a $\widehat{T}$-surface $\Sigma \in \Sigma\bigl(X(w), u\bigr)$ containing C and D. We will now focus our attention on the weights of $\tau_C(\Sigma, u)$. First of all, by construction,

$$\tau_C(\Sigma, u) \subseteq T_u(\Sigma)$$

so that, in particular, any weight of $\tau_C(\Sigma, u)$ is also a weight $T_u(\Sigma)$. Secondly, Lemma 2.12.1 gives us that

$$\tau_C(\Sigma, u) \subseteq \tau_C\bigl(X(w), u\bigr).$$

This not only tells us that $-\hat{\beta}$ is not a weight of $\tau_C(\Sigma, u)$, but also that Σ must be singular at u, since otherwise

$$T_u(D) \subseteq T_u(\Sigma) = \tau_C(\Sigma, u) \subseteq \tau_C\bigl(X(w), u\bigr).$$

An immediate result is that $\dim T_u(\Sigma) \geq 3$. Thirdly, since

$$\dim \tau_C(\Sigma, u) \geq \dim \Sigma = 2$$

we know that $\tau_C(\Sigma, u)$ has at least one weight (not equal to $-\hat{\beta}$) in addition to $-\hat{\alpha}$. As this weight is also a weight of $T_u(\Sigma)$, we learn from Lemma 4.9.3 that it is a negative weight , say $-\hat{\gamma}$, which is equal to

$$-\hat{\alpha} - l\delta, \qquad -l\delta, \qquad \text{or} \quad -\hat{\beta} - l\delta,$$

for some integer $l \geq 1$. Finally, as $-\hat{\gamma}$ is a weight of $\tau_C(\Sigma, u)$, it is also a weight of $\tau_{C_{-\hat{\alpha}}}\big(X(w), u\big)$, as required. This completes the proof of the first claim in the statement of this lemma.

Furthermore, by relabeling $C = C_{-\hat{\beta}}$ and $D = C_{-\hat{\alpha}}$ and then repeating the above argument, we also obtain that $\tau_{C_{-\hat{\beta}}}\big(X(w), u\big)$ does not have $-\hat{\alpha}$ as a weight, but does have a weight $-\hat{\gamma} = \hat{\alpha} + l\delta$, $l\delta$, or $\hat{\beta} + l\delta$, for some integer $l \geq 1$, which is not equal to $-\hat{\beta}$. $\square$

In the following two lemmas, we will prove that, for a maximal singularity u, if $E^-(X(w), u)$ contains a strong Type I pair of $\widehat{T}$-curves, then u satisfies at least one of the two kite properties in $X(w)$. Note, however, that this is true for any singular Schubert variety $X(w)$. The condition that $X(w)$ be rationally smooth is not required.

Lemma 5.9.2. *Let $X(w)$ be a singular Schubert variety in $\mathcal{G}/\mathcal{B}$. Suppose that $u \in X(w)^{\widehat{T}}$ is a maximal singularity such that $E^-(X(w), u)$ contains a pair of strong Type I $\widehat{T}$-curves $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ for which $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, where $k \geq 2$. Then u satisfies the $(-\alpha)$–kite property in $X(w)$.*

Proof. Let $v = s_{\hat{\alpha}} u = s_\alpha u$, let $y = s_{\hat{\beta}} u = s_{\alpha - k\delta} u$, and let $x = s_{\alpha-\delta} s_\alpha u$. We first note that $u < v \leq w$ and $u < y \leq w$, since $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^-(X(w), u)$. Furthermore, since u is a maximal singularity of $X(w)$, we know that v and y are nonsingular points of $X(w)$ and hence $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ are good by Remark 4.7.4. We also note that, since

$$s_{\alpha-\delta} s_\alpha = s_{\alpha-(k+1)\delta} s_{\alpha-k\delta},$$

by Lemma 4.10.4, we can alternatively describe

$$x = s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u).$$

Additionally, we have that $s_{\alpha-\delta}u < u$. Suppose not, then $u < s_{\alpha-\delta}u$ and since $C_{-\hat{\beta}} = C_{\alpha-k\delta} = \overline{\mathcal{U}_{\alpha-k\delta}u} \in E^{-}(X(w),u)$, according to Lemma 4.13.1, the $\widehat{T}$-curve $C_{\alpha-\delta} = \overline{\mathcal{U}_{\alpha-\delta}u}$ is also an element of $E^{-}(X(w),u)$. However, we know from Lemma 5.6.2 that is impossible, since $E^{-}(X(w),u)$ contains the strong Type I pair of $\widehat{T}$-curves $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\alpha}u}$ and $C_{-\hat{\beta}} = \overline{\mathcal{U}_{\alpha-k\delta}u}$. Thus

$$s_{\alpha-\delta}u < u.$$

Furthermore, since $u < v = s_{\alpha}u$, Lemma 4.11.1 yields that

$$u < s_{\alpha}u < s_{\alpha-\delta}s_{\alpha}u$$

Finally, we claim that

$$y = s_{\alpha-k\delta}u < s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u) = x.$$

Assume not, then $y > x = s_{\alpha-(k+1)\delta}y$ and hence by Lemma 4.8.12, $-\alpha+(k+1)\delta$ is a weight of $\tau_{C_{-\hat{\beta}}}\big(X(w),y\big)$. As such, Lemma 4.8.10 guarantees that at least one member of the $(-\alpha+k\delta)$-string through $-\alpha+(k+1)\delta$, i.e.

$$-\alpha+(k+1)\delta \qquad \delta \qquad \alpha-(k-1)\delta,$$

is a weight of $\tau_{C_{-\hat{\beta}}}\big(X(w),u\big)$ and hence a weight of $T_u\big(X(w)\big)$.

If $-\alpha+(k+1)\delta$ appears as a weight of $T_u\big(X(w)\big)$, then Lemma 4.8.12 tells us that $u > s_{\alpha-(k+1)\delta}u$. However, since $u < y = s_{\alpha-k\delta}u$, we know from Lemma 4.12.3 that $u < s_{\alpha-(k+1)\delta}u$. Thus, $-\alpha+(k+1)\delta$ is not a weight of $T_u\big(X(w)\big)$.

We also know by Remark 4.4.1 that δ is not a weight of $T_u\big(X(w)\big)$.

Furthermore, if $\alpha-(k-1)\delta$ is a weight of $T_u\big(X(w)\big)$, then Lemma 4.3.3 indicates that $u < s_{\alpha-(k-1)\delta}u$. From this and the fact that $C_{-\hat{\beta}} = \overline{\mathcal{U}_{\alpha-k\delta}u}$ is in $E^{-}(X(w), u)$, it would follow from Lemma 4.13.1 that $\overline{\mathcal{U}_{\alpha-(k-1)\delta}u}$ is also in $E^{-}(X(w), u)$. As this is impossible by Lemma 5.6.2, we know $\alpha - (k-1)\delta$ is not a weight of $T_u\big(X(w)\big)$.

Thus, since none of the members of the $(-\alpha+k\delta)$-string through $-\alpha+(k+1)\delta$ are weights of $T_u\big(X(w)\big)$, we have obtained a contradiction and conclude that

$$y < x,$$

as claimed.

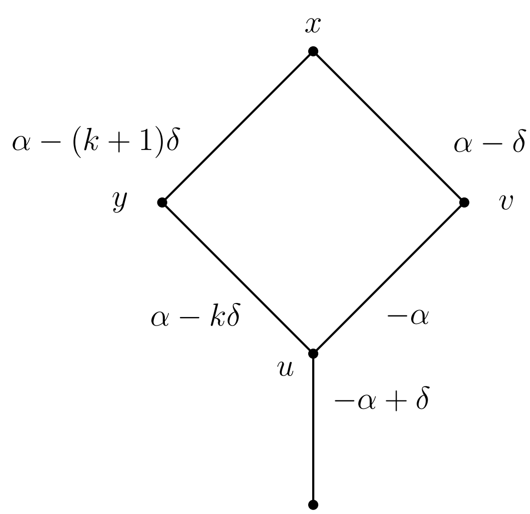
In summary, we have determined that

$$s_{\alpha-\delta}u < u < s_\alpha u < s_{\alpha-\delta}s_\alpha u = x$$

and

$$u < s_{\alpha-k\delta}u < s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta}u) = x$$

from which, in $\mathcal{G}/\mathcal{B}$, we construct the Bruhat graph:



Thus u satisfies the $(-\alpha)$–kite property in $\mathcal{G}/\mathcal{B}$, however, we still require that u satisfies this property in $X(w)$. For this, it suffices to show that $x \leq w$.

Let $C = C_{-\hat{\alpha}}$. To show that $x \leq w$, we will use the fact, presented in Lemma

5.9.1, that $\Omega\Big(\tau_C\big(X(w),u\big)\Big)$ contains a negative weight $-\hat{\gamma} =$

$$-\hat{\alpha} - l\delta, \qquad -l\delta, \qquad \text{or} \quad -\hat{\beta} - l\delta,$$

for some integer $l \geq 1$, which is not equal to $-\hat{\alpha}$ or $-\hat{\beta}$. Also, for $-\hat{\gamma}$ to be a weight of $\tau_C\big(X(w),u\big)$ it must first be a weight of $T_u\big(\mathcal{G}/\mathcal{B}\big)$ and, as such, in the event that $-\hat{\gamma}$ is real, $u < s_{\hat{\gamma}}u$, by Lemma 4.3.3.

Given that $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, where $k \geq 2$, in this context $-\hat{\gamma}$ becomes one of:

$$-\alpha - l\delta \qquad -l\delta \qquad \alpha - (k+l)\delta$$

from some integer $l \geq 1$. We will examine what happens when $-\hat{\gamma}$ assumes each of these three values.

If $-\hat{\gamma} = -\alpha - l\delta$, then by Lemma 4.8.7,

$$\hat{\alpha} + (-\hat{\gamma}) = \alpha + (-\alpha - l\delta) = -l\delta$$

is also in $\Omega\Big(\tau_C\big(X(w),u\big)\Big)$. Thus, by Lemma 4.8.10, $\Omega\Big(\tau_C\big(X(w),v\big)\Big)$ must contain at least two members of the α-string through $-\alpha - l\delta$:

$$\alpha - l\delta \qquad -l\delta \qquad -\alpha - l\delta$$

which are then weights of $T_v\big(X(w)\big)$. In light of Lemma 4.7.5, we can eliminate $-l\delta$, since $X(w)$ is nonsingular at v. This means that $\alpha - l\delta$ and $-\alpha - l\delta$ are both weights of $T_v\big(X(w)\big)$. In particular, as $\alpha - l\delta$ is a weight of $T_v\big(X(w)\big)$ and $X(w)$ is nonsingular at v, Lemma 4.8.13 gives us that $v < s_{\alpha - l\delta}v \leq w$. It now follows that $x = s_{\alpha-\delta}v \leq w$, either directly if $l = 1$ or from Lemma 4.12.2, if $l > 1$.

Secondly, suppose that $-\hat{\gamma} = -l\delta$.

If $u^{-1}(\alpha - l\delta) < 0$, then by Lemma 4.8.7,

$$\hat{\alpha} + (-\hat{\gamma}) = \alpha + (-l\delta) = \alpha - l\delta$$

is a weight of $\tau_C\big(X(w), u\big)$. Therefore, for this case we know that $-l\delta$ and $\alpha - l\delta$ are weights of $\tau_C\big(X(w), u\big)$.

If $u^{-1}(\alpha - l\delta) > 0$, then $u^{-1}(-\alpha + l\delta) < 0$, which means that the positive root $-\alpha + l\delta$ is a weight of $\tau_C\big(X(w), u\big)$, by Lemma 4.8.12. Thus, for this case, we have that $-l\delta$ and $-\alpha + l\delta$ are weights of $\tau_C\big(X(w), u\big)$.

Note that, since we have already determined that $s_{\alpha-\delta}u < u$ above, Lemma 4.8.12 yields that $u^{-1}(-\alpha + \delta) < 0$. Thus, if $l = 1$, then we are dealing with the latter of these two options.

If $-l\delta$ and $\alpha - l\delta$ are weights of $\tau_C\big(X(w), u\big)$ (so $l > 1$), then by Lemma 4.8.10, the set $\Omega\Big(\tau_C\big(X(w), v\big)\Big)$ contains at least two of the members of the α-string through $-l\delta$:

$$\alpha - l\delta \qquad\qquad - l\delta \qquad\qquad - \alpha - l\delta$$

Once again, as weights of $\tau_C\big(X(w), v\big)$, they are also weights of $T_v\big(X(w)\big)$. The two member must be $\alpha - l\delta$ and $-\alpha - l\delta$, since, by Lemma 4.7.5, the tangent space of $X(w)$ at a nonsingular point does not have any negative imaginary weights. Specifically, we have that $\alpha - l\delta$ is a weight of $T_v\big(X(w)\big)$ and hence, using the same argument (for $l > 1$) as presented in the previous case, we determine that $x = s_{\alpha-\delta}v \leq w$.

If instead $-l\delta$ and $-\alpha + l\delta$ are weights of $\tau_C\big(X(w), u\big)$, then we know from Lemma 4.8.10 that $\Omega\Big(\tau_C\big(X(w), v\big)\Big)$, and hence $\Omega\Big(T_v\big(X(w)\big)\Big)$, contains at least one of the elements in the α-string through $-l\delta$:

$$\alpha - l\delta \qquad\qquad - l\delta \qquad\qquad - \alpha - l\delta$$

and at least one of the elements in the α-string through $-\alpha + l\delta$:

$$-\alpha + l\delta \qquad\qquad l\delta \qquad\qquad \alpha + l\delta$$

Using Lemma 4.7.5 and Remark 4.4.1, we can exclude $-l\delta$ and $l\delta$ as possibilities from their respective strings.

Since $u < s_\alpha u = v$, we also have $s_\alpha v < v$ and therefore by Lemma 4.8.12,

$$v^{-1}(\alpha) < 0$$

Thus

$$v^{-1}(-\alpha) > 0.$$

and hence

$$v^{-1}(-\alpha + l\delta) = v^{-1}(-\alpha) + l\delta > 0.$$

Lemma 4.8.12 now indicates that $-\alpha + l\delta$ is not a weight of $T_v\big(X(w)\big)$ which leaves $\alpha + l\delta$ as the member of the α-string through $-\alpha + l\delta$ which appears as a weight of $T_v\big(X(w)\big)$. Therefore, by Lemma 4.8.12

$$v^{-1}(\alpha + l\delta) < 0$$

so that

$$v^{-1}(-\alpha - l\delta) > 0.$$

As such, Lemma 4.8.12 indicates that $-\alpha - l\delta$ is not a weight of $T_v\big(X(w)\big)$. This elimination means that $\alpha - l\delta$ has to be the member of the α-string through $-l\delta$ contained in $\Omega\Big(T_v\big(X(w)\big)\Big)$. Therefore, as above, we deduce that $x = s_{\alpha-\delta} v \leq w$.

