University of Alberta

GEOMETRY AND FIXED POINT PROPERTIES FOR A CLASS OF BANACH ALGEBRAS ASSOCIATED TO LOCALLY COMPACT GROUPS

by

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To My Father, Brian

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ABSTRACT

In this thesis, we discuss two separate topics from the theory of abstract harmonic analysis. The first topic revolves around a locally compact group (Part I), the second one deals with the more abstract setting of a Lau algebra (Part II).

Part I is primarily concerned with the study of Dunford-Pettis operators related to the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group. We provide simpler proofs of known results relating to $L^1(G)$ when G is first countable. We then investigate the corresponding properties for $A(G)$ when G is non-abelian. A new Banach space $DP(\widehat{G})$ associated to G is introduced and we investigate some of its properties. We compare $DP(\widehat{G})$ to important subspaces of $A(G)^*$. In addition, it is shown that $DP(\widehat{G})^*$ has a natural multiplication, turning it into a Banach algebra. Stronger properties are developed for discrete groups, where weak convergence implies multiplier convergence for sequences. Afterwards, we investigate the tensor product of two abstract Segal algebras and subsequently introduce the concept of a vector-valued Segal algebra.

Part II is of a more abstract nature and relies heavily on the powerful properties of von Neumann algebras. We prove several fixed point theorems characterizing the left amenability of a Lau algebra. We also prove several hereditary properties for left amenable Lau algebras, with applications to semigroups.

The notion of operator left amenability for a Lau algebras is discussed and it is shown to be equivalent to left amenability. To finish, we introduce several new notions of amenability for a Lau algebra.

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Part I

Dunford-Pettis Operators and the Fourier Algebra

Chapter 1

Introduction - Abstract Harmonic Analysis

The phenomena of locally compact groups and their associated algebras has attracted many researchers over the years. A locally compact group is a group which is equipped with a locally compact Hausdorff topology which is compatible with the group operations, i.e., both the multiplication and inversion operations are continuous. In addition to the usual spaces of functions associated with a topological space, the group structure plays an essential role in the construction of the almost periodic functions $AP(G)$, the weakly almost periodic functions $WAP(G)$, and the left uniformly continuous functions $LUC(G)$ over G.

Arguably, the most important feature of a locally compact group is the existence of a positive, regular, Borel measure that is invariant under left translation by group elements. This allows us to employ tools from measure theory and functional analysis to study both the topological and geometrical properties of a locally compact group. This thesis will continue with that trend.

Compact and weakly compact operators are very important notions in functional analysis. They can be used to characterize the spaces $AP(G)$ and $WAP(G)$. A Dunford-Pettis operator, also known as a completely continuous operator, is very closely related to a compact operator. In fact, these two classes of operators coincide on reflexive Banach spaces and originally competed for popularity. However, compact operators have much nicer hereditary properties and applications.

Chapters 2 and 3 of this thesis will deal with how Dunford-Pettis operators can be applied to abstract harmonic analysis. The first appearance of a Dunford-Pettis operator relating to abstract harmonic analysis appears to be Crombez and Govaerts [9]. They attempted to characterize when convolution maps from $L^1(G)$ into $L^\infty(G)$ are completely continuous. They succeeded in finding a measure theoretic condition to describe this situation for metrizable groups. In chapter 3, we use the notion of a Dunford-Pettis operator to define a new function space associated to G. We provide new proofs to several results found in [9] for metrizable groups from functional analytic point of view; this provides new techniques for developing the theory in a noncommutative setting.

In chapter 4, we turn our focus to the noncommutative analogue of chapter 3. For abelian groups, the Fourier transform allows us to identify $L^1(G)$ with a subspace of $C_0(\widehat{G})$, where $\widehat{G}$ is the dual group of G, called the Fourier algebra of $\widehat{G}$. In [22], Eymard defined the Fourier algebra for an arbitrary locally compact group. Its dual space, denoted by $VN(G)$, is a noncommutative von Neumann algebra whenever G is non-abelian. By using functional analytic techniques to identify various subspace of $L^\infty(G)$, researchers have nicely developed a parallel non-commutative theory of function spaces. In section 4.1, we define a new subspace of $VN(G)$, which we call $DP(\widehat{G})$. We show that its dual space has a natural multiplication turning it into a Banach algebra and make comparison with other non-commutative function spaces. In section 4.2, we investigate certain ideals of $A(G)$, whose geometrical properties appear to be essential for a deeper understanding of $DP(\widehat{G})$. In section 4.3, we look at discrete groups in order to obtain some stronger results.

Chapter 5 deals with the study of Segal algebras. In section 5.2, we provide a technique to merge two abstract Segal algebras. This technique is then used in section 5.3 to construct a vector-valued Segal algebra.

Chapter 2

Preliminaries

2.1 Geometric Properties of Banach Spaces

Let X be a Banach space and let X^* denote its dual Banach space. The closed unit ball of X will be denoted by B_X. Given a subset Y of X^*, the $\sigma(X, Y)$ topology on X is the weakest topology on X which renders each linear functional in Y continuous. As usual, we will call the $\sigma(X, X^*)$ the weak topology on X. A sequence which converges to 0 in weak topology will be called *weakly null*. When we regard X as naturally embedded into X^{**}, the $\sigma(X^*, X)$ topology is called the weak star topology on X^*. When we do not mention a topology, we always refer to the norm topology. For duality reasons, we will normally denote $\phi(x)$ by $\langle \phi, x \rangle$ whenever $x \in X$ and $\phi \in X^*$.

Definition 2.1. Let X and Z be Banach spaces and let $T : X \to Z$ be a bounded linear operator. Then:

(i) T is *weakly compact* if the closure of $T(B_X)$ is weakly compact in Z.

(ii) T is *compact* if the closure of $T(B_X)$ is compact in Z.

(iii) T is a *Dunford-Pettis operator* if T maps weakly convergent sequences to norm convergent sequences.

It is clear that each compact operator must also be weakly compact. Less obvious is the fact that each compact operator is a Dunford-Pettis operator. In general, there is no relation between a weakly compact operator and a Dunford-Pettis operator. However, for some Banach spaces each weakly compact operator from X into any Banach space Y is a Dunford-Pettis operator; when this happens X is said to have the *Dunford-Pettis property.* An important equivalent definition was shown by Grothendieck in [36]; he proved that a Banach space X has the Dunford-Pettis property if and only if for every weakly null sequence (x_n) in X and weakly null sequence (ϕ_n) in X^*, we have $\langle \phi_n, x_n \rangle \to 0$, as $n \to \infty$.

Another geometric property, which we are interested in, is the Schur property. A Banach space is said to have the *Schur property* whenever each weakly null sequence is norm convergent. This is much stronger condition than the Dunford-Pettis property. For instance, $L^1(G)$ has the Dunford-Pettis property for any locally compact group, while it has the Schur property only when G is discrete, see [36].

2.2 Banach Algebras

A Banach algebra is an algebra $\mathcal{A}$ equipped with a complete norm such that

$$\|ab\| \leq \|a\| \, \|b\| \,, \qquad a, b \in \mathcal{A}.$$

An involution on $\mathcal{A}$ is a map $a \mapsto a^*$ from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

(i) $(a+b)^* = a^* + b^*$

(ii) $(\lambda a)^* = \overline{\lambda} a^*$

(iii) $(ab)^* = b^* a^*$

(iv) $a^{**} = a$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A Banach algebra equipped with an isometric involution, i.e. $\|a^*\| = \|a\|$, is called an *involutive Banach algebra.* An involution Banach algebra which satisfies

$$\|a^* a\| = \|a\|^2$$

for all $a \in \mathcal{A}$ is called a $C^* - algebra.$

Let $\mathcal{H}$ be a Hilbert space, whose inner product will be denoted by $\langle \cdot | \cdot \rangle$. The Banach algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. With the involution operation defined by $\langle T^* \eta | \xi \rangle = \langle \eta | T \xi \rangle$, for $T \in B(\mathcal{H}), \eta, \xi \in \mathcal{H}$, $B(\mathcal{H})$ is the prime example of a C^*-algebra. In fact, every C^*-algebra can be thought of as a C^*-subalgebra of some $B(\mathcal{H})$. When a unital C^*-subalgebra of $B(\mathcal{H})$ is closed in the weak operator topology it is called a von Neumann algebra. A W^*-algebra is a C^*-algebra $\mathcal{M}$ which can be identified as a dual Banach space; in this case, the predual of $\mathcal{M}$ is unique. A famous theorem of Sakai states that each von Neumann algebra is isometrically isomorphic to a W^*-algebra, and vice versa. Thus, W^*-algebras are always unital; the identity of a W^*-algebra B will always be denoted by 1_B.

Given a Banach algebra $\mathcal{A}$, we let $\mathcal{A}$ act on $\mathcal{A}^*$ in the natural way. For $a, b \in \mathcal{A}$ and $\varphi \in \mathcal{A}^*$, define

$$\langle a \cdot \varphi, b \rangle = \langle \varphi, ba \rangle, \quad \langle \varphi \cdot a, b \rangle = \langle \varphi, ab \rangle. \tag{2.1}$$

We say that a subspace X of $\mathcal{A}^*$ is *topologically left (resp. right) invariant* if $x \cdot a \in X$ (resp. $a \cdot x \in X$) for each $a \in \mathcal{A}$ and $x \in X$. X is called *topologically invariant* when it is both topologically left and right invariant.

Let X be a topologically left invariant subspace of $\mathcal{A}^*$ and $m \in X^*$. We define the operator m_L from X into $\mathcal{A}^*$ by

$$\langle m_L(x), a\rangle = \langle m, x \cdot a\rangle, \qquad x \in X, a \in \mathcal{A}.$$

X is called *topologically left introverted* if $m_L(X) \subseteq X$.

In [1], Arens defined a product on $\mathcal{A}^{**}$ given by

$$\langle m \odot n, \varphi\rangle = \langle m, n_L(\varphi)\rangle$$

for each $m, n \in \mathcal{A}^{**}$ and $\varphi \in \mathcal{A}^*$. When equipped with this multiplication, $\mathcal{A}^{**}$ becomes a Banach algebra. In fact, whenever X is a topologically left introverted subspace of $\mathcal{A}^*$, the Arens product construction still makes sense on X^* and renders it into a Banach algebra.

2.3 Locally Compact Groups

Let G be a locally compact group with a fixed left Haar measure λ and let Δ denote the modular function associated to G. For a Borel measurable function f on G and a measurable subset X of G, we denote the Lebesgue integral of f over X by

$$\int_X f(x)dx.$$

For each $1 \leq p \leq \infty$, let $(L^p(G), \|\cdot\|_p)$ denote the usual Banach spaces associated with G and λ. When $p = 2$, we will denote the inner product of the Hilbert space $L^2(G)$ by $\langle\cdot|\cdot\rangle$. The characteristic function of a subset H of G will be denoted by χ_H. For a complex valued function f on G and $y \in G$, define

(i) $L_y f(x) = f(yx)$,

(ii) $R_y f(x) = f(xy)$,

(iii) $\check{f}(x) = f(x^{-1})$,

(iv) $\tilde{f}(x) = \overline{f(x^{-1})}$.

There are several important subspaces of $L^\infty(G)$. Let $CB(G)$ denote the space of all bounded complex valued continuous functions on G. For $f \in CB(G)$, we identify f with its equivalence class in $L^\infty(G)$. Let $C_0(G)$ and $C_c(G)$ denote the set of all $f \in CB(G)$ which vanish at infinity and have compact support respectively. Let $AP(G)$ and $WAP(G)$ denote the subspaces of $CB(G)$ for which the *left orbit of f*

$$L_G f = \{L_y f \mid y \in G\}$$

is relatively norm compact and relatively weakly compact in $CB(G)$ respectively. Lastly, let $LUC(G)$ denote the space of all $f \in CB(G)$ for which the map $x \mapsto L_x g$ from G to $CB(G)$ is continuous.

In general, the following inclusions hold

$$AP(G) \oplus C_0(G) \subseteq WAP(G) \subseteq LUC(G) \subseteq CB(G) \subseteq L^\infty(G).$$

Moreover, with pointwise multiplication and involution given by $f \mapsto \overline{f}$, each of these spaces is commutative C^*-algebra.

2.4 Harmonic Analysis

The algebraic structure of a locally compact group enables us to add more structure to $L^1(G)$. With the convolution product

$$(2.2) \qquad f * g(x) = \int_G f(xy)g(y^{-1})dy \qquad f, g \in L^1(G),$$

$L^1(G)$ becomes a Banach algebra. Moreover, with the involution

$$f^*(x) = \Delta(x^{-1})\tilde{f}(x)$$

$L^1(G)$ becomes an involutive Banach algebra, called the *group algebra.*

The convolution product and translation by group elements behave nicely when combined. For instance, see [37, 20.11], we have

$$(L_z f) * g = L_z(f * g), \tag{2.3}$$

$$f * (R_z g) = R_z(f * g), \tag{2.4}$$

$$f * (L_z g) = \Delta(z)(R_z f) * g. \tag{2.5}$$

A *continuous unitary representation* of G is a group homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}$ that is continuous with respect to the weak operator topology. That is, for each $\xi, \eta \in \mathcal{H}_\pi$, the coefficient function

$$\omega_{(\pi,\xi,\eta)}(x) = \langle \pi(x)\xi | \eta \rangle$$

is a continuous function on G. Two representations π_1 and π_2 of G are called unitarily equivalent if there is a unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $U\pi_1(x) = \pi_2(x)U$ for all $x \in G$. The collection of all equivalence classes of continuous unitary representations of G will be denoted by Σ_G.

As an example, we will consider the *left regular representation* of G on the Hilbert space $L^2(G)$, which is given by

$$\rho(x) = L_{x^{-1}}$$

for $x \in G$. The C^*−subalgebra of $B(L^2(G))$ generated by $\rho(G) = \{L_x \mid x \in G\}$ is denoted by $C^*_\delta(G)$.

Each continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ can be lifted to a non-degenerate *-representation of $L^1(G)$, still represented by π, in the following way. For $f \in L^1(G)$, $\pi(f)$ is defined on $\mathcal{H}$ via

$$\langle \pi(f)\xi|\eta\rangle = \int_G \langle \pi(x)\xi|\eta\rangle f(x)dx$$

for each $\xi, \eta \in \mathcal{H}$. Remarkably, each non-degenerate *-representation of $L^1(G)$ can be derived in this manner.

Returning to the example of the left regular representation ρ of G. Let $f \in L^1(G)$, then $\rho(f)$ can be identified with the left convolution operator induced by f. That is, for each $g \in L^2(G)$, we have

$$\rho(f) : g \mapsto f * g \in L^2(G).$$

The C^*−subalgebra of $B(L^2(G))$ generated by $\rho(L^1(G))$ is denoted by $C^*_\rho(G)$ and called the *reduced group C^*-algebra* of G.

Note that each $\pi \in \Sigma_G$ satisfies

$$\|\pi(f)\| \leq \|f\|_1$$

for all $f \in L^1(G)$. Therefore, we may define a new norm on $L^1(G)$ by

$$\|f\|_{C^*(G)} = \sup_{\pi \in \Sigma_G} \|\pi(f)\|.$$

It turns out that $\|\cdot\|_{C^*(G)}$ is a C^*−norm, thus the completion of $L^1(G)$ with respect to $\|\cdot\|_{C^*(G)}$ is a C^*−algebra. It is called the *group C^*-algebra* and it is denoted by $C^*(G)$.

The dual space of $C^*(G)$ will be denoted by $B(G)$. It can be identified with space of all coefficient functions from Σ_G. That is,

$$B(G) = \{\omega_{(\pi,\xi,\eta)} \mid \pi \in \Sigma_G, \quad \xi, \eta \in \mathcal{H}_\pi\}.$$

Let $\|\cdot\|_{B(G)}$ denote the norm on $B(G)$ induced by $C^*(G)$ under the duality formula

$$\langle u, f\rangle = \int_G u(x)f(x)dx, \qquad u \in B(G),\ f \in L^1(G).$$

With this norm and pointwise operations, $B(G)$ becomes a commutative, regular, semisimple Banach algebra. It is called the *Fourier-Stieltjes algebra* of G.

The collection of all coefficient functions of the left regular representation is denoted by $A(G)$. That is,

$$\begin{aligned} A(G) &= \{u(x) = \omega_{(\rho,g,h)} \mid g, h \in L^2(G)\} \\ &= \{u(x) = \int_G (\rho(x)g)(y)\overline{h(y)}\, dy \mid g, h \in L^2(G)\} \\ &= \{u(x) = (g * \tilde{h})\check{}(x) \mid g, h \in L^2(G)\}. \end{aligned}$$

Alternatively, $A(G)$ can be defined as the norm closure of $B(G) \bigcap C_c(G)$ in $B(G)$, thus it is also a closed ideal of $B(G)$. With the norm inherited from $B(G)$, it becomes a commutative, regular, semisimple Banach algebra whose Gelfand spectrum is homeomorphic to G. $A(G)$ is called the *Fourier algebra* of G.

The left regular representation is also intimately related to the dual space of $A(G)$. Let $VN(G)$ denote the von Neumann subalgebra of $B(L^2(G))$ generated by either $\{\rho(x) \mid x \in G\}$ or $\{\rho(f) \mid f \in L^1(G)\}$. Then $VN(G)$ can be identified with dual space of $A(G)$ via the formula

$$\langle T, (g * \tilde{h})\check{}\rangle = \langle Tg|h\rangle$$

for $T \in VN(G)$ and $u(x) = (g * \tilde{h})\check{}(x) \in A(G)$.

When G is abelian, $L^1(G)$ is a commutative Banach algebra. In this case, the Gelfand spectrum of $L^1(G)$ can be identified with set of all continuous group homomorphisms from G into the unit circle $\mathbb{T}$ of the complex plane; it is denoted by $\widehat{G}$. With pointwise multiplication and the weak star topology inherited from $L^\infty(G)$, $\widehat{G}$ becomes an abelian locally compact group; it is called the *dual group* of G. The *Fourier transform*

$$\Gamma(f)(\alpha) = \int_G f(x)\alpha(x^{-1})dx, \quad \alpha \in \widehat{G},$$

embeds $L^1(G)$ into $C_0(\widehat{G})$. The image of $L^1(G)$ is precisely $A(\widehat{G})$ and the Fourier transform is an isometric isomorphism. For more information about Fourier analysis on abelian groups, we refer the reader to [66].

The Fourier and Fourier-Stieltjes algebras are two of the central objects in modern abstract harmonic analysis. The definition for these algebras are due to Eymard in [22]. For a comprehensive journey into abstract harmonic analysis, we refer the reader to Hewitt and Ross [37]; for a more friendly introduction, see Folland [24].

2.5 The Projective Tensor Product

Let X and Y be Banach space and let $X \otimes Y$ be their algebraic tensor product. The *projective tensor product norm* on $X \otimes Y$ is defined by

$$\|u\|_\pi = \inf\left\{\Sigma_{i=1}^n \|x_i\|_X \|y_i\|_Y \,|u = \Sigma_{i=1}^n x_i \otimes y_i, x_i \in X, y_i \in Y\right\}.$$

The completion of $X \otimes Y$ with respect to $\|\cdot\|_\pi$ is called the *projective tensor product of X and Y*, it is denoted by $X \widehat{\otimes} Y$.

The projective tensor product has many attractive properties. For instance, it is the largest cross norm, i.e. a norm satisfying $\|x \otimes y\| = \|x\|_X \|y\|_Y$, on $X \otimes Y$. When A and B are Banach algebras, the natural multiplication induced by

$$a_1 \otimes b_1 \bullet a_2 \otimes b_2 = a_1 a_2 \otimes b_1 b_2, \tag{2.6}$$

turns $A\widehat{\otimes}B$ into a Banach algebra.

One of the most useful features of this norm is the following theorem, see [69, Theorem 2.9].

Theorem 2.2. *Let $T : X \times Y \to Z$ be a bounded bilinear mapping. Then there exists a unique linear operator $\tilde{T} : X\widehat{\otimes}Y \to Z$ satisfying $\tilde{T}(x \otimes y) = T(x, y)$ for every $x \in X, y \in Y$. Moreover, $\|T\| = \left\|\tilde{T}\right\|$.*

Our interest in the projective tensor product arises from the following example.

Example 2.3. Let G be a locally compact group and let $\mathcal{A}$ be a Banach algebra. The Bochner-Lebesgue space $L^1(G, \mathcal{A})$ is the Banach space of all equivalence classes of Bochner integrable functions $f : G \to \mathcal{A}$, with the norm

$$\|f\|_1 = \int_G \|f(x)\|_{\mathcal{A}}\, dx.$$

The convolution product given by (2.2) still makes sense on $L^1(G, \mathcal{A})$ and turns it into a Banach algebra.

