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ON INTENSE GRAVITATIONAL FIELDS

BY

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled ON INTENSE GRAVITATIONAL FIELDS, submitted by Vicente de la Cruz, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The object of this thesis is to study several topics in the general theory of relativity which involve space-time regions of intense gravitational field.

Salient features of the Schwarzschild-Kruskal manifold are reviewed. A system of null coordinates of Kruskal's type for a collapsing spherical cloud of dust is presented which gives a singularity-free description of the complete space-time right down to the singular event of maximum implosion. Questions regarding the qualitative effects of departures from sphericity on gravitational collapse are discussed in the light of some recent advances. A solution of Einstein's field equations is derived which represents a class of thin spherical shells of charged dust collapsing (and, in general, bouncing) in the field of a charged spherical body placed at the center. A method is described which develops the interior field and physical properties of a slowly spinning, nearly spherical mass shell as a power series in the angular velocity ω , on the presumption that the exterior field is the Kerr metric; results are worked out explicitly, correct to the third order in ω .

ACKNOWLEDGEMENTS

I am deeply indebted to Professor W. Israel for his supervision throughout the course of this work.

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CHAPTER I

INTRODUCTION

In scope, depth, and logical simplicity, the general theory of relativity [1](GR) today still stands as the best (classical) theory of space, time, and gravitation. Created in 1915 by Einstein, it has since stood its ground against the tests of empirical observations (few though there are) and arguments of principle. And although on empirical grounds there might appear little to distinguish GR from some other theories[2,3] (of gravitation), in assessing the inherent plausibility of these later theories one must bear in mind that they were formulated with full knowledge of the important effects (such as the deflection of light near a mass center) already known to exist, after having been first predicted by GR and subsequently confirmed.

By showing how gravitation may be reduced to being an aspect of space-time geometry, and by bringing down, in turn, this geometry from the lofty height of the *a priori*, Einstein has effected a profound modification on our concept of the external world. But owing partly to the intrinsic mathematical difficulties of the theory, as manifested most characteristically in the nonlinearity of its field equations, and partly to the extreme sophistication needed to experimentally check those of its predictions that can be worked out, subsequent progress of the theory has not been as great as the initial enthusiasm which it had generated might have led one to expect. Moreover, since under normal situations the gravitational field is incomparably weaker than the other so-called fundamental fields, many physicists are inclined to believe that GR would never become important on a microscopic level. Such a consideration, however, in view of our present state of understanding, would appear somewhat premature; in particular, the essential non-linear character of GR may yet lead to unsuspected subtleties.[4] It may be mentioned that one of the reasons responsible for the considerable rise in renewed interest in GR during the last fifteen years or so is precisely the lack of essential progress in other branches of theoretical physics: some hope that, perhaps by suitably amalgamating GR and quantum theory, a satisfactory theory of elementary particles will emerge.[4]

Another reason has to do with recent or imminent technological advances which will permit one to increase the accuracy of earlier experiments and also to test other predictions of GR.[5] The interaction between theories and experiments is certainly vital to the growth of any empirical science. But it is well to note again that, except in cosmology, almost all of the physical predictions of GR, as normally envisaged, involve geometries that deviate extremely little from flatness —— a condition sufficient for the Newtonian gravitational theory to hold with tremendous accuracies (provided the velocities of material particles are small in comparison with the speed of light c). Only when the gravitational field is intense, i.e., when the Newtonian potential ϕ becomes comparable with c^2, do we expect significant differences between the predictions of GR on the one hand, and, on the other hand, those of Newtonian and indeed other gravitational theories. For a collection of matter of mass m and characteristic dimension R,

$$|\phi|/c^2 \simeq Gm/c^2R$$

at the surface, where G is the Newtonian gravitational constant. At the surface of the sun for example, we have $Gm/c^2R \sim 10^{-6}$. One would therefore look to astrophysics for possible processes where general relativistic effects might play an essential role.

Such a process could conceivably be taking place in "Quasi-stellar objects" * —— astrophysical objects discovered since 1960 and probably belonging to a class of their own with no parallel in known observations. Though these objects are now believed to be much smaller in size than a typical galaxy, their radio power is about 10^{44} ergs/sec

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* For a recent monograph on Quasi-stellar objects, see(6).

and their optical power about 10^{46} ergs/sec, which is 100 times the total energy output of a giant galaxy. Other peculiarities such as that connected with their light variations have also been observed, but the central problem of the moment is to find a mechanism to account for the energy output. Reasons can be adduced to practically rule out thermonuclear reactions as being responsible for the energy output, and so attention has turned to gravitation. The reason is that gravitational energy depends on Gm^2/R, i.e., on the square of the mass, so that it can play a decisive role in the evolution of a very massive body ($m \sim 10^5 - 10^8$ solar masses). Thus a model for quasi-stellar objects has been proposed(7) which consists of a very massive body undergoing violent gravitational contraction (collapse). The gravitational collapse of this super-star could perhaps supply the necessary energy if it were to shrink to dimensions such that the Newtonian potential became comparable with c^2 —— thus necessitating the employment of the full machinery of GR.

Relativistic gravitational collapse(8) by itself forms a fascinating study, for it has brought to light peculiar features wholly unexpected on the basis of Newtonian theory. Chapter III gives a quantitative summary of the collapse of a model which is particularly simple but generally thought to be representative: the collapse of a spherical ball of dust. In that connection we shall introduce a coordinate system of the Kruskal[9] type, so as to present the complete history of the collapse in a unified picture. The chapter ends with a discussion on whether or not our conventional picture of collapse can still be upheld when the assumption of strict spherical symmetry —— which is of course never realized in actual bodies —— is dropped.

A collapsing ball of dust eventually ends in a state of singularity characterized by infinite curvature and energy density, at which the analysis must come to halt. This state of affairs can be avoided in the case of charged bodies, though apparent paradoxes may arise. The collapse of charged spherical shells is dealt with in Chapter IV.

Chapter II contains a summary of some of the most important aspects of the Schwarzschild (exterior) solution[10], including its analytic completion.[11] This remarkable solution represents the vacuum field outside any spherically symmetric (uncharged) distribution of matter[12], and is one of the only two solutions of Einstein's gravitational equations which have any experimental basis (the other being the Friedmann solution[13] in cosmology). The trajectories of test particles and light rays have already been worked out[14] and may be compared with the corresponding Newtonian situation. In regions of intense field, qualitative differences occur. The Schwarzschild solution is also interesting in that it may be used to illustrate such basic ideas as completeness of space-time manifold and singularities, and the peculiar difficulties associated with the problem of coordinates in GR —— all of which are trivial in the Newtonian picture. Finally, the solution is of great importance for both the qualitative and the quantitative analyses of relativistic gravitational collapse.

While the Schwarzschild metric has been in existence for more than fifty years and may be complemented by some interior solution to form the complete geometry of a space-time manifold due to, for example, a static sphere of perfect fluid, no exact solution representing the complete field of some isolated rotating object of a realistic nature has yet been exhibited, although at large distances from such an object, where the field is assumed weak, the form of the metric can be obtained[15] from the linearized Einstein field equations. However, in 1963 was discovered a remarkable two-parameter vacuum field solution (the Kerr solution[16]) which turned out to have features that might be expected of a space-time exterior to some object(s) possessing mass and angular momentum[17,18] the exact nature being still unknown. In Chapter V, a method is described which develops the interior field and the physical properties of a slowly spinning, nearly spherical thin mass shell as a power series in the angular velocity ω, on the presumption that the exterior field is the Kerr metric.

Results are worked out explicitly to the third order in ω, and we extend a recent strong-field study by Brill and Cohen[19] on the classical problem of the "relativity of inertia" in the sense of Mach[20] (see Part A, Chapter V), first formulated and worked out by H. Thirring[21] in 1918 within the framework of GR, but using the weak field approximations.

CHAPTER II

SPHERICALLY SYMMETRIC VACUUM GRAVITATIONAL FIELD

1. General Relativity: A Summary.

The two most important postulates underlying Einstein's theory of gravitation are, firstly, that the geometry of space-time is Riemannian, and, secondly, that of all mathematically possible Riemannian spaces, only those that are related to matter distributions in a specific way can be physically realized. A few standard geometrical concepts and relations will be needed to make the statement more precise.

A Riemannian space may be defined as a geometrical space in which the "distance" or "interval" ds between two neighbouring points x^α and $x^\alpha + dx^\alpha$ (referred to a suitable coordinate system) is given by a (not necessarily positive-definite) quadratic differential form

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta \tag{1}$$

where $g_{\alpha\beta}$, the metric tensor, behaves like a covariant tensor of the second rank under coordinate transformations, and is assumed non-singular in the sense of matrix theory:

$$g \equiv \det(g_{\alpha\beta}) \neq 0 \,. \tag{2}$$

Condition (2) enables us to define the contravariant metric tensor $g^{\alpha\beta}$ by

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma} . \tag{3}$$

By convention, $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are used to lower or raise tensorial indices; for example, with a covariant vector A_α one can associate a contravariant vector $A^\alpha \equiv g^{\alpha\beta} A_\beta$.

The following entities are constructed out of $g_{\alpha\beta}$ and $g^{\alpha\beta}$ and their derivatives:

Christoffel symbols of the first and second kind (respectively):

$$\Gamma_{\alpha,\beta\gamma} = \tfrac{1}{2}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) , \tag{4}$$

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\delta} \Gamma_{\delta,\beta\gamma} , \tag{5}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha$.

Riemann tensor:

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma(\Gamma^{\alpha}_{\beta\delta}) - \partial_\delta(\Gamma^{\alpha}_{\beta\gamma}) + \Gamma^{\lambda}_{\beta\delta}\Gamma^{\alpha}_{\lambda\gamma} - \Gamma^{\lambda}_{\beta\gamma}\Gamma^{\alpha}_{\lambda\delta} . \tag{6}$$

Ricci tensor:

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\beta\gamma} = \tfrac{1}{2}\partial_\alpha\partial_\beta[\ln(-g)] - \tfrac{1}{2}\Gamma^{\gamma}_{\alpha\beta}\partial_\gamma[\ln(-g)] - \partial_\gamma\Gamma^{\gamma}_{\alpha\beta} + \Gamma^{\gamma}_{\delta\alpha}\Gamma^{\delta}_{\gamma\beta} . \tag{7}$$

Curvature invariant:

$$R = g^{\alpha\beta}R_{\alpha\beta} \ . \tag{8}$$

Einstein's tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta}R \ . \tag{9}$$

For vector and tensor fields defined throughout a domain of space-time, the covariant derivatives (indicated by a vertical stroke) are as follows:

$$A^{\alpha}{}_{|\beta} = \partial_{\beta}A^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}A^{\gamma} \ , \tag{10}$$

$$A_{\alpha|\beta} = \partial_{\beta}A_{\alpha} - \Gamma^{\gamma}_{\alpha\beta}A_{\gamma} \ , \tag{11}$$

$$A^{\alpha\beta}{}_{|\gamma} = \partial_{\gamma}A^{\alpha\beta} + \Gamma^{\alpha}_{\gamma\delta}A^{\beta\delta} + \Gamma^{\beta}_{\gamma\delta}A^{\alpha\delta} \ , \text{ (etc.).} \tag{12}$$

We now give a brief summary of general relativity.

Space-time is represented by a four-dimensional normal hyperbolic Riemannian space of signature (+++-). This implies that at any given event E, the metric tensor* $g_{\alpha\beta}$ can be reduced to the Minkowskian form

$$(g_{\alpha\beta})_E = \text{diag}\ (1,1,1,-c^2)$$

* Here and in the following, greek indices run from 1 to 4.

by a suitable coordinate transformation (c is the fundamental velocity).

The physical interpretation of the metric is given as follows. Let E and E' be neighbouring events x^μ and $x^\mu + dx^\mu$. An observer (with four-velocity V^μ) momentarily at E measures the spatial separation of the two events as

$$[(g_{\mu\nu} + V_\mu V_\nu/c^2)dx^\mu dx^\nu]^{\frac{1}{2}}$$

and their time-difference as

$$-\frac{1}{c^2} V_\mu dx^\mu .$$

The geometry of space-time is related locally to the distribution of matter (characterized phenomenologically by an energy-momentum tensor $T_{\alpha\beta}$) through Einstein's gravitational field equations

$$G_{\alpha\beta} = -\kappa\, T_{\alpha\beta} \quad , \quad \kappa \equiv 8\pi G/c^4 \quad , \tag{13}$$

in which G is the Newtonian gravitational constant. The existence of four differential identities

$$G^{\alpha\beta}{}_{|\beta} \equiv 0 \qquad \text{(Contracted Bianchi Identities)}$$

means that a consistent choice of $T_{\alpha\beta}$ must satisfy the "conservation equations"

$$T^{\alpha\beta}{}_{|\beta} = 0 . \tag{14}$$

Thus, given $T_{\alpha\beta}$, eqs. (13) are a system of six (non-linear) partial differential equations for the ten independent components of $g_{\alpha\beta}$. This amount of indeterminacy is, in fact, just right, on account of the arbitrariness of the coordinate system.

The field equations (13) are the analogue of the Poisson equation of Newtonian gravitational theory,

$$\nabla^2\phi = 4\pi G\mu$$

(where ∇^2 is the 3-dimensional Laplacian operator, ϕ the gravitational potential, and μ the mass density), which satisfyingly represents a limiting case of (13) for weak, quasi-static fields, for which the general relativistic line-element reduces to

$$ds^2 = (1-2\phi/c^2)(dx^2+dy^2+dz^2)-(1+2\phi/c^2)c^2dt^2 \tag{15}$$

in appropriate coordinates.

For vacuum space-times ($T_{\alpha\beta}$=0), the field equations reduce to $G_{\alpha\beta}$=0 or, equivalently,

$$R_{\alpha\beta} = 0 . \tag{16}$$

An important material system figuring in the study of general relativity is the perfect fluid, formally defined by $T^{\alpha\beta}$ having the form

$$T^{\alpha\beta} = (\mu + P/c^2)u^\alpha u^\beta + P\, g^{\alpha\beta} , \tag{17}$$

in which $u^\alpha = dx^\alpha(\tau)/d\tau$ (where τ is the proper time) is the four-velocity of macroscopic fluid particles, P the isotropic pressure, and μ the mass-energy density as measured by an observer moving with the fluid element. When $P \equiv 0$, we speak of "dusts".

From (17) we observe that

$$T^{\alpha\beta} u_\beta = -\mu c^2 u^\alpha \qquad (u_\alpha u^\alpha = -c^2) . \tag{18}$$

Thus u^α is the time-like normalized (i.e., $u_\alpha u^\alpha = -c^2$) eigenvector of $T^{\alpha\beta}$, with corresponding eigenvalue $-\mu c^2$. For a continuous medium having an <u>arbitrary</u> energy-momentum tensor $T^{\alpha\beta}$ we shall, following Synge, take this statement as a definition of its velocity field u^α and proper energy density μc^2.

Each space-time contains a class of geometrically distinguished curves which form the closest analogue of straight lines in ordinary Euclidean spaces. They are the geodesics, i.e., space-time curves whose parametric equations $x^\mu = x^\mu(\lambda)$ satisfy

$$d^2x^\mu/d\lambda^2 + \Gamma^\mu_{\alpha\beta}\, (dx^\alpha/d\lambda)(dx^\beta/d\lambda) = 0 . \tag{19}$$

Here λ is known as an affine parameter, and is determined up to linear transformations. The geodesic equations (19) possess the first integral

$$g_{\alpha\beta}(dx^{\alpha}/d\lambda)(dx^{\beta}/d\lambda) = \text{const} \quad , \tag{20}$$

i.e.,

$$v_{\alpha}v^{\alpha} = \text{const} \ ,$$

where $v^{\alpha} = dx^{\alpha}(\lambda)/d\lambda$ is the tangent vector. (Since the metric of space-time is not positive-definite, the norm $A_{\alpha}A^{\alpha}$ of a vector A_{α} can be positive, zero, or negative; A_{α} is then designated as "space-like", "null", or "time-like", respectively). Thus there are three subclasses of geodesics: space-like, null, and time-like geodesics. The latter two are of fundamental importance, as we shall presently see.

Unlike any other known field theory, general relativity is one in which the equations of motion for localized sources may not be postulated separately, but rather follow from its field equations. In particular, we have the following basic result:

In vacuo, the world line of a (non-spinning and uncharged) test particle is a time-like geodesic; the world line of a "test photon" (or neutrino) is a null geodesic.

If non-gravitational forces and internal interactions can safely be ignored, a proof of this "geodesic hypothesis" becomes very simple. Consider a small body of dust moving in a vacuum domain. The energy-momentum tensor is thus $T^{\alpha\beta} = \mu u^{\alpha}u^{\beta}$, with $\mu = 0$ outside the world tube representing the history of the object. From the conservation equations (14) one immediately obtains

$$(\mu u^{\beta})_{|\beta} u^{\alpha} + u^{\alpha}{}_{|\beta} \mu u^{\beta} = 0 ,$$

which yields, upon contracting with u_{α} ,

$$(\mu u^{\beta})_{|\beta} = 0 ,$$

since $u_{\alpha} u^{\alpha} = -c^2$ and consequently $u^{\alpha}{}_{|\beta} u_{\alpha} = 0$. Thus, provided $\mu \neq 0$,

$$\begin{aligned} 0 &= u^{\alpha}{}_{|\beta} u^{\beta} \\ &= (\partial_{\beta} u^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} u^{\gamma}) u^{\beta} \\ &= d^2 x^{\alpha}/d\tau^2 + \Gamma^{\alpha}_{\beta\gamma} (dx^{\beta}/d\tau)(dx^{\gamma}/d\tau) , \end{aligned}$$

which shows that the dust particles follow geodesics.

* * *

The remaining part of this chapter is devoted to a review of the salient features of the Schwarzschild geometry: trajectories of test particles and photons (Sect. 2) and the Kruskal completion of the Schwarzschild manifold (Sect. 3).

We shall employ "geometrized units" (c=G=1) throughout the remainder of the thesis. In this system, mass, time, and length all have the same dimension.