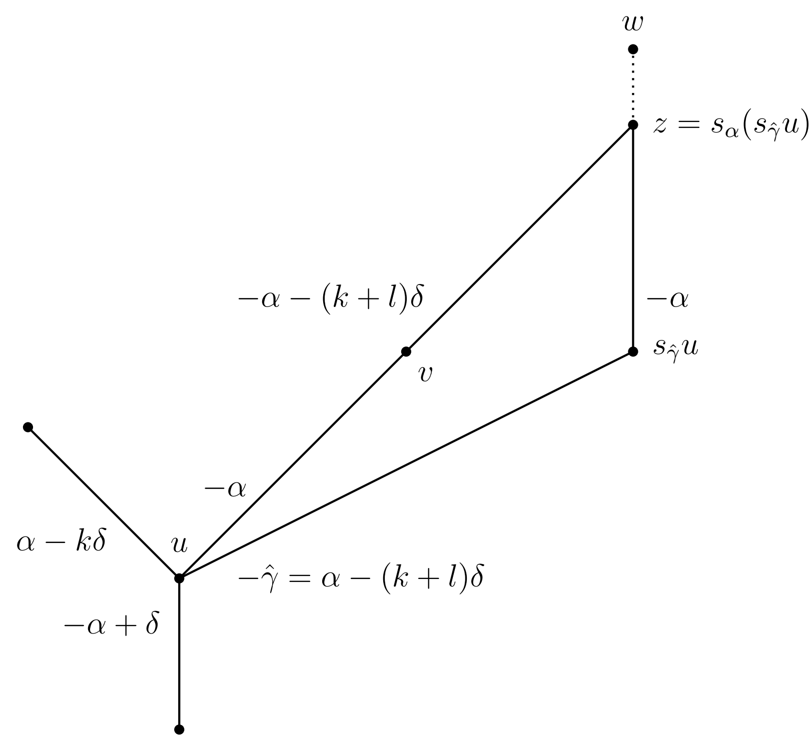
Finally, we consider the third case in which $-\hat{\gamma} = \alpha - (k+l)\delta$. According to Lemma 4.8.10, at least one of the members of the α-string through $\alpha - (k+l)\delta$ is a weight of $\tau_C\big(X(w), v\big)$ and so also of $T_v\big(X(w)\big)$:

$$\alpha - (k+l)\delta \qquad\qquad -(k+l)\delta \qquad\qquad -\alpha - (k+l)\delta$$

We remove $-(k+l)\delta$ from this list using the now familiar argument that $T_v\big(X(w)\big)$ has no negative imaginary weights (Lemma 4.7.5).

We claim that $-\alpha - (k+l)\delta$ is also not weight of $T_v\big(X(w)\big)$. To prove this, we suppose that $-\alpha - (k+l)\delta$ is weight of $T_v\big(X(w)\big)$.

A Bruhat graph representing our argument is:



We construct this Bruhat graph as follows: since $X(w)$ is nonsingular at v, from Lemma 4.8.13 we that determine that

$$v < s_{\alpha+(k+l)\delta}v \leq w.$$

Let $z = s_{\alpha+(k+l)\delta}v$ and hence we have $v \leq z \leq w$. By Remark 4.10.3, we know that

$$s_{\hat{\gamma}}u = s_{\alpha-(k+l)\delta}u = (s_{\alpha-\delta}s_\alpha)^{k+l-1}s_{\alpha-\delta}u$$

Remark 4.10.3 also gives us that

$$z = s_{\alpha+(k+l)\delta}v = (s_\alpha s_{\alpha-\delta})^{k+l}s_\alpha v = (s_\alpha s_{\alpha-\delta})^{k+l}s_\alpha(s_\alpha u) = (s_\alpha s_{\alpha-\delta})^{k+l}u$$

and therefore,

$$z = (s_\alpha s_{\alpha-\delta})^{k+l}u = s_\alpha s_{\alpha-\delta}(s_\alpha s_{\alpha-\delta})^{k+l-1}u = s_\alpha(s_{\alpha-\delta}s_\alpha)^{k+l-1}s_{\alpha-\delta}u = s_\alpha(s_{\hat{\gamma}}u).$$

Furthermore, since $u < s_{\hat{\gamma}}u = s_{\alpha-(k+l)\delta}u$, where $k+l \geq 3$, it follows from Lemma 4.12.4 that

$$s_{\hat{\gamma}}u = s_{\alpha-(k+l)\delta}u < s_{\alpha}\big(s_{\alpha-(k+l)\delta}u\big) = s_{\alpha}\big(s_{\hat{\gamma}}u\big) = z$$

Thus, since $z \leq w$, we have that

$$u < s_{\hat{\gamma}}u < w.$$

As a result, $\overline{\mathcal{U}_{-\hat{\gamma}}u} = \overline{\mathcal{U}_{\alpha-(k+l)\delta}u}$ is a $\widehat{T}$-curve in $E^{-}(X(w),u)$, which contradicts the fact that $E^{-}(X(w),u)$ does not contain any $\widehat{T}$-curves of this form, other than $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$, by Lemma 5.6.2. Thus, $-\alpha-(k+l)\delta$ is also not weight of $T_v\big(X(w)\big)$.

Therefore, $\alpha-(k+l)\delta$ is the member of the α-string through $\alpha-(k+l)\delta$ which is present in $\Omega\Big(T_v\big(X(w)\big)\Big)$. As such, we obtain from Lemma 4.8.13 that

$$v < s_{\alpha-(k+l)\delta}v \leq w,$$

where $k+l \geq 3$. Lemma 4.12.2 now yields that $s_{\alpha-\delta}v < s_{\alpha-(k+l)\delta}v$ and hence, for the last time, we deduce that $x = s_{\alpha-\delta}v < w$. □

Remark 5.9.3. In order to show that $x \leq w$ in the proof of Lemma 5.9.2, we used the fact that $\Omega\Big(\tau_C\big(X(w),u\big)\Big)$ contains a weight $-\hat{\gamma}$ as described in Lemma 5.9.1, where $C = C_{-\hat{\alpha}}$. However, we could have used $C = C_{-\hat{\beta}}$ instead. The proof using $C = C_{-\hat{\beta}}$ is very similar to the proof provided for Lemma 5.9.4.

Lemma 5.9.4. *Let $X(w)$ be a singular Schubert variety in $\mathcal{G}/\mathcal{B}$. Suppose that $u \in X(w)^{\widehat{T}}$ is a maximal singularity such that $E^{-}(X(w),u)$ contains a pair of strong Type I $\widehat{T}$-curves $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ for which $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 1$. Then u satisfies the $(\alpha-\delta)$–kite property in $X(w)$.*

Proof. Let $v = s_{\hat{\alpha}}u = s_{\alpha+k\delta}u$, let $y = s_{\hat{\beta}}u = s_{\alpha-\delta}u$, and let $x = s_{\alpha}s_{\alpha-\delta}u$ ($= s_{\alpha+(k+1)\delta}s_{\alpha+k\delta}u$, by Lemma 4.10.4). As $C_{-\hat{\alpha}}, C_{-\hat{\beta}} \in E^{-}(X(w),u)$, we know that

$$u < v \leq w \qquad \text{and} \qquad u < y \leq w.$$

Also, since u is a maximal singularity of $X(w)$, both v and y are nonsingular points of $X(w)$, which implies that $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ are good by Remark 4.7.4.

As well, if $u < s_\alpha u$, then Lemma 4.13.1 would indicate that $C_{-\alpha} = \overline{\mathcal{U}_{-\alpha} u}$ is an element of $E^-(X(w), u)$, since $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\alpha-k\delta} u} \in E^-(X(w), u)$, where $k \geq 1$. As this would contradict Lemma 5.6.2, it must be the case that

$$s_\alpha u < u.$$

In addition, since $u < y = s_{\alpha-\delta} u$, we determine from Lemma 4.11.1 that

$$u < s_{\alpha-\delta} u < s_\alpha s_{\alpha-\delta} u.$$

Finally, we claim that

$$v = s_{\alpha+k\delta} u < s_{\alpha+(k+1)\delta}(s_{\alpha+k\delta} u) = x.$$

To prove the claim, we assume that $v > x = s_{\alpha+(k+1)\delta} v$. Hence, from Lemma 4.8.12, we know that $\alpha + (k+1)\delta$ is a weight of $\tau_{C_{-\hat{\alpha}}}\big(X(w), v\big)$. Consequently, Lemma 4.8.10 implies that $\Omega\Big(\tau_{C_{-\hat{\alpha}}}\big(X(w), u\big)\Big)$, and hence $\Omega\Big(T_u\big(X(w)\big)\Big)$, contains at least one member of the $(\alpha + k\delta)$-string through $\alpha + (k+1)\delta$:

$$\alpha + (k+1)\delta \qquad \delta \qquad -\alpha - (k-1)\delta,$$

However, none of these options are weights of $T_u\big(X(w)\big)$. Indeed, if $\alpha + (k+1)\delta$ is a weight of $T_u\big(X(w)\big)$, then Lemma 4.8.12 stipulates that $u > s_{\alpha+(k+1)\delta} u$. On the contrary, as $u < v = s_{\alpha+k\delta} u$, Lemma 4.12.3 gives us that $u < s_{\alpha+(k+1)\delta} u$. Hence, $\alpha + (k+1)\delta$ is not a weight of $T_u\big(X(w)\big)$.

Remark 4.4.1 indicates that δ is not a weight of $T_u\big(X(w)\big)$.

Finally, if $-\alpha - (k-1)\delta$ is a weight of $T_u\big(X(w)\big)$, then Lemma 4.3.3 would yield that $u < s_{-\alpha-(k-1)\delta} u$. As a result, since $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\alpha-k\delta} u} \in E^-(X(w), u)$, Lemma 4.13.1 specifies that $\overline{\mathcal{U}_{-\alpha-(k-1)\delta}u}$ is also a $\widehat{T}$-curve in $E^{-}(X(w),u)$. However, according to Lemma 5.6.2 this cannot occur, since the strong Type I pair $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ are already present in $E^{-}(X(w),u)$. Therefore, $-\alpha-(k-1)\delta$ is not a weight of $T_u\big(X(w)\big)$.

Hence, since none of these three possibilities are weights of $T_u\big(X(w)\big)$, we have obtained a contradiction.

Therefore, as claimed, we have

$$v < x.$$

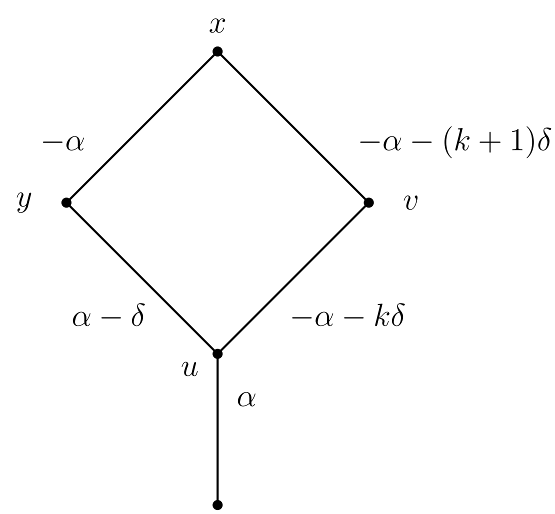
In summary, we have deduced that

$$s_\alpha u < u < y = s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u = x$$

and

$$u < v = s_{\alpha+k\delta}u < s_{\alpha+(k+1)\delta}(s_{\alpha+k\delta}u) = x$$

from which, in $\mathcal{G}/\mathcal{B}$, we construct the Bruhat graph:



Thus u satisfies the $(\alpha-\delta)$–kite property in $\mathcal{G}/\mathcal{B}$. To show that u satisfies this property in $X(w)$, we will show that $x \leq w$.

To that end, suppose that $-\hat{\gamma}$ is the weight of $\tau_C\big(X(w), u\big)$, for $C := C_{-\hat{\alpha}}$, specified by Lemma 5.9.1. As such, $-\hat{\gamma}$ and has three possible forms:

$$-\alpha - (k+l)\delta \qquad\qquad -l\delta \qquad\qquad \alpha - (l+1)\delta,$$

where $l \geq 1$ is an integer.

We will first consider the case in which $-\hat{\gamma} = -\alpha - (k+l)\delta$. Using Lemma 4.8.7, we obtain that

$$\hat{\alpha} + (-\hat{\gamma}) = \alpha + k\delta + (-\alpha - (k+l)\delta) = -l\delta \in \Omega\Big(\tau_C\big(X(w), u\big)\Big).$$

As such, Lemma 4.8.10 establishes that $\Omega\Big(\tau_C\big(X(w), v\big)\Big)$, and subsequently $\Omega\Big(T_v\big(X(w)\big)\Big)$, contains at least two members of the $(\alpha + k\delta)$-string through $-\alpha - (k+l)\delta$:

$$-\alpha - (k+l)\delta \qquad\qquad -l\delta \qquad\qquad \alpha + (k-l)\delta$$

Lemma 4.7.5 allows us to eliminate $-l\delta$ as a possibility and therefore the two aforementioned weights are $-\alpha - (k+l)\delta$ and $\alpha + (k-l)\delta$. As $-\alpha - (k+l)\delta$ is a weight of $T_v\big(X(w)\big)$ and $X(w)$ is nonsingular at v, Lemma 4.8.13 tells us that

$$v < s_{\alpha+(k+l)\delta}v \leq w.$$

If $l = 1$, then we have already obtained that $x = s_{\alpha+(k+1)\delta}v \leq w$. If $l > 1$, then we use the fact that $k+l > k+1 \geq 2$ to determine that $s_{\alpha+(k+1)\delta}v < s_{\alpha+(k+l)\delta}v$, by Lemma 4.12.2. Thus, $x = s_{\alpha+(k+1)\delta}v < w$.

We will now consider the case in which $-\hat{\gamma} = -l\delta$. For this, we will take into account the values of l and k. We first note that $l \neq k+1$, since otherwise, by Lemma 4.8.7,

$$\hat{\alpha} + (-\hat{\gamma}) = \alpha + k\delta - (k+1)\delta = \alpha - \delta = -\hat{\beta}$$

would be a weight of $\tau_C\big(X(w), u\big)$, which is impossible by Lemma 5.9.1.

Now, assume $l \geq k$, but $l \neq k+1$. Thus $k-l \leq 0$. Also, since $s_\alpha u < u$, we know from Lemma 4.8.12, that

$$u^{-1}(\alpha) < 0$$

and so

$$u^{-1}(\alpha + (k-l)\delta) = u^{-1}(\alpha) + (k-l)\delta < 0.$$

Therefore, by Lemma 4.8.7, we know that

$$\hat{\alpha} + (-\hat{\gamma}) = \alpha + k\delta - l\delta = \alpha + (k-l)\delta$$

is also a weight of $\tau_C\big(X(w), u\big)$. Hence Lemma 4.8.10 yields that the set $\Omega\Big(\tau_C\big(X(w), v\big)\Big)$ contains at least two of the members of the $(\alpha + k\delta)$-string through $-l\delta$:

$$\alpha + (k-l)\delta \qquad - l\delta \qquad - \alpha - (k+l)\delta$$

At this point, we may proceed exactly as we did in when $-\hat{\gamma} = -\alpha - (k+l)\delta$ to once again determine that $x = s_{\alpha+(k+1)\delta} v \leq w$.

Now assume that $l < k$, so that $k - l > 0$. If it is still true that

$$u^{-1}(\alpha + (k-l)\delta) < 0,$$

then, as in the previous case, we ascertain that $x = s_{\alpha+(k+1)\delta} v \leq w$. Otherwise,

$$u^{-1}(-\alpha - (k-l)\delta) < 0,$$

which indicates that the negative root $-\alpha - (k-l)\delta$ is a weight of $T_u\big(\mathcal{G}/\mathcal{B}\big)$ for which $u < s_{-\alpha-(k-l)\delta} u$, by Lemma 4.3.3. Moreover, as $C_{-\hat{\alpha}} = \overline{\mathcal{U}_{-\alpha-k\delta} u}$ is in $E^-(X(w), u)$ and $1 \leq k - l < k$, Lemma 4.13.1 establishes that $\overline{\mathcal{U}_{-\alpha-(k-l)\delta} u}$ is also in $E^-(X(w), u)$. However, this leads to a contradiction, since by Lemma 5.6.2, $E^-(X(w), u)$ does not contain any $\widehat{T}$-curves other than $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ of this form.