The map

$$f \otimes a \mapsto f(\cdot)a$$

induces an isometric isomorphism from $L^1(G)\widehat{\otimes}\mathcal{A}$ onto $L^1(G, \mathcal{A})$. Moreover, if $L^1(G)\widehat{\otimes}\mathcal{A}$ is equipped with the convolution product and $L^1(G)\widehat{\otimes}\mathcal{A}$ has the multiplication induced from (2.6), then this map is also an algebraic isomorphism.

For more information about tensor products in Banach spaces or the Lebesgue-Bochner space $L^1(G, \mathcal{A})$, we refer the reader to the book [69].

Chapter 3

Dunford-Pettis Operators on the Group Algebra

3.1 Introduction

Each of the classical functions spaces defined in Section 2.3 are intimately related to the convolution product. It is shown by Dunkl and Ramirez in [18] that $WAP(G)$ is precisely the set of all functions $g \in L^\infty(G)$ for which the map $f \mapsto f * g$ from $L^1(G)$ into $L^\infty(G)$ is a weakly compact operator. Similarly, $AP(G)$ is the collection of all functions $g \in WAP(G)$ for which the map $f \mapsto f * g$ from $L^1(G)$ into $L^\infty(G)$ is actually a compact operator. Also, it follows from the Cohen Factorization Theorem that $LUC(G) = \{f * g \mid f \in L^1(G) \text{ and } g \in L^\infty(G)\}$.

In section 3.2, we will use the notion of a Dunford-Pettis operator and the convolution product to introduce a new function space $DP(G)$ related to a locally compact group G. Closely related to this space is the class of uniformly measurable functions, which was introduced by Crombez and Govaerts in [9].

Definition 3.1. Let U be a measurable subset of G and let $\mathfrak{U} = (U_j)_{j=1}^n$ be a measurable partition of U. A function on U is called an *$\mathfrak{U}$-step function* if h is constant on each U_j. The set of all $\mathfrak{U}$-step functions will be denoted by *Step* $(\mathfrak{U})$.

Definition 3.2. A function $g \in L^\infty(G)$ is called *uniformly measurable* if for each $\varepsilon > 0$ and each compact K of G, there exists a measurable partition $\mathfrak{K} = (K_j)_{j=1}^n$ of K such that to each $x \in G$, there corresponds a function $g_x \in$ *Step* $(\mathfrak{K})$ with $\int_K |g(a^{-1}x) - g_x(a)|\, da < \varepsilon$. The space of all uniformly measurable function will be denoted by $\mathcal{UM}(G)$.

In their paper [9], Crombez and Govaerts showed that $\mathcal{UM}(G)$ is always a Banach space. We will demonstrate that this space is actually a C^*−subalgebra of $L^\infty(G)$.

In section 3.3, we will use Dunford-Pettis operators in conjunction with the Dunford-Pettis property to provide new, less measure theoretic, proofs of several nice theorems found in [9] for a first countable group.

3.2 Convolution Maps from $L^1(G)$ into $L^\infty(G)$

As we mentioned earlier, the convolution operation can be used to identify various important function spaces related to a locally compact group from a functional analytic viewpoint.

Definition 3.3. Let G be a locally compact group. Let $DP(G)$ denote the set of all $g \in L^\infty(G)$ such that the map $f \mapsto f * g$ from $L^1(G)$ to $L^\infty(G)$ is a Dunford-Pettis operator.

Equivalently, a function $g \in L^\infty(G)$ lies inside $DP(G)$ if and only if $f_n * g$ converges uniformly to zero whenever f_n is a weakly null sequence in $L^1(G)$.

Proposition 3.4. *Let G be a locally compact group.*

(i) $DP(G)$ is a norm closed subspace of $L^\infty(G)$.

(ii) If $g \in DP(G)$ and $z \in G$, then $L_z g$ and $R_z g$ are also in $DP(G)$.

(iii) If $g \in DP(G)$, then $\overline{g} \in DP(G)$.

(iv) The constant functions are inside $DP(G)$.

Proof. (i) Since the convolution product is bilinear, we see that $DP(G)$ is a linear subspace of $L^\infty(G)$. Suppose that g_n is a sequence in $DP(G)$ which converges uniformly to function $g \in L^\infty(G)$. We must show that $g \in DP(G)$. To this end, let f_n be any weakly null sequence in $L^1(G)$. A simple application of the Banach-Steinhaus theorem shows that the f_n's are uniformly bounded. That is, there exists an $M > 0$ such that $\|f_n\|_1 < M$ for every n. Let $\varepsilon > 0$ be arbitrary and choose n_0 so that $\|g_{n_0} - g\|_\infty < \varepsilon/2M$. Since $g_{n_0} \in DP(G)$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies that $\|f_n * g_{n_0}\|_\infty < \varepsilon/2$. Thus for $n \geq N$ we have

$$\begin{aligned} \|f_n * g\|_\infty &= \|f_n * (g - g_{n_0} + g_{n_0})\|_\infty \\ &\leq \|f_n * (g - g_{n_0})\|_\infty + \|f_n * g_{n_0}\|_\infty \\ &< \|f_n\|_1 \ \|g - g_{n_0}\|_\infty + \varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

(ii) Let f_n a weakly null sequence of $L^1(G)$. Notice that the sequences $L_z f_n$ and $\Delta(z) R_z f_n$ are also weakly null in $L^1(G)$. Thus, it follows from equation (2.4) that

$$\|f_n * (R_z g)\|_\infty = \|R_z(f_n * g)\|_\infty \to 0, \text{ as } n \to \infty.$$

Similarly, it follows from equation (2.5) that

$$\|f_n * (L_z g)\|_\infty = \|\Delta(z)(R_z f_n) * g\|_\infty \to 0, \text{ as } n \to \infty.$$

Hence, $DP(G)$ is translation invariant.

(iii) Let f_n be a weakly null sequence of $L^1(G)$. Note that its conjugate sequence $\overline{f_n}$ is also weakly null in $L^1(G)$. Therefore, we have

$$\|f_n * \overline{g}\|_\infty = \left\|\overline{\overline{f_n} * g}\right\|_\infty \to 0, \text{ as } n \to \infty.$$

Thus, $DP(G)$ is a conjugate closed subspace of $L^\infty(G)$.

(iv) It will suffice to show that the constant function χ_G lies inside $DP(G)$. Again, let f_n be a weakly null sequence in $L^1(G)$. Then for any $x \in G$, we have

$$f_n * \chi_G(x) = \int_G f_n(y)\chi_G(y^{-1}x)dy = \int_G f_n(y)dy = \langle \chi_G, f_n \rangle.$$

Thus, we have

$$\|f_n * \chi_G\|_\infty = |\langle \chi_G, f_n \rangle| \to 0, \text{ as } n \to \infty.$$

□

This space is actually quite large. Since $L^1(G)$ has the Dunford-Pettis property, we can see that $WAP(G) \subseteq DP(G)$. When G is discrete, we know that $\ell^1(G)$ has the Schur property. In this case, we have $DP(G) = \ell^\infty(G)$. The following example is taken from [9]. It demonstrates that there can be continuous and bounded functions which fail to lie inside $DP(G)$.

Example 3.5. Consider the additive group $\mathbb{R}$. Define $g \in C(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ by $g(t + 2\pi m) = e^{imt}$ for $0 < t \leq 2\pi$, $m \in \mathbb{Z}$. Conceptually, g is winding around the unit circle m times on the interval $[m, m+1]$. We will show that the convolution operator $T_g : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is not completely continuous.

For each $n \in \mathbb{N}$ set $f_n(t) = e^{int}\chi_{[0,2\pi]}(t)$. Then each $f_n \in L^1(\mathbb{R})$ with $||f_n||_1 = 2\pi$. Moreover, $f_n \to 0$ weakly in $L^1(\mathbb{R})$. Indeed, let $h \in L^\infty(\mathbb{R})$ be arbitrary and set $k = h\chi_{[0,2\pi]}$. By viewing k as a bounded continuous function on the unit circle $\mathbb{T}$, its Fourier transform $\widehat{k}$ lies inside $C_0(\mathbb{Z})$ by the Riemann-Lebesgue Lemma. Thus, we have

$$\int_{\mathbb{R}} f_n(t)k(t)dt = \int_0^{2\pi} e^{int}k(t)dt = \widehat{k}(-n) \to 0.$$

To finish, let $x_n = 2\pi(n+1)$. For $0 \leq t \leq 2\pi$, $x_n - t = (2\pi - t) + 2\pi n$. Thus for $0 < t < 2\pi$ we have $g(x_n - t) = e^{in(2\pi - t)}$; hence this holds almost everywhere on the interval $[0, 2\pi]$. Consequently,

$$f_n * g(x_n) = \int_0^{2\pi} e^{int}e^{in(2\pi-t)}dx = 2\pi e^{2\pi in}.$$

This shows that $T_g(f_n) = f_n * g \in C(\mathbb{R})$ does not converge uniformly to zero. As a result, we see that $CB(\mathbb{R}) \not\subset DP(\mathbb{R})$.

We will now show that the space $\mathcal{UM}(G)$ is always a commutative C^*-algebra. First, we will need the following lemma.

Lemma 3.6. *Let g be a uniformly measurable function. For each $x \in G$, the step function g_x from Definition 3.2 can be taken such that $\|g_x\|_\infty \leq 2\|g\|_\infty$.*

Proof. Let $\varepsilon > 0$ and let K be a compact subset of G. Choose a measurable partition $\mathfrak{K} = (K_j)_{j=1}^n$ of K which satisfies Definition 3.2. Let $x \in G$ be arbitrary and choose a function $g_x = \Sigma\lambda_j\chi_{K_j} \in$ *Step* $(\mathfrak{K})$ such that $\int_K |g(a^{-1}x) - g_x(a)|da < \varepsilon$.

For each j such that $|\lambda_j| \geq 2\|g\|_\infty$, we have

$$\|g\|_\infty \leq |g(a^{-1}x) - g_x(a)|$$

for almost all $a \in K_j$. It follows that

$$\int_{K_j} |g(a^{-1}x)|\, da \leq \int_{K_j} \|g\|_\infty\, da \leq \int_{K_j} |g(a^{-1}x) - g_x(a)|\, da.$$

To finish, set

$$\beta_j = \begin{cases} \lambda_j, & \text{if } |\lambda_j| \leq 2\,\|g\|_\infty; \\ 0, & \text{otherwise,} \end{cases}$$

for each j. Now take the function $h_x = \Sigma \beta_j \chi_{K_j} \in Step\ (\mathfrak{K})$. □

Proposition 3.7. *Let G be any locally compact group. Then $\mathcal{UM}(G)$ is a C^*-algebra.*

Proof. It was shown in [9] that $\mathcal{UM}(G)$ is a Banach space. Based on Definition 3.2, it is trivial to see that $\mathcal{UM}(G)$ is conjugate closed. Thus, we only need to show that $\mathcal{UM}(G)$ is closed under multiplication.

Let $f, g \in \mathcal{UM}(G)$ be nonzero, $\varepsilon > 0$, and let K be a compact subset of G. For $\varepsilon_f = \varepsilon/(4\|g\|_\infty)$, there exists a measurable partition $\mathfrak{A} = \{A_i\}$ of K such that for each $x \in G$, there is a function $f_x \in Step\ \mathfrak{A}$ with

$$\int_K |f(a^{-1}x) - f_x(a)|\ da < \varepsilon_f.$$

Similarly, for $\varepsilon_g = \varepsilon/(2\|f\|_\infty)$, there exists a measurable partition $\mathfrak{B} = \{B_i\}$ of K such that for each $x \in G$, there is a function $g_x \in Step\ \mathfrak{B}$ with

$$\int_K |g(a^{-1}x) - g_x(a)|\ da < \varepsilon_g.$$

Apply Lemma 3.6 to ensure that $\|g_x\|_\infty \leq 2\|g\|_\infty$.

Combine $\mathfrak{A}$ and $\mathfrak{B}$ to get a new partition $\mathfrak{C} = \{A_i \bigcap B_j\}$. Then for any $x \in G$ we may take $h_x = f_x g_x \in Step\ \mathfrak{C}$. It follows that,

$$
\begin{aligned}
&\int_K |f(a^{-1}x)g(a^{-1}x) - f_x(a)g_x(a)|\, da \\
&\leq \int_K |f(a^{-1}x)g(a^{-1}x) - f(a^{-1}x)g_x(a)| + |f(a^{-1}x)g_x(a) - f_x(a)g_x(a)|\, da \\
&= \int_K |f(a^{-1}x)||g(a^{-1}x) - g_x(a)| + |g_x(a)||f(a^{-1}x) - f_x(a)|\, da \\
&\leq \|f\|_\infty \int_K |g(a^{-1}x) - g_x(a)|\, da + \|g_x\|_\infty \int_K |f(a^{-1}x) - f_x(a)|\, da \\
&\leq \varepsilon_g \|f\|_\infty + \varepsilon_f \|g_x\|_\infty \\
&\leq \varepsilon.
\end{aligned}
$$

□

Amongst other things, Crombez and Govaerts were able to show that for metrizable groups, $DP(G)$ is the same space as $\mathcal{UM}(G)$. This allows us to use the tools from measure theory to gain some insight on the algebraic structure of $DP(G)$.

Corollary 3.8. *Let G be a metrizable locally compact group. Then $DP(G)$ is a translation invariant C^*–subalgebra of $L^\infty(G)$.*

3.3 Multiplier Convergence in $L^1(G)$

In this section, G is always assumed to be first countable. This will allow us to use properties of the space $DP(G)$ to provide a new more direct proof of a very nice theorem found in [9]. Our proof takes a functional analytic approach.

Proposition 3.9. *$C_0(G)$ is a subspace of $DP(G)$.*

Proof. It will suffice to show that $C_{00}(G) \subseteq DP(G)$. Let $g \in C_{00}(G)$ and suppose that $g \notin DP(G)$. Then there exists a weakly null sequence f_n in

$L^1(G)$ such that $f_n * g$ does not converge uniformly to zero. By passing to a subsequence if necessary, we can find an $\varepsilon > 0$ and a sequence (x_n) in G such that $|f_n * g(x_n)| \geq \varepsilon$, for each $n \in \mathbb{N}$.

First suppose that the sequence x_n has a cluster point. Again, by passing to a subsequence if necessary, we may further assume that $x_n \longrightarrow x$ in G. However, we also know that

$$\begin{aligned}
|f_n * g(x_n)| &= \left|\int_G f_n(y) g(y^{-1}x_n)dy\right| \\
&= \left|\langle f_n, L_{x_n^{-1}}\check{g}\rangle\right| \\
&= \left|\langle f_n, L_{x_n^{-1}}\check{g} - L_{x^{-1}}\check{g} + L_{x^{-1}}\check{g}\rangle\right| \\
&\leq \left|\langle f_n, L_{x_n^{-1}}\check{g} - L_{x^{-1}}\check{g}\rangle\right| + \left|\langle f_n, L_{x^{-1}}\check{g}\rangle\right| \\
&\leq \|f_n\|_1 \|L_{x_n^{-1}}\check{g} - L_{x^{-1}}\check{g}\|_\infty + \left|\langle f_n, L_{x^{-1}}\check{g}\rangle\right|.
\end{aligned}$$

Which converges to zero since $f_n \longrightarrow 0$ weakly in $L^1(G)$, and $\check{g}$ is uniformly continuous. Thus, our sequence cannot have a cluster point.

Now suppose that x_n has no cluster point. Since the support of g is compact, for any fixed $y \in G$ we must have that $y^{-1}x_n \notin$ supp (g) eventually. In other words, the sequence of functions $L_{x_n^{-1}}\check{g}$ converges pointwise to 0. Since $\|L_{x_n^{-1}}\check{g}\|_\infty = \|g\|_\infty$, we see that $\sup_{n\in\mathbb{N}} \|L_{x_n^{-1}}\check{g}\|_\infty$ is finite. Whence, $L_{x_n^{-1}}\check{g} \longrightarrow 0$ weakly in $C_0(G)$, see [19, Theorem 4.10.2]. Moreover, it is well known that $C_0(G)$ has the DPP, see [19, Theorem 9.4.4], so that $|f_n * g(x_n)| = |\langle f_n, L_{x_n^{-1}}\check{g}\rangle| \longrightarrow 0$ contradicting our assumption. □

Corollary 3.10. *Let G be a first countable locally compact group and let f_n be a weakly null sequence in $L^1(G)$. Then for any $g \in L^1(G)$ we have $\|f_n * g\|_1 \longrightarrow 0$ in $L^1(G)$.*

Proof. Since $C_c(G)$ is dense in $L^1(G)$, we may assume that $g \in C_c(G) \subseteq DP(G)$. Let $\varepsilon > 0$ be arbitrary. It is not hard to see that $f_n * g$ is also weakly null in $L^1(G)$. Thus, we may apply [19, Proposition 4.21.1] to get a compact subset K of G such that $\int_{G\setminus K} |f_n * g(x)|dx < \varepsilon$. Thus we have $\|f_n * g\|_1 = \int_{G\setminus K} |f_n*g(x)|dx + \int_K |f_n*g(x)|dx \leq |K|\|f_n*g\|_\infty + \varepsilon$. Lastly, since $g \in DP(G)$, we have $\|f_n * g\|_\infty \to 0$. From which it follows that $\|f_n * g\|_1 \to 0$. □

Corollary 3.11. *Let G be a first countable locally compact group. Then $LUC(G) \subseteq DP(G)$.*

Proof. Recall that $LUC(G) = L^1(G) * L^\infty(G)$. Thus, for any weakly null sequence (f_n) in $L^1(G)$ and $u = f*g \in LUC(G)$, then we have $\|f_n * (f * g)\|_\infty = \|(f_n * f) * g\|_\infty \leq \|f_n * f\|_1 \|g\|_\infty \to 0$. □

Remark 3.12. By using an intense measure theoretic argument, Crombez and Govaerts have proved Corollary 3.10 for any locally compact group in [9].

Remark 3.13. Convolution operators relating to $L^p(G)$ have also attracted the attention of researchers in recent years, see for instance [15]. At first glance, it might be interesting to investigate when convolution map into other L^p spaces is a Dunford-Pettis operator. Let $1 < p < \infty$. Since $L^p(G)$ is reflexive, each continuous linear operator from $L^1(G)$ into $L^p(G)$ is weakly compact. Furthermore, since $L^1(G)$ has the Dunford-Pettis property, each continuous linear operator must then be a Dunford-Pettis operator. Consequently, for any weakly null sequence f_n in $L^1(G)$ and $h \in L^p(G)$, we have $\|f_n * h\|_p \to 0$. When $p = 1$, Corollary 3.10 implies that each convolution operator from $L^1(G)$ into $L^1(G)$ is also a Dunford-Pettis operator. As a result, there is nothing to investigate in these settings.

Chapter 4

Dunford-Pettis operators on the Fourier Algebra

When G is abelian, the adjoint of the Fourier transform allows us to identify $L^\infty(\widehat{G})$ with $VN(G)$. By taking a functional analytic approach to the subspaces of $L^\infty(\widehat{G})$ defined in Section 2.3, researchers have been able to develop their non-commutative analogues.

Let $AP(\widehat{G})$ denote the set of operators $T \in VN(G)$ such that the linear operator $T^A : u \mapsto u \cdot T$ from $A(G)$ to $VN(G)$ is compact. Similarly, let $WAP(\widehat{G})$ denote the set of operators $T \in VN(G)$ such that the linear $u \mapsto u{\cdot}T$ from $A(G)$ to $VN(G)$ is weakly compact. Lastly, let $UCB(\widehat{G})$ denote the norm closure of the linear span of $A(G){\cdot}VN(G) = \{u{\cdot}T : u \in A(G), T \in VN(G)\}$ in $VN(G)$. Each of these spaces are norm closed self adjoint subspaces of $VN(G)$. It is known that $UCB(\widehat{G})$ is always a C^*-algebra, see [31, 43]. However, it remains unknown precisely when $AP(\widehat{G})$ and $WAP(\widehat{G})$ are C^*-algebras.

For an abelian group G, $AP(\widehat{G})$ and $WAP(\widehat{G})$ are the usual $C*-$algebras of almost periodic and weakly almost periodic functions on the dual group $\widehat{G}$. Meanwhile, $UCB(\widehat{G})$ can be identified with the left uniformly continuous functions on $\widehat{G}$.