2. The Schwarzschild Solution. Motions of Test Particles and Photons.

The best known non-trivial solution of the Einstein's gravitational field equations is the Schwarzschild solution. Its traditional form is[10]

$$ds^2 = (1 - 2m/r)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - (1 - 2m/r)dt^2. \tag{21}$$

The line-element is static in the sense that the coefficients are independent of time t and it is invariant under time inversion $(t\to -t)$. The Schwarzschild radial coordinate r has the geometrically invariant meaning that a 2-space S: $r,t = \text{const}$ has an area $4\pi r^2$ (independently of t), and (θ,ϕ) are polar coordinates on S. This 2-space is intrinsically indistinguishable from an ordinary spherical surface of radius r. Thus the line-element is also spherically symmetric. At large spatial distances $(r \gg 2m)$, it is approximately Minkowskian, and by requiring that Newton's law should hold at such a region of weak field, we can identify the constant m with the total mass of the body producing the Schwarzschild field. (Cf. the linearized line-element written down previously, eq.(15)). Accordingly we assume $m > 0$.

The Schwarzschild solution satisfies the vacuum field equations (16). It owes its great importance to a theorem of Birkhoff[12], according to which it represents the external vacuum field of any spherically symmetric uncharged (static or non-static) distribution of matter. On this solution are based two of the three famous experimental tests of the theory of general relativity. (However, these do not check the full range of validity of (21), since the ratio $2m/r << 1$ outside the sun). Recently it has regained attention in connection with the study of relativistic gravitational collapse.

One always gets a firmer grasp of the geometry of any solution if one knows how particles and light rays behave in it. For the Schwarzschild geometry, investigations on the motions of test particles and photons have been thoroughly carried out in the past[14]. We shall review some of the results obtained, as they not only illustrate the qualitative differences from corresponding Newtonian predictions that can arise when the geometry of space-time deviates strongly from flatness, but are also relevant for an understanding of certain features of gravitational collapse.

Within the framework of Newtonian theory, the motion of a particle in the gravitational field of a mass center is well-known. If the angular momentum is not zero, the path of the particle is either an ellipse or a hyperbola, depending on the total energy. Gravitational capture is impossible except for radial motions.

Since the paths of test particles in general relativity are time-like geodesics, they are in principle completely determined once the metric is known. For the Schwarzschild field, the geodesic equations (19) possess the following first integrals:

$$(dr/d\tau)^2 = E^2 - U(r,\ell) \;, \; U(r,\ell) \equiv (1-2m/r)(1+\ell^2/r^2), \quad (22)$$

$$r^2(d\phi/d\tau) \equiv \ell = \text{const} \;, \quad (23)$$

and without loss of generality θ can be set equal to $\pi/2$. The affine parameter τ is the proper time of the particle, $(d\tau^2 = -ds^2)$; the constant E is the "total energy" (including the rest mass energy $\mu = 1$), and the constant ℓ the "angular momentum".

Fig. 1 (p.26) is a schematic plot of $U(r,\ell)$ vs. r for three typical values of ℓ.. One gets a qualitative idea of how the radius of a moving particle varies with time from an examination of eq.(22) and fig. 1. For $\ell > 4m$ (i.e., $4m\mu G/c$), in which case U has a maximum value $U_{max} > 1$, the situation is as follows.

(1) $E < 1$. The particle neither falls to the center $r = 0$ nor flies off to spatial infinity ("finite motion"). The radius of closest approach is $r_{min} > 4m$. Unlike the Newtonian case, here the orbit is in general not a closed

curve. (This is manifested in the secular shift of the perihelion of Mercury).

(2) $1 < E < U_{max}$. The particle, coming from infinity, goes off again to infinity in analogy to the hyperbolic motion in the Newtonian case. $r_{min} > 3m$.

(3) $E > U_{max}$. The particle, coming from infinity, spirals pass $r = 2m$ and is eventually captured.

For $\ell < 4m$, all particles coming from infinity $(E > 1)$ are captured; with $E < 1$ and $\sqrt{3} \cdot 2m < \ell < 4m$, the particles either are captured or execute finite motions $(r_{min} > 4m)$, depending on E and ℓ. However, when $\ell < \sqrt{3} \cdot 2m$, there exists no finite motion, and a particle inevitably ends up at $r = 0$.

We now consider light rays. According to Newtonian theory, their motions are, of course, uniform along straight lines. In general relativity, they are represented by null geodesics, and in the present case can easily be obtained from (22), (23) by a limiting process.

Each light ray may be characterized by its impact distance b at infinity. For $b > 3\sqrt{3}\,m$, the light ray is deflected from its linear motion, but never comes closer than $r = 3m$, and eventually goes off to spatial infinity. (For a ray grazing the surface of the sun, the deflection amounts to 1.75". This prediction of Einstein was, as is universally known, brilliantly confirmed during the time of total solar eclipse of 1919). For $b < 3\sqrt{3}\,m$, all photons are gravitationally captured. A photon can move in a circular orbit at $r = 3m$, but that orbit is unstable.

Finally, we mention that a light ray emitted by a stationary source located at a radius $r = \text{const} > 2m$ can escape (and hence be observed by some observer at spatial infinity) only if its inclination to the radial direction is less than a critical value $\Psi_0(r)$ determined by

$$\sin^2 \Psi_0 = 27\, m^2(r-2m)/r^3 \quad , \quad \Psi_0(2m) = 0 \ . \qquad (24)$$

For $r \geq 3m$, so that $\Psi_0 \geq \pi/2$, all outward going rays escape to infinity. As r decreases from $3m$, the cone with half-angle Ψ_0 begins to narrow. In the limit $r \to 2m$, only rays emitted in the radial directions can escape; all other emitted rays, though some of them may go a long way out, are ultimately "pulled back" by the intense gravitational field.

3. The Schwarzschild-Kruskal Manifold.

The beautiful simplicity of the Schwarzschild metric (21) is in a way deceptive, for it exhibits pathological features that have troubled physicists since the early days of general relativity. Clearly, (21) breaks down at $r = 0$ and $r = 2m$. The latter, known as the

"Schwarzschild surface", is of great interest. Its nature had not been fully understood until quite recently. When $r \to 2m$, the metric components $g_{11} \to \infty$ and $g_{44} \to 0$, but $\det(g_{\alpha\beta})$ remains finite. For lower values of r, the curves $r = \text{const}$ lose their time-like character, while the curves $t = \text{const}$ are no longer space-like. But perhaps the Schwarzschild surface is not physically realizable? Thus it can be shown(22), for example, that no static fluid sphere with any reasonable equation of state can have a radius r less than $\frac{9}{8}(2m)$, which means that for this broad class of sources, the Schwarzschild surface is actually buried inside the matter where, of course, the vacuum field equations no longer hold. Nevertheless, one may wish to explore the Schwarzschild solution from a geometrical point of view when there is no "real mass". Nor are physical motivations in fact lacking. For there are dynamical objects which, starting from large radii, continuously contract until they actually cross the Schwarzschild surface (gravitational collapse). A full study of the dynamics requires an understanding of the geometry on and inside the Schwarzschild surface.

Modern advances in this area reveal the Schwarzschild surface as a regular surface, though possessing the most remarkable physical properties. The apparently singular behaviour of the metric at $r = 2m$ merely reflects a poor choice of coordinates. This can be demonstrated by embedding the entire Schwarzschild geometry in higher dimensional pseudo-Euclidean spaces[23], or by exhibiting suitable coordinate transformations[11]. On the other hand, the singularity at $r = 0$ is intrinsic (i.e., coordinate-independent), as can be seen, for example, from the expression for the invariant scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48m^2/r^6 \ . \tag{25}$$

A maximal analytic extension of the Schwarzschild space-time has been discovered by Kruskal[9], and independently by Szekeres[11]. In terms of "null coordinates" (u,v), related to (r,t) by

$$\begin{aligned} uv &= (r/2m-1)\exp(r/2m) \ , \\ u/v &= \exp(t/2m) \ , \end{aligned} \tag{26}$$

metric (21) takes on the form

$$ds^2 = (32m^3/r)\exp(-r/2m)dudv + r^2 d\Omega^2 \ , \tag{27}$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \ , \tag{28}$$

which is manifestly regular for all $r > 0$, i.e., for $\infty > uv > -1$. Since the transformation (26) is well-defined for $r > 2m$, and $G_{\alpha\beta}$ is a tensor, it follows that $G_{\alpha\beta}$, calculated from (27), vanishes when $r > 2m$. But $G_{\alpha\beta}$ is analytic in the domain of regularity of (27). Hence (27) is a solution of the vacuum field equations for all $r > 0$.

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$$ds^2 = (32m^3/r)\exp(-r/2m)dudv + r^2 d\Omega^2 \ , \tag{27}$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \ , \tag{28}$$

which is manifestly regular for all $r > 0$, i.e., for $\infty > uv > -1$. Since the transformation (26) is well-defined for $r > 2m$, and $G_{\alpha\beta}$ is a tensor, it follows that $G_{\alpha\beta}$, calculated from (27), vanishes when $r > 2m$. But $G_{\alpha\beta}$ is analytic in the domain of regularity of (27). Hence (27) is a solution of the vacuum field equations for all $r > 0$.

Figure 2 (p.26) is a sketch of a section (θ,ϕ = const) of the Schwarzschild-Kruskal manifold. Lines of constant r are hyperbolas in the (u,v) map, and lines of constant t are straight lines passing through the origin. However, the Schwarzschild time t runs off to infinity on the coordinate axes ($u = 0$ or $v = 0$) and is undefined for $uv < 0$ (quadrants II and IV, fig. 2). The pathology of the Schwarzschild metric (21) is intimately connected with this inadequacy of the Schwarzschild time coordinate. The extended manifold is maximal in the sense that any geodesic of (27) is extendable to infinite value of its affine parameter unless it runs into the intrinsic singularity at $r = 0$.

One sees at once from (27) that lines (u,θ,ϕ) = const and (v,θ,ϕ) = const are null ($ds^2 = 0$). Hence the Kruskal null coordinates have the important property that the coordinate lines represent the paths of radial light rays. The lines u = const are radial incoming light rays in the sense that $dr/dt < 0$ whenever t is defined. Similarly, v = const represent outgoing radial light rays.

The extended Schwarzschild manifold manifests a curious duplicity, in that <u>each</u> of quadrants I and III (fig. 2) corresponds to the entire region ($2m < r < \infty$, $-\infty < t < \infty$) where the Schwarzschild metric (21) is regular. Moreover, the metric (27) and equations (26) are invariant under the transformation ("reflection") $(u,v) \rightarrow (-u,-v)$. Consequently the idea of identifying the points (u,v,θ,ϕ) with $(-u,-v,\theta,\phi)$ have been proposed and examined[24]. This freedom of identifying points without disturbing the analyticity of the metric (aside from a sort of "conical" singularity at the origin $u,v = 0$) is allowed because the gravitational field equations only determine the local geometry. The resulting topology of the manifold, however, is such that it is impossible to maintain a global distinction between past and future in the region $r < 2m$. Analyses of the paths of radial particles show the existence of self-intersecting time-like lines, thus suggesting the annoying possibility of causal violations.

The conventional interpretation[25] of the topology of the Schwarzschild-Kruskal manifold, in contrast to the "elliptic interpretation" just mentioned, assumes different points of the Kruskal map to be physically distinct, but imposes on the manifold a global time-direction. This is achieved by introducing, at the expense of the reflectional symmetry intrinsic to the manifold, a time coordinate T and an orthogonal space coordinate X (see fig. 2), where

$$T = (u-v)/2 \;, \qquad X = (u+v)/2 \;, \tag{29}$$

in terms of which metric (27) becomes

$$ds^2 = (32m^3/r)\exp(-r/2m)(dX^2-dT^2) + r^2 d\Omega^2 \ , \qquad (30)$$

with

$$(r/2m-1)\exp(r/2m) = X^2 - T^2 \ . \qquad (31)$$

The conventional interpretation, though apparently accepted by most relativists, is not free of objections. For a critique, see[(26)].

Many of the features of the Schwarzschild-Kruskal manifold manifest themselves in the phenomenon of relativistic gravitational collapse, and will be further discussed in that context (Chapter III).

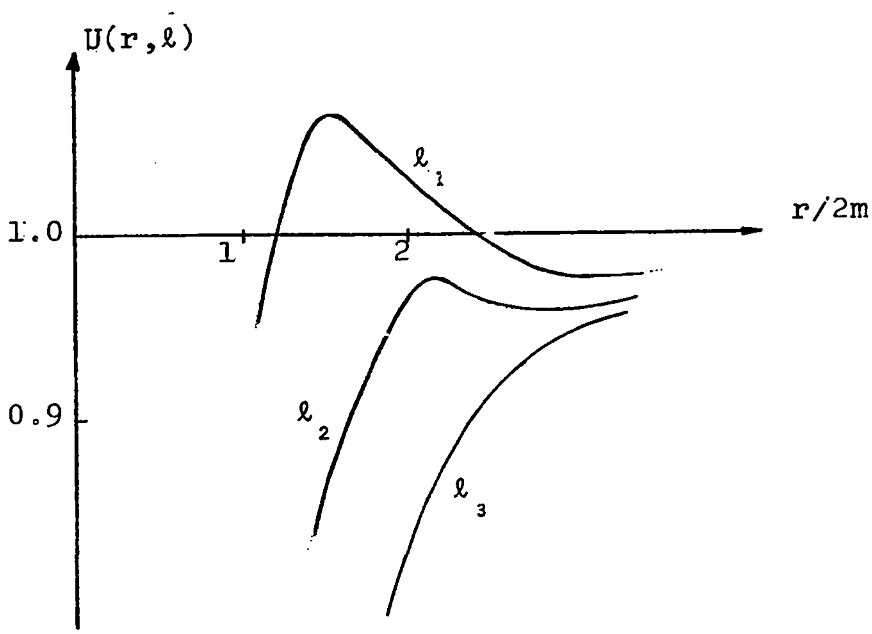

Fig. 1. Effective radial potential curves [cf. eq.(22)] for different angular momenta ℓ. ($\ell_3 < 2\sqrt{3}m < \ell_2 < 4m < \ell_1$).

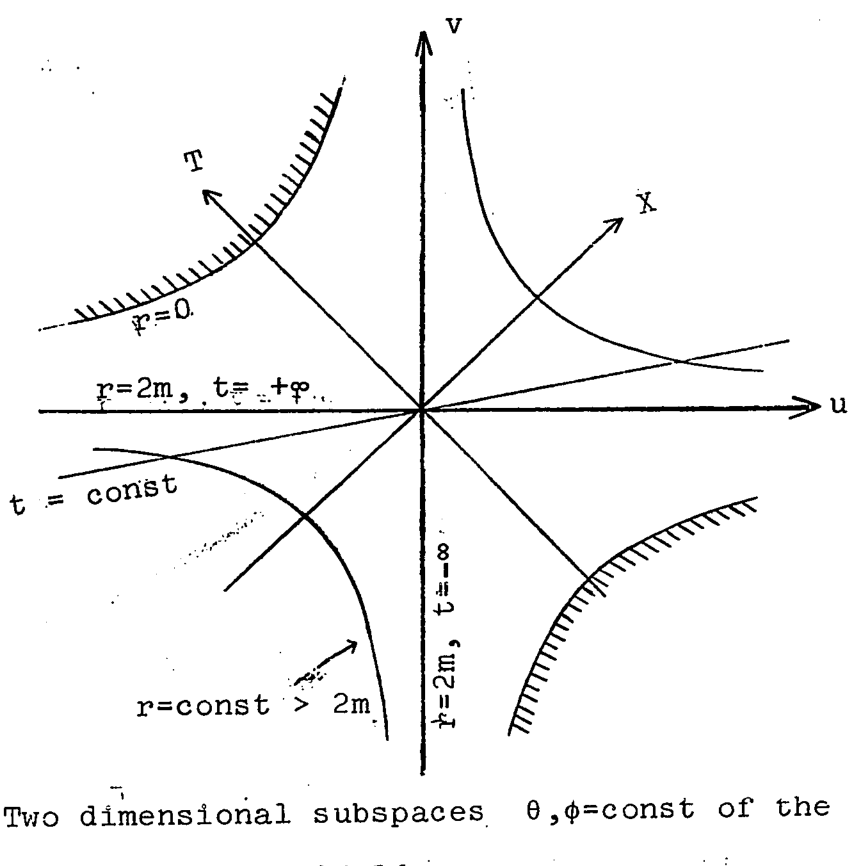

Fig. 2. Two dimensional subspaces θ,ϕ=const of the Schwarzschild-Kruskal manifold.

CHAPTER III

GRAVITATIONAL COLLAPSE

1. Introduction

Any collection of matter ("star") will contract under its own weight unless prevented by non-gravitational forces or rotation. The question is whether the balance between the outward push of pressure and the inward pull of gravitation may not be eventually upset if the mass, and hence the resulting gravitational force, of a system is increased indefinitely. In other words, given that a collection of matter of a specified nature is in a state of equilibrium, can we infer that its mass must therefore be less than a certain finite value?

According to Chandrasekhar[(27)], who uses the Newtonian equations of hydrostatic equilibrium, there is indeed an upper limit to the mass of a cold star (white dwarf) maintained in equilibrium by the pressure of a degenerate electron gas, of the order of one solar mass. Within the framework of general relativity, Oppenheimer and Volkoff[(28)] have shown that for systems of degenerate neutron gas, the maximum mass for equilibrium is also of the order of one solar mass. Recently, Wheeler and his coworkers[(29)] examined exhaustively the case of a static sphere subject to an equation of state thought to represent matter at the end of thermonuclear evolution. Again there exists a critical mass of the same order, above which there is no equilibrium configuration.

Thus when a star which in the course of its evolution has sufficiently cooled down possesses a mass appreciably larger than the critical value for equilibrium, it will start to collapse. The study of relativistic gravitational collapse was pioneered by Oppenheimer and his student Snyder in 1939[8], but until about five years ago, little further progress had been made. The recent discovery of astronomical objects known as "quasi-stellar" objects[6] has stimulated new interest in the problem, and marked the first time that general relativistic effects had been taken seriously by astrophysicists. However, for many relativists the fascination of the collapse problem lies primarily in the weird consequences which general relativity appears to predict.

