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HIGHER SPIN FIELD INTERACTIONS

by

© AHMAD SHAMALY

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled HIGHER SPIN FIELD INTERACTIONS submitted by AHMAD SHAMALY in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date July 4, 1972

ABSTRACT

A general method of finding first-order wave equations without subsidiary conditions for integer spin and finite mass is described. Spin 0, 1, 2 and 3 are examples that we use to illustrate this method.

We also show that for interacting fields: The spin one theory of Takahashi and Palmer leads to acausal propagation when minimally coupled to an electromagnetic field. The addition of an anomalous magnetic moment (Pauli term) allows one to recover causal propagation. In the case of Rasita-Schwinger spin 3/2 theory it is impossible to obtain causal propagation even when all possible Pauli terms are included. Coupling of a massive spin 1/2 and massive spin 1 fields leads to causal propagation for all nonderivative scalar, pseudo-scalar, vector, pseudo-vector, and the derivative coupling of vector and tensor types. While the derivative coupling of the scalar, pseudo-scalar, pseudo-vector lead always to acausal propagation.

Finally we show that the causal Takahashi-Palmer field leads to the same vacuum polarization as that of the Proca field.

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CHAPTER I

INTRODUCTION

The theory of relativistic wave equations appropriate for the description of fields with arbitrary spin has proceeded along several lines of development. In addition to the requirement of relativistic invariance and the quantum mechanical principle of superposition (i.e. linearity in the states) which is shared by all theories, different theories define the term elementary field according to several postulates.

Generally speaking, these definitions are equivalent to various combination of the following physical postulates: (A) the requirement of a unique rest mass; (B) the requirement of a unique spin; (C) the requirement that either the total energy or the total charge of the field is positive definite.

Requirements (A) and (B) follow from the postulate that the field shall transform under an irreducible representation of the Poincáre group specified by the mass and spin (m, s).[W.] Requirement (A) is reflected in the condition that the field satisfies the Klein-Gordon equation $(\Box + m^2)\psi = 0$[†], while (B) imposes certain conditions

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[†] Throughout this thesis we adopt the units $\hbar = c = 1$.

on the finite dimensional representations of the homogeneous Lorentz group which are used to implement the representations of the Poincaré group. The third requirement (C) comes from the q-number theory and is necessary if the field is to be quantized. It is related to the connection between spin and statistics. Postulates (B) and (C) are intimately related as shown in[(Co.)].

Among various theories are the Fierz-Pauli[(F.1)], they satisfy all three requirements and the field equations are of second order with supplementary conditions (divergence conditions) on the ψ's in order to insure a unique spin. For spin >1, however, these theories cannot be derived from a simple Lagrangian form (i.e. without supplementary conditions) except through rather involved procedures. Therefore in general, the transition from the free field to the interacting field by the minimal electromagnetic coupling does not lead to algebraically consistent field equations when the spin is >1.

Another theory which has been proposed is the Harish-Chandra[(H.)] spin 3/2 theory. It satisfies the requirements (A) and (C), and is a first order system of differential equations of the Dirac type: $(-i\beta^{\mu}\partial_{\mu}+m)\psi=0$, here the introduction of minimal coupling does not lead to algebraic inconsistency difficulties since there are no other conditions on the field equations. This theory, however, leads to what is called compound spin character (both spin 3/2 and spin 1/2 states appear).

Bhabha[(B.)] introduced a theory which adopts the above linear form to avoid inconsistency of the electromagnetic interaction but rejects the first requirement which is found to be equivalent to rejecting all three requirements for spin >1, because by iterating the equations one does not arrive at the Klein-Gordon equation but rather at an equation of higher order which corresponds to the fact that elementary field so defined has compound spin and mass. Moreover requirement (C) is not satisfied in the general case, so that quantization is not possible - except through the introduction of new procedure, the so called quantization on indefinite metric in the system Hilbert space.[(D.1,2)]

All the above theories, however, agree in the cases of spin 0, 1/2, 1.[(T.1)]

In this thesis we derive first order wave equations with unique integer spin and mass by extending the technique used by Capri[(Ca.1)] for half-odd integer spin fields to integer spin fields. The field equations are first order so as to avoid the algebraic inconsistency of the electromagnetic magnetic interaction encountered in Fierz-Pauli theory. This, however, imposes no restrictions on our theory since any system of higher-order differential equations can be reduced to a first order one. Hence by formulating the theory in a first order system from the start, the subsidiary conditions are incorporated in the equations of motion and do not have to be specified separately thus leading to an algebraically consistent system when minimally coupled to the (e.m) field.

An elementary state in this theory is defined as a state which transforms under an irreducible representation of the Poincáre group specified by the mass and spin (m, s) thus satisfying requirements (A) and (B). Requirement (C) on the other hand is satisfied on account of (B) so that quantization is straightforward in the free field case.[T.1] We also require that the field equations be derivable from a Lagrangian so that a current and an energy-momentum tensor can be defined.

Unlike other recent higher spin theories[Ch.] (†), our formulation yields many different inequivalent theories for a given integer spin. The term inequivalent implies inequivalence under the homogeneous Lorentz group, however as far as the free field is concerned all these different theories for a given mass and spin are physically indistinguishable because they transform under the same irreducible representation of the Poincáre group. When interaction is present, on the other hand, these different theories might describe different physically distinguishable fields depending on the nature of the interaction.

(†) Throughout this work the term higher spin refers to $s \geq 1$.

The abundance of theories for a given integer spin is mainly due to the general form of the homogeneous Lorentz group representation which we introduce. In particular the presence of the irreducible representation (k,k) which is absent in the case of half odd integer spin allows more freedom for a given integer spin as compared with half odd integer spin representation given by Capri.(Ca.1)

Our formulation of integer spin and that of Capri for half odd integer spin constitute a classification of higher spin free fields. When interaction is present, however, many problems arise. The second part of this work is mainly an investigation into the nature of these problems.

It has been known for some time that fields with spin greater than 1 show acausal propagation in the presence of interaction and in some cases the equations of motion cease to be hyperbolic. This difficulty arises from the constraints inherent in the equations of motion and is directly connected with the properties of differential equations as follows: Wave propagation is usually associated with a hyperbolic system of partial differential equations.(Cou.) Such systems of equations allow an initial value problem to be posed on a class of surfaces called space-like with respect to the equations and possess solutions with wave front (or disturbance) that travel along rays of finite velocities (light cones). That is the rays through any point from a ray cone which is determined entirely by the coefficient of the highest derivatives. When coupling for a hyperbolic system occurs only in lower derivatives the ray cone is the same in the interacting as well as the free case. Klein-Gordon and Dirac equations are well known examples of hyperbolic system and the characteristic surfaces remain the light cone when they are coupled through lower order derivatives. For spin greater than one, on the other hand, the free equations of motion are not hyperbolic but constitute instead a degenerate system because they imply the constraints. However, it can be easily shown (existence of the Klein-Gordon devisor)(T.1) that they are equivalent to a system of hyperbolic equations which describe the wave propagation supplemented by constraints which are conserved in time. If low or nonderivative coupling terms are added to the free higher spin Lagrangian the resulting equations of motion may or may not remain equivalent to a hyperbolic system depending on the nature of the interaction. Even when the system remains hyperbolic the propagation velocity may exceed the velocity of light. While in other cases the system may lose hyperbolicity; this makes it unsuitable for the description of wave propagation. The constraints may also propagate in some cases thus increasing the number of degrees of freedom of the field. These difficulties were found by Velo and Zwanziger[V.1,2] when they investigated spin 1, 2 and 3/2.

In this work we have investigated the following fields: The spin 1 theory of an antisymmetric second rank tensor given by Takahashi and Palmer[T.2], the Rarita-Schwinger spin 3/2[Ra.] theory and the Proca field[P.] interacting with a Dirac[4] field. The motivation and results are as follows:

Spin 1

Takahashi and Palmer[T.2] [(†)] recently constructed a Lagrangian which yields an equation of motion for a divergenceless skew-symmetric second rank tensor that describes a massive spin one field. Although the field equations were known for some time, no satisfactory Lagrangian was known which yields the field equations and hence the general quantization procedure[T.1] could not be carried out. The importance of this field stems from the fact that the free field equations are invariant under a gauge transformation of the second kind whereas the other known spin 1 theories are not. In fact the Proca field results when the gauge of this field is fixed. We have considered the following types of interactions:

(†) Hereafter we refer to it as the T.P field.

a) minimal coupling to an external electromagnetic source and b) self coupling of the form $q(\psi^{\mu\nu}\psi_{\mu\nu})^2$. For a) we found that the field will always propagate acausally in contrast with the Proca field which is causal.[V.2] Further analysis of the equations indicates that this behavior is due to a different intrinsic magnetic moment. Therefore we repeated the calculations with an added arbitrary Pauli term and found that the propagation will always be causal if the added Pauli term has a definite fixed strength. This results from the fact that since all spin one theories are equivalent in the free case they will generally only differ by Pauli terms when minimally coupled to the electromagnetic field. So if any spin one theory is causal when coupled minimally the other theories will also be causal provided that we add the proper Pauli terms. In b) we found that the self coupling of the form $q(\psi^{\mu\nu}\psi_{\mu\nu})^2$ will give rise to exactly the same difficulty as that of the Proca field[V.2], in fact the dependence of the propagation on the field strength is exactly the same as that for the Proca field.

Spin 3/2

It has been shown[V.1] that when the Rarita-Schwinger spin 3/2 field is coupled minimally to an external electromagnetic field, then, no matter how weak the field is there will always be, in addition to the light cones, space-like characteristic surfaces, for which the field propagates acausally. Analysis of this behavior shows that the nature of the difficulty lies in the intrinsic magnetic dipole moment of the spin 3/2 field. This situation is analogous to that of the T.P spin 1 field. Hence, it is interesting to know whether propagation can be made causal by adding appropriate Pauli terms. For if one succeeds in making the field propagate causally by the addition of Pauli terms (anomalous dipole moment) etc., then there must exist at least one spin 3/2 theory which is causal when minimally coupled to the electromagnetic field without any Pauli terms. On the other hand if by adding the most general Pauli terms along with the minimal coupling, fails to yield causal propagation then there exists no spin 3/2 theory which is causal when minimally coupled to the (e.m) field. We have calculated the propagation with the most general Pauli terms and found that there is no way of making the Rarita-Schwinger field causal. Hence, we conclude that there exists no spin 3/2 theory which is causal when minimally coupled to an external electromagnetic field. We repeat our argument. Because all spin 3/2 theories are equivalent up to Pauli terms when minimally coupled, and if one theory cannot be made causal by the most general Pauli terms then there exists no spin 3/2 theory which is causal when minimally coupled to the electromagnetic field.

Proca field and Dirac field

It is physically interesting to investigate the propagation of spin 1 Proca field when coupled to a spin 1/2 Dirac field, as would be the case in an intermediate vector boson theory of weak interactions and other theories when spin 1 Proca field interacts with a spin 1/2 Dirac field. To this end we have calculated the characteristic surfaces for various types of coupling and found the following results: In direct coupling of scalar, pseudo-scalar, vector and pseudo-vector type all these couplings are causal. For the derivative coupling on the other hand we found that only the vector and tensor couplings are causal, while the scalar, pseudo-scalar and pseudo-vector show acausal behaviors because the characteristic surfaces in this case depend on both the fields strength and their derivatives. Even the most general combination of these interactions (i.e. taken all at once) fails to make the acausal ones with any combination of the coupling constants causal.

Vacuum polarization

We have indicated above, when discussing the T.P field, that it will propagate causally when minimally coupled to the (e.m) field provided a fixed dipole moment interaction is added to the minimally coupled Lagrangian[†].

(†) This is equivalent to adding a four-divergence to the free T.P Lagrangian as shown in a later chapter.

On the other hand the Proca field is causal when coupled minimally even when an arbitrary dipole moment interaction is added to it.(V.2) The T.P field is gauge invariant while the Proca field is not. Thus the advantage of gauge independence restricts the T.P field to a fixed dipole moment while the disadvantage of noninvariance under a gauge transformation of the second kind allows the Proca field to have an arbitrary dipole moment without disturbing its propagation characteristics. It is interesting to see whether, the minimally coupled Proca field and the minimally coupled causal T.P field, will yield the same physical quantities or not. In the free case the two fields are indistinguishable, while in the interacting case one does not know à priori whether they are indistinguishable or not. An interesting and simple physical quantity to investigate is the vacuum polarization. Here we have calculated the vacuum polarization of the minimally coupled causal T.P field and found that it is identical with that of the minimally coupled Proca field. Thus to this extent the two interacting fields are again physically indistinguishable.

CHAPTER II

FIRST ORDER WAVE EQUATIONS FOR INTEGRAL SPIN

1. Introduction

If one desires to have a reasonable theory of higher spin, it is necessary to be able to describe interactions at least formally. The most important and best understood interaction is the electromagnetic. Wave equations with subsidiary conditions lead to consistency difficulties when minimally coupled to an electromagnetic field. It is best, therefore, to avoid the subsidiary conditions from the start.

First order wave equations for half-odd integer spin without subsidiary conditions have been given.[Ca.1] The method of derivation[Ca.1] is based on finding a representation, D, of the homogeneous Lorentz group, under which the field of a particle with a given mass and half-odd integer spin, transforms. The form of D is chosen in such a manner as to insure the existence of a Lagrangian and hence it is possible to define a current and an energy momentum tensor.

In this chapter we extend the method of[Ca.1] to the case of integer spin. We discuss the general theory and give the representation D of the homogeneous Lorentz group under which integer spin fields transform followed by the construction of the β matrices.

2. General Theory

We are concerned with finding equations of the form

$$(\beta_\mu p^\mu - m)\psi = 0 \qquad \text{II.1}$$

that are form-invariant under the Lorentz transformations, irreducible, derivable from a Lagrangian, and whose solutions transform according to mass m spin s representations of the inhomogeneous Lorentz group. We shall now discuss each of these requirements in turns.

(i) Form invariance of equation II.1 demands that if ψ transforms under Lorentz transformation according to

$$\psi(x) \longrightarrow \psi'(x') = D(\Lambda)\psi(\Lambda^{-1} x')$$

then the β's satisfy

$$D(\Lambda)^{-1}\beta^\mu D(\Lambda) = \Lambda^\mu{}_\nu \beta^\nu \quad . \qquad \text{II.2}$$

In terms of generators this reads

$$[\beta^\mu, M^{\rho\sigma}] = g^{\mu\sigma}\beta^\rho - g^{\mu\rho}\beta^\sigma \qquad \text{II.3}$$

so that

$$\beta^k = [\beta^o, M^{k,o}] \qquad k=1,2,3 \ . \qquad \text{II.4}$$

This shows that β^k are defined in terms of β^o and M^{ko}. Bhabha[(B.)] found the most general solutions of equation II.3 in terms of certain matrices $u^\alpha(k)$, $v^\beta(k)$ first introduced by Dirac[D.3] and further studied by Fierz.[F.2] These matrices are useful in reducing the representation $\mathcal{D}^{(k)} \otimes \mathcal{D}^{(\frac{1}{2})}$ of the rotation group into a direct sum of irreducible parts $\mathcal{D}^{(k+\frac{1}{2})} \oplus \mathcal{D}^{(k-\frac{1}{2})}$. The similarity transformation which does this reduction is

$$U = (2k+1)^{-\frac{1}{2}} \left\| \begin{matrix} u^1(k+\frac{1}{2}) & u^2(k+\frac{1}{2}) \\ v^1(k) & v^2(k) \end{matrix} \right\|$$

and II.5

$$U^{-1} = (2k+1)^{-\frac{1}{2}} \left\| \begin{matrix} v_1(k+\frac{1}{2}) & u_1(k) \\ v_2(k+\frac{1}{2}) & u_2(k) \end{matrix} \right\| \quad (-1)^{2k+1}$$

From the condition that U performs the reduction and that U and U^{-1} are inverses we get

$$u^\alpha(k)v_\beta(k) = (-1)^{2k+1}[k\delta_\beta{}^\alpha - J^\alpha{}_\beta(k)]$$

$$v^\alpha(k)u_\beta(k) = (-1)^{2k}\ [(k+\tfrac{1}{2})\delta^\alpha{}_\beta + J^\alpha{}_\beta(k-\tfrac{1}{2})] \qquad \text{II.6}$$

$$u^\alpha(k+\tfrac{1}{2})v_\alpha(k+\tfrac{1}{2}) = v^\alpha(k)u_\alpha(k) = (-1)^{2k+1}\ (2k+1)$$

$$u^\alpha(k+\tfrac{1}{2})u_\alpha(k) = v^\alpha(k)v_\alpha(k+\tfrac{1}{2}) = 0$$

Here $J^\alpha{}_\beta(k)$ is the spinor associated with the vector $\vec{J}$; the generator of rotation

$$J_{21} = -J_{12} = J_z$$

$$J_{22} = J_- = J_x - iJ_y \qquad \text{II.7}$$

$$-J_{11} = J_+ = J_x + iJ_y$$

Choosing

$$u^{\alpha}(k) = u_{\dot{\alpha}}(k) \quad ,$$

$$v^{\alpha}(k) = v_{\dot{\alpha}}(k)^{(\dagger)}, \qquad \text{II.8}$$

and

$$[u^{\alpha}(k)]^{\dagger} = (-1)^{2k+1} v_{\alpha}(k)$$

an explicit representation of $u^{\alpha}(k)$, $v^{\alpha}(k)$ which satisfies the above conditions is

$$||u_1(k)||_{s,r} = (s)^{\frac{1}{2}} \delta_{s-1,r} \qquad ||v^1(k)||_{s,r} = (-1)^{2k+1} \sqrt{r+1}\, \delta_{r+1,s}$$

$$||u_2(k)||_{s,r} = \sqrt{2k-s}\, \delta_{s,r} \qquad ||v^2(k)||_{s,r} = (-1)^{2k+1} \sqrt{2k-r}\, \delta_{r,s}$$

II.9

where r denotes an index which gives the number of index 1 of a spinor and ranges from 0 to 2k-1, s is a similar index ranging from 0 to 2k.

Bhabha has shown that given a representation of the homogeneous Lorentz group of the form

$$D = \mathcal{D}^{(k_1,\ell_1)} \oplus \mathcal{D}^{(k_2,\ell_2)} \oplus \ldots \qquad \text{II.10}$$

(†) The definition of the dotted spinors as well as the representation of the homogeneous Lorentz group are given in the appendix.

then the solutions of equation II.3 are such that the spinor components $\beta^{\alpha\dot{\beta}}$ of β_μ are of the form

$$<(k,\ell)_r|\beta^{\alpha\dot{\beta}}|(k-\tfrac{1}{2},\ell+\tfrac{1}{2})_s> = c_{rs}u^{\alpha}(k) \otimes v^{\dot{\beta}}(\ell+\tfrac{1}{2})$$

$$<(k,\ell)_r|\beta^{\alpha\dot{\beta}}|(k+\tfrac{1}{2},\ell-\tfrac{1}{2})_s> = c_{rs}v^{\alpha}(k+\tfrac{1}{2}) \otimes u^{\dot{\beta}}(\ell) \qquad \text{II.11}$$

$$<(k,\ell)_r|\beta^{\alpha\dot{\beta}}|(k-\tfrac{1}{2},\ell-\tfrac{1}{2})_s> = c_{rs}u^{\alpha}(k) \otimes u^{\dot{\beta}}(\ell)$$

$$<(k,\ell)_r|\beta^{\alpha\dot{\beta}}|(k+\tfrac{1}{2},\ell+\tfrac{1}{2})_s> = c_{rs}v^{\alpha}(k+\tfrac{1}{2}) \otimes v^{\dot{\beta}}(\ell+\tfrac{1}{2}) .$$

Thus $<(k,\ell)_r|\beta^{\alpha\dot{\beta}}|(k',\ell')_s> = 0$ unless $k' = k\pm\frac{1}{2}$ $\ell' = \ell\pm\frac{1}{2}$. Here r,s denote two different irreducible representations $(k,\ell)_r$, $(k',\ell')_s$. Hence the solutions are nonzero only if the representation D is such that for every irreducible representation $\mathcal{D}^{(k,\ell)}$ in D there also occurs at least one irreducible representation $\mathcal{D}^{(k',\ell')}$ such that $|\ell-\ell'| = |k-k'| = \frac{1}{2}$. Two such irreducible representations are said to be linked. If the irreducible representations occurring in D can be split into two or more independently linked sets of representations with no cross-linkage between the sets then the resultant β's as well as equation II.1 are reducible. The coefficients c_{rs} are complex numbers. They satisfy certain properties which will be discussed later.

(ii) Unique mass: Harish-Chandra(H.) has shown that the necessary and sufficient conditions for equation II.1 to yield a representation of mass m (i.e., imply the Klein-Gordon equation) are that the β's satisfy

$$(\beta^\mu p_\mu)^{n+2} = p^2(\beta^\mu p_\mu)^n \qquad \text{II.12}$$

for some integer n. Umezawa and Visconti(U.1) have shown that n = 2q-1 where q is the highest spin contained in the representation under which ψ transforms.

Letting $p = (1,\vec{0})$ we get

$$\beta_o^{2q+1} = \beta_o^{2q-1} \quad . \qquad \text{II.13}$$

From equation II.13 it follows that β_o has eigenvalues 0, ±1. The physical solutions of equation II.1 correspond to the ±1 eigenvalues.

(iii) In order to be able to derive equation II.1 from a Lagrangian and to be able to define a current and energy momentum tensor, it is necessary to have a Hermitianizing matrix η. That is we require that there exists an η such that

$$\eta\beta^{\mu\dagger}\eta^{-1} = \beta^\mu \quad . \qquad \text{II.14}$$

For this we prove the following theorem.(Ca.1)

<u>Theorem</u>: If in II.2 D(Λ) is equivalent to a direct sum of self conjugate and pairs of conjugate representations,

and if each representation occurs with unit multiplicity, then there exists one and only one Hermitianizing matrix.

<u>Proof</u>: Under the conditions of the theorem there exists an S such that

$$S D^{\dagger^{-1}}(\Lambda)\, S^{-1} = D(\Lambda) \quad . \tag{II.15}$$

From equation II.2 we get

$$D^{\dagger}(\Lambda)\beta^{\mu^{\dagger}} D^{\dagger^{-1}}(\Lambda) = \Lambda^{\mu}{}_{\nu}\beta^{\nu^{\dagger}} \quad . \tag{II.16}$$

Combining these, we get

$$D^{-1}(\Lambda)[S\beta^{\mu^{\dagger}}S^{-1}]D(\Lambda) = \Lambda^{\mu}{}_{\nu}[S\beta^{\nu^{\dagger}}S^{-1}] \quad .$$

Now by a theorem of Gårding (G.) and under the conditions of this theorem, this equation has one and only one solution up to factors and equivalence. Therefore there exists a nonsingular matrix B such that

$$S\beta^{\mu^{\dagger}}S^{-1} = cB^{-1}\beta^{\mu}B \quad . \tag{II.17}$$

Since β^{o} has at least one nonzero eigenvalue and since eigenvalues are invariant under similarity transformations, $|c| = 1$ so $c = e^{i\theta}$ and therefore

$$(BS)\beta^{\mu^{\dagger}}(BS)^{-1} = e^{i\theta}\beta^{\mu} \ .$$

Redefining

$$\gamma^{\mu} = e^{i\frac{\theta}{2}} \beta^{\mu}$$

we get

$$(BS)\gamma^{\mu\dagger}(BS)^{-1} = \gamma^{\mu}.$$

So

$$\eta = BS \qquad \text{II.18}$$

is the required unique Hermitianizing matrix for the matrices γ^{μ}.

The method of proof permits us to display η explicitly for the cases considered in the theorem. We need only consider three cases since all others are equivalent to direct sum of cases of this kind.

(a) $D(\Lambda) = \mathcal{D}^{(\ell,o)} \oplus \mathcal{D}^{(o,\ell)}$

Then

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{II.19}$$

since by A.10

$$\mathcal{D}^{(\ell,o)\dagger^{-1}} = \mathcal{D}^{(o,\ell)} .$$

(b) $D(\Lambda) = \mathcal{D}^{(\ell,k)}(\Lambda) \oplus \mathcal{D}^{(k,\ell)}$ $\qquad \ell \neq k$

Now by A.9

$$\mathcal{D}^{(\ell,k)} = \mathcal{L}^{(\ell,o)} \otimes \mathcal{L}^{(o,k)} .$$

Therefore,

$$\mathcal{D}^{(\ell,k)\dagger^{-1}} = \mathcal{D}^{(o,\ell)} \otimes \mathcal{D}^{(k,o)}$$

$$= P_{k\ell}\, \mathcal{D}^{(k,o)} \otimes \mathcal{D}^{(o,\ell)}\, P_{k\ell}^{-1}$$

$$= P_{k\ell}\, \mathcal{D}^{(k,j)}\, P_{k\ell}^{-1}$$

where $P_{k\ell}$ is a permutation matrix.[(M.)] Then

$$\eta = \begin{pmatrix} 0 & P_{k\ell} \\ P_{\ell k} & 0 \end{pmatrix} \qquad \text{II.20}$$

The $P_{\ell k}$ have the properties

$$P_{\ell k}^{-1} = P_{k\ell} \qquad \text{II.21}$$

$$P_{\ell k}^{\dagger} = P_{k\ell} \quad . \qquad \text{II.22}$$

Therefore $\eta^2 = 1$, $\eta^{\dagger} = \eta$.

(c) $D(\Lambda) = \mathcal{D}^{(k,k)}$

Then

$$\eta = P_{kk} \quad . \qquad \text{II.23}$$

In case the representations do not occur with unit multiplicity as specified in the theorem, Gårding's theorem still applies. In this case, however, equation II.17 has several solutions and we cannot à priori specify the Hermitianizing matrix.

(iv) For equation II.1 to yield solutions of pure spin s, it is necessary that operating on solutions of equation II.1 in the rest frame yields

$$J^2 = s(s + 1) \quad . \qquad \text{II.24}$$

That is, if

$$\beta_o \psi_o = \pm\psi_o \qquad \text{II.25}$$

then

$$J^2 \psi_o = s(s + 1)\psi_o \quad . \qquad \text{II.26}$$

This can be guaranteed by requiring that

$$J^2 E = s(s + 1)E \qquad \text{II.27}$$

where

$$E = E_+ + E_- \qquad \text{II.28}$$

and $E_\pm$ are the projections onto spaces spanned by the eigenvectors of β_o corresponding to the eigenvalues ±1. The required projections are

$$E_\pm = \frac{1}{2}(1 \pm \beta_o)\beta_o^{2s} \quad . \qquad \text{II.29}$$

For integer spin, so that

$$E = \beta_o^{2s} \quad . \qquad \text{II.30}$$

<u>Proof</u>: That $E_\pm$ are projection operators is clear once we realize that β_o is Hermitian with respect to the inner product

$$(u,u) = u^\dagger \eta\, u \quad . \qquad \text{II.31}$$

Thus we need only show that they are the desired projection operators. But if $\beta_o u = \pm u$, then

$$E_\pm u = \frac{1}{2}\,(1 \pm \beta_o)\beta_o^{2s} u = (\pm 1)^{2s}\,\frac{1}{2}(1+1)u = u \ .$$

Also if $E_\pm u = u$, then

$$\beta_o u = \beta_o E_\pm u = \beta_o\,\frac{1}{2}(1 \pm \beta_o)\beta_o^{2s} u = \frac{1}{2}(\beta_o \pm \beta_o^2)\beta_o^{2s} u$$

$$= \pm\,\frac{1}{2}(1 \pm \beta_o)\beta_o^{2s} u = \pm E_\pm u = \pm u \ .$$

3. <u>The representation D and the β matrices</u>

We assume that the field transforms under Lorentz transformation according to

$$D = \bigoplus_{j=0}^{s+1} n_j^{(j)}(j,0) \oplus n_j^{(j-1)}(j-\tfrac{1}{2},\tfrac{1}{2}) \oplus \ldots \oplus$$

$$n_j^{(o)}(\tfrac{j}{2},\tfrac{j}{2}) \oplus \ldots \oplus n_j^{(j-1)}(\tfrac{1}{2},j-\tfrac{1}{2}) \oplus n_j^{(j)}(0,j). \qquad \text{II.32}$$

Each bracket (j,k) in the above representation refers to an irreducible representation of the homogeneous Lorentz group. $n_j^{(\ell)}$ are integers which represent the multiplicity of the representation $(\frac{j-\ell}{2}, \frac{j+\ell}{2})$ and the conjugate i.e., $(\frac{j+\ell}{2}, \frac{j-\ell}{2})$, is to be determined in such a manner that β_o satisfies the mass and spin conditions. According to the theorem proved in section (iii) the above form of D guarantees the existence of a Hermitianizing matrix η. Therefore, equation II.1 is derivable from a Lagrangian and a current and an energy momentum tensor can be defined.

To determine the coefficients c_{rs} of equation II.11, we transform to a basis in which J^2 is diagonal, the $\beta^{\alpha\dot{\beta}}$ will be labelled.

$$<(j,m)_r|\beta^{\alpha\dot{\beta}}|(j',m')_s> .$$

Wild[(Wi.)] has explicitly written down such a transformations in terms of the $u^{\alpha}(k)$ and $v^{\beta}(k)$. The result of applying this transformation to β_o yields

$$<(k,\ell)_r|\beta_o|(k-\tfrac{1}{2},\ell+\tfrac{1}{2})_s> \longrightarrow c_{rs}[(k+j-\ell)(j+\ell-k+1)]^{\frac{1}{2}}\ \delta_{jj'} \otimes 1$$

$$<(k,\ell)_r|\beta_o|(k-\tfrac{1}{2},\ell-\tfrac{1}{2})_s> \longrightarrow c_{rs}(-1)^{k+\ell+j}[(k+\ell-j)(k+\ell+j+1)]^{\frac{1}{2}}\delta_{jj'}\otimes 1$$

$$<(k,k+\tfrac{1}{2})_r|\beta_o|(k+\tfrac{1}{2},k)_s> \longrightarrow c_{rs}(-1)^{\{j\}+1}(j+\tfrac{1}{2})\,\delta_{jj'} \otimes 1 \quad .$$

II.33

Here {j} means the integer part of j and 1 represents a $(2j+1)\times(2j+1)$ unit matrix, $|k-\ell| \le k+\ell$, $|k-\ell| \le j' \le k+\ell$. All other components that we need are obtained from

$$\langle k,\ell|\beta_o|k',\ell'\rangle = (-1)^{2k+2\ell}\langle k',\ell'|\beta_o|k,\ell\rangle$$

II.34

$$\langle k,\ell|\beta_o|k',\ell'\rangle = -\langle \ell,k|\beta_o|\ell',k'\rangle \quad .$$

These formulas do not apply for $\ell=k$ or $\ell'=k'$. For this case we have the formula

$$\langle(k-\tfrac{1}{2},k+\tfrac{1}{2})_r|\beta_o|(k,k)_s\rangle \longrightarrow c_{rs}(-1)^{2k+j-1}[j(j+1)]^{\frac{1}{2}}\delta_{jj'} \otimes 1.$$

II.35

Furthermore if r,t and s,u denote two pairs of inequivalent irreducible representations of the proper Lorentz group, such that r goes into t and s into u by reflexion then

$$c_{rs} = -c_{tu} \quad . \qquad \text{II.36}$$

In case the representation is $(k,k)_r$ then

$$c_{rs} = \pm(-1)^{2k}\, c_{ru} \qquad \text{II.37}$$

where the sign is fixed by the η matrix. These conditions are a consequence of requiring the existence of a parity operator. This is tantamount to requiring an η matrix since as Harish-Chandra[H.] has shown

$$\eta = AP$$

where P is the parity operator and A is a non-singular matrix depending on the specific representation chosen. Thus in one fixed representation we can get $\eta = cP$ where c is a constant.

Now since matrix elements with different j are not linked, β_o can be written in block diagonal form such that different blocks correspond to different j. According to our D, j lies in the range $s+1 \geq j \geq 0$ so that the corresponding β_o is:

$$\beta_o = \begin{pmatrix} \beta_o(s+1) & & & & \\ & \beta_o(s) & & & \\ & & \beta_o(s-1) & & \\ & & & \ddots & \\ & & & & \beta_o(0) \end{pmatrix} \qquad \text{II.38}$$

For this form of β_o the square of the generators of rotations J^2 takes the form

$$J^2 = \begin{pmatrix} (s+1)(s+2) & & & & & \\ & \ddots & & & & \\ & & s(s+1) & & & \\ & & & (s-1)s & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad \text{II.39}$$

We select the arbitrary parameters in β_o so that

$$\beta_o^{2(s+1)+1}(s) = \beta_o^{2(s+1)-1}(s)$$

and II.40

$$\beta_o^{2(s+1)-1}(j) = 0 \qquad j \neq s$$

Here q, the highest spin contained in D, is equal to s+1. If, however, the $\eta_{s+1}^{(\ell)}$ are identically zero then q=s and equations II.40 become

$$\beta_o^{2s+1}(s) = \beta_o^{2s-1}(s)$$

and II.41

$$\beta_o^{2s-1}(j) = 0 \qquad j \neq s$$

Equations II.40 or II.41 imply the mass and spin conditions.

CHAPTER III

EXAMPLES

1. Introduction

In this chapter we apply the theory developed in Chapter II to construct explicitly the β matrices for spin 0, 1, 2 and 3. We find that there exist several theories for a given higher spin field i.e. $s \geq 1$. For spin zero we find the Duffin-Kemmer theory. For spin one we consider among many allowable theories the following: 10, 14, 20 and 26 component theories. The 10 component theory is identical with the Duffin-Kemmer spin 1 theory while the other three are new spin one theories that have not been constructed before. Among many possible theories for spin 2, we only consider the 30 and 35 component theories. For spin 3, only the 70 component theory is considered.

2. Spin Zero (Duffin-Kemmer)

According to equation II.32 the representation under which spin zero transforms is

$$D = n_o^{(o)}(0,0) \oplus n_1^{(1)}(1,0) \oplus n_1^{(o)}(\tfrac{1}{2},\tfrac{1}{2}) \oplus n_1^{(1)}(0,1). \quad \text{III.1}$$

Now since the representations (1,0) and (0,1) do not contain spin zero it follows that $n_1^{(1)}$ is identically zero. Furthermore we choose $n_o^{(o)} = n_1^{(o)} = 1$, so finally:

$$D = (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,0) \ . \qquad \text{III.1a}$$

In spinor form

$$\beta^{\alpha\dot{\beta}} = \begin{bmatrix} 0 & c_1 u^{\alpha}(\tfrac{1}{2})\, u^{\dot{\beta}}(\tfrac{1}{2}) \\ c_1 v^{\alpha}(\tfrac{1}{2})\, v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \end{bmatrix}$$

where the products inside the matrix are direct products. After the transformation due to Wild and omitting $\delta_{mm'}$ we have

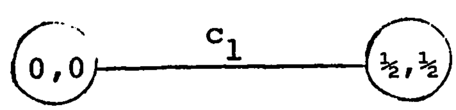
$$\beta_o = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -\sqrt{2}\, c_1 \\ 0 & -\sqrt{2}\, c_1 & 0 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{2}, \tfrac{1}{2} \; \} \; j = 1 \\ \hline \left.\begin{array}{l} \tfrac{1}{2}, \tfrac{1}{2} \\ 0, 0 \end{array}\right\} j = 0 \end{array}$$

Equation II.40 in this case give

$$\beta_o^3(0) = \beta_o(0) \ .$$

This is satisfied if $c_1 = \pm\frac{1}{\sqrt{2}}$.

Schematically the components of the field belonging to different irreducible representations are linked by the β matrices as follows:



Using the results of Chapter II, the η matrix is readily constructed. Thus for this representation of spin zero the η matrix is

$$\eta = \left(\begin{array}{c|c} P_{\frac{1}{2}\frac{1}{2}} & 0 \\ \hline 0 & -P_{oo} \end{array} \right)$$

where P_{kj} is a permutation matrix.

3. Spin 1

According to equation II.32 spin one transforms under the representation:

$$D = n_o^{(o)}(0,0) \oplus n_1^{(1)}(1,0) \oplus n_1^{(o)}(\tfrac{1}{2},\tfrac{1}{2}) \oplus n_1^{(1)}(0,1)$$
$$\oplus\, n_2^{(2)}(2,0) \oplus n_2^{(1)}(\tfrac{3}{2},\tfrac{1}{2}) \oplus n_2^{(o)}(1,1) \oplus n_2^{(1)}(\tfrac{1}{2},\tfrac{3}{2})$$
$$\oplus\, n_2^{(2)}(0,2) \quad . \qquad \text{III.2}$$

There are many spin one theories according to this representation. Various spin one theories may be constructed by the proper choice of $n_j^{(\ell)}$. In this chapter we will consider four of these spin one theories.

a) 10-component theory (Duffin-Kemmer):

If we choose

$$n_o^{(o)} = n_2^{(2)} = n_2^{(1)} = n_2^{(o)} = 0$$

and

$$n_1^{(1)} = n_1^{(o)} = 1$$

then D becomes

$$D = (1,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \quad . \qquad \text{III.2a}$$

The corresponding β's in spinor form are:

$$\beta^{\alpha\dot{\beta}} = \begin{pmatrix} 0 & c_1 u^{\alpha}(1) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \\ c_1 v^{\alpha}(1) u^{\dot{\beta}}(\tfrac{1}{2}) & & c_1 u^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(1) \\ 0 & c_1 v^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(1) & 0 \end{pmatrix}$$

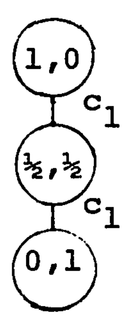
After the transformation due to Wild we have

$$\beta_o = \left(\begin{array}{ccc|c} 0 & \sqrt{2}c_1 & 0 & \\ \sqrt{2}c_1 & 0 & \sqrt{2}c_1 & \\ 0 & \sqrt{2}c_1 & 0 & \\ \hline & & & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} 1,0 \\ \tfrac{1}{2},\tfrac{1}{2} \\ 0,1 \end{array}\right\} j = 1 \\ \tfrac{1}{2},\tfrac{1}{2}\} \; j = 0 \end{array}$$

Since $q = s$, β_o must satisfy

$$\beta_o^3 = \beta_o \quad .$$

This condition gives $c_1 = \pm\tfrac{1}{2}$. Schematically the field components for different irreducible representations are linked as shown:



The η matrix for this theory is

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -P_{\frac{1}{2}\frac{1}{2}} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

b) 14-component theory:

If we choose

$$n_1^{(1)} = n_2^{(2)} = n_2^{(1)} = 0$$

and

$$n_o^{(o)} = n_1^{(o)} = n_2^{(o)} = 1$$

then D becomes

$$D = (1,1) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,0) \;. \qquad \text{III.2b}$$

Corresponding to this

$$\beta^{\alpha\dot{\beta}} = \begin{pmatrix} 0 & c_1 u^{\alpha}(1) u^{\dot{\beta}}(1) & 0 \\ c_1 v^{\alpha}(1) v^{\dot{\beta}}(1) & 0 & c_2 u^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(\tfrac{1}{2}) \\ 0 & c_2 v^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \end{pmatrix}$$

After Wild's transformation β_o becomes

$$\beta_o = \left(\begin{array}{c|cc|ccc} 0 & & & & & \\ \hline & 0 & -2c_1 & & & \\ & -2c_1 & 0 & & & \\ \hline & & & 0 & \sqrt{6}c_1 & 0 \\ & & & \sqrt{6}c_1 & 0 & -\sqrt{2}c_2 \\ & & & 0 & -\sqrt{2}c_2 & 0 \end{array}\right) \begin{array}{l} 1,1 \;\}\; j = 2 \\ \left.\begin{array}{l} 1,1 \\ \frac{1}{2},\frac{1}{2} \end{array}\right\} j = 1 \\ \left.\begin{array}{l} 1,1 \\ \frac{1}{2},\frac{1}{2} \\ 0,0 \end{array}\right\} j = 0 \end{array}$$

Since $q = s + 1 = 2$, β_o must satisfy

$$\beta_o^5(1) = \beta_o^3(1)$$

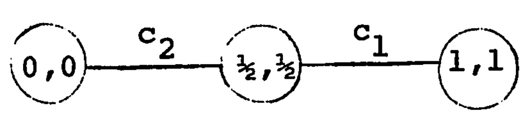
$$\beta_o^3(2) = 0$$

$$\beta_o^3(0) = 0 \quad .$$

The first condition is satisfied for $c_1 = \pm\frac{1}{2}$. The second condition is already satisfied while the third condition is satisfied if $6c_1^2 + 2c_2^2 = 0$ which gives

$$c_2 = \pm\, i \,\frac{\sqrt{3}}{2} \quad .$$

The schematic diagram for this theory is as shown:



The η matrix for this theory is

$$\eta = \begin{pmatrix} P_{11} & & \\ & -P_{\frac{1}{2}\frac{1}{2}} & \\ & & -P_{oo} \end{pmatrix}$$

c) 20-component spin theory:

For this theory we select

$$n_2^{(2)} = n_2^{(1)} = 0$$

and

$$n_o^{(o)} = n_1^{(1)} = n_1^{(o)} = n_2^{(o)} = 1$$

so that D is:

$$D = (1,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \oplus (1,1) \oplus (0,0). \qquad \text{III.2c}$$

Corresponding to this we have

$$\beta^{\alpha\dot{\beta}} = \begin{pmatrix} 0 & c_1 u^{\alpha}(1)\, v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & 0 \\ c_1 v^{\alpha}(1)\, u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & c_1 u^{\alpha}(\tfrac{1}{2})\, v^{\dot{\beta}}(1) & c_2 v^{\alpha}(1)\, v^{\dot{\beta}}(1) & c_3 u^{\alpha}(\tfrac{1}{2})\, u^{\dot{\beta}}(\tfrac{1}{2}) \\ 0 & c_1 v^{\alpha}(\tfrac{1}{2})\, u^{\dot{\beta}}(1) & 0 & 0 & 0 \\ 0 & c_2 u^{\alpha}(1)\, u^{\dot{\beta}}(1) & 0 & 0 & 0 \\ 0 & c_3 v^{\alpha}(\tfrac{1}{2})\, v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & 0 \end{pmatrix}$$

And after Wild's transformation we get

$$\beta_o = \left(\begin{array}{c|cccc|ccc}
0 & & & & & & & \\
\hline
 & 0 & 0 & \sqrt{2}c_1 & 0 & & & \\
 & 0 & 0 & -2c_2 & 0 & & & \\
 & \sqrt{2}c_1 & -2c_2 & 0 & \sqrt{2}c_1 & & & \\
 & 0 & 0 & \sqrt{2}c_1 & 0 & & & \\
\hline
 & & & & & 0 & \sqrt{6}c_2 & 0 \\
 & & & & & \sqrt{6}c_2 & 0 & -\sqrt{2}c_3 \\
 & & & & & 0 & -\sqrt{2}c_3 & 0
\end{array}\right)
\begin{array}{l}
1,1 \;\}\; j=2 \\
\left.\begin{array}{l} 1,0 \\ 1,1 \\ \tfrac{1}{2},\tfrac{1}{2} \\ 0,1 \end{array}\right\} j=1 \\
\left.\begin{array}{l} 1,1 \\ \tfrac{1}{2},\tfrac{1}{2} \\ 0,0 \end{array}\right\} j=0
\end{array}$$

β_o must satisfy

$$\beta_o^3(2) = 0$$

$$\beta_o^5(1) = \beta_o^3(1)$$

$$\beta_o^3(0) = 0 \quad .$$

The first condition is automatically satisfied while the second and third conditions are satisfied if

$$4c_1^2 + 4c_2^2 = 1$$

and

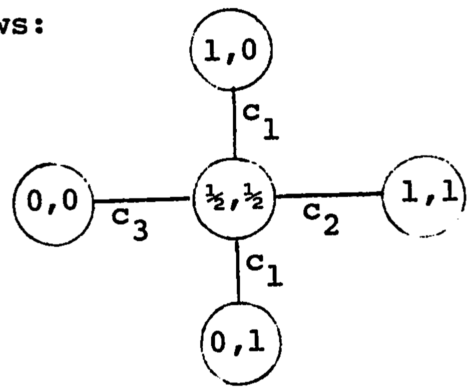
$$6c_2^2 + 2c_3^2 = 0 \quad ,$$

i.e.

$$c_1 = \pm \sqrt{\frac{1}{4} + \frac{c_3^2}{3}}$$

$$c_2 = \pm \frac{i}{\sqrt{3}} \; c_3$$

and c_3 is arbitrary. Schematically, the field components for different irreducible representations are linked as follows:



The η matrix for this theory is:

$$\eta = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & & \\ 0 & -P_{\frac{1}{2}\frac{1}{2}} & 0 & & \\ 1 & 0 & 0 & & \\ \hline & & & \pm P_{11} & 0 \\ & & & 0 & \mp P_{oo} \end{array}\right)$$

Here the upper (lower) signs are taken when c_2 is real (imaginary) respectively.

d) The last spin one theory that we consider is a 26-component theory. This can be arrived at by choosing

$$n_o^{(o)} = n_2^{(o)} = n_2^{(2)} = 0$$

and

$$n_1^{(1)} = n_1^{(o)} = n_2^{(1)} = 1 \ .$$

With this choice D takes the form

$$D = (\tfrac{3}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},\tfrac{3}{2}) \oplus (1,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \ . \qquad \text{III.2d}$$

The corresponding β's are

$$\beta^{\alpha\dot{\beta}} = \begin{pmatrix} 0 & 0 & c_1 u^{\alpha}(\tfrac{3}{2}) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_1 u^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(\tfrac{3}{2}) \\ c_1 v^{\alpha}(\tfrac{3}{2}) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & c_2 u^{\alpha}(1) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 \\ 0 & 0 & c_2 v^{\alpha}(1) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & c_2 u^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(1) \\ 0 & -c_1 v^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(\tfrac{3}{2}) & 0 & c_2 v^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(1) & 0 \end{pmatrix}$$

After Wild's transformation we have

$$\beta_o = \left(\begin{array}{cc|ccccc|c} 0 & 0 & & & & & & \\ 0 & 0 & & & & & & \\ \hline & & 0 & -2c_1 & 0 & 0 & 0 & \\ & & -2c_1 & 0 & \sqrt{2}c_2 & 0 & 0 & \\ & & 0 & \sqrt{2}c_2 & 0 & \sqrt{2}c_2 & 0 & \\ & & 0 & 0 & \sqrt{2}c_2 & 0 & 2c_1 & \\ & & 0 & 0 & 0 & 2c_1 & 0 & \\ \hline & & & & & & & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \frac{3}{2},\frac{1}{2} \\ \frac{1}{2},\frac{3}{2} \end{array}\right\} j = 2 \\ \left.\begin{array}{l} \frac{3}{2},\frac{1}{2} \\ 1,0 \\ \frac{1}{2},\frac{1}{2} \\ 0,1 \\ \frac{1}{2},\frac{3}{2} \end{array}\right\} j = 1 \\ \left.\frac{1}{2},\frac{1}{2}\right\} j = 0 \end{array}$$

β_o must satisfy

$$\beta_o^3(1) = \beta_o^5(1)$$

$$\beta_o^3(2) = 0$$

$$\beta_o^3(0) = 0 \quad .$$

These conditions give

$$4c_1^2 + 4c_2^2 = 0$$

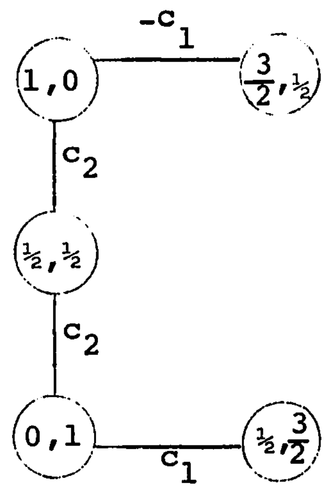
and

$$4c_1^2 + 2c_2^2 = \frac{1}{2} \quad ,$$

i.e.

$$c_1 = \pm\,\frac{1}{2} \quad \text{and} \quad c_2 = \pm\,\frac{i}{2} \quad .$$

The schematic diagram for this theory is:



The matrix η corresponding to this theory is

$$\eta = \left(\begin{array}{cc|ccc} 0 & -P_{\frac{1}{2}\frac{3}{2}} & & & \\ -P_{\frac{3}{2}\frac{1}{2}} & 0 & & & \\ \hline & & 0 & 0 & 1 \\ & & 0 & P_{\frac{1}{2}\frac{1}{2}} & 0 \\ & & 1 & 0 & 0 \end{array}\right)$$

Thus to summarize spin one, we have found the following theories:

a) 10-component theory

b) 14-component theory

c) 20-component theory

d) 26-component theory .

4. Spin 2

The representation under which spin 2 transforms according to equation II.32 is

$$D = \bigoplus_{j=0}^{3} n_j^{(j)}(j,0) \oplus n_j^{(j-1)}(j-\tfrac{1}{2},\tfrac{1}{2}) \oplus \dots \oplus n_j^{(o)}(\tfrac{j}{2},\tfrac{j}{2})$$

$$\oplus \dots \oplus n_j^{(j-1)}(\tfrac{1}{2},j-\tfrac{1}{2}) \oplus n_j^{(j)}(0,j) \quad . \qquad \text{III.3}$$

We will only consider two theories, although just as in spin 1 this does not exhaust all possibilities.

a) 35-component theory:

For this theory we choose

$$n_3^{(o)} = n_3^{(1)} = n_3^{(2)} = n_3^{(3)} = n_2^{(2)} = n_o^{(o)} = 0$$

and

$$n_1^{(o)} = n_1^{(1)} = n_2^{(o)} = n_2^{(1)} = 1$$

then D reduces to

$$D = (\tfrac{3}{2},\tfrac{1}{2}) \oplus (1,1) \oplus (\tfrac{1}{2},\tfrac{3}{2}) \oplus (1,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \quad .$$

The corresponding β's are

$$
\beta^{\alpha\dot{\beta}} = \begin{pmatrix}
0 & c_1 u^{\alpha}(\tfrac{3}{2}) v^{\dot{\beta}}(1) & 0 & c_2 u^{\alpha}(\tfrac{3}{2}) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 \\
c_1 v^{\alpha}(\tfrac{3}{2}) u^{\dot{\beta}}(1) & 0 & c_1 u^{\alpha}(1) v^{\dot{\beta}}(\tfrac{3}{2}) & 0 & c_4 u^{\alpha}(1) u^{\dot{\beta}}(1) & 0 \\
0 & c_1 v^{\alpha}(1) u^{\dot{\beta}}(\tfrac{3}{2}) & 0 & 0 & 0 & -c_2 u^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(\tfrac{3}{2}) \\
c_2 v^{\alpha}(\tfrac{3}{2}) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & 0 & c_3 u^{\alpha}(1) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \\
0 & c_4 v^{\alpha}(1) v^{\dot{\beta}}(1) & 0 & c_3 v^{\alpha}(1) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & c_3 u^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(1) \\
0 & 0 & -c_2 v^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(\tfrac{3}{2}) & 0 & c_3 v^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(1) & 0
\end{pmatrix}
$$

After wild's transformation β_o becomes

$$
\beta_o = \left(\begin{array}{ccc|ccc|ccc|cc}
0 & \sqrt{6}c_1 & 0 & & & & & & & & \\
\sqrt{6}c_1 & 0 & \sqrt{6}c_1 & & & & & & & & \\
0 & \sqrt{6}c_1 & 0 & & & & & & & & \\
\hline
 & & & & & & -2c_2 & \sqrt{2}c_1 & 0 & & \\
 & & & & 0 & & \sqrt{2}c_3 & -2c_4 & \sqrt{2}c_3 & & \\
 & & & & & & 0 & \sqrt{2}c_1 & 2c_2 & & \\
\hline
 & & & -2c_2 & \sqrt{2}c_3 & 0 & & & & & \\
 & & & \sqrt{2}c_1 & -2c_4 & \sqrt{2}c_1 & & 0 & & & \\
 & & & 0 & \sqrt{2}c_3 & 2c_2 & & & & & \\
\hline
 & & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0 & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \tfrac{3}{2},\tfrac{1}{2} \\ 1,1 \\ \tfrac{1}{2},\tfrac{3}{2} \end{array}\right\} j=2 \\
\left.\begin{array}{l} \tfrac{3}{2},\tfrac{1}{2} \\ \tfrac{1}{2},\tfrac{1}{2} \\ \tfrac{1}{2},\tfrac{3}{2} \\ 1,0 \\ 1,1 \\ 0,1 \end{array}\right\} j=1 \\
\left.\begin{array}{l} 1,1 \\ \tfrac{1}{2},\tfrac{1}{2} \end{array}\right\} j=0
\end{array}
$$

Since $q = s = 2$ β_o must satisfy

$$\beta_o^5(2) = \beta_o^3(2)$$

and

$$\beta_o^3(1) = 0 \quad , \qquad \beta_o^3(0) = 0 \; .$$

These conditions are satisfied if

$$c_1 = \pm \frac{1}{\sqrt{12}}$$

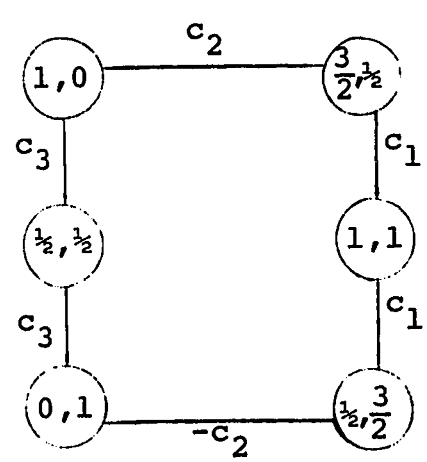
and

$$\left.\begin{array}{l} 4c_2^2 + \frac{1}{3} = 0 \\ 2c_3^2 = \frac{1}{6} \end{array}\right\} \text{ i.e. } c_2 = \pm \frac{i}{\sqrt{12}} \quad , \; c_3 = \pm \frac{1}{\sqrt{12}}$$

and

$$c_4 = 0 \quad .$$

The schematic diagram for this theory is:



The η matrix corresponding to the above theory is

$$\eta = \left(\begin{array}{ccc|ccc} 0 & 0 & P_{\frac{1}{2}\frac{3}{2}} & & & \\ 0 & -P_{11} & 0 & & & \\ P_{\frac{3}{2}\frac{1}{2}} & 0 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & & & 0 & -P_{\frac{1}{2}\frac{1}{2}} & 0 \\ & & & 1 & 0 & 0 \end{array}\right)$$

b) 30-component spin 2 theory:

For this we select

$$n_3^{(3)} = n_3^{(2)} = n_3^{(1)} = n_3^{(o)} = n_1^{(1)} = n_2^{(2)} = 0$$

and

$$n_o^{(o)} = n_1^{(o)} = n_2^{(o)} = n_2^{(1)} = 1$$

then D reduces to

$$D = (\tfrac{3}{2},\tfrac{1}{2}) \oplus (1,1) \oplus (\tfrac{1}{2},\tfrac{3}{2}) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,0). \qquad \text{III.3b}$$

The corresponding β's are

$$\beta^{\alpha\dot{\beta}}=\begin{pmatrix} 0 & c_1u^{\alpha}(\tfrac{3}{2})v^{\dot{\beta}}(1) & 0 & 0 & 0 \\ c_1v^{\alpha}(\tfrac{3}{2})u^{\dot{\beta}}(1) & 0 & c_1u^{\alpha}(1)u^{\dot{\beta}}(\tfrac{3}{2}) & c_2u^{\alpha}(1)u^{\dot{\beta}}(1) & 0 \\ 0 & c_1v^{\alpha}(1)u^{\dot{\beta}}(\tfrac{3}{2}) & 0 & 0 & 0 \\ 0 & c_2v^{\alpha}(1)v^{\dot{\beta}}(1) & 0 & 0 & c_3u^{\alpha}(\tfrac{1}{2})u^{\dot{\beta}}(\tfrac{1}{2}) \\ 0 & 0 & 0 & c_3v^{\alpha}(\tfrac{1}{2})v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \end{pmatrix}$$

After the transformation due to Wild β_o becomes

$$\beta_o=\left(\begin{array}{ccc|cccc|ccc} 0 & \sqrt{6}c_1 & 0 & & & & & & & \\ \sqrt{6}c_1 & 0 & \sqrt{6}c_1 & & & & & & & \\ 0 & \sqrt{6}c_1 & 0 & & & & & & & \\ \hline & & & 0 & \sqrt{2}c_1 & 0 & 0 & & & \\ & & & \sqrt{2}c_1 & 0 & \sqrt{2}c_1 & -2c_2 & & & \\ & & & 0 & \sqrt{2}c_1 & 0 & 0 & & & \\ & & & 0 & -2c_2 & 0 & 0 & & & \\ \hline & & & & & & & 0 & \sqrt{6}c_2 & 0 \\ & & & & & & & \sqrt{6}c_2 & 0 & -\sqrt{2}c_3 \\ & & & & & & & 0 & -\sqrt{2}c_3 & 0 \end{array}\right) \begin{array}{l} \tfrac{3}{2},\tfrac{1}{2} \\ 1,1 \\ \tfrac{1}{2},\tfrac{3}{2} \\ \tfrac{3}{2},\tfrac{1}{2} \\ 1,1 \\ \tfrac{1}{2},\tfrac{3}{2} \\ \tfrac{1}{2},\tfrac{1}{2} \\ 1,1 \\ \tfrac{1}{2},\tfrac{1}{2} \\ 0,0 \end{array} \begin{array}{l} \left.\right\} j=2 \\ \\ \left.\right\} j=1 \\ \\ \left.\right\} j=0 \end{array}$$

β_o must satisfy

1) $\beta_o^5(2) = \beta_o^3(2)$

which gives $c_1 = \pm \frac{1}{\sqrt{12}}$

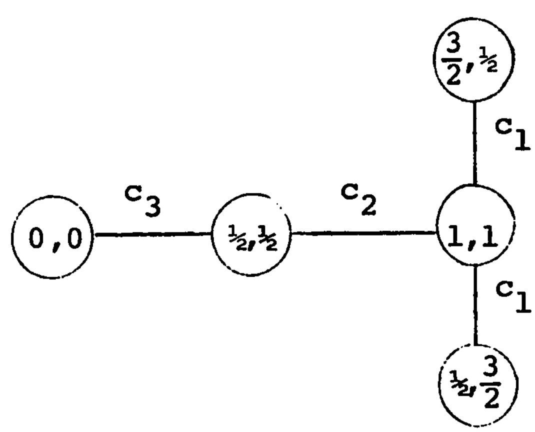
2) $\beta_o^3(1) = 0$

which gives $4c_1^2 + 4c_2^2 = 0$, i.e. $c_2 = \pm \frac{i}{\sqrt{12}}$

3) $\beta_o^3(0) = 0$

which gives $6c_2^2 + 2c_3^2 = 0$, i.e. $c_3 = \pm \frac{1}{2}$.

The schematic diagram for this theory is:



The corresponding η is

$$\eta = \left(\begin{array}{ccc|cc} 0 & 0 & P_{\frac{1}{2}\frac{3}{2}} & & \\ 0 & -P_{11} & 0 & & \\ P_{\frac{3}{2}\frac{1}{2}} & 0 & 0 & & \\ \hline & & & -P_{\frac{1}{2}\frac{1}{2}} & 0 \\ & & & 0 & P_{oo} \end{array}\right)$$

Thus to summarize, we have obtained the following theories for spin 2

a) 35-component spin 2 theory

b) 30-component spin 2 theory .

5. Spin 3

According to equation II.32 the representation under which spin 3 transforms is

$$D = \bigoplus_{j=0}^{4} n_j^{(j)}(j,0) \oplus n_j^{(j-1)}(j-\tfrac{1}{2},\tfrac{1}{2}) \oplus \ldots \oplus n_j^{(o)}(\tfrac{j}{2},\tfrac{j}{2})$$

$$\oplus \ldots \oplus n_j^{(j-1)}(\tfrac{1}{2},j-\tfrac{1}{2}) \oplus n_j^{(j)}(0,j) \ . \qquad \text{III.4}$$

Although many spin 3 theories may be derived from the above representation, we will only consider one theory namely a 70-component spin 3 theory. This theory is derived by selecting

$$n_4^{(4)} = n_4^{(3)} = n_4^{(2)} = n_4^{(1)} = n_4^{(o)} = n_3^{(3)}$$

$$= n_3^{(2)} = n_2^{(2)} = n_2^{(1)} = 0$$

$$n_3^{(o)} = n_3^{(1)} = n_2^{(o)} = n_1^{(1)} = n_o^{(o)} = 1$$

and

$$n_1^{(o)} = 2 \ .$$

Thus the representation under which this spin 3 theory transforms is

$$D = (2,1) \oplus (\tfrac{3}{2},\tfrac{3}{2}) \oplus (1,2) \oplus (1,0) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,1) \oplus$$

$$(1,1) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,0) \ . \qquad \text{III.4a}$$

The corresponding β's are

$$
\beta^{\alpha\dot{\beta}} = \left[\begin{array}{ccc|ccc|ccc}
0 & c_1 u^{\alpha}(2) v^{\dot{\beta}}(\tfrac{3}{2}) & 0 & & & & 0 & 0 & 0 \\
c_1 v^{\alpha}(2) u^{\dot{\beta}}(\tfrac{3}{2}) & 0 & c_1 u^{\alpha}(\tfrac{3}{2}) v^{\dot{\beta}}(2) & & 0 & & c_2 u^{\alpha}(\tfrac{3}{2}) u^{\dot{\beta}}(\tfrac{3}{2}) & 0 & 0 \\
0 & c_1 v^{\alpha}(\tfrac{3}{2}) u^{\dot{\beta}}(2) & 0 & & & & 0 & 0 & 0 \\
\hline
 & & & 0 & c_4 u^{\alpha}(1) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & c_8 u^{\alpha}(1) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 \\
 & 0 & & c_4 v^{\alpha}(1) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & c_4 u^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(1) & c_3 v^{\alpha}(1) v^{\dot{\beta}}(1) & 0 & c_5 u^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(\tfrac{1}{2}) \\
 & & & 0 & c_4 v^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(1) & 0 & 0 & c_8 v^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(1) & 0 \\
\hline
0 & c_2 v^{\alpha}(\tfrac{3}{2}) v^{\dot{\beta}}(\tfrac{3}{2}) & 0 & 0 & c_3 u^{\alpha}(1) u^{\dot{\beta}}(1) & 0 & 0 & c_7 u^{\alpha}(1) u^{\dot{\beta}}(1) & 0 \\
0 & 0 & 0 & c_8 v^{\alpha}(1) u^{\dot{\beta}}(\tfrac{1}{2}) & 0 & c_8 u^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(1) & c_7 v^{\alpha}(1) v^{\dot{\beta}}(1) & 0 & c_6 u^{\alpha}(\tfrac{1}{2}) u^{\dot{\beta}}(\tfrac{1}{2}) \\
0 & 0 & 0 & 0 & c_5 v^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(\tfrac{1}{2}) & 0 & 0 & c_8 v^{\alpha}(\tfrac{1}{2}) v^{\dot{\beta}}(\tfrac{1}{2}) & 0
\end{array}\right]
$$

After Wild's transformation β_o becomes

$$\beta_o = \left(\begin{array}{cccc}
\begin{array}{|ccc|} \hline 0 & \sqrt{12}c_1 & 0 \\ \sqrt{12}c_1 & 0 & \sqrt{12}c_1 \\ 0 & \sqrt{12}c_1 & 0 \\ \hline \end{array} & & & \\
& \begin{array}{|cccc|} \hline 0 & \sqrt{6}c_1 & 0 & 0 \\ \sqrt{6}c_1 & 0 & \sqrt{6}c_1 & -\sqrt{6}c_2 \\ 0 & \sqrt{6}c_1 & 0 & 0 \\ 0 & -\sqrt{6}c_2 & 0 & 0 \\ \hline \end{array} & & \\
& & \begin{array}{|cccccccc|} \hline 0 & 0 & \sqrt{2}c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}c_1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}c_1 & \sqrt{2}c_1 & 0 & \sqrt{10}c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{10}c_2 & 0 & 0 & 0 & -2c_3 & -2c_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}c_4 & \sqrt{2}c_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}c_4 & \sqrt{2}c_8 \\ 0 & 0 & 0 & -2c_3 & \sqrt{2}c_4 & \sqrt{2}c_4 & 0 & 0 \\ 0 & 0 & 0 & -2c_7 & \sqrt{2}c_8 & \sqrt{2}c_8 & 0 & 0 \\ \hline \end{array} & \\
& & & \begin{array}{|ccccc|} \hline 0 & -\sqrt{12}c_2 & 0 & 0 & 0 \\ -\sqrt{12}c_2 & 0 & \sqrt{6}c_3 & 0 & \sqrt{6}c_7 \\ 0 & \sqrt{6}c_3 & 0 & -\sqrt{2}c_5 & 0 \\ 0 & 0 & -\sqrt{2}c_5 & 0 & -\sqrt{2}c_6 \\ 0 & \sqrt{6}c_7 & 0 & -\sqrt{2}c_6 & 0 \\ \hline \end{array}
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} 2,1 \\ \tfrac{3}{2},\tfrac{3}{2} \\ 1,2 \end{array}\right\} j=3 \\
\left.\begin{array}{c} 2,1 \\ \tfrac{3}{2},\tfrac{3}{2} \\ 1,2 \\ 1,1 \end{array}\right\} j=2 \\
\left.\begin{array}{c} 2,1 \\ 1,2 \\ \tfrac{3}{2},\tfrac{3}{2} \\ 1,1 \\ 1,0 \\ 0,1 \\ \tfrac{1}{2},\tfrac{1}{2} \\ \tfrac{1}{2},\tfrac{1}{2} \end{array}\right\} j=1 \\
\left.\begin{array}{c} \tfrac{3}{2},\tfrac{3}{2} \\ 1,1 \\ \tfrac{1}{2},\tfrac{1}{2} \\ 0,0 \\ \tfrac{1}{2},\tfrac{1}{2} \end{array}\right\} j=0
\end{array}$$

Since $q = s = 3$ β_o must satisfy

1) $\beta_o^7(3) = \beta_o^5(3)$

which gives $c_1 = \pm \frac{1}{\sqrt{24}}$

2) $\beta_o^5(2) = 0$

which gives $c_2 = \pm \frac{i}{\sqrt{12}}$

3) $\beta_o^3(1) = 0$

which is satisfied by:

a) $c_3 = \pm \frac{2}{\sqrt{30}}$, $c_4 = \pm \frac{1}{\sqrt{30}}$, $c_7 = c_8 = 0$

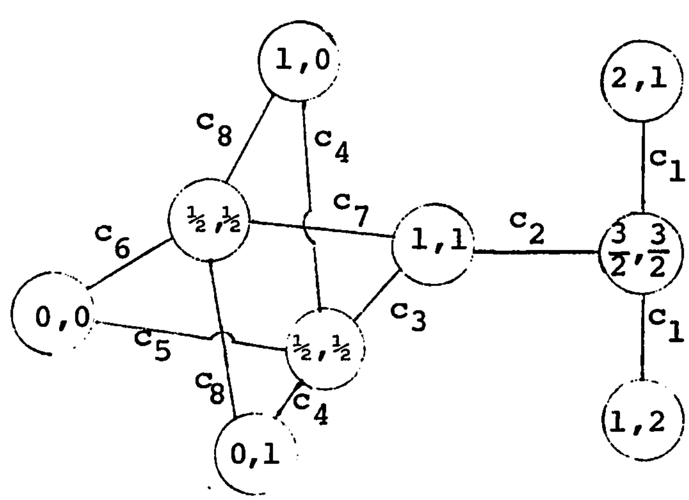
or

b) $c_7 = \pm \frac{2}{\sqrt{30}}$, $c_8 = \pm \frac{i}{\sqrt{30}}$, $c_3 = c_4 = 0$.

4) $\beta_o^5 = 0$

which gives $c_5 = \pm \frac{i}{\sqrt{40}}$ and $c_6 = \pm \frac{1}{\sqrt{8}}$.

The schematic diagram for this theory is



The η matrix corresponding to this theory is

$$\eta = \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & P_{12} & & & & & & \\
0 & -P_{\frac{3}{2}\frac{3}{2}} & 0 & & & & & & \\
P_{21} & 0 & 0 & & & & & & \\
\hline
 & & & 0 & 0 & -1 & & & \\
 & & & 0 & P_{\frac{1}{2}\frac{1}{2}} & 0 & & & \\
 & & & -1 & 0 & 0 & & & \\
\hline
 & & & & & & -P_{11} & 0 & 0 \\
 & & & & & & 0 & -P_{\frac{1}{2}\frac{1}{2}} & 0 \\
 & & & & & & 0 & 0 & P_{oo}
\end{array}\right)$$

CHAPTER IV

PROPAGATION OF INTERACTING FIELDS

1. Introduction

The analysis of hyperbolic systems of partial differential equations using the method of characteristic surfaces as described in Courant and Hilbert[Cou.] was applied recently by Velo and Zwanziger[V.1,2] to determine the causal nature of the Rarita-Schwinger[Ra.] spin 3/2 field and the spin 1 Proca field[P.].

In this chapter we investigate the causal nature of the following fields: The Takahashi-Palmer[T.2] spin one field, the minimally coupled Rarita-Schwinger spin 3/2 field with added Pauli terms and finally the spin 1 Proca field interacting with the spin 1/2 Dirac field via various types of coupling. We summarise our results:

The minimally coupled Takahashi-Palmer field is acausal. However, by adding a fixed magnetic dipole moment interaction to its minimally coupled Lagrangian, causal propagation can be restored. The results for the self coupled T.P field are identical with those for the Proca field.

The minimally coupled Rarita-Schwinger spin 3/2 field with the most general Pauli interaction terms is always acausal. As a result, all spin 3/2 theories are acausal when minimally coupled to the electromagnetic field.

For spin 1 (Proca) interacting with spin 1/2 (Dirac), only the direct coupling of the scalar, pseudo-scalar, vector, pseudo-vector and the derivative coupling of the tensor and vector types are causal.

Before proceeding to show our results quantitatively, we review briefly, for the sake of completeness, the method of characteristic surfaces as discussed in Courant and Hilbert (Cou.).

2. Characteristic Surfaces

The concept of characteristic surface originates from the problem of extending initial values of a function $\psi(x)$ on a surface σ to the solution of the partial differential equation which ψ satisfies.

A partial differential equation for a field $\psi(x)$ is defined by a functional F, satisfying the equation,

$$F(x,\ \psi(x),\ \partial_\mu\psi(x),\ \partial_\mu\partial_\nu\partial_\rho\ldots\psi(x)) = 0 \qquad \text{IV.1}$$

where ψ is possibly a multi-component object, $x = (x_o,\ x_1\ \ldots\ x_n)$ and $\partial_\mu \equiv \partial/\partial x_\mu$. The indices $\mu,\nu,\rho\ldots$ take the values $0,1,\ldots n$.

The differential equation is called linear if F is linear in the variables ψ, $\partial_\mu\psi\ldots$ etc. with coefficients depending only on x. If F is linear in the highest order derivatives, say m, with coefficients depending on x and possibly upon ψ and its derivatives up to order m-1, then the differential equation is called quasi-linear. We now define the terms outward and interior derivatives.

The derivative of a function ψ at a point $p = (x_o, x_1, \ldots x_n)$ in the direction of a vector $n = (n_o, n_1, \ldots n_n)$ is

$$\frac{\partial\psi}{\partial s} = n^{\mu}\partial_{\mu}\psi \quad . \tag{IV.2}$$

Here s is a parameter which characterizes the line passing through p in the direction of the vector n.

If p is on the surface $\sigma:\Phi(x) = 0$ with $\partial_{\mu}\Phi \neq 0$ then the directional derivative is an outward derivative provided $n^{\mu}\partial_{\mu}\Phi \neq 0$. In particular if

$$n_{\mu} = \partial_{\mu}\Phi \quad , \tag{IV.3}$$

then the outward derivative is normal to the surface. If, on the other hand, $n_{\mu}\partial^{\mu}\Phi = 0$ then the directional derivative lies on the surface and is called an interior derivative.

An interior derivative is known if the data are given on the surface. The expression $(\partial_{\mu}\Phi\partial_{\nu}\psi - \partial_{\nu}\Phi\partial_{\mu}\psi)$ represents first order interior derivatives. Similarly

the expression $(\partial_\mu\Phi\partial_\nu\partial_\rho\psi - \partial_\rho\Phi\partial_\mu\partial_\nu\psi)$ represents second order interior derivatives and the same is true for $(\partial_\sigma\Phi\partial_\mu\partial_\nu\psi - \partial_\nu\Phi\partial_\mu\partial_\sigma\psi)$. The linear combination of these two $(\partial_\mu\Phi\partial_\sigma\Phi\partial_\nu\partial_\rho\psi - \partial_\nu\Phi\partial_\rho\Phi\partial_\mu\partial_\sigma\psi)$ is the most general expression for the second order interior derivatives.
And in general the interior derivatives of an m-th order differential equation are given by

$$\underbrace{\partial_\mu\Phi\partial_\nu\Phi\ldots}_{m}\underbrace{\partial_\alpha\partial_\beta\ldots\psi}_{m} - \underbrace{\partial_\alpha\Phi\partial_\beta\Phi\ldots}_{m}\underbrace{\partial_\mu\partial_\nu\ldots\psi}_{m} = \text{data} \qquad \text{IV.4}$$

To extend initial data to the solution of a partial differential equation, the highest outward derivative, say m, must be determined by the differential equation, when the function together with all of its m-1 outward derivatives are given as data on the surface. If the determinant of the coefficients of the highest outward derivative does not vanish on a given surface, then the given surface is called free and the initial data are uniquely extended to the solution of the differential equation. If, on the other hand, the coefficients are such that the highest outward derivative cannot be determined by the differential equation on a given surface, then the surface is called characteristic. In this case the differential equation represents interior differentiation and the data cannot be specified arbitrarily but must satisfy the constraints imposed by the differential equation on the surface. Moreover the highest outward derivative is discontinuous across the surface. Characteristic surfaces play a role as "wave fronts". Such "wave fronts" occur as frontiers beyond which no excitation can occur. The solution representing a discontinuity vanishes identically on one side of this frontier but not on the other.

The coefficients of the highest derivatives determine whether the differential equation possesses characteristic surfaces or not as follows:

Given a partial differential equation of the form

$$\sum_{|\alpha|\leq m} A^{\alpha}p^{\alpha}\psi = f \qquad \text{IV.5}$$

where $p^{\alpha} = p_o^{\alpha_o}\cdots p_n^{\alpha_m}$, $|\alpha| = \alpha_o + \ldots + \alpha_m$, $p_{\mu}^{\alpha_i} \equiv i\frac{\partial^{\alpha_i}}{\partial^{\alpha_i}x}$ and A^{α}, f are $k \times k$ square matrices (k is dimension of ψ) which may be constants or may depend on x; for quasi-linear equations they may depend on ψ and its partial derivatives up to order m-1. Then the roots n of the characteristic form $D(n) = \left|\sum_{|\alpha|=m} A^{\alpha}n^{\alpha}\right|$; i.e. the solutions of

$$D(n) = 0 \qquad \text{IV.6}$$

define the characteristic surfaces of the differential equation. Here, n is a vector with components $n_\mu = \partial_\mu \Phi$ and is normal to the surface σ by definition. For real characteristic surfaces the solutions of IV.6 must yield real values for the components of n. This assertion can be proved as follows:

On the surface σ we introduce the interior coordinates $(x_1 \ldots x_n)$ and the normal coordinate λ. The interior derivative expression which contains the highest order outward derivative is

$$(n_\lambda)^m \partial_i \partial_j \partial_k \ldots \psi - n_i n_j n_k \ldots \ \partial_\lambda^m \psi = \text{data} \qquad \text{IV.7}$$

$$i,j,k \ldots = 1,\ldots,n \ .$$

Solving for $\partial_i \partial_j \partial_k \ldots \psi$ from the above equation and substituting in equation IV.5 we get, after multiplication by $(n_\lambda)^m$

$$\sum_{|\alpha|=m} A^\alpha n^\alpha \partial_\lambda^m \psi + \ldots = 0 \qquad \text{IV.8}$$

where the dots represent quantities which depend only on the data. The above equation can be solved uniquely for $\partial_\lambda{}^m \psi$ provided the determinant, $|\sum_{|\alpha|=m} A^\alpha n^\alpha|$ is not zero. If the determinant vanishes on a surface σ then $\partial_\lambda{}^m \psi$ is not defined by the differential equation and this is precisely the condition for σ to be a characteristic surface.

Thus to determine the characteristic condition we replace $i\partial_\mu$ by n_μ in the highest derivatives and calculate the determinant of the resulting expression. However, for higher spin differential equations, the equations are not true equations of motion because they imply the constraints and one cannot use the above procedure mechanically to compute the characteristic form unless the equations are true equations of motion. In this case the true equations of motion are obtained when the constraints are found and substituted back in the original equations.

We now turn to the question of hyperbolicity. A system of equations is hyperbolic if all the roots of its characteristic form are real. If some of the roots are complex the system loses hyperbolicity and propagation ceases.

The solution of the Cauchy problem requires the concept of space-like surfaces. The above definition is a special case of a more general definition:

At a point p an operator (of highest degree m for a system of k components) is called hyperbolic with respect to a vector ξ passing through p if every two dimensional plane π through ξ intersects the normal cone defined by $D(n) = 0$ in m k real lines. Algebraically, if θ is an arbitrary vector (not parallel to ξ) then the line $n = \tau\xi + \theta$, τ being a parameter must intersect the normal cone in mk real points; i.e. the equation for τ $D(\tau\xi + \theta) = 0$ must have mk real roots. Space elements at p orthogonal to the vector ξ are called space-like and ξ is called the space-like normal. Space-like surfaces are free surfaces; they separate the forward parts of ray cones from the backward parts. (A ray cone is the cone orthogonal to the normal cone). By the invariance property of the characteristic we may pick $\xi = (1,0,0,0,0..)$. Then $D(\tau,\theta_1 \ldots \theta_m) = 0$ must have m k real roots and this is equivalent to our special definition of hyperbolicity. Cauchy data must be specified on a free surface; i.e. a space-like surface. Otherwise the differential equation reduces to an interior differentiation.

After these preliminaries we proceed to calculate the characteristic form of the fields cited in the introduction.

3. Spin 1 (T.P) Field

In this section we analyse the propagation of a divergenceless skew-symmetric second rank tensor field describing a massive spin 1 field when a) coupled to an external electromagnetic field and b) coupled to itself. The Lagrangian for this field was given by Takahashi and Palmer[(T.2)] and so we refer to it as the T.P field in contrast to the Proca field. The results for the Proca field were given by Velo and Zwanziger[(V.2)].

The free Lagrangian given in[(T.2)] is:[(†)]

$$\mathcal{L} = \frac{1}{2}\,[p^\lambda\bar\psi^{\mu\nu}p_\lambda\psi_{\mu\nu} - p^\lambda\bar\psi^{\mu\nu}p_\lambda\psi_{\nu\mu} - p_\mu\bar\psi^{\mu\nu}p^\lambda\psi_{\lambda\nu} + p_\mu\bar\psi^{\mu\nu}p^\lambda\psi_{\nu\lambda} + p_\nu\bar\psi^{\mu\nu}p^\lambda\psi_{\lambda\mu} - p_\nu\bar\psi^{\mu\nu}p^\lambda\psi_{\mu\lambda}] + m^2\bar\psi^{\mu\nu}\psi_{\mu\nu} \quad . \qquad \text{IV.9}$$

a) Minimal Coupling

The minimally coupled Lagrangian results when we replace

$$p^\lambda\bar\psi^{\mu\nu} \longrightarrow (p^\lambda - eA^\lambda)\bar\psi^{\mu\nu}$$

and

$$p^\lambda\psi^{\mu\nu} \longrightarrow (p^\lambda + eA^\lambda)\psi^{\mu\nu} \equiv \pi^\lambda\psi^{\mu\nu}.$$

(†) Here we adopt the notation $p_\nu = i\partial_\nu$ and diag $g_{\mu\nu}$ = $(1,-1,-1,-1)$, $\pi^\lambda = (p^\lambda + eA^\lambda)$.

Making this replacement and varying with respect to $\bar{\psi}^{\mu\nu}$ we get:

$$\pi^2\psi_{\mu\nu} - \pi_\mu\pi^\lambda\psi_{\lambda\nu} + \pi_\nu\pi^\lambda\psi_{\lambda\mu} - m^2\psi_{\mu\nu} = 0 \qquad \text{IV.10}$$

$$\psi_{\mu\nu} = -\psi_{\nu\mu} \quad \text{for} \quad m^2 \neq 0 . \qquad \text{IV.10a}$$

By inspection, equation IV.10 is not the true equation of motion because the 2nd time derivatives of $\psi_{o\nu}$, $\psi_{\mu o}$ never appear in it. This implies three primary constraints. The true equation of motion will be obtained if we differentiate equation IV.10 in a covariant way, to get the secondary constraint, and substitute the result back in equation IV.10. Thus contracting equation IV.10 with π^μ we get:

$$\pi^\mu\pi^2\psi_{\mu\nu} - \pi^\mu\pi_\mu\pi^\lambda\psi_{\lambda\nu} + \pi^\mu\pi_\nu\pi^\lambda\psi_{\lambda\mu} - m^2\pi^\mu\psi_{\mu\nu} = 0 \ . \qquad \text{IV.11}$$

Using:

$$[\pi^\mu,\pi^\nu] = i\ e\ F^{\mu\nu}$$

and

$$\pi^\mu\pi_\nu\pi^\lambda - \pi^\lambda\pi_\nu\pi^\mu = ie\ [\pi^\mu F_\nu{}^\lambda + F^{\mu\lambda}\pi_\nu + \pi^\lambda F^\mu{}_\nu]$$

and after some rearrangement equation IV.11 becomes:

$$\pi^\lambda\psi_{\lambda\nu} = \frac{ie}{m^2}[(F^{\mu\sigma}\pi_\sigma + \pi^\sigma F^\mu{}_\sigma)\psi_{\mu\nu} + \frac{1}{2}[\pi^\mu F_\nu{}^\lambda + F^{\mu\lambda}\pi_\nu + \pi^\lambda F^\mu{}_\nu)\psi_{\lambda\mu}] .$$

Substituting in equation IV.10 we find

$$(\pi^2 - m^2)\psi_{\mu\nu} - ie\pi_\mu[(F^{\rho\sigma}\pi_\sigma + \pi^\sigma F^\rho{}_\sigma)\psi_{\rho\nu}$$
$$+ \frac{1}{2}(\pi^\rho F_\nu{}^\varepsilon + F^{\rho\varepsilon}\pi_\nu + \pi^\varepsilon F^\rho{}_\nu)\psi_{\varepsilon\rho}]$$
$$+ ie\pi_\nu[(F^{\rho\sigma}\pi_\sigma + \pi^\sigma F^\rho{}_\nu)\psi_{\rho\mu}$$
$$+ \frac{1}{2}(\pi^\rho F_\mu{}^\varepsilon + F^{\rho\varepsilon}\pi_\mu + \pi^\varepsilon F^\rho{}_\mu)\psi_{\varepsilon\rho}] = 0 \,. \qquad \text{IV.12}$$

This is the true equation of motion since all second time derivatives of $\psi_{\mu\nu}$ occur in it. To get the characteristic determinant we replace π_μ by a Lorentz four-vector n_μ in the highest derivatives. Furthermore, since the determinant is a polynomial in n_μ we may pick n_μ = (n, 0, 0, 0) and by a Lorentz transformation get to a general frame. Calculation of the determinant gives:

$$D(n) = (n^2)^6 \left[1 - \frac{e^2}{m^4} B^2\right] . \qquad \text{IV.13}$$

In a covariant form (general n) this becomes:

$$D(n) = (n^2)^5 \left[n^2 + \frac{e^2}{m^4} (n.F^d)^2\right] \qquad \text{IV.14}$$

where $F^{d\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

$\varepsilon^{\mu\nu\rho\sigma}$ = +1 for even permutations of 0, 1, 2, 3

= −1 for odd permutations of 0, 1, 2, 3

= 0 when two indices are equal.

The characteristic surfaces are normal to n_μ. Solving for n_μ we find:

$$(n^2)^5 \; [n_o^2 - n^2 - n_o^2 \frac{e^2}{m^2} B^2] = 0 \; .$$

This has the solutions $(n^2)^5 = 0 \Rightarrow n_o = \pm |\vec{n}|$ so for this the characteristic surfaces are the light cones.

We also have for the second factor in the brackets, the solution:

$$n_o = \pm \frac{|\vec{n}|}{\sqrt{1 - \frac{e^2}{m^4} B^2}} \; .$$

Here if $1 - \frac{e^2}{m^4} B^2 > 0$ then we have real solutions for n_μ which lie inside the light cones, therefore the characteristic surfaces in this case are space-like and the propagation is acausal. If however $1 - \frac{e^2}{m^4} B^2 < 0$ then n_μ is complex and the equations of motion cease to be hyperbolic. This analysis indicates that the intrinsic dipole moment is responsible for the acausal behaviour and even the loss of hyperbolicity. We therefore add an arbitrary dipole interaction to the minimally coupled Lagrangian. This dipole interaction has the form:

$$\mathcal{L}_I = ig(\bar{\psi}^{\mu\nu} F^\sigma{}_\nu \psi_{\mu\sigma} - \bar{\psi}^{\mu\nu} F^\sigma{}_\mu \psi_{\nu\sigma}) \qquad \text{IV.15}$$

where g is real (from Hermiticity) and this form is mandatory because of the anti-symmetry properties of $\psi_{\mu\nu}$.

Calculation of the characteristic determinant yields

$$D(n) = (n^2)^5 \; [n^2 + \frac{(e-g)^2}{m^4} (n.F^d)^2]$$

or

$$D(n) = (n^2)^5 \; [n^2 - \frac{(e-g)^2}{m^4} B^2 \, n_o^2] \quad . \qquad \text{IV.16}$$

Therefore if we take $e = g$ the propagation will always be causal since the characteristic determinant becomes $D(n) = (n^2)^6$ so every characteristic surface is the light cone, which is the same result as that for the Proca field coupled minimally to the electromagnetic source. Thus we conclude that the T.P field carries an "incorrect" intrinsic dipole moment which has to be accounted for when an interaction is present.

b) Self Coupling:

Next we carry the comparison between the T.P field and the Proca field a little further by considering the self interaction of the form

$$\mathcal{L}_I = q/2 \; (\psi^{\mu\nu}\psi_{\mu\nu})^2 \quad .$$

Here we take a neutral spin field i.e. $\bar{\psi}^{\mu\nu} = \psi^{\mu\nu}$. Using the same procedure as above we find that the characteristic determinant is given by:

$$D(n) = (n^2)^5 \left[n^2 + \frac{2q/m^2}{1 + \frac{q}{m^2}\psi^2} (n^\mu \psi_{\mu\nu})(n_\rho \psi^{\rho\nu})\right] \qquad \text{IV.17}$$

with

$$m^2 + q\psi^2 \neq 0$$

where

$$\psi^2 = \psi^{\mu\nu}\psi_{\mu\nu} \quad .$$

Analysis of equation IV.17 indicates that the last factor of this equation determines characteristic surfaces with normal n_μ satisfying:

$$n^2\left(1 + \frac{q}{m^2}\psi^2\right) = -\frac{2q}{m^2}(n^\mu \psi_{\mu\nu})(n_\rho \psi^{\rho\nu}) \quad . \qquad \text{IV.18}$$

Here we notice that these characteristic surfaces are not a property of the equations above but depend on the particular solution $\psi^{\mu\nu}$. Equation IV.18 indicates that if $\frac{q}{m^2}\psi^2 > -1$ which is true if $|\psi^2|$ is sufficiently small then n_μ will be space-like for $q > 0$ and time-like for $q < 0$, hence with the initial value of ψ^2 sufficiently small and consistent with the constraint the initial propagation will be causal if $q > 0$. This behaviour is identical with that of the Proca field when coupled to itself by $q/2(\omega^2)^2$ as discussed in [(V.2)].

4. Spin 3/2

The Rarita-Schwinger(†) spin 3/2 equations are derived from the Lagrangian:

$$\mathcal{L} = \bar{\psi}\ (\Gamma\cdot p - B)\psi \qquad \text{IV.19}$$

where

$$[\Gamma\cdot p]_{\kappa}{}^{\lambda} = g_{\kappa}{}^{\lambda}\gamma\cdot p - (\gamma_{\kappa}p^{\lambda} + p_{\kappa}\gamma^{\lambda}) + \gamma_{\kappa}\gamma\cdot p\gamma^{\lambda}$$

$$B_{\kappa}{}^{\lambda} = m(g_{\kappa}{}^{\lambda} - \gamma_{\kappa}\gamma^{\lambda}) \quad .$$

Here ψ is the R.S vector spinor ψ^{μ} with $\bar{\psi}^{\mu} = \psi^{\mu+}\gamma^{o}$. Velo and Zwanziger(V.1) found that minimal coupling always leads to acausal propagation. We attempt to remedy this by the same method used for spin 1 T.P field. We introduce the minimal coupling $p^{\mu} \rightarrow \pi^{\mu}$ and a general Pauli term:

$$\mathcal{L}_{I} = \bar{\psi}^{\mu}T_{\mu}{}^{\nu}\psi_{\nu} \quad .$$

Here $T_{\mu}{}^{\nu}$ is a second rank tensor with spinor indices composed from the Dirac matrices and the electromagnetic

---

(†) For the Dirac matrices $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I$ we take the following representation:

$$\gamma^{o} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \quad , \quad \gamma^{\kappa} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\vec{\alpha} = \gamma^{o}\vec{\gamma} \qquad \beta = \gamma^{o} \quad .$$

field F. The consistency of the R.S equation with spin 3/2 i.e. eight independent components restricts the form of $T_\mu{}^\nu$ as we shall see.

Varying the total Lagrangian with respect to $\bar{\psi}$ gives:

$$\gamma\cdot\pi\psi_\kappa - (\gamma_\kappa\pi^\lambda + \pi_\kappa\gamma^\lambda)\psi_\lambda + \gamma_\kappa\gamma\cdot\pi\gamma^\lambda\psi_\lambda - m(g_\kappa{}^\lambda - \gamma_\kappa\gamma^\lambda)\psi_\lambda + T_\kappa{}^\lambda\psi_\lambda = 0. \qquad \text{IV.20}$$

Equation IV.20 is not the true equation of motion since it implies the constraints. In fact we see that when $\kappa = 0$ equation IV.20 contains no time derivatives, but yields instead the primary constraint equation:

$$(\vec{\pi} - h\vec{\alpha})\cdot\vec{\psi} + T_o{}^\lambda\psi_\lambda = 0 \qquad \text{IV.21}$$

where

$$\vec{\pi} = (\pi^i),\ \vec{\psi} = (\psi^i),\ i = 1,2,3 \quad \text{and} \quad h = \vec{\alpha}\cdot\vec{\pi} + \beta m.$$

Furthermore, we see that the time derivative of ψ^o never appears at all in equation IV.20, nor is ψ^o determined from equation IV.21 if $T_o{}^o = 0$; if $T_o{}^o \neq 0$, however, ψ^o would be determined by IV.21 which would then imply that the R.S field contains 12 independent components and four constraints, hence $T_o{}^o$ must be identically zero. To get an equation for ψ_o i.e. the secondary constraint, we contract equation IV.20 with π^κ and γ^κ respectively:

$$ieF^{\lambda\kappa}\gamma_\kappa\psi_\lambda - m(\pi^\lambda - \pi^\kappa\gamma_\kappa\gamma^\lambda)\psi_\lambda + \pi^\kappa T_\kappa{}^\lambda\psi_\lambda$$

$$+ \frac{ie}{2} F_{\nu\mu}\gamma^\nu\gamma^\mu\gamma^\lambda\psi_\lambda = 0 \qquad \text{IV.22}$$

and

$$-2[\pi^\lambda - \gamma\cdot\pi\gamma^\lambda]\psi_\lambda + 3m\gamma^\lambda\psi_\lambda + \gamma^\kappa T_\kappa{}^\lambda\psi_\lambda = 0 \ . \qquad \text{IV.23}$$

Solving for $\gamma^\lambda\psi_\lambda$ and $\pi^\lambda\psi_\lambda$ we find:

$$\gamma^\lambda\psi_\lambda = \frac{2}{3}\frac{\pi^\kappa}{m^2} T_\kappa{}^\lambda\psi_\lambda + \frac{2ie}{3m^2} F^{\lambda\kappa}\gamma_\kappa\psi_\lambda - \frac{\gamma^\kappa}{3m} T_\kappa{}^\lambda\psi_\lambda$$

$$+ \frac{ie}{3m} F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^\lambda\psi_\lambda \qquad \text{IV.24}$$

and

$$\pi^\lambda\psi_\lambda = \frac{ie}{m} F^{\lambda\kappa}\gamma_\kappa\psi_\lambda + \pi^\rho\gamma_\rho \ [\frac{2}{3}\frac{\pi^\kappa}{m^2} T_\kappa{}^\lambda\psi_\lambda + \frac{2ie}{3m} F^{\lambda\kappa}\gamma_\kappa\psi_\lambda$$

$$- \gamma^\kappa T_\kappa{}^\lambda\psi_\lambda + \frac{ie}{3m} F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^\lambda\psi_\lambda] + \frac{\pi^\kappa}{m} T_\kappa{}^\lambda\psi_\lambda \ . \qquad \text{IV.25}$$

Analysis of equations IV.24 and IV.25 indicates that if $T_o{}^o \neq 0$ these constraint equations will turn into equations of motion and the field will have twelve independent components, thus reemphasizing the fact that $T_o{}^o$ must vanish. When this condition is met equations IV.24 and IV.25 are not equations of motion nor are they the true constraints; the true constraints in this case would be obtained if equation IV.20 were used in these equations. This procedure is quite tedious and instead we decompose the field ψ^μ into a transverse field V^μ and the gradient of a spinor field B. This technique has been used for the Proca field$^{(V.2)}$ with quadrapole interaction.

We write:

$$\psi_\mu = V_\mu + \pi_\mu B \qquad \text{IV.26a}$$

with

$$\pi_\mu V^\mu = 0 \ . \qquad \text{IV.26b}$$

Even though B is a new dynamical variable the number of components of V_μ and B is still sixteen because of the invariance of the equations under the gauge transformation:

$$\begin{aligned} V^\mu &\to V^\mu + \pi^\mu \Lambda \\ B &\to B - \Lambda \ . \end{aligned} \qquad \text{IV.27}$$

With Λ an arbitrary spinor solution of the wave equation

$$\pi^2 \Lambda = 0 \ .$$

Upon substitution of equations IV.26a and IV.26b in equations IV.20, IV.22 and IV.23 we find:

$$\begin{aligned} &\gamma\cdot\pi V_\kappa + ieF_{\mu\kappa}\gamma^\mu B + \gamma_\kappa \frac{ie}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu B - \pi_\kappa\gamma^\lambda V_\lambda + \gamma_\kappa\gamma\cdot\pi\gamma^\lambda V_\lambda \\ &\quad - m(V_\kappa + \pi_\kappa B) + m(\gamma_\kappa\gamma^\lambda V_\lambda + \gamma_\kappa\gamma^\lambda\pi_\lambda B) \\ &\quad + T_\kappa{}^\lambda V_\lambda + T_\kappa{}^\lambda \pi_\lambda B = 0 \ . \end{aligned} \qquad \text{IV.28}$$

$$ie\gamma_\mu F^{\kappa\mu}V_\kappa + ie\pi^\kappa F_{\mu\kappa}\gamma^\mu B + \frac{ie}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^\lambda V_\lambda$$

$$+ \frac{ie}{2} mF_{\mu\nu}\gamma^\mu\gamma^\nu B + \frac{ie}{2} \pi^\kappa\gamma_\kappa F_{\mu\nu}\gamma^\mu\gamma^\nu B + \pi^\kappa T_\kappa{}^\lambda V_\lambda$$

$$+ \pi^\kappa T_\kappa{}^\lambda \pi_\lambda B + m\pi^\kappa\gamma_\kappa\gamma^\lambda V_\lambda = 0 \qquad \text{IV.29}$$

$$2\gamma\cdot\pi\gamma^\lambda V_\lambda + 3m(\gamma^\lambda V_\lambda + \gamma^\lambda\pi_\lambda B) + ieF_{\mu\nu}\gamma^\mu\gamma^\nu B$$

$$+ \gamma^\kappa T_\kappa{}^\lambda \pi_\lambda B + \gamma^\kappa T_\kappa{}^\lambda V_\lambda = 0 \ . \qquad \text{IV.30}$$

Analysis of these equations indicates that equation IV.28 contains the primary constraints; these we obtain by setting $\kappa = 0$:

$$(\vec{\pi} - h\vec{\alpha})(\vec{V} + \vec{\pi}B) + T_o{}^i(V_i + \pi_i B) = 0 \ .$$

It also contains the time derivatives of V_o but does not contain the time derivative of B when the interaction is removed.

Hence it is mainly the kinematical equation of motion for the vector-spinor V_μ. Equation IV.30, a useful relation as will be seen shortly, is satisfied identically by IV.28. A kinematical equation for B will be obtained from IV.28 and IV.29. However, IV.29 does not carry any vector indices and this is where the usefulness of IV.30 appears. Solving for $\gamma\cdot\pi\gamma^\lambda V_\lambda$ from IV.30 and substituting in IV.29 we find:

$$ie\gamma_\mu F^{\kappa\mu}V_\kappa + ie\pi^\kappa F_{\mu\kappa}\gamma^\mu B + \frac{ie}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu\gamma^\lambda V_\lambda + \frac{ie}{2} mF_{\mu\nu}\gamma^\mu\gamma^\nu B$$

$$+ \frac{ie}{2} \pi^\kappa\gamma_\kappa F_{\mu\nu}\gamma^\mu\gamma^\nu B + \pi^\kappa T_\kappa{}^\lambda V_\lambda + (\pi^\kappa T_\kappa{}^\lambda)\pi_\lambda B$$

$$+ T_\kappa{}^\lambda\pi^\kappa\pi_\lambda B + m(-\frac{3}{2} m(\gamma^\lambda V_\lambda + \gamma^\lambda\pi_\lambda B) - \frac{ie}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu B$$

$$- \frac{\gamma^\kappa}{2} T_\kappa{}^\lambda\pi_\lambda B - \frac{\gamma^\kappa}{2} T_\kappa{}^\lambda V_\lambda) = 0 \quad . \qquad \text{IV.31}$$

Equation IV.31 is still not the kinematical equation for B. It is dominated by the dynamical term $T_\kappa{}^\lambda\pi_\kappa\pi^\lambda B$ which makes it second order dynamically and hence will never produce a kinematical equation for B which would imply inconsistency. The only way to avoid this inconsistency is by demanding that $T_\kappa{}^\lambda$ be antisymmetric. With this condition on $T_\kappa{}^\lambda$ equation IV.31 becomes an equation of motion for B. The secondary constraints required to eliminate the rest of the redundant components are implied by equations IV.26b and IV.30. The preservation of these constraints in time as well as the fact that equations IV.26, IV.28, IV.30 and IV.31 imply the original Rarita-Schwinger equations i.e. equation IV.20 can be shown in a manner analogous to that in[V.1,2]. Omitting the terms which do not contain derivatives in equations IV.28 and IV.31 we find:

$$\gamma\cdot\pi V_\kappa - \pi_\kappa\gamma^\lambda V_\lambda + \gamma_\kappa\gamma\cdot\pi\gamma^\lambda V_\lambda - m\pi_\kappa B + m\gamma_\kappa\gamma^\lambda\pi_\lambda B$$

$$+ T_\kappa{}^\lambda\pi_\lambda B = 0 \qquad \text{IV.32}$$

and

$$ie\pi^{\kappa}F_{\mu\kappa}\gamma^{\mu}B + \frac{ie}{2}\,\pi^{\kappa}\gamma_{\kappa}F_{\mu\nu}\gamma^{\mu}\gamma^{\nu}B - \frac{3}{2}\,m^{2}\gamma.\pi B$$

$$- \frac{m}{2}\,\gamma^{\kappa}T_{\kappa}{}^{\lambda}\pi_{\lambda}B + \pi^{\kappa}T_{\kappa}{}^{\lambda}V_{\lambda} + (\pi^{\kappa}T_{\kappa}{}^{\lambda})\pi_{\lambda}B = 0 \,. \qquad \text{IV.33}$$

These are the equations which determine the characteristic determinant of the fields V_{μ} and B when we replace π^{λ} by n^{λ}. Now we determine the most general form of the anti-symmetric tensor-spinor $T_{\kappa}{}^{\lambda}$ i.e. the most general Pauli tensors that can be added to the Lagrangian that are consistent with the spin 3/2 R.S equations. The Lorentz tensors are $g_{\mu\nu}$, $\varepsilon^{\mu\nu\rho\sigma}$, $F^{\mu\nu}$ and the spinors (scalar, P.S., vectors, P.V. and tensors) are 1, γ^{5}, γ^{μ}, $\gamma^{5}\gamma^{\mu}$ and $\sigma^{\mu\nu}$. The most general anti-symmetric (in vector indices) parity conserving spinor tensor that can be constructed out of these is:

$$T_{\kappa}{}^{\lambda} = iq_{2}F_{\kappa}{}^{\lambda} + iq_{3}\gamma^{5}F_{\kappa}{}^{\lambda^{d}} + iq_{1}(F_{\kappa\mu}\sigma^{\mu\lambda} - F^{\lambda\mu}\sigma_{\mu\kappa}) \,. \qquad \text{IV.34}$$

Here q_1, q_2, and q_3 are real and have the dimension e/m. We remark here that the anti-symmetric term $q_4(\gamma^{5}F^{d}{}_{\sigma}{}^{\mu\lambda} - F^{d\lambda\mu}\gamma^{5}\sigma_{\mu\kappa})$ which seems to be absent from equation IV.34

is actually identical with the last term as can be easily seen from writing $\gamma^5\sigma^{\mu\lambda}$, $\gamma^5\sigma_{\mu\kappa}$ in terms of the σ's and F^d in terms of F.

Replacing π^λ by n^λ in equations IV.32 and IV.33 we see that the determinant will be a polynomial in n_μ. Therefore we pick $n_\mu \equiv (n, 0, 0, 0)$ to calculate the determinant in the rest frame and then by a Lorentz transformation we may go back to a general frame. With these equations IV.32 and IV.33 become:

$$n\gamma^o V_\kappa - ng_{\kappa,o} + n\gamma_\kappa\gamma^o\gamma^\lambda V_\lambda - mng_{\kappa,o} + mn\gamma_\kappa\gamma^o B$$

$$+ nT_\kappa{}^o B = 0 \qquad \text{IV.32'}$$

and

$$ienF_{\mu o}\gamma^\mu B + \frac{ie}{2} n\gamma_o F_{\mu\nu}\gamma^\mu\gamma^\nu B - \frac{3}{2} m^2 n\gamma^o B$$

$$- \frac{mn}{2} \gamma^\kappa T_\kappa{}^o B + nT_o{}^\lambda V_\lambda + n(\pi^\kappa T_\kappa{}^o)B = 0 \ . \qquad \text{IV.33'}$$

The characteristic determinant is then:

$$
D(n) = \left|\begin{array}{c|c|c|c|l}
n\gamma^{o} & 0 & 0 & 0 & 0 \\
\hline
n\gamma_{1} & 0 & n\gamma_{1}\gamma_{o}\gamma^{2} & n\gamma_{1}\gamma_{o}\gamma^{3} & nm\gamma_{1}\gamma^{o} + nT_{1}{}^{o} \\
\hline
n\gamma_{2} & n\gamma_{2}\gamma_{o}\gamma^{1} & 0 & n\gamma_{2}\gamma_{o}\gamma^{3} & nm\gamma_{2}\gamma^{o} + nT_{2}{}^{o} \\
\hline
n\gamma_{3} & n\gamma_{3}\gamma_{o}\gamma^{1} & n\gamma_{3}\gamma_{o}\gamma^{2} & 0 & nm\gamma_{3}\gamma^{o} + nT_{3}{}^{o} \\
\hline
0 & nT_{o}{}^{1} & nT_{o}{}^{2} & nT_{o}{}^{3} & ineF_{\mu o}\gamma^{\mu} + \frac{ie}{2}\, n\gamma^{\mu}\gamma^{\nu}F_{\mu\nu} \\
 & & & & - \frac{3}{2}\, nm^{2}\gamma^{o} - \frac{nm}{2}\, \gamma^{\kappa}T_{\kappa}{}^{o} \\
 & & & & + n(\pi^{\kappa}T_{\kappa}{}^{o})
\end{array}\right|
$$

Before we carry on the calculation we briefly comment on the term $(\pi^{\kappa}T_{\kappa}{}^{o})$ which enters the determinant. This term written explicitly reads $(i\partial^{\kappa} + eA^{\kappa})T_{\kappa}{}^{o}$ and thus involved the source of the e.m field as well as the vector potential A_{μ} and its derivatives. The presence of these terms will ultimately lead to characteristic surfaces that are dependent on the source of the electromagnetic field and the vector potential and its derivatives. Further examination of equation IV.34 indicates that this type of dependence cannot be compensated for by the electric or the magnetic terms present in the rest of the determinant, hence there will be characteristic surfaces which involve the vector potential and its derivatives as well as the source except of course for the special case of equation IV.34 when $q_1 = q_2 = 0$. With $q_1 = q_2 = 0$ $(\pi^\kappa T_\kappa{}^o)$ is identically zero because

$$\pi^\kappa T_\kappa{}^o = iq_3(eA^\kappa + e\partial^\kappa)F^d{}_\kappa{}^o$$

$$= iq_3[e\vec{A}.(\vec{\nabla}\times\vec{A}) + i\vec{\nabla}.\vec{B}]$$

$$= 0$$

and the Pauli term $iq_3\bar{\psi}^\mu\gamma^5 F^d{}_{\mu\nu}\psi^\nu$ seems to be the most natural term that can be added to the Lagrangian. Calculation of the characteristic determinant for

$$T_\kappa{}^\lambda = iq_3\gamma^5 F^d{}_\kappa{}^\lambda$$

gives

$$D(n) = n^{20}[(\tfrac{3}{2})^2 m^4 - (\tfrac{5}{2} m^2 q_3{}^2 + e^2 + 2mq_3 e)B^2 + q_3{}^4{}_{/4}B^4][(\tfrac{3}{2})^2 m^4 - (\tfrac{5}{2} m^2 q_3{}^2 + e^2 - 2mq_3 e)B^2 + q_3{}^4{}_{/4}B^4] .$$

Written in covariant form it becomes:(†)

(†) This reduces to the characteristic determinant given in (V.1) when $q_3 = 0$.

$$D(n) = n^{12}[(\tfrac{3}{2})^2 m^4n^4 + (\tfrac{5}{2} m^2q_3{}^2 + e^2 + 2mq_3e)n^2(n.F^d)^2 + q_{3/4}^4(n.F^d)^2)^2]\ [(\tfrac{3}{2})^2 m^4n^4 + (\tfrac{5}{2} m^2q_3{}^2 + e^2 - 2mq_3e)n^2 \times (n.F^d)^2 + q_{3/4}^4((n.F^d)^2)^2] \ . \qquad \text{IV.35}$$

This can be written as:

$$D(n) = n^{12}[(\tfrac{3}{2})^2 m^4(n_o{}^2 - \bar{n}^2)^2 - (\tfrac{5}{2} m^2q_3{}^2 + e^2 + 2mq_3e) \times (n_o{}^2 - \bar{n}^2)n_o{}^2B^2 + n_o{}^4q_{3/4}^4B^4]\ [(\tfrac{3}{2})^2m^4(n_o{}^2 - \bar{n}^2)^2 - (\tfrac{5}{2} m^2q_3{}^2 + e^2 - 2mq_3e)(n_o{}^2 - \bar{n}^2)n_o{}^2B^2 + n_o{}^4q_{3/4}^4B^4] \ .$$

Here we see that in addition to the light cone characteristic surfaces given by $n^{12} = 0$ we have characteristic surfaces determined by:

$$[(\tfrac{3}{2})^2 m^4(n_o{}^2 - \bar{n}^2)^2 - (\tfrac{5}{2} m^2q_3{}^2 + e^2 + 2mq_3e)(n_o{}^2 - \bar{n}^2) \times n_o{}^2B^2 + n_o{}^4q_{3/4}^4B^4] = 0 \qquad \text{IV.36}$$

$$[(\tfrac{3}{2})^2m^4(n_o{}^2 - \bar{n}^2)^2 - (\tfrac{5}{2} m^2q_3{}^2 + e^2 - 2mq_3e)(n_o{}^2 - \bar{n}^2) \times n_o{}^2B^2 + n_o{}^4q_{3/4}^4B^4] = 0 \ .$$

We divide equation IV.36 by $n_o{}^4$, $n_o \neq 0$ and let

$$1 - \frac{\bar{n}^2}{n_o{}^2} \equiv x \ . \qquad \text{IV.37}$$

Then propagation will be causal if and only if $x < 0$, when $0 < x < 1$ propagation is acausal and for $x > 1$ the system ceases to be hyperbolic. Substituting equation IV.37 in IV.36 gives:

$$(\tfrac{3}{2})^2 m^4 X^2 - (\tfrac{5}{2} m^2 q_3{}^2 + e^2 + 2mq_3 e) B^2 X + q_3{}^4{}_{/4} B^4 = 0$$

$$(\tfrac{3}{2})^2 m^4 X^2 - (\tfrac{5}{2} m^2 q_3{}^2 + e^2 - 2mq_3 e) B^2 X + q_3{}^4{}_{/4} B^4 = 0 \; . \qquad \text{IV.38}$$

An examination of equation IV.38 indicates that the maximum number of negative roots is two and the minimum number of positive roots is two,hence there will always be space-like characteristic surfaces and propagation will always be acausal no matter how small (but finite) the e.m field is. Furthermore, the system will lose its hyperbolicity for a certain range of the coupling constant q_3 and the magnitude of the e.m field. Next we calculate the characteristic determinant for the most general Pauli term given by IV.34. Here we must à priori insist that $(\pi^\lambda T_\lambda{}^o)$ be identically zero otherwise some characteristic surfaces will depend on the source, the vector potential and its derivatives. Therefore, granted that $\pi^\lambda T_\lambda{}^o = 0$ for a special class of sources, then the characteristic determinant (in covariant form) is:

$$D(n) = n^{12} [-n^2e^2(F^d.n)^2 + 4q_1{}^2(q_2-q_3)^2(n.F^d.F.n)^2$$

$$- (-q^2{}_{2/2}(F.n)^2 - q^2{}_{3/2}(F^d.n)^2 - \tfrac{3}{2}\, n^2m^2)^2]^2$$

$$- 2[4q_1{}^2(q_2-q_3)^2(n.F^d.F.n)^2 + (-q^2{}_{2/2}(F.n)^2$$

$$- q^2{}_{3/2}\ (F^d.n)^2 - \tfrac{3}{2}\, n^2m^2)^2]$$

$$\times [-4m^2n^2q_1{}^2(F.n)^2 - m^2n^2q_3{}^2(F^d.n)^2 + (2q_1{}^2 + q_2q_3)^2$$

$$\times (n.F)^2(n.F^d)^2 - (2q_1{}^2 + q_2q_3)(n.F_d.F.n)^2]$$

$$+ 8m^2n^2q_1{}^2q_3(q_2-q_3)(n.F^d.F.n)^2[4(-q^2{}_{2/2}(n.F)^2$$

$$- q^2{}_{3/2}(n.F^d)^2 - 6n^2m^2] + 4m^2n^4e^2[4q_1{}^2(n.F^d.F.n)^2$$

$$- q_3{}^2(F^d.n)^2)^2] - 16m^4q_1{}^2q_3{}^2n^4(n.F^d.F.n)^2$$

$$- 2e^2n^2(F^d.n)^2[-4m^2q_1{}^2n^2(n.F)^2 + m^2n^2q_3{}^2(n.F^d)^2$$

$$+ (2q_1{}^2 + q_2q_3)(n.F)^2(n.F^d)^2 - (n.F^d.F.n)^2)]$$

$$+ [-4m^2n^2q_1{}^2(n.F)^2 - m^2n^2q_3{}^2(n.F^d)^2 + (2q_1{}^2 + q_2q_3)^2$$

$$\times (n.F)^2(n.F^d)^2 - (n.F^d.F.n)^2)]^2 . \qquad \text{IV.39}$$

This can be written as:

$$
\begin{aligned}
D(n) = n^{12}\ & [n^2 n_o{}^2 e^2 B^2 + 4q_1{}^2 (q_2 - q_3)^2 n_o{}^4 (E.B)^2 \\
& - (q^2{}_{2/2} n_o{}^2 E^2 + q^2{}_{3/2} n_o{}^2 B^2 - \tfrac{3}{2} n^2 m^2)^2]^2 \\
& - 2[4q_1{}^2 (q_2 - q_3)^2 n_o{}^4 (E.B)^2 + q^2{}_{2/2} n_o{}^2 E^2 \\
& + q^2{}_{3/2} n_o{}^2 B^2 - \tfrac{3}{2} n^2 m^2)^2] \\
& \times [4n^2 m^2 n_o{}^2 q_1{}^2 E^2 + m^2 n^2 q_3{}^2 n_o{}^2 B^2 \\
& + (2q_1{}^2 + q_2 q_3)[n_o{}^4 E^2 B^2 - n_o{}^4 (E.B)^2]] \\
& + 8m^2 n^2 q_1{}^2 q_3 (q_2 - q_3) n_o{}^4 (E.B)^2 \\
& \times [4n_o{}^2 (q^2{}_{2/2} E^2 + q^2{}_{3/2} B^2) - 6n^2 m^2] \\
& + 4m^2 n^4 e^2 [4q_1{}^2 n_o{}^4 (E.B)^2 - q_3{}^2 n_o{}^4 B^4] \\
& - 16m^4 q_1{}^2 q_3{}^2 n^4 n_o{}^4 (E.B)^2 - 2e^2 n^2 n_o{}^2 B^2 \\
& \times [4m^2 n^2 q_1{}^2 n_o{}^2 E^2 - m^2 n^2 q_3{}^2 n_o{}^2 B^2 \\
& + n_o{}^4 (2q_1{}^2 + q_2 q_3)(E^2 B^2 - (E.B)^2)] \\
& + [4m^2 n^2 q_1{}^2 n_o{}^2 E^2 + m^2 n^2 q_3{}^2 n_o{}^2 B^2 \\
& + n_o{}^4 (2q_1{}^2 + q_2 q_3)(E^2 B^2 - (E.B)^2)]^2 \qquad \text{IV.39'}
\end{aligned}
$$

Dividing equation IV.39' by n_o with $n_o \neq 0$ and using equation IV.37 the determinant becomes a fourth degree polynomial in X. We emphasize again that a necessary and sufficient condition for the propagation to be causal is that all the roots of the polynomial be negative and that if some of the roots are nonnegative then propagation will be acausal and loss of hyperbolicity may occur. To study the roots of this polynomial we use "Descartes Rule of Signs" technique. According to this technique the maximum number of positive roots for a polynomial with its terms listed in a descending degree is equal to the number of variations of sign of the coefficients. Similarly, the maximum number of negative roots is equal to the number of variations of sign when X is replaced by (-X) in the polynomial. Thus, if there is a change of sign in the first two terms of a quartic polynomial then there would be one positive root regardless of the coefficients of the other terms. If, however, there is no change in sign in the first two terms we need to go to a third term and so on. The first two terms of our polynomials are:

$$(\tfrac{3}{2}\, m^2)^4 X^4 - (\tfrac{3}{2}\, m^2)^2 [m^2 (8q_1{}^2 + 3q_2{}^2) E^2 + (5m^2 q_3{}^2 + 2e^2) B^2] X^3.$$

Here we see that the coefficients of X^3 are always negative, therefore the maximum number of negative roots of our polynomial is three regardless of the coefficient of the other terms. Hence, we conclude that the spin 3/2 R.S theory coupled to an (e.m) field is always acausal and may even cease to be hyperbolic or both. Therefore, as mentioned in the introduction we conclude that all spin 3/2 theories are acausal and may lose hyperbolicity or both when minimally coupled to the (e.m) field.

5. Spin 1/2 (Dirac) Coupled to Spin 1 (Proca)

If the intermediate vector boson theory of weak interaction is correct, it is interesting to know what restrictions causality places on the possible form of the boson-lepton interaction.

The free Proca field and Dirac field Lagrangians are:

$$\mathcal{L}_P = \frac{1}{2}\,(p_\mu \omega^\nu p^\mu \omega_\nu - p_\mu \omega_\nu p^\nu \omega^\mu) + \frac{1}{2}\, m^2 \omega_\nu \omega^\nu$$

$$\mathcal{L}_D = -\,\bar{\psi}(-\not{p} + M)\psi \quad .$$

For the interaction Lagrangians we take the following:

$$\mathcal{L}_I : g_V \omega^\mu \bar{\psi}\gamma_\mu \psi,\ i\,\frac{g}{2}{}_{DV}\,(p_\mu \omega^2)\psi\gamma^\mu\psi,\ ig_A \omega^\mu \bar{\psi}\gamma_\mu\gamma^5\psi,$$

$$\frac{1}{2}\, g_{DA}(p_\mu \omega^2)\bar{\psi}\gamma^\mu\gamma^5\psi,\ \ \frac{g}{2}S\ \omega^2\bar{\psi}\psi,\ ig_{DS}(p_\mu\omega^\mu)\bar{\psi}\psi,$$

$$\frac{g}{2}\ p\omega^2\bar{\psi}\gamma^5\psi,\ ig_{DP}(p_\mu\omega^\mu)\bar{\psi}\gamma^5\psi,\ +\ ig_{DT}(p_\mu\omega_\nu)\bar{\psi}\sigma^{\mu\nu}\psi \ .$$

We will examine each of these interactions individually, then a combination of all of them. We start with direct vector coupling type of interaction:

a) $\mathcal{L}_V = g_V \omega^\mu \bar{\psi} \gamma_\mu \psi .$

The Lagrangian equations are

$$p^2 \omega^\nu - p_\mu p^\nu \omega^\mu - m^2 \omega^\nu - g_V \bar{\psi} \gamma^\lambda \psi = 0 \qquad \text{IV.40}$$

$$\left.\begin{aligned} (-\not{p} + M)\psi - g_V \omega^\mu \gamma_\mu \psi &= 0 \qquad \text{IV.41a} \\ \bar{\psi}(\not{p} + M) - g_V \bar{\psi} \gamma_\mu \omega^\mu &= 0 \;. \qquad \text{IV.41b} \end{aligned}\right\} \quad \text{IV.41}$$

Examination of equation IV.40 shows that the second time derivative of ω^o does not appear which implies the primary constraint of the Proca field. Contracting equation IV.40 with p_ν we get the secondary constraint:

$$p_\nu \omega^\nu = - \frac{g_V}{m^2} \; p_\nu (\bar{\psi} \gamma^\nu \psi) \;.$$

Using equation IV.41 we find that

$$p_\nu (\bar{\psi} \gamma^\nu \psi) = 0 \;,$$

so the secondary constraint becomes:

$$p_\nu \omega^\nu = 0 \;.$$

Substituting in equation IV.40 and multiplying equation IV.41a and IV.41b by $\vec{\not p}$ and $\overleftarrow{\not p}$ respectively then replacing p_ν by n_ν in the highest derivative for the whole system of equations we find:

$$n^2\omega^\nu = 0$$

$$n^2\psi = 0$$

$$n^2\,\bar{\psi} = 0\;.$$

The characteristic determinant of the above system is:

$$D(n) = (n^2)^{12}\;.$$

Hence all the characteristic surfaces are the light cones and propagation is always causal. Thus, the direct vector type of interaction is always causal.

Next we consider the derivative coupling vector type of interaction:

b) $\mathcal{L}_{DV} = \frac{ig}{2}{}_{DV}\,(p_\mu\omega^2)\bar{\psi}\gamma^\mu\psi.$

The Lagrangian equations are:

$$p^2\omega^\nu - p_\mu p^\nu\omega^\mu - m^2\omega^\nu + ig_{DV}\omega^\nu p_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \qquad \text{IV.42}$$

$$\left.\begin{aligned} &(-\not p + M)\psi - ig_{DV}\omega^\nu(p_\mu\omega_\nu)\gamma^\mu\psi = 0 && \text{IV.43a} \\ &\bar{\psi}(\not p + M) - ig_{DV}\omega^\nu(p_\mu\omega_\nu)\bar{\psi}\gamma^\mu = 0 && \text{IV.43b} \end{aligned}\right\} \quad \text{IV.43}$$

but by the use of the last two equations we find that:

$$p_\mu(\bar{\psi}\gamma^\mu\psi) \equiv 0 \ .$$

Hence the secondary constraint becomes:

$$p_\nu\omega^\nu = 0 \ .$$

Inserting in equation IV.42 and replacing p_ν by n_ν in the highest derivatives after multiplying equations IV.43a and IV.43b by $\vec{\not{p}}$ and $\overleftarrow{\not{p}}$ respectively we find:

$$n^2\omega^\nu = 0$$

$$n^2\psi + ig_{DV}(\omega^\nu n_\rho n_\mu \gamma^\rho\gamma^\mu\psi)\omega_\nu = 0$$

$$n^2\bar{\psi} - ig_{DV}(\omega^\nu n_\rho n_\mu \bar{\psi}\gamma^\mu\gamma^\rho)\omega_\nu = 0 \ .$$

The characteristic determinant of the above system is $D(n) = (n^2)^{12}$, therefore the characteristic surfaces are the light cones and propagation of the derivative coupling vector types of interaction is also always causal.

c) Direct Coupling Axial Vector Interaction:

Here $\mathcal{L}_A = ig_A\omega^\mu\bar{\psi}\gamma_\mu\gamma^5\psi$.

The Lagrangian equations are:

$$p^2\omega^\nu - p^2 p_\mu\omega^\mu - m^2\omega^\nu - ig_A\bar{\psi}\gamma^\nu\gamma^5\psi = 0 \qquad \text{IV.44}$$

$$(-\not{p} + M)\psi - ig_A\omega^\mu\gamma_\mu\gamma^5\psi = 0 \qquad \text{IV.45a}$$

$$\bar{\psi}(\not{p} + M) - ig_A\omega^\mu\bar{\psi}\gamma_\mu\gamma^5 = 0 \ . \qquad \text{IV.45b}$$

IV.45

The secondary constraints are:

$$p_\nu\omega^\nu = -\frac{ig_A}{m^2} p_\nu(\bar{\psi}\gamma^\nu\gamma^5\psi) \ .$$

However, by the use of equation IV.45 these are reduced to

$$p_\nu\omega^\nu = \frac{2ig_AM}{m^2} \bar{\psi}\gamma^5\psi \ .$$

By the same procedure used above we find the characteristic determinant to be $D(n) = (n^2)^{12}$, hence the direct coupling axial vector type of interaction is also always causal.

Now we consider the derivative coupling axial vector type of interaction:

d) $$\mathcal{L}_{DA} = \frac{1}{2} g_{DA}(p_\mu\omega^2)\bar{\psi}\gamma^\mu\gamma^5\psi$$

$$= g_{DA}\omega_\nu(p_\mu\omega^\nu)\bar{\psi}\gamma^\mu\gamma^5\psi \ .$$

The equations of motion are:

$$p^2\omega^\nu - p^\nu p_\mu\omega^\mu - m^2\omega^\nu + g_{DA}\omega^\nu p_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = 0 \qquad \text{IV.46}$$

$$(-\not{p} + M)\psi - g_{DA}\omega^\nu(p_\mu\omega_\nu)\gamma^\mu\gamma^5\psi = 0 \qquad \text{IV.47a}$$

$$\bar{\psi}(\not{p} + M) - g_{DA}\omega^\nu(p_\mu\omega_\nu)\bar{\psi}\gamma^\mu\gamma^5 = 0 \ . \qquad \text{IV.47b}$$

IV.47

The secondary constraints are:

$$p_\nu \omega^\nu = \frac{g_{DA}}{m^2} p_\nu (\omega^\nu p_\mu (\bar{\psi}\gamma^\mu \gamma^5 \psi)).$$

By the use of equation IV.47 this can be reduced to

$$p_\nu \omega^\nu = - \frac{2Mg_{DA}}{m^2} p_\nu (\omega^\nu \bar{\psi}\gamma^5 \psi) .$$

By the same procedure used above the equations which determine the characteristic determinants are:

$$n^2\omega^\nu + \frac{2Mg_{DA}\omega^\mu n_\mu n^\nu \bar{\psi}\gamma^5\psi}{m^2[1 + \frac{2Mg_{DA}}{m^2}\bar{\psi}\gamma^5\psi]} + \frac{\bar{\psi}2Mg_{DA}\omega^\mu n_\mu n^\nu \gamma^5\psi}{m^2[1 + \frac{Mg_{DA}}{m^2}\bar{\psi}\gamma^5\psi]} = 0$$

$$n^2\psi + g_{DA}(\omega_\nu n_\mu n_\rho \gamma^\rho \gamma^\mu \gamma^5 \psi)\omega^\nu = 0$$

$$n^2\bar{\psi} - g_{DA}(\omega_\nu n_\mu n_\rho \bar{\psi}\gamma^\mu \gamma^5 \gamma^\rho)\omega^\nu = 0 .$$

The characteristic determinant is:

$$D(n) = (n^2)^{10}\left\{(n^2)^2 + \frac{4Mg^2_{DA}(n_\nu\omega^\nu)^2 n_\rho n_\delta \bar{\psi}\gamma^\rho\gamma^\delta\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi}\right\}$$

$$= (n^2)^{10}\left\{(n^2)^2 + n_o^{\,4}\;\frac{4Mg^2_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi}\right\} .$$

As in spin 3/2 we let $X = 1 - (\bar{n}^2/n_o^{\,2})$. The last term in the bracket becomes:

$$X^2 + \frac{4Mg^2{}_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi} \quad .$$

If

$$\frac{4Mg^2{}_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi} > 0$$

all the roots are complex and the system loses hyperbolicity, but if

$$\frac{4Mg^2{}_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi} < 0$$

then we have one negative and one positive root. In this case there will always be space-like characteristic surfaces and the system will be acausal. Hence the D.A type of coupling will always lead to acausal propagation if

$$\frac{4Mg^2{}_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi} < 0$$

and loss of hyperbolicity if

$$\frac{4Mg^2{}_{DA}\omega_o\omega^o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi} > 0 \quad .$$

e) Next we consider direct scalar coupling:

$$\mathcal{L}_S = \frac{g_S}{2}\,\omega^2\bar{\psi}\psi \quad .$$

The equations of motion are:

$$p^2\omega^\nu - p_\mu p^\nu \omega^\mu - m^2\omega^\nu - g_S\omega^\nu\bar{\psi}\psi = 0 \qquad \text{IV.48}$$

$$\left.\begin{aligned} &(-\not{p} + M)\psi - \frac{g_S}{2}\,\omega^2\psi = 0 && \text{IV.49a} \\ &\bar{\psi}(\not{p} + M) - \frac{g_S}{2}\,\omega^2\bar{\psi} = 0\,. && \text{IV.49b} \end{aligned}\right\} \quad \text{IV.49}$$

The secondary constraints are:

$$p_\nu\omega^\nu = -\frac{g_S\omega^\nu p_\nu(\bar{\psi}\psi)}{m^2(1 + \frac{g_S}{m^2}\,\bar{\psi}\psi)}\quad .$$

By the same procedure used above we find the characteristic determinant to be $D(n) = (n^2)^{12}$, hence the characteristic surfaces are the light cones and propagation is always causal.

f) Next we consider the derivative scalar type of interaction:

$$\mathcal{L}_{DS} = ig_{DS}(p_\mu\omega^\mu)\bar{\psi}\psi \quad .$$

The equations of motion are:

$$p^2\omega^\nu - p_\mu p^\nu\omega^\mu - m^2\omega^\nu + ig_{DS}p^\nu(\bar{\psi}\psi) = 0 \qquad \text{IV.50}$$

$$\left.\begin{aligned} &(-\not{p} + M)\psi - ig_{DS}(p_\mu\omega^\mu)\psi = 0 && \text{IV.51a} \\ &\bar{\psi}(\not{p} + M) - ig_{DS}(p_\mu\omega^\mu)\bar{\psi} = 0\,. && \text{IV.51b} \end{aligned}\right\} \quad \text{IV.51}$$

The secondary constraints are:

$$p_\nu \omega^\nu = \frac{ig_{DS}}{m^2} p^2(\bar{\psi}\psi) = \frac{ig_{DS}}{m^2}[(p^2\bar{\psi})\psi + \bar{\psi}(p^2\psi) + 2(p_\nu\bar{\psi})(p^\nu\psi)].$$

Although the second derivative of $\bar{\psi}$ and ψ appear in this equation it is still of first order and it is still a constraint equation. The reduction is done by the use of equation IV.51 and the result is

$$p_\nu \omega^\nu = \frac{2ig_{DS}}{m^2} [(M - ig_{DS}(p_\mu \omega^\mu))^2 \bar{\psi}\psi + p_\nu \bar{\psi} p^\nu \psi] .$$

The equations of motion which determine the characteristic determinant are:

$$n^2\omega^\nu - \frac{2ig_{DS}}{\alpha m^2} \bar{\psi}(n^\nu n_\mu p^\mu \psi) - \frac{2ig_{DS}}{\alpha m^2} (p^\mu \bar{\psi} n^\nu n_\mu)\psi = 0$$

$$p^2\psi + ig_{DS}(n_\rho n_\mu \gamma^\rho \psi)\omega^\mu = 0$$

$$p^2\bar{\psi} - ig_{DS}(\bar{\psi}\gamma^\rho n_\rho n_\mu)\omega^\mu = 0$$

where

$$\alpha = \frac{1}{1 - \frac{4g^2_{DS}}{m^2} (M - ig_{DS} p_\mu \omega^\mu)} .$$

The characteristic determinant is

$$D(n) = (n^2)^{11}[n^2 + \frac{2g^2_{DS}}{\alpha m^2}\{\bar{\psi}\gamma^{\rho}n_{\rho}n_{\nu}(p^{\nu}\psi) - (p_{\mu}\bar{\psi}\gamma^{\rho}n_{\rho}n^{\mu}\psi\}] .$$

Here we see that the term in brackets determines the characteristic surface which depends on the Dirac field and its derivatives and also on the derivatives of the Proca field through α. Thus, if

$$\frac{2g^2_{DS}}{\alpha m^2} \{\bar{\psi}\gamma^{o}(p^{o}\psi) - (p_{o}\bar{\psi})\gamma^{o}\psi\} > 0$$

then propagation is causal. If this is not satisfied, the propagation is acausal whenever the above term lies between 0 and -1. If it is less than -1 the equations lose their hyperbolicity.

g) Next we investigate the direct pseudo-scalar type of interaction:

$$\mathcal{L}_P = \frac{1}{2} g_P \omega^2 \bar{\psi}\gamma^5\psi .$$

The equations of motion are:

$$p^2\omega^{\nu} - p_{\mu}p^{\nu}\omega - g_P\omega^{\nu}\bar{\psi}\gamma^5\psi = 0 \qquad \text{IV.52}$$

$$(-\not{p} + M)\psi - \frac{g_P}{2}\omega^{\nu}\gamma^5\psi = 0 \qquad \text{IV.53a}$$

$$\bar{\psi}(\not{p} + M) - \frac{g_P}{2}\omega^2\bar{\psi}\gamma^5 = 0 . \qquad \text{IV.53b}$$

(IV.53)

The secondary constraints are:

$$p_\nu \omega^\nu = - \frac{g_P}{m^2} \frac{\omega^\nu p_\nu (\bar{\psi}\gamma^5\psi)}{(1 + \frac{g_P}{m^2} \bar{\psi}\gamma^5\psi)}$$

and the equations which determine the characteristic determinant are:

$$n^2\psi = 0$$

$$n^2\bar{\psi} = 0$$

$$n^2\omega^\nu + \frac{(g_P\omega^\mu n_\mu n^\nu \bar{\psi}\gamma^5)\psi}{m^2 + g_P\bar{\psi}\gamma^5\psi} + \bar{\psi} \frac{(g_P\omega^\mu n_\mu n^\nu \gamma^5\psi)}{m^2 + g_P\bar{\psi}\gamma^5\psi} = 0 \ .$$

The characteristic determinant is $D(n) = (n^2)^{12}$ and propagation is always causal with the light cone as the characteristic surfaces.

h) Now we consider the derivative coupling pseudoscalar type of interaction:

$$\mathcal{L}_{DP} = ig_{DP}(p_\mu \omega^\mu)\bar{\psi}\gamma^5\psi \quad .$$

Here the equations of motion are:

$$p^2\omega^\nu - p_\mu p^\nu \omega^\mu - m^2\omega^\nu + ig_{DP}p^\nu(\bar{\psi}\gamma^5\psi) = 0 \qquad \text{IV.54}$$

$$\left.\begin{aligned} (-\not{p} + M)\psi - ig_{DP}(p_\mu\omega^\mu)\gamma^5\psi &= 0 \qquad \text{IV.55a} \\ \bar{\psi}(\not{p} + M) - ig_{DP}(p_\mu\omega^\mu)\bar{\psi}\gamma^5 &= 0 \ . \qquad \text{IV.55b} \end{aligned}\right\} \quad \text{IV.55}$$

The secondary constraints are:

$$p_\nu \omega^\nu = \frac{ig_{DP}}{m^2} p^2(\bar{\psi}\gamma^5\psi) = \frac{ig_{DP}}{m^2}[(p^2\bar{\psi})\gamma^5\psi + \bar{\psi}\gamma^5(p^2\psi)$$

$$+ 2(p_\nu\bar{\psi})\gamma^5(p^\nu\psi)] \quad .$$

The second derivative must be eliminated by the use of the equation of motion. Using the second and third equations of motion we get for the secondary constraint:

$$p_\nu \omega^\nu = \frac{ig_{DP}}{m^2}[(2M^2 - 2g^2_{DP}(p_\mu\omega^\mu)^2)\bar{\psi}\gamma^5\psi + 2(p^\nu\bar{\psi})\gamma^5(p_\nu\psi)] .$$

By the same procedure used before, the equations of motion written in terms of n_μ become:

$$n^2\omega^\nu - \frac{2ig_{DP}}{m^2 + 4ig^3_{DP}(p_\mu\omega^\nu)\bar{\psi}\gamma^5\psi} \{\bar{\psi}(n^\nu n^\mu \gamma^5(p_\mu\psi))$$

$$+ ((p_\mu\bar{\psi})\gamma^5 n^\nu n^\mu)\psi\} = 0$$

$$n^2\psi + ig_{DP}(n_\mu n_\rho \gamma^\rho\gamma^5\psi)\omega^\mu = 0$$

$$n^2\bar{\psi} - ig_{DP}(n_\mu n_\rho \bar{\psi}\gamma^5\gamma^\rho)\omega^\mu = 0 \ .$$

The characteristic determinant is

$$D(n) = (n^2)^{11} \left(n^2 + \frac{2g^2_{DP}}{m^2 + 4ig^3_{DP}(p_\mu\omega^\mu)\bar{\psi}\gamma^5\psi} \right.$$

$$\left. \times \{\bar{\psi}\gamma^\rho n_\rho n_\mu (p^\mu\psi) - (p_\mu\bar{\psi})n^\mu n_\rho\gamma^\rho\psi\} \right) \ .$$

Here again as in (e) the characteristic surfaces determined by the term in the bracket are dependent on the Dirac field and its derivative and on the derivative of the Proca field. Again if

$$\frac{2g^2_{DP}\{\bar{\psi}\gamma^o(p^o\psi) - (p_o\bar{\psi})\gamma^o\psi\}}{m^2 + 4ig^3_{DP}(p_\mu\omega^\mu)\bar{\psi}\gamma^5\psi}$$

is positive then the system is causal, propagation is acausal if this quantity is between 0 and -1, and the system loses hyperbolicity if this quantity is less than -1.

i) Next we consider a derivative coupling tensor of interaction:

$$\mathcal{L}_{DT} = ig_{DT}(p_\mu\omega_\nu)\bar{\psi}\sigma^{\mu\nu}\psi \ .$$

The equations of motion are

$$p^2\omega^\nu - p_\mu p^\nu\omega^\nu - m^2\omega^\nu + ig_{DT}p_\mu(\bar{\psi}\sigma^{\mu\nu}\psi) = 0 \qquad \text{IV.56}$$

$$\left.\begin{aligned} &(-\not{p} + M)\psi - ig_{DT}(p_\mu\omega_\nu)\sigma^{\mu\nu}\psi = 0 &\quad \text{IV.57a} \\ &\bar{\psi}(\not{p} + M) - ig_{DT}(p_\mu\omega_\nu)\bar{\psi}\sigma^{\mu\nu} = 0 \ . &\quad \text{IV.57b} \end{aligned}\right\} \text{IV.57}$$

The secondary constraints are:

$$p_\nu\omega^\nu = \frac{ig_{DT}}{m^2} \ p_\nu p_\mu(\bar{\psi}\sigma^{\mu\nu}\psi) \equiv 0 \ .$$

The equation of motion written in terms of n_μ reads

$$n^2\omega^\nu = 0$$

$$n^2\psi + ig_{DT}(n_\rho n_\mu \gamma^\rho \sigma^{\mu\nu}\psi)\omega_\nu = 0$$

$$n^2\bar{\psi} - ig_{DT}(n_\rho n_\mu \bar{\psi}\sigma^{\mu\nu}\gamma^\rho)\omega_\nu = 0 \ .$$

The characteristic determinant is $D(n) = (n^2)^{12}$ so the system is always causal with the light cone as the characteristic surfaces.

So far we have considered only individual types of interactions and found some to be causal and others acausal. It would be interesting to know if there exists a relation between the coupling constants for which an interaction, which includes all of these, i.e. the causal and the acausal, will always have causal propagation. Here the interaction Lagrangian is

$$\mathcal{L}_I = \omega^\mu\bar{\psi}(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi + \tfrac{1}{2}(p_\mu\omega^2)\bar{\psi}(g_{DV}\gamma^\mu + g_{DA}\gamma^\mu\gamma^5)\psi$$
$$+ \tfrac{1}{2}\,\omega^2\bar{\psi}(g_S + g_P\gamma^5)\psi + (p_\mu\omega^\mu)\bar{\psi}(g_{DS} + g_{DP}\gamma^5)\psi$$
$$+ g_{DT}(p_\mu\omega_\nu)\bar{\psi}\sigma^{\mu\nu}\psi \quad . \qquad \text{IV.58}$$

We proceed in the same manner as before and get the equations of motion, the constraints, and the characteristic determinant.

The calculations are tedious but straightforward so we only give the essential results here.

The constraint equations on the Proca field become equations of motion and cannot be reduced to a constraint equation by the use of the equation of motion unless $g_{DS} = g_{DP} = 0$ or $g_{DV} = g_{DT} = 0$. This implies that the system will gain more degrees of freedom and hence will not be a spin 1 field interacting with a spin 1/2 field, i.e. the derivative coupling of the scalar and pseudo-scalar type of interaction or the derivative coupling of the vector and tensor types must be contained in the over-all Lagrangian if we are to conserve the contraints of the system.

Now we proceed to investigate the nature of the propagation. If we take the first possibility, i.e. $g_{DS} = g_{DP} = 0$, then calculation of the characteristic determinant indicates the presence of space-like characteristic surfaces as well as a loss of hyperbolicity unless g_{DA} is identically zero. The other possibility i.e. $g_{DV} = g_{DT} = 0$ gives a characteristic determinant which immediately requires that $g_{DS} = g_{DP} = 0$ for causal propagation. A necessary and sufficient condition for causal propagation in this case will therefore be $g_{DS} = g_{DP} = g_{DA} = 0$. This means that the second alternative is a special case of the first one. Hence we conclude that for causal propagation it is necessary and sufficient to have only $g_{DS} = g_{DP} = g_{DA} = 0$, i.e. the acausal types of interactions, derivative couplings of scalar, pseudo-scalar and pseudo-vector, g_{DS}, g_{DP} , G_{DA} must not be present in the interaction Lagrangian if propagation is to be causal, regardless of the strength of the coupling constants for the causal ones. Therefore, in this case interactions which are acausal will always lead to acausal propagation and even to inconsistency when they are further coupled with causal or acausal ones, while causal interactions remain causal even when they are taken together.

CHAPTER V

VACUUM POLARIZATION OF GAUGE INDEPENDENT SPIN 1 FIELD

1. Introduction

In Chapter IV we have found that when the T.P field is coupled minimally to the electromagnetic field it propagates acausally, that is, some of the characteristic surfaces are space-like. Furthermore, we have shown that by adding an appropriate magnetic dipole moment interaction to the minimally coupled Lagrangian, causal propagation will be recovered providing the strength of the coupling constant is equal to e, any other value will always lead to acausal propagation. This clearly indicates that the T.P spin 1 field possesses a fixed dipole moment and does not admit an anomalous arbitrary dipole moment if causality is to be met. This fixed dipole moment interaction can be introduced in a more natural way namely, by adding a fixed four-divergence to the free Lagrangian as shown in section 2 of this chapter.

We now compare the T.P field with the well known spin 1 Proca field. It is known[(V.2)] that the Proca field is causal when minimally coupled to the electromagnetic field and remains so even when an arbitrary dipole moment interaction is added to it. The Proca field, on the other hand, is not invariant under a gauge transformation of the second kind while the T.P field is gauge independent. Thus the advantage of gauge independence restricts the T.P field to a fixed dipole moment while the disadvantage of non-invariance under a gauge transformation of the second kind allows the Proca field to have an arbitrary dipole moment without disturbing its propagation characteristics.

It is interesting to see whether the minimally coupled Proca field and the minimally coupled T.P field with the added dipole moment, will yield the same physical quantities or not. In the free case the two fields are physically indistinguishable while in the interacting case one does not know à priori whether they are indistinguishable or not. An interesting and simple physical quantity to investigate is the vacuum polarization. Here we have calculated the vacuum polarization of the minimally coupled causal T.P field and found that it is identical with that of the minimally coupled Proca field. Hence to this extent the two interacting fields are again physically indistinguishable.

2. Modified Spin 1 T.P Field Lagrangian

The Lagrangian given by Takahashi and Palmer[T.2] is

$$\mathcal{L} = -\frac{1}{2}\,[\partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\nu}(x) - \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\mu}(x)$$

$$- \partial_\mu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\nu}(x) + \partial_\mu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\lambda}(x)$$

$$+ \partial_\nu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\mu}(x) - \partial_\nu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\lambda}(x)]$$

$$- m^2\bar{\psi}_{\mu\nu}(x)\psi_{\mu\nu}(x) \quad . \qquad \text{V.1}$$

Here

$$\bar{\psi}_{\mu\nu}(x) = \psi^\dagger_{\lambda\tau}(x)g_{\lambda\mu}g_{\tau\nu}\ , \quad g_{\mu\nu} = 1 \text{ for } \mu=\nu=1,2,3$$
$$= -1 \text{ for } \mu=\nu=4$$
$$= 0 \text{ otherwise.}$$

It was shown in Chapter IV that the above Lagrangian leads to acausal propagation when minimally coupled to the electromagnetic field. Moreover, causality can be recovered providing the fixed dipole moment interaction(†)

$$\mathcal{L}_{int} = -ie(\bar{\psi}_{\mu\nu}(x)F_{\sigma\nu}\psi_{\mu\sigma}(x) - \bar{\psi}_{\mu\nu}(x)F_{\sigma\mu}\psi_{\nu\sigma}(x)) \qquad \text{V.2}$$

is added to the minimally coupled T.P Lagrangian.

The above dipole moment interaction can be produced by the following four divergences:

(†) $\mathcal{L}_{int}$ has a different sign from that of Chapter IV because a different $g_{\mu\nu}$ is employed throughout this chapter.

$$- \frac{1}{2}\,\partial_\lambda[-\bar{\psi}_{\mu\nu}(x)\partial_\mu\psi_{\lambda\nu}(x) + \bar{\psi}_{\mu\nu}(x)\partial_\mu\psi_{\nu\lambda}(x)$$

$$+ \bar{\psi}_{\mu\nu}(x)\partial_\nu\psi_{\lambda\mu}(x) - \bar{\psi}_{\mu\nu}(x)\partial_\nu\psi_{\mu\lambda}(x)]$$

$$+ \frac{1}{2}\,\partial_\mu[-\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\nu}(x) + \bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\lambda}(x)]$$

$$+ \frac{1}{2}\,\partial_\nu[\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\mu}(x) - \bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\lambda}(x)] \;.$$

Adding this to the Lagrangian of equation V.1 and collecting terms we get the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,[\partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\nu}(x) - \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\mu}(x)$$

$$- \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\mu\psi_{\lambda\nu}(x) + \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\mu\psi_{\nu\lambda}(x)$$

$$+ \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\nu\psi_{\lambda\mu}(x) - \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\nu\psi_{\mu\lambda}(x)]$$

$$- m^2\bar{\psi}_{\mu\nu}(x)\psi_{\mu\nu}. \qquad \text{V.3}$$

The characteristic determinant of the equations of motion derived from the above Lagrangian, according to the method of Chapter IV, is:

$$D(n) = (n^2)^6 \;. \qquad \text{V.4}$$

Thus, every characteristic surface is the light cone and propagation is always causal.

3. Vacuum Polarization

Following Schwinger(S.) and Feldman(Fel.), the equations of motion in the interaction representation have the form:

$$i \frac{\delta\psi(\sigma)}{\delta\sigma} = H(x,\sigma)\psi(\sigma) \; . \qquad \text{V.5}$$

Here $H(x,\sigma)$ is the Hamiltonian density for the interaction between the field in question and the electromagnetic field. To first order in the coupling constant we may write the solution of equation V.5 as:

$$\psi(\sigma) = (1 - i \int_{-\infty}^{\sigma} H(x',\sigma', dx')\psi(-\infty) \qquad \text{V.6}$$

where $\psi(-\infty)$ is the state vector characterizing the initially undisturbed vacuum state of the system. For the induced current we find:

$$\begin{aligned} \delta j_\mu(x) &= (\psi(\sigma), \; j_\mu(x,\sigma)\psi(\sigma)) - \langle 0|j_\mu(x,\sigma)_{A=0}|0\rangle \\ &= \langle 0|j_\mu(x,\sigma) - (j_\mu(x,\sigma))_{A=0}|0\rangle \\ &\quad - i\int_{-\infty}^{\sigma} \langle 0|[j_\mu(x,\sigma), \; H(x',\sigma')]|0\rangle dx' \; . \end{aligned} \qquad \text{V.7}$$

Since we are interested in corrections to the current density of order e^2, it is only the linear part of the current density operator $j_\mu(x)$ which enters the commutator. Moreover we discard that part of $<0|j_\mu(x,\sigma)|0>$ which does not involve the external field. Assuming that the external electromagnetic field cannot create pairs we may write equation V.7 as:

$$\delta j_\mu(x) = <0|j_\mu(x,\sigma) - (j_\mu(x,\sigma))_{A=0}|0>$$
$$- \frac{i}{2}\int <0|[j_\mu(x),H(x')]|0>\varepsilon(x-x')dx' \qquad \text{V.8}$$

Feldman[(Fel.)] and Umezawa[(U.2)] calculated the vacuum polarization of the spin 1 Proca field and got the following expression for the induced current density:

$$\delta j_\mu(x) = -e^2 \int K_{\mu\nu}(x-x')A_\nu(x')\ dx' \qquad \text{V.9}$$

where

$$K_{\mu\nu} = 3[-\partial_\mu\Delta^{(1)}.\partial_\nu\bar{\Delta} - \partial_\nu\Delta^{(1)}.\partial_\mu\bar{\Delta} + \partial_\mu\partial_\nu\Delta^{(1)}.\bar{\Delta} + \Delta^{(1)}.\partial_\mu\partial_\nu\bar{\Delta}$$
$$+ \delta_{\mu\nu}(-\Delta^{(1)}.\Box\bar{\Delta} - \Box\Delta^{(1)}.\Delta + 2m^2\Delta^{(1)}\bar{\Delta})]$$
$$- \frac{2}{m^2}[\partial_\mu\partial_\nu\Delta^{(1)}.\Box\bar{\Delta} + \Box\Delta^{(1)}.\partial_\mu\partial_\nu\bar{\Delta} - \partial_\mu\partial_\rho\Delta^{(1)}.\partial_\nu\partial_\rho\bar{\Delta}$$
$$- \partial_\nu\partial_\rho\Delta^{(1)}.\partial_\mu\partial_\rho\bar{\Delta} + \delta_{\mu\nu}(\partial_\rho\partial_\sigma\Delta^{(1)}.\partial_\rho\partial_\sigma\bar{\Delta} - \Box\Delta^{(1)}.\Box\bar{\Delta})].$$

with

$$\bar{\Delta}(x-x') = -\frac{1}{2}\varepsilon(x-x')\Delta(x-x')$$

where Δ is the solution of the homogeneous Klein-Gordon equation, $(\Box - m^2)\Delta = 0$, which vanishes outside the light cone. Furthermore $\bar{\Delta}$ satisfies $(\Box - m^2)\bar{\Delta}(x) = -\delta^4(x)$. $\Delta^{(1)}$ is the other solution of the homogeneous Klein-Gordon equation which does not vanish outside the light cone.

Here we calculate the induced current for the minimally coupled T.P field with the added dipole moment interaction or equivalently the field resulting from the minimally coupled Lagrangian of equation V.3. The current in the interaction representation is:

$$j_\lambda(x,\sigma) = -e^2(2A_\lambda\bar{\psi}_{\mu\nu}\psi_{\mu\nu} - 2A_\mu\bar{\psi}_{\mu\nu}\psi_{\lambda\nu} - 2A_\mu\bar{\psi}_{\lambda\nu}\psi_{\mu\nu})$$
$$-ie(\bar{\psi}_{\mu\nu}\partial_\lambda\psi_{\mu\nu} - \partial_\lambda\bar{\psi}_{\mu\nu}\psi_{\mu\nu} + 2\partial_\mu\bar{\psi}_{\lambda\nu}\psi_{\mu\nu} - 2\bar{\psi}_{\mu\nu}\partial_\mu\psi_{\lambda\nu})$$
$$+ \text{surface terms} . \qquad \text{V.10}$$

The surface terms in the above current which enter the vacuum polarization through the term $<0|j_\lambda(x,\sigma) - (j_\lambda(x,\sigma))_{A=0}|0>$ have not been written explicitly here; their only contribution is to change $(\partial_\mu\partial_\nu\ldots\Delta)\,\frac{\varepsilon(x-x')}{-2}$ into $\partial_\mu\partial_\nu\ldots\bar{\Delta}$ (T.1,Ma.) in the term

$$\int <0|[j_\lambda(x),H(x')]|0>\varepsilon(x-x')dx' .$$

For the linear terms in the Hamiltonian and current density we have:

$$H(x) = ie(\bar{\psi}_{\mu\nu}\partial_\rho\bar{\psi}_{\mu\nu} - \partial_\rho\bar{\psi}_{\mu\nu}\psi_{\mu\nu} + 2\partial_\mu\bar{\psi}_{\rho\nu}\psi_{\mu\nu} - 2\bar{\psi}_{\mu\nu}\partial_\mu\psi_{\rho\nu})A_\rho$$

$$j_\lambda(x) = -ie(\bar{\psi}_{\mu\nu}\partial_\lambda\psi_{\mu\nu} - \partial_\lambda\bar{\psi}_{\mu\nu}\psi_{\mu\nu} + 2\partial_\mu\bar{\psi}_{\lambda\nu}\psi_{\mu\nu} - 2\bar{\psi}_{\mu\nu}\partial_\mu\psi_{\lambda\nu}) \ . \qquad \text{V.11}$$

The commutator of the T.P field is

$$[\psi_{\mu\nu}(x),\bar{\psi}_{\rho\sigma}(x')] = [\bar{\psi}_{\mu\nu}(x),\psi_{\rho\sigma}(x')] = id_{\mu\nu\rho\sigma}(\partial)\Delta(x-x') . \qquad \text{V.12}$$

We also have the following relation:

$$<0|\bar{\psi}_{\mu\nu}(x)\psi_{\rho\sigma}(x') + \bar{\psi}_{\rho\sigma}(x')\psi_{\mu\nu}(x)|0> = d_{\mu\nu\rho\sigma}(\partial)\Delta^{(1)}(x-x') . \qquad \text{V.13}$$

$d_{\mu\nu\alpha\beta}(\partial)$ is the Klein-Gordon divisor and is given by (T.2)

$$d_{\mu\nu\alpha\beta}(\partial) = \frac{1}{2}[(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) - \frac{1}{m^2}(\delta_{\nu\beta}\partial_\mu\partial_\alpha - \delta_{\mu\beta}\partial_\nu\partial_\alpha - \delta_{\nu\alpha}\partial_\mu\partial_\beta + \delta_{\mu\alpha}\partial_\nu\partial_\beta)] . \qquad \text{V.14}$$

The induced current given by equation V.8 may be considered as the sum of two parts

$$\delta j_\lambda(x) = \delta j_\lambda^{(1)}(x) + \delta j_\lambda^{(2)}(x)$$

where

$$\delta j_\lambda^{(1)}(x) = -\frac{i}{2}\int <0|[j_\lambda(x),H(x')]|0>\varepsilon(x-x')dx' + \text{surface terms} \qquad \text{V.15}$$

and

$$\delta j_\lambda^{(2)}(x) = \langle 0|j_\mu(x,\sigma) - (j_\mu(x,\sigma))_{A=0}|0\rangle$$

$$- \text{ surface terms } . \qquad \text{V.16}$$

We now evaluate $\delta j_\lambda^{(1)}(x)$.

$$\delta j_\lambda^{(1)}(x) = +\frac{i}{2}\int \langle 0|[j_\lambda(x),j_\rho(x')]|0\rangle A_\rho(x')\varepsilon(x-x')dx'$$

$$+ \text{ surface terms}$$

$$= -e^2\int \{d_{\mu\nu\alpha\beta}(\partial)\bar{\Delta}(x-x')(4\partial_\mu\partial_\alpha d_{\lambda\nu\rho\beta}(\partial)$$

$$+ \partial_\lambda\partial_\rho d_{\mu\nu\alpha\beta}(\partial) - 2\partial_\lambda\partial_\alpha d_{\mu\nu\rho\beta}(\partial) - 2\partial_\mu\partial_\rho d_{\lambda\nu\alpha\beta}(\partial))$$

$$\times \Delta^{(1)}(x-x') + d_{\mu\nu\alpha\beta}(\partial)\Delta^{(1)}(x-x')(4\partial_\mu\partial_\alpha d_{\mu\nu\rho\beta}(\partial)$$

$$+ \partial_\lambda\partial_\rho d_{\mu\nu\alpha\beta}(\partial) - 2\partial_\lambda\partial_\alpha d_{\mu\nu\rho\beta}(\partial) - 2\partial_\mu\partial_\rho d_{\lambda\nu\alpha\beta}(\partial))$$

$$\times \bar{\Delta}(x-x') + (2\partial_\alpha d_{\mu\nu\rho\beta}(\partial) - \partial_\rho d_{\mu\nu\alpha\beta}(\partial))\bar{\Delta}(x-x')$$

$$\times (\partial_\lambda d_{\mu\nu\alpha\beta}(\partial) - 2\partial_\mu d_{\lambda\nu\alpha\beta}(\partial))\Delta^{(1)}(x-x')$$

$$+ (2\partial_\alpha d_{\mu\nu\rho\beta}(\partial) - \partial_\rho d_{\mu\nu\alpha\beta}(\partial))\Delta^{(1)}(x-x')$$

$$\times (\partial_\lambda d_{\mu\nu\alpha\beta}(\partial) - 2\partial_\mu d_{\lambda\nu\alpha\beta}(\partial))\bar{\Delta}(x-x')\}A_\rho(x')dx'.$$

V.17

Here, use has been made of equations V.11, V.12, V.13 and the property of the surface terms mentioned above. Using equation V.14 and after tedious but straightforward calculations equation V.17 reduces to:

$$\delta j_\lambda^{(1)}(x) = -e^2\int\{\bar{\Delta}\partial_\lambda\partial_\rho\Delta^{(1)} + \Delta^{(1)}\partial_\lambda\partial_\rho\bar{\Delta} + 2\bar{\Delta}\Box\Delta^{(1)}\delta_{\lambda\rho}$$

$$- 3\partial_\lambda\Delta^{(1)}\partial_\rho\bar{\Delta} - 3\partial_\lambda\bar{\Delta}\partial_\rho\Delta^{(1)}$$

$$+ \frac{1}{m^2}(2\partial_\lambda\partial_\mu\bar{\Delta}\partial_\mu\partial_\rho\Delta^{(1)} + 2\partial_\mu\partial_\rho\bar{\Delta}\partial_\lambda\partial_\mu\Delta^{(1)}$$

$$- 2\delta_{\lambda\rho}\partial_\mu\partial_\alpha\bar{\Delta}\partial_\mu\partial_\alpha\Delta^{(1)})\}A_\rho(x')dx' \; . \qquad \text{V.18}$$

The second part of the induced current, using equations V.10 and V.16, is

$$\delta j_\lambda^{(2)}(x) = -e^2\langle 0|(2A_\rho(x)\bar{\psi}_{\mu\nu}(x)\psi_{\mu\nu}(x)\delta_{\rho\lambda}$$

$$- 2A_\rho(x)\bar{\psi}_{\rho\nu}(x)\psi_{\lambda\nu}(x) - 2A_\rho(x)\bar{\psi}_{\lambda\nu}(x)\psi_{\rho\nu}(x))|0\rangle$$

$$= -e^2\langle 0|\int\{(\bar{\psi}_{\mu\nu}(x)\psi_{\mu\nu}(x') + \bar{\psi}_{\mu\nu}(x')\psi_{\mu\nu}(x))\delta_{\rho\lambda}$$

$$- 2(\bar{\psi}_{\rho\nu}(x)\psi_{\lambda\nu}(x') + \bar{\psi}_{\lambda\nu}(x')\psi_{\rho\nu}(x))\}$$

$$\times A_\rho(x')\delta(x-x')dx' \qquad \text{V.19}$$

where the integral must have the proper boundary conditions so that no pair creation can occur. This can be guaranteed if $\delta(x-x')$ is replaced by the Green function $\bar{\Delta}$, where

$$(\Box - m^2)\bar{\Delta}(x-x') = -\delta(x-x') \; . \qquad \text{V.20}$$

Substituting for $\delta(x-x')$ and using equation V.13, equation V.19 becomes:

$$\delta j_\lambda^{(2)}(x) = -e^2\int(\delta_{\rho\lambda}d_{\mu\nu\mu\nu}(\partial)\Delta^{(1)}.(m^2-\Box)\bar{\Delta}$$

$$- 2d_{\rho\nu\lambda\nu}(\partial)\Delta^{(1)}.(m^2-\Box)\bar{\Delta})A_\rho(x')dx' \;.$$

With the use of equation V.14 the above equation reduces to:

$$\delta j_\lambda^{(2)}(x) = -e^2\int(2\partial_\rho\partial_\lambda\Delta^{(1)}.\bar{\Delta} + m^2\Delta^{(1)}.\bar{\Delta}\delta_{\rho\lambda}$$

$$- \Delta^{(1)}.\Box\bar{\Delta}\delta_{\rho\lambda} - \frac{2\partial_\rho\partial_\lambda\Delta^{(1)}\Box\bar{\Delta}}{m^2})A_\rho(x')dx'. \qquad \text{V.21}$$

Combining equations V.18 and V.21 and with the use of the following equation for $\Delta^{(1)}$

$$(\Box - m^2)\Delta^{(1)}(x-x') = 0 \qquad \text{V.22}$$

the induced current density may be written:

$$\delta j_\lambda(x) = -e^2\int K_{\lambda\rho}(x-x')A_\rho(x')dx' \qquad \text{V.23}$$

where

$$K_{\lambda\rho} = 3[-\partial_\lambda\Delta^{(1)}.\partial_\rho\bar{\Delta} - \partial_\rho\Delta^{(1)}.\partial_\lambda\bar{\Delta} + \partial_\lambda\partial_\rho\Delta^{(1)}.\bar{\Delta} + \Delta^{(1)}.\partial_\lambda\partial_\rho\bar{\Delta}$$

$$+ \delta_{\lambda\rho}(-\Delta^{(1)}.\Box\bar{\Delta} - \Box\Delta^{(1)}.\bar{\Delta} + 2m^2\Delta^{(1)}.\bar{\Delta})]$$

$$- \frac{2}{m^2}[\partial_\lambda\partial_\rho\Delta^{(1)}.\Box\bar{\Delta} + \Box\Delta^{(1)}\partial_\lambda\partial_\rho\bar{\Delta} - \partial_\lambda\partial_\alpha\Delta^{(1)}.\partial_\rho\partial_\alpha\bar{\Delta}$$

$$- \partial_\rho\partial_\alpha\Delta^{(1)}.\partial_\lambda\partial_\alpha\bar{\Delta} + \delta_{\lambda\rho}(\partial_\alpha\partial_\beta\Delta^{(1)}.\partial_\alpha\partial_\beta\bar{\Delta} - \Box\Delta^{(1)}.\Box\bar{\Delta}].$$

This is identical with induced current for the Proca field given by equation V.9.

Thus we conclude that as far as the vacuum polarization is concerned the Proca field and the causal T.P field are physically indistinguishable.

CHAPTER VI

CONCLUSIONS

In this work, we have derived first order wave equations without subsidiary conditions, which are form invariant under the homogeneous Lorentz group, and describe a field with a fixed integer spin and mass. Furthermore, we have shown that there exist many theories for a given integer spin field, as, for example, in the case of spin one, where we have considered four of these, namely the 10, 14, 20 and 26 component ones. The form of D chosen in this work is less restrictive, as compared to the form of D chosen for half-odd integer spin[(Ca.1)]. Here the presence of the irreducible representation (k,k) in D, which is absent for half odd integer spin, allows more free parameters in β_o.

Difficulties with propagation arising from interaction for higher spin fields were analysed and in some cases overcome. We considered the spin 1 theory of Takahashi and Palmer and showed that in certain cases, it is possible to combine interaction terms which separately lead to acausal propagation, so that the result is causal. This led us to try to get a causal spin 3/2 theory coupled to an electromagnetic field. In this case, however, no choice of the coupling constants leads to causal propagation. We also considered a Proca field coupled to a Dirac field. The interactions considered were scalar, pseudo-scalar, vector, axial vector as well as derivative coupling of these types and of the tensor type. All the direct couplings lead to causal propagation. For the derivative coupling, only the vector and tensor couplings yielded causal propagation. Furthermore, when taken together, all individually causal terms lead to causal propagation, whereas no choice of the coupling constants (other than zero) among all the terms led to causal propagation with one or two or all three of the individually acausal terms included. Thus both types of situations occur: one in which acausal terms may be combined to yield causal propagation as in the Takahashi-Palmer spin 1 theory or else as in the case of the Proca and Dirac field combination, where acausal terms cannot be combined to yield causal propagation.

With regard to the above mentioned difficulties of higher spin interacting field, we note the following: Wightman[(Wig.)] indicated that the acausal behaviour in the c-number problem will have no bearing on the q-number problem, as long as the system of equations remains hyperbolic. This is a consequence of Capri's[(Ca.2)] investigation with respect to the existence of the interacting Green's function. Capri[(Ca.2)] has proved the existence of the interpolating field, providing the system of equations is hyperbolic on the classical level. In the examples presented in this work we find that in all acausal cases, the system will lose hyperbolicity if the external sources or the fields exceed a certain range. The range is determined by the quantity x of equation IV.37. The system is acausal if $0 < x < 1$ and loss of hyperbolicity occurs when x is a complex number or a real number larger than one. For the T.P self coupled field

$$x = -\frac{2g}{m^2}\psi_{o\nu}\psi^{o\nu}\Big/\left(1 + \frac{g}{m^2}\psi^2\right).$$

For the Dirac field interacting with spin 1 Proca field by the derivative coupling of the scalar, pseudo-scalar and pseudo-vector types, x is equal to

$$-\frac{g_{DS}\{\bar{\psi}\sigma^o(p_o\psi) - (p_o\bar{\psi})\gamma^o\psi\}}{m^2 - 4g_{DS}^2(M - ig_{DS}p_\mu\omega^\mu)},$$

$$-\frac{2g_{DP}^2\{\bar{\psi}\gamma^o(p_o\psi) - (p_o\bar{\psi})\gamma^o\psi\}}{m^2 + 4ig_{DP}^3(p_\mu\omega^\mu)\bar{\psi}\gamma^5\psi},$$

$$\left(\frac{-4Mg_{DA}\omega^o\omega_o\bar{\psi}\psi}{m^2 + 2Mg_{DA}\bar{\psi}\gamma^5\psi}\right)^{\frac{1}{2}},$$

respectively. For the minimally coupled spin 3/2 case

$x = (2e/3m^2)^2 B^2$; and for the minimally coupled spin 3/2 field with added Pauli terms, x was shown to have complex values as well as real values larger than one depending on the strength of the external (e.m) source. Examination of the above values for x indicates that to the extent of our examples, acausal systems will lose hyperbolicity when the fields or the sources exceed a certain range and thus the existence of the interpolating field is questionable.

We have also shown in this work that the dipole moment interaction needed to make the T.P field causal can be introduced by adding a fixed four-divergence to the free T.P Lagrangian.

Finally, we have calculated the vacuum polarization of the T.P causal spin 1 field, and have shown that it is identical with that of the Proca field. Thus to this extent, the T.P causal field and the Proca field are physically indistinguishable.

APPENDIX A

In this section we review some results regarding the representations of SL(2,c) and its relation to the Lorentz group.

The Lorentz group $\mathcal{L}$ consists of a set of real linear transformations which leaves the form

$$x.y = x^o y^o - \vec{x}.\vec{y} = g_{\mu\nu} x^\mu y^\nu$$

invariant. Under $\Lambda \in \mathcal{L}$ a four vector transforms

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \ . \qquad \text{A.1}$$

The connected component $\mathcal{L}_+^\uparrow$ is called the proper orthochronous Lorentz group which have det $\Lambda = 1$ and $\Lambda_o{}^o > 0$. There are three more connected components:

$$\mathcal{L}_-^\uparrow \ \det\Lambda = -1 \qquad \Lambda_o{}^o > 0$$

$$\mathcal{L}_+^\downarrow \ \det\Lambda = +1 \qquad \Lambda_o{}^o < 0$$

$$\mathcal{L}_-^\downarrow \ \det\Lambda = -1 \qquad \Lambda_o{}^o < 0 \ .$$

The universal covering group of $\mathcal{L}_+^\uparrow$ is SL(2,c), the group of 2×2 complex matrices.

The connection between SL(2,c) and $\mathcal{L}_+^\uparrow$ is established by the Pauli matrices

$$\sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{A.2}$$

We denote

$$\tilde{\sigma}_\mu = (\sigma_o, -\vec{\sigma}) = \sigma^\mu .$$

Then the one to two homomorphism between $\mathcal{L}_+^\uparrow$ and SL(2,c) is given by

$$\Lambda_{\mu\nu} = (\pm A) = \frac{1}{2}\,\mathrm{Tr}(\tilde{\sigma}_\mu A \sigma_\nu A^\dagger)$$

$$A \propto SL(2,c) \quad , \quad \Lambda \propto \mathcal{L}_+^\uparrow \quad . \qquad \text{A.3}$$

We use the convention that the spinor indices transforming according to A, $A^\dagger$ are written as lower undotted and upper dotted, respectively, while those transforming according to the contragradient transformation A^{-1t}, $A^{-1\dagger}$ (†) are written as upper undotted and lower dotted. For example σ_μ, $\tilde{\sigma}_\mu$ have the indices $\sigma_{\mu\alpha\dot{\beta}}$ and $\tilde{\sigma}_\mu{}^{\dot{\alpha}\beta}$.

For every four vector x^μ is associated the Hermitian matrix

$$X = x^\mu \sigma_\mu$$

$$\det X.X = \det[x^\mu \sigma_\mu x^\nu \sigma_\nu] = x.x \ . \qquad \text{A.4}$$

The finite dimensional representations of SL(2,c) are defined by their action on the space of homogeneous polynomials

---

(†) We use † for Hermitian adjoint, * for complex conjugate and t for transpose.

$$\chi_{\alpha\dot{\beta}}^{(j_1 j_2)} = \frac{(\xi_{\frac{1}{2}})^{j+\alpha}(\xi_{-\frac{1}{2}})^{j-\alpha}(\eta_{\frac{1}{2}})^{k+\dot{\beta}}(\eta_{-\frac{1}{2}})^{k-\dot{\beta}}}{[(j+\alpha)!(j-\alpha)!(k+\dot{\beta})!(k-\dot{\beta})!]^{\frac{1}{2}}} \qquad \text{A.5}$$

where

$$\alpha_1 = j,\ j-1,\ \ldots,\ -j$$

$$\dot{\beta} = k,\ k-1,\ \ldots,\ -k \quad .$$

Here ξ and η are two-component spinors transforming according to

$$\xi \to \xi' = A\xi$$

$$\eta \to \eta' = A^*\eta \ . \qquad \text{A.6}$$

By substituting these in A.5 we get the (2j+1)(2k+1) dimensional representation $\mathcal{D}^{(j,k)}$ of SL(2,c) defined by

$$\chi_{\alpha'\dot{\beta}'}^{'(j,k)} = \mathcal{D}^{(j,k)}(A)_{\alpha'\dot{\beta}'}{}^{\alpha\dot{\beta}}\chi^{(j,k)}{}_{\alpha\dot{\beta}} \qquad \text{A.7}$$

By convention we therefore have

$$\mathcal{D}^{(\frac{1}{2},0)}(A) = A \ , \qquad \text{A.8}$$

also

$$\mathcal{D}^{(j,k)}(A) = \mathcal{D}^{(j,o)} \otimes \mathcal{D}^{(o,k)}(A) \qquad \text{A.9}$$

$$\mathcal{D}^{(s,o)}(A) = \mathcal{D}^{(o,s)}(A^{-1})^{\dagger} \qquad \text{A.10}$$

$$\mathcal{D}^{(s,o)}(A^{\dagger}) = \mathcal{D}^{(s,o)}(A)^{\dagger} \qquad \text{A.11}$$

$$\mathcal{D}^{(s,o)}(A^t) = \mathcal{D}^{(s,o)}(A)^t \quad . \qquad \text{A.12}$$

If A is restricted to $U \propto SU(2)$, then

$$\mathcal{D}^{(j,o)}(U) = \mathcal{D}^{(o,j)}(U) = \mathcal{D}^{j}(U) \qquad \text{A.13}$$

and the representation is unitary; furthermore $\mathcal{D}^{(j,k)}(U)$ is reducible since $\mathcal{D}^{j}(U) \otimes \mathcal{D}^{k}(U)$ decomposes into

$$\bigoplus_{\ell=|j-k|}^{j+k} \mathcal{D}^{\ell}(U) \quad .$$

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