For more information on these spaces, we refer the reader to [3, 7, 8, 18, 31, 32, 39, 43, 46, 47].

4.1 Actions from $A(G)$ into $VN(G)$

In this section, we will look at the natural analogue of $DP(G)$ as a subspace of $VN(G)$. Our main goal is to investigate how it relates to other non-commutative function spaces.

Definition 4.1. Let G be any locally compact group. By $DP(\widehat{G})$ we will denote the set of all $T \in VN(G)$ such that the map $u \mapsto u \cdot T$ from $A(G)$ into $VN(G)$ is a Dunford-Pettis operator.

As we saw with $DP(G)$, the space $DP(\widehat{G})$ is also a Banach space.

Proposition 4.2. *Let G be a locally compact group. Then $DP(\widehat{G})$ is a closed subspace of $VN(G)$.*

Proof. It is trivial to see that $DP(\widehat{G})$ is a linear subspace of $VN(G)$. Suppose $T_n \in DP(\widehat{G})$ is a sequence which converges to an operator $T \in VN(G)$. Let u_n be a weakly null sequence in $A(G)$. It follows from the Banach-Steinhaus theorem that the u_n's are uniformly bounded. Let $M \in \mathbb{N}$ be such that $||u_n|| < M$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose n_0 so that $||T_{n_0} - T|| < \varepsilon/2M$. Since $T_{n_0} \in DP(\widehat{G})$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies that

$||u_n \cdot T_{no}|| < \varepsilon/2$. Thus for $n \geq N$ we have

$$\begin{aligned} ||u_n \cdot T|| &= ||u_n \cdot (T - T_{no} + T_{no})|| \\ &\leq ||u_n \cdot (T - T_{no})|| + ||u_n \cdot T_{no}|| \\ &< ||u_n|| \; ||(T - T_{no})|| + \varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

It now follows that $T \in DP(\widehat{G})$.

□

Proposition 4.3. *Let G be a locally compact group. Then $DP(\widehat{G})$ is a topologically invariant and topologically left introverted subspace of $VN(G)$. In particular, $DP(\widehat{G})^*$ is a Banach algebra when it is equipped with the Arens product.*

Proof. Suppose that $T \in DP(\widehat{G})$, $u \in A(G)$ and v_n is a weakly null sequence in $A(G)$. Then $||v_n \cdot (u \cdot T)|| = ||u \cdot (v_n \cdot T)|| \leq ||u|| \; ||v_n \cdot T|| \to 0$. Thus, $DP(\widehat{G})$ is topologically invariant.

Now let $\phi \in DP(\widehat{G})^*$ and let $T \in DP(\widehat{G})$. We must show that $\phi_L T \in DP(\widehat{G})$. We remind the reader that $\phi_L T \in VN(G)$ is defined by $\langle \phi_L T, u\rangle = \langle \phi, u \cdot T\rangle$.

Suppose $u_n \to 0$ weakly in $A(G)$, and let $v \in A(G)$ have norm one. Then $|\langle u_n \cdot \phi_L T, v\rangle| = |\langle \phi_L T, u_n v\rangle| = |\langle \phi, u_n v \cdot T\rangle| \leq ||\phi|| \; ||v|| \; ||u_n \cdot T|| = ||\phi|| \; ||u_n \cdot T||$. Whence, $||u_n \cdot \phi_L T|| \leq ||\phi|| \; ||u_n \cdot T|| \to 0$ for each $n \in \mathbb{N}$. Thus, $\phi_L T \in DP(\widehat{G})$. □

Remark 4.4. Since $A(G)$ is a closed ideal of $B(G)$, the previous proof can be easily modified to show that $DP(\widehat{G})$ is a Banach $B(G)-$bimodule.

We will now take a brief look at how $DP(\widehat{G})$ compares to some of the non-commutative function spaces. Recall that a locally compact group is said to be a *Moore group* whenever every irreducible continuous unitary representation of G is finite dimensional.

Proposition 4.5. *For any locally compact group G we have $AP(\widehat{G}) \subseteq DP(\widehat{G})$. Moreover, when every irreducible unitary representation of G is finite dimensional, we have $WAP(\widehat{G}) \subseteq DP(\widehat{G})$.*

Proof. Recall that any compact operator is a Dunford-Pettis operator. This yields that $AP(\widehat{G}) \subseteq DP(\widehat{G})$ instantly.

It follows from a theorem of Lau-Ülger [47] and Bunce [5] that G is a Moore group if and only if $A(G)$ has the Dunford-Pettis property. Thus, it follows that every weakly compact operator from $A(G)$ is a Dunford-Pettis operator. □

It was shown in [47] that G is compact if and only $A(G)$ has the Schur property. This yields the following proposition.

Proposition 4.6. *Suppose G is a compact group, then $DP(\widehat{G}) = VN(G)$.*

Proof. When G is compact, $A(G)$ has the Schur property. Thus, every continuous operator from $A(G)$ into any Banach space is a Dunford-Pettis operator. In particular, we have $DP(\widehat{G}) = VN(G)$. □

Proposition 4.7. *Let G be a locally compact group. Then $C^*_\delta(G) \subseteq DP(\widehat{G})$*

Proof. Recall that $C^*_\delta(G)$ is generated by the left translation operators on $L^2(G)$ by elements of G. Thus, it will suffice to show that $\rho(x) \in DP(\widehat{G})$ for each $x \in G$. To this end, let u_n be a weakly null sequence in $A(G)$ and let $x \in G$ be arbitrary. Notice that $u \cdot \rho(x) = \langle \rho(x), u \rangle \rho(x)$ for each $u \in A(G)$. Thus, we have $\|u_n \cdot \rho(x)\| = \|\langle \rho(x), u_n \rangle \rho(x)\| = |\langle \rho(x), u_n \rangle| \, \|\rho(x)\| \to 0$. Hence, $\rho(x) \in DP(\widehat{G})$. □

The following lemma will be useful for us to show that the reduced group C^*-algebra lies inside $DP(\widehat{G})$.

Lemma 4.8. *$C_c(G) * C_c(G)$ is norm dense in $A(G)$.*

Proof. Let $f, g \in L^2(G)$ and let $f_n, g_n \in C_c(G)$ be such that that $f_n \to f$ and $g_n \to g$ in $L^2(G)$. Since for any $u \in A(G)$ we have $\|u\|_{A(G)} = \inf \|\xi\|_2 \|\eta\|_2$, where the infimum is taken of all $\xi, \eta \in L^2(G)$ such that $u = \xi * \tilde{\eta}$. Thus it follows that $\|f * g - f_n * g_n\|_{A(G)} \leq \|f * (g - g_n)\|_{A(G)} + \|(f - f_n) * g_n\|_{A(G)} \leq \|f\|_2 \|g - g_n\|_2 + \|f - f_n\|_2 \|g_n\|_2 \to 0$. □

Proposition 4.9. *Let G be a locally compact group. Then $C^*_\rho(G) \subseteq DP(\widehat{G})$.*

Proof. Since $\rho(C_c(G))$ is dense in $C^*_\rho(G)$, it will suffice to show that $\rho(C_c(G)) \subseteq DP(\widehat{G})$. Let $f \in C_c(G)$ and suppose that $u_n \in A(G)$ is a weakly null sequence. Since any weakly convergent sequence is norm bounded, we may assume that $\|u_n\| \leq 1$ for each $n \in \mathbb{N}$. Also, since the left translation operators lie inside $VN(G)$, we see that $u_n \to 0$ pointwise. We will now show that $\|u_n \cdot \rho(f)\| \to 0$.

Let $\varepsilon > 0$. By viewing f as an element of $M(G)$, we may apply Egoroff's theorem to obtain a measurable set E such that $\int_E |f| d\lambda < \varepsilon/2$ and $u_n \to 0$ uniformly on $G \backslash E$. Now choose N so that $|u_n(x)| < \varepsilon/(2\|f\|_1)$ for each $n \geq N$ and each $x \notin E$. Then for $n \geq N$ we have

$$\begin{aligned}
\|u_n \cdot \rho(f)\| &= \sup_{\|v\| \leq 1} |\langle u_n \cdot \rho(f), v\rangle| = \sup_{\|v\| \leq 1} |\langle \rho(f), u_n v\rangle| \\
&= \sup_{\|v\| \leq 1} \left| \int f u_n v \, d\lambda \right| \leq \sup_{\|v\|_\infty \leq 1} \int |f u_n v| \, d\lambda \\
&= \int |f u_n| \, d\lambda = \int_E |u_n||f| \, d\lambda + \int_{G \backslash E} |u_n||f| \, d\lambda \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

□

4.2 The Ideal $A_K(G)$

Examining the structure of certain ideals of the Fourier algebra can help to achieve a better understanding of geometric properties of some subspaces of $VN(G)$. This was demonstrated by Belanger and Forrest in [3]. In this section, we will look at various topologies on $A(G)$ that are related to geometric properties relating to the following class of ideas.

Definition 4.10. Let G be a locally compact group and let K be a closed subset of G. We let $A_K(G)$ denote the subspace of $A(G)$ consisting of all functions whose support lies inside K. That is $A_K(G) = \{u \in A(G) \mid \text{supp}\, u \subseteq K\}$.

Since the multiplication of two elements of $A(G)$ is simply their pointwise product, it is trivial to see that $A_K(G)$ is an ideal of $A(G)$. For our purposes, we are only interested when K is a compact subset of G with nonempty interior. Geometric properties of $A_K(G)$ in this situation has already been initiated by Granirer and Leinert in [34]. By viewing $A_K(G)$ as a subset of $B(G) \cong C^*(G)^*$, we can identify $A_K(G)$ with the dual space of the quotient space $C^*(G)/A_K(G)_\perp$, where $A_K(G)_\perp = \{\phi \in C^*(G) \mid \langle \phi, u \rangle = 0 \text{ for all } u \in A_K(G)\}$. For more general results concerning various topologies on the Fourier-Stieltjes algebra we refer the reader to [14].

Before we begin our inquest about the geometric properties of $A_K(G)$, we will first identify the Gelfand spectrum of these ideals.

Lemma 4.11. *Let B be a commutative Banach algebra and let A be a closed ideal of B. Then the Gelfand spectrum of A is the set $\Sigma_A = \{\phi|_A : \phi \in \Sigma_B$ and $\phi \notin A^\perp\}$.*

Proof. Clearly, $\{\phi|_A : \phi \in \Sigma_B \text{ and } \phi \notin A^{\perp}\} \subseteq \Sigma_A$. To show that these are all of them, suppose $\phi \in \Sigma_A$. Since ϕ is nonzero, there is an $a_0 \in A$ such that $\langle \phi, a_0 \rangle = 1$. For $b \in B$, define $\langle \tilde{\phi}, b \rangle := \langle \phi, a_0 b \rangle$, note that this is well defined since A is an ideal in B. Since ϕ is multiplicative, linear and B is commutative, we have $\tilde{\phi} \in \Sigma_B$. We will include the details for the sake of completeness; let $b, c \in B$ and $\lambda \in \mathbb{C}$, then $\langle \tilde{\phi}, \lambda b + c \rangle = \langle \phi, a_0(\lambda b + c) \rangle = \langle \phi, \lambda a_0 b \rangle + \langle \phi, a_0 c \rangle = \lambda \langle \tilde{\phi}, b \rangle + \langle \tilde{\phi}, c \rangle$, and $\langle \tilde{\phi}, bc \rangle = \langle \phi, a_0 bc \rangle = \langle \phi, a_0 bc \rangle \langle \phi, a_0 \rangle = \langle \phi, a_0 b a_0 c \rangle = \langle \phi, a_0 b \rangle \langle \phi, a_0 c \rangle = \langle \tilde{\phi}, b \rangle \langle \tilde{\phi}, c \rangle$. Lastly, for each $a \in A$, we have have $\langle \tilde{\phi}, a \rangle := \langle \phi, a_0 a \rangle = \langle \phi, a_0 \rangle \langle \phi, a \rangle = \langle \phi, a \rangle$. Thus, $\tilde{\phi}$ is an extension of ϕ which lies inside Σ_B.. □

Proposition 4.12. *Let $K \subseteq G$ be closed. Then the Gelfand spectrum of $A_K(G)$ is homeomorphic to the interior of K.*

Proof. It was shown by Eymard in [22] that the map, $x \mapsto \delta_x$, is a homeomorphism from G onto $\Sigma_{A(G)}$. Applying the previous lemma to the ideal $A_K(G)$ of $A(G)$, we see that $\Sigma_{A_K(G)} = \{\chi_a : \chi_a \notin (A_K(G))^{\perp}\}$. To finish the proof, we will show that these are precisely the characters induced by elements of the interior of K.

Let $a \in \text{int } K$, and let $U \subseteq K$ be a neighborhood of a. It now follows from [22, Lemma 3.2] that we can find an element $u \in A_K(G)$ such that $u(a) = 1$. In particular, $\chi_a(u) = 1 \neq 0$. Thus, $\chi_a \in \Sigma_{A_K(G)}$ for each $a \in \text{int } K$.

Now suppose that $a \notin \text{int } K$, that there is a net $x_\alpha \in K^C$ such that $x_\alpha \to a$. Since each $u \in A_K(G)$ is continuous, we have $u(x_\alpha) \to u(a)$. Thus, $\chi_a(u) = u(a) = 0$ for every $u \in A_K(G)$. Hence, $\chi_a \notin \Sigma_{A_K(G)}$ for each $a \notin \text{int } K$. □

We now turn our attention to a geometric property of $A_K(G)$ which would have nice applications to $DP(\widehat{G})$. When G is an abelian, compact, or discrete group, then it can be shown that $A_K(G)$ has the Schur property whenever K

is compact. We conjecture that this should be the case for all locally compact groups. However, we have not yet succeeded in finding a proof. The rest of this section will take steps towards a possible technique for attacking this problem.

The following proposition shows the link between the ideals $A_K(G)$ and different topologies on $A(G)$.

Proposition 4.13. *Let G be a locally compact group. The following are equivalent:*

1. *For each compact subset K of G. The ideal $A_K(G)$ has the Schur property.*

2. *For each weakly null sequence u_n in $A(G)$ and for each $u \in A(G)$ we have $\|uu_n\| \to 0$.*

Proof. Note that it follows from the Hahn-Banach extension theorem that the $\sigma(A_K(G), A_K(G)^*)$ and the $\sigma(A_K(G), A(G)^*)$ topologies coincide.

(1) $\Rightarrow$ (2). Suppose that there exists a weakly null sequence u_n in $A(G)$ and an element $v \in A(G)$ for which vu_n does not converge to 0. Then there exists an $\varepsilon > 0$ such $\|vu_n\| \geq \varepsilon$ frequently. By passing to a subsequence if necessary, we may assume that $\|vu_n\| \geq \varepsilon$ for each $n \in \mathbb{N}$.

Let $v_0 \in A(G) \bigcap C_c(G)$ be such that $\|v - v_0\| < \varepsilon/(2\sup\{\|u_n\| : n \in \mathbb{N}\})$. This can be done since weakly null sequences are norm bounded and $A(G) \bigcap C_c(G)$ is dense in $A(G)$. Let $K = \text{supp } v_0$. Then the sequence $u_n v_0$ is weakly null inside $A_K(G)$. Thus, $\|u_n v_0\| \to 0$, as $n \to \infty$. Thus, we may choose $n_0 \in N$ such that $\|u_n v_0\| < \varepsilon/2$. This leads to

$$\begin{aligned} \|u_n v\| &= \|u_n v - u_n v_0 + u_n v_0\| \\ &\leq \|u_n v - u_n v_0\| + \|u_n v_0\| \\ &\leq \|u_n\|\,\|v - v_0\| + \|u_n v_0\| \\ &< \varepsilon. \end{aligned}$$

However, this contradicts our earlier assumption.

(2) $\Rightarrow$ (1). Suppose that there is a compact subset K of G for which $A_K(G)$ does not have the Schur property. Then we can find a weakly null sequence $u_n \in A_K(G)$ such that $\|u_n\| = 1$ for each n. It follows from [22, Lemma 3.2] that there exists an element $u \in A(G)$ such that $u(x) = 1$ for every $x \in K$. In particular, we have $u_n u = u_n$. Thus, (2) cannot hold. □

Proposition 4.14. *Let u_n be a weakly null sequence of positive definite functions in $A(G)$. Then for any $v \in A(G)$ we have $\|vu_n\| \to 0$.*

Proof. It is well known that any continuous positive definite function u on G can be written as $u(x) = \langle \pi(x)\xi|\xi\rangle$ for some unitary representation π and some $\xi \in H_\pi$, see [17, Theorem 13.4.5]. Then for any $x \in K$ we have $|u(x)| = |\langle \pi(x)\xi|\xi\rangle| \leq \|\xi\|^2 = \langle \xi|\xi\rangle = u(e)$. Thus, in this situation, we have $\|u_n\|_\infty = u_n(e) = \langle \delta_e, u_n\rangle \to 0$. The result now follows from [6, Proposition 5.1]. □

Although the positive definite functions span a dense subspace of $A(G)$, we are unable to use the previous proposition to pass to the whole space. A more promising direction is to exploit the geometric properties of $A_K(G)$.

Our goal is to eventually apply the following theorem, which can be found in [16].

Theorem 4.15. *Let X be a Banach space, then X^* has the Schur property if and only if X has the Dunford-Pettis property and does not contain a copy of ℓ^1.*

Proposition 4.16. *Let K be a compact subset of G. $C^*(G)/(A_K)_\perp$ does not contain a copy of $\ell_1(\mathbb{N})$.*

Proof. It will suffice to show that each weak*-compact convex subset of $A_K(G)$ is the norm-closed convex hull of its extreme points, see [16, pg. 215].

It is shown in [34] that the weak* topology coincides with the norm topology on the unit sphere of $A_K(G)$. Thus, we may apply a result of Namioka, [54, Proposition 4.11], to see that each norm-closed, bounded, convex subset of $A_K(G)$ is the norm-closed convex hull of its extreme points. To finish, note that each weak*-compact subset of $A_K(G)$ must also be closed and bounded. □

To summarize, we have proved the following.

Proposition 4.17. *Let G a locally compact group and let K be a compact subset of G. Then $A_K(G)$ has the Schur property if and only it has the Dunford-Pettis property.*

The Dunford-Pettis property is a much weaker condition then the Schur property. We hope that in the future, this progress will help to show that the equivalent condition for 4.13 are true. If this is indeed true, the technique used in Proposition 4.20 would carry over verbatim to all locally compact groups.

4.3 Discrete Groups

In this section, we take a brief look at the setting of discrete groups. When G has the discrete topology, each compact subset of G consists of finitely many elements. This bring us to the following lemma.

Lemma 4.18. *Let K be a compact subset of G. Let u_n be a sequence in $A_K(G)$. Then the following are equivalent:*

(i) $u_n \to 0$ weakly,

(ii) $\|u_n\|_\infty \to 0$,

(iii) $\|u_n\|_{A(G)} \to 0$.

Proof. Since G is discrete, $A_K(G)$ is finite dimensional. □

Corollary 4.19. *Let G be a discrete group and suppose that $u_n \to u$ weakly in $A(G)$. Then for any $v \in A(G)$ we have $u_n v \to uv$.*

Proof. This follow directly from the previous lemma and Proposition 4.13. □

Applying this to $DP(\widehat{G})$ yields the following.

Proposition 4.20. *Let G be discrete group. Then $UCB(\widehat{G}) \subseteq DP(\widehat{G})$.*

Proof. Let $u \in A(G), T \in VN(G)$, and let u_n be a weakly null sequence in $A(G)$. Then $|\langle u \cdot T, u_n \rangle| = |\langle T, uu_n \rangle| \leq \|T\|\|uu_n\| \to 0$. This shows that $u \cdot T \in DP(\widehat{G})$. Since $UCB(\widehat{G})$ is generated by operators of this form and $DP(\widehat{G})$ is a Banach space, it follows that $UCB(\widehat{G}) \subseteq DP(\widehat{G})$. □

By using the Fourier transform and the results from [9], we have the following result for discrete abelian groups.

Proposition 4.21. *Suppose that G is a discrete abelian group, then $DP(\widehat{G}) = VN(G)$.*

Proof. It was shown in [9] that $DP(G) = L^\infty(G)$ for all compact groups. In our situation, the adjoint of the Fourier transform allows us to identify $DP(\widehat{G})$ with the function space as defined in 3.3 on the dual group $\widehat{G}$ of G. It is well known that $\widehat{G}$ is compact whenever G is discrete. □

The following proposition should be helpful for determining when $DP(\widehat{G})$ is a proper subspace of $VN(G)$.