Consider a spherical star collapsing adiabatically (i.e., no energy transfer). Its interior geometry is described by a certain solution to the Einstein's field equations depending, of course, on the specific energy-momentum tensor and equation of state assumed. On the other hand, by Birkhoff theorem, its exterior geometry is uniquely described by the Schwarzschild-Kruskal line-element. Hence the history of (an element of) the star's <u>surface</u>

must be a time-like world line of the Schwarzschild-Kruskal manifold. For a star which collapses to a radius smaller than its Schwarzschild radius $r = 2m$ (where m is the mass of the star), such a world line may be qualitatively represented by the curve labelled $p = p_b$ in fig. 4 (p.43c), with the portion to the left of the curve representing the exterior region. From the dispositions of the future null cones shown in the figure, it is easy to see that, once the star has collapsed past the Schwarzschild surface ($v = 0$, fig. 4), curious things happen: (1) no signals or particles emitted on the star's surface can ever escape to the external world ($r > 2m$), and (2) the collapse cannot be stopped but must rather proceed inexorably until the entire star gets "swallowed up" at the singularity $r = 0$. Paradoxically, to an external observer who stays out of the enigmatic Schwarzschild surface, the star never shrinks to a radius less than $2m$ at all. This we conclude from the fact that the world line of the star's surface cuts the Schwarzschild surface at time $t = +\infty$, where t is the Schwarzschild coordinate time, essentially the time measured by our observer. Thus what he sees is that the star begins to slow down as the radius $r = 2m$ is approached, reaching it only asymptotically.

We shall return to a more detailed discussion of these peculiar features of collapse and the important question of their generality in Sects. 3 and 4, turning our attention now to the interior of the collapsing star and the precise world lines followed by its particles. The collapse of a fluid sphere of uniform density and zero pressure ("homogeneous ball of dust"), which is instructive because of its simplicity, has been thoroughly analyzed[8]. Our aim here is to render the results more physically transparent by introducing a unified system of null coordinates which covers both the interior and the exterior domains of the collapsing object (Sect.2).

2. Null Coordinates for Spherically Symmetric Gravitational Collapse.*

For the interior domain of a homogeneous ball of dust, the metric, as determined by the gravitational field equations and the symmetry of the problem, turns out to be[30] identical with that of a Friedmann universe:

$$(ds^2)_{Int.} = S^2(\tau)\{(1-k\bar{r}^2)^{-1}\, d\bar{r}^2 + \bar{r}^2\, d\Omega^2\} - d\tau^2, \quad (k=0,\pm1) \tag{1}$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \;, \tag{2}$$

where $S(\tau)$, which essentially measures the radius $(=\bar{r}S)$ of the ball, is determined by

$$(dS/d\tau)^2 = \beta/S - k \;, \tag{3}$$

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* Based on V. de la Cruz, reference 33.

β being a positive constant. Here τ measures the proper time along the world lines of the dust particles, and $\bar{r}$, θ, ϕ are co-moving coordinates, i.e., the particles are at rest in this coordinate system, so that the four-velocity of the dust has the form** $u^\alpha = \delta^\alpha_4$. The only non-vanishing component of the dust energy-momentum tensor $T^\alpha{}_\beta = \mu u^\alpha u_\beta$ gives the energy density

$$
\begin{aligned}
- T^4_4 &= \mu \\
&= 3\beta/(8\pi S^3) \ . \qquad (4)
\end{aligned}
$$

From eq.(3), we easily see that the constant k has the following significance, as well as the usual interpretation in term of the curvature of the spatial section τ = const. $k = 0$: collapse from a state of rest at infinity; $k = +1$: collapse from a state of rest at a finite radius; $k = -1$: collapse from infinity with non-zero initial velocity.

The exterior metric of the ball of dust, complementing (1), is simply the Schwarzschild metric, eq.(II.21):

$$(ds^2)_{Ext.} = (1-2m/r)^{-1} dr^2 + r^2 d\Omega^2 - (1-2m/r)dt^2 \ , \qquad (5)$$

where m is the total mass of the dust.

To get a unified picture, the interior co-moving coordinate system can be extended to the exterior

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** Greek indices take values 1 - 4 ; latin, 1 - 3.

region[8] ; however, it retains no direct geometrical significance there. Extensions of Schwarzschild's coordinates to the interior[31] break down when the collapse has proceeded to the Schwarzschild radius $r = 2m$.

A system of null coordinates of Kruskal's type covering both interior and exterior of a ball of collapsing dust will now be given. It is defined down to the singular event of maximum implosion, and gives a direct picture of the paths of radial light rays and the dispositions of null cones throughout the manifold. For simplicity, attention is confined to the special solution of Oppenheimer and Snyder ($k = 0$), but the analysis can be readily extended to the other cases ($k = \pm 1$), with qualitatively similar results.

For $k = 0$, eq. (3) yields, after a simple integration,

$$S^{3/2} = -\frac{3}{2}\beta^{\frac{1}{2}}\tau + \text{const} , \tag{6}$$

where the constant of integration will be set to zero by adjusting the zero point of τ. Substituting (6) into (1), we obtain the interior metric for the case $k = 0$:[32]

$$(ds^2)_{\text{Int.}} = \tau^{4/3}(dp^2 + p^2 d\Omega^2) - d\tau^2 \tag{7}$$

in which

$$p \equiv \left(\frac{9}{4}\beta\right)^{1/3}\bar{r} \tag{8}$$

is also a co-moving radial coordinate. Eqs. (4) and (6) yield the expression for the homogeneous density as a function of time,

$$\mu = 1/(6\pi\tau^2) \ . \tag{9}$$

Thus (in summary) the line-element (7) represents, in terms of co-moving coordinates, the field in the interior of a homogeneous ball of dust, with density given by (9), collapsing (for $\tau < 0$) from a state of rest at infinity.

We will first seek to express the line-element (7) in terms of the Schwarzschild radial coordinate

$$r = p\tau^{2/3} \tag{10}$$

(the right hand side expression being the square root of the coefficient of $d\Omega^2$ in eq.(7)), and an orthogonal time coordinate $\bar{t} = \bar{t}(p,\tau)$. Using the general tensorial transformation equations

$$g^{\alpha'\beta'} = (\partial x^{\alpha'}/\partial x^{\alpha})(\partial x^{\beta'}/\partial x^{\beta})g^{\alpha\beta} \ , \tag{11}$$

we find

$$\begin{aligned} g^{rr} &= (r,_{\tau})^2 \ g^{\tau\tau} + 2r,_{\tau}r,_{p} \ g^{p\tau} + (r,_{p})^2 \ g^{pp} \\ &= 1 - \frac{4}{9} \ p^2\tau^{-2/3} \\ &= 1 - \frac{4}{9} \ p^3/r \end{aligned} \tag{12A}$$

where use has been made of (10) and (7),

$$g^{\bar{t}\bar{t}} = -(\bar{t}_{,\tau})^2 + \tau^{-4/3}(\bar{t}_{,p})^2 , \qquad (12B)$$

and

$$g^{r\bar{t}} = 0 = -\frac{2}{3} p\tau^{-1/3}\,\bar{t}_{,\tau} + \tau^{-2/3}\,\bar{t}_{,p} . \qquad (12C)$$

A particular solution of (12C) is

$$\bar{t} = p^2 + \frac{9}{2}\tau^{2/3} , \qquad (13)$$

with which $g^{\bar{t}\bar{t}}$ can be calculated from (12B). Finally, the line-element (7) takes the form

$$(ds^2)_{Int.} = (1-2M(p)/r)^{-1}\,dr^2 - \frac{1}{9}\tau^{2/3}(1-2M(p)/r)^{-1}\,d\bar{t}^2 + $$

$$+ r^2\,d\Omega^2 , \qquad (14)$$

where

$$M(p) \equiv \frac{4}{3}\pi r^3\mu(\tau) = \frac{2}{9}p^3 \qquad (15)$$

is the "mass" of a sphere interior to a given dust particle with co-moving coordinate p. The coefficients in (14) could be written as functions of the new coordinates r and $\bar{t}$ by using (10) and (13).

Let

$$p = p_b = \text{const} , \qquad (16)$$

i.e.,

$$r = r_b(\tau) = P_b\tau^{2/3} \qquad (17)$$

represent the boundary of the ball of dust, at which (14) must join continuously to the exterior metric (5). The condition is automatically satisfied by the coefficients of $d\Omega^2$. For those of dr^2 to match, $M(p_b)$ must be identified with the total mass of the dust ball, i.e.,

$$m = M(p_b) = \frac{2}{9} p_b^3 \,. \tag{18}$$

Finally, one requires

$$\frac{1}{9} \tau^{2/3} (1-2m/r_b)^{-1} d\bar{t}^2 = (1-2m/r_b)dt^2$$

for $p = p_b$, i.e.,

$$\begin{aligned} dt/d\bar{t} &= \frac{1}{3} (-\tau)^{1/3} (1-2m/r_b)^{-1} \\ &= \frac{1}{3} [\tfrac{2}{9}(\bar{t}-p_b{}^2)]^{3/2}[\tfrac{2}{9}(\bar{t}-p_b{}^2) - \tfrac{4}{9} p_b{}^2]^{-1}, \end{aligned} \tag{19}$$

the last equation being obtained with the help of (13), (16), and (17). Integration of (19) yields

$$\bar{t} = -T^3 - \frac{4}{3} p_b{}^2 T + \frac{4}{9} p_b{}^3 \ln[(T + \tfrac{2}{3} p_b)/(T - \tfrac{2}{3} p_b)] \,, \tag{20}$$

where

$$T(\bar{t}) \equiv [\, \frac{2}{9} (\bar{t}-p_b{}^2) \,]^{\frac{1}{2}} \tag{21}$$

and the constant of integration has been set to zero.

In this way, Schwarzschild coordinates (r,θ,ϕ,t)

are extended continuously into the interior —— the resulting metric is (14), with $\bar{t} = \bar{t}(t)$ defined implicitly by (20) and (21). However, (20) shows that the Schwarzschild time coordinate becomes infinite (and useless) when

$$[\tfrac{2}{9}(\bar{t}-p_b{}^2)]^{\frac{1}{2}} \pm \tfrac{2}{3}\, p_b = 0 \ ,$$

i.e.,

$$p^2 + \tfrac{9}{2}\,\tau^{2/3} = 3\, p_b{}^2 \ . \qquad (22)$$

For the boundary $p = p_b$, this happens when

$$\tfrac{9}{2}\,\tau^{2/3} = 2\, p_b{}^2 \ ,$$

i.e.,

$$\begin{aligned} r_b &= p_b \tau^{2/3} \\ &= \tfrac{4}{9}\, p_b{}^3 \\ &= 2m \ , \end{aligned} \qquad (23)$$

that is, when the radius of the ball is equal to its Schwarzschild radius, as should be expected. Thus some other system of coordinates, such as null coordinates of Kruskal's type, will be needed to cover the entire manifold.

Recall (Chap.II) that the exterior metric, maximally extended, can be expressed in terms of Kruskal null coordinates (u,v) by

$$(ds^2)_{Ext.} = (32m^3/r) \exp(-r/2m)dudv + r^2 d\Omega^2 \; , \qquad (24)$$

(u,v) being related to (r,t) by

$$u^2 = (r/2m-1) \exp(r+t)/2m \; , \qquad (25)$$

$$v^2 = (r/2m-1) \exp(r-t)/2m \; . \qquad (26)$$

The lines $u = \text{const}$ and $v = \text{const}$ represent the paths of incoming and outgoing radial light rays, respectively.

To find the radial null lines in the interior, we set $d\Omega^2 = 0$ in (7) and solve $ds^2 = 0$, obtaining

$$x \equiv (3\tau^{1/3} + p - p_b)/2p_b = \text{const (incoming)} \; , \qquad (27)$$

and

$$y \equiv (3\tau^{1/3} - p + p_b)/2p_b = \text{const (outgoing)} \; . \qquad (28)$$

Thus (x,y) are null coordinates for the interior, analytically related to p and τ everywhere except at the geometrical singularity (instant of maximum implosion) $\tau = 0$. Since $0 \leq p \leq p_b$ and $\tau < 0$, we have $x < 0$ and $y < \frac{1}{2}$. Along the curves $p = \text{const}$, y extends to $\frac{1}{2}(1-p/p_b)$ before reaching maximum implosion. See fig. 3 (p.43a).

Now an external, say incoming radial null line $u = u_1 = \text{const}$ (for fixed θ,ϕ) is uniquely determined by the event (r_1,t_1) at which it meets the boundary, u_1 being found by substituting $r = r_1$, $t = t_1$ into (25). In the interior, we now seek an analytic, monotonic function $u = u(x)$ such that $u(x) = u_1$ represents the continuation of the external null line $u = u_1$, i.e. such that $u = \text{const}$ labels the same incoming light rays throughout the space-time manifold.

Let the event (r_1, t_1) be also specified in terms of the co-moving coordinates by (p_b, τ_1). Then, from (17) and (20),

$$r_1 = p_b \tau_1^{2/3} \tag{29}$$

and

$$t_1 = \tau_1 + \frac{4}{3} p_b^2 \tau_1^{1/3} + \frac{4}{9} p_b^3 \ln[(\tau_1^{1/3} - \frac{2}{3} p_b)/(\tau_1^{1/3} + \frac{2}{3} p_b)]. \tag{30}$$

Substituting these into (25) and making use of (27), we obtain

$$\begin{aligned} u_1^2 &= (r_1/2m-1) \exp[(r_1 + t_1)/2m] \\ &= (x_1 - 1)^2 \exp(2x_1 + x_1^2 + \frac{2}{3} x_1^3) \;, \end{aligned} \tag{31}$$

where

$$x_1 \equiv x(p_b, \tau_1) \;. \tag{32}$$

Hence the desired function $u = u(x)$ is

$$u = (x-1) \exp(x + \frac{1}{2} x^2 + \frac{1}{3} x^3) \;, \quad (x < 0) \;. \tag{33}$$

Similarly, the other function $v = v(x)$ can be found to be

$$v \equiv v_1 = (y+1) \exp\left(-y+\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\right) \quad , \quad (y < 0) \ . \qquad (34)$$

Eqs. (33) and (34) together define a coordinate chart (chart I), in terms of which the interior metric becomes

$$(ds^2)_I = m^2(x+y)^4(xy)^{-3}\, dudv \ \times$$

$$\times \exp\{-[\tfrac{1}{3}(x^3-y^3) + \tfrac{1}{2}(x^2-y^2) + (x-y)]\} + r^2 d\Omega^2 \ . \qquad (35)$$

At the boundary $p = p_b$, $x = y$, this joins continuously to the exterior Kruskal solution (24).

Since $v(y)$ ceases to be monotonic at $y = 0$, the domain of validity of I does not include the whole of the interior. Another chart is needed to cover the domain $0 \leq y \leq \frac{1}{2}$, i.e., the region AOB in fig. 3. Define chart II by

$$u = (x-1) \exp\left(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3\right) \quad , \quad (x < 0) \ , \qquad (36)$$

$$v \equiv v_2 = (y+1)^{-1} \exp\left(y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3\right) \quad , \quad (0 < y < \tfrac{1}{2}) . \qquad (37)$$

Then the interior metric in terms of chart II is

$$(ds^2)_{II} = -m^2(1+y)^2(xy)^{-3}(x+y)^4 \, dudv \times$$
$$\times \exp\{-[\tfrac{1}{3}(x^3+y^3) + \tfrac{1}{2}(x^2-y^2) + (x+y)]\} + r^2 d\Omega^2 \, . \qquad (38)$$

The coordinates v defined by charts I and II match continuously (though not analytically) at $y = 0$, so that no concusion could arise from using the same labels for the coordinates. Moreover, the lines $v = \text{const}$ for $0 < y < \frac{1}{2}$ (for $1 < v < 1.0013$)* do not reach the boundary $p = p_b$, so there is no question of continuity of v across this boundary to the exterior v of eq.(24). The charts I and II, which are adjacent instead of overlapping, are properly connected to each other by the (x,y) chart, which overlaps both.

Figure 4 is a sketch in which we drew, in a (u,v) diagram, not exactly true to scale, the curves $p = 0$ and $p = p_b$ until they meet the curve $\tau = 0$ ($uv_2 = -1$). The exterior curve $uv = -1$ and the interior curves $uv_2 = -1$ and $p = 0$ together form the curve $r = 0$ of vanishing spherical area (shaded along its side). The

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* I am indebted to Professor F.J. Belinfante for the numerical data presented here, as well as many useful suggestions.

dotted line at $v = 1$ $(y = 0)$ separates the region of chart II from the region of chart I. The size of the region of chart II is exaggerated in the figure. Actually, its three corners lie at $(u = -0.8692,\ v = 1\ ;\ x = -1,\ y = 0\ ;\ p = 0\ ,\ \tau = -m/6)$, at $(u = -0.9889,\ v = 1.0113\ ;\ -x = y = 0.5;\ p = \tau = 0)$, and at $(-u = v = 1\ ;\ x = y = 0\ ;\ p = p_b,\ \tau = 0)$. The relation between the x,y scale and the u,v charts are indicated by the following little table.

x	0.00	-0.50	-1.00	-1.50	-2.00	-2.50
u	-1.0000	-0.9889	-0.8692	-0.5578	-0.2085	-0.0358

y	-2.00	-1.50	-1.00	-0.50	0.00	0.50
v	-785.77	-21.2605	0.00	0.9739	1.00	1.0113

The curves $\tau = \text{const}$ and $p = \text{const}$ (with the exception of $\tau = 0$, $p = 0$, and $p = p_b$) are not shown in fig. 4. The curves $p = \text{const}$ rise and bend over till they are tangent to $v = 1$ $(y = 0)$. From here, we continue these curves for $y > 0$ into the region of chart II, so that in the u,v plane these curves show an inflection point at $v = 1$. The only exception is the curve* $p = p_b$ which meets $v = 1$ at $u = -1$ under 45^o $(dv/du = -1)$, and which does not continue beyond $y = 0$. The apparent light-like direction which all the other curves $p = \text{const}$ instantaneously seem

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* The curve $p = p_b$ is simultaneously a geodesic of the exterior and interior geometries.

to take at $v = 1$ $(y = 0)$ has no physical reality and may be regarded as a local breakdown of charts I and II where they meet, as (fig. 3) we always have $dy/dx = 1$ along $p = \text{const}$. Similarly, always $dy/dx = -1$ along the curves $\tau = \text{const}$, so that the lines $u = \text{const}$ and $v = \text{const}$ bisect the angles made by the curves $p = \text{const}$ and $\tau = \text{const}$.

