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SHAPE PRESERVING APPROXIMATION

BY

KIRILL A. KOPOTUN ©

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ABSTRACT

During the last few decades, there have been many developments in the areas of Polynomial and Spline Approximation, and the active study continues with unremitting interest. At the same time, there are still quite a few problems which can not be resolved even though they are well known to the specialists in the field.

The thesis focuses on some of such problems in the area of Shape Preserving (Constrained) Approximation of real functions on finite intervals. In many applications, it is desirable that the mathematical model preserve certain geometric properties of the data (such as monotonicity and convexity). These are the subjects dealt with by Constrained Approximation.

The estimates of different rates of approximation obtained in the thesis are given in terms of the usual and the Ditzian-Totik moduli of smoothness (both of which, in a sense, "measure smoothness" of the functions being approximated). In particular, one of the main results in the thesis is the closure of a gap that was open for more than 10 years. Namely, we estimate the degree of approximation of convex functions by convex polynomials in terms of the third moduli of smoothness. The weaker estimates in terms of the moduli of order two were obtained by D. Leviatan in 1986. On the other hand, it was shown by A. S. Shvedov in 1981 that the estimates of this type involving the fourth moduli are no longer correct. Thus, the obtained estimate is exact in a certain sense.

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Contents

CHAPTER 1

INTRODUCTION

1.1 Preamble

In many applications, it is desirable that the mathematical model preserve certain geometric properties of the data (such as monotonicity and convexity). These are the subjects dealt with by Shape Preserving (or Constrained) Approximation. More precisely, Shape Preserving Approximation is the approximation of functions f for which the mth forward difference, given by

$$\bar{\Delta}_h^m(f, x, [a, b]) := \begin{cases} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f(x+ih), & \text{if } [x, x+mh] \subset [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

is nonnegative for given $m \in \mathcal{N}$, for all $0 < h \leq (b-a)/m$ and $x \in [a, b]$, by polynomials (or splines) with nonnegative mth derivatives. Let $\Delta^m [a, b]$ be the set of such functions f (note that if $f \in \mathbf{C}^m [a, b]$, then $f \in \Delta^m [a, b]$ if and only if $f^{(m)}(x) \geq 0$, $x \in [a, b]$).

The rates of the best nth degree unconstrained and shape preserving polynomial approximation of a function f are defined by

$$E_n(f) = \inf_{p_n \in \mathbf{\Pi}_n} \|f - p_n\|_\infty$$

and

$$E_n^{(m)}(f) = \inf_{p_n \in \mathbf{\Pi}_n \cap \Delta^m} \|f - p_n\|_\infty, \qquad m \in \mathcal{N},$$

respectively, where $\mathbf{\Pi}_n$ denotes the set of algebraic polynomials of degree $\leq n$.

It is very well known that, for every $m \in \mathcal{N}$, the set $\bigcup_{n \in \mathcal{N}} \mathbf{\Pi}_n \cap \Delta^m$ is dense in Δ^m, *i.e.*, $E_n^{(m)}(f) \to 0$ as $n \to \infty$. This immediately follows, for example, from the fact that, $\|B_n(f, x) - f\| \to 0$ for any $f \in \mathbf{C}[0, 1]$, and $B_n^{(m)}(f, x) \geq 0$ if $f \in \mathbf{C}[0, 1] \cap \Delta^m$, where $B_n(f, x)$ denotes the Bernstein polynomial of degree $\leq n$:

$$B_n(f, x) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}.$$

At the same time, in 1969, G. G. Lorentz and K. L. Zeller [6] showed that there exists a function $f \in \Delta^m$ such that

$$\limsup_{n \to \infty} \frac{E_n^{(m)}(f)}{E_n(f)} = \infty.$$

Thus, the problems on Shape Preserving Approximation are not trivial consequences of those on Unconstrained Approximation, and require a special treatment.

In this thesis we investigate how well the "m-monotone" functions can be approximated by the "m-monotone" polynomials. We use the so called Jackson type estimates in terms of the usual and the Ditzian–Totik moduli of smoothness. The usual modulus of smoothness in the uniform metric is defined by

$$\omega^k(f, t, [a, b]) := \omega^k(f, t, [a, b])_\infty := \sup_{0<h\leq t} \|\bar{\Delta}_h^k(f, \cdot, [a, b])\|_{C[a,b]} .$$

The modulus of smoothness ω_φ^k introduced and used extensively by Z. Ditzian and V. Totik [2] is given by

$$\omega_\varphi^k(f, t)_p = \sup_{0<h\leq t} \|\Delta_{h\varphi(\cdot)}^k(f, \cdot)\|_p ,$$

where

$$\Delta_h^k(f, x) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + kh/2 - ih), & \text{if } x \pm kh/2 \in I, \\ 0, & \text{otherwise} \end{cases}$$

($\Delta_h^k(f, x)$ is the kth symmetric difference). It is obvious that under the proper conditions on the function $\varphi = \varphi(x)$ (for example, for $\varphi(x) = \sqrt{1 - x^2}$) the step of the difference is decreasing near the endpoints of $[-1, 1]$. So uniform estimates in terms of the Ditzian–Totik modulus of smoothness are more exact than the usual ones (see [2]).

We are now ready to give a brief outline of the thesis.

1.2 Outline of the thesis

The second chapter of the thesis closes a gap that was open for more than 10 years. Namely, it was shown by D. Leviatan [5] in 1986 that it is possible to estimate the rate of convex approximation in terms of the second usual modulus of smoothness, *i.e.*, for a convex function $f \in C(I)$ there exists a sequence of polynomials p_n of degree $\leq n$ such that

$$(1) \qquad |f(x) - p_n(x)| \leq C\omega^2(f, \Delta_n(x)), \qquad x \in I,$$

where $\Delta_n(x) := n^{-1}\sqrt{1 - x^2} + n^{-2}$. It follows from A. S. Shvedov [9] that ω^2 in (1) can not be replaced by ω^4. We show in the second chapter that not only $\omega^2(f, \Delta_n)$ can be replaced by $\omega^3(f, \Delta_n)$, but also the estimate in terms of $\omega_\varphi^3(f, 1/n)$ holds true. This allows constructive characterization of approximation properties of convex functions in terms of their moduli of smoothness (involving both pointwise and uniform estimates). We also obtain an estimate for the convex polynomial approximation in $\mathbf{L}_p$, $1 \le p < \infty$, metric in terms of $\tau^3(f, 1/n)_p$ (the third Sendov-Popov τ modulus in $\mathbf{L}_p$).

In the third chapter, we obtain uniform estimates for monotone and convex approximations in terms of the weighted Ditzian-Totik moduli of smoothness. Together with known results they complement the investigation of the rate of shape preserving approximation in the sense of the orders of these moduli.

In the fourth chapter, we prove a theorem on coconvex polynomial approximation which is the first result on this subject without any extra restrictions on the function being approximated. The only previously known direct results on coconvex approximation are due to R. K. Beatson and D. Leviatan [1] (who remarked that it is possible to obtain Jackson type theorems for functions with only one inflection point) and X. M. Yu [10] (who obtained a Jackson type estimate for a function with one *regular convexity-turning point*, and also proved a result on approximation of (at least three times) differentiable functions with some extra conditions on convexity-turning points). We prove that if $f \in \mathbf{C}^2[-1,1]$ has finitely many inflection points, then, for sufficiently large n, there exists a polynomial p_n of degree $\le n$ satisfying $f''(x)p_n''(x) \ge 0, x \in [-1,1]$ and such that

$$\|f - p_n\|_\infty \le C\, n^{-2}\, \omega_\varphi(f'', n^{-1})\,.$$

Finally, in the fifth chapter, we obtain a direct estimate for copositive polynomial approximation in terms of ω_φ^3, the third Ditzian-Totik modulus of smoothness. Namely, for a function $f \in C[-1,1]$ with finitely many sign changes we construct a sequence of polynomials p_n which are copositive with f (*i.e.*, $f(x)p_n(x) \ge 0$, $-1 \le x \le 1$) and such that

$$\|f - p_n\|_\infty \le C\omega_\varphi^3(f, n^{-1})\,. \tag{2}$$

It is known (see S. P. Zhou [11]) that ω_φ^3 can be replaced neither by ω_φ^4, nor even by ω^4. Thus, the above estimate is exact in a certain sense, and completes the investigation of this type of constrained approximation in the uniform metric. Also, together with converse theorems in terms of the Ditzian-Totik moduli the estimate (2) immediately implies the equivalence between $E_n(f) = O(n^{-\alpha})$ and $E_n^{(0)}(f,r) = O(n^{-\alpha})$ in the case $0 < \alpha < 3$.

1.3 Definitions and Notation

The following notation (see [4] and [7]–[8]) is used throughout the thesis:

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}, \qquad x \in I := [-1, 1];$$

$$x_j := x_{j,n} := \cos\frac{j\pi}{n},\ 0 \le j \le n\ ;$$

$$\bar{x}_j := \bar{x}_{j,n} := \cos\left(\frac{j\pi}{n} - \frac{\pi}{2n}\right),\ 1 \le j \le n\ ;$$

$$x_j^0 := x_{j,n}^0 := \cos\left(\frac{j\pi}{n} - \frac{\pi}{4n}\right) \text{ if } j < n/2,$$

$$x_j^0 := x_{j,n}^0 := \cos\left(\frac{j\pi}{n} - \frac{3\pi}{4n}\right) \text{ if } j \ge n/2\ ;$$

$$I_j := I_{j,n} := [x_j, x_{j-1}],\ h_j := h_{j,n} := x_{j-1} - x_j,\ 1 \le j \le n$$

(note that $h_{j\pm1} < 3h_j$ and $\Delta_n(x) < h_j < 5\Delta_n(x)$ for $x \in I_j$.)

Also,

$$t_j(x) := t_{j,n}(x) := (x - x_j^0)^{-2}\cos^2 2n \arccos x + (x - \bar{x}_j)^{-2}\sin^2 2n \arccos x$$

is an algebraic polynomial of degree $4n-2$ (see [3] and [7]).

Let

$$\Pi_j(n, \mu; a_1, \dots, a_m; b_1, \dots, b_k) := \int_{-1}^{1} t_j(y)^\mu \prod_{i=1}^{m}(y - a_i) \prod_{i=1}^{k}(b_i - y)\, dy\ ,$$

then for $a_i \le x_j$, $1 \le i \le m$, $b_i \ge x_{j-1}$, $1 \le i \le k$ and sufficiently large μ

$$T_j(n, \mu; a_1, \dots, a_m; b_1, \dots, b_k)(x) := \frac{\int_{-1}^{x} t_j(y)^\mu \prod_{i=1}^{m}(y - a_i) \prod_{i=1}^{k}(b_i - y)\, dy}{\Pi_j(n, \mu; a_1, \dots, a_m; b_1, \dots, b_k)}$$

is an algebraic polynomial of degree $2\mu(2n-1) + m + k + 1$, which is well defined because $\Pi_j(n, \mu; a_1, \dots, a_m; b_1, \dots, b_k)$ is positive for large μ (see Proposition 10).

If $m = 0$ or $k = 0$, *i.e.*, if there are no a_i's or b_i's in the definition of T_j, we use the notation $T_j(n, \mu; \emptyset; b_1, \dots, b_k)(x)$ or $T_j(n, \mu; a_1, \dots, a_m; \emptyset)(x)$, respectively. Thus, for example,

$$T_j(n, \mu; \emptyset; b_1, \dots, b_k)(x) := \frac{\int_{-1}^{x} t_j(y)^\mu \prod_{i=1}^{k}(b_i - y)\, dy}{\int_{-1}^{1} t_j(y)^\mu \prod_{i=1}^{k}(b_i - y)\, dy}.$$

For brevity, we also denote

$$\psi_j := \psi_{j,n} := \psi_{j,n}(x) := \frac{h_j}{|x - x_j| + h_j},\quad \chi[a, b](x) := \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

and $\chi_j(x) := \chi[x_j, 1](x)$.

Also,

$$\operatorname{sgn}(f(x)) := \begin{cases} -1 & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) > 0. \end{cases} \qquad \text{and} \qquad \operatorname{sgn}_\alpha(x) := \operatorname{sgn}(x - \alpha)\ .$$

$L(x, f; a_0, \ldots, a_k)$ denotes the Lagrange polynomial, of degree $\leq k$, which interpolates a function f at the points $a_0, \ldots, a_k$. We also denote

$$L(x, f; x_i) := L(x, f; x_i, x_{i-1}, x_{i-2})$$

and

$$\tilde{L}(x, f; [a, b]) := L\left(x, f; a, (a+b)/2, b\right) .$$

C represents positive constants which are not necessarily the same even when they occur in the same line. In order to emphasize that C depends only on the parameters $\nu_1, \ldots, \nu_k$ we use the notation $C(\nu_1, \ldots, \nu_k)$. At the same time, $A(\mu)$ and $C_0(\mu)$ denote constants which depend only on μ and remain fixed in the proofs.

References

[1] R. K. Beatson and D. Leviatan (1983), *On comonotone approximation*, Canad. Math.Bull., **26**: 220–224.

[2] Z. Ditzian and V. Totik (1987): Moduli of Smoothness. Berlin: Springer-Verlag.

[3] V. K. Dzjadyk (1977): Introduction to the Theory of Uniform Approximation of Functions by Polynomials. Moscow: Izdat. "Nauka".

[4] K. A. Kopotun (1992), *Uniform estimates of convex approximation of functions by polynomials*, Mat. Zametki, **51**(3): 35–46. (English translation: Math. Notes, **51**: 245–254.)

[5] D. Leviatan (1986), *Pointwise estimates for convex polynomial approximation*, Proc. Amer. Math. Soc., **98**(3): 471–474.

[6] G. G. Lorentz and K. L. Zeller (1969), *Degree of approximation by monotone polynomials II*, JAT, **2**(3): 265–269.

[7] I. A. Shevchuk (1989), *On coapproximation of monotone functions*, Dokl. Akad. Naul SSSR, **308**(3): 537–541. (English translation: Soviet. Math. Dokl., **40**(2): 349–354.)

[8] I. A. Shevchuk (1992), Approximation by Polynomials and Traces of the Functions Continuous on an Interval, Kiev: Naukova Dumka.

[9] A. S. Shvedov (1981), *Orders of coapproximation of functions by algebraic polynomials*, Mat. Zametki, **29**(1): 117–130. (English translation: Math. Notes, **30**: 63–70.)

[10] X. M. Yu (1990), *Coconvex approximation of continuous functions*, Acta Math. Sinica, **6**(3): 220–229.

[11] S. P. Zhou (1992), *A counterexample in copositive approximation*, Israel J. Math., **78**, 75–83.

CHAPTER 2

POINTWISE AND UNIFORM ESTIMATES FOR CONVEX APPROXIMATION OF FUNCTIONS BY ALGEBRAIC POLYNOMIALS [1]

2.1 Introduction

Let $I := [-1, 1]$, $\Delta^m := \Delta^m(I)$, $\bar{\Delta}_h^m(f, x) := \bar{\Delta}_h^m(f, x, I)$. For $k \in \mathcal{N}$ we denote by H_k^ϕ the class of all functions $f \in C(I)$ whose kth modulus of smoothness does not exceed the k-majorant $\phi = \phi(t)$ (*i.e.*, ϕ is a continuous and nondecreasing function such that $\phi(0) = 0$, and $t^{-k}\phi(t)$ is nonincreasing), that is,

$$\omega^k(f, t) := \omega^k(f, t, I) \le \phi(t)\,.$$

Also, let $W^r H_k^\phi := \{f : f^{(r)} \in H_k^\phi\}$.

In the monotone case (*i.e.*, when the shape preserving approximation of functions from Δ^1 is considered) the following analog of direct theorems for unconstrained polynomial approximation is known.

Theorem A *Let $k \in \mathcal{N}$ if $r \in \mathcal{N}$, and $k = 1$ or 2 if $r = 0$. Then, for $f \in W^r H_k^\phi \cap \Delta^1$ and an arbitrary $n \in \mathcal{N}$, $n \ge k + r - 1$, a polynomial $p_n \in \Pi_n \cap \Delta^1$ satisfying*

$$|f(x) - p_n(x)| \le C\Delta_n(x)^r \phi(\Delta_n(x)), \qquad C = C(k, r),\ x \in I, \tag{3}$$

exists.

An immediate consequence of A. S. Shvedov [12] is the fact that the statement of Theorem A is not correct for $r = 0$, $k \ge 3$. For $r = 0$, $k = 1$ or 2 Theorem A is a consequence of the work of R. A. DeVore and X. M. Yu [1] who constructed the sequence of polynomials $p_n \in \Pi_n \cap \Delta^1$ which approximate a function $f \in C(I) \cap \Delta^1$ so that

$$|f(x) - p_n(x)| \le C\omega^2(f, n^{-1}\sqrt{1 - x^2}), \qquad x \in I. \tag{4}$$

For $r \in \mathcal{N}$, $k \in \mathcal{N}$ Theorem A was proved by I. A. Shevchuk [9], [10].

For convex approximation the following result is known.

[1] A version of this chapter has been published in Constr. Approx. (1994) 10: 153–178.

Theorem B *Let $k \in \mathcal{N}$ if $r \geq 2$, $k = 1$ if $r = 1$, and $k = 1$ or 2 if $r = 0$. Then, for $f \in W^r H_k^\varphi \cap \Delta^2$ and an arbitrary $n \in \mathcal{N}$, $n \geq k + r - 1$, a polynomial $p_n \in \mathbf{\Pi}_n \cap \Delta^2$ satisfying (3) exists.*

A. S. Shvedov showed in [12] (see also [11]) that the statement of Theorem B is not correct for $r = 0$, $k \geq 4$ and $r = 1$, $k \geq 3$. For $r = 0$, $k = 1$ or 2, and $r = k = 1$ Theorem B is a consequence of D. Leviatan [6], where the estimate (4) was obtained for convex approximation. For $r \geq 2$, $k \in \mathcal{N}$ Theorem B was proved by S. P. Manya and I. A. Shevchuk (see [11], for example). There is a gap in Theorem B as nothing is known for $r = 0$, $k = 3$, and $r = 1$, $k = 2$.

Let us write a number $\alpha > 0$ as the sum $\alpha = r + \beta$ where r is a nonnegative integer and $0 < \beta \leq 1$. Denote by $\mathrm{Lip}^*\alpha$ the class of all functions $f(x)$ on I such that

$$\omega^2(f^{(r)}, t) = O(t^\beta)\,.$$

A concequence of Theorem B and also classical converse theorems (see, for example, p. 263 of [3]) is

Theorem C *For $\alpha > 0$, $\alpha \neq 2$, a function $f = f(x)$ is convex on I and belongs to $\mathrm{Lip}^*\alpha$ if and only if, for each $n \geq r + 1$, a convex polynomial on I, $p_n = p_n(x)$ of degree $\leq n$, exists such that*

$$|f(x) - p_n(x)| \leq C\Delta_n(x)^\alpha, \qquad x \in I\,. \tag{5}$$

For $\alpha = 2$ the result of Theorem C is not complete as this case corresponds to $r = 1$, $k = 2$ in Theorem B.

In this chapter it is shown that Theorem B is correct for $r = 0$, $k = 3$ (and, therefore, for $r = 1$, $k = 2$), and hence Theorem C is correct for $\alpha = 2$. Namely, they are consequences of the following theorem.

Theorem 1 *For a convex function $f \in \mathbf{C}(I)$ and every $n \geq 2$ a convex polynomial $p_n = p_n(x)$ of degree $\leq n$ exists such that*

$$|f(x) - p_n(x)| \leq C\omega^3(f, \Delta_n(x)), \qquad x \in I. \tag{6}$$

If $f \in \mathbf{C}^1(I)$, then the following estimate also holds:

$$|f'(x) - p_n'(x)| \leq C\omega^2(f', \Delta_n(x)), \qquad x \in I. \tag{7}$$

Moreover, for $f \in \mathbf{C}^2(I)$ there is also the following estimate:

$$|f''(x) - p_n''(x)| \leq C\omega(f'', \Delta_n(x)), \qquad x \in I. \tag{8}$$

Corollary 2 *If $f \in \mathbf{C}^1(I) \cap \Delta^2$, then, for every $n \geq 2$, a polynomial $p_n \in \mathbf{\Pi}_n \cap \Delta^2$ exists such that*

$$|f(x) - p_n(x)| \leq C\Delta_n(x)\omega^2(f', \Delta_n(x)), \qquad x \in I. \tag{9}$$

Remark . Estimate (6) can be improved to some degree (see the method in [11], for example). Namely,

$$|f(x) - p_n(x)| \tag{10}$$

$$\le C \begin{cases} \omega^3(f, n^{-1}\sqrt{1-x^2}), & x \in [-1+n^{-2},\, 1-n^{-2}], \\ \omega^3(f, n^{-4/3}(1-x^2)^{1/3}), & x \in [-1,\, -1+n^{-2}) \cup (1-n^{-2},\, 1]. \end{cases}$$

All the estimates above are pointwise. The uniform estimates in terms of the usual moduli of smoothness are rather imperfect because, as can be seen from inequalities (3)–(10) the degree of approximation improves as the endpoints of the interval I are approached.

Theorem 3 *For a function $f \in \mathbf{C}(I) \cap \Delta^2$ and every $n \ge 2$ a polynomial $p_n = p_n(x) \in \mathbf{C}(I) \cap \Delta^2$ exists such that*

$$\|f - p_n\|_\infty \le C\bar{\omega}^3_\varphi(f, n^{-1}). \tag{11}$$

If $f \in \mathbf{C}^1(I)$, then the following estimate also holds:

$$\|f' - p'_n\|_\infty \le C\bar{\omega}^2_\varphi(f', n^{-1}). \tag{12}$$

Moreover, for $f \in \mathbf{C}^2(I)$ there is also the following estimate:

$$\|f'' - p''_n\|_\infty \le C\bar{\omega}_\varphi(f'', n^{-1}). \tag{13}$$

Theorem 3 improves the estimate of convex approximation

$$\|f - p_n\|_\infty \le C\bar{\omega}^2_\varphi(f, n^{-1}), \tag{14}$$

which was obtained by D. Leviatan [6].

Corollary 4 *If $f \in \mathbf{C}^1(I) \cap \Delta^2$, then, for every $n \ge 2$, a polynomial $p_n \in \mathbf{\Pi}_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_\infty \le Cn^{-1}\bar{\omega}^2_\varphi(f', n^{-1}). \tag{15}$$

Let us recall that the kth integral modulus of $f \in \mathbf{L}_p[-1,\, 1]$, $1 \le p \le \infty$, is the function

$$\omega^k(f, t)_p := \sup_{0<h\le t} \left\{ \frac{1}{2} \int_{-1}^{1-kh} |\bar{\Delta}^k_h(f, x)|^p \, dx \right\}^{1/p}, \qquad t \in [0,\, 2k^{-1}].$$

For a function f bounded on $[-1,\, 1]$ the local modulus of smoothness of order k at the point $x \in [-1,\, 1]$ is the function (see Definition 1.4 of [8])

$$\omega^k(f, x, \delta) := \sup \left\{ |\bar{\Delta}^k_h(f, t)| : t, t + kh \in [x - k\delta/2,\, x + k\delta/2] \right\}.$$

The kth averaged Sendov–Popov modulus of smoothness of a function f bounded and measurable on $[-1, 1]$ is (see Definition 1.5 of [8])

$$\tau^k(f,\delta)_p := \|\omega^k(f,\cdot,\delta)\|_p = \left\{\frac{1}{2}\int_{-1}^{1} \omega^k(f,x,\delta)^p\,dx\right\}^{1/p}, \qquad \delta \in [0,\, 2k^{-1}].$$

The following properties of τ^k are used (see Theorems 1.4 and 1.5 of [8]):

$$\omega^k(f,\delta)_p \le \tau^k(f,\delta)_p \le \omega^k(f,\delta). \tag{16}$$

A constant $C(k)$ depending only on $k \ge 2$ exists such that, for each function f absolutely continuous on $[a,\, b]$, the following inequality holds:

$$\tau^k(f,\delta)_p \le C(k)\delta\omega^{k-1}(f',\delta)_p. \tag{17}$$

In this paper the following theorem is proved.

Theorem 5 *Let $1 \le p \le \infty$. For a function $f \in \mathbf{C}(I) \cap \Delta^2$ and every $n \ge 2$ a polynomial $p_n = p_n(x) \in \mathbf{C}(I) \cap \Delta^2$ exists such that*

$$\|f - p_n\|_p \le C\tau^3(f, n^{-1})_p. \tag{18}$$

If $f \in \mathbf{C}^1(I)$, then the following estimate also holds:

$$\|f' - p_n'\|_p \le C\tau^2(f', n^{-1})_p. \tag{19}$$

Moreover, for $f \in \mathbf{C}^2(I)$ there is also the following estimate:

$$\|f'' - p_n''\|_p \le C\tau(f'', n^{-1})_p. \tag{20}$$

Corollary 6 *By (17) and (18), for $f \in \mathbf{C}^1(I) \cap \Delta^2$ and every $n \ge 2$, a polynomial $p_n \in \mathbf{\Pi}_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_p \le Cn^{-1}\omega^2(f', n^{-1})_p, \qquad 1 \le p < \infty. \tag{21}$$

By the method of [12], using (17) and also estimate (18), it is easy to prove:

Theorem 7 *For a function $f \in \mathbf{C}^2(I) \cap \Delta^3$ and every $n \ge 2$ a polynomial $p_n \in \mathbf{\Pi}_n \cap \Delta^3$ exists such that*

$$\|f - p_n\|_\infty \le Cn^{-1}\omega^2(f', n^{-1})_\infty. \tag{22}$$

2.2 Auxiliary Results

Proposition 8 *(see [9] and [10], for example). The following inequalities hold:*

(23)
$$\min\left\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\right\} \le t_j(x)$$
$$\le \max\left\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\right\}, \qquad x \in I,$$

(24)
$$t_j(x) \le 10^3 h_j^{-2}, \qquad x \in I_j,$$

(25)
$$x_j^0 - x_j > \frac{\bar{x}_j - x_j}{2} > \frac{1}{4}h_j,\ x_{j-1} - \bar{x}_j > \frac{1}{4}h_j,\ \bar{x}_j - x_j^0 \le \frac{3}{8}h_j \text{ for } j \le \frac{n}{2},$$

(26)
$$x_{j-1} - x_j^0 > \frac{x_{j-1} - \bar{x}_j}{2} > \frac{1}{4}h_j,\ \bar{x}_j - x_j > \frac{1}{4}h_j,\ x_j^0 - \bar{x}_j \le \frac{3}{8}h_j \text{ for } j > \frac{n}{2},$$

(27)
$$\max\left\{(x-x_j^0)^{-2},(x-\bar{x}_j)^{-2}\right\} \le 64(|x-x_j|+h_j)^{-2}, \qquad x \notin I_j$$

and

(28)
$$(|x-x_j|+h_j)^{-2} \le t_j(x) \le 4\cdot 10^3\,(|x-x_j|+h_j)^{-2}, \qquad x \in I.$$

Proposition 8 can be verified by simple calculations using the definitions of the points x_j, x_j^0, $\bar{x}_j$ and properties of the trigonometric functions sin and cos.

The following proposition can be easily verified either by straightforward computations or by induction on ν.

Proposition 9 *Let ν be an integer, $p \ge \nu + 2$ and $c_i \ge t_0 > 0$, $1 \le i \le \nu$. Then the following estimates are valid:*

$$\frac{1}{p-1}\,c_1\dots c_\nu\,t_0^{1-p} \le \int_{t_0}^{\infty} t^{-p}\prod_{i=1}^{\nu}(c_i+t)\,dt \le \frac{2^\nu}{p-\nu-1}\,c_1\dots c_\nu\,t_0^{1-p}.$$

The following result permits us to define the polynomials

$$T_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k).$$

Proposition 10 *Let $1 \le j \le n$ be fixed. Then the following inequalities hold:*

$$C_0(\mu) \le \Pi_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k)\,h_j^{2\mu-1}\left(\prod_{i=1}^{m}(x_{j-1}-a_i)\prod_{i=1}^{k}(b_i-x_j)\right)^{-1}$$
$$\le C_0(\mu),$$

where $a_i \le x_j$, $1 \le i \le m$, $b_i \ge x_{j-1}$, $1 \le i \le k$ and μ is sufficiently large in comparison with k and m (for example, $\mu \ge 5(m+k+1)$ will do).

Proof (cf. [10]). The proposition will be proved for $j \le n/2$. For $j > n/2$ the proof is analogous with the only difference that instead of (25) one should use (26).

We write

$$\Pi_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k) = \left\{ \int_{-1}^{x_j} + \int_{x_j}^{x_{j-1}} + \int_{x_{j-1}}^{1} \right\} = \Theta_1 + \Theta_2 + \Theta_3 \,.$$

Now denoting $\prod_{i=1}^m (x_{j-1}-a_i)\,\prod_{i=1}^k (b_i - x_j)$ by Γ_{jmk} and using estimates (23)-(25), we have

$$\Theta_2 \;\le\; \Gamma_{jmk}\,(x_{j-1}-x_j)\,10^{3\mu}\,h_j^{-2\mu} \;=\; 10^{3\mu}\Gamma_{jmk}h_j^{-2\mu+1}$$

and

$$\begin{aligned}
\Theta_2 \;&\ge\; \int_{x_j^0}^{\bar{x}_j} \prod_{i=1}^m (x_j^0 - a_i) \prod_{i=1}^k (b_i - \bar{x}_j)\, \min\left\{(y-x_j^0)^{-2\mu}, (y-\bar{x}_j)^{-2\mu}\right\} dy \\
&\ge\; 4^{-m-k}\Gamma_{jmk} \int_{x_j^0}^{\bar{x}_j} \min\left\{(y-x_j^0)^{-2\mu}, (y-\bar{x}_j)^{-2\mu}\right\} dy \\
&\ge\; \frac{2}{2\mu-1} 4^{-m-k}(2^{2\mu-1}-1)\Gamma_{jmk}\,(\bar{x}_j - x_j^0)^{-2\mu+1} \\
&\ge\; \frac{2}{2\mu-1} 4^{-m-k}(2^{2\mu-1}-1)\left(\frac{8}{3}\right)^{2\mu-1} \Gamma_{jmk}\,h_j^{-2\mu+1} \,.
\end{aligned}$$

Similarly, Proposition 9 and the inequalities (25) yield

$$\begin{aligned}
|\Theta_1| \;&\le\; \int_{-1}^{x_j} t_j(y)^{\mu} \prod_{i=1}^m |y-a_i| \prod_{i=1}^k (b_i - y)\, dy \\
&\le\; \int_{-1}^{x_j} (x_j^0 - y)^{-2\mu} \prod_{i=1}^m (|x_{j-1}-a_i| + x_j^0 - y) \prod_{i=1}^k (|b_i - x_j| + x_j^0 - y)\, dy \\
&\le\; \int_{x_j^0 - x_j}^{\infty} t^{-2\mu} \prod_{i=1}^m (|x_{j-1}-a_i| + t) \prod_{i=1}^k (|b_i - x_j| + t)\, dt \\
&\le\; \frac{2^{m+k}}{2\mu - m - k - 1}\,\Gamma_{jmk}\,(x_j^0 - x_j)^{-2\mu+1} \\
&\le\; 4^{2\mu-1}\,\frac{2^{m+k}}{2\mu - m - k - 1}\,\Gamma_{jmk}\,h_j^{-2\mu+1}
\end{aligned}$$

and

$$\begin{aligned}
|\Theta_3| \;&\le\; \int_{x_{j-1}}^{1} t_j(y)^{\mu} \prod_{i=1}^m (y-a_i) \prod_{i=1}^k |b_i - y|\, dy \\
&\le\; \int_{x_{j-1}}^{1} (y-\bar{x}_j)^{-2\mu} \prod_{i=1}^m (|x_{j-1}-a_i| + y - \bar{x}_j) \prod_{i=1}^k (|b_i - x_j| + y - \bar{x}_j)\, dy
\end{aligned}$$

$$\leq \int_{x_{j-1}-\bar{x}_j}^{\infty} t^{-2\mu} \prod_{i=1}^{m}(|x_{j-1}-a_i|+t) \prod_{i=1}^{k}(|b_i-x_j|+t)\,dt$$
$$\leq \frac{2^{m+k}}{2\mu-m-k-1}\,\Gamma_{jmk}\,(x_{j-1}-\bar{x}_j)^{-2\mu+1}$$
$$\leq 4^{2\mu-1}\,\frac{2^{m+k}}{2\mu-m-k-1}\,\Gamma_{jmk}\,h_j^{-2\mu+1}\,.$$

Hence,

$$\Pi_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k)\,h_j^{2\mu-1}\,(\Gamma_{jmk})^{-1}$$
$$\leq 10^{3\mu} + \frac{2^{4\mu+m+k-1}}{2\mu-m-k-1} \leq 10^{3\mu+1}\,.$$

Finally, the inequalities in the other direction are the following

$$\Pi_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k)\,h_j^{2\mu-1}\,(\Gamma_{jmk})^{-1}$$
$$\geq \frac{2}{2\mu-1}4^{-m-k}(2^{2\mu-1}-1)\left(\frac{8}{3}\right)^{2\mu-1} - \frac{2^{4\mu+m+k-1}}{2\mu-m-k-1} \geq \frac{1}{\mu}\,.$$

■

Lemma 11
Let $a_i \leq x_j, 1 \leq i \leq m,\ b_i \geq x_{j-1}, 1 \leq i \leq k$ and $1 \leq j \leq n$ be a fixed index. Then for the polynomial $T_j(x) := T_j(n,\mu;a_1,\dots,a_m;b_1,\dots,b_k)(x)$ of degree $\leq 4\mu n+m+k$ the following inequalities hold:

$$\left|T_j'(x)\right| \leq C(\mu)\,\psi_j^{2\mu-m-k}\,h_j^{-1}\,, \tag{29}$$

$$1-x_{j-1} \leq \int_{-1}^{1} T_j(y)\,dy \leq 1-x_j \tag{30}$$

and

$$|\chi_j(x)-T_j(x)| \leq C(\mu)\,\psi_j^{2\mu-m-k-1}\,, \tag{31}$$

where $\mu \geq 5(m+k+1)$, $x \in I$.

Proof Proposition 10 and the inequalities (28) imply for any $x \in I$

$$\left|T_j'(x)\right| \leq C(\mu)\,t_j(x)^{\mu}\,h_j^{2\mu-1} \prod_{i=1}^{m}\frac{|x-a_i|}{|x_{j-1}-a_i|} \prod_{i=1}^{k}\frac{|b_i-x|}{|b_i-x_j|}$$
$$\leq C(\mu)\,\psi_j^{2\mu}\,h_j^{-1} \prod_{i=1}^{m}\left(1+\psi_j^{-1}\frac{h_j}{|x_{j-1}-a_i|}\right) \prod_{i=1}^{k}\left(1+\psi_j^{-1}\frac{h_j}{|b_i-x_j|}\right)$$
$$\leq C(\mu)\,\psi_j^{2\mu}\,h_j^{-1}\left(1+\psi_j^{-1}\right)^{m+k} \leq C(\mu)\,\psi_j^{2\mu-m-k}\,h_j^{-1}\,,$$

which is the inequality (29).

To prove the inequality (31), first, we consider the case $x < x_j$. The estimate (29) implies

$$|\chi_j(x) - T_j(x)| = |T_j(x)| = \left|\int_{-1}^{x} T_j'(y)\,dy\right|$$

$$\leq C(\mu)\,h_j^{2\mu-m-k-1} \int_{-\infty}^{x} (x_j - y + h_j)^{-2\mu+m+k}\,dy \leq C(\mu)\,\psi_j^{2\mu-m-k-1}.$$

For $x \geq x_j$, similarly, we have

$$|\chi_j(x) - T_j(x)| = |1 - T_j(x)| = \left|\int_{x}^{1} T_j'(y)\,dy\right|$$

$$\leq C(\mu)\,h_j^{2\mu-m-k-1} \int_{x}^{\infty} (y - x_j + h_j)^{-2\mu+m+k}\,dy \leq C(\mu)\,\psi_j^{2\mu-m-k-1}.$$

Thus, the inequality (31) is also verified.

To prove the remaining inequality (30), denoting for brevity $\prod_{i=1}^{m}(y-a_i)\prod_{i=1}^{k}(b_i-y)$ by $\Gamma_j(y)$, and using integration by parts,we get the following identities:

$$\int_{-1}^{1} T_j(y)\,dy \leq 1 - x_j$$

$$\Longleftrightarrow \int_{-1}^{1}\int_{-1}^{x} t_j(y)^{\mu}\,\Gamma_j(y)\,dy\,dx \leq (1 - x_j)\int_{-1}^{1} t_j(y)^{\mu}\,\Gamma_j(y)\,dy$$

$$\Longleftrightarrow \int_{-1}^{1}(1-y)t_j(y)^{\mu}\,\Gamma_j(y)\,dy \leq (1 - x_j)\int_{-1}^{1} t_j(y)^{\mu}\,\Gamma_j(y)\,dy$$

$$\Longleftrightarrow \Pi_j(n,\mu;x_j,a_1,\ldots,a_m;b_1,\ldots,b_k) = \int_{-1}^{1}(y - x_j)t_j(y)^{\mu}\,\Gamma_j(y)\,dy \geq 0,$$

and, analogously,

$$\int_{-1}^{1} T_j(y)\,dy \geq 1 - x_{j-1}$$

$$\Longleftrightarrow \Pi_j(n,\mu;a_1,\ldots,a_m;x_{j-1},b_1,\ldots,b_k) = \int_{-1}^{1}(x_{j-1} - y)t_j(y)^{\mu}\,\Gamma_j(y)\,dy \geq 0.$$

Together with Proposition 10 this yields (30). ∎

Let

$$Q_j(x) := T_j(n,10;x_j;\emptyset),$$

$$\bar{Q}_j(x) := T_j(n,10;\emptyset;x_{j-1})$$

and

$$\tilde{Q}_j(x) := T_j(n,9;\emptyset;\emptyset).$$

The following lemma is a corollary of the above results.

Lemma 12 *The following inequalities hold for all $x \in I$:*

(32) $$0 \le \tilde{Q}_j'(x) \le C\psi_j^{18} h_j^{-1},$$

(33) $$|\chi_j(x) - \tilde{Q}_j(x)| \le C\psi_j^{17},$$

(34) $$1 - x_{j-1} \le \int_{-1}^{1} \tilde{Q}_j(y)\,dy \le 1 - x_j,$$

(35) $$|Q_j'(x)| \le C\psi_j^{19} h_j^{-1},$$

(36) $$0 \le \chi_j(x) - Q_j(x) \le C\psi_j^{18},$$

(37) $$1 - x_{j-1} \le \int_{-1}^{1} Q_j(y)\,dy \le 1 - x_j,$$

(38) $$|\bar{Q}_j'(x)| \le C\psi_j^{19} h_j^{-1},$$

(39) $$0 \le \bar{Q}_j(x) - \chi_{j-1}(x) \le C\psi_j^{18}$$

and

(40) $$1 - x_{j-1} \le \int_{-1}^{1} \bar{Q}_j(y)\,dy \le 1 - x_j.$$

Proof Let us note that $Q_j'(x) < 0$, $x < x_j$; $Q_j'(x) > 0$, $x > x_j$; $Q_j(-1) = 0$; $Q_j(1) = 1$ and $\bar{Q}_j'(x) > 0$, $x < x_{j-1}$; $\bar{Q}_j'(x) < 0$, $x > x_{j-1}$; $\bar{Q}_j(-1) = 0$; $\bar{Q}_j(1) = 1$. This yields $Q_j(x) \le \chi_j(x)$ and $\chi_{j-1}(x) \le \bar{Q}_j(x)$, which are the left-hand side inequalities in (36) and (39), respectively. The other inequalities follow from Lemma 11. ■

It follows from inequalities (34), (37) and (40) that $\alpha, \beta, \gamma \in [0, 1]$ can be chosen so that for polynomials

$$\delta_j(x) := \int_{-1}^{x} (\alpha Q_j(y) + (1-\alpha) Q_{j+1}(y))\,dy,$$

$$\bar{\delta}_j(x) := \int_{-1}^{x} \left(\beta \bar{Q}_j(y) + (1-\beta)\bar{Q}_{j+1}(y)\right)\,dy$$

and

$$\sigma_j(x) := \int_{-1}^{x} \left(\gamma \tilde{Q}_j(y) + (1-\gamma)\tilde{Q}_{j+1}(y)\right)\,dy$$

the following equations occur:

$$\delta_j(1) = \bar{\delta}_j(1) = \sigma_j(1) = 1 - x_j.$$

Let $R_j(x) := (x - x_j)\delta_j(x)$ and $\bar{R}_j(x) := (x - x_j)\bar{\delta}_j(x)$. Polynomials $R_j(x)$, $\bar{R}_j(x)$, and $\sigma_j(x)$ and their derivatives give sufficient approximation of the truncated power functions $\chi_j(x)$, $(x - x_j)_+$, and $(x - x_j)_+^2$ (definitions of the truncated power functions are given in the next lemma). Taking into account the fact that analytic representation of any spline of degree 2 contains only these functions, this enables us to obtain a good approximation of any spline of second degree by polynomials with controlled derivatives (see Section 2.3).

Lemma 13 *The following inequalities hold for all $x \in I$:*

$$|\delta_j''(x)| \leq C\psi_j^{19}h_j^{-1}, \tag{41}$$

$$|\bar{\delta}_j''(x)| \leq C\psi_j^{19}h_j^{-1}, \tag{42}$$

$$0 \leq \sigma_j''(x)| \leq C\psi_j^{18}h_j^{-1}, \tag{43}$$

$$|\delta_j'(x) - \chi_j(x)| \leq C\psi_j^{18}, \tag{44}$$

$$|\bar{\delta}_j'(x) - \chi_j(x)| \leq C\psi_j^{18}, \tag{45}$$

$$|\sigma_j'(x) - \chi_j(x)| \leq C\psi_j^{17}, \tag{46}$$

$$\delta_j'(x) \leq \chi_{j+1}(x),\ \bar{\delta}_j'(x) \geq \chi_{j-1}(x), \tag{47}$$

$$|(x - x_j)_+ - \delta_j(x)| \leq C\psi_j^{17}h_j, \tag{48}$$

$$|(x - x_j)_+ - \bar{\delta}_j(x)| \leq C\psi_j^{17}h_j, \tag{49}$$

$$|(x - x_j)_+ - \sigma_j(x)| \leq C\psi_j^{16}h_j, \tag{50}$$

$$|(x - x_j)_+^2 - R_j(x)| \leq C\psi_j^{16}h_j^2, \tag{51}$$

$$|(x - x_j)_+^2 - \bar{R}_j(x)| \leq C\psi_j^{16}h_j^2, \tag{52}$$

$$|2(x - x_j)_+ - R_j'(x)| \leq C\psi_j^{17}h_j, \tag{53}$$

$$|2(x - x_j)_+ - \bar{R}_j'(x)| \leq C\psi_j^{17}h_j, \tag{54}$$

$$|R_j''(x) - 2\chi_j(x)| \leq C\psi_j^{18} \tag{55}$$

and

$$|\bar{R}_j''(x) - 2\chi_j(x)| \leq C\psi_j^{18}, \tag{56}$$

where $(x - x_j)_+^k := (x - x_j)^k \chi_j(x)$.

Proof First, (47) is a consequence of the left-hand side inequalities in (36) and (39) as

$$\begin{aligned} \delta_j'(x) &= \alpha Q_j(x) + (1-\alpha)Q_{j+1}(x) \\ &\leq \alpha\chi_j(x) + (1-\alpha)\chi_{j+1}(x) \\ &\leq \chi_{j+1}(x) \end{aligned}$$

and

$$\begin{aligned} \bar{\delta}_j'(x) &= \beta\bar{Q}_j(x) + (1-\beta)\bar{Q}_{j+1}(x) \\ &\geq \beta\chi_{j-1}(x) + (1-\beta)\chi_j(x) \\ &\geq \chi_{j-1}(x). \end{aligned}$$

Inequalities (41)-(43) are immediate consequences of (35), (38), and (32) and the observation that $\psi_{j\pm1} < 18\psi_j$. Also, since $\psi_j \sim C$ if $x \in I_j$, then inequalities (44)-(46) are consequences of (36), (39), and (33) as

$$\begin{aligned} |\delta_j'(x) - \chi_j(x)| &\le |Q_j(x) - \chi_j(x)| + |Q_{j+1}(x) - \chi_j(x)| \\ &\le |Q_j(x) - \chi_j(x)| + |Q_{j+1}(x) - \chi_{j+1}(x)| + \chi\,[x_{j+1}, x_j]\,(x) \\ &\le C\psi_j^{18}. \end{aligned}$$

The proofs of (48)-(50) are similar so we only prove (48). For $x \le x_j$ we have

$$\begin{aligned} &|(x - x_j)_+ - \delta_j(x)| \\ &\quad\le \int_{-1}^{x} (\alpha|Q_j(y) - \chi_j(y)| + (1-\alpha)|Q_{j+1}(y) - \chi_j(y)|)\, dy \\ &\quad\le C \int_{-1}^{x} \left(\frac{h_j}{|y - x_j| + h_j}\right)^{18} dy \\ &\quad\le C h_j^{18}(|x - x_j| + h_j)^{-17}. \end{aligned}$$

For $x \ge x_j$ we have the estimate

$$\begin{aligned} &|(x - x_j)_+ - \delta_j(x)| \\ &\quad= |(\delta_j(1) - \delta_j(x)) - ((1 - x_j) - (x - x_j))| \\ &\quad= |\delta_j(1) - \delta_j(x) - (1 - x)| \\ &\quad= \left|\int_x^1 (\alpha Q_j(y) + (1-\alpha)Q_{j+1}(y) - 1)\, dy\right| \\ &\quad\le \int_x^1 (\alpha|Q_j(y) - \chi_j(y)| + (1-\alpha)|Q_{j+1}(y) - \chi_{j+1}(y)|)\, dy \\ &\quad\le C \int_x^\infty \left(\frac{h_j}{|y - x_j| + h_j}\right)^{18} dy \le C h_j \psi_j^{17}. \end{aligned}$$

The inequality (48) is proved.

Inequalities (51) and (52) follow immediately from (48) and (49), respectively, and inequalities (53) and (54) are consequences of (48), (44), (49), and (45). Finally, (55) and (56) follow from (41), (44), (42), and (45). The proof of the lemma is now complete. ∎

2.3 Pointwise Estimates of Convex Approximation

Following the ideas of [1] we construct a convex spline $S(x)$ of degree ≤ 2 which sufficiently approximates the convex function $f = f(x)$, $f \in \mathbf{C}(I)$, that is,

$$|f(x) - S(x)| \le C\omega^3(f, \Delta_n(x)), \qquad x \in I.$$

Then we approximate $S(x)$ by a convex algebraic polynomial so that

$$|S(x) - p_n(x)| \le C\omega^3(f, \Delta_n(x)), \qquad x \in I.$$

This proves the estimate (6).

Construction of the Convex Spline

Let

$$S(x) := \max\{L(x,f;x_j), L(x,f;x_{j+1})\}, \qquad x \in I_j,\ 2 \le j \le n-1,$$

$$S(x) := L(x,f;x_2), \qquad x \in I_1,$$

and

$$S(x) := L(x,f;x_n), \qquad x \in I_n.$$

It is easy to see that $S(x)$ is a convex spline of degree ≤ 2 with knots x_j, $0 \le j \le n$.

Now we consider the index j to be fixed and denote

$$a_\nu := L'(x_j, f; x_\nu), \qquad \nu = j, j+1, j+2.$$

Let us call a knot x_j, $2 \le j \le n-2$, "*a knot of type* **I**" if

$$a_{j+1} \le a_{j+2}, \qquad a_{j+1} < a_j. \tag{57}$$

That is,

$$S(x) = L(x,f;x_\nu), \qquad x \in [x_\nu,\ x_{\nu-1}],\ \ \nu = j+1, j.$$

Note that inequalities (57) are equivalent to the following ones:

$$[x_j, x_{j-1}, x_{j-2}; f] < [x_{j+1}, x_j, x_{j-1}; f] \le [x_{j+2}, x_{j+1}, x_j; f],$$

where square brackets denote the divided difference of f.

A knot x_j is "*a knot of type* **II**" if

$$a_{j+2} < a_{j+1}, \qquad a_j \le a_{j+1}, \tag{58}$$

which is equivalent to

$$[x_{j+2}, x_{j+1}, x_j; f] < [x_{j+1}, x_j, x_{j-1}; f] \le [x_j, x_{j-1}, x_{j-2}; f].$$

In this case

$$S(x) = L(x,f;x_{\nu+1}),\ x \in [x_\nu,\ x_{\nu-1}], \nu = j+1, j.$$

Let x_j be "*a knot of type* **III**" if

$$a_{j+2} < a_{j+1} < a_j \tag{59}$$

or, equivalently,

$$[x_{j+2}, x_{j+1}, x_j; f] < [x_{j+1}, x_j, x_{j-1}; f]$$

and

$$[x_j, x_{j-1}, x_{j-2}; f] < [x_{j+1}, x_j, x_{j-1}; f].$$

In this case

$$S(x) = L(x, f; x_{j+2}), \qquad x \in [x_{j+1}, x_j]$$

and

$$S(x) = L(x, f; x_j), \qquad x \in [x_j, x_{j-1}].$$

Let the knots, which are not knots of type **I**, **II**, or **III**, be "*knots of type* **IV**". It is not difficult to see that if x_j is a knot of type **IV**, then

$$S(x) = L(x, f; x_{j+1}), \; x \in [x_{j+1}, x_{j-1}]. \tag{60}$$

Let x_1 be a knot of type **II** if $a_3 < a_2$, x_{n-1} is a knot of type **I** if $a_n < a_{n-1}$, otherwise they are knots of type **IV**.

From (57)–(60) it follows that the spline $S(x)$ has defect 2 at knots **I**, **II**, and **III** (*i.e.*, the first derivative of the continuous spline $S(x)$ does not exist at these knots) and does not have it at knots of type **IV** (S, S', and S'' exist and are continuous at these knots). Taking this into consideration, we get the following analytic representation of the spline $S(x)$ in terms of the truncated power functions $(x - x_j)_+$ and $(x - x_j)_+^2$ (for an analytic representation of splines see, for example, Section 2.3 of [5]):

$$\begin{aligned} S(x) &= f(-1) + A_0(x+1) + [x_n, x_{n-1}, x_{n-2}; f](x+1)^2 \\ &+ \sum_{2 \le i \le n-1,\; x_i \in \mathbf{I} \cup \mathbf{III}} A_i \left\{ (x_{i-1} - x_i)(x - x_i)_+ - (x - x_i)_+^2 \right\} \\ &+ \sum_{1 \le i \le n-2,\; x_i \in \mathbf{II} \cup \mathbf{III}} B_i \left\{ (x_i - x_{i+1})(x - x_i)_+ + (x - x_i)_+^2 \right\}, \end{aligned}$$

where

$$A_0 := [x_n, x_{n-1}; f] - [x_{n-1}, x_{n-2}; f] + [x_{n-2}, x_n; f],$$

$$A_i := [x_{i+1}, x_i, x_{i-1}; f] - [x_i, x_{i-1}, x_{i-2}; f], \qquad 2 \le i \le n-1,$$

and

$$B_i := -A_{i+1}, \qquad 1 \le i \le n-2.$$

Note that $A_i > 0$ for $x_i \in \mathbf{I} \cup \mathbf{III}$ (*i.e.*, if knot x_i is a knot of type **I** or **III**) and $B_i > 0$ if $x_i \in \mathbf{II} \cup \mathbf{III}$.

Now let us estimate the value $|f(x) - S(x)|$, $x \in I$. For this we need the following well-known Whitney inequality (see Section 2.1 of [8], for example):

$$|g(x) - L(x, g; a_0, \ldots, a_k)| \le C(k)\omega^{k+1}(g, a_k - a_0, [a_0, a_k]), \tag{61}$$

where $g \in \mathbf{C}[a, b]$, $a_{i+1} - a_i = a_i - a_{i-1}$, $1 \le i \le k-1$, and $x \in [a_0, a_k]$.

For $x \in [x_i, x_{i-2}]$ we have

$$
\begin{aligned}
& |f(x) - L(x, f; x_i)| \\
(62) \qquad &= |f(x) - \tilde{L}(x, f; [x_i, x_{i-2}]) - L(x, f - \tilde{L}; x_i)| \\
&\leq \|f - \tilde{L}\|_{C[x_i, x_{i-2}]} \left(1 + \left|\frac{(x - x_i)(x - x_{i-2})}{(x_{i-1} - x_i)(x_{i-1} - x_{i-2})}\right|\right) \\
&\leq C\omega^3(f, h_i + h_{i-1}, [x_i, x_{i-2}]) \left(1 + \frac{(h_i + h_{i-1})^2}{4h_i h_{i-1}}\right) \\
&\leq C\omega^3(f, \Delta_n(x)) .
\end{aligned}
$$

This yields

$$
(63) \qquad |f(x) - S(x)| \leq C\omega^3(f, \Delta_n(x)), \qquad x \in I .
$$

Construction of the Convex Polynomial

Let us fix n, denote $n_1 := Mn$, where an absolute constant M is an integer and will be chosen later, and choose i_1 so that $x_{i_1, n_1} = x_{i,n}$.

Using the analytic representation of $S(x)$ and also the approximation of the truncated power functions given in Lemma 13, we write the following algebraic polynomial of degree $\leq 50Mn$:

$$
\begin{aligned}
p_n(x) &= f(-1) + A_0(x+1) + [x_n, x_{n-1}, x_{n-2}; f](x+1)^2 \\
&+ \sum_{2 \leq i \leq n-1,\ x_i \in \mathbf{I} \cup \mathbf{III}} A_i \{(x_{i-1} - x_i)\sigma_{i_1, n_1}(x) - R_{i_1, n_1}(x)\} \\
&+ \sum_{1 \leq i \leq n-2,\ x_i \in \mathbf{II} \cup \mathbf{III}} B_i \left\{(x_i - x_{i+1})\sigma_{i_1, n_1}(x) + \bar{R}_{i_1, n_1}(x)\right\} .
\end{aligned}
$$

(The distance between $S(x)$ and this polynomial is estimated in inequality (69) below.)

Now we show that it is possible to choose M so that this polynomial will be convex on I. For this it is enough to choose M so that the following inequalities hold:

$$
\begin{aligned}
(64) \qquad & (x_{i-1} - x_i)\sigma''_{i_1, n_1}(x) - R''_{i_1, n_1}(x) \geq -2\chi_i(x), \qquad x \in I, \\
& (x_i - x_{i+1})\sigma''_{i_1, n_1}(x) + \bar{R}''_{i_1, n_1}(x) \geq 2\chi_i(x), \qquad x \in I .
\end{aligned}
$$

Indeed, using (64) and taking into account inequalities $A_i > 0$ for $x_i \in \mathbf{I} \cup \mathbf{III}$ and $B_i > 0$ for $x_i \in \mathbf{II} \cup \mathbf{III}$, we have, for $x \in I \setminus \{x_1, \ldots, x_{n-1}\}$,

$$
p''_n(x) \geq 2[x_n, x_{n-1}, x_{n-2}; f] + \sum_{2 \leq i \leq n-1,\ x_i \in \mathbf{I} \cup \mathbf{III}} A_i \{-2\chi_i(x)\}
$$

$$+ \sum_{1\le i\le n-2,\ x_i\in \mathrm{II}\cup\mathrm{III}} 2B_i\chi_i(x) = S''(x).$$

As $S(x)$ is convex on each interval I_j, $1 \le j \le n$, then $S''(x) \ge 0$ for $x \in (x_i, x_{i-1})$, $1 \le i \le n$ and, hence, $p_n''(x) \ge 0$ for $x \in I \setminus \{x_1, \dots, x_{n-1}\}$. As $p_n(x)$ is a polynomial, *i.e.*, it has a continuous second derivative, then $p_n''(x) \ge 0$ for $x \in I$, and therefore $p_n \in \Delta^2$.

Thus, it is sufficient to prove (64). Inequalities (64) are consequences of the following estimates:

$$\min\{\tilde{Q}'_{i_1,n_1}(x), \tilde{Q}'_{i_1+1,n_1}(x)\} \min\{h_i, h_{i+1}\} > 4,\ x \in I_{i_1+1,n_1} \cup I_{i_1,n_1} \tag{65}$$

and

$$\begin{aligned} &\min\{\tilde{Q}'_{i_1,n_1}(x), \tilde{Q}'_{i_1+1,n_1}(x)\} \min\{h_i, h_{i+1}\} \\ &\ge 2|x - x_i| \max\{|Q'_{i_1,n_1}(x)|, |Q'_{i_1+1,n_1}(x)|, |\bar{Q}'_{i_1,n_1}(x)|, |\bar{Q}'_{i_1+1,n_1}(x)|\} \end{aligned} \tag{66}$$

for every $x \in I$.

Indeed, suppose that (65) and (66) are true. Then for any $x \in I$ together with (47) we have, for $x \notin I_{i_1+1,n_1} \cup I_{i_1,n_1}$,

$$\begin{aligned} &(x_{i-1} - x_i)\sigma''_{i_1,n_1}(x) - R''_{i_1,n_1}(x) \\ &= h_i \left(\gamma\tilde{Q}'_{i_1,n_1}(x) + (1-\gamma)\tilde{Q}'_{i_1+1,n_1}(x)\right) \\ &\quad -(x - x_i)\left(\alpha Q'_{i_1,n_1}(x) + (1-\alpha)Q'_{i_1+1,n_1}(x)\right) - 2\delta'_{i_1,n_1}(x) \\ &\ge h_i \min\{\tilde{Q}'_{i_1,n_1}(x), \tilde{Q}'_{i_1+1,n_1}(x)\} \\ &\quad -|x - x_i| \max\{|Q'_{i_1,n_1}(x)|, |Q'_{i_1+1,n_1}(x)|\} - 2\chi[x_{i_1+1,n_1}, 1](x) \\ &\ge -2\chi[x_{i_1+1,n_1}, 1](x) = -2\chi_i(x). \end{aligned}$$

For $x \in I_{i_1+1,n_1} \cup I_{i_1,n_1}$ taking into account (65) and (66) we have the following estimate:

$$\begin{aligned} &(x_{i-1} - x_i)\sigma''_{i_1,n_1}(x) - R''_{i_1,n_1}(x) \\ &\ge h_i/2 \min\{\tilde{Q}'_{i_1,n_1}(x), \tilde{Q}'_{i_1+1,n_1}(x)\} - 2\chi[x_{i_1+1,n_1}, 1](x) \\ &\ge 2 - 2\chi[x_{i_1+1,n_1}, 1](x) \ge -2\chi_i(x). \end{aligned}$$

This proves the first estimate in (64). Considerations for the proof of the second estimate in (64) are analogous.

Thus, our problem is reduced to the following one: find such an integer constant M that, for $n_1 := Mn$, inequalities (65) and (66) are valid.

It follows from (23) that, for $x \in I_{i_1+1,n_1} \cup I_{i_1,n_1}$,

$$\begin{aligned} t_{i_1,n_1}(x) &> \min\{(x - x^0_{i_1,n_1})^{-2}, (x - \bar{x}_{i_1,n_1})^{-2}\} \\ &> (h_{i_1+1,n_1} + h_{i_1,n_1})^{-2} > h^{-2}_{i_1,n_1}/16. \end{aligned}$$

From Proposition 10 we have

$$\int_{-1}^{1} t_{i_1,n_1}(y)^9\,dy \;<\; C_0(9)h_{i_1,n_1}^{-17}\,.$$

Thus, we have, for example, the following estimate:

$$\tilde{Q}'_{i_1,n_1}(x)h_i \;=\; \frac{t_{i_1,n_1}(x)^9}{\int_{-1}^{1} t_{i_1,n_1}(y)^9\,dy} \;>\; C_0(9)^{-1}16^{-9}\frac{h_i}{h_{i_1,n_1}}\,.$$

Now it is sufficient to choose the number $n_1 \in \mathcal{N}$ so that

$$h_i \geq 4\cdot 16^9 C_0(9)h_{i_1,n_1}\,.$$

This verifies (65).

Using the same idea, Proposition 10, and also inequalities (28) to prove (66) we write, for example,

$$\begin{aligned}
&h_i\tilde{Q}'_{i_1,n_1}(x) - 2|x-x_i||\tilde{Q}'_{i_1,n_1}(x)|\\
&\geq\; C_0(9)^{-1}h_i h_{i_1,n_1}^{17} t_{i_1,n_1}(x)^9\\
&\quad -2C_0(10)|x-x_i||x_{i_1-1,n_1}-x|h_{i_1,n_1}^{18}t_{i_1,n_1}(x)^{10}\\
&\geq\; C_0(9)^{-1}h_i h_{i_1,n_1}^{17}(|x-x_{i_1,n_1}|+h_{i_1,n_1})^{-18}\\
&\quad -2\cdot 4^{10}\cdot 10^{30}C_0(10)h_{i_1,n_1}^{18}(|x-x_{i_1,n_1}|+h_{i_1,n_1})^{-18}\\
&\geq\; \psi_{i_1,n_1}^{18}h_{i_1,n_1}^{-1}\left\{C_0(9)^{-1}h_i - 2\cdot 4^{10}\cdot 10^{30}C_0(10)h_{i_1,n_1}\right\}\,.
\end{aligned}$$

Thus, (66) is verified if the number $n_1 \in \mathcal{N}$ is chosen so that

$$h_i \;\geq\; 2\cdot 4^{10}\cdot 10^{30}C_0(9)C_0(10)h_{i_1,n_1}\,.$$

Taking into account that $h_i/h_{i_1,n_1} \geq n_1/5n$ we conclude that inequalities (65) and (66) are true for $n_1 = [10^{50}C_0(9)C_0(10)]n =: Mn$.

It remains to estimate $|p_n(x) - S(x)|$. Similarly to (62) using

$$[x_i,x_{i-1},x_{i-2},x_{i-3};f] \;=\; \frac{f(x_{i-1}) - L(x_{i-1},f;x_i)}{(x_{i-1}-x_i)(x_{i-1}-x_{i-2})(x_{i-1}-x_{i-3})}$$

we have

$$[x_i,x_{i-1},x_{i-2},x_{i-3};f] \;\leq\; Ch_i^{-3}\omega^3(f,\Delta_n(x_i))$$

and hence

$$|A_i| \;=\; |[x_{i+1},x_i,x_{i-1},x_{i-2};f](x_{i-2}-x_{i+1})| \;\leq\; Ch_i^{-2}\omega^3(f,\Delta_n(x_i))\,.$$

It also follows from the last inequality that

$$|A_i| \;\leq\; Ch_i^{-1}\omega^2(f',\Delta_n(x_i)) \quad \text{for} \quad f\in \mathbf{C}^1(I)$$

and

$$|A_i| \le C\omega(f'', \Delta_n(x_i)) \quad \text{for} \quad f \in \mathbf{C}^2(I).$$

Now let us note that the estimate $h_{i_1,n_1} < h_i < (M^2/5)h_{i_1,n_1}$ implies that

$$\psi_{i_1,n_1} < 5M^2\psi_i. \tag{67}$$

Using the inequalities (see [9] and [10])

$$\Delta_n^2(y) < 4\Delta_n(x)(|x-y| + \Delta_n(x))$$

and

$$2(|x-y| + \Delta_n(x)) > |x-y| + \Delta_n(y) > (|x-y| + \Delta_n(x))/2, \qquad x, y \in I,$$

and also the properties of the modulus of smoothness we have

$$\begin{aligned}
\omega^3(f, \Delta_n(x_i)) &\le \omega^3\left(f, 2\sqrt{\Delta_n(x)(|x-x_i| + \Delta_n(x))}\right) \\
&= \omega^3\left(f, 2\Delta_n(x)\sqrt{\frac{|x-x_i| + \Delta_n(x)}{\Delta_n(x)}}\right) \\
&\le 64\left(\frac{|x-x_i| + \Delta_n(x)}{\Delta_n(x)}\right)^{3/2} \omega^3(f, \Delta_n(x)) \\
&\le 10^6\left(\frac{|x-x_i| + h_i}{h_i}\right)^3 \omega^3(f, \Delta_n(x)).
\end{aligned} \tag{68}$$

From (50)-(52), (67), and (68) we now have

$$\begin{aligned}
&|p_n(x) - S(x)| \\
&\le \sum_{2\le i\le n-1,\, x_i \in \mathrm{I}\cup\mathrm{III}} |A_i| \{h_i|(x-x_i)_+ - \sigma_{i_1,n_1}(x)| \\
&\quad + |R_{i_1,n_1}(x) - (x-x_i)_+^2|\} \\
&\quad + \sum_{1\le i\le n-2,\, x_i \in \mathrm{II}\cup\mathrm{III}} |B_i| \{h_{i+1}|(x-x_i)_+ - \sigma_{i_1,n_1}(x)| \\
&\quad + |\bar{R}_{i_1,n_1}(x) - (x-x_i)_+^2|\} \\
&\le C\sum_{i=1}^{n-1} \omega^3(f, \Delta_n(x))\psi_i^{13} \\
&\le C\omega^3(f, \Delta_n(x)).
\end{aligned} \tag{69}$$

Inequalities (63) and (69) complete the proof of the estimate (6) as

$$|f(x) - p_n(x)| \le |f(x) - S(x)| + |S(x) - p_n(x)| \le \omega^3(f, \Delta_n(x)), \qquad x \in I.$$

To prove (7) the following equations are used:

$$\begin{aligned} S'(x) &= A_0 + 2[x_n, x_{n-1}, x_{n-2}; f](x+1) \\ &+ \sum_{2\le i\le n-1,\ x_i\in \mathrm{I}\cup\mathrm{III}} A_i \{h_i\chi_i(x) - 2(x-x_i)_+\} \\ &+ \sum_{1\le i\le n-2,\ x_i\in \mathrm{II}\cup\mathrm{III}} B_i \{h_{i+1}\chi_i(x) + 2(x-x_i)_+\}\,. \end{aligned}$$

It follows from (46), (53), and (54) that

$$\begin{aligned} (70)\qquad & |p_n'(x) - S'(x)| \\ \le & \sum_{2\le i\le n-1,\ x_i\in \mathrm{I}\cup\mathrm{III}} |A_i| \left\{h_i|\sigma'_{i_1,n_1}(x) - \chi_i(x)| \right. \\ & \left. + |R'_{i_1,n_1}(x) - 2(x-x_i)_+|\right\} \\ & + \sum_{1\le i\le n-2,\ x_i\in \mathrm{II}\cup\mathrm{III}} |B_i| \left\{h_{i+1}|\sigma'_{i_1,n_1}(x) - \chi_i(x)| \right. \\ & \left. + |\bar R'_{i_1,n_1}(x) - 2(x-x_i)_+|\right\} \\ \le & \; C\sum_{i=1}^{n-1} \omega^2(f', \Delta_n(x))\psi_i^{14} \\ \le & \; C\,\omega^2(f', \Delta_n(x))\,. \end{aligned}$$

Now, (7) follows from the following estimate for $x \in \tilde I_j := [x_j, x_{j-2}]$:

$$(71)\qquad |f'(x) - L'(x, f; x_j)| \le C\,\omega^2(f', h_j, \tilde I_j)\,.$$

In order to prove inequality (71) (see Lemma 1.4.2 of [11]) let us denote

$$\bar L(x) := f(x_j) + \int_{x_j}^{x} L(u, f', x_j, x_{j-2})\,du$$

($\bar L(x)$ is an algebraic polynomial of degree 2) and note that

$$f'(x) - L'(x, f; x_j) = f'(x) - \bar L'(x) - L'(x - \bar L; x_j)\,.$$

The following estimate is a consequence of the Whitney inequality (61):

$$|f'(x) - L(x, f'; x_j, x_{j-2})| \le C\,\omega^2(f', h_j, \tilde I_j), \qquad x \in \tilde I_j\,.$$

This implies, for any $x \in \tilde I_j$,

$$|f(x) - \bar L(x)| = \left|\int_{x_j}^{x} (f'(u) - L(u, f', x_j, x_{j-2}))\,du\right| \le C\,h_j\,\omega^2(f', h_j, \tilde I_j)\,.$$

Now, together with the estimate for $x \in \tilde I_j$,

$$|L'(x - \bar L; x_j)| \le C\,h_j^{-1}\,\|f - \bar L\|_{\tilde I_j} \le C\,\omega^2(f', h_j, \tilde I_j)\,,$$

the following inequalities complete the proof of (71):

$$\begin{aligned} |f'(x) - L'(x,f;x_j)| &\le |f'(x) - \bar{L}'(x)| + C\,\omega^2(f',h_j,\tilde{I}_j) \\ &= |f'(x) - L(x,f';x_j,x_{j-2})| + C\,\omega^2(f',h_j,\tilde{I}_j) \\ &\le C\,\omega^2(f',h_j,\tilde{I}_j), \qquad x \in \tilde{I}_j\,. \end{aligned}$$

To prove (8) we use the following equations:

$$\begin{aligned} S''(x) &= 2[x_n, x_{n-1}, x_{n-2}; f] \\ &\quad + \sum_{2\le i\le n-1,\ x_i\in \mathrm{I}\cup\mathrm{III}} A_i\,\{-2\chi_i(x)\} + \sum_{1\le i\le n-2,\ x_i\in \mathrm{II}\cup\mathrm{III}} 2B_i\chi_i(x) \end{aligned}$$

and

$$S''(x) = 2[x_j, x_{j-1}, x_{j-2}; f] \quad \text{or} \quad S''(x) = 2[x_{j+1}, x_j, x_{j-1}; f], \quad \text{if } x \in I_j\,.$$

It follows from (43), (55), (56), and (68) that

$$\begin{aligned} (72) \quad & |p_n''(x) - S''(x)| \\ &\le \sum_{2\le i\le n-1,\ x_i\in \mathrm{I}\cup\mathrm{III}} |A_i|\left\{h_i|\sigma''_{i_1,n_1}(x)| + |2\chi_i(x) - R''_{i_1,n_1}(x)|\right\} \\ &\quad + \sum_{1\le i\le n-2,\ x_i\in \mathrm{II}\cup\mathrm{III}} |B_i|\left\{h_{i+1}|\sigma''_{i_1,n_1}(x)| + |2\chi_i(x) - \bar{R}''_{i_1,n_1}(x)|\right\} \\ &\le C\sum_{i=1}^{n-1} \omega(f'',\Delta_n(x))\psi_i^{15} \\ &\le C\,\omega(f'',\Delta_n(x))\,. \end{aligned}$$

Now, (8) follows from the following estimate for $x \in I_j$ (see also Lemma 1.4.2 of [11]):

$$\begin{aligned} (73) \quad & |f''(x) - 2[x_j, x_{j-1}, x_{j-2}; f]| \\ &= \left|2\int_0^1\int_0^{t_1} \{f''(x) - f''(x_j + (x_{j-1}-x_j)t_1 + (x_{j-2}-x_{j-1})t_2)\}\,dt_2\,dt_1\right| \\ &\le \omega(f'', h_j + h_{j-1}, I_j) \\ &\le C\,\omega(f'',\Delta_n(x))\,. \end{aligned}$$

Thus, Theorem 1 is proved for all $n \ge 2$.

2.4 Estimates in Terms of the Ditzian-Totik Moduli

To prove inequalities (11)-(13) it is sufficient to estimate $|f^{(\nu)}(x) - S^{(\nu)}(x)|$ and $|p_n^{(\nu)}(x) - S^{(\nu)}(x)|$ in terms of the $\bar{\omega}_\varphi^{3-\nu}$ moduli with $\nu = 0, 1$, and 2, respectively.

First, let us note the following:

For the interval $[x_i, x_{i-2}]$ we denote $\xi_i := x_i$ if $|x_{i-2}| < |x_i|$ and $\xi_i = x_{i-2}$ otherwise. Then for any $y \in [x_i, x_{i-2}]$ and $0 < h \le n^{-1}$ the inequality $\rho(\xi_i, h) \le \rho(y, h)$ is valid.

If $h = \sqrt{1-x^2}\tilde{h} + \tilde{h}^2 = \rho(x, \tilde{h})$, then $0 < h \le \Delta_n(x) \iff 0 < \tilde{h} \le n^{-1}$. Using (62) and the above we get, for a fixed $x \in [x_i, x_{i-2}]$,

$$\begin{aligned}
|f(x) - L(x, f; x_i)| &\le 10^5 \omega^3(f, \Delta_n(x); [x_i, x_{i-2}]) \\
&\le 10^5 \omega^3(f, 15\Delta_n(\xi_i); [x_i, x_{i-2}]) \\
&\le C\, \omega^3(f, \Delta_n(\xi_i); [x_i, x_{i-2}]) \\
&= C \sup_{0<h\le\Delta_n(\xi_i)} \left\| \bar{\Delta}_h^3(f, y, [x_i, x_{i-2}]) \right\|_{C[x_i, x_{i-2}]} \\
&\le C \sup_{0<h\le\Delta_n(\xi_i)} \left\| \bar{\Delta}_h^3(f, y) \right\|_{C[x_i, x_{i-2}]} \\
&= C \sup_{0<\tilde{h}\le n^{-1}} \left\| \bar{\Delta}^3_{\rho(\xi_i, \tilde{h})}(f, y) \right\|_{C[x_i, x_{i-2}]} \\
&\le C \left| \bar{\Delta}^3_{\rho(\xi_i, \tilde{h}_0)}(f, \zeta_i) \right|
\end{aligned}$$

for some $0 < \tilde{h}_0 \le n^{-1}$ and $\zeta_i \in [x_i, x_{i-2}]$. (Actually, using the compactness argument the last inequality can be replaced by an equality.)

Now, using the inequality $\rho(\xi_i, \tilde{h}_0) \le \rho(\zeta_i, \tilde{h}_0)$, continuity of $\rho(\zeta_i, h)$, and the fact that $\rho(\zeta_i, h) \to 0$ as $h \to 0$ we can conclude that a number $\tilde{h}_1$, $0 < \tilde{h}_1 \le \tilde{h}_0 \le n^{-1}$, exists such that $\rho(\xi_i, \tilde{h}_0) = \rho(\zeta_i, \tilde{h}_1)$. Thus,

$$\begin{aligned}
|f(x) - L(x, f; x_i)| &\le C \left| \bar{\Delta}^3_{\rho(\zeta_i, \tilde{h}_1)}(f, \zeta_i) \right| \\
&\le C \sup_{0<\tilde{h}\le n^{-1}} \left\| \bar{\Delta}^3_{\rho(y, \tilde{h})}(f, y) \right\|_{C[x_i, x_{i-2}]} \\
&\le C\, \bar{\omega}^3_\varphi(f, n^{-1}).
\end{aligned}$$

This implies

$$|f(x) - S(x)| \le C\bar{\omega}^3_\varphi(f, n^{-1}), \qquad x \in I.$$

Now we can use the same considerations as in (69) to estimate $|p_n(x) - S(x)|$. For this we need the estimates of the coefficients A_i, which appeared in the constructions of $S(x)$ and $p_n(x)$, in terms of the "nonuniform" moduli $\bar{\omega}^3_\varphi$.

Using the same method as above we have the following estimate of $|A_i|$:

$$\begin{aligned}
|A_i| &= |[x_{i+1}, x_i, x_{i-1}, x_{i-2}; f](x_{i-2} - x_{i+1})| \\
&= \left| \frac{f(x_i) - L(x_i, f; x_{i+1}, x_{i-1}, x_{i-2})}{(x_i - x_{i+1})(x_i - x_{i-1})(x_i - x_{i-2})} (x_{i-2} - x_{i+1}) \right| \\
&\le C\, h_i^{-2}\, \omega^3(f, \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\
&\le C\, h_i^{-2}\, \bar{\omega}^3_\varphi(f, n^{-1}).
\end{aligned}$$

Moreover, if $f \in \mathbf{C}^1(I)$, then

$$\begin{aligned} |A_i| &\leq C\, h_i^{-2}\, \omega^3(f, \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\leq C\, h_i^{-2}\, \Delta_n(x_i)\, \omega^2(f', \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\leq C\, h_i^{-1}\, \bar{\omega}_\varphi^2(f', n^{-1}) . \end{aligned}$$

Similarly, if $f \in \mathbf{C}^2(I)$, then

$$\begin{aligned} |A_i| &\leq C\, h_i^{-2}\, \Delta_n(x_i)^2\, \omega(f'', \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\leq C\, \bar{\omega}_\varphi(f'', n^{-1}) . \end{aligned}$$

Now, analogously to (69), we have

$$|p_n(x) - S(x)| \leq C\bar{\omega}_\varphi^3(f, n^{-1}) \sum_{i=1}^{n-1} \psi_i^{16} \leq C\bar{\omega}_\varphi^3(f, n^{-1}) .$$

This completes the proof of the estimate (11).

Analogously to (70) and (72) in the cases $f \in \mathbf{C}^1(I)$ and $f \in \mathbf{C}^2(I)$, we have the estimates

$$|p_n'(x) - S'(x)| \leq C\bar{\omega}_\varphi^2(f', n^{-1}) \sum_{i=1}^{n-1} \psi_i^{17} \leq C\bar{\omega}_\varphi^2(f', n^{-1})$$

and

$$|p_n''(x) - S''(x)| \leq C\bar{\omega}_\varphi(f'', n^{-1}) \sum_{i=1}^{n-1} \psi_i^{18} \leq C\bar{\omega}_\varphi(f'', n^{-1}) ,$$

respectively.

Now inequalities (71) and (73) imply, for $x \in I_j$,

$$|f'(x) - S'(x)| \leq C\omega^2(f', \Delta_n(x_j); \tilde{I}_j) \leq C\bar{\omega}_\varphi^2(f', n^{-1})$$

and

$$|f''(x) - S''(x)| \leq C\omega(f'', \Delta_n(x_j); \tilde{I}_j) \leq C\bar{\omega}_\varphi(f'', n^{-1})$$

in the cases $f \in \mathbf{C}^1(I)$ and $f \in \mathbf{C}^2(I)$, respectively. Thus, inequalities (12) and (13) are also proved. ∎

2.5 Estimates in Terms of the Sendov-Popov τ-moduli in L_p norm

We need the well-known Jensen inequality, that is,

$$|a_1 b_1 + \ldots + a_n b_n|^p \leq a_1 |b_1|^p + \ldots + a_n |b_n|^p ,$$

where $a_i \geq 0$, $1 \leq i \leq n$, and $\sum_{i=1}^n a_i = 1$, $p \geq 1$, and also the following lemma.

Lemma D *(see Lemma 2.5 of [8]).*
Let $\{z_i : -1 = z_0 < z_1 < \ldots < z_{n+1} = 1\}$ *be a partition of the interval* $[-1, 1]$ *into* $n+1$ *subintervals and let* $r \geq 1$ *be an integer. If* $\delta_i = z_{i+1} - z_{i-1}$, $1 \leq i \leq n$, $d_n = \max\{\delta_i : 1 \leq i \leq n\}$, *then*

$$\left(\frac{1}{2}\sum_{i=1}^{n}(\omega^r(f, z_i; 2h))^p\,\delta_i\right)^{1/p} \leq 2^{1/p+2(r+1)}\,\tau^r\,(f; h + d_n/r)_p\,.$$

Now, let us estimate $\|f - S\|_p$:

$$\begin{aligned}
\|f - S\|_p &= \left(\frac{1}{2}\int_{-1}^{1}|f(x) - S(x)|^p\,dx\right)^{1/p} \\
&= \left(\frac{1}{2}\sum_{j=1}^{n}\int_{I_j}|f(x) - S(x)|^p\,dx\right)^{1/p} \\
&\leq \left(\frac{1}{2}\sum_{j=1}^{n}\int_{I_j}C^p\,\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})^p\,dx\right)^{1/p} \\
&\leq C\left(\sum_{j=1}^{n}\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})^p\,h_j\right)^{1/p} \\
&\leq C\,\tau^3(f; n^{-1})_p\,,
\end{aligned}$$

where $h_0 := h_{n+1} := 0$.

Using the estimate $\sum_{j=1}^{n-1}\psi_j^{16} \leq C$ (see the proof of Lemma 2 in [4], for example) and the Jensen inequality we get

$$\begin{aligned}
\|S - p_n\|_p &= \left(\frac{1}{2}\int_{-1}^{1}|p_n(x) - S(x)|^p\,dx\right)^{1/p} \\
&\leq C\left(\int_{-1}^{1}\left(\sum_{j=1}^{n}\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})\psi_j^{16}\right)^p\,dx\right)^{1/p} \\
&\leq C\left(\int_{-1}^{1}\sum_{j=1}^{n}\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})^p\psi_j^{16}\,dx\right)^{1/p} \\
&\leq C\left(\sum_{j=1}^{n}\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})^p\int_{0}^{\infty}\left(\frac{h_j}{t + h_j}\right)^{16}\,dt\right)^{1/p} \\
&\leq C\left(\sum_{j=1}^{n}\omega^3(f, x_j; h_{j-1} + h_j + h_{j+1})^p\,h_j\right)^{1/p} \\
&\leq C\,\tau^3(f; n^{-1})_p\,.
\end{aligned}$$

Thus, using Minkowski's inequality we have

$$\|f - p_n\|_p \leq \|f - S\|_p + \|S - p_n\|_p \leq C\tau^3(f;n^{-1})_p.$$

The proof of (18) is complete. The proofs of (19) and (20) are analogous. ∎

Note that it is possible to relax some of the conditions put on f in Theorems 1-7. This is connected with the fact that a convex function on the interval $[-1,1]$ (at least in terms of divided differences) is continuous on the open interval $(-1,1)$ and has left and right derivatives at every point of this interval (see, for example, Sections 11 and 72 of [7]).

2.6 Final Remark

Recently, Y. Hu, D. Leviatan and X. M. Yu obtained the uniform estimate for convex polynomial approximation in terms of $\omega^3(f,n^{-1})$. Their paper "Convex Polynomial and Spline Approximation in $\mathbf{C}[-1,1]$" appeared in *Constructive Approximation*, **10**: 31-64.

References

[1] R. A. DeVore and X. M. Yu (1985): *Pointwise estimates for monotone polynomial approximation.* Constr. Approx., **1**: 323–331.

[2] Z. Ditzian and V. Totik (1987): Moduli of Smoothness. Berlin: Springer-Verlag.

[3] V. K. Dzjadyk (1977): Introduction to the Theory of Uniform Approximation of Functions by Polynomials. Moscow: Izdat. "Nauka".

[4] K. A. Kopotun (1992), *Uniform estimates of convex approximation of functions by polynomials*, Mat. Zametki, **51**(3): 35–46. (English translation: Math. Notes, **51**: 245–254.)

[5] N. P. Kornejchuk (1987), Exact Constants in Approximation Theory, Moskow: Izdat. "Nauka". (English translation: Cambridge: Cambridge University Press, 1991.)

[6] D. Leviatan (1986), *Pointwise estimates for convex polynomial approximation*, Proc. Amer. Math. Soc., **98**(3): 471–474.

[7] A. W. Roberts and D. E. Varberg (1973): Convex Functions. New York: Academic Press.

[8] B. Sendov and V. A. Popov (1983): The Averaged Moduli of Smoothness. Sofia: Bulgarian Academy of Science. (English translation: New York: Wiley, 1988.)

[9] I. A. Shevchuk (1989), *On coapproximation of monotone functions*, Dokl. Akad. Naul SSSR, **308**(3): 537–541. (English translation: Soviet. Math. Dokl., **40**(2): 349–354.)

[10] I. A. Shevchuk (1989), Comonotone Approximation and Polynomial Kernels of Dzjadyk, Kiev: Institut Matematiki AN USSR (Preprint).

[11] I. A. Shevchuk (1992), Approximation by Polynomials and Traces of the Functions Continuous on an Interval, Kiev: Naukova Dumka.

[12] A. S. Shvedov (1981), *Orders of coapproximation of functions by algebraic polynomials*, Mat. Zametki, **29**(1): 117–130. (English translation: Math. Notes, **30**: 63–70.)

CHAPTER 3

UNIFORM ESTIMATES OF MONOTONE AND CONVEX APPROXIMATION OF SMOOTH FUNCTIONS [2]

3.1 Introduction and Main Results

The present chapter is devoted to the investigation of monotone and convex approximation of smooth functions, *i.e.*, cases for $q = 1$ and $q = 2$.

The rates of the best nth degree unconstrained and shape preserving polynomial approximation of a function f are defined by

$$E_n(f) = \inf_{p_n \in \Pi_n} \|f - p_n\|_\infty$$

and

$$E_n^{(q)}(f) = \inf_{p_n \in \Pi_n \cap \Delta^q} \|f - p_n\|_\infty, \qquad q \in \mathcal{N},$$

respectively.

We recall that

$$\omega^k(f, t, [a, b]) := \sup_{0<h\le t} \max_{[x, x+kh]\subset[a,b]} |\bar{\Delta}_h^k(f, x)|$$

denotes the usual kth modulus of smoothness of f.

The Ditzian-Totik modulus of smoothness is given by

$$\omega_\varphi^k(f, t)_p = \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^k(f, \cdot)\|_p, \qquad \varphi(x) = \sqrt{1 - x^2}.$$

The Ditzian-Totik weighted modulus of smoothness with weight φ^r is

$$\omega_\varphi^k(f, t)_{\varphi^r, p}$$

$$:= \sup_{0<h\le t} \left\|(1 - x - kh\varphi(x)/2)^{r/2} (1 + x - kh\varphi(x)/2)^{r/2} \Delta_{h\varphi(x)}^k(f, x)\right\|_p,$$

where

$$\Delta_h^k(f, x) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & \text{if } |x \pm kh/2| \le 1, \\ 0, & \text{otherwise} \end{cases}$$

[2] A version of this chapter has been published in J. Approx. Theory (1995) 80: 76–107.

is the kth symmetric difference.

Let $I := [-1,1]$, $\rho(h,y) := h\sqrt{1-y^2}+h^2$, $y \in I$, $\rho := \rho(h,x)$, $x \in I$.

For the sake of brevity and convenience of exposition in the uniform metric we will use the following definition of the "nonuniform" modulus of smoothness $\bar{\omega}_\varphi^k(f,t)$ and the "nonuniform" weighted modulus $\bar{\omega}_{\varphi,r}^k(f,t)$, which are equivalent to $\omega_\varphi^k(f,t)_\infty$ and $\omega_\varphi^k(f,t)_{\varphi^r,\infty}$, respectively (see [5] and [14]):

$$\bar{\omega}_\varphi^k(f,t,[a,b]) := \sup_{0<h\le t}\ \max_{[x,x+k\rho]\subset[a,b]} |\bar{\Delta}_\rho^k(f,x)|\,, \qquad t \ge 0\,,$$

$$\bar{\omega}_\varphi^k(f,t) := \bar{\omega}_\varphi^k(f,t,I)\,,$$

$$\bar{\omega}_{\varphi,r}^k(f,t) := \sup_{0<h\le t}\ \sup_{[x,x+k\rho]\subset I} \left|w_r(x,k,h)\bar{\Delta}_\rho^k(f,x)\right|\,,$$

where $w_r(x,k,h) := (1+x)^{r/2}(1-x-k\rho)^{r/2}$, $(r+1) \in \mathcal{N}$. Obviously, $\bar{\omega}_{\varphi,0}^k(f,t) = \bar{\omega}_\varphi^k(f,t)$.

For $k=0$, let

$$\bar{\omega}_{\varphi,r}^0(f,t) := \operatorname{ess\,sup}_{x\in(-1,1)}|(1-x^2)^{r/2}f(x)|\,.$$

For arbitrary $f \in \mathbf{C}(-1,1)$, the function $\bar{\omega}_{\varphi,r}^k(f,t)$ can be unbounded. However, it was shown in [5] (see also [14]) that the necessary and sufficient condition for $\bar{\omega}_{\varphi,r}^k(f,t)$ to be bounded for all $t > 0$ is the existence of a constant $M < \infty$ such that

$$\left|(1-x^2)^{r/2}f(x)\right| < M, \qquad x \in (-1,1)\,.$$

Let $\mathbf{B}^r$, $(r+1) \in \mathcal{N}$, denote the space of all functions f such that $f \in \mathbf{C}[-1,1] \cap \mathbf{C}^r(-1,1)$ and $\left|(1-x^2)^{r/2}f^{(r)}(x)\right| < \infty$, $x \in (-1,1)$. Thus,

$$\bar{\omega}_{\varphi,r}^k(f^{(r)},t) < \infty,\ t>0 \qquad \Longleftrightarrow \qquad f \in \mathbf{B}^r\,.$$

In order to avoid considerations of trivial cases (when the right-hand sides of estimates are equal to infinity) we will have the restriction on f that it be from the $\mathbf{B}^r$ class. Also, let us note that for such functions f, any $0 \le l \le r$, and $k \ge 0$, the following inequality holds (see also Lemma H below):

$$\bar{\omega}_{\varphi,l}^{k+r-l}(f^{(l)},t) \le C_0(r,k)\,t^{r-l}\,\bar{\omega}_{\varphi,r}^k(f^{(r)},t) \qquad t>0\,. \tag{74}$$

For unconstrained approximation the following direct result is known.

Theorem E *(see [5] and [14], for example). Let $k \in \mathcal{N}$, $(r+1) \in \mathcal{N}$. Then for a given function $f \in \mathbf{B}^r$ on I and each $n \ge k+r-1$,*

$$E_n(f) \le C\,n^{-r}\,\bar{\omega}_{\varphi,r}^k(f^{(r)},n^{-1}) \qquad C = C(r,k)\,. \tag{75}$$

Our goal is to investigate the possibility of obtaining the estimate (75) for shape preserving approximation.

First of all, the following negative results are known.

Lemma 14 *There is no such constant C that for every nondecreasing function f on I, $f \in \mathbf{B}^2$, the estimate*

$$E_n^{(1)}(f) \le C\, n^{-2}\, \bar{\omega}_{\varphi,2}^{1}(f'', n^{-1})$$

is valid.

Moreover, even the estimate $E_n^{(1)}(f) \le C\, \bar{\omega}_{\varphi,2}^{1}(f'', 1)$ is false.

Thus, the estimates

$$E_n^{(1)}(f) \le C\, n^{-l}\, \bar{\omega}_{\varphi,l}^{k+2-l}(f^{(l)}, n^{-1}) \qquad C = C(k),$$

generally speaking, are not correct for $0 \le l \le 2$ and $k \in \mathcal{N}$.

Lemma 15 *Let $\nu \ge 0$ be fixed. There is no such constant C that for every convex function f on I, $f \in \mathbf{B}^4$, the estimate*

$$E_n^{(2)}(f) \le C\, n^{-4}\, \bar{\omega}_{\varphi,4}^{\nu}(f^{(4)}, n^{-1})$$

is valid.

Moreover, even the estimate $E_n^{(2)}(f) \le C\, \bar{\omega}_{\varphi,4}^{\nu}(f^{(4)}, 1)$ is false.

Thus, the estimates

$$E_n^{(2)}(f) \le C\, n^{-l}\, \bar{\omega}_{\varphi,l}^{k+3-l}(f^{(l)}, n^{-1}) \qquad C = C(k),$$

generally speaking, are not correct for $0 \le l \le 4$ and $k \in \mathcal{N}$.

Proof Lemma 14 follows from Lemma 2 of [10] and the estimate

$$\bar{\omega}_{\varphi,2}^{1}(g_b'', t) \le 4$$

from the proof of Lemma 3 in [10]. Lemma 15 is a consequence of Theorem 2 of [8]. It is worth mentioning that for the particular cases $l = 0$ and $l = 0$ or 1 in Lemmas 14 and 15, respectively, they follow from A. S. Shvedov's work [15]. ∎

It will be shown in the present chapter that the estimate (75) can be obtained for shape preserving approximation of functions $f \in \mathbf{B}^r$ with $r \ge 3$ and $r \ge 5$ in the monotone and convex cases, respectively. For the other r such direct results are known (see [7]-[9], [11] and [12]).

Namely, the following theorems will be proved.

Theorem 16 *Let $k \in \mathcal{N}$, $r \in \mathcal{N}$, $r \ge 3$, and $f \in \mathbf{B}^r$. If f is nondecreasing on I, then for every $n \ge k + r - 1$,*

$$E_n^{(1)}(f) \le C\, n^{-r}\, \bar{\omega}_{\varphi,r}^{k}(f^{(r)}, n^{-1}), \qquad C = C(r,k).$$

Theorem 17 *Let $k \in \mathcal{N}$, $r \in \mathcal{N}$, $r \geq 5$, and $f \in \mathbf{B}^r$. If f is convex on I, then for every $n \geq k+r-1$,*

$$E_n^{(2)}(f) \leq C\,n^{-r}\,\bar{\omega}_{\varphi,r}^k(f^{(r)}, n^{-1}), \qquad C = C(r,k).$$

Now one can summarize estimates of monotone and convex approximation in terms of $\bar{\omega}_{\varphi,r}^k$. For the sake of convenience we will present the results obtained in the form of Figs. 1 and 2. A disk in the position (k,r) means that for a monotone ($i=1$) or convex ($i=2$) function f from the $\mathbf{B}^r$ class the estimate

$$E_n^{(i)}(f) \leq C\,n^{-r}\,\bar{\omega}_{\varphi,r}^k(f^{(r)}, n^{-1}), \qquad C = C(r,k)$$

holds. A circle means that this estimate is correct for not all $f \in \mathbf{B}^r$.

These results are obtained or are derived from the following papers.

Positive results (monotone case)	
$r=0$, $k=2$, and, consequently, also for $\{(k,r)\|1 \leq k+r \leq 2\}$	D. Leviatan [11]
$r \geq 3$, $k=0$	G. A. Dzubenko, V. V. Listopad and I. A. Shevchuk [7]
$r \geq 3$, $k \geq 1$	present chapter
Negative results (monotone case)	
$r=0$, $k \geq 3$	A. S. Shvedov [15]
$r=2$, $k \geq 1$ and $r=1$, $k \geq 2$	K. A. Kopotun and V. V. Listopad [10]
Positive results (convex case)	
$r=0$, $k=2$, and, consequently, also for $\{(k,r)\|1 \leq k+r \leq 2\}$	D. Leviatan [12]
$r=0$, $k=3$, and, consequently, also for $\{(k,r)\|k+r=3\}$	K. A. Kopotun [9]
$r \geq 5$, $k=0$	K. A. Kopotun [8]
$r \geq 5$, $k \geq 1$	present chapter
Negative results (convex case)	
$r=0$, $k \geq 4$ and $r=1$, $k \geq 3$	A. S. Shvedov [15]
$r=4$, $k=0$, and, consequently, also for $\{(k,r)\|k+r \geq 4, r \leq 4\}$	K. A. Kopotun [8], see also Lemma 15

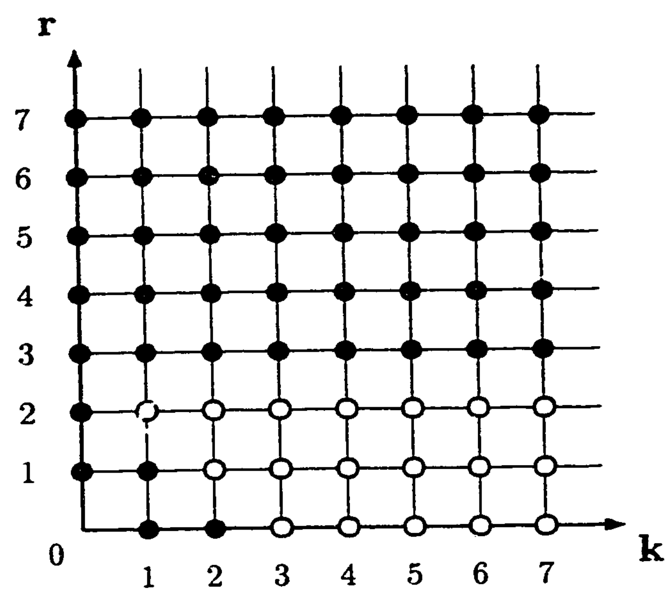

FIG. 1. MONOTONE APPROXIMATION

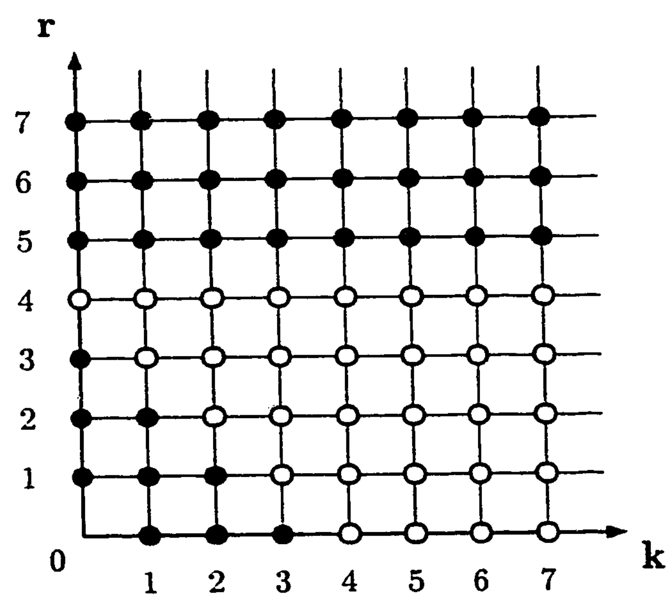

FIG. 2. CONVEX APPROXIMATION

Thus, investigation of the rate of shape preserving approximation of functions from $\mathbf{B}^r$ classes in terms of $n^{-r}\bar{\omega}^k_{\varphi,r}(f^{(r)}, n^{-1})$ is complete in the sense of the orders of moduli of smoothness.

However, more detailed consideration shows that some extra conditions on the smoothness of f sometimes allow direct results in the cases for which the general estimate is not correct. These conditions are given by the relation of f to $\mathbf{B}^r\bar{H}[k,\psi]$ classes. The necessary definitions and detailed discussions are given in the following section.

3.2 Shape Preserving Approximation of Functions from $\mathbf{B}^r\bar{H}[k,\psi]$ Classes

The following construction of Φ^k classes which was created by Stechkin (see [16], for example) will be useful.

Let Φ^k be the class of all k majorant functions, *i.e.*, continuous nondecreasing functions $\psi = \psi(t)$ on $[0,\infty)$ such that $\psi(0) = 0$ and $t^{-k}\psi(t)$ does not increase on $[0,\infty)$.

Obviously, $\bar{\omega}^k_{\varphi,r}(f,t)$ does not have to belong to Φ^k class. However, the following result is valid:

For the function

$$\omega^*(t) := \sup_{u>t} \frac{t^k\bar{\omega}^k_{\varphi,r}(f,u)}{u^k}, \qquad t \geq 0,$$

the inequalities

$$\bar{\omega}^k_{\varphi,r}(f,t) \leq \omega^*(t) \leq C(k)\,\bar{\omega}^k_{\varphi,r}(f,t)$$

hold, and if $\bar{\omega}^k_{\varphi,r}(f,t) \to 0$ as $t \to 0$, then $\omega^* \in \Phi^k$.

Also, for any $\psi \in \Phi^k$ or $\psi \sim 1$, $k \in \mathcal{N}$, and $r+1 \in \mathcal{N}$ there exists a function $f \in \mathbf{C}(-1,1)$ such that

$$C(k)\,\psi(t) \leq \bar{\omega}^k_{\varphi,r}(f,t) \leq C(k)\,\psi(t).$$

(Proofs of these statements can be found, for example, in [14].)

Now, let $\mathbf{B}^r\bar{H}[k,\psi]$ be the set of functions $f \in \mathbf{B}^r$ such that

$$\bar{\omega}^k_{\varphi,r}(f^{(r)},t) \leq \psi(t) \qquad \text{where } \psi \in \Phi^k \text{ or } \psi \sim 1.$$

Thus, we have the decomposition of the $\mathbf{B}^r$ class

$$\mathbf{B}^r = \bigcup_{\psi \in \{\psi | \psi \in \Phi^k \text{ or } \psi \sim 1\}} \mathbf{B}^r\bar{H}[k,\psi],$$

which, first of all, is complete, as for any function $f \in \mathbf{B}^r$ there exists $\psi \in \Phi^k$ or $\psi \equiv \mathrm{const}$ such that $f \in \mathbf{B}^r \bar{H}[k,\psi]$ and $\bar{\omega}^k_{\varphi,r}(f^{(r)},t) \sim \psi(t)$, $t > 0$. Second, this decomposition is "relatively precise," as for any $\psi \in \Phi^k$ or $\psi \sim 1$ there exists a function $f \in \mathbf{B}^r$ such that $\psi(t) \sim \bar{\omega}^k_{\varphi,r}(f^{(r)},t)$, $t > 0$.

Taking into account all this and also the inequality (74), one can conclude that Theorems 16 and 17 are corollaries of the following Theorems 18 and 19.

Theorem 18 *Let $k \in \mathcal{N}$, $\psi \in \Phi^k$ or $\psi \equiv 1$, and let $f \in \mathbf{B}^3\bar{H}[k,\psi]$ be a nondecreasing function on $[-1,1]$. Then for every $n \geq k+2$*

$$E_n^{(1)}(f) \leq C\,n^{-3}\,\psi(n^{-1}), \qquad C = C(k).$$

Theorem 19 *Let $k \in \mathcal{N}$, $\psi \in \Phi^k$ or $\psi \equiv 1$, and let $f \in \mathbf{B}^5\bar{H}[k,\psi]$ be a convex function on $[-1,1]$. Then for every $n \geq k+4$*

$$E_n^{(2)}(f) \leq C\,n^{-5}\,\psi(n^{-1}), \qquad C = C(k).$$

Remark . For $\psi \equiv 1$, Theorems 18 and 19 are consequences of the results obtained in [7] and [8], respectively. In this chapter only the case $\psi \in \Phi^k$ will be considered.

Also, as was shown in [10] (see also [8]), functions f which are being discussed in Lemmas 14 and 15 belong to the classes $\mathbf{B}^2\bar{H}[1,\mathrm{const}]$ and $\mathbf{B}^4\bar{H}[1,\mathrm{const}] \cap \mathbf{B}^3\bar{H}[1,Ct]$, respectively.

Now, using the following inclusions which are consequences of (74),

$$\mathbf{B}^2\bar{H}[1,1] \subset \mathbf{B}^l\bar{H}[3-l,Ct^{2-l}], \qquad l = 0 \text{ or } 1,$$

and

$$\mathbf{B}^3\bar{H}[1,t] \subset \mathbf{B}^l\bar{H}[4-l,Ct^{4-l}], \qquad 0 \leq l \leq 2,$$

and also the fact that

$$\psi \in \Phi^k \qquad \Rightarrow \qquad \psi(t) \geq Ct^k,\ 0 < t \leq 1,\ C = \mathrm{const},$$

one can obtain the following lemmas.

Lemma 20
There is no such constant C that the estimate $E_n^{(1)}(f) \leq Cn^{-2}\psi(n^{-1})$ is valid for every nondecreasing function from the $\mathbf{B}^2\bar{H}[1,\psi]$ class with $\psi(t) \sim 1$, $0 < t \leq 1$. Thus, for $0 \leq l \leq 2$ and any $k \geq 3-l$ the estimate $E_n^{(1)}(f) \leq Cn^{-l}\psi(n^{-1})$, $C = C(k)$, generally speaking, is not correct for nondecreasing functions from $\mathbf{B}^l\bar{H}[k,\psi]$ with $\psi(t) \geq t^{2-l}$, $0 < t \leq 1$.

Lemma 21
There is no such constant C that the estimate $E_n^{(2)}(f) \le Cn^{-3}\psi(n^{-1})$ is valid for every convex function from the $\mathbf{B}^3\bar{H}[1,\psi]$ class with arbitrary $\psi(t) \in \Phi^1$. Thus, for $0 \le l \le 3$ and any $k \ge 4-l$ the estimate $E_n^{(2)}(f) \le Cn^{-l}\psi(n^{-1})$, $C = C(k)$, generally speaking, is not correct for convex functions from $\mathbf{B}^l\bar{H}[k,\psi]$ with arbitrary $\psi(t) \in \Phi^k$.

For $l = 4$ and any $k \ge 1$ the estimate $E_n^{(2)}(f) \le Cn^{-4}\psi(n^{-1})$ is not correct for convex functions from $\mathbf{B}^4\bar{H}[1,\psi]$ with $\psi(t) \sim 1$, $0 < t \le 1$.

At the same time, the following theorems are valid.

Theorem 22 *Let $k \in \mathcal{N}$, and let f be a nondecreasing function such that $f \in \mathbf{B}^2\bar{H}[k,\psi]$ where $\psi(t) = t^\beta$, $t > 0$, $0 < \beta \le k$, and $\beta < 2$. Then for every $n \ge k+1$*

$$E_n^{(1)}(f) \le C\,n^{-2}\,\psi(n^{-1}) \qquad (\text{i.e., } E_n^{(1)}(f) \le C\,n^{-2-\beta}),$$

where $C = C(k)/(2-\beta)$.

Theorem 23 *Let $k \in \mathcal{N}$, and let f be a convex function such that $f \in \mathbf{B}^4\bar{H}[k,\psi]$ where $\psi(t) = t^\beta$, $t > 0$, $0 < \beta \le k$, and $\beta < 4$. Then for every $n \ge k+3$*

$$E_n^{(2)}(f) \le C\,n^{-4}\,\psi(n^{-1}) \qquad (\text{i.e., } E_n^{(2)}(f) \le C\,n^{-4-\beta}),$$

where $C = C(k)/(4-\beta)$.

It is worth mentioning that Theorems 18-23 are genaralizations of the direct result for

$$\hat{H}^\alpha := \begin{cases} \mathbf{B}^r\bar{H}[1,t^\beta] & \text{if } \alpha \notin \mathcal{N}, \text{ where } r := [\alpha] \text{ and } \beta := \alpha - r \\ \mathbf{B}^r\bar{H}[2,t] & \text{if } \alpha \in \mathcal{N}, \text{ where } r := \alpha - 1 \end{cases}$$

classes which are obtained in [10].

3.3 Characterization of $\mathbf{B}^r\bar{H}[k,\psi]$ Classes

We will write $\psi \in \mathcal{S}(r,k)$ (Conditions Z and Z_k of [1], see also [16]) if

$$r\int_0^t \psi(u)\,u^{-1}\,du + t^k \int_t^1 \psi(u)\,u^{-k-1}\,du = O(\psi(t)), \qquad t \in (0,1].$$

The following inverse theorem is known (see [5] and [14], for example).

Theorem F *Let $k \in \mathcal{N}$, $r+1 \in \mathcal{N}$, and $\psi \in \Phi^k \cap \mathcal{S}(r,k)$. If for a given function f on $[-1,1]$ and each $n \geq k+r-1$ the inequality*

$$E_n(f) \leq n^{-r}\, \psi(n^{-1})$$

holds (if $n = 0$ then we define the right-hand side of this inequality to be an absolute constant), then

$$f \in \mathbf{B}^r \bar{H}[k, C\psi], \qquad C = C(r,k).$$

Now a consequence of Theorem F and the direct results for the shape preserving approximation which are given above is the following constructive characteristic of $\mathbf{B}^r \bar{H}[k,\psi]$ classes with $\psi \in \Phi^k \cap \mathcal{S}(r,k)$.

Theorem 24 *Let $\psi \in \Phi^k \cap \mathcal{S}(r,k)$, where*

$$(k,r) \in \{(k,r) | k \in \mathcal{N},\ r \geq 3\} \cup \{(k,r) | k+r \leq 2,\ k \in \mathcal{N},\ r+1 \in \mathcal{N}\}\,.$$

A function f is nondecreasing and in the $\mathbf{B}^r \bar{H}[k, C\psi]$ class, if and only if for each $n \geq k+r-1$

$$E_n^{(1)}(f) \leq C\, n^{-r}\, \psi(n^{-1}), \qquad \text{where } C = C(r,k)\,.$$

Theorem 25 *Let $\psi \in \Phi^k \cap \mathcal{S}(r,k)$, where*

$$(k,r) \in \{(k,r) | k \in \mathcal{N},\ r \geq 5\} \cup \{(k,r) | k+r \leq 3,\ k \in \mathcal{N},\ r+1 \in \mathcal{N}\}\,.$$

A function f is convex and in the $\mathbf{B}^r \bar{H}[k, C\psi]$ class, if and only if for each $n \geq k+r-1$

$$E_n^{(1)}(f) \leq C\, n^{-r}\, \psi(n^{-1}), \qquad \text{where } C = C(r,k)\,.$$

Remark. For $(k,r) \in \{(k,r) | k+r \geq 4,\ 0 \leq r \leq 3\}$ and any $\psi \in \Phi^k \cap \mathcal{S}(r,k)$ Theorem 25 is false.

For $(k,r) \in \{(k,0) | k \geq 3\} \cup \{(k,1) | k \geq 2\}$ and $\psi \in \Phi^k \cap \mathcal{S}(r,k)$ such that $\psi(t) \geq C t^{k-1}$, $t > 0$, Theorem 24 is false.

For $r = 2$ Theorem 24 is valid in the case $\psi(t) = t^\beta \in \Phi^k$ and $\beta < 2$, and for $r = 4$ Theorem 25 is valid in the case $\psi(t) = t^\beta \in \Phi^k$ and $\beta < 4$, with the same dependance of the constants C on k and β as in Theorems 22 and 23, respectively. In the other cases this question is still open.

3.4 Notation

Throughout this chapter the following notation and definitions are used (cf. [7]-[10], [13], [14]):

$$\Delta := \rho(n^{-1}, x) = \Delta_n(x), \qquad x \in I,$$

$$T_{j,n}(x) := T_j(n, 3\chi; \emptyset; \emptyset)(x),$$

$$\tilde{T}_{j,n}(x) := T_j(n, 3\chi + 1; x_j; x_{j-1})(x)$$

are algebraic polynomials of degree $6\chi(2n-1)+1$ and $6\chi(2n-1)+4n+1$, respectively.

$$\sigma_{j,n}(x) := \int_{-1}^{x} (\alpha_1 T_{j,n}(y) + (1-\alpha_1) T_{j+1,n}(y))\, dy$$

and

$$\tilde{\sigma}_{j,n}(x) := \int_{-1}^{x} (\alpha_2 \tilde{T}_{j,n}(y) + (1-\alpha_2)\tilde{T}_{j+1,n}(y))\, dy, \qquad 1 \le j \le n-1,$$

where numbers α_1 and α_2, $0 \le \alpha_1 \le 1$, $0 \le \alpha_2 \le 1$, are chosen so that $\sigma_{j,n}(1) = \tilde{\sigma}_{j,n}(1) = 1 - x_j$ (see [8]), are polynomials of degree $6\chi(2n-1)+2$ and $6\chi(2n-1)+4n+2$, respectively.

$$J_{n,\xi}(t) = \left(\frac{\sin nt/2}{\sin t/2}\right)^{2\xi+2} \left(\int_{-\pi}^{\pi} \left(\frac{\sin nt/2}{\sin t/2}\right)^{2\xi+2} dt\right)^{-1}$$

is the Jackson type kernel.

$$D_{\zeta,n,\xi}(y,x) = \frac{1}{(\zeta-1)!}\frac{\partial^\zeta}{\partial x^\zeta}(x-y)^{\zeta-1} \int_{\arccos x - \arccos y}^{\arccos x + \arccos y} J_{n,\xi}(t)\, dt, \qquad x, y \in I$$

is the Dzjadyk type kernel.

χ, ξ, and ζ in the above definitions are integers which will be chosen later.

For brevity, we denote

$$L_k(x, f, \{x_0, h\}) := L_k(x, f, x_0, x_0 + \rho(x_0, h), \ldots, x_0 + k\rho(x_0, h)).$$

Also, for $a \ne b$ and $(x-a)(x-b) \le 0$ let

$$S(x, l; a, b) := \int_a^x (y-a)^l (b-y)^l\, dy \left(\int_a^b (y-a)^l (b-y)^l\, dy\right)^{-1},$$

$S(x,l;a,b) = 0$ if $(x-a)(x-b) > 0$ and $|x-a| < |x-b|$, and $S(x,l;a,b) = 1$ otherwise.

3.5 Auxiliary Statements

In our proofs we will use the method from [13] (see also [7] and [8]) which is a modification of DeVore's ideas concerning the decomposition of the approximated function (see [3] and [4]).

The following analog of Whitney's theorem in terms of "nonuniform" moduli of smoothness $\bar{\omega}_\varphi^k$ will be important for the proofs given below.

Lemma G *(see Lemma 18.2 and ineq. (18.13) of [14]). Denote* $\rho_0 := \rho(h, x_0)$. *Let* $[x_0, x_0 + (k-1)\rho_0] \subset [a,b] \subset I$. *Then for every* $x \in [a,b]$, *the following inequality holds:*

$$|f(x) - L_{k-1}(x, f; \{x_0, h\})| \le C\,(|x - x_0| + \rho_0)^{2k} \rho_0^{-2k} \bar{\omega}_\varphi^k(f, h, [a,b]) .$$

In particular, for every $x \in [x_0, x_0 + (k-1)\rho_0]$,

$$|f(x) - L_{k-1}(x, f; \{x_0, h\})| \le C\,\bar{\omega}_\varphi^k(f, h, [x_0, x_0 + (k-1)\rho_0]) ,$$

where $C = C(k)$.

The following lemma shows the connection between moduli of smoothness of different orders.

Lemma H *(see Lemma 18.4 of [14]). Let* $k+1 \in \mathcal{N}$, $r \in \mathcal{N}$, $0 \le l \le r-1$, $(x, x + (k+r-l)\rho) \in I$, *and*

$$G_{k,l,r}(x,h) := \begin{cases} h^{2r-2l}\rho^{l-r}(w_{2l-r}(x, k+r-l, h))^{-1} & \text{if } l > r/2 , \\ h^r \rho^{-r/2} |\ln(h w_1(x, k + r/2, h)\rho^{-1})| & \text{if } l = r/2 , \\ h^r \rho^{-l} & \text{if } l < r/2 . \end{cases}$$

If $f \in \mathbf{B}^r$, *then*

$$\bar{\Delta}_\rho^{k+r-l}(f^{(l)}, x) \le C\, G_{k,l,r}(x,h) \bar{\omega}_{\varphi,r}^k(f^{(r)}, h) , \qquad h > 0 , \tag{76}$$

where $C = C(r,k)$. *In particular, inequality (74) holds.*

Remark . Another consequence of (76) is the estimate

$$\bar{\omega}_\varphi^{k+r-l}(f^{(l)}, t) \le C\, t^{r-2l} \bar{\omega}_{\varphi,r}^k(f^{(r)}, t) , \qquad t > 0 \qquad \text{and} \qquad l < r/2 .$$

For $2l = r$ we have $\sup_x G_{k,l,r}(x,h) = +\infty$ which presents the main difficulty in this case. In fact, these will be the cases for $r = 2$ and $r = 4$ for monotone and convex approximation, respectively.

It turns out that the estimate (76) can be improved for some classes of functions. Namely, the following result is valid.

Lemma 26 *Let $l \in \mathcal{N}$, $f \in \mathbf{B}^{2l}\bar{H}[k,\psi]$ with $\psi(t) = t^\beta \in \Phi^k$ (i.e., $0 < \beta \le k$) and such that $\beta < 2l$. Then the following estimate holds:*

$$|\bar{\Delta}_\rho^{k+l}(f^{(l)},x)| \le C\,h^{2l}\rho^{-1}\,\psi(h)\,, \qquad (x, x+(k+l)\rho) \subset (-1,1)$$

and $h > 0$, where $C = C(r,l)/(2l-\beta)$.

Proof The beginning of the proof is analogous to that of Lemma 18.4 in [14]. First, using the formula for integral presentation of the usual differences we get

$$\begin{aligned}
\bar{\Delta}_\rho^{k+l}(f^{(l)},x) &= \bar{\Delta}_\rho^k\left(\int_0^\rho \cdots \int_0^\rho f^{(2l)}(\cdot + u_1 + u_2 + \cdots + u_l)\,du_1 \ldots du_l, x\right) \\
&= \int_0^\rho \cdots \int_0^\rho \bar{\Delta}_\rho^k\left(f^{(2l)}, x + u_1 + u_2 + \cdots + u_l\right)\,du_1 \ldots du_l \\
&=: \Theta(x)\,.
\end{aligned}$$

Now, if $[x, x+k\rho] \subset [-1+h^2, 1-h^2]$ and $x < 0$ (for $x > 0$ considerations are analogous), then

$$\begin{aligned}
|\Theta(x)| \le \int_0^\rho \cdots \int_0^\rho &\left|\bar{\Delta}^k_{\rho(\theta h,y)}(f^{(2l)},y)(1+y)^l(1-y-k\rho(\theta h,y))^l\right| \\
&\times (1+y)^{-l}(1-y-k\rho(\theta h,y))^{-l}\,du_1 \ldots du_l\,,
\end{aligned}$$

where $y := x + u_1 + u_2 + \cdots + u_l$, and $\theta = \theta(y)$ is chosen so that $\rho(h,x) = \rho(\theta h, x + u_1 + u_2 + \cdots + u_l)$.

This yields

$$\begin{aligned}
|\Theta(x)| &\le C\int_0^\rho \cdots \int_0^\rho \sup_{0<\tilde{h}<Ch}\ \sup_{y\in[-1+h^2,1-h^2]} \left|\bar{\Delta}^k_{\rho(\tilde{h},y)}(f^{(2l)},y)\,w_{2l}(y,k,\tilde{h})\right| \\
&\qquad \times (1+x)^{-l}\,du_1 \ldots du_l \\
&\le C\int_0^\rho \cdots \int_0^\rho \psi(Ch)\,(1+x)^{-l}\,du_1 \ldots du_l \\
&\le C\,\rho^l\,(1+x)^{-l}\,\psi(Ch) \\
&\le C\,h^{2l}\rho^{-l}\,\psi(h)\,, \qquad C = C(k,l)\,.
\end{aligned}$$

Now, let us consider the case $-1 < x < -1 + h^2$ (if $[x, x+k\rho] \cap [1-h^2, 1] \ne \emptyset$, considerations are analogous).

The following Shevchuk's identity will be employed (see (1.27) of [14]):

> Let $N+1$ given points $y_0, y_1, \ldots, y_N$ be such that $M+1$ of them coincide with $x_0, x_1, \ldots, x_M$, where $N \ge M \ge 2$. Then the following identity holds:
>
> $$\begin{aligned} &[x_0, x_1, \ldots, x_M; f] \\ &= \sum_{n=0}^{N-M}(y_{n+M} - y_n)\,[y_n, \ldots, y_{n+M}; f]\,[x_0, \ldots, x_M; \Pi_{n,M}]\,, \end{aligned}$$
>
> where $\Pi_{n,M}(x_s) := \prod_{j=1}^{M-1}(x_s - y_{n+j})_+$.

We fix $y := x + u_1 + u_2 + \cdots + u_l$, choose $m \geq 1$ so that $\rho/(2^m - 1) \leq y + 1 < \rho/(2^{m-1} - 1)$ and denote $\nu := \rho/(2^m - 1)$.

Let $m + k$ points z_i be defined by

$$z_i = y + (2^i - 1)\nu, \qquad 0 \leq i \leq m,$$

$$z_i = y + (i - m + 1)\rho, \qquad m + 1 \leq i \leq m + k - 1.$$

Now, $\bar{\Delta}_\rho^k(f^{(2l)}, y) = [y, y + \rho, \ldots, y + k\rho; f^{(2l)}]\rho^k k!$.

At the same time, for $k \geq 2$, choosing $M = k$ and $N = m + k - 1$, we have

$$[y, y + \rho, \ldots, y + k\rho; f^{(2l)}]$$

$$= \sum_{n=0}^{m-1} (z_{n+k} - z_n)\,[z_n, \ldots, z_{n+k}; f^{(2l)}]\,[y, y + \rho, \ldots, y + k\rho; \Pi_{n,k}],$$

where

(77) $$\Pi_{n,k}(y + i\rho) := \prod_{j=1}^{k-1} (y + i\rho - z_{n+j})_+, \qquad 0 \leq i \leq k.$$

Each term of this sum will be examined below.

First of all, let us note that $(z_{n+k} - z_n) \sim 2^n\nu$ for all $0 \leq n \leq m - 1$. Now,

$$\begin{aligned}
&\left|[z_n, \ldots, z_{n+k}; f^{(2l)}]\right| \\
&= \left|\frac{f^{(2l)}(z_{n+1}) - L_{k-1}(z_{n+1}; f^{(2l)}; z_n, z_{n+2}, z_{n+3}, \ldots, z_{n+k})}{(z_{n+1} - z_n)(z_{n+1} - z_{n+2})(z_{n+1} - z_{n+3}) \ldots (z_{n+1} - z_{n+k})}\right| \\
&\leq C(2^n\nu)^{-k}\left|f^{(2l)}(z_{n+1}) - \tilde{L}_{k-1}(z_{n+1}; f^{(2l)}; \{z_n, \tilde{h}\})\right. \\
&\qquad \left. - L_{k-1}(z_{n+1}; f^{(2l)} - \tilde{L}_{k-1}; z_n, z_{n+2}, z_{n+3}, \ldots, z_{n+k})\right|
\end{aligned}$$

where $\tilde{h}$ is chosen so that $z_n + (k-1)\rho(\tilde{h}, z_n) = z_{n+k}$, which implies that $\tilde{h} \sim \sqrt{2^n\nu}$.

Taking this into consideration and using Lemma G we have

$$\begin{aligned}
\left|[z_n, \ldots, z_{n+k}; f^{(2l)}]\right| &\leq C(2^n\nu)^{-k}\,\bar{\omega}_\varphi^k(f^{(2l)}, \sqrt{2^n\nu}, [z_n, z_{n+k}]) \\
&= C(2^n\nu)^{-k} \sup_{0<\bar{h}\leq\sqrt{2^n\nu}}\ \sup_{z:[z, z+k\rho(z,\bar{h})]\subset[z_n, z_{n+k}]} \\
&\qquad \times \left|\bar{\Delta}^k_{\rho(\bar{h},z)}(f^{(2l)}, z)\right| \\
&\leq C(2^n\nu)^{-k-l} \sup_{\bar{h}} \sup_{z} \left|w_{2l}(z, k, \bar{h})\bar{\Delta}^k_{\rho(\bar{h},z)}(f^{(2l)}, z)\right| \\
&\leq C(2^n\nu)^{-k-l}\,\psi\left(\sqrt{2^n\nu}\right), \qquad C = C(k, l).
\end{aligned}$$

Now, $|[y, y + \rho, \ldots, y + k\rho; \Pi_{n,k}]|$ with $0 \leq n \leq m - 1$ will be estimated. First, let us consider the case $n \leq m - k + 1$. Let

$$p_{k-1}(z) := \prod_{j=1}^{k-1} (z - z_{n+j}), \qquad z \in [y, y + k\rho]$$

and

$$\tilde{p}_{k-1}(z) := \begin{cases} p_{k-1}(z) & \text{if } z \le z_{n+k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the equality

$$\Pi_{n,k}(z) := \prod_{j=1}^{k-1} (z - z_{n+j})_+ = p_{k-1}(z) - \tilde{p}_{k-1}(z)$$

gives

$$\begin{aligned} |[y, y+\rho, \dots, y+k\rho; \Pi_{n,k}]| &= |[y, y+\rho, \dots, y+k\rho; \tilde{p}_{k-1}]| \\ &= |\tilde{p}_{k-1}(y)|\,(k!\rho^k)^{-1} \\ &\le C\rho^{-k}(2^n\nu)^{k-1}. \end{aligned}$$

Now, if $m - k + 1 < n \le m - 1$, then $2^n\nu \sim 2^m\nu \sim \rho$. This yields

$$\begin{aligned} |[y, y+\rho, \dots, y+k\rho; \Pi_{n,k}]| &\le C\rho^{-k} \sum_{i=0}^{k} \prod_{j=1}^{k-1} (y + i\rho - z_{n+j})_+ \\ &\le C\rho^{-1} \\ &\le C\rho^{-k}(2^n\nu)^{k-1}, \qquad C = C(k). \end{aligned}$$

Putting all these estimates together we get the following:

$$\left|[y, y+\rho, \dots, y+k\rho; f^{(2l)}]\right| \le C(k,l)\,\rho^{-k} \sum_{n=0}^{m-1} (2^n\nu)^{-l}\psi(\sqrt{2^n\nu}).$$

Thus, using the inequality $\beta < 2l$, one has

$$\begin{aligned} \left|\bar{\Delta}_\rho^k(f^{(2l)}, y)\right| &\le C(k,l) \sum_{n=0}^{\infty} (2^n(y+1))^{-l}\psi(\sqrt{2^n(y+1)}) \\ &\le C(k,l) \sum_{n=0}^{\infty} (2^n(y+1))^{-l+\beta/2} \\ &\le C(k,l)\,(y+1)^{-l+\beta/2}\,(1 - 2^{-l+\beta/2})^{-1} \\ &\le C(k,l)\,(y+1)^{-l+\beta/2}\,\frac{1}{2l-\beta}. \end{aligned}$$

And now the desired estimate emerges as

$$\begin{aligned} |\Theta(x)| &\le C \int_0^\rho \dots \int_0^\rho (x + u_1 + \dots + u_l + 1)^{-l+\beta/2}\, du_1 \dots du_l \\ &\le C \begin{Bmatrix} |\bar{\Delta}_\rho^l((x+1)^{\beta/2}, x)| & \text{if } \beta/2 \notin \mathcal{N} \\ |\bar{\Delta}_\rho^l((x+1)^{\beta/2}\ln(x+1), x)| & \text{otherwise} \end{Bmatrix} \\ &\le C\,h^\beta \qquad \text{where } C = C(k,l)\frac{1}{2l-\beta}. \end{aligned}$$

The last inequality is a consequence of Dzjadyk's [6, p. 160-161] understanding that if $\beta/2 \in \mathcal{N}$ then $l \geq \beta/2+1$.

For $k=1$ considerations are simpler. The difference is that instead of (77) one should consider the identity

$$[y, y+\rho; f^{(2l)}] = \rho^{-1} \sum_{n=0}^{m-1} (z_{n+1}-z_n)[z_{n+1}, z_n; f^{(2l)}].$$

Thus, the lemma is proved. ∎

In our proofs we will deal with the first derivative of a function f in the monotone case and with the second one in the convex case. Obviously, the condition $f \in \mathbf{B}^r$, $r \in \mathcal{N}$ implies that $f \in \mathbf{C}^\mu(-1,1)$ for all $\mu \leq r$. However, it would be more convenient to have the continuity of the derivatives on the closed interval I.

The following lemma gives sufficient conditions for a function f to have continuous derivatives on $[-1,1]$.

Lemma 27
Let $k \in \mathcal{N}$, $(r+1) \in \mathcal{N}$, $\mu \in \mathcal{N}$, $\psi \in \Phi^k$ be such that $\int_0^1 \psi(u)\, u^{-2\mu+r-1}\, du < \infty$. If for a function f and each $n \geq k+r-1$ there exists a polynomial $p_n \in \Pi_n$ such that

$$|f(x)-p_n(x)| \leq n^{-r}\, \psi(n^{-1}), \qquad x \in I,$$

then $f \in \mathbf{C}^\mu[-1,1]$.

Despite the fact that this lemma is probably known to the reader, its proof is adduced here since the author failed to find any references to it.

Proof For any $n_0 \in \mathcal{N}$ the series $\sum_{n=n_0}^{M} (p_{2^{n+1}}(x) - p_{2^n}(x))$ converges uniformly to $f(x) - p_{2^{n_0}}(x)$ as $M \to \infty$, and

$$|p_{2^{n+1}}(x) - p_{2^n}(x)| \leq 2^{1-nr}\, \psi(2^{-n}), \qquad x \in I.$$

Applying Markov's inequality one has

$$\left|p^{(\mu)}_{2^{n+1}}(x) - p^{(\mu)}_{2^n}(x)\right| \leq C\, 2^{2n\mu-nr}\, \psi(2^{-n}), \qquad x \in I.$$

This implies

$$\sum_{n=n_0}^{\infty} \left|p^{(\mu)}_{2^{n+1}}(x) - p^{(\mu)}_{2^n}(x)\right| \leq C \sum_{n=n_0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} u^{-2u+r-1}\, \psi(u)\, du$$

$$= C \int_0^{2^{-n_0}} u^{-2u+r-1}\, \psi(u)\, du < \infty.$$

Thus, $f \in \mathbf{C}^\mu[-1,1]$, and the proof of the lemma is complete. ∎

The following corollary is a consequence of Lemma 27 and Theorem E.

Corollary I *The following implications are valid:*

$$\begin{array}{lll} f \in \mathbf{B}^3\bar{H}[k,\psi], & \psi\in\Phi^k & \Rightarrow\ f\in \mathbf{C}^1[-1,1],\\ f \in \mathbf{B}^2\bar{H}[k,t^\beta], & 0<\beta\le k & \Rightarrow\ f\in \mathbf{C}^1[-1,1],\\ f \in \mathbf{B}^5\bar{H}[k,\psi], & \psi\in\Phi^k & \Rightarrow\ f\in \mathbf{C}^2[-1,1],\\ f \in \mathbf{B}^4\bar{H}[k,t^\beta], & 0<\beta\le k & \Rightarrow\ f\in \mathbf{C}^2[-1,1]. \end{array}$$

Lemma J *(see [13], for example). Let $p+1\in\mathcal{N}$ and $q+1\in\mathcal{N}$. The Dzjadyk-type kernel $D_{\zeta,n,\xi}(y,x)$ is a polynomial in x of degree $<(\xi+1)(n-1)$, and the following inequalities hold:*

$$\left|\frac{\partial^p}{\partial x^p}D_{\zeta,n,\xi}(y,x)\right| \le C\,\Delta^{\xi-p-1}(|x-y|+\Delta)^{-\xi}, \qquad C=C(p,\xi,\zeta),$$

$$\left|\frac{1}{p!}\int_{-1}^{1}(y-x)^q\frac{\partial^p}{\partial x^p}D_{\zeta,n,\xi}(y,x)\,dy \;-\; \delta_{p,q}\right| \le C\,n^{-\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\}},$$

$C=C(p,q,\xi,\zeta)$, where $\delta_{p,q}$ is the Kronecker symbol, and the integral in the last inequality is a polynomial of degree $\le q-p$ (it is identically equal to zero if $q<p$).

Now, let us note that the methods of proofs of Theorems 18-23 as well as of all auxiliary statements are the same. In connection with all this it would be inexpedient to give their proofs separately. We will give the complete statements of auxiliary propositions for all four cases, using the following abridgements. For the sake of convenience throughout the chapter, in the wording $\mathbf{B}^2\bar{H}[k,\psi]$ and $\mathbf{B}^4\bar{H}[k,\psi]$ it will be implied that $\psi(t)=t^\beta$, $0<\beta\le k$, $\beta<2$ and $\psi(t)=t^\beta$, $0<\beta\le k$, $\beta<4$, respectively (however, most of the statements are true for all functions $\psi\in\Phi^k$). We will also use the notation [m_i], [m_ii], [c_i], and [c_ii] in order to emphasize the cases designed for the proofs of Theorems 22, 18, 23, and 19, respectively. Also, we set variables Ξ and Λ to have the following values in these cases:

$$\begin{array}{llll} [\text{m\_i}] & \Xi=1,\ \Lambda=2, & [\text{m\_ii}] & \Xi=1,\ \Lambda=3,\\ [\text{c\_i}] & \Xi=2,\ \Lambda=4, & [\text{c\_ii}] & \Xi=2,\ \Lambda=5. \end{array}$$

Thus, in order to follow the proof of Theorem 22, for example, it is enough to pay attention to the statements marked by [m_i], understanding that in this case $\Xi=1$ and $\Lambda=2$.

The following theorem is a generalization of the direct theorem (Theorem E) for $\mathbf{B}^\Lambda\bar{H}[k,\psi]$ classes.

Theorem 28 *Let a set $F\subset I$ and a function Q be such that $Q\in\mathbf{B}^\Lambda\bar{H}[k,\psi]$ and $Q^{(\Xi)}(x)=0$ for $x\in F$. Then the polynomial*

$$d_n(x,Q) := \int_{-1}^{1}(Q(y)-\Omega(y,Q))\,D_{\zeta,n,\xi}(y,x)\,dy + \Omega(x,Q)$$

approximates Q and its derivatives so that

$$\left|Q^{(p)}(x) - d_n^{(p)}(x,Q)\right| \le C_1 n^{-\Lambda}\Delta^{-p}\psi(n^{-1})\left(\frac{\Delta}{\Delta + \operatorname{dist}(x, I\setminus F)}\right)^{\xi-2k-2\Lambda+\Xi-1},$$

$$x \in I,\ p+1 \in \mathcal{N},\quad \text{and} \quad 0 \le p \le \Xi,$$

where the polynomial $\Omega(x,Q)$ *is defined by*

$$\begin{aligned}
[\mathrm{m}_]\quad \Omega(x,Q) &:= Q(-1) + \textstyle\int_{-1}^{x} L_{k+\Lambda-2}(z, Q', \{-1, \sqrt{2(k+\Lambda-2)^{-1}}\})\,dz\,,\\
[\mathrm{c}_]\quad \Omega(x,Q) &:= Q(-1) + Q'(-1)(x+1)\\
&\quad + \textstyle\int_{-1}^{x}\int_{-1}^{t} L_{k+\Lambda-3}(z, Q'', \{-1, \sqrt{2(k+\Lambda-3)^{-1}}\})\,dz\,dt\,.
\end{aligned}$$

Proof In order to avoid overloading of the text by unnecessary notation let us give the proof in the case [c_i]. The proofs for the other cases are analogous.

Denote $g(x) = Q(x) - \Omega(x,Q)$. Then $g \in \mathbf{B}^4\bar{H}[k,\psi]$ and applying Lemmas G, H, and 26 we get

$$\begin{aligned}
|g(x)| &= \left|\int_{-1}^{x}\int_{-1}^{t}\left(Q''(z) - L_{k+1}(z, Q'', \{-1, \sqrt{2(k+1)^{-1}}\})\right)\,dz\,dt\right|\\
&\le C\,\bar{\omega}^{k+2}(Q'', 1) \le C\,\psi(1)\,.
\end{aligned}$$

Thus,

$$Q^{(p)}(x) - d_n^{(p)}(x,Q) = g^{(p)}(x) - \int_{-1}^{1} g(y)\frac{\partial^p}{\partial x^p}D_{\zeta,n,\xi}(y,x)\,dy\,.$$

Now, let x be fixed and, for convenience, such that $x + (k+1)\Delta \le 1$. Denote

$$l(y) := g(x) + g'(x)(y-x) + \int_x^y\int_x^t L_{k+1}(z, g'', \{x, n^{-1}\})\,dz\,dt\,,$$

and note that $l^{(p)}(x) = g^{(p)}(x)$, $p = 0,1,2$.

For $y \in [x, x+(k+1)\Delta]$ we have the estimate

$$\begin{aligned}
|l(y)| &\le |l(y) - g(y)| + |g(y)|\\
&\le C\,\psi(1) + \int_x^y\int_x^t\left|L_{k+1}(z, g'', \{x, n^{-1}\}) - g''(z)\right|\,dz\,dt\\
&\le C\,\psi(1) + C\,\bar{\omega}_\varphi^{k+2}(g'', n^{-1}) \le C\,\psi(1)\,.
\end{aligned}$$

Therefore, applying Markov's inequality for all $0 \le j \le k+3$ we have

$$|l^{(j)}(y)| \le C\Delta^{-j}\psi(1)\,, \qquad y \in [x, x+(k+1)\Delta]$$

and, in particular, $|l^{(j)}(x)| \le C\Delta^{-j}\psi(1)$.

We expand the polynomial $l(y)$ into Taylor series

$$l(y) = l(x) + \sum_{j=1}^{k+3}(y-x)^j l^{(j)}(x)/j!\,.$$

Thus,

$$
\begin{aligned}
& g^{(p)}(x) - \int_{-1}^{1} g(y)\frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy \\
= \; & \int_{-1}^{1} (l(y)-g(y))\,\frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy \\
& + \sum_{j=0}^{k+3}\frac{1}{j!} l^{(j)}(x)\left(\delta_{j,p}p! - \int_{-1}^{1}(y-x)^j\,\frac{\partial^p}{\partial x^p} D_{\zeta,n,\xi}(y,x)\,dy\right) \\
= \; & A(x) + \sum_{j=0}^{k+3}\frac{1}{j!}B(x,j)\,.
\end{aligned}
$$

Using Lemma J and the above estimate for $l^{(j)}(x)$ we have

$$
\begin{aligned}
\left|\sum_{j=0}^{k+3}\frac{1}{j!}B(x,j)\right| \;\le\; & \sum_{j=0}^{k+3} C\Delta^{-j}\psi(1)n^{-\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\}} \\
\le\; & C\Delta^{-k-3}\psi(1)n^{-\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\}} \\
\le\; & C\Delta^{-k-3}\psi(n^{-1})n^{k-\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\}} \\
\le\; & Cn^{-4}\Delta^{\xi-2k-p-7}\psi(n^{-1})\,.
\end{aligned}
$$

The last inequality is true if

$$
\min\{2\xi+1,\zeta+(1-(-1)^\zeta)/2\} \;\ge\; 2\xi-k-4 \qquad \text{and} \qquad \xi\ge k+6\,.
$$

Now, let us estimate $A(x)$, using Lemma J and the following estimate:

$$
\begin{aligned}
|l(y)-g(y)| \;\le\; & \int_x^y\int_x^t \left|L_{k+1}(z,g'',\{x,n^{-1}\}) - g''(z)\right|\,dz\,dt \\
\le\; & C|y-x|^2\bar{\omega}_\varphi^{k+2}(g'',n^{-1})\left(\frac{|y-x|+\Delta}{\Delta}\right)^{2k+4} \\
\le\; & C|y-x|^2\left(\frac{|y-x|+\Delta}{\Delta}\right)^{2k+4} n^{-4}\Delta^{-2}\psi(n^{-1})\,, \qquad y\in I\,.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|A(x)| \;\le\; & C\int_{-1}^{1}|l(y)-g(y)|\,\Delta^{\xi-p-1}(|x-y|+\Delta)^{-\xi}\,dy \\
\le\; & C\int_{-1}^{1}(|x-y|+\Delta)^{2k-\xi+6}\Delta^{\xi-p-2k-7}n^{-4}\psi(n^{-1})\,dy \\
\le\; & C\Delta^{\xi-p-2k-7}n^{-4}\psi(n^{-1})\int_0^\infty (t+\Delta)^{2k-\xi+6}\,dt \\
\le\; & C\Delta^{-p}n^{-4}\psi(n^{-1})\,, \qquad \text{with} \qquad \xi-2k-8\ge 0\,.
\end{aligned}
$$

The estimate of the theorem is proved in the case $x \notin F$.

Now, if $x \in F$, then $g''(x) = -L_{k+1}(z, Q'', \{-1, \sqrt{2(k+1)^{-1}}\})$, *i.e.*, $g(x)$ is a polynomial of degree $\le k+1$ on F. Thus, if $[x, x+(k+1)\Delta] \subset F$, then $l(y) = g(y)$ for $y \in F$, and, therefore, for these x

$$\begin{aligned} |A(x)| &\le C \int_{I \setminus F} (|x-y| + \Delta)^{2k-\xi+6} \Delta^{\xi-p-2k-7} n^{-4} \psi(n^{-1})\, dy \\ &\le Cn^{-4}\Delta^{-p}\psi(n^{-1}) \left(\frac{\Delta}{\Delta + \operatorname{dist}(x, I \setminus F)} \right)^{\xi-2k-7}. \end{aligned}$$

The case when $x \in F$ and $(x + (k+1)\Delta) \notin F$ follows from the above, since in this case $\operatorname{dist}(x, I \setminus F) \sim \Delta$.

The proof is complete. ∎

Remark . In the proofs of Theorem 28 for the other cases it is sufficient to have the following inequalities:

$$\min\{2\xi+1, \zeta + (1-(-1)^\zeta)/2\} \ge 2\xi - k - 4 \qquad \text{and} \qquad \xi \ge 2k+10\,.$$

Lemma 29 [m_]*([7], see also [13]). Let E be a union of some intervals I_j. Then the polynomial*

$$\tilde{Q}_n(x, E) := \sum_{i \in \{i | I_{j_i} \in E\}} \left(T_{j_i,n}(x) - \tilde{T}_{j_i,n}(x) \right)$$

of degree $\le 6\chi(2n-1) + 4n + 1$ satisfies the following inequalities:

$$\begin{aligned} &(1) \quad \left|\tilde{Q}_n(x, E)\right| \le C_2\,, && x \in I\,, \\ &(2) \quad \tilde{Q}_n'(x, E) \ge -C_3\Delta^{-1}\,, && x \in E\,, \\ &(3) \quad \tilde{Q}_n'(x, E) \ge C_4\Delta^{-1}\left(\Delta/(\Delta + \operatorname{dist}(x, E))\right)^{12\chi-1}\,, && x \in I \setminus E\,. \end{aligned}$$

Lemma 29 [c_]*([8]). Let E be a union of some intervals I_j. Then the polynomial*

$$\tilde{Q}_n(x, E) := \sum_{i \in \{i | I_{j_i} \in E, I_{j_i+1} \in E\}} (x_{j_i-1} - x_{j_i})^{-1} \left(\sigma_{j_i,n}(x) - \tilde{\sigma}_{j_i,n}(x)\right)$$

of degree $\le 2(3\chi+1)(2n+1)$ satisfies the inequalities:

$$\begin{aligned} &(1) \quad \left|\tilde{Q}_n(x, E)\right| \le C_2\,, && x \in I\,, \\ &(2) \quad \tilde{Q}_n''(x, E) \ge -C_3\Delta^{-2}\,, && x \in E\,, \\ &(3) \quad \tilde{Q}_n''(x, E) \ge C_4\Delta^{-2}\left(\Delta/(\Delta + \operatorname{dist}(x, \tilde{E}))\right)^{12\chi-2}\,, && x \in I \setminus E\,, \end{aligned}$$

where $\tilde{E} := E \setminus \{I_{j_i} | I_{j_i \pm 1} \notin E\}$.

Lemma 30 **[m_]**_([7], see also [13])._ *Let* $0 \le g'(x) \le \Delta^{-1}$, $x \in I$, *then the polynomial*

$$R_n(x,g) := g(-1) + \sum_{j=1}^{n} (g(x_{j-1}) - g(x_j))\, T_{j,n}(x)$$

of degree $\le 6\chi(2n-1)+1$ *is nondecreasing on* I, *and the following inequality holds:*

$$|g(x) - R_n(x,g)| \le C_5, \qquad x \in I.$$

Lemma 30 **[c_]**_([8])._ *Let* $0 \le g''(x) \le \Delta^{-2}$, $x \in I$, *then the polynomial*

$$\begin{aligned} R_n(x,g) &:= g(x_{n-1}) + [x_n, x_{n-1}; g](x - x_{n-1}) \\ &+ \sum_{j=1}^{n-1} [x_{j+1}, x_j, x_{j-1}; g](x_{j-1} - x_{j+1})\sigma_{j,n}(x) \end{aligned}$$

of degree $\le 6\chi(2n-1)+2$ *is convex on* I, *and the following inequality holds:*

$$|g(x) - R_n(x,g)| \le C_5, \qquad x \in I.$$

Lemma 31 *Let a function* $g \in \mathbf{B}^{\Lambda}\bar{H}[k,\psi]$ *and a set* $\Im_j$, *which contains* $2k + 2\Lambda - 2\Xi - 1$ *neighboring intervals* I_j, i.e., $\Im_j = I_j \cup I_{j+1} \cup \ldots \cup I_{j+2(k+\Lambda-\Xi-1)}$, *be given. If for every* $0 \le i \le 2(k+\Lambda-\Xi-1)$ *there exists a point* $\tilde{x}_i \in I_{j+i}$ *at which*

$$|g^{(\Xi)}(\tilde{x}_i)| \le n^{-\Lambda}\psi(n^{-1})(\rho(n^{-1},\tilde{x}_i))^{-\Xi},$$

then

$$|g^{(\Xi)}(x)| \le C_6 n^{-\Lambda} \psi(n^{-1})\Delta^{-\Xi}$$

for all $x \in \Im_j$.

Proof The identity

$$\begin{aligned} g^{(\Xi)}(x) \;=\; & \left(g^{(\Xi)}(x) - L_{k+\Lambda-\Xi-1}\left(x, g^{(\Xi)}, \left\{x_{j+2(k+\Lambda-\Xi-1)}, n^{-1}\right\}\right)\right) \\ & - \tilde{L}_{k+\Lambda-\Xi-1}\left(x, g^{(\Xi)} - L_{k+\Lambda-\Xi-1}, \tilde{x}_0, \tilde{x}_2, \tilde{x}_4, \ldots, \tilde{x}_{2(k+\Lambda-\Xi-1)}\right) \\ & + \tilde{L}_{k+\Lambda-\Xi-1}\left(x, g^{(\Xi)}, \tilde{x}_0, \tilde{x}_2, \tilde{x}_4, \ldots, \tilde{x}_{2(k+\Lambda-\Xi-1)}\right), \end{aligned}$$

the inequality

$$\begin{aligned} & \left|g^{(\Xi)}(x) - L_{k+\Lambda-\Xi-1}\left(x, g^{(\Xi)}, \left\{x_{j+2(k+\Lambda-\Xi-1)}, n^{-1}\right\}\right)\right| \\ & \le C\,\bar{\omega}_{\varphi}^{k+\Lambda-\Xi}\left(g^{(\Xi)}, n^{-1}, \Im_j\right) \le C\, n^{-\Lambda}\Delta^{-\Xi}\psi(n^{-1}), \qquad x \in \Im_j, \end{aligned}$$

which is a consequence of Lemmas G, H, and 26, and the estimate

$$\begin{aligned} & |L_m(x, f; a_0, a_1, \ldots, a_m)| \\ & \le \left(\max_{0\le i,j\le m} |a_i - a_j|\right)^m \left(\min_{0\le i,j\le m} |a_i - a_j|\right)^{-m} \max_{0\le i\le m} |f(a_i)| \end{aligned}$$

complete the proof of the lemma. ∎

3.6 Decomposition of Approximated Functions

Let a function f belong to $\Delta^{\Xi} \cap \mathbf{B}^{\Lambda}\bar{H}[k,\psi]$.

Definition 1 *The interval I_j is called an* **interval of type I** *if, for all $x \in I_j$,*

$$f^{(\Xi)}(x) \leq C_6(C_3 + C_4)n^{-\Lambda}\Delta^{-\Xi}\psi(n^{-1}),$$

an **interval of type II** *if it is not an interval of type* **I** *and, for all $x \in I_j$,*

$$f^{(\Xi)}(x) \geq (C_3 + C_4)n^{-\Lambda}\Delta^{-\Xi}\psi(n^{-1}).$$

Let all other intervals be of **type III**.

We denote intervals of types **I**, **II**, and **III** by E_1, E_2, and E_3, respectively.

Remark . It follows from Lemma 31 that there cannot be more than $2(k + \Lambda - \Xi - 1)$ neighboring intervals of type **III**, *i.e.*, each set $\Im_j$ contains at least one interval of type **I** or **II**.

Now, let the set [m_] $E_1 \cup E_3$ [c_] $E_1 \cup E_3 \cup \{I_j \in E_2 | I_{j\pm 1} \notin E_2\}$ be presented as a finite union of nonintersecting intervals. Let G_1 be the set containing all those intervals which include not less that $40k + 10$ intervals I_j:

$$G_1 = [x_{j_1}, x_{j_0}] \cup [x_{j_3}, x_{j_3}] \cup \dots, \qquad 0 < j_\nu < j_{\nu+1} \leq n.$$

Let us denote $\bar{j}_\nu := j_\nu + \frac{1}{2}(1 + (-1)^\nu)$ and let $S_\nu(x) := 1$ if $|x_{j_\nu}| = 1$, and $S_\nu(x) := S(x, k+4; x_{j_\nu}, \bar{x}_{\bar{j}_\nu}$ if $|x_{j_\nu}| \neq 1$ (see Section 3.4 for the definition of $S(x, l; a, b)$).

Definition 2 *Let $g_1(x) := 0$ for $x \notin G_1$,*

$$g_1(x) := f^{(\Xi)}(x)S_\nu(x) \qquad \text{for} \qquad x \in [x_{j_\nu}, \bar{x}_{\bar{j}_\nu}]$$

and $g_1(x) := f^{(\Xi)}(x)$ in all other cases.

Denote $g_2(x) := f^{(\Xi)}(x) - g_1(x)$ and

$$\begin{array}{lrcl} \text{[m\_]} & f_1(x) & := & f(-1) + \int_{-1}^{x} g_1(y)\,dy, \\ & f_2(x) & := & \int_{-1}^{x} g_2(y)\,dy; \\ \text{[c\_]} & f_1(x) & := & f(-1) + f'(-1)(x+1) + \int_{-1}^{x}\int_{-1}^{t} g_1(y)\,dy\,dt, \\ & f_2(x) & := & \int_{-1}^{x}\int_{-1}^{t} g_2(y)\,dy\,dt. \end{array}$$

Obviously, the following correlations hold:

$$f_1(x) + f_2(x) = f(x)$$

$$g_1(x) \geq 0 \qquad \text{and} \qquad g_2(x) \geq 0 \qquad \text{for all} \qquad x \in I.$$

Lemma 32 *The following inequality holds:*

$$g_1(x) \le C_7 n^{-\Lambda} \psi(n^{-1}) \Delta^{-\Xi}, \qquad x \in I.$$

Proof Analogously to the proof of Lemma 31, one can show the validity of the estimate $f^{(\Xi)} \le Cn^{-\Lambda}\psi(n^{-1})\Delta^{-\Xi}$, $x \in G_1$. Together with $0 \le S_\nu(x) \le 1$, this proves the lemma. ∎

Lemma 33 *The function f_2 belongs to $\mathbf{B}^\Lambda \bar{H}[k, C_8\psi]$.*

Lemma 33 is a consequence of the following lemmas.

Lemma 34 [m_] *Let the interval $[a,b] \subset [-1+n^{-2}, 1-n^{-2}]$ be such that $|a-b| \sim \sqrt{1-a^2}/n$, where n is a fixed natural number which is sufficiently large. And let a given function $g \in \mathbf{B}^r\bar{H}[k,\psi]$, $r \ge 1$, be such that*

$$|g'(x)| \le n^{-r}\psi(n^{-1})(b-a)^{-1} \qquad x \in [a,b].$$

Define the function G so that

$$G'(x) := g'(x)\, S(x,l;a,b) \qquad -1 \le x \le 1$$

and $G(-1) = g(-1)$, where $l \ge k+r$.

Then $G \in \mathbf{B}^r\bar{H}[k, C\psi]$ with C independent of n.

Lemma 34 [c_] *Let the interval $[a,b] \subset [-1+n^{-2}, 1-n^{-2}]$ be such that $|a-b| \sim \sqrt{1-a^2}/n$, where n is a fixed natural number which is sufficiently large. And let a given function $g \in \mathbf{B}^r\bar{H}[k,\psi]$, $r \ge 2$, be such that*

$$|g''(x)| \le n^{-r}\psi(n^{-1})(b-a)^{-2} \qquad x \in [a,b].$$

Define the function G so that $G''(x) := g''(x)\, S(x,l;a,b)$, $-1 \le x \le 1$, $G'(-1) = g'(-1)$, and $G(-1) = g(-1)$, where $l \ge k+r$.

Then $G \in \mathbf{B}^r\bar{H}[k, C\psi]$ with C independent of n.

We will prove Lemma 34 [c_] for $[a,b] \subset [-1+n^{-2}, 0]$ (for $b \ge 0$ considerations are similar). The case [m_] is analogous with the only difference being that instead of the second derivatives one should deal with the first ones.

Proof of Lemma 34 [c_] We will use the fact that the interval $[a,b]$ is separated from the endpoints of the interval $[-1,1]$. Also, it is enough to consider the behavior of G "near" the interval $[a,b]$, as outside of $[a,b]$ G either is a linear function or it coincides with g.

Namely, it is sufficient to prove that

$$\sup_{0<h\le t}\ \sup_{[x,x+k\rho]\cap[a,b]\ne\emptyset} \left|(1+x)^{r/2}\bar{\Delta}^k_\rho(G^{(r)},x)\right| \le C\psi(t),$$

where t is such that $t < (10kn)^{-1}$.

Now, let $0 < h \le t$ be fixed and note that if $[x, x + k\rho] \cap [a, b] \neq \emptyset$, then $x \in \left[a - 3k\sqrt{1+a}h, b\right]$, and the following holds:

$$(78) \qquad (1+x) \sim (1+a) \qquad \text{and} \qquad \rho \sim \sqrt{1+x}h \sim \sqrt{1+a}h\,.$$

For convenience, denote $S(x) := S(x, l; a, b)$ and note that

$$(79) \qquad |S^{(p)}(x)| \le C\,(b-a)^{-p}\,, \qquad 0 \le p \le l\,, \qquad x \in I\,.$$

(In fact, $S^{(p)}(x) = 0$ for $x \notin [a, b]$ and $p \ge 1$.)

We will use the Marchaud inequality for the usual moduli of smoothness (see, for example, [5], [6], and [14])

$$\begin{aligned}\omega^j(g, t; [a,b]) \;&\le\; C\,t^j \left(\int_t^{b-a} u^{-j-1}\,\omega^k(g, u; [a,b])\,du \right.\\ &\left. \quad +(b-a)^{-j}\|g\|_{[a,b]}\right)\,, \qquad 1 \le j \le k-1\,,\end{aligned}$$

and the Besov inequality ([2]) which is given by

$$\begin{aligned}\|g^{(j)}\|_{[a,b]} \;&\le\; C\left((b-a)^{r-j}\omega^k(g^{(r)}, b-a; [a,b]) \right.\\ &\left. \quad +(b-a)^{-j}\|g\|_{[a,b]}\right)\,, \qquad 0 \le j \le r\,.\end{aligned}$$

Using (79) and also the identities

$$G^{(r)} = (g''S)^{(r-2)} = \sum_{i=0}^{r-2}\binom{r-2}{i} g^{(i+2)}S^{(r-i-2)}$$

and

$$\bar\Delta_h^k(g_1 g_2, x_0) = \sum_{j=0}^{k}\binom{k}{j}\bar\Delta_h^j(g_1, x_0)\bar\Delta_h^{k-j}(g_2, x_0 + jh)\,,$$

we have for $x_0 \in \left[a - 3k\sqrt{1+a}h, b\right]$

$$\begin{aligned}&\left|\bar\Delta_\rho^k(G^{(r)}, x_0)\right|\\ &\le C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}\rho^{k-j}(b-a)^{i+j+2-r-k}|\bar\Delta_\rho^j(g^{(2+i)}, x_0)|\,.\end{aligned}$$

Now, (78) and the Besov inequality yield, for $0 \le i \le r-2$,

$$\begin{aligned}\|g^{(i+2)}\|_{[a,b]} \;&\le\; C\left((b-a)^{r-i-2}\omega^k(g^{(r)}, b-a; [a,b]) + (b-a)^{-i}\|g''\|_{[a,b]}\right)\\ &\le\; C\left((b-a)^{r-i-2}(1+a)^{-r/2}\psi\left((b-a)(1+a)^{-1/2}\right)\right.\\ &\left.\quad +(b-a)^{-i-2}n^{-r}\psi(n^{-1})\right)\\ &\le\; C(b-a)^{r-i-2}(1+a)^{-r/2}\psi(n^{-1})\,.\end{aligned}$$

Using the last estimate, (78), and the Marchaud inequality, we have for $j < k + r - i - 2$:

$$
\begin{aligned}
(80)\quad \bar{\omega}_\varphi^j(g^{(i+2)}, t; [a,b]) &\sim \omega^j\left(g^{(i+2)}, \sqrt{1+a}t; [a,b]\right) \\
&\le C(1+a)^{j/2}t^j\left(\int_{\sqrt{1+a}t}^{b-a} u^{-j-1}\omega^{k+r-i-2}(g^{(i+2)}, u; [a,b])\,du\right. \\
&\quad \left. + (b-a)^{-j}\|g^{(i+2)}\|_{[a,b]}\right) \\
&\le C(1+a)^{j/2}t^j\left(\int_{\sqrt{1+a}t}^{b-a} u^{r-i-j-3}(1+a)^{-r/2}\psi\left(u(1+a)^{-1/2}\right)du\right. \\
&\quad \left. + (b-a)^{r-i-j-2}(1+a)^{-r/2}\psi(n^{-1})\right) \\
&\le C(1+a)^{(j-r)/2}t^{j-k}\psi(t) \\
&\quad \times\left((1+a)^{-k/2}\int_{\sqrt{1+a}t}^{b-a} u^{k+r-i-j-3}\,du + n^{-k}(b-a)^{r-i-j-2}\right) \\
&\le C(1+a)^{(j-r)/2}t^{j-k}\psi(t)n^{-k}(b-a)^{r-i-j-2}\,.
\end{aligned}
$$

Note that $k + r - i - j = 2$ only if $i = r - 2$ and $j = k$, and, thus, $\bar{\omega}_\varphi^k(g^{(r)}, t; [a,b]) \le C(1+a)^{-r/2}\psi(t)$, *i.e.*, (80) is true in this case also.

Now, putting all these estimates together we have the following inequalities for any x_0, such that $[x_0, x_0 + k\rho(h, x_0)] \cap [a,b] \ne \emptyset$:

$$
\begin{aligned}
&\left|(1+x_0)^{r/2}\bar{\Delta}_\rho^k(G^{(r)}, x_0)\right| \\
&\quad \le C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}\rho^{k-j}(b-a)^{-k}(1+a)^{j/2}n^{-k}t^{j-k}\psi(t) \\
&\quad \le C\sum_{i=0}^{r-2}\sum_{j=0}^{k}\binom{r-2}{i}\binom{k}{j}(b-a)^{-k}(1+a)^{k/2}n^{-k}\psi(t) \\
&\quad \le C\psi(t)\,.
\end{aligned}
$$

Thus, the lemma is proved. ∎

Denote $\mathcal{E} := \{I_j | I_j \in E_2, I_j \notin G_1\}$ (clearly, $\mathcal{E} = E_2$ in the case [m_]) and $G_2 := \{x | \mathrm{dist}(x, \mathcal{E}) \le 3^{2k+7}\Delta\}$.

It follows from Definition 2 that $g_2(x) = 0$ for $x \in I \setminus G_2$. Note, that for $n_1 \ge n$ the following inequality holds:

$$
\frac{\rho(n_1^{-1}, x)}{\mathrm{dist}(x, G_2) + \rho(n_1^{-1}, x)} \le C_9\frac{\Delta}{\mathrm{dist}(x, \mathcal{E}) + \Delta}\,.
$$

Now, we choose ξ, ζ, and χ so that all the conditions in the proofs above are valid. For example, $\xi = 24k$, $\zeta = 48k$, and $\chi = k$ will do.

The following lemma is a consequence of Theorem 28 and Lemma 33.

Lemma 35 *For any integer $n_1 \geq n$ the polynomial $d_{n_1}(x, f_2)$ has the following properties:*

$$|f_2(x) - d_{n_1}(x, f_2)| \leq C_{10} n^{-\Lambda} \psi(n^{-1}),$$

$$d_{n_1}^{(\Xi)}(x, f_2) \geq -C_{11} n_1^{-\Lambda} \psi(n_1^{-1}) (\rho(n_1^{-1}, x))^{-\Xi} \left(\frac{\Delta}{\operatorname{dist}(x, \mathcal{E}) + \Delta} \right)^{12k}, \ x \in I \setminus \mathcal{E},$$

and

$$d_{n_1}^{(\Xi)}(x, f_2) \geq (C_3 + C_4) n^{-\Lambda} \psi(n^{-1}) \Delta^{-\Xi} - C_{11} n_1^{-\Lambda} \psi(n_1^{-1}) (\rho(n_1^{-1}, x))^{-\Xi}, \ x \in \mathcal{E},$$

where $C_{10} = C_1 C_8$ and $C_{11} = C_1 C_8 C_9^{12k}$.

3.7 Proofs of Theorems 18-23

Let $n_1 \in \mathcal{N}$, $n_1 \geq n$. Denote

$$\pi_{n_1}(x) := n^{-\Lambda} \psi(n^{-1}) \tilde{Q}_n(x, \mathcal{E}) + d_{n_1}(x, f_2) + R_n(x, f_1).$$

Then $\pi_{n_1}(x)$ is a polynomial of degree $< 50kn_1$.

It follows from Lemmas 29, 30, 32, and 35 that

$$|f(x) - \pi_{n_1}(x)| \leq (C_2 + C_{10} + C_5 C_7) n^{-\Lambda} \psi(n^{-1}), \ x \in I,$$

$$\begin{aligned} \pi_{n_1}^{(\Xi)}(x) \ \geq \ & \left(C_4 C_{12} n^{-\Lambda} \psi(n^{-1}) \Delta^{-\Xi} - C_{11} n_1^{-\Lambda} \psi(n_1^{-1}) (\rho(n_1^{-1}, x))^{-\Xi} \right) \\ & \times \left(\frac{\Delta}{\operatorname{dist}(x, \mathcal{E}) + \Delta} \right)^{12k}, \ x \in I \setminus \mathcal{E}, \end{aligned}$$

and

$$\pi_{n_1}^{(\Xi)}(x) \geq C_4 n^{-\Lambda} \psi(n^{-1}) \Delta^{-\Xi} - C_{11} n_1^{-\Lambda} \psi(n_1^{-1}) (\rho(n_1^{-1}, x))^{-\Xi}, \ x \in \mathcal{E},$$

where $C_{12} = 3^{12k(8k+23)}$.

Now, let us choose n_1 so that $n_1 = C$[illegible], where $C_{13} := \{[4C_{11}/C_4 C_{12}] + 2\} \in \mathcal{N}$. Then the following inequalities hold.

$$\begin{array}{lll} [\mathrm{m\_i}] & \pi'_{n_1}(x) > 0, & x \in I \setminus (I_1 \cup I_n), \\ & \pi'_{n_1}(x) > -C_{11} \psi(n^{-1}), & x \in I_1 \cup I_n; \\ [\mathrm{m\_ii}] & \pi'_{n_1}(x) > 0, & x \in I; \\ [\mathrm{c\_i}] & \pi''_{n_1}(x) > 0, & x \in I \setminus (I_1 \cup I_n), \\ & \pi''_{n_1}(x) > -C_{11} \psi(n^{-1}), & x \in I_1 \cup I_n; \\ [\mathrm{c\_ii}] & \pi''_{n_1}(x) > 0, & x \in I. \end{array}$$

Thus, Theorems 18 and 19 are proved for $n \geq C_{13}$.

In order to obtain analogous results for Theorems 22 and 23, the following lemmas will be useful.

Lemma 36 [m_] *For the algebraic polynomial of degree* $< 5n$,

$$M_n(x) := \int_{-1}^{x} \left(\frac{\sin(n/2 \arccos t)}{n \sin(1/2 \arccos t)} \right)^{10} dt \,,$$

the following inequalities hold:

$$\begin{array}{ll} M_n'(x) \geq 0 \,, & x \in I \,, \\ 0 \leq M_n(x) \leq 10^5 n^{-2} \,, & x \in I \,, \\ M_n'(x) \geq 2^{-10} \,, & x \in I_1 \,. \end{array}$$

Lemma 36 [c_] *For the algebraic polynomial of degree* $< 5n$,

$$\mathcal{M}_n(x) := \int_{-1}^{x} \int_{-1}^{y} \left(\frac{\sin(n/2 \arccos t)}{n \sin(1/2 \arccos t)} \right)^{10} dt \, dy \,,$$

the following inequalities hold:

$$\begin{array}{ll} \mathcal{M}_n''(x) \geq 0 \,, & x \in I \,, \\ 0 \leq \mathcal{M}_n(x) \leq 2 \times 10^4 n^{-4} \,, & x \in I \,, \\ \mathcal{M}_n''(x) \geq 2^{-10} \,, & x \in I_1 \,. \end{array}$$

Proof Lemma 36 [c_] is Lemma 8 from [8]. Lemma 36 [m_] can be verified by direct computations with the use of the inequalities $2t/\pi \leq \sin t \leq t$, $0 \leq t \leq \pi/2$, or, applying Markov's inequality, it can be immediately derived from Lemma 36 [c_]. ∎

Now, the polynomial

$$\begin{array}{ll} [\mathrm{m\_}] & \bar{\pi}_n(x) := \pi_{n_1}(x) + 2^{10} C_{11} \psi(n^{-1})(M_n(x) - M_n(-x)) \,, \\ [\mathrm{c\_}] & \bar{\pi}_n(x) := \pi_{n_1}(x) + 2^{10} C_{11} \psi(n^{-1})(\mathcal{M}_n(x) + \mathcal{M}_n(-x)) \,, \end{array}$$

of degree $< 50kn_1$ satisfies Theorem 22 in the monotone case and Theorem 23 in the convex one.

Thus, Theorems 22 and 23 are proved for $n \geq C_{13}$.

For the other n, the theorems are consequences of the cases $n = k + 1$ for Theorem 22, $n = k + 2$ for Theorem 18, $n = k + 3$ for Theorem 23, and $n = k + 4$ for Theorem 19, for which it is sufficient to choose

$$\pi_n(x) := \Omega(x, f) + 5 \max\{C_0(2, k), C_0(3, k), C_0(4, k), C_0(5, k)\} \psi(\sqrt{2/k}) x^{\Xi} \,.$$

The proofs of Theorems 18-23 are now complete.

References

[1] N. K. Bari and S. B. Stechkin (1956), *Best approximations and differential properties of two conjugate functions*, Trud. Moskov. Mat. Obsch., **5**: 483–522.

[2] O. V. Besov (1963), *Continuation of functions with conservation of their differential and difference properties in L_p*, Dokl. Akad. Nauk USSR, **150**(3): 963–966.

[3] R. A. DeVore (1977), *Monotone approximation by polynomials*, SIAM J. Math. Anal., **8**(5): 906–921.

[4] R. A. DeVore (1977), *Monotone approximation by splines*, SIAM J. Math. Anal., **8**(5): 891–905.

[5] Z. Ditzian and V. Totik (1987): Moduli of Smoothness. Berlin: Springer-Verlag.

[6] V. K. Dzjadyk (1977): Introduction to the Theory of Uniform Approximation of Functions by Polynomials. Moscow: Izdat. "Nauka".

[7] G. A. Dzubenko, V. V. Listopad and I. A. Shevchuk (1993), *Uniform estimates of monotone polynomial approximation*, Ukr. Mat. Zhurnal, **45**(1): 38–43.

[8] K. A. Kopotun (1992), *Uniform estimates of convex approximation of functions by polynomials*, Mat. Zametki, **51**(3): 35–46. (English translation: Math. Notes, **51**: 245–254.)

[9] K. A. Kopotun (1994), *Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials*, Constr. Approx., **10**(2): 153–178.

[10] K. A. Kopotun and V. V. Listopad (1994), *Remarks on monotone and convex approximation by algebraic polynomials*, Ukr. Mat. Zhurnal, **46**(9).

[11] D. Leviatan (1988), *Monotone and comonotone approximation revisited*, J. Approx. Theory, **53**: 1–16.

[12] D. Leviatan (1986), *Pointwise estimates for convex polynomial approximation*, Proc. Amer. Math. Soc., **98**(3): 471–474.

[13] I. A. Shevchuk (1989), *On coapproximation of monotone functions*, Dokl. Akad. Naul SSSR, **308**(3): 537–541. (English translation: Soviet. Math. Dokl., **40**(2): 349–354.)

[14] I. A. Shevchuk (1992), Approximation by Polynomials and Traces of the Functions Continuous on an Interval, Kiev: Naukova Dumka.

[15] A. S. Shvedov (1981), *Orders of coapproximation of functions by algebraic polynomials*, Mat. Zametki, **29**(1): 117–130. (English translation: Math. Notes, **30**: 63–70.)

[16] S. B. Stechkin (1951), *On the order of best approximations of continuous functions*, Izv. Akad. Nauk USSR, Ser. Mat., **15**(3): 219–242.

CHAPTER 4

COCONVEX POLYNOMIAL APPROXIMATION OF TWICE DIFFERENTIABLE FUNCTIONS [3]

4.1 Introduction and Main Result

Our primary interest in this chapter is comonotone and coconvex polynomial approximation, that is, approximation of a function f which is piecewise monotone (or piecewise convex, *i.e.*, has finitely many inflection points) by polynomials which are comonotone (or coconvex) with f. Let $\|\cdot\| := \|\cdot\|_\infty$ denote the uniform norm and $\varphi(x) := \sqrt{1-x^2}$.

The following result on comonotone approximation of continuous functions is known.

Theorem K *Let $f \in \mathrm{C}[-1,1]$ have $1 \le r < \infty$ changes of monotonicity at the points $\{y_i\}_{i=1}^r : -1 < y_1 < \ldots < y_r < 1$. Then there exist polynomials $p_n^*, p_n^{**} \in \Pi_n$ which are comonotone with f on $[-1,1]$ and such that*

$$\|f - p_n^*\| \le C^*(r, d(r))\, \omega^2(f, n^{-1}) \tag{81}$$

and

$$\|f - p_n^{**}\| \le C^{**}(r, d_0)\, \omega_\varphi(f, n^{-1}), \tag{82}$$

where $d(r) := \min\{y_1+1, y_2-y_1, \ldots, y_r-y_{r-1}, 1-y_r\}$ and $d_0 := \min\{y_1+1, 1-y_r\}$.

For piecewise monotone differentiable functions we have the following.

Theorem L *Let $f \in \mathrm{C}^1[-1,1]$ have $1 \le r < \infty$ changes of monotonicity at the points $\{y_i\}_{i=1}^r : -1 < y_1 < \ldots < y_r < 1$. Then for each $n \ge 1$ there is a polynomial $p_n \in \Pi_n$ comonotone with f and such that*

$$\|f - p_n\| \le C(r, d_0)\, n^{-1}\, \omega_\varphi(f', n^{-1}) \tag{83}$$

and

$$\|f' - p_n'\| \le C(r, d_0)\, \omega_\varphi(f', n^{-1}), \tag{84}$$

where $d_0 := \min\{y_1 + 1, 1 - y_r\}$.

[3] A version of this chapter has been published in J. Approx. Theory (1995) **83**: 141–156.

Theorem L and the estimate (82) in Theorem K were proved by D. Leviatan [5]. Estimate (81) is due to A. S. Shvedov [13] and X. M. Yu [16]. It was also shown by A. S. Shvedov [13] that the constant C^* in (81) can not be replaced by that independent of $d(r)$. Moreover, the estimate (81) is exact in the sense that ω^2 can not be replaced by ω^3. This is an immediate consequence of S. P. Zhou [17].

Other relevant results can be found in [1], [3], [6]–[12], for example.

Thus, for comonotone polynomial approximation there are quite a few satisfactory results. At the same time, it seems that little is known about coconvex approximation. The only direct results of this type which we are aware of at present are the following.

i) R. K. Beatson and D. Leviatan remarked in [1] that it is possible to obtain Jackson type theorems for coconvex approximation of functions with only one inflection point.

ii) X. M. Yu [15] obtained a Jackson type estimate of coconvex approximation of a function with one *regular convexity-turning* point.

iii) Also, in [15] X. M. Yu quoted her result on coconvex approximation of differentiable functions (which are at least in $\mathbf{C}^3[-1,1]$) with some extra conditions on convexity-turning points.

The goal of this chapter is to present a result on coconvex approximation which is analogous to Theorem L. Namely, we prove the following theorem.

Theorem 37 (coconvex approximation) *Let $f \in \mathbf{C}^2[-1,1]$ have $1 \le r < \infty$ inflection points at $\{y_i\}_{i=1}^r : -1 < y_1 < \ldots < y_r < 1$, $d_0 := \min\{y_1+1, 1-y_r\}$ and $d(r) := \min\{y_1+1, y_2-y_1, \ldots, y_r-y_{r-1}, 1-y_r\}$. Then there exists a constant $A = A(r)$ such that for each $n > \frac{A(r)}{d(r)}$ there is a polynomial $p_n \in \mathbf{\Pi}_n$ satisfying $f''(x)p_n''(x) \ge 0, x \in [-1,1]$ and such that*

(85) $$\|f - p_n\| \le C(r)\, n^{-2}\, \omega_\varphi(f'', n^{-1})\,,$$

(86) $$\|f' - p_n'\| \le C(r)\, n^{-1}\, \omega_\varphi(f'', n^{-1})$$

and

(87) $$\|f'' - p_n''\| \le \frac{C(r)}{\sqrt{d_0}}\, \omega_\varphi(f'', n^{-1})\,.$$

Corollary 38 (comonotone approximation)
Let f be the same as in Theorem L. Then there exists a constant $A = A(r)$ such that for each $n > \frac{A(r)}{d(r)}$ there is a polynomial $p_n \in \mathbf{\Pi}_n$ satisfying $f'(x)p_n'(x) \ge 0, x \in [-1,1]$, and the following inequalities hold:

(88) $$\|f - p_n\| \le C(r)\, n^{-1}\, \omega_\varphi(f', n^{-1})$$

and

(89) $$\|f' - p_n'\| \le \frac{C(r)}{\sqrt{d_0}}\, \omega_\varphi(f', n^{-1}).$$

4.2 Auxiliary Statements and Results

Using the identity $\mathrm{sgn}_{x_j}(x) = 2\chi_j(x) - 1$ a.e. we conclude that the polynomial $\tilde{T}_j(x) := 2T_j(x) - 1$ (see Lemma 11) sufficiently approximates $\mathrm{sgn}_{x_j}(x)$. Also, it is easy to see that $\tilde{T}_j(x)$ is increasing on I_j. Later on we will need similar polynomial (it will be denoted by $Q_j(x)$) which satisfies one extra condition: $\mathrm{sgn}(Q_j(x)) = \mathrm{sgn}_\alpha(x)$ *for some* $\alpha \in I_j$ (In other words, we want the polynomial not only approximate $\mathrm{sgn}_\alpha(x)$ and be increasing on I_j, but also be copositive with $\mathrm{sgn}_\alpha(x)$). Our construction of $\tilde{T}_j(x)$ does not immediately yield this equality. However, $\tilde{T}_j(x)$ can be refined to satisfy it. Namely, the following lemma is valid (note that we assume $a_i \le x_{j+1}, 1 \le i \le m,\ b_i \ge x_{j-2}, 1 \le i \le k$).

Lemma 39 *Let n and $1 \le j \le n$ be fixed, $a_i \le x_{j+1}, 1 \le i \le m,\ b_i \ge x_{j-2}, 1 \le i \le k$ and $\alpha \in I_j$. Then there exist numbers $A = 2^\nu, \nu \in \mathcal{N}$ and $0 \le \xi \le 1$ such that for the polynomial*

$$\begin{aligned} Q_j(x) &:= Q_j(n,\mu,\alpha;a_1,\dots,a_m;b_1,\dots,b_k)(x) \\ &= 2\,\{\xi\, T_{j_1}(An,\mu;a_1,\dots,a_m;b_1,\dots,b_k)(x) \\ &\quad + (1-\xi)\, T_{j_2}(An,\mu;a_1,\dots,a_m;b_1,\dots,b_k)(x)\} - 1\,, \end{aligned}$$

where indices $j_1 = j_1(A)$ and $j_2 = j_2(A)$ are chosen so that $x_{j_1,An} = \bar{x}_{j+1}$ and $x_{j_2,An} = \bar{x}_{j-1}$, the following is true:

$$Q_j'(x)\prod_{i=1}^{m}(x-a_i)\prod_{i=1}^{k}(b_i - x) \ge 0\,, \tag{90}$$

in particular, $Q_j(x)$ increases on $I_{j+1}\cup I_j \cup I_{j-1}$,

$$|\mathrm{sgn}_\alpha(x) - Q_j(x)| \le C(\mu)\,\psi_j^{2\mu-m-k-1}\,, \tag{91}$$

$$|\mathrm{sgn}_\alpha(x) - Q_j(x)| \le 2/3,\ x \notin I_{j+1}\cup I_j\cup I_{j-1}\,, \tag{92}$$

$$Q_j(x)\mathrm{sgn}_\alpha(x) \ge 0 \ \ \textit{for all}\ \ x\in[-1,1] \tag{93}$$

and

$$\left|Q_j'(x)\right| \le C(\mu)\,\psi_j^{2\mu-m-k}\,h_j^{-1}\,. \tag{94}$$

Proof First of all, denoting $n_1 := An$ we obtain the following consequence of Lemma 11 for any $x \notin I_{j+1}$ (note that $\chi_{j_1,n_1}(x) = \chi_j(x)$ for $x \notin I_{j+1}$):

$$\begin{aligned} &|\chi_j(x) - T_{j_1}(x)| \\ &\le C\left(\frac{h_{j_1,n_1}}{|x-\bar{x}_{j+1}| + h_{j_1,n_1}}\right)^{2\mu-m-k-1} \le C\left(\frac{h_{j_1,n_1}}{h_j/12 + h_{j_1,n_1}}\right)^{2\mu-m-k-1} \\ &\le C\left(\frac{h_{j_1,n_1}}{h_j}\right)^{2\mu-m-k-1} \le C\left(\frac{n}{n_1}\right)^{2\mu-m-k-1} = \frac{C}{A^{2\mu-m-k-1}} \le \frac{1}{3} \end{aligned}$$

for sufficiently large A.

Analogously, for any $x \notin I_{j-1}$ choosing n_1 to be large in comparison with n one has

$$|\chi_{j-1}(x) - T_{j_2}(x)| \le C\left(\frac{h_{j_2,n_1}}{|x-\bar{x}_{j-1}| + h_{j_2,n_1}}\right)^{2\mu-m-k-1}$$

$$\le C\left(\frac{h_{j_2,n_1}}{h_j}\right)^{2\mu-m-k-1} \le C\left(\frac{n}{n_1}\right)^{2\mu-m-k-1} = \frac{C}{A^{2\mu-m-k-1}} \le \frac{1}{3}.$$

Now let A be fixed and such that the above inequalities are satisfied.

Since $\alpha \in I_j$ we have, in particular, $T_{j_1}(\alpha) \ge 2/3$ and $T_{j_2}(\alpha) \le 1/3$. Hence, there exists $0 \le \xi \le 1$ such that $Q_j(\alpha) = 2\{\xi T_{j_1}(\alpha) + (1-\xi)T_{j_2}(\alpha)\} - 1 = 0$.

The above estimates yield

$$|\mathrm{sgn}_\alpha(x) - Q_j(x)| \le 2/3, \; x \notin I_{j+1} \cup I_j \cup I_{j-1},$$

which is the inequality (92).

Now note that (91) and (94) are immediate corollaries of Lemma 11, the fact that

$$\psi_j \sim C, \; x \in I_{j+1} \cup I_j \cup I_{j-1}$$

and the observation that

$$\max\{\psi_{j_1,n_1}, \psi_{j_2,n_1}\} \le 10A^2\psi_j$$

(see inequality (62) of [4], for example).

Finally, using the definitions of $Q_j(x)$ and $T_j(x)$ we have

$$\begin{aligned}\frac{1}{2}Q_j'(x) &= \big\{\xi T_{j_1}'(An,\mu;a_1,\ldots,a_m;b_1,\ldots,b_k)(x) \\ &\quad + (1-\xi)T_{j_2}'(An,\mu;a_1,\ldots,a_m;b_1,\ldots,b_k)(x)\big\} \\ &= \prod_{i=1}^m (x-a_i)\prod_{i=1}^k (b_i - x)\left[\frac{\xi t_{j_1,An}(x)^\mu}{\Pi_{j_1}(An,\mu;a_1,\ldots,a_m;b_1,\ldots,b_k)}\right. \\ &\quad \left. + \frac{(1-\xi)t_{j_2,An}(x)^\mu}{\Pi_{j_2}(An,\mu;a_1,\ldots,a_m;b_1,\ldots,b_k)}\right].\end{aligned}$$

Since the expression in square brackets is always positive (see inequalities (28) and Proposition 10) we conclude that $Q_j'(x)$ is copositive with $\prod_{i=1}^m(x - a_i)\prod_{i=1}^k(b_i - x)$. In particular, since $a_i \le x_{j+1}, 1 \le i \le m$ and $b_i \ge x_{j-2}, 1 \le i \le k$, $Q_j(x)$ increases on $I_{j+1} \cup I_j \cup I_{j-1}$. Together with (92) this implies (93). ∎

4.3 Coconvex Polynomial Approximation

We use the method from [1] and [5] and prove Theorem 37 by induction on r, the number of inflection points. For $r = 0$ Theorem 37 becomes a theorem on convex approximation which is a simple consequence of Theorem 2 of [4]. Let us assume that (85)-(87) are valid for functions with $r-1 \geq 0$ inflection points. Let $f \in \mathbf{C}^2[-1,1]$ have $r < \infty$ inflection points at $\{y_i\}_{i=1}^r : -1 < y_1 < \ldots < y_r < 1$. Without loss of generality we can assume that $f''(x) \geq 0, x \in [-1, y_1]$. We fix one of y_i's. In fact, it is not important which one to fix, but notation and considerations are simpler for $y_1 =: \alpha$ (also, if $r = 1$ it is convenient to denote $y_2 := y_r := 1$). We can assume that $f(\alpha) = f'(\alpha) = 0$ (subtract a linear function from f which, obviously, has no effect on convexity). Since $f \in \mathbf{C}^2$ and α is an inflection point, then $f''(\alpha) = 0$.

Following [1] we define the "flipped" function

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \geq \alpha, \\ -f(x) & \text{if } x < \alpha. \end{cases}$$

Then $\hat{f} \in \mathbf{C}^2[-1,1]$, $\hat{f}(\alpha) = \hat{f}'(\alpha) = \hat{f}''(\alpha) = 0$ and $\hat{f}$ has $r-1$ inflection points at $y_2, \ldots, y_r$, and also, as was shown in [5],

$$\omega_\varphi(\hat{f}'', t) \leq C\,\omega_\varphi(f'', t),\ t > 0. \tag{95}$$

By induction hypothesis there exists a constant $A(r-1)$ such that for each $n > \frac{A(r-1)}{d(r)} \geq \frac{A(r-1)}{d(r-1)}$ there is a polynomial $q_n \in \mathbf{\Pi}_n$ such that $\hat{f}''(x) q_n''(x) \geq 0$, $x \in I$, and the inequalities (85)-(87) hold for $\hat{f}$ and q_n (since $\hat{f}(\alpha) = 0$, increasing the constant in (85) we can assume that $q_n(\alpha) = 0$).

Now we fix $n > \max\left\{\frac{A(r-1)}{d(r)}, \frac{50}{y_2-\alpha}, \frac{50}{\alpha+1}\right\}$ and consider corresponding decomposition of $[-1,1]$: $I = \cup_{j=1}^n I_j = \cup_{j=1}^n [x_j, x_{j-1}]$. Let index j_0 be such that $\alpha \in [x_{j_0}, x_{j_0-1})$. Then $x_{j_0+3} \geq -1$ and $x_{j_0-4} \leq y_2$, *i.e.*, $[-1,\alpha]$ and $[\alpha, y_2]$ contain at least three intervals I_j each. This implies, in particular, that $\varphi(\alpha) \geq n^{-1}$ and, therefore, $2\varphi(\alpha) \geq n\Delta_n(\alpha)$.

Now we consider the algebraic polynomial $p_n(x) := \int_\alpha^x p_n'(y)\,dy$ such that

$$p_n'(x) = (q_n'(x) - q_n'(\alpha))V_n(x) + q_n'(\alpha)W_n(x)\,,$$

and show that it is possible to choose polynomials $V_n(x)$ and $W_n(x)$ so that p_n is coconvex with f and the inequalities (85)-(87) are satisfied. We claim that the following properties of V_n and W_n are sufficient for coconvexity of p_n with f:

(i) $V_n(x)\,\mathrm{sgn}_\alpha(x) \geq 0$, $x \in I$,

(ii) V_n' is copositive with $(q_n'(x) - q_n'(\alpha))\, q_n''(x)\,\mathrm{sgn}_\alpha(x)$, *i.e.*, $(q_n'(x) - q_n'(\alpha))\, q_n''(x)\, V_n'(x)\,\mathrm{sgn}_\alpha(x) \geq 0$, $x \in I$,

(iii) W_n' is copositive with $f''(x)\,\mathrm{sgn}(q_n'(\alpha))$, *i.e.*, $f''(x)\,W_n'(x)\,\mathrm{sgn}(q_n'(\alpha)) \geq 0$, $x \in I$.

Indeed, using these properties, the inequality $\hat{f}''(x)q_n''(x) \geq 0$ and the definition of $\hat{f}$ we have

$$\begin{aligned}
&\mathrm{sgn}\{p_n''(x)f''(x)\} \\
&= \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))V_n'(x)f''(x) + q_n''(x)V_n(x)f''(x) \\
&\quad + q_n'(\alpha)W_n'(x)f''(x)\} \\
&\geq \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))V_n'(x)f''(x) + q_n''(x)V_n(x)f''(x)\} \\
&= \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))V_n'(x)q_n''(x)\mathrm{sgn}_\alpha(x) + (q_n''(x))^2V_n(x)\mathrm{sgn}_\alpha(x)\} \geq 0\,.
\end{aligned}$$

Therefore, it is sufficient to construct polynomials $V_n(x)$ and $W_n(x)$ which satisfy conditions (i)-(iii) and also (as we will see later) sufficiently approximate $\mathrm{sgn}_\alpha(x)$.

Using Lemma 39 we conclude that the polynomial

$$W_n(x) := \begin{cases} Q_{j_0+2}(n,\mu,x_{j_0+2};\emptyset;y_1,\ldots,y_r)(x) & \text{if } q_n'(\alpha) \geq 0, \\ Q_{j_0-2}(n,\mu,x_{j_0-2};y_1;y_2,\ldots,y_r)(x) & \text{if } q_n'(\alpha) < 0. \end{cases}$$

satisfies condition (iii).

Indeed, it is clear that $f''(x)$ is copositive with $\prod_{i=1}^r(y_i - x)$. Lemma 39 yields that if $q_n'(\alpha) \geq 0$, then $W_n'(x) = Q_{j_0+2}'(n,\mu,x_{j_0+2};\emptyset;y_1,\ldots,y_r)(x)$ is also copositive with $\prod_{i=1}^r(y_i - x)$ and, therefore, with $f''(x)$. If $q_n'(\alpha) < 0$, then $W_n'(x) = Q_{j_0-2}'(n,\mu,x_{j_0-2};y_1;y_2,\ldots,y_r)(x)$ is copositive with $(x-y_1)\prod_{i=2}^r(y_i-x)$ and, hence,

$$\begin{aligned}
&\mathrm{sgn}\{W_n'(x)f''(x)\}\mathrm{sgn}\{q_n'(\alpha)\} \\
&= -\mathrm{sgn}\{W_n'(x)f''(x)\} = -\mathrm{sgn}\{-\prod_{i=1}^r(y_i - x)^2\} \geq 0.
\end{aligned}$$

Thus, (iii) is satisfied.

To construct $V_n(x)$, first, we note that since $q_n''(x)$ changes sign only at $y_2,\ldots,y_r$, the function $q_n'(x)-q_n'(\alpha)$ is monotone on each of the intervals $[-1,y_2]$, $[y_r,1]$ and $[y_i,y_{i+1}], 2 \leq i \leq r-1$. Thus, $q_n'(x) - q_n'(\alpha)$ has at most one zero in each of these intervals. Moreover, it changes sign at every zero different from $y_i, 2 \leq i \leq r$ (Note that $q_n'(x) - q_n'(\alpha)$ vanishes on some subinterval only if $q_n(x)$ is a linear function. Since this case is trivial, everywhere below we assume that $q_n(x)$ is a polynomial of degree ≥ 2). Using this and also the inequality $q_n''(x) \leq 0, -1 \leq x \leq y_2$ we conclude that the function $(q_n'(x) - q_n'(\alpha))q_n''(x)$ is nonpositive for $-1 \leq x \leq \alpha$, nonnegative for $\alpha \leq x \leq y_2$ and has at most $2(r-1)$ changes of sign on $[y_2,1]$ (we denote these points in increasing order by $\beta_1,\beta_2,\ldots,\beta_l, l \leq 2r-2$ and note that $\beta_1 = y_2$). Hence, $(q_n'(x) - q_n'(\alpha))q_n''(x)$ is copositive with $(x-\alpha)\prod_{i=1}^l(\beta_i - x)$.

Now we define

$$V_n(x) := Q_{j_0}(n, \mu, \alpha; \emptyset; \beta_1, \ldots, \beta_l)(x) .$$

Condition (i) immediately follows from (93). Using (90) we conclude that $V_n'(x)$ is copositive with $\prod_{i=1}^{l}(\beta_i - x)$. Therefore,

$$\begin{aligned} &\operatorname{sgn}\{(q_n'(x) - q_n'(\alpha))q_n''(x)V_n'(x)\operatorname{sgn}_\alpha(x)\} \\ &= \operatorname{sgn}\{(x-\alpha)\operatorname{sgn}_\alpha(x)\prod_{i=1}^{l}(\beta_i - x)^2\} \geq 0 , \end{aligned}$$

and (ii) is also satisfied.

Thus, $p_n(x)$ is coconvex with $f(x)$ and it remains to verify the inequalities (85)-(87).

Using [5], properties of ω_φ, inequality (95) and recalling that $2\varphi(\alpha) \geq n\Delta_n(\alpha)$ we have the following estimates for any $x \in I$:

$$\begin{aligned} (96) \qquad |\hat{f}''(x)| &= |\hat{f}''(x) - \hat{f}''(\alpha)| \leq \omega_\varphi\left(\hat{f}'', \frac{2|x-\alpha|}{\varphi(\alpha)}\right) \\ &\leq \omega_\varphi\left(\hat{f}'', 4\frac{|x-\alpha| + h_{j_0}}{n\Delta_n(\alpha)}\right) \leq C\,\frac{|x - x_{j_0}| + h_{j_0}}{h_{j_0}}\omega_\varphi\left(\hat{f}'', n^{-1}\right) \\ &= C\,\psi_{j_0}^{-1}\,\omega_\varphi\left(\hat{f}'', n^{-1}\right) \leq C\,\psi_{j_0}^{-1}\,\omega_\varphi\left(f'', n^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} (97) \qquad |\hat{f}'(x)| &= |\hat{f}'(x) - \hat{f}'(\alpha)| = |x-\alpha||\hat{f}''(\zeta)| \\ &\leq C\,|x-\alpha|\,\frac{|\zeta - x_{j_0}| + h_{j_0}}{h_{j_0}}\,\omega_\varphi\left(f'', n^{-1}\right) \\ &\leq C\left(|x - x_{j_0}| + h_{j_0}\right)\frac{|x - x_{j_0}| + h_{j_0}}{h_{j_0}}\,\omega_\varphi\left(f'', n^{-1}\right) \\ &\leq C\,n^{-1}\,\psi_{j_0}^{-2}\,\omega_\varphi\left(f'', n^{-1}\right) , \end{aligned}$$

since $|x-\alpha| \geq |\zeta - \alpha|$.

Now we choose μ so that all the conditions above are satisfied. For example, $\mu = 15r$ will do. However, because this choice of μ is not important we will continue to write μ keeping in mind that $\mu = \mu(r)$.

Using (96), (97) and also the inequalities

$$\begin{aligned} |V_n'(x)| &\leq C(\mu)\,\psi_{j_0}^{\mu} h_{j_0}^{-1} , \\ |W_n'(x)| &\leq C(\mu)\,\psi_{j_0}^{\mu} h_{j_0}^{-1} \\ |\operatorname{sgn}_\alpha(x) - V_n(x)| &\leq C(\mu)\,\psi_{j_0}^{\mu} \end{aligned}$$

and

$$|\mathrm{sgn}_\alpha(x) - W_n(x)| \le C(\mu)\,\psi_{j_0}^\mu\,,$$

which follow from the definitions of V_n and W_n and Lemma 39 (since $\psi_{j_0} \sim \psi_{j_0\pm i}$, $i = 1, 2$ and $|\mathrm{sgn}_\alpha(x) - \mathrm{sgn}_{x_{j_0\pm 2}}(x)| \le C(\mu)\,\psi_{j_0}^\mu$), we have the following estimates:

$$\begin{aligned}
&|f''(x) - p_n''(x)| \\
&= \Big|(\hat{f}''(x) - q_n''(x))\mathrm{sgn}_\alpha(x) + q_n''(x)(\mathrm{sgn}_\alpha(x) - V_n(x)) \\
&\quad -(q_n'(x) - q_n'(\alpha))V_n'(x) - q_n'(\alpha)W_n'(x)| \\
&\le |\hat{f}''(x) - q_n''(x)|(1 + |\mathrm{sgn}_\alpha(x) - V_n(x)|) + |\hat{f}''(x)||\mathrm{sgn}_\alpha(x) - V_n(x)| \\
&\quad + |\hat{f}'(x) - q_n'(x)||V_n'(x)| + |\hat{f}'(\alpha) - q_n'(\alpha)|(|V_n'(x)| + |W_n'(x)|) \\
&\quad + |\hat{f}'(x)||V_n'(x)| \\
&\le C(r)\,\omega_\varphi(f'', n^{-1}) \left\{ \frac{1 + \psi_{j_0}^\mu}{\min(\sqrt{y_2 + 1}, \sqrt{1 - y_r})} + \psi_{j_0}^{\mu-1} + \frac{\psi_{j_0}^\mu}{nh_{j_0}} + \frac{\psi_{j_0}^{\mu-2}}{nh_{j_0}} \right\} \\
&\le C(r)\,\omega_\varphi(f'', n^{-1}) \left\{ \frac{1}{\min(\sqrt{y_2 + 1}, \sqrt{1 - y_r})} + \frac{1}{\sqrt{1 - \alpha^2}} \right\},
\end{aligned}$$

since $nh_{j_0} \ge \sqrt{1 - \alpha^2}$. Therefore,

$$|f''(x) - p_n''(x)| \le \frac{C(r)}{\min(\sqrt{\alpha + 1}, \sqrt{1 - y_r})}\,\omega_\varphi(f'', n^{-1}) = \frac{C(r)}{\sqrt{d_0}}\,\omega_\varphi(f'', n^{-1}).$$

Similarly,

$$\begin{aligned}
&|f'(x) - p_n'(x)| \\
&= \Big|(\hat{f}'(x) - q_n'(x))\mathrm{sgn}_\alpha(x) + q_n'(x)(\mathrm{sgn}_\alpha(x) - V_n(x)) \\
&\quad + q_n'(\alpha)(V_n(x) - W_n(x))| \\
&\le C(r)\,n^{-1}\,\omega_\varphi(f'', n^{-1}) \left\{1 + \psi_{j_0}^\mu + \psi_{j_0}^{\mu-2}\right\} \le C(r)\,n^{-1}\,\omega_\varphi(f'', n^{-1})
\end{aligned}$$

and, using the identity $\int_\alpha^x q_n'(y)\,\mathrm{sgn}_\alpha(y)\,dy = q_n(x)\,\mathrm{sgn}_\alpha(x)$,

$$\begin{aligned}
&|f(x) - p_n(x)| \\
&= \Big|(\hat{f}(x) - q_n(x))\mathrm{sgn}_\alpha(x) + \int_\alpha^x q_n'(y)(\mathrm{sgn}_\alpha(y) - V_n(y))dy \\
&\quad + q_n'(\alpha)\int_\alpha^x (V_n(y) - W_n(y))dy\Big| \\
&\le C(r)\,n^{-2}\,\omega_\varphi(f'', n^{-1}) \left\{1 + n\left|\int_\alpha^x (1 + \psi_{j_0}(y)^{-2})\psi_{j_0}(y)^\mu\,dy\right| \right. \\
&\quad \left. + n\left|\int_\alpha^x \psi_{j_0}(y)^\mu\,dy\right|\right\}
\end{aligned}$$

$$\leq C(r)\, n^{-2}\, \omega_\varphi(f'', n^{-1}) \left\{1 + n \left| \int_\alpha^x \left(\frac{h_{j_0}}{|y - x_{j_0}| + h_{j_0}} \right)^{\mu-2} dy \right| \right\}$$

$$\leq C(r)\, n^{-2}\, \omega_\varphi(f'', n^{-1}) \left\{1 + n\, h_{j_0}^{\mu-2} \left| \int_\alpha^x (|y - \alpha| + h_{j_0})^{2-\mu}\, dy \right| \right\}$$

$$\leq C(r)\, n^{-2}\, \omega_\varphi(f'', n^{-1}) \{1 + n h_{j_0}\} \leq C(r)\, n^{-2}\, \omega_\varphi(f'', n^{-1}).$$

Finally, to complete the proof of Theorem 37 it is sufficient to recall that $p_n \in \Pi_{C(r)n}$ and use properties of ω_φ modulus.

Remark . Although, all the proofs were given in the case when f has finitely many inflection points, the considerations will not change if we allow f to be linear on some subintervals. For example, if $f''(x) = 0$ for $x \in [\alpha, \beta] \subset (-1, 1)$, it is sufficient to fix any $x_0 \in [\alpha, \beta]$ as an "inflection" point. Thus, Theorem 37 is valid for any function with finite number of convexity changes.

References

[1] R. K. Beatson and D. Leviatan (1983), *On comonotone approximation*, Canad. Math.Bull., **26**: 220–224.

[2] Z. Ditzian and V. Totik (1987): Moduli of Smoothness. Berlin: Springer-Verlag.

[3] G. L. Iliev (1978), *Exact estimates for partially monotone approximation*, Anal. Math., **4**: 181–197.

[4] K. A. Kopotun (1994), *Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials*, Constr. Approx., **10**: 153–178.

[5] D. Leviatan (1988), *Monotone and comonotone polynomial approximation revisited*, J. Approx. Theory, **53**: 1–16.

[6] D. J. Newman (1979), *Efficient comonotone approximation*, J. Approx. Theory, **25**: 189–192.

[7] D. J. Newman, E. Passow and L. Raymon (1972), *Piecewise monotone polynomial approximation*, Trans. Am. Math. Soc., **172**: 465–472.

[8] E. Passow and L. Raymon (1974), *Monotone and comonotone approximation*, Proc. Amer. Math. Soc., **42**: 390–394.

[9] E. Passow , L. Raymon and J. A. Roulier (1974), *Comonotone polynomial approximation*, J. Approx. Theory, **11**: 221-224.

[10] J. Roulier (1968), *Monotone approximation of certain classes of functions*, J. Approx. Theory, **1**: 319–324.

[11] I. A. Shevchuk (1989), *On coapproximation of monotone functions*, Dokl. Akad. Naul SSSR, **308**(3): 537–541. (English translation: Soviet. Math. Dokl., **40**(2): 349–354.)

[12] I. A. Shevchuk (1992), Approximation by Polynomials and Traces of the Functions Continuous on an Interval, Kiev: Naukova Dumka.

[13] A. S. Shvedov (1981), *Coapproximation of piecewise–monotone functions by polynomials*, Mat. Zametki, **30**(6): 839–846.

[14] A. S. Shvedov (1981), *Orders of coapproximation of functions by algebraic polynomials*, Mat. Zametki, **29**(1): 117–130. (English translation: Math. Notes, **30**: 63–70.)

[15] X. M. Yu (1990), *Coconvex approximation of continuous functions*, Acta Math. Sinica, **6**(3): 220–229.

[16] X. M. Yu (1988), *Degree of comonotone polynomial approximation*, Approx. Theory Appl., **4**(3): 73–78.

[17] S. P. Zhou (1993), *On comonotone approximation by polynomials in L^p space*, Analysis, **13**: 363–376.

CHAPTER 5

ON COPOSITIVE APPROXIMATION BY ALGEBRAIC POLYNOMIALS [4]

5.1 Introduction and main result

We are interested in how well one can approximate a function $f \in C[-1,1]$ with finitely many sign changes by polynomials p_n such that $f(x)p_n(x) \geq 0$, $x \in [-1,1]$ (we say that in this case f and p_n are copositive in $[-1,1]$).

One of recent results on copositive approximation is due to D. Leviatan [5] who proved the following theorem.

Theorem M *There exists an absolute constant $C = C(r)$ such that for every $f \in C[-1,1]$ which alternates in sign r times in $[-1,1]$, $0 < r < \infty$, and each $n \geq 1$, there is a polynomial $p_n \in \mathbf{\Pi}_n$ which is copositive with f and satisfies*

$$\|f - p_n\| \leq C\,\omega(f, n^{-1}). \tag{98}$$

This result was later improved. In particular, modulus ω was replaced by ω_φ, and the dependance of C on the set of points of sign change was investigated (see [6], for example). However, it was not known for a long time whether ω in (98) can be replaced by the modulus of smoothness of higher order.

S. P. Zhou showed that estimate (98) can not hold with ω^4 instead of ω. He also considered copositive approximation in L_p, $1 < p < \infty$ metric, and proved that the estimate by the second integral modulus of smoothness $\omega^2(f, n^{-1})_p$ is not correct in this case. These results can be summarized in the following theorem (see [9] and [10]).

Theorem N *There are functions f_1 and f_2 in $C^1[-1,1]$ with $r \geq 1$ sign changes such that*

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f_1, r)_\infty}{\omega^4(f_1, n^{-1})} = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{E_n^{(0)}(f_2, r)_p}{\omega^2(f_2, n^{-1})_p} = \infty\,, 1 < p < \infty\,,$$

where $E_n^{(0)}(f, r)_p$ is the error of the best copositive L_p (C if $p = \infty$) approximation to f by polynomials from $\mathbf{\Pi}_n$.

[4] A version of this chapter has been published in Analysis Mathematica (1995) **21**: 269–283.

Recently, Y. Hu, D. Leviatan and X. M. Yu [4] showed that Theorem M can be considerably improved. They were able to replace ω in (98) by ω^2, thus, together with Theorem N, revealing an interesting and unexpected difference between the cases $p = \infty$ and $1 < p < \infty$ for copositive polynomial approximation. Their result is stated as follows.

Theorem O *Let $f \in C[-1,1]$ change sign r times at $-1 < y_1 < \ldots < y_r < 1$, and let $\delta := \min_{0 \le i \le r} |y_{i+1} - y_i|$, where $y_0 := -1$ and $y_{r+1} := 1$. Then there exists a constant $C = C(r,\delta)$ but otherwise independent of f and n such that for each $n > 4\delta^{-1}$ there is a polynomial $p_n \in \Pi_{Cn}$, copositive with f, satisfying*

$$\|f - p_n\| \le C\omega^2(f, n^{-1}) . \tag{99}$$

In fact, it is not very difficult to show that ω^2 in (99) can be replaced by ω^3. However, there is still room for improvement. It is well known that if one wants to characterize approximation properties of a function f in terms of its moduli of smoothness, then this characterization should involve either $\omega^m(f, \Delta_n(x))$ or $\omega_\varphi^m(f, n^{-1})$ (or equivalent quantities). Thus, in a sense, "exact" estimates for algebraic polynomial approximation are those in terms of the above mentioned quantities.

The following theorem is the main result of this chapter.

Theorem 40 *Let $f \in C[-1,1]$ change sign $r \ge 1$ times at $-1 < y_1 < \ldots < y_r < 1$, and let $\delta := \min_{0 \le i \le r} |y_{i+1} - y_i|$, where $y_0 := -1$ and $y_{r+1} := 1$. Then there exists a constant $C_1 = C_1(r,\delta)$ such that for each $n > C_1$ there is a polynomial $P_n \in \Pi_n$, copositive with f, satisfying*

$$\|f - P_n\| \le C(r)\, \omega_\varphi^3(f, n^{-1}) . \tag{100}$$

Theorem 40 implies

Theorem 40′ *Let f be the same as in Theorem 40. Then for each $n \ge 0$ in the case $r \ge 3$, and $n \ge 2$ if $r = 1$ or 2, there is a polynomial $P_n \in \Pi_n$, copositive with f, such that*

$$\|f - P_n\| \le C(r,\delta)\, \omega_\varphi^3(f, n^{-1}) ,$$

where $0^{-1} := 1$.

An immediate consequence of Theorem(s) 40(40′) and converse theorems in terms of the Ditzian-Totik moduli ([3], [8]) is the following result.

Corollary 41 *Let $0 < \alpha < 3$, and let a function $f \in C[-1,1]$ change sign $r < \infty$ times in $[-1,1]$. Then*

$$E_n(f) = O(n^{-\alpha}) \;\Leftrightarrow\; E_n^{(0)}(f,r) = O(n^{-\alpha}) , \tag{101}$$

where $E_n(f) = \inf_{p_n \in \Pi_n} \|f - p_n\|$ and $E_n^{(0)}(f,r) = \inf_{p_n \in \Pi_n,\, p_n f \ge 0} \|f - p_n\|$.

5.2 Notations and auxiliary results

Suppose $f \in C[-1,1]$ satisfies the conditions of Theorem 40, *i.e.*, it changes sign $1 \le r < \infty$ times at $-1 < y_1 < \ldots < y_r < 1$. Also, let n be fixed and sufficiently large. If $y_i \in I_{j(i)}$, $i = 1, 2, \ldots, r$, then it is convenient to denote

$$y_i' := x_{j(i)+1}\,,\ y_i'' := x_{j(i)-2}\,,$$

$$\mathcal{I}_i := [y_i', y_i''] := I_{j(i)+1} \cup I_{j(i)} \cup I_{j(i)-1} = [x_{j(i)+1}, x_{j(i)-2}]$$

and

$$\mathcal{Y}_i := \left[\frac{y_i + y_i'}{2}, \frac{y_i + y_i''}{2}\right] \text{ for } i = 1, 2, \ldots, r\,.$$

Then $5/3h_{j(i)} < |\mathcal{I}_i| = 2|\mathcal{Y}_i| < 7h_{j(i)}$, $i = 1, \ldots, r$ and, therefore,

$$|\mathcal{I}_i| \sim |\mathcal{Y}_i| \sim h_{j(i)} \sim \Delta_n(x) \text{ for } x \in \mathcal{I}_i\,.$$

Throughout the chapter K_i, $i \ge 1$ denote constants which are independent of f and n and remain fixed everywhere in the proofs.

While proving Theorem 40 we will need to smooth the function f which is only assumed to be continuous on $[-1,1]$. The idea to consider a smooth approximation instead of the original function is very well known. It is frequently used in different areas of approximation theory. In particular, the construction of such an approximation is crucial in the proofs of theorems on the equivalence of K-functionals and the appropriate moduli of smoothness (see [1], [3] and [7], for example). There are numerous approaches to this problem. Thus, it is often convenient first to extend the function to a larger interval preserving some of its smoothness characteristics. In particular, this idea was employed in the proof of the main lemma in [4]. In our proof we will avoid the problems of smoothing and extending f (though, this approach is possible) simply by considering an algebraic polynomial which sufficiently approximates f and satisfies some extra conditions (in fact, the polynomial of best approximation to f in $C[-1,1]$ will do). Then, we will modify this polynomial near the points of sign change obtaining a smooth piecewise polynomial approximation f_n with controlled first and third derivatives. The following lemma is crucial for the proof of Theorem 40.

Lemma 42 *Let f be the same as in Theorem 40. Then for each $n \ge 4\delta^{-1}$ there exists a function $f_n \in C^3[-1,1]$, copositive with f in $Y := \cup_{i=1}^{r}\mathcal{Y}_i$, such that*

(102) $$\|f - f_n\| \le C(r)\omega_\varphi^3(f, n^{-1})\,,$$

(103) $$\|\varphi(x)^3 f_n'''(x)\| \le K_1(r)\, n^3\, \omega_\varphi^3(f, n^{-1})$$

and

(104) $$|\Delta_n(x) f_n'(x)| \ge \omega_\varphi^3(f, n^{-1}) \text{ for } x \in Y\,.$$

Proof Let $n \geq 4\delta^{-1}$ and index $1 \leq i \leq r$ be fixed. For $x \in \mathcal{I}_i$ we set $\tilde{f}_i$ to be the polynomial of degree ≤ 2 which vanishes at y_i:

$$\tilde{f}_i(x) := \frac{x - y_i}{y_i'' - y_i'} \left\{ \frac{x - y_i'}{y_i'' - y_i} \tilde{f}_i(y_i'') + \frac{x - y_i''}{y_i - y_i'} \tilde{f}_i(y_i') \right\},$$

where $\tilde{f}_i(y_i')$ and $\tilde{f}_i(y_i'')$ are chosen so that

$$\tilde{f}_i(y_i') = \begin{cases} 60\omega_\varphi^3(f, n^{-1})\mathrm{sgn}(f(y_i')), & \text{if } |f(y_i')| \leq 60\omega_\varphi^3(f, n^{-1}), \\ f(y_i'), & \text{otherwise.} \end{cases}$$

and

$$\tilde{f}_i(y_i'') = \begin{cases} 60\omega_\varphi^3(f, n^{-1})\mathrm{sgn}(f(y_i'')), & \text{if } |f(y_i'')| \leq 60\omega_\varphi^3(f, n^{-1}), \\ f(y_i''), & \text{otherwise.} \end{cases}$$

(If $f(y_i') = 0$, *e.g.*, then $\mathrm{sgn}(f(y_i'))$ equals the sign of f on (y_{i-1}, y_i).)

Since $\tilde{f}_i \in \Pi_2$, and $\tilde{f}_i(y_i')$ and $\tilde{f}_i(y_i'')$ have opposite signs, then the only zero of $\tilde{f}_i$ in $\mathcal{I}_i$ is y_i. Hence, $\tilde{f}_i$ is copositive with f in $\mathcal{I}_i$. Also, the first derivative of $\tilde{f}_i$

$$\tilde{f}_i'(x) = \frac{2x - y_i - y_i'}{(y_i'' - y_i')(y_i'' - y_i)} \tilde{f}_i(y_i'') + \frac{2x - y_i - y_i''}{(y_i'' - y_i')(y_i - y_i')} \tilde{f}_i(y_i')$$

is a linear function, and

$$\tilde{f}_i'\left(\frac{y_i + y_i'}{2}\right) = -\frac{\tilde{f}_i(y_i')}{y_i - y_i'} \text{ and } \tilde{f}_i'\left(\frac{y_i + y_i''}{2}\right) = \frac{\tilde{f}_i(y_i'')}{y_i'' - y_i}$$

are of the same sign, which implies that $\tilde{f}_i'$ does not change sign in $\mathcal{Y}_i$, and for any $x \in \mathcal{Y}_i$

$$\begin{aligned} |\tilde{f}_i'(x)| &\geq \min\left\{ \left|\tilde{f}_i'\left(\frac{y_i + y_i'}{2}\right)\right|, \left|\tilde{f}_i'\left(\frac{y_i + y_i''}{2}\right)\right| \right\} \\ &= \min\left\{ \frac{|\tilde{f}_i(y_i')|}{y_i - y_i'}, \frac{|\tilde{f}_i(y_i'')|}{y_i'' - y_i} \right\} \\ &\geq \frac{1}{60\Delta_n(x)} \min\left\{ |\tilde{f}_i(y_i')|, |\tilde{f}_i(y_i'')| \right\} \\ &\geq \Delta_n(x)^{-1}\omega_\varphi^3(f, n^{-1}). \end{aligned} \tag{105}$$

Now we will show that

$$|\tilde{f}_i(x) - f(x)| \leq C\omega_\varphi^3(f, n^{-1}), \; x \in \mathcal{I}_i. \tag{106}$$

We use the fact that

$$|f(x) - L(x, f)| \leq C\omega_\varphi^3(f, n^{-1}), \; x \in \mathcal{I}_i, \tag{107}$$

where

$$L(x,f) := L(x,f,y_i',y_i,y_i'') = \frac{x-y_i}{y_i''-y_i'}\left\{\frac{x-y_i'}{y_i''-y_i}f(y_i'') + \frac{x-y_i''}{y_i-y_i'}f(y_i')\right\}$$

is the Lagrange polynomial of degree ≤ 2, which interpolates f at y_i', y_i and y_i''. Inequality (107) is an analog of Whitney's inequality for Ditzian-Totik moduli and can be found in I. A. Shevchuk [8, Lemma 18.2], for example.

Using (107) and the above presentations of $\tilde{f}_i(x)$ and $L(x,f)$ we write for $x \in \mathcal{I}_i$

$$\begin{aligned} |\tilde{f}_i(x) - f(x)| &\le |\tilde{f}_i(x) - L(x,f)| + |L(x,f) - f(x)| \\ &\le \left|\frac{(x-y_i)(x-y_i')}{(y_i''-y_i')(y_i''-y_i)}\right| \left|\tilde{f}_i(y_i'') - f(y_i'')\right| \\ &\quad + \left|\frac{(x-y_i)(x-y_i'')}{(y_i''-y_i')(y_i-y_i')}\right| \left|\tilde{f}_i(y_i') - f(y_i')\right| + C\,\omega_\varphi^3(f,n^{-1}) \\ &\le C\,\omega_\varphi^3(f,n^{-1}), \end{aligned}$$

and (106) is proved.

At this stage it is worth mentioning that the ability to construct a function $\tilde{f}_i$ for which (105) and (106) hold determines the possibility to obtain the estimates in terms of the third modulus of smoothness in Theorem 40. For instance, if we could find a polynomial $\tilde{g}_i$ of degree ≤ 3, copositive with f in $\mathcal{I}_i$, and such that inequalities (105) and (106) held with ω_φ^4 instead of ω_φ^3, then we would be able to replace ω_φ^3 in Theorem 40 by ω_φ^4. However, because of Theorem N, the creation of such $\tilde{g}_i$ is impossible in general. Function $\tilde{f}_i$ is the best of what one can construct. At the same time, if we add some conditions on the behavior of f near the points of sign change, then estimate (100) can be improved.

Now, let us continue with the proof of the lemma.

It is well known (see [3, Theorems 7.2.1 and 7.3.1], for example) that there exists a polynomial $Q(x)$ of degree $\le n$ (the polynomial of best approximation to f in $C[-1,1]$ will do) satisfying

$$\|f - Q\| \le C\omega_\varphi^3(f,n-1) \tag{108}$$

and

$$\|\varphi(x)^3 Q'''(x)\| \le C\,n^3\,\omega_\varphi^3(f,n-1). \tag{109}$$

Now, we define the piecewise polynomial function $\mathcal{S}(x)$ as follows:

$$\mathcal{S}(x) :=$$

$$\begin{cases} 1, & \text{if } x \notin \cup_{i=1}^r \mathcal{I}_i\,, \\ 0, & \text{if } x \in \cup_{i=1}^r \mathcal{Y}_i\,, \\ \lambda_i \displaystyle\int_{(y_i+y_i')/2}^{x} (y-y_i')^3\left(\tfrac{y_i+y_i'}{2} - y\right)^3 dy\,, & \text{if } x \in [y_i', \tfrac{y_i+y_i'}{2}]\,,\ i = 1,\dots,r\,, \\ \tilde{\lambda}_i \displaystyle\int_{(y_i+y_i'')/2}^{x} \left(y - \tfrac{y_i+y_i''}{2}\right)^3 (y_i''-y)^3 dy\,, & \text{if } x \in [\tfrac{y_i+y_i''}{2}, y_i'']\,,\ i = 1,\dots,r\,. \end{cases}$$

where normalizing constants λ_i and $\tilde{\lambda}_i$ are chosen so that $\mathcal{S}$ is a continuous function, *i.e.*,

$$\lambda_i = \left(\int_{(y_i+y_i')/2}^{y_i'} (y - y_i')^3 (\tfrac{y_i+y_i'}{2} - y)^3 dy \right)^{-1}$$

and

$$\tilde{\lambda}_i = \left(\int_{(y_i+y_i'')/2}^{y_i''} (y - \tfrac{y_i+y_i''}{2})^3 (y_i'' - y)^3 dy \right)^{-1} .$$

Moreover, it is easy to see that not only $\mathcal{S}$ is continuous, but also $\mathcal{S} \in C^3[-1,1]$.

Finally, the function $f_n(x)$ such that

$$f_n(x) := \begin{cases} (Q(x) - \tilde{f}_i(x))\mathcal{S}(x) + \tilde{f}_i(x), & \text{if } x \in \mathcal{I}_i, \\ Q(x), & \text{otherwise.} \end{cases}$$

is copositive with f in $Y = \cup_{i=1}^r \mathcal{Y}_i$, and the inequalities (102)-(104) are satisfied.

Indeed, f_n coincides with $\tilde{f}_i$ on $\mathcal{Y}_i$ and, hence, it is copositive with f in Y, and (104) holds. Also, $\mathcal{S} \in C^3[-1,1]$ and $\mathcal{S}(x) = 1, x \notin \cup_{i=1}^r \mathcal{I}_i$ imply that f_n is in $C^3[-1,1]$. Inequality (102) follows from (106), (108) and the observation that for every fixed $x \in \mathcal{I}_i \setminus \mathcal{Y}_i$ $f_n(x)$ is a convex combination of $Q(x)$ and $\tilde{f}_i(x)$ (since $0 \le \mathcal{S}(x) \le 1$).

To prove the remaining inequality (103) we need the following well known Kolmogorov type inequality (see, *e.g.*, [1] or [8]):

$$\|g^{(\nu)}\|_{[a,b]} \le C\left((b-a)^{r-\nu}\|g^{(r)}\|_{[a,b]} + (b-a)^{-\nu}\|g\|_{[a,b]}\right), \quad g \in C^r[a,b] \text{ and } 0 \le \nu \le r. \tag{110}$$

For $x \in [y_i', \frac{y_i+y_i'}{2}]$, $i = 1, \dots, r$ (for $x \in [\frac{y_i+y_i''}{2}, y_i'']$ considerations are similar, for $x \notin \cup_{i=1}^r \mathcal{I}_i$ (103) follows from (109), and for $x \in Y$ it is trivial), using the fact that $\varphi(x) \sim n\Delta_n(x) \sim n|\mathcal{I}_i|$ for $x \in \mathcal{I}_i$, we have

$$|\varphi(x)^3 f_n'''(x)| \le C\, n^3 |\mathcal{I}_i|^3 \sum_{\nu=0}^{3} |Q^{(\nu)}(x) - \tilde{f}_i^{(\nu)}(x)||\mathcal{S}^{(3-\nu)}(x)| .$$

Applying (110) for $|Q^{(\nu)}(x) - \tilde{f}_i^{(\nu)}(x)|$ and Markov's inequality for $|\mathcal{S}^{(3-\nu)}(x)|$, together with (106), (108) and (109), we obtain

$$\begin{aligned} &|\varphi(x)^3 f_n'''(x)| \\ &\le\ C\, n^3 |\mathcal{I}_i|^3 \sum_{\nu=0}^{3} \left(|\mathcal{I}_i|^{3-\nu}\|Q'''\|_{\mathcal{I}_i} + |\mathcal{I}_i|^{-\nu}\|Q - \tilde{f}_i\|_{\mathcal{I}_i}\right) |\mathcal{I}_i|^{\nu-3}\|\mathcal{S}\|_{\mathcal{I}_i} \\ &\le\ C\, n^3 \omega_\varphi^3(f, n-1) . \end{aligned}$$

This completes the proof of (103). ∎

The following lemma is due to I. A. Shevchuk [8, Theorem 18.2]. It will be used to construct a polynomial approximant to f_n established in Lemma 42.

Lemma P *If $g \in C^m[-1,1]$ is such that $|(1-x^2)^{m/2} g^{(m)}(x)| \le M$, $x \in [-1,1]$, then for every $n \ge m-1$ there exists a polynomial $q_n(g) \in \mathbf{\Pi}_n$ satisfying*

$$\|g - q_n(g)\| \le CMn^{-m}, \tag{111}$$

and

$$\left|\Delta_n(x)^\nu \left(g^{(\nu)}(x) - q_n^{(\nu)}(g,x)\right)\right| \le CMn^{-m}, \; 0 < \nu < m/2. \tag{112}$$

Corollary 43 *If $g \in C^3[-1,1]$ is such that $|(1-x^2)^{3/2} g'''(x)| \le M$, $x \in [-1,1]$, $-1 < y_1 < \ldots < y_r < 1$ and $\delta := \min_{1\le i\le r-1} |y_{i+1} - y_i|$, then for every $n \ge C(r,\delta)$ there exists a polynomial $p_n(g) \in \mathbf{\Pi}_n$, which interpolates g at $y_1, \ldots, y_r$ and such that*

$$\|g - p_n(g)\| \le K_2(r,\delta)Mn^{-3}, \tag{113}$$

and

$$\|\Delta_n(x)\,(g'(x) - p_n'(g,x))\,\| \le K_3(r)Mn^{-3}. \tag{114}$$

Proof Let $q_n \in \mathbf{\Pi}_n$ satisfy (111) and (112) with $m = 3$ and $\nu = 1$. Then the polynomial $p_n(g,x)$, given by

$$p_n(g,x) := q_n(x) + L(x, g - q_n, y_1, \ldots, y_r),$$

interpolates g at $y_1, \ldots, y_r$ and satisfies (113) and (114). Indeed,

$$\|g - p_n(g)\| \le \|g - q_n\| + \|L(x, g - q_n, y_1, \ldots, y_r)\| \le C\,\|g - q_n\| \le CMn^{-3},$$

which is the inequality (113). Inequality (114) is valid since

$$\begin{aligned} &\|\Delta_n(x)\,(g'(x) - p_n'(g,x))\,\| \\ &\le \|\Delta_n(x)(g'(x) - q_n'(x))\| + 2n^{-1}\|L'(x, g - q_n, y_1, \ldots, y_r)\| \\ &\le CMn^{-3} + \frac{2^{r-1}r(r-1)}{\delta^{r-1}} n^{-1}\|g - q_n\| \le CMn^{-3} \end{aligned}$$

for sufficiently large n ($n > C\delta^{1-r}$). The proof of the corollary is complete. ∎

Proposition 44 *For every y_i, $i = 1, \ldots, r$ there exists an increasing polynomial $T_n(y_i, x)$ of degree $\le n$, copositive with $\operatorname{sgn}(x - y_i)$ in $[-1,1]$, satisfying $T_n(y_i, -1) = -1$, $T_n(y_i, 1) = 1$, and such that*

$$|\operatorname{sgn}(x - y_i) - T_n(y_i, x)| \le K_4 \left(\frac{\Delta_n(y_i)}{|x - y_i| + \Delta_n(y_i)}\right)^2. \tag{115}$$

Proof Let index $i = 1, \ldots, r$ and integer n be fixed. It is known (see, *e.g.*, [8, Lemma 17.2]) that for every $N = Cn \in \mathcal{N}$ there exist increasing polynomials $\tilde{T}_N(y_i', x)$ and $\tilde{T}_N(y_i'', x)$ of degree $\leq N$ such that $\tilde{T}_N(y_i', -1) = \tilde{T}_N(y_i'', -1) = -1$, $\tilde{T}_N(y_i', 1) = \tilde{T}_N(y_i'', 1) = 1$, and satisfying

$$|\mathrm{sgn}(x - y_i') - \tilde{T}_N(y_i', x)| \leq K_5 \left(\frac{\Delta_N(y_i')}{|x - y_i'| + \Delta_N(y_i')} \right)^2, \tag{116}$$

and

$$|\mathrm{sgn}(x - y_i'') - \tilde{T}_N(y_i'', x)| \leq K_5 \left(\frac{\Delta_N(y_i'')}{|x - y_i''| + \Delta_N(y_i'')} \right)^2. \tag{117}$$

Now, we choose N to be sufficiently large, say, $N := [2\sqrt{K_5} + 1]n$. Then, the following inequalities hold

$$\begin{aligned} \tilde{T}_N(y_i', y_i) &\geq 1 - K_5 \left(\frac{\Delta_N(y_i')}{y_i - y_i' + \Delta_N(y_i')} \right)^2 \\ &\geq 1 - K_5 \left(\frac{\Delta_N(y_i')}{\Delta_n(y_i')} \right)^2 \\ &\geq 1 - 4K_5 \left(\frac{n}{N} \right)^2 > 0 \end{aligned}$$

and, similarly,

$$\begin{aligned} \tilde{T}_N(y_i'', y_i) &\leq -1 + K_5 \left(\frac{\Delta_N(y_i'')}{y_i'' - y_i + \Delta_N(y_i'')} \right)^2 \\ &\leq -1 + K_5 \left(\frac{\Delta_N(y_i'')}{\Delta_n(y_i'')} \right)^2 \\ &\leq -1 + 4K_5 \left(\frac{n}{N} \right)^2 < 0. \end{aligned}$$

Therefore, there exists $0 < \alpha_i < 1$ such that

$$\alpha_i \tilde{T}_N(y_i', y_i) + (1 - \alpha_i)\tilde{T}_N(y_i'', y_i) = 0.$$

Now, let

$$T_n(y_i, x) := \alpha_i \tilde{T}_N(y_i', x) + (1 - \alpha_i)\tilde{T}_N(y_i'', x).$$

Then T_n is an increasing polynomial of degree $\leq Cn$ such that $T_n(y_i, y_i) = 0$ (this implies that T_n is copositive with $\mathrm{sgn}(y_i - x)$), and the following inequalities hold:

$$\begin{aligned} &|\mathrm{sgn}(x - y_i) - T_n(y_i, x)| \\ &\leq |\mathrm{sgn}(x - y_i) - \mathrm{sgn}(x - y_i')| + |\mathrm{sgn}(x - y_i') - \tilde{T}_N(y_i', x)| \\ &\quad + |\mathrm{sgn}(x - y_i) - \mathrm{sgn}(x - y_i'')| + |\mathrm{sgn}(x - y_i'') - \tilde{T}_N(y_i'', x)| \end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\frac{\Delta_n(y_i)}{|x-y_i|+\Delta_n(y_i)}\right)^2 + C\left(\frac{\Delta_N(y_i')}{|x-y_i'|+\Delta_N(y_i')}\right)^2 \\
&\quad + C\left(\frac{\Delta_N(y_i'')}{|x-y_i''|+\Delta_N(y_i'')}\right)^2 \\
&\leq C\left(\frac{\Delta_n(y_i)}{|x-y_i|+\Delta_n(y_i)}\right)^2 .
\end{aligned}
$$

The proof of the proposition is complete. ∎

The following result is a generalization of [4, Lemma 1].

Lemma 45 *Let f be as in Theorem 40. If for $n \geq 4\delta^{-1}$ there exists $p_n \in \Pi_n$, copositive with f in $Y = \cup_{i=1}^r \mathcal{Y}_i$, then there is a polynomial $P_n \in \Pi_{K_6 n}$, $K_6 = K_6(r)$ which is copositive with f in $[-1,1]$ and such that*

$$\|f - P_n\| \leq C(r)\|f - p_n\| . \tag{118}$$

Sketch of the proof. The polynomial

$$P_n(x) := p_n(x) + 2^r \|f - p_n\| \eta \prod_{i=1}^r T_N(y_i, x)$$

where N is sufficiently large ($N = ([18\sqrt{K_4}] + 1)n$ will do), and $\eta = \pm 1$ is such that $\mathrm{sgn}(P_n(x)) = \eta \prod_{i=1}^r \mathrm{sgn}(x - y_i)$, satisfies the assertion of the lemma. The verification of this fact is similar to the proof of [4, Lemma 1]. The only difference is that instead of $(y_i - \frac{1}{2n}, y_i + \frac{1}{2n})$ intervals $\mathcal{Y}_i, i = 1, \ldots, r$ are considered. Namely, $P_n(x)f(x) \geq 0$ in $\cup_{i=1}^r \mathcal{Y}_i$ since both p_n and $\eta \prod_{i=1}^r T_N(y_i)$ are copositive with f in this set. Also, $P_n(x)f(x) \geq 0$ in $[-1,1] \setminus \cup_{i=1}^r \mathcal{Y}_i$ since $\eta \prod_{i=1}^r T_N(y_i, x) f(x) \geq 0$ and $|\prod_{i=1}^r T_N(y_i, x)| \geq 2^{-r}$. Finally, (118) holds since $|\prod_{i=1}^r T_N(y_i, x)| \leq 1$. ∎

5.3 Copositive Polynomial Approximation

Proof of Theorem 40. The proof of Theorem 40 is based on a modification of the ideas used by Y. Hu, D. Leviatan and X. M. Yu [4].

Let $n \geq 4\delta^{-1}$ be fixed, and let $N = N(n) \geq n$ be an integer (we will prescribe its exact value later). Also, let $f_n \in C^3[-1,1]$ be a function which was described in Lemma 42. Inequality (103) can be written as

$$\|(1-x^2)^{3/2} f_n'''(x)\| \leq M \text{ with } M := K_1 n^3 \omega_\varphi^3(f, n-1) .$$

It follows from Corollary 43 that there exists a polynomial $p_N(f_n, x) \in \Pi_N$, which interpolates f_n at $y_1, \ldots, y_r$ (*i.e.*, $p_N(f_n, y_i) = 0, i = 1, \ldots, r$), and such that

$$\|f_n - p_N(f_n)\| \leq K_1 K_2 \left(\frac{n}{N}\right)^3 \omega_\varphi^3(f, n-1) \tag{119}$$

and

$$\|\Delta_N(x)\,(f_n'(x)-p_N'(f_n,x))\| \le K_1K_3\left(\frac{n}{N}\right)^3 \omega_\varphi^3(f,n-1). \tag{120}$$

We prescribe N to be such that $K_1K_2\left(\frac{n}{N}\right)^3 \le 1$ and $K_1K_3\left(\frac{n}{N}\right)^2 \le \frac{1}{4}$. For instance,

$$N := K_7 n := \left([(K_1K_2)^{1/3}] + [2\sqrt{K_1K_3}] + 2\right) n.$$

It follows from (120) that for $x \in \mathcal{Y}_i, i = 1, \ldots, r$ the following estimate is valid

$$\begin{aligned} |f_n'(x) - p_N'(f_n,x)| &\le K_1K_3\,\frac{n}{N\Delta_N(x)}\left(\frac{n}{N}\right)^2 \omega_\varphi^3(f,n-1) \\ &\le K_1K_3\,\frac{n}{\sqrt{1-x^2}}\left(\frac{n}{N}\right)^2 \omega_\varphi^3(f,n-1) \\ &\le \frac{1}{2}\Delta_n(x)^{-1}\,\omega_\varphi^3(f,n-1). \end{aligned}$$

Together with (104) this implies that $\operatorname{sgn}(p_N(f_n,x)) = \operatorname{sgn}(f_n(x))$, $x \in \cup_{i=1}^r \mathcal{Y}_i$. In turn, it follows that $p_N(f_n)$ is copositive with f in $\cup_{i=1}^r \mathcal{Y}_i$, and also by (102) and (119)

$$\|f - p_N(f_n)\| \le \|f - f_n\| + \|f_n - p_N(f_n)\| \le C\,\omega_\varphi^3(f,n-1).$$

Together with Lemma 45 this yields the assertion of Theorem 40 for $n > K_8 := 4\delta^{-1}K_6K_7$, $K_8 = K_8(r,\delta)$. ■

Proof of Theorem 40′. Clearly, we only have to prove Theorem 40′ for $0 \le n \le K_8$. If $r \ge 3$, then it is sufficient to choose

$$P_n(x) := L(x,f,y_1,\ldots,y_r) \equiv 0.$$

In this case denoting

$$\tilde{L}(x) := L(x,f,-1,-1+\frac{2}{r-1},\ldots,-1+\frac{2(r-2)}{r-1},1)$$

we have for any $x \in [-1,1]$

$$\begin{aligned} |f(x) - P_n(x)| &= |f(x) - L(x,f,y_1,\ldots,y_r)| \\ &= |f(x) - \tilde{L}(x) - L(x,f-\tilde{L},y_1,\ldots,y_r)| \\ &\le \left(1 + \frac{2^{r-1}r}{\delta^{r-1}}\right)\|f - \tilde{L}\|. \end{aligned}$$

Now using Whitney's inequality we conclude that

$$\begin{aligned} \|f - P_n\| &\le C\,\omega^r(f,1) \le C\,\omega^3(f,1) \le C\,\omega_\varphi^3(f,1) \\ &\le C\,\omega_\varphi^3(f,K_8^{-1}) \le C\,\omega_\varphi^3(f,n-1), \end{aligned}$$

where $C = C(r, \delta)$.

In the cases $r = 1$ and $r = 2$ for $2 \le n \le K_8$ one should apply similar consideration for the polynomials of the second degree $P_n(x) := L(x, f, -1, y_1, 1)$ and $P_n(x) := L(x, f, -1, y_1, y_2)$, respectively. The proof of Theorem 40′ is now complete. ■

5.4 Remarks

Remark 1. All the considerations will remain the same if f vanishes on some subinterval(s), say, $[\alpha_i, \beta_i] \subset [-1, 1]$. In this case, if $[\alpha_i, \beta_i]$ is an interval of sign change (*i.e.*, if $f(\alpha_i - \varepsilon)f(\beta_i + \varepsilon) < 0$ for small ε), then it is sufficient to fix any $x_0 \in [\alpha_i, \beta_i]$ as a *point of sign change.* Thus, Theorem 40 is valid for any $f \in C[-1, 1]$ with finitely many changes of sign. In fact, if f vanishes in all the intervals of sign change, then Theorem 40(40′) can be considerably improved (see the next remark).

Remark 2. As we mentioned in the proof of Lemma 42, estimate (100) can be improved if f satisfies extra conditions near the points of sign change. For example, the following theorem is valid.

Theorem 46 *Suppose $f \in C[-1, 1]$ changes sign $1 \le r \le \infty$ times in $[-1, 1]$ and vanishes in the intervals of sign change,* i.e., *suppose that $f(x) = 0, x \in \cup_{i=1}^r [y_i - \delta_i, y_i + \delta_i]$ and $f(y_i - \delta_i - \varepsilon)\, f(y_i + \delta_i + \varepsilon) < 0$ for $i = 1, \dots, r$ and all sufficiently small ε. Let $\delta := \min\{\delta_1, \dots, \delta_r\}$ and $m \in \mathcal{N}$. Then the exists a sequence of polynomials $P_n \in \mathbf{\Pi}_n$, copositive with f, such that*

$$\|f - P_n\| \le C(r, m, \delta)\, \omega_\varphi^m(f, n^{-1}) . \tag{121}$$

Proof Theorem 46 can be proved using considerations similar to those in the proof of Theorem 40. The most important difference which makes estimate (121) possible, is that $\tilde{f}_i$ in the proof of Lemma 42 can be replaced by the linear polynomial

$$\begin{aligned} \tilde{l}_i(x) \;:=\; & \frac{60\, \omega_\varphi^m(f, n^{-1})}{y_i'' - y_i'} \left((x - y_i')\operatorname{sgn} f(y_i + \delta_i + \varepsilon)\right. \\ & \left. +(y_i'' - x)\operatorname{sgn} f(y_i - \delta_i - \varepsilon)\right) . \end{aligned}$$

However, there is a trivial proof. For sufficiently large $n \in \mathcal{N}$, let $\mathcal{P}_n \in \mathbf{\Pi}_n$ be the best approximant to f in $C[-1, 1]$. Then $\|f - \mathcal{P}_n\| \le C\, \omega_\varphi^m(f, n^{-1})$ and $f(x)\mathcal{P}_n(x) = 0$, $x \in Y := \cup_{i=1}^r \mathcal{Y}_i$, *i.e.*, by the definition $\mathcal{P}_n$ is copositive with f in Y. Now, Lemma 45 implies that there exists a polynomial P_n, copositive with f in $[-1, 1]$, such that

$$\|f - P_n\| \le C\, \|f - \mathcal{P}_n\| \le C\, \omega_\varphi^m(f, n^{-1}) .$$

■

Another example of how the behavior of f near the points of sign change determines the rate of copositive polynomial approximation, is the following theorem.

Theorem 47 *Let $f \in C[-1,1]$ change sign $r \geq 1$ times at $-1 < y_1 < \ldots < y_r < 1$, and let $f(x) \in \mathbf{\Pi}_1$ for $x \in [y_i - \delta_i, y_i + \delta_i], i = 1, \ldots, r$. Then, for any $m \in \mathcal{N}$, a sequence of polynomials $P_n \in \mathbf{\Pi}_n$, copositive with f in $[-1,1]$, exists such that (121) holds with $\delta := \min\{\delta_1, \ldots, \delta_r\}$.*

The proof of Theorem 47 is less trivial than that of Theorem 46. We omit its details, and only mention that, again, the crucial idea is to replace $\tilde{f}_i$ in the proof of Lemma 42 by a rapidly increasing or decreasing linear polynomial.

References

[1] R. DeVore and G. G. Lorentz, Constructive Approximation, Berlin: Springer-Verlag 1993.

[2] Z. Ditzian, *Polynomials of best approximation in $C[-1,1]$*, Israel J. Math., **52(4)**, 341-354 (1985).

[3] Z. Ditzian and V. Totik, Moduli of Smoothness, Berlin: Springer-Verlag 1987.

[4] Y. Hu, D. Leviatan and X. M. Yu, *Copositive polynomial approximation in $C[0,1]$*, J. Anal., **1**, 85-90 (1993).

[5] D. Leviatan, *The degree of copositive approximation by polynomials*, Proc. Amer. Math. Soc., **88**, 101-105 (1983).

[6] D. Leviatan, *Monotone and comonotone approximation revisited*, J. Approx. Theory, **53**, 1-16 (1988).

[7] P. Petrushev and V. Popov, Rational Approximation of Real Functions, Cambridge University Press, Cambridge, England 1987.

[8] I. A. Shevchuk, Approximation by Polynomials and Traces of the Functions Continuous on an Interval, Kiev: Naukova Dumka 1992 (in Russian).

[9] S. P. Zhou, *A counterexample in copositive approximation*, Israel J. Math., **78**, 75-83 (1992).

[10] S. P. Zhou, *On copositive approximation*, Approx. Theory Appl. **9(2)**, 104-110 (1993).