Proposition 4.22. *Let G be any locally compact group. Then $DP(\widehat{G}) = VN(G)$ if and only if whenever u_n is a weakly null sequence in $A(G)$ and v_n is any sequence on the unit sphere of $A(G)$ we have $u_n v_n \to 0$ weakly.*

Proof. ($\Rightarrow$) Let $u_n \to 0$ weakly in $A(G)$ and let $v_n \in A(G)$ all have norm one. Then for each $T \in VN(G)$ we have

$$|\langle T, u_n v_n \rangle| = |\langle u_n \cdot T, v_n \rangle| \leq \|u_n \cdot T\| \|v_n\| = \|u_n \cdot T\| \to 0.$$

($\Leftarrow$) Suppose that there is an element $T \in VN(G)$ which is not in $DP(\widehat{G})$. Then there is a weakly null sequence u_n such that $\|u_n \cdot T\|$ does not converge to zero. Thus, there is a sequence v_n in the unit sphere of $A(G)$ such that $\langle T, u_n v_n \rangle = \langle u_n \cdot T, v_n \rangle$ does not converge to zero. □

Chapter 5

Vector-Valued Segal Algebras

5.1 Segal Algebras

As we mentioned earlier, the Lebesgue space $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group are the central objects of study in abstract harmonic analysis. The intersection of these two spaces $L^1(G) \cap A(G)$, with the norm $|||u||| = \|u\|_1 + \|u\|_{A(G)}$, is called the *Fourier-Lebesgue algebra* of G and it is denoted by $\mathcal{L}A(G)$. It turns out that $\mathcal{L}A(G)$ is a Banach algebra when it is equipped with either pointwise multiplication or convolution. Moveover, is a dense left ideal in both $L^1(G)$ and $A(G)$. This space was intensely studied by Lau and Ghahramani in [28].

The Fourier-Lebesgue algebra is an important example of two related concepts. The first was introduced by Reiter in [63].

A dense subspace $S^1(G)$ of $L^1(G)$ is said to be a *Segal algebra* if it satisfies following conditions:

(SA_1) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$, and for each $s \in S^1(G)$ we have:

$$\|s\|_S \geq \|s\|_1 ;$$

(SA_2) $S^1(G)$ is left translation invariant and the map $x \mapsto L_x s$ from G into $S^1(G)$ is continuous for each $s \in S^1(G)$;

(SA_3) For each $s \in S^1(G)$ and $x \in G$ we have

$$\|L_x s\|_S = \|s\|_S .$$

The second one, which is a generalization of Reiter's work, is due to Burnham [4].

An *abstract Segal algebra* $\mathcal{B}$ with respect to a Banach algebra $\mathcal{A}$ is a dense left ideal that satisfies the following conditions:

(ASA_1) $\mathcal{B}$ is a Banach space with respect to a norm $\|\cdot\|_{\mathcal{B}}$;

(ASA_2) $\exists\, C > 0$ such that for each $b \in \mathcal{B}$ we have

$$\|b\|_{\mathcal{A}} \leq C \|b\|_{\mathcal{B}} ;$$

(ASA_3) $\exists\, M > 0$ such that for each $a, b \in \mathcal{B}$ we have

$$\|ab\|_{\mathcal{B}} \leq M \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}} .$$

We will say that $\mathcal{B}$ is a *contractive abstract Segal algebra with respect to $\mathcal{A}$* when we can take $C \leq 1$ and $M \leq 1$.

Reiter proved that every Segal algebra of $L^1(G)$ is an abstract Segal algebra with respect to $L^1(G)$. However, as the next example illustrates, the converse is not true.

Example 5.1. Consider $\mathcal{B} = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with the norm $\|\cdot\|_{\mathcal{B}} := \|\cdot\|_1 + \|\cdot\|_\infty$. Then $\mathcal{B}$ is an abstract Segal algebra of $L^1(\mathbb{R})$, however it is not a Segal algebra of $L^1(\mathbb{R})$.

Proof. Since $C_c(\mathbb{R}) \subseteq \mathcal{B}$, it follows that $\mathcal{B}$ is a dense in $L^1(\mathbb{R})$. Let f_n be a Cauchy sequence in $\mathcal{B}$, then f_n is Cauchy in both $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. Thus there exists $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ such that $\|f_n - f\|_1 \to 0$ and $\|f_n - g\|_\infty \to 0$. Consequently, there is a subsequence f_{n_k} of f_n such that $f_{n_k} \to f$ pointwise almost everywhere. This means that $f = g$ almost everywhere. As a result, we have $f \in \mathcal{B}$ and $\|f_n - f\|_{\mathcal{B}} \to 0$.

Since $L^1(\mathbb{R}) * L^\infty(\mathbb{R}) = LUC(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, it easily seen that $\mathcal{B}$ is a ideal in $L^1(\mathbb{R})$. Lastly, we must show that $\mathcal{B}$ is a $L^1(\mathbb{R})$-module. To this end, let $f \in L^1(\mathbb{R})$ and $g \in \mathcal{B}$. Then

$$\|f * g\|_{\mathcal{B}} = \|f * g\|_1 + \|f * g\|_\infty \leq \|f\|_1 \|g\|_1 + \|f\|_1 \|g\|_\infty = \|f\|_1 \|g\|_{\mathcal{B}} .$$

Thus $\mathcal{B}$ is an abstract Segal algebra of a $L^1(\mathbb{R})$. However, there are many functions in $\mathcal{B}$ which are not left uniformly continuous. As a result (SA_2) cannot hold for $\mathcal{B}$. □

Segal algebras and abstract Segal algebras have been studied by several authors including Reiter [63], Burnham [4], Feichtinger [23], Dales and Pandey [10], Zhang [75], Ghahramani and Lau [28, 29], Granirer [33], Forrest, Spronk and Wood [25], Spronk [70], and many others.

In this chapter, we will provide a constructive technique for merging two abstract Segal algebras. Using the identification of $L^1(G, \mathcal{A})$ with $L^1(G)\widehat{\otimes}\mathcal{A}$, we will also develop the notion of a vector-valued Segal algebra.

5.2 Tensor products of Abstract Segal Algebras

In this section, we will investigate the projection tensor product of two abstract Segal algebras. These results will be applied in the next section when we investigate the notion of a vector-valued Segal algebra.

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ be an abstract Segal algebras of $\mathcal{A}$ and $\mathfrak{A}$ respectively. Set $C_{\mathcal{B}} = \sup\{\|b\|_{\mathcal{A}} : b \in \mathcal{B},\ \|b\|_{\mathcal{B}} \leq 1\}$ and $M_{\mathcal{B}} = \sup\{\|ab\|_{\mathcal{B}} : a \in \mathcal{A},\ b \in \mathcal{B}, \|b\|_{\mathcal{B}} \leq 1, \|a\|_{\mathcal{A}} \leq 1\}$. Define $C_{\mathfrak{B}}$ and $M_{\mathfrak{B}}$ similarly.

We are interested to know when $(\mathcal{B}\widehat{\otimes}\mathfrak{B}, \|\cdot\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}})$ can be viewed as a abstract algebra of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$. The main problem to overcome is to determine when $\mathcal{B}\widehat{\otimes}\mathfrak{B}$ can be embedded into $\mathcal{A}\widehat{\otimes}\mathfrak{A}$. However, by passing to a quotient space, we can avoid this downfall.

Lemma 5.2. *Let X and $\mathcal{B}$ be dense subspaces of Banach space Y and $\mathcal{A}$, respectively. Then $X \otimes \mathcal{B}$ is dense in $Y\widehat{\otimes}\mathcal{A}$*

Proof. It will suffice to show that $Y \otimes \mathcal{A} \subseteq \overline{X \otimes \mathcal{B}}^{\|\cdot\|_{Y\widehat{\otimes}\mathcal{A}}}$. To this end, let $z = \sum_{i=1}^{n} y_i \otimes a_i \in Y \otimes \mathcal{A}$ and let $\varepsilon > 0$. Set $M = \max\{\|y_i\|_Y, \|a_i\|_{\mathcal{A}}\} + 1$. Now choose $x_i \in X$ such that $\|x_i - y_i\|_Y < \varepsilon/(2nM)$ and $b_i \in \mathcal{B}$ such that $\|a_i - b_i\|_{\mathcal{A}} < \min\{1, \varepsilon/(2nM)\}$. Then we have

$$\begin{aligned}
\left\|\sum_{i=1}^{n} y_i \otimes a_i - \sum_{i=1}^{n} x_i \otimes b_i\right\|_{Y\widehat{\otimes}\mathcal{A}} &\leq \sum_{i=1}^{n} \|y_i \otimes a_i - x_i \otimes b_i\|_{Y\widehat{\otimes}\mathcal{A}} \\
&= \sum_{i=1}^{n} \|y_i \otimes a_i - y_i \otimes b_i + y_i \otimes b_i - x_i \otimes b_i\|_{Y\widehat{\otimes}\mathcal{A}} \\
&= \sum_{i=1}^{n} \|y_i \otimes (a_i - b_i) + (y_i - x_i) \otimes b_i\|_{Y\widehat{\otimes}\mathcal{A}} \\
&\leq \sum_{i=1}^{n} \|y_i\|_Y \|(a_i - b_i)\|_{\mathcal{A}} + \|(y_i - x_i)\|_Y \|b_i\|_{\mathcal{A}} \\
&\leq \sum_{i=1}^{n} \|y_i\|_Y \|(a_i - b_i)\|_{\mathcal{A}} + \|(y_i - x_i)\|_Y (\|a_i\|_{\mathcal{A}} + 1) \\
&\leq \varepsilon.
\end{aligned}$$

□

Lemma 5.3. *The identity map*

$$I_d : (\mathcal{B} \otimes \mathfrak{B}, \|\cdot\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}}) \to (\mathcal{A} \otimes \mathfrak{A}, \|\cdot\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}})$$

is a continuous linear map which extends uniquely to a continuous multiplicative map

$$\Phi : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{A}\widehat{\otimes}\mathfrak{A}, \textit{ with } \|\Phi\| \leq C_{\mathcal{B}} C_{\mathfrak{B}}.$$

Proof. First suppose that $u \in \mathcal{B} \otimes \mathfrak{B}$, then we have

$$\begin{aligned}
\|u\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}} &= \inf\left\{\sum_{i=1}^{n} \|y_i\|_{\mathcal{A}} \|a_i\|_{\mathfrak{A}} : u = \sum_{i=1}^{n} y_i \otimes a_i, \quad y_i \in \mathcal{A}, a_i \in \mathfrak{A}\right\} \\
&\leq \inf\left\{\sum_{i=1}^{n} \|x_i\|_{\mathcal{A}} \|b_i\|_{\mathfrak{A}} : u = \sum_{i=1}^{n} x_i \otimes b_i, \quad x_i \in \mathcal{B}, b_i \in \mathfrak{B}\right\} \\
&\leq \inf\left\{\sum_{i=1}^{n} C_{\mathcal{B}} \|x_i\|_{\mathcal{B}} C_{\mathfrak{B}} \|b_i\|_{\mathfrak{B}} : u = \sum_{i=1}^{n} x_i \otimes b_i, \quad x_i \in \mathcal{B}, b_i \in \mathfrak{B}\right\} \\
&= C_{\mathcal{B}} C_{\mathfrak{B}} \|u\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}}.
\end{aligned}$$

Thus the identity map

$$I_d : (\mathcal{B} \otimes \mathfrak{B}, \|\cdot\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}}) \to (\mathcal{A}\widehat{\otimes}\mathfrak{A}, \|\cdot\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}})$$

is a continuous linear map and therefore it extends uniquely to a continuous map

$$\Phi : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{A}\widehat{\otimes}\mathfrak{A}, \qquad \|\Phi\| \leq C_{\mathcal{B}}C_{\mathfrak{B}}.$$

Now for each $u, v \in \mathcal{B} \otimes \mathfrak{B}$, we have $\Phi(uv) = uv = \Phi(u)\Phi(v)$. Now suppose that $u, v \in \mathcal{B}\widehat{\otimes}\mathfrak{B}$. It follows for Lemma 5.2 that we can find sequences (u_n) and (v_n) in $\mathcal{B} \otimes \mathfrak{B}$ such that $u_n \to u$ and $v_n \to v$. The continuity of Φ implies that

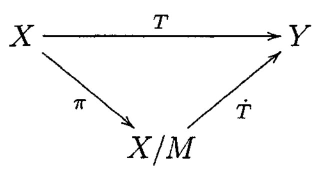
$$\Phi(uv) = \Phi(\lim u_n v_n) = \lim \Phi(u_n v_n) = \lim \Phi(u_n)\Phi(v_n) = \Phi(u)\Phi(v).$$

Thus, Φ is multiplicative. □

Consider $Z = \Phi(\mathcal{B}\widehat{\otimes}\mathfrak{B}) \subseteq \mathcal{A}\widehat{\otimes}\mathfrak{A}$. Then Z is linearly isomorphic to $\mathcal{B}\widehat{\otimes}\mathfrak{B}/Ker(\Phi)$. We will equip Z with the quotient norm inherited from this identification. That is, Z is isometrically isomorphic to $\mathcal{B}\widehat{\otimes}\mathfrak{B}/Ker(\Phi)$. Moreover, Z is a subalgebra of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$ since Φ is multiplicative.

The following useful theorem, which can be found in [50], will help us to show that Z is an abstract Segal algebra with respect to $\mathcal{A}\widehat{\otimes}\mathfrak{A}$.

Theorem 5.4. *Suppose X and Y are normed spaces and $T : X \to Y$ is a bounded linear map. Let M be a closed subspace of X such that $M \subseteq Ker(T)$ and let $\pi : X \to X/M$ be the quotient map. Then there is a unique function $\dot{T} : X/M \to Y$ such that $T = \dot{T} \circ \pi$, that is, such that the following diagram commutes.*



Moreover, $\dot{T}$ is linear and $\left\|\dot{T}\right\| = \|T\|$.

For our purposes, we will regularly have $X = \mathcal{B}\widehat{\otimes}\mathfrak{B}$, $Y = \mathcal{A}\widehat{\otimes}\mathfrak{A}$, and $M = Ker(\Phi)$.

Lemma 5.5. *For each $w \in \mathcal{A}\widehat{\otimes}\mathfrak{A}$. Then the map*

$$L_w : \mathcal{B} \otimes \mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B}, \qquad u \mapsto wu,$$

extends to a continuous linear map

$$\widetilde{L_w} : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B},$$

with $\|L_w\| \leq M_{\mathcal{B}} M_{\mathfrak{B}} \|w\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}}$.

Proof. Since $\mathcal{A} \otimes \mathfrak{A}$ is dense in $\mathcal{A}\widehat{\otimes}\mathfrak{A}$, it will suffice to only consider $w = \sum_{i=1}^{n} f_i \otimes a_i \in \mathcal{A} \otimes \mathfrak{A}$. In this case, the map

$$L_w : \mathcal{B} \times \mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B}, \qquad (s, b) \mapsto \sum_{i=1}^{n} f_i s \otimes a_i b,$$

is a bilinear map. Moreover, we have

$$\begin{aligned} \|L_w(s,b)\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}} &= \left\| \sum_{i=1}^{n} f_i s \otimes a_i b \right\|_{\mathcal{B}\widehat{\otimes}\mathfrak{B}} \\ &\leq \sum_{i=1}^{n} \|f_i s\|_{\mathcal{B}} \|a_i b\|_{\mathfrak{B}} \\ &\leq \sum_{i=1}^{n} M_{\mathcal{B}} M_{\mathfrak{B}} \|f_i\|_{\mathcal{A}} \|s\|_{\mathcal{B}} \|a_i\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}} \\ &= M_{\mathcal{B}} M_{\mathfrak{B}} \|s\|_{\mathcal{B}} \|b\|_{\mathfrak{B}} \sum_{i=1}^{n} \|f_i\|_{\mathcal{A}} \|a_i\|_{\mathfrak{A}}. \end{aligned}$$

Thus, it is also continuous. By taking the infimum over all possible representations of w, we also see that $\|L_w\| \leq M_{\mathcal{B}} M_{\mathfrak{B}} \|w\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}}$.

Now by the universal property of the tensor product, i.e. Theorem 2.2, L_w factors uniquely through to a linear map

$$\widetilde{L_w} : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B},$$

while preserving the norm. $\square$

In order for us to show that Z is a abstract Segal algebra of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$, it will be necessary for us to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}\widehat{\otimes}\mathfrak{B} & \xrightarrow{\Phi} & \mathcal{A}\widehat{\otimes}\mathfrak{A} \\ {\scriptstyle \widetilde{L_w}}\downarrow & & {\scriptstyle L_w}\downarrow \\ \mathcal{B}\widehat{\otimes}\mathfrak{B} & \xrightarrow{\Phi} & \mathcal{A}\widehat{\otimes}\mathfrak{A} \end{array}$$

This is indeed the case.

Lemma 5.6. *For each $w \in \mathcal{A}\widehat{\otimes}\mathfrak{A}$ and $u \in \mathcal{B}\widehat{\otimes}\mathfrak{B}$ we have $\Phi(\widetilde{L_w}u) = L_w\Phi(u)$.*

Proof. First suppose that $w \in \mathcal{A}\otimes\mathfrak{A}$ and $u \in \mathcal{B}\otimes\mathfrak{B}$. Then we have $\Phi(\widetilde{L_w}u) = \Phi(w \cdot u) = w \cdot u = L_w u = L_w\Phi(u)$. Now let $w \in \mathcal{A}\widehat{\otimes}\mathfrak{A}$, $u \in \mathcal{B}\widehat{\otimes}\mathfrak{B}$ and choose sequences $w_n \in \mathcal{A}\otimes\mathfrak{A}$ converging to w and $u_n \in \mathcal{B}\otimes\mathfrak{B}$ converging to u. Since The Φ is continuous, we have

$$\Phi(\widetilde{L_w}u) = \Phi(\lim \widetilde{L_{w_n}}u_n) = \lim \Phi(\widetilde{L_{w_n}}u_n) = \lim L_{w_n}\Phi(u_n) = L_w\Phi(u).$$

$\square$

We may now use the notation L_w for both spaces. We are now ready to look the associated abstract Segal algebra of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$.

Theorem 5.7. *The space $(Z, \|\cdot\|_Z)$ is an abstract Segal algebra of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$.*

Proof. Since Φ is a continuous algebra homomorphism, $(Z, \|\cdot\|_Z)$ is isometrically isomorphic to $\mathcal{B}\widehat{\otimes}\mathfrak{B}/Ker(\Phi)$, thus it is a Banach space. It follows directly from Lemma 5.2 that Z is a dense subspace of $\mathcal{A}\widehat{\otimes}\mathfrak{A}$. Moreover, for each $w \in \mathcal{A}\widehat{\otimes}\mathfrak{A}$ and $z \in Z$, there exists $u \in \mathcal{B}\widehat{\otimes}\mathfrak{B}$ with $z = \Phi(u)$. Applying Lemma 5.6 yields

$$wz = w\Phi(u) = \Phi(wu) \in Z.$$

Thus (ASA_1) holds.

Combining Lemma 5.3 with Theorem 5.4 we see that

$$\|z\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}} \leq C_{\mathcal{B}} C_{\mathfrak{B}} \|z\|_Z$$

is valid for each $z \in Z$. Therefore, (ASA_2) holds.

Similarly, for $w \in \mathcal{A}\widehat{\otimes}\mathfrak{A}$ and the map $L_w : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B}$ we may apply Lemma 5.5 and Theorem 5.4 to conclude that

$$\|wz\|_Z \leq \|L_w\| \|z\|_Z \leq M_{\mathcal{B}} M_{\mathfrak{B}} \|w\|_{\mathcal{A}\widehat{\otimes}\mathfrak{A}} \|z\|_Z .$$

□

5.3 Vector-valued Segal Algebras

In this section, we will further develop the results of the previous section when one of our abstract Segal algebras is a Segal algebra of $L^1(G)$. Recall that there is an isometric algebraic isomorphism between $L^1(G, \mathcal{A})$ and $L^1(G)\widehat{\otimes}\mathcal{A}$.

Throughout this section, let $S^1(G)$ be a Segal algebra of $L^1(G)$ and $\mathcal{B}$ a contractive, i.e. $C_{\mathcal{B}} = M_{\mathcal{B}} = 1$, abstract Segal algebra of $\mathcal{A}$. Let $(Z, \|\cdot\|_Z)$ denote the corresponding abstract Segal algebra of $L^1(G, \mathcal{A})$ with respect to the multiplicative contraction $\Phi : S^1(G)\widehat{\otimes}\mathcal{B} \to L^1(G, \mathcal{A})$.

The following is a natural vector-valued analogue of a Segal algebra for $L^1(G, \mathcal{A})$.

Definition 5.8. A linear subspace $S^1(G, \mathcal{A})$ of $L^1(G, \mathcal{A})$ is said to be an *vector-Segal subspace of* $L^1(G, \mathcal{A})$ if it satisfies following conditions.

(VSA_0) $S^1(G, \mathcal{A})$ is dense in $L^1(G, \mathcal{A})$;

(VSA_1) $S^1(G,\mathcal{A})$ is a Banach space under some norm $\|\cdot\|_S$, and for each $u \in S^1(G,\mathcal{A})$ we have:

$$\|u\|_1 \leq \|u\|_S;$$

(VSA_2) $S^1(G,\mathcal{A})$ is left translation invariant and the map $x \mapsto L_x s$ from G to $S^1(G,\mathcal{A})$ is continuous for each $s \in S^1(G,\mathcal{A})$;

(VSA_3) For each $s \in S^1(G,\mathcal{A})$ and $x \in G$ we have

$$\|L_x s\|_S = \|s\|_S;$$

(VSA_4) $S^1(G,\mathcal{A})$ is a Banach left $\mathcal{A}$-module under the action induced by

$$a \cdot f \otimes b = f \otimes (ab)$$

for $f \in L^1(G)$ and $a, b \in \mathcal{A}$.

The following lemmas will help us to show that Z satisfies these axioms.

Lemma 5.9. *For each $a \in \mathcal{A}$, the map*

$$L_a : S^1(G) \otimes \mathcal{B} \to S^1(G)\widehat{\otimes}\mathcal{B}, \qquad s \otimes b \mapsto s \otimes ab,$$

extends to a contractive map from $S^1(G)\widehat{\otimes}\mathcal{B}$ into $S^1(G)\widehat{\otimes}\mathcal{B}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} S^1(G)\widehat{\otimes}\mathcal{B} & \xrightarrow{\ \Phi\ } & L^1(G)\widehat{\otimes}\mathcal{A} \\ {\scriptstyle L_a}\downarrow & & {\scriptstyle L_a}\downarrow \\ S^1(G)\widehat{\otimes}\mathcal{B} & \xrightarrow{\ \Phi\ } & L^1(G)\widehat{\otimes}\mathcal{A} \end{array}$$

and

$$\|L_a\| \leq \|a\|_{\mathcal{A}}.$$

Proof. Consider the contractive bilinear map

$$L_a : S^1(G) \times \mathfrak{B} \to S^1(G)\widehat{\otimes}\mathcal{B}, \qquad (s,b) \mapsto s \otimes ab,$$

then we have

$$\|s \otimes ab\|_{S^1(G)\widehat{\otimes}\mathcal{B}} = \|s\|_S \|ab\|_{\mathcal{B}} \leq \|s\|_S \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}} \, .$$

Thus, L_a factors uniquely to a continuous linear map

$$L_a : \mathcal{B}\widehat{\otimes}\mathfrak{B} \to \mathcal{B}\widehat{\otimes}\mathfrak{B},$$

with $\|L_a\| \leq \|a\|_{\mathcal{A}}$. To see that the diagram is commutative, we can use a technique similar to that in Lemma 5.6. □

Lemma 5.10. *For each $x \in G$, the map*

$$L_x : S^1(G) \otimes \mathcal{B} \to S^1(G)\widehat{\otimes}\mathcal{B}, \qquad s \otimes b \mapsto (L_x s) \otimes b,$$

extends to an isometric map

$$L_x : S^1(G)\widehat{\otimes}\mathcal{B} \to S^1(G)\widehat{\otimes}\mathcal{B}.$$

Moreover, the following diagram commutes.

$$\begin{array}{ccc} S^1(G)\widehat{\otimes}\mathcal{B} & \xrightarrow{\Phi} & L^1(G)\widehat{\otimes}\mathcal{A} \\ {\scriptstyle L_x}\downarrow & & {\scriptstyle L_x}\downarrow \\ S^1(G)\widehat{\otimes}\mathcal{B} & \xrightarrow{\Phi} & L^1(G)\widehat{\otimes}\mathcal{A} \end{array}$$

Proof. Note that the map

$$L_x : S^1(G) \times \mathcal{B} \to S^1(G)\widehat{\otimes}\mathcal{B}, \qquad (s, b) \mapsto L_x s \otimes b,$$

is continuous and bilinear with $\|L_x\| \leq 1$. By the universal property of the tensor product L_x factors uniquely through to a contractive map

$$L_x : S^1(G)\widehat{\otimes}\mathcal{B} \to S^1(G)\widehat{\otimes}\mathcal{B}.$$

Since for any $u \in S^1(G)\widehat{\otimes}\mathcal{B}$ we have

$$\|L_x u\| \leq \|u\| = \|L_{x^{-1}} L_x u\| \leq \|L_x u\| \, ,$$

it follows that L_x is an isometry.

We will now show that $L_x\Phi = \Phi L_x$. First suppose that $u \in s_i \otimes b_i \in S^1(G) \otimes \mathcal{B}$. Then we have $\Phi(L_x u) = L_x u = L_x \Phi(u)$. Now let $u \in S^1(G)\widehat{\otimes}\mathcal{B}$, so that we can find a sequence $u_n \in S^1(G) \otimes \mathcal{B}$ converging to u. In this case we have

$$\Phi(L_x u) = \Phi(\lim L_x u_n) = \lim \Phi(L_x u_n) = \lim L_x \Phi(u_n) = L_x \Phi(u).$$

□

Lemma 5.11. *For each $u \in S^1(G)\widehat{\otimes}\mathcal{B}$ the map from $G \to S^1(G)\widehat{\otimes}\mathcal{B}$, $x \mapsto L_x u$ is continuous.*

Proof. Fix $v = \sum_{i=1}^n s_i \otimes b_i \in S^1(G) \otimes \mathcal{B}$. Since the map $x \mapsto L_x s_i$ from $G \to S^1(G)$ is continuous for each s_i, we see that the map

$$x \mapsto L_x v \text{ from } G \to S^1(G)\widehat{\otimes}\mathcal{B}$$

is continuous. We are now in a position to prove the general case. Suppose $u \in S^1(G)\widehat{\otimes}\mathcal{B}$ is arbitrary. Let $x_\alpha \to x$ and let $\varepsilon > 0$. Then we can choose a $v \in S^1(G) \otimes \mathcal{B}$ such that $\|u - v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} < \varepsilon/3$. Since $v \in S^1(G) \otimes \mathcal{B}$ we can find a β such that for each $\alpha \geq \beta$ we have $\|L_{x_\alpha} v - L_x v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} < \varepsilon/3$. Hence for each $\alpha \geq \beta$ we have

$$\begin{aligned}
\|L_{x_\alpha} u - L_x u\|_{S^1(G)\widehat{\otimes}\mathcal{B}} &\leq \|L_{x_\alpha} u - L_{x_\alpha} v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} + \|L_{x_\alpha} v - L_x v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&\quad + \|L_x v - L_x u\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&= \|L_{x_\alpha}(u - v)\|_{S^1(G)\widehat{\otimes}\mathcal{B}} + \|L_{x_\alpha} v - L_x v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&\quad + \|L_x(v - u)\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&= \|u - v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} + \|L_{x_\alpha} v - L_x v\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&\quad + \|v - u\|_{S^1(G)\widehat{\otimes}\mathcal{B}} \\
&< \varepsilon.
\end{aligned}$$

□

Theorem 5.12. *Let $S^1(G)$ be a Segal algebra of $L^1(G)$ and let $\mathcal{B}$ be a contractive abstract Segal subalgebra of $\mathcal{A}$. Then $Z = \Phi(S^1(G)\widehat{\otimes}\mathcal{B})/ker(\Phi)$ is a vector-Segal subspace of $L^1(G, \mathcal{A})$. Moreover, Z is also a dense left ideal of $L^1(G, \mathcal{A})$.*

Proof. (VSA_0) and (VSA_1) follow directly from Theorem 5.7.

We can see from Lemma 5.10 that Z is left translation invariant and an application of Theorem 5.4 shows that each L_x is an isometry on Z. For each fixed $z \in Z$, the map $z \mapsto L_x z$ from $G \to Z$ is continuous since is it the composition of two continuous maps. Therefore, (VSA_2) and (VSA_3) hold.

Lastly, (VSA_4) follows directly from Lemma 5.9 and 5.7. □

This provides us with an ample amount of examples of vector-valued Segal algebras of $L^1(G, \mathcal{A})$. We will finish this chapter with a few analogues of the some classical Segal algebras. The proofs are similar to the classical cases, which can be found in [64].

Let $CB(G, \mathcal{A})$ denote the Banach space of all bounded complex valued continuous functions from G into $\mathcal{A}$, equipped which the supremum norm.

Example 5.13. Let $C_0(G, \mathcal{A})$ denote the Banach space of all continuous functions from G into $\mathcal{A}$, vanishing at infinity and equipped with the supremum norm. Consider $S := L^1(G, \mathcal{A}) \bigcap C_0(G, \mathcal{A})$, with the norm $\|f\| = \|f\|_1 + \|f\|_\infty$. Then S is a vector-valued Segal algebra of $L^1(G, \mathcal{A})$.

Example 5.14. Let $LUC(G, \mathcal{A})$ to be the Banach space of all $f \in CB(G, \mathcal{A})$ such that the maps $x \mapsto L_x f$ from G into $CB(G, \mathcal{A})$ is continuous, equipped with the supremum norm. Set $S := L^1(G, \mathcal{A}) \bigcap LUC(G, \mathcal{A})$, with the norm $\|f\| = \|f\|_1 + \|f\|_\infty$. Then S is a vector-valued Segal algebra of $L^1(G, \mathcal{A})$.

Part II

Lau-Algebras, Amenability and Fixed Points

Chapter 6

Introduction, Lau Algebras

This part of the thesis is devoted to the study of certain amenability conditions pertaining to Lau algebras.

As we saw in Part I, it is often convenient to use the powerful tools from functional analysis to study Banach algebras associated to a locally compact group instead of the group itself.

The group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$, and the Fourier-Stieltjes algebra $B(G)$ are prime examples of how the structure of the Banach algebra can be used to retrieve information about the underlying group, as each of these Banach algebras determines G; see [72, 40, 73].

In his monograph [41], Johnson introduced the notion of an amenable Banach algebra. He had discovered a homological condition on the group algebra that was equivalent to the underlying group being amenable. In some sense, amenability can be considered its own branch of analysis. It is very broad and found in many unexpected places. For more references on this remarkable subject, we refer the reader to Pier's book [61], A. Paterson's A.M.S. monograph [60], or Runde's book [68].

In his pioneering paper [44], Lau noticed that many of the important Banach algebras arising in abstract harmonic analysis shared a common structure: each of them is the unique predual of a W^*-algebra and the identity of the W^*-algebra is a multiplicative linear functional on the Banach algebra. Lau called algebras of this form $F-$*algebras*; the term was later changed to *Lau algebras* by Pier in his monograph [62]. This broad class of Banach algebras includes the group algebra and the Fourier algebra of a locally compact group. In fact, it includes all Hopf-von Neumann algebras (see [21]).

Lau algebras have been the center of attention by many researchers. For instance, see[2, 27, 45, 48, 51, 52, 53, 55, 56, 57, 58, 59].

We will now briefly describe the layout of the rest of this thesis. The rest of this chapter will be used to give a brief introduction into the basic notations and terminologies about Lau algebras and left amenability that will be used throughout the rest of the thesis.

In Chapter 7, we study left amenability of Lau algebras as well as several new notions of amenabilities for this class of Banach algebras. In section 1, we characterize left amenability in terms of various fixed point properties. Section 2 is devoted to hereditary properties of left amenable Lau algebras, using in part our fixed point theorems established in section 1. In section 3, we investigate the notion of operator left amenability in the setting of a Lau algebra. We show that a Lau algebra is left amenable if and only if it is operator left amenable. In the final section, we introduce a new concept of amenability, which fits naturally in the setting of Lau algebras.

6.1 Amenability

Let G be a locally compact group. A *mean* on $L^\infty(G)$ is an element $M \in L^\infty(G)^*$ such that

$$M(\chi_G) = 1 = \|M\|.$$

A mean is called *left invariant* if

$$M(L_x(f)) = M(f)$$

for all $f \in L^\infty(G)$ and $x \in G$. A locally compact group G for which there is a left invariant mean on $L^\infty(G)$ is called *amenable.*

Let $\mathcal{A}$ be a Banach algebra. A *left Banach $\mathcal{A}$-module* is a Banach space X equipped with a bounded bilinear map from $\mathcal{A} \times X \to X$, $(a, x) \mapsto a \cdot x$, such that for any $a, b \in \mathcal{A}$ and $x \in X$ we have $a \cdot (b \cdot x) = (ab) \cdot x$. A *right Banach $\mathcal{A}$-module* is a Banach space X equipped with a bounded bilinear map from $X \times \mathcal{A} \to X$, $(x, a) \mapsto x \cdot a$, such that for any $a, b \in \mathcal{A}$ and $x \in X$ we have $(x \cdot a) \cdot b = x \cdot (ab)$. By a *Banach $\mathcal{A}$-bimodule* we mean a Banach space X which is both a left and right Banach $\mathcal{A}$-module such that for any $a, b \in \mathcal{A}$ and $x \in X$ we have $a \cdot (x \cdot b) = (a \cdot x) \cdot b$.

The dual space X^* of a Banach $\mathcal{A}$-bimodule X becomes a Banach $\mathcal{A}$-bimodule with

$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad \text{and} \quad \langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle \tag{6.1}$$

for all $a \in \mathcal{A}$, $x \in X$, and $f \in X^*$. For obvious reasons, we will often denote $f \cdot a$ by $\ell_a f$.

A *derivation* from $\mathcal{A}$ into X^* is a linear map $D : \mathcal{A} \to X^*$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for all $a, b \in \mathcal{A}$. Each $f \in X^*$ gives rise to a continuous derivation $ad_f : \mathcal{A} \to X$ defined by

$$ad_f(a) = a \cdot f - f \cdot a.$$

Any derivation of this form is called *inner*.

A Banach algebra $\mathcal{A}$ is called *amenable* if every bounded derivation from $\mathcal{A}$ into X^* is inner, for any Banach $\mathcal{A}$-bimodule X. The justification for this terminology comes from the following theorem.

Theorem 6.1 (Johnson [41]). *Let G be a locally compact group. Then G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra.*

Two other important algebras to abstract harmonic analysis that have not been discussed in this thesis include the measure algebra $M(G)$ of a locally compact group, see [37], and the convolution algebra $\ell^1(S)$ of a semigroup S as defined in [11] or [13]. Unfortunately, Johnson's theorem does not hold for neither $\ell^1(S)$ nor $M(G)$. For instance, the measure algebra $M(\mathbb{R})$ of the topological group $\mathbb{R}$ and the semigroup algebra $\ell^1(\mathbb{N})$ on the additive semigroup of positive integers both fail to be amenable as Banach algebras.

6.2 Lau algebras

When $\mathcal{A}$ is the pre-dual of a von Neumann algebra, we will let $\mathcal{A}_+$ denote the set of all $a \in \mathcal{A}$ which induce positive functionals on the W*-algebra $\mathcal{A}^*$. That is, $\mathcal{A}_+ = \{a \in \mathcal{A} : \langle a, \phi^*\phi\rangle \geq 0,\ \forall \phi \in \mathcal{A}^*\} = \{a \in \mathcal{A} : \langle a, 1_{\mathcal{A}^*}\rangle = \|a\|\}$. $S_{\mathcal{A}}$ will denote the set of all $a \in \mathcal{A}_+$ whose norm is equal to one. In the terminology of C^*-algebras, $S_{\mathcal{A}}$ is set of all normal states on $\mathcal{A}^*$.

A *Lau algebra* is a pair $(\mathcal{A}, \mathcal{M})$ such that $\mathcal{A}$ is a Banach algebra, $\mathcal{A}^* = \mathcal{M}$ is a W^*-algebra, and the identity of $\mathcal{M}$ is a multiplicative linear functional on

$\mathcal{A}$. When there is no confusion, we will simply refer $\mathcal{A}$ as a Lau algebra. Note that the multiplicativity of $1_{\mathcal{A}^*}$ implies that $S_{\mathcal{A}}$ is a topological semigroup.

Examples of Lau algebras in abstract harmonic analysis include the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$, and the measure algebra $M(G)$, as well as the semigroup algebra $\ell^1(S)$ of a semigroup.

In the paper [44], Lau noticed that, with minor variant to Johnson's definition, there would be a generalization of Theorem 6.1 that includes the measure algebra of a locally compact group, as well as the semigroup algebra of a discrete semigroup.

Definition 6.2. [44, Lau] A Lau algebra $\mathcal{A}$ is called *left amenable* if for any Banach $\mathcal{A}$-bimodule X such that $a \cdot x = \langle a, 1_{\mathcal{A}^*}\rangle x$ for all $x \in X$ and $a \in \mathcal{A}$, each bounded derivation from $\mathcal{A}$ into X^* is inner.

From this definition, it is clear that any amenable Lau algebra is left amenable. In some situations these two notions coincide. For instance the group algebra of a locally compact group G is amenable if and only if it is left amenable. In general, though, left amenability is a much weaker condition than amenability. For example, it follows from the Markov- Kakutani fixed point theorem that commutative Lau algebra is left amenable. In particular, for any locally compact group G, the the Fourier and Fourier-Stieltjes algebras are both left amenable.

Parallel to the theory of locally compact groups; left amenable Lau algebras can be characterized by the existence of special invariant means on $\mathcal{A}^*$. An element $m \in A^{**}$ is call a *topologically left invariant mean* provided that

1. $\|m\| = 1 = \langle m, 1_{A^*}\rangle$
2. $\langle m, \ell_\varphi x\rangle = \langle m, x\rangle$ for all $\varphi \in S_{\mathcal{A}}$ and all $x \in \mathcal{A}^*$.

Similar to the study of amenable locally compact groups, it is sometimes useful to know when various subspaces of $\mathcal{A}^*$ admit a topological left invariant mean; see [35, Section 2.2] for the situation when $\mathcal{A} = L^1(G)$. Given a topologically left invariant subspace X of $\mathcal{A}^*$ that contains $1_{\mathcal{A}^*}$, a topological left invariant mean on X is an element $m \in X^*$ such that

1. $\|m\| = 1 = \langle m, 1_{\mathcal{A}^*} \rangle$
2. $\langle m, \ell_\varphi x \rangle = \langle m, x \rangle$ for all $\varphi \in S_{\mathcal{A}}$ and all $x \in X$.

We will need the following useful characterization of left amenability.

Theorem 6.3 ([44] A. T.-M. Lau). *Let $\mathcal{A}$ be a Lau algebra. Then the following are equivalent:*

(1) $\mathcal{A}$ is left amenable.

(2) $\mathcal{A}^*$ has a topologically left invariant mean.

(3) There exists a net $\varphi_\alpha \in S_{\mathcal{A}}$ such that $||\varphi\varphi_\alpha - \varphi_\alpha|| \to 0$ for each $\varphi \in S_{\mathcal{A}}$.

In this setting, there is a natural analogue of Theorem 6.1. A semigroup S is left amenable if and only if $\ell_1(S)$ is left amenable as a Lau algebra. Also, a locally compact group G is amenable if and only if the measure algebra $M(G)$ is left amenable as a Lau algebra.

Chapter 7

Left Amenability and Fixed Point Properties

7.1 Left Amenability and Fixed Points

Fixed point theorems have far-reaching applications, both in mathematics and the social sciences. For example, the Banach fixed point theorem is used to prove the Picard-Lindelöf theorem about the existence and uniqueness of solutions to certain ordinary differential equations; the Brouwer fixed point theorem has important application to advanced economics and game theory; the Ryll-Nardzewski fixed point theorem can be used to show the existence of a left invariant mean on the space of weakly almost periodic functions on any locally compact group; the Markov-Kakutani theorem can be used to show that all abelian semigroups are left amenable; and of course, there is Day's fixed point theorem [12] which extends the Markov-Kakutani theorem to all left amenable semigroups. For information about fixed point theorems and their applications, we refer the reader to [30].

The motivation for this section arises from various fixed point theorems relating to the group algebra of a locally compact group; see for instance Wong [74], Ganesan [26], and Lau-Wong [48]. For more recent results concerning relationships between amenability and fixed point properties, see Lau and Zhang [49].

Throughout this section, let E be a separated locally convex vector space, i.e., E is a complex vector space equipped with a compatible Hausdorff topology, which is generated by a family of semi-norms on E. A representation of an algebra $\mathcal{A}$ is an algebra homomorphism T of $\mathcal{A}$ into the algebra $B(E)$ of all continuous linear operators on E. Given $a \in \mathcal{A}$ and $x \in E$, we will normally denote $\langle T(a), x\rangle$ by either $T_a(x)$ or simply $a \cdot x$ when there is no confusion. A subset S of E is called $S_{\mathcal{A}}$−invariant if $\phi \cdot s \in S$ for any $\phi \in S_{\mathcal{A}}$ and any $s \in S$. In the special case that S is convex, each representation induces an affine action of $S_{\mathcal{A}}$ on S, that is,

$$s \cdot (\lambda y_1 + (1-\lambda)y_2) = \lambda s \cdot y_1 + (1-\lambda)s \cdot y_2,$$

whenever $s \in S_{\mathcal{A}}, y_1, y_2 \in S$, and $0 \leq \lambda \leq 1$.

Let τ be any topology on $\mathcal{A}$. A representation T on $\mathcal{A}$ is said to be τ-separately continuous if the map $\mathcal{A} \times E \to E$ is separately continuous when $\mathcal{A}$ has the topology τ. Whenever we do not mention a topology we always refer to the norm or usual topology.

Lemma 7.1. *Let $\mathcal{A}$ be a Lau algebra and X be a topologically left invariant, topologically left introverted subspace of $\mathcal{A}^*$ that contains $1_{\mathcal{A}^*}$. Then X has a topological left invariant mean if and only if there exists a net $\varphi_\alpha \in S_{\mathcal{A}}$ such that $(\phi\varphi_\alpha - \varphi_\alpha) \to 0$ in the $\sigma(\mathcal{A}_X, X)$ topology for each $\phi \in S_{\mathcal{A}}$, where $\mathcal{A}_X$ denotes the restriction of $\mathcal{A}$ to X.*

Proof. Suppose that m is a topological left invariant mean on X. Since $m \in X^*$ is a mean, we have

$$m(1_{A^*}) = 1 = \|m\|.$$

Thus we can apply the Hahn-Banach theorem to extend m to a state $\widetilde{m}$ on $\mathcal{A}^*$. Moreover, it was shown in Lau [44] that $S_{\mathcal{A}}$ is weak*-dense in the set of all states on $\mathcal{A}^*$. Thus we can find a net $\varphi_\alpha \in S_{\mathcal{A}}$ which converges m in the weak star topology. To see that this is our desired net, let $\phi \in S_{\mathcal{A}}$ and $x \in X$ be arbitrary. Then we have

$$\begin{aligned}
\langle \phi\varphi_\alpha - \varphi_\alpha, x\rangle &= \langle \phi\varphi_\alpha, x\rangle - \langle \varphi_\alpha, x\rangle \\
&= \langle \varphi_\alpha, \ell_\phi x\rangle - \langle \varphi_\alpha, x\rangle \\
&\to \langle \widetilde{m}, \ell_\phi x\rangle - \langle \widetilde{m}, x\rangle \\
&= \langle m, \ell_\phi x\rangle - \langle m, x\rangle \\
&= 0.
\end{aligned}$$

This last step follows from the fact that m is topological left invariant.

Conversely, let $\varphi_\alpha \in S_{\mathcal{A}}$ be a net which satisfies the hypothesis. The Banach-Alaoglu theorem tells us that the closed unit ball of $\mathcal{A}^{**}$ is weak*-compact. Since the set of means on $\mathcal{A}^*$ is weak*-closed and lies inside the unit ball of $\mathcal{A}^{**}$, we may pass to a subnet if necessary and assume that $\varphi_\alpha \xrightarrow{w^*} m \in P_1(\mathcal{A}^{**})$. We claim that m is our desired topologically left invariant mean. To verify this, let $a \in S_{\mathcal{A}}$ and $x \in X$, then

$$m(x \cdot a) = \lim_\alpha \langle x \cdot a, \varphi_\alpha\rangle = \lim_\alpha \langle x, a\varphi_\alpha\rangle = \lim_\alpha \langle x, \varphi_\alpha\rangle = m(x).$$

□

We are now in a position to characterize the left amenability of a Lau algebra $\mathcal{A}$ in terms of a fixed point theorem for the topological semigroup $S_{\mathcal{A}}$.

Theorem 7.2. *Let $\mathcal{A}$ be a Lau algebra and let X be a topologically left invariant, topologically left introverted subspace of $\mathcal{A}^*$ which contains $1_{\mathcal{A}^*}$. Then the following are equivalent:*

(1) X has a topological left invariant mean.

(2) For any $\sigma(\mathcal{A}_X, X)$-separately continuous representation T of $\mathcal{A}_X$ on a separated locally convex space E and any compact convex $S_{\mathcal{A}}$-invariant subset S of E, the induced action $T : S_{\mathcal{A}} \times S \to S$, $(\phi, s) \mapsto \phi \cdot s$, has a fixed point.

Proof. (1) $\Rightarrow$ (2). Suppose X has a topological left invariant mean. By Lemma 7.1, there is a net $\varphi_\alpha \in S_{\mathcal{A}}$ such that $(\varphi\varphi_\alpha - \varphi_\alpha) \to 0$ in the $\sigma(\mathcal{A}_X, X)$-topology for each $\varphi \in S_{\mathcal{A}}$. Fix $s \in S$ and consider the net $\varphi_\alpha \cdot s \in S$. Since S is compact, by passing to a subnet if necessary, we may assume that $\varphi_\alpha \cdot s \to s_0 \in S$. We claim that s_0 is the desired fixed point. Indeed, let $\phi \in S_{\mathcal{A}}$, then

$$\begin{aligned} \phi \cdot s_0 &= \phi \cdot (\lim_\alpha \varphi_\alpha \cdot s) = \lim_\alpha \phi \cdot (\varphi_\alpha \cdot s) = \lim_\alpha (\phi\varphi_\alpha) \cdot s \\ &= \lim_\alpha (\phi\varphi_\alpha - \varphi_\alpha) \cdot s + \varphi_\alpha \cdot s = \lim_\alpha \varphi_\alpha \cdot s = s_0. \end{aligned}$$

(2) $\Rightarrow$ (1). Let $E = X^*$ with the weak* topology. Since X is topologically left invariant, for each $a \in \mathcal{A}$ we may define the map $\ell_a : X \to X$, $x \mapsto \ell_a x$. Now define $T : \mathcal{A} \times E \to E$ by $T_a(m) = \ell_a^*(m)$, where ℓ_a^* is the adjoint operator of ℓ_a. Since $\ell_{a_1 a_2} = \ell_{a_2}\ell_{a_1}$, T is clearly a representation of $\mathcal{A}$. Moreover, T is $\sigma(\mathcal{A}, X)$-separately continuous. Indeed, let $a_\alpha \to a$ in the $\sigma(\mathcal{A}_X, X)$-topology on $\mathcal{A}$ and let $m \in E = X^*$ be fixed. We must show that $T_{a_\alpha}(m) \to T_a(m)$ weakly* in X^*. To see this, let $\phi \in X$ be arbitrary, then

$$\begin{aligned} \langle T_{a_\alpha}(m), \phi\rangle - \langle T_a(m), \phi\rangle &= \langle \ell_{a_\alpha}^*(m), \phi\rangle - \langle \ell_a^*(m), \phi\rangle = \langle m, \ell_{a_\alpha}(\phi)\rangle - \langle m, \ell_{a_\alpha}(\phi)\rangle \\ &= \langle m, \ell_{a_\alpha}(\phi) - \ell_a(\phi)\rangle = \langle m, \ell_{a_\alpha - a}(\phi)\rangle \\ &= \langle m_L(\phi), a_\alpha - a\rangle, \end{aligned}$$

which converges to zero since X is left introverted.

Now suppose that $a \in \mathcal{A}$ is fixed and let $m_\beta \xrightarrow{w^*} m$ in X^*. We need to show that $T_a(m_\beta) \xrightarrow{w^*} T_a(m)$ in X^*. To this end, let $x \in X$ be arbitrary. Then

$$\begin{aligned} \langle T_a(m_\beta), x \rangle &= \langle \ell_a^*(m_\beta), x \rangle \\ &= \langle m_\beta, \ell_a(x) \rangle \\ &\to \langle m, \ell_a(x) \rangle \\ &= \langle \ell_a^*(m), x \rangle \\ &= \langle T_a(m), x \rangle. \end{aligned}$$

To finish, take S to be the set of all means in X^*, then S is weak*-compact, convex, and $S_{\mathcal{A}}$-invariant under the induced action from T. Consequently, the restriction of T to $S_{\mathcal{A}} \times S \to S$ has a fixed point, say m_0. It is clear by our construction that m_0 is a desired topological left invariant mean on X. □

Theorem 7.3. *Let $\mathcal{A}$ be a Lau algebra. Then the following are equivalent:*
(1) $\mathcal{A}$ is left amenable.
(2) For any $\sigma(\mathcal{A}, \mathcal{A}^*)$-separately continuous representation T of $\mathcal{A}$ on a separated locally convex space E and any compact convex $S_{\mathcal{A}}$-invariant subset S of E, the induced action $T : S_{\mathcal{A}} \times S \to S$ has a fixed point.
(3) For any separately continuous representation T of $\mathcal{A}$ on a separated locally convex space E and any compact convex $S_{\mathcal{A}}$-invariant subset S of E, the induced action $T : S_{\mathcal{A}} \times S \to S$ has a fixed point.

Proof. (1) $\Leftrightarrow$ (2) follows directly from Theorem 7.2.

(2) $\Rightarrow$ (3) holds since any separately continuous representation of $\mathcal{A}$ is automatically a $\sigma(\mathcal{A}, \mathcal{A}^*)$-separately continuous representation.

(3) $\Rightarrow$ (1)

Let $E = \mathcal{A}^{**}$ with the weak* topology. Let $T : A \times \mathcal{A}^{**} \to \mathcal{A}^{**}$ be the first Aren's product. Recall that the Aren's product actually yields $\mathcal{A}^{**}$ into a Banach algebra and it is also weak*-continuous in the second variable. Hence, T is a separately continuous representation of $\mathcal{A}$.

Now take S to be the set of all means in $\mathcal{A}^{**}$, then S is weak*-compact, convex, and $S_{\mathcal{A}}$-invariant under the induced action from T. Consequently, the restriction of T to $S_{\mathcal{A}} \times S \to S$ has a fixed point, say m_0. Again, it is clear by our construction that m_0 is a desired topological left invariant mean on $\mathcal{A}^*$. □

We will now turn our attention to the additively uniformly continuous functions on $S_{\mathcal{A}}$. The importance of this space was demonstrated by Lau-Wong in [48], where they showed that it could be used to characterize left amenability. Using their result and techniques similar to those introduced by Ganesan in [26], we will derive another fixed point characterization of left amenable Lau algebras.

Definition 7.4. A continuous and bounded function f on $S_{\mathcal{A}}$ is called *additively uniformly continuous* if for each $\varepsilon > 0$, there is a $\delta > 0$ such that whenever a and b are elements of $S_{\mathcal{A}}$ with $\|a - b\| < \delta$ we have $|f(a) - f(b)| < \varepsilon$. We will denote the set of all additively uniformly continuous functions on $S_{\mathcal{A}}$ by $AUC(S_{\mathcal{A}})$.

Then $AUC(S_{\mathcal{A}})$ is a $C*-$subalgebra of $CB(S_{\mathcal{A}})$ which is translation invariant and contains the restriction of any bounded linear functional on $\mathcal{A}$. A very interesting property about the space $AUC(S_{\mathcal{A}})$ is that it is independent of the multiplication of $\mathcal{A}$, thus it only depends on the Banach space structure of $\mathcal{A}$. However, it always lies inside the space of uniformly continuous functions, which is normally denoted by $UCB(S_{\mathcal{A}})$. Recall that a function $f \in CB(S_{\mathcal{A}})$

is uniformly continuous if whenever s_α converges s in $S_\mathcal{A}$, then $R_{s_\alpha}f$ converges uniformly to $R_s f$ and $\ell_{s_\alpha}f$ converges uniformly to $\ell_s f$.

Lemma 7.5. *Let $\mathcal{A}$ be a Lau algebra and $f \in AUC(S_\mathcal{A})$. Then for any $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that $a, b \in S_\mathcal{A}$ and $\|a - b\| < \delta$ imply that $\|\ell_a f - \ell_b f\|_\infty < \varepsilon$. In particular, any additively uniformly continuous function is uniformly continuous function. That is, $AUC(S_\mathcal{A}) \subseteq UCB(S_\mathcal{A})$.*

Proof. Let $f \in AUC(S_\mathcal{A})$ and let $\varepsilon > 0$. Choose $\delta > 0$ so that $|f(a) - f(b)| < \varepsilon/2$ whenever $a, b \in S_\mathcal{A}$ and $\|a-b\| < \delta$. Then for any $t \in S_\mathcal{A}$ we have $\|t\| = 1$, so that $\|at - bt\| \leq \|a - b\| \, \|t\| = \|a - b\| < \delta$. From which it follows that $\|\ell_a f - \ell_b f\|_\infty < \varepsilon$.

Now suppose that $a_\alpha \to a$ in $S_\mathcal{A}$. Then $\|a_\alpha - a\| < \delta$ eventually. Thus $\|\ell_{a_\alpha} f - \ell_a f\|_\infty < \varepsilon$ eventually. The proof for right translation is similar. □

Remark 7.6. The size of $UC(S_\mathcal{A})$ depends on the multiplication of $\mathcal{A}$. For example, when $\mathcal{A}$ is the pre-dual of any W^*-algebra and we define the multiplication on $\mathcal{A}$ by $ab = \langle a, 1_{A^*}\rangle \, b$, then we have $UC(S_\mathcal{A}) = CB(S_\mathcal{A})$.

Whenever Q is a family of seminorms which generates the topology of E, there is a natural notion of a $Q-$unform action. We will say that an action of $S_\mathcal{A}$ on a convex subset Y of E is *$Q-$uniform* if for each $y \in Y$ and $\rho \in Q$, the map from $S_\mathcal{A}$ into Y, given by $s \mapsto s \cdot y$, is uniformly continuous with respect to ρ. More precisely, for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that $\rho(a \cdot y - b \cdot y) < \varepsilon$ for all a and b in $S_\mathcal{A}$ for which $\|a - b\| < \delta$.

Theorem 7.7. *Let $\mathcal{A}$ be a Lau algebra. Then TFAE:*

(1) $\mathcal{A}$ is left amenable.

(2) $AUC(S_\mathcal{A})$ has a LIM.

(3) Let E be a separated locally convex space, and let Q be a family of seminorms which generates the topology of E. Whenever $S_{\mathcal{A}}$ acts affinely on a compact convex subset Y of E, where the action $S_{\mathcal{A}} \times Y \to Y$ is separately continuous and $Q-$uniform, Y has a fixed point under the action of $S_{\mathcal{A}}$.

(4) Let E be a separated locally convex space, and let Q be a family of seminorms which generates the topology of E. Whenever $S_{\mathcal{A}}$ acts affinely on a compact convex subset Y of E, where the action $S_{\mathcal{A}} \times Y \to Y$ is jointly continuous and $Q-$uniform then Y has a fixed point under the action of $S_{\mathcal{A}}$.

Proof. (1) $\Leftrightarrow$ (2) is due to Lau-Wong [48]. To see (1) $\Rightarrow$ (3). Suppose $\mathcal{A}$ is left amenable, then there is a net $\varphi_\alpha \in S_{\mathcal{A}}$ such that $||\varphi\varphi_\alpha - \varphi_\alpha|| \to 0$ for each $\varphi \in S_{\mathcal{A}}$ [44, Theorem 4.6]. Fix $y \in Y$ and consider the net $\varphi_\alpha \cdot y \in Y$. Since Y is compact, by passing to a subnet if necessary, we may assume that $\varphi_\alpha \cdot y \to y_0$. We claim that y_0 is the desired fixed point. Indeed, let $\phi \in S_{\mathcal{A}}$ be arbitrary. Since the topology of E is generated by Q and the action is $Q-$uniform, we know that $\varphi\varphi_\alpha \cdot y - \varphi_\alpha \cdot y \to 0$ in E. Thus, it follows that

$$\begin{aligned}
\phi \cdot y_0 &= \phi \cdot (\lim_\alpha \varphi_\alpha \cdot y) \\
&= \lim_\alpha \phi \cdot (\varphi_\alpha \cdot y) \\
&= \lim_\alpha (\phi\varphi_\alpha) \cdot y \\
&= \lim_\alpha \varphi_\alpha \cdot y \\
&= y_0.
\end{aligned}$$

(3) $\Rightarrow$ (4) is obvious. To finish the proof, we will show that (4) $\Rightarrow$ (2).

Let $E = AUC(S_{\mathcal{A}})^*$ with the weak* topology, let Q the set of seminorms induced by elements of $AUC(S_{\mathcal{A}})$, and let Y equal the set of all states in E.

Then Y is compact and convex in E. Define an action of $S_{\mathcal{A}}$ on Y by

$$a \cdot m = \ell_a^*(m), \text{ for } a \in S_{\mathcal{A}} \text{ and } m \in Y,$$

where ℓ_a^* is the adjoint operator of $\ell_a : AUC(S_{\mathcal{A}}) \to AUC(S_{\mathcal{A}})$. Since $\ell_{a_1 a_2} = \ell_{a_2}\ell_{a_1}$, this action is clearly affine. It follows from Lemma 7.5 that this action is $Q-$uniform. Moreover, the action is jointly continuous. To see this, let $a_\alpha \to a$ in $S_{\mathcal{A}}$ and let $m_\beta \to m$ in Y, and let $f \in AUC(S_{\mathcal{A}})$ be arbitrary. Then

$$\begin{aligned}
0 &\leq \lim_{\alpha,\beta} |\langle a_\alpha \cdot m_\beta, f\rangle - \langle a \cdot m, f\rangle| \\
&= \lim_{\alpha,\beta} |\langle m_\beta, \ell_{a_\alpha} f\rangle - \langle m, \ell_a f\rangle| \\
&= \lim_{\alpha,\beta} |\langle m_\beta, \ell_{a_\alpha} f\rangle - \langle m_\beta, \ell_a f\rangle + \langle m_\beta, \ell_a f\rangle - \langle m, \ell_a f\rangle| \\
&\leq \lim_{\alpha,\beta} |\langle m_\beta, \ell_{a_\alpha} f - \ell_a f\rangle + \langle m_\beta - m, \ell_a f\rangle| \\
&\leq \lim_{\alpha} \|\ell_{a_\alpha} f - \ell_a f\|_\infty + \lim_{\beta} |\langle m_\beta - m, \ell_a f\rangle| \\
&= 0.
\end{aligned}$$

Thus, for any $f \in AUC(S_{\mathcal{A}})$ we have $\lim_{\alpha,\beta}\langle a_\alpha \cdot m_\beta, f\rangle = \langle a \cdot m, f\rangle$, which means that $a_\alpha \cdot m_\beta \to a \cdot m$ weakly* in $AUC(S_{\mathcal{A}})^*$, i.e., $a_\alpha \cdot m_\beta \to a \cdot m$ in Y.

Consequently, the action $S_{\mathcal{A}} \times Y \to Y$ has a fixed point, say $m_0 \in Y$. Again, by our construction it is easily seen that that m_0 is a left invariant mean on $AUC(S_{\mathcal{A}})$. □

We will finish this section with a few miscellaneous results related to $AUC(S_{\mathcal{A}})$.

Proposition 7.8. *Let $\mathcal{A}$ be any Lau algebra. Then the space $AUC(S_{\mathcal{A}})^*$ has a natural multiplication which renders it into a Banach algebra.*

Proof. Let $n, m \in AUC(S_{\mathcal{A}})^*$, $f \in AUC(S_{\mathcal{A}})$. Since $AUC(S_{\mathcal{A}})$ is stable under translations, we can define the functional $n \odot f \in AUC(S_{\mathcal{A}})$

$$\langle n \odot f, s\rangle := \langle n, \ell_s f\rangle, \text{ where } s \in S_{\mathcal{A}}.$$

To see that $n \odot f$ is indeed additively uniformly continuous, let $\varepsilon > 0$ and use Lemma 7.5 to choose a δ so that $\|\ell_a f - \ell_b f\|_\infty \leq \varepsilon/\|n\|$ whenever $a, b \in S_\mathcal{A}$ with $\|a - b\| < \delta$. Then we have

$$|\langle n \odot f, a\rangle - \langle n \odot f, b\rangle| = |\langle n, \ell_a f - \ell_b f\rangle| \leq \|n\|\|\ell_a f - \ell_b f\|_\infty < \varepsilon.$$

It now follows easily that $\langle m \odot n, f\rangle := \langle m, n \odot f\rangle$ renders $AUC(S_\mathcal{A})^*$ into a Banach algebra. □

Corollary 7.9. *Let $\mathcal{A}$ be any Lau algebra. If $AUC(S_\mathcal{A})$ has both a left invariant mean and a right invariant mean, then it has a two-sided invariant mean.*

Proof. Suppose that m_L is a left invariant mean and m_R is a right invariant mean for $AUC(S_\mathcal{A})$. Then $m_L \odot m_R$ a two-sided invariant mean.

Indeed, $\langle m_L \odot m_R, 1\rangle = \langle m_L, m_R \odot 1\rangle = \langle m_L, 1\rangle = 1$. Since $\|m_L \odot m_R\| \leq \|m_L\|\|m_R\|$, $m_L \odot m_R$ is a mean. To see invariance, note that $m_R \odot \ell_s f = \ell_s(m_R \odot f)$ and that $m_R \odot r_s f = m_R \odot f$. □

In many situations a Lau algebra, $\mathcal{A}$, is also an involutive Banach algebra. In the event that $S_\mathcal{A}$ is self-adjoint, we will call $\mathcal{A}$ an involutive Lau algebra.

Theorem 7.10. *Suppose $\mathcal{A}$ is an involutive Lau algebra such that $S_\mathcal{A}$ is self-adjoint. Then $\mathcal{A}$ is left amenable if and only if the space $AUC(S_\mathcal{A})$ has a two-sided invariance mean.*

Proof. Since every left amenable Lau algebra is equipped with a left invariant mean, it will suffice to show that a left invariant mean can be used to produce a right invariant mean on $AUC(S_\mathcal{A})$.

To this end, let m be a left invariant mean on $AUC(S_\mathcal{A})$. Define $\tilde{m} \in AUC(S_\mathcal{A})^*$ by

$$\langle \tilde{m}, f\rangle := \langle m, f^*\rangle,$$

where $f^* \in AUC(S_{\mathcal{A}})$ is defined by

$$f^*(t) := f(t^*), t \in S_{\mathcal{A}}.$$

It is clear that $\tilde{m}$ is also a mean on $AUC(S_{\mathcal{A}})$. Moreover, since $(r_a f)^*(t) = r_a f(t^*) = \ell_{a^*} f^*(t)$, we have $\langle \tilde{m}, r_a f \rangle = \langle m, (r_a f)^* \rangle = \langle m, \ell_{a^*} f^* \rangle = \langle m, f^* \rangle = \langle \tilde{m}, f \rangle$. Thus $\tilde{m}$ is a right invariant mean. □

7.2 Hereditary Properties

In the study of amenability, researchers are often interested with various hereditary properties, see [11, 13, 35, 37, 41, 60, 61, 68]. As pointed out by Lau [44], the theory of left amenable Lau algebras contains that of semigroups. Unfortunately, due to the abstractness of semigroups, there is lack of strong hereditary properties for amenable semigroups. In fact, Hochster [38] showed that it is possible to embed the non-amenable free semigroup on two generators in a solvable, hence amenable, group. Thus, one cannot expect Lau algebras to have hereditary properties paralleling those to locally compact groups or Banach algebras. For instance, $\ell^1(\mathbb{F}_2)$ is not left amenable since the free group on two generators is not left amenable as a semigroup. However, as Lau [44] demonstrated, $\mathbb{C} \oplus \ell^1(\mathbb{F}_2)$ is left amenable.

Occasionally one Lau algebra can be embedded into another one in a nice way. In order to investigate this phenomena, we will need to define a suitable notion of a Lau-subalgebra. Once this is done, we will develop some hereditary properties that are analogous to the theory of amenable Banach algebras.

Suppose that B is a complemented subalgebra of a Lau algebra $\mathcal{A}$. That is, B is a closed subalgebra of $\mathcal{A}$ and there is a continuous projection $Q_B : \mathcal{A} \to \mathcal{A}$ such that $B = Q_B(\mathcal{A})$. In this situation, we will let V denote the topological complement of B, and Q_V its projection. We will call B an *admissible subalgebra* of $\mathcal{A}$ when B is non-trivial, $\mathcal{A} = B \oplus_1 V$, $B^* \cong Q_B^*\mathcal{A}^*$ and $V^* \cong Q_V^*\mathcal{A}^*$ are both W^*-subalgebras of $\mathcal{A}^*$ such that $\mathcal{A}^* = B^* \oplus_\infty V^*$. The identities of B^* and V^* will be denoted by 1_{B^*} and 1_{V^*} respectively. In this case (B, B^*) is also a Lau algebra, while V need not be a Lau algebra. It will often be convenient to regard both B and V as their own Banach space as well as a subspace of $\mathcal{A}$.

At times we will be dealing with up to three Banach spaces simultaneously, as well as their dual spaces. In order to keep track of variables more easily, we will adopt the following notation. Let X be a Banach space. Elements of X will be denoted by x, elements of its dual space X^* will be denoted by x^*, and elements of its bidual space X^{**} will be denoted by x^{**}.

Lemma 7.11. *Let $\mathcal{A}$ be a Lau algebra and B be an admissible subalgebra of $\mathcal{A}$. If m is a positive linear functional on $\mathcal{A}^*$, then the functionals $Q_V^{**}m$ and $Q_B^{**}m$ are also positive. Moreover for any positive normal functional b on B, the functional $b \odot Q_V^{**}m \in \mathcal{A}^{**}$ is positive.*

Proof. Since projections between C^*-algebras preserve positivity (see [71, III.3.4]) we see that the maps Q_B^* and Q_V^* preserve positivity. Thus for any positive elements $a^* \in \mathcal{A}^*$ and $b^* \in B^*$ we have $\langle Q_V^{**}m, a^*\rangle = \langle m, Q_V^* a^*\rangle \geq 0$. Similarly, $\langle Q_B^{**}m, b^*\rangle = \langle m, Q_B^* b^*\rangle \geq 0$.

Now let b be a positive element of B, viewed as an element of $\mathcal{A}$. Let a_α be a net in $P(\mathcal{A})$ converging weakly* to $Q_V^{**}m$ in $\mathcal{A}^{**}$. Then $ba_\alpha \in P(\mathcal{A})$ and $ba_\alpha \xrightarrow{w^*} b \odot m$ in $\mathcal{A}^{**}$. Since $P(\mathcal{A}^{**})$ is weak* closed, we are done. ☐

Corollary 7.12. *Let $\mathcal{A}$ be a Lau algebra and B be an admissible subalgebra of $\mathcal{A}$. Then $P_1(\mathcal{A}) = conv\{P_1(B), P_1(V)\}$.*

Proof. Let $b \in P_1(B)$, $v \in P_1(V)$, and $\lambda \in [0,1]$. Then $\|\lambda b + (1-\lambda)v\| \leq \lambda\|b\| + (1-\lambda)\|v\| = 1$. We also have, $\langle 1_{B^*} + 1_{V^*}, \lambda b + (1-\lambda)v\rangle = \lambda\langle 1_{B^*}, b\rangle + (1-\lambda)\langle 1_{V^*}, v\rangle = 1$.

Now suppose $a \in P_1(\mathcal{A})$, then by Lemma 7.11 we see that $b := Q_B(a) \geq 0$ and $v := Q_V(a) \geq 0$. Without loss of generality, we will assume that neither b nor v are zero. Since $a \in P_1(\mathcal{A})$ it follows that $1 = \langle 1_{B^*} + 1_{V^*}, b + v\rangle = \langle 1_{B^*}, b\rangle + \langle 1_{V^*}, v\rangle = \|b\| + \|v\|$. So that $a = \|b\|(b/\|b\|) + \|v\|(v/\|v\|) \in conv\{P_1(B), P_1(V)\}$. □

Lemma 7.13. *Let $\mathcal{A}$ be a Lau algebra and B be an admissible subalgebra of $\mathcal{A}$. Then for any $\tilde{b} \in P_1(B)$ and $b^* \in B^*$ we have*

$$(Q_B^* b^*) \cdot \tilde{b} = Q_B^*(b^* \cdot \tilde{b}) \oplus Q_V^*[(Q_B^* b^*) \cdot \tilde{b}].$$

Proof. Let $a \in \mathcal{A}$ be arbitrary and choose $b \in B, v \in V$ with $a = b + v$. Then

$$\begin{aligned}
\langle (Q_B^* b^*) \cdot \tilde{b}, b + v\rangle &= \langle Q_B^* b^*, \tilde{b}b + \tilde{b}v\rangle \\
&= \langle b^*, \tilde{b}b\rangle + \langle Q_B^* b^*, \tilde{b}v\rangle \\
&= \langle b^* \cdot \tilde{b}, b\rangle + \langle (Q_B^* b^*) \cdot \tilde{b}, v\rangle \\
&= \langle Q_B^*(b^* \cdot \tilde{b}), b + v\rangle + \langle Q_V^*[(Q_B^* b^*) \cdot \tilde{b}], b + v\rangle \\
&= \langle Q_B^*(b^* \cdot \tilde{b}) \oplus Q_V^*[(Q_B^* b^*) \cdot \tilde{b}], b + v\rangle.
\end{aligned}$$

□

Lemma 7.14. *Let $\mathcal{A}$ be a Lau algebra and let B be an admissible subalgebra of $\mathcal{A}$. If m is a topological left invariant mean on $\mathcal{A}^*$, then*

$$\langle m, Q_V^*[(Q_B^* 1_{B^*}) \cdot \tilde{b}]\rangle = 0$$

for each $\tilde{b} \in P_1(B)$.

Proof. By applying Lemma 7.13 to the element $1_{B^*} \in B^*$. We have,

$$\begin{aligned}\langle m, Q_B^* 1_{B^*}\rangle &= \langle m, (Q_B^* 1_{B^*}) \cdot \tilde{b}\rangle \\ &= \langle m, Q_B^*(1_{B^*} \cdot \tilde{b}) \oplus Q_V^*[(Q_B^* 1_{B^*}) \cdot \tilde{b}]\rangle \\ &= \langle m, Q_B^* 1_{B^*} \oplus Q_V^*[(Q_B^* 1_{B^*}) \cdot \tilde{b}]\rangle \\ &= \langle m, Q_B^* 1_{B^*}\rangle + \langle m, Q_V^*[(Q_B^* 1_{B^*}) \cdot \tilde{b}]\rangle.\end{aligned}$$

Whence it follows that $\langle m, Q_V^*[(Q_B^* 1_{B^*}) \cdot \tilde{b}]\rangle = 0$. ☐

Lemma 7.15. *Let $\mathcal{A}$ be a Lau algebra and B be an admissible subalgebra of $\mathcal{A}$. If m is a topological left invariant mean on $\mathcal{A}^*$, then*

$$\langle m, Q_V^*[(Q_B^* b^*) \cdot \tilde{b}]\rangle = 0$$

for each $\tilde{b} \in P_1(B)$ and $b^* \in B^*$.

Proof. Notice that

$$\begin{aligned}\langle m, Q_V^*[(Q_B^* b^*) \cdot \tilde{b}]\rangle &= \langle Q_V^{**} m, (Q_B^* b^*) \cdot \tilde{b}\rangle \\ &= \langle \tilde{b} \odot Q_V^{**} m, Q_B^* b^*\rangle \\ &= \langle Q_B^{**}[\tilde{b} \odot Q_V^{**} m], b^*\rangle.\end{aligned}$$

By viewing our equation this way, we can apply Lemma 7.11 and see that $Q_B^{**}[\tilde{b} \odot Q_V^{**} m]$ is a positive linear functional on B^*. Moreover for any positive element $b^* \in B^*$, we have $b^* \leq \|b^*\| 1_{B^*}$. Using this fact and Lemma 7.14, we get

$$0 \leq \langle Q_B^{**}[\tilde{b} \odot Q_V^{**} m], b^*\rangle \leq \langle Q_B^{**}[\tilde{b} \odot Q_V^{**} m], \|b^*\| 1_{B^*}\rangle = 0.$$

Since any element in a C^*–algebra can be written as a linear combination of positive elements we are done. ☐

Theorem 7.16. *Let $\mathcal{A}$ be a left amenable Lau algebra and let B be an admissible subalgebra of $\mathcal{A}$. Suppose that there is a topological left invariant mean m on $\mathcal{A}^*$ such that $Q_B^{**}m \neq 0$. Then B is left amenable.*

Proof. Since $Q_B^{**}m$ is a non-zero positive functional, it will suffice to show that it is topologically left invariant. Let $\tilde{b} \in P_1(B)$ and $b^* \in B^*$ be arbitrary. Applying Lemma 7.13 and Lemma 7.14, we have

$$\begin{aligned}
\langle Q_B^{**}m, b^* \cdot \tilde{b} \rangle &= \langle m, Q_B^*(b^* \cdot \tilde{b}) \rangle \\
&= \langle m, (Q_B^* b^*) \cdot \tilde{b} \rangle - \langle m, Q_V^*[(Q_B^* b^*) \cdot \tilde{b}] \rangle \\
&= \langle m, (Q_B^* b^*) \cdot \tilde{b} \rangle \\
&= \langle m, Q_B^* b^* \rangle \\
&= \langle Q_B^{**}m, b^* \rangle.
\end{aligned}$$

Thus $Q_B^{**}m$ is a non-zero positive topologically left invariant linear functional on B^*. By normalizing it, we obtain a topological left invariant mean on B^*. □

We mentioned earlier that, in general, sub-semigroups of a left amenable semigroup need not be left amenable. Day in [11] gave sufficient conditions in order to preserve amenability. Next we show that Theorem 7.16 can be used to achieve the same result.

Corollary 7.17 (Day)**.** *Let S be a sub-semigroup of a left amenable semigroup T. If there is a left invariant mean m on $\ell^\infty(T)$ such that $\langle m, \chi_S \rangle \neq 0$ then S is also left amenable.*

Proof. First note that any left invariant mean on $\ell^\infty(T)$ is also a topological left invariant mean. To see this, it will suffice to show that for each $t \in T$ and

$f \in \ell^\infty(T)$, we have $\ell_t f = \ell_{\delta_t} f$. To this end, let $w \in \ell^1(T)$ be arbitrary. Then we have

$$\begin{aligned}
\langle \ell_{\delta_t} f, w\rangle &= \langle f, \delta_t * w\rangle \\
&= \sum_{v\in T} f(v)\ \delta_t * w(v) \\
&= \sum_{v\in T} f(v) \sum_{xy=v} \delta_t(x) w(y) \\
&= \sum_{v\in T} f(v) \sum_{ty=v} w(y) \\
&= \sum_{v\in T} \sum_{ty=v} f(v)\ w(y) \\
&= \sum_{y\in T} f(ty)\ w(y) \\
&= \langle \ell_t f, w\rangle.
\end{aligned}$$

Since the convex hull of $\{\delta_t \mid t \in T\}$ is dense in the set of all normal means on $\ell^\infty(T)$, we see that m is topological left invariant. We may now apply Theorem 7.16 to see that S is left amenable. □

To finish this section, we will derive a few more miscellaneous hereditary properties, using in part our fixed point theorem established in section 7.1.

Proposition 7.18. *Let $\mathcal{A}$ be a left amenable Lau algebra and let B be an admissible subalgebra of $\mathcal{A}$. If B is a left ideal in $\mathcal{A}$ then B is also left amenable.*

Proof. Let B^{**} be equipped with the weak* topology. Let $\mathcal{A}$ act on B^{**} via the relative Arens product, i.e., for $a \in \mathcal{A}$ and $b^{**} \in B^{**}$ define

$$\langle a \odot b^{**}, b^*\rangle = \langle b^{**}, b^* \cdot a\rangle$$

where

$$\langle b^* \cdot a, b\rangle = \langle b^*, ab\rangle.$$

Then we get a separately continuous representation on the separated locally convex space $(B^{**}, w*)$.

Now set $K = \{m \in X^{**}) : \langle m, 1_{B^*}\rangle = 1 = \|m\|\}$. Then X is a weak*-compact and convex subset of B^{**}. Moreover, it is $S_{\mathcal{A}}$ invariant under the action defined above. Consequently, by Theorem 7.3, the induced action $S_{\mathcal{A}} \times K \to K$ has a fixed point, say m_0. It is clear by our construction that m_0 is a topologically $S_{\mathcal{A}}$ invariant. Since $P_1(B) \subseteq S_{\mathcal{A}}$ are done. □

Proposition 7.19. *Let $\mathcal{A}$ be a left amenable Lau algebra and let B be an admissible subalgebra of $\mathcal{A}$. If B is a right ideal in $\mathcal{A}$ then B is also left amenable.*

Proof. Let $\tilde{b} \in P_1(B)$, then $Q_B^* 1_{B^*} \cdot \tilde{b} = 1_{\mathcal{A}^*} \cdot \tilde{b}$. Indeed, for any $b + v \in \mathcal{A}$ we have

$$\begin{aligned}
\langle Q_B^* 1_{B^*} \cdot \tilde{b}, b + v\rangle &= \langle Q_B^* 1_{B^*}, \tilde{b}(b+v)\rangle \\
&= \langle 1_{B^*}, Q_B[\tilde{b}(b+v)]\rangle \\
&= \langle 1_{B^*} + 1_{V^*}, Q_B[\tilde{b}(b+v)]\rangle \\
&= \langle 1_{\mathcal{A}^*}, \tilde{b}(b+v)\rangle \\
&= \langle 1_{\mathcal{A}^*} \cdot \tilde{b}, (b+v)\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\langle Q_B^{**} m, 1_{B^*}\rangle &= \langle m, Q_B^* 1_{B^*}\rangle \\
&= \langle m, (Q_B^* 1_{B^*}) \cdot \tilde{b}\rangle \\
&= \langle m, 1_{\mathcal{A}^*} \cdot \tilde{b}\rangle \\
&= \langle m, 1_{\mathcal{A}^*}\rangle \\
&= 1.
\end{aligned}$$

So we may apply Theorem 7.16 to finish the proof. □

Proposition 7.20. *Let $\mathcal{A}$ be a left amenable Lau algebra, let B be another Lau algebra, and let $\pi : \mathcal{A} \to B$ be a continuous homomorphism with dense range such that $\pi^* 1_{B^*} = 1_{\mathcal{A}^*}$. Then B is is left-amenable as well.*

Proof. Let X be a Banach B-bimodule such that $b \cdot x = \langle b, 1_{B^*} \rangle x$, and let $D : B \to X^*$ be a bounded derivation. We must show that D is inner. In order to do this we will filter D through $\mathcal{A}$.

Note that X can be viewed as Banach $\mathcal{A}$-bimodule with operations defined by

$$a \cdot x = \pi(a) \cdot x \quad \text{and} \quad x \cdot a = x \cdot \pi(a),$$

for $a \in \mathcal{A}$ and $x \in X$. By looking at the left modulo action closer, we see that

$$\begin{aligned} a \cdot x &= \pi(a) \cdot x \\ &= \langle \pi(a), 1_{B^*} \rangle x \\ &= \langle a, \pi^*(1_{B^*}) \rangle x \\ &= \langle a, 1_{A^*} \rangle x. \end{aligned}$$

Since $D : B \to X^*$ is a bounded derivation, we can see that the map $D \circ \pi : \mathcal{A} \to X^*$ is also a bounded derivation. Thus there is an $x^* \in X^*$ such that for any $a \in \mathcal{A}$ we have $D \circ \pi(a) = a \cdot x^* - x^* \cdot a$. Thus for any $b \in Ran(\pi)$ we have $D(b) = D(\pi(a)) = a \cdot x^* - x^* \cdot a = \pi(a) \cdot x^* - x^* \cdot \pi(a) = b \cdot x^* - x^* \cdot b$. Lastly, since $\pi(\mathcal{A})$ is dense in B and π is continuous we see that for any $b \in B$, $D(b) = b \cdot x^* - x^* \cdot b$. ☐

Proposition 7.21. *Let $\mathcal{A}$ be a Lau algebra and let B be an admissible subalgebra of $\mathcal{A}$. If B is a right ideal in $\mathcal{A}$ which is left amenable, then $\mathcal{A}$ is left amenable.*

Proof. Let $P_1(\mathcal{A})$ act affinely on a compact convex subset Y of a separated locally convex space E, where the action $P_1(\mathcal{A}) \times Y \to Y$ is jointly continuous and uniform. Then induced action $P_1(B)$ also acts affinely of Y and of course is jointly continuous and uniform. By Theorem 7.7, we see that there is a $y_0 \in Y$ which is a fixed point for $P_1(B)$. We claim that y_0 is also a fixed point for $P_1(\mathcal{A})$. Indeed let $b \in P_1(B)$, then for any $a \in P_1(\mathcal{A})$

$$a \cdot y_0 = a \cdot (b \cdot y_0) = ab \cdot y_0 = y_0.$$

Thus it follows from Theorem 7.7 that $\mathcal{A}$ is left amenable. $\square$

7.3 Operator Left-amenability

The importance of operator space theory to abstract harmonic analysis was illustrated by Ruan in 1995. In the paper [65], Ruan introduced a notion of operator amenability and proved that the Fourier algebra of a locally compact group is operator amenable if and only if the underlying group is amenable. This result is very important since Johnson had demonstrated in [42] that the amenability of the Fourier algebra was not a suitable match for the amenability of the underlying group.

In this section, we investigate the natural extension of left amenability to the category of operator spaces. We will not attempt to give a survey on a subject as deep as operator space theory, but merely outline the concepts which are necessary. For a more detailed approach and any concepts not thoroughly defined, we refer the reader to the excellent book of Effros and Ruan, [20]. More applications of operator space theory to abstract harmonic analysis can be found in the survey paper [67].

An *operator space* is a linear space V together with a matrix norm, $\|\cdot\| = \{\|\cdot\|_n\}$; where $\|\cdot\|_n$ is a norm on the matrix space $M_n(V)$, $(V, \|\cdot\|_1)$ is complete and

(M1) $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \ \|w\|_n\}$,

(M2) $\|\alpha v \beta\| \leq \|\alpha\| \|v\|_m \|\beta\|$

for all $v \in M_m(V)$, $w \in M_n(V)$ and $\alpha \in M_{n,m}(\mathbb{C}), \beta \in M_{m,n}(\mathbb{C})$.

Let $\mathcal{H}$ be a Hilbert space, then any closed subspace of $B(\mathcal{H})$ is an operator space in the natural way. In fact, all operator spaces occur in this fashion. Also any dual space of an operator space is an operator space. Thus, the pre-dual of a von Neumann algebra is an operator space in the canonical way.

Given two operator spaces V and W and a linear mapping $T : V \to W$, the n-th amplification $T^{(n)} : M_n(V) \to M_n(W)$ of T is given by

$$T^{(n)}([v_{k,l}]) = [T(v_{k,l})].$$

If $\sup_{n \in \mathbb{N}} \|T^{(n)}\|$ is finite, then T is called *completely bounded* and

$$\|T\|_{cb} = \sup_{n \in \mathbb{N}} \|T^{(n)}\|$$

is the *completely bounded norm* of T.

Definition 7.22. Let V and W be operator spaces and let $T : X \to Y$ be a linear map. Then:

(a) T is *completely bounded* if $\|T\|_{cb} < \infty$.

(b) T is *completely contractive* if $\|T\|_{cb} \leq 1$.

(c) T is a *complete isometry* if each $T^{(n)}$ is an isometry.

An associative algebra $\mathcal{A}$ with multiplication m is called a *completely contractive Banach algebra* if $\mathcal{A}$ is an operator space such that the multiplication $m : \mathcal{A} \widehat{\otimes}_{op} \mathcal{A} \longrightarrow \mathcal{A}$ is completely contractive. Where $\mathcal{A} \widehat{\otimes}_{op} \mathcal{A}$ is the operator space projective tensor product.

Equivalently, $\mathcal{A}$ is a completely contractive Banach algebra if

$$\|[a_{i,j}b_{k,l}]\| \leq \|a\|\|b\|$$

for all $a = [a_{i,j}] \in M_m(\mathcal{A})$ and $b = [b_{k,l}] \in M_n(\mathcal{A})$, where $m, n \in \mathbb{N}$ are arbitrary.

Let $\mathcal{A}$ be a completely contractive Banach algebra and V an $\mathcal{A}$-bimodule. Then V is called an *operator $\mathcal{A}$-bimodule* if V is an operator space and the $\mathcal{A}$-bimodule operation

$$\mathcal{A}\widehat{\otimes}_{op} V \longrightarrow V; \qquad a \otimes v \longmapsto a \cdot v,$$

and

$$V\widehat{\otimes}_{op} \mathcal{A} \longrightarrow V; \qquad a \otimes v \longmapsto v \cdot a$$

are completely bounded. Equivalently, there exists a $C > 0$ such that

$$\|[a_{i,j} \cdot v_{k,l}]\| \leq C\|a\|\|v\|$$

and

$$\|[v_{k,l} \cdot a_{i,j}]\| \leq C\|v\|\|a\|$$

for all $a = [a_{i,j}] \in M_m(\mathcal{A})$ and $v = [v_{k,l}] \in M_n(V)$, with $m, n \in \mathbb{N}$ arbitrary.

Let V be an operator $\mathcal{A}$–bimodule and V^* the operator dual of V. Then the $\mathcal{A}$–bimodule operations on V^* given by (6.1) are completely bounded.

A completely contractive Banach algebra $\mathcal{A}$ is called *operator amenable* if for any operator $\mathcal{A}$–bimodule X, every completely bounded derivation from $\mathcal{A}$ into X^* is inner.

Now each Lau algebra is the pre-dual of a von Neumann algebra. Thus, each Lau algebra $\mathcal{A}$ comes with an operator space structure via the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}^{**}$. Examples of completely contractive Lau algebras include each pre-dual of a Hopf-von Neumann algebra. In particular, the group algebra and the Fourier algebra of a locally compact group are examples of a completely contractive Lau algebra.

Definition 7.23. A completely contractive Lau algebra $\mathcal{A}$ is called *operator left-amenable* if for any operator $\mathcal{A}$-bimodule X such that $a \cdot x = \langle a, 1_{A^*} \rangle x$ for all $x \in X$ and $a \in \mathcal{A}$, each completely bounded derivation from $\mathcal{A}$ into X^* is inner.

It is not hard to see that any completely contractive Lau algebra that is left amenable must also be operator left amenable. Surprisingly, we will see that the converse is also true.

Lemma 7.24. *Let $\mathcal{A}$ be a completely contractive Lau algebra that is operator left amenable. Then there is a nonzero $n \in \mathcal{A}^{**}$ such that $n(x \cdot a) = a(1_{A^*})x$ for each $a \in \mathcal{A}$ and $x \in \mathcal{A}^*$, where $a \cdot x$ is the usual action of $\mathcal{A}$ on $\mathcal{A}^*$ and 1_{A^*} is identity of $\mathcal{A}^*$.*

Proof. Let $X = \mathcal{A}^*$ and set

$$\langle x \cdot a, b \rangle := \langle x, ab \rangle \text{ and } a \cdot x := \langle a, 1_{A^*} \rangle x$$

for each $a, b \in \mathcal{A}$ and $x \in X$. Then X is an operator $\mathcal{A}$-bimodule.

To see this, note that the right action is the dual left action of multiplication of $\mathcal{A}$ on $\mathcal{A}$ which is completely bounded. Also for any $a \in M_n(\mathcal{A})$ and $x \in M_m(X)$ we have

$$\|[a_{i,j} \cdot x_{k,l}]\| = \|[\langle a_{i,j}, 1_{A^*} \rangle x_{k,l}]\| = \|1_{A^*}^{(n)}[a_{i,j}] \otimes [x_{k,l}]\| \leq \|1_{A^*}\|_{cb} \|a\| \|x\| = \|a\| \|x\|.$$

Since 1_{A^*} is a multiplicative linear functional on $\mathcal{A}$, for any $a, b \in \mathcal{A}$ we have

$$\langle 1_{A^*} \cdot a, b \rangle = \langle 1_{A^*}, ab \rangle = \langle 1_{A^*}, a \rangle \langle 1_{A^*}, b \rangle = \langle \langle 1_{A^*}, a \rangle 1_{A^*}, b \rangle = \langle a \cdot 1_{A^*}, b \rangle.$$

Thus $Y = \mathbb{C}1_{A^*}$ is a closed operator $\mathcal{A}$-submodule of X. Put $Z = X/Y$ and let $\pi : X \longrightarrow Z$ be the complete quotient map (see [20]). Then the dual mapping $\pi^* : Z^* \longrightarrow X^*$ is a completely isometric [20, Corollary 4.1.9] $\mathcal{A}$-bimodule homomorphism. We remark that the range of π^* is precisely the set $Y^\perp = \{x^* \in X^* : \langle x^*, y\rangle = 0, \forall y \in Y\} = \{x^* \in X^* : \langle x^*, 1_{A^*}\rangle = 0\}$.

Choose a $v \in X^*$ with $\langle 1_{A^*}, v\rangle = 1$ and let D_v be the inner derivation from $\mathcal{A}$ into X^*, $a \mapsto a \cdot v - v \cdot a$. Then for any $a \in \mathcal{A}$ we have

$$\begin{aligned}
\langle D_v(a), 1_{A^*}\rangle &= \langle a \cdot v - v \cdot a, 1_{A^*}\rangle \\
&= \langle a \cdot v, 1_{A^*}\rangle - \langle v \cdot a, 1_{A^*}\rangle \\
&= \langle v, 1_{A^*} \cdot a\rangle - \langle v, a \cdot 1_{A^*}\rangle \\
&= \langle v, 1_{A^*} \cdot a - a \cdot 1_{A^*}\rangle \\
&= 0.
\end{aligned}$$

So that $D_v(a) \in \pi^*(Z^*)$. Now since π^* is injective there is a unique $Da \in Z^*$ such that $D_v(a) = \pi^* Da$. Moreover, since π^* is completely isometric module-homomorphism, the map $D : \mathcal{A} \longrightarrow Z^*$, $a \mapsto Da$ is a completly bounded derivation.

Since $\mathcal{A}$ is operator left-amenable and Z^* is an admissible dual operator $\mathcal{A}$-bimodule, there is an $z^* \in Z^*$ with

$$Da = a \cdot z^* - z^* \cdot a$$

for each $a \in \mathcal{A}$. Thus for any $a \in \mathcal{A}$ we have

$$\begin{aligned} a \cdot \pi^* z^* - \pi^* z^* \cdot a &= \pi^*(a \cdot z^* - z^* \cdot a) \\ &= \pi^*(D_{z^*}(a)) \\ &= \pi^*(Da) \\ &= D_v(a) \\ &= a \cdot v - v \cdot a. \end{aligned}$$

Or in other words we have

$$a \cdot (v - \pi^* z^*) = (v - \pi^* z^*) \cdot a.$$

Finally, set $n = v - \pi^* z^* \in \mathcal{A}^{**}$. Then we have $n(1_{A^*}) = v(1_{A^*}) - \langle \pi^* z^*, 1_{A^*} \rangle = 1 - 0 = 1$ and for any $a \in \mathcal{A}$ we have $n \cdot a = a \cdot n$. and $x \in X$ we have

$$n(x \cdot a) = n(a \cdot x) = \langle a, 1_{A^*} \rangle n(x).$$

□

Theorem 7.25. *Let $\mathcal{A}$ be a completely contractive Lau algebra. Then $\mathcal{A}$ is left amenable if and only if $\mathcal{A}$ is operator left amenable.*

Proof. It is shown by Lau in [44] that each Lau algebra that admits a non-zero topologically invariant functional on its dual space must left-amenable. The existence of such a functional when $\mathcal{A}$ is operator left amenable follows from Lemma 7.24. The converse is trivial. □

7.4 P-amenability

We will end this chapter by exploiting the semigroup structure of $S_{\mathcal{A}}$. The study of topological semigroups and invariant means on various function spaces is very difficult and has drawn the attention of many researchers. Unlike in the group setting, the existence of invariant means on one subspace relative to another is not pleasant. For this reason, the notion of a left amenable topological semigroup is not universal. We will call a topological semigroup S *amenable* when the space of continuous and bounded functions on S admits a left invariant mean. We are also interested in the spaces of left uniformly continuous functions on S, as well as the space of all bounded functions on S. When equipped with the supremum norm and pointwise operations, each of these spaces is a left translation invariant $C*$-algebra.

The notion of a P-amenable group was introduced by Ganesan in [26]. He was interested to study when $P_1(G)$ was amenable as a topological semigroup. Although there are still many unsettled problems for group algebras, we will take a brief look at the Lau algebra setting. This will allow us to discuss several other plausible notions of amenability that one might be interested in when studying a Lau algebra. All of which are interesting roads for future research.

Definition 7.26. Let $\mathcal{A}$ be a Lau algebra.

(i) $\mathcal{A}$ is called *P-amenable* whenever $CB(S_{\mathcal{A}})$ possesses a left invariant mean.

(ii) $\mathcal{A}$ is called *weakly P-amenable* whenever $LUC(S_{\mathcal{A}})$ possesses a left invariant mean.

(iii) $\mathcal{A}$ is called *strongly P-amenable* whenever $\ell^\infty(S_{\mathcal{A}})$ possesses a left invariant mean.

It is not hard to see that any strongly P-amenable Lau algebra is P-amenable, and any P-amenable Lau algebra must be weakly P-amenable.

Moreover, it follows from Lemma 7.5 and Theorem 7.7 that any weakly P-amenable Lau algebra must be left amenable.

Example 7.27. It follows from the Markov-Kakutani Fixed Point Theorem that any abelian semigroup is left amenable. Hence, any commutative Lau algebra is strongly P-amenable.

Example 7.28. Let M be any W*-algebra and $\mathcal{A} = M_*$. If we define the multiplication on $\mathcal{A}$ by $ab = \langle a, 1_{\mathcal{A}^*}\rangle\, b$ for $a, b \in \mathcal{A}$, then $\mathcal{A}$ is strongly P-amenable. Note that in this case, for any $f \in \ell^\infty(S_\mathcal{A})$, and $a \in S_\mathcal{A}$ we have $l_a f = f$. From which it follows that any mean on $\ell^\infty(S_\mathcal{A})$ is left invariant.

Since the norm is multiplicative on the set $\mathcal{A}_+$ of all normal positive functionals on $\mathcal{A}^*$, it may be interesting to also study $\mathcal{A}_+$ as topological semigroup. However, in many situations this larger semigroup does not provide any new information about the left amenability of $\mathcal{A}$.

Proposition 7.29. *Let $\mathcal{A}$ be a Lau algebra with a bounded approximate identity for $\mathcal{A}_+$ lying inside $S_\mathcal{A}$. Then $\mathcal{A}$ is P-amenable if and only if $\mathcal{A}^+$ is left amenable.*

Proof. Suppose $\mathcal{A}_+$ is left amenable. Let $\Phi : \mathcal{A}_+ \to S_\mathcal{A}$ be defined by $a \mapsto a/\|a\|$. Then Φ is a continuous surjective homomorphism and thus $S_\mathcal{A}$ is left amenable (see Day [13]).

Now suppose that m is a LIM on $CB(S_\mathcal{A})$. Let $f \in CB(\mathcal{A}_+)$ and define

$$f'(a) := \langle m, (\ell_a f)|_{S_\mathcal{A}}\rangle, \qquad a \in \mathcal{A}_+. \tag{7.1}$$

Then we have

$$f'(ab) = f'(a),\ (\ell_a f)' = \ell_a f' \quad a \in \mathcal{A}_+,\ b \in S_\mathcal{A}. \tag{7.2}$$

Now let e_α be a bounded approximate identity for $\mathcal{A}_+$ in $S_\mathcal{A}$. Then for any $a \in \mathcal{A}_+$ we have

$$\begin{aligned} f'(a) &= \lim_\alpha f'(e_\alpha a) \\ &= \lim_\alpha f'(\|a\| e_\alpha a / \|a\|) \\ &\overset{(7.2)}{=} \lim_\alpha f'(\|a\| e_\alpha). \end{aligned}$$

Thus, $\dot{f}(\omega) := \lim_\alpha f'(\omega e_\alpha)$ is well defined for any $\omega \in \mathbb{R}^+$. Also note that we have $f'(a) = \dot{f}(\|a\|)$ for any $a \in \mathcal{A}_+$.

Let M be a LIM on $CB(\mathbb{R}^+, \cdot)$ and define $\mu \in CB(\mathcal{A}_+)^*$ by

$$\langle \mu, f \rangle := \langle M, \dot{f} \rangle.$$

Then μ is our desired LIM. We will included the details for completeness sake.

Clearly we have $(cf + g)^\cdot = c\dot{f} + \dot{g}$, so that μ is indeed linear. Also $\|\dot{f}\|_\infty \leq \|f'\|_\infty \leq \|f\|_\infty$ shows that $\|\mu\| \leq 1$. Since $\dot{1} = 1$, we have $\langle \mu, 1 \rangle = \langle M, \dot{1} \rangle = 1$. So that $\|\mu\| = 1 = \langle \mu, 1 \rangle$ and μ is a mean. We now proceed to show that μ is left invariant. First note that for $a, b \in \mathcal{A}_+$ we have

$$(\ell_a f)'(b) = \ell_a f'(b) = f'(ab) = \dot{f}(\|ab\|) = \dot{f}(\|a\|\|b\|).$$

Thus for $a \in \mathcal{A}_+$ and $\omega \in \mathbb{R}^+$ we have:

$$(\ell_a f)^\cdot(\omega) = \lim_\alpha (\ell_a f)'(\omega e_\alpha) = \lim_\alpha \dot{f}(\|a\|\|\omega e_\alpha\|) = \dot{f}(\|a\|\omega) = \ell_{\|a\|}\dot{f}(\omega).$$

i.e. $(\ell_a f)^\cdot = \ell_{\|a\|}\dot{f}$. Thus for any $f \in CB(\mathcal{A}_+)$ and $a \in \mathcal{A}_+$ we have $\langle \mu, \ell_a f \rangle = \langle M, (\ell_a f)^\cdot \rangle = \langle M, \ell_{\|a\|}\dot{f} \rangle = \langle M, \dot{f} \rangle = \langle \mu, f \rangle$. □

Minor changes to the proof above shows that a similar result holds for strong and weak P-amenability. The relationship between the various forms of P-amenability still needs to be investigated. Currently, we do not know of any left amenable Lau algebras which are not P-amenable.

Chapter 8

Open Questions

Problem 1. For which groups is $DP(G) = \mathcal{UM}(G)$?

It was shown by Crombez and Govaerts that $DP(G) = \mathcal{UM}(G)$ for all metrizable groups. It would be interesting to know whether this relation holds for non-metrizable groups as well.

Problem 2. When is $DP(G)$ a C^*−algebra?

It was shown in Proposition 3.7 that $\mathcal{UM}(G)$ is always a C^*−algebra. Consequently, for metrizable groups, $DP(G)$ is a C^*−algebra.

Problem 3.

(a) For which groups is $UCB(\widehat{G})$ contained in $DP(\widehat{G})$?

(b) For which groups is $WAP(\widehat{G}) \subseteq DP(\widehat{G})$?

(c) For which groups is $DP(\widehat{G})$ closed under multiplication and involution? In particular, when is $DP(\widehat{G})$ a C^*−algebra.

Our results from Section 4.3 tells us that (a) is true for at least discrete or compact groups. An affirmative answer to (a) for amenable groups would imply that (b) is also valid for amenable groups. Our quest to solve (a) has led us to the next problem.

Problem 4. Let G be a locally compact group and let K be a compact subset of G. Does $A_K(G)$ have the Schur property?

We have shown that this is actually equivalent to asking whether $A_K(G)$ has the Dunford-Pettis property. An affirmative answer would imply that $UCB(\widehat{G}) \subseteq DP(\widehat{G})$ always. A deeper understanding of weak convergence in the Fourier algebra would be beneficial for tackling this problem.

Problem 5. What are necessary and sufficient conditions for weak convergence of sequences in $A(G)$?

Problem 6. Are vector Segal subspaces always Banach algebras?

We feel that some technical computations still need to be worked out to ensure that each vector-Segal subspace is always a left Banach-$L^1(G, \mathcal{A})$ module.

Problem 7. Are the different forms of P-amenability introduced in section 7.4 equivalent?

For a topological semigroup, the conditions of Definition 7.26 are not equivalent. However, for Lau algebras, the topological semigroup is more pleasant than many other topological semigroups.

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