We have sketched in a few curves $r = \text{const}$ and $t = \text{const}$. The Schwarzschild time t is defined, for the exterior, by (25), (26), and, for the interior, by (20), (21). Again the lines $u = \text{const}$ and $v = \text{const}$ bisect the angles between the intersecting curves $r = \text{const}$ and $t = \text{const}$.

As noted before, the Schwarzschild chart breaks down, for the exterior, when $u/v < 0$, and, for the interior, when $9\tau^{2/3} + 2p^2 < 6\,p_b{}^2$, that is, when $3(x-y)^2 < 4(1+y)(1-x)$. The present null coordinates enable us to give a unified map (fig. 4) of the complete exterior and interior manifolds.

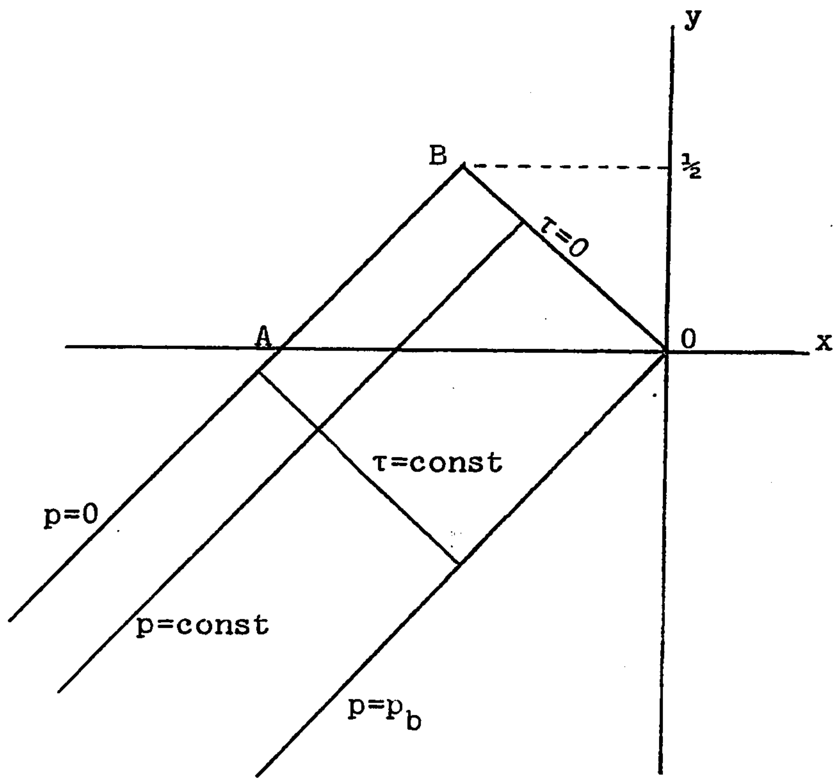

Fig. 3. The interior domain (bounded by the lines $p = 0$, $p = p_b$, and $\tau = 0$) of a collapsing ball of dust in terms of null coordinates (x,y) [cf. eqs.(27,28)]. Also cf. fig. 4.

LEGEND FOR FIGURE 4

Unified map of the complete exterior and interior manifolds of a collapsing ball of dust. The interior is bounded by the outside boundary (the curve $p = p_b$), the history of the center of the sphere (the curve $p = 0$), and the instant of maximum implosion (the curve $\tau = 0$). On $\tau = 0$ as well as on $p = 0$, we have $r = 0$ (zero spherical area). I (II) denotes the region of validity of chart I (chart II). The dotted line at $v = 1$ separates the two regions. The size of the region of chart II is exaggerated in the figure. Actually, its three corners lie at ($u = -0.8692$, $v = 1$), at ($u = -0.9889$, $v = 1.0113$), and at ($u = -1$, $v = 1$). Typical lines of constant Schwarzschild radial coordinate r and time coordinate t are shown [cf.eq.(20)]. Wavy lines represent radial signals emitted on the surface of the dust sphere at regular proper time intervals. Also shown are some future null cones. Parts of the figure are not drawn to scale in order to make some of the features of this diagram qualitatively more conspicuous.

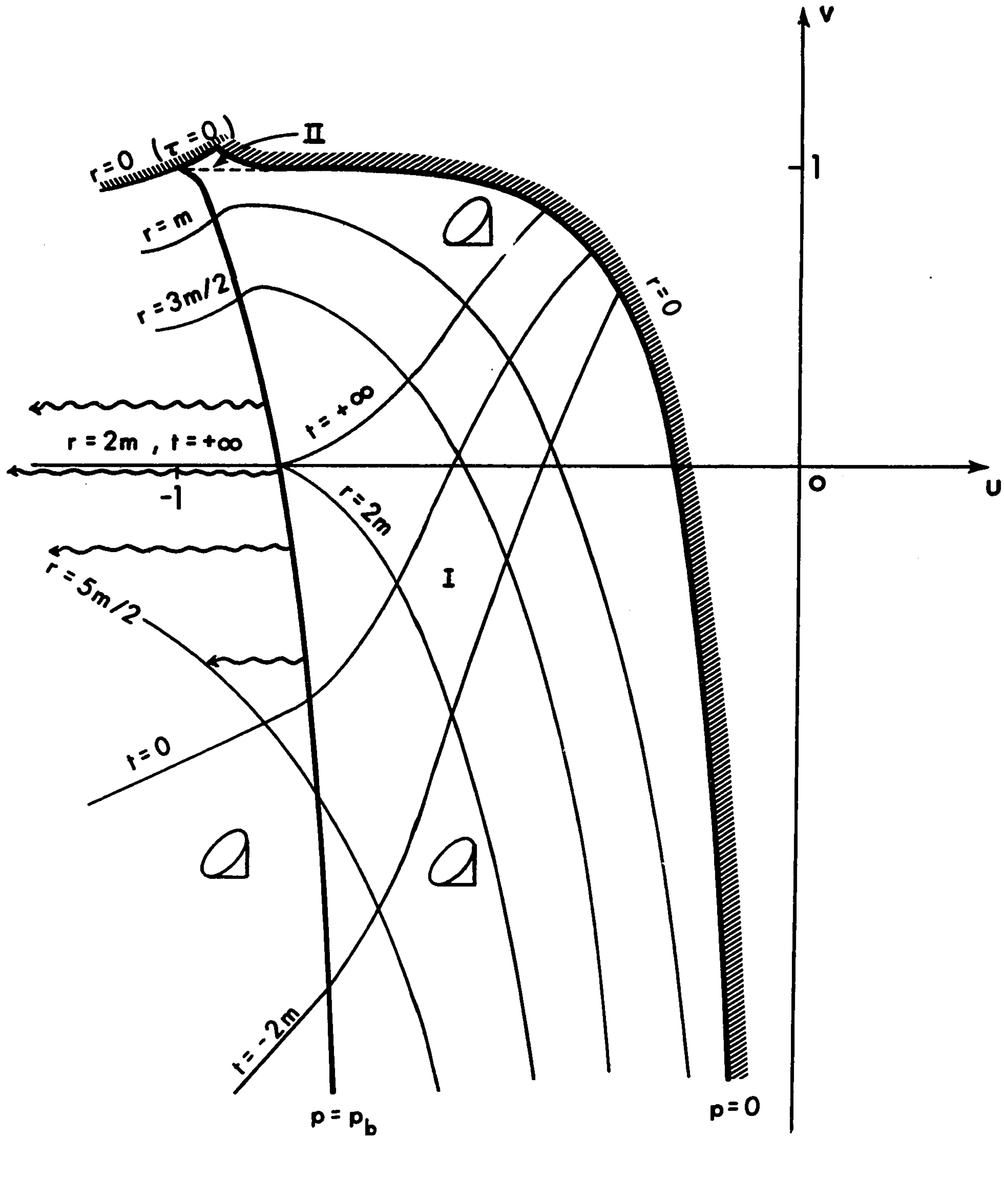

Figure 4

3. Features of Spherically Symmetric Gravitational Collapse.

Before going on to the question of the generality of the features of collapse outlined in the introductory section, we pause to describe these features in greater detail, basing our discussion on the dust model, which is historically the first example studied of an object that actually collapses past its Schwarzschild radius.

The picture as seen by an external stationary observer (world line $r = r_1 = \text{const} > 2m$), whose clock registers time according to $t_1 = (1-2m/r_1)^{\frac{1}{2}} t$, is as follows. At first, when the stellar radius is still large, the picture is qualitatively Newtonian, since the distortion of geometry produced by the star is then negligible. As the star shrinks toward the Schwarzschild surface $r = 2m$, its motion, instead of being accelerated as one would expect from the Newtonian description, appears to an external observer to slow down, reaching the radius $r = 2m$ only asymptotically. At the same time, the star becomes redder and redder. Fig. 4 gives a qualitative explanation. The wavy lines represent radial light rays sent out at regular proper time intervals by an audacious astrophysicist riding on the surface of the star and falling with it ("co-moving observer"). By this means, the external observer ($r = 5m/2$ in the figure) is informed of the progress of the collapse. We see that, as the collapse proceeds toward the Schwarzschild surface, the light rays are received at increasingly longer intervals; it will take an infinite time to receive the signal sent out at the instant when the co-moving observer crosses the Schwarzschild surface, while all later signals could never reach the external observer at all. At the same time, the signals will also be progressively red-shifted (partly because of Doppler effect, partly because the photons will have to spend more and more energy to overcome the gravitational pull) —— infinitely so for the last signal received. In addition (but this we cannot deduce from fig. 4), more and more of the light rays emitted at the stellar surface at oblique angles will be recaptured by the growingly intense field, in a way similar to the case of light rays emitted by stationary observers (cf. Chapter II). Thus the luminosity drops to zero asymptotically.

Nevertheless, to the co-moving observer it takes a perfectly finite time for the star to collapse from an arbitrary radius $r_o > 2m$ to the Schwarzschild surface and thence to the singular state $r = 0$. Indeed, if the density corresponding to radius r_o is μ_o, then (9) gives at once

$$\Delta\tau_o = (6\pi\mu_o)^{-\frac{1}{2}} \qquad (\text{i.e.}, = (6\pi G\mu_o)^{-\frac{1}{2}}) \qquad (39)$$

for the time of collapse to the singularity. (By coincidence, this is identical with the prediction of Newtonian theory). For $\mu_o \sim 1\ g/cm^3$, $\Delta\tau_o$ is of the order of 10^3 sec. Since the proper time τ has an invariant meaning in general relativity ($d\tau^2 = -ds^2$), this description by the co-moving observer is essentially coordinate-independent.

Thus the star does cross the schwarzschild surface in a finite proper time from any arbitrary radius, and in fact that event will not appear to be specially remarkable to an innocent co-moving observer.* However, when this has happened, gravitational self-closure takes place: the star loses all possibilities of communication with the external world. In other words, the Schwarzschild surface is an "event horizon" for all external observers. Only the gravitational field of the star will be felt. The star itself reaches a state of zero volume and infinite density at proper time $\tau = 0$, producing a singularity of infinite curvature. Further analysis would not seem to be possible or meaningful**. However, if our ordinary idea of causality is to be preserved, the star can never re-emerge to the same external world.

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* The physical conditions on the Schwarzschild surface are in general quite normal, except for the intense gravitational field. However, according to the Principle of Equivalence, this would not give rise to any local change of physical laws.

** The entire star probably gets "crushed out of existence" by the infinite tidal forces at the singularity [34]. For a different point of view, cf. Israel [54].

Whether these peculiarities are qualitatively specific to the present much-too-idealized model is a question that naturally comes to mind. Certainly they appear to be more general than the model itself, as the discussion in the introductory section indicates. For a sufficiently massive star, neither the density nor the pressure is immoderately large when the radius is of the order $r = 2m$, and so perhaps the idealization of zero pressure is not important. But once the Schwarzschild radius is passed, we saw that the collapse must proceed irreversibly to zero volume, provided the assumption of spherical symmetry is not violated, and provided no energy-transfer occurs throughout. That pressure forces, which are expected to increase with compression, are powerless to stop the collapse at these late stages from progressing to completion might appear strange. But we have to bear in mind the fact that for normal matter a larger pressure is always accompanied by a larger internal energy, which contributes to the gravitational mass and hence to self-attraction.

Such considerations have been confirmed by recent work. For spherically symmetric collapse of (non-rotating) fluid spheres, it is now known that the employment of more realistic physical conditions (inhomogeneous matter[(35)], non-vanishing pressure and pressure-gradients[(36)], or even the emission of isotropic radiation[(37)]) generally preserves the qualitative features presented above.

The situation with regard to non-spherical collapse, with or without rotation, is much less clear. The problem is still too complicated to be handled by the analytic or numerical techniques now at our disposal. It has been claimed[38] that small departures from sphericity or small rotations are likewise without qualitative effects. We note that on the basis of Newtonian theory, even a small rotation could prevent unlimited contraction.[39] This corresponds to the fact that a test particle with an arbitrarily small angular momentum can (according to Newtonian theory) move along a stable orbit around a gravitational force center (of mass m, say). On the other hand, as was noted in Chap.II, general relativity predicts that for an angular momentum less than $2\sqrt{3}m$, the particle will inevitably be captured. In other words[39], the gravitational field can be strong enough even to crush rotation, a situation that never occurs in the Newtonian case. Heuristic arguments of this nature, however, though they can be refined[38], cannot be trusted completely . The next section deals with the question of non-spherical collapse, citing a number of interesting works.

In Chap. IV we shall study the collapse of <u>charged</u> bodies.

4. On Asymmetric Gravitational Collapse and Singularities.

We have seen that for spherical collapse, an event horizon generally develops which seals off the star from the external world. The subsequent fate of the star can be followed until it ends in a singularity characterized by infinite curvature and density.*

The occurence of singularities is a prevalent feature of the known exact solutions of Einstein's gravitational field equations. Some of the best known of these are the Schwarzschild solution, the Reissner-Nordström solution(41), the Kerr solution(16), and the Friedmann's cosmological solutions. The prediction of physical singularities by a theory for possibly physically realizable systems (e.g., cosmological models** and models for gravitational collapse) is a matter of serious concern†, perhaps signalling the breakdown of the theory itself. However, all exact

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* Presumably quantum mechanical fluctuations in the curvature of space-time must be taken into account when the gravitational field becomes extremely intense. However, these probably would not become important until a density of the order of 10^{93} g/cm³ is reached [40].

** For opposing views on whether singularities can occur in realistic cosmological models, see Lifschitz et al [44] and Hawking[47].

† "Physicists abhor singularities, and they like to think that Nature does too" —— K.S. Thorne.

solutions involve some special assumptions concerning the metrics or the distributions of matter, which are often motivated by mathematical convenience rather than physical necessity. To what extent, in particular, is the assumption of strict sphericity responsible for the appearance of the singularity in gravitational collapse?

Closely connected with the concept of singularity is the notion of "geodesic incompleteness." Although no entirely satisfactory general definition of a singularity has yet been given, a region where the energy density or some invariants of the Riemann curvature tensor (e.g., the Petrov scalars[(42)]) become infinite must presumably be regarded as singular. However, there is some difficulty in defining the concept — needed for global analyses — of a singular space-time manifold as one containing singular points, since a new manifold may simply be defined without these points. Nevertheless, the two manifolds would share a common feature: there are geodesics which cannot be extended to arbitrary values of their affine parameters. The three types of geodesic incompleteness — that of time-like, null, and space-like geodesics, respectively — are however not equivalent[(43)]. There are space-times which are, for example, space-like complete but neither time-like nor null complete. But because time-like and null geodesics represent world lines of test particles and light rays, time-like and null completeness would seem to be a minimum condition for a space-time manifold to be regarded as "singularity-free".

In what follows we shall quote and briefly assess the important works of Lifschitz and Khalatnikov, of Penrose, and of Israel, which together form the basis on which present day speculations on the qualitative effects of asymmetry on collapse are made.

(A) Lifschitz-Khalatnikov solution near a singularity.

In 1959 and 1960, Lifschitz and Khalatnikov[44] sought to solve the following problem which they posed: Assuming a singularity to exist, it is required to find near it the form of the broadest class of solutions of the gravitational equations, so as to judge from the number of arbitrary functions (of the spatial coordinates) it contains, whether this solution is general. We remind ourselves that among the arbitrary functions contained in a solution, generally there are some whose arbitrariness is connected with the arbitrariness of the coordinate system itself, and are hence not directly associated with the physics of the problem . The number of physically arbitrary functions in the general solution of the gravitational equations for a perfect fluid is 8, corresponding to the 8 pieces of initial data which must be specified to fix the future evolution of the system. (These are the spatial distribution of the fluid density, the three components of the fluid velocity, and the amplitudes and rates of change of amplitudes for two modes of gravitational waves.)[13]

For their purpose, Lifschitz and Khalatnikov make use of synchronous coordinate systems, characterized by the properties* $g_{4i} = 0$, $g_{44} = -1$, and assume the equation of state of an ultra-relativistic gas $P = \frac{1}{3}\mu$ near the singularity (at which the pressure and density supposedly become infinite). Furthermore, they confine their explicit analysis to solutions whose singularities occur on a space-like hypersurface, contending that this is the type of singularity which is of major interest here. By a suitable choice of the synchronous frame, one can reduce the equation of the hypersurface to the form $\tau = 0$. Lifschitz and Khalatnikov found that, near the singularity, the geometry of space-time and the fluid behave in the following way:

$$ds^2 = [\tau^{2p_1} l_i l_j + \tau^{2p_2} m_i m_j + \tau^{2p_3} n_i n_j] dx^i dx^j - d\tau^2 , \tag{40A}$$

$$u = u^{(0)} \tau^{-2(1-p_3)} , \quad u_j = u_j^{(0)} \tau^{(1-p_3)/2} , \quad u_4 = u_4^{(0)} \tau^{-(3p_3-1)/2} . \tag{40B}$$

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* Latin indices run from 1 to 3; greek, 1 to 4. $x^4 \equiv \tau$ is the proper time of the matter.