Finally, suppose that $-\hat{\gamma} = \alpha - (l+1)\delta$. According to Lemma 4.8.10, the set $\Omega\Big(\tau_C\big(X(w), v\big)\Big)$ must contain at least one member of the $(\alpha + k\delta)$-string through $\alpha - (l+1)\delta$:

$$\alpha - (l+1)\delta \qquad -(k+l+1)\delta \qquad -\alpha - (2k+l+1)\delta$$

which in turn is an element of $\Omega\Big(T_v\big(X(w)\big)\Big)$. The negative imaginary weight $-(k+l+1)\delta$ is removed from this list, by Lemma 4.7.5, as $X(w)$ is nonsingular at v.

If $\alpha - (l+1)\delta$ is the member that appears as a weight of $T_v\big(X(w)\big)$, then Lemma 4.8.13 guarantees that $\overline{\mathcal{U}_{\alpha-(l+1)\delta}v}$ is in $E^-(X(w), v)$. Further, since $u < s_{\alpha+k\delta}u = v$, Lemma 4.12.4 gives us that

$$v = s_{\alpha+k\delta}u < s_{\alpha-\delta}\big(s_{\alpha+k\delta}u\big) = s_{\alpha-\delta}v.$$

Therefore, by Lemma 4.13.1, we deduce that $\overline{\mathcal{U}_{\alpha-\delta}v}$ is in $E^-(X(w), v)$. Consequently, $E^-(X(w), v)$ contains $\overline{\mathcal{U}_{\alpha-(l+1)\delta}v}$ and $\overline{\mathcal{U}_{\alpha-\delta}v}$, which form either a pseudo Type II or a Type II pair of $\widehat{T}$-curves. Either way, Lemmas 5.3.2 and 5.4.1 specify that v is a singular point of $X(w)$. As this contradicts the fact that $X(w)$ is nonsingular at v, we conclude that $\alpha - (l+1)\delta$ is not a weight of $T_v\big(X(w)\big)$.

For this reason, the member of the $(\alpha + k\delta)$-string through $\alpha - (l+1)\delta$ in $\Omega\Big(T_v\big(X(w)\big)\Big)$ must be $-\alpha - (2k+l+1)\delta$. We now use Lemma 4.8.13 to obtain that

$$v < s_{\alpha+(2k+l+1)\delta}v \leq w,$$

where $2k + l + 1 \geq 4$. Furthermore, since $2k + l + 1 > k + 1 \geq 2$, Lemma 4.12.2 establishes that $s_{\alpha+(k+1)\delta}v < s_{\alpha+(2k+l+1)\delta}v$. Thus, $x = s_{\alpha+(k+1)\delta}v < w$, as required. $\square$

With these two lemmas in hand, we are now in a position to prove the main result of this chapter:

Theorem 5.9.5. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$. If $u \in X(w)^{\widehat{T}}$ is a maximal singularity, then u satisfies either the $(-\alpha)$–kite property in $X(w)$ or the $(\alpha - \delta)$–kite property in $X(w)$, for some $\alpha \in \Phi^+$.*

Proof. From Theorem 5.7.2, we know that $E^-(X(w), u)$ contains a strong Type I pair of $\widehat{T}$-curves, $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$. As $\hat{\alpha}$ and $\hat{\beta}$ form a strong Type I pair of roots, we have either $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, for some $k \geq 2$, or $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 1$. If $\hat{\alpha} = \alpha$ and $\hat{\beta} = -\alpha + k\delta$, for some $k \geq 2$, then by Lemma 5.9.2, u satisfies the $(-\alpha)$–kite property in $X(w)$. Otherwise, $\hat{\alpha} = \alpha + k\delta$ and $\hat{\beta} = -\alpha + \delta$, where $k \geq 1$ and it follows from Lemma 5.9.4 that u satisfies the $(\alpha - \delta)$–kite property in $X(w)$. □

Remark 5.9.6. As indicated in Remark 5.8.3, if $u \in X(w)$ satisfies one of the kite properties in $X(w)$, it also satisfies the corresponding chain property. The specific version of each property that appears (either $(-\alpha)$ or $(\alpha - \delta)$) is determined by which version of a strong Type I pair of $\widehat{T}$-curves is contained in $E^-(X(w), u)$: if $C_{-\alpha}$ and $C_{\alpha-k\delta}$ are present in $E^-(X(w), u)$, then u satisfies the $(-\alpha)$–kite property and $(-\alpha)$–chain property. If $C_{-\alpha-k\delta}$ and $C_{\alpha-\delta}$ occur in $E^-(X(w), u)$, then u satisfies the $(\alpha - \delta)$–kite property and $(\alpha - \delta)$–chain property. Thus, as a consequence of Theorem 5.9.5, if $u \in X(w)^{\widehat{T}}$ is a maximal singularity, then u satisfies either the $(-\alpha)$–chain property or the $(\alpha-\delta)$–chain property, for some $\alpha \in \Phi^+$.

Remark 5.9.7. We think that it is highly likely that the ideas of Dale Peterson presented in his proof of his ADE-Theorem (see [15]) can be used to give a more geometric and shorter proof of Lemmas 5.9.2 and 5.9.4, and Theorem 5.9.5 than what we have provided in this thesis.

5.10 An Additional Condition for Maximality

In previous sections, we have only considered elements of $E^-(X(w), u)$ whose tangent spaces at u have weights with the same real parts, up to sign. We know from Theorem 5.6.3 that, for a maximal singularity u, there can be at most two such $\widehat{T}$-curves in $E^{-}(X(w), u)$ for each $\alpha \in \Phi^{+}$. For a singular rationally smooth Schubert variety $X(w)$, we know from Theorem 5.7.2 that $E^{-}(X(w), u)$ contains a strong Type I pair of $\widehat{T}$-curves. We have determined that this pair of $\widehat{T}$-curves limits which other $\widehat{T}$-curves may appear in $E^{-}(X(w), u)$, based upon the real part of the weights of their tangent spaces at u.

Lemma 5.10.1. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$, let $u \in X(w)^{\widehat{T}}$ be a maximal singularity of $X(w)$, and let $C_{-\hat{\alpha}} := \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} := \overline{\mathcal{U}_{-\hat{\beta}} u}$ be a strong Type I pair of $\widehat{T}$-curves in $E^{-}(X(w), u)$, where* $\mathrm{Re}(\hat{\alpha}) = \alpha = (ij) \in \Phi^{+}$. *If $\overline{\mathcal{U}_{-\hat{\gamma}} u} \in E^{-}(X(w), u)$, where $\hat{\gamma} = \pm\gamma + h_{\gamma}\delta$ is a positive root such that $\gamma = (lk) \in \Phi^{+}$, then $l = i, j$ or $k = i, j$.*

Proof. Let $\alpha = (ij) \in \Phi^{+}$ (with $i < j$) and assume that

$$D = \overline{\mathcal{U}_{-\hat{\gamma}} u} \in E^{-}(X(w), u),$$

where $\hat{\gamma} = \pm\gamma + h_{\gamma}\delta$ is a positive root such that $\gamma = (lk) \in \Phi^{+}$ (with $l < k$) and $\{l, k\} \cap \{i, j\} = \emptyset$. Let $u' = s_{\hat{\gamma}} u$. Therefore, we know that

$$u < u' \leq w$$

and that $X(w)$ is nonsingular at u', since u is a maximal singularity of $X(w)$.

We know from Remark 5.9.6 that u satisfies either the $(-\alpha)$–chain property in $X(w)$ or the $(\alpha - \delta)$–chain property in $X(w)$. For our initial case, suppose that u satisfies the $(-\alpha)$–chain property in $X(w)$. Thus

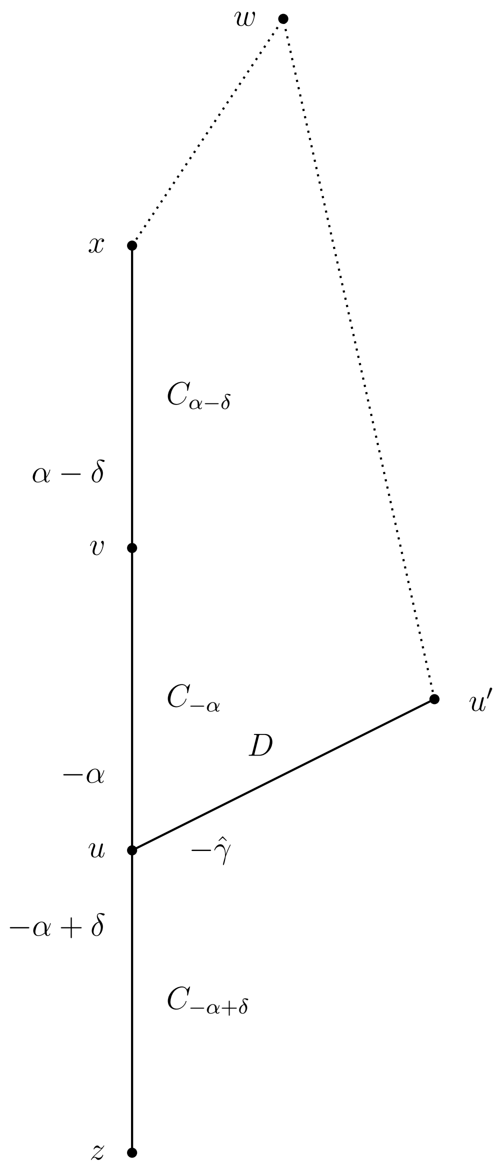
$$s_{\alpha-\delta} u < u < s_{\alpha} u < s_{\alpha-\delta} s_{\alpha} u \leq w.$$

Let $v = s_{\alpha} u$, let $z = s_{\alpha-\delta} u$, and let $x = s_{\alpha-\delta} s_{\alpha} u$ $(= s_{\alpha-(k+1)\delta}(s_{\alpha-k\delta} u))$. Using this notation, the $(-\alpha)$–chain property becomes:

$$z < u < v < x \leq w$$

In addition, let $C_{-\alpha} = \overline{\mathcal{U}_{-\alpha} u}$ in $E^{-}(X(w), u)$, let $C_{-\alpha+\delta} = \overline{\mathcal{U}_{-\alpha+\delta} u}$ in $E^{+}(X(w), u)$, and let $C_{\alpha-\delta} = \overline{\mathcal{U}_{\alpha-\delta} v}$ in $E^{-}\big(X(w), v\big)$.

The Bruhat graph depicting this situation is:



As u is a maximal singularity of $X(w)$, the points v and x are nonsingular points of $X(w)$. As $z < u$, we know that z is a singular point of $X(w)$, by Lemma 4.7.3.

Since $\pm\gamma \neq \pm\alpha$, we obtain from Lemma 4.9.4 that $-\hat{\gamma}$ is a weight of $\tau_{C_{-\alpha}}\big(X(w), u\big)$ and, as such, we apply Lemma 4.8.10 to determine that $\tau_{C_{-\alpha}}\big(X(w), v\big)$ has at least one weight which is an element of the α-string through $-\hat{\gamma}$. However, as $\{l, k\} \cap \{i, j\} = \emptyset$, we know that $-\gamma + c\alpha$ is not a root for any $c \in \mathbb{Z} \setminus \{0\}$ and hence $-\hat{\gamma} + c\alpha$ is not a root for any $c \in \mathbb{Z} \setminus \{0\}$. Therefore, the α-string through

$-\hat{\gamma}$ consists solely of $-\hat{\gamma}$, which implies that $-\hat{\gamma}$ is a weight of $\tau_{C_{-\alpha}}\big(X(w), v\big)$ and hence

$$v < s_{\hat{\gamma}} v \leq w,$$

by Lemma 4.8.13, since $X(w)$ is nonsingular at v.

Repeating the above argument using $\tau_{C_{\alpha-\delta}}\big(X(w), v\big)$ and the point x in place of $\tau_{C_{-\alpha}}\big(X(w), u\big)$ and v, we determine that

$$x < s_{\hat{\gamma}} x \leq w.$$

We can repeat most of the above argument for $\tau_{C_{-\alpha+\delta}}\big(X(w), u\big)$ and the point z, but, since z is a singular point of $X(w)$, we can only go as far as to deduce that $-\hat{\gamma}$ is a weight of $\tau_{C_{-\alpha+\delta}}\big(X(w), z\big)$ and hence of $T_z\big(X(w)\big)$. Although we have yet to determine if $s_{\hat{\gamma}} z$ is a point of $X(w)$, we do know by Lemma 4.3.3 that

$$z < s_{\hat{\gamma}} z$$

in $\mathcal{G}/\mathcal{B}$, since $-\hat{\gamma}$ is also a weight of $T_z\big(\mathcal{G}/\mathcal{B}\big)$.

Let $v' := s_{\hat{\gamma}} v$, let $x' := s_{\hat{\gamma}} x$, and let $z' := s_{\hat{\gamma}} z$. Summarizing our work above in terms of these assignments, we have:

$$v < v' \leq w,$$

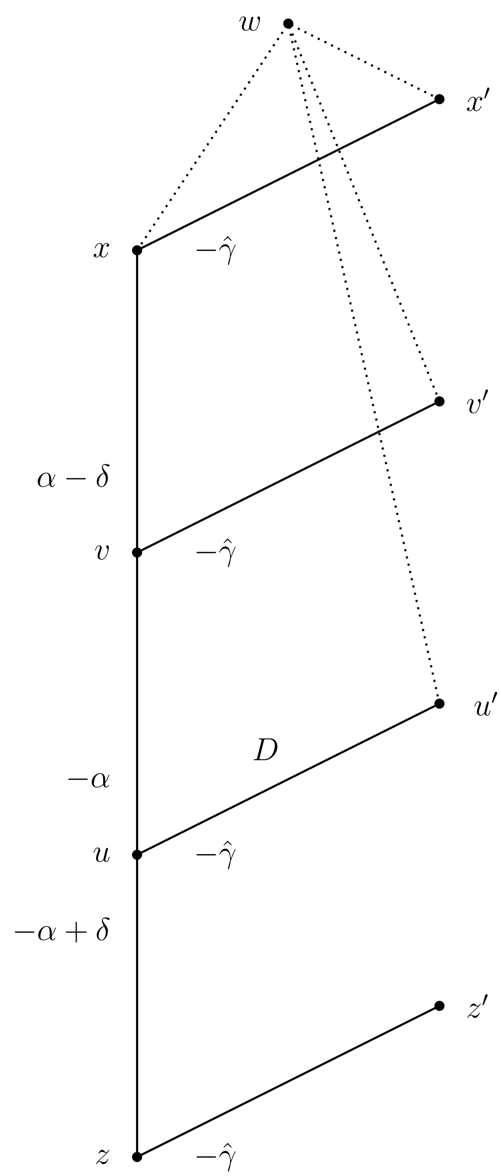
$$x < x' \leq w,$$

and

$$z < z'.$$

As an immediate consequence, we observe that $X(w)$ is nonsingular at v' and x', since u is a maximal singularity of $X(w)$ and $u < v < x$.

