University of Alberta

Filtering and Estimation in Modern Computer Control Systems

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

in

Controls

Department of Electrical and Computer Engineering

Edmonton, Alberta
Spring 2007

Your file Votre référence
ISBN: 978-0-494-29734-6
Our file Notre référence
ISBN: 978-0-494-29734-6

University of Alberta

Library Release Form

Name of Author: Mehrdad Sahebsara

Title of Thesis: Filtering and Estimation in Modern Computer Control Systems

Degree: Doctor of Philosophy

Year this Degree Granted: 2007

Mehrdad Sahebsara

Date: ________________

In memory of my father

To my wife, Payvand, and sons, Mehrshad and Kusha
and my mother, brothers and sisters
for their support and encouragement

Abstract

This thesis studies problems of filtering and estimation in modern computer control systems. The main focus is on filtering in control systems employing communication networks and in discrete-time control systems with multi-rate sampled data. More specifically, filtering problems are studied for systems with random sensor delays, multiple sensor data packet dropout, uncertainty in observation, and networked control systems with multiple packet dropout. Multi-rate Kalman filtering and the problem of parameter estimation for some general multi-rate systems are also explored.

The random sensor delay, multiple packet dropout, or uncertainty in observation is transformed into a stochastic parameter problem in the state-space framework. A new formulation is employed to model the multiple packet dropout in the sensor data, and in networked control systems. The formulation also can be used to assign separate dropout rates from the sensors to the controller and from the controller to the actuators. Based on the stochastic definition of the $\mathcal{H}_2$-norm, new relations for the $\mathcal{H}_2$-norm of a stochastic parameter system with both stochastic and deterministic inputs are derived. Also, a generalized $\mathcal{H}_\infty$-norm is studied for this type of system. The stochastic $\mathcal{H}_2$-norm or $\mathcal{H}_\infty$-norm of the filtering error is used as a criterion for filter design. The relations derived for the new norm definition are used to obtain a set of linear matrix inequalities to solve the corresponding filter design problem.

A *state lifting* method is introduced that can be used to generalize the minimum variance Kalman filtering method to the multi-rate case for fast-rate state filtering and estimation. Based on multi-rate input-output data and fast-rate system models, the optimal Kalman gains and covariance matrices are found at the fast rate.

This thesis also studies the parameter estimation of a general multi-input, multi-output multi-rate system in the frequency domain. Two methods, *dividing to sub-systems* and *input extension*, are introduced for dealing with multi-rate systems, and the later method is used to convert a multi-input, multi-output multi-rate system into several sub-problems with fast input updating and slow output sampling. In this framework, all frequency-domain parameter estimation methods can be applied. In this work, a least-square parameter estimation method is generalized for parameter estimation in the multi-input, multi-output case.

Several examples, including one with industrial data, are provided to show the effectiveness and applicability of the proposed methods.

Acknowledgements

I am deeply indebted to my supervisor, Professor Tongwen Chen, for his patient guidance, encouragement, excellent advice and constant support throughout my study. Without his help, this work would not be possible. I would also like to express my sincere thanks to Professor Sirish L. Shah for his invaluable experience and advice, support and encouragement throughout my study.

Special thanks to Professor Asok Ray who accepted to be my external examiner and gave valuable comments and suggestions. I also acknowledge my committee members, Dr. Bob Koch, Dr. Alan Lynch and Dr. Qing Zhao for their time, judgment and comments.

I also acknowledge the financial support by the Ministry of Science, Research and Technology of Iran, and the Natural Sciences and Engineering Research Council of Canada (NSERC).

Words fail me to express my appreciation to my wife Payvad for her dedication, support, encouragement, patience, companionship and understanding during this time. Great thanks to my son, Mehrshad, for his patience and understanding as well. I moved him to a new country and culture at his childhood, hope my education deserve all these changes to his life. One of the best experiences in this period was the birth of our son, Kusha, who provided an additional and joyful dimension to our life.

I am deeply and forever indebted to my mother for her support and encouragement throughout my entire life. I am also very grateful to my nice brothers and sisters for their great and continues support and encouragement.

Finally, I would like to thank the good friends in Edmonton who were a mental support to me, as well as expressing my apology that I could not mention personally one by one.

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Chapter 1

Introduction

1.1 Background

Due to the rapid developments in digital computers and microcomputers, control systems are changing drastically. Nowadays, powerful microprocessor systems are used to control even the most basic control loops. Because of the higher flexibility and lower cost of digitally implemented control systems, digital controllers normally outperform their analog counterparts. Microcomputer-based smart sensors and actuators are also employed in control systems. Advances in digital computer technology combined with control systems theory have led to new developments in modern computer control systems. These new techniques give us flexibility of implementation in different frameworks but also raise new challenges in design. Since digital computers are used for control system implementation, carrying out modelling and design in discrete time is naturally more convenient. Discrete-time modelling can be used either because of its simplicity in modelling and simulation or because of the system characteristics, for example, in a radar system, the information is naturally obtained once per each revolution of the antenna; an internal combustion engine is another example of a sampled system. The main focus in this thesis is on discrete-time models.

With the increasing presence of wired and wireless communication networks, control systems using communication networks and the internet are also an important emerging technology. Network-based control systems as executed over internet communication channels have become an increasingly challenging research area (see, e.g., [18, 22, 28, 37, 38, 42, 51, 56–58, 60, 61]). Even though using a distributed control system over a network provides flexibility in installation and maintenance and results in cost savings, it makes the design problem much more challenging. In such distributed control systems, data travel through different networks and communication channels from the sensors to the controller and from the controller to the actuators. As a direct consequence of the finite bandwidth for data transmission over networks, time delay and packet dropout are inevitable in networked systems where a common medium is used for data transfers. In most network-based systems, time delay or data packet dropout is random. Similar problems arise in other practical systems such as wireless sensor networks or target tracking systems, where only sensor delay or dropout exists. Filtering and control in these systems are more challenging due to the stochastic nature of the delay or dropouts. Classical (non-delay-based) estimation and designs may not satisfy the performance and stability requirements and are not optimal for the delay or dropout cases. Clearly, new methods are needed in these cases.

State feedback is the most common strategy used in modern control systems for the stabilization and control of complex physical systems. In practice, especially in networked control systems, not all of the state variables are always available for direct measurement, so state filtering and estimation play a key role in state feedback methods. The filtering problem is to estimate the states or a linear combination of them by using the measured system inputs and outputs. With the introduction of state estimators by Luenberger [34], state estimators have been used to estimate state variables from readily available measurements. One of the early optimal estimators in a Gaussian noise environment is the so-called *Kalman filter* [23].

To work with sensor delay, uncertain observation or networked control systems, stochastic delay, dropout and uncertainty in observation are transformed into stochastic parameters in the state-space system representation. The commonly used transformation [36, 53, 56, 57] can be used to model a maximum of one sampling delay, whereas the new model proposed in this thesis allows for multiple packet dropouts. Some existing methods try to generalize Kalman filtering to stochastic parameter cases, but these methods make complicated formulations and fail to provide the optimal solution [45]. Also, multiple packet dropouts in sensor data have not yet been well studied. In this thesis, by using a new formulation, estimation and filtering with multiple packet dropout are cast in the same framework as the single delay and uncertain observation problem. By introducing new notion of stochastic $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms, the filtering and estimation involving random sensor delay, multiple packet dropout, uncertain observation, and networked control systems can all be treated in a unified framework and therefore are presented as a generalization of the classical case.

In conventional computer control, input updating and output sampling are performed in discrete time instants by using samplers and zero-order holds. In such discrete-time control systems, the plant input updating and output sampling are at the same rate. However, updating the control input and sampling the output at the same rate are not always possible due to various limitations such as the cost of fast-rate sensors and actuators. Moreover, sometimes the plant dynamics are such that sampling the different plant signals at the same rate is not economical and useful. As a result, a multi-rate sampling scheme should be considered for such cases. Of course, this scheme introduces the complication of mixed time steps. Such systems are often used in the chemical process industry (see, e.g., [19, 39]). The Kalman filtering and parameter and output estimation for these systems will be studied in Chapters 5 and 6.

1.2 Summary of Contributions

The main contributions of this thesis are summarized as follows:

- A new formulation is proposed to model the multiple packet dropout in sensor information.

- The new formulation is generalized to model the multiple packet dropout in networked control systems. This formulation enables us to assign separate dropout rates from the sensors to the controller and from the controller to the actuators.

- Based on the new definition of the $\mathcal{H}_2$-norm of a system with stochastic parameters, new relations for the stochastic $\mathcal{H}_2$-norm of a stochastic parameter system with both stochastic and deterministic inputs are derived.

- A generalized $\mathcal{H}_\infty$-norm is studied for the systems with stochastic parameters and both stochastic and deterministic inputs.

- By using stochastic $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm definitions, a general framework is provided to study filtering for different problems such as sensor delay, multiple sensor data packet dropout, uncertain observation, and networked control systems with multiple packet dropout. The stochastic $\mathcal{H}_2$-norm or $\mathcal{H}_\infty$-norm of the filtering error is used as a criterion for filter design. The relations derived for the new norm definition are used to obtain a set of linear matrix inequalities to solve the corresponding filter design problem.

- A *state lifting* method is introduced that can be easily used to generalize the minimum variance Kalman filtering method to the multi-rate case for fast-rate state filtering. The optimal Kalman gains and covariance matrices are found at the fast rate by using multi-rate input-output data and fast-rate system models.

- The parameter estimation of a general multi-input, multi-output multi-rate system in the frequency domain is studied. Two methods, *dividing to subsystems* and *input extension*, are introduced for dealing with multi-rate systems. Finally, a least-square parameter estimation method is generalized for parameter estimation in the multi-input, multi-output multi-rate case.

1.3 Outline of the Thesis

The remainder of this thesis is organized as follows.

In Chapter 2, the problem of optimal filtering for discrete-time systems with random sensor delay, multiple sensor data packet dropout, or uncertain observation is studied. The random sensor delay, multiple packet dropout or uncertainty in observation is transformed into a stochastic parameter in the system representation. A new formulation enables us to design an optimal filter for a system with multiple packet dropout in the sensor data. Based on a stochastic definition of the $\mathcal{H}_2$-norm of a system with a stochastic parameter, new relations for the stochastic $\mathcal{H}_2$-norm are derived. The stochastic $\mathcal{H}_2$-norm of the estimation error is used as a criterion for the filter design. The relations derived for the new norm definition are used to obtain a set of linear matrix inequalities (LMIs) to solve the filter design problem. Simulation examples show the effectiveness of the proposed method.

Chapter 3 studies the problem of optimal $\mathcal{H}_2$ filtering in networked control systems (NCSs) with multiple packet dropout. A new formulation is employed to model the multiple packet dropout case, where the random dropout rates are transformed into stochastic parameters in the system's representation. By generalization of the $\mathcal{H}_2$-norm definition, new relations for the stochastic $\mathcal{H}_2$-norm of a linear discrete-time stochastic parameter system represented in the state-space form are derived. The stochastic $\mathcal{H}_2$-norm of the estimation error is used as a criterion for filter design in the NCS framework. A set of LMIs is provided to solve the corresponding filter design problem. A simulation example supports the theory.

By using the same formulation as in Chapter 3, the problem of $\mathcal{H}_\infty$ filtering in networked control systems with multiple packet dropout is studied in Chapter 4. Again, by employing the new formulation, random dropout rates are transformed into stochastic parameters in the system's representation. A generalized $\mathcal{H}_\infty$-norm for systems with stochastic parameters and both stochastic and deterministic inputs is derived. The stochastic $\mathcal{H}_\infty$-norm of the filtering error is used as a criterion for filter design in the NCS framework. A set of LMIs is provided to solve the corresponding filter design problem. A simulation example supports the theory.

Chapter 5 studies the problem of optimal Kalman filtering for multi-rate processes. A *state lifting* method is introduced that can be easily used to generalize the minimum variance Kalman filtering method to the multi-rate case for fast-rate state estimation. The optimal Kalman gains and covariance matrices are found at the fast rate by using multi-rate input-output data and fast-rate system models. Some examples, especially the one taken from a real mechanical system for air-fuel ratio control, validate the applicability of the proposed method to Kalman filter design in dual-rate and multi-rate processes represented in the state-space form.

In Chapter 6, the parameter estimation of a general multi-input, multi-output multi-rate system in the frequency domain is studied. Two methods, *dividing to sub-*

systems and *input extension*, are introduced for dealing with multi-rate systems, and the later method is used to convert a multi-input, multi-output multi-rate system into several sub-problems with fast input updating and slow output sampling. In this case, frequency-domain parameter estimation methods can be applied for identification purposes. Here, a least-squares parameter estimation method is generalized for parameter estimation in the multi-input, multi-output case. Several examples, including one with real industrial data, are provided to show the effectiveness of the methods proposed.

The thesis ends in Chapter 7 with our conclusions and some suggestions for directions for future work.

Chapter 2

Optimal $\mathcal{H}_2$ Filtering for some Discrete-time Stochastic Systems

2.1 Introduction

For stabilization and control of complex physical systems, modern control methods use the state-space formulation. Several state-space control system design techniques have been developed to implement more sophisticated controllers. Remarkably, in almost all of these methods, control is in the form of a state feedback, which is applicable under the implicit assumption that all state variables are available for feedback. This assumption may not hold in practice, either because not all state variables are accessible for direct connection or because sensing devices or transducers are not available or are very expensive for all state variables. In this case, in order to apply state feedback, a state estimator must be designed to estimate the states from measurable signals. With the introduction of state estimators by Luenberger [34], they have been used to estimate state variables from readily available measurements. One of the early optimal estimators used is the so-called *Kalman*

The main material of this chapter was reported in [45].

filter [23].

During the past few years, interest in delay systems has increased, mainly because of the advantages of and new advances in networked control systems (NCSs) (see, e.g., [18, 22, 37, 38, 42, 51, 56–58]). In this control scheme, data travel through the communication networks from the sensors to the controller and from the controller to the actuators. Time delay and packet dropout are inevitable in networked systems where a common medium is used for data transfers. In most network-based systems, time delay is random. Also, a random packet dropout could occur due to network congestion. Estimation and control in these systems are more challenging due to the stochastic delay or dropouts. Classical (non-delay-based) estimation and designs do not satisfy the performance and stability requirements and optimality in the delay or dropout cases. Therefore, new methods are needed in these cases. Generally speaking, three types of delay can be considered:

- Delay in states
- Input delay
- Output (sensor) delay

Amongst all of these delays, the sensor delay has not received much attention even though it exists and is challenging. Even by considering the NCS with delay both from the sensor to the controller and from the controller to the actuator, the two delays can be lumped together to have a single delay [18]. As well, some practical problems such as those involving wireless sensor networks or target tracking systems present the problem of stochastic sensor delay. Packet dropout is another problem that can arise in networked systems and is closely related to the sensor delay problem.

To work with sensor network systems, stochastic delay or uncertain information are usually transformed into a stochastic parameter of the system. The commonly used transformation [36,53,56,57] can be used to model a maximum of one sampling delay, while the new proposed representation in this chapter allows for multiple packet dropouts as well.

The derivation used in [56, 57] is based on instantaneous error variance minimization, which is a generalization of Kalman filtering in the delay case. In these studies, the augmented noise vectors are incorrectly assumed to be white, so that some dependencies between the noise terms seem to have been ignored, thus making the designs suboptimal.

Data packet dropout is another common problem in networked systems. This dropout is a kind of uncertainty that may happen due to node failures or network congestion. In real-time feedback control systems, discarding the old packets and considering new packets so that the controller always receives fresh data for control calculation are normally advantageous. The problem of packet dropout has been studied before (see, e.g., [28, 60, 61] and the references therein). While most previous work has used switched systems and Markov chains, the proposed method handles the problem of multiple packet dropouts in an easier way, by using the same framework used for the sensor delay systems.

Another problem closely related to sensor delay systems is the so-called *uncertain observation* system. In some cases, there is an uncertainty about the measurements: the measurements are either the current system output or just the noise. For example, this problem occurs in some cases like tracking systems. This problem has also been studied in some papers [36, 58]. The new general framework proposed in this chapter can also handle the problem of measurement uncertainty.

Uncertainty and delays in sensor information are more common and have been the focus of attention in the literature [36, 56–58]. Classical methods fail to solve the filtering and estimation problems for cases with delays, uncertainty or packet dropout. Existing methods that try to generalize the Kalman filtering to delay cases make complicated formulations and fail to provide optimal solutions. In addition, multiple packet dropouts in sensor data have not been well studied yet.

This chapter makes two main contributions. First, by using a new formulation, estimation and filtering with multiple packet dropout are cast in the same framework as that for a single delay and uncertain observation problem. Secondly, by introducing a new notion of the stochastic $\mathcal{H}_2$-norm, the estimation and filtering of random sensor delay, multiple packet dropout, and uncertain observation cases can all be treated in the same framework and therefore are presented as a generalization of the classical cases.

In this chapter, we consider the system as depicted in Figure 2.1. The input, $\tilde{\omega}$, is an exogenous signal, a random white noise. z is the signal to be estimated, and $\hat{z}$ is its estimate. The output to be minimized is the filtering error $\tilde{z} = z - \hat{z}$. The aim is to minimize the $\mathcal{H}_2$-norm of the filtering error. In the sensor delay and packet dropout cases, due to network effects, the filter input, y, is either the current plant output, $\tilde{y}$, or the previous one. In the case of an uncertain observation system, the filter input, y, is either the current plant output, $\tilde{y}$, or just the noise.

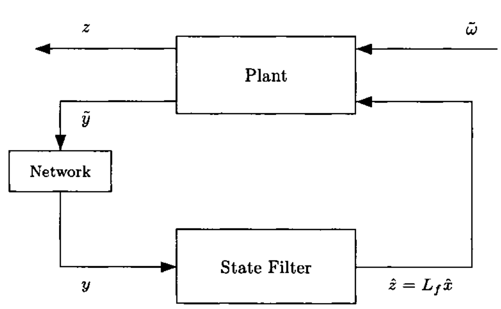

Figure 2.1: Filtering with sensor delay, dropout or uncertain observation

To solve this problem, the filter gains are designed so that the $\mathcal{H}_2$-norm of the estimation error is minimized. This minimization is transformed into the minimization of the $\mathcal{H}_2$-norm of the transfer function from the input, $\tilde{\omega}$, to the filtering error, $\tilde{z}$. As the delay, dropout, or uncertainty in observation are stochastic, we face a stochastic parameter system, so the stochastic $\mathcal{H}_2$-norm ($\mathcal{H}_{2s}$-norm) is defined and problems are solved by using LMI techniques [4].

2.2 Preliminaries

2.2.1 Norms in Deterministic Parameter Systems

Many goals in controller and filter design can be expressed in terms of the size of a signal. For example, in control problems, a controller without large actuator signals is desirable. In filtering and estimation problems, filter gains are designed so that the filtering or estimation error signal is as small as possible. Even though a signal's size can be defined in different ways, the methods that satisfy certain properties have proven to be most useful and are called *norms.*

Definition 1. *Suppose V is a vector space and $\phi : V \rightarrow \mathbb{R}$. ϕ is a norm on V if it satisfies [3]*

Nonnegativity: $\phi(v) \geq 0$, *with* $\phi(v) = 0 \Leftrightarrow v = 0$,

Homogeneity: $\phi(cv) = |c|\phi(v)$,

Triangle inequality: $\phi(v + w) \leq \phi(v) + \phi(w)$,

for all $c \in \mathbb{R}$ *and* $v, w \in V$.

One of the most commonly used norms is the so-called *2-norm*, which (actually, its square) is associated with energy. For the signal $v = \{v(0), v(1), \cdots\}$, the 2-norm is defined as follows [6]:

$$\|v\|_2 = [v(0)^2 + v(1)^2 + \cdots]^{1/2}. \tag{2.1}$$

A notion closely related to the signal norm is the size of a transfer function or linear system, known as the system norm, which has to satisfy properties similar to those that must be satisfied by the signal norm. One of the most commonly used system norms is the 2-norm which is defined as

$$\|G\|_2 = (\frac{1}{2\pi}\int_0^{2\pi} |G(e^{j\theta})|^2 d\theta)^{1/2}, \tag{2.2}$$

where $G(z)$ is the system transfer function. As an immediate consequence of Parseval's equality, if the system's input is a unit impulse, the output's 2-norm equals the systems's 2-norm. As well, if the input is standard white noise, then the output's root-mean-square value will be $\|G\|_2$ [6]. If system G is represented in the state space form with A, B, C and D matrices, then in the SISO case, the $\mathcal{H}_2$-norm of the system will be

$$\|G\|_2 = (D^2 + CLC')^{1/2} \tag{2.3}$$

with L being the controllability Gramian satisfying

$$L = ALA' + BB'. \tag{2.4}$$

As well, in the MIMO case [6],

$$\|G\|_2 = [\text{trace}(D^2 + CLC')]^{1/2} \tag{2.5}$$

with

$$L = ALA' + BB'. \tag{2.6}$$

2.2.2 Filtering in Deterministic Parameter Systems

The filtering problem is to estimate the states or a linear combination of them by using the measured system inputs and outputs. With the introduction of state estimators by Luenberger [34], they have been used to estimate state variables by using readily available measurements. Consider the system represented by the following equations:

$$\begin{cases} x_{k+1} &= Ax_k + B\omega_k \\ y_k &= Cx_k + D\omega_k. \end{cases} \tag{2.7}$$

The Luenberger-type estimator will be of the following form:

$$\hat{x}_{k+1} = A\hat{x}_k + K(y_k - C\hat{x}_k), \tag{2.8}$$

where $\hat{x}$ is an estimate of x, and K is the estimator gain. This estimator has been extensively used in the literature. The Kalman filter [23] uses this type of estimator. As will be seen later, in the stochastic parameter case, the parameter matrices are not deterministic, so the Luenberger estimator cannot be directly used. The common estimator used in that case is provided in (2.30). Besides the Kalman filter, other optimal filters such as optimal $\mathcal{H}_2$ filters have been introduced [7, 40]. The details are omitted here, but the adaptation and generalization for the stochastic parameter systems will be discussed in the following sections.

2.2.3 Linear Matrix Inequalities

A wide variety of problems in systems and control theory can be converted into optimization problems involving linear matrix equalities (LMIs). The resulting LMIs can be numerically solved efficiently by using the existing solver packages.

An LMI has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{2.9}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}, i = 0, \cdots, m$, are given. The inequality symbol means that $F(x)$ is positive definite. The LMI in (2.9) is a convex constraint on x; i.e., the set $\{x|F(x) > 0\}$ is convex. Multiple LMIs $F^1(x) > 0, \cdots, F^p(x) > 0$ can be expressed as a single LMI: $\text{diag}(F^1(x), \cdots, F^p(x)) > 0$. The most important tool used to convert nonlinear convex inequalities into LMIs is the so-called *Schur complements*: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{2.10}$$

is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0, \tag{2.11}$$

where symmetric matrices $Q(x)$ and $R(x)$ and the matrix $S(x)$ are affine on x. In most cases, variables are matrices. The most famous example is the Lyapunov inequality

$$A^T P + PA < 0, \tag{2.12}$$

where $A \in \mathbb{R}^{n\times n}$ is given, and the symmetric matrix P is the variable. Even if the last inequality can be converted into an LMI as expressed in the form of (2.9), leaving the LMI in a condensed form saves notation and leads to more efficient computation [4].

LMIs have been used to formulate different problems in filtering and control. The problem of optimal $\mathcal{H}_2$ filtering has also been studied in the LMI framework (see, e.g., [40]). Using LMIs gives us the numerical solution while other methods such as the Kalman method, provide closed form analytical solutions. However, efficient numerical methods can be used to solve the LMI problems, and as will be discussed later, in some cases as in stochastic parameter systems, the explicit solutions either do not exist or have a complex form.

2.3 Problem Formulation

2.3.1 Sensor Delay and Multiple Packet Dropout

We consider the system depicted in Figure 2.1. Here, the plant is a discrete-time linear time-invariant (LTI) plant subject to random disturbances. Also, the case is considered where sensor data is contaminated with noise:

$$\begin{cases} \tilde{x}_{k+1} &= \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k \\ \tilde{y}_k &= \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k. \end{cases} \tag{2.13}$$

Here, $\tilde{x}_k \in \mathbb{R}^n$ is the plant state vector, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are system parameter matrices with appropriate dimensions. $\tilde{\omega}_k$ is an exogenous input with

$$\begin{aligned} \mathcal{E}\{\tilde{\omega}_k\} &= 0, \\ \mathcal{E}\{\tilde{\omega}_j\tilde{\omega}_k'\} &= \begin{cases} 0, & j \neq k, \\ 1, & j = k \end{cases} \end{aligned} \tag{2.14}$$

where $\mathcal{E}\{.\}$ stands for the mathematical expectation. For simplicity in the derivations, the single-input, single-output (SISO) case is considered, but as will be mentioned later, generalizing the results to the multiple-input, multiple-output (MIMO) case is not difficult. In the SISO case, $\tilde{y}_k \in \mathbb{R}$ is the system output contaminated with zero-mean noise, $\tilde{\omega}_k$. In the following sub-sections, formulations regarding this system is considered to represent the delayed observations, multiple packet dropout, and uncertain observation.

One Sampling Delay Formulation

The formulation used in this section was first introduced in [36] and has become a common formulation for sensor delay systems [53, 56–58]. Consider a system as described by (2.13). Consider that the current observation, y_k, is the current system output, $\tilde{y}_k$, with probability of α or the previous one, $\tilde{y}_{k-1}$, with probability of $(1-\alpha)$. These expressions can be represented by the following relation:

$$y_k = \delta_k \tilde{y}_k + (1-\delta_k)\tilde{y}_{k-1}, \tag{2.15}$$

where the stochastic parameter δ_k is a Bernoulli distributed white sequence taking the values 0 and 1 with

$$\text{prob}\{\delta_k = 1\} = \mathcal{E}\{\delta_k\} = \alpha, \tag{2.16}$$

where $\alpha \in \mathbb{R}$ is a known constant. It is also supposed that δ_k is uncorrelated with $\tilde{\omega}_k$ and the initial state values, so

$$\begin{aligned} &\text{prob}\{\delta_k = 0\} = 1-\alpha \\ &\text{var}\{\delta_k\} = (1-\alpha)\alpha. \end{aligned} \tag{2.17}$$

Now, by putting Equations 2.13 and 2.15 together, we have the sensor delay representation as

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k \\ y_k & = \delta_k \tilde{y}_k + (1-\delta_k)\tilde{y}_{k-1}. \end{cases} \tag{2.18}$$

To obtain a compact representation of the plant and measurement system, the system states can be augmented:

$$x_{k+1} = \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{x}_k \end{bmatrix}, \tag{2.19}$$

so we get

$$\begin{cases} x_{k+1} & = \mathbf{a}_k x_k + \mathbf{b}_k \omega_k \\ y_k & = \mathbf{c}_k x_k + \mathbf{d}_k \omega_k \\ z_k & = L x_k, \end{cases} \tag{2.20}$$

where z_k is the signal to be estimated and

$$\begin{aligned} &\mathbf{a}_k = \begin{bmatrix} \mathbf{a} & 0 \\ I & 0 \end{bmatrix}, \quad \mathbf{b}_k = \begin{bmatrix} \mathbf{b} & 0 \\ 0 & 0 \end{bmatrix}, \\ &\mathbf{c}_k = \begin{bmatrix} \delta_k \mathbf{c} & (1-\delta_k)\mathbf{c} \end{bmatrix}, \quad \mathbf{d}_k = \begin{bmatrix} \delta_k \mathbf{d} & (1-\delta_k)\mathbf{d} \end{bmatrix}, \quad \omega_k = \begin{bmatrix} \tilde{\omega}_k \\ \tilde{\omega}_{k-1} \end{bmatrix}. \end{aligned} \tag{2.21}$$

This formulation allows us to have a maximum of one sampling delay for observations. If $\delta_k = 0$, then $y_k = \tilde{y}_{k-1}$ for every $k > 1$. Note that $\mathbf{c}_k$ and $\mathbf{d}_k$ are functions of δ_k, but for simplicity in notation, $\mathbf{c}_k$ and $\mathbf{d}_k$ instead of $\mathbf{c}(\delta_k)$ and $\mathbf{d}(\delta_k)$ are used. $\mathbf{a}_k$ and $\mathbf{b}_k$ are constant matrices, but the subscript k is used for consistency with the following formulation.

Multiple Packet Dropout Formulation

In order to treat the case of multiple packet dropout, the following formulation is suggested:

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k \\ y_k & = \delta_k \tilde{y}_k + (1-\delta_k) y_{k-1}. \end{cases} \tag{2.22}$$

In this case, if $\delta_k = 1$, the observation is the current output. Otherwise, the observation is the previous observation, as can be seen from (2.22). With successive 0's in δ_k, multiple packet dropouts can be modelled.

Now, to have a compact representation, the system states and the observation can be augmented as

$$x_{k+1} = \begin{bmatrix} \tilde{x}_{k+1} \\ y_k \end{bmatrix}. \tag{2.23}$$

Thus,

$$\begin{cases} x_{k+1} & = \mathbf{a}_k x_k + \mathbf{b}_k \tilde{\omega}_k \\ y_k & = \mathbf{c}_k x_k + \mathbf{d}_k \tilde{\omega}_k \\ z_k & = L x_k, \end{cases} \tag{2.24}$$

where

$$\begin{aligned} \mathbf{a}_k &= \begin{bmatrix} \mathbf{a} & 0 \\ \delta_k \mathbf{c} & (1-\delta_k) \end{bmatrix}, \quad \mathbf{b}_k = \begin{bmatrix} \mathbf{b} \\ \delta_k \mathbf{d} \end{bmatrix} \\ \mathbf{c}_k &= \begin{bmatrix} \delta_k \mathbf{c} & (1-\delta_k) \end{bmatrix}, \quad \mathbf{d}_k = \delta_k \mathbf{d}. \end{aligned} \tag{2.25}$$

Note that in this case, all matrices of the augmented system are functions of δ_k, but for simplicity, still $\mathbf{a}_k$, $\mathbf{b}_k$, $\mathbf{c}_k$ and $\mathbf{d}_k$ are used instead.

2.3.2 Uncertain Observation Formulation

Normally, in filtering and estimation theory, the observation is always assumed to contain some information from either the current system output or the delayed one, as described in the previous subsection. In some practical cases, observations may contain either the actual output contaminated with noise or the noise alone, only the probability of the occurrence of such cases is available to the estimator. An example is trajectory tracking [36].

Here, we have the same discrete-time linear time-invariant (LTI) plant as before:

$$\tilde{x}_{k+1} = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k, \tag{2.26}$$

with similar definitions for the states, system matrices and noise. Now, consider that the observation, y_k, is the current system output with the probability of α or the noise alone, with the probability of $(1-\alpha)$. This expression can be represented by the following relation:

$$y_k = \delta_k \mathbf{c} x_k + \mathbf{d}\tilde{\omega}_k, \tag{2.27}$$

where the stochastic parameter δ_k and matrices $\mathbf{c}$ and $\mathbf{d}$ are as defined before. Now, all the relations can be put in the following form to provide a representation similar to those of the above-mentioned cases:

$$\begin{cases} x_{k+1} &= \mathbf{a}_k x_k + \mathbf{b}_k \tilde{\omega}_k \\ y_k &= \mathbf{c}_k x_k + \mathbf{d}_k \tilde{\omega}_k \\ z_k &= L x_k, \end{cases} \tag{2.28}$$

where

$$\mathbf{a}_k = \mathbf{a}, \quad \mathbf{b}_k = \mathbf{b}, \quad \mathbf{c}_k = \delta_k \mathbf{c}, \quad \mathbf{d}_k = \mathbf{d}. \tag{2.29}$$

In summary, a unifying framework, as in (2.20), can be used to model the sensor delay, multiple packet dropout, and uncertain observation. This unifying model can be easily used with the new derivations for the stochastic $\mathcal{H}_2$-norm to provide a unified optimal filter design for all of the above-mentioned cases.

2.3.3 Filter Formulation

Consider the linear stochastic time-varying discrete-time system represented in (2.20) where x_k is the state vector, y_k is the measurement output, ω_k is the noise signal and z_k is the signal to be estimated. The goal in the filtering problem is to find the estimate $\hat{z}_k$ of z_k such that a performance criterion such as the stochastic $\mathcal{H}_2$-norm of the estimation error is minimized. Now, consider the following filter:

$$F : \begin{cases} \hat{x}_{k+1} & = \mathbf{a}_f \hat{x}_k + \mathbf{b}_f y_k \\ \hat{z}_k & = L_f \hat{x}_k, \end{cases} \tag{2.30}$$

where $\hat{x}_k$ is an estimate of the state, and $\mathbf{a}_f$, $\mathbf{b}_f$ and L_f are the filter parameters to be designed. The filtering error is defined as $\tilde{z}_k = z_k - \hat{z}_k$. Now, the system states, x_k, and the filter states, $\hat{x}_k$, can be augmented to obtain the following augmented system:

$$H : \begin{cases} \zeta_{k+1} & = A_k \zeta_k + B_k \omega_k \\ \tilde{z}_k & = C \zeta_k, \end{cases} \tag{2.31}$$

where

$$\zeta_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \quad A_k = \begin{bmatrix} \mathbf{a}_k & 0 \\ \mathbf{b}_f \mathbf{c}_k & \mathbf{a}_f \end{bmatrix}, \quad B_k = \begin{bmatrix} \mathbf{b}_k \\ \mathbf{b}_f \mathbf{d}_k \end{bmatrix}, \quad C = [L \quad -L_f]. \tag{2.32}$$

The filtering problem is to design a filter F as in (2.30) such that the time-averaged filtering error variance is minimized.

2.4 $\mathcal{H}_2$-norm of Systems with a Stochastic Parameter

As was shown in the previous section, the representation of a system with stochastic sensor delay, multiple packet dropout, or uncertain observation can be reformulated as a system with a stochastic parameter. The problem of filter design for systems with deterministic parameters has been fully studied in the literature (see, e.g., [7,40] and references therein). Attempts have been made to solve the filtering problem in the Kalman filtering framework [56, 57], but complex formulations do not seem to provide an optimal solution. An important objective of this chapter is to introduce the stochastic $\mathcal{H}_2$-norm of the filtering error system as a performance index. To solve such a problem, a LMI formulation of the performance index and corresponding constraints are introduced. As a first step, it is tried to define the $\mathcal{H}_2$-norm of a system with a stochastic parameter, and the corresponding LMIs are considered in the next section.

For a deterministic stable discrete-time linear time-invariant (LTI) system, "if the input is a standard (unit variance) white noise, then the root-mean-square value of the output equals the $\mathcal{H}_2$-norm of the system" [6].

As the time delay system under consideration is transformed into a time-varying system, the classical norm definition needs to be modified to be applicable in this case.

Let us consider a general stable discrete-time linear time-varying system G:

$$G : \begin{cases} \zeta_{k+1} & = A_k \zeta_k + B_k \omega_k \\ \tilde{z}_k & = C_k \zeta_k + D_k \omega_k, \end{cases} \tag{2.33}$$

where A_k, B_k, C_k and D_k are time-dependent matrices (through the stochastic parameter, δ_k) with appropriate dimensions. Following the general definition of the $\mathcal{H}_2$-norm of a time-invariant system, we define the $\mathcal{H}_2$-norm of the stochastic time variant system G, belonging to a class of systems represented in (2.33), as

$$\|G\|_{2s}^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{E}\{\tilde{z}_k^2\}, \tag{2.34}$$

with ω_k a unit variance white noise input.

Remark. *The expression in (2.34) can be shown to satisfy the norm properties for a class of systems represented in (2.33). If the system is time-invariant, (2.34)* *reduces to the standard $\mathcal{H}_2$-norm. Hence, it is regarded as a generalization of the $\mathcal{H}_2$-norm to the stochastic parameter systems.*

Suppose that $\zeta(0) = 0$, then by using system representation we can write:

$$\begin{aligned}
\zeta(1) &= C_1 B_0 \omega_0 + D_1 \omega_1 \\
\zeta(2) &= C_2 A_1 B_0 \omega_0 + C_2 B_1 \omega_1 + D_2 \omega_2 \\
\zeta(3) &= C_3 A_2 A_1 B_0 \omega_0 + C_3 A_2 B_1 \omega_1 + C_3 B_2 \omega_2 + D_3 \omega_3 \\
&\vdots
\end{aligned}$$

Then it is easy to derive

$$\begin{aligned}
\mathcal{E}\{\zeta(1)^2\} &= \mathcal{E}\{C_1 B_0 \omega_0 \omega_0' B_0' C_1' + D_1 \omega_1 \omega_1' D_1'\} = \mathcal{E}\{C_1 B_0 B_0' C_1' + D_1 D_1'\} \\
\mathcal{E}\{\zeta(2)^2\} &= \mathcal{E}\{C_2 A_1 B_0 \omega_0 \omega_0' B_0' A_1' C_2' + C_2 B_1 \omega_1 \omega_1' B_1' C_2' + D_2 \omega_2 \omega_2' D_2'\} \\
&= \mathcal{E}\{C_2 A_1 B_0' B_0' A_1' C_2' + C_2 B_1 B_1' C_2' + D_2 D_2'\} \qquad (2.35) \\
&\vdots,
\end{aligned}$$

where $(.)'$ is the transpose of $(.)$. Now, suppose that A_k, B_k, C_k and D_k are affine in a stochastic parameter δ_k and

$$\mathcal{E}\{\delta_k\} = \alpha, \quad \text{var}\{\delta_k\} = q^2. \qquad (2.36)$$

Thus, δ_k can be written as the sum of its mean value and a zero-mean stochastic variable λ_k with the same variance:

$$\delta_k = \alpha + \lambda_k, \qquad (2.37)$$

where

$$\mathcal{E}\{\lambda_k\} = 0, \quad \text{var}\{\lambda_k\} = q^2, \quad \mathcal{E}\{\lambda_k \lambda_s\} = 0, \; s \neq k, \qquad (2.38)$$

and

$$\begin{cases} A_k &= A + \lambda_k \tilde{A} \\ B_k &= B + \lambda_k \tilde{B} \\ C_k &= C + \lambda_k \tilde{C} \\ D_k &= D + \lambda_k \tilde{D}, \end{cases} \qquad (2.39)$$

where $A, \tilde{A}, ...$ are constant known matrices. Now, define

$$A_q = q\tilde{A}, \quad B_q = q\tilde{B}, \quad C_q = q\tilde{C}, \quad D_q = q\tilde{D}, \tag{2.40}$$

then

$$\mathcal{E}\{A_k A_k'\} = \mathcal{E}\{(A + \lambda_k \tilde{A})(A' + \lambda_k \tilde{A}')\} = AA' + A_q A_q', \tag{2.41}$$

and similarly,

$$\begin{cases} \mathcal{E}\{B_k B_k'\} & = BB' + B_q B_q' \\ \mathcal{E}\{C_k C_k'\} & = CC' + C_q C_q' \\ \mathcal{E}\{D_k D_k'\} & = DD' + D_q D_q'. \end{cases} \tag{2.42}$$

Thus,

$$\begin{aligned} \mathcal{E}\{\zeta(1)^2\} &= CBB'C' + C_qBB'C_q' + CB_qB_q'C' + C_qB_qB_q'C_q' + DD' + D_qD_q' \\ \mathcal{E}\{\zeta(2)^2\} &= CBB'C' + C_qBB'C_q' + CB_qB_q'C' + C_qB_qB_q'C_q' + \\ &\quad +CABB'A'C' + CAB_qB_q'A'C' + CA_qBB'A_q'C' + CA_qB_qB_q'A_q'C' + \\ &\quad +C_qABB'A'C_q' + C_qAB_qB_q'A'C_q' + C_qA_qBB'A_q'C_q' + \\ &\quad +C_qA_qB_qB_q'A_q'C_q' + DD' + D_qD_q' \\ &\vdots \end{aligned} \tag{2.43}$$

Putting all these relations into equation (2.34), we have

$$\|G\|_{2s}^2 = C[L_1 + L_2]C' + C_q[L_1 + L_2]C_q' + \bar{D}\bar{D}', \tag{2.44}$$

where

$$\begin{aligned} L_1 &= \bar{B}\bar{B}' + A\bar{B}\bar{B}'A' + A^2\bar{B}\bar{B}'A^{2'} + A^3\bar{B}\bar{B}'A^{3'} + ... \\ &\quad +AA_q\bar{B}\bar{B}'A_q'A' + AA_q^2\bar{B}\bar{B}'A_q^{2'}A' + AA_qA\bar{B}\bar{B}'A'A_q'A' + ... \\ L_2 &= A_q\bar{B}\bar{B}'A_q' + A_q^2\bar{B}\bar{B}'A_q^{2'} + A_q^3\bar{B}\bar{B}'A_q^{3'} + ... \\ &\quad +A_qA\bar{B}\bar{B}'A'A_q' + A_qA^2\bar{B}\bar{B}'A^{2'}A_q' + A_q^2A\bar{B}\bar{B}'A'A_q^{2'} + ... \\ \bar{B}\bar{B}' &= BB' + B_qB_q' \\ \bar{D}\bar{D}' &= DD' + D_qD_q'. \end{aligned} \tag{2.45}$$

It is not difficult to see that

$$\begin{cases} L_1 &= \bar{B}\bar{B}' + A(L_1 + L_2)A' \\ L_2 &= A_q(L_1 + L_2)A_q'. \end{cases} \tag{2.46}$$

Adding relations of L_1 and L_2 together, we get

$$L_c = \bar{B}\bar{B}' + AL_cA' + A_qL_cA_q', \tag{2.47}$$

where

$$L_c = L_1 + L_2. \tag{2.48}$$

Therefore,

$$\|G\|_{2s}^2 = CL_cC' + C_qL_cC_q' + \bar{D}\bar{D}'. \tag{2.49}$$

Thus, the result can be summarized as follows:

Theorem 1. **($\mathcal{H}_{2s}$-norm)**: *Consider G, the discrete-time linear stochastic parameter system represented in (2.33). The stochastic $\mathcal{H}_2$-norm ($\mathcal{H}_{2s}$-norm) of the system defined by (2.34) is*

$$\|G\|_{2s}^2 = CL_cC' + C_qL_cC_q' + \bar{D}\bar{D}' \tag{2.50}$$

with

$$L_c = \bar{B}\bar{B}' + AL_cA' + A_qL_cA_q'. \tag{2.51}$$

Corollary 1. **(MIMO $\mathcal{H}_{2s}$-norm)**: *Consider G, the discrete-time linear stochastic parameter system represented in (2.33) in the MIMO case. The stochastic $\mathcal{H}_2$-norm of the system is given by*

$$\|G\|_{2s}^2 = trace\{CL_cC' + C_qL_cC_q' + \bar{D}\bar{D}'\}, \tag{2.52}$$

where

$$L_c = \bar{B}\bar{B}' + AL_cA' + A_qL_cA_q'. \tag{2.53}$$

Equations (2.52) and (2.53) are a generalization of the SISO case similar to the classical one as in [6] and the proof is straightforward.

Corollary 2. *Consider G, the stochastic parameter system represented in (2.33). The stochastic $\mathcal{H}_2$-norm of the SISO system is given by*

$$\|G\|_{2s}^2 = B'L_bB + B_q'L_bB_q + \bar{D}'\bar{D}, \tag{2.54}$$

and for the MIMO case it is

$$\|G\|_{2s}^2 = trace\{B'L_bB + B_q'L_bB_q + \bar{D}'\bar{D}\}, \tag{2.55}$$

where

$$\begin{aligned} L_b &= \bar{C}'\bar{C} + A'L_bA + A_q'L_bA_q \\ \bar{C}'\bar{C} &= C'C + C_q'C_q. \end{aligned} \tag{2.56}$$

Proof. Let us define

$$\mathcal{A} = A^{i_1}A_q^{i_2}A^{i_3}A_q^{i_4}\cdots A^{i_j}\cdots, \quad i_j = 0, 1, \cdots, \quad j = 1, 2, \cdots \tag{2.57}$$

as this term appears in (2.43). In the SISO case, the terms $C\mathcal{A}B$ and $B'\mathcal{A}'C'$ appearing in (2.43) are scalars. Thus,

$$(C\mathcal{A}B)(B'\mathcal{A}'C') = (B'\mathcal{A}'C')(C\mathcal{A}B). \tag{2.58}$$

For the MIMO case,

$$\text{trace}\{(C\mathcal{A}B)(B'\mathcal{A}'C')\} = \text{trace}\{(B'\mathcal{A}'C')(C\mathcal{A}B)\}. \tag{2.59}$$

Thus, the new derivations in (2.54), (2.55) and (2.56) will require rearrangement in the proof of Theorem 1 and Corollary 1. □

2.5 Optimal Filter Design

Now, we have the required tools to solve the optimal filtering problem which has been solved for the deterministic parameter case before. Based on the new derivations for the $\mathcal{H}_{2s}$-norm as in the previous section, the design methods will be generalized to the stochastic parameter case. In the case of no stochastic parameter in the system, the results will be the same as those for the deterministic case. The optimal filtering problem can be stated as follows:

The optimal $\mathcal{H}_{2s}$ filtering problem: Design a filter F as in (2.30) such that the time-averaged estimation error variance is minimized.

Based on the $\mathcal{H}_{2s}$-norm definition, we need to minimize the $\mathcal{H}_{2s}$-norm of the estimation error dynamics to solve the filtering problem.

If A_k and B_k contain a stochastic parameter as described before, we can write:

$$\begin{aligned} A_k &= \begin{bmatrix} \mathbf{a} & 0 \\ \mathbf{b}_f\mathbf{c} & \mathbf{a}_f \end{bmatrix} + \lambda_k \begin{bmatrix} \tilde{\mathbf{a}} & 0 \\ \mathbf{b}_f\tilde{\mathbf{c}} & 0 \end{bmatrix} = A + \lambda_k \tilde{A} \\ B_k &= \begin{bmatrix} \mathbf{b} \\ \mathbf{b}_f\mathbf{d} \end{bmatrix} + \lambda_k \begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{b}_f\tilde{\mathbf{d}} \end{bmatrix} = B + \lambda_k \tilde{B}. \end{aligned} \tag{2.60}$$

Let us define A_q and B_q as

$$A_q = q\tilde{A}, \quad B_q = q\tilde{B}, \tag{2.61}$$

where $q^2 = \mathrm{var}\{\lambda_k\}$. Define the transfer function from ω to $\tilde{z}$ as $H_{\tilde{z}\omega}$. Now, using Theorem 3, we can write

$$\|H_{\tilde{z}\omega}\|_{2s}^2 = CL_{\mathbf{c}}C', \tag{2.62}$$

where

$$L_{\mathbf{c}} = BB' + B_qB_q' + AL_{\mathbf{c}}A' + A_qL_{\mathbf{c}}A_q'. \tag{2.63}$$

The problem of $\mathcal{H}_2$ filtering for deterministic discrete time systems has been studied in the literature [14, 40]. (See also [9] for a detailed discussion of the continuous time case.) In the following, we try to adapt the deterministic filter design problem to the stochastic cases by using the tools defined in the previous section.

The problem of filtering error variance minimization can be formulated as follows where the matrix variables J and P are symmetric:

$$\min_{\mathbf{a}_f, \mathbf{b}_f, L_f, P} \quad \text{trace}(J) \tag{2.64}$$

s.t.

$$\begin{bmatrix} P & PC' \\ * & J \end{bmatrix} > 0 \tag{2.65}$$

$$\begin{bmatrix} P & AP & A_qP & B & B_q \\ * & P & 0 & 0 & 0 \\ * & * & P & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} > 0. \tag{2.66}$$

Next, we want to convert the two matrix inequalities in (2.65) and (2.66) into LMIs. Then, the filter design problem turns out to be a convex programming problem that can be solved efficiently by using the existing numerical methods available.

Let us partition P and its inverse as

$$P = \begin{bmatrix} X & U \\ U' & X_2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & V \\ V' & Y_2 \end{bmatrix}, \tag{2.67}$$

where X, Y, X_2 and Y_2 are symmetric and positive definite matrices. As $PP^{-1} = I$, the following relations hold:

$$\begin{aligned} &XY + UV' = I \\ &Y^{-1} = X - UX_2^{-1}U' \\ &Y'U + VX_2 = 0. \end{aligned} \tag{2.68}$$

Now, we define the following nonsingular matrices:

$$\bar{T} = \begin{bmatrix} Z & Y \\ 0 & V' \end{bmatrix}, \quad T_1 = \begin{bmatrix} \bar{T} & 0 \\ 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} \bar{T} & & & & \\ & \bar{T} & & 0 & \\ & & \bar{T} & & \\ & 0 & & I & \\ & & & & I \end{bmatrix} \tag{2.69}$$

with $Z = X^{-1}$. By applying the congruence transformation to (2.65) by T_1, we get the following LMI:

$$T_1' \begin{bmatrix} P & PC' \\ * & J \end{bmatrix} T_1 = \begin{bmatrix} Z & Z & L' - G' \\ * & Y & L' \\ * & * & J \end{bmatrix} > 0, \tag{2.70}$$

where $G = L_f U' Z$. We can also get a LMI by applying the congruence transformation to (2.66) by T_2:

$$\begin{bmatrix} \bar{T}'P\bar{T} & \bar{T}'AP\bar{T} & \bar{T}'A_qP\bar{T} & \bar{T}'B & \bar{T}'B_q \\ * & \bar{T}'P\bar{T} & 0 & 0 & 0 \\ * & * & \bar{T}'P\bar{T} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} > 0, \tag{2.71}$$

where it is not difficult to see that

$$\begin{aligned} \bar{T}'P\bar{T} &= \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}, \\ \bar{T}'AP\bar{T} &= \begin{bmatrix} Z\mathbf{a} & Z\mathbf{a} \\ Y\mathbf{a} + F\mathbf{c} + Q & Y\mathbf{a} + F\mathbf{c} \end{bmatrix}, \\ \bar{T}'A_qP\bar{T} &= \begin{bmatrix} Z\mathbf{a}_q & Z\mathbf{a}_q \\ Y\mathbf{a}_q + F\mathbf{c}_q & Y\mathbf{a}_q + F\mathbf{c}_q \end{bmatrix}, \\ \bar{T}'B &= \begin{bmatrix} Z\mathbf{b} \\ Y\mathbf{b} + F\mathbf{d} \end{bmatrix}, \\ \bar{T}'B_q &= \begin{bmatrix} Z\mathbf{b}_q \\ Y\mathbf{b}_q + F\mathbf{d}_q \end{bmatrix}, \end{aligned} \tag{2.72}$$

with $F = V\mathbf{b}_f$ and $Q = V\mathbf{a}_f U' Z$. Thus, the results can be summarized as follows.

Theorem 2. **($\mathcal{H}_{2s}$-filtering)** *The filter design problem of (2.64)-(2.66) is equivalent* *to the following convex programming problem:*

$$\min_{Z,Y,Q,F,G} \quad trace(J)$$

$$s.t. \tag{2.73}$$

$$(2.70) \text{ and } (2.71).$$

To find the filter parameters, $\mathbf{a}_f$, $\mathbf{b}_f$ and L_f, we need to know U and V, which do not appear in the LMIs. One of the matrices U or V can be defined freely. Different choices give us different filter state-space realizations. One logical choice is to set $L_f = L$ that can come from setting $V = V' = -Y$, leading to $U = U' = Z^{-1} - Y^{-1}$.

2.6 Examples

In this section, three simulation examples are provided to show the effectiveness and applicability of the proposed method for optimal filter design in systems with sensor delay, multiple packet dropout, and uncertain observation. All the simulations are based on Monte Carlo simulations with 100 runs for each value of α. The first example simulates the sensor delay system. The second example uses the same system as in Example 1, but multiple packet dropouts are simulated by using the new formulation. The third example studies the uncertain observation case. In Examples 1 and 3 the existing methods are compared with the proposed method, whose superiority is demonstrated.

Example 1 - Consider a discrete-time LTI system represented by

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k \end{cases} \tag{2.74}$$

with the following matrix values:

$$\mathbf{a} = \begin{bmatrix} 1.7240 & -0.7788 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{c} = \begin{bmatrix} 0.0286 & 0.0264 \end{bmatrix}, \quad \mathbf{d} = 0.2. \tag{2.75}$$

Also, consider that the observations are governed by

$$y_k = \delta_k \tilde{y}_k + (1 - \delta_k)\tilde{y}_{k-1}. \tag{2.76}$$

The initial state values are $\tilde{x}(0) = [0 \quad 0]'$ and $\hat{\tilde{x}}(0) = [2 \quad -2]'$. System states and their estimates due to the unit variance white noise input are plotted in the following figures where x_1 and x_2 refer to the first and second states, respectively. The simulations are done by using the method presented in [56], referred to as the *Yaz and Ray method.* Figure 2.2 shows the simulation results obtained by using the proposed stochastic $\mathcal{H}_2$ and the Yaz and Ray method when the average observation uncertainty rate is $\alpha = 0.2$. Figure 2.3 compares the estimation error variance of the two methods, where e_1 and e_2 refer to the estimation error for the first and second states, respectively. The graphs show the effectiveness of the proposed method with less overshoot and less filtering error variance.

Example 2 - In this example, the same system as in Example 1 is used, but the observations in this case can have multiple packet dropout as in (2.22). The simulation results are given in Figure 2.4 for the classical and stochastic $\mathcal{H}_2$ filtering with $\alpha = 0.2$. The estimation error variances in two cases are given in Figure 2.5, which shows that the stochastic design method provides superior results compared to those from the classical one. Figure 2.5 also shows that the proposed stochastic method has the same results when no randomness occurs in the observation ($\alpha = 1$).

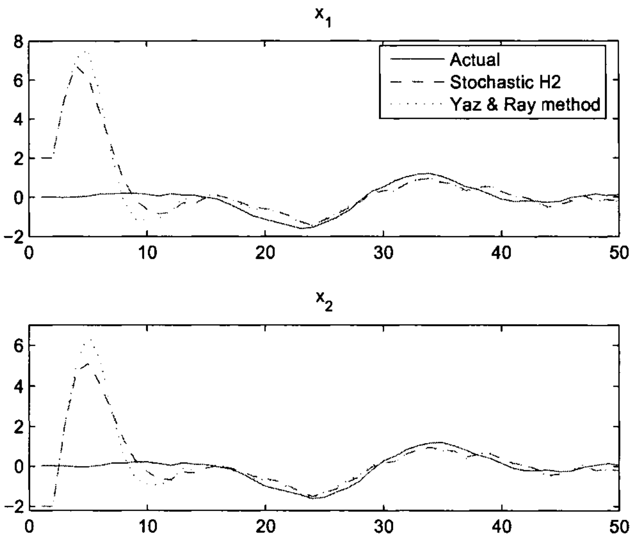

Figure 2.2: Actual and estimated states for the stochastic $\mathcal{H}_2$ and Yaz and Ray methods, $\alpha = 0.2$, single delay case

Example 3 - Consider a discrete-time LTI system represented by

$$\tilde{x}_{k+1} = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k \tag{2.77}$$

with uncertain observations as

$$y_k = \delta_k \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k \tag{2.78}$$

with the same definitions for **a**, **b**, **c** and **d**, the initial state values, the noise, and the uncertainty parameter δ, as in Example 1. The system states and their estimates due to the unit variance white noise input are plotted in the following figures. The simulations are also done by using the method presented in [36], referred to as *Nahi's method.* Figure 2.6 shows the simulation results for the stochastic $\mathcal{H}_2$ and Nahi's method when the average observation uncertainty rate is $\alpha = 0.2$. Figure 2.7 shows the estimation error variance for the two cases. The graphs show the effectiveness of the proposed method.

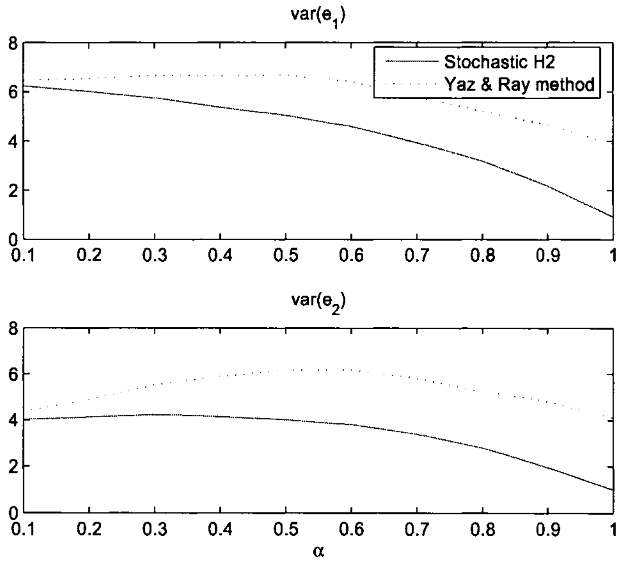

Figure 2.3: Estimation error variance vs. α (the stochastic $\mathcal{H}_2$ and Yaz and Ray methods)

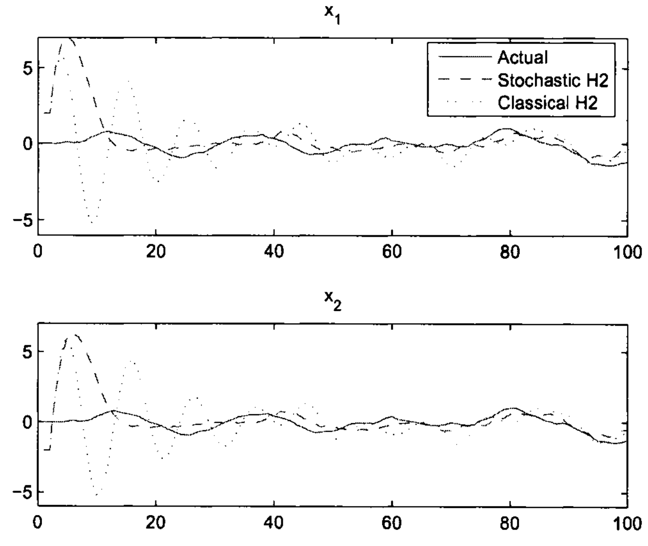

Figure 2.4: Actual and estimated states for the classical and stochastic $\mathcal{H}_2$ filtering, $\alpha = 0.2$, multiple packet dropout

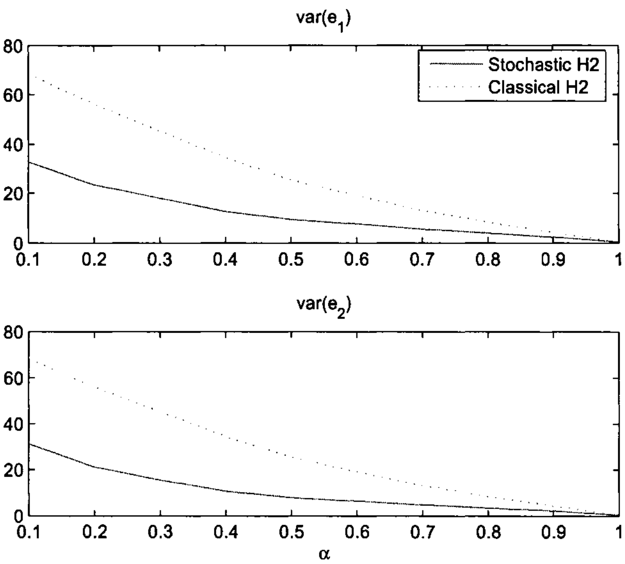

Figure 2.5: Estimation error variance vs. α (the stochastic and classical $\mathcal{H}_2$ methods)

Remark. *In Examples 1 and 3, the two methods converge when the observation noise tends to zero. The proposed method works better even in the deterministic case because this method uses a performance index considering the averaged error variance over all the times, not just the instantaneous error variance.*

2.7 Conclusions

In this chapter, the problem of optimal filtering in the sensor delay, multiple packet dropout, and uncertain observation case has been studied. A new formulation was introduced to model the multiple packet dropout in the sensor data. To solve the problems, the stochastic $\mathcal{H}_2$-norm for systems containing a stochastic parameter was defined and the relations were developed. Based on this new derivations, the problem was transformed into a set of LMIs that can be solved by using existing solver

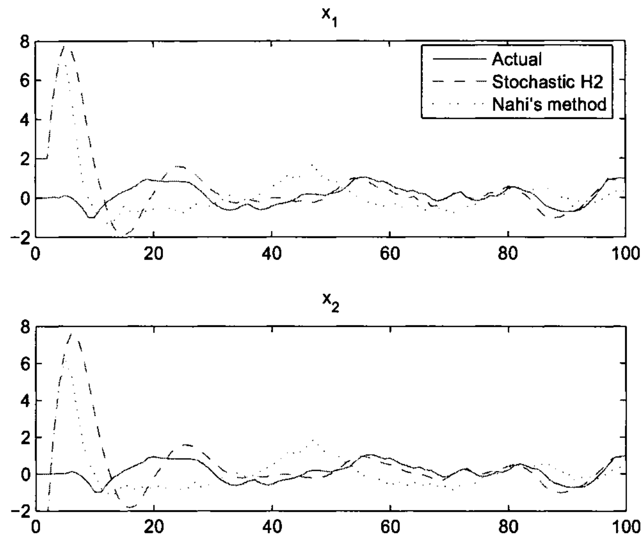

Figure 2.6: Actual and estimated states for the stochastic $\mathcal{H}_2$ and Nahi's methods, $\alpha = 0.2$, uncertain observation

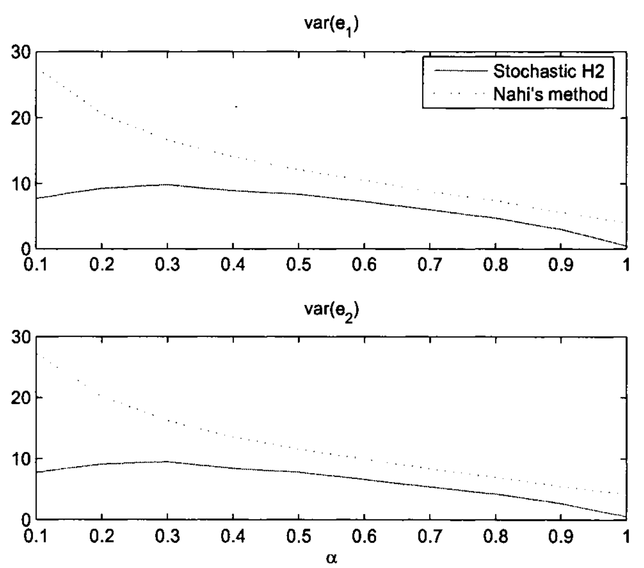

Figure 2.7: Estimation error variance vs. α (the stochastic $\mathcal{H}_2$ and Nahi's methods)

packages. Some simulation examples showed the effectiveness and applicability of the proposed method.

Chapter 3

Optimal $\mathcal{H}_2$ Filtering in NCS with Multiple Packet Dropout

3.1 Introduction

As discussed in the previous chapter, many modern control methods employing the state feedback strategy use state-space formulation. State feedback is applicable under the implicit assumption that all state variables are measurable. However, in practice, some state variables may not be directly accessible or the corresponding sensing devices may be unavailable or very expensive. In such cases, state filters or state estimators are used to give an estimate of the unavailable states. Luenberger [34] first introduced state estimators, and many later studies have been conducted in this area for different practical scenarios.

Networked control systems (NCSs) have gained attention during last few years (see, e.g., [18, 22, 33, 37, 42, 51] and references therein). Compared to using the conventional point-to-point system connection, using a NCS has advantages like easy installation and reduced set-up, wiring and maintenance costs. In a NCS, data travel through the communication channels from the sensors to the controller and

The material of this chapter was reported in [46].

from the controller to the actuators. Data packet dropout, a kind of uncertainty that may happen due to node failures or network congestion, is a common problem in networked systems. In real-time feedback control systems, discarding the old packets and considering new packets so that the controller always receives fresh data for control calculation are normally advantageous. The dropouts happen randomly. Because of random dropout, classical estimation and control methods cannot be used directly. Dropouts can degrade system performance and increase the difficulty of filtering and estimation.

Even though most research conducted on NCSs considers random delay, the closely related random packet dropout has not been well studied and only in last few years has been the focus of some research studies. To the best of our knowledge, no work has been conducted regarding filtering in packet dropout systems, but the problem of stabilization and control has been studied recently in packet dropout systems (see, e.g., [27, 28, 59–61] and references therein). In some of these studies, only sensor data dropouts are studied ([27, 60]). While [27] considers adaptive genetic algorithms and simulated annealing algorithms, guaranteed cost control, and the state feedback controller, other references consider switched systems and Markov chains to solve the problem. The main problem in working with Markov chains is the unknown Markov states. Identifying the number of states of the Markov chain and their transient probability by using hidden Markov models are other issues in the research on NCSs.

The problem of optimal $\mathcal{H}_2$ filtering has been tackled in deterministic cases (see, e.g., [14, 40]), but, to the best of our knowledge, optimal $\mathcal{H}_2$ filtering has not been studied in NCSs with multiple packet dropout. In this chapter, the problem of optimal $\mathcal{H}_2$ filtering in a NCS with multiple packet dropout has been considered. A new formulation is proposed to formulate the NCS with multiple random packet dropout.

By generalization of the $\mathcal{H}_2$-norm definition, new relations for the stochastic $\mathcal{H}_2$-norm of a linear discrete-time stochastic parameter system represented in the state space form are derived. The new derivations enable us to consider estimation and filtering of the NCS as a generalization of the classical case. To solve the filtering problem, the filter gains are designed so that the $\mathcal{H}_2$-norm of the estimation error is minimized. As dropout rates are stochastic, the problem formulation leads to a system with stochastic parameters. Thus, the stochastic $\mathcal{H}_2$-norm ($\mathcal{H}_{2s}$-norm) of the estimation error is considered as a measure to minimize. With both deterministic and stochastic inputs present in the NCS framework, a weighted $\mathcal{H}_2$-norm is defined and used. The filtering problem is transformed into a convex optimization problem through a set of LMIs that can be solved by using existing numerical techniques [4]. The design method proposed in this chapter gives a general framework to study other scenarios like NCSs with random delays.

3.2 Problem Formulation

The schematic of the NCS under study is depicted in Figure 3.1. We suppose that the controller is already designed. The exogenous input, $\tilde{\omega}$, is a random white noise signal with unity variance. z is the signal to be estimated, and $\hat{z}$ is its estimate. We want to minimize the variance of the filtering error, $\tilde{z}$. The plant is a discrete-time linear time-invariant (LTI) one subject to random disturbances. Also, the sensor data are contaminated with noise. The plant can be represented by the following equations:

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}_1\tilde{u}_k + \mathbf{b}_2\tilde{\omega}_k \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}_1\tilde{u}_k + \mathbf{d}_2\tilde{\omega}_k, \end{cases} \tag{3.1}$$

where $\tilde{x}_k \in \mathbb{R}^n$ is the plant state vector, and $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{c}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ are system parameter matrices with appropriate dimensions. For simplicity in the derivations,

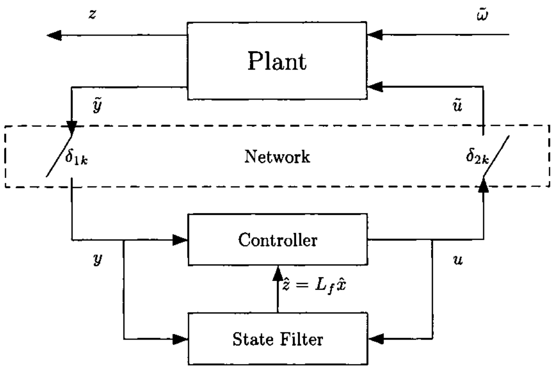

Figure 3.1: NCS schematic with packet dropout

the single-input, single-output (SISO) case is considered, but the results can be easily generalized to the multiple-input, multiple-output (MIMO) case, as will be mentioned later. In the SISO case, $\tilde{y}_k \in \mathbb{R}$ is the system output contaminated with zero-mean noise, $\tilde{\omega}_k$. Also, $\tilde{u}_k \in \mathbb{R}$ is the system command input.

Consider the system described by (3.1). The system output, $\tilde{y}$, is passed through the network and there may be random dropouts, only the probability of the dropouts, α_1, is known. Thus, the current observation, y_k, is the current system output, $\tilde{y}_k$, with the probability of α_1. In the case of no new data, previous data will be used, so the previous data, y_{k-1}, will be used with the probability of $(1 - \alpha_1)$. The filter has knowledge of the current control command, but the system input, $\tilde{u}_k$, is the current controller output, u_k, with the probability of α_2 or the previous one, $\tilde{u}_{k-1}$, with the probability of $(1 - \alpha_2)$. These expressions can be represented by the following relations:

$$\begin{cases} y_k &= \delta_{1k}\tilde{y}_k + (1-\delta_{1k})y_{k-1} \\ \tilde{u}_k &= \delta_{2k}u_k + (1-\delta_{2k})\tilde{u}_{k-1}, \end{cases} \tag{3.2}$$

where the stochastic parameters δ_{ik}'s are Bernoulli distributed white sequences taking the values of 0 or 1 with

$$\text{prob}\{\delta_{ik} = 1\} = \mathcal{E}\{\delta_{ik}\} = \alpha_i, \quad 0 \le \alpha_i \le 1, \quad i = 1, 2, \tag{3.3}$$

where α_i's are known constants. Also suppose that δ_{ik}'s are uncorrelated with each other, $\tilde{\omega}_k$, and the initial state values, so

$$\text{prob}\{\delta_{ik} = 0\} = 1 - \alpha_i, \quad \text{var}\{\delta_{ik}\} = \alpha_i(1 - \alpha_i) = q_i^2. \tag{3.4}$$

Now, Equations (3.1) and (3.2) can be put together to have the NCS formulation with multiple packet dropout as follows:

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}_1\tilde{u}_k + \mathbf{b}_2\tilde{\omega}_k \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}_1\tilde{u}_k + \mathbf{d}_2\tilde{\omega}_k \\ y_k & = \delta_{1k}\tilde{y}_k + (1 - \delta_{1k})y_{k-1} \\ \tilde{u}_k & = \delta_{2k}u_k + (1 - \delta_{2k})\tilde{u}_{k-1}, \end{cases} \tag{3.5}$$

In order to get a compact representation, we augment the system states, measurement and the system input:

$$x_{k+1} = \begin{bmatrix} \tilde{x}_{k+1} \\ y_k \\ \tilde{u}_k \end{bmatrix}, \tag{3.6}$$

thus,

$$\begin{cases} x_{k+1} & = \mathbf{a}_k x_k + \mathbf{b}_{1k}u_k + \mathbf{b}_{2k}\tilde{\omega}_k \\ y_k & = \mathbf{c}_k x_k + \mathbf{d}_{1k}u_k + \mathbf{d}_{2k}\tilde{\omega}_k \\ z_k & = Lx_k \end{cases} \tag{3.7}$$

where z_k is the signal to be estimated and

$$\mathbf{a}_k = \begin{bmatrix} \mathbf{a} & 0 & (1 - \delta_{2k})\mathbf{b}_1 \\ \delta_{1k}\mathbf{c} & 1 - \delta_{1k} & \delta_{1k}(1 - \delta_{2k})\mathbf{d}_1 \\ 0 & 0 & 1 - \delta_{2k} \end{bmatrix}, \quad \mathbf{b}_{1k} = \begin{bmatrix} \delta_{2k}\mathbf{b}_1 \\ \delta_{1k}\delta_{2k}\mathbf{d}_1 \\ \delta_{2k} \end{bmatrix}, \quad \mathbf{b}_{2k} = \begin{bmatrix} \mathbf{b}_2 \\ \delta_{1k}\mathbf{d}_2 \\ 0 \end{bmatrix}$$
$$\mathbf{c}_k = \begin{bmatrix} \delta_{1k}\mathbf{c} & 1 - \delta_{1k} & \delta_{1k}(1 - \delta_{2k})\mathbf{d}_1 \end{bmatrix}, \quad \mathbf{d}_{1k} = \delta_{1k}\delta_{2k}\mathbf{d}_1, \qquad \mathbf{d}_{2k} = \delta_{1k}\mathbf{d}_2. \tag{3.8}$$

Note that $\mathbf{a}_k$, $\mathbf{b}_{1k}$, $\mathbf{b}_{2k}$, $\mathbf{c}_k$, $\mathbf{d}_{1k}$ and $\mathbf{d}_{2k}$ are functions of δ_{ik}'s, but for simplicity, we use $\mathbf{a}_k$, $\mathbf{b}_{1k}$, $\mathbf{b}_{2k}$, $\mathbf{c}_k$, $\mathbf{d}_{1k}$ and $\mathbf{d}_{2k}$ instead.

Considering the linear stochastic discrete-time system as in (3.7), we want to find the estimate $\hat{z}_k$ of z_k such that the variance of the filtering error is minimized. Now, consider the following filter:

$$F : \begin{cases} \hat{x}_{k+1} & = \mathbf{a}_f\hat{x}_k + \mathbf{b}_f u_k + \mathbf{c}_f y_k \\ \hat{z}_k & = L_f\hat{x}_k, \end{cases} \tag{3.9}$$

where $\hat{x}_k$ is an estimate of the state, and $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f are the filter parameters to be designed. The filtering error is defined as $\tilde{z}_k = z_k - \hat{z}_k$. Now, the system states, x_k, and the filter states, $\hat{x}_k$, can be augmented to get the following augmented system:

$$H : \begin{cases} \zeta_{k+1} &= A_k \zeta_k + B_{1k} u_k + B_{2k} \tilde{\omega}_k \\ \tilde{z}_k &= C \zeta_k, \end{cases} \tag{3.10}$$

where

$$A_k = \begin{bmatrix} \mathbf{a}_k & 0 \\ \mathbf{c}_f \mathbf{c}_k & \mathbf{a}_f \end{bmatrix}, \quad B_{1k} = \begin{bmatrix} \mathbf{b}_{1k} \\ \mathbf{b}_f + \mathbf{c}_f \mathbf{d}_{1k} \end{bmatrix}, \quad B_{2k} = \begin{bmatrix} \mathbf{b}_{2k} \\ \mathbf{c}_f \mathbf{d}_{2k} \end{bmatrix}, \quad C = \begin{bmatrix} L & -L_f \end{bmatrix}, \tag{3.11}$$

and

$$\zeta_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}. \tag{3.12}$$

In the next section, to design the optimal $\mathcal{H}_2$ filter, first the relations for the $\mathcal{H}_2$-norm of systems with stochastic parameters are found.

3.3 $\mathcal{H}_2$-norm of Systems with Stochastic Parameters

As was shown in the previous section, the formulation of state estimation in the NCSs with random packet dropout leads to the state space representation of a system with stochastic parameters. The problem of state filtering for systems with deterministic parameters has been studied before (see, e.g., [7, 40] and references therein). Also, the problem was studied in the previous chapter for the case where only one stochastic parameter is present. In this section, to extend the problem to stochastic parameter systems with several parameters, the $\mathcal{H}_2$-norm of a system with stochastic parameters and both stochastic and deterministic inputs is defined. The stochastic $\mathcal{H}_2$-norm ($\mathcal{H}_{2s}$-norm) of the filtering error dynamics can be used as a performance index for the filter design. The LMI formulation of the performance index and corresponding constraints are presented in the next section.

For a deterministic stable discrete-time linear time-invariant (LTI) system, we have the following two facts:

Fact 1: *If the input is standard (unit variance) white noise, then the root-mean-square value of the output equals the $\mathcal{H}_2$-norm of the system [6].*

Fact 2: *An immediate consequence of Parseval's equality is that if the input is the unit impulse, then the 2-norm of the output equals the $\mathcal{H}_2$-norm of the system [6].*

As the NCS under consideration is reformulated as a time-varying stochastic system, the classical norm definition needs to be modified to be applicable in this case. To study an even more general case, consider a general stable time-varying stochastic system G with both deterministic input, u_k, and unit variance white noise input, ω_k:

$$G: \begin{cases} \zeta_{k+1} & = A_k\zeta_k + B_{1k}u_k + B_{2k}\omega_k \\ \tilde{z}_k & = C_k\zeta_k + D_{1k}u_k + D_{2k}\omega_k, \end{cases} \tag{3.13}$$

where A_k, B_{1k}, B_{2k}, C_k, D_{1k} and D_{2k} are stochastic time dependent matrices. Note that (3.13) is a generalization of the equations defined in (3.10).

To handle the problem of both the deterministic and stochastic inputs, the linearity property of the system is used to write

$$\tilde{z}_k = \tilde{z}_{1k} + \tilde{z}_{2k} = G_1u_k + G_2\omega_k, \tag{3.14}$$

where

$$G1: \begin{cases} \zeta_{1,k+1} & = A_k\zeta_{1,k} + B_{1k}u_k \\ \tilde{z}_{1k} & = C_k\zeta_{1,k} + D_{1k}u_k \end{cases} \tag{3.15}$$

and

$$G2: \begin{cases} \zeta_{2,k+1} & = A_k\zeta_{2,k} + B_{2k}\omega_k \\ \tilde{z}_{2k} & = C_k\zeta_{2,k} + D_{2k}\omega_k. \end{cases} \tag{3.16}$$

Following the general definition of the $\mathcal{H}_2$-norm of a time-invariant system, the $\mathcal{H}_2$-norm of the stable stochastic time varying system G_1 is defined as

$$\|G_1\|_{2s}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{E}\{\tilde{z}_{1k}^2\}, \tag{3.17}$$

where the input u_k is a unit impulse. Similarly, we can define

$$\|G_2\|_{2s}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{E}\{\tilde{z}_{2k}^2\}, \tag{3.18}$$

where the input ω_k is standard (unit variance) white noise.

Remark. *The expressions in (3.17) and (3.18) can be shown to satisfy the norm properties. If the system is time-invariant, the expressions (3.17) and (3.18) reduce to the standard $\mathcal{H}_2$-norm. Hence, they are regarded as a generalization of the norm to stochastic parameter systems with deterministic or stochastic inputs.*

In the previous chapter, a derivation was given for the $\mathcal{H}_2$-norm of a system containing only one stochastic parameter. Even though the same method can be generalized to the cases with more than one stochastic parameters, the formulation will be more complex when the number of parameters increases. In the following section, relations are derived in a closed form for the $\mathcal{H}_2$-norm of G_2 that can be easily used for the cases with more than two parameters. The derivations for G_1 will be very similar and will be discussed later. A discussion of the $\mathcal{H}_2$-norm of G follows.

By using the G_2 subsystem representations in (3.16),

$$\mathcal{E}\{\tilde{z}_{2k}^2\} = \mathcal{E}\{(C_k\zeta_{2k} + D_{2k}\omega_k)(\zeta_{2k}'C_k' + \omega_k'D_{2k}')\}. \tag{3.19}$$

As ω_k is unit variance white noise,

$$\mathcal{E}\{\tilde{z}_{2k}^2\} = \mathcal{E}\{C_kL_{2k}C_k' + D_{2k}D_{2k}'\}, \tag{3.20}$$

where

$$L_{2,k+1} = \mathcal{E}\{\zeta_{2,k+1}\zeta_{2,k+1}'\}. \tag{3.21}$$

Thus,

$$L_{2,k+1} = \mathcal{E}\{(A_k\zeta_{2k} + B_{2k}\omega_k)(A_k\zeta_{2k} + B_{2k}\omega_k)'\} = \mathcal{E}\{A_kL_{2k}A_k' + B_{2k}B_{2k}'\}. \quad (3.22)$$

Matrices A_k, B_{2k}, C_k and D_{2k} are dependent on stochastic parameters δ_{ik}'s. The δ_{ik}'s are Bernoulli distributed white sequences with a known mean value of α_i and variance of q_i^2. Therefore, the δ_{ik}'s can be written as the sum of their mean value and the zero mean stochastic variables λ_{ik}'s with the same variance:

$$\delta_{ik} = \alpha_i + \lambda_{ik}, \quad (3.23)$$

where

$$\mathcal{E}\{\lambda_{ik}\} = 0, \ \text{var}\{\lambda_{ik}\} = q_i^2, \ \mathcal{E}\{\lambda_{1k}\lambda_{2s}\} = 0, \ \forall\ k \neq s. \quad (3.24)$$

Now, from (3.11) and (3.8), we can write

$$A_k = A + \lambda_{1k}\tilde{A}_1 + \lambda_{2k}\tilde{A}_2 + \lambda_{1k}\lambda_{2k}\tilde{A}_{12}, \quad (3.25)$$

where A, $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_{12}$ are known constant matrices. Define

$$A_{q1} = q_1\tilde{A}_1, \quad A_{q2} = q_2\tilde{A}_2, \quad A_{q12} = q_1q_2\tilde{A}_{12}, \quad (3.26)$$

then

$$\mathcal{E}\{A_kL_{2k}A_k'\} = AL_{2k}A' + A_{q1}L_{2k}A_{q1}' + A_{q2}L_{2k}A_{q2}' + A_{q12}L_{2k}A_{q12}'. \quad (3.27)$$

Similar relations can be found for $\mathcal{E}\{B_{2k}B_{2k}'\}$, $\mathcal{E}\{C_kL_{2k}C_k'\}$ and $\mathcal{E}\{D_{2k}D_{2k}'\}$ as follows:

$$\begin{aligned}
\mathcal{E}\{B_{2k}B_{2k}'\} &= B_{2k}B_{2k}' + B_{2,q1}B_{2,q1}' + B_{2,q2}B_{2,q2}' + B_{2,q12}B_{2,q12}' \\
\mathcal{E}\{C_kL_{2k}C_k'\} &= CL_{2k}C' + C_{q1}L_{2k}C_{q1}' + C_{q2}L_{2k}C_{q2}' + C_{q12}L_{2k}C_{q12}' \quad (3.28) \\
\mathcal{E}\{D_{2k}D_{2k}'\} &= D_{2k}D_{2k}' + D_{2,q1}D_{2,q1}' + D_{2,q2}D_{2,q2}' + D_{2,q12}D_{2,q12}'.
\end{aligned}$$

Let us define:

$$\begin{aligned}
\bar{B}_2\bar{B}_2' &= B_2B_2' + B_{2,q1}B_{2,q1}' + B_{2,q2}B_{2,q2}' + B_{2,q12}B_{2,q12}' \\
\bar{D}_2\bar{D}_2' &= D_2D_2' + D_{2,q1}D_{2,q1}' + D_{2,q2}D_{2,q2}' + D_{2,q12}D_{2,q12}'. \quad (3.29)
\end{aligned}$$

Putting all these relations into equation (3.18), we have the following theorem:

Theorem 3. **($\mathcal{H}_{2s}$-norm)**- *Consider G_2, the stable discrete-time linear stochastic parameter system represented in (3.16). The $\mathcal{H}_{2s}$-norm of the system defined by (3.18) is*

$$
\begin{aligned}
\|G_2\|_{2s}^2 &= \mathcal{E}\{C_k L_c C_k' + D_{2k} D_{2k}'\} \\
&= C L_c C' + C_{q1} L_c C_{q1}' + C_{q2} L_c C_{q2}' + C_{q12} L_c C_{q12}' + \bar{D}_2 \bar{D}_2'
\end{aligned}
\tag{3.30}
$$

with

$$
\begin{aligned}
L_c &= \mathcal{E}\{A_k L_c A_k' + B_k B_k'\} \\
&= \bar{B}_2 \bar{B}_2' + A L_c A' + A_{q1} L_c A_{q1}' + A_{q2} L_c A_{q2}' + A_{q11} L_c A_{q12}'.
\end{aligned}
\tag{3.31}
$$

The generalization of the results to the MIMO case is provided in the following Corollary.

Corollary 3. **(MIMO $\mathcal{H}_{2s}$-norm)**: *Consider G_2, the stable discrete-time linear stochastic parameter system represented in (3.16) in the MIMO case. The $\mathcal{H}_{2s}$-norm of the system is given by*

$$
\begin{aligned}
\|G_2\|_{2s}^2 &= \mathcal{E}\{trace(C_k L_c C_k' + D_{2k} D_{2k}')\} \\
&= trace\{C L_c C' + C_{q1} L_c C_{q1}' + C_{q2} L_c C_{q2}' + C_{q12} L_c C_{q12}' + \bar{D}_2 \bar{D}_2'\},
\end{aligned}
\tag{3.32}
$$

where

$$
\begin{aligned}
L_c &= \mathcal{E}\{A_k L_c A_k' + B_k B_k'\} \\
&= \bar{B}_2 \bar{B}_2' + A L_c A' + A_{q1} L_c A_{q1}' + A_{q2} L_c A_{q2}' + A_{q11} L_c A_{q12}'.
\end{aligned}
\tag{3.33}
$$

This corollary is a generalization of the SISO case similar to the classical one as in [6] and the proof is quite straightforward.

So far, we have found the $\mathcal{H}_{2s}$-norm of system G_2 with stochastic input ω_k as in (3.16). Following the same method, similar relations are obtained for system G_1 in (3.15) with a deterministic input. The results are given in the following corollary.

Corollary 4. *Consider G_1, the stable discrete-time linear stochastic parameter system represented in (3.15). The $\mathcal{H}_{2s}$-norm of the system defined by (3.17) is*

$$\begin{aligned} \|G_1\|_{2s}^2 &= \mathcal{E}\{trace(C_k L_c C_k' + D_{2k} D_{2k}')\} \\ &= trace\{C L_c C' + C_{q1} L_c C_{q1}' + C_{q2} L_c C_{q2}' + C_{q12} L_c C_{q12}' + \bar{D}_1 \bar{D}_1'\}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} L_c &= \mathcal{E}\{A_k L_c A_k' + B_k B_k'\} \\ &= \bar{B}_1 \bar{B}_1' + A L_c A' + A_{q1} L_c A_{q1}' + + A_{q2} L_c A_{q2}' + A_{q11} L_c A_{q12}' \end{aligned} \tag{3.35}$$

with

$$\begin{aligned} \bar{B}_1 \bar{B}_1' &= B_1 B_1' + B_{1,q1} B_{1,q1}' + B_{1,q2} B_{1,q2}' + B_{1,q12} B_{1,q12}' \\ \bar{D}_1 \bar{D}_1' &= D_1 D_1' + D_{1,q1} D_{1,q1}' + D_{1,q2} D_{1,q2}' + D_{1,q12} D_{1,q12}'. \end{aligned} \tag{3.36}$$

Now, to combine the stochastic and deterministic inputs, the weighted $\mathcal{H}_2$-norm of G is defined as follows:

$$\|G\|_{2s}^2 = \|G_1\|_{2s}^2 + \rho \|G_2\|_{2s}^2, \tag{3.37}$$

where $\rho \in \mathbb{R}$ is a weighting factor.

Remark. *The expression in (3.37) can be shown to satisfy the norm properties. It can be regarded as a generalization of the norm of systems with both deterministic and stochastic inputs.*

The following theorem gives the relations for the stochastic weighted $\mathcal{H}_2$-norm of the system G. The proof is straightforward and is omitted.

Theorem 4. *Consider G, the stable discrete-time linear stochastic parameter system represented in (3.13). The $\mathcal{H}_{2s}$-norm of the system defined by (3.37) is*

$$\|G\|_{2s}^2 = trace\{CL_cC' + C_{q1}L_cC_{q1}' + C_{q2}L_cC_{q2}' + C_{q12}L_cC_{q12}' + \bar{D}_1\bar{D}_1' + \bar{\bar{D}}_2\bar{\bar{D}}_2'\}, \quad (3.38)$$

where

$$L_c = \bar{B}_1\bar{B}_1' + \bar{\bar{B}}_2\bar{\bar{B}}_2' + AL_cA' + A_{q1}L_cA_{q1}' + A_{q2}L_cA_{q2}' + A_{q12}L_cA_{q12}' \quad (3.39)$$

with $\bar{B}_1\bar{B}_1'$, $\bar{D}_1\bar{D}_1'$, $\bar{B}_2\bar{B}_2'$ and $\bar{D}_2\bar{D}_2'$ as defined in (3.36) and (3.29) and

$$\bar{\bar{B}}_2 = \rho\bar{B}_2, \quad \bar{\bar{D}}_2 = \rho\bar{D}_2. \quad (3.40)$$

3.4 Optimal Filter Design

Now, we have the required tools to solve the optimal $\mathcal{H}_2$ filtering problem in the NCS framework with multiple packet dropout. We want to design a filter F as in (3.9) such that the estimation error variance is minimized. Based on the $\mathcal{H}_{2s}$-norm definition, it is needed to minimize the $\mathcal{H}_{2s}$-norm of the filtering error dynamics to solve the filtering problem.

The problem of $\mathcal{H}_2$ filtering for deterministic discrete-time systems has been studied in the literature (see, e.g., [14,40] and references therein). In the following, we try to adapt the deterministic filter design problem to the stochastic cases by using the tools developed in the previous section.

Consider H, the filtering error dynamics defined in (3.10). By using Theorem 4

$$
\begin{aligned}
\|H\|_{2s}^2 &= \text{trace}\{CL_cC'\} && (3.41)\\
L_c &= \bar{B}_1\bar{B}_1' + \bar{\bar{B}}_2\bar{\bar{B}}_2' + \\
&\quad +AL_cA' + A_{q1}L_cA_{q1}' + A_{q2}L_cA_{q2}' + A_{q12}L_cA_{q12}'. && (3.42)
\end{aligned}
$$

As a generalization of the $\mathcal{H}_2$ filtering in the deterministic ([14,40]) and dropout ([45]) cases, the $\mathcal{H}_{2s}$ filtering in the NCS can be formulated as follows:

$$
\min_{\mathbf{a}_f,\mathbf{b}_f,\mathbf{c}_f,L_f,P} \quad \text{trace}(J) \tag{3.43}
$$

s.t.

$$
\begin{bmatrix} P & PC' \\ CP & J \end{bmatrix} > 0 \tag{3.44}
$$

$$
\begin{bmatrix} P & \Xi_A & \Xi_{B_1} & \Xi_{B_2} \\ * & \Xi_P & 0 & 0 \\ * & * & \Xi_I & 0 \\ * & * & * & \Xi_I \end{bmatrix} > 0 \tag{3.45}
$$

where the matrix variables J and P and the matrix inequalities are symmetric, ρ is known, and

$$
\begin{aligned}
\Xi_A &= \begin{bmatrix} AP & A_{q1}P & A_{q2}P & A_{q12}P \end{bmatrix} \\
\Xi_{B_1} &= \begin{bmatrix} B_1 & B_{1,q1} & B_{1,q2} & B_{1,q12} \end{bmatrix} \\
\Xi_{B_2} &= \rho \begin{bmatrix} B_2 & B_{2,q1} & B_{2,q2} & B_{2,q12} \end{bmatrix} \qquad (3.46)\\
\Xi_P &= \text{diag}(P,P,P,P), \quad \Xi_I = \text{diag}(I,I,I,I).
\end{aligned}
$$

Now, it is desirable to convert the two matrix inequalities in (3.44) and (3.45) into LMIs. Then, the filter design problem turns out into a convex programming problem that can be solved efficiently by the numerical methods available.

Let us partition P and its inverse as

$$P = \begin{bmatrix} X & U \\ U' & X_2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & V \\ V' & Y_2 \end{bmatrix}, \tag{3.47}$$

where X, Y, X_2 and Y_2 are symmetric and positive definite matrices. Now, we define the following nonsingular matrices:

$$\bar{T} = \begin{bmatrix} Z & Y \\ 0 & V' \end{bmatrix}, \quad T_1 = \begin{bmatrix} \bar{T} & 0 \\ 0 & I \end{bmatrix}, \quad T_2 = \text{diag}(\bar{T}, \bar{T}, \bar{T}, \bar{T}, \bar{T}, I, I, I, I, I, I, I, I), \tag{3.48}$$

where $Z = X^{-1}$, and I is the identity matrix with appropriate dimension. By applying the congruence transformation with T_1 to (3.44), we get the following LMI:

$$T_1' \begin{bmatrix} P & PC' \\ * & J \end{bmatrix} T_1 = \begin{bmatrix} Z & Z & L' - G' \\ * & Y & L' \\ * & * & J \end{bmatrix} > 0, \tag{3.49}$$

where $G = L_f U' Z$. We can also get an LMI by applying the congruence transformation with T_2 to (3.45):

$$T_2' \, [3.45] \, T_2 > 0, \tag{3.50}$$

where it is easy to see that

$$\begin{aligned}
\bar{T}'P\bar{T} &= \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}, \\
\bar{T}'AP\bar{T} &= \begin{bmatrix} Z\mathbf{a} & Z\mathbf{a} \\ Y\mathbf{a} + F\mathbf{c} + Q & Y\mathbf{a} + F\mathbf{c} \end{bmatrix} \\
\bar{T}'A_{q*}P\bar{T} &= \begin{bmatrix} Z\mathbf{a}_{q*} & Z\mathbf{a}_{q*} \\ Y\mathbf{a}_{q*} + F\mathbf{c}_{q*} & Y\mathbf{a}_{q*} + F\mathbf{c}_{q*} \end{bmatrix} \\
\bar{T}'B_1 &= \begin{bmatrix} Z\mathbf{b}_1 \\ Y\mathbf{b}_1 + M + F\mathbf{d}_1 \end{bmatrix}, \\
\bar{T}'B_2 &= \begin{bmatrix} Z\mathbf{b}_2 \\ Y\mathbf{b}_2 + F\mathbf{d}_2 \end{bmatrix} \\
\bar{T}'B_{1,q*} &= \begin{bmatrix} Z\mathbf{b}_{1q*} \\ Y\mathbf{b}_{1q*} + F\mathbf{d}_{1q*} \end{bmatrix}, \\
\bar{T}'B_{2,q*} &= \begin{bmatrix} Z\mathbf{b}_{2q*} \\ Y\mathbf{b}_{2q*} + F\mathbf{d}_{2q*} \end{bmatrix},
\end{aligned} \tag{3.51}$$

where $F = V\mathbf{c}_f$, $Q = V\mathbf{a}_f U' Z$ and $M = V\mathbf{b}_f$. Also, $q*$ stands for q_1, q_2 and q_{12}. Thus, the results can be summarized as follows.

Theorem 5. **($\mathcal{H}_{2s}$-filtering)** *The filter design problem of (3.43)-(3.45) is equivalent to the following convex programming problem:*

$$\begin{aligned} &\min_{Z,Y,Q,F,G,M} \quad trace(J) \\ &s.t. \\ &\quad (3.49) \text{ and } (3.50). \end{aligned} \tag{3.52}$$

To find the filter parameters, $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f, we need to know U and V, which do not appear in the LMIs. One of the matrices U or V can be defined freely. Different choices give us different filter state-space realizations. One logical choice is to set $L_f = L$ that can come from setting $V = V' = -Y$, leading to $U = U' = Z^{-1} - Y^{-1}$.

Remark. *Even though the problem of multiple packet dropout in the NCS framework have been studied, the general tools used are capable of solving the other related problems as well. A delay problem is a direct reformulation of the augmentation procedure and can be solved similarly. More discussion on this topic will appear in Chapter 7.*

3.5 Example

A simulation example is given in this section to support the developed theory. Consider a discrete-time LTI system represented by (3.7) with the following matrix values:

$$\mathbf{a} = \begin{bmatrix} 1.7240 & -0.7788 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$
$$\mathbf{c} = \begin{bmatrix} 0.0286 & 0.0264 \end{bmatrix}, \quad \mathbf{d}_1 = 1, \ \mathbf{d}_2 = 1, \ L = I_2, \tag{3.53}$$

where I_2 is an identity matrix with the size of 2. The initial state values are $\tilde{x}(0) = [0 \ \ 0]'$ and $\hat{\tilde{x}}(0) = [2 \ \ -2]'$. The system states and their estimates due to sinusoidal input are plotted in the following figures. Note that the controller is not designed here. It is assumed that it simply sends some sinusoidal commands. Figure 3.2 shows the simulation results for the case when the average sensor to the controller and the controller to the actuator dropout rate are 0.2 and 0.8, respectively, with a weighting factor of $\rho = 2$. This result shows the superiority of the proposed $\mathcal{H}_{2s}$ filtering over the classical one. Figure 3.3 shows the variance of the estimation error of x_1 when α_1 and α_2 change from 0.1 to 1 with the step size of 0.1. The weighting factor ρ, is set to 1. When less dropout occurs, a smaller estimation error variance is achieved. In the case of no dropout ($\alpha_1 = \alpha_2 = 1$), as expected, the least estimation error equal to that in the deterministic case is achieved. This example demonstrates the effectiveness of the proposed method.

3.6 Conclusions

In this chapter, the problem of optimal $\mathcal{H}_2$ filtering in the NCS environment with multiple packet dropouts has been studied. The stochastic $\mathcal{H}_2$-norm of systems containing stochastic parameters was defined, and the relations were developed. A weighted $\mathcal{H}_2$-norm was defined to be used in systems with both deterministic and stochastic inputs. Based on the new derivations, the problem was transformed into a set of LMIs that can be easily solved by existing software packages. A simulation example showed the effectiveness and applicability of the proposed method.

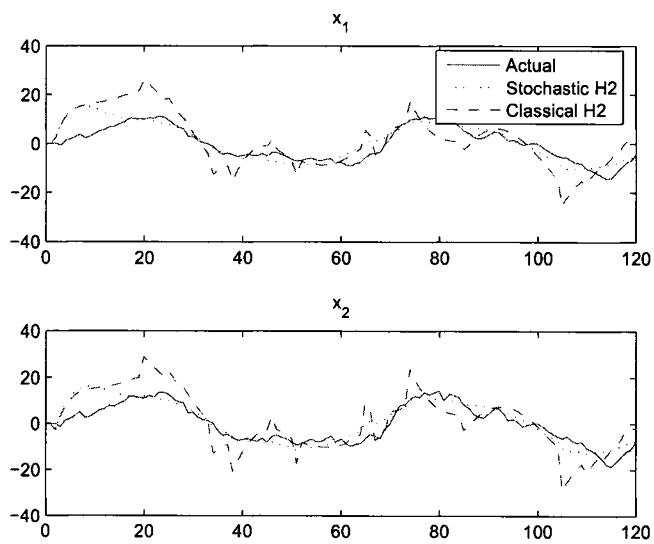

Figure 3.2: Actual and estimated states for the classical and stochastic $\mathcal{H}_2$ filtering, $\alpha_1 = 0.2$, $\alpha_2 = 0.8$, $\rho = 2$

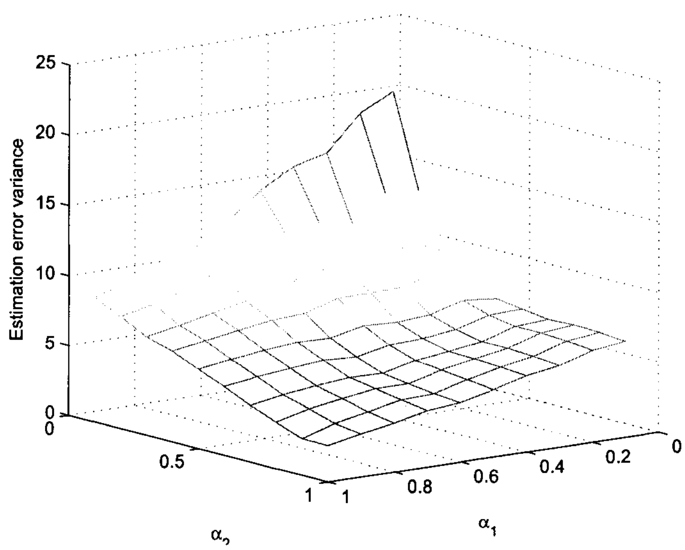

Figure 3.3: Estimation error variance vs. α_1 and α_2 for x_1, stochastic $\mathcal{H}_2$ with $\rho = 1$

Chapter 4

Optimal $\mathcal{H}_\infty$ Filtering in NCS with Multiple Packet Dropout

4.1 Introduction

As discussed before, state feedback is the most common strategy used in modern control systems. However in practice, not all of the state variables are always available for direct measurement, therefore state filtering and estimation play a key role in state feedback methods. The filtering problem is to estimate the states or a linear combination of them by using the measured system inputs and outputs.

The importance of networked control systems was also discussed in Chapter 3. Networked control systems (NCSs) have been the focus of several research studies over the last few years (see, e.g., [18, 37, 51] and references therein). Compared to using the conventional point-to-point system connection, using a NCS has advantages in installation, wiring, and maintenance cost and time. In a NCS, data travel through the communication channels from the sensors to the controller and from the controller to the actuators. Data packet dropout can occur due to node failures or network congestion and is a common problem in networked systems. In

The material of this chapter was contained in [47].

real-time feedback control systems, it is normally advantageous to discard the old packets and consider the new ones so that the controller always receives fresh data for control calculation. Packet dropouts usually occur randomly. Because of random packet dropout, classical estimation and control methods cannot be used directly in NCS systems. Dropouts degrade system performance and make the filtering and estimation more difficult and challenging.

The random packet dropout has been the focus of some research studies in the last few years. The problem of filtering in multiple packet dropout systems has been studied in Chapter 3 and in [45, 46] in the $\mathcal{H}_2$ framework. Also, the problem of stabilization and control has been studied recently in this type of system (see, e.g., [27,28,60,61] and references therein). In some of these studies, only sensor data dropouts are studied [27, 60].

The problem of $\mathcal{H}_\infty$ filtering and control of the deterministic parameter systems has been fully studied (see, e.g., [8, 12–14, 40] and references therein). This problem has also been studied in the stochastic cases [11, 54, 55]. In all of these references, the stochastic $\mathcal{H}_\infty$ problem is studied when only stochastic inputs are present. The problem of stochastic packet dropout has also been studied in the context of sensor delay and NCS [45,46] in the $\mathcal{H}_2$ setting. But, to the best of our knowledge, optimal $\mathcal{H}_\infty$ filtering has not been studied in NCSs with multiple packet dropout.

In this chapter, the problem of optimal $\mathcal{H}_\infty$ filtering in a NCS with multiple packet dropout is considered. By using the proposed formulation, we can formulate the NCS with multiple random packet dropouts both from the sensors to the controller and from the controller to the actuators. The $\mathcal{H}_\infty$-norm definition is generalized to derive new relations for the stochastic $\mathcal{H}_\infty$-norm of a linear discrete-time stochastic parameter system represented in the state-space form with both deterministic and stochastic inputs. The new derivations give us a general framework so that the same tool can be used to study some other problems like random sensor delay or uncertain observation. To solve the filtering problem, the filter gains are designed so that the $\mathcal{H}_\infty$-norm of the estimation error dynamics is minimized. As dropout rates are random, the problem formulation leads to a system with stochastic parameters. Thus, the stochastic $\mathcal{H}_\infty$-norm of the estimation error dynamics is considered as a measure to minimize. The filtering problem is transformed into a convex optimization problem through a set of LMIs that can be solved by using existing numerical techniques [4].

4.2 Problem Formulation

Figure 4.1 shows the schematic of the NCS under study in which the controller is already designed. The plant is a discrete-time linear time-invariant (LTI) system subject to random disturbances. Also, the sensor data are contaminated with noise. The plant can be represented by the following equations:

$$\begin{cases} \tilde{x}_{k+1} &= \mathbf{a}\tilde{x}_k + \mathbf{b}_1\tilde{u}_k + \mathbf{b}_2\tilde{\omega}_k \\ \tilde{y}_k &= \mathbf{c}\tilde{x}_k + \mathbf{d}_1\tilde{u}_k + \mathbf{d}_2\tilde{\omega}_k, \end{cases} \tag{4.1}$$

where $\tilde{x}_k \in \mathbb{R}^n$ is the plant state vector, and $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{c}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ are system parameter matrices with appropriate dimensions. The exogenous input vector, $\tilde{\omega}$, is the zero mean stochastic disturbance input belonging to the space of mean square summable vectors. $\tilde{y}_k$ is the system output contaminated with $\tilde{\omega}_k$. Also, $\tilde{u}_k$ is the system command input. z is the signal to be estimated, and $\hat{z}$ is its estimate.

Consider the system described by (4.1). The system output, $\tilde{y}$, is passed through the network and there may be random dropouts, only the probability of the dropouts, α_1, is known. Thus, the current observation, y_k, is the noise corrupted current sys-

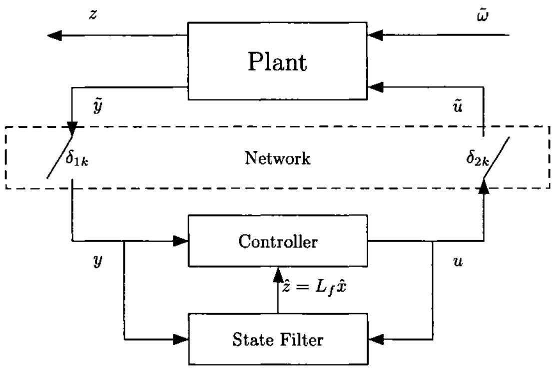

Figure 4.1: NCS schematic with packet dropout

tem output, $\tilde{y}_k$, with the probability of α_1. In the case of no new data, previous data will be used, so the previous data, y_{k-1}, will be used with the probability of $(1-\alpha_1)$. The filter has knowledge of the current control command, but the plant input, $\tilde{u}_k$, is the current controller output, u_k, with the probability of α_2 or the previous one, $\tilde{u}_{k-1}$, with the probability of $(1-\alpha_2)$. These expressions can be represented by the following relations:

$$\begin{cases} y_k &= \delta_{1k}\tilde{y}_k + (1-\delta_{1k})y_{k-1} \\ \tilde{u}_k &= \delta_{2k}u_k + (1-\delta_{2k})\tilde{u}_{k-1}, \end{cases} \tag{4.2}$$

where the stochastic parameters δ_{ik}'s are Bernoulli distributed white sequences taking the values of 0 or 1 with

$$\text{prob}\{\delta_{ik}=1\} = \mathcal{E}\{\delta_{ik}\} = \alpha_i, \quad 0 \le \alpha_i \le 1, \quad i = 1,2, \tag{4.3}$$

where α_i's are known constants. We also suppose that δ_{ik}'s are uncorrelated with each other, $\tilde{\omega}_k$, and the initial state values, so

$$\begin{aligned} &\text{prob}\{\delta_{ik}=0\} = 1-\alpha_i, \\ &\text{var}\{\delta_{ik}\} = \alpha_i(1-\alpha_i). \end{aligned} \tag{4.4}$$

Now, we put equations (4.1) and (4.2) together to have the NCS formulation with multiple packet dropout as follows:

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}_1\tilde{u}_k + \mathbf{b}_2\tilde{\omega}_k \\ \tilde{y} & = \mathbf{c}\tilde{x}_k + \mathbf{d}_1\tilde{u}_k + \mathbf{d}_2\tilde{\omega}_k \\ y_k & = \delta_{1k}\tilde{y}_k + (1-\delta_{1k})y_{k-1} \\ \tilde{u}_k & = \delta_{2k}u_k + (1-\delta_{2k})\tilde{u}_{k-1}. \end{cases} \tag{4.5}$$

In order to get a compact representation, the system states, measurement and the system input can be augmented:

$$x_{k+1} = \begin{bmatrix} \tilde{x}_{k+1} \\ y_k \\ \tilde{u}_k \end{bmatrix}. \tag{4.6}$$

Thus,

$$\begin{cases} x_{k+1} & = \mathbf{a}_k x_k + \mathbf{b}_{1k}u_k + \mathbf{b}_{2k}\tilde{\omega}_k \\ y_k & = \mathbf{c}_k x_k + \mathbf{d}_{1k}u_k + \mathbf{d}_{2k}\tilde{\omega}_k \\ z_k & = Lx_k, \end{cases} \tag{4.7}$$

where z_k is the signal to be estimated and

$$\mathbf{a}_k = \begin{bmatrix} \mathbf{a} & 0 & (1-\delta_{2k})\mathbf{b}_1 \\ \delta_{1k}\mathbf{c} & 1-\delta_{1k} & \delta_{1k}(1-\delta_{2k})\mathbf{d}_1 \\ 0 & 0 & 1-\delta_{2k} \end{bmatrix}, \quad \mathbf{b}_{1k} = \begin{bmatrix} \delta_{2k}\mathbf{b}_1 \\ \delta_{1k}\delta_{2k}\mathbf{d}_1 \\ \delta_{2k} \end{bmatrix}, \quad \mathbf{b}_{2k} = \begin{bmatrix} \mathbf{b}_2 \\ \delta_{1k}\mathbf{d}_2 \\ 0 \end{bmatrix}$$
$$\mathbf{c}_k = \begin{bmatrix} \delta_{1k}\mathbf{c} & 1-\delta_{1k} & \delta_{1k}(1-\delta_{2k})\mathbf{d}_1 \end{bmatrix}, \quad \mathbf{d}_{1k} = \delta_{1k}\delta_{2k}\mathbf{d}_1, \qquad \mathbf{d}_{2k} = \delta_{1k}\mathbf{d}_2. \tag{4.8}$$

Note that $\mathbf{a}_k$, $\mathbf{b}_{1k}$, $\mathbf{b}_{2k}$, $\mathbf{c}_k$, $\mathbf{d}_{1k}$ and $\mathbf{d}_{2k}$ are functions of δ_{ik}'s, but for simplicity, $\mathbf{a}_k$, $\mathbf{b}_{1k}$, $\mathbf{b}_{2k}$, $\mathbf{c}_k$, $\mathbf{d}_{1k}$ and $\mathbf{d}_{2k}$ are used instead.

Considering the linear stochastic discrete-time system in (4.7), we want to find the estimate $\hat{z}_k$ of z_k such that the $\mathcal{H}_\infty$-norm of the filtering error dynamics is minimized. Now, consider the following filter:

$$F: \begin{cases} \hat{x}_{k+1} & = \mathbf{a}_f\hat{x}_k + \mathbf{b}_f u_k + \mathbf{c}_f y_k \\ \hat{z}_k & = L_f\hat{x}_k, \end{cases} \tag{4.9}$$

where $\hat{x}_k$ is an estimate of the state, and $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f are the filter parameters to be designed. The filtering error is defined as $\tilde{z}_k = z_k - \hat{z}_k$. Now, the system states, x_k, and the filter states, $\hat{x}_k$, can be augmented to get the following augmented system:

$$H: \begin{cases} \zeta_{k+1} & = A_k\zeta_k + B_{1k}u_k + B_{2k}\tilde{\omega}_k \\ \tilde{z}_k & = C\zeta_k, \end{cases} \tag{4.10}$$

where

$$A_k = \begin{bmatrix} \mathbf{a}_k & 0 \\ \mathbf{c}_f \mathbf{c}_k & \mathbf{a}_f \end{bmatrix}, \quad B_{1k} = \begin{bmatrix} \mathbf{b}_{1k} \\ \mathbf{b}_f + \mathbf{c}_f \mathbf{d}_{1k} \end{bmatrix}, \quad B_{2k} = \begin{bmatrix} \mathbf{b}_{2k} \\ \mathbf{c}_f \mathbf{d}_{2k} \end{bmatrix}, \quad C = \begin{bmatrix} L & -L_f \end{bmatrix}. \tag{4.11}$$

Note that system H has two types of inputs, deterministic, u_k, and stochastic, $\tilde{\omega}_k$.

4.3 $\mathcal{H}_\infty$-Norm for Deterministic Parameter Systems

In this section, a brief introduction to the $\mathcal{H}_\infty$-norm for systems with deterministic parameters will be given. For the signal $v = \{v(0), v(1), \cdots\}$, the ∞-norm is defined as follows [6]:

$$\|v\|_\infty = \sup_k |v(k)|. \tag{4.12}$$

In other words, the ∞-norm of a signal is the least upper bound on the amplitude. The $\mathcal{H}_\infty$-norm of a system is defined as follows:

$$\|G\|_\infty = \max_\theta |G(e^{j\theta})|, \tag{4.13}$$

where $G(z)$ is the system transfer function. The fact is that the best bound on the 2-norm of the system output over all inputs of unit 2-norm equals the ∞-norm of G [6]. If we name the system inputs as $\tilde{u}$ and the system outputs as $\tilde{y}$, then

$$\|G\|_\infty = \sup\{\|\tilde{y}\|_2 : \|\tilde{u}\|_2 = 1\}. \tag{4.14}$$

4.4 $\mathcal{H}_\infty$-Norm for Stochastic Parameter Systems

As was shown in the problem formulation, the formulation of filtering in the NCS framework leads to state-space representation of a system with stochastic parameters with both deterministic and stochastic inputs. In this section, the definition of the $\mathcal{H}_\infty$-norm is extended to a system with stochastic parameters. The stochastic $\mathcal{H}_\infty$-norm of the filtering error dynamics is used as a performance index, and corresponding LMI formulations are given in the next section.

In order to study an even more general case, consider a general stable time-varying stochastic parameter system G as follows:

$$G: \left\{ \begin{array}{ll} \zeta_{k+1} & = A_k\zeta_k + B_{1k}u_k + B_{2k}\omega_k \\ \tilde{z}_k & = C_k\zeta_k + D_{1k}u_k + D_{2k}\omega_k, \end{array} \right. \tag{4.15}$$

where A_k, B_{1k}, B_{2k}, C_k, D_{1k} and D_{2k} are stochastic time dependent matrices. Note that (4.15) is a generalization of the equations defined in (4.10).

In the deterministic parameter case, the $\mathcal{H}_\infty$-norm of a linear discrete time-invariant system is the maximum bound on the 2-norm of the output over all inputs of unit 2-norm [6]. Now, to combine the stochastic and deterministic inputs, the weighted stochastic $\mathcal{H}_\infty$-norm of G (referred to as the stochastic $\mathcal{H}_\infty$-norm for simplicity) is defined as follows:

$$\|G\|_\infty^2 = \sup \frac{\sum_{k=0}^{\infty} \mathcal{E}\{\|\tilde{z}_k\|^2\}}{\sum_{k=0}^{\infty} \mathcal{E}\{\|\omega_k\|^2 + \rho\|u_k\|^2\}}, \tag{4.16}$$

where $\rho \in \mathbb{R}$ is a known weighting factor.

Before proceeding to the main theorem, we consider the following definitions and proposition adapted from [11,35]:

Definition 2. *The stochastic parameter system in (4.15) is exponentially stable in the mean-square sense or internally stable if there exist $\beta > 0$ and $0 < \tau < 1$ such that with zero inputs ($\omega_k = 0$ and $u_k = 0$),*

$$\mathcal{E}\{\|\zeta_k\|^2\} < \beta\tau^k\mathcal{E}\{\|\zeta_0\|^2\}, \quad \forall k > 0. \tag{4.17}$$

Proposition 1. *The stochastic parameter system in (4.15) is exponentially stable in the mean-square sense if there exists a positive definite matrix P such that*

$$\mathcal{E}\{A_k' P A_k\} - P < 0. \tag{4.18}$$

Proof. Suppose that (4.18) holds. Since $P > 0$, there exist $\kappa_1 > 0$ and $\kappa_2 > 0$ such that

$$\kappa_1 I \leq P \leq \kappa_2 I, \tag{4.19}$$

where I is a unitary matrix with appropriate dimension. Thus we can write,

$$\kappa_1 \mathcal{E}\{\|\zeta_k\|^2\} \leq \mathcal{E}\{\zeta_k' P \zeta_k\} \leq \kappa_2 \mathcal{E}\{\|\zeta_k\|^2\}. \tag{4.20}$$

Now, we can see that

$$\mathcal{E}\{\zeta_{k+1}' P \zeta_{k+1}\} = \mathcal{E}\{\zeta_k' A_k' P A_k \zeta_k\} = \mathcal{E}\{\zeta_k'(A_k' P A_k - P)\zeta_k\} + \mathcal{E}\{\zeta_k' P \zeta_k\}. \tag{4.21}$$

As $(A_k' P A_k - P) < 0$, there exists $\kappa_3,\ 0 < \kappa_3 < \kappa_2$, such that

$$\mathcal{E}\{\zeta_{k+1}' P \zeta_{k+1}\} \leq -\kappa_3 \mathcal{E}\{\|\zeta_k\|^2\} + \mathcal{E}\{\zeta_k' P \zeta_k\} \leq \mathcal{E}\{\zeta_k' P \zeta_k\} - \frac{\kappa_3}{\kappa_2}\mathcal{E}\{\zeta_k' P \zeta_k\}. \tag{4.22}$$

Thus,

$$\begin{aligned} \kappa_1 \mathcal{E}\{\|\zeta_k\|^2\} &\leq \mathcal{E}\{\zeta_k' P \zeta_k\} \leq (1 - \frac{\kappa_3}{\kappa_2})\mathcal{E}\{\zeta_{k-1}' P \zeta_{k-1}\} \\ &\leq (1 - \frac{\kappa_3}{\kappa_2})^k \mathcal{E}\{\zeta_0' P \zeta_0\} \leq \kappa_2 (1 - \frac{\kappa_3}{\kappa_2})^k \mathcal{E}\{\|\zeta_0\|^2\}, \end{aligned} \tag{4.23}$$

or

$$\mathcal{E}\{\|\zeta_k\|^2\} \leq \beta \tau^k \mathcal{E}\{\|\zeta_0\|^2\} \tag{4.24}$$

with $\beta = \frac{\kappa_2}{\kappa_1}$ and $\tau = 1 - \frac{\kappa_3}{\kappa_2}$. □

Definition 3. *The stochastic parameter system in (4.15) is input-output stable if there exists a constant* $\gamma > 0$ *such that*

$$\sum_{k=0}^{\infty} \mathcal{E}\{\|\tilde{z}_k\|^2\} \leq \gamma^2 \sum_{k=0}^{\infty} \mathcal{E}\{\|\omega_k\|^2 + \rho\|u_k\|^2\}. \tag{4.25}$$

Now, consider system G as in (4.15). Suppose that the system matrix parameters are functions of stochastic variables δ_{1k} and δ_{2k} as follows:

$$A_k = A + \lambda_{1k}\tilde{A}_1 + \lambda_{2k}\tilde{A}_2 + \lambda_{1k}\lambda_{2k}\tilde{A}_{12}, \tag{4.26}$$

where

$$\delta_{ik} = \alpha_i + \lambda_{ik}, \quad i = 1, 2 \tag{4.27}$$

and

$$\mathcal{E}\{\lambda_{ik}\} = 0, \quad \mathrm{var}\{\lambda_{ik}\} = q_i^2 \tag{4.28}$$

and A, $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_{12}$ are known constant matrices. Define

$$A_{q1} = q_1\tilde{A}_1, \quad A_{q2} = q_2\tilde{A}_2, \quad A_{q12} = q_1 q_2 \tilde{A}_{12}, \tag{4.29}$$

then,

$$\mathcal{E}\{A_k'A_k\} = A'A + A_{q1}'A_{q1} + A_{q2}'A_{q2} + A_{q12}'A_{q12}. \tag{4.30}$$

Using a similar approach,

$$\begin{aligned} \mathcal{E}\{B_{1k}'B_{1k}\} &= B_1'B_1 + B_{1,q1}'B_{1,q1} + B_{1,q2}'B_{1,q2} + B_{1,q12}'B_{1,q12} \\ \mathcal{E}\{B_{2k}'B_{2k}\} &= B_2'B_2 + B_{2,q1}'B_{2,q1} + B_{2,q2}'B_{2,q2} + B_{2,q12}'B_{2,q12} \\ \mathcal{E}\{C_k'C_k\} &= C'C + C_{q1}'C_{q1} + C_{q2}'C_{q2} + C_{q12}'C_{q12} \\ \mathcal{E}\{D_{1k}'D_{1k}\} &= D_1'D_1 + D_{1,q1}'D_{1,q1} + D_{1,q2}'D_{1,q2} + D_{1,q12}'D_{1,q12} \\ \mathcal{E}\{D_{2k}'D_{2k}\} &= D_2'D_2 + D_{2,q1}'D_{2,q1} + D_{2,q2}'D_{2,q2} + D_{2,q12}'D_{2,q12}. \end{aligned} \tag{4.31}$$

Remark. *It is straightforward to extend the number of stochastic parameters in the relations, but for simplicity in notation, only two variables as δ_1 and δ_2 are considered.*

The following theorem gives sufficient condition for the system in (4.15) to be both internally and input-output stable.

Theorem 6. *For $\gamma > 0$, the system G in (4.15) is exponentially stable in the mean-square sense and satisfies the $\mathcal{H}_\infty$ condition in (4.25) if there exists $P = P' > 0$ such that*

$$\begin{bmatrix} \Xi & \mathcal{E}\{A_k'PB_{1k} + C_k'D_{1k}\} & \mathcal{E}\{A_k'PB_{2k} + C_k'D_{2k}\} \\ * & \mathcal{E}\{B_{1k}'PB_{1k} + D_{1k}'D_{1k}\} - \rho\gamma^2 I & \mathcal{E}\{B_{1k}'PB_{2k} + D_{1k}'D_{2k}\} \\ * & * & \mathcal{E}\{B_{2k}'PB_{2k} + D_{2k}'D_{2k}\} - \gamma^2 I \end{bmatrix} < 0, \quad (4.32)$$

with $\Xi = \mathcal{E}\{A_k'PA_k + C_k'C_k\} - P$.

Proof. Suppose that condition (4.32) holds, thus, $\Xi < 0$. As $\mathcal{E}\{C_k'C_k\} > 0$, we conclude that

$$\mathcal{E}\{A_k'PA_k\} - P < 0. \quad (4.33)$$

Thus, based on Proposition 1, system G in (4.15) is exponentially stable in the mean-square sense.

Now, let us define the Lyapunov function as

$$V_k = \mathcal{E}\{\zeta_k'P\zeta_k\}, \quad P = P' > 0. \quad (4.34)$$

Thus,

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k = \mathcal{E}\{\zeta_{k+1}'P\zeta_{k+1} - \zeta_k'P\zeta_k\} \\ &= \mathcal{E}\{\zeta_k'(A_k'PA_k - P)\zeta_k + u_k'B_{1k}'PB_{1k}u_k + \omega_k'B_{2k}'PB_{2k}\omega_k + \\ &\quad + \zeta_k'A_k'PB_{1k}u_k + \zeta_k'A_k'PB_{2k}\omega_k + u_k'B_{1k}'PA_k\zeta_k + u_k'B_{1k}'PB_{2k}\omega_k + \\ &\quad + \omega_k'B_{2k}'PA_k\zeta_k + \omega_k'B_{2k}'PB_{1k}u_k\}. \end{aligned} \quad (4.35)$$

Also,

$$\mathcal{E}\{\|\tilde{z}_k\|^2\} = \mathcal{E}\{\zeta'\zeta\} = \mathcal{E}\{(C_k\zeta_k + D_{1k}u_k + D_{2k}\omega_k)'(C_k\zeta_k + D_{1k}u_k + D_{2k}\omega_k)\}$$

$$= \mathcal{E}\{\zeta_k'C_k'C_k\zeta_k + u_k'D_{1k}'D_{1k}u_k + \omega_k'D_{2k}'D_{2k}\omega + \zeta_k'C_k'D_{1k}u_k +$$

$$+ \zeta_k'C_k'D_{2k}\omega_k + u_k'D_{1k}'C\zeta_k + u_k'D_{1k}'D_{2k}\omega_k + \omega_k'D_{2k}'C\zeta_k + \omega_k'D_{2k}'D_{1k}u_k\}. \tag{4.36}$$

By adding the term

$$\mathcal{E}\{\gamma^2(\|\omega_k\|^2 - \|\omega_k\|^2) + \rho\gamma^2(\|u_k\|^2 - \|u_k\|^2) + (\|\tilde{z}_k\|^2 - \|\tilde{z}_k\|^2)\} \tag{4.37}$$

to the right hand side of the equation in (4.35), we can write:

$$\Delta V_k = \mathcal{E}\{-\|\tilde{z}_k\|^2 + \gamma^2(\|\omega_k\|^2 + \rho\|u_k\|^2) + \begin{bmatrix}\tilde{z}_k' & u_k' & \omega_k'\end{bmatrix} \Phi \begin{bmatrix}\tilde{z}_k \\ u_k \\ \omega_k\end{bmatrix}\}, \tag{4.38}$$

where Φ is the matrix defined in (4.32). As $\Phi < 0$,

$$\Delta V_k - \mathcal{E}\{-\|\tilde{z}_k\|^2 + \gamma^2(\|\omega_k\|^2 + \rho\|u_k\|^2)\} < 0, \tag{4.39}$$

or

$$\mathcal{E}\{\|\tilde{z}_k\|^2\} < \gamma^2(\|\omega_k\|^2 + \rho\|u_k\|^2) - \Delta V_k. \tag{4.40}$$

Now we sum up both sides of (4.40) for $k = 0, ..., \infty$ to get

$$\sum_{k=0}^{\infty} \mathcal{E}\{\|\tilde{z}_k\|^2\} < \sum_{k=0}^{\infty} \gamma^2(\|\omega_k\|^2 + \rho\|u_k\|^2) + V_0 - V_\infty. \tag{4.41}$$

Considering zero initial conditions, it can be concluded that

$$\sum_{k=0}^{\infty} \mathcal{E}\{\|\tilde{z}_k\|^2\} < \sum_{k=0}^{\infty} \gamma^2(\|\omega_k\|^2 + \rho\|u_k\|^2). \tag{4.42}$$

□

4.5 Optimal $\mathcal{H}_\infty$ Filter Design

Now, we have the required tools to design the optimal $\mathcal{H}_\infty$ filter in the NCS framework with multiple packet dropout. The filtering problem can be stated as follows:

Optimal $\mathcal{H}_\infty$ filtering problem: Design a filter as in (4.9) with minimum γ such that the filtering error dynamics in (4.10) are exponentially stable in the mean-square sense and the $\mathcal{H}_\infty$ criterion in (4.25) is satisfied.

Consider H, the filtering error dynamics defined in (4.10). By using Theorem 6, the optimal $\mathcal{H}_\infty$ filtering problem can be formulated as follows:

$$\min_{\mathbf{a}_f,\mathbf{b}_f,\mathbf{c}_f,L_f,P} \gamma$$
$$s.t.$$
$$\begin{bmatrix} \mathcal{E}\{A_k'PA_k + C_k'C_k - P\} & \mathcal{E}\{A_k'PB_{1k}\} & \mathcal{E}\{A_k'PB_{2k}\} \\ * & \mathcal{E}\{B_{1k}'PB_{1k}\} - \rho\gamma^2 I & \mathcal{E}\{B_{1k}'PB_{2k}\} \\ * & * & \mathcal{E}\{B_{2k}'PB_{2k}\} - \gamma^2 I \end{bmatrix} < 0. \tag{4.43}$$

Now, it is desirable to convert the matrix inequality in (4.43) into an LMI. Then, the filter design problem turns into a convex programming problem that can be solved efficiently by using available numerical methods. By using simple matrix manipulations, the matrix inequality in (4.43) can be shown to be equivalent to the following matrix inequality:

$$\Psi'\Pi^{-1}\Psi - \Gamma < 0, \tag{4.44}$$

where

$$\Psi' = \begin{bmatrix} A' & A_{q1}' & A_{q2}' & A_{q12}' & C' \\ B_1' & B_{1q1}' & B_{1q2}' & B_{1q12}' & 0 \\ B_2' & B_{2q1}' & B_{2q2}' & B_{2q12}' & 0 \end{bmatrix} \tag{4.45}$$

and

$$\Pi = \mathrm{diag}(-P^{-1}, -P^{-1}, -P^{-1}, -P^{-1}, I), \quad \Gamma = \mathrm{diag}(-P, -\rho\gamma^2 I, -\gamma^2 I). \tag{4.46}$$

By using the Schur complement [4], the last inequality is equivalent to the following matrix inequality:

$$\begin{bmatrix} \Pi & \Psi \\ \Psi' & \Gamma \end{bmatrix} < 0. \tag{4.47}$$

As P^{-1} exists, we put $Q = P^{-1}$ in (4.47) and apply the congruence transformation with $\text{diag}(I, I, I, I, I, Q, I, I)$ to get the following matrix inequality:

$$\left[\begin{array}{ccccc:ccc} -Q & 0 & 0 & 0 & 0 & AQ & B_1 & B_2 \\ 0 & -Q & 0 & 0 & 0 & A_{q1}Q & B_{1q1} & B_{2q1} \\ * & * & -Q & 0 & 0 & A_{q2}Q & B_{1q2} & B_{2q2} \\ * & * & * & -Q & 0 & A_{q12}Q & B_{1q12} & B_{2q12} \\ * & * & * & * & I & CQ & 0 & 0 \\ \hdashline * & * & * & * & * & -Q & 0 & 0 \\ * & * & * & * & * & * & -\rho\gamma^2 I & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I \end{array}\right] < 0. \tag{4.48}$$

Let us partition Q and P as

$$Q = \begin{bmatrix} X & U \\ U' & X_2 \end{bmatrix}, \quad P = Q^{-1} = \begin{bmatrix} Y & V \\ V' & Y_2 \end{bmatrix}, \tag{4.49}$$

where X, Y, X_2 and Y_2 are symmetric and positive definite matrices. We define the following nonsingular matrices:

$$T = \begin{bmatrix} Z & Y \\ 0 & V' \end{bmatrix}, \quad \bar{T} = \text{diag}(T, T, T, T, I, T, I, I). \tag{4.50}$$

By applying the congruence transformation with $\bar{T}$ to (4.48), we get the following LMI:

$$\left[\begin{array}{ccccc:ccc} -T'QT & 0 & 0 & 0 & 0 & T'AQT & T'B_1 & T'B_2 \\ 0 & -T'QT & 0 & 0 & 0 & T'A_{q1}QT & T'B_{1q1} & T'B_{2q1} \\ * & * & -T'QT & 0 & 0 & T'A_{q2}QT & T'B_{1q2} & T'B_{2q2} \\ * & * & * & -T'QT & 0 & T'A_{q12}QT & T'B_{1q12} & T'B_{2q12} \\ * & * & * & * & I & CQT & 0 & 0 \\ \hdashline * & * & * & * & * & -T'QT & 0 & 0 \\ * & * & * & * & * & * & -\rho\gamma^2 I & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I \end{array}\right] < 0, \tag{4.51}$$

where it is easy to verify that

$$
\begin{aligned}
T'QT &= \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}, \\
T'AQT &= \begin{bmatrix} Z\mathbf{a} & Z\mathbf{a} \\ Y\mathbf{a} + F\mathbf{c} + J & Y\mathbf{a} + F\mathbf{c} \end{bmatrix}, \\
T'A_{q*}QT &= \begin{bmatrix} Z\mathbf{a}_{q*} & Z\mathbf{a}_{q*} \\ Y\mathbf{a}_{q*} + F\mathbf{c}_{q*} & Y\mathbf{a}_{q*} + F\mathbf{c}_{q*} \end{bmatrix}, \\
T'B_1 &= \begin{bmatrix} Z\mathbf{b}_1 \\ Y\mathbf{b}_1 + M + F\mathbf{d}_1 \end{bmatrix}, \\
T'B_2 &= \begin{bmatrix} Z\mathbf{b}_2 \\ Y\mathbf{b}_2 + F\mathbf{d}_2 \end{bmatrix} \\
T'B_{1,q*} &= \begin{bmatrix} Z\mathbf{b}_{1q*} \\ Y\mathbf{b}_{1q*} + F\mathbf{d}_{1q*} \end{bmatrix}, \\
T'B_{2,q*} &= \begin{bmatrix} Z\mathbf{b}_{2q*} \\ Y\mathbf{b}_{2q*} + F\mathbf{d}_{2q*} \end{bmatrix} \\
CQT &= \begin{bmatrix} L - G & L \end{bmatrix},
\end{aligned}
\tag{4.52}
$$

with $F = V\mathbf{c}_f$, $G = \mathbf{c}_f U'Z$, $J = V\mathbf{a}_f U'Z$ and $M = V\mathbf{b}_f$. Also, $q*$ stands for q_1, q_2 and q_{12}. Thus, the results can be summarized as follows.

Theorem 7. **($\mathcal{H}_\infty$-filtering)** *The $\mathcal{H}_\infty$ filter design problem of (4.43) is equivalent to the following convex programming problem:*

$$
\begin{aligned}
&\min_{J,Z,Y,F,G,M} \quad \gamma \\
&s.t. \quad (4.51).
\end{aligned}
\tag{4.53}
$$

To find the filter parameters, $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f, we need to know U and V, which do not appear in the LMIs. One of the matrices U or V can be defined freely. Different choices give us different filter state-space realizations. One logical choice is to set $L_f = L$ that can come from setting $V = V' = -Y$, leading to $U = U' = Z^{-1} - Y^{-1}$.

Remark. *Even though we studied the problem of $\mathcal{H}_\infty$ filtering with multiple packet dropout in the NCS framework, the general tools used can be extended to solving problems in other frameworks such as sensor delay, multiple packet dropout of sensor information, and uncertain observations. These mentioned end up with the same formulation as in (4.10). For detailed derivations in the $\mathcal{H}_2$ case involving see Chapter 3 or [45].*

4.6 Example

In this section, a simulation example is provided to show the applicability and effectiveness of the proposed filtering method. Consider a discrete-time LTI system represented by (4.5) with the following matrix values:

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} 1.7240 & -0.7788 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ \mathbf{c} &= \begin{bmatrix} 0.0286 & 0.0264 \end{bmatrix}, \quad \mathbf{d}_1 = 1, \quad \mathbf{d}_2 = 1, \quad L = I_2, \end{aligned} \tag{4.54}$$

where I_2 is a 2×2 identity matrix. The initial state values are $\tilde{x}(0) = [0 \;\; 0]'$ and $\hat{\tilde{x}}(0) = [1 \;\; -1]'$. We assume that the controller is already designed and simply sends some sinusoidal commands to the actuator. The system states and their estimates are plotted in the following figures. Figure 4.2 shows the simulation results for the case when the average sensor to the controller and the controller to the actuator dropout rates are 0.2 and 0.8, respectively, with a weighting factor of $\rho = 0.5$. This result shows the superiority of the proposed $\mathcal{H}_\infty$ filtering over the classical one where no compensation is applied for dropouts. Figure 4.3 shows the changes of γ when α_1 and α_2 change from 0.1 to 1 with the step size of 0.1 for $\rho = 1$. With less dropout rate, a smaller γ is achieved. This example demonstrates the effectiveness of the proposed method.

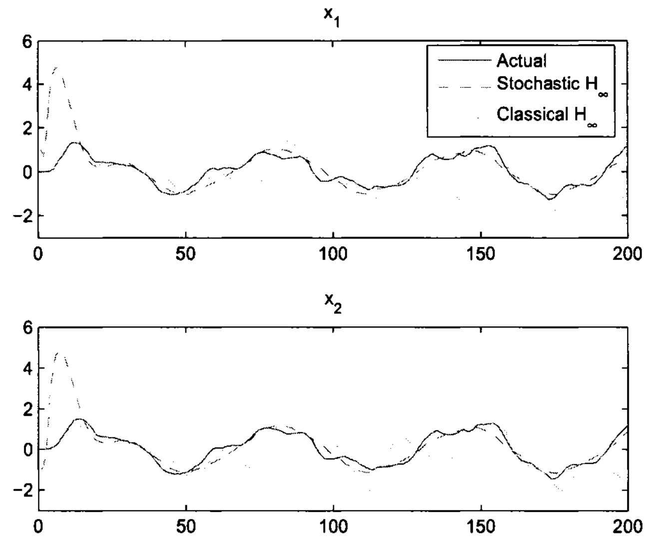

Figure 4.2: Actual and estimated states for the classical and stochastic $\mathcal{H}_\infty$ filtering, $\alpha_1 = 0.2$, $\alpha_2 = 0.8$, $\rho = 0.5$

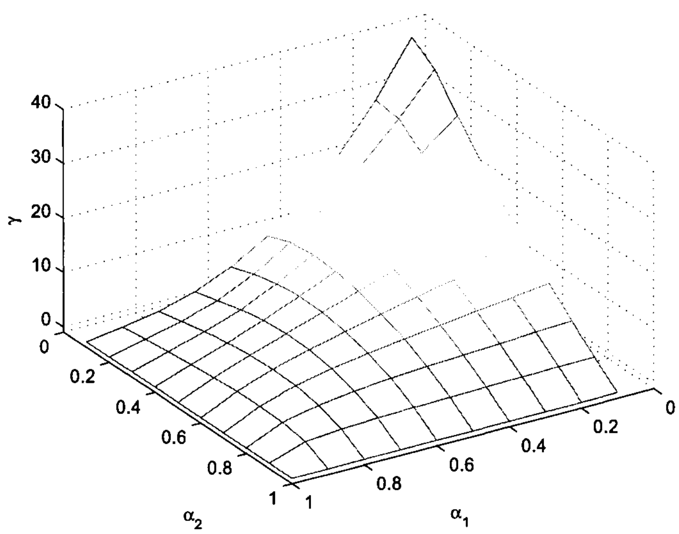

Figure 4.3: γ vs. α_1 and α_2 for $\rho = 1$

4.7 Conclusions

In this chapter, the problem of optimal $\mathcal{H}_\infty$ filtering in the NCS framework with multiple packet dropouts has been studied. The relations for the stochastic $\mathcal{H}_\infty$-norm of systems containing stochastic parameters and with both stochastic and deterministic inputs were developed. Based on the new derivations, the problem was transformed into a convex programming problem by using LMIs that can be easily solved by existing software packages. A simulation example showed the effectiveness and applicability of the proposed method.

Chapter 5

Kalman Filtering for Multi-rate Systems

5.1 Introduction

The importance of state filters was discussed in the previous chapters. One of the early optimal state filters is the Kalman filter, named after Rudolph E. Kalman, who published a famous paper describing a recursive solution to the discrete-data linear filtering problem [23]. The Kalman filter is essentially a set of mathematical relations implementing an estimator that minimizes the estimation error covariance. The Kalman filter is applicable where system models are known or can be found based on physical rules.

Effective control and monitoring of a process requires frequent information on essential process variables and states. In many processes, however, the essential variables are either not measured or are measured infrequently. In particular, in processes where measurements are available at different frequencies, multi-rate state estimators can provide frequent estimates of the variables. Infrequent measurements are usually related to key process variables, and thus their use in estimation leads

Some material of this chapter was reported in [44].

to more reliable estimates, especially in the presence of measurement noise.

State estimation has been studied in multi-rate systems [1, 16, 17, 20, 25, 48–50]. Andrisani and Gau [1] consider the case where the measurements are sampled at two different rates; the proposed optimal filter consists of two parallel Kalman filters: one processing the fast-rate measurements and the other processing the slow-rate ones. Hara and Tomizuka [20] use lifting to build a lifted discrete-time plant model, and then a discrete-time dual-rate estimator is designed; with it, estimation is based on the information from the output given at the output sampling (slow-rate) and the same values are used at inter-sample instants. It is proposed to use constant estimation gains in inter-sample instants; also a relation between the single-rate and dual-rate estimation gains is provided in order to have the same set of eigenvalues; however, the proposed method is not optimal.

Thein *et al.* [49, 50] introduce the idea of parallel observer systems, which have two separate observers running parallel with each other. The slow observer system performs during the output measurement period, and the fast observer system runs at the control input period (the fast rate). Both systems are Luenberger-type observers. The estimated states of the slow-rate observer are used to feed the fast-rate state estimates.

In the most recent work, Sheng *et al.* [48] consider dual-rate systems. They design dual-rate filters in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ settings so that the estimation error at the fast-rate can be minimized by satisfying a pre-specified criterion. As well, LMI techniques are employed for solving the problems.

In this chapter, the state lifting technique is introduced to include inter-sample states in a lifted system representation. Then, by generalizing the Kalman filter to dual-rate and multi-rate cases, optimal Kalman gains are designed that can be used to estimate the fast-rate states based on the multi-rate measurements. The optimal gains in the inter-sample instants are found to be zero; it means that in the optimal sense, the output estimates should be used to drive the Luenberger-type estimator. Simulation results support this conclusion.

5.2 State Lifting

One of the most important and widely used techniques for handling multi-rate systems is the so-called *lifting* method [6, 24], which converts a periodic discrete-time system into a time-invariant system. By lifting, a multi-rate system is converted into a single-rate one, which can be analyzed by using many different methods. By lifting, a multi-rate system can easily be transformed into a single-rate system, but the main drawback is the dimension increase. For example, consider the case involving an underlying clock with frame period qT, with a discrete-time signal $v(k)$ available every T, where q is some positive integer; that is, $v(0)$ occurs at time $t = 0$, $v(1)$ at $t = T$, etc. The lifted signal, $\underline{v}$, is defined as follows: If $v = \{v(0), v(1), \cdots\}$, then

$$\underline{v} = \left\{ \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(q-1) \end{bmatrix}, \begin{bmatrix} v(q) \\ v(q+1) \\ \vdots \\ v(2q-1) \end{bmatrix}, \ldots \right\}. \tag{5.1}$$

Thus, the dimension of $\underline{v}(k)$ equals q times that of $v(k)$, and $\underline{v}$ operates with the frame period qT. The lifting operator L is defined to be the map $v \to \underline{v}$. The inverse lifting, L^{-1}, is defined in an obvious way.

For a multi-rate system, we can lift the input and output to obtain a single-rate system with a period which is the least common multiple (LCM) of the input and output periods. Then the lifted model will be LTI [6, 26].

Consider a dual-rate system, with input updating at the fast rate, T, and output sampling at the slow rate, qT, where q is an integer and T is a real number. By lifting, the input and output are lifted to get a time-invariant system. If the fast-rate system is

$$\begin{cases} x_{k+1} & = Ax_k + Bu_k \\ y_k & = Cx_k + Du_k \end{cases} \tag{5.2}$$

with k an integer and A, B, C and D matrices of appropriate dimensions, then the lifted model will be

$$\begin{cases} x_{(k+1)q} & = \underline{\mathrm{A}}\ x_{kq} + \underline{\mathrm{B}}\ \underline{\mathrm{u}}_{kq} \\ y_{kq} & = \underline{\mathrm{C}}\ x_{kq} + \underline{\mathrm{D}}\ \underline{\mathrm{u}}_{kq}, \end{cases} \tag{5.3}$$

with

$$\begin{aligned} &\underline{\mathrm{A}} = A^n, \quad \underline{\mathrm{B}} = [A^{n-1}B \quad A^{n-2}B \quad \cdots \quad B], \\ &\underline{\mathrm{C}} = C, \quad \underline{\mathrm{D}} = [D \quad 0 \quad \cdots \quad 0], \end{aligned} \tag{5.4}$$

and

$$\underline{\mathrm{u}}_{kq} = \begin{pmatrix} u_{kq} \\ u_{kq+1} \\ \vdots \\ u_{kq+q-1} \end{pmatrix}. \tag{5.5}$$

Using this method, the slow-rate states can be related to the fast-rate input and slow-rate output. In equation (5.3), we have a relation for only the states in time instants where the output data are available, but in our study, we are interested in state estimation at the inter-sample instants and the fast-rate state estimates as well, thus we need to establish relations between the fast-rate states and the input-outputs. The following relations for estimated states show how such a lifted model can be obtained. To emphasize the incorporation of inter-sample states, this kind of lifting is called *state lifting.* To derive the equations, we consider the Luenberger-type state estimation and constant output during the inter-sample instants (by using a zero-order hold).

Based on the fast-rate system relations and by considering the Luenberger-type estimator, we can easily write the following relations. Here, the estimation gains are named L^i_{kq} for the i^{th} inter-sample instant in the kq^{th} frame period:

$$\begin{aligned}
\hat{x}_{kq+1} &= A\hat{x}_{kq} + Bu_{kq} + L^1_{kq}(y_{kq} - C\hat{x}_{kq}) \\
\hat{x}_{kq+2} &= A\hat{x}_{kq+1} + Bu_{kq+1} + L^2_{kq}(y_{kq} - C\hat{x}_{kq}) \\
&= A^2\hat{x}_{kq} + ABu_{kq} + Bu_{kq+1} + [AL^1_{kq} + L^2_{kq}](y_{kq} - C\hat{x}_{kq}) \\
&\vdots \\
\hat{x}_{kq+i} &= A\hat{x}_{kq+i-1} + Bu_{kq+i-1} + L^i_{kq}(y_{kq} - C\hat{x}_{kq}) \qquad (5.6) \\
&= A^i\hat{x}_{kq} + A^{i-1}Bu_{kq} + \cdots + Bu_{kq+i-1} + [A^{i-1}L^1_{kq} + \cdots + L^i_{kq}](y_{kq} - C\hat{x}_{kq}),
\end{aligned}$$

or

$$\underline{\hat{X}}_{(k+1)q} = \underline{A}\ \underline{\hat{X}}_{kq} + \underline{B}\ \underline{U}_{kq} + \underline{L}_{kq}\ (y_{kq} - \underline{C}\hat{X}_{kq}), \qquad (5.7)$$

where

$$\begin{aligned}
\underline{\hat{X}}_{(k+1)q} &= \begin{pmatrix} \hat{x}_{kq+1} \\ \hat{x}_{kq+2} \\ \vdots \\ \hat{x}_{kq+q} \end{pmatrix}, \quad \underline{U}(kq) = \begin{pmatrix} u_{kq} \\ u_{kq+1} \\ \vdots \\ u_{kq+q-1} \end{pmatrix}, \\
\underline{A} &= \begin{pmatrix} 0 & 0 & \cdots & A \\ 0 & 0 & \cdots & A^2 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & A^q \end{pmatrix}, \\
\underline{B} &= \begin{pmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & & \vdots & \\ A^{q-2}B & A^{q-3}B & \cdots & B \end{pmatrix}, \qquad (5.8) \\
\underline{C} &= [0 \ \ 0 \ \ \cdots \ \ C], \\
\underline{L}_{kq} &= \begin{pmatrix} I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & & \vdots & \\ A^{q-1} & A^{q-2} & \cdots & I \end{pmatrix} \begin{pmatrix} L^1_{kq} \\ L^2_{kq} \\ \vdots \\ L^q_{kq} \end{pmatrix}.
\end{aligned}$$

Using a similar approach, we can state lift the system to find the following relations:

$$\underline{X}_{(k+1)q} = \underline{A}\ \underline{X}_{kq} + \underline{B}\ \underline{U}_{kq}, \qquad (5.9)$$

with the same $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ as in (5.7) and

$$\underline{\mathrm{X}}_{(k+1)q} = \begin{pmatrix} x_{kq+1} \\ x_{kq+2} \\ \vdots \\ x_{kq+q} \end{pmatrix}. \tag{5.10}$$

Similarly, the state lifted representation for multi-rate systems can be found.

5.3 Dual-rate and Multi-rate Kalman Filter

Kalman [23] introduced the classical Kalman filter as a means of recursively solving the discrete-data linear filtering problem by minimizing a mean squared error. To the best of our knowledge, no general method has been proposed for using the Kalman filter to estimate the fast-rate states in the case of multi-rate observations. Here, the state lifted model derived in the previous section is used to find the Kalman filter gains for the fast-rate estimates in dual-rate and multi-rate cases. Consider a system with the fast-rate model represented by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + v_k \\ y_k = Cx_k + \omega_k, \end{cases} \tag{5.11}$$

where v_k and ω_k are discrete-time Gaussian white-noise processes with zero mean and

$$\mathcal{E}\{v_k v_k^T\} = R_1, \quad \mathcal{E}\{\omega_k \omega_k^T\} = R_2. \tag{5.12}$$

Then

$$\underline{\hat{\mathrm{X}}}_{(k+1)q} = \underline{\mathrm{A}}\ \underline{\hat{\mathrm{X}}}_{kq} + \underline{\mathrm{B}}\ \underline{\mathrm{U}}_{kq} + \underline{\mathrm{L}}_{kq}\ (y_{kq} - \underline{\mathrm{C}}\ \underline{\hat{\mathrm{X}}}_{kq}) \tag{5.13}$$

and

$$\underline{\mathrm{X}}_{(k+1)q} = \underline{\mathrm{A}}\ \underline{\mathrm{X}}_{kq} + \underline{\mathrm{B}}\ \underline{\mathrm{U}}_{kq} + \underline{\mathrm{v}} \tag{5.14}$$

with $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}$, $\underline{\mathrm{U}}$ and $\underline{\mathrm{L}}_{kq}$ as introduced before and

$$\underline{\mathrm{v}} = \begin{pmatrix} I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & & \vdots & \\ A^{q-1} & A^{q-2} & \cdots & I \end{pmatrix} \begin{pmatrix} v_{kq} \\ v_{kq+1} \\ \vdots \\ v_{kq+q-1} \end{pmatrix}. \tag{5.15}$$

Now, the optimal states can be found by applying the Kalman filtering method to the state lifted model. The following theorem gives the optimal state reconstruction in the dual-rate case where the input is updated at the fast rate and the output is sampled at the slow rate.

Theorem 8. (Dual-rate Kalman filter)- *Consider the system in (5.11) with a dual-rate output observation. The state estimation in (5.13) is optimal in the sense of providing a minimum variance of the estimation error if the estimation gain is given by the following relations:*

$$\begin{aligned} \underline{L}_{kq} &= \underline{A}\ \underline{P}_{kq}(R_2 + \underline{C}\ \underline{P}_{kq}\ \underline{C}^T)^{-1} \\ \underline{P}_{(k+1)q} &= \underline{A}\ \underline{P}_{kq}\ \underline{A}^T + \underline{R}_1 - \underline{A}\ \underline{P}_{kq}\ \underline{C}^T(R_2 + \underline{C}\ \underline{P}_{kq}\ \underline{C}^T)^{-1}\underline{C}\ \underline{P}_{kq}\ \underline{A}^T. \end{aligned} \tag{5.16}$$

Proof. The proof of the theorem is a straightforward generalization of the single-rate one [2] and is omitted. □

Corollary 5. *The optimal Kalman gains are* 0 *at the inter-sample instants, and at the time instants where the output measurements are available, the gains are given by*

$$\begin{aligned} L^1_{kq} &= AP^{nn}_{kq}C^T(R_2 + CP^{nn}_{kq}C^T)^{-1} \\ P^{nn}_{(k+1)q} &= A^nP^{nn}_{kq}A^n + (A^{n-1}R_1A^{n-1^T} + \cdots + R_1) \\ &\quad -A^nP^{nn}_{kq}C^T(R_2 + CP^{nn}_{kq}C^T)^{-1} \times CP^{nn}_{kq}A^n. \end{aligned} \tag{5.17}$$

Proof. Suppose that

$$\underline{P}_{kq} = \begin{pmatrix} P_{kq}^{11} & P_{kq}^{12} & \cdots & P_{kq}^{1n} \\ P_{kq}^{21} & P_{kq}^{22} & \cdots & P_{kq}^{2n} \\ \vdots & & \vdots & \\ P_{kq}^{n1} & P_{kq}^{n2} & \cdots & P_{kq}^{nn} \end{pmatrix}. \tag{5.18}$$

Now we can write

$$\underline{A}\ \underline{P}_{kq}\ \underline{C}^T = \begin{pmatrix} AP_{kq}^{n1} & AP_{kq}^{n2} & \cdots & AP_{kq}^{nn} \\ A^2P_{kq}^{n1} & A^2P_{kq}^{n2} & \cdots & A^2P_{kq}^{nn} \\ \vdots & & \vdots & \\ A^nP_{kq}^{n1} & A^nP_{kq}^{n2} & \cdots & A^nP_{kq}^{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ C^T \end{pmatrix} = \begin{pmatrix} AP_{kq}^{nn}C^T \\ A^2P_{kq}^{nn}C^T \\ \vdots \\ A^nP_{kq}^{nn}C^T \end{pmatrix} \tag{5.19}$$

and

$$R_2 + \underline{C}\ \underline{P}_{kq}\ \underline{C}^T = R_2 + [0\ \ 0\ \ \cdots\ \ C]\underline{P}_{kq}[0\ \ 0\ \ \cdots\ \ C]^T = R_2 + CP_{kq}^{nn}C^T. \tag{5.20}$$

Thus,

$$\underline{L}_{kq} = \begin{pmatrix} AP_{kq}^{nn}C^T \\ A^2P_{kq}^{nn}C^T \\ \vdots \\ A^nP_{kq}^{nn}C^T \end{pmatrix} (R_2 + CP_{kq}^{nn}C^T)^{-1}, \tag{5.21}$$

or

$$\begin{cases} L_{kq}^1 = AP_{kq}^{nn}C^T(R_2 + CP_{kq}^{nn}C^T)^{-1} \\ L_{kq}^2 = \cdots L_{kq}^n = 0 \end{cases} \tag{5.22}$$

and

$$\begin{aligned} P_{(k+1)q}^{nn} &= A^nP_{kq}^{nn}A^n + (A^{n-1}R_1A^{n-1^T} + \cdots + R_1) \\ &\quad - A^nP_{kq}^{nn}C^T \times (R_2 + CP^{nn}kqC^T)^{-1}CP_{kq}^{nn}A^n. \end{aligned} \tag{5.23}$$

□

5.3.1 Multi-rate Case

The idea of input extension [43] can be used to solve the Kalman filter problem for the multi-rate case. For simplicity, consider a multi-rate system where the inputs are updated at every $pT = 2T$ and the output samples are available at every $qT = 3T$. Then we can write:

$$\underline{\hat{X}}_{(k+1)pq} = \underline{A}\ \underline{\hat{X}}_{kpq} + \underline{B}\ \underline{U}_{kpq} + \underline{L}_{kpq}\ (\underline{Y}_{kpq} - \underline{C}\ \underline{\hat{X}}_{kpq}), \tag{5.24}$$

with

$$\underline{\hat{X}}_{(k+1)pq} = \begin{pmatrix} \hat{x}_{kpq+1} \\ \hat{x}_{kpq+2} \\ \vdots \\ \hat{x}_{kpq+pq} \end{pmatrix}, \quad \underline{Y}_{kpq} = \begin{pmatrix} y_{kpq} \\ y_{kpq+q} \end{pmatrix} \tag{5.25}$$

$$\underline{U}_{kpq} = \begin{pmatrix} u_{kpq} \\ u_{kpq+2} \\ u_{kpq+4} \end{pmatrix}, \quad \underline{A} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ A^2 & 0 & 0 & 0 & 0 & 0 \\ A^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^2 & 0 & 0 \\ 0 & 0 & 0 & A^3 & 0 & 0 \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} B & 0 & 0 \\ AB & 0 & 0 \\ A^2B + AB & B & 0 \\ 0 & B & 0 \\ 0 & AB & B \\ 0 & A^2B & AB + B \end{pmatrix},$$

$$\underline{L}_{kpq} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ A & I & 0 & 0 & 0 & 0 \\ A^2 & A & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & A & I & 0 \\ 0 & 0 & 0 & A^2 & A & I \end{pmatrix} \begin{pmatrix} L^1_{kpq} & 0 \\ L^2_{kpq} & 0 \\ L^3_{kpq} & 0 \\ 0 & L^4_{kpq} \\ 0 & L^5_{kpq} \\ 0 & L^6_{kpq} \end{pmatrix}. \tag{5.26}$$

and

$$\underline{X}_{(k+1)pq} = \underline{A}\ \underline{X}_{kpq} + \underline{B}\ \underline{U}_{kpq} + \underline{v} \tag{5.27}$$

where

$$\underline{\mathrm{X}}_{(k+1)pq} = \begin{pmatrix} x_{kpq+1} \\ x_{kpq+2} \\ \vdots \\ x_{kpq+6} \end{pmatrix},$$

$$\underline{\mathrm{v}}_{kpq} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ A & I & 0 & 0 & 0 & 0 \\ A^2 & A & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & A & I & 0 \\ 0 & 0 & 0 & A^2 & A & I \end{pmatrix} \begin{pmatrix} v_{kpq} \\ v_{kpq+1} \\ v_{kpq+2} \\ v_{kpq+3} \\ v_{kpq+4} \\ v_{kpq+5} \end{pmatrix}. \tag{5.28}$$

Again, by using a similar approach, we reach the following Corollary:

Corollary 6. *The optimal Kalman gains are 0 at the inter-sample instants, and at time instants where the output measurements are available, the gains are given by*

$$\begin{cases} L^1_{kpq} = AP^{11}_{kpq}C^T(R_2 + CP^{11}_{kpq}C^T)^{-1} \\ L^4_{kpq} = AP^{44}_{kpq}C^T(R_2 + CP^{44}_{kpq}C^T)^{-1} \end{cases} \tag{5.29}$$

and

$$\begin{aligned} P^{11}_{(k+1)pq} =& A^3P^{11}_{kpq}(A^3)^T + A^2R_1(A^2)^T + \\ &+ AR_1A^T + R_1 - A^3P^{11}_{kpq}C^T(R_2 + CP^{11}_{kpq}C^T)^{-1}CP^{11}_{kpq}(A^3)^T, \\ P^{44}_{(k+1)pq} =& A^3P^{44}_{kpq}(A^3)^T + A^2R_1(A^2)^T + \\ &+ AR_1A^T + R_1 - A^3P^{44}_{kpq}C^T(R_2 + CP^{44}_{kpq}C^T)^{-1}CP^{44}_{kpq}(A^3)^T. \end{aligned} \tag{5.30}$$

The optimal solution is found to be zero estimation gains at the inter-sample instants. These results suggest that the estimated output could be used in the Luenberger type state estimation of equation (5.7) at the inter-sample instants.

5.4 Examples

In this section, two examples are given to show the applicability of the multi-rate Kalman filter in dual-rate and multi-rate cases. The first example is based on a mathematical model, and the second one comes from a real mechanical system.

Example 1 - Consider a fast-rate system model represented by the following matrices:

$$\begin{aligned} A &= \begin{pmatrix} 1.0168 & 0.2059 \\ -1.8117 & 0.3991 \end{pmatrix}, \quad & B &= \begin{pmatrix} 0.0317 \\ 0.0111 \end{pmatrix} \\ C &= \begin{pmatrix} -0.8 & 0.6 \end{pmatrix}, & D &= 0 \end{aligned} \tag{5.31}$$

with an input updating period of $pT = 0.5$ and an output sampling period of $qT = 0.75$ (multi-rate ratio of 3/2, frame period of 1.5). Also consider the initial states of $x(0) = [0.0 \quad 0.1]'$ and $\hat{x}(0) = [0.1 \quad 0.0]'$. Figure 5.1 shows the actual, slow-rate and multi-rate estimated variables for a step input with magnitude of 2. Figure 5.1 reveals that the multi-rate Kalman filter gives good state estimates and outperforms the slow single-rate one.

Example 2 - In this example, an air-fuel ratio control system in spark-ignition engines is considered. The complete modelling and data are given in [5]. The problem is modified for ease of understanding and to show the applicability of the proposed multi-rate Kalman filter design method. The following paragraphs summarize the problem.

Current strict emission standards require accurate and fast air-fuel control in automotive systems. To achieve a desired air-fuel ratio, we need to know some engine variables, especially the liquid fuel puddle mass, which cannot be measured directly due to technical and economical constraints. Therefore, a model of the system is derived and a Kalman filter is designed to estimate the desired variable.

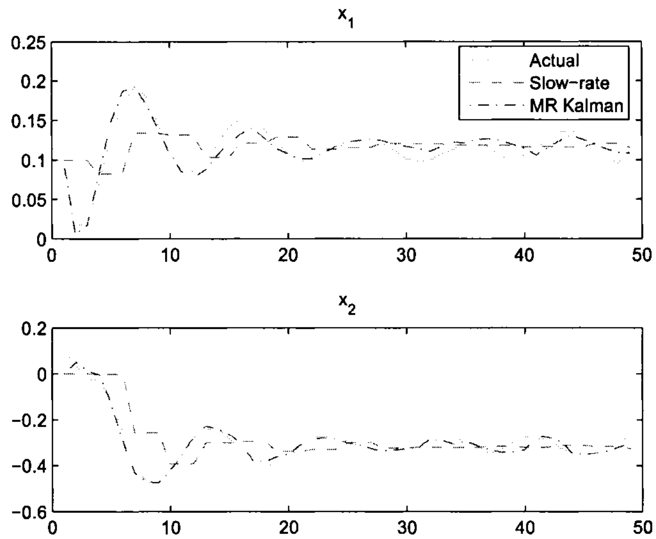

Figure 5.1: Actual, slow single-rate and multi-rate Kalman filter estimates

Different models are suggested for spark-ignition engines, mostly in the time domain, but the modelling here is based on the engine cycle.

This system has two parameters: air flow and fuel flow. The mass of air flow entering the cylinder per cycle, m_a, is a function of engine speed, N, and throttle angle, α, specified by the driver, that is: $m_a = m_a(\alpha, N)$. For the fuel flow, we have the following relations:

$$\begin{cases} m_{f_p}(k+1) = (1-f_\beta)m_{f_p}(k) + (1-f_\alpha)m_{f_i}(k) \\ m_f(k) = f_\beta m_{f_p}(k) + f_\alpha m_{f_i}(k), \end{cases} \tag{5.32}$$

where m_{f_p} is the liquid fuel puddle mass, m_{f_i} is the injected fuel mass, m_f the fuel mass into the cylinder per cycle, f_α is the fraction of injected fuel that enters the cylinder directly each cycle and f_β is the fraction of fuel puddle that evaporates and enters the cylinder each cycle. In [5], a universal air-fuel ratio heated exhaust gas oxygen (UEGO) sensor is used to establish the observer design and then an economical sensor is used in practice. The dynamics of the UEGO system are described as follows:

$$\tau_e \dot{\phi}_m(t) + \phi_m(t) = \phi_e(t - t_d) \tag{5.33}$$

with ϕ_e as the equivalence ratio in the exhaust manifold, ϕ_m as the equivalence ratio measured by the UEGO, τ_e the lag time constant and t_d the transport delay. In discrete time, Equation (5.33) can be represented as

$$\phi_m(k+1) = \gamma_0 \phi_m(k) + \gamma_1 \phi_e(k-1) + \gamma_2 \phi_e(k), \tag{5.34}$$

where

$$\begin{cases} \gamma_0 = e^{-T/\tau_e} \\ \gamma_1 = e^{-mT/\tau_e} - e^{-T/\tau_e} \\ \gamma_2 = 1 - e^{-mT/\tau_e} \\ m = 2 - (\theta_{EVO}/720) - (t_d/T) \end{cases} \tag{5.35}$$

with T as the sampling time, and θ_{EVO} the crank angle at which the exhaust valve opens. Putting all this together and considering ϕ_m as the output, we reach the following state space relations for the discrete engine model:

$$\begin{aligned} \begin{pmatrix} m_{f_p} \\ \phi_e \\ \phi_d \\ \phi_m \end{pmatrix}_{k+1} &= \begin{pmatrix} 1 - f_\beta & 0 & 0 & 0 \\ \frac{AF_s}{m_a} f_\beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}_k \begin{pmatrix} m_{f_p} \\ \phi_e \\ \phi_d \\ \phi_m \end{pmatrix}_k + \begin{pmatrix} 1 - f_\alpha \\ \frac{AF_s}{m_a} f_\alpha \\ 0 \\ 0 \end{pmatrix}_k (m_{f_i})_k \\ \phi_m(k) &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{f_p} \\ \phi_e \\ \phi_d \\ \phi_m \end{pmatrix}_k . \end{aligned} \tag{5.36}$$

This time-varying representation is considered at a fixed engine speed of $N = 1200$, which the relations are derived for. These relations are for the single-rate case. Now, consider the dual-rate case with ratio 2, so that an even simpler and slower sensor are used. To close the control loop, an optimal dual-rate Kalman filter is designed because the m_{f_p} cannot be measured economically. The throttle angle, α, is changed between 40 and 50 degrees and the estimated m_{f_p} and the actual one are depicted in Figure 5.2. Also, the equivalence ratio is plotted, showing it to be desirable. The results given in this example do not completely agree with those in [5], because not

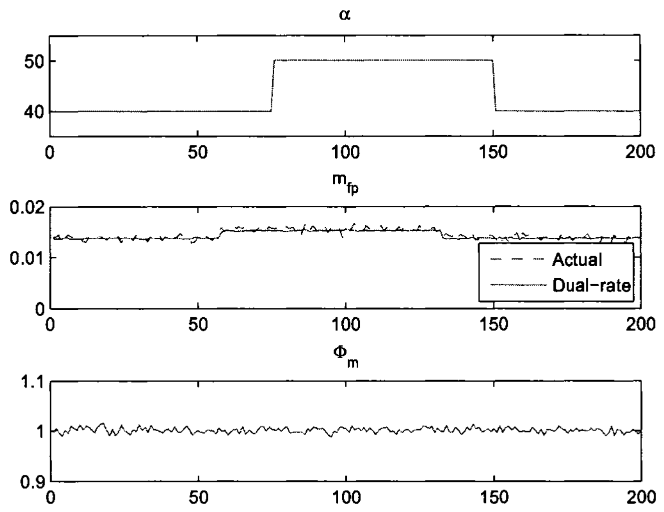

Figure 5.2: Throttle angle, actual and estimated m_{f_p} and measured ratio

all data are given in that paper, so some values are assumed here.

5.5 Conclusion

The problem of Kalman filter design for dual-rate and multi-rate processes was considered. The optimal Kalman gains were found to be zero at the inter-sample instants where the output samples were not available. Some examples were given to support the proposed method.

Chapter 6

Frequency-domain Parameter Estimation of General Multi-rate Systems

6.1 Introduction

In a conventional sampled-data control system, the plant input updating and output sampling are at the same rate. However, it is not always possible to update the control input and sample the output at the same rate due to various limitations such as the cost of fast-rate sensors and actuators. Also, sometimes the plant dynamics are such that sampling the different plant signals at the same rate is not economical and useful. As a result, a multi-rate sampling scheme should be considered for such cases. Of course, this scheme introduces the complication of mixed time steps.

Figure 6.1 shows a general multi-input, multi-output (MIMO), multi-rate system, where every input has its own updating rate and every output is sampled at its own rate. Continuous arrows are used for the continuous signals and dotted arrows for the discrete signals. Here, P_c is a continuous-time plant, $\mathbb{H}$ is a multi-rate zero-order hold, and $\mathbb{S}$ is a multi-rate output sampling device which can be defined

The material of this chapter was reported in [43].

as follows:

$$\mathbb{H} = \begin{pmatrix} H_{p_1h} & & \\ & \ddots & \\ & & H_{p_mh} \end{pmatrix}, \quad \mathbb{S} = \begin{pmatrix} S_{q_1h} & & \\ & \ddots & \\ & & S_{q_nh} \end{pmatrix}. \tag{6.1}$$

These correspond to holding m input channels of u with periods p_ih, $i = 1, \cdots, m$, and sampling n channels of output y_c with periods q_jh, $j = 1, \cdots, n$, respectively. Here, p_i and q_j are different integers and h is a real number called the *base period.* If we partition the signals accordingly,

$$y_c = \begin{pmatrix} y_{c_1} \\ \vdots \\ y_{c_n} \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \tag{6.2}$$

$$u_c = \begin{pmatrix} u_{c_1} \\ \vdots \\ u_{c_m} \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \tag{6.3}$$

then

$$\begin{aligned} u_{c_i}(t) &= u_i(k), \quad kp_ih \le t < (k+1)p_ih, \quad i = 1, \cdots, m \\ y_j(k) &= y_{c_j}(kq_jh), \quad j = 1, \cdots, n. \end{aligned} \tag{6.4}$$

We use u to denote a fictitious case in which all inputs are at the fast rate; that is, $p_i = 1$, $i = 1, \cdots, m$. Similarly, y denotes the output when all outputs are sampled at the fast rate, or $q_j = 1$, $j = 1, \cdots, n$.

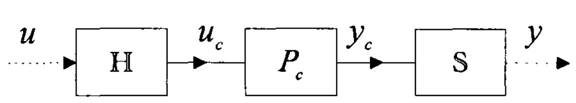

Figure 6.1: General MIMO multi-rate system

Such systems are often used in the chemical process industry. For example, in polymer reactors [39], the composition and density measurements are typically obtained after several minutes of analysis, whereas the control inputs can be applied at relatively fast rate. For another example, consider an industrial bleaching process [19] that is a chemical process applied to cellulose materials to increase their brightness and usefulness; in this process, some output variables, like brightness, need laboratory analysis and are in the slow rate and are irregularly sampled, while inputs can be applied at a relatively fast rate. One of the problems with such a system is finding the estimation of the system parameters and the output at those time instants when measurements are not available.

One application of this work is output monitoring at the fast rate. Another interesting application is the use of output estimates to run an inferential control scheme, as most inferential control algorithms need the parameters of fast single-rate models, which are not usually available. Some work has been done in this area. The existing multi-rate identification methods can be divided into two main categories: state-space identification and frequency-domain identification. Li *et al.* [26] studied the identification of a multi-rate sampled-data system consisting of a continuous-time process with or without time delay, a sampler with period pT, and a zero-order hold with period qT ($p < q$), and the problem of identifying a fast-rate model with sampling period pT. The method used was state-space based, employing the lifting technique. Their work is continued by Wang *et al.* [53] where a fast-rate model with sampling period T is identified. Lu and Fisher [29–32] studied the parameter and output estimation of dual-rate systems in the frequency domain; they proposed least-square and projection-based algorithms for dual-rate noise-free systems. Ding and Chen [10] studied the problem of parameter and output estimation for the dual-rate case for stochastic systems.

In this chapter, two simple methods are introduced for dealing with general multi-rate systems. These methods are named *dividing to subsystems* and *input extension* and are useful in the frequency domain. In the first method, a multi-rate system can be divided into some dual-rate subsystems and existing estimation methods can be used for the parameter estimation of each subsystem; then, the parameters of the original system can be extracted. In the second method, a multi-rate system can easily be converted to a dual-rate system with all input updating at a fast-rate. A least-square parameter estimation algorithm is derived for such systems.

When system parameters are estimated, they can be used for different applications like inter-sample output estimation as shown in Figure 6.2. Here, the process input(s) and sampled output(s) are fed into a parameter estimation engine that produces estimates of the assumed model's parameters at the fast rate. Based on the estimated parameters and known inputs, the output can be estimated at the fast rate. In Figure 6.2, $\hat{\theta}$ and $\hat{y}$ are used to show the estimates of parameters and y.

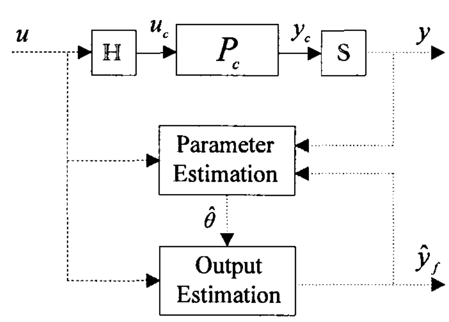

Figure 6.2: Multi-rate output estimation algorithm

6.2 Problem Transformation

To estimate a fast-rate model of the system, we assume a model structure for the system and try to estimate the parameters. This model is transformed into some multi-input, single-output, dual-rate subsystems and their parameter estimation is studied in the frequency domain.

Consider that we have a MIMO multi-rate system as in Figure 6.1 and that a series of input and output values are given. Then we assume a fast-rate frequency-domain model (transfer function) for this system and want to estimate the model parameters based on these multi-rate data.

Now, suppose that the fast-rate system model with period h in the frequency domain is P; that is,

$$y = P(z)u \ , \tag{6.5}$$

or

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} P_{11}(z) & P_{12}(z) & \cdots & P_{1m}(z) \\ P_{21}(z) & P_{22}(z) & \cdots & P_{2m}(z) \\ & & \ddots & \\ P_{n1}(z) & P_{n2}(z) & \cdots & P_{nm}(z) \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \tag{6.6}$$

where

$$P_{ij}(z) = \frac{b_{ij}(z)}{a_{ij}(z)} \ , \tag{6.7}$$

with

$$\begin{aligned} a_{ij}(z) &= 1 + a_{ij}^1 z^{-1} + a_{ij}^2 z^{-2} + \cdots + a_{ij}^N z^{-N} \\ b_{ij}(z) &= b_{ij}^0 + b_{ij}^1 z^{-1} + b_{ij}^2 z^{-2} + \cdots + b_{ij}^N z^{-N}. \end{aligned} \tag{6.8}$$

To deal with this MIMO problem, it can be divided to n MISO subsystems. For each one, we can write,

$$y_i(k) = P_{i1}u_1(k) + P_{i2}u_2(k) + \cdots + P_{im}u_m(k). \tag{6.9}$$

Without loss of generality, it can be assumed that all $a_{ij}(z)$ are equal to $a(z)$ with

$$a(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}, \tag{6.10}$$

thus

$$y_i(k) = \frac{b_{i1}(z)}{a(z)}u_1(k) + \frac{b_{i2}(z)}{a(z)}u_2(k) + \cdots + \frac{b_{im}(z)}{a(z)}u_m(k), \tag{6.11}$$

or

$$a(z)y_i(k) = b_{i1}(z)u_1(k) + b_{i2}(z)u_2(k) + \cdots + b_{im}(z)u_m(k). \tag{6.12}$$

This relation, even if available, does not help in the multi-rate problem, because if we expand this relation in the time domain, we have

$$\begin{aligned} y_i(k) = &- a_1 y_i(k-1) - \cdots - a_N y_i(k-N) + b_{i1}^0 u_1(k) + b_{i1}^1 u_1(k-1) + \cdots + \\ &+ b_{i1}^N u_1(k-N) + \cdots + b_{im}^0 u_m(k) + b_{im}^1 u_m(k-1) + \cdots + b_{im}^N u_m(k-N). \end{aligned} \tag{6.13}$$

However, we have no information about $y_i(k-j)$, $j \neq lq_i$, supposing k is an integer multiple of q_i. To obtain a recursive equation by directly using the available multi-rate data, equation (6.13) needs to be transformed into a form so that the $a(z)$ is a polynomial in z^{-q_i} instead of z^{-1}, and $b_{ij}(z)$ is a polynomial in z^{-p_i}. By using properties of zero-order holds and by the method suggested later, $b_{ij}(z)$'s can still be polynomials in z^{-1}.

For a general discussion, let the roots of $a(z)$ be λ_i to get

$$a(z) = \prod_{i=1}^{N} (1 - \lambda_i z^{-1}). \tag{6.14}$$

Define

$$\phi_{q_i}(z) = \prod_{i=1}^{N} (1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{q_i-1} z^{-q_i+1}) \tag{6.15}$$

Then

$$\begin{aligned} a(z)\phi_{q_i}(z) &= \prod_{i=1}^{N} (1 - \lambda_i z^{-1}) \prod_{i=1}^{N} (1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{q_i-1} z^{-q_i+1}) \\ &= \prod_{i=1}^{N} (1 - \lambda_i z^{-1})(1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{q_i-1} z^{-q_i+1}) \\ &= \prod_{i=1}^{N} (1 - \lambda_i^{q_i} z^{-q_i}) \\ &= 1 - \lambda_1^{q_i} z^{-q_i} - \lambda_2^{q_i} z^{-q_i} \cdots + (-1)^N \lambda_1 \lambda_2 \cdots \lambda_N z^{-Nq_i} \\ &= 1 - (\lambda_1^{q_i} + \lambda_2^{q_i} \cdots + \lambda_N^{q_i}) z^{-q_i} + \cdots + (-1)^N \lambda_1 \lambda_2 \cdots \lambda_N z^{-Nq_i} \\ &=: 1 + \alpha_{i1} z^{-q_i} + \alpha_{i2} z^{-2q_i} + \cdots + \alpha_{iN} z^{-Nq_i}. \end{aligned} \tag{6.16}$$

However, in the multiplication of $b_{ij}(z)\phi_{q_i}(z)$, generally all the coefficients are nonzero:

$$\begin{aligned} b_{ij}(z)\phi_{q_i}(z) &= (b_{ij}^0 + b_{ij}^1 z^{-1} + \cdots + b_{ij}^N z^{-N}) \prod_{i=1}^{N} (1 + \lambda_i z^{-1} + \cdots + \lambda_i^{q_i-1} z^{-q_i+1}) \\ &=: \beta_{ij}^0 + \beta_{ij}^1 z^{-1} + \beta_{ij}^2 z^{-2} + \cdots + \beta_{ij}^{q_i N} z^{-Nq_i}. \end{aligned} \tag{6.17}$$

Thus, multiplying the numerator and denominator of $P_{ij}(z)$ by $\phi_{q_i}(z)$ transforms the denominator into the desired form where the denominator is a polynomial in z^{-q_i}:

$$P_{ij}(z) = \frac{b_{ij}(z)\phi_{q_i}(z)}{a(z)\phi_{q_i}(z)} =: \frac{\beta_{ij}(z)}{\alpha_i(z)}. \tag{6.18}$$

Now, equation (6.13) can be written as

$$y_i(k) = -\sum_{j=1}^{N} \alpha_{ij} y_i(k - jq_i) + \sum_{j=1}^{m} \sum_{l=0}^{Nq_i} \beta_{ij}^l u_j(k - l). \tag{6.19}$$

For the input data, we can consider a zero-order hold property. Figure 6.3 shows that by using the zero-order hold, we have input information to the plant at every h instant, because the output of the zero-order hold remains the same until the next update. Using a zero-order hold property, we propose two methods for dealing with multi-rate systems: dividing the multi-rate data to p subsets, and extending the input such that the input updating rate becomes h.

6.2.1 Dividing to Subsystems

This method can be used to convert a multi-rate system into some dual-rate subsystems. After estimating the parameters of the subsystems, the parameters of the original multi-rate system can be extracted by using simple least-squares methods.

To discuss this method, consider a second-order SISO system with transfer function $P(z)$:

$$P(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}, \tag{6.20}$$

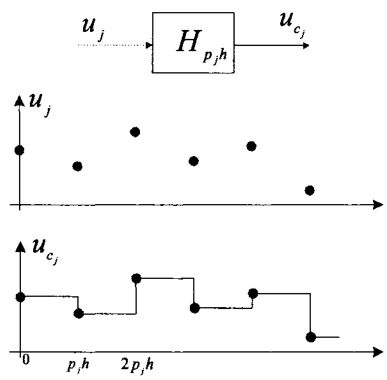

Figure 6.3: Zero-order hold

with an input updating period of $2h$ or $p = 2$, and an output sampling period of $3h$, or $q = 3$. Using the method just discussed, a polynomial, $\phi_q(z)$, can be found such that

$$P(z) = \frac{y(k)}{u(k)} = \frac{b(z)\phi_q(z)}{a(z)\phi_q(z)} = \frac{\beta(z)}{\alpha(z)}, \tag{6.21}$$

with

$$\begin{aligned} \alpha(z) &= 1 + \alpha_1 z^{-3} + \alpha_2 z^{-6}, \\ \beta(z) &= \beta_0 + \beta_1 z^{-1} + \cdots + \beta_6 z^{-6}. \end{aligned} \tag{6.22}$$

Expanding equation (6.21) in the time domain gives

$$y(k) = -\alpha_1 y(k-3) - \alpha_2 y(k-6) + \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_6 u(k-6). \tag{6.23}$$

Suppose that a series of input values $\{u(k), u(k-p), \cdots\}$ and output values $\{y(k), y(k-q), \cdots\}$ are given. We cannot simply use equation (6.23) as the input values are not given at every time instant, but considering the zero-order hold property we can write:

$$\begin{aligned} u(lpq-1) &= u(lpq-2), \\ u(lpq-3) &= u(lpq-4), \\ u(lpq-5) &= u(lpq-6), \end{aligned} \tag{6.24}$$

and

$$u(lpq+q) = u(lpq+q-1),$$
$$u(lpq+q-2) = u(lpq+q-3), \tag{6.25}$$
$$u(lpq+q-4) = u(lpq+q-5).$$

Thus, the system can be divided to two (in general p) subsystems and write equation (6.23) for each one; one for $k = lpq$ and the other one for $k = lpq + q$ with l a positive integer. Thus, for subsystem 1 we can write:

$$y(kpq) + \alpha_1 y(kpq-q) + \alpha_2 y(kpq-2q) =$$
$$\beta_0 u(kpq) + (\beta_1+\beta_2)u(kpq-1) + (\beta_3+\beta_4)u(kpq-3) + (\beta_5+\beta_6)u_s(kpq-5)$$
$$=: \beta_{01}u(kpq) + \beta_{11}u(kpq-1) + \beta_{21}u(kpq-3) + \beta_{31}u(kpq-5). \tag{6.26}$$

For subsystem 2;

$$y(kpq+q) + \alpha_1 y(kpq) + \alpha_2 y(kpq-q) =$$
$$(\beta_0+\beta_1)u(kpq+q) + (\beta_2+\beta_2)u(kpq+q-2) +$$
$$(\beta_4+\beta_5)u(kpq+q-4) + \beta_6 u(kpq+q-6) =:$$
$$\beta_{02}u(kpq+q) + \beta_{12}u(kpq+q-2) + \beta_{22}u(kpq+q-4) + \beta_{32}u(kpq+q-6). \tag{6.27}$$

For each of the two subsystems, any parameter estimation method can be used to find an estimation of α_1, α_2 and $\beta_{01}, \beta_{11}, \cdots, \beta_{32}$. Then, by using the simple least-square method, the estimates of α_i and β_i can be found.

6.2.2 Input Extension

The simpler method given here can easily be used to convert a multi-rate MIMO system into a dual-rate MIMO system with all input updating at the fast rate.

Consider Figure 6.3, which shows that we have information about the input at every time instant, by simply taking the same input until the next update. Thus, if we name the slow rate input, which updates every $p_j h$ instant as u_s^j, and the fast rate input, which updates every h instant as u, the following relation is obtained:

$$u_j(k) = u_s^j(ip_j) \quad \text{for} \quad k = ip_j, ip_j + 1, \cdots, ip_j + p_j - 1. \tag{6.28}$$

6.2.3 Parameter Estimation

When the multi-rate models are transformed properly, any frequency-domain identification method can be used for parameter estimation. Here, we use the dual-rate least squares method suggested in [10] and adapt it for MISO systems.

If we consider the stochastic case, equation (6.9) can be written as

$$y_i(k) = \sum_{1}^{m} P_{ij}(z).u_j(k) + v_i(k), \quad i = 1 \cdots n, \tag{6.29}$$

where $v_i(k)$ is assumed to be a white Gaussian and zero-mean random signal. Substituting the polynomial $\alpha(z)$ in z^{-q_i} from (6.16) and β_{ij} in z^{-1} from (6.17) leads to the following regression equation,

$$y_i(k) = \varphi_i^T(k)\theta_i + v_i(k), \tag{6.30}$$

where the superscript T denotes the matrix transpose, and the parameter vector θ_i and information vector $\varphi_i(k)$ are defined by

$$\theta_i = [\alpha_{i1} \;\; \alpha_{i2} \;\; \cdots \;\; \alpha_{iN} \;\; \beta_{i1}^0 \;\; \beta_{i2}^0 \;\; \cdots \;\; \beta_{im}^0 \;\; \cdots \;\; \beta_{i1}^{Nq_i} \;\; \beta_{i2}^{Nq_i} \;\; \cdots \;\; \beta_{im}^{Nq_i}] \tag{6.31}$$

and

$$\begin{aligned} \varphi_i(k) \;\; = \;\; & [-y_i(k-q_i) \;\; -y_i(k-2q_i) \;\; \cdots \;\; -y_i(k-Nq_i) \;\; u_1(k) \;\; u_2(k) \;\; \cdots \\ & u_m(k) \;\; \cdots \;\; u_1(k-Nq_i) \;\; u_2(k-Nq_i) \;\; \cdots \;\; u_m(k-Nq_i)]^T. \end{aligned} \tag{6.32}$$

Notice that θ_i contains all the parameters to be estimated in the model in (6.9), and if k is an integer multiple of q_i, then $\varphi_i(k)$ contains only the available data which are the past output measurements (slow-rate) and the past and current inputs (fast-rate).

Let $\hat{\theta}_i(kq_i)$ be the estimate of θ_i at time kq_i. The following recursive least squares algorithm is proposed for estimating the parameter vector θ_i of the dual-rate system:

$$\begin{aligned}
\hat{\theta}_i(kq_i) &= \hat{\theta}_i(kq_i - q_i) + P_i(kq_i)\varphi_i(kq_i)[y_i(kq_i) - \varphi_i^T(kq_i)\hat{\theta}_i(kq_i - q_i)], \\
\hat{\theta}_i(kq_i + l) &= \hat{\theta}_i(kq_i); \quad l = 0, 1, \cdots, q_i - 1, \\
P_i^{-1}(kq_i) &= P_i^{-1}(kq_i - q_i) + \varphi_i(kq_i)\varphi_i^T(kq_i), \quad P_i(0) = P_i^T(0).
\end{aligned} \tag{6.33}$$

To initialize the algorithm, we take $P_i(0) = p_{i_0} I$ with p_{i_0} normally a large positive number, and $\hat{\theta}_i(0) = \hat{\theta}_{i_0}$, some real vector. Notice that the parameter estimate $\hat{\theta}_i$ is updated at every q_i samples, namely, at the slow rate, as is the matrix P_i. In between the slow samples, $\hat{\theta}_i$ is simply held unchanged. It is easy to see that by defining

$$L_i(kq_i) := P_i(kq_i)\varphi_i(kq_i) = \frac{P_i(kq_i - q_i)\varphi_i(kq_i)}{1 + \varphi_i^T(kq_i)P_i(kq_i - q_i)\varphi_i(kq_i)}, \tag{6.34}$$

the covariance matrix P_i can be updated as follows:

$$\begin{aligned}
P_i(kq_i) &= P_i(kq_i - q_i) - \frac{P_i(kq_i - q_i)\varphi_i(kq_i)\varphi_i^T(kq_i)P_i(kq_i - q_i)}{1 + \varphi_i^T(kq_i)P_i(kq_i - q_i)\varphi_i(kq_i)} \\
&= [I - L_i(kq_i)\varphi_i^T(kq_i)]P_i(kq_i - q_i).
\end{aligned} \tag{6.35}$$

6.3 Model Reconciliation

When a parameter estimation method is applied, the α and β parameters can be estimated, and normally, a fast-rate model as in equation (6.5) is desired. Here, we study the problem of extracting the a and b parameters from the estimated α and β parameters obtained from n MISO subsystems.

One solution to this problem is to use the concept of model order reduction [15], as we want to reduce the order of the model represented by the α and β parameters. However, the model order reduction methods cannot guarantee the convergence of the parameters. Actually, for practical cases, when no real model of the system exists, and only input-output data are available, the concept of model order reduction can be used to find a reduced order fast-rate model.

Another solution to the problem can be constructed by using equation (6.18). If we denote the estimate of parameters by $(\hat{.})$, we can write:

$$\frac{\hat{b_{ij}}(z)}{\hat{a}(z)} = \frac{\hat{\beta_{ij}}(z)}{\hat{\alpha_i}(z)}, \tag{6.36}$$

or

$$\frac{\hat{b}_{ij}^0 + \hat{b}_{ij}^1 z^{-1} + \cdots + \hat{b}_{ij}^N z^{-N}}{1 + \hat{a}_1 z^{-1} + \cdots + \hat{a}_N z^{-N}} = \frac{\hat{\beta}_{ij}^0 + \hat{\beta}_{ij}^1 z^{-1} + \cdots + \hat{\beta}_{ij}^{Nq_i} z^{-Nq_i}}{1 + \hat{\alpha}_{i1} z^{-q_i} + \cdots + \hat{\alpha}_{iN} z^{-Nq_i}}, \tag{6.37}$$

where $\hat{\alpha_i}$ and $\hat{\beta_{ij}}$ are given by the estimation algorithm. By multiplying the polynomials in the nominator and denominator of Equation (6.37), equation (6.36) can be converted to a set of polynomial equations and the parameters $\hat{a}$ and $\hat{b}_{ij}$ can be extracted. Suppose that the vector of parameters is defined as

$$\hat{\theta} = [a_1\ a_2\ \cdots\ a_N\ b_{11}^0\ \cdots\ b_{11}^N\ \cdots\ b_{nm}^0\ \cdots\ b_{nm}^N]; \tag{6.38}$$

then we can write:

$$S\hat{\theta} = \begin{pmatrix} S_\beta^{11} & S_\alpha^{11} & 0 & \cdots & 0 \\ S_\beta^{12} & 0 & S_\alpha^{12} & \cdots & 0 \\ & & \ddots & & \\ S_\beta^{nm} & 0 & 0 & \cdots & S_\alpha^{nm} \end{pmatrix} \hat{\theta} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \vdots \\ \rho_{nm} \end{pmatrix} = \rho, \tag{6.39}$$

where

$$S_\alpha^{ij} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \gamma_1^{ij} & 1 & & \vdots \\ \gamma_2^{ij} & \gamma_1^{ij} & \ddots & 0 \\ \vdots & \gamma_2^{ij} & \ddots & 1 \\ \vdots & & \ddots & \vdots \\ \gamma_{q_iN}^{ij} & & & \gamma_2^{ij} \\ 0 & \gamma_{q_iN}^{ij} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_{q_iN}^{ij} \end{pmatrix}, \quad S_\beta^{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\hat{\beta}_{ij}^0 & 0 & & \vdots \\ -\hat{\beta}_{ij}^1 & -\hat{\beta}_{ij}^0 & \ddots & 0 \\ \vdots & -\hat{\beta}_{ij}^1 & \ddots & 0 \\ \vdots & & \ddots & -\hat{\beta}_{ij}^0 \\ -\hat{\beta}_{ij}^{q_iN} & & & -\hat{\beta}_{ij}^1 \\ 0 & -\hat{\beta}_{ij}^{q_iN} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & -\hat{\beta}_{ij}^{q_iN} \end{pmatrix} \tag{6.40}$$

$$\gamma_l^{ij} = \begin{cases} \hat{\alpha}_{ik} & l = kq_i, \quad k = 1, 2, \cdots, N \\ 0 & else \end{cases}, \tag{6.41}$$

$$\rho_{ij} = [\hat{\beta}_{ij}^0 \ \hat{\beta}_{ij}^1 \ \cdots \ \hat{\beta}_{ij}^{q_iN} \ 0 \ \cdots \ 0]^T. \tag{6.42}$$

The least squares solution of equation (6.39) is given by

$$\hat{\theta} = [S^TS]^{-1}S^T\rho. \tag{6.43}$$

6.4 Examples

To show the applicability of the proposed method, three illustrative examples are given, one for the SISO case and two for the MIMO systems, including one with real industrial data.

Example 1 - Consider a system with the fast-rate transfer function $P(z)$:

$$P(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{0.412z^{-1} + 0.309z^{-2}}{1 - 1.6z^{-1} + 0.80z^{-2}}. \tag{6.44}$$

Several sampling cases are considered for this system:

- Single-rate case ($p = q = 1$),
- Dual-rate case with output sampling at $q = 2$,
- Multi-rate case with $p = 2$ and $q = 3$.

For the dual-rate case, $\phi(z) = 1 - a_1 z^{-1} + a_2 z^{-2}$ is used to get

$$P(z) = \frac{b(z)\phi(z)}{a(z)\phi(z)} = \frac{0.4120z^{-1} + 0.9682z^{-2} + 0.8240z^{-3} + 0.2472z^{-4}}{1 - 0.96z^{-2} + 0.64z^{-4}}. \tag{6.45}$$

Also, $\phi(z) = 1 - a_1 z^{-1} + (a_1^2 - a_2) z^{-2} - a_1 a_2 z^{-3} + a_2^2 z^{-4}$ can be used for the multi-rate case to get

$$P(z) = \frac{0.412z^{-1} + 0.968z^{-2} + 1.2195z^{-3} + 1.071z^{-4} + 0.659z^{-5} + 0.1978z^{-6}}{1 - 0.2560z^{-3} + 0.5120z^{-6}}. \tag{6.46}$$

To deal with the multi-rate case, both suggested methods are used. To provide a fair comparison, the same total data points of the input-output data are maintained for all cases. The relative parameter estimation error (PEE) measured in the Euclidean norm is defined as

$$PEE = \|\hat{\theta} - \theta\| / \|\theta\|. \tag{6.47}$$

A lower PEE is obtained for the single-rate case and a higher for the multi-rate case, as information in between the sampling instants is lacking compared to that available in the single-rate case. As the noise is different for different cases, and in order to reduce its effects on our comparison, simulations are done several times

Table 6.1: Estimated Parameters, single-rate case

	$\hat{a}_1$	$\hat{a}_2$	$\hat{b}_1$	$\hat{b}_2$	$PEE\%$
SR	-1.5077	0.7114	0.4222	0.3903	8.3
DR	-1.4786	0.6819	0.3971	0.2601	9.5
MR1	-1.5550	0.7612	0.6530	0.0486	19.33
MR2	-1.3317	0.5609	0.4257	0.3688	19.59
True Value	-1.6	0.8	0.412	0.309	-

Table 6.2: Estimated Parameters, dual-rate case

	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$PEE\%$
DR	-0.9555	0.6048	0.5193	0.9243	0.7988	0.1657	8.5
True Value	-0.96	0.64	0.4120	0.9682	0.8240	0.2472	-

for each case (100 times in this example) and the mean value of PEE is extracted. To run the simulations, a persistent excitation input sequence, a random binary sequence (RBS) in the frequency range of 0 to $1/2\pi$, is applied as an input, and an additive white noise with zero mean and unity variance is considered at the output. The total number of input-output data is considered to be 2000. The following tables show the results. SR stands for single-rate, DR for dual-rate, MR1 for the multi-rate case with input extension, and MR2 for the multi-rate case of dividing into subsystems.

Table 6.3 shows that the two suggested methods for multi-rate systems have very similar results, while the input extension method can easily be applied to MIMO systems. Figure 6.4 shows the step response of the system and the estimated model

Table 6.3: Estimated Parameters, multi-rate case

	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$	$PEE\%$
MR1	-0.1892	0.4691	0.5712	0.8963	1.2240	1.2271	0.6560	0.1539	18.43
MR2	-0.1173	0.4300	0.3279	0.8724	1.3622	1.1358	0.8916	0.0463	18.15
True Value	-0.2560	0.5120	0.4120	0.9682	1.2195	1.0712	0.6592	0.1978	-

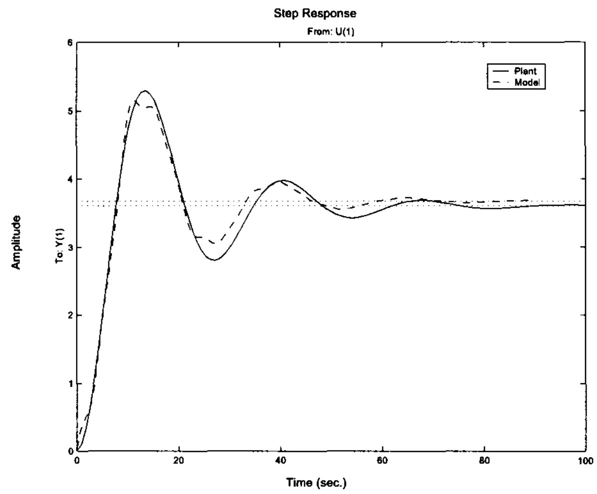

Figure 6.4: Step response of the system and the estimated model for Example 1

using the dividing to subsystems method.

Example 2 - Consider a two-input, two-output system as shown in Figure 6.5. The input u_1 is updated every $2h$ period, while the input u_2 is updated at the fast rate. The outputs y_1 and y_2 are sampled every $3h$ and $2h$, respectively. The following fast-rate model is used for the plant P:

$$\begin{aligned} P &= \frac{1}{1+a_1z^{-1}+a_2z^{-2}}\begin{pmatrix} b_{11}^1z^{-1}+b_{11}^2z^{-2} & b_{12}^1z^{-1}+b_{12}^2z^{-2} \\ b_{21}^0+b_{21}^1z^{-1}+b_{21}^2z^{-2} & b_{22}^0+b_{22}^1z^{-1}+b_{22}^2z^{-2} \end{pmatrix} \\ &= \frac{1}{1-1.6z^{-1}+0.8z^{-2}}\begin{pmatrix} 0.412z^{-1}+0.309z^{-2} & 0.1z^{-1}+0.3z^{-2} \\ 1+0.2z^{-1}+0.1z^{-2} & 1+0.8z^{-1}+0.6z^{-2} \end{pmatrix}. \end{aligned} \tag{6.48}$$

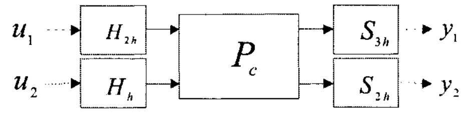

Figure 6.5: MIMO multi-rate system of Example 2

This system is transformed into two two-input, one-output subsystems as follows:

$$
\begin{aligned}
y_1 &= \frac{b_{11}^1 z^{-1} + b_{11}^2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_1 + \frac{b_{12}^1 z^{-1} + b_{12}^2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_2 = \frac{b_{11}(z)u_1 + b_{12}(z)u_2}{a(z)} \\
y_2 &= \frac{b_{21}^0 + b_{21}^1 z^{-1} + b_{21}^2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_1 + \frac{b_{22}^0 + b_{22}^1 z^{-1} + b_{22}^2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_2 = \frac{b_{21}(z)u_1 + b_{22}(z)u_2}{a(z)}
\end{aligned}
\tag{6.49}
$$

Now we can use $\phi_1(z) = 1 - a_1 z^{-1} - (a_2 + a_1^2) z^{-2} - a_1 a_2 z^{-3} + a_2^2 z^{-4}$ for subsystem 1 and $\phi_2(z) = 1 - a_1 z^{-1} + a_2 z^{-2}$ for subsystem 2 to get

$$
\begin{aligned}
y_1 &= \frac{b_{11}(z)\phi_1(z)u_1 + b_{12}(z)\phi_1(z)u_2}{a(z)\phi_1(z)} = \frac{\beta_{11}(z)u_1 + \beta_{12}(z)u_2}{\alpha_1(z)}, \\
y_2 &= \frac{b_{21}(z)\phi_2(z)u_1 + b_{22}(z)\phi_2(z)u_2}{a(z)\phi_2(z)} = \frac{\beta_{21}(z)u_1 + \beta_{22}(z)u_2}{\alpha_2(z)}.
\end{aligned}
\tag{6.50}
$$

Now, the suggested least-squares algorithm is applied to each one. To run the simulations, persistent excitation input sequences, random binary sequences (RBS) in the frequency range of 0 and $1/2\pi$, are applied as inputs, and additive white noise with zero mean and unity variance is considered at the outputs. Then the average estimation error for 100 simulations for all of the parameters is calculated as $PEE = 22.81$. Figures 6.6 and 6.7 show the step responses of the model and the actual system.

Example 3 - The results obtained in this chapter can be extended to MIMO cases with irregularly sampled systems as well. Here, a real industrial system shown

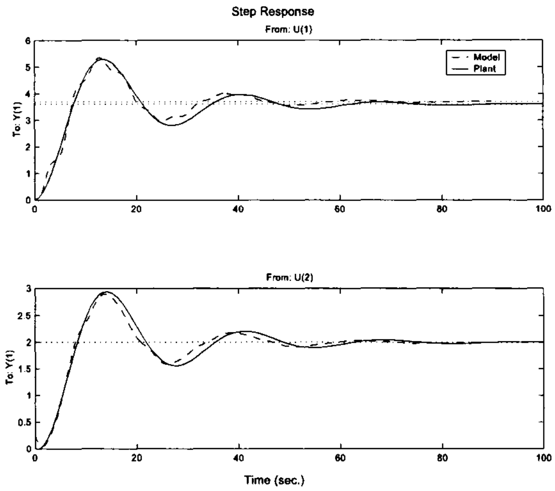

Figure 6.6: Step responses of the system and the estimated model for Example 2 (for y_1)

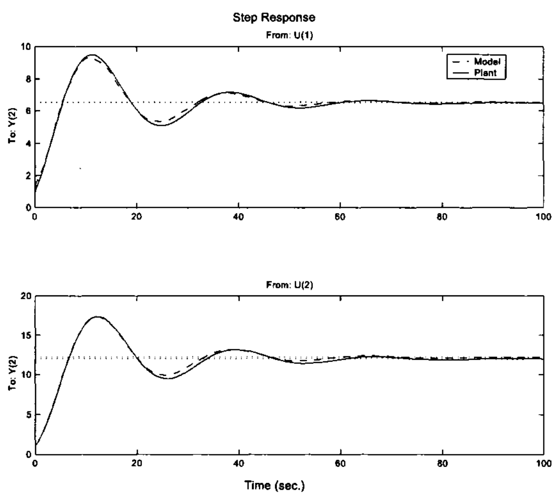

Figure 6.7: Step responses of the system and the estimated model for Example 2 (for y_2)

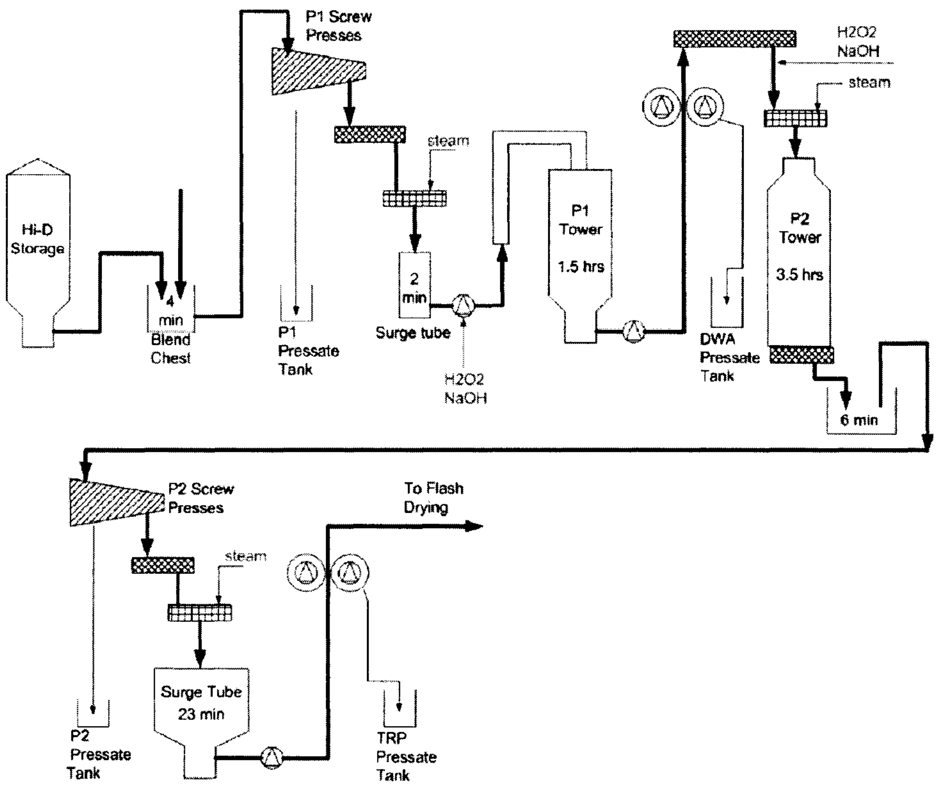

Figure 6.8: Millar Western bleaching process [19]

in Figure 6.8 is studied. This system is used for an industrial bleaching process at the Millar Western, Alberta. Pulp bleaching is a chemical process applied to cellulose materials to increase their brightness and also to increase the capacity of paper to accept printed or written images, thus increasing its usefulness. The bleaching process at Millar Western uses hydrogen peroxide as a bleaching agent. In this process, the cleaned and filtered pulp is squeezed in presses and heated before entering the bleaching tower $P1$, where the pulp sits for about one and half hour in a hydrogen peroxide bleach solution. The resulting semi-bleached pulp is de-watered in another press and additional hydrogen peroxide is added in a mixer. This stage takes about three and half to five hours. The pulp is washed and pressed to extract a bleach solution [19]. There are nine different inputs consisting of the peroxide and caustic add rate at $P1$ and $P2$, the $P2$ discharge temperature and correction, the $Na2SO3$ and caustic add on chips, and the PQM freeness. Among the outputs, we use the most important one, which is the brightness, to show the effectiveness of the proposed method. Information for the inputs is ready every ten minutes, while the output is sampled completely irregularly. To overcome the problem, first-order interpolation is used to estimate the output in between the samples and then re-sample it to obtain a regularly sampled output. Moreover, as the inputs are either constant or have low changing rates, the inputs are re-sampled to obtain a lower input-output rate ratio. Using this method, a 9-input, one-output system is obtained with an input updating period of 50 minutes and an output sampling period of 100 minutes. As the inputs have either no or slow changes, and the system is in almost steady state conditions, not enough input excitation is available to find a dynamic model of the system. Therefore, only a steady state model (zero order model) is extracted. The data from April 1st to June 17, 2001 are used for estimation, and data from July 5 to July 30, 2001 for validation. Figures 6.9 and 6.10 show the results for the real output and the estimated one for the two different time intervals.

As we do not have the real model of the system, we cannot use the PEE as defined earlier as an evaluation benchmark. To evaluate the model quality, the Cross-correlation Coefficient (CC) and Mean Squared Error (MSE) are used. Both of the benchmarks are based on the measured output (y) and the estimated one ($\hat{y}$) and are defined as follows:

$$CC(y, \hat{y}) = \frac{cov(y, \hat{y})}{\sqrt{var(y)var(\hat{y})}} \,, \tag{6.51}$$

$$MSE(y, \hat{y}) = \sqrt{\frac{\sum_{i=1}^{N} (y(i) - \hat{y}(i))^2}{N}} \,. \tag{6.52}$$

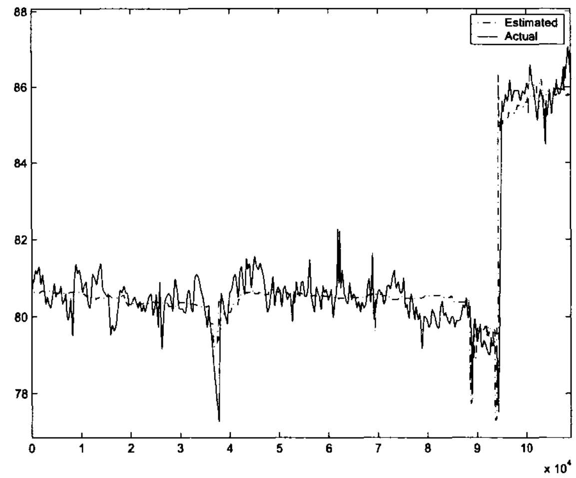

Figure 6.9: Actual output and estimated one for estimation interval

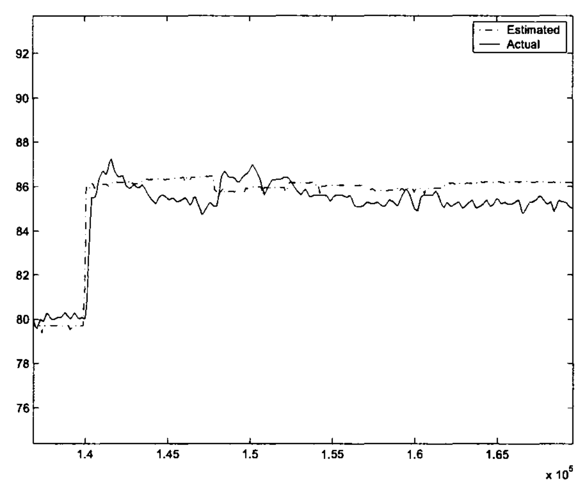

Figure 6.10: Actual output and estimated one for validation interval

Using these benchmarks, we get $CC = 0.9464$ and $MSE = 0.9484$ for the data in the evaluation time interval. Given the difficulties with the problem (irregularity in output samples and lack of input excitation), we conclude that the model obtained works well.

6.5 Conclusion

In this chapter, parameter estimation methods for general multi-input, multi-output multi-rate systems in the frequency domain were studied. Two methods for dealing with multi-rate systems were proposed, and a multi-rate least squares estimation was derived. Simulation examples showed the applicability of the proposed methods to both SISO and MIMO systems.

Chapter 7

Conclusions and Future Work

In this thesis, some basic problems in filtering and estimation in networked control systems and multi-rate systems have been studied. This chapter summarizes the main contributions of this work and proposes some future research directions to expand the present study and to apply the results to other related problems.

7.1 Conclusions

In this thesis, the problems of filtering and estimation in modern control systems were investigated. The sensor delay systems, networked control systems, systems with uncertain observation, and multi-rate systems were considered.

The problem of optimal $\mathcal{H}_2$ filtering for discrete-time systems with random sensor delay, multiple sensor data packet dropout, uncertain observation, or networked systems with multiple packet dropout was studied. To find the filter gains, the stochastic variables arising from the random sensor delay, multiple packet dropout or uncertainty in observation were transformed into the stochastic parameters in the system representation. New formulations were employed to model the multiple packet dropout in sensor data and networked control systems. A stochastic definition of the $\mathcal{H}_2$-norm of a system with stochastic parameters was given and new relations for the stochastic $\mathcal{H}_2$-norm were derived. The stochastic $\mathcal{H}_2$-norm of the estimation error was used as a criterion for the filter design. The relations derived for the new norm definition were used to obtain a set of linear matrix inequalities (LMIs) to solve the filter design problem.

As an alternative to the $\mathcal{H}_2$-norm filter design, the problem of $\mathcal{H}_\infty$ filtering in networked control systems with multiple packet dropouts was studied. Again, by employing the new formulation, random dropout rates were transformed into stochastic parameters in the system's representation. A generalized $\mathcal{H}_\infty$-norm for systems with stochastic parameters and both stochastic and deterministic inputs was derived and the stochastic $\mathcal{H}_\infty$-norm of the filtering error was used as a criterion for the filter design in a NCS framework. A set of linear matrix inequalities (LMIs) was provided to solve the corresponding filter design problem.

Another problem arising in computer control systems is the problem of multi-rate sampling. The problem of optimal Kalman filtering for multi-rate processes was also studied in this thesis. A *state lifting* method was introduced and used to generalize the minimum variance Kalman filtering method to the multi-rate case for the fast-rate state estimation. The optimal Kalman gains and covariance matrices were found at the fast rate, based on multi-rate input-output data and fast-rate system models.

Also, the parameter estimation problem of a general multi-input, multi-output multi-rate system in the frequency domain was studied. Two methods, *dividing to subsystems* and *input extension*, were introduced for dealing with multi-rate systems, and the later method was used to convert a multi-input, multi-output multi-rate system into several sub-problems with fast input updating and slow output sampling. Then, a least-squares parameter estimation method was generalized for parameter estimation in the multi-input, multi-output multi-rate case.

Several examples, including one with real industrial data, were used to show the effectiveness and applicability of the proposed methods.

7.2 Extensions and Future Work

The use of computers and communication networks in control systems is growing extensively, but many open problems remain. Based on the research discussed in this thesis, some directions for future research are suggested:

- In this thesis, the problem of filtering with sensor delay was studied. The formulation used to model the delay can handle at most one sampling delay. The problem was also studied for the multiple packet dropout case. A closely related problem is that of multiple delay in both sensor delay and networked control systems. Consider the following system

$$\begin{cases} \tilde{x}_{k+1} & = \mathbf{a}\tilde{x}_k + \mathbf{b}\tilde{\omega}_k, \\ \tilde{y}_k & = \mathbf{c}\tilde{x}_k + \mathbf{d}\tilde{\omega}_k, \end{cases} \tag{7.1}$$

  with the same definitions for the states, inputs and outputs and system parameters as in Chapter 2. The single delay model used for the sensor delay system is as follows:

$$y_k = \delta_k \tilde{y}_k + (1 - \delta_k)\tilde{y}_{k-1}, \tag{7.2}$$

  where δ_k is a Bernoulli distributed white sequence taking the values of 0 or 1 with

$$\text{prob}\{\delta_k = 1\} = \mathcal{E}\{\delta_k\} = \alpha, \quad 0 \leq \alpha \leq 1, \tag{7.3}$$

  where α is a known constant. A direct generalization to the two-sampling delay case may be considered as follows:

$$y_k = \delta_{1k}\tilde{y}_k + \bar{\delta}_{1k}(\delta_{2k}\tilde{y}_{k-1} + \bar{\delta}_{2k}\tilde{y}_{k-2}). \tag{7.4}$$

Here $\bar{\delta} = 1 - \delta$. The difficulty in using the relation in (7.4) is that it cannot guarantee that the delay will not jump from no-delay to a two-sampling delay: if $\delta_{1k} = 1$, then $y_k = \tilde{y}_k$. Now, if $\delta_{1,k+1} = 0$ and $\delta_{2,k+1} = 0$, then $y_{k+1} = \tilde{y}_{k-1}$, while we already know the information of $\tilde{y}_k$. To overcome the problem, the following relation can be used:

$$y_k = \delta_{1k}\tilde{y}_k + \bar{\delta}_{1k}(\delta_{1,k-1} + \bar{\delta}_{1,k-1}\delta_{2k})\tilde{y}_{k-1} + \bar{\delta}_{1k}\bar{\delta}_{2k}\bar{\delta}_{1,k-1}\tilde{y}_{k-2}. \tag{7.5}$$

In a more general case, the following relation can model the multiple delays for up to M sampling delays:

$$\begin{aligned} y_k = \delta_{1k}\tilde{y}_k + \sum_{l=0}^{M-1}\prod_{j=1}^{l}\prod_{i=0}^{l-j}\bar{\delta}_{j,k-i}(\delta_{1,k-l} + \bar{\delta}_{1,k-l}\delta_{l+1,k})\tilde{y}_{k-l} + \\ + \prod_{j=1}^{M}\prod_{i=0}^{M-j}\bar{\delta}_{j,k-i}\tilde{y}_{k-M}. \end{aligned} \tag{7.6}$$

The delayed states can be easily augmented to get a compact stochastic parameter state-space form for the multiple delay systems. However, even for a small delay size (say, 3), a complicated formulation will result. More investigation is needed to find a more useful delay modelling for the multiple delay case.

- The most important use of estimated states is in control applications. The proposed methods in this thesis were given to estimate the system states and variables. It will be very interesting to use the estimated variables in a control action, and study related closed-loop issues.

- A very interesting problem is that of identification with randomly sampled or uncertain information. The situation of randomly sampled data is common in industry, especially when the results are coming from manually sampled outputs or are analytical results. A similar scenario exists in networked control systems where the random delays and packet dropouts occur due to limited channel bandwidth. The stochastic $\mathcal{H}_2$-norm formulation may be used as an entry to solving this problem. A stochastic least squares method will be more beneficial. To the best of our knowledge, this problem has not been solved yet.

- In this thesis, the problems of estimation in NCS and in multi-rate systems have been studied. An interesting problem would be to consider the multi-rate sampling in networked systems. This sampling scheme could be used to decrease the total amount of information passed through the communication channel. This scenario will lead to less channel traffic and a lower delay and packet dropout rate would be achieved.

- One of the direct uses of state filtering and estimation is system monitoring. Estimation in the sensor delay and sensor packet dropout cases can be regarded as a monitoring system. A more interesting case would be the case in which both inputs and outputs are passed through communication channels for monitoring purposes.

- The stochastic $\mathcal{H}_\infty$-norm with both stochastic and deterministic inputs was studied in this thesis. Only the sufficiency requirement was given in the proof. Proof of necessity would be beneficial.

- In the filtering and estimation study in this thesis, the stochastic delays or dropouts were considered while the parameters of the plant were assumed to be known without any uncertainty. Adding uncertainty to the plant parameters makes the problems more realistic, but more challenging.

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