Here μ is the mass-energy density and u_α the four-velocity of the fluid. The quantities p_i, l_i, m_i, n_i, $\mu^{(0)}$, and $u_\alpha^{(0)}$ are 17 functions of the space coordinates x^i, which are connected by 7 algebraic relations*. Lifschitz and Khalatnikov further asserted[45] that "It can be shown that the higher terms of the expansion of the metric contain no other arbitrary functions." If this is correct, then the Lifschitz-Khalatnikov solution contains altogether ten (17-7) arbitrary functions of the space coordinates, three of which can be fixed at will by performing a co-ordinate transformation among the x^i, leaving us with 7 physically arbitrary functions, which is one short of the number required of the general solution.

Thus the results of the investigation of Lifschitz and Khalatnikov show** that space-like singularities characterized by an infinite density (such as that which occurs in the spherical collapse of dust-cf.Sect.2) cannot be created by realistic gravitational collapse; in particular, a slight perturbation of the sphericity will introduce qualitative changes.

** But we note here that whether the employment of synchronons frames might entail some loss of generality is a question that has been raised[34].

* Two of these relations read: $\Sigma p_i = \Sigma p_i^2 = 1$. Also, by convention, $-1/3 \leq p_1 \leq 0 \leq p_2 \leq 2/3 \leq p_3 \leq 1$.

(B) Penrose theorem.

The temptation (which one might feel) to generalize the results of Lifschitz and Khalatnikov was somewhat checked by the more recent piece of work of Penrose[46] and its further ramifications, mainly in the hand of Hawking[47]. Penrose theorem states that space-time (containing a collapsing system) cannot be null-complete if

(1) At some initial instant of time, the universe has an infinite volume;

(2) There exists a Cauchy hypersurface H. (A Cauchy hypersurface is defined as a complete connected space-like 3-surface which intersects every time-like or null line once and only once. This means that data given on the hypersurface are sufficient to fix the entire future evolution of the universe). Condition (1) implies that H is open;

(3) In the manifold M_+ which represents the future time development of H, a time direction can be globally assigned. (Causality condition);

(4) At every point of M_+, the local energy measured by any observer is non-negative. That is, if $T_{\mu\nu}$ is the energy-momentum tensor, then $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors t^μ ;

and

(5) There exists in M_+ a "trapped surface" T —— defined generally as a closed, spacelike 2-surface with the property that the null geodesics of each of the two systems (of outgoing and incoming null geodesics) which meet T orthogonally converge locally in future directions at T.

For spherically symmetric collapse, trapped surfaces exist everywhere inside the event horizon (cf. fig. 4). One could plausibly argue that*, provided deviation from sphericity is not too large, trapped surfaces will still exist. Hence, if conditions (1) to (4) are accepted, it appears that singularities cannot be avoided by destroying the spherical symmetry.

In case space-time does not admit a Cauchy surface (as happens for example with the geometries of Kerr and of Reissner-Nordström), Hawking and Ellis[48] have given a modified version of the Penrose theorem.

The rigorous proof[46] of the Penrose theorem requires advanced techniques from differential geometry and topology, which lie beyond the scope of the present work. But essentially, the proof runs as follows[34]:

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* But see (C), below.

Assume that all the hypotheses of the theorem are satisfied, and also that space-time is null-geodesic complete. Consider the null geodesics which issue perpendicularly from the trapped surface T. Since T is trapped, adjacent null geodesics will start to converge, and by extending them to indefinite values of their affine parameters (null-geodesic completeness), they will continue to converge (energy condition (4)) until they cross. In this way, the totality of such null geodesic segments forms a closed 3-dimensional null hypersurface, which can be approximated arbitrarily closely by a closed space-like 3-surface B. Construct all time-like geodesics orthogonal to B, and extend them into the past until they reach the Cauchy hypersurface H (condition (2)). This family of timelike geodesics provides a smooth, nearly one-to-one mapping of the closed hypersurface B onto the open hypersurface H. But such a mapping is impossible according to a fundamental theorem of topology. Hence a contradiction is produced, and the theorem is proved.

The Penrose-type theorems are not incompatible with the implication of the works of Lifschitz and Khalatnikov as we stated it, for these theorems say nothing about the nature of the singularities that may occur. The Schwarzschild $r = 0$ singularity, which is hit by every time-like and null geodesic that passes inside the Schwarzschild surface, is actually not typical. For the

Reissner-Nordström solution, for instance, only radial null or space-like geodesics can reach the singularity. Grischuk[49] has found a general solution to the gravitational field equations for dust which contains a time-like singularity. All these, together with the results of Lifschitz and Khalatnikov, indicate[48] that the singularities, if they do occur in asymmetric gravitational collapse, should consist of isolated points or timelike surfaces which most of the world lines managed to avoid.

(C) Israel's Theorem.

The significance of the Penrose-type theorems hinges on the assumption that when the strict sphericity is only very slightly perturbed, the Schwarzschild surface will —— though distorted —— still retain its essential characteristics as an event horizon, since otherwise the occurence of trapped surfaces for the general case cannot be plausibly argued for. For our purpose, an event horizon (with respect to observers far from the collapsing system) may be defined as a hypersurface which nowhere extends to spatial infinity, and which divides space-time events into two non-empty sets, one consisting of events that in principle could be observed by some of the observers, the other consisting of events that forever lie outside their power of observation. In the special case of static space-times (i.e., those independent of time and time-reversal) which are not necessarily spherically symmetric, the surfaces $g_{44} = 0$ are the natural analogue of the Schwarzschild surface.(50)

The production of an event horizon is itself a remarkable feature of spherical collapse, and perhaps the assumption of its continual existence for more general cases should not be lightly made. Indeed, Israel(51) has very recently shown that any static perturbation that destroys the spherical symmetry of a source of the Schwarzschild field, also destroys the event horizon. More precisely, Israel's theorem(51) states that

> Among all static, asymptotically flat vacuum fields with closed, simply-connected equipotential surfaces $g_{44} = \text{const}$, Schwarzschild's solution is the only one which has a non-singular event horizon $g_{44} = 0$.

What is the bearing of Israel's theorem on the question of asymmetric collapse(52)? Suppose that an event horizon forms during the collapse of an asymmetric (non-rotating) star. Because of the slowing down of processes for the external observer, it seems permissible to regard the limiting external field as static. Thus, if the regular event horizon is to be retained, the star must, before passing through it, divest itself of all quadrupole and higher moments; then the Penrose-type theorems may become operable, and the collapse should proceed in qualitatively much the same way as previously described, ending in a state of singularity*. The chief theoretical difficulty is to find a plausible mechanism to accomplish this task(52). But unless this is accomplished, Israel's theorem either prohibits (due to the formation of singularities on the event horizon(53)) or renders unlikely (due to the absence of an event horizon) the development of trapped surfaces. There would then be no reasons why the star should not eventually bounce out or, indeed, pulsate†, as happens generally in the corresponding Newtonian case.

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* Or the star may re-emerge into another universe. Cf. Chapter IV.

† For specific considerations in favor of pulsating stars as a model for the baffling quasi-stellar sources, see Israel [26].

CHAPTER IV

GRAVITATIONAL BOUNCE*

1. Introduction.

We shall be concerned in this chapter with the relativistic gravitational collapse of a charged spherical shell falling in a spherically symmetric external field. The study of such simple artificial problems, while of no direct relevance to astrophysics, can nonetheless serve a useful purpose, since it brings into relief basic issues of principle in the general relativistic theory of collapse which are still far from understood.

As we saw in the last chapter, general relativity leads to the following picture for the evolution of a contracting spherical body. Once the compression passes a certain critical limit, characterized roughly by the Newtonian potential becoming comparable with c^2, the subsequent history is one of continuing collapse which cannot be halted by pressure forces. The irreversibility of this picture is surprising, and differs radically from the corresponding Newtonian picture, where the motion is in general oscillatory. If one examines the relativistic

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* Based on V. de la Cruz and W. Israel(55).

derivation to see how the element of irreversibility enters, one finds that it stems from two largely unconnected causes.

(A) External irreversibility: development of an event horizon at $r = 2m$. The surface of the contracting spherical body passes (in finite proper time) within the critical Schwarzschild sphere $r = 2m$. To an external observer, light emitted from this sphere suffers infinite gravitational and Doppler red-shift, and $r = 2m$ therefore appears as an event horizon which the contracting body seems to be approaching asymptotically as $t \to \infty$. If ordinary ideas of causality are to be maintained, he can never see the body re-emerge from this sphere.

(B) Intrinsic irreversibility: spacelike character of the curves $r = \text{const}$ near $r = 0$. The exterior (Schwarzschild) field of the body, analytically extended to $r = 0$, has the property that the curves $r = \text{const} < 2m$ are space-like. The history of a particle on the surface of the body is a time-like curve of the exterior manifold. It is easy to see that the particle can reverse its inward motion at $r = r_0 < 2m$ only if (a) its world-line is momentarily space-like, and (b) it subsequently travels into the past. Assuming that classical general relativity remains valid even under the extreme conditions prevailing near $r = 0$, one is forced to the conclusion that no rebound is possible and that the entire mass piles up irreversibly on the singular curve $r = 0$.

Of these two arguments, (A) seems (at least in so far as the condition of spherical symmetry is strictly met ; cf. Sect.4, Chap.III) on surer ground, since it does not depend on extrapolation to extreme conditions. For masses of astrophysical interest, compression to $r = 2m$ does not produce immoderate densities or curvatures. Modification due to quantized gravitation, possible inapplicability of Einstein's field equations in regions of extremely intense gravitational fields, or other new physical effects of an unanticipated kind might profoundly affect the situation near $r = 0$, but should not be important near $r = 2m$. The exact nature of such modifications is, of course, unknown. One could try to take their effects into account in a crude way by supposing that the standard Schwarzschild metric is modified to

$$ds^2 = (1-2m/r+a/r^2+\cdots)^{-1}dr^2-(1-2m/r+b/r^2+\cdots)dt^2 + r^2 d\Omega^2 ,$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2 .$$

If the usual astronomical predictions of Einstein's theory are to be preserved, the constants a and b would have to be small compared with m^2. In that case, we still have an event horizon (the sphere on which $g_{44} = 0$), but the arguments of (B) are clearly liable to break down. The possibility cannot be ruled out the a collapsing spherical body reverses its motion near $r = 0$ and re-expands. The intriguing question is how such a picture can be reconciled with the apparent irreversibility of the collapse as seen by an external observer.

We shall illustrate some of the possibilities by focussing attention on the special line-element†

$$ds^2 = f^{-1}\,dr^2 - f dt^2 + r^2 d\Omega^2 \,, \qquad f(r) \equiv 1 - 2m/r + e^2/r^2 \,, \tag{1}$$

which is formally the Reissner-Nordström(41) metric for the gravitational field of a charged particle. In Sects. 2 and 3 we derive the equation of motion of a thin spherical shell in such a field. It is found (Sects.4,7) that bounce can occur under a great variety of conditions. In particular, the shell can bounce inside an event horizon. In that case the manifold represented by (1) is incomplete. If it is extended analytically (Sect.5) in the manner of Graves and Brill(41), the extended space-time appears as a periodic lattice of geometrically similar asymptotically flat spaces, joined by "tunnels" in which space is closed. Re-emergence of the rebounding shell from the event horizon then takes place in a new, distinct space (Sects.6,7).

Our results are similar to those recently obtained by Novikov(56), who has considered the homologous collapse of a uniformly charged ball of dust. The shell model has

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† An an illustration, it may be remarked that (1) represents the external field of an (uncharged) spherical body in the theory of F. Hoyle and J.V. Narlikar [3] .

the advantage that complete solution can be given in a simple explicit form. Novikov's results can be recovered as a special case (Sect.7).

Sect. 8 concludes the chapter with some general remarks.

2. Dynamics of a Thin Shell.

We shall adopt the approach(57) to thin shells in general relativity in which their histories are characterized in a purely geometrical way by the extrinsic curvatures of their imbeddings in space-time.

Let V be a four-dimensional Riemannian manifold of metric $(+++-)$, with coordinates x^α and metric tensor $g_{\alpha\beta}$. Let Σ be a time-like hypersurface in V, i.e., the unit normal n^α to Σ is space-like:

$$n_\alpha n^\alpha = +1 \ . \tag{2}$$

In terms of the coordinates x^α in V, the parametric equations of Σ are of the form

$$x^\alpha = f^\alpha\ (\xi^1, \xi^2, \xi^3) \ , \tag{3}$$

where* ξ^i are intrinsic coordinates of Σ. An infi-

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* Throughout this chapter, Greek indices refer to 4-dimensional, Latin indices to 3-dimensional quantities.

nitesimal displacement dx^α in Σ is given by

$$dx^\alpha = (\partial f^\alpha/\partial \xi^i) d\xi^i \ , \tag{4}$$

which defines the triad of holonomic basis vectors tangent to Σ and associated with ξ^i :

$$e^\alpha_{(i)} = \partial f^\alpha/\partial \xi^i \ . \tag{5}$$

The intrinsic geometrical properties of Σ are completely determined by its metric 3-tensor $g_{ij} = e^\alpha_{(i)}\, e_{(j)\alpha}$, and intrinsic operations such as covariant differentiation may be analogously defined in Σ . Thus, if A^α is a vector field tangent to Σ , we can associate with it a 3-vector field A^i in Σ given by

$$A_i = A_\alpha e^\alpha_{(i)} \ , \ A^\alpha = A^i e^\alpha_{(i)} \ , \tag{6}$$

and define the covariant derivative of A_i with respect to ξ^i to be

$$A_{i;j} = \partial A_i/\partial \xi^j - A^k \, \Gamma_{k,ij} \tag{7}$$

where

$$\Gamma_{k,ij} = \tfrac{1}{2} (\partial \, g_{ik}/\partial \xi^j + \partial \, g_{jk}/\partial \xi^i - \partial \, g_{ij}/\partial \xi^k)$$

are the 3-dimensional analogue of the 4-dimensional Christoffel symbols $\Gamma_{\alpha,\beta\gamma}$.

Extrinsic properties of Σ associated with its imbedding in V are measured by the absolute derivatives

$\delta n^{\alpha}/\delta\xi^{i}$ of the unit normal. We recall that the absolute derivative of a vector function B^{α} defined on a curve parametrized by t is

$$\delta B^{\alpha}/\delta t = \partial B^{\alpha}/\partial t + B^{\lambda}\Gamma^{\alpha}_{\lambda\mu}\, dx^{\mu}/dt \ . \tag{8}$$

Since, from (2),

$$n_{\alpha}\delta n^{\alpha}/\delta\xi^{i} = 0 \ , \tag{9}$$

i.e., $\delta n^{\alpha}/\delta\xi^{i}$ are perpendicular to n^{α}, we can write

$$\delta n^{\alpha}/\delta\xi^{i} = K_{i}{}^{j} e^{\alpha}_{(j)} \tag{10}$$

thus defining the extrinsic curvature 3-tensor K_{ij} of Σ . Explicitly,

$$\begin{aligned} K_{ij} &= - n_{\alpha}\delta e^{\alpha}_{(j)}/\delta\xi i \\ &= - n_{\alpha}\{\partial^2 f^{\alpha}/\partial\xi^{i}\partial\xi^{j} + \\ &\quad + \Gamma^{\alpha}_{\lambda\mu}(\partial f^{\lambda}/\partial\xi^{i})(\partial f^{\mu}/\partial\xi^{i})\} \\ &= K_{ji} \ . \end{aligned} \tag{11}$$

If A^{α} is a vector field tangent to Σ, then[57]

$$\delta A^{\alpha}/\delta\xi^{i} = A^{i}{}_{;j}\, e^{\alpha}_{(i)} - A^{i} K_{ij}\, n^{\alpha} \ . \tag{12}$$

In general, one has the following four relations[57] between the extrinsic curvature 3-tensor K_{ij} and the normal components of the Einstein tensor $G_{\alpha\beta}$ on Σ :

$$ {}^3R - K_{ab}K^{ab} + K^2 = -2\, G_{\alpha\beta}n^{\alpha}n^{\beta} \, , \tag{13} $$

$$ K^{b}_{a;b} - \partial_a K = - G_{\alpha\beta}e^{\alpha}_{(a)}\, n^{\beta} \, . \tag{14} $$

Here, $K \equiv g^{ij} K_{ij}$, and ${}^3R$ is the intrinsic curvature invariant of Σ . Each term of (13) and (14) is a 4-scalar, i.e. independent of the coordinates x^{α} .

Now let the time-like hypersurface Σ divide space-time into two parts V_- , V_+ which both contain Σ as part of their boundaries and are otherwise disjoint. Let n^{α} (directed from V_- to V_+) be the unit space-like normal to Σ , and let $K_{ab}{}^-$, $K_{ab}{}^+$ denote the extrinsic curvatures of Σ associated with its imbeddings in V_-, V_+ respectively. Then[57] Σ is the history of a thin shell if

$$ \gamma_{ab} \equiv K_{ab}{}^+ - K^-_{ab} \tag{15} $$

is nonvanishing. The surface energy tensor S_{ab} of the shell is given by the Lanczos equations[57]

$$ \gamma_{ab} - g_{ab}\,\gamma = - 8\pi\, S_{ab} \qquad (\gamma \equiv g^{ab}\gamma_{ab}) \, , \tag{16} $$

the analogue of Einstein's field equations

$$ G_{\alpha\beta} = - 8\pi\, T_{\alpha\beta} \tag{17} $$

for the surrounding continuous medium.