With the addition of these newly defined points our Bruhat graph including the weight $-\hat{\gamma}$ of $T_z\big(X(w)\big)$ is:



Our goal now is to show that u' also satisfies the $(-\alpha)$–chain property in $X(w)$ (involving the points z', u', v', and x').

To that end, keeping in mind that $\alpha \neq \pm\gamma$, we know from Lemma 4.9.4 that $-\alpha + \delta$ and $-\alpha$ are weights of $\tau_D\big(X(w), u\big)$ and hence also weights of $\tau_D\big(X(w), u'\big)$, by Lemma 4.8.10, since $-\alpha + c\hat{\gamma}$ and $-\alpha + \delta + c'\hat{\gamma}$ are not roots for any $c, c' \in \mathbb{Z} \setminus \{0\}$. Lemma 4.8.12 now indicates that

$$s_{\alpha-\delta}u' < u' \leq w,$$

and, since $X(w)$ is nonsingular at u', we learn from Lemma 4.8.13 that

$$u' < s_{\alpha}u' \leq w.$$

Using the same method, we also obtain that

$$v' < s_{\alpha-\delta}v' \leq w.$$

Further, according to Remark 4.10.1, as $\{l, k\} \cap \{i, j\} = \emptyset$, we have the following:

$$s_{\hat{\gamma}}s_{\alpha-\delta}s_{\hat{\gamma}} = s_{s_{\hat{\gamma}}(\alpha-\delta)} = s_{\alpha-\delta}$$

$$s_{\hat{\gamma}}s_{\alpha}s_{\hat{\gamma}} = s_{s_{\hat{\gamma}}(\alpha)} = s_{\alpha}$$

Therefore,

$$s_{\alpha-\delta}u' = s_{\hat{\gamma}}s_{\alpha-\delta}s_{\hat{\gamma}}u' = s_{\hat{\gamma}}s_{\alpha-\delta}u = s_{\hat{\gamma}}z = z'$$

$$s_{\alpha}u' = s_{\hat{\gamma}}s_{\alpha}s_{\hat{\gamma}}u' = s_{\hat{\gamma}}s_{\alpha}u = s_{\hat{\gamma}}v = v'$$

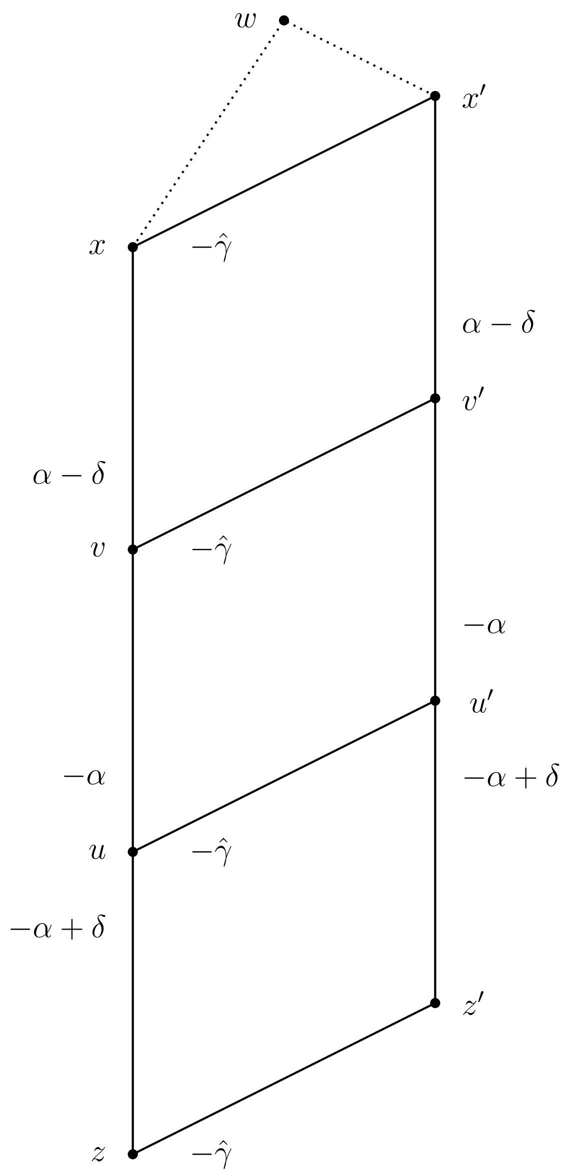
$$s_{\alpha-\delta}v' = s_{\hat{\gamma}}s_{\alpha-\delta}s_{\hat{\gamma}}v' = s_{\hat{\gamma}}s_{\alpha-\delta}v = s_{\hat{\gamma}}x = x'$$

Collectively, these computations yield

$$z' = s_{\alpha-\delta}u' < u' < s_{\alpha}u' = v' < s_{\alpha-\delta}v' = x' \leq w$$

and hence u' satisfies the $(-\alpha)$–chain property in $X(w)$. We have also confirmed that z' is indeed a point of $X(w)$.

With this new information our Bruhat graph becomes:



Therefore, by Lemma 5.1.3, u' is a singular point of $X(w)$, which contradicts the fact that $X(w)$ is nonsingular at u', since $u < u'$ and u is a maximal singularity.

For our second case, suppose instead that u satisfies the $(\alpha-\delta)$–chain property in $X(w)$, that is,

$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u \leq w.$$

Redefining from the previous case, let $z = s_\alpha u$, let $v = s_{\alpha-\delta}u$, and also let

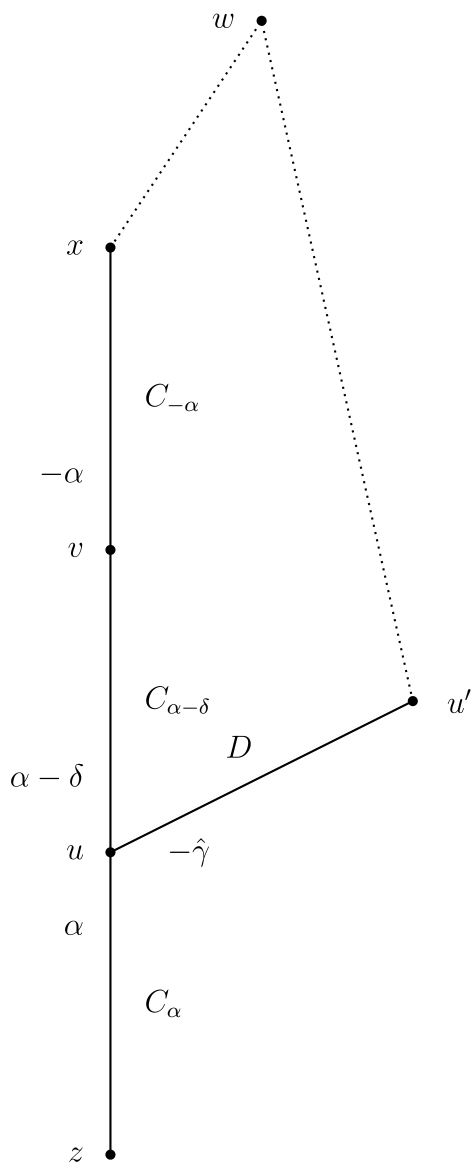
$x = s_\alpha s_{\alpha-\delta} u$. Using these, the $(\alpha - \delta)$–chain property condition above can be expressed as:

$$z < u < v < x \leq w.$$

Once again, since u is a maximal singularity of $X(w)$, we observe that $X(w)$ is nonsingular at v and x, but singular at z (Lemma 4.7.3).

This time, let $C_{\alpha-\delta} = \overline{\mathcal{U}_{\alpha-\delta} u}$ in $E^-(X(w), u)$, let $C_\alpha = \overline{\mathcal{U}_\alpha u}$ in $E^+(X(w), u)$, and let $C_{-\alpha} = \overline{\mathcal{U}_{-\alpha} v}$ in $E^-\big(X(w), v\big)$.

The Bruhat graph illustrating the set-up of our second case is:



Since $\pm\gamma \neq \pm\alpha$, Lemma 4.9.4 indicates that $-\hat{\gamma}$ is a weight of $\tau_{C_{\alpha-\delta}}\big(X(w), u\big)$. Therefore, from Lemma 4.8.10 we are given that $\tau_{C_{\alpha-\delta}}\big(X(w), v\big)$ has at least one weight which is in the $(-\alpha+\delta)$-string through $-\hat{\gamma}$. However, since $\{l, k\} \cap \{i, j\} = \emptyset$, we have that $-\hat{\gamma} + c(-\alpha + \delta)$ is not a root for any $c \in \mathbb{Z} \setminus \{0\}$ and thus the $(-\alpha + \delta)$-string through $-\hat{\gamma}$ is composed only of $-\hat{\gamma}$. Therefore, $-\hat{\gamma}$ is a weight of $\tau_{C_{\alpha-\delta}}\big(X(w), v\big)$. Furthermore, since $X(w)$ is nonsingular at v, Lemma 4.8.13 yields that

$$v < s_{\hat{\gamma}} v \leq w.$$

Likewise, using $\tau_{C_{-\alpha}}\big(X(w), v\big)$ and the point x as replacements for $\tau_{C_{\alpha-\delta}}\big(X(w), u\big)$ and v in the above argument, we establish that

$$x < s_{\hat{\gamma}} x \leq w.$$

Regarding $\tau_{C_\alpha}\big(X(w), u\big)$ and the point z, since z is a singular point of $X(w)$, we can only ascertain that $-\hat{\gamma}$ is a weight of $\tau_{C_\alpha}\big(X(w), z\big)$ and hence of $T_z\big(X(w)\big)$. At this point, we have not yet demonstrated that $s_{\hat{\gamma}} z$ is a point of $X(w)$, however, since $-\hat{\gamma}$ is also a weight of $T_z\big(\mathcal{G}/\mathcal{B}\big)$, we know from Lemma 4.3.3 that

$$z < s_{\hat{\gamma}} z$$

in $\mathcal{G}/\mathcal{B}$.

Let $v' := s_{\hat{\gamma}} v$, and let $x' := s_{\hat{\gamma}} x$, and let $z' := s_{\hat{\gamma}} z$. In terms of these labels, our deductions above become:

$$v < v' \leq w,$$

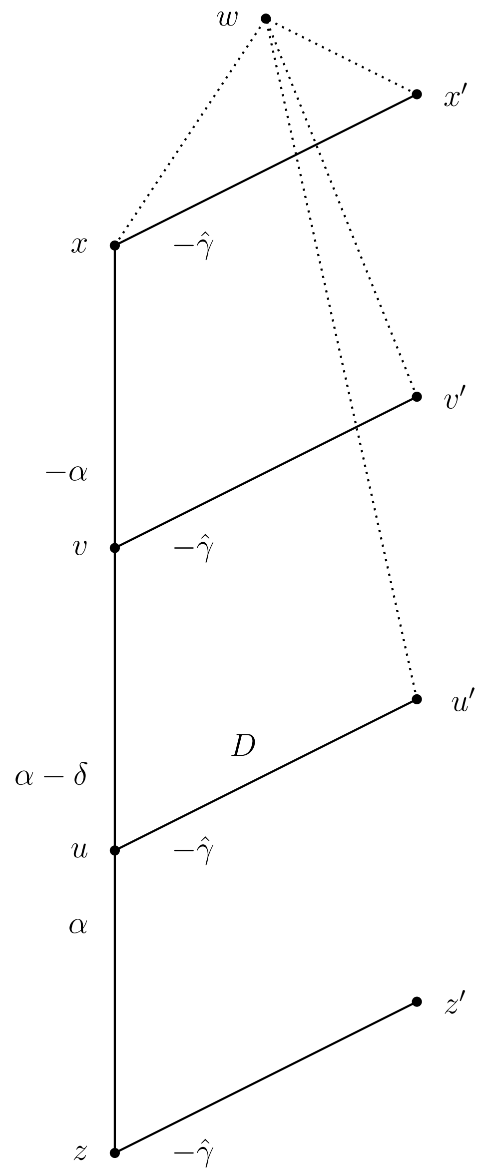
$$x < x' \leq w,$$

and

$$z < z'.$$

Since u is a maximal singularity of $X(w)$ and $u < v, y < x$, we obtain that $X(w)$ is nonsingular at v' and x'.

Adding this information to our Bruhat graph (including the weight $-\hat{\gamma}$ of $T_z\big(X(w)\big)$) yields:



At this point, we will show that u' satisfies the $(\alpha-\delta)$–chain property in $X(w)$.

Accordingly, since $-\alpha \neq \pm\gamma$, we determine from Lemma 4.9.4 that α and $\alpha-\delta$ are weights of $\tau_D\big(X(w), u\big)$. As well, since $\alpha + c\hat{\gamma}$ and $\alpha - \delta + c'\hat{\gamma}$ are not roots for any $c, c' \in \mathbb{Z} \setminus \{0\}$, it follows from Lemma 4.8.10 that α and $\alpha - \delta$ are weights of $\tau_D\big(X(w), u'\big)$.

Thus,

$$s_\alpha u' < u' \leq w,$$

by Lemma 4.8.12, and

$$u' < s_{\alpha-\delta} u' \leq w$$

by Lemma 4.8.13, since $X(w)$ is nonsingular at u'.

Similarly, we ascertain that

$$v' < s_\alpha v' \leq w.$$

Moreover, using Remark 4.10.1 and the fact that $\{l, k\} \cap \{i, j\} = \emptyset$, we compute:

$$s_{\hat{\gamma}} s_\alpha s_{\hat{\gamma}} = s_{s_{\hat{\gamma}}(\alpha)} = s_\alpha$$

$$s_{\hat{\gamma}} s_{\alpha-\delta} s_{\hat{\gamma}} = s_{s_{\hat{\gamma}}(\alpha-\delta)} = s_{\alpha-\delta}$$

Consequently,

$$s_\alpha u' = s_{\hat{\gamma}} s_\alpha s_{\hat{\gamma}} u' = s_{\hat{\gamma}} s_\alpha u = s_{\hat{\gamma}} z = z'$$

$$s_{\alpha-\delta} u' = s_{\hat{\gamma}} s_{\alpha-\delta} s_{\hat{\gamma}} u' = s_{\hat{\gamma}} s_{\alpha-\delta} u = s_{\hat{\gamma}} v = v'$$

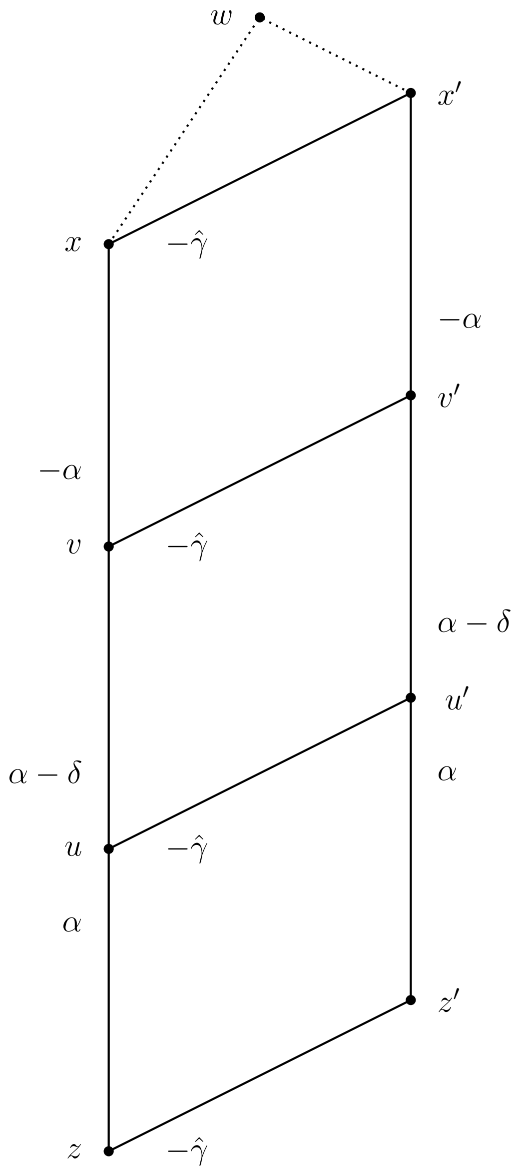
$$s_\alpha v' = s_{\hat{\gamma}} s_\alpha s_{\hat{\gamma}} v' = s_{\hat{\gamma}} s_\alpha v = s_{\hat{\gamma}} x = x'$$

Putting all of this together, we obtain

$$z' = s_\alpha u' < u' < s_{\alpha-\delta} u' = v' < s_\alpha v' = x' \leq w$$

and hence u' satisfies the $(\alpha - \delta)$–chain property in $X(w)$. This also verifies that z' is a point of $X(w)$.

Thus our finalized Bruhat graph is:



Therefore, by Lemma 5.1.3, u' is a singular point of $X(w)$. Thus, since $u < u'$, we have obtained a contradiction to the fact that u is a maximal singularity of $X(w)$. $\square$

As a consequence of this lemma, we obtain a bound on the size of $E^{-}(X(w), u)$ and hence also the codimension of $X(u)$ in $X(w)$.

Corollary 5.10.2. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$. If $u \in X(w)^{\widehat{T}}$ is a maximal singularity of $X(w)$, then*

$$|E^{-}(X(w), u)| \leq 4n - 6.$$

Proof. Let $C_{-\hat{\alpha}} := \overline{\mathcal{U}_{-\hat{\alpha}} u}$ and $C_{-\hat{\beta}} := \overline{\mathcal{U}_{-\hat{\beta}} u}$ be a strong Type I pair of $\widehat{T}$-curves in $E^{-}(X(w), u)$, where $\mathrm{Re}(\hat{\alpha}) = \alpha = (ij) \in \Phi^{+}$, with $i < j$. If $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ are the only $\widehat{T}$-curves in $E^{-}(X(w), u)$, then

$$|E^{-}(X(w), u)| = 2 \leq 4n - 6,$$

for $n \geq 2$, as required. Now suppose that $E^{-}(X(w), u)$ contains at least three $\widehat{T}$-curves. Let $\overline{\mathcal{U}_{-\hat{\gamma}} u}$ be any $\widehat{T}$-curve in $E^{-}(X(w), u)$ other than $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$. Thus, by Lemma 5.10.1, $\hat{\gamma} = \pm\gamma + h_{\gamma}\delta$ is a positive root such that $\gamma \in \Phi^{+}$ of the form

$$\gamma = (li), (ik), (l'j), \text{or}\, (jk'),$$

where l, l', k, k' are integers, none of which are equal to i or j, satisfying

$$1 \leq l < i < k \leq n$$

and

$$1 \leq l' < j < k' \leq n.$$

There are $i - 1$ possible values for l, $n - i - 1$ possible values for k, $j - 2$ possible values for l', and $n - j$ possible values for k'. Thus, there are

$$i - 1 + n - i - 1 + j - 2 + n - j = 2n - 4$$

possibilities for γ. By Theorem 5.6.3, $E^{-}(X(w), u)$ contains at most $2(2n-4)$ $\widehat{T}$-curves other than $C_{-\hat{\alpha}}$ and $C_{-\hat{\beta}}$ and hence $E^{-}(X(w), u)$ contains at most $4n - 6$ $\widehat{T}$-curves, as required. ∎

Corollary 5.10.3. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$. If $u \in X(w)^{\widehat{T}}$ is a maximal singularity of $X(w)$, then the codimension of u is at most $4n-6$.*

Proof. This follows from Corollary 5.10.2 and the fact that

$$|E^{-}(X(w), u)| = \dim X(w) - \dim X(u)$$

(see Remark 4.6.2). $\square$

Chapter 6

Smooth Schubert Varieties and Pattern Avoidance

In this chapter, we will provide a proof of a conjecture due to Billey-Crites which states that a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ is smooth if and only if w avoids the patterns 3412 and 4231. Using the results of Billey-Crites in [2], to prove this conjecture it suffices to prove that the affine permutation w indexing any singular rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ contains the pattern 3412. To that end, we will introduce the concept of a wide affine permutation (see Definition 6.3.1 below). These permutations are useful to us since they all contain the pattern 3412 (see Lemma 6.3.2 below). We know from Theorem 5.9.5 above that every maximal singularity u of a singular rationally smooth Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ satisfies either the $(-\alpha)$–kite property or the $(\alpha-\delta)$–kite property in $X(w)$. We will show that, in this situation, the maximal $\widehat{T}$-fixed point appearing in either of the two kite patterns is wide and hence contains the pattern 3412 (see Theorem 6.4.1 below). Using this fact, we will then prove that w itself is wide and hence also contains the pattern 3412, as required (see Theorem 6.5.1). We would like to note that some of the proofs that we have given in this chapter are long and technical in nature, and that someone with a more extensive background in combinatorics may be able to find a shorter method to prove Theorem 6.5.1.

6.1 The Billey-Crites Conjecture

In this section we describe the research of Billey and Crites presented in [2]. Unless otherwise specified, the source of information given in this section is [2].

Our work on maximal singularities of Schubert varieties in $\mathcal{G}/\mathcal{B}$ was initiated, in part, to prove Conjecture 1 in [2] (given below as Conjecture 6.1.6). Conjecture 6.1.6 is an affine analogue of an earlier results by Lakshmibai and Sandhya in the classical algebraic group setting regarding Schubert varieties in G/B, where $G = \mathrm{SL}_n(\mathbb{C})$ and B is the set of upper triangular matrices in G. Specifically, in [29], Lakshmibai and Sandhya proved the following important result:

Theorem 6.1.1. *Let $X(w)$, where $w = [a_1, a_2, \ldots, a_n] \in S_n$, be a Schubert variety in G/B. Then $X(w)$ is singular if and only if there are indices i, j, k, l, where $1 \leq i < j < k < l \leq n$, for which at least one of the following holds:*

1) $a_k < a_l < a_i < a_j$

2) $a_l < a_j < a_k < a_i$.

Using the natural numbers 1 through 4 to represent the relative size of the numbers in each inequality, where 1 represents the smallest number and 4 the largest, these two conditions in one-line permutation notation can be expressed as:

$$\begin{array}{cccccccccc} & & 3 & & 4 & & 1 & & 2 & \\ w = [& \cdots & a_i & \cdots & a_j & \cdots & a_k & \cdots & a_l & \cdots] \end{array}$$

and

$$\begin{array}{cccccccccc} & & 4 & & 2 & & 3 & & 1 & \\ w = [& \cdots & a_i & \cdots & a_j & \cdots & a_k & \cdots & a_l & \cdots]. \end{array}$$

In the first case, we say w contains the pattern 3412 and in the second we say w contains the pattern 4231. If a permutation w does not satisfy condition 1), we say it avoids the pattern 3412 and if w does not satisfy condition 2), we say it avoids the pattern 4231. Thus Theorem 6.1.1 can be restated as: $X(w)$ is smooth if and only if w avoids the patterns 3412 and 4231. Furthermore, Dale Peterson, when working over $\mathbb{C}$, proved that for simply laced types all rationally smooth Schubert varieties are smooth and, moreover, that the rationally smooth locus and the smooth locus coincide (see [15]). As such, Theorem 6.1.1 still holds if smooth is replaced by rationally smooth.

The concept of pattern avoidance has been generalize to the affine setting. In order to examine this, we start with a definition given in [2]:

Definition 6.1.2. Let $p \in S_k$ and let $w \in \widehat{W} = \widehat{S}_n$. Then w is said to contain the pattern p if there are indices $i_1 < i_2 < \cdots < i_k$ such that the substring $w_{i_1} w_{i_2} \cdots w_{i_k}$ follows that same relative order as the entries of p. Furthermore, w is said to avoid the pattern p if no such indices / substring exits.

For example, in $\widehat{S}_5$,

$$w = [-3, 6, 0, 8, 4]2, 11, 5, 13, 9, 7, \ldots$$

contains the pattern 3412

$$\begin{array}{lccccccccc} & & 3 & & 4 & & 1 & & 2 & \\ w = [-3, & & 6, & 0, & 8, & 4] & 2, & 11, & 5, & 13, \ldots \end{array}$$

As in the classical case, the patterns of interest in the affine context are 3412 and 4231. Indeed, Billey and Crites show that if $w \in \widehat{W}$ avoids these two patterns, then $X(w) \subset \mathcal{G}/\mathcal{B}$ is rationally smooth. However, in the affine setting, pattern avoidance no longer gives the full characterization of rational smoothness. One additional concept is required, namely, the notion of a twisted spiral permutation. As twisted spiral permutations are not explicitly used in this thesis, we direct the interested reader to Section 2.5 of [2] for the definition. Two features of twisted spiral permutations that are relevant to our work are that they all contain the pattern 3412 and that the affine Schubert varieties they index are singular (see [2]).

The central result of [2] is Theorem 1.1 (stated here as Theorem 6.1.3), in which they provide a means of identifying all rationally smooth Schubert varieties in $\mathcal{G}/\mathcal{B}$:

Theorem 6.1.3. *Let $w \in \widehat{W} = \widehat{S}_n$, where $n \geq 3$ and let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$. Then $X(w)$ is rationally smooth if and only if either:*

1) w avoids the patterns 3412 and 4231, or

2) w is a twisted spiral permutation.

Remark 6.1.4. Theorem 6.1.3 applies to the case in which $n \geq 3$. For $n = 2$, all Schubert varieties $X(w)$ are rationally smooth (see Example 4.6.3).

The corollary stated below (which is Corollary 1.2 in [2]) follows from Theorem 6.1.3, since any Schubert variety which is not rationally smooth is also not smooth (see Remark 4.6.4):

Corollary 6.1.5. *Let $w \in \widehat{W} = \widehat{S}_n$, where $n \geq 3$ and let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$. If w contains either a 3412 pattern or a 4231 pattern, then $X(w)$ is singular.*

In Conjecture 1 in [2] (stated here as Conjecture 6.1.6), Billey and Crites speculate that Corollary 6.1.5 is, in fact, an equivalence.

Conjecture 6.1.6. *Let $w \in \widehat{W}$ and let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$. Then $X(w)$ is smooth if and only if w avoids 3412 and 4231.*

Billey and Crites indicate in [2] that they have verified this conjecture up to $n = 5$. The case for $n = 2$ is handled in [17].

In this thesis, we will provide a proof for Conjecture 6.1.6 (See Theorem 6.5.2).

6.2 The Bruhat-Chevalley Order on $\widehat{W}$ and One-line Notation

In order to prove Conjecture 6.1.6, we will show that any Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ which is both rationally smooth and singular is indexed by an affine permutation w which contains the pattern 3412 (see Theorem 6.5.1

below). As such, our focus in on pattern detection, as opposed to pattern avoidance. The method that we have developed involves the entries which appear in the standard window of an affine permutation expressed in one-line notation:

$$w = [w_1, w_2, \ldots, w_n]$$

We want to be able to detect the pattern 3412 in w, however, our work in Chapter 5 involved maximal singularities $u \in X(w)^{\widehat{T}}$. In order to link the two, we want to understand how left multiplication with a reflection changes the entries which appear in the standard window of u.

In this section, we prove a rather technical lemma which compares the entries of two affine permutations u and $s_{\hat{\alpha}}u$ when $u < s_{\hat{\alpha}}u$. Although we did not find this result in the literature, there is a possibility that this result was previously known. Before we state this lemma, we will establish some notation.

Since $s_{\hat{\alpha}} = s_{-\hat{\alpha}}$ for any $\hat{\alpha} \in \widehat{\Phi}$, henceforth, we will let $\hat{\alpha} = \alpha + h\delta$, where $h \in \mathbb{Z}$ and $\alpha = (ab) \in \Phi^+$, with $1 \leq a < b \leq n$. Let $u = [u_1, u_2, \ldots, u_n] \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$ and let $1 \leq i, j \leq n$, with $i \neq j$ be the indices for which $\overline{u_i}, \overline{u_j} \in \{a, b\}$. Let $v = s_{\hat{\alpha}}u$. Thus $v = [v_1, v_2, \ldots, v_n]$, where $v_k = s_{\hat{\alpha}}(u_k)$, for every integer k.

Set

$$u_i = \overline{u_i} + nq,$$

where $1 \leq \overline{u_i} \leq n$ and $q \in \mathbb{Z}$ and

$$u_j = \overline{u_j} + np,$$

where $1 \leq \overline{u_j} \leq n$ and $p \in \mathbb{Z}$. Since $q, p \in \mathbb{Z}$, we have that $q = p + c$, for some $c \in \mathbb{Z}$.

Thus

$$v_i = s_{\hat{\alpha}}(u_i) \qquad\qquad v_j = s_{\hat{\alpha}}(u_j)$$

$$\begin{aligned}
v_i &= s_{\hat{\alpha}}(u_i) \\
&= s_{\hat{\alpha}}(\overline{u_i} + nq) \\
&= \begin{cases} \overline{u_j} + n(q+h) & \text{if } \ \overline{u_i} = a \\ \overline{u_j} + n(q-h) & \text{if } \ \overline{u_i} = b \end{cases} \\
&= \overline{u_j} + n(q \pm h) \\
&= \overline{u_j} + n(p + c \pm h) \\
&= u_j + n(c \pm h) \\
&= u_j + nd
\end{aligned}
\qquad\qquad
\begin{aligned}
v_j &= s_{\hat{\alpha}}(u_j) \\
&= s_{\hat{\alpha}}(\overline{u_j} + np) \\
&= \begin{cases} \overline{u_i} + n(p-h) & \text{if } \ \overline{u_j} = b \\ \overline{u_i} + n(p+h) & \text{if } \ \overline{u_j} = a \end{cases} \\
&= \overline{u_i} + n(p \mp h) \\
&= \overline{u_i} + n(q - c \mp h) \\
&= u_i - n(c \pm h) \\
&= u_i - nd
\end{aligned}$$

where $d = c \pm h \in \mathbb{Z}$. From these descriptions we obtain that

$$u_i > v_i \iff u_i > u_j + nd \iff u_i - nd > u_j \iff v_j > u_j,$$

equivalently (since $u_i \neq v_i$ and $u_j \neq v_j$),

$$u_i < v_i \iff v_j < u_j.$$

Furthermore,

$$v_j \leq u_i \iff u_i - nd \leq u_i \iff d \geq 0 \iff u_j \leq u_j + nd \iff u_j \leq v_i$$

and likewise,

$$v_j \geq u_i \iff u_j \geq v_i$$

Relabel $d = |c \pm h|$. So now, when $v_j \leq u_i$ and $u_j \leq v_i$ we have

$$v_j = u_i - nd = u_{i-nd} \quad \text{and} \quad v_i = u_j + dn = u_{j+dn}, \quad \text{for some } d \geq 0.$$

If $v_j < u_i$ and $u_j < v_i$, then $d \geq 1$.

If $v_j \geq u_i$ and $u_j \geq v_i$, then

$$v_j = u_i + nd = u_{i+nd} \quad \text{and} \quad v_i = u_j - dn = u_{j-dn}, \quad \text{for some } d \geq 0.$$

Again, if $v_j > u_i$ and $u_j > v_i$, then $d \geq 1$.

Consequently, there are only eight possible ways that u_i, u_j, v_i, and v_j can be ordered:

$$\begin{array}{ll} v_j < u_j < u_i < v_i & \qquad u_j < v_j < v_i < u_i \\ v_j \leq u_i < u_j \leq v_i & \qquad u_j \leq v_i < v_j \leq u_i \\ v_i \leq u_j < u_i \leq v_j & \qquad u_i \leq v_j < v_i \leq u_j \\ v_i < u_i < u_j < v_j & \qquad u_i < v_i < v_j < u_j \end{array}$$

Lemma 6.2.1. *Let $\hat{\alpha} = \alpha + h\delta = (ab) + h\delta$ be a real root with associated reflection $s_{\hat{\alpha}} \in \widehat{W}$ and let $u = [u_1, u_2, \ldots, u_n] \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$. Let $1 \leq i < j \leq n$ be the indices for which $\overline{u_i}, \overline{u_j} \in \{a, b\}$. Then $u < v = s_{\hat{\alpha}}u$ if and only if*

$$\begin{array}{lll} |u_i - u_j| < |v_i - v_j| & \textit{if} \;\; u_i > u_j & \qquad \textit{or} \\ |u_i - u_j| \leq |v_i - v_j| & \textit{if} \;\; u_j > u_i & \end{array}$$

Proof. We will first prove the "if" direction of this lemma. Suppose that

$$\begin{array}{lll} |u_i - u_j| < |v_i - v_j| & \text{if} \;\; u_i > u_j & \qquad \text{or} \\ |u_i - u_j| \leq |v_i - v_j| & \text{if} \;\; u_j > u_i & \end{array}$$

Given the eight possible orders for u_i, u_j, v_i, and v_j stated above, our assumption implies that exactly one of the following holds:

$$\begin{array}{ll} u_i - dn = v_j < u_j < u_i < v_i = u_j + dn, & \text{where } d \in \mathbb{Z}_{\geq 1}, \\ u_i - dn = v_j \leq u_i < u_j \leq v_i = u_j + dn, & \text{where } d \in \mathbb{Z}_{\geq 0} \\ u_j - dn = v_i < u_j < u_i < v_j = u_i + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \\ u_j - dn = v_i < u_i < u_j < v_j = u_i + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \end{array}$$

As $u < s_{\hat{\alpha}}u$ if and only if $\ell(u) < \ell(s_{\hat{\alpha}}u)$, we will show that $\ell(u) < \ell(s_{\hat{\alpha}}u)$. To that end, for each of these cases, we will construct a bijection ρ between $\mathrm{Inv}_{\widehat{S}_n}(u)$ and a proper subset of $\mathrm{Inv}_{\widehat{S}_n}(v)$. We first consider the initial two options above:

$$u_i - dn = v_j < u_j < u_i < v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}, \quad \text{or}$$
$$u_i - dn = v_j \leq u_i < u_j \leq v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 0}$$

Let k be an integer such that $1 \leq k \leq n$, but that $k \neq i, j$. Subsequently, $v_k = s_{\hat{\alpha}}(u_k) = u_k$. For every such k, we define ρ as follows, if applicable.

For any $(k, l) \in \mathrm{Inv}_{\widehat{S}_n}(u)$ with $l \not\equiv i, j \mod n$,

$$v_k = u_k > u_l = v_l$$

and so $(k, l) \in \mathrm{Inv}_{\widehat{S}_n}(v)$. Consequently, we define

$$\rho(k, l) = (k, l).$$

If $(k, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, then $k < j + ln$ and $u_k > u_{j+ln}$. Since $u_j > v_j$,

$$v_k = u_k > u_{j+ln} = u_j + ln > v_j + ln = v_{j+ln},$$

we know $(k, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ and hence we may define

$$\rho(k, j + ln) = (k, j + ln).$$

If $(k, i + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, then $k < i + ln$ and $u_k > u_{i+ln}$. If $v_k > v_{i+ln}$, then $(k, i + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ and we may set

$$\rho(k, i + ln) = (k, i + ln).$$

If $v_k < v_{i+ln}$, then $(k, i + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(v)$. However, as

$$v_k = u_k > u_{i+ln} = u_i + ln = u_i - dn + ln + dn = u_{i-dn} + (l+d)n = v_j + (l+d)n = v_{j+(l+d)n}$$

and

$$k < i + ln < j + ln \leq j + (l + d)n$$

we have $\big(k, j + (l + d)n\big) \in \mathrm{Inv}_{\widehat{S}_n}(v)$. Furthermore, since

$$u_k = v_k < v_{i+ln} = v_i + ln = u_{j+dn} + ln = u_{j+(l+d)n},$$

$\big(k, j + (l + d)n\big) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$, which means that $\big(k, j + (l + d)n\big) \notin \mathrm{Im}(\rho)$ as currently defined. Subsequently, we may set

$$\rho(k, i + ln) = \big(k, j + (l + d)n\big).$$

If $(i, k+ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, then $i < k+ln$ and $u_i > u_{k+ln}$. Since

$$v_i > u_i > u_{k+ln} = u_k + ln = v_k + ln = v_{k+ln},$$

$(i, k + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$. Hence, we define

$$\rho(i, k + ln) = (i, k + ln)$$

For any $(j, k + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, where $l \geq 0$ is an integer, we have $j < k + ln$ and $u_j > u_{k+ln}$. If $v_j > v_{k+ln}$, then $(j, k + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ and we let

$$\rho(j, k + ln) = (j, k + ln)$$

However, if $v_j < v_{k+ln}$, then $(j, k + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(v)$, but fortunately we have $\big(i, k + (d + l)n\big) \in \mathrm{Inv}_{\widehat{S}_n}(v)$, as

$$v_i = u_{j+dn} = u_j + dn > u_{k+ln} + dn = u_k + (l + d)n = v_k + (l + d)n = v_{k+(l+d)n}$$

and

$$i < j < k + ln \leq k + (l + d)n$$

In addition,

$$u_i = u_i - dn + dn = u_{i-dn} + dn = v_j + dn < v_{k+ln} + dn = v_{k+(l+d)n}$$

so that $\big(i, k + (l + d)n\big) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$. Consequently, as defined thus far,

$\big(i, k + (l + d)n\big) \notin \mathrm{Im}(\rho)$. So let

$$\rho(j, k + ln) = \big(i, k + (l + d)n\big)$$

At this point, we consider the cases where $u_j < u_i$ and $u_i < u_j$ separately in order to define ρ for the remaining elements of $\mathrm{Inv}_{\widehat{S}_n}(u)$.

To that end, suppose that $u_j < u_i$, that is,

$$u_i - dn = v_j < u_j < u_i < v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$

This means that $u_j < u_i + ln$, for all $l \geq 0$, and hence $(j, i + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$ for any $l \geq 0$.

Since $v_i = u_{j+dn} > u_i$, if $\mathrm{Inv}_{\widehat{S}_n}(u)$ contains an element of the form $(i, j + ln)$, then it must be the case that $0 \leq l < d$. So if $(i, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $0 \leq l < d$, then $i < j + ln$ and $u_i > u_{j+ln}$. From,

$$v_i > u_i > u_{j+ln} = u_j + ln > v_j + ln = v_{j+ln},$$

we obtain that $(i, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ and so we may set

$$\rho(i, j + ln) = (i, j + ln).$$

At this stage, we have defined the image under ρ of all possible elements of $\mathrm{Inv}_{\widehat{S}_n}(u)$. As $l < d$ and $d \geq 1$, we observe that $(i, j + dn) \notin \mathrm{Im}(\rho)$, but we do have $(i, j + dn) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ since

$$v_i > u_i = u_i - dn + dn = u_{i-dn} + dn = v_j + dn = v_{j+dn}$$

and

$$i < j < j + dn.$$

Thus, ρ is an injective map onto a proper subset of $\mathrm{Inv}_{\widehat{S}_n}(v)$ and hence

$$\ell(u) = \left|\mathrm{Inv}_{\widehat{S}_n}(u)\right| < \left|\mathrm{Inv}_{\widehat{S}_n}(v)\right| = \ell(v),$$

as required.

Now suppose that $u_i < u_j$, that is,

$$u_i - dn = v_j \leq u_i < u_j \leq v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 0}$$

Since $u_i < u_j + ln = u_{j+ln}$, for all $l \geq 0$, we have that $(i, j + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$ for any $l \geq 0$.

If $(j, i + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$ for some $l \geq 1$, then $j < i + ln$ and $u_j > u_{i+ln}$. (Note that $l \neq 0$, since $i < j$.)

As

$$v_j < v_i < v_i + ln = v_{i+ln}$$

we obtain that $(j, i + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(v)$. However,

$$v_i \geq u_j > u_{i+ln} = u_i + ln \geq v_j + ln = v_{j+ln}$$

and

$$i < j < j + ln.$$

Thus, $(i, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ and we may define

$$\rho(j, i + ln) = (i, j + ln).$$

Since $i < j$ and $v_i > v_j$, $(i, j) \in \mathrm{Inv}_{\widehat{S}_n}(v)$. However, since $l \geq 1$ in the final assignment above, we see that $(i, j) \notin \mathrm{Im}(\rho)$. Thus, as in the previous case, ρ is an injective map onto a proper subset of $\mathrm{Inv}_{\widehat{S}_n}(v)$. As a result,

$$\ell(u) = \left|\mathrm{Inv}_{\widehat{S}_n}(u)\right| < \left|\mathrm{Inv}_{\widehat{S}_n}(v)\right| = \ell(v),$$

as required.

We now move on to the last two options given at the beginning of this proof.

Accordingly, suppose that

$$\begin{aligned}
&u_j - dn = v_i < u_j < u_i < v_j = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}, \quad \text{or} \\
&u_j - dn = v_i < u_i < u_j < v_j = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}
\end{aligned}$$

As before, we will define a bijection ρ from $\mathrm{Inv}_{\widehat{S}_n}(u)$ to a proper subset of $\mathrm{Inv}_{\widehat{S}_n}(v)$. However, as the justification that ρ is well defined and injective is straightforward and very similar to the previous case, in most of what follows, we will only give the assignment rules defining ρ.

Now, for every integer k such that $1 \leq k \leq n$ and $k \neq i, j$, wherever applicable, we define ρ in the following manner:

For any $(k, l) \in \mathrm{Inv}_{\widehat{S}_n}(u)$ with $l \not\equiv i, j \mod n$, we define

$$\rho(k, l) = (k, l).$$

If $(k, i + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, we set

$$\rho(k, i + ln) = (k, i + ln)$$

For any $(k, j + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, we let

$$\rho(k, j + ln) = \begin{cases} (k, j + ln) & \text{if} \quad v_k > v_{j+ln} \\ \big(k, i + (l + d)n\big) & \text{if} \quad v_k < v_{j+ln} \end{cases}$$

If $(j, k + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, we set

$$\rho(j, k + ln) = (j, k + ln)$$

If $(i, k + ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, we define

$$\rho(i, k + ln) = \begin{cases} (i, k + ln) & \text{if} \quad v_i > v_{k+ln} \\ \big(j, k + (l + d)n\big) & \text{if} \quad v_i < v_{k+ln} \end{cases}$$

In the case that $u_i < u_j$, we have $(i, j + ln) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$, for any integer $l \geq 0$.

However, if $(j, i+ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $0 < l < d$, then we define

$$\rho(j, i+ln) = (j, i+ln)$$

For this definition of ρ, $(j, i+dn)$ is an element of $\mathrm{Inv}_{\widehat{S}_n}(v)$ which is not in $\mathrm{Im}(\rho)$.

If instead $u_j < u_i$, we know $(j, i+ln) \notin \mathrm{Inv}_{\widehat{S}_n}(u)$, for any integer $l \geq 0$. If $(i, j+ln) \in \mathrm{Inv}_{\widehat{S}_n}(u)$, for some integer $l \geq 0$, then $i < j+ln$ and $u_i > u_{j+ln}$. As

$$v_{j+ln} = v_j + ln > v_i + ln = v_{i+ln},$$

$(i, j+ln) \notin \mathrm{Inv}_{\widehat{S}_n}(v)$. However, since $d \geq 1$, we have

$$v_j = u_i + dn \geq u_i + n > u_{j+ln} + n = u_j + ln + n - dn + dn = v_i + (l+d+1)n = v_{i+(l+d+1)n},$$

and

$$1 \leq i < j \leq n < i + (l+d+1)n$$

so that $\big(j, i+(l+d+1)n\big) \in \mathrm{Inv}_{\widehat{S}_n}(v)$. As such, we define

$$\rho(i, j+ln) = \big(j, i+(l+d+1)n\big).$$

We have that $(j, i+dn) \in \mathrm{Inv}_{\widehat{S}_n}(v)$ as a result of

$$v_j = u_i + dn > v_i + dn = v_{i+dn}$$

and, since $d \geq 1$,

$$1 \leq i < j \leq n < i + dn.$$

It is clear from our definition of ρ that $(j, i+dn) \notin \mathrm{Im}(\rho)$.

Consequently, once again we have that ρ is an injective map onto a proper subset of $\mathrm{Inv}_{\widehat{S}_n}(v)$ and this yields

$$\ell(u) = \left|\mathrm{Inv}_{\widehat{S}_n}(u)\right| < \left|\mathrm{Inv}_{\widehat{S}_n}(v)\right| = \ell(v),$$

as required. With this, we have now completed the proof that if

$$\begin{aligned} |u_i - u_j| &< |v_i - v_j| \quad \text{if} \quad u_i > u_j \qquad \text{or} \\ |u_i - u_j| &\leq |v_i - v_j| \quad \text{if} \quad u_j > u_i \end{aligned}$$

then $u < s_{\hat{\alpha}} u$.

To verify the "only if" direction of this lemma, we prove its contrapositive. Assume

$$\begin{aligned} |u_i - u_j| &\geq |v_i - v_j| \quad \text{if} \quad u_i > u_j \qquad \text{or} \\ |u_i - u_j| &> |v_i - v_j| \quad \text{if} \quad u_j > u_i \end{aligned}$$

We will show that $s_{\hat{\alpha}} u < u$. Given the eight possible ways to order u_i, u_j, v_i, v_j, we only have to consider the following:

$$\begin{gathered} u_j < v_j < v_i < u_i \\ u_j \leq v_i < v_j \leq u_i \\ u_i < v_j < v_i < u_j \\ u_i < v_i < v_j < u_j \end{gathered}$$

From options one and three, we obtain

$$|v_i - v_j| < |u_i - u_j| \quad \text{if} \;\; v_i > v_j$$

and from options two and four, we know

$$|v_i - v_j| \leq |u_i - u_j| \quad \text{if} \;\; v_j > v_i$$

Since $v = s_{\hat{\alpha}} u$, we also have $s_{\hat{\alpha}} v = u$, where, in particular, $u_i = s_{\hat{\alpha}}(v_i)$ and $u_j = s_{\hat{\alpha}}(v_j)$. As a result, we have

$$v < s_{\hat{\alpha}} v$$

(by applying the "if " part of this lemma, which was proved above, to v as opposed to u), which yields

$$s_{\hat{\alpha}} u < u$$

as required. □

In the proof of 6.2.1, we identified some relationships which will be useful in subsequent proofs. As such, we restate these in the following remark, for ease of reference.

Remark 6.2.2. Let $u = [u_1, u_2, \ldots, u_n] \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$ and $\hat{\alpha} = \alpha + h\delta = (ab) + h\delta$ be a real root with associated reflection $s_{\hat{\alpha}} \in \widehat{W}$ such that $u < v = s_{\hat{\alpha}}u$. Let $1 \leq i < j \leq n$ be the indices for which $\overline{u_i}, \overline{u_j} \in \{a, b\}$ and let $v_i = s_{\hat{\alpha}}(u_i)$ and $v_i = s_{\hat{\alpha}}(u_i)$.

Thus, by Lemma 6.2.1 we have

$$\begin{aligned} |u_i - u_j| &< |v_i - v_j| \quad \text{if} \quad u_i > u_j \qquad \text{or} \\ |u_i - u_j| &\leq |v_i - v_j| \quad \text{if} \quad u_j > u_i \end{aligned}$$

which yields

$$\begin{aligned} &u_j - dn = v_i < u_j < u_i < v_j = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1} \\ &u_i - dn = v_j < u_j < u_i < v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1} \\ &u_j - dn = v_i < u_i < u_j < v_j = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1} \qquad \text{or} \\ &u_i - dn = v_j \leq u_i < u_j \leq v_i = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 0} \end{aligned}$$

6.3 Pattern Detection and Wide $\widehat{T}$-Fixed Points

In this section, we will introduce our method for detecting the pattern 3412 in the one-line notation of an affine permutation. We begin with the following definition.

Definition 6.3.1. A $\widehat{T}$-fixed point $u = [u_1, u_2, \ldots, u_n]$ in $\mathcal{G}/\mathcal{B}$ is said to be *wide* if there exists indices $1 \leq i < j \leq n$ such that either $u_i - u_j > 2n$ or $u_j - u_i > 3n$.

The significance of this definition is made clear in the next lemma.

Lemma 6.3.2. *Let $u = [u_1, u_2, \ldots, u_n] \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$. If u is wide, then u contains the pattern 3412.*

Proof. Suppose there exists indices $1 \le i < j \le n$ such that $u_i - u_j > 2n$. Thus

$$u_i > u_j + 2n = u_{j+2n}$$

and so

$$u_{j+n} < u_{j+2n} < u_i < u_{i+n}$$

Given that $i < j$, we have $i < i+n < j+n < j+2n$. Consequently,

$$u_i u_{i+n} u_{j+n} u_{j+2n}$$

is a subword of u which forms a 3412 pattern.

Likewise, if there exists indices $1 \le i < j \le n$ for which $u_j - u_i > 3n$, we have

$$u_j > u_i + 3n = u_{i+3n}$$

and hence

$$u_{i+2n} < u_{i+3n} < u_j < u_{j+n}$$

Since $1 \le i < j \le n$, we know $1+n < j+n \le 2n$ and $1+2n \le i+2n < 3n$, so that $j+n < i+2n$.

Thus, $j < j+n < i+2n < i+3n$ and hence we have obtained a subword

$$u_j u_{j+n} u_{i+2n} u_{i+3n}$$

of u which forms a 3412 pattern. □

From Lemma 6.3.2, it should be clear that our goal is to prove that any w indexing a rationally smooth, but singular Schubert variety $X(w)$ is wide. To do this, we first show that if an affine permutation is u is wide, then any $\widehat{T}$-fixed point above u in the Bruhat-Chevalley order on $\widehat{W}$ is also wide.

Lemma 6.3.3. *Let $\hat{\alpha}$ be a real root with associated reflection $s_{\hat{\alpha}} \in \widehat{W}$ and let $u \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$. If u is wide and $u < v = s_{\hat{\alpha}} u$, then v is wide.*

Proof. Suppose that $u = [u_1, u_2, \ldots, u_n]$ is wide and, as such, let $1 \leq i < j \leq n$ be indices for which either $u_i - u_j > 2n$ or $u_j - u_i > 3n$. Set $v = [v_1, v_2, \ldots, v_n]$, so that $v_i = s_{\hat{\alpha}}(u_i) = u_i$, set $v_j = s_{\hat{\alpha}}(u_j) = u_j$. Finally, let $\hat{\alpha} = (ab) + h\delta$.

We will consider four main cases: $\overline{u_i}, \overline{u_j} \notin \{a, b\}$, $\overline{u_i}, \overline{u_j} \in \{a, b\}$, $\overline{u_i}, \in \{a, b\}$, but $\overline{u_j} \notin \{a, b\}$, and $\overline{u_j} \in \{a, b\}$, but $\overline{u_i} \notin \{a, b\}$,

We will first deal with the case in which $\overline{u_i}, \overline{u_j} \notin \{a, b\}$. Thus $v_i = u_i$ and $v_j = u_j$. Consequently, if $u_i - u_j > 2n$, then $v_i - v_j > 2n$ and if $u_j - u_i > 3n$, then $v_j - v_i > 3n$. Hence v is wide.

Now suppose that $\overline{u_i}, \overline{u_j} \in \{a, b\}$. By Remark 6.2.2 there are four possibilities. If $u_i - u_j > 2n$ (that is, $u_i > u_j$), there are two options:

$$u_j - dn = v_i < u_j < u_i < v_j = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$
$$v_j < u_j < u_i < v_i$$

Hence,

$$v_j - v_i = u_i + dn - (u_j - dn) = u_i - u_j + 2dn > 2n + 2dn \geq 4n > 3n,$$

since $d \geq 1$, or

$$v_i - v_j > u_i - u_j > 2n,$$

respectively. As $i < j$, either way, we obtain that v is wide. If $u_j - u_i > 3n$ (that is, $u_j > u_i$), then the remaining two possibilities from Remark 6.2.2 are

$$v_i < u_i < u_j < v_j$$
$$v_j \leq u_i < u_j \leq v_i$$

Thus,

$$v_j - v_i > u_j - u_i > 3n$$

or

$$v_i - v_j \geq u_j - u_i > 3n > 2n$$

Once again, both of these situations yield that v is wide.

Now for case three, we assume that $\overline{u_i}, \overline{u_k} \in \{a, b\}$, for some $1 \leq k \leq n$ such that $k \neq i, j$ (so $\overline{u_j} \notin \{a, b\}$). Thus $v_j = u_j$. By Remark 6.2.2, one of the following must hold:

$$u_k - dn = v_i < u_i < u_k < v_k = u_i + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$

$$\begin{cases} u_k - dn = v_i < u_k < u_i < v_k = u_i + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \ \text{ if } \ i < k \\ u_k - dn = v_i \leq u_k < u_i \leq v_k = u_i + dn, & \text{where } d \in \mathbb{Z}_{\geq 0} \ \text{ if } \ k < i \end{cases}$$

$$\begin{cases} u_i - dn = v_k \leq u_i < u_k \leq v_i = u_k + dn, & \text{where } d \in \mathbb{Z}_{\geq 0} \ \text{ if } \ i < k \\ u_i - dn = v_k < u_i < u_k < v_i = u_k + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \ \text{ if } \ k < i \end{cases}$$

$$u_i - dn = v_k < u_k < u_i < v_i = u_k + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$

We will go through each option in the order given above. We know $i < j$, but we could have $k > j$ or $k < j$. However, for computations involving v_j and v_k, if the difference computed is at least $3n$, then v is wide for either case.

Assuming $u_i - u_j > 2n$:

$v_k - v_j = u_i + dn - u_j > 2n + dn > 3n$ (since $d \geq 1$)

$$v_k - v_j = u_i + dn - u_j > 2n + dn > \begin{cases} 3n & \text{if } \ i < k & (\text{so } \ d \geq 1) \\ 2n & \text{if } \ k < i < j & (\text{so } \ d \geq 0) \end{cases}$$

For the third and forth possibilities we have $v_i - v_j > u_i - u_j > 2n$.

Assuming $u_j - u_i > 3n$:

For options one and two we have $v_j - v_i > u_j - u_i > 3n$.

For possibility three we obtain

$$v_j - v_k \quad \begin{cases} \geq u_j - u_i > 3n & \text{if } \ i < k \\ > u_j - u_i > 3n & \text{if } \ k < i \end{cases}$$

For the fourth, we have $v_j - v_k > u_j - u_i > 3n$.

Collectively, these computations show that v is wide in this case.

For our fourth and final case, we assume that $\overline{u_j}, \overline{u_k} \in \{a, b\}$, for some $1 \leq k \leq n$ such that $k \neq i, j$ (so $\overline{u_i} \notin \{a, b\}$). Therefore, $v_i = u_i$. As in the previous case, Remark 6.2.2 gives us the only possible relationships:

$$u_k - dn = v_j < u_j < u_k < v_k = u_j + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$

$$\begin{cases} u_k - dn = v_j < u_k < u_j < v_k = u_j + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \quad \text{if} \quad j < k \\ u_k - dn = v_j \leq u_k < u_j \leq v_k = u_j + dn, & \text{where } d \in \mathbb{Z}_{\geq 0} \quad \text{if} \quad k < j \end{cases}$$

$$\begin{cases} u_j - dn = v_k \leq u_j < u_k \leq v_j = u_k + dn, & \text{where } d \in \mathbb{Z}_{\geq 0} \quad \text{if} \quad j < k \\ u_j - dn = v_k < u_j < u_k < v_j = u_k + dn, & \text{where } d \in \mathbb{Z}_{\geq 1} \quad \text{if} \quad k < j \end{cases}$$

$$u_j - dn = v_k < u_k < u_j < v_j = u_k + dn, \quad \text{where } d \in \mathbb{Z}_{\geq 1}$$

As above, we have that $i < j$, but either $k > i$ or $k < i$ is possible. Once again, for computations involving v_i and v_k, if the difference computed is at least $3n$, then v is wide for either case.

If $u_i - u_j > 2n$:

For the first two options, we have $v_i - v_j = u_i - u_j > 2n$.

For possibility three:

$$v_i - v_k = u_i - (u_j - dn) > 2n + dn > \begin{cases} 2n & \text{if} \quad i < j < k \quad (\text{so} \ \ d \geq 0) \\ 3n & \text{if} \quad k < j \qquad\quad (\text{so} \ \ d \geq 1) \end{cases}$$

For the fourth:

$$v_i - v_k = u_i - (u_j - dn) > 2n + dn > 3n \ \ (\text{since} \ \ d \geq 1)$$

If $u_j - u_i > 3n$:

The first situations yields $v_k - v_i > u_j - u_i > 3n$.

The second gives

$$v_k - v_i \quad \begin{cases} > u_j - u_i > 3n & \text{if} \quad j < k \\ \geq u_j - u_i > 3n & \text{if} \quad k < j \end{cases}$$

For options three and four we have $v_j - v_i > u_j - u_i > 3n$.

Together, these give that v is wide. □

Corollary 6.3.4.

Let $u, v \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$ such that $u < v$. If u is wide, then so is v.

Proof. Since $u < v$, we have that

$$u < s_{\hat{\alpha}_1} u < s_{\hat{\alpha}_2} s_{\hat{\alpha}_1} u < \cdots < s_{\hat{\alpha}_k} s_{\hat{\alpha}_{k-1}} \cdots s_{\hat{\alpha}_2} s_{\hat{\alpha}_1} u = v$$

for some $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_k \in \widetilde{\Phi}$, for some $k \geq 1$. From repeated applications of Lemma 6.3.3, we obtain that each of the $\widehat{T}$-fixed points in this chain are wide, and hence v is wide. □

The previous result required that we started with a wide $\widehat{T}$-fixed point. However, in the next lemma, we give a condition on the real root $\hat{\alpha}$ associated with a reflection $s_{\hat{\alpha}}$ which guarantees that $s_{\hat{\alpha}} u$ is wide when $s_{\hat{\alpha}} u > u$, even if u is not.

Lemma 6.3.5. *Let $\hat{\alpha} = \alpha + h\delta$, where $|h| \geq 3$, be a real root with associated reflection $s_{\hat{\alpha}} \in \widehat{W}$ and let $u \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$. If $u < v = s_{\hat{\alpha}} u$, then v is wide.*

Proof. Let $\alpha = (ab) \in \Phi^+$, where $1 \leq a < b \leq n$ and let $h \in \mathbb{Z}$. Let $u = [u_1, u_2, \ldots, u_n]$ and let $u_i = \overline{u_i} + nq$, for some $q \in \mathbb{Z}$ and $u_j = \overline{u_j} + np$, for some $p \in \mathbb{Z}$, be the entries in u with $1 \leq i < j \leq n$ such that $\overline{u_i}, \overline{u_j} \in \{a, b\}$. Let Set $v = [v_1, v_2, \ldots, v_n] = s_{\hat{\alpha}} u$, so that $v_i = s_{\alpha+h\delta}(u_i)$ and $v_j = s_{\alpha+h\delta}(u_j)$.

By Remark 6.2.2, we know that

$$|u_i - u_j| < |v_i - v_j| \quad \text{if} \quad u_i > u_j$$
$$|u_i - u_j| \leq |v_i - v_j| \quad \text{if} \quad u_j > u_i$$

and, as such, there are four ways to order u_i, u_j, v_i, and v_j:

$$v_i < u_j < u_i < v_j$$
$$v_j < u_j < u_i < v_i$$
$$v_i < u_i < u_j < v_j$$
$$v_j \leq u_i < u_j \leq v_i$$

Suppose that

$$u_i = a + nq$$
$$u_j = b + np$$

Thus

$$v_i = b + n(q + h)$$
$$v_j = a + n(p - h)$$

If $h \geq 3$, then $v_i > u_i$ and $v_j < u_j$, so one of

$$v_j < u_j < u_i < v_i$$
$$v_j \leq u_i < u_j \leq v_i$$

holds. Either way, $v_i > v_j$, so that $v_i - v_j > 0$. Since $1 \leq a < b \leq n$, we have $b - a \geq 1$. If $u_i > u_j$, then, since $a < b$, we know that $q > p$ and hence $q - p \geq 1$. If $u_j > u_i$, then

$$\begin{aligned} & u_j - u_i = |u_i - u_j| \leq |v_i - v_j| = v_i - v_j \\ \implies & b + np - (a + nq) \leq b + n(q + h) - (a + n(p - h)) \\ \implies & b - a + n(p - q) \leq b - a + n(q - p) + 2nh \\ \implies & 2n(p - q) \leq 2nh \\ \implies & q - p \geq -h \end{aligned}$$

Combining these facts, we obtain

$$\begin{aligned}
v_i - v_j &= b + n(q+h) - \big(a + n(p-h)\big) \\
&= b - a + n(q-p) + 2nh \\
&\geq \begin{cases} 1 + n + 2nh & \text{if} \quad u_i > u_j \\ 1 - nh + 2nh & \text{if} \quad u_j > u_i \end{cases} \\
&\geq \begin{cases} 1 + 7n & \text{if} \quad u_i > u_j \\ 1 + 3n & \text{if} \quad u_j > u_i \end{cases}
\end{aligned}$$

with the final inequality resulting from the fact that $h \geq 3$.

In both cases, we have $v_i - v_j > 2n$ and hence v is wide.

If instead $h \leq -3$, then $v_i < u_i$ and $v_j > u_j$, in which case one of

$$v_i < u_j < u_i < v_j$$
$$v_i < u_i < u_j < v_j$$

holds. As such, $v_j > v_i$, so that $v_j - v_i > 0$. Since $1 \leq a < b \leq n$, we know that $a - b \geq 1 - n$. If $u_j > u_i$, then, as $a < b$, we have $p \geq q$ and so $p - q \geq 0$. If $u_i > u_j$, then

$$\begin{aligned}
u_i - u_j = |u_i - u_j| &< |v_i - v_j| = v_j - v_i \\
\implies a + nq - (b + np) &< a + n(p-h) - (b + n(q+h)) \\
\implies a - b + n(q-p) &< a - b + n(p-q) - 2nh \\
\implies 2n(q-p) &< -2nh \\
\implies p - q &> h \\
\implies p - q &\geq h + 1
\end{aligned}$$

since $p, q \in \mathbb{Z}$.

Subsequently,

$$\begin{aligned} v_j - v_i &= a + n(p-h) - \big(b + n(q+h)\big) \\ &= a - b + n(p-q) - 2nh \\ &\geq \begin{cases} 1 - n + 0 - 2nh = 1 - n(2h+1) & \text{if} \quad u_j > u_i \\ 1 - n + n(h+1) - 2nh = 1 - nh & \text{if} \quad u_i > u_j \end{cases} \\ &\geq \begin{cases} 1 + 5n & \text{if} \quad u_j > u_i \\ 1 + 3n & \text{if} \quad u_i > u_j \end{cases} \end{aligned}$$

with the final inequality owing to the fact that $h \leq -3$.

Therefore, $v_j - v_i > 3n$ and subsequently v is wide.

The case in which

$$u_i = b + nq$$
$$u_j = a + np$$

follows essentially the same argument as the other case. Here we have

$$v_i = a + n(q-h)$$
$$v_j = b + n(p+h)$$

If $h \geq 3$, we know one of the following is true:

$$v_i < u_j < u_i < v_j$$
$$v_i < u_i < u_j < v_j$$

Hence, $v_j > v_i$ and the end result is that

$$v_j - v_i \geq \begin{cases} 1 + 7n & \text{if} \quad u_j > u_i \\ 1 + 4n & \text{if} \quad u_i > u_j \end{cases}$$

Thus, $v_j - v_j > 3n$ and therefore v is wide.

Finally, if $h \leq -3$, it follows that one of

$$v_j < u_j < u_i < v_i$$
$$v_j < u_i < u_j < v_i$$

is true. Hence $v_i > v_j$ and

$$v_i - v_j \geq \begin{cases} 1 + 5n & \text{if} \quad u_i > u_j \\ 1 + 2n & \text{if} \quad u_j > u_i \end{cases}$$

Hence $v_i - v_j > 2n$ and as a result, v is wide. $\square$

6.4 Wide $\widehat{T}$-Fixed Points and Kite Patterns

In Section 5.9, we proved that every maximal singularity of a singular rationally smooth Schubert variety $X(w)$ satisfies either the $(-\alpha)$–kite property in $X(w)$ or $(\alpha - \delta)$– kite property in $X(w)$. In this section, we prove that the $\widehat{T}$-fixed point at the top of either kite pattern is wide.

Theorem 6.4.1. *Let $u \in (\mathcal{G}/\mathcal{B})^{\widehat{T}}$. If u satisfies the $(\alpha - \delta)$–kite property for some $\alpha \in \Phi^+$, then $s_\alpha s_{\alpha-\delta} u$ is wide and hence contains the pattern 3412. If u satisfies the $(-\alpha)$–kite property for some $\alpha \in \Phi^+$, then $s_{\alpha-\delta} s_\alpha u$ is wide and hence contains the pattern* 3412.

Proof. Suppose that u satisfies the $(-\alpha)$–kite property. Thus,

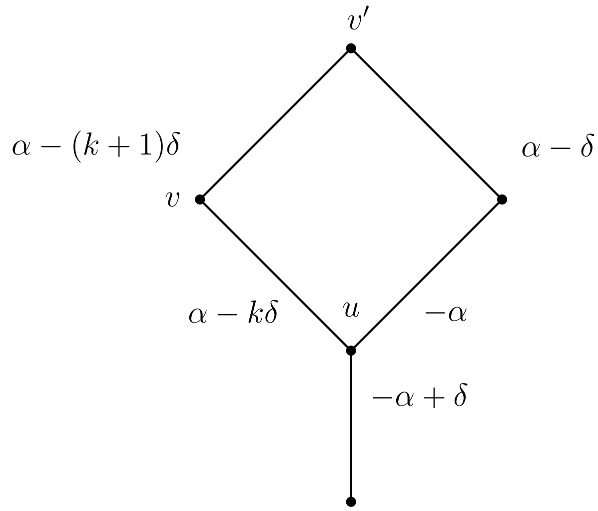
$$s_{\alpha-\delta} u < u < s_\alpha u < s_{\alpha-\delta} s_\alpha u$$

and

$$u < s_{(-\alpha+k\delta)} u < s_{(-\alpha+(k+1)\delta)}\, s_{(-\alpha+k\delta)} u$$

for some integer $k \geq 2$. Let $v = s_{(-\alpha+k\delta)} u$ and $v' = s_{(-\alpha+(k+1)\delta)} v$ $(= s_{\alpha-\delta} s_\alpha u)$.

The Bruhat graph showing these relations is:



Since $k \geq 2$, we know that $k + 1 \geq 3$. It follows from Lemma 6.3.5 that $v' = s_{(-\alpha+(k+1)\delta)}v$ is wide and, as such, contains the pattern 3412, by Lemma 6.3.2.

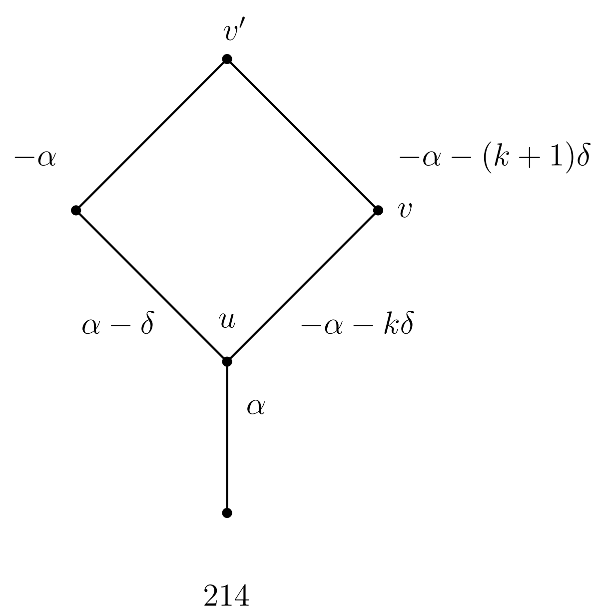
Now suppose that u satisfies the $(\alpha - \delta)$–*kite property*. Therefore,

$$s_\alpha u < u < s_{\alpha-\delta}u < s_\alpha s_{\alpha-\delta}u$$

and

$$u < s_{(\alpha+k\delta)}u < s_{(\alpha+(k+1)\delta)}s_{(\alpha+k\delta)}u$$

for some integer $k \geq 1$. Let $v = s_{(\alpha+k\delta)}u$ and $v' = s_{(\alpha+(k+1)\delta)}v$. The Bruhat graph showing these relations is:



For $k \geq 2$, we have that $k+1 \geq 3$. Therefore, by Lemma 6.3.5, $v' = s_{(\alpha+(k+1)\delta)}v$ is wide and subsequently, by Lemma 6.3.2, v' contains the pattern 3412.

We are only left to consider $k = 1$, in which case, $v = s_{(\alpha+\delta)}u$ and $v' = s_{(\alpha+2\delta)}v$. Let $\alpha = (ab) \in \Phi^+$, where $1 \leq a < b \leq n$ and set $u = [u_1, u_2, \ldots, u_n]$. Let $u_i = \overline{u_i} + nq$, for some $q \in \mathbb{Z}$ and $u_j = \overline{u_j} + np$, for some $p \in \mathbb{Z}$, be the entries in u with $1 \leq i < j \leq n$ such that $\overline{u_i}, \overline{u_j} \in \{a, b\}$. Let $v = [v_1, v_2, \ldots, v_n]$ and $v' = [v'_1, v'_2, \ldots, v'_n]$.

If $\overline{u_i} = a$ and $\overline{u_j} = b$ (so that $\overline{u_i} < \overline{u_j}$), then

$$\begin{aligned} v_i &= \overline{u_j} + n(q+1) \\ v_j &= \overline{u_i} + n(p-1) \end{aligned}$$

and

$$\begin{aligned} v'_i &= \overline{u_i} + n(q+1-2) = \overline{u_i} + n(q-1) \\ v'_j &= \overline{u_j} + n(p-1+2) = \overline{u_j} + n(p+1) \end{aligned}$$

It follows that

$$v_i - v_j = \overline{u_j} - \overline{u_j} + n(q-p+2)$$

and

$$v'_j - v'_i = \overline{u_j} - \overline{u_i} + n(p-q+2)$$

Also, since $\overline{u_i} < \overline{u_j}$, we have that $\overline{u_j} - \overline{u_i} \geq 1$.

If $q - p \geq 0$, then

$$v_i - v_j = \overline{u_j} - \overline{u_j} + n(q-p+2) \geq 1 + 2n > 2n$$

As a result, v is wide and thus, from Lemma 6.3.3, we obtain that v' is wide. Therefore, by Lemma 6.3.2, v' contains the pattern 3412.

Otherwise, $p - q \geq 1$, from which we obtain

$$v'_j - v'_i = \overline{u_j} - \overline{u_i} + n(p-q+2) \geq 1 + 3n > 3n$$

Hence, v' is wide and so, by Lemma 6.3.2, v' contains the pattern 3412.

On the other hand, if $\overline{u_i} = b$ and $\overline{u_j} = a$ (so that $\overline{u_i} > \overline{u_j}$), then

$$\begin{aligned}
v_i &= \overline{u_j} + n(q-1) \\
v_j &= \overline{u_i} + n(p+1) \\
v'_i &= \overline{u_i} + n(q-1+2) = \overline{u_i} + n(q+1) \\
v'_j &= \overline{u_j} + n(p+1-2) = \overline{u_j} + n(p-1)
\end{aligned}$$

which gives

$$v_j - v_i = \overline{u_j} - \overline{u_j} + n(p-q+2)$$

and

$$v'_i - v'_j = \overline{u_i} - \overline{u_j} + n(q-p+2)$$

Again, as $\overline{u_i} > \overline{u_j}$, we have that $\overline{u_i} - \overline{u_j} \geq 1$.

This time, if $q - p \geq 0$, then

$$v'_i - v'_j = \overline{u_i} - \overline{u_j} + n(q-p+2) \geq 1 + 2n > 2n$$

Consequently, v' is wide and hence v' contains the pattern 3412, by Lemma 6.3.2.

However, if $p - q \geq 1$, we compute that

$$v'_j - v'_i = \overline{u_j} - \overline{u_i} + n(p-q+2) \geq 1 + 3n > 3n$$

which guarantees that v is wide. Due to Lemma 6.3.3, v' is also wide and thus, owing to Lemma 6.3.2 we have that v' contains the pattern 3412. $\square$

6.5 Producing the Pattern 3412

With the previous lemma in hand, we are now in a position to prove our main result of this chapter, specifically, that the $\widehat{T}$-fixed point w that indexes a Schubert variety $X(w)$ in $\mathcal{G}/\mathcal{B}$ which is both rationally smooth and singular contains the pattern 3412.

Theorem 6.5.1. *Let $X(w)$ be a singular rationally smooth Schubert variety in $\mathcal{G}/\mathcal{B}$. Then w contains the pattern* 3412.

Proof. Let $u \in X(w)^{\widehat{T}}$ be a maximal singularity of $X(w)$. By Theorem 5.9.5, u satisfies either the $(-\alpha)$–kite property in $X(w)$ or the $(\alpha - \delta)$–kite property in $X(w)$. From Theorem 6.4.1 we obtain that v' is wide, where $v' = s_{\alpha-\delta}s_{\alpha}u$ in the case the u satisfies the $(-\alpha)$–kite property in $X(w)$ or $v' = s_{\alpha}s_{\alpha-\delta}u$ in the case that u satisfies the $(\alpha - \delta)$–kite property in $X(w)$. Since v' is wide and $v' < w$, it follows from Corollary 6.3.4, that w is wide. Subsequently, by Lemma 6.3.2, w contains the pattern 3412. □

To conclude this chapter, we provide a proof of Conjecture 6.1.6. This conjecture was independently proven by Richmond-Slofstra in [35].

Theorem 6.5.2. *Let $X(w)$ be a Schubert variety in $\mathcal{G}/\mathcal{B}$. Then $X(w)$ is smooth if and only if w avoids the patterns* 3412 *and* 4231.

Proof. By Corollary 6.1.5, we know that if w contains either the pattern 3412 or the pattern 4231, then $X(w)$ is singular.

Now suppose that $X(w)$ is singular. If $X(w)$ is not rationally smooth (so $n \geq 3$), then we obtain from Theorem 6.1.3 that w must contain either the pattern 3412 or the pattern 4231. If $X(w)$ is rationally smooth, then Theorem 6.5.1 yields that w contains the pattern 3412. Therefore, if $X(w)$ is singular, then w contains either the pattern 3412 or the pattern 4231. □

Chapter 7

Conclusion

In this thesis, we investigated the smoothness of certain rationally smooth subvarieties of flag varieties in two different contexts: the classical G/B context and affine $\mathcal{G}/\mathcal{B}$ context.

In our work on T-orbit closures in G/B in the type D case, we obtained a description of the weights of $T_y(\Sigma)$, where $\Sigma \in \Sigma_S(Y, y)$ (see Lemma 3.10.5 above). We would like to extend this result to a complete characterization of the singularities of S-surfaces in Y, similar in nature to the results obtained by Carrell-Kurth for T-surfaces in G/P (see Section 6 in [14]) and by Carrell-Kuttler for T-surfaces in G/B (see Proposition 5.2 in [15]). We would also like to further investigate the connectedness of S. In addition, we would like to use the approach applied in the type A and D cases to study the singular locus of rationally smooth T-orbit closures in other types. It would also be interesting to see how far our result on T-orbit closures in the type A case carries over to simply laced affine Kac-Moody groups.

In the classical backdrop, we would like to develop an explicit algorithm for computing Peterson translates for T-orbit closures in G/B. Furthermore, it would be interesting to determine whether the method we used to study T-orbit closures can be applied to study more general varieties (eg. spherical and symmetric varieties).

In terms of our work on Schubert varieties $X(w)$ in $\mathcal{G}/\mathcal{B}$, in Theorem 5.9.5 and Lemma 5.10.1 we provide necessary conditions for a $\widehat{T}$-fixed point of a rationally smooth Schubert variety to be a maximal singularity, however, we have not obtained a sufficient condition. We would like to obtain a complete characterization of the maximal singularities of rationally smooth Schubert varieties X in $\mathcal{G}/\mathcal{B}$. It would be of particular interest to see if there is a combinatorial relationship between the maximal singularities of $X(w)$ and the affine permutation w, as was the case in the classical G/B setting (see [29], [5], [16], [22], [32]). Furthermore, we would like to develop an explicit algorithm for computing Peterson translates for Schubert varieties in $\mathcal{G}/\mathcal{B}$.

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