University of Alberta

Selected Topics in Asymptotic Geometric Analysis and Approximation Theory

by

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Abstract

This thesis is mostly based on six papers on selected topics in Asymptotic Geometric Analysis, Wavelet Analysis and Applied Fourier Analysis.

The first two papers are devoted to Ball's integral inequality. We prove this inequality via spline functions. We also provide a method for computing all terms in the asymptotic expansion of the integral in Ball's inequality, and indicate how to derive an asymptotically sharp form of a generalized Ball's integral inequality.

The third paper deals with a Khinchine type inequality for weakly dependent random variables. We prove the Khinchine inequality under the assumption that the sum of the Rademacher random variables is zero. We also discuss other approaches to the problem. In particular, one may use simple random walks on graph, concentration and the chaining argument. As a special case of Khinchine's type inequality, we provide a tail estimate for a random variable with hypergeometric distribution, improving previously known estimates.

The fourth paper devoted to the quantitative version of a Silverstein's Theorem on the 4-th moment condition for convergence in probability of the norm of a random matrix. More precisely, we show that for a random matrix with i.i.d. entries, satisfying certain natural conditions, its norm cannot be small.

The fifth paper deals with Bernstein's type inequalities and estimation of wavelet coefficients. We establish Bernstein's inequality associated with wavelets. We also prove an asymptotically sharp form of Bernstein's type inequality for splines. We study the asymptotic behavior of wavelet coefficients for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets. We provide comparison of these two families.

The sixth paper is on prolate spheroidal function. We prove that a function that is almost time and band limited is well represented by a certain truncation of its expansion in the Hermite basis.

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Contents

Introduction

Special functions

Sinc function

The **sinc function** is a real valued function defined on the real line by the following expression

$$sinc(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

This function and its L_p-norm play an important role in many areas of Approximation Theory, Numerical Analysis, Computing Sciences. In particular, it is used in interpolation and approximation of functions; approximate evaluation of Hilbert, Fourier, Laplace, Mellon and Hankel transforms; in finding approximate of solutions of differential and integral equations; it is widely used in image processing, signal processing and information theory (see e.g. [14, 33, 34, 35] for more applications).

In the present work we deal with the following expression

$$I(p) = \sqrt{p} \int_0^{+\infty} \left| \frac{sin\, x}{x} \right|^p dx, \tag{1}$$

for $1 < p < \infty$. We define $I(p)$ for $p > 1$, as $I(1) = \infty$. Note that $\displaystyle\int_0^{+\infty} \frac{sin\, x}{x}\, dx = \frac{\pi}{2}$ (see [35] for details).

Even though the function $I(p)$ is important and used in many approximation problems, there are many open questions on its behaviour. It was proved in [7] that for all $p > 0$ one has

$$I(p) > \sqrt{\frac{3\pi}{2}} \frac{2p}{2p+1} > \sqrt{\frac{3\pi}{2}} \left(1 - \frac{1}{2p}\right).$$

Moreover, $\lim_{p \longrightarrow \infty} I(p) = \sqrt{\frac{3\pi}{2}}$.

Much of effort (see Chapter 1 and 2 as well as [1, 16, 23]) was required to prove the following upper bound

$$\sqrt{p} \int_{-\infty}^{\infty} \left| \frac{\sin x}{x} \right|^p dx \leq \sqrt{2}\pi, \quad p \geq 2, \tag{2}$$

known in Asymptotic Geometric Analysis as Ball's integral inequality.

In the present work we prove inequality (2) using B-spline functions (see Chapter 1 for definitions and the result). We use the fact that the sinc function is the Fourier transform of a symmetric B-spline, as well as the property that the B-spline, together with its Fourier transform, converge to the probability density function of normal distribution.

It is known (see e.g. [4, 5, 6] for more applications) that for p a positive integer the integral $\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^p$ can be calculated explicitly. In particular, for p an even integer, $I(p)$ have the closed form expression,

$$I(p) = \sqrt{p}\frac{1}{(p-1)!}\frac{\pi}{2^p}\sum_{k=0}^{\lfloor \frac{p}{2} \rfloor}(-1)^k \binom{p}{k}(p-2k)^{p-1}.$$

The behavior of $I(p)$ for intermediate values of p is not fully established. It has been conjectured in [7] that $I(p)$ is increasing for p on $[p_0, \infty)$ and concave on $[p_1, \infty)$, where $p_0 \approx 3.36$ is the point of global minimum and $p_1 \approx 4.469$ is an inflection point.

In [7] the authors establish the existence of real constants c_j, such that

$$I(p) \sim \sqrt{\frac{3\pi}{2}} - \frac{3}{20}\sqrt{\frac{3\pi}{2}}\frac{1}{p} + \sum_{j=2}^{\infty} c_j \frac{1}{p^j}, \quad \text{as}\, p \longrightarrow \infty.$$

From this one may deduce that $I(p)$ is concave and increasing for sufficiently large p. D. Borwein, J.M. Borwein and I.E. Leonard posed the problem of determining the second order term in the asymptotic expansion of $I(p)$. In the present work (Chapter 2) we provide a method by which one can compute any term in the expansion. We also indicate how to derive an asymptotically sharp form of generalized Ball's integral inequality.

Prolate Spheroidal Wave function

Definition 0.0.1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be **band-limited**, if there exists $c > 0$ and $\sigma \in L_2([-1, 1])$, such that

$$f(x) = \int_{-1}^{1} e^{icxt}\sigma(t)\, dt.$$

Band-limited functions appear naturally as the result of the measurement and generation of physical signals. Indeed, measurements of electromagnetic or acoustic data are band-limited due to the oscillatory character of the processes that have generated the quantities being measured.

For band-limited functions, that are well behaved on the whole real line, numerical tools (for example classical Fourier Analysis) have been well studied. However, in many cases, one deals with band-limited functions defined on intervals (or, more generally, on compact sets of $\mathbb{R}^n$). In this environment, standard tools based on polynomials are effective, but not optimal. In fact, the optimal approach was discovered more than 30 years ago by Slepian and his co-authors, who observed that for the analysis of band-limited functions on intervals, prolate spheroidal wave functions are a natural tool. They built the analytical apparatus and applied it in the areas of signal processing, statistics, antenna theory, among others. However, their efforts did not lead to numerical techniques (the principal reason appears to be the lack at this time of effective numerical algorithms for the evaluation of prolate spheroidal wave functions and relate quantities).

Definition 0.0.2. Given real number $c > 0$, called the bandwidth, the **prolate spheroidal wave functions (PSWFs)**, denoted by $(\psi_{n,c}(\cdot))_{n\geq 0}$, are the eigenfunctions of the Sturm-Liouville operator's L_c, defined on $C^2([-1,1])$ by

$$L_c(\psi) = (1-x^2)\frac{d^2\psi}{d\,x^2} - 2x\frac{d\psi}{d\,x} - c^2x^2\psi. \tag{3}$$

As early as 1880, C. Niven [24] gave a remarkably detailed theoretical and computational study of the eigenfunctions. Later, in their pioneering works [18, 19, 30, 31, 32] on almost time and band limited functions, D. Slepian, H. Landau and H. Pollak have shown various important properties of the PSWFs and their associated spectra. Among these properties, they have proved that the PSWFs are also the eigenfunctions of the compact integral operators F_c and Q_c, defined on $L^2([-1,1])$ by

$$F_c(\psi)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\sin c(x-y)}{x-y}\,\psi(y)\,\mathrm{d}y, \quad Q_c(f)(x) = \int_{-1}^{1} e^{i\,c\,x\,y} f(y)\,\mathrm{d}y. \tag{4}$$

As a result, they have shown that the PSWFs exhibit the unique properties to form an orthogonal basis of $L^2([-1,1])$, an orthonormal system of $L^2(\mathbf{R})$ and an orthonormal basis of B_c, the Paley-Wiener space of c-band-limited functions defined by

$$B_c = \left\{ f \in L^2(\mathbb{R}),\ \text{Support}\,\widehat{f} \subset [-c,c] \right\}.$$

The PSWFs are normalized by using the following rule,

$$\int_{-1}^{1} |\psi_{n,c}(x)|^2\,\mathrm{d}x = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2\,\mathrm{d}x = \frac{1}{\lambda_n(c)}, \quad n \geq 0, \tag{5}$$

where $(\lambda_n(c))_n$ is the infinite sequence of the eigenvalues of F_c, arranged in the decreasing order $1 > \lambda_0(c) > \lambda_1(c) > \cdots > \lambda_n(c) > \cdots$.

Numerical evidence (*see e.g.* [27]) suggests that $\lambda_n(c) \leq \frac{c}{2}\left(\frac{ec}{4n}\right)^{2n}$, which implies super-exponential decay for $n \geq ec/4$. The best result to date is the following theorem.

Theorem 0.0.3. *Bonami-Karoui [3]. Let $\delta > 0$. There exists N_δ and κ_δ such that, for all $c \geq 0$ and $n \geq \max(N_\delta, \kappa_\delta c)$,*

$$\lambda_n(c) \leq e^{-\delta(n-\kappa c)}.$$

In the present work, we prove that a function that is almost time and band limited is well represented by a certain of its expansions in the Hermite basis (see Chapter 6 for the results).

Wavelets

Definition 0.0.4. A function ψ is called a **wavelet** if there exists a dual function $\widetilde{\psi}$, such that any function $f \in L_2(\mathbb{R})$ can be expressed in the form

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,\nu} \rangle \psi_{j,\nu}(t).$$

The development of wavelets goes back to A. Haar's work in early 20-th century and to D. Gabor's work (1946), who constructed functions similar to wavelets. Notable contributions to wavelet theory can be attributed to G. Zweig's discovery of the continuous wavelet transform in 1975; D. Goupilland, A. Grossmann and J. Morlet's formulation of the cosine wavelet transform (CWT) in 1982; J. Strömberg's work on discrete wavelets (1983); I. Daubechies' orthogonal wavelets with compact support (1988); S. Mallat's multiresolution framework (1989); and many others.

Wavelets are used in signal analysis, molecular dynamics, density-matrix localisation, optics, quantum mechanics, image processing, DNA analysis, speech recognition, to name few. Wavelets have such a wide variety of applications mainly because of their ability to encode a signal using onl a few of the larger coefficients. The numbers of large coefficients depends on

- the size of the support of the signal: the shorter support the better;
- the number of vanishing moments: the more vanishing moments a wavelet has, the more it oscillates. (The number of vanishing moments determines what the wavelet does not see).
- regularity (smoothness) of the signal: the number of continuous derivatives.

In the present work we deal with two families of wavelets – orthogonal Daubechies wavelets and semiorthogonal spline wavelets (see e.g. [10, 11]). These wavelets are very important for practical use, as they have minimal support length for a given numbers of vanishing moments. (For compact support with length m, the number of vanishing moments is $2m - 1$ for both orthogonal Daubechies wavelets and semiorthogonal spline wavelets, so these two families are comparable.)

In Chapter 5, we study the asymptotic behavior of wavelet coefficients for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets, respectively. Comparison of these two families is done by using the quantity

$$C_{k,p}(\psi) := \sup \left\{ \frac{\langle f, \psi \rangle}{\|\widehat{\psi}\|_p} : f \in A_k^{p'} \right\}, \quad \frac{1}{p'} + \frac{1}{p} = 1, \tag{6}$$

where A_k^p indicates the function space defined by

$$A_k^p := \{ f : \|(i\omega)^k \widehat{f}(\omega)\|_p \leq 1 \},$$

with a nonnegative integer k and $p \in (1, \infty)$ (see Section 5.1 for definitions and explanations).

Some of the results on the comparison of the two families of wavelets extend the early study by Ehrich in [12] on the functions in $L^2(R)$ to those in $L^p(R)$ for $p \in (1, \infty)$ and higher dimensions (see Sections 5.3 and 5.4).

The quantity $C_{k,p}(\psi)$ in (6) is the best possible constant in the following Bernstein type inequality

$$|\langle f, \psi_{j,\nu} \rangle| \leqslant C_{k,p}(\psi) 2^{-j(k+1/p-1/2)} \|\hat{\psi}\|_p \|(i\omega)^k \hat{f}(\omega)\|_{p'}. \tag{7}$$

Such type of inequalities plays an important role in wavelet algorithms for the numerical solution of integral equations (see e.g [2, 25]), where wavelet coefficients arise by applying an integral operator to a wavelet and a bound of the type (7) gives apriori information on the size of the wavelet coefficients.

This inequality (7) gives us a way of investigating the magnitude of the coefficients in the wavelet decomposition of a function f. We have in particular, obtained a lower bound of the quantity $C_{k,p}(\psi)$ with ψ the semiorthogonal spline wavelets (see Proposition 5.2.2). In fact, this bound is just a simple consequence of the result on the upper bound of the Bernstein-type inequalities for splines in the sense of L_p with $p \in (1, \infty)$ (see Section 5.2 for the definitions and results).

Khinchine Inequality

The Khinchine inequality plays a crucial role in many deep results of Probability and Analysis (see [13, 17, 21, 22, 25, 37] among others). It says that L_p and L_2 norms of sums of weighted independent Rademacher random variables are comparable.

Let $a \in \mathbb{R}^N$ and let $\varepsilon_i, i \leq N$, be independent Rademacher random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$, for $i \leq N$. The Khinchine inequality (see e.g. Theorem 2.b.3 in [21], Theorem 12.3.1 in [13] or survey [25]) states that for any $p \geq 2$

$$\left(\mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p \right)^{\frac{1}{p}} \leq \sqrt{p} \|a\|_2. \tag{8}$$

Sometimes, it is very convenient to talk about inequalities using notion of ψ_α-estimate. Let us, first, give some definitions.

Definition 0.0.5. An *Orlicz function* is a convex, increasing function $\psi : [0, \infty) \longrightarrow [0, \infty]$, such that $\psi(0) = 0$ and $\psi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Classical examples of Orlicz functions are

$$\varphi_p(x) = x^p, \quad \text{for some } p \geq 1, \forall x \geq 0 \tag{9}$$

and

$$\psi_\alpha(x) = e^{x^\alpha} - 1, \quad \text{for some } \alpha \geq 1, \forall x \geq 0. \tag{10}$$

Definition 0.0.6. Let ψ be an Orlicz function. For any real random variable X on a measurable space (Ω, σ, μ), define its L_ψ*-norm* by

$$\|X\|_\psi := \inf\{c > 0 : \mathbb{E}\,\psi\left(|X|/c\right) \leq 1\}.$$

We say X is ψ-variable if $\|X\|_\psi < \infty$.

The space $L_\psi(\Omega, \sigma, \mu) = \{X : \|X\|_\psi < \infty\}$ is the Orlicz space associated to ψ. Note that the Orlicz space associated to function φ_p, defined by (9), is the classical L_p-space.

The following well-known theorem describes the behaviour of a random variable with bounded ψ_2-norm (see for example [8]).

Theorem 0.0.7. *Let X be real-valued random variable and $\alpha \geq 1$. The following assertions are equivalent:*

1. *There exists $K_1 > 0$, such that $\|X\|_{\psi_\alpha} \leq K_1$.*
2. *There exists $K_2 > 0$, such that for every $p \geq \alpha$,*
$$\left(\mathbb{E}|X|^p\right)^{1/p} \leq K_2 p^{1/\alpha}.$$
3. *There exists $K_3, K_3' > 0$, such that for every $t > K_3'$,*
$$\mathbb{P}(|X| \geq t) \leq \exp\left(-t^\alpha / K_3^\alpha\right).$$
Note, $K_2 \leq 2eK_1$, $K_3 \leq eK_2$, $K_3' \leq e^2 K_2$, $K_1 \leq 2\max(K_2, K_3')$.
4. *In the case $\alpha > 1$, let β be such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. There exist $K_4, K_4' > 0$ such that for every $\lambda \geq 1/K_4'$,*
$$\mathbb{E}\,\exp\left(\lambda|X|\right) \leq \exp\left(\lambda K_4\right)^\beta.$$
Note, $K_4 \leq K_1$, $K_4' \leq K_1$, $K_3' \leq 2K_4^\beta/(K_4')^{\beta-1}$.

In particular, the classical Khinchine inequality is equivalent to the boundness of ψ_2-norm of the corresponding random sum (see e.g. [26]).

Our aim in the present work is to prove something like inequality (8) under additional assumption that the Rademacher random variables are not independent anymore, in particular, when sum of them is zero (see Chapter 3 for more explanations):

$$S = \sum_{i=1}^{2n} \varepsilon_i = 0. \tag{11}$$

In Section 3.2 we prove the ψ_2-estimate for the random sum $\sum_{i=1}^{N} a_i \varepsilon_i$ under assumption (11). Section 3.3 is devoted to the special case of our problem, when vector a is such that its coordinates are either ones or zeros. This particular case leads to the hypergeometric distribution. In Section 3.5, we obtain a ψ_1-norm estimate by looking at our problem from the point of view of simple random walk on graph. We also establish bounds on ψ_2-norm using different techniques than in Section 3.2. First, using the notion of Lévy family we get an estimate of the order $\sqrt{n}$. Then, using a chaining argument, we improve this to $\sqrt{\log n}$.

Random Matrices

Random Matrix Theory (in statistics known as an Asymptotic Random Matrix Theory) is now a big subject with applications in many disciplines of science, engineering and finance.

Let $F = \mathbb{R}$ (or $\mathbb{C}$). By $Mat_n(F)$ we denote the space of $n \times n$ matrices with entries in F. Let $(\Omega, F, \mathbb{P})$ be a probability space.

Definition 0.0.8. A **random matrix** Γ_n is a measurable map from $(\Omega, \mathbb{P})$ to $Mat_n(F)$.

One of the central problems is to estimate the operator norm of random matrix:

$$\|\Gamma\| := \sup_{x \in \mathbb{C}^n : |x|=1} |\Gamma x|.$$

Note that $\|\Gamma\|$ is also the largest singular value $\sigma_1(\Gamma)$ of Γ. Thus, it dominates all other singular values; as well as all eigenvalues of Γ, $\lambda_i(\Gamma)$. It is used to estimate other parameters of Γ as well.

In order to gain some intuition about the order of the operator norm of matrix Γ, let us consider possible general cases. If Γ is $n \times n$ matrix with all entries equal 1, then $\|\Gamma\| = n$ (it follows from Cauchy-Schwartz inequality). A matrix whose entries are all uniformly $\mathcal{O}(1)$, has operator norm $\mathcal{O}(n)$. From analogy with concentration of measure, when entries of matrix Γ are all independent, one would expect that the operator norm is of size $\mathcal{O}(\sqrt{n})$. From the R. Latala's result below [20] we will see that this intuition is correct (see e.g. [9, 15, 28, 36, 38, 39] for results on Gaussian case as well as results for matrices with general i.i.d. entries).

Theorem 0.0.9. *Let $\Gamma = (\xi_{i,j})_{1 \le i,j, \le n}$ be a matrix with independent zero mean entries obeying the second moment bounds*

$$\sup_i \sum_{j=1}^n \mathbb{E}|\xi_{ij}|^2 \le K^2 n, \quad \textit{and} \quad \sup_j \sum_{i=1}^n \mathbb{E}|\xi_{ij}|^2 \le K^2 n$$

Assume also the fourth moment bound

$$\sum_{i=1}^n \sum_{j=1}^n \mathbb{E}|\xi_{ij}|^4 \le K^4 n^2,$$

for some $K > 0$. Then,

$$\mathbb{E}\|\Gamma\| = \mathcal{O}(K\sqrt{n}).$$

As a corollary of this Theorem we see, that if Γ is a matrix with entries which are i.i.d. mean zero random variables with fourth moment of $\mathcal{O}(1)$, then the expected operator norm is of $\mathcal{O}(\sqrt{n})$.

The presence of the fourth moment in Latala's theorem is not surprising. The result of Silverstein [29] states that if ξ_{ij} are i.i.d. mean zero random variables such that the norms of random matrices $n^{-n/2}\|\Gamma\|$ are bounded, then $\mathbb{E}|\xi_{i,j}|^p < \infty$, for any $p < 4$.

We observe that the operator norm $\|\Gamma\|$ dominates all its entries, i.e.

$$\|\Gamma\| \geq \sup_{i,j} |\xi_{ij}|,$$

or, equivalently,

$$\mathbb{P}(\|\Gamma\| \leq t) \leq \mathbb{P}(\sup_{i,j} |\xi_{ij}| \leq t).$$

Suppose now that all $\xi_{i,j} \sim \xi$ are i.i.d. Taking $t = K\sqrt{n}, K > 0$, we have

$$\mathbb{P}(\|\Gamma\| \leq K\sqrt{n}) \leq \mathbb{P}(|\xi| \leq K\sqrt{n})^{n^2}. \tag{12}$$

Now, from dominated convergence Theorem, assuming fourth moment hypothesis, we obtain

$$\mathbb{P}(|\xi| \leq K\sqrt{n}) \geq 1 - o_K\left(\frac{1}{n^2}\right).$$

Thus, the right hand side of (12) is asymptotically correct. If one would weaken the fourth moment hypothesis, then it happened that the rate of convergence of $\mathbb{P}(|\xi| \leq K\sqrt{n})$ to 1 can be slower, and the right hand side of (12) is of order $o_K(1)$, for every $K > 0$. It forces that the size of $\|\Gamma\|$ would be much larger than $\sqrt{n}$ on the average.

So, the fourth moment assumption is pretty sharp.

In the present work we prove the quantitative version of Silverstein's result about fourth moment assumption (see Theorem 4.2.1).

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Chapter 1

A proof of Ball's integral inequality using splines *

1.1 Introduction

In his work [1] K. Ball used the following inequality

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^p dt \leq \frac{1}{\sqrt{p}}, \quad p \geq 1, \tag{1.1}$$

in which equality holds if and only if $p = 1$ (see Chapter 2 for more details).

As we will see, the right side of (1.1) has the correct rate of decay though the limit of the ratio of the right and left side is $\sqrt{\frac{3}{\pi}}$ rather then $\sqrt{2}$. Applying Ball's methods we put all of this into the following improved form of (1.1).

Theorem 1.1.1. *Let*

$$C(p) := \begin{cases} \sqrt{\frac{3}{\pi}}, & 1 \leq p \leq p_0 \\ 1 + \frac{1}{\sqrt{3\pi}}\frac{\left(\sqrt{5}/6\right)^{2p-1}}{\sqrt{p}-1/2\sqrt{p}}, & p > p_0, \end{cases}$$

where

$$\frac{\left(\sqrt{5}/6\right)^{2p_0-1}}{\sqrt{p}_0 - 1/2\sqrt{p}_0} = \left(1 - \sqrt{3/\pi}\right)\pi$$

so that $p_0 = 1.8414\ldots$.

Then,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^p dt \leq C(p)\frac{\sqrt{3/\pi}}{\sqrt{p}}, \quad p \geq 1, \tag{1.2}$$

Note, $C(p)\frac{\sqrt{3/\pi}}{\sqrt{p}} \leq \frac{1}{\sqrt{p}}, \quad p \geq 1,$ *where equality hold if only if* $p = 1$.

*A version of this chapter has been published online. R. Kerman and S. Spektor. An asymptotically sharp form of Ball's integral inequality. arXive:1208.3799v1.

Furthermore,

$$\lim_{p\longrightarrow\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^p dt\Big/\frac{\sqrt{3/\pi}}{\sqrt{p}}\leq \lim_{p\longrightarrow\infty} C(p)=1.$$

1.2 Symmetric B-splines and the integral $\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^p dt$

The symmetric B-splines, β^n, are defined inductively by

$$\beta^0(x):=\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)\quad\text{and}\quad \beta^n(x):=\int_{-\frac{1}{2}}^{\frac{1}{2}}\beta^{n-1}(x-y)dy,$$

$n=1,2,...$

Using known properties of these B-splines we obtain an asymptotic formula for our integral as $p\longrightarrow\infty$, namely

Proposition 1.2.1.

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^p dt\sim\frac{\sqrt{3/\pi}}{\sqrt{p}},\qquad \textit{as}\quad p\longrightarrow\infty. \tag{1.3}$$

Proof. Suppose to begin with that $p\in\mathbb{Z}_+$, say $p=n$. Now,

$$\widehat{\beta^n}(t):=\int_{-\infty}^{\infty}\beta^n(s)e^{-2\pi its}ds=\left(\frac{\sin\pi t}{\pi t}\right)^n,$$

so Plancherel's theorem yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^n dt=\int_{-\infty}^{\infty}\left(\frac{\sin\pi t}{t}\right)^{2n}dt=\int_{-\infty}^{\infty}|\beta^n(s)|^2ds.$$

Further, by [2],

$$\int_{-\infty}^{\infty}\beta^n(s)^2ds=\int_{-\infty}^{\infty}\beta_n(s)\beta_n(1)ds=\beta^{2n}(0).$$

Again, according to Theorem 1 in [3],

$$\beta^{2n}\left(\sqrt{\frac{2n+1}{12}}x\right)\sim\sqrt{\frac{6}{\pi(2n+1)}}\exp(-x^2/2),$$

so in particular,

$$\beta^{2n}(0)\sim\sqrt{\frac{6}{\pi(2n+1)}}\sim\frac{\sqrt{3/\pi}}{\sqrt{\pi}},\quad\text{as}\quad n\longrightarrow\infty.$$

Finally, $\int_{-\infty}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^p dt$ is a decreasing function of p, so one has

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^{[p]+1} dt \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^{p} dt \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^{[p]} dt \tag{1.4}$$

and hence (1.3), since the extreme term in (1.4) are both asymptotically equal to $\frac{\sqrt{3/\pi}}{\sqrt{p}}$. □

1.3 Proof of the main result

Ball shows that

$$\frac{1}{\pi}\int_{-6/\sqrt{5}}^{6/\sqrt{5}} \left(\frac{\sin^2 t}{t^2}\right)^p dt \leq \frac{\sqrt{3/\pi}}{\sqrt{p}}.$$

Further,

$$\frac{1}{\pi}\int_{|t|\geq 6/\sqrt{5}}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^p dt \leq \frac{2}{\pi}\int_{\geq 6/\sqrt{5}}^{\infty} t^{-2p} dt = \frac{1}{\pi}\frac{(\sqrt{5}/6)^{2p-1}}{p-\frac{1}{2}}.$$

Altogether, then,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^p dt \leq \left(1 + \frac{1}{\sqrt{3\pi}}\frac{(\sqrt{5}/6)^{2p-1}}{\sqrt{p}-1/2\sqrt{p}}\right)\frac{\sqrt{3/\pi}}{\sqrt{\pi}}.$$

Finally,

$$1 + \frac{1}{\sqrt{3\pi}}\frac{(\sqrt{5}/6)^{2p-1}}{\sqrt{p}-1/2\sqrt{p}} \leq \sqrt{\pi/3}, \quad \text{for} \quad p \geq p_0. \qquad \square$$

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Chapter 2

An Asymptotically Sharp form of Ball's integral inequality*

2.1 Introduction

To prove that every $(n-1)$-dimensional section of the unit cube in $\mathbb{R}^n$ has volume at most $\sqrt{2}$, K. Ball [1] made essential use of the inequality

$$\sqrt{n}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^n dt \leq \sqrt{2}\pi, \quad n \geq 2, \tag{2.1}$$

in which equality holds if and only if $n = 2$.

Later, Ball's integral inequality (2.1) was proved using different methods; see [2, 6] (also see [4] for an analogue of Ball's inequality). Independently of Ball, D. Borwein, J. M. Borwein and I. E. Leonard investigated, in [2], the asymptotic expansion of the left side of (2.1). They established the existence of real constants, c_j, such that

$$\sqrt{n}\int_{0}^{\infty}\left|\frac{\sin t}{t}\right|^n dt \sim \sqrt{\frac{3\pi}{2}} - \frac{3}{20}\sqrt{\frac{3\pi}{2}}\frac{1}{n} + \sum_{j=2}^{\infty}\frac{c_j}{n^j}, \quad as \quad n \longrightarrow \infty, \tag{2.2}$$

and posed the problem of determining the value of c_2.

K. Oleszkiewicz and A. Pelczyński, in [7], proved the following variant of Ball's inequality, namely,

$$n\int_{0}^{\infty}\left(\frac{2|J_1(t)|}{t}\right)^n t dt \leq 4, \quad n \geq 2, \tag{2.3}$$

involving a special case of

$$J_\nu(t) := \sum_{j=0}^{\infty}(-1)^j\left(\frac{t}{2}\right)^{2j+\nu}\frac{1}{j!\Gamma(j+\nu+1)}, \quad t \geq 0, \nu \geq \frac{1}{2},$$

*A version of this chapter has been submitted for publication. R. Kerman, R. Ol'hava and S. Spektor. An asymptotically sharp form of Ball's integral inequality. Proceedings of the AMS. July 15, 2013. PROC 130715.

the Bessel function of order ν. They showed that, with the method used to establish their inequality (2.3), one can prove (2.1). Also, they discussed the more general inequality

$$n^{\nu} \int_0^{\infty} \left(2^{\nu}\Gamma(\nu+1)\frac{|J_{\nu}(t)|}{t^{\nu}} \right)^n t^{2\nu-1} dt < 2^{\nu} \left(\int_0^{\infty} \left(2^{\nu}\Gamma(\nu+1)\frac{J_{\nu}(t)}{t^{\nu}} \right)^2 t^{2\nu-1} dt \right), \quad n > 2.$$

They conjectured it holds if and only if $\frac{1}{2} \leq \nu \leq 1$. In this connection, they pointed out that H. König has noticed the inequality is false when $\nu = \frac{k}{2}, k = 3, 4, \ldots$.

This Chapter is divided into three sections and an appendix. The first section is an Introduction. The second section is devoted to calculating the c_2 in (2.2), thereby solving the problem posed by D. Borwein, J. M. Borwein and I. E. Leonard. The method that gives c_2 can be used to derive *any* term in the asymptotic expansion in (2.2). In the third section, we indicate how the method of Section 2.2 enables one to determine the asymptotic expansion of

$$n^{\nu} \int_0^{\infty} \left(2^{\nu}\Gamma(\nu+1)\frac{|J_{\nu}(t)|}{t^{\nu}} \right)^n t^{2\nu-1} dt, \quad n \geq 2,$$

for all $\nu \geq 1/2$.

2.2 An Asymptotically Sharp Form of Ball's Integral Inequality

In this section we answer the open question of D. Borwein, J. M. Borwein and I. E. Leonard in the following theorem.

Theorem 2.2.1. *Let*

$$I(n) := \sqrt{n} \int_0^{\infty} \left| \frac{\sin t}{t} \right|^n dt,$$

$n \geq 2$, *and fix* $m \in \mathbb{Z}_+, m \geq 3$. *Then, there exist constants* $c_3, c_4, \ldots, c_m$ *such that*

$$I(n) = \sqrt{\frac{3\pi}{2}} \left[1 - \frac{3}{20}\frac{1}{n} - \frac{13}{1120}\frac{1}{n^2} + \sum_{j=3}^{m} \frac{c_j}{n^j} \right] + O\left(\frac{1}{n^{m+1}} \right).$$

Proof. We first observe that $I(n)$ can be replaced by

$$J(n) := \sqrt{n} \int_0^{\sqrt{6}} \left| \frac{\sin t}{t} \right|^n dt.$$

Indeed,

$$\sqrt{n} \int_{\sqrt{6}}^{\infty} \left| \frac{\sin t}{t} \right|^n dt \leq \sqrt{n} \int_{\sqrt{6}}^{\infty} t^{-n} dt = \frac{\sqrt{6n}}{n-1} 6^{-n/2}.$$

Next, set

$$T_k(t) := \sum_{j=0}^{k} \frac{(-1)^j}{(2j+1)!} t^{2j}.$$

Then, for k odd, one has

$$0 \leqslant T_k(t) \leq \frac{\sin t}{t} \leq T_{k+1}(t),$$

$t \in (0, \sqrt{6})$, so,

$$\int_0^{\sqrt{6}} T_k(t)^n dt \leq \int_0^{\sqrt{6}} \left(\frac{\sin t}{t}\right)^n dt \leq \int_0^{\sqrt{6}} T_{k+1}(t)^n dt.$$

Therefore, it suffices to show there exist constants $c_3, c_4, \ldots, c_m$ such that

$$K(n) := \sqrt{n} \int_0^{\sqrt{6}} T_k(t)^n dt = \sqrt{\frac{3\pi}{2}} \left[1 - \frac{3}{20}\frac{1}{n} - \frac{13}{1120}\frac{1}{n^2} + \sum_{j=3}^{m} \frac{c_j}{n^j}\right] + O\left(\frac{1}{n^{m+1}}\right),$$

whenever $k \geq m+1$.

Making the change of variable $s = \dfrac{t}{\sqrt{n}}$ in $\displaystyle\int_0^{\sqrt{6}} T_k(s)^n ds$ we obtain

$$K(n) = \int_0^{\sqrt{6n}} T_k\left(\frac{t}{\sqrt{n}}\right)^n dt = \int_0^{\sqrt{6n}} e^{-t^2/6} \left[e^{t^2/6n} T_k\left(\frac{t}{\sqrt{n}}\right)\right]^n dt.$$

Now,

$$e^{t^2/6n} = \sum_{j=0}^{\infty} \frac{\left(\frac{t^2}{6n}\right)^j}{j!},$$

whence

$$e^{t^2/6n} T_k\left(\frac{t}{\sqrt{n}}\right) = 1 + \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j},$$

in which

$$a_j = \sum_{i=0}^{\min[j,k]} \frac{1}{i! 6^i} \frac{(-1)^{j-i}}{(2(j-i)+1)!}.$$

Using Newton's Binomial Formula we obtain

$$\left[e^{t^2/6n}T_k\left(\frac{t}{\sqrt{n}}\right)\right]^n = 1 + n\left[\sum_{j=2}^{\infty}\frac{a_j}{n^j}t^{2j}\right] + \frac{n(n-1)}{2}\left[\sum_{j=2}^{\infty}\frac{a_j}{n^j}t^{2j}\right]^2 + \dots$$
$$+ \frac{n(n-1)\dots(n-m+1)}{m!}\left[\sum_{j=2}^{\infty}\frac{a_j}{n^j}t^{2j}\right]^m + \dots. \tag{2.4}$$

We observe that, for $t \in [0, \sqrt{6n}]$,

$$\left|\sum_{j=2}^{\infty}\frac{a_j}{n^j}t^{2j}\right| = \left|e^{\frac{t^2}{6n}}T_k\left(\frac{t}{\sqrt{n}}\right) - 1\right| < 1.$$

Only the first $m+1$ terms on the right-hand side of (2.4) yield the powers $\frac{1}{n^0}, \frac{1}{n}, \frac{1}{n^2}, \dots, \frac{1}{n^m}$. We get

$$\frac{1}{n^0}, \quad \frac{a_2t^4}{n} = -\frac{t^4}{180n}, \quad \left(a_3t^6 + \frac{1}{2}a_2^2t^8\right)\frac{1}{n^2} = \left(-\frac{t^6}{2835} + \frac{t^8}{64800}\right)\frac{1}{n^2}$$

and so on.

The highest power of t yielding $\frac{1}{n^m}$ is t^{4m}. Accordingly, we write

$$\left[e^{t^2/6n}T_k\left(\frac{t}{\sqrt{n}}\right)\right]^n = \sum_{j=0}^{2m} b_j\left(\frac{t}{\sqrt{n}}\right)^{2j} + R_{2m}\left(\frac{t}{\sqrt{n}}\right), \quad b_j = b_j(n), \tag{2.5}$$

in which

$$\left|R_{2m}\left(\frac{t}{\sqrt{n}}\right)\right| \le C\frac{t^{4m+2}}{n^{2m+1}},$$

the constant $C > 1$ being independent of $t \in [0, \sqrt{6n}]$.

For concreteness, we now work with the polynomial of degree 28 in (2.5) corresponding to $m = 7$. It is given in the Appendix. Formula (2.5) becomes

$$\left[e^{t^2/6n}T_8\left(\frac{t}{\sqrt{n}}\right)\right]^n = \sum_{j=0}^{14} b_j\left(\frac{t}{\sqrt{n}}\right)^{2j} + O\left(\frac{t^{30}}{n^{15}}\right), \quad b_j = b_j(n), \tag{2.6}$$

and gives all the correct terms in the asymptotic expansion up to $\frac{1}{n^7}$. Multiplying the polynomial in (2.6) by $e^{-t^2/6}$, integrating the product from 0 to $\sqrt{6n}$ and using the fact that

$$\int_0^{\sqrt{6n}} e^{-t^2/6} t^{2j} dt = 6^{j+\frac{1}{2}} \int_0^{\infty} e^{-t^2} t^{2j} dt + O\left(\frac{1}{n^8}\right)$$
$$= 3^j (2j-1)(2j-3)\dots 1 \sqrt{\frac{3\pi}{2}} + O\left(\frac{1}{n^8}\right),$$

$j = 1, 2, \dots, 2m$, we obtain, with an error of $O\left(\frac{1}{n^8}\right)$,

$$K(n) = \sqrt{\frac{3\pi}{2}} \left[1 - \frac{3}{20}\frac{1}{n} - \frac{13}{1120}\frac{1}{n^2} + \sum_{j=3}^{7} \frac{c_j}{n^j}\right].$$

□

Remark 2.2.2. Working with the polynomial of degree 28 in the Appendix one can show

$$c_3 = \frac{27}{3200}, \quad c_4 = \frac{52791}{3942400}, \quad c_5 = -\frac{5270328789}{136478720000},$$
$$c_6 = -\frac{124996631}{10035200000}, \quad c_7 = -\frac{625651892383657}{5250746735410176000000000}.$$

Remark 2.2.3. A proof using splines that $I(n) \sim \sqrt{\frac{3\pi}{2}}$ is given in [3].

2.3 A generalized Ball's integral inequality

We indicate how to determine constants $c_0, c_1, c_2, c_3, \dots, c_m$ so that, with $n \geq 2$,

$$I_\nu(n) := n^\nu \int_0^\infty \left(\frac{2^\nu \Gamma(\nu+1)|J_\nu(t)|}{t^\nu}\right)^n t^{2\nu-1} dt = c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \dots + \frac{c_m}{n^m} + O\left(\frac{1}{n^{m+1}}\right). \tag{2.7}$$

For definiteness, we do this when $m = 3$.

Our first observation is that $I_\nu(n)$ may be replaced by

$$n^\nu \int_0^{2^\nu \Gamma(\nu+1)} \left(\frac{2^\nu \Gamma(\nu+1)|J_\nu(t)|}{t^\nu}\right)^n t^{2\nu-1} dt. \tag{2.8}$$

Indeed, using the estimate

$$|J_\nu(t)| \leq c t^{-\frac{1}{3}}, \quad t \in \mathbb{R}_+, \nu \geq 1, c = 0.7857468704\dots,$$

given in [5], we get, for n sufficiently large,

$$\begin{aligned} n^\nu \int_x^\infty \left(\frac{2^\nu \Gamma(\nu+1)|J_\nu(t)|}{t^\nu}\right)^n t^{2\nu-1}dt &\leq n^\nu \int_x^\infty \left(2^\nu \Gamma(\nu+1) c t^{-\nu-\frac{1}{3}}\right)^n t^{2\nu-1} dt \\ &= n^\nu \left(2^\nu \Gamma(\nu+1) c\right)^n \int_x^\infty t^{-\left(\nu+\frac{1}{3}\right)n+2\nu-1} dt \\ &= \frac{n^\nu \left(2^\nu \Gamma(\nu+1) c\right)^n x^{-\left(\nu+\frac{1}{3}\right)n+2\nu}}{n\left(\nu+\frac{1}{3}\right)-2\nu} \\ &\leq n^{\nu-1} c^\nu, \end{aligned}$$

with $x = 2^\nu \Gamma(\nu+1)$.

As we did in Section 2 for $\dfrac{\sin t}{t}$, we approximate $2^\nu t^{-\nu}\Gamma(\nu+1)J_\nu(t)$ in (2.8) by the k-th partial sum of its Maclaurin series, namely,

$$T_k(t) := \sum_{j=0}^{k} \frac{\left(-\frac{t^2}{4}\right)^j \Gamma(\nu+1)}{j!\Gamma(\nu+j+1)}, \tag{2.9}$$

where $k \geq m+1$.

The change of variable $t \longrightarrow \dfrac{t}{\sqrt{n}}$ in the integral of

$$K_\nu(n) := n^\nu \int_0^{2^\nu \Gamma(\nu+1)} T_k(t)^n t^{2\nu-1} dt$$

yields

$$\int_0^{2^\nu \Gamma(\nu+1)\sqrt{n}} T_k\left(\frac{t}{\sqrt{n}}\right)^n t^{2\nu-1} dt.$$

Using the Maclaurin expansion of $\exp\left(\dfrac{t^2}{4n(\nu+1)}\right)$, together with (2.9), we obtain

$$\begin{aligned} K_\nu(n) &= \int_0^{2^\nu\Gamma(\nu+1)\sqrt{n}} \exp\left(\frac{-t^2}{4(\nu+1)}\right) \left[\exp\left(\frac{t^2}{4n(\nu+1)}\right) T_k\left(\frac{t}{\sqrt{n}}\right)\right]^n t^{2\nu-1}dt \\ &= \int_0^{2^\nu\Gamma(\nu+1)\sqrt{n}} \exp\left(\frac{-t^2}{4(\nu+1)}\right) \left[1 + \sum_{j=2}^\infty a_j \left(\frac{t^2}{4n}\right)^j\right]^n t^{2\nu-1}dt \\ &= \int_0^{2^\nu\Gamma(\nu+1)\sqrt{n}} \exp\left(\frac{-t^2}{4(\nu+1)}\right) \left(1 + n\left[\sum_{j=2}^\infty a_j \left(\frac{t^2}{4n}\right)^j\right] + \dots \right. \\ &\quad \left. + \frac{n(n-1)\dots(n-m+1)}{m!} \left[\sum_{j=2}^\infty a_j \left(\frac{t^2}{4n}\right)^j\right]^m + \dots\right) t^{2\nu-1}dt, \end{aligned}$$

in which

$$a_j = \sum_{i=0}^{\min[j,k]} (-1)^i \frac{\Gamma(\nu+1)}{(\nu+1)^{j-1}(j-i)!i!\Gamma(\nu+i+1)}.$$

One finds that

$$\begin{aligned} a_2 &= \frac{-1}{2(\nu+1)^2(\nu+2)} \\ a_3 &= \frac{-2}{3(\nu+1)^3(\nu+2)(\nu+3)} \\ a_4 &= \frac{\nu-5}{8(\nu+1)^4(\nu+2)(\nu+3)(\nu+4)} \end{aligned}$$

and that, moreover,

$$\begin{aligned} c_0 &= \int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^{2\nu-1} dt = \frac{4^\nu}{2}(\nu+1)^\nu \Gamma(\nu) \\ c_1 &= \frac{a_2}{16}\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^4 t^{2\nu-1} dt = \frac{-4^{\nu-1}(\nu+1)^\nu\Gamma(\nu+2)}{\nu+2} \\ c_2 &= \frac{a_3}{64}\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^6 t^{2\nu-1} dt + \frac{a_2^2}{512}\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^8 t^{2\nu-1} dt \\ &= 4^{\nu-2}(\nu+1)^\nu\Gamma(\nu+2)\frac{3\nu^2+2\nu-5}{3(\nu+2)(\nu+3)} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \left(\frac{a_4}{256} - \frac{a_2^2}{512}\right)\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^8 t^{2\nu-1} dt + \frac{a_2 a_3}{1024}\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^{10} t^{2\nu-1} dt \\ &\quad + \frac{a_2^3}{24576}\int_0^\infty \exp\left(\frac{-t^2}{4(\nu+1)}\right) t^{12} t^{2\nu-1} dt \\ &= -4^{\nu-2}(\nu+1)^{\nu+1}\Gamma(\nu+2)\frac{\nu^3-\nu^2-4\nu-8}{6(\nu+2)^2(\nu+4)}. \end{aligned}$$

We observe that, when $\nu = 1$, $c_0 = 4$, so

$$\lim_{n\longrightarrow\infty} n \int_0^\infty \left(\frac{2|J_1(t)|}{t}\right)^n t dt = 4,$$

which means the maximum value of $I_1(n)$ occurs at $n = 2$ and in the limit as n approaches infinity.

However, when $\nu > 1$ and $n \geq 2$, the c_0 in (2.7) is greater than the $I_\nu(n)$. In particular,

$$I_\nu(2) = 2^{3\nu}\Gamma(\nu+1)^2 \int_0^\infty \frac{J_\nu(t)^2}{t} dt = 2^{3\nu-1}\nu!(\nu-1)! < 2^{2\nu-1}(\nu+1)^\nu(\nu-1)! = c_0.$$

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2.4 Appendix

The polynomial in (2.5) corresponding to $m = 7$ is

$$
\begin{aligned}
&1 - \frac{1}{180\,n}t^4 - \frac{1}{2835\,n^2}t^6 + \left(\frac{1}{64800\,n^2} - \frac{1}{37800\,n^3}\right)t^8 + \left(\frac{1}{510300\,n^3} - \frac{1}{467775\,n^4}\right)t^{10} \\
&+ \left(-\frac{1}{34992000\,n^3} + \frac{269}{1285956000\,n^4} - \frac{691}{3831077250\,n^5}\right)t^{12} + \left(-\frac{1}{183708000\,n^4} + \frac{1}{47151720\,n^5}\right. \\
&\left. - \frac{2}{127702575\,n^6}\right)t^{14} + \left(\frac{1}{25194240000\,n^4} - \frac{349}{462944160000\,n^5} + \frac{23237}{11033502480000\,n^6}\right. \\
&\left. - \frac{3617}{2605132530000\,n^7}\right)t^{16} + \left(\frac{1}{99202320000\,n^5} - \frac{5543}{60153806790000\,n^6} + \frac{9001}{43444416015000\,n^7}\right. \\
&\left. - \frac{43867}{350813659321125\,n^8}\right)t^{18} + \left(-\frac{1}{22674816000000\,n^5} + \frac{143}{83329948800000\,n^6} - \frac{146843}{13902213124800000\,n^7}\right. \\
&\left. + \frac{62809}{3094897445640000\,n^8} - \frac{174611}{15313294652906250\,n^9}\right)t^{20} + \left(\frac{-1}{71425670400000\,n^6} + \frac{10643}{43310740888800000\,n^7}\right. \\
&\left. - \frac{17}{14582741040000\,n^8} + \frac{1621577}{817189465242150000\,n^9} - \frac{155366}{147926426347074375\,n^{10}}\right)t^{22} \\
&+ \left(\frac{1}{24488801280000000\,n^6} - \frac{509}{179992689408000000\,n^7} + \frac{90749797}{2837719743034176000000\,n^8}\right. \\
&\left. - \frac{370206979}{29480755108188384000000\,n^9} + \frac{441301082837}{2275545784913290890000000\,n^{10}} - \frac{236364091}{2423034863565078262500\,n^{11}}\right)t^{24} \\
&+ \left(\frac{1}{64283103360000000\,n^7} - \frac{7241}{15591866719968000000\,n^8} + \frac{463523}{118238322626424000000\,n^9}\right. \\
&\left. - \frac{465818341}{35008396690973706000000\,n^{10}} + \frac{41342265857}{21807313772085704362500000\,n^{11}} - \frac{1315862}{1442282656883975156250\,n^{12}}\right)t^{26} \\
&+ \left(-\frac{1}{308558896128000000000\,n^7} + \frac{589}{1619934204672000000000\,n^8} - \frac{6915119}{1021579107492303360000000\,n^9}\right. \\
&+ \frac{3673793561}{7959803879210863680000000\,n^{10}} - \frac{570787478291}{409598241284392360020000000\,n^{11}} \\
&\left. + \frac{27997256387}{150973710729824107125000000\,n^{12}} - \frac{3392780147}{395257562119053391570312\,5\,n^{13}}\right)t^{28}.
\end{aligned}
$$

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Chapter 3

Khinchine inequality for Slightly dependent random variables*

3.1 Introduction

The Khinchine inequality plays a crucial role in many deep results of Probability and Analysis (see [11, 13, 15, 17, 19, 22] among others). It says that L_p and L_2 norms of sums of weighted independent Rademacher random variables are comparable. More precisely, we say that ε_0 is a Rademacher random variable if $\mathbb{P}(\varepsilon_0 = 1) = \mathbb{P}(\varepsilon_0 = -1) = \frac{1}{2}$. Let ε_i, $i \leq N$, be independent copies of ε_0 and $a \in \mathbb{R}^N$. The Khinchine inequality (see e.g. Theorem 2.b.3 in [15] or Theorem 12.3.1 in [11]) states that for any $p \geq 2$ one has

$$\left(\mathbb{E}\left|\sum_{i=1}^{N} a_i\varepsilon_i\right|^p\right)^{\frac{1}{p}} \leq \sqrt{p}\,\|a\|_2 = \sqrt{p}\left(\mathbb{E}\left|\sum_{i=1}^{N} a_i\varepsilon_i\right|^2\right)^{\frac{1}{2}}. \tag{3.1}$$

Note that the (Rademacher) random vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ in the Khinchine inequality has independent coordinates. However in many problems of Analysis and Probability it is important to consider random vectors with dependent coordinates, e.g. so-called log-concave random vectors, which in general have dependent coordinates, but whose behaviour is similar to that of Rademacher random vector or to the Gaussian random vector (see e.g. [9] and references there in). In [**?**] the S. O'Rourke considered random matrices, whose rows are independent random vectors satisfying certain conditions (so the vectors may have dependent coordinates). He studied limiting empirical distribution of eigenvalues of such matrices. As an example of such a vector, showing that the conditions cover large class of natural distributions, not covered by previously known results, O'Rurke considered the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$, whose coordinates are Rademacher random variables under the

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additional condition

$$S = \sum_{i=1}^{N} \varepsilon_i = 0 \tag{3.2}$$

(see Examples 1.3 and 1.10 in [18]). For such vectors he proved a Khintchine type inequality with the factor $C\sqrt{N}p/\log N$ in front of $\|a\|_2$, which was enough for his purposes. The goal of this paper is to show that such random variables satisfy a Khintchine type inequality with the same factor $\sqrt{p}$ as in the standard inequality. To shorten notation, by $\mathbb{E}_S$ we denote an expectation with assumption (3.2). Note that the corresponding probability space is

$$\Omega = \left\{ \varepsilon \in \{-1,1\}^N \mid \sum_{i=1}^{N} \varepsilon_i = 0 \right\} = \left\{ \varepsilon \in \{-1,1\}^N \mid \text{card}\{i : \varepsilon_i = 1\} = \frac{N}{2} \right\}. \tag{3.3}$$

Our main result is the following theorem.

Theorem 3.1.1. *Let $\varepsilon_i, i \le N$, be Rademacher random variables satisfying condition (3.2). Let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ and $b = \frac{1}{N}\sum_{i=1}^{N} a_i$. Then*

$$\left(\mathbb{E}_S \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p \right)^{1/p} \le \sqrt{2p} \left(\|a\|_2^2 - N\, b^2 \right)^{1/2} \le \sqrt{2p} \left(\mathbb{E}_S \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^2 \right)^{1/2}. \tag{3.4}$$

The first step in the proof is a reformulation in terms of random variables on the permutation group as follows. Let $N = 2n$. For the set Ω defined in (3.3), we put into correspondence the group Π_N of all permutations of the set $\{1, ..., N\}$ as

$$\sigma \in \Pi_N \longleftrightarrow A_\sigma = \{ \varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le n;\ \varepsilon_i = -1 \text{ if } \sigma(i) > n \}.$$

Given $a \in \mathbb{R}^N$, define $f_a : \Pi_N \longrightarrow \mathbb{R}$ by

$$f_a(\sigma) := \left| \sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} \right|. \tag{3.5}$$

By $\mathbb{E}_\Pi$ we denote the average over Π_N, i.e. the expectation with respect to the normalized counting measure on Π_N. Note, that $\mathbb{E}_S \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p = \mathbb{E}_\Pi |f|^p$. Therefore Theorem 3.1.1 is equvivalent to the following theorem.

Theorem 3.1.2. *Let $N = 2n$, $a \in \mathbb{R}^N$. Let f_a be the function defined in (3.5). Let $b = \frac{1}{N}\sum_{i=1}^{N} a_i$. Then, for $p \ge 2$*

$$\left(\mathbb{E}_\Pi |f_a|^p \right)^{1/p} \le \sqrt{2p} \left(\sum_{i=1}^{N} a_i^2 - N\, b^2 \right)^{1/2} \le \sqrt{2p} \left(\mathbb{E}_\Pi |f_a|^2 \right)^{1/2}. \tag{3.6}$$

In Section 3.2 we prove Theorem 3.1.2. Then, in Section 3.3, we consider a special case of our problem, when the coordinates of the vector a are either ones or zeros. This particular case leads to the hypergeometric distribution. We obtain new bounds for the p-th central moments of such variables.

Finally let us note the setting of Theorem 3.1.2 can be extended to a more general case. We pose the following problem.

Problem 3.1.3. Let $a, b \in \mathbb{R}^N$. What is the best possible constant $C(a, b, N)$ in the inequality

$$\left(\mathbb{E}_\Pi \left|\sum_{i=1}^{N} a_{\sigma(i)} b_i\right|^p\right)^{1/p} \leq C(a, b, p, N) \left(\mathbb{E}_\Pi \left|\sum_{i=1}^{N} a_{\sigma(i)} b_i\right|^2\right)^{1/2}.$$

We discuss this problem in the last Section.

3.2 Proof of Theorem 3.1.2

Direct calculations show that

$$\mathbb{E}_\Pi |f|^2 = \frac{N\|a\|_2^2 - \left(\sum_{i=1}^N a_i\right)^2}{N(N-1)}.$$

Thus, without loss of generality we may assume that $\sum\limits_{i=1}^{N} a_i = 0$.

For $k \leq n$ denote $b_{k,\sigma} := a_{\sigma(k)} - a_{\sigma(n+k)}$ and by $H_{k,\sigma} := \sum\limits_{i=k+1}^{n} a_{\sigma(i)} - \sum\limits_{i=n+k+1}^{2n} a_{\sigma(i)}$ (with $H_{n,\sigma} = 0$). Clearly,

$$\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} = b_{1,\sigma} + H_{1,\sigma} = b_{1,\sigma} + b_{2,\sigma} + H_{2,\sigma} = \ldots = \sum_{i=1}^{n} b_{i,\sigma}.$$

Note, that $\mathbb{E}_\Pi |b_{1,\sigma} + H_{1,\sigma}|^p = \mathbb{E}_\Pi |-b_{1,\sigma} + H_{1,\sigma}|^p$. Hence,

$$\mathbb{E}_\Pi |f_a(\sigma)|^p = \mathbb{E}_\Pi \left|\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\right|^p = \frac{\mathbb{E}_\Pi |b_{1,\sigma} + H_{1,\sigma}|^p + \mathbb{E}_\Pi |-b_{1,\sigma} + H_{1,\sigma}|^p}{2}.$$

Thus, denoting by $\delta_i, i \leq n$, i.i.d. Rademacher random variables independent of

$\varepsilon_1, \ldots, \varepsilon_N$, and using Khinchine inequality (3.1), we obtain

$$\begin{aligned}
\mathbb{E}_\Pi |f_a(\sigma)|^p &= \mathbb{E}_\Pi \mathbb{E}_{\delta_1} |\delta_1 \, b_{1,\sigma} + H_{1,\sigma}|^p \\
&= \mathbb{E}_\Pi \mathbb{E}_{\delta_1} \mathbb{E}_{\delta_2} |\delta_1 \, b_{1,\sigma} + \delta_2 \, b_{2,\sigma} + H_{2,\sigma}|^p = \ldots = \mathbb{E}_\Pi \mathbb{E}_{\delta_1} \mathbb{E}_{\delta_2} \ldots \mathbb{E}_{\delta_n} \left| \sum_{i=1}^n \delta_i \, b_{i,\sigma} \right|^p \\
&\leq \mathbb{E}_\Pi \left[\sqrt{p} \left(\sum_{i=1}^n b_{i,\sigma}^2 \right)^{1/2} \right]^p = p^{p/2} \, \mathbb{E}_\Pi \left(\sum_{i=1}^n \left| a_{\sigma(i)} - a_{\sigma(i+n)} \right|^2 \right)^{p/2} \\
&\leq p^{p/2} \, \mathbb{E}_\Pi \left(2 \sum_{i=1}^n \left(a_{\sigma(i)}^2 + a_{\sigma(i+n)}^2 \right) \right)^{p/2} \leq (2p)^{p/2} \, \|a\|_2^p,
\end{aligned}$$

which completes the proof. □

3.3 Hypergeometric distribution

In this section we discuss a specific case of hypergeometric distribution and show how it is related to our problem. Recall that hypergeometric random variable with parameters (N, n, ℓ) is a random variable ξ which takes values $k = 0, \ldots, \ell$ with probability

$$p_k = \frac{\binom{\ell}{k}\binom{N-\ell}{n-k}}{\binom{N}{n}}.$$

In this section we consider only the case $N = 2n$, $\ell \leq n$. It is well known that $\mathbb{E}\, \xi = \ell/2$. In the next proposition we estimate the central moment of ξ.

Proposition 3.3.1. *Let $1 \leq \ell \leq n$, $p \geq 2$. Let ξ be $(2n, n, \ell)$ hypergeometric random variable. Then,*

$$\mathbb{E}\, |\xi - \mathbb{E}\, \xi|^p \leq \sqrt{2} \left(\frac{p\, \ell}{4} \right)^{\frac{p}{2}}.$$

Remark 3.3.2. It is well known (see e.g. Theorem 1.1.5 in [7]) that the conclusion of Proposition 3.3.1 is equivalent to the following deviation inequality.

$$\forall t \geq 1 \qquad \mathbb{P}(|\xi - \mathbb{E}\, \xi| > t) \leq \exp\left(\frac{-ct^2}{\ell} \right).$$

This estimate is of independent interest, in particular it is better than the one from [12] (see Section 6.5 there) or in [21] (formulas (10) and (14) there), where the bound $\exp(-ct^2/n))$ was observed.

Remark 3.3.3. One can use Theorem 3.1.2 to estimate $\mathbb{E}_S |\sum_{i=1}^{2n} a_i \varepsilon_i|^p$ in the case that the vector a has 0/1 coordinates with ℓ ones. Indeed, without loss of generality assume that $a_1 = a_2 = \ldots = a_\ell = 1$ and $a_{\ell+1} = a_{\ell+2} = \ldots = a_{2n} = 0$. Then, $\sum_{i=1}^{2n} a_i \varepsilon_i = \sum_{i=1}^{\ell} \varepsilon_i$. Theorem 3.1.2 implies the following estimate.

Corollary 3.3.4. *Let $a \in \mathbb{R}^N$, $N = 2n$, be a vector with ℓ coordinates equals to one and $N - \ell$ zero coordinates. Then, for $p \geq 2$,*

$$\mathbb{E}_S \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p \leq (2\, p\, \ell)^{p/2}.$$

Proof of Proposition 3.3.1. Denote $X := \sum_{i=1}^{2n} a_i \varepsilon_i = \sum_{i=1}^{\ell} a_i \varepsilon_i$. Since the vector a has 0/1 coordinates with ℓ ones, $\|a\|_2 = \sqrt{\ell}$. For every $0 \leq k \leq \ell$ we compute the probability q_k that exactly k of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ equals to one (in that case $X = 2k - \ell$). Since $S = \sum_{i=1}^{2n} \varepsilon_i = 0$, in order to get k ones, we have to choose k ones out of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ and $n - k$ ones out of $\varepsilon_{\ell+1}, \varepsilon_{\ell+2}, \ldots, \varepsilon_{2n}$. This gives us $\binom{\ell}{k}\binom{2n-\ell}{n-k}$ choices. Since $|\Omega| = \left| \left\{ \varepsilon \in \{-1,1\}^{2n} \mid \sum_{i=1}^{2n} \varepsilon_i = 0 \right\} \right| = \binom{2n}{n}$, we obtain that $q_k = p_k$, i.e. $X = 2(\xi - \mathbb{E}\,\xi)$, where ξ has hypergeometric distribution with parameters $(2n, n, \ell)$. Therefore, Corollary 3.3.4 implies

$$(\mathbb{E}|\xi - \mathbb{E}\xi|^p)^{1/p} \leq \sqrt{2\, p\, \ell}.$$

$\square$

We would also like to note that Proposition 3.3.1 can be proved directly. Below we provide such a direct proof, which gives 2 in place of $\sqrt{2}$ in front of $\left(\frac{p\ell}{4}\right)^{p/2}$. This proof is of interest as it can be extended to slightly more general case (see Remark 3.3.5) and can be used in another approach to the main problem (see Remark 3.4.3).

Direct proof of the Proposition 3.3.1. From Stirling's formula together with the observation that $\sqrt{\pi n}\binom{2n}{n}/4^n$ increases, we observe that

$$\frac{2^{2n}}{\sqrt{2\pi n}} \leq \binom{2n}{n} \leq \frac{2^{2n}}{\sqrt{\pi n}}.$$

Using this, we obtain

$$\frac{\binom{2n-\ell}{n-k}}{\binom{2n}{n}} \leq \frac{\binom{2n-\ell}{n-\lfloor \frac{\ell}{2} \rfloor}}{\binom{2n}{n}} \leq \frac{2^{2n-\ell}}{\sqrt{\pi(n - \lfloor \frac{\ell}{2} \rfloor)}} \frac{\sqrt{2\pi n}}{2^{2n}} \leq \frac{2}{2^\ell} \leq 1. \tag{3.7}$$

Therefore

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p = \frac{1}{2^p} \sum_{k=0}^{\ell} |2k - \ell|^p \frac{\binom{\ell}{k}\binom{2n-\ell}{n-k}}{\binom{2n}{n}} \leq \frac{2}{2^{\ell+p}} \sum_{k=0}^{\ell} |2k - \ell|^p \binom{\ell}{k} = \frac{2}{2^p} \mathbb{E}|2S_\ell|^p,$$

where S_ℓ is a sum of ℓ i.i.d. Rademacher random variables. By Khinchine inequality (3.1), we have

$$(\mathbb{E}|S_\ell|^p)^{1/p} \leq \sqrt{p}\,\sqrt{\ell}.$$

Thus,

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p \le 2\left(\frac{p\,\ell}{4}\right)^{p/2}.$$

$\square$

Remark 3.3.5. The above proof can be extended to slightly larger class of hypergeometric random variables. Note that the proof works whenever $\binom{N-\ell}{n-k}\Big/\binom{N}{n} \le 1$. Thus, if $\ell \ge N - \log_2\left[\sqrt{\pi}\binom{N}{n}\right]$, then

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p \le 2\,(p\,\ell/4)^{\frac{p}{2}}$$

for a (N, n, ℓ) hypergeometric random variable ξ.

3.4 Concluding Remarks

In this section we discuss Problem 3.1.3. A possible approach to this problem is to use the concentration on the group Π_N (endowed with the distance $d_N(\sigma,\pi) = |\{i : \sigma(i) \ne \pi(i)\}|$). The following Theorem was proved by Maurey ([16], see also [20]).

Theorem 3.4.1. *Let $f : \Pi_N \longrightarrow \mathbb{R}$ be 1-Lipschitz function. Then for all $t > 0$ and probability measure μ*

$$\mu\left(\{\sigma : |f(\sigma) - \mathbb{E}f| \ge t\}\right) \le 2e^{-t^2/(32N)}. \tag{3.8}$$

Let us mention here, the following open question, posed by G. Schechtman in [20]: "*Is there an equivalent (with constants independent of N) metric on Π_N for which the isoperimetric problem can be solved?*"

Theorem 3.4.1 implies the following estimate.

Corollary 3.4.2. *Let $a, b \in \mathbb{R}^N$. Let $f : \Pi_N \longrightarrow \mathbb{R}$ be defined by*

$$f(\sigma) := \left|\sum_{i=1}^{N} a_{\sigma(i)} b_i\right|. \tag{3.9}$$

Then

$$(\mathbb{E}|f|^p)^{1/p} \le \mathbb{E}|f| + 4\sqrt{p}\,\sqrt{N}\|a\|_\infty\|b\|_\infty. \tag{3.10}$$

Proof. It is easy to see that f is a Lipschitz function with Lipschitz constant $2\|a\|_\infty\|b\|_\infty$, indeed,

$$\begin{aligned}|f(\sigma) - f(\pi)| &\le \left|\sum_{i=1}^{N} a_{\sigma(i)} b_i - \sum_{i=1}^{N} a_{\pi(i)} b_i\right| \\ &\le \sum_{i=1}^{N} |b_i||a_{\sigma(i)} - a_{\pi(i)}| \le 2\|a\|_\infty\|b\|_\infty d_N(\sigma,\pi).\end{aligned}$$

Using Theorem **??** and the bound $\Gamma(x) \le x^{x-1}$ for all $x \ge 1$ (see for example [6]), we obtain

$$\mathbb{E}|f-\mathbb{E}f|^p = \int_0^\infty \mu_N(|f-\mathbb{E}f|^p \ge t^p)dt^p \le 2p\int_0^\infty e^{-t^2/(32N\|a\|_\infty^2\|b\|_\infty^2)}t^{p-1}dt$$
$$\le 4^p\,\Gamma\left(\frac{p}{2}\right) N^{p/2}\|a\|_\infty^p\|b\|_\infty^p$$
$$\le 4^p\, N^{p/2}p^{p/2}\|a\|_\infty^p\|b\|_\infty^p.$$

Thus,

$$(\mathbb{E}|f|^p)^{1/p} \le \mathbb{E}|f| + 4\sqrt{p}\sqrt{N}\|a\|_\infty\|b\|_\infty \le \sqrt{\mathbb{E}|f|^2} + 4\sqrt{p}\sqrt{N}\|a\|_\infty\|b\|_\infty.$$

□

Remark 3.4.3. In the case when $b_i = \pm 1$ with condition $\sum_{i=1}^N b_i = 0$, Corollary 3.4.2 gives an additional factor $\sqrt{N}$ in the upper estimate in (3.5). Using the chaining argument similar to the one used in [2, 3, 4] and Proposition 3.3.1, the factor $\sqrt{N}$ can be reduced to $\sqrt{\ln N}$ (the details will be provided in the next section).

Remark 3.4.4. It would be nice to obtain the upper bound in Corollary 3.4.2 with constant independent of N.

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3.5 Other Techniques to prove Khinchine inequality

In this section we show different proofs of Khinchine inequality. The results are slightly weaker than in Section 3.2, but the technique is interesting by itself and can be useful in future. We establish bounds on ψ_2-norm. First, using the notion of Lévy family we get an estimate of the order $\sqrt{n}$. Then, using a chaining argument, we improve this to $\sqrt{\log n}$. Also, we obtain a ψ_1-norm estimate by looking at our problem from the point of view of simple random walk on graph.

In the next two Subsections we give some basic facts about Lévy family and simple random walk on graph.

3.5.1 Lévy families

Let (X, ρ, μ) be metric space (X, ρ) with a Borel probability measure μ. For a subset $A \subseteq X$ and $\delta > 0$ we define δ-neighborhood of A as $A_\delta = \{x \in X : \rho(x, A) \le \delta\}$. We also assume that diameter $\mathrm{diam}(X) \ge 1$.

For $\delta > 0$ the function

$$\alpha(X,\delta) = 1 - \inf\{\mu(A_\delta) : A \subseteq X \text{ Borel set with } \mu(A) \ge 1/2\} \tag{3.11}$$

is called a *concentration function*.

A family (X_n, ρ_n, μ_n), $n = 1, 2, ...$, of metric probability spaces is called a *Lévy family* if for every $\delta > 0$

$$\alpha\left(X_n, \delta \operatorname{diam}(X_n)\right) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

The family is called a *normal Lévy family* with constants c_1, c_2, if for any $n = 1, 2, ...$

$$\alpha\left(X_n, \delta\right) \leq c_1 e^{-c_2 \delta^2 n}. \tag{3.12}$$

Note also, that any normal Lévy family is a Lévy family ([17]).

It is known that in each Lévy family we have phenomenon of concentration of measure around one value of function. Let us recall some definitions first.

Let μ be a probability measure on the Borel subsets of (X, ρ). For a measurable real valued function on (X, ρ), its **median** is defined as a number M_f satisfying

$$\mu\{x \in X : f(x \leq M_f)\} \geq \frac{1}{2} \quad \text{and} \quad \mu\{x \in X : f(x \geq M_f)\} \geq \frac{1}{2}.$$

For a continuous function f on (X, ρ) and $\delta > 0$ we denote by

$$\omega_f(\delta) = \sup\{|f(x) - f(y)|; \rho(x, y) \leq \delta\}$$

its **modulus of continuity**.

Since M_f is a median of f and if $A = \{f \leq M_f\}$ (so we have that $\mu(A) \geq 1/2$), for some $y \in A$ and for $x \in X$, such that $\rho(x, y) \leq \delta$

$$f(x) \leq f(y) + \omega_f(\delta) \leq M_f + \omega_f(\delta).$$

Hence,

$$\mu\left(|f - M_f| \leq \omega_f(\delta)\right) \geq 1 - 2\alpha\left(X, \delta\right). \tag{3.13}$$

This inequality is the concentration inequality of f around its median with rate $\alpha\left(X, \delta\right)$ (see for example [14, 17] for more information).

Concentration of measure is closely related to a special type of functions called Lipschitz functions. A general idea is that a Lipschitz fucntion depending on many variables is almost a constant on a set of large measure. A natural choice for that constant is either a median or the average of the function.

A real-valued function f on (X, ρ) is a **Lipschitz function** if

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)} < \infty.$$

We say that f is 1-Lipschitz if $\|f\|_{Lip} \leq 1$.

Clearly, $\omega_f(\delta) \leq \delta \|f\|_{Lip}$ for every $\delta > 0$. Thus, concentration of measure inequality (3.13) can be rewritten (see for example [14]) as

$$\mu\left(|f - M_f| \leq t\right) \geq 1 - 2\alpha\left(X, t/\|f\|_{Lip}\right), \quad \text{for every} \quad t > 0. \tag{3.14}$$

We will deal with the group Π_{2n} of all permutations of the set $\{1, ..., 2n\}$. On this group we consider the normalized counting measure

$$\mu_{2n}(A) = \frac{\text{card}(A)}{(2n)!}, \quad A \subseteq \Pi_{2n}$$

and normalized metric

$$\bar{d}_{2n}(\sigma_1, \sigma_2) = \frac{1}{2n} d_{2n},$$

where $d_{2n} = \#\{i : \sigma_1(i) \neq \sigma_2(i); \sigma_1, \sigma_2 \in \Pi_{2n}\}$.

It was proved by Maurey in [16] that $(\Pi_{2n}, \bar{d}_{2n}, \mu_{2n})$ is a normal Lévy family with constants $c_1 = 2, c_2 = 1/64$. Later, Maurey's method was developed (see for example [20]), and the following isoperimetric inequality for $(\Pi_{2n}, d_{2n}, \mu_{2n})$ was proved (see Theorem 3.4.1)

3.5.2 Simple random walk on graph

To get a ψ_1-estimate on the sum $\sum_{i=1}^N a_i \varepsilon_i$, we will look at our problem from the point of view of simple random walk on graph. Let $G(V, E)$ be a connected undirected graph, where V stands for a set of vertices and E is a set of edges. A *simple random walk* is a sequence of vertices $v_0, v_1, \ldots, v_t$, where $v_i \sim v_{i+1}$ (that is $\{v_i, v_{i+1}\} \in E$) for $i = 0, 1, \ldots, t-1$. That is, given an initial vertex v_0, select randomly an adjacent vertex v_1, and move to its neighbor. Then, select randomly a neighbor v_2 of v_1, and move to it, etc. The probability of movement from vertex v_i to vertex v_{i+1} is given by

$$p(v_i, v_{i+1}) = \begin{cases} \dfrac{1}{deg(v_i)}, & \text{if } v_i \sim v_{i+1} \\ \\ 0, & \text{otherwise,} \end{cases} \tag{3.15}$$

where $deg(v_i)$ denotes the degree of vertex v_i. This is a walk using a transition probability matrix, $P = \left(p(v_i, v_{i+1})\right)_{v_i, v_{i+1} \in V}$. The transition probability (3.15) has a reversible equilibrium probability distribution $\mu(v_i)$. That is,

$$\mu(v_i) p(v_i, v_{i+1}) = \mu(v_{i+1}) p(v_{i+1}, v_i)$$

and $\mu(v_i)$ is proportional to $deg(v_i)$.

Let I be the $V \times V$ identity matrix. The discrete Laplacian is the matrix $L = P - I$ with its eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, ordered in non-increasing order. The smallest eigenvalue, $\lambda_1 > 0$, is called the *spectral gap* of the random walk.

For $f : V \longrightarrow \mathbb{R}$ define

$$|||f|||_\infty^2 = \frac{1}{2} \sup_{v_i \in V} \sum_{v_{i+1} \in V} |f(v_i) - f(v_{i+1})|^2 p(v_i, v_{i+1}). \tag{3.16}$$

We will use the following concentration inequality (see [5] or [14])

Theorem 3.5.1. *Assume that (p, μ) is reversible on the finite graph $G(V, E)$, and let $\lambda_1 > 0$ be the spectral gap. Then, if $|||f|||_\infty^2 < \infty$, we have*

$$\mu\left(f > \int f d\mu + t\right) \leq 3 \exp\left(\frac{-t\sqrt{\lambda_1}}{2|||f|||_\infty^2}\right). \tag{3.17}$$

Let us now specialize to $V = \Pi_{2n}$, the group of all permutations σ of the set $\{1, \dots, 2n\}$, and to $E = \{(\sigma, \sigma\tau) \mid \tau \text{ is a transposition on } \Pi_{2n}\}$. The transition probability $p(\sigma, \sigma\tau)$ on $G = (\Pi_{2n}, E)$ is

$$p(\sigma, \sigma\tau) = \frac{2}{(2n)^2}, \tag{3.18}$$

and reversible equilibrium distribution μ on Π_{2n} is a unique invariant measure for p (see for example [8] for these facts). Also, as proved in [10], the spectral gap of the random transposition walk on Π_{2n} is $\lambda_1 = \frac{2}{2n} = \frac{1}{n}$. Thus, the concentration inequality (3.17) for simple random walk on $G(\Pi_{2n}, E)$ can be rewritten as

$$\mu(\{\sigma : f(\sigma) - \mathbb{E}f \geq t\}) \leq \exp\left(\frac{-t}{2|||f|||_\infty^2 \sqrt{n}}\right). \tag{3.19}$$

3.5.3 ψ_2-estimate using connection with permutations.

In this subsection we would like to consider a more general assumption on Rademacher random variables, namely,

$$\sum_{i=1}^{N} \varepsilon_i = M, \quad -N \leq M \leq N. \tag{3.20}$$

For shorter notation, by $\mathbb{E}_M$ we denote an expectation with assumption (3.20).

Note here, that with condition (3.20), for $M \geq \sqrt{N}$ (or $M \leq -\sqrt{N}$), we have strongly dependent Rademacher random variables, and straightforward use of Proposition 3.1.2 wold not work. Our Lemma 3.5.2 below gives the result. Unfortunately, for $M = 0$, the result is weaker then in the Proposition 3.1.2.

As before, let $a \in \mathbb{R}^N$ and let $\varepsilon_i, i = 1, \dots, N$ be independent Rademacher random variables. As usual for $\varepsilon \in \{\pm 1\}^N$ by $\varepsilon_1, \dots, \varepsilon_N$ we denote coordinates of ε.

Consider the following set

$$\Omega = \left\{ \varepsilon \in \{-1,1\}^N \mid \sum_{i=1}^{N} \varepsilon_i = M \right\} = \left\{ \varepsilon \in \{-1,1\}^N \mid \mathrm{card}\{i:\ \varepsilon_i = 1\} = m = \left[\frac{M+N}{2} \right] \right\}. \tag{3.21}$$

Thus, for $\varepsilon \in \Omega$ the sequence of its coordinates is a sequence of dependent Rademacher random variables.

For set Ω, defined by (3.21), we put into correspondence the group Π_N of all permutations of set $\{1, ..., N\}$ as

$$\sigma \in \Pi_N \longleftrightarrow A_\sigma = \{\varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le m;\ \varepsilon_i = -1 \text{ if } \sigma(i) > m\}.$$

Define $f : \Pi_N \longrightarrow \mathbb{R}$ by

$$f(\sigma) := \left| \sum_{i=1}^{m} a_{\sigma(i)} - \sum_{i=m+1}^{N} a_{\sigma(i)} \right|, \tag{3.22}$$

where $\sum_{i=1}^{0} a_{\sigma(i)} = 0$ and $\sum_{i=N+1}^{N} a_{\sigma(i)} = 0$.

Note, that $\mathbb{E}_M \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p = \mathbb{E}|f|^p$. Thus, it is enough to estimate p-th moments of f. It is easy to see that f is a Lipschitz function with Lipschitz constant $2\|a\|_\infty$, indeed,

$$\begin{aligned} |f(\sigma) - f(\pi)| &\le \left| \sum_{i=1}^{m} (a_{\sigma(i)} - a_{\pi(i)}) - \sum_{i=m+1}^{N} (a_{\sigma(i)} - a_{\pi(i)}) \right| \\ &\le \sum_{i=1}^{N} |a_{\sigma(i)} - a_{\pi(i)}| \le 2\|a\|_\infty d_N(\sigma, \pi). \end{aligned}$$

Using concentration inequality (3.8) we prove the following lemma.

Lemma 3.5.2. *Let function f be defined as above. Then, for $p \ge 2$*

$$\begin{aligned} (\mathbb{E}|f|^p)^{1/p} &\le \mathbb{E}|f| + C\sqrt{p}\sqrt{N}\|a\|_\infty \\ &\le \sqrt{\frac{(N^2 - M^2)\|a\|_2^2 - (N - M^2)\left(\sum_{i=1}^{N} a_i\right)^2}{N(N-1)}} + 4\sqrt{p}\sqrt{N}\|a\|_\infty. \end{aligned} \tag{3.23}$$

Proof. Using Theorem **??** and the bound $\Gamma(x) \le x^{x-1}$ for all $x \ge 1$ (see for example [6]), we obtain

$$\begin{aligned} \mathbb{E}|f - \mathbb{E}f|^p = \int_0^\infty \mu_N(|f - \mathbb{E}f|^p \ge t^p) dt^p &\le 2p \int_0^\infty e^{-t^2/(32N\|a\|_\infty^2)} t^{p-1} dt \\ &\le 4^p\, \Gamma\left(\frac{p}{2}\right) N^{p/2} \|a\|_\infty^p \\ &\le 4^p\, N^{p/2} p^{p/2} \|a\|_\infty^p. \end{aligned}$$

Then

$$(\mathbb{E}|f|^p)^{1/p} \le \mathbb{E}|f| + 4\sqrt{p}\sqrt{N}\|a\|_\infty \le \sqrt{\mathbb{E}|f|^2} + 4\sqrt{p}\sqrt{N}\|a\|_\infty. \qquad (3.24)$$

We compute now $\mathbb{E}|f|^2 = \mathbb{E}\left|\sum_{i=1}^{m} a_{\sigma(i)} - \sum_{i=m+1}^{N} a_{\sigma(i)}\right|^2$. Note first, that for every i and every $i \neq j$ expectations over all permutations respectively are

$$\mathbb{E}(a^2_{\sigma(i)}) = \frac{\|a\|_2^2}{N} \quad \text{and} \quad \mathbb{E}(a_{\sigma(i)}a_{\sigma(j)}) = \frac{\left(\sum_{i=1}^N a_i\right)^2 - \|a\|_2^2}{N(N-1)}.$$

Expanding square in the expectation, we obtain

$$\mathbb{E}|f(\sigma)|^2 = \mathbb{E}\Bigg(\sum_{i=1}^{N} a^2_{\sigma(i)} + \sum_{i=1}^{m} \sum_{i\neq j, j=1}^{m} a_{\sigma(i)}a_{\sigma(j)} + \sum_{i=m+1}^{N} \sum_{i\neq j, j=m+1}^{N} a_{\sigma(i)}a_{\sigma(j)} - 2\sum_{i=1}^{m} \sum_{j=m+1}^{N} a_{\sigma(i)}a_{\sigma(j)} \Bigg).$$

The last sum contains the square of each entry – it gives us a term $\|a\|_2$. Also it contains $m(m-1) + (N-m)(N-m-1)$ positive and $2m(N-m)$ negative terms with pairs of different entries. Thus, with $m = \left[\frac{M+N}{2}\right]$,

$$\mathbb{E}|f|^2 = \frac{(N^2 - M^2)\|a\|_2^2 - (N - M^2)\left(\sum_{i=1}^N a_i\right)^2}{N(N-1)}. \qquad (3.25)$$

This completes the proof. □

Remark 3.5.3. It would be nice to obtain the upper bound in (**??**) with constant independent of N.

3.5.4 Improvement of Lemma 3.5.2

Comparing results of Lemma 3.5.2, when $M = 0$, and Proposition 3.1.2, we see that lemma gives an estimate of order $\sqrt{n}$. We improve this estimate to $\sqrt{\log n}$ using chaining argument. In our technique we are going to use result of Corollary 3.3.4 (which is essentially the same as result for a hypergeometric distribution), thus we can only obtain our improvement for the case when Rademacher random variables are slightly dependent, i.e. when we work under assumption (3.2).

Suppose vector $a^\ell = (a^\ell_1, a^\ell_2, \dots, a^\ell_{2n})$ is a $(2n)$-dimensional real valued vector with only $\ell, \ell \le n$, non-zero coordinates. Without loss of generality, assume that $a^\ell_i = 0$ for $i > \ell$. Then, in the definition (3.22) of the function $f : \Pi_{2n} \longrightarrow \mathbb{R}$, one can reduce the number of permutations from $2n$ to ℓ (as the sum $\sum_{i=1}^{2n} a_i\varepsilon_i$ contains only ℓ non-zero terms).

Consider

$$f_a(\varepsilon) := |\sum_{i=1}^{\ell} a_i^{\ell}\varepsilon_i| \quad \text{on} \quad \Omega = \Big\{\varepsilon \,|\, S = \textstyle\sum_{i=1}^{2n}\varepsilon_i = 0\Big\}.$$

Let $0 \le k \le \ell$ be the number of $+1$ Rademacher random variables in the sum of f_a. For each k we denote $\Omega_k := \{\varepsilon \in \Omega \,|\, \#\{\varepsilon_i = 1, i \le \ell\} = k\}$. Note that the cardinality $|\Omega_k| = \binom{\ell}{k}\binom{2n-\ell}{n-k}$. Note also, $\Omega = \bigcup_{k=0}^{\ell}\Omega_k$. For each sample set $\Omega_k, k = 0, \ldots, \ell$, we put into correspondence the group Π_ℓ of all permutations of set $\{1, \ldots, \ell\}$ as

$$\sigma \in \Pi_{2n} \longleftrightarrow A_\sigma = \{\varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le k;\ \varepsilon_i = -1 \text{ if } \sigma(i) > k\}\,.$$

and define $g_k : \Pi_\ell \longrightarrow \mathbb{R}$ by

$$g_k(\sigma) := \left|\sum_{i=1}^{k} a_{\sigma(i)} - \sum_{i=k+1}^{\ell} a_{\sigma(i)}\right|. \tag{3.26}$$

Let μ_ℓ denote the normalized counting measure on the group Π_ℓ.

Consider now

$$\begin{aligned}
\mathbb{E}_S|f_a|^p &= \frac{1}{\binom{2n}{n}}\sum_{\omega\in\Omega}|f_a(\omega)|^p \\
&= \frac{1}{\binom{2n}{n}}\sum_{k=0}^{\ell}\binom{2n-\ell}{n-k}\frac{1}{k!(\ell-k)!}\sum_{\omega\in\Omega_k}|g_k(\omega)|^p \\
&= \sum_{k=0}^{\ell} d_k \sum_{\omega\in\Omega_k}|g_k(\omega)|^p/\ell! = \sum_{k=0}^{\ell} d_k \int_{\Pi_\ell}|g_k(\sigma)|^p d\sigma,
\end{aligned} \tag{3.27}$$

where

$$d_k = \frac{\binom{\ell}{k}\binom{2n-\ell}{n-k}}{\binom{2n}{n}}. \tag{3.28}$$

Note also that $\sum_{k=0}^{\ell} d_k = 1$.

Lemma 3.5.4. *Let g_k be the function defined in (3.26). Then, for $p \ge 2$,*

$$\begin{aligned}
(\mathbb{E}|g_k|^p)^{1/p} &\le \sqrt{\mathbb{E}|g_k|^2} + 4\sqrt{p}\,\sqrt{\ell}\,\|a^\ell\|_\infty \\
&\le \sqrt{\frac{(4\ell k - 4k^2)\|a^\ell\|_2^2 + (4k^2 + \ell^2 - 4k\ell - \ell)\left(\sum_{i=1}^{\ell} a_i^\ell\right)^2}{\ell(\ell-1)}} + 4\sqrt{p}\sqrt{\ell}\|a^\ell\|_\infty.
\end{aligned} \tag{3.29}$$

Proof. Since for each $k = 0, \ldots, \ell$

$$|g_k(\sigma) - g_k(\pi)| \le 2\|a\|_\infty d_\ell(\sigma, \pi),$$

functions g^k are Lipschitz functions with Lipschitz constant $2\|a\|_\infty$. We will use the same procedure as in Lemma 3.5.2, namely, for each g_k, $k=0,\dots,\ell$ we apply Theorem **??** and the bound $\Gamma(x)\le x^{x-1}$, $x\ge 1$, to obtain

$$\begin{aligned}\mathbb{E}|g_k-\mathbb{E}g_k|^p=\int_0^\infty \mu_\ell(|g_k-\mathbb{E}g_k|^p\ge t^p)dt^p &\le 2p\int_0^\infty e^{-t^2/(32\ell\|a\|_\infty^2)}t^{p-1}dt\\ &\le 2^{\frac{5}{2}p}p\,\Gamma\left(\frac{p}{2}\right)\ell^{p/2}\|a\|_\infty^p\\ &\le 4^p\ell^{p/2}p^{p/2}\|a\|_\infty^p.\end{aligned}$$

Thus, we get

$$(\mathbb{E}|g_k|^p)^{1/p}\le \mathbb{E}|g_k|+4\sqrt{p}\sqrt{\ell}\|a^\ell\|_\infty\le\sqrt{\mathbb{E}|g_k|^2}+4\sqrt{p}\sqrt{\ell}\|a^\ell\|_\infty.$$

We compute now

$$\mathbb{E}\left|g_k(\sigma)\right|^2=\mathbb{E}\left|\sum_{i=1}^k a^\ell_{\sigma(i)}-\sum_{i=k+1}^{\ell}a^\ell_{\sigma(i)}\right|^2.$$

Note first, that for every i and $i\neq j$,

$$\begin{aligned}\mathbb{E}((a^\ell_{\sigma(i)})^2)&=\frac{\sum_{i=1}^\ell(a_i^\ell)^2}{\ell}=\frac{\|a^\ell\|_2^2}{\ell},\\ \mathbb{E}(a^\ell_{\sigma(i)}a^\ell_{\sigma(j)})&=\frac{\left(\sum_{i=1}^\ell a_i^\ell\right)^2-\sum_{i=1}^\ell(a_i^\ell)^2}{\ell(\ell-1)}=\frac{\left(\sum_{i=1}^\ell a_i^\ell\right)^2-\|a^\ell\|_2^2}{\ell(\ell-1)}.\end{aligned}$$

Expanding the square in the expectation, we obtain

$$\begin{aligned}\mathbb{E}|g_k(\sigma)|^2&=\mathbb{E}\left(\left(\sum_{i=1}^k a^\ell_{\sigma(i)}\right)^2+\left(\sum_{i=k+1}^\ell a^\ell_{\sigma(i)}\right)^2-2\sum_{i=1}^k\sum_{j=k+1}^\ell a^\ell_{\sigma(i)}a^\ell_{\sigma(j)}\right)\\ &=\mathbb{E}\left(\left(\sum_{i=1}^\ell a^\ell_{\sigma(i)}\right)^2+\sum_{i=1}^k\sum_{i\neq j,j=1}^k a^\ell_{\sigma(i)}a^\ell_{\sigma(j)}+\sum_{i=k+1}^\ell\sum_{i\neq j,j=k+1}^\ell a^\ell_{\sigma(i)}a^\ell_{\sigma(j)}-2\sum_{i=1}^k\sum_{j=k+1}^\ell a^\ell_{\sigma(i)}a^\ell_{\sigma(j)}\right).\end{aligned}$$

The last sum contains the square of each entry – it gives us a term $\|a^\ell\|_2$. Also it contains $k(k-1)+(\ell-k)(\ell-k-1)$ positive and $2k(\ell-k)$ negative terms with pairs of different entries. Thus,

$$\begin{aligned}\mathbb{E}|g_k(\sigma)|^2 &= \|a^\ell\|_2^2 + (k(k-1)+(\ell-k)(\ell-k-1)-2k(\ell-k))\frac{(\sum_{i=1}^\ell a_i^\ell)^2 - \|a^\ell\|_2^2}{\ell(\ell-1)} \\ &= \frac{\ell(\ell-1)\|a^\ell\|_2^2 - (k(k-1)+(\ell-k)(\ell-k-1)-2k(\ell-k))\|a^\ell\|_2^2}{\ell(\ell-1)} \\ &+ \frac{(k(k-1)+(\ell-k)(\ell-k-1)-2k(\ell-k))\left(\sum_{i=1}^\ell a_i^\ell\right)^2}{\ell(\ell-1)} \\ &= \frac{(4\ell k-4k^2)\|a^\ell\|_2^2 + (\ell^2+4k^2-\ell-4k\ell)\left(\sum_{i=1}^\ell a_i^\ell\right)^2}{\ell(\ell-1)} \\ &= \|a^\ell\|_2^2 + \frac{(\ell-2k)^2-\ell}{\ell(\ell-1)}\sum_{i,j=1,i\neq j}^{\ell} a_i^\ell a_j^\ell.\end{aligned}$$

So,

$$\begin{aligned}\sqrt{\mathbb{E}|g_k(\sigma)|^2} &= \sqrt{\frac{(4\ell k-4k^2)\|a^\ell\|_2^2 + (\ell^2+4k^2-\ell-4k\ell)\left(\sum_{i=1}^\ell a_i^\ell\right)^2}{\ell(\ell-1)}} \\ &= \sqrt{\|a^\ell\|_2^2 + \frac{(\ell-2k)^2-\ell}{\ell(\ell-1)}\sum_{i,j=1,i\neq j}^{\ell} a_i^\ell a_j^\ell}.\end{aligned}$$

This implies the desired result. □

The following is an immediate consequence of Lemma 3.5.4 and (3.27).

Corollary 3.5.5. *Let* $f_a(\varepsilon) = \left|\sum_{i=1}^\ell a_i^\ell \varepsilon_i\right|$, *then*

$$(\mathbb{E}|f_a|^p)^{1/p} \le \sum_{k=0}^{\ell} d_k \sqrt{\frac{(4\ell k-4k^2)\|a^\ell\|_2^2 + (4k^2+\ell^2-4k\ell-\ell)\left(\sum_{i=1}^\ell a_i^\ell\right)^2}{\ell(\ell-1)}} + 4\sqrt{p}\sqrt{\ell}\|a^\ell\|_\infty. \tag{3.30}$$

Proposition 3.5.6. *Let* $a = (a_1, \dots, a_{2n}) \in \mathbb{R}^{2n}$. *Then,*

$$\left(\mathbb{E}_S\left|\sum_{i=1}^{2n} a_i\varepsilon_i\right|^p\right)^{1/p} \le C\sqrt{p}\,\sqrt{\ln n}\|a\|_2. \tag{3.31}$$

Proof. Without loss of generality, we assume that $a \in S^{2n-1}$ and $|a_1| \ge |a_2| \ge \ldots \ge$

$|a_{2n}|$. We decompose vector a into m vectors, choosing $m = \lceil \log_2(2n) \rceil$,

$$\begin{aligned} a^1 &= (a_1, \ldots, a_{n_1}, 0, \ldots, 0), \\ a^2 &= (0, \ldots, 0, a_{n_1+1}, \ldots, a_{n_2}, 0, \ldots, 0), \\ &\vdots \\ a^j &= (0, \ldots, 0, a_{n_{j-1}+1}, \ldots, a_{n_j}, 0, \ldots, 0), \\ &\vdots \\ a^m &= (0, \ldots, 0, a_{n_{m-1}+1}, \ldots, a_{n_m}), \end{aligned}$$

where $n_0 = 0, n_m = 2n$ and the cardinality of the support of each vector $|supp\,(a^{\ell_j})| \leq \ell_j = n_j - n_{j-1} + 1 = \left[\frac{2n}{2^{m-j}}\right] - \left[\frac{2n}{2^{m-j+1}}\right]$, $j = 1, \ldots m$. So,

$$\max\left\{1, \frac{2n}{2^{m-j+1}}\right\} \leq \ell_j \leq \frac{2n}{2^{m-j}} \quad \text{and} \quad \sum_{j=1}^{m} \ell_j = 2n.$$

Note, that $\|a^1\|_\infty = |a_1| \leq 1$, and $\|a^{j+1}\|_\infty = |a_{n_j+1}| \leq |a_{n_j}|$, $j = 2, \ldots, m-1$. Since for any $j = 1, \ldots, m$, $1 = \sum_{i=1}^{2n} a_i \geq \sum_{i=n_{j-1}+1}^{n_j} (a_i)^2 \geq \ell_j |a_{n_j}|^2$, we have that

$$\begin{aligned} \|a^{j+1}\|_\infty &\leq \frac{\|a^j\|_2}{\sqrt{\ell_j}} \leq \frac{1}{\sqrt{\ell_j}} \leq \sqrt{\frac{2^{m-j+1}}{2n}}, \quad j = 1, \ldots, m-1 \\ \|a^1\|_\infty &\leq \sqrt{\frac{2^m}{2n}} = 1. \end{aligned} \tag{3.32}$$

(From the last two inequalities, we have $\|a^j\|_\infty \leq \sqrt{\frac{2^{m-j+2}}{2n}}$, and $\|a^j\|_2 \leq \sqrt{\ell_j}\sqrt{\frac{2^{m-j+2}}{2n}}$ for $j = 1, \ldots, m$).

For $j = 1, \ldots, m$ we let $A_j = \{i\}_{n_{j-1}+1 \leq i \leq n_j}$. Also, we let $b_j, j = 1, \ldots, m+1$ be numbers, defined by $b_j = \frac{\sum_{i \in A_j} a_i^{\ell_j}}{\ell_j}$. Note here, $b_j \leq \frac{\ell_j |a_{n_{j-1}+1}|}{\ell_j} = |a_{n_{j-1}+1}| = \|a^j\|_\infty$.

Consider

$$\sum_{i=1}^{2n} a_i \varepsilon_i = \sum_{j=1}^{m} \sum_{i \in A_j} a_i^j \varepsilon_i = \sum_{j=1}^{m} \sum_{i \in A_j} \left(a_i^j - b_j\right) \varepsilon_i + \sum_{j=1}^{m} b_j \sum_{i \in A_j} \varepsilon_i.$$

By triangle inequality observe

$$\begin{aligned} \left(\mathbb{E}_S \left|\sum_{i=1}^{2n} a_i \varepsilon_i\right|^p\right)^{1/p} &= \left(\mathbb{E}_S \left|\sum_{j=1}^{m} \sum_{i \in A_j} (a_i - b_j)\, \varepsilon_i + \sum_{j=1}^{m} b_j \sum_{i \in A_j} \varepsilon_i\right|^p\right)^{1/p} \\ &\leq \sum_{j=1}^{m} \left(\mathbb{E}_S \left|\sum_{i \in A_j} (a_i - b_j)\, \varepsilon_i\right|^p\right)^{1/p} + \sum_{j=1}^{m} |b_j| \left(\mathbb{E}_S \left|\sum_{i \in A_j} \varepsilon_i\right|^p\right)^{1/p}. \end{aligned}$$

Applying Corollary 3.5.5 for each summand in the first sum and denoting $\bar{b}_j = (b_j, \ldots, b_j)$, we obtain

$$A := \sum_{j=1}^{m} \left(\mathbb{E}_S \left| \sum_{i \in A_j} (a_i - b_j)\, \varepsilon_i \right|^p \right)^{1/p}$$
$$\leq \sum_{j=1}^{m} \sum_{k_j=0}^{\ell_j - \ell_{j-1}} d_{k_j} \sqrt{\frac{(4\ell_j k_j - 4k_j^2)\|a^j - \bar{b}_j\|_2^2 + (\ell_j^2 + 4k_j^2 - 4k_j\ell_j - \ell_j)\left(\sum_{i\in A_j}(a_i - b_j)\right)^2}{\ell_j(\ell_j - 1)}}$$
$$+ \sum_{j=1}^{m+1} 4\sqrt{p}\sqrt{\ell_j}\max_{i\in A_j} |a_i - b_j|.$$

Note, that by the choice of b_j, $\sum\limits_{i \in A_j} (a_i - b_j) = 0$. Also, it is clear that $\|a^j - \bar{b}_j\|_2 \leq 2\|a^j\|_2$ and $\|a^j - b_j\|_\infty \leq 2\|a^j\|_\infty$. Thus,

$$A \leq \sum_{j=1}^{m} \sum_{k_j=0}^{\ell_j - \ell_{j-1}} d_{k_j} \sqrt{\frac{8(\ell_j k_j - k_j^2)\|a^j\|_2^2}{\ell_j(\ell_j - 1)}} + \sum_{j=1}^{m} 8\sqrt{p}\sqrt{\ell_j}\|a^j\|_\infty.$$

Noticing, that for each $1 \leq j \leq m+1$, $\sum\limits_{k_j=0}^{\ell_j - \ell_{j-1}} d_{k_j} = 1$ and $\ell_j k_j - k_j^2 \leq \frac{3\ell_j^2}{4}$, and using (3.32), we obtain

$$A \leq \sqrt{6}\sum_{j=1}^{m} \sqrt{\frac{\ell_j \|a^j\|_2^2}{(\ell_j - 1)}} + 8\sqrt{p}\sum_{j=1}^{m} \sqrt{\ell_j}\|a^{\ell_j}\|_\infty$$
$$\leq \sqrt{12}\sum_{j=1}^{m} \|a^j\|_2 + 8\sqrt{p}\sum_{j=1}^{m} \sqrt{\ell_j}\|a^{\ell_j}\|_\infty$$
$$\leq \sqrt{12}\sum_{j=1}^{m} \sqrt{\ell_j}\sqrt{\frac{2^{m-j+2}}{2n}} + 8\sqrt{p}\sum_{j=1}^{m} \sqrt{\ell_j}\sqrt{\frac{2^{m-j+2}}{2n}}$$
$$\leq 12\sqrt{p}\sum_{j=1}^{m} \sqrt{\ell_j}\sqrt{\frac{2^{m-j+2}}{2n}}.$$

Now, using Hölder inequality, we get

$$
\begin{aligned}
A &\le 12\sqrt{p}\,\left(\sum_{j=1}^{m}\ell_j\,\frac{2^{m-j+2}}{2n}\right)^{1/2}\left(\sum_{j=1}^{m+1}1\right)^{1/2} \\
&\le 12\sqrt{p}\,\sqrt{m}\left(4\sum_{j=1}^{m}\ell_j\,\frac{2^{m-j}}{2n}\right)^{1/2} \\
&\le 24\sqrt{p}\,\sqrt{m}\left(\sum_{j=1}^{m-1}\|a^{j+1}\|_\infty^2+1\right)^{1/2} \le 34\sqrt{p}\,\sqrt{m}\,\|a\|_2 \\
&\le 34\sqrt{p}\,\sqrt{\log_2(2n)}\|a\|_2 \le 68\sqrt{p}\sqrt{\ln n}\|a\|_2.
\end{aligned} \tag{3.33}
$$

Consider now $\sum_{j=1}^{m+1}|b_j|\left(\mathbb{E}_S\left|\sum_{i\in A_j}\varepsilon_i\right|^p\right)^{1/p}$. First, we fix j and calculate $\mathbb{E}_S\left|\sum_{i\in A_j}\varepsilon_i\right|^p$. In other words, we would like to calculate $\mathbb{E}_S\left|\sum_{i=1}^{2n}a_i\varepsilon_i\right|^p$, where vector a consists of ℓ_j ones, $\ell_j \le n$, and $2n-\ell_j$ zeros. By Corollary 3.3, we have

$$
\mathbb{E}_S\left|\sum_{i\in A_j}\varepsilon_i\right|^p \le 2\sqrt{2}(\sqrt{p}\,\sqrt{\ell_j})^p.
$$

By our choice of b_j and using Hölder inequality, as above, we obtain

$$
\begin{aligned}
\sum_{j=1}^{m+1}|b_j|\left(\mathbb{E}_S\left|\sum_{i\in A_j}\varepsilon_i\right|^p\right)^{1/p} &\le (2\sqrt{2})^{1/p}\sqrt{p}\sum_{j=1}^{m}\sqrt{\ell_j}|b_j| \le (2\sqrt{2})^{1/p}\sqrt{p}\sum_{j=1}^{m}\sqrt{\ell_j}\sqrt{\frac{2^{m-j+2}}{2n}} \\
&\le (2\sqrt{2})^{1/p+1}\sqrt{p}\,\sqrt{\log_2(2n)}\|a\|_2 \le (2\sqrt{2})^{1/p+2}\sqrt{p}\,\sqrt{\ln n}\|a\|_2.
\end{aligned} \tag{3.34}
$$

Now, from (3.33) and (3.34), we get

$$
\left(\mathbb{E}_S\left|\sum_{i=1}^{2n}a_i\varepsilon_i\right|^p\right)^{1/p} \le C\sqrt{p}\,\sqrt{\ln n}\|a\|_2.
$$

□

3.5.5 ψ_1-estimate

In this section we obtain a ψ_1-estimate for $\left|\sum_{i=1}^{2n}a_i\varepsilon_i\right|$ under assumption (3.2).

Theorem 3.5.7. *Let $f:\Pi_{2n}\longrightarrow\mathbb{R}$ be defined by (15). Then, for $p\ge 2$,*

$$
(\mathbb{E}|f|^p)^{1/p} \le \mathbb{E}|f| + 24p\|a\|_2. \tag{3.35}
$$

Remark: Note that $\mathbb{E}|f| \le \left(\mathbb{E}f^2\right)^{1/2}$ and $\mathbb{E}f^2$ was calculated in (3.25).

Proof. We are going to use inequality (3.19). We calculate first

$$|||f|||_\infty^2 = \frac{1}{2} \sup_{\sigma\in\Pi_{2n}} \sum_{\tau:\sigma\tau\in\Pi_{2n}} |f(\sigma) - f(\sigma\tau)|^2 p(\sigma, \sigma\tau),$$

where $p(\sigma, \sigma\tau)$ is defined in (3.18).

Consider $g(\sigma) = \sum_{i=1}^n a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}$. Since $\tau(i,j)$ is a random transposition with i,j chosen uniformly from the set $\{1,\dots,2n\}$, we obtain

$$g(\sigma) - g(\sigma\tau) = 2(a_i - a_j)h(i,j),$$

where

$$h(i,j) = \begin{cases} 1 & , \quad \text{if } j \le n < i \le 2n \\ -1 & , \quad \text{if } i \le n < j \le 2n \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus, $|f(\sigma) - f(\sigma\tau)|^2 = 4(a_i - a_j)^2 h^2(i,j)$. And we can calculate

$$\begin{aligned} |||f|||_\infty^2 &= \frac{1}{n^2} \sum_{\tau(i,j)} (a_i - a_j)^2 h^2(i,j) \\ &= \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} (a_i - a_j)^2 h^2(i,j) \\ &= \frac{2}{n^2} \left(n\|a\|_2^2 - 2\sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \right) \end{aligned}$$

Since

$$-\sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \le \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \frac{a_i^2 + a_j^2}{2} = \frac{n}{2}\|a\|_2^2,$$

the last equation can be bounded by

$$|||f|||_\infty^2 \le \frac{4}{n}\|a\|_2^2. \tag{3.36}$$

Now, using (3.19), (3.36) and an upper bound, as before, for the Gamma function (see [6]), we obtain

$$\begin{aligned} \mathbb{E}(f - \mathbb{E}f)^p = \int_0^\infty \mu((f(\sigma) - \mathbb{E}f)^p \ge t^p) dt^p &\le 6p \int_0^\infty e^{-t/(4\|a\|_2)} t^{p-1} dt \\ &= 6\, p\, 4^p \Gamma(p) \|a\|_2^p \le 4^p\, 6p^p \|a\|_2^p. \end{aligned}$$

Hence

$$(\mathbb{E}f^p)^{1/p} \le \mathbb{E}|f| + 24p\|a\|_2.$$

□

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Chapter 4

Quantitative version of a Silverstein's result*

Let w be a real random variable with $\mathbb{E}w = 0$ and $\mathbb{E}w^2 = 1$, and let w_{ij}, $i, j \geq 1$ be its i.i.d. copies. For integers n and $p = p(n)$ consider the $p \times n$ matrix $W_n = \{w_{ij}\}_{i \leq p, j \leq n}$, and consider its sample covariance matrix $\Gamma_n := \frac{1}{n} W_n W_n^T$. We also denote by $X_j = (w_{j1}, \ldots, w_{jn})$, $j \leq p$, the rows of W_n.

The questions on behavior of eigenvalues are of great importance in random matrix theory. We refer to [4, 5, 6, 12] for the relevant results, history and references.

In this chapter we study lower bounds on $\max_{i \leq p} |X_i|$ and on the operator (spectral) norms of matrices W_n and Γ_n. Note, as Γ_n is symmetric, its largest singular value $\lambda_{\max}$ is equal to the norm and that in general we have

$$\lambda_{\max}(\Gamma_n) = \|\Gamma_n\| = \frac{1}{n}\|W_n\|^2 \geq \frac{1}{n} \max_{i \leq p} |X_i|^2. \tag{4.1}$$

Assume that $p(n)/n \to \beta > 0$ as $n \to \infty$. In [17] it was proved that if $\mathbb{E}w^4 < \infty$ then $\|\Gamma_n\| \to (1 + \sqrt{\beta})^2$ a.s., while in [6] it was shown that $\limsup_{n \to \infty} \|\Gamma_n\| = \infty$ a.s. if $\mathbb{E}w^4 = \infty$.

In [14] Silverstein studied the weak behavior of $\|\Gamma_n\|$. In particular, he proved that assuming $p(n)/n \to \beta > 0$ as $n \to \infty$, $\|\Gamma_n\|$ converges to a non-random quantity (which must be $(1 + \sqrt{\beta})^2$) in probability if and only if $n^4 \mathbb{P}(|w| \geq n) = o(1)$.

The purpose of this work is to provide the quantitative counterpart of Silverstein's result. More precisely, we want to show an estimate of the type $\mathbb{P}\big(\|\Gamma_n\| \geq K\big) \geq \delta = \delta(K)$ for an arbitrary large K, provided that w has heavy tails (in particular, provided that w does not have 4-th moment). Our proof essentially follows ideas of [14]. It gives a lower bound on $\max_{i \leq p} |X_i|$ as well.

Theorem 4.0.8. *Let $\alpha \geq 2$, $c_0 > 0$. Let w be a random variable satisfying $\mathbb{E}w = 0$, $\mathbb{E}w^2 = 1$ and*

$$\forall t \geq 1 \qquad \mathbb{P}(|w| \geq t) \geq \frac{c_0}{t^\alpha}. \tag{4.2}$$

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Let $W_n = \{w_{ij}\}_{i\le p,\, j\le n}$ be a $p\times n$ matrix whose entries are i.i.d. copies of w and let $X_i, i\le p$, be the rows of W_n. Then, for every $K\ge 1$,

$$\mathbb{P}\left(\max_{i\le p}|X_i|\ge \sqrt{Kn}\right)\ge \min\left\{\frac{c_0 p}{4n^{(\alpha-2)/2}K^{\alpha/2}},\ \frac{1}{2}\right\}. \tag{4.3}$$

In particular, $\Gamma_n = \frac{1}{n}W_nW_n^T$ satisfies for every $K\ge 1$,

$$\mathbb{P}\left(\|\Gamma_n\|\ge K\right)\ge \min\left\{\frac{c_0 p}{4n^{(\alpha-2)/2}K^{\alpha/2}},\ \frac{1}{2}\right\}.$$

Remark 4.0.9. If p is proportional to n, say $p=\beta n$, the theorem gives

$$\mathbb{P}(\|\Gamma_n\|\ge K)\ge \mathbb{P}\left(\max_{i\le p}|X_i|\ge\sqrt{Kn}\right)\ge\min\left\{\frac{c_0\,\beta}{4n^{(\alpha-4)/2}K^{\alpha/2}},\ \frac{1}{2}\right\}.$$

Remark 4.0.10. Note that by Chebychev's inequality one has $\mathbb{P}(|w|\ge t)\le t^{-2}$. Note also that we use condition (4.2) in the proof only once, with $t=\sqrt{Kn}$.

Remark 4.0.11. If $p\ge (2/c_0)K^{\alpha/2}n^{(\alpha-2)/2}$, then, by condition (4.2), we have

$$\frac{n}{2}\mathbb{P}(w^2\ge Kn)\ge \frac{nc_0}{2(Kn)^{\alpha/2}}=\frac{c_0}{2K^{\alpha/2}n^{(\alpha-2)/2}}\ge\frac{1}{p}.$$

Therefore in this case the proof below gives

$$\mathbb{P}(\|\Gamma_n\|\ge K)\ge\mathbb{P}\left(\max_{i\le p}|X_i|\ge\sqrt{Kn}\right)\ge\frac{1}{2}.$$

In particular, if $\alpha=4$ and $p\ge(2K^2/c_0)n$ then $\|\Gamma_n\|\ge K$ with probability at least 1/2.

Before we prove the theorem we would like to mention that last decade many works appeared on non-limit behavior of the norms of random matrices with random entries. In most of them $\max_{i\le p}|X_i|$ appears naturally (or $\sqrt{n}$, when X_i is with high probability bounded by $\sqrt{n}$). For earlier works on Gaussian matrices we refer to [7, 8, 16] and references therein. For the general case of centered i.i.d. $w_{i,j}$ (as in our setting) Seginer [13] proved that

$$\mathbb{E}\|W_n\|\le C\left(\mathbb{E}\max_{i\le p}|X_i|+\mathbb{E}\max_{j\le n}|Y_j|\right),$$

where Y_j, $j\le n$, are the columns of W_n. Later Latała [9] was able to remove the condition that $w_{i,j}$ are identically distributed (his formula involves 4-th moments). Moreover, Mendelson and Paouris [10] have recently proved that for centered i.i.d. $w_{i,j}$ of variance one satisfying $\mathbb{E}|w_{1,1}|^q\le L$ for some $q>4$ and $L>0$ with high probability one has

$$\mathbb{E}\|W_n\|\le\max\{\sqrt{p},\sqrt{n}\}+C(q,L)\min\{\sqrt{p},\sqrt{n}\}.$$

In [1, 3, 9, 10, 11, 15] matrices with independent columns (which can have dependent coordinates) were investigated. In particular, in [1] (see Theorem 3.13 there) it was shown that if columns of $p\times n$ matrix A satisfy

$$\sup_{q\ge 1}\sup_{i\le p}\sup_{y\in S^{n-1}}\tfrac{1}{q}\left(\mathbb{E}|\langle X_i,y\rangle|^p\right)^{1/q}\le\psi$$

then with probability at least $1-\exp\left(-c\sqrt{p}\right)$ one has

$$\|A\|\le 6\max_{i\le p}|X_i|+C\psi\sqrt{p}$$

(using Theorem 5.1 in [2] the factor 6 can be substituted by $(1+\varepsilon)$ in which case constants C and c will be substituted with $C\ln(2/\varepsilon)$ and $c\ln(2/\varepsilon)$ correspondingly).

Proof of the Theorem. By (4.1) the "In particular" part of the Theorem follows immediately from (4.3). Thus, it is enough to prove (4.3).

Since $X_1,\dots,X_p$ are i.i.d. random vectors and since $|X_1|^2$ is distributed as $\sum_{j=1}^n w_{1,j}^2$, we observe for every $K\ge 1$,

$$\begin{aligned}\mathbb{P}\Big(\max_{i\le p}|X_i|\ge\sqrt{Kn}\Big)&=1-\mathbb{P}\Big(\max_{i\le p}|X_i|<\sqrt{Kn}\Big)=1-\mathbb{P}\Big(\{\forall i\,:\,|X_i|<\sqrt{Kn}\}\Big)\\&=1-\Big(\mathbb{P}(|X_i|<K)\Big)^p=1-\Big(\mathbb{P}\Big(\sum_{j=1}^n w_{1,j}^2<Kn\Big)\Big)^p.\end{aligned}\tag{4.4}$$

For $j\le n$ consider the events $A_j:=\{w_{1,j}^2\ge nK\}$. Clearly,

$$A:=\left\{\sum_{j=1}^n w_{1,j}^2\ge nK\right\}\ \supset\ \bigcup_{j=1}^n A_j.$$

By the inclusion-exclusion principle, we have

$$\begin{aligned}\mathbb{P}(A)\ge\mathbb{P}\Big\{\bigcup_{j=1}^n A_j\Big\}&\ge\sum_{j=1}^n\mathbb{P}(A_j)-\sum_{j\ne k}\mathbb{P}\left(A_j\cap A_k\right)=\sum_{j=1}^n\mathbb{P}\left(w^2\ge nK\right)-\sum_{j\ne k}\big(\mathbb{P}\left(w^2\ge nK\right)\big)^2\\&=n\mathbb{P}\left(w^2\ge nK\right)-\frac{n^2-n}{2}\big(\mathbb{P}\left(w^2\ge nK\right)\big)^2\\&=\frac{n}{2}\mathbb{P}(w^2\ge nK)(2-(n-1)\mathbb{P}(w^2\ge nK)).\end{aligned}$$

By Chebychev's inequality we have $\mathbb{P}(w^2\ge nK)\le\dfrac{1}{nK}$, hence, $2-(n-1)\mathbb{P}(w^2\ge nK)\ge 1$. Thus, by (4.4),

$$\mathbb{P}\left(\max_{i\le p}|X_i|\ge\sqrt{Kn}\right)\ge 1-\left(1-\mathbb{P}\left(\frac{1}{n}\sum_{j=1}^n w_{1,j}^2\ge K\right)\right)^p\ge 1-\left(1-\frac{n}{2}\mathbb{P}\left(w^2\ge nK\right)\right)^p.$$

If $\frac{n}{2}\mathbb{P}(w^2 \geq Kn) \geq \frac{1}{p}$, then

$$\mathbb{P}\left(\max_{i\leq p}|X_i| \geq \sqrt{Kn}\right) \geq 1 - \left(1-\frac{1}{p}\right)^p \geq 1-\frac{1}{e} \geq \frac{1}{2}.$$

Finally assume that

$$\frac{n}{2}\mathbb{P}(w^2 \geq Kn) \leq \frac{1}{p}. \tag{4.5}$$

Using that $(1-x)^p \leq (1+px)^{-1}$ on $[0,1]$, we get

$$\mathbb{P}\left(\max_{i\leq p}|X_i| \geq \sqrt{Kn}\right) \geq 1 - \frac{1}{(np/2)\mathbb{P}(w^2 \geq Kn)+1}.$$

Applying condition (4.2) with $t = \sqrt{Kn}$ and using (4.5) again, we observe

$$1 \geq \frac{np}{2}\mathbb{P}(w^2 \geq Kn) \geq \frac{np}{2}\frac{c_0}{(Kn)^{\alpha/2}}.$$

Thus,

$$\mathbb{P}\left(\max_{i\leq p}|X_i| \geq \sqrt{Kn}\right) \geq \frac{c_0 p}{4n^{(\alpha-2)/2}K^{\alpha/2}},$$

which completes the proof. □

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Chapter 5

Asymptotic Bernstein type inequalities and estimation of wavelets coefficients*

5.1 Introduction and motivations

Almost any kind of practical sciences requires the analysis of data. Depending on the specific application, the collection of data may consist of measurements, signals, or images. In mathematical framework, all of those objects are functions. One way to analyze them is by representing them into wavelet decomposition. Such methods are not only used in mathematics, but also in physics, electrical engineering, and medical imaging [6, 7, 10, 12, 14, 15, 20]. Wavelets provide reconstruction (approximation) of the original function (the collection of data). In order to characterize the approximation class, one has to establish Bernstein inequality. We will first give some basic definitions before representing the importance and applications of the mentioned inequality.

We say that $\varphi : \mathbb{R} \to \mathbb{C}$ is a 2-*scaling function* if

$$\varphi = 2 \sum_{\nu \in \mathbb{Z}} a(\nu) \varphi(2 \cdot - \nu), \tag{5.1}$$

where $a : \mathbb{Z} \to \mathbb{C}$ is a finitely supported sequence of complex numbers on $\mathbb{Z}$, called the *mask (or low-pass filter)* for φ. In frequency domain, the refinement equation in (5.1) can be rewritten as

$$\hat{\varphi}(2\omega) = \hat{a}(\omega)\hat{\varphi}(\omega), \quad \omega \in \mathbb{R}, \tag{5.2}$$

where $\hat{a}$ is the *Fourier series* of a given by

$$\hat{a}(\omega) := \sum_{\nu \in \mathbb{Z}} a(\nu) e^{-i\nu\omega}, \quad \omega \in \mathbb{R}. \tag{5.3}$$

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The Fourier transform $\hat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\omega x}dx$ and can be extended to square integrable functions and tempered distributions.

Usually, a wavelet system is generated by some wavelet function ψ from a 2-scaling function φ as follows:

$$\psi = 2\sum_{\nu\in\mathbb{Z}} b(\nu)\varphi(2\cdot - \nu) \quad \text{or} \quad \hat{\psi}(2\cdot) = \hat{b}(\cdot)\hat{\varphi}(\cdot), \tag{5.4}$$

where $b : \mathbb{Z} \to \mathbb{C}$ is a finitely supported sequence of complex numbers on $\mathbb{Z}$, called the *mask (or band-pass filter)* for ψ. For a more general approach on obtaining wavelet functions from a d-scaling function, see [8, 17].

Many wavelet applications, for example, image/signal compression, denoising, inpainting, compressive sensing, and so on, are based on investigation of the wavelet coefficients $\langle f, \varphi_{j,\nu}\rangle$ and $\langle f, \psi_{j,\nu}\rangle$ for $j, \nu \in \mathbb{Z}$, where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$ and $\varphi_{j,\nu} := 2^{j/2}\varphi(2^j \cdot - \nu), \psi_{j,\nu} := 2^{j/2}\psi(2^j \cdot - \nu)$. The magnitude of the wavelet coefficients depends on both the smoothness of the function f and the wavelet ψ. In this paper, we shall investigate the quantity

$$C_{k,p}(\psi) = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle f, \psi\rangle|}{\|\hat{\psi}\|_p}, \tag{5.5}$$

where $1 < p, p' < \infty$, $1/p' + 1/p = 1$, $k \in \mathbb{N} \cup \{0\}$, and $\mathcal{A}_k^{p'} := \{f \in L_{p'}(\mathbb{R}) : \|(i\omega)^k \hat{f}(\omega)\|_{p'} \leqslant 1\}$. This quantity is closely related to Bernstein type inequality in wavelet analysis. The classical Bernstein inequality states that for any $\alpha \in \{\mathbb{N} \cup \{0\}\}^n$, one has $\|\partial^\alpha f\|_p \leqslant R^{|\alpha|}\|f\|_p$, where $f \in L_p(\mathbb{R}^n)$ is an arbitrary function whose Fourier transform $\hat{f}$ is supported in the ball $|\omega| \leqslant R$. The quantity $C_{k,p}(\psi)$ in (5.5) is the best possible constant in the following Bernstein type inequality

$$|\langle f, \psi_{j,\nu}\rangle| \leqslant C_{k,p}(\psi)2^{-j(k+1/p-1/2)}\|\hat{\psi}\|_p\|(i\omega)^k\hat{f}(\omega)\|_{p'}. \tag{5.6}$$

This inequality gives us a way of investigating the magnitude of the coefficients in wavelet decomposition of the function. The coefficients tell in what way the analyzing function needs to be modified in order to reconstruct the data (see [13]). On the other hand, bound of type (5.6) gives a-priori information on the size of wavelet coefficients which is important for such application as compression of data (see e.g. [6, 20]). Also, such types of inequalities play an important role in wavelet algorithms for the numerical solution of integral equations (see e.g. [5, 25]), where wavelet coefficients arise by applying an integral operator to a wavelet; and for the estimation of wavelet coefficients of the space of distributions with bounded variations derivatives (see [4]).

Note that

$$C_{k,p}(\psi) = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle f, \psi\rangle|}{\|\hat{\psi}\|_p} = \sup_{f\in\mathcal{A}_k^{p'}} \frac{|\langle \hat{f}, \hat{\psi}\rangle|}{\|\hat{\psi}\|_p} = \frac{\|\widehat{_k\psi}\|_p}{\|\hat{\psi}\|_p}, \tag{5.7}$$

where for a function $f \in L_1(\mathbb{R})$, ${}_kf$ is defined to be the function such that

$$\widehat{{}_kf}(\omega) = (i\omega)^{-k}\hat{f}(\omega), \quad \omega \in \mathbb{R}. \tag{5.8}$$

For ψ that is compactly supported, it is easily shown that the quantity $C_{k,p}(\psi) < \infty$ is equivalent to

$$\int_{\mathbb{R}} \psi(x)x^\nu dx = 0 \quad \text{or} \quad \frac{d^\nu}{dx^\nu}\hat{\psi}(0) =: \hat{\psi}^{(\nu)}(0) = 0 \tag{5.9}$$

for $\nu = 0, \dots, m-1$. That is, ψ has m *vanishing moments*. Consequently, for a wavelet ψ with m vanishing moments, we can investigate the magnitude of the wavelet coefficients in the function spaces $\mathcal{A}_1^{p'}, \dots, \mathcal{A}_m^{p'}$ for $1 < p' < \infty$ using the quantity $C_{k,p}(\psi)$.

On the other hand, a fundamental question in wavelet application is which type of wavelets one should choose for a specific purpose. In [18], Keinert used a constant G_M in the following approximation for comparison of wavelets.

$$\int_{\mathbb{R}} f(x)\psi_{j,\nu}(x)dx \approx 2^{-(j+1)(M+1/2)}\frac{G_M}{M!}f^{(M)}(2^{-j}\nu), \tag{5.10}$$

where f is sufficient smooth, ψ has exactly M vanishing moments, and G_M depends only on ψ. Keinert presented numerical values of G_M for some commonly used wavelets and provided constructions for wavelets with short support and minimal G_M, which lead to better compression in practical calculation. By considering the quantity $C_{k,p}(\psi)$, the "$\approx$" in (5.10) can be replaced by precise inequality. In [16], Ehrich investigated the quantity $C_{k,p}(\psi)$ for $p = 2$ and for two important families of wavelets, namely, Daubechies orthonormal wavelets and semiorthogonal wavelets. Precise asymptotic relations of quantities $C_{k,2}(\psi)$ are established in [16] showing that the quantity for the family of semiorthogonal spline wavelets is generally smaller than that for the family of Daubechies orthonormal wavelets.

In this paper, we shall investigate the quantity $C_{k,p}(\psi), p \in (1,\infty)$ mainly for the family of Daubechies orthonormal wavelets (see [10]) and the family of semiorthogonal spline wavelets (see [9]). We next give a brief introduction of these two families.

Let m be a positive integer. Let a_m^D and b_m^D be two masks determined by:

$$|\widehat{a_m^D}(\omega)|^2 = \cos^{2m}(\omega/2)\sum_{\nu=0}^{m-1}\binom{m-1+\nu}{\nu}\sin^{2\nu}(\omega/2), \tag{5.11}$$

and

$$\widehat{b_m^D}(\omega) = e^{i\omega}\overline{\widehat{a_m^D}(\omega+\pi)}. \tag{5.12}$$

It is well-known that $|\widehat{a_m^D}(\omega)|^2$ is the Dubuc-Deslauriers interpolatory mask of order m ([11]) and a_m^D can be obtained by factoring (5.11) via Riesz Lemma ([10]). The Daubechies 2-scaling function φ_m^D of order m associated with mask a_m^D and Daubechies orthonormal wavelet ψ_m^D of order m associated with mask b_m^D are then given by

$$\widehat{\phi_m^D} = \frac{1}{\sqrt{2\pi}}\prod_{\ell=1}^{\infty}\widehat{a_m^D}(2^{-\ell}\cdot) \quad \text{and} \quad \widehat{\psi_m^D} = \widehat{b_m^D}(\cdot/2)\widehat{\phi_m^D}(\cdot/2).$$

The semiorthogonal spline wavelet ψ_m^S of order m is given by

$$\psi_m^S(x) = \sum_{\nu=0}^{2m-2} \frac{(-1)^\nu}{2^{m-1}} N_{2m}(\nu+1) N_{2m}^{(m)}(2x-\nu), \quad x \in \mathbb{R}, \tag{5.13}$$

where N_m is the B-spline of order m. That is,

$$N_m(x) = \frac{1}{(m-1)!} \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} (x-\nu)_+^{m-1}, \quad x \in \mathbb{R} \tag{5.14}$$

or equivalently,

$$\widehat{N_m}(\omega) = \frac{1}{\sqrt{2\pi}} \left(e^{-i\omega/2} \frac{\sin(\omega/2)}{\omega/2} \right)^m = \frac{1}{\sqrt{2\pi}} \left(\frac{1-e^{i\omega}}{i\omega} \right)^m, \quad \omega \in \mathbb{R}. \tag{5.15}$$

Here for $k \geqslant 1$,

$$(y)_+^k = \begin{cases} y^k & y > 0, \\ 0, & y \leqslant 0, \end{cases} \quad \text{and} \quad (y)_+^0 = \begin{cases} 1 & y > 0, \\ \frac{1}{2}, & y = 0, \\ 0, & y < 0. \end{cases}$$

Note that ψ_m^S is generated from the 2-scaling function $\varphi_m^S := N_m$ via (5.4) by some mask b_m^S (cf. [7, 9]).

These two families are widely used in many applications. For example, see [1, 5, 12, 21, 23, 24, 25] for their applications on numerical solution of PDE and signal/image processing. Both of the Daubechies orthonormal wavelet ψ_m^D and the semiorthogonal spline wavelet ψ_m^S have vanishing moments of order m and support length $2m-1$. The Daubechies orthonormal wavelet ψ_m^D generates an orthonormal basis $\{2^{j/2}\psi_m^D(2^j \cdot -\nu) : j, \nu \in \mathbb{Z}\}$ for $L_2(\mathbb{R})$ (see [10]). However, the wavelet function ψ_m^D is implicitly defined and the coefficients in the mask for ψ_m^D are not rational numbers. Though the semiorthogonal spline wavelets generated by ψ_m^S are not orthogonal in the same level j, they are orthogonal on different levels. And more importantly, the semiorthogonal spline wavelet ψ_m^S is explicitly defined and the coefficients for its mask are indeed rational numbers, which is a very much desirable property in the implementation of fast wavelet algorithms. We shall see that these two families significantly differ with respect to the magnitude of their wavelet coefficients in terms of $C_{k,p}(\psi_m^D)$ and $C_{k,p}(\psi_m^S)$.

We are using several strategies to study asymptotic and non-asymptotic behavior of different kind wavelets. In Section 2, for $k, m \in \mathbb{N}$ fixed and $p \in (1, \infty)$, we shall investigate the quantity $C_{k,p}(\psi_m^S)$ in the Bernstein type inequality in (5.6) for the family of semiorthogonal spline wavelets. One of crucial new ingredients is Proposition 1, which is essential in the setting of non-asymptotic behaviour of semiorthogonal spline wavelets. In Section 3, we shall establish results on the asymptotic behaviors ($m \to \infty$) of the quantities $C_{k,p}(\varphi)$ and $C_{k,p}(\psi)$ for both the scaling function φ and wavelet function ψ and for both the two families of wavelets. We shall generalize our results to high-dimensional wavelets in Section 4. The last section are some technical proofs for some results in previous sections.

5.2 Bernstein type inequalities for splines

In this section, we shall first establish a sharp result on the Bernstein type inequality for splines and then present a lower bound for the quantity $C_{k,p}(\psi_m^S)$.

Recall that a function s is a *spline of order m of minimal defect with nodes* $\ell h, h > 0, \ell \in \mathbb{Z}$, if

(1) s is a polynomial with real coefficients of the degree $< m$ on each interval $(h(\ell-1), h\ell)$, $\ell \in \mathbb{Z}$;

(2) $s \in C^{m-2}(\mathbb{R})$.

The collection of all such splines is denoted by $S_{m,h}$. It is well known that any spline $s \in S_{m,h}$ can be uniquely represented by

$$s(x) = \sum_{\nu\in\mathbb{Z}} c_\nu N_m(x/h - \nu), \qquad x \in \mathbb{R}. \tag{5.16}$$

Here N_m is the B-spline of order m given in (5.14). One can show that for $m \geqslant 2$,

$$N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1), \qquad x \in \mathbb{R}. \tag{5.17}$$

The following result provides an exact upper bound for the Bernstein type inequality for any spline $s \in S_{m,h}$ (also cf. [3] for a special case $p = 2$).

Theorem 5.2.1. *Let $k, m \in \mathbb{N} \cup \{0\}$, $0 \leqslant k < m$, and $h > 0$. Let $p \in (1,\infty)$. Then, for any spline function $s \in S_{m,h}$ such that $\hat{s} \in L_p(\mathbb{R})$, the following inequality holds:*

$$\|\widehat{s^{(k)}}\|_p \leq K_{p,m,k}\left(\frac{2\pi}{h}\right)^k \|\hat{s}\|_p, \tag{5.18}$$

where $K_{p,m,k}$ is a constant depending on p, m, and k and is defined to be

$$K_{p,m,k} := \max_{\omega\in[0,2\pi]}\left(\frac{\sum_{\ell\in\mathbb{Z}}|\frac{\omega}{2\pi}+\ell|^{-p(m-k)}}{\sum_{\ell\in\mathbb{Z}}|\frac{\omega}{2\pi}+\ell|^{-pm}}\right)^{1/p}. \tag{5.19}$$

Moreover, the constant $K_{p,m,k}$ is sharp in the sense that there exists a sequence $s_j \in S_{m,h}$ such that

$$\frac{\|\widehat{s_j^{(k)}}\|_p}{\|\widehat{s_j}\|_p} \to (2\pi/h)^k K_{p,m,k}, \quad j \to \infty.$$

Proof. We first show that (5.18) is true for $h = 1$.

Recursively applying (5.17), we can deduce that

$$\widehat{s^{(k)}}(\omega) = \sum_{\nu\in\mathbb{Z}} c_\nu e^{-i\nu\omega}(1-e^{-i\omega})^k \widehat{N_{m-k}}(\omega), \quad \omega \in \mathbb{R}; 0 \leqslant k < m.$$

Consequently,

$$\begin{aligned}
\|\widehat{s^{(k)}}\|_p^p &= \int_{\mathbb{R}} \left| \sum_{\nu\in\mathbb{Z}} c_\nu e^{-i\nu\omega} \widehat{N_{m-k}}(\omega)(1-e^{-i\omega})^k \right|^p d\omega \\
&= \int_0^{2\pi} \left| \widehat{a_s}(\omega)(1-e^{-i\omega})^k \right|^p \sum_{\ell\in\mathbb{Z}} \left| \widehat{N_{m-k}}(\omega+2\pi\ell) \right|^p d\omega \\
&= \int_0^{2\pi} \frac{\left|1-e^{-i\omega}\right|^{kp} \sum_{\ell\in\mathbb{Z}} \left| \widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}} \left| \widehat{N_m}(\omega+2\pi\ell)\right|^p} |\widehat{a_s}(\omega)|^p \sum_{\ell\in\mathbb{Z}} \left| \widehat{N}_m(\omega+2\pi\ell)\right|^p d\omega \\
&\leq \max_{\omega\in[0,2\pi]} \frac{\left|1-e^{-i\omega}\right|^{kp} \sum_{\ell\in\mathbb{Z}} \left| \widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}} \left| \widehat{N_m}(\omega+2\pi\ell)\right|^p} \|\hat{s}\|_p^p.
\end{aligned}$$

Here $\widehat{a_s}(\omega) = \sum_{\nu\in\mathbb{Z}} c_\nu e^{-i\nu\omega}$. Define

$$L(\omega) := \frac{\left|1-e^{-i\omega}\right|^{kp} \sum_{\ell\in\mathbb{Z}} \left| \widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}} \left| \widehat{N_m}(\omega+2\pi\ell)\right|^p}, \quad \omega \in [0,2\pi]. \tag{5.20}$$

Then, we obtain

$$\|\widehat{s^{(k)}}\|_p^p \leqslant \max_{\omega\in[0,2\pi]} L(\omega)\cdot \|\hat{s}\|_p^p.$$

Since $\widehat{N_m}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{1-e^{i\omega}}{i\omega} \right)^m$, we have

$$\sum_{\ell\in\mathbb{Z}} \left| \widehat{N_m}(\omega+2\pi\ell)\right|^p = \frac{1}{(\sqrt{2\pi})^p} \left|1-e^{-i\omega}\right|^{pm} \sum_{\ell\in\mathbb{Z}} |\omega+2\pi\ell|^{-pm},$$

and, similarly,

$$\sum_{\ell\in\mathbb{Z}} \left| \widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p = \frac{1}{(\sqrt{2\pi})^p} \left|1-e^{-i\omega}\right|^{p(m-k)} \sum_{\ell\in\mathbb{Z}} |\omega+2\pi\ell|^{-p(m-k)}.$$

Hence,

$$\max_{\omega\in[0,2\pi]} L(\omega)^{1/p} = \max_{\omega\in[0,2\pi]} \left(\frac{\sum_{\ell\in\mathbb{Z}} |\omega+2\pi\ell|^{-p(m-k)}}{\sum_{l\in\mathbb{Z}} |\omega+2\pi\ell|^{-pm}} \right)^{1/p} = (2\pi)^k K_{p,m,k}.$$

Therefore

$$\|\widehat{s^{(k)}}\|_p \leq (2\pi)^k K_{p,m,k} \|\hat{s}\|_p.$$

Next, for any $h > 0$ and $s \in S_{m,h}$, we have $s = \sum_{\nu\in\mathbb{Z}} c_\nu N_m(\cdot/h+\nu)$. Let $s_1 := s(h\cdot)$. Then $s_1 \in S_{m,1}$ and it is easy to deduce that

$$\hat{s}(\omega) = h\widehat{s_1}(h\omega) \quad \text{and} \quad \widehat{s^{(k)}}(\omega) = h^{-k+1}\widehat{s_1^{(k)}}(h\omega). \tag{5.21}$$

By what we have been proved, we get

$$\|\widehat{s_1^{(k)}}\|_p \le (2\pi)^k K_{p,m,k} \|\widehat{s_1}\|_p.$$

Now, it is straightforward to deduce (5.18) from above inequality using (5.21).

Finally, we show that the constant in (5.18) is the best possible one.

Let $|\widehat{a_s}(\omega)|^p := \frac{1}{2\pi}\Phi_j(\omega-\omega_0)$ and $\hat{s}(\omega) := \widehat{a_s}(\omega)\widehat{N_m}(\omega)$, $\omega \in \mathbb{R}$, where $\Phi_j(\omega)$ is a Feyer's kernel of order j and ω_0 is the point which realizes the maximum of the function $L(\omega)$ on $[0, 2\pi]$. Note, $\frac{1}{2\pi}\int_0^{2\pi}\Phi_j(\omega)d\omega = 1$. Then,

$$\begin{aligned}
\|\widehat{s^{(k)}}\|_p^p &= \int_{\mathbb{R}} \left|\sum_{\nu\in\mathbb{Z}} c_\nu e^{-i\nu x}\widehat{N_{m-k}}(\omega)(1-e^{-i\omega})^k\right|^p d\omega \\
&= \int_0^{2\pi} |\widehat{a_s}(\omega)(1-e^{-i\omega})^k|^p \sum_{\ell\in\mathbb{Z}} \left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{|1-e^{-i\omega}|^{kp}\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p} |\Phi_j(\omega-\omega_0)| \sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p d\omega \\
&\to \frac{|1-e^{-i\omega_0}|^{kp}\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega_0+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega_0+2\pi\ell)\right|^p}\|\hat{s}\|_p^p, \qquad j\to\infty.
\end{aligned}$$

Consequently,

$$\frac{\|\widehat{s^{(k)}}\|_p}{\|\hat{s}\|_p} \to \max_{\omega\in[0,2\pi]} \left(\frac{|1-e^{-i\omega}|^{kp}\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_{m-k}}(\omega+2\pi\ell)\right|^p}{\sum_{\ell\in\mathbb{Z}}\left|\widehat{N_m}(\omega+2\pi\ell)\right|^p}\right)^{1/p} = (2\pi)^k K_{p,m,k}, \quad j\to\infty$$

which completes the proof. □

We remark that for $p = 2$, the function $L(\omega), \omega \in [0, 2\pi]$ defined in (5.20) assumes its maximal value at $\omega = \pi$ and the constant $K_{2,m,k}$ can be obtained explicitly as follows:

$$K_{2,m,k} = \pi^2 \frac{\sum_{\ell\in\mathbb{Z}}|1+2\ell|^{-2(m-k)}}{\sum_{\ell\in\mathbb{Z}}|1+2\ell|^{-(2m)}},$$

which is related to the Favard's constant (see [3]). For general p, L can be expressed as

$$L(\omega) = (2\pi)^{kp}\frac{\zeta(p(m-k),\frac{\omega}{2\pi}) + \zeta(p(m-k),-\frac{\omega}{2\pi}) - \left(-\frac{\omega}{2\pi}\right)^{-p(m-k)}}{\zeta(pm,\frac{\omega}{2\pi}) + \zeta(pm,-\frac{\omega}{2\pi}) - \left(-\frac{\omega}{2\pi}\right)^{-pm}}, \quad \omega\in[0,2\pi],$$

where $\zeta(x,y) := \sum_{\ell=0}^{\infty}(\ell+y)^{-x}$ is the Hurwitz zeta function.

For $s \in S_{m,h}$. Let $f :=_k s$. Then $f^{(k)} = s$, which implies $f \in S_{m+k,h}$. By Theorem 5.2.1 and the definition of $C_{k,p}(f)$ in (5.7), we have,

$$C_{k,p}(s) = \frac{\|\widehat{_k s}\|_p}{\|\hat{s}\|_p} = \frac{\|\hat{f}\|_p}{\|\widehat{f^{(k)}}\|_p} \geqslant \left(\frac{h}{2\pi}\right)^k \frac{1}{K_{p,m+k,k}}.$$

Moreover, by the definition of ψ_m^S in (5.13), we have the following result.

Proposition 5.2.2. *Let ψ_m^S be the semiorthogonal spline wavelet of order m defined in* (5.13)*. Let k be an nonnegative integer such that $k \leqslant m$. Then*

$$C_{k,p}(\psi_m^S) \geqslant \left(\frac{1}{4\pi}\right)^k \frac{1}{K_{p,m+k,k}}.$$

Proof. Let $f :=_k \psi_m^S$. Then $\widehat{f^{(k)}} = \widehat{\psi_m^S}$. By (5.13),

$$f(x) =_k \psi_m^S(x) = \sum_{\nu=0}^{2m-2} \frac{(-1)^\nu}{2^{m+k-1}} N_{2m}(\nu+1) N_{2m}^{(m-k)}(2x-\nu).$$

Consequently, $f \in S_{m+k,1/2}$. In view of Theorem 5.2.1, we have

$$\frac{\|\widehat{f^{(k)}}\|_p}{\|\hat{f}\|_p} \leqslant \left(\frac{2\pi}{\frac{1}{2}}\right)^k K_{p,m+k,k} = (4\pi)^k K_{p,m+k,k}.$$

Now, by that $C_{k,p}(\psi_m^S) = \frac{\|\hat{f}\|_p}{\|\widehat{f^{(k)}}\|_p}$, we are done. □

From Proposition 5.2.2, when m is large enough, we see that $C_{k,p}(\psi_m^S) \approx (4\pi)^{-k}$. In next section, we shall study the exact asymptotic behavior of these types of quantities as $m \to \infty$ for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets.

5.3 Asymptotic estimation of wavelet coefficients

In this section, we shall study the asymptotic behavior of wavelet coefficients for both Daubechies orthonormal wavelets and semiorthogonal spline wavelets (also see [19] for the asymptotic behavior of Battle-Lemari wavelet family). We shall discuss the asymptotic behavior of the wavelet coefficients for Daubechies orthonormal wavelets in the first subsection. In the second subsection, we shall investigate the asymptotic behavior of the wavelet coefficients for semiorthogonal spline wavelets. In the last subsection, we shall compare the asymptotic behaviors of wavelet coefficients for these two families based on the quantities obtained in previous two subsections.

5.3.1 Wavelet coefficients of Daubechies orthonormal wavelets

In this subsection, we shall discuss the asymptotic behavior of the following quantities: $\|\widehat{_{-k}\varphi_m^D}\|_p$, $\|\widehat{_k\psi_m^D}\|_p$, and $\|\widehat{_m\psi_m^D}\|_p$, $p \in (1,\infty)$.

To facilitate our investigation on the asymptotic behavior of Daubechies orthonormal wavelets, let us rewrite the mask a_m^D in another equivalent form. Let $H_m(t)$ be a 2π-periodic trigonometric function defined by

$$H_m(t) = \sum_{\nu=0}^{L} h_\nu e^{-i\nu t}, \quad |H_m(t)|^2 = 1 - c_m \int_0^t \sin^{2m-1} \omega d\omega, \tag{5.22}$$

where $c_m = \left(\int_0^\pi \sin^{2m-1} \omega d\omega \right)^{-1} = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}\Gamma(m)} \sim \sqrt{\frac{m}{\pi}}$. Then, we have $|H_m(\cdot)|^2 = |\widehat{a_m^D}(\cdot)|^2$. Hence, H_m is the Daubechies orthonormal mask of order m (cf. [16]).

To compare with the semiorthogonal spline wavelets, we need the following result for the Daubechies scaling function φ_m^D.

Theorem 5.3.1. *Let φ_m^D be the Daubechies orthonormal scaling function of order m, i.e., $\widehat{\varphi_m^D}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{\ell=1}^\infty H_m(2^{-\ell}\omega)$. Then, for $p \in (1, \infty)$,*

$$\lim_{m\to\infty} \|\widehat{{}_{-k}\varphi_m^D}\|_p = \pi^k \frac{(2\pi)^{1/p-1/2}}{(1+pk)^{1/p}}, \quad k \in \mathbb{N} \cup \{0\}. \tag{5.23}$$

Proof. Let $\Phi := \frac{1}{\sqrt{2\pi}} \chi_{[-\pi,\pi]}$. We have

$$\|\widehat{{}_{-k}\varphi_m^D}\|_p^p = \int_{\mathbb{R}} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) \right|^p d\omega = \int_{\mathbb{R}} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) - \Phi(\omega) + \Phi(\omega) \right|^p d\omega.$$

Note that

$$\int_{\mathbb{R}} |\omega|^{pk} |\Phi(\omega)|^p d\omega = \pi^{pk} \frac{(2\pi)^{1-p/2}}{1+pk}.$$

We next prove that

$$I := \int_{\mathbb{R}} |\omega|^{pk} \left| \widehat{\varphi_m^D} - \Phi(\omega) \right|^p d\omega \to 0, \quad \text{as } m \to \infty.$$

In fact,

$$I = \int_{|\omega|>\pi} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) \right|^p d\omega + \int_{|\omega|\leqslant\pi} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) - \Phi(\omega) \right|^p d\omega =: I_1 + I_2.$$

By the regularity of φ_m^D, i.e., $|\widehat{\varphi_m^D}(\omega)| \leqslant C_1 |\omega|^{-C_2 \log(m)}$ (see [10]), obviously, $I_1 \to 0$ as $m \to \infty$. For I_2, let $I := [-\pi, \pi]$, $\delta > 0$ be fixed, and $I_\delta := [-\pi+\delta, \pi-\delta]$. Then

$$I_2 = \int_{I_\delta} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) - \Phi(\omega) \right|^p d\omega + \int_{I \setminus I_\delta} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) - \Phi(\omega) \right|^p d\omega := I_{21} + I_{22}.$$

For I_{22}, we have $I_{22} \leqslant C\delta$ for some C depending only on p, k, since $\widehat{\varphi_m^D}$ and Φ are both bounded. For I_{21}, we have

$$\begin{aligned} I_{21} \leqslant & \int_{I_\delta} |\omega|^{pk} \left| \widehat{\varphi_m^D}(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2) \right|^p d\omega + \int_{I_\delta} |\omega|^{pk} \left| \frac{1}{\sqrt{2\pi}} H_m(\omega/2) - \Phi(\omega) \right|^p d\omega \\ & =: I_{31} + I_{32}. \end{aligned}$$

$I_{32} \to 0$ as $m \to \infty$ since $\frac{1}{\sqrt{2\pi}} H_m(\omega/2)$ converges to Φ uniformly in I_δ. To see that

$I_{31} \to 0$ as $m \to \infty$, by the definition of H_m, for $\omega \in [0, \pi]$, we have

$$\begin{aligned}\left|H_m\left(\frac{\omega}{4}\right)\right|^2 &\geqslant 1 - c_m \frac{\omega}{4} \sin^{2m-1}\left(\frac{\omega}{4}\right) \\ &\geqslant 1 - c_m \frac{\pi}{4} \sin^{2m-1}\left(\frac{\pi}{4}\right) \\ &\geqslant 1 - \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(m)}\left(\frac{\pi}{4}\right)^{2m}\end{aligned} \tag{5.24}$$

and

$$\left|H_m\left(\frac{\omega}{8}\right)\right|^2 \geqslant 1 - c_m \left(\frac{\pi}{4}\right)^{2m}. \tag{5.25}$$

Moreover, by (cf. [16, Lemma 2])

$$\begin{aligned}\prod_{\ell=1}^{\infty}\left|H_m\left(2^{-\ell-3}\omega\right)\right|^2 &\geqslant \prod_{\ell=1}^{\infty}\left|1 - c_m\left(2^{-\ell-3}\omega\right)^{2m}\right| \geqslant \prod_{\ell=1}^{\infty}\left|1 - c_m\left(\frac{\pi}{4}\right)^{2m}\left(2^{-2m}\right)^{\ell}\right| \\ &\geqslant \prod_{\ell=1}^{\infty}\left|1 - \left(2^{-2m}\right)^{\ell}\right| \geqslant \left(1 - 2^{-2m}\right)^{1/(1-2^{-2m})}.\end{aligned} \tag{5.26}$$

In view of (5.24) – (5.26), we have $1 \geqslant \left|\prod_{\ell=1}^{\infty} H_m\left(2^{-l-1}\omega\right)\right| \geqslant 1 - o(1)$. Consequently,

$$\begin{aligned}I_{31} &= \int_{I_\delta} |\omega|^{pk}\left|\widehat{\varphi_m^D}(\omega) - \frac{1}{\sqrt{2\pi}} H_m(\omega/2)\right|^p d\omega \\ &\leqslant \int_{I_\delta} |\omega|^{pk}\left|\frac{1}{\sqrt{2\pi}} H_m(\omega/2)\left(\prod_{\ell=1}^{\infty} H_m(2^{-l-1}\omega) - 1\right)\right|^p d\omega \\ &\leqslant \int_{I_\delta} |\omega|^{pk}\left|\frac{1}{\sqrt{2\pi}}\left(\prod_{\ell=1}^{\infty} H_m(2^{-l-1}\omega) - 1\right)\right|^p d\omega \to 0, \quad m \to \infty.\end{aligned}$$

Therefore, we obtain

$$\lim_{m\to\infty} \|\widehat{{}_{-k}\varphi_m^D}\|_p = \pi^k \frac{(2\pi)^{1/p-1/2}}{(1+pk)^{1/p}}, \quad k \in \mathbb{N} \cup \{0\}.$$

□

More generally, one can show that for $\alpha \in \mathbb{R}$ such that $1 - p\alpha > 0$,

$$\lim_{m\to\infty} \|\widehat{{}_{\alpha}\varphi_m^D}\|_p = \pi^{-\alpha} \frac{(2\pi)^{1/p-1/2}}{(1-p\alpha)^{1/p}}, \tag{5.27}$$

where for a real number $\alpha \in \mathbb{R}$, the function ${}_{\alpha}\varphi_m^D$ is similarly defined as in (5.8). However, when $1 - p\alpha \leqslant 0$, i.e., $\alpha \geqslant 1/p$, the constant $\|\widehat{{}_{\alpha}\varphi_m^D}\|_p \to \infty$ as $m \to \infty$.

When k is fixed and $m \to \infty$, Babenko and Spektor ([2]) show that, for the

Daubechies orthonormal wavelet function ψ_m^D with m vanishing moments, one has

$$\lim_{m\to\infty}\|\widehat{{}_k\psi_m^D}\|_p = \frac{(2\pi)^{1/p-1/2}}{\pi^k}\left(\frac{1-2^{1-pk}}{pk-1}\right)^{1/p}, \quad k\in\mathbb{N}. \tag{5.28}$$

For $k=m$ and $m\to\infty$, we can deduce the following estimation, which in turn gives rise to the asymptotic behavior of the constant $(C_{m,p}(\psi_m^D))^{1/m}$ (see Subsection 3.3).

Theorem 5.3.2. *Let ψ_m^D be the Daubechies wavelet with m vanishing moments, i.e., $\widehat{\psi_m^D}(\omega) = \frac{1}{\sqrt{2\pi}}H_m(\omega/2+\pi)\prod_{\ell=1}^{\infty}H_m(2^{-l-1}\omega)$. Then, for $p\in(1,\infty)$,*

$$\|\widehat{{}_m\psi_m^D}\|_p = C\cdot\frac{2^{1/p}}{\sqrt{2\pi}}\cdot\frac{2^{-m}\cdot A(m)}{(\sqrt{mp/2})^{1/p}}\cdot(1+\mathcal{O}(m^{-1/2})), \tag{5.29}$$

where C is a positive constant independent of m,p and $\sqrt{\frac{c_m}{2m}}\leqslant A(m)\leqslant\sqrt{\frac{1}{2}}$, where $c_m = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}\Gamma(m)}\sim\sqrt{\frac{m}{\pi}}$.

Proof. By definition,

$$\begin{aligned}\|\widehat{{}_m\psi_m^D}\|_p^p &= \int_{\mathbb{R}}|\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega\\ &= \int_{|\omega|\leqslant\pi}|\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega + \int_{|\omega|>\pi}|\omega|^{-mp}\left|\widehat{\psi_m^D}(\omega)\right|^p d\omega =: I_1+I_2.\end{aligned}$$

We first estimate I_2. Since $|H_m(t)|\leqslant 1$,

$$I_2\leqslant\frac{2}{(\sqrt{2\pi})^p}\int_{\pi}^{\infty}\omega^{-mp}d\omega\leqslant\frac{2}{(\sqrt{2\pi})^p}\cdot\frac{1}{mp-1}\left(\frac{1}{\pi}\right)^{mp-1}, \quad mp>1.$$

Next, we show that $I_1\sim C\cdot c_m^{p/2}\cdot(\sqrt{mp/2})^{-1}\cdot 2^{-mp}$. Again, by (5.24) – (5.26),

$$\begin{aligned}I_1 &= \frac{1}{(\sqrt{2\pi})^p}\int_{|\omega|\leqslant\pi}|\omega|^{-mp}\Bigg[|H_m(\omega/2+\pi)|^2|H_m(\omega/4)|^2|H_m(\omega/8)|^2\\ &\qquad\times\prod_{\ell=1}^{\infty}\left|H_m\left(2^{-l-3}\omega\right)\right|^2\Bigg]^{p/2}d\omega\\ &\geqslant(1-o(1))\frac{1}{(\sqrt{2\pi})^p}\int_{|\omega|\leqslant\pi}|\omega|^{-mp}|H_m(\omega/2+\pi)|^p d\omega.\end{aligned}$$

Obviously,

$$I_1\leqslant\frac{1}{(\sqrt{2\pi})^p}\int_{|\omega|\leqslant\pi}|\omega|^{-mp}|H_m(\omega/2+\pi)|^p d\omega.$$

Now, we use the property of H_m to deduce the asymptotic behavior of

$$I_{11} := \int_{|\omega|\leqslant\pi}|\omega|^{-mp}|H_m(\omega/2+\pi)|^p d\omega.$$

Let $u = \dfrac{\sin^2 t}{\sin^2(\omega/2)}$. We have

$$
\begin{aligned}
|H_m(\omega/2+\pi)|^2 &= c_m \int_0^{\omega/2} \sin^{2m-1} t dt \\
&= \frac{c_m}{2}\sin^{2m}(\omega/2)\int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2}du.
\end{aligned}
$$

Since

$$
\frac{1}{m} = \int_0^1 u^{m-1}du \leqslant \int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2}du \leqslant \int_0^1 u^{m-1}(1-u)^{-1/2}du = c_m^{-1}
$$

and

$$
I_{11} = 2\int_0^{\pi} |\omega|^{-mp}\cdot\left[\frac{c_m}{2}\sin^{2m}(\omega/2)\int_0^1 u^{m-1}(1-u\sin^2(\omega/2))^{-1/2}du\right]^{p/2} d\omega,
$$

we obtain

$$
\left(\frac{c_m}{2m}\right)^{p/2}\cdot 2^{-mp}\cdot\int_0^{\pi}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{mp} d\omega \leqslant \frac{1}{2}I_{11} \leqslant \left(\frac{1}{2}\right)^{-p/2}\cdot 2^{-mp}\cdot\int_0^{\pi}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{mp} d\omega.
$$

Now using that $\int_{-\pi}^{\pi}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{2\cdot mp/2} d\omega = C(\sqrt{mp/2})^{-1}(1+\mathcal{O}(m^{-1/2}))$ and $\frac{1}{\pi} < \frac{1}{2}$, we conclude

$$
\|\widehat{{}_m\psi_m^D}\|_p^p = C\cdot\frac{2}{(\sqrt{2\pi})^p}\cdot\frac{2^{-mp}\cdot A(m)^p}{\sqrt{mp/2}}\cdot(1+\mathcal{O}(m^{-1/2})),
$$

which completes our proof. □

5.3.2 Wavelet coefficients of semiorthogonal spline wavelets

In this subsection, we mainly focus on the asymptotic behavior of wavelet coefficients for the semiorthogonal spline wavelets. Next three theorems present the asymptotic estimations of the following quantities: $\|\widehat{{}_k\varphi_m^S}\|_p$, $\|\widehat{{}_k\psi_m^S}\|_p$, and $\|\widehat{{}_m\psi_m^S}\|_p$, $p\in(1,\infty)$. Since the proofs of the main results in this subsection share the similar idea of proofs in previous subsection but with more technical treatment, we therefore postpone their proofs to the last section.

For the scaling function φ_m^S, which is the B-spline N_m of order m, we have the following result gives the asymptotic estimate of $\|\widehat{{}_k\varphi_m^S}\|_p$.

Theorem 5.3.3. *Let $\varphi_m^S := N_m$ be the B-Spline of order m. Let $k \geqslant 0$ be a nonnegative integer. Then, for $p\in(1,\infty)$,*

$$
\|\widehat{{}_k\varphi_m^S}\|_p = \frac{4^{1/p}}{(\sqrt{2\pi})^{1-1/p}}\cdot\frac{1}{(\sqrt{\Lambda_1 mp})^{1/p}}\cdot(2\xi_1)^{-k}\cdot(\lambda_1/\xi_1)^{m/2}\cdot(1+\mathcal{O}(m^{-1/2})), \quad (5.30)
$$

where

$$\lambda_1 = \frac{\sin^2(\xi_1)}{\xi_1} \approx 0.72461, \\ \Lambda_1 = -\frac{1}{2}\frac{d^2}{d\omega^2}\ln\frac{\sin^2(\xi_1-\omega)}{\xi_1-\omega}\Big|_{\omega=0} \approx 0.81597, \tag{5.31}$$

and $\xi_1 \approx 1.1655$ *is the unique solution of the transcendental equation* $\xi_1 - 2\cot(\xi_1) = 0$ *in the interval* $(0,\pi)$.

Similarly, for the spline wavelet function ψ_m^S, we have the following theorem:

Theorem 5.3.4. *Let* $k \geqslant 0$ *be a nonnegative integer. Let* ψ_m^S *be the semiorthogonal spline wavelet of order* m. *Then, for* $p \in (1,\infty)$,

$$\|\widehat{{}_k\psi_m^S}\|_p = \frac{2^{3/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \frac{(2\pi - 4\xi_2)^{-k}}{(\sqrt{2\Lambda_2 mp})^{1/p}} \cdot \lambda_2^m \cdot (1 + \mathcal{O}(m^{-1/2})). \tag{5.32}$$

where

$$\lambda_2 = \frac{\sin^2(\xi_2 - \pi/2)\sin^2(\xi_2)}{(\pi/2 - \xi_2)\xi_2^2} \approx 0.69706 \\ \Lambda_2 = -\frac{1}{2}\frac{d^2}{du^2}\ln\frac{\sin^2(u-\pi/2)\sin^2(u)}{(\pi/2-u)u^2}\Big|_{u=\xi_2} \approx 1.2229, \tag{5.33}$$

and $\xi_2 = 0.2853...$ *is the unique solution of the transcendental equation*

$$(2\pi\xi - 4\xi^2)\cos(2\xi) + (3\xi - \pi)\sin(2\xi) = 0, \quad \xi \in (0, \pi/2).$$

Finally, to compare with the wavelet case for $k = m$, we also provide the following estimation for the spline case with $k = m$:

Theorem 5.3.5. *Let* ψ_m^S *be the semiorthogonal spline wavelet of order* m. *Then, for* $p \in (1,\infty)$,

$$\|\widehat{{}_m\psi_m^S}\|_p = \frac{2^{1/p}}{(\sqrt{2\pi})^{1-1/p}} \cdot \left(\frac{\pi}{\sqrt{\pi^2-8}}\right)^{1/p} \cdot \frac{1}{(\sqrt{2mp})^{1/p}} \cdot \left(\frac{16}{\pi^4}\right)^m \cdot (1 + \mathcal{O}(m^{-1/2})). \tag{5.34}$$

5.3.3 Comparison of Daubechies orthonormal wavelets and semiorthogonal wavelets

Now, by the results we obtained in the above two subsections, we can compare the Daubechies orthonormal wavelets and the semiorthogonal spline wavelets using the constants $C_{k,p}(f)$. Note that both Daubechies orthonormal wavelets and the semiorthogonal spline wavelets have the same support length and number of vanishing moments, thereby a comparison is possible in this respect.

We first consider the situation when k is fixed and let $m \to \infty$. For Daubechies family, by Theorem 5.3.1 and (5.28), we can deduce the following result.

Corollary 5.3.6. *Let φ_m^D and ψ_m^D be the Daubechies orthonormal scaling function and wavelet function of order m, respectively. Let $k \geqslant 0$ be a nonnegative integer. Then, for $p \in (1, \infty)$,*

$$\lim_{m\to\infty} C_{-k,p}(\varphi_m^D) = \lim_{m\to\infty} \frac{\|\widehat{{}_{-k}\varphi_m^D}\|_p}{\|\widehat{\varphi_m^D}\|_p} = \frac{\pi^k}{(1+pk)^{1/p}} \tag{5.35}$$

and

$$\lim_{m\to\infty} C_{k,p}(\psi_m^D) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\psi_m^D}\|_p}{\|\widehat{\psi_m^D}\|_p} = \pi^{-k}\left(\frac{1-2^{1-pk}}{pk-1}\right)^{1/p}. \tag{5.36}$$

For the semiorthogonal spline wavelet family, by Theorems 5.3.3 and 5.3.4, similarly, we have the following result.

Corollary 5.3.7. *Let φ_m^S and ψ_m^S be the semiorthogonal spline wavelet of order m, respectively. Let $k \geqslant 0$ be an integer. Then, for $p \in (1, \infty)$,*

$$\lim_{m\to\infty} C_{k,p}(\varphi_m^S) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\varphi_m^S}\|_p}{\|\widehat{\varphi_m^S}\|_p} = (2\xi_1)^{-k} \approx (2.331)^{-k} \tag{5.37}$$

and

$$\lim_{m\to\infty} C_{k,p}(\psi_m^S) = \lim_{m\to\infty} \frac{\|\widehat{{}_k\psi_m^S}\|_p}{\|\widehat{\psi_m^S}\|_p} = (2\pi - 4\xi_2)^{-k} \approx (5.1419)^{-k}, \tag{5.38}$$

where $\xi_1 \approx 1.1655$ and $\xi_2 \approx 0.2853$ are constants given in Theorems 5.3.3 and 5.3.4.

Comparing Corollaries 5.3.6 and 5.3.7, we see that for every $k \in \mathbb{N} \cup \{0\}$, the semiorthogonal spline wavelets are better than the Daubechies orthonormal wavelets in the sense of asymptotically smaller constants. More precisely, we have

Corollary 5.3.8. *Let ψ_m^D and ψ_m^S be Daubechies orthonormal wavelet and the semiorthogonal spline wavelet of order m, respectively. Then, for $p \in (1, \infty)$,*

$$\lim_{k\to\infty}\lim_{m\to\infty} \left(\frac{C_{k,p}(\psi_m^S)}{C_{k,p}(\psi_m^D)}\right)^{1/k} = \frac{\pi}{2\pi - 4\xi_2}. \tag{5.39}$$

Note that $\frac{\pi}{2\pi-4\xi_2} \approx 0.61098 < 1$. In other words, (5.39) shows that the semiorthogonal spline wavelet constant $C_k(\psi_m^S)$ is exponentially better than Daubechies orthonormal wavelet constant $C_k(\psi_m^D)$ for increasing k.

Since the number of vanishing moments increases with m, it is natural to consider the behavior of the constants $C_k(\psi_m^S)$ with $k = k(m) = m$. In this situation, from Theorems 5.3.2 and 5.3.5, we have the following result, which shows that for smooth functions, the ratio in (5.39) when $k = m$ is even more in favor of the semiorthogonal spline wavelets.

Corollary 5.3.9. *Let ψ_m^D and ψ_m^S be Daubechies orthonormal wavelet and the semiorthogonal spline wavelet of order m, respectively. Then*

$$\lim_{m\to\infty} \left(C_{m,p}(\psi_m^D)\right)^{1/m} = \frac{1}{2}, \quad \lim_{m\to\infty} \left(C_{m,p}(\psi_m^S)\right)^{1/m} = \frac{16}{\lambda_2 \pi^4}, \tag{5.40}$$

and

$$\lim_{m\to\infty}\left(\frac{C_{m,p}(\psi_m^S)}{C_{m,p}(\psi_m^D)}\right)^{1/m} = \frac{32}{\lambda_2\pi^4}. \tag{5.41}$$

We end this section by comparing the asymptotic behaviors between the scaling function φ and the wavelet function ψ for both the Daubechies orthonormal wavelets and semiorthogonal spline wavelets.

For the Daubechies orthonormal wavelets, again, by Theorems 5.3.1 and (5.28), we have the following result.

Corollary 5.3.10. *Let φ_m^D and ψ_m^D be the Daubechies orthonormal scaling function and wavelet function of order m, respectively. Let $k_1, k_2 \geqslant 0$ be nonnegative integers. Then, for $p \in (1,\infty)$*

$$\lim_{m\to\infty}\frac{\|\widehat{{}_{-k_1}\varphi_m^D}\|_p}{\|\widehat{{}_{k_2}\psi_m^D}\|_p} = \frac{\pi^{k_1+k_2}}{(1-2^{1-pk_2})^{1/p}}\left(\frac{pk_1+1}{pk_2-1}\right)^{1/p}. \tag{5.42}$$

For the semiorthogonal wavelets, similarly, using the results of Theorems 5.3.3 and 5.3.4, we have

Corollary 5.3.11. *Let φ_m^S and ψ_m^S be the semiorthogonal spline scaling function and wavelet function of order m, respectively. Let $k_1, k_2 \geqslant 0$ be nonnegative integers. Then, for $p \in (1,\infty)$*

$$\lim_{m\to\infty}\left(\frac{\|\widehat{{}_{k_1}\varphi_m^S}\|_p}{\|\widehat{{}_{k_2}\psi_m^S}\|_p}\right)^{1/m} = \sqrt{\frac{\lambda_1}{\xi_1\lambda_2^2}}. \tag{5.43}$$

5.4 High-dimensional wavelet coefficients

One of the simple ways to construct high-dimensional wavelets is using tensor product. In this section, we discuss the wavelet coefficients for high-dimensional tensor product wavelets. We shall mainly focus on dimension two while results of higher dimensions can be similarly obtained using the properties of tensor product.

Let φ, ψ be the one-dimensional scaling function and wavelet function that generates a wavelet basis in $L_2(\mathbb{R})$. Then, in two-dimensional case, the scaling function $\mathbf{\Phi}(x_1,x_2) = \varphi(x_1)\varphi(x_2)$ and we have three wavelets instead of one,

$$\begin{aligned}\mathbf{\Psi}^1(x_1,x_2) &:= \psi(x_1)\varphi(x_2),\\ \mathbf{\Psi}^2(x_1,x_2) &:= \varphi(x_1)\psi(x_2),\\ \mathbf{\Psi}^3(x_1,x_2) &:= \psi(x_1)\psi(x_2).\end{aligned} \tag{5.44}$$

Let $k = (k_1,k_2) \in \mathbb{Z}^2$ be a two-dimensional index. Then, for a two-dimensional wavelet function $\mathbf{\Psi}$, we can define $C_{k,p}(\mathbf{\Psi})$ similar to (5.5) by

$$C_{k,p}(\mathbf{\Psi}) = \sup_{f\in\mathcal{A}_k^{p'}}\frac{|\langle f,\mathbf{\Psi}\rangle|}{\|\widehat{\mathbf{\Psi}}\|_p} = \frac{\|\widehat{{}_k\mathbf{\Psi}}\|_p}{\|\widehat{\mathbf{\Psi}}\|_p}, \tag{5.45}$$

where $1 < p,p' < \infty$, $1/p' + 1/p = 1$ and $\mathcal{A}_k^{p'} := \{f \in L_{p'}(\mathbb{R}^2) : \|(i\omega)^k\hat{f}(\omega)\|_{p'} \leqslant 1\}$. Here, for $x = (x_1,x_2) \in \mathbb{R}^2, k = (k_1,k_2) \in \mathbb{Z}^2$, $x^k := x_1^{k_1}x_2^{k_2}$. And for a function $f \in L_1(\mathbb{R}^2)$, ${}_kf$ is defined to be a function such that $\widehat{{}_kf}(\omega) = (i\omega)^k \widehat{f}$, $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. In particular, when $\mathbf{\Psi}(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, one can easily show that $C_{k,p}(\mathbf{\Psi}) = C_{k_1,p}(\psi_1)C_{k_2,p}(\psi_2)$.

In two-dimensional case, the semiorthogonal wavelets can be represented by

$$\begin{aligned} \mathbf{\Psi}_m^{S,1}(x_1,x_2) &:= \psi_m^S(x_1)\varphi_m^S(x_2) = \psi_m^S(x_1)N_m(x_2), \\ \mathbf{\Psi}_m^{S,2}(x_1,x_2) &:= \varphi_m^S(x_1)\psi_m^S(x_2) = N_m(x_1)\psi_m^S(x_2), \\ \mathbf{\Psi}_m^{S,3}(x_1,x_2) &:= \psi_m^S(x_1)\psi_m^S(x_2). \end{aligned} \tag{5.46}$$

We can obtain that following corollaries using results in previous sections and the properties of tensor product.

Corollary 5.4.1. *Let $\mathbf{\Psi}_m^{S,1}$, $\mathbf{\Psi}_m^{S,2}$, and $\mathbf{\Psi}_m^{S,3}$ be defined in (5.46). Let $k = (k_1, k_2) \in \mathbb{N}^2$. Then,*

$$\begin{aligned} C_{k,p}(\mathbf{\Psi}_m^{S,1}) &\geqslant \frac{1}{2^{k_1}}\left(\frac{1}{2\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}}, \\ C_{k,p}(\mathbf{\Psi}_m^{S,2}) &\geqslant \frac{1}{2^{k_2}}\left(\frac{1}{2\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}}, \\ C_{k,p}(\mathbf{\Psi}_m^{S,3}) &\geqslant \left(\frac{1}{4\pi}\right)^{k_1+k_2} \frac{1}{K_{p,m+k_1,k_1}K_{p,m+k_2,k_2}}, \end{aligned} \tag{5.47}$$

where $K_{p,m,k}$'s are constants defined by (5.19). Moreover,

$$\begin{aligned} \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,1}) &= (2\pi - 4\xi_2)^{-k_1}(2\xi_1)^{-k_2}, \\ \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,2}) &= (2\pi - 4\xi_2)^{-k_2}(2\xi_1)^{-k_1}, \\ \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{S,3}) &= (2\pi - 4\xi_2)^{-k_1-k_2}, \end{aligned} \tag{5.48}$$

where ξ_1, ξ_2 are constants given in Corollary 5.3.7.

In two-dimensional case, Daubechies wavelets can be represented by

$$\begin{aligned} \mathbf{\Psi}_m^{D,1}(x_1,x_2) &:= \psi_m^D(x_1)\varphi_m^D(x_2), \\ \mathbf{\Psi}_m^{D,2}(x_1,x_2) &:= \varphi_m^D(x_1)\psi_m^D(x_2), \\ \mathbf{\Psi}_m^{D,3}(x_1,x_2) &:= \psi_m^D(x_1)\psi_m^D(x_2). \end{aligned}$$

Similarly, we have the following result.

Corollary 5.4.2. *Let $\mathbf{\Psi}_m^{D,1}$, $\mathbf{\Psi}_m^{D,2}$, and $\mathbf{\Psi}_m^{D,3}$ be defined in (5.48). Then,*

$$\begin{aligned} \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,1}) &= \pi^{-k_1}\left(\frac{1-2^{1-pk_1}}{pk_1-1}\right)^{1/p} \frac{\pi^{-k_2}}{(1-pk_2)^{1/p}},\ k_2 \leqslant 1/p, k_1 \in \mathbb{N}, \\ \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,2}) &= \pi^{-k_2}\left(\frac{1-2^{1-pk_2}}{pk_2-1}\right)^{1/p} \frac{\pi^{-k_1}}{(1-pk_1)^{1/p}},\ k_1 \leqslant 1/p, k_2 \in \mathbb{N}, \\ \lim_{m\to\infty} C_{k,p}(\mathbf{\Psi}_m^{D,3}) &= \pi^{-k_1-k_2}\left(\frac{1-2^{1-pk_1}}{pk_1-1}\cdot\frac{1-2^{1-pk_2}}{pk_2-1}\right)^{1/p},\ (k_1,k_2) \in \{\mathbb{N}\cup\{0\}\}^2. \end{aligned} \tag{5.49}$$

5.5 Appendix

In this section, we give the proofs of Theorem 5.3.3, Theorem 5.3.4, and Theorem 5.3.5.

Proof of Theorem 5.3.3. By (5.14),

$$\|\widehat{{}_k\varphi_m^S}\|_p^p = \int_{\mathbb{R}} |\omega|^{-kp} \cdot \Big|\frac{\sin(\omega/2)}{\omega/2}\Big|^{mp} d\omega = \frac{2^{1-kp}}{(\sqrt{2\pi})^p} \int_R |\omega|^{-kp} \cdot \Big|\frac{\sin(\omega)}{\omega}\Big|^{mp} d\omega$$

$$= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \int_0^\infty \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega$$

$$= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \Big(\int_0^\pi \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega + \int_\pi^\infty \omega^{-p(m/2+k)} \cdot \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} d\omega \Big)$$

$$=: \frac{2^{2-kp}}{(\sqrt{2\pi})^p} (I_1 + I_2).$$

For I_2 with $mp > 1$, we have

$$I_2 \leqslant \int_\pi^\infty \omega^{-mp} d\omega = \frac{1}{mp-1} \left(\frac{1}{\pi}\right)^{mp-1}.$$

To estimate I_1, we use the same technique as in the proof of [16, Lemma 4]. Let ξ_1 be the point where $\sin^2(\omega)/\omega$ takes its maximum value λ_1 in $(0,\pi)$, i.e., $\xi_1 \approx 1.1655$ is the root of the transcendental equation $\xi_1^{-1} - 2\cot(\xi_1) = 0$ and $\lambda_1 = \dfrac{\sin^2(\xi_1)}{\xi_1} \approx 0.72461$. Separate I_1 to two parts as follows

$$I_1 = \int_0^{\xi_1} \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega + \int_{\xi_1}^{\pi} \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega =: I_{11} + I_{12}.$$

We first estimate I_{11}. Let

$$t = t(\omega) = \ln \frac{\xi_1 - \omega}{\sin^2(\xi_1 - \omega)} - \ln \frac{\xi_1}{\sin^2(\xi_1)} = \ln \frac{\lambda_1(\xi_1 - \omega)}{\sin^2(\xi_1 - \omega)}, \quad \omega \in (0, \xi_1).$$

Then, we have

$$t(\omega) \sim a_2\omega^2 + a_3\omega^3 + \cdots \sim a_2\omega^2 \left(1 + \frac{a_3}{a_2}\omega + \cdots\right), \quad \omega \to 0,$$

where

$$a_2 = \Lambda_1 = -\frac{1}{2}\frac{d^2}{d\omega^2} \ln \frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega}\Big|_{\omega=0} \approx 0.81597\,.$$

Then, similar to the proof of [16, Lemma 4], we can obtain

$$\begin{aligned}
\omega = \omega(t) &\sim (\Lambda_1)^{-1/2}\sqrt{t}(1 + c_1 t^{1/2} + c_2 t + \cdots),\\
\frac{d\omega}{dt} &\sim \frac{1}{2\sqrt{\Lambda_1 t}}(1 + d_1 t^{1/2} + d_2 t + \cdots),\\
\xi_1 - \omega(t) &\sim \xi_1(1 - e_1 t^{1/2} - e_2 t - \cdots),
\end{aligned}$$

for $t \to 0$. Changing the variable of I_{11}, we have

$$\begin{aligned}
I_{11} &= \int_0^{\xi_1} \left(\frac{\sin^2(\omega)}{\omega}\right)^{mp/2} \cdot \omega^{-p(m/2+k)} d\omega\\
&= \int_0^{\xi_1} \left(\frac{\sin^2(\xi_1 - \omega)}{\xi_1 - \omega}\right)^{mp/2} \cdot (\xi_1 - \omega)^{-p(m/2+k)} d\omega\\
&= \lambda_1^{mp/2} \int_0^{\infty} e^{-\frac{mp}{2}t} q(t) dt,
\end{aligned}$$

where

$$q(t) \sim \left(\xi_1^{p(m/2+k)} 2\sqrt{\Lambda_1 t}\right)^{-1} (1 + f_1 t^{1/2} + f_2 t + \cdots).$$

Now by Watson's lemma, i.e.,

$$\int_0^T e^{-xt} t^s f(t) dt \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\Gamma(s+n+1)}{n! x^{s+n+1}}, x \to \infty \tag{5.50}$$

for function f having an infinite number of derivatives in the neighborhood of $t = 0$ (c.f. [22, Theorem 3.1]), we have

$$\begin{aligned}
I_{11} &= \lambda_1^{mp/2} \cdot \left(\xi_1^{p(m/2+k)} 2\sqrt{\Lambda_1}\right)^{-1} \cdot \frac{\sqrt{\pi}}{\sqrt{mp/2}} \cdot (1 + \mathcal{O}(m^{-1/2}))\\
&= \frac{\sqrt{2\pi}}{2\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left(\frac{\lambda_1}{\xi_1}\right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})).
\end{aligned}$$

For I_{12}, we use

$$t = t(\omega) = \ln \frac{\xi_1 + \omega}{\sin^2(\xi_1 + \omega)} - \ln \frac{\xi_1}{\sin^2(\xi_1)} = \ln \frac{\lambda_1(\xi_1 + \omega)}{\sin^2(\xi_1 + \omega)}, \quad \omega \in (0, \pi - \xi_1).$$

Similarly, we have $I_{12} = \frac{\sqrt{2\pi}}{2\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left(\frac{\lambda_1}{\xi_1}\right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2}))$. Consequently,

$$I_1 = \frac{\sqrt{2\pi}}{\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left(\frac{\lambda_1}{\xi_1}\right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Noting that $\frac{1}{\pi} \approx 0.31830 < \left(\frac{\lambda_1}{\xi_1}\right)^{1/2} \approx 0.78846$, we conclude

$$\begin{aligned}\|\widehat{{}_k\varphi^S_m}\|_p^p &= \frac{2^{2-kp}}{(\sqrt{2\pi})^p} \cdot \frac{2\sqrt{2\pi}}{\sqrt{\Lambda_1 mp}} \cdot (\xi_1)^{-kp} \cdot \left(\frac{\lambda_1}{\xi_1}\right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})) \\ &= \frac{4}{(\sqrt{2\pi})^{p-1}} \cdot \frac{1}{\sqrt{\Lambda_1 mp}} \cdot (2\xi_1)^{-kp} \cdot \left(\frac{\lambda_1}{\xi_1}\right)^{mp/2} \cdot (1 + \mathcal{O}(m^{-1/2})),\end{aligned}$$

which completes our proof. □

Proof of Theorem 5.3.4. Using the Fourier transform of the B-spline and the definition of Euler-Frobenius polynomial $E_{2m-1}(z)$ for $z = e^{i\omega}$:

$$\frac{E_{2m-1}(z)}{(2m-1)!} = \sum_{\nu=0}^{2m-2} N_{2m}(\nu+1) z^\nu = e^{-i(m-1)\omega} (2\sin(\omega/2))^{2m} \sum_{\ell=-\infty}^{\infty} \frac{1}{(\omega + 2\pi\ell)^{2m}}, \tag{5.51}$$

we can derive (c.f. [16, Lemma 4])

$$\begin{aligned}|\widehat{{}_k\psi^S_m}(\omega)| &= \frac{2^{-2k}}{\sqrt{2\pi}} \Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^m \Big|\frac{\omega}{4}\Big|^{-k} \Big|\frac{E_{2m-1}(\tilde{z})}{(2m-1)!}\Big| \\ &= \frac{2^{-2k}}{\sqrt{2\pi}} \Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^m \Big|\frac{\omega}{4}\Big|^{-k} \Big|2\sin(\tilde{\omega}/2)\Big|^{2m} \Big|\sum_{\ell=-\infty}^{\infty} \frac{1}{(\tilde{\omega} + 2\pi\ell)^{2m}}\Big|,\end{aligned}$$

where $\tilde{z} = e^{i\tilde{\omega}}$ and $\tilde{\omega} = \pi - \omega/2$. Then, we obtain

$$\begin{aligned}\|\widehat{{}_k\psi^S_m}\|_p^p &= \frac{2^{-2kp}}{(\sqrt{2\pi})^p} \int_{\mathbb{R}} \Big|\frac{\sin^2(\omega/4)}{\omega/4}\Big|^{mp} \Big|\frac{\omega}{4}\Big|^{-kp} \Big|2\sin(\tilde{\omega}/2)\Big|^{2mp} \Big|\sum_{\ell=-\infty}^{\infty} \frac{1}{(\tilde{\omega} + 2\pi\ell)^{2m}}\Big|^p d\omega \\ &= \frac{2^{-2kp}}{(\sqrt{2\pi})^p} \int_{\mathbb{R}} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\right)^{2m} (u-\pi/2)^{-2k} (2\sin(u))^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(2u+2\pi\ell)^{2m}}\right)^2\right]^{p/2} 4du \\ &= \frac{4 \cdot 2^{-2kp}}{(\sqrt{2\pi})^p} \Big(\int_{-\infty}^{-\pi/2} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\right)^{2m} (u-\pi/2)^{-2k} (\sin(u))^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\ &+ \int_{-\pi/2}^{\xi_2} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\right)^{2m} (u-\pi/2)^{-2k} (\sin(u))^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\ &+ \int_{\xi_2}^{\pi/2} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\right)^{2m} (u-\pi/2)^{-2k} (\sin(u))^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \\ &+ \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{\infty} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2}\right)^{2m} (u-\pi/2)^{-2k} (\sin(u))^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}\right)^2\right]^{p/2} du \Big) \\ &=: \frac{4 \cdot 2^{-2kp}}{(\sqrt{2\pi})^p} (I_1 + I_2 + I_3 + I_4 + I_5).\end{aligned}$$

Here, ξ_2 is the point where the function

$$g(u) := \frac{\sin^2(u-\pi/2)\sin^2(u)}{(\pi/2-u)u^2}$$

takes its maximum value in $(0,\pi/2)$, i.e. point at which $g'(u)=0$. One can show that $\xi_2 \approx 0.28532$ is the root of the transcendental equation

$$h(u) := (2\pi u - 4u^2)\cos(2u) + (3u-\pi)\sin(2u).$$

Note that $g'(u) = \frac{\sin(2u)}{4(\pi/2-u)^2u^4}\cdot h(u)$ and $\lambda_2 = g(\xi_2) \approx 0.69706$.

We first estimate I_2. By [16, Lemma 3], we have

$$I_2 = \int_{-\pi/2}^{\xi_2}\left[g(u)^{2m}(u-\pi/2)^{-2k}(1+R_1+r(u))^2\right]^{p/2}du =: I_{21}+\tilde{R},$$

where $|R_1| \leqslant (2m-1)^{-1}$,

$$r(u) = \begin{cases} \left(\frac{u}{\pi+u}\right)^{2m}, & -\pi/2 < u \leqslant 0, \\ \left(\frac{u}{\pi-u}\right)^{2m}, & 0 \leqslant u < \xi_2. \end{cases}$$

$$I_{21} := \int_{-\pi/2}^{\xi_2}\left[g(u)^{2m}(u-\pi/2)^{-2k}(1+R_2(u))^2\right]^{p/2}du,$$

where

$$R_2(u) = \begin{cases} R_1 + r(u), & -\pi/2+\delta < u < \xi_2, \\ R_1, & -\pi/2 < u < -\pi/2+\delta. \end{cases}$$

$0 < \delta < \pi/2 - \xi_2$ is fixed. Hence

$$|R_2(u)| \leqslant \frac{1}{2m-1} + \left(\frac{\pi/2-\delta}{\pi/2+\delta}\right)^{2m},$$

and

$$\begin{aligned}\tilde{R} &= \int_{-\pi/2}^{-\pi/2+\delta}\left[g(u)^{2m}(u-\pi/2)^{-2k}\right]^{p/2}\cdot\left[(1+R_1+r(u))^p - (1+R_1)^p\right]du \\ &\leqslant (p2^p+o(1))\int_{-\pi/2}^{-\pi/2+\delta}\left[g(u)^{2m}(u-\pi/2)^{-2k}\right]^{p/2}du \\ &\leqslant (p2^p+o(1))\delta\cdot\left[\frac{\sin^{4m}\delta}{(\pi-\delta)^{2m+2k}}\right]^{p/2} \leqslant (p2^p+o(1))\delta\cdot\frac{\sin^{2mp}\delta}{(\pi-\delta)^{p(m+k)}}.\end{aligned}$$

For the estimation of I_{21}, we shall employ the Watson's lemma. We introduce

$$t = t(v) := \ln g(\xi_2) - \ln g(\xi_2 - v) = \ln\frac{\lambda_2}{g(\xi_2-v)}, \quad \frac{dt}{dv} = \frac{g'(\xi-v)}{g(\xi-v)},$$

for $v \in [0, \pi/2+\xi_2]$. We have $t \to 0$ as $v \to 0$ and t goes from 0 to ∞ monotonically as v increases from 0 to $\pi/2+\xi_2$. We can state the asymptotic expansion of $t(v)$

near $v = 0$ as follows:

$$t(v) \sim a_2 v^2 + a_3 v^3 + \cdots \sim a_2 v^2(1 + a_3/a_2 v + \cdots),$$

where

$$a_2 = \Lambda_2 = -\frac{1}{2}\frac{d^2}{dv^2}\ln g(\xi_2 - v)\Big|_{v=0} = -\frac{h'(\xi_2)}{2\xi_2(\pi/2 - \xi_2)\sin(2\xi_2)} \approx 1.2229.$$

Let $s = \sqrt{t}$. Then

$$s(v) \sim \sqrt{\Lambda_2} v(1 + b_1 v + \cdots), \quad v \to 0.$$

Now $s'(v) \neq 0$, we can reverse this expansion,

$$v = v(t) \sim \Lambda_2^{-1/2} s(1 + c_1 s + c_2 s^2 + \cdots) \sim \Lambda_2^{-1/2} t^{1/2}(1 + c_1 t^{-1/2} + c_2 t + \cdots).$$

Also,

$$\frac{dv}{dt} = \frac{(\pi/2 + v - \xi_2)(\xi_2 - v)\sin 2(\xi_2 - v)}{h(\xi_2 - v)}$$

Asymptotic expansion of numerator and denominator at $v = 0$ and division yields

$$\begin{aligned}\frac{dv}{dt} &\sim \frac{(\pi/2 - \xi_2)\xi_2 \sin(2\xi_2)}{-h'(\xi_2)v(t)}(1 + d_1 v(t)^2 + \cdots) \\ &\sim \frac{1}{2\Lambda_2 v(t)}(1 + d_1 v(t)^2 + \cdots) \\ &\sim \frac{1}{2\sqrt{\Lambda_2 t}}(1 + e_1 t^{1/2} + e_2 t + \cdots).\end{aligned}$$

Now changing the variable in I_{21} and noting $g(\xi_2 - v) = \lambda_2 e^{-t}$, we have

$$\begin{aligned}I_{21} &\sim \int_{-\pi/2}^{\xi_2} [g(u)^{2m}(u - \pi/2)^{-2k}]^{p/2} du \\ &= \lambda_2^{mp} \int_0^{\xi_2 + \pi/2} [(g(\xi_2 - v)/\lambda_2)^{2m}(\xi_2 - v - \pi/2)^{-2k}]^{p/2} dv \\ &= \lambda_2^{mp} \int_0^{\infty} e^{-mpt} q(t) dt,\end{aligned}$$

where

$$\begin{aligned}q(t) &= (\pi/2 + v(t) - \xi_2)^{-kp} \cdot \frac{dv}{dt} \\ &\sim \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2 t}}(1 + f_1 t^{1/2} + f_2 t + \cdots)^{-kp}(1 + e_1 t^{1/2} + e_2 t + \cdots) \\ &\sim \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2 t}}(1 + g_1 t^{1/2} + g_2 t + \cdots).\end{aligned}$$

By Watson's lemma and choosing δ such that $\sin^2(\delta/(\pi - \delta)) < \lambda_2$, we conclude that

$$I_2 \sim I_{21} \sim \lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Similarly, we can estimate the asymptotic behavior of I_3. We use

$$t = t(v) = \ln g(\xi_2) - \ln g(\xi + v) = \ln \frac{\lambda_2}{g(\xi + v)}, \quad v \in (0, \pi/2 - \xi_2).$$

Same technique implies

$$I_3 \sim \lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Next, for I_4, observing the period of $\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}}$ is π, we have

$$\begin{aligned}
I_4 &= \int_{\pi/2}^{3\pi/2} \left[\left(\frac{\sin^2(u-\pi/2)\sin^2(u)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du \\
&\overset{u \to \pi - u}{=} \int_{-\pi/2}^{\pi/2} \left[\left(\frac{\sin^2(u-\pi/2)\sin^2(u)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du \\
&= I_2 + I_3.
\end{aligned}$$

Consequently,

$$I_4 \sim 2\lambda_2^{mp} \cdot \frac{(\pi/2 - \xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1 + \mathcal{O}(m^{-1/2})).$$

Next, we estimate I_5. By $E_{2m-1}(z) = (2m-1)! \sum_{\nu=0}^{2m-2} N_{2m}(\nu+1) z^{\nu}$, we derive that $|E_{2m-1}(z)| \leqslant (2m-1)!$ for $|z| = 1$ and

$$\begin{aligned}
I_5 &= \int_{3\pi/2}^{\infty} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \frac{|E_{2m-1}(e^{2iu})|}{(2m-1)!} \right]^{p/2} du \\
&\leqslant \int_{\pi}^{\infty} \left[\left(\frac{\sin^2(u)}{u} \right)^{2m} u^{-2k} \right]^{p/2} du \leqslant \frac{1}{\pi^{2k}} \int_{\pi}^{\infty} u^{-mp} du \\
&\leqslant \frac{1}{(mp-1)\pi^{2k}} \left(\frac{1}{\pi} \right)^{mp-1}, \quad mp - 1 > 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_1 &= \int_{-\infty}^{-\pi/2} \left[\left(\frac{\sin^2(u-\pi/2)}{u-\pi/2} \right)^{2m} (u-\pi/2)^{-2k} \frac{|E_{2m-1}(e^{2iu})|}{(2m-1)!} \right]^{p/2} du \\
&\leqslant \int_{-\infty}^{-\pi} \left[\left(\frac{\sin^2(u)}{u} \right)^{2m} u^{-2k} \right]^{p/2} du \leqslant \frac{1}{(mp-1)\pi^{2k}} \left(\frac{1}{\pi} \right)^{mp-1}, \quad mp - 1 > 0.
\end{aligned}$$

In summary, we have

$$I_1 \sim I_5 \leqslant \frac{1}{(mp-1)\pi^{2k}} \left(\frac{1}{\pi}\right)^{mp-1}$$

and

$$I_2 \sim I_3 \sim \frac{1}{2} I_4 \sim \lambda_2^{mp} \cdot \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1+\mathcal{O}(m^{-1/2})).$$

Due to $\frac{1}{\pi} \approx 0.31830 \leqslant \lambda_2 \approx 0.69706$, we conclude that

$$\begin{aligned}
\|\widehat{{}_k\psi_m^S}\|_p^p &= \frac{4 \cdot 2^{-2kp}}{\sqrt{2\pi}^p} \cdot 4\lambda_2^{mp} \cdot \frac{(\pi/2-\xi_2)^{-kp}}{2\sqrt{\Lambda_2}} \cdot \frac{\sqrt{\pi}}{\sqrt{mp}} \cdot (1+\mathcal{O}(m^{-1/2})) \\
&= \frac{8}{\sqrt{2\pi}^{p-1}} \cdot \frac{(2\pi-4\xi_2)^{-kp}}{\sqrt{2\Lambda_2 mp}} \cdot \lambda_2^{mp} \cdot (1+\mathcal{O}(m^{-1/2})),
\end{aligned}$$

which completes our proof. □

Proof of Theorem 5.3.5. By definition, ${}_m\psi_m^S(x) = \sum_{\nu=0}^{2m-2} \frac{(-1)^\nu}{2^{m-1}} N_{2m}(\nu+1) N_{2m}(2x-\nu)$. Hence,

$$\begin{aligned}
|\widehat{{}_m\psi_m^S}(\omega)| &= \frac{2^{-2m}}{\sqrt{2\pi}} \left(\frac{\sin(\omega/4)}{\omega/4}\right)^{2m} \frac{|E_{2m-1}(\tilde{z})|}{(2m-1)!} \\
&= \frac{2^{-2m}}{\sqrt{2\pi}} \left(\frac{\sin(\omega/4)}{\omega/4}\right)^{2m} (2\sin(\tilde{\omega}/2))^{2m} \left|\sum_{l=-\infty}^{\infty} \frac{1}{(\tilde{\omega}+2\pi l)^{2m}}\right|,
\end{aligned}$$

where E_{2m-1} is the Euler-Frobenius polynomial, $\tilde{z} = e^{i\tilde{\omega}}$, and $\tilde{\omega} = \pi - \omega/2$. Setting

$u = \tilde{\omega}/2 = \pi/2 - \omega/4$, we obtain

$$\|\widehat{{}_m\psi_m^S}\|_p^p = \frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\mathbb{R}} \left[\left(\frac{\sin(u-\pi/2)}{u-\pi/2} \right)^{4m} \cdot (\sin(u))^{4m} \cdot \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du$$

$$= \frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} \Big(\int_{-\infty}^{-\pi/2} \left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2} \right)^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du$$

$$+ \int_{-\pi/2}^{\pi/4} \left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2} \right)^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du$$

$$+ \int_{\pi/4}^{\pi} \left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2} \right)^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du$$

$$+ \int_{\pi}^{\infty} \left[\left(\frac{\sin(u-\pi/2)\sin(u)}{u-\pi/2} \right)^{4m} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{(u+\pi\ell)^{2m}} \right)^2 \right]^{p/2} du \Big)$$

$$=: \frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} (I_1 + I_2 + I_3 + I_4).$$

Let

$$g(u) := \left(\frac{\sin(u-\pi/2)\sin(u)}{(u-\pi/2)u} \right)^2.$$

Then g is symmetric about $u = \pi/4$ and $g(u) \leqslant g(\pi/4) = 64/\pi^4$. Similarly, using [16, Lemma 3], we have

$$I_2 \sim \int_{-\pi/2}^{\pi/4} (g(u))^{mp} du.$$

Introducing

$$t = t(v) = \ln \frac{g(\pi/4)}{g(\pi/4 - v)}, \quad v \in [0, \frac{3}{4}\pi],$$

we can derive

$$q(t) := \frac{dv}{dt} \sim \frac{\pi}{4}(\pi^2 - 8)^{-1/2} t^{-1/2} (1 + e_1 t^{1/2} + e_2 t + \cdots).$$

Changing the variable $u \to \pi/4 - v$ in I_2 and using Watson's lemma, we deduce

$$\frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} I_2 \sim \frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} [g(\pi/4)]^{mp} \int_0^{\infty} e^{-mpt} q(t) dt$$

$$\sim \frac{1}{\sqrt{2\pi}^{p-1}} \cdot \frac{\pi}{\sqrt{\pi^2 - 8}} \left(\frac{16}{\pi^4} \right)^{mp} \cdot \frac{1}{\sqrt{2mp}} \cdot (1 + \mathcal{O}(m^{-1/2}))$$

It is easily seen that $I_3 = I_2$ due to the symmetry of $g(u)$. Also, by the symmetry, we have $I_1 = I_4$. Using the fact that $|E_{2m-1}(z)| \leqslant (2m-1)!$ for $|z| = 1$, we have

$$\begin{aligned}
\frac{4 \cdot 2^{-2mp}}{(\sqrt{2\pi})^p} I_4 &= \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi}^{\infty} \left(\frac{\sin(u - \pi/2)}{u - \pi/2} \right)^{2mp} \left(\frac{|E_{2m-1}(\tilde{z})|}{(2m-1)!} \right)^p d\omega \\
&\leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi}^{\infty} \left(\frac{\sin(u - \pi/2)}{u - \pi/2} \right)^{2mp} d\omega \leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \int_{\pi/2}^{\infty} \left(\frac{1}{\omega} \right)^{2mp} d\omega \\
&\leqslant \frac{2^{-2mp}}{(\sqrt{2\pi})^p} \frac{1}{2mp-1} \left(\frac{2}{\pi} \right)^{2mp-1} = \frac{1}{2mp-1} \frac{1}{(\sqrt{2\pi})^{p-2}} \left(\frac{1}{\pi^2} \right)^{mp}.
\end{aligned}$$

Noting that $1/\pi^2 \leqslant 16/\pi^4$, we conclude

$$\|\widehat{{}_m\psi_m^S}\|_p^p = \frac{2}{\sqrt{2\pi}^{p-1}} \cdot \frac{\pi}{\sqrt{\pi^2 - 8}} \cdot \frac{1}{\sqrt{2mp}} \cdot \left(\frac{16}{\pi^4} \right)^{mp} \cdot (1 + \mathcal{O}(m^{-1/2})),$$

which completes our proof. □

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Chapter 6

Approximation of almost time and band limited functions by finite Hermite series*

6.1 Introduction

We consider $L^2(\mathbb{R})$-normalized functions which are almost time and band limited; more specifically, there exist (not too large) T, $\Omega > 0$ and (small) positive constants, ε_T and ε_Ω, such that, with $\chi_T := \chi_{[-T,T]}$,

$$\int_{|t|>T} |f(t)|^2 \,\mathrm{d}t \le \varepsilon_T^2 \qquad \text{and} \qquad \int_{|\omega|>\Omega} |(\chi_T f)^\wedge(\omega)|^2 \,\mathrm{d}\omega \le \varepsilon_\Omega^2. \tag{6.1}$$

Here, when $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, its Fourier transform, $g^\wedge$, is defined by

$$g^\wedge(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) e^{-i\omega t} \,\mathrm{d}t, \quad \omega \in \mathbb{R},$$

the transform being extended to all $g \in L^2(\mathbb{R})$ in the usual way.

Recall that the k-th Hermite function, h_k, is given at $t \in \mathbb{R}$ by

$$h_k(t) := (-1)^k \gamma_k e^{\frac{t^2}{2}} \frac{d^k e^{-t^2}}{dt^k}, \quad k = 0, 1, \ldots,$$

where $\gamma_k = \pi^{-\frac{1}{4}} 2^{-\frac{k}{2}} (k!)^{-\frac{1}{2}}$.

We intend to prove, in the context of Hermite functions, an analogue of the "$2\Omega T$ Theorem" for expansions in the prolate spheroidal wave functions of bandwidth $c := \Omega T$, denoted $\psi_{m,c}$, $m = 0, 1, \ldots$. These $\psi_{m,c}$ have been shown to be the $L^2(I)$-normalized eigenfunctions of the compact integral operator, F_c, with

$$(F_c\psi)(t) := \frac{1}{\pi} \int_{-1}^{1} \frac{\sin c(t-s)}{t-s} \psi(s) \,\mathrm{d}s, \quad \psi \in L^2(I),\, t \in I := [-1, 1].$$

*A version of this chapter is a pre-print of the following paper: R. Kerman and S. Spektor. Approximation of almost time and band limited functions by finite Hermite series.

The "$2\Omega T$ Theorem" essentially asserts of the f in (6.1) that

$$f \approx \sum_{0\le m<2\Omega T} d_m \psi^T_{m,c} + \varepsilon_T + \varepsilon_\Omega, \quad |t| \le T,$$

in which $\psi^T_{m,c}(t) := T^{-\frac{1}{2}}\psi_{m,c}\left(\frac{t}{T}\right)$ and $d_m := \int_{-T}^{T} f\psi^T_{m,c}\, dt$ (see [4]).

6.2 The main result

Our result is given in the following

Theorem 6.2.1. *Suppose f is an $L^2(\mathbb{R})$-normalized function satisfying (6.1). Let K be the least positive integer such that*

$$2\,\Omega\,T \le \sqrt{2K+1} + \sqrt{2K+3}.$$

Define

$$h^T_k(t) = T^{-\frac{1}{2}} h_k\left(\frac{t}{T}\right), \quad |t| \le T,$$

and set

$$c_k := \int_{-T}^{T} f(t) h^T_k(t)\, dt, \quad k = 0, 1, \ldots, K.$$

Then, with $S_K f := \sum_{k=0}^{K} c_k h^T_k$, one has

$$\left[\int_{-T}^{T} |f - S_K f|^2\, dt\right]^{\frac{1}{2}} \le \varepsilon_\Omega + \frac{2L}{\pi c},$$

and hence

$$\left[\int_{-\infty}^{\infty} |f - \chi_T S_K f|^2\, dt\right]^{\frac{1}{2}} \le \varepsilon_T + \varepsilon_\Omega + \frac{2L}{\pi c}.$$

Here, $L > 0$ is independent of f, K and T.

Proof. Let $f_T(t) := T^{\frac{1}{2}} f(Tt)\chi_I(t)$, so that $c_k = \int_{-1}^{1} f_T h_k\, dt$, and take

$$\left(\sum\nolimits_K f_T\right)(t) := \sum_{k=0}^{K} c_k h_k(t), \quad t \in \mathbb{R}.$$

J.V. Uspensky has shown that

$$\left(\sum\nolimits_K f_T\right)(t) = (F_c f_T)(t) + (R_K f_T)(t), \quad t \in \mathbb{R},$$

where

$$(R_K f_T)(t) := \frac{1}{\pi c}\int_{-1}^{1} T^{(K)}(t,s) f_T(s)\, ds,$$

moreover, for $s, t \in I$

$$|T^{(K)}(t,s)| \le L,$$

the positive constant L being independent of $K \in Z_+$, as well. See [2, p. 372 (5) and p.377 (14) and (16)].

Now,

$$(F_c f_T)(t) = \sqrt{\frac{2}{\pi}} c \left(\left(\frac{\sin cy}{cy} \right) * f_T \right)(t)^\dagger, \quad t \in \mathbb{R}.$$

whence

$$\begin{aligned}(F_c f_T)^\wedge(\omega) &= \sqrt{\frac{2}{\pi}} c \left(\frac{\sin cy}{cy} \right)^\wedge (\omega) f_T^\wedge(\omega) \\ &= \chi_{(-c,c)}(\omega) f_T^\wedge(\omega), \quad \omega \in \mathbb{R}.\end{aligned}$$

Thus,

$$\begin{aligned}\left[\int_{-T}^{T} |f - S_K f|^2 \, dt \right]^{1/2} &= \left[\int_{-1}^{1} \left| f_T - \sum\nolimits_K f_T \right|^2 \, dt \right]^{1/2} \\ &\le \left[\int_{-\infty}^{\infty} |f_T - F_c f_T|^2 \, dt \right]^{1/2} + \left[\int_{-1}^{1} |R_K f_T|^2 \, dt \right]^{1/2} \\ &\le \left[\int_{-\infty}^{\infty} |f_T - F_c f_T|^2 \, dt \right]^{1/2} + \frac{2L}{\pi c} \\ &= \left[\int_{-\infty}^{\infty} \left| f_T^\wedge - \chi_{(-c,c)} f_T^\wedge \right|^2 \, d\omega \right]^{1/2} + \frac{2L}{\pi c} \\ &= \left[\int_{|\omega| \ge c} |f_T^\wedge|^2 \, d\omega \right]^{\frac{1}{2}} + \frac{2L}{\pi c} \\ &= \left[\int_{|\omega| \ge c} \left| \frac{T^{-\frac{1}{2}}}{\sqrt{2\pi}} \int_{-T}^{T} f(y) e^{-i \frac{\omega}{T} y} \, dy \right|^2 \, d\omega \right]^{\frac{1}{2}} + \frac{2L}{\pi c} \\ &= \left[\int_{|\omega| > \Omega} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_T(y) f(y) e^{-i\xi y} \, dy \right|^2 \, d\xi \right]^{\frac{1}{2}} + \frac{2L}{\pi c} \\ &\le \varepsilon_\Omega + \frac{2L}{\pi c},\end{aligned}$$

as asserted. □

Remark 6.2.2. The inequality $\int_{|\omega| > \Omega} |f^\wedge(\omega)|^2 \, d\omega \le \varepsilon_\Omega^2$ implies

$$\left[\int_{|\omega| > \Omega} |(\chi_T f)^\wedge(\omega)|^2 \, d\omega \right]^{\frac{1}{2}} \le \varepsilon'_\Omega + \varepsilon_T.$$

† The convolution, $g * h$, of g and h in $L^2(\mathbb{R})$ is here defined by $(g * h) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t-s) h(s) \, ds$, $t \in \mathbb{R}$. One has $(g * h)^\wedge(\omega) = g^\wedge(\omega) h^\wedge(\omega)$, $\omega \in \mathbb{R}$. See [3].

So that in (6.1) we can take $\varepsilon_\Omega = \varepsilon'_\Omega + \varepsilon_T$. Indeed,

$$\begin{aligned}
\left[\int_{|\omega|>\Omega} |(\chi_T f)^\wedge(\omega)|^2\, d\omega\right]^{\frac{1}{2}} &\le \left[\int_{|\omega|>\Omega} |f^\wedge(\omega)|^2\, d\omega\right]^{\frac{1}{2}} + \left[\int_{|\omega|>\Omega} |(\chi_{|t|>T} f)^\wedge(\omega)|^2\, d\omega\right]^{\frac{1}{2}} \\
&\le \varepsilon'_\Omega + \left[\int_{\mathbb{R}} |(\chi_{|t|>T} f)^\wedge(\omega)|^2\, d\omega\right]^{\frac{1}{2}} \\
&\le \varepsilon'_\Omega + \left[\int_{\mathbb{R}} |\chi_{|t|>T}(t) f(t)|^2\, dt\right]^{\frac{1}{2}} \\
&= \varepsilon'_\Omega + \varepsilon_T.
\end{aligned}$$

6.3 Appendix

In the Introduction (see Section about Prolate Spheroidal Wave Function) we mentioned known upper bounds for the eigenvalue $\lambda_n(c)$ of the compact integral operator F_c. In the Theorem below we obtain a lower bound.

Theorem 6.3.1. *Let $\lambda_n(c)$, $n = 0, 1, 2, \ldots,$ be the eigenvalues, in decreasing order, of the operator F_c, $c = \Omega T$. Then, for n an odd integer n, with $\frac{2c}{n} << 1$, one has*

$$\lambda_n(c) \ge \left(1 - \frac{c^2}{6\, n^2}\right) \sqrt{n} \left(\frac{2c}{7en}\right)^{n-1}.$$

Proof. According to the Rayleigh-Ritz Principle, we must find an n-dimensional subspace, V, of $L^2(I)$ such that, for all $f \in V$,

$$\int_{-1}^{1} |(F_c f)(t)|^2\, dt \ge \left(1 - \frac{c^2}{6\, n^2}\right)^2 n \left(\frac{2c}{7en}\right)^{2(n-1)} \int_{-1}^{1} f^2\, ds.$$

To this end, define V to be the space of functions f taking on each

$$I_j := \left(-\frac{1}{n}, \frac{1}{n}\right) + \frac{2j}{n}, \quad j = -\frac{n+1}{2}, \ldots, 0, \ldots, \frac{n-1}{2},$$

a constant value, f_j, so that

$$f = \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j \chi_{I_j}.$$

We have then to show that whenever $\int_{-1}^{1} f(s)^2\, ds = 1$, or $\sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j^2 = \frac{n}{2}$, there holds

$$\int_{-1}^{1} (F_c f)(t)^2\, dt \ge \left(1 - \frac{c^2}{6\, n^2}\right)^2 n \left(\frac{2c}{7en}\right)^{2(n-1)},$$

which, by Parseval's identity, amounts to

$$\int_{-c}^{c} |\widehat{f}|^2 \, d\xi \geq \left(1 - \frac{c^2}{6\, n^2}\right)^2 n \left(\frac{2c}{7en}\right)^{2(n-1)}.$$

But,

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(t) e^{-i\xi t} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j \int_{-\frac{1}{n}+\frac{2j}{n}}^{\frac{1}{n}+\frac{2j}{n}} e^{-i\xi t} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j e^{-2ij\frac{\xi}{n}} \int_{-\frac{1}{n}}^{\frac{1}{n}} e^{-i\xi t} \, dt \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin \frac{\xi}{n}}{\frac{\xi}{n}} \frac{1}{n} \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j e^{-2ij\frac{\xi}{n}}.
\end{aligned}$$

So,

$$\begin{aligned}
\int_{-c}^{c} |\widehat{f}(\xi)|^2 \, d\xi &\geq \frac{2}{\pi} \left(1 - \frac{c^2}{6n^2}\right)^2 \frac{1}{n} \int_{-c}^{c} \left| \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j e^{-2ij\frac{\xi}{n}} \right|^2 d\xi \\
&= \frac{1}{\pi} \left(1 - \frac{c}{6n^2}\right)^2 \int_{-\frac{2c}{n}}^{\frac{2c}{n}} \left| \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j e^{ij\xi} \right|^2 d\xi \\
&\geq \frac{1}{\pi} \left(1 - \frac{c^2}{6n^2}\right)^2 \left(\frac{\frac{4c}{n}}{14e}\right)^{2(n-1)} \int_{-\pi}^{\pi} \left| \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j e^{ij\xi} \right|^2 d\xi \\
&= \frac{1}{\pi} \left(1 - \frac{c^2}{6n^2}\right)^2 \left(\frac{2c}{7en}\right)^{2(n-1)} 2\pi \sum_{j=-\frac{n+1}{2}}^{\frac{n-1}{2}} f_j^2 \\
&= \left(1 - \frac{c^2}{6n^2}\right)^2 n \left(\frac{2c}{7en}\right)^{2(n-1)},
\end{aligned}$$

where we have used an inequality from [1] in the third step. □

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Chapter 7

Concluding Discussion

7.1 Concluding Remarks

The present thesis has deepened our understanding of sinc functions, splines and wavelets and their use in Functional Analysis, Approximation Theory and Asymptotic Geometric Analysis. We developed our knowledge through study of Ball's integral inequality and the Khinchine type inequality. In particular, we used splines to prove Ball's integral inequality. Also, we developed a method by which one can compute all terms in the asymptotic expansion of the integral in Ball's inequality. We used various topics in mathematics to provide different techniques for proving Khinchine type inequality. We involved Probability Theory, Graph Theory, Theory of Permutations and such notions as Lévy family and chaining argument. We also have been interested in conditions for boundness of the norm of a matrix and proved a quantitative version of the known result in limiting case. With the help of Bernstein type inequality we have been able to study the asymptotic behavior of a wavelet coefficients for both the family of Daubechies orthonormal wavelets and the family of semiorthogonal spline wavelets. Finally, we proved that an almost time and bandlimited function is well represented by the truncation of its expansions in the Hermite basis.

The focus has essentially been on the connections between various areas of mathematics. In that regard, our results establish more connections between Functional Analysis, Approximation Theory and Asymptotic Geometric Analysis, by showing that these areas overlap not only in pure theoretical aspects, but in applications, such as Compressed Sensing, Signal Processing, Wireless Communication.

7.2 Directions for Future Work

Below we list some questions and, possibly, new directions of research raised by this work.

7.2.1 Sinc function. Ball's Integral Inequality

Lately, the sinc function and its p-integral, $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^p dx, \quad p \geq 1$, have been applied in Approximation Theory, Numerical Analysis and, indeed, in many computational problems. Given that, it is surprising how little is known about them. A number of open problems regarding their properties have been rased, for example, in [3, 4].

Recently, a breakthrough in study of Ball's integral inequality has been made. H. König and A. Koldobsky in [14] generalized K. Ball's known result (see [2]) that central sections of the n-dimensional cube perpendicular to the vector $(1, 1, 0, \ldots, 0)$ have maximal volume to certain product measures which include Gaussian type measures. Technically, the main difficulty was to generalize Ball's integral inequality for $\left|\frac{\sin x}{x}\right|^p$ to more general settings. They overcame this difficulty in the high dimensional case, which, from the viewpoint of Asymptotic Geometrical Analysis, was sufficient. But, H. König, [15], in order to complete all possible cases in the Ball's integral inequality, raised an open question about that inequality in lower dimensions.

Another idea for generalizing of Ball's inequality is an estimating/calculating an integral of the generalized sinc function (see [20] for definitions, properties and open questions about this function):

$$sinc(n, x) := \Gamma\left(1 + \frac{n}{2}\right) 2^{n/2} \frac{J_{n/2}(x)}{x^{n/2}}, \quad n \in \mathbb{Z},$$

where $J_{n/2}$ is a half-integer Bessel function of the first kind. Note, $sinc(1, x) = \frac{\sin x}{x}$.

We would like to bound from above the integral

$$\int_{-\infty}^{+\infty} (sinc(n, x))^p \, dx, \quad p \geq 1. \tag{7.1}$$

We share our idea on how to find an upper bound of the integral (7.1) with n being an odd integer. Let $n = 2k + 1$, $k \in N$. We calculate the following integral

$$\int_{-\infty}^{+\infty} (sinc(2k + 1, x))^p \, dx, \quad p \in \mathbb{N}, k \in \mathbb{N}. \tag{7.2}$$

Let $f_n(x) = sinc(n, x)$. The inverse Fourier transform of $f_n(x)$ is

$$f_n^{\vee}(t) = \sqrt{\frac{\pi}{2}}\, c_n \,(t^2 - 1)^n [sgn(t - 1) - sgn(t + 1)],$$

where c_n is an absolute constant which changes with n. (One can compute, that for $sinc(3, x), c_3 = \frac{3}{4}$; $sinc(5, x), c_5 = -\frac{15}{16}$; $sinc(7, x), c_7 = \frac{35}{32}$ and so on.) Thus, to bound from above the integral (7.2) is sufficient to calculate

$$\left(\sqrt{\frac{\pi}{2}}\, c_n\right)^p \int_{-\infty}^{+\infty} (t^2 - 1)^{pn} [sgn(t - 1) - sgn(t + 1)]^p \, dt.$$

Note, $[sgn(t - 1) - sgn(t + 1)] = \begin{cases} 0, & |t| \geq 1 \\ -2, & |t| < 1 \end{cases}$. So, essentially, the problem involves breaking up the integral into relevant pieces and looking the signum function in this pieces. We get

$$\int_{-\infty}^{+\infty} (t^2-1)^{pn}[sgn(t-1)-sgn(t+1)]^p\,dt = \int_{-\infty}^{+\infty} (t^2-1)^{pn}(-2)^p\,dt$$
$$= \sqrt{\pi}(-1)^{pn}(-2)^p\frac{\Gamma(pn+1)}{\Gamma(pn+3/2)}.$$

Thus, we have that

$$\int_{-\infty}^{+\infty} (sinc(2k+1,x))^p\ dx \le 2^{p/2}\, c_{2k+1}^p \frac{\Gamma(p(2k+1)+1)}{\Gamma(p(2k+1)+3/2)}.$$

Problem 7.2.1. Can similar techniques be used to calculate or bound the integral (7.1) when n is an even integer; a negative integer?

7.2.2 Khinchine Type Inequality

In this section we talk about various possibilities for generalization of Khinchine type inequality. Along with some ideas we pose open questions, solutions of which can be interesting by themselves as well as can be a good tool for applications. In Chapter 3 we study Khinchine type inequality under assumption that the Rademacher random variables, ε_i, are not independent, precisely, when we have condition that $\sum_{i=1}^{N} \varepsilon_i = 0$. It is naturally to ask now if the inequality would be true under more general assumption on Rademacher random variables, say, if

$$\sum_{i=1}^{N} \varepsilon_i = M, \quad -N \le M \le N. \tag{7.3}$$

For shorter notation, by $\mathbb{E}_M$ we denote an expectation with assumption (7.3).

As before, let $a \in \mathbb{R}^N$ and let $\varepsilon_i, i = 1, \ldots, N$ be independent Rademacher random variables. As usual for $\varepsilon \in \{\pm 1\}^N$ by $\varepsilon_1, \ldots, \varepsilon_N$ we denote coordinates of ε.

Consider the following set

$$\Omega = \left\{ \varepsilon \in \{-1,1\}^N \mid \sum_{i=1}^{N} \varepsilon_i = M \right\} = \left\{ \varepsilon \in \{-1,1\}^N \mid \text{card}\{i : \varepsilon_i = 1\} = m = \left[\frac{M+N}{2}\right] \right\}. \tag{7.4}$$

Thus, for $\varepsilon \in \Omega$ the sequence of its coordinates is a sequence of dependent Rademacher random variables.

For set Ω, defined by (7.4), we put into correspondence the group Π_N of all permutations of set $\{1, ..., N\}$ as

$$\sigma \in \Pi_N \longleftrightarrow A_\sigma = \{\varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le m;\ \varepsilon_i = -1 \text{ if } \sigma(i) > m\}.$$

Define $f : \Pi_N \longrightarrow \mathbb{R}$ by

$$f(\sigma) := \left| \sum_{i=1}^{m} a_{\sigma(i)} - \sum_{i=m+1}^{N} a_{\sigma(i)} \right|, \tag{7.5}$$

where $\sum_{i=1}^{0} a_{\sigma(i)} = 0$ and $\sum_{i=N+1}^{N} a_{\sigma(i)} = 0$.

Note, that $\mathbb{E}_M \left| \sum_{i=1}^N a_i \varepsilon_i \right|^p = \mathbb{E}|f|^p$. Thus, it is enough to estimate p-th moments of f.

Without loss of generality assume that $M \in [0, N]$. Denote $q = N - m$. Consider

$$\begin{aligned} |f(\sigma)|^p &= \left| \sum_{i=1}^{m} a_{\sigma(i)} - \sum_{i=m+1}^{N} a_{\sigma(i)} \right|^p = \left| \sum_{i=1}^{M} a_{\sigma(i)} + \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p \\ &\leq 2^p \left| \sum_{i=1}^{M} a_{\sigma(i)} \right|^p + 2^p \left| \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p \\ &= 2^p |g_M|^p + 2^p |f_M|^p. \end{aligned}$$

We would like to estimate

$$\begin{aligned} (\mathbb{E}|f(\sigma)|^p)^{1/p} &\leq \left(\mathbb{E} \left| \sum_{i=1}^{M} a_{\sigma(i)} \right|^p \right)^{1/p} + \left(\mathbb{E} \left| \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p \right)^{1/p} \\ &= (\mathbb{E}|g_M|^p)^{1/p} + (\mathbb{E}|f_M|^p)^{1/p}. \end{aligned} \tag{7.6}$$

Estimation of $\mathbb{E}|f_M|^p$: We show that $\mathbb{E}|f_M|^p \leq (2p)^{p/2} \|a\|_2$.

Denote $U = \{1, \dots, N\}$. For all $I \subset U$ with cardinality $|I| = M$ and for all $\sigma \in \Pi_N$ consider $\sigma_I : \{M+1, \dots, N\} \longrightarrow U/I$, defined by $\sigma_I(i) = \sigma(i)$. Denote $B_I := \{\sigma_I \,|\, \sigma \in \Pi_N\}$.

Consider

$$\begin{aligned} \mathbb{E}|f_M|^p &= \frac{1}{N!} \sum_{\sigma \in \Pi_N} \left| \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p \\ &= \frac{1}{N!} \sum_{|I|=M} \sum_{\sigma_I \in B_I} \left| \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p. \end{aligned}$$

Applying Proposition 3.1.2, we obtain

$$\begin{aligned} \sum_{\sigma_I \in B_I} \left| \sum_{i=M+1}^{M+q} a_{\sigma(i)} - \sum_{i=M+q+1}^{M+2q} a_{\sigma(i)} \right|^p &\leq (N-M)!(2p)^{p/2} \left(\sum_{i \in I} a_i^2 \right)^{1/2} \\ &\leq (N-M)!(2p)^{p/2} \|a\|_2. \end{aligned}$$

Thus,

$$\mathbb{E}|f_M|^p \le \frac{(N-M)!}{N!}\sum_{i\in I}(2p)^{p/2}\|a\|_2 = (2p)^{p/2}\|a\|_2. \tag{7.7}$$

Estimation of $\mathbb{E}|g_M|^p$:
It is easy to see that $\mathbb{E}|g_M|^p$ is bounded by ℓ_1-norm of the vector a. More precisely,

$$\mathbb{E}|g_M|^p = \mathbb{E}\left|\sum_{i=1}^{M} a_{\sigma(i)}\right|^p \le \mathbb{E}\left|\sum_{i=1}^{N} |a_{\sigma(i)}|\right|^p = \sum_{i=1}^{N}|a_i| = \|a\|_1^p.$$

With such estimate we have

$$(\mathbb{E}|f|^p)^{1/p} \le \|a\|_1 + \sqrt{2p}\|a\|_2. \tag{7.8}$$

This estimate is too weak. In order to get better bound we are going to use notion of Lévy family.

It is easy to see that $g_M : \Pi_N \longrightarrow \mathbb{R}$ is a Lipschitz function with Lipschitz constant $2\|a\|_\infty$. Using Theorem **??** and the bound $\Gamma(x) \le x^{x-1}$ for all $x \ge 1$ (see for example [1]), we obtain

$$\begin{aligned}
\mathbb{E}|g_M - \mathbb{E}g_M|^p = \int_0^\infty \mu_N(|g_M - \mathbb{E}g_M|^p \ge t^p)dt^p &\le 2p\int_0^\infty e^{-t^2/(32N\|a\|_\infty^2)}t^{p-1}dt \\
&\le 4^p\,\Gamma\left(\frac{p}{2}\right)N^{p/2}\|a\|_\infty^p \\
&\le 4^p\,N^{p/2}p^{p/2}\|a\|_\infty^p.
\end{aligned}$$

Thus, we have

$$(\mathbb{E}|g_M|^p)^{1/p} \le \mathbb{E}|g_M| + 4\sqrt{p}\sqrt{N}\|a\|_\infty \le \sqrt{\mathbb{E}|g_M|^2} + 4\sqrt{p}\sqrt{N}\|a\|_\infty. \tag{7.9}$$

Its only left to calculate

$$\mathbb{E}|g_M|^2 = \mathbb{E}\left|\sum_{i=1}^{M} a_{\sigma(i)}\right|^2 = \frac{M(N-M)}{N(N-1)}\sum_{i=1}^{N}a_i^2 + \frac{M(M+1)}{N(N-1)}\left(\sum_{i=1}^{N}a_i\right)^2.$$

With such estimate we have

$$(\mathbb{E}|f|^p)^{1/p} \le \sqrt{\frac{M(N-M)}{N(N-1)}\sum_{i=1}^{N}a_i^2 + \frac{M(M+1)}{N(N-1)}\left(\sum_{i=1}^{N}a_i\right)^2} + 4\sqrt{p}\sqrt{N}\|a\|_\infty + \sqrt{2p}\|a\|_2. \tag{7.10}$$

This estimate is better then (7.7), but the right hand side of (7.10) depend on the number of functions, N.

Problem 7.2.2. It would be nice to obtain the upper bound in (7.10) with constant independent on N.

Another possibility for generalization of the Khinchine type inequality with dependent Rademacher random variables is to consider the case when $a_1, \dots, a_N$ are elements of a Banach space $(X, \|\cdot\|)$. Under assumption that $\sum_{i=1}^{N} \varepsilon_i = 0$, the proof of Theorem 3.1.2 gives the following estimate

$$\left(\mathbb{E}_\Pi \left\| \sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} \right\|^p \right)^{1/p} = \left(\mathbb{E}_\Pi \mathbb{E}_\delta \left\| \sum_{i=1}^{n} \delta_i \, b_{i,\sigma} \right\|^p \right)^{1/p}$$
$$\leq \sqrt{p} \left(\mathbb{E}_\Pi \left(\mathbb{E}_\delta \left\| \sum_{i=1}^{n} \delta_i \, b_{i,\sigma} \right\| \right)^p \right)^{1/p}. \quad (7.11)$$

Problem 7.2.3. Whether one can get better estimate then in (7.11), with the best possibility of Kahane's inequality (see e.g. [11] for the definition of classical Kahane's inequality)?

Problem 7.2.4. What about estimate of type (7.11) under assumption that $\sum_{i=1}^{N} \varepsilon_i = M$, with $-N \leq M \leq N$?

The scalar Khinchine inequality above can be generalized to the case where the coefficients are matrices. In this case one would obtain, so-called, non-commutative Khinchie inequality, which is a powerful tool in Random Matrix Theory. We have to introduce Shatten class norms, first.

Definition 7.2.5. For a matrix Γ we let $\sigma(\Gamma) = (\sigma_1(\Gamma), \dots, \sigma_n(\Gamma))$ be its sequence of singular values. Then, the **Shatten p-norm** is defined as

$$\|\Gamma\|_{S_p} := \|\sigma(\Gamma)\|_p, \quad 1 \leq p \leq \infty.$$

We state now the non-commutative Khinchine inequality for matrix-values Rademacher sums. This inequality was introduced first by F. Lust-Piquard, [17], with unspecified constants. Later, A. Buchholz in [5, 6] provided the optimal constants for the inequality.

Theorem 7.2.6. *Let $\varepsilon_i, i = 1, \dots, N$ be independent Rademacher random variables. Let $A_i, i = 1, \dots, N$ be real (or complex) matrices of the same dimension. Choose $n \in \mathbb{N}$. Then,*

$$\mathbb{E} \left\| \sum_{i=1}^{N} \varepsilon_i A_i \right\|_{S_{2n}}^{2n} \leq \frac{(2n)!}{2^n n!} \max \left\{ \left\| \left(\sum_{i=1}^{N} A_i A_i^* \right)^{1/2} \right\|_{S_{2n}}^{2n}, \left\| \left(\sum_{i=1}^{N} A_i^* A_i \right)^{1/2} \right\|_{S_{2n}}^{2n} \right\}. \quad (7.12)$$

Note, that $A_i A_i^*$ and $A_i^* A_i$ are positive and self-adjoint matrices, so the square roots in (7.12) are well-defined.

Problem 7.2.7. It would be nice to obtain something like inequality (7.12) under assumption that $\sum_{i=1}^{N} \varepsilon_i = M, \quad -N \leq M \leq N.$

7.2.3 Splines and Wavelets

In Chapter 5 of present work we obtain few main results. One of them is a sharp form of Bernstein type inequality for splines (see Theorem 5.2.1). This inequality gives a bound for the norm of the derivative of spline in terms of the norm of spline function itself. This type of estimates are very important in many questions of Functional Analysis and Approximation Theory. In this section we provide some ideas for the future work in this direction.

It is known that splines can be generalized to fractional orders. The construction of polynomial splines was first extended to fractional degrees in [22] by M. Unser and T. Blu. It is interesting to obtain a sharp inequality of Bernstein type for fractional splines and wavelets with fractional derivatives (see e.g. [21] for the definition of fractional derivative). This result can be applied in image compression (to compress roentgenograph images in medicine, for example).

Problem 7.2.8. Whether similar techniques to those in Theorem 5.2.1 can be used to obtain Bernstein type inequality for fractional splines and semiorthogonal spline wavelets with fractional derivatives?

In the Proposition 5.2.2, which is the consequences of Theorem 5.2.1, we obtain a lower bound for the quantity $C_{k,p}(\psi_m^S)$. It is naturally to ask now the question about upper bound of $C_{k,p}(\psi_m^S)$. One of the possible approach is to prove Reverse Bernstein inequality for splines $s \in S_{m,h}$, i.e. to find constant $c_{p,m,k}$, depending on p, m and k, such that $\|\widehat{s^{(k)}}\|_p \geq c_{p,m,k}\|\hat{s}\|_p$. Than, in the spirit of Proposition 5.2.2, one would almost automatically obtain the result.

It turns out that the question about Reverse Bernstein inequality is interesting by itself. We don't know how to proceed with this problem, but one of the good idea would be to check if the techniques for Reverse Bernstein inequality for polynomials would work in order to get such inequality for splines (see e.g. [13] for the reference on the Reverse Bernstein inequality for polynomials).

Problem 7.2.9. To obtain Reverse Bernstein inequality for splines $s \in S_{m,h}$.

7.2.4 Random Matrices

One of the application which uses both the Wavelet Approximation Theory and non-limiting approximation Random Matrix Theory is Compressed Sensing. This theory represents technique for finding sparse solutions to under-determined linear systems. The field exists for a few decades, but lately has caught significant attention. The papers by E. Candes, J. Romberg and T. Tao [7] and by D. Donoho [10] have triggered a lot of research activities after their appearance. The rational for Compressed Sensing is the fact that many signals can be approximated by sparse signals. In other words, real-world- and audio- data can be approximated by an expansion in terms of a suitable basis, which has only a relatively few non-vanishing terms. To obtain a compressed representation one has to compute the coefficients in the basis (for instance wavelet basis) and then keeps only the largest coefficients. When the compressed signal will be recovered only these coefficients will be stored while the rest of them will be substituted by zero.

Compressed Sensing, in order to compress signal, use only a small number of linear and non-adaptive measurements. Each measurement can be represented as an inner product of the signal $x \in \mathbb{R}^N$ and a some vector $\psi_k \in \mathbb{R}^N$ *(or* $\mathbb{C}^N$*)*. With m $(m < N)$ such measurements we may consider the $m \times N$ measurement matrix Γ with column-vectors ψ_k. Then, the sparse recovering problem can be viewed as the recovery of the s-sparse signal $x \in \mathbb{R}^N$ from its measurement vector $y = \Gamma x \in \mathbb{R}^m$ *(or* $\mathbb{C}^m$*)*.

One way to guarantee exact recovery of s-sparse signals, is so called RIP: the restricted isometry property, defined as follows.

For each integers $s = 1, 2, ..,$ define the isometry constant δ_s of a matrix Γ as the smallest number, such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|\Gamma x\|_2^2 \leq (1 + \delta_s)\|x\|_2^2,$$

holds for all s-sparse vectors x. We say that matrix Γ is RIP if δ_s is small for reasonably large s.

Our research in this area fits within the framework of structured random matrices. An important class of structured random matrices is connected with random sampling of functions in certain finite dimensional functional space. We require an orthonormal basis of functions which are uniformly bounded in the L_∞-norm. The most prominent example consists of the trigonometric system [7, 18].

Let $D \subset \mathbb{R}^n$ be endowed with a probability measure ν. Further, let $\psi_1, .., \psi_N$ be an orthonormal system of complex-valued functions on D, that is, for $j, k \in \mathbb{N}$,

$$\int_D \psi_j(t)\overline{\psi_k(t)}d\nu(t) = \delta_{j,k}. \tag{7.13}$$

The orthonormal system will be assumed to be uniformly bounded in L_∞. We consider a vector f of the form

$$f(t) = \sum_{k=1}^{N} x_k \psi_k(t),\ t \in D \tag{7.14}$$

with coefficients $x_1, .., x_N \in \mathbb{C}$.

Let $t_1, .., t_m \in D$ be some points and suppose we are given the measurements

$$y_\ell = f(t_\ell) = \sum_{k=1}^{N} x_k \psi_k(t_\ell),\ \ell = 1, .., m.$$

We may consider the measurement matrix $\Gamma \in \mathbb{C}^{m \times N}$ with entries

$$\Gamma_{\ell,k} = \psi_k(t_\ell),\ \ell = 1, .., m, \quad k = 1, .., N, \tag{7.15}$$

the vector $y = (y_1, .., y_m)^T$ of simple values can be written in the form

$$y = \Gamma x, \tag{7.16}$$

where x is the vector of coefficients in (7.14).

The main goal of compressive sensing is to reconstruct the polynomial f – or equivalently its vector x of coefficients – from the vector of measurements y. For this we assume sparsity. A polynomial f of the form (7.14) is called s-sparse if its coefficient vector x is s-sparse. The problem of recovering an s-sparse polynomial from m measurement values reduces then to solving (7.16) with a sparsity constant, where Γ is the matrix in (7.15). Now, assuming that the points $t_1, .., t_m$ are selected independently at random according to probability measure ν. This means in particular that probability $P(t_\ell \in B) = \nu(B)$, $\ell = 1, .., m$, for a measurable subset $B \subset D$. The matrix Γ in (7.15) becomes then a structured random matrix [19].

One of the example of the bounded orthonormal system which can be used to build a structured random matrix is the system constructed using Haar-Wavelets and Noiselets [8]. Such orthonormal system is potentially useful for image processing applications.

The possibilities for the future work is to consider a basis of functions a-priori different from the Haar system, and to understand whether we can use something similar to noiselets in order to build a structured random matrix. The reason is that the Haar system does not lead to good approximation error rates. Thus, maybe Daubechies wavelets or spline-wavelets would do the job. The good idea in consideration towards this question is to check if the noiselets associated to the Haar basis also works for more general wavelets.

In his work [16] V. Kolev presented a simple approach for orthogonal wavelets in Compressed Sensing. He compared efficient algorithm for different orthogonal wavelets measurement matrices in Compressed Sensing for image processing from scanned photographic plates (SPP). The analysis shows that one of the best choice for image of SPP is the Daubechies wavelets.

In Chapter 5 of present work we compare two families of wavelets–the orthonormal Daubechies wavelets and semiorthogonal spline wavelets. We conclude, that the semiorthogonal spline wavelets gives better approximation. It is naturally to ask now the following applied problem: if algorithm provided in [16] would work when one would weaken condition on the orthogonality of wavelets and consider semiorthogonal spline wavelets, which would give a better result for image quality analysis in Compresse Sensing method.

Problem 7.2.10. Construct a bounded semiorthogonal system (possibly, using semiorthogonal spline wavelets) which can be used to build a structured random matrix.

Problem 7.2.11. To present a method for image compression of SPP which leads to simple compressed sensing algorithm in semiorthogonal spline wavelet domain.

7.2.5 Prolate Spheroidal Wave Function

In this section we talk about various possibilities for future work in the topic of Prolate Spheroidal Wave Function (PSWF).

In Chapter 6 of the present work we prove that a function that is almost time and band limited is well represented by the truncation of its expansions in the Hermite basis (see Theorem 6.2.1). Unlikely, the $2L/(\pi c)$ term appears in the result.

Problem 7.2.12. Can one get rid of the $2L/(\pi c)$ term in Theorem 6.2.1?

Problem 7.2.13. To prove that a function that is almost time and band limited is well represented by the truncation of its expansion in the Legendre basis.

In Section 6.3 we provide a lower bound for the eigenvalue $\lambda_n(c)$ of the operator F_c (see Theorem 6.3.1). We are wondering, if one can get better estimate, meaning estimate closer to the known upper bound (see Theorem 0.0.3 in the Introduction).

Problem 7.2.14. To obtain a better estimate than in Theorem 6.3.1 for the lower bound of $\lambda_n(c)$.

Traditionally, the PSWF have been used to solve various problems from physics and signal processing. Nowadays, more and more techniques and algorithms appears where this function is applied. In particular, PSWF have been used to sample a time-limited and nearly band-limited signal. Also, this function can be applied for sampling theory to reduce the aliasing error of the recovered signal (see e.g. [9]). Recently, PSWF has appeared in the Random Matrix Theory. The work [12] performed Compressed Sensing with a sensing matrix build from the PSWF. The author provided a proof of the Restricted Isometry Property of such sensing matrix and gave an algorithm for the exact recovery of sparse signals.

In our Theorem 6.2.1 we show that the PSWF is well represented in terms of Hermite functions, for which all kinds of properties and estimation techniques are known. Thus, we would like to ask the following question:

Problem 7.2.15. Can one can build the sensing matrix in [12] using representation of the PSWF in terms of Hermite functions?

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