Corresponding to (13), (14) are the following eight relations[†]

$$^3R - K_{ab}K^{ab} + K^2|^{\pm} = -2\, G_{\alpha\beta}n^{\alpha}n^{\beta}|^{\pm} \tag{18}$$

$$K^{b}{}_{a;b} - \partial_a K|^{\pm} = -\, G_{\alpha\beta}\, e_{(a)}^{\alpha} n^{\beta}|^{\pm} \tag{19}$$

The jump of (18) across Σ is

$$\begin{aligned}
&- 2\, [G_{\alpha\beta}n^{\alpha}n^{\beta}] \\
&= (-K_{ab}K^{ab}+K^2)\Big|^{+} - (-K_{ab}K^{ab}+K^2)\Big|^{-} \\
&= -\,(K_{ab}{}^{+} - K_{ab}{}^{-})(K^{ab}_{+} + K^{ab}_{-}) + (K^{+}-K^{-})(K^{+}+K^{-}) \\
&= -\gamma_{ab}\,(K^{ab}_{+} + K^{ab}_{-}) + \gamma(K^{+}+K^{-}) \qquad \text{by (15)} \\
&= -(\gamma_{ab}-\gamma g_{ab})(K^{ab}_{+} + K^{ab}_{-}) \\
&= 8\pi\, S_{ab}\,(K^{ab}_{+} + K^{ab}_{-}) \qquad \text{by (16)},
\end{aligned}$$

or, in view of (17),

$$2\,[T_{\alpha\beta}n^{\alpha}n^{\beta}] = S^{ab}\,(K_{ab}{}^{+} + K_{ab}^{-}). \tag{20}$$

Similarly, (19) yields

$$[T_{\alpha\beta}\, e_{(a)}^{\alpha}\, n^{\beta}] = -\, S^{b}{}_{a;b}\,. \tag{21}$$

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† Limits of a field quantity Ψ as an event P on Σ is approached from V_-, V_+ respectively are denoted by $\Psi|^{-}$, $\Psi|^{+}$. In Sects. 2 and 3 square brackets denote jump discontinuities: $[\Psi] \equiv \Psi|^{+} - \Psi|^{-}$.

We now specialize to the case of a coherent shell of dust, characterized by the surface energy tensor

$$S^{ab} = \sigma\ u^a u^b \ , \qquad (22)$$

where the unit time-like vector $u^a = d\xi^a/d\tau$ tangent to Σ represents (τ being the proper time) the 4-velocity of the dust particles, and σ is the sum of their rest masses per unit area. The dynamics of the shell in vacuo was considered in[57]. The case where the shell falls in a continuous medium with nonvanishing energy tensor is a straight forward generalization.

From (22) ,

$$S_a{}^b{}_{;b} = u_{a;b}\ \sigma\ u^b + (\sigma\ u^b)_{;b} u_a \ ,$$

$$u^a\ S_a{}^b{}_{;b} = u_{a;b} u^a\ \sigma\ u^b + (\sigma\ u^b)_{;b} u_a u^a$$

$$= -\ (\sigma\ u^b)_{;b} \ ,$$

and by use of (21), we obtain

$$(\sigma\ u^b);b = u^a [T_{\alpha\beta}\ e^\alpha_{(a)}\ n^\beta] = [T_{\alpha\beta}\ u^\alpha\ n^\beta] \qquad (23)$$

and

$$\sigma\ u_{a;b}\ u^b = -\ (\delta^c_a + u_a u^c)\ [T_{\alpha\beta}\ e^\alpha_{(c)}\ n^\beta] \ , \qquad (24)$$

where

$$u^\alpha|^\pm = u^a\ e^\alpha_{(a)}|^\pm$$

$$= dx^\alpha/d\tau|^\pm$$

represent the 4-velocity of the dust particles as measured in V_+ , V_- . The 4-acceleration $\delta u^\alpha/\delta\tau|^\pm$ can be written, according to (12), as

$$\delta u^\alpha/\delta\tau|^\pm = (\delta u^\alpha/\delta\xi^i)\, d\xi^i/d\tau$$

$$= u^a{}_{;b}\, u^b\, e^\alpha_{(a)} - K_{ab}\, u^a\, u^b\, n^\alpha|^\pm \quad .$$

From this and (20) we immediately obtain

$$\sigma n_\alpha \delta u^\alpha/\delta\tau|^\pm + \sigma n_\alpha \delta u^\alpha/\delta\tau|^- = -\ 2[T_{\alpha\beta}n^\alpha n^\beta] \qquad (25)$$

and

$$n_\alpha \delta u^\alpha/\delta\tau|^+ - n_\alpha \delta u^\alpha/\delta\tau|^- = -\ \gamma_{ab}u^a u^b = 4\pi\sigma \qquad (26)$$

with the aid of (16) and (22).

3. Charged Spherical Shell in a Spheri-symmetric Electrovac Field.

We consider a charged spherical shell of dust falling in the electrovac field produced by a spherically symmetric concentration of mass and charge at its center. For such a spherically symmetric (static or nonstatic) universe, an extension[58] of Birkhoff's theorem shows that the line element is reducible to the standard Reissner-Nordström metric (1) —— with appropriate parameters e,m —— in any region free of matter.

Let $r = R(\tau)$ be the equation of Σ , the history of the shell, and

$$(ds^2)_\Sigma = \{R(\tau)\}^2\, d\Omega^2 - d\tau^2 \tag{27}$$

be its intrinsic metric, so that τ is the proper time measured along the streamlines θ,ϕ = const. The interior and exterior line elements may be written

$$(ds^2)_- = \{f_-(r)\}^{-1}\, dr^2 + r^2 d\Omega^2 - f_-(r) dt_-^2, \quad (r<R(\tau)), \tag{28}$$

$$(ds^2)_+ = \{f_+(r)\}^{-1}\, dr^2 + r^2 d\Omega^2 - f_+(r) dt_+^2, \quad (r>R(\tau)), \tag{29}$$

where

$$f_-(r) \equiv 1 - 2m_1/r + e_1{}^2/r^2 ,$$
$$f_+(r) \equiv 1 - 2m_2/r + e_2{}^2/r^2 . \tag{30}$$

Thus, the shell has charge $e_2 - e_1$ and gravitational mass $m_2 - m_1$.

Both (28) and (29) must induce the same intrinsic metric, namely (27), on Σ . Comparison of the coefficients of $d\Omega^2$ confirms that the interior and exterior radial coordinates agree on Σ . Further,

$$d\tau^2 = f_-(R) dt_-^2 - \{f_-(R)\}^{-1} dR^2 = f_+(R) dt_+^2 - \{f_+(R)\}^{-1} dR^2 . \tag{31}$$

This fixes the relation between t_- and t_+ on Σ , and verifies, as expected, that the simultaneous imbedding of

Σ in V_- , V_+ is possible.

We proceed to write out explicitly the dynamical equations (23) to (26). Since u^α, n^α are orthogonal unit vectors in the 2-space of $r(\equiv x^1_\pm)$, $t_\pm(\equiv x^4_\pm)$, we have

$$u^\alpha_+ = dx^\alpha_+/d\tau = (\dot{R},o,o,X_+) \quad , \quad \dot{R} \equiv dR/d\tau \ , \tag{32}$$

$$n^+_\alpha = (X_+,o,o,-\dot{R}) \ , \tag{33}$$

where

$$X_+ \equiv dt_+/d\tau$$

$$= \{f_+(R) + \dot{R}^2\}^{\frac{1}{2}}/f_+ \text{ by (31)}, \tag{34}$$

with corresponding expressions for u^α_- , n^-_α . By intrinsic differentiation of $u^+_\alpha u^\alpha_+ = -1$ and use of (32) we find

$$0 = u_\alpha \delta u^\alpha/\delta\tau|^+ = f^{-1}_+ \dot{R}\ \delta u^1/\delta\tau - f_+\ X_+ \delta u^4/\delta\tau \ ,$$

and this may be used to eliminate $\delta u^4_+/\delta\tau$ from

$$n_\alpha\ \delta u^\alpha/\delta\tau|^+ = X\ \delta u^1/\delta\tau - \dot{R}\ \delta u^4/\delta\tau|^+$$

yielding

$$n_\alpha\ \delta u^\alpha/\delta\tau|^+ = (fX)^{-1}\ \delta u^1/\delta\tau$$

$$= (fX)^{-1}\ \{\ddot{R} + \Gamma^1_{\lambda\mu}u^\lambda u^\mu\}|^+$$

$$= (f_+X_+)^{-1}\ \{\ddot{R} + \tfrac{1}{2}\ df_+(R)/dR\} \ , \tag{35}$$

with a corresponding expression for $n_\alpha\ \delta u^\alpha/\delta\tau|^-$.

The Reissner-Nordström metric (1) is associated[59] with the energy tensor

$$-T^4_4 = -T^1_1 = T^2_2 = T^3_3 = e^2/8\pi r^4 \qquad (36)$$

(other components zero), so that in our case

$$T^\beta_\alpha u^\alpha n_\beta \Big|^{\pm} = 0 \ , \qquad (37)$$

$$[T^\beta_\alpha\ n^\alpha n_\beta] = [T^1_1\ n^1 n_1 + T^4_4\ n^4 n_4]$$

$$= [T^1_1\ n_\alpha n^\alpha]$$

$$= (e_1{}^2 - e_2{}^2)/8\pi r^4 \ . \qquad (38)$$

From (23) and (37) ,

$$(\sigma u^b)_{;b} = 0 \qquad (39)$$

expressing the conservation of the proper mass of the shell. For the co-moving coordinates employed in (27), $u^b = (0,0,1)$ and (39) symplifies to

$$d\ \ln(R^2\sigma)/d\tau = 0 \ ,$$

or

$$4\pi R^2\sigma \equiv k = \text{const} \ . \qquad (40)$$

Equations (24) are here identically satisfied. Equations (25) and (26) yield,when (35), (38), and (40) are substituted into them,

$$(e_2{}^2-e_1{}^2)/kR^2 = (f_+X_+)^{-1}\{\ddot{R} + m_2/R^2 - e_2{}^2/R^3\} + $$
$$+ (f_-X_-)^{-1}\{\ddot{R} + m_1/R^2 - e_1{}^2/R^3\}, \quad (41)$$

$$k/R^2 = (f_+X_+)^{-1}\{\ddot{R} + m_2/R^2 - e_2{}^2/R^3\} -$$
$$- (f_-X_-)^{-1}\{\ddot{R} + m_1/R^2 - e_1{}^2/R^3\} . \quad (42)$$

We recall from (34) that

$$f_+X_+ = \{1 + \dot{R}^2 - 2m_2/R + e_2{}^2/R^2\}^{\frac{1}{2}} ,$$

$$f_-X_- = \{1 + \dot{R}^2 - 2m_1/R + e_1{}^2/R^2\}^{\frac{1}{2}} .$$

Setting $y \equiv \frac{1}{2}(1 + \dot{R}^2)$ so that $\ddot{R} = dy/dR$, and employing $x \equiv 1/R$ as independent variable, we obtain from (41), (42) the equivalent set of equations

$$(e_2{}^2-e_1{}^2)/k + k = - 2 \frac{d}{dx} (2y - 2m_2x + e_2{}^2x^2)^{\frac{1}{2}} , \quad (43)$$

$$(e_2{}^2-e_1{}^2)/k - k = - 2 \frac{d}{dx} (2y - 2m_1x + e_1{}^2x^2)^{\frac{1}{2}} . \quad (44)$$

Either of these may be integrated at once and the constant of integration determined from the other. In this way we arrive at the first integral*

---

* This may be compared with the corresponding equation of motion in Newtonian theory:

$$\tfrac{1}{2}(m_2-m_1)(dR/dt)^2 + \tfrac{1}{2}[e_1{}^2+e_2{}^2-(m_1^2+m_2^2)]/R = \text{const} .$$

$$1 + (dR/d\tau)^2 = A + B/R + C/R^2 \ , \tag{45}$$

where

$$A \equiv (m_2 - m_1)^2/k^2 \tag{46}$$

is introduced for convenience, and

$$B = m_1 + m_2 - A(e_2{}^2 - e_1{}^2)/(m_2 - m_1) \ , \tag{47}$$

$$4C = A(e_2{}^2 - e_1{}^2)^2/(m_2 - m_1)^2 - 2(e_1{}^2 + e_2{}^2) + A^{-1}(m_2 - m_1)^2 \ . \tag{48}$$

From (40) and (46) ,

$$4\pi R^2 \sigma = A^{-\frac{1}{2}}(m_2 - m_1) \ ,$$

which enables us to interpret the constant A in terms of the binding energy W , since

$$- W \equiv (m_2 - m_1)(1 - A^{-\frac{1}{2}})$$

represents the difference between the gravitational mass $m_2 - m_1$ of the shell (i.e., its total energy) and the sum of the rest masses of its constituent particles. It thus represents the contribution to the shell's gravitational mass due to its kinetic and potential energies.

The further integration of (45) would be elementary. However, the various physical possibilities emerge more clearly from a qualitative description of a few representative special cases. This will be our aim in the next few sections.

4. Charged Shell in Vacuo.

Suppose that no mass or charge is present apart from the shell itself (mass m, charge e). Then $e_1 = m_1 = 0$, $e_2 = e$, $m_2 = m$ and the equation of motion (45) reduces to

$$\{1 + (dR/d\tau)^2\}^{\frac{1}{2}} = a - b/R \ , \tag{49}$$

where we have written $A^{\frac{1}{2}} \equiv a$ and

$$b \equiv (a^2e^2-m^2)/2am. \tag{50}$$

If $b < 0$, we can imagine the shell as starting, either from infinity with initial velocity $\dot{R} = -(a^2-1)^{\frac{1}{2}}$ (for $a \geq 1$), or from rest at a finite maximal radius $R_{max} = |b|/(1-a)$ (for $0 < a < 1$). It accelerates as it falls inward and, upon reaching $R = 0$, produces a singularity. The subsequent history is therefore a matter of conjecture (the possibility of a rebound is, of course, not excluded).

More definite conclusions can be drawn when $b > 0$ (always obtainable for a shell with given mass and nonvanishing charge by taking a sufficiently large). In this case, it is necessary that $a > 1$. The shell is impelled inwards from infinity with initial velocity

$(a^2-1)^{\frac{1}{2}}$. It is decelerated, and comes to rest at a finite radius $R_{min} = b/(a-1)$, then re-expands symmetrically to infinity. As measured by a co-moving observer, who uses the proper time τ, the time required to implode from any given radius $R_o > R_{min}$ and re-expand to this radius is

$$\Delta\tau = 2(a^2-1)^{-1} \sqrt{[(a+1)R_o-b][(a-1)R_o-b]} + 4ab(a^2-1)^{-3/2} \ln\{\sqrt{(a-1)[(a+1)R_o-b]/2b} + \sqrt{(a+1)[(a-1)R_o-b]/2b}\} \quad (51)$$

and is therefore finite.

Since τ is related to the time t_- of stationary observers in the interior flat domain by, according to (31),

$$dt_-^2 = (1+\dot{R}^2)d\tau^2 ,$$

we obtain from (49)

$$(dR/dt_-)^2 = [(a-1)R-b][(a+1)R-b]/(aR-b)^2 . \quad (52)$$

The denominator does not vanish for $R \geq R_{min}$. The motion as seen by an interior observer is thus qualitatively similar to the intrinsic description just given.

To an external observer, however, the sequence of events may appear quite different. From (49) and (31) we obtain

$$\left(\frac{dt_+}{dR}\right)^2 = \frac{1}{f^2} + \frac{R^2}{f[(a-1)R-b][(a+1)R-b]} ,$$

$$f(R) \equiv [(R-m)^2 + e^2-m^2]/R^2 , \tag{53}$$

as the equation of motion in terms of the exterior time coordinate t_+ (essentially the proper time of a stationary observer with large radial coordinate). For a shell with $e^2 > m^2$, f never vanishes and the coordinate r, t_+ cover the complete exterior manifold: qualitatively the motion seen externally is as previously described. However, for $m^2 \geq e^2$, f vanishes at

$$R_1 \equiv m + (m^2-e^2)^{\frac{1}{2}}$$

before $R_{min} = m(a+1)/2a - a(m^2-e^2)/2m(a-1) \leq m$ is reached. The time Δt_+ needed to implode from any given radius $R_0 > R_1$ to R_1 is

$$\Delta t_+ \approx \lim_{\varepsilon\to 0^+} \tfrac{1}{2} R_1{}^2(m^2-e^2)^{-\frac{1}{2}} \ln(1/\varepsilon) \quad \text{for} \quad m^2 > e^2 ,$$

$$\approx \lim_{\varepsilon\to 0^+} (m^2/\varepsilon) \quad \text{for} \quad m^2 = e^2 . \tag{54}$$

Thus we reach the curious conclusion that an external observer never sees the re-expanding shell if $e^2 \leq m^2$; re-expansion to arbitrarily large radius nevertheless occurs in finite time according to an interior or a co-moving observer.

5. Analytic Completion of Reissner-Nordström Manifold for $e^2 \leq m^2$.

To resolve the apparent paradox of the previous Section, we require a picture of the exterior manifold when $e^2 \leq m^2$. The coordinates r, θ, ϕ, t then no longer furnish a complete map. The problem of analytically completing the Reissner-Nordström manifold has been dealt with by Graves and Brill[41] (for $e^2 < m^2$) and by Carter[41] (for $e^2 = m^2$) . We shall present a somewhat simplified review.

(i) The case $e^2 = m^2$. In this case the function $f(r)$ in the Reissner-Nordström metric (1) becomes

$$f(r) = (1-m/r)^2 ,$$

and the coordinate t remains time-like for all r. Introduce an angular time-like coordinate Θ , with range $-\infty < \Theta < \infty$, such that $t/2m = \mathrm{tg}\,\Theta$ for $-\pi/2 < \Theta < \pi/2$. The (formally) extended line element

$$ds^2 = (1-m/r)^{-2}\, dr^2 + r^2\, d\Omega^2 - 4m^2(1-m/r)^2(d\, \mathrm{tg}\Theta)^2 \tag{55}$$

represents a periodic space-time which has a geometrical singularity at $r = 0$ and is otherwise free of singularities. The r, Θ map is subject to local breakdown on the lines $r = m$, $\Theta = (n+\frac{1}{2})\pi$. That $r = m$ is actually a regular part of the manifold can be verified by expressing the line element in a form which is manifestly regular for $r > 0$:

$$ds^2 = 2\,dv'dr - (1-m/r)^2 dv'^2 + r^2 d\Omega^2 \tag{56}$$

where the advanced time parameter v' is analytically related to r, t by

$$dv' = (1-m/r)^{-2}\,dr + dt \qquad (r > m)\,. \tag{57}$$

In the v', r chart one can, for instance, follow any incoming radial null geodesic $v' = \text{const}$ originating in a region with $r > m$ (e.g. region Ia of fig.5) down to $r = 0$ (in region IIIa). This chart thus provides a regular mapping of two adjoining regions such as Ia and IIIa. By analogous use of a retarded time parameter we can construct a chart for IIIa and Ib. An infinite chain of such overlapping coordinate patches enables us to follow any null or time-like geodesic down to the singularity at $r = 0$ or to indefinite values of its affine parameter. Use of the r, Θ map means that allowance must be made for local breakdowns, but has the advantage of providing a clearer over-all picture.

(ii) <u>The case</u> $e^2 < m^2$. In this case, the quadratic coefficient $f(r)$ in the Reissner-Nordström metric (1) has real unequal factors

$$f(r) = (r-r_1)(r-r_2)/r^2 , \qquad (0 < r_2 < r_1) , \tag{58}$$

$$r_{1,2} \equiv m \pm (m^2-e^2)^{\frac{1}{2}} . \tag{59}$$

Writing the metric in the form

$$ds^2 = f\ (f^{-1}dr+dt)(f^{-1}dr-dt) + r^2d\Omega^2 ,$$

we see that outgoing and incoming radial null geodesics may be represented by equations $u = \text{const}$ and $v = \text{const}$ respectively, where

$$2ku^{-1}du = f^{-1}dr - dt \tag{60}$$

$$2kv^{-1}dv = f^{-1}dr + dt \tag{61}$$

and k is an adjustable constant. In the u, v chart the line element (1) takes the form

$$ds^2 = (4k^2f/uv)dudv + r^2d\Omega^2 . \tag{62}$$

Integration of (60) and (61) yields

$$r + \frac{r_1{}^2}{r_1-r_2} \ln|1-r/r_1| - \frac{r_2{}^2}{r_1-r_2} \ln|1-r/r_2| = k \ln|uv| , \tag{63}$$

$$t = k \ln |v/u| \qquad (r > r_1 \text{ or } r < r_2) . \tag{64}$$

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$$ds^2 = (4k^2 f/uv)dudv + r^2 d\Omega^2 . \tag{62}$$

Integration of (60) and (61) yields

$$r + \frac{r_1^{\,2}}{r_1 - r_2} \ln|1-r/r_1| - \frac{r_2^{\,2}}{r_1 - r_2} \ln|1-r/r_2| = k \ln|uv| , \tag{63}$$

$$t = k \ln |v/u| \qquad (r > r_1 \ \text{ or } \ r < r_2) . \tag{64}$$

The constants of integration have been set to zero for convenience.

Consider now the chart u_1, v_1 obtained by setting $k = k_1 \equiv r_1^2/(r_1-r_2)$. We find from (63)

$$u_1 v_1 = (\frac{r}{r_1} - 1)(\frac{r}{r_2} - 1)^{-(r_2/r_1)^2} \exp[\frac{r_1-r_2}{r_1^2} r] \quad (r > r_2) \, , \tag{65}$$

and (62) exhibits no singularity at $r = r_1$. The chart u_1, v_1 in fact gives a regular mapping of any given subregion of the manifold which has $r > r_2$. A (coordinate) singularity does develop at $r = r_2$, however, and it is necessary to go over to another chart before that happens.

Define the chart u_2, v_2 by setting $k = k_2 \equiv - r_2^2/(r_1-r_2)$ in (63) and (64). Then

$$u_2 v_2 = (\frac{r}{r_2} - 1)(1 - \frac{r}{r_1})^{-(r_1/r_2)^2} \exp[- \frac{r_1-r_2}{r_2^2} r] \quad (r < r_1) \, , \tag{66}$$

and this provides a regular covering for any subregion with $r < r_1$.

In the domain of overlap $r_2 < r < r_1$ we have

$$r_1^2 \ln|u_1| + r_2^2 \ln|u_2| = - (r_1^2 \ln|v_1| + r_2^2 \ln|v_2|) \, . \tag{67}$$

Since $u_1 = \text{const.}$ represents an incoming radial geodesic and therefore must correspond to a constant u_2, we have $u_1 = u_1(u_2)$. Similarly $v_1 = v_1(v_2)$. Hence from (67),

$$|u_1|^{r_1^2} \cdot |u_2|^{r_2^2} = \text{const} = e^{\alpha} \text{, say,}$$

$$|v_1|^{r_1^2} \cdot |v_2|^{r_2^2} = e^{-\alpha} .$$

By adjusting the scales we can set $\alpha = 0$ and obtain

$$|u_1|^{r_1^2} = |u_2|^{-r_2^2} , \; |v_1|^{r_1^2} = |v_2|^{-r_2^2} \qquad (r_2 < r < r_1) . \tag{68}$$

The complete manifold for $e^2 < m^2$ is a periodic lattice of alternating regions of type I $(r > r_1)$, type II $(r_2 < r < r_1)$, and type III $(r < r_2)$. Figure 6 (due to Carter[41]) is a schematic over-all map with local singularities at some of the lattice points. Figures 7(a) and (b) are Kruskal-type diagrams which together give a faithful map of any subregion covered by a pair of overlapping charts u_1, v_1 and u_2, v_2.

Because of the cyclic character of the extended manifold, it is natural to raise the question of possible topological identifications. For instance, in Fig. 5 for $e^2 = m^2$, one might postulate that all points $(r, \theta + 2n\pi)$, $n = 0, \pm 1, \ldots$, represent the same physical event.

Such "space-saving" devices are tempting, but they lead to causal paradoxes. In addition, there would be dynamical difficulties connected with gravitational self-interaction, since a world tube would then intersect a space $t = \text{const}$ more than once. These possibilities will not be considered further here.

6. Charged Shell with $e^2 \leq m^2$ in Vacuo.

We now return to the discussion, begun in Sect. 4, of the charged shell in empty space, and proceed to consider the exterior view of the motion for $b \geq 0$, $e^2 \leq m^2$, when an event horizon exists.

For a shell with $e^2 = m^2$, there is always a special solution ($a = 1$ in (49) and (50)) which is static. The shell is then at rest (in neutral equilibrium) at any radius R. The world-line ST (Fig. 5) represents the history of such a shell with $R = \text{const} < m$. The extended manifold displays an infinite sequence of $r = 0$ physical singularities, e.g. for $\frac{1}{2}\pi < \theta < 3/2\ \pi$. If we wish, we can remove these singularities and maintain strict periodicity by introducing an endless number of "re-incarnations" of the shell, e.g. at S'T'. Space-time is then flat for $r < R$ and all θ. The result is of some interest mathematically, since it represents a universe containing an event horizon ($r=m$) which is

LEGENDS FOR FIGURES

Figure 5. Schematic representation of the extended Reissner-Nordström manifold for $e^2=m^2$. Shaded sections of the map are not part of the manifold. Dashed lines represent radial null geodesics; the apparent constriction of these lines at $r=m$ is due to local defectiveness of the coordinates. The time-like curve KLM represents the history of a thin shell, which implodes in the space Ia, reverses its motion at L after passing through the event horizon $r=m$, then re-expands in the space Ib.

Figure 6. Schematic representation of the extended Reissner-Nordström manifold for $e^2<m^2$. Null lines are inclined at 45°. FGHJM is the history of a shell which collapses from infinity in the asymptotically flat space Ia, passes through the event horizon $r=r_1$, comes to rest at J with a minimal radius smaller than r_2, then re-expands into the asymptotically flat space Ic. ABCDE is the history of an oscillatory shell or uniformly charged sphere. Shading on the curves distinguishes the interior domain.

Figure 7. Kruskal-type diagrams for portions of the overall map of figure 6, showing the same curves ABCDE and FGHJM. Figures 7(a) and (b) overlap in the region IIb, and may be regarded as linked together along the curve $r=r_0$, where r_0 is any convenient value between r_1 and r_2.

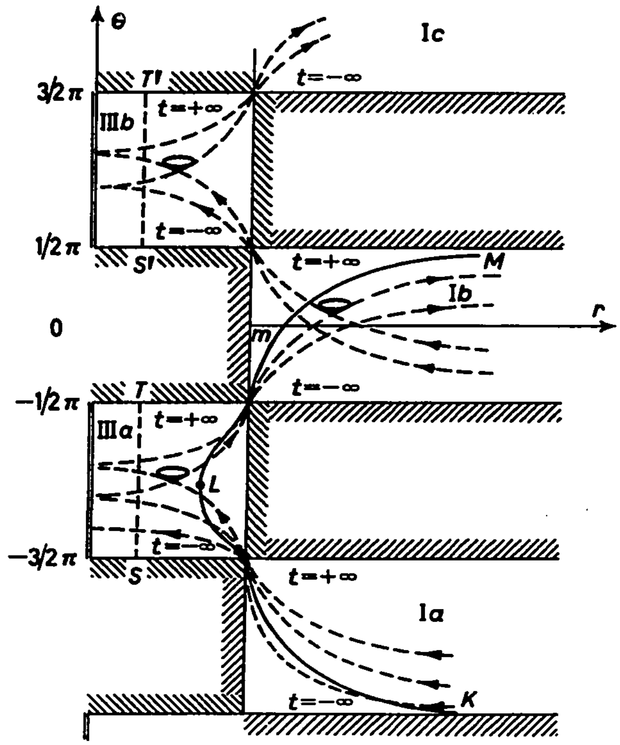


Figure 5

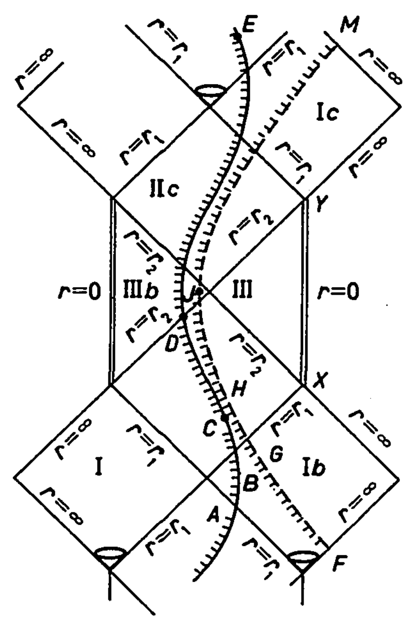
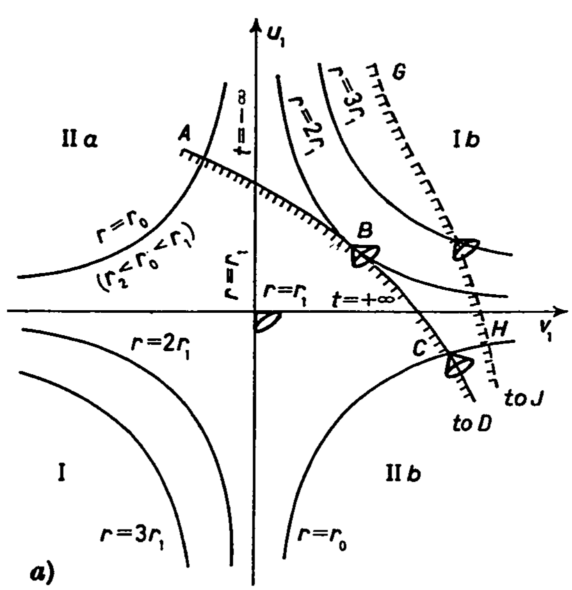
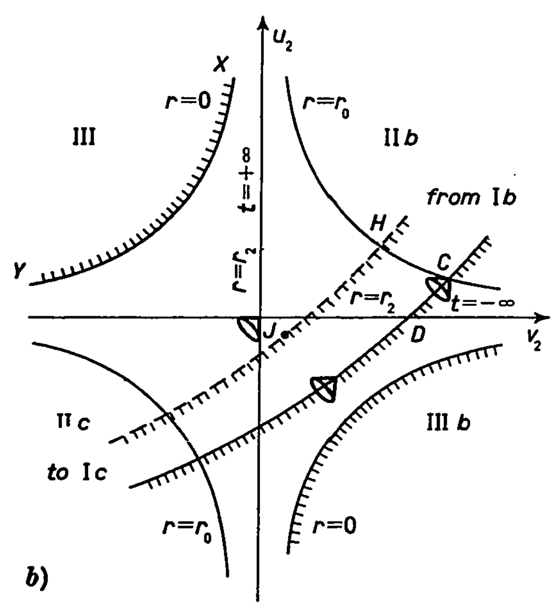


Figure 6





Figures 7

everywhere free of singularity.†

The history of a shell with $e^2 = m^2$, $a > 1$ is represented by the time-like curve KLM in fig. 5. To an external observer in the asymptotically flat space Ia the shell implodes, then appears to slow down as it approaches the observer's event horizon $r = m$, reaching it only asymptotically as $t \to \infty$. On the other hand, an observer moving with the shell finds that it passes rapidly and uneventfully through $r = m$, contracts to a nonzero minimal radius at L , then re-expands into a new space Ib , identical with Ia in its geometrical properties, but physically distinct from it. It appears that we are forced to accept this resolution of the paradox encountered in Sect. 4.

The path FGHJM (Figs. 6 and 7) of a bouncing shell with $b > 0$, $e^2 < m^2$ has a similar general character: The bounce carries the shell into a different space. A new and peculiar feature is the appearance of a time-like singular curve $r = 0$ (the curve XY) in the region outside the shell. It has to be interpreted as the history of a particle with mass m and charge $-e$.

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† This does not contradict the Penrose theorem quoted in Chapter III, since two of the hypotheses of that theorem are not satisfied here. In the first place, the manifold with $e^2 = m^2$ contains no "trapped surface" (even though it contains an event horizon), since outgoing radial null geodesics have $dr/dt = (1-m/r)^2 \geq 0$ and do not converge anywhere. Secondly, the manifold with $e^2 \leq m^2$ does not admit a Cauchy hypersurface.

7. Test Shell; Uniformly Charged Ball of Dust.

We now turn briefly to the situation where the hollow interior of the shell contains nonvanishing charge e_1 and mass m_1. We shall confine our discussion to the case where the mass $\mu = m_2 - m_1$ and charge $\varepsilon = e_2 - e_1$ of the shell itself are small compared with m_1 and e_1, and for a qualitative description it will be sufficient to consider the limit of a "test shell" ($\mu \to 0$, $\varepsilon \to 0$ with ε/μ finite). In this limit we easily obtain from (45) to (48)

$$\left(\frac{dR}{d\tau}\right)^2 = -\left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right) + A\left(1 - \frac{\varepsilon}{\mu}\frac{e}{R}\right)^2 \qquad (69)$$

where we have written $e_1 \equiv e$, $m_1 \equiv m$.

As is to be expected, (69) agrees with the equation of motion of a radially moving test particle of mass $\mu A^{-\frac{1}{2}}$ and charge ε in the Reissner-Nordström field (1). The latter may be obtained either[(60)] from the equations[(61)]

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -(\varepsilon/\mu A^{-\frac{1}{2}})F^\nu_\alpha\frac{dx^\alpha}{d\tau}$$

where the electromagnetic field tensor F^ν_α has the non-vanishing components

$$F_{rt} = -F_{tr} = e/r^2 ,$$

or from the Lagrangian

$$L(R,dR/dt) \equiv -\mu A^{-\frac{1}{2}}\, d\tau/dt + \varepsilon\phi_\mu\, dx^\mu/dt$$

$$= -\mu A^{-\frac{1}{2}}\{f - f^{-1}(dR/dt)^2\}^{\frac{1}{2}} + \varepsilon e/R$$

(where $\phi_\mu = (0,0,0,e/r)$ is the electromagnetic vector potential) by forming the Hamiltonian integral $H = \mu$.

If $|e/m|$, $|\varepsilon/\mu|$ and A are each less than unity, (69) shows that the shell's radius oscillates between a maximum larger than $r_1 = m + \sqrt{m^2-e^2}$ and a minimum smaller than $r_2 = m - \sqrt{m^2-e^2}$. The history of the shell is represented by the curve ABCDE... in figs. 6 and 7. In each oscillation the shell enters a new space. As viewed by a co-moving or an interior observer the oscillation is strictly periodic; however, the path ABCDE... in the exterior space-time is not cyclic, but subject to a systematic time-shift. If a given maximum occurs for $t = t_0$ in the space Ib (say), then succeeding maxima (in Ic, etc.) occur for $t = t_0 + C$, $t_0 + 2C$, etc. For large maximal radius, the constant C is nearly equal to the proper period of pulsation, and both agree closely with the corresponding period calculated from Newtonian theory*.

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* In the Newtonian description the pulsating shell of course always remains in the same space.

The occurrence of a bounce is independent of the relative sign of ε and e, so it clearly has little to do with a contest between gravitational attraction and electrostatic repulsion. For a neutral shell ($\varepsilon=0$) we obtain from (69) by differentiation,

$$d^2R/d\tau^2 = -M(R)/R^2 \quad , \quad M(R) \equiv m - e^2/R \; .$$

This brings out clearly the physical mechanism responsible for the bounce. Because the electrostatic field energy of the internal charge e is diffused throughout space, less and less of it contributes to the effective interior gravitational mass $M(R)$ as the shell contracts. Ultimately $M(R)$ becomes negative and there is a gravitational repulsion.

Finally, let us note another interesting special case. If we set $\varepsilon/\mu = e/m$, (69) may be regarded as the equation of motion of a particle on the outer surface $r = R(\tau)$ of a uniformly charged ball of dust with total charge e and mass m, which is collapsing homologously. For $e^2 < m^2$, $A < 1$, the motion is again oscillatory, and the history of the surface is given qualitatively by the curve ABCDE... in figs. 6 and 7. This example has been discussed by Novikov[56].

8. Concluding Remarks.

The collapse of a spherically symmetric body to an event horizon appears as an irreversible process to an external observer. As we have seen, the possibility cannot be ruled out that the body reverses its motion within the event horizon and re-expands symmetrically. It then appears necessary to believe in the existence of other asymptotically flat spaces geometrically similar to but distinct from ours, which will accomodate the re-expansion. This seems at least as fantastic as the alternative of irreversible collapse to virtually point-like dimensions.

In assessing the possible relevance of these results to realistic gravitational collapse, it is, of course, necessary to keep in mind the various idealizations and hypotheses involved (exaet spherical symmetry, asymptotic flatness, analytic continuability of the manifold, etc.; see also Chapter III), each of which could be questioned.

The Israel Theorem (Chapter III), which applies to fields due to eleetrically neutral objects, can be extended[62] to read:

> Among all static, asymptotically flat electrovac space-times with closed, simply connected equipotential surfaces g_{44} = const., the only ones which have regular event horizon $g_{44} = 0$ are the

Reissner-Nordström family of spherically symmetric solutions with $m \geq |e|$.

(The adjective "eleetrovac" means "devoid of matter but containing eleetromagnetic fields"). Thus it appears that the implication of the Israel theorem for the question of asymmetric collapse as discussed in Sect. 4 of the last Chapter will still be retained if charges are added to the collapsing objects.

As a null hypersurface (i.e., a characteristic hypersurface of the field equations), an event horizon is a possible locus of discontinuities of the field. It is not necessary, and perhaps not physically justified, to insist on analytic continuation of a manifold through an event horizon[63].

As to the idealization of asymptotic flatness for the collapse of a stellar mass in our expanding universe, it is justified in our present epoch, but clearly not in the remote past. It will not always be justified in the future if the universe happens to be oscillatory. In fact, a lattice structure for space-time of the general type we have been considering here would find a natural interpretation in terms of an oscillatory universe.

CHAPTER V

MACH'S PRINCIPLE. SPINNING SHELL AS A SOURCE OF THE KERR METRIC†.

This chapter is conveniently divided into two parts. Part A gives an introductory account of Mach's principle, with particular reference to the theory of general relativity, which aside from its inherent interest, will also prove useful for an appreciation of the subsequent work. Part B opens with a brief derivation of the Kerr metric, in the simple and natural way recently discovered by Ernst. It is then followed by the main body of the chapter, dealing with the interiors of rotating shells and their physical properties, on the presumption that the exterior geometry is described by the Kerr line-element.

PART A. MACH'S PRINCIPLE.

Mach's idea[20] of inertia as something causally determined by the distribution of matter of the universe was one of the great currents of thought that guided Einstein to the creation of his general theory of relativity.[65] It is based on the kinematic dictum, already advocated about one hundred years earlier than Mach by the

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† Based on V. de la Cruz and W. Israel[64].

Irish philosopher Bishop Berkeley[66], that the only meaningful concept of motion is that of relative motions between masses. In direct contrast, Newton had postulated the existence of an absolute space, and in support cited his famous bucket experiment[67]:

An empty bucket is hung along its axis of symmetry by a rope. After the rope is twisted a number of times, the bucket is held at rest, half-filled with water, and then released. The bucket, which is set into rotational motion by the tortion of the rope, in turn drags along the water. Initially, when both are at rest, the shape of the water is a plane. Thereafter, it changes into that of a paraboloid, reaching a maximum curvature when the water catches up with the bucket, and finally becomes plane again.

Newton's contention that, since the relative angular velocity of the water and the bucket does not determine the shape of the water surface, one must and can meaningfully ascribe to the water an absolute motion, was criticized by Mach, who pointed out that the distant "fixed stars", in particular, are all the while there. According to Mach, the centrifugal force acting on the water arises from the rotation of the water with respect to the fixed stars or, indifferently, the rotation of the fixed stars with respect to the water. <u>A priori</u>, it appears impossible to decide who is more nearly correct. But in so far as Mach's idea represents an endeavor to explain in physical terms —— i.e., without recourse to the somewhat metaphysical concept of absolute space —— why certain reference frames are in fact inertial, it is to be preferred. Moreover, whereas the observational fact that the fixed stars are indeed fixed (i.e., non-rotating) with respect to our inertial frames cannot be accounted for on the basis of Newtonian theory as otherwise than accidental, it can be taken as an empirical support for the Mach's principle.

After the advent of the general theory of relativity, works on Mach's principle have centered on the question of whether, or to what extent, Mach's principle has been incorporated into general relativity. Of particular importance in this connection is the quantitative investigation of H. Thirring(21). Using the weak field approximations to Einstein's field equations, he studied the motions of test particles located near the center of a slowly rotating mass shell —— his version of the Newtonian rotating bucket. The result shows that such a particle will be subjected to forces which are completely analogous to the Coriolis and centrifugal forces of classical mechanics, but differ in magnitudes from these by factors of the order of GM/c^2R, where M and R are the mass and radius of the shell, respectively. Because of the approximations used, this dimensionless quantity has to be much less than unity. Nevertheless the Thirring effect must be regarded as a strong manifestation of Mach's principle in general relativity, and it led to the speculation that, when strong-field calculations were done, so that the ratio GM/c^2R could be made more nearly equal to unity, the interior inertial frame would be rigidly dragged around by the rotating shell, in accord with the expectation of Mach. This is indeed the case for a slowly rotating shell, to the first order in its angular velocity, as Brill and Cohen(19) recently demonstrated. In Part B, we shall summarize and extend the results of Brill and Cohen, in conjunction with our study of rotating mass shells as possible sources of the Kerr metric.

While emphasizing the significance of the work of Thirring, Einstein himself was the first to recognize that the gravitational field equations contain solutions which, from a Machian point of view, would be unacceptable. Thus, for example, the flat space-time ($R_{\alpha\beta\gamma\delta}=0$), while consistent with the vacuum field equations, would endow a test particle with inertia which, ipso facto, is not caused by other masses. But since the gravitational equations, like those of Maxwell, are local differential equations, a large number of solutions could be ruled out by admitting only certain boundary conditions, perhaps differing according to the matter distributions, so that Mach's principle would act as a supplementary "selection rule". Einstein, however, rejected this idea as ad hoc and artificial. Instead, he argued that only spatially closed solutions —— which do not require special boundary conditions to be imposed —— are to be accepted. But whether even this criterion is sufficiently restrictive is opened to doubt(68).

Results of this nature has indicated to some(69) as calling for modifications of the general theory of relativity along lines more in accord with the spirit of Mach. Others(70) have revived the suggestion that Mach's principle should serve as a selection rule supplementary to the field equations, but in a more general sense so as to include spatially closed solutions. In this connection we may perhaps remark that, strictly speaking, Mach's principle has no relevance outside the context of cosmology. If we are prepared to believe that the existence of our actual world is not accidental but should rather follow from some further synthesis of physical theories, then the role played by Mach's principle would appear somewhat obscure. Finally, there are some who prefer to regard Mach's principle as a historical relic —— pregnant in its own time with suggestive ideas —— which is now properly superseded by general relativity and can be forgotten except for heuristic purposes.

PART B. SPINNING SHELL AS A SOURCE OF THE KERR METRIC

1. The Kerr Metric: Derivation.

As we noted in Chapter I, the Kerr solution[16] is of considerable interest because it probably represents the geometry exterior to some finite rotating object. This intepretation was first deduced from the form of the metric itself, since its original derivation was very formal, being discovered by Kerr "accidentally" in the course of his investigation of algebraically-special vacuum fields. But recently Ernst[71] was able to present an alternative derivation which employs only elementary techniques of analysis, and has the additional merit that it proceeds from the assumption of axial symmetry rather than the assumption that the metric is algebraically-special. We will give here an outline of Ernst's derivation, which will also serve to introduce his elegant and promising $\mathcal{E}$-function method.

The line-element of a stationary, axially-symmetric vacuum field can generally be cast into the "canonical form" of Lewis[72],

$$ds^2 = F^{-1}\,[e^{2\nu}(d\rho^2+dz^2) + \rho^2 d\phi^2] - F(dt-\psi d\phi)^2 \,, \quad (1)$$

where the coefficients are independent of t (stationary) and of ϕ (axial symmetry). Some of Einstein's vacuum field equations $G_{\alpha\beta} = 0$ had been employed in the reduction of the metric to the form (1), and the rest are coupled equations for the three functions ν, F, and ψ. It turns out that, once F and ψ are known, ν can be obtained in a relatively straightforward manner. The equations governing F and ψ are

$$F\nabla^2 F - \vec{\nabla}F \cdot \vec{\nabla}F + \rho^{-2}F^4\vec{\nabla}\psi \cdot \vec{\nabla}\psi = 0 , \qquad (2)$$

$$\vec{\nabla} \cdot (\rho^{-2}F^2\vec{\nabla}\psi) = 0 . \qquad (3)$$

Here, the <u>three-dimensional</u> gradient operator is to be understood, treating (ρ,z,ϕ) formally as the usual cylindrical coordinates in ordinary Euclidean 3-space. For ν, we have (subscripts indicating partial differentiations)

$$\nu_\rho = \rho(\lambda_\rho{}^2-\lambda_z{}^2) - \tfrac{1}{4}\rho^{-1}e^{4\lambda}(\psi_\rho{}^2-\psi_z{}^2) , \qquad (4)$$

$$\nu_z = 2\rho\lambda_\rho\lambda_z - \tfrac{1}{2}\rho^{-1}e^{4\lambda}\psi_\rho\psi_z , \qquad (5)$$

(where $e^{2\nu} \equiv F$) of which (2), (3) are the integrability conditions.

Let $\hat{\phi}$ denote the unit normal vector in the azimuthal direction and Q be any reasonable function. Then by virtue of the identity

$$\vec{\nabla} \cdot (\rho^{-1}\, \hat{\phi} \times \vec{\nabla}\, Q) = 0 , \qquad (6)$$

eq. (3) will be automatically satisfied if we set

$$\rho^{-1}F^2\vec{\nabla}\psi = \hat{\phi} \times \vec{\nabla}Q \ . \tag{7}$$

Provided ψ is independent of ϕ (which we now assume), (7) is equivalent to

$$F^{-2}\vec{\nabla}Q = -\rho^{-1}\,\hat{\phi} \times \vec{\nabla}\psi \ ,$$

and hence the identity (6) implies the field equation

$$\vec{\nabla}\cdot(F^{-2}\vec{\nabla}Q) = 0 \tag{8}$$

for the new potential Q. When (2) is expressed in term of Q one then finds that the complex function ("$\mathcal{E}$-function")

$$\mathcal{E} \equiv F + i\,Q \tag{9}$$

satisfies the simple homogeneous quadratic differential equation

$$(\mathrm{Re}\,\mathcal{E})\nabla^2\mathcal{E} = \vec{\nabla}\mathcal{E}\cdot\vec{\nabla}\mathcal{E} \,. \tag{10}$$

A modification of the $\mathcal{E}$-equation (10), convenient in certain cases, is obtained by the substitution

$$\mathcal{E} = (\xi-1)/(\xi+1) \ , \tag{11}$$

whereby (10) becomes

$$(\xi\xi^*-1)\nabla^2\xi = 2\xi^*\vec{\nabla}\xi\cdot\vec{\nabla}\xi \ . \tag{12}$$

We now particularize our discussion to the derivation of the Kerr metric. For this purpose, it is advantageous to introduce "prolate spheroidal" coordinates (x,y) , where

$$\rho = (x^2-1)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} , \qquad (13)$$

$$z = xy .$$

In terms of x, y, we have the expressions

$$\nabla^2 A \equiv \frac{1}{x^2-y^2} \left[\frac{\partial}{\partial x} (x^2-1) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} (1-y^2) \frac{\partial}{\partial y} \right] A ,$$

$$\vec{\nabla} A \cdot \vec{\nabla} B \equiv \frac{1}{x^2-y^2} \left[(x^2-1) \frac{\partial A}{\partial x} \frac{\partial B}{\partial x} + (1-y^2) \frac{\partial A}{\partial y} \frac{\partial B}{\partial y} \right] , \qquad (14)$$

for arbitrary functions A, B independent of the azimuth.

With the help of (14), it is easy to verify that $\xi = x$ is a particular solution* of (12). But since the operations (14) are symmetric in x and y, $\xi = y$ is also a solution. If we now look for a linear combination of $\xi = x$ and $\xi = y$ which will again satisfy (12), one arrives at

$$\xi = x \cos \alpha + iy \sin \alpha , \qquad (15)$$

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* This solution gives rise to the Schwarzschild solution, e.q.(II.21). ($F = (x-1)/(x+1)$, $Q = 0$, $e^{2\nu} = (x^2-1)/(x^2-y^2)$; $r = x+1$, $\cos \theta = y$; lengths measured in units of m) .

where α is a real parameter. Solution (15) corresponds to the Kerr solution. To obtain the metric in its familiar form(73), introduce $a = \tan\alpha$ and $m = \sec\alpha$, noting that lengths are measured in units of $(m^2-a^2)^{\frac{1}{2}}$ [since $\sec^2\alpha - \tan^2\alpha \equiv 1$], and identify the radial and polar coordinates by

$$r = x(m^2-a^2)^{\frac{1}{2}} + m \quad , \quad \cos\theta = y \; . \tag{16}$$

When the entire metric is constructed one obtains

$$\begin{aligned} ds^2 = {} & (r^2+a^2\cos^2\theta)[dr^2/(r^2-2mr+a^2) + d\theta^2] + \\ & + [r^2+a^2+2mra^2\sin^2\theta/(r^2+a^2\cos^2\theta)]\sin^2\theta \; d\phi^2 + \\ & + [4amr \sin^2\theta/(r^2+a^2\cos^2\theta)]d\phi dt - \\ & - [1-2mr/(r^2+a^2\cos^2\theta)]dt^2 \; , \end{aligned} \tag{17}$$

which is the Kerr line-element.

2. Spinning Shell as a Source of the Kerr Metric: Introduction.

The Kerr metric (17) is, as we have just seen, a 2-parameter particular solution of Einstein's vacuum equations which is axially symmetric and stationary (but not static: $g_{\phi t} \neq 0$ unless $a = 0$, in which case it reduces to the Schwarzschild line-element). One can also easily verify that it is asymptotically Minkowskian.

In a way reminiscent of the cases with the Schwarzschild and the Reissner-Nordström solutions, the Kerr line-element in the form given is regular only for $r > m + (m^2-a^2)^{\frac{1}{2}}$ when $|a| \leq m$. Its maximal analytic completion, found by Boyer and Lindquist(73), turns out to be vastly more complicated than the other two cases. However, for our purpose here, only the regular part of (17) will be needed.

The Kerr metric is generally believed to represent the exterior gravitational field of an isolated, uniformly spinning, spherical (or nearly spherical) body of mass m and angular velocity proportional to $(-a)$. Until quite recently, the sole basis for this belief was the agreement between the asymptotic form of (17) and the weak gravitational field of such a body, computed from the linearized Einstein's equations(74):

$$ds^2 \approx (1+2m/r)dr^2 + r^2 d\Omega^2 - (4J/r)\sin^2\theta \, d\phi dt -$$

$$- (1-2m/r)dt^2 ,$$

where J is the angular momentum. Despite several attempts and partial results(75), no one has yet been able to match the Kerr metric to an exact interior solution.

However, a significant advance towards this goal was achieved recently by Brill and Cohen(19). These authors derived an approximate expression for the field of a slowly spinning thin spherical shell by considering it as a linearized perturbation of the static spherically symmetric solution in which the angular velocity ω is treated as a small parameter, but without restrictions on the mass or radius of the shell. Cohen(17) was then able to show that the exterior field agrees with the Kerr metric to first order in a. Thus, if a possible source of Kerr's metric is taken to be a thin shell held together by a surface pressure and hoop stresses, then it follows from the results of Brill and Cohen that (to first order in a) (i) the shell is spherical and of uniform density, (ii) it spins rigidly (independent of latitude), and (iii) the interior space-time is flat. The last result means that inertial frames can be defined globally in the interior. In the limit when the radius of the shell approaches its Schwarzschild radius, Brill and Cohen found that the internal inertial frames are dragged around rigidly by the shell. This conclusion appears to support the Machian philosophy underlying the investigations of Thirring, as discussed in Part A.

It is of considerable interest to find out whether these elegant results continue to be valid beyond the first-order approximation of Brill and Cohen. In the present paper we shall describe a systematic iterative procedure for developing the interior field of a spinning shell as a power series in ω on the assumption that the exterior field is the Kerr metric. Our method —— described in more detail in Sect. 3 —— is to connect the Kerr metric continuously across a nearly spherical boundary Σ (whose shape can be chosen arbitrarily to within quantities of order a^2) to an interior axisymmetric vacuum solution which is developed by successive approximations in a power series in a^2. The physical characteristics of the shell are then inferred _a posteriori_ from the discontinuity in the normal derivative of the metric tensor across Σ. In lowest approximation (Sect. 4) we merely recover the results of Brill and Cohen in a new and very simple way. The next approximation (Sect. 5) gives results correct to order a^3 and again requires only simple calculations, because departures from flatness in the interior are linear to this order. Details of calculations can be found in the Appendix.

We shall find that the elegant properties (i), (ii) and (iii) all break down in the next approximation. But in the (strong field) limit of compression to the Schwarzschild radius, a "Machian" feature survives. The rotation tends to a rigid motion in this limit, and —— provided the shape of Σ is suitably chosen —— the interior metric tends to flatness, with the internal inertial frames rigidly attached to the spinning shell.

3. Outline of Method.

We recall (see Chap. IV) that thin shells in general relativity may be characterized in a geometrical way as follows.[57] Consider two distinct space-time manifolds V^- and V^+ which are partially bounded by time-like, cylindrical hypersurfaces Σ^- and Σ^+ respectively. We could describe V^- and V^+ as "compatible" if Σ^- and Σ^+ can be put into isometric correspondence. In that case, by making the identification $\Sigma^- \equiv \Sigma^+$ $(=\Sigma)$, we can consider V^- and V^+ together as comprising a single space-time which contains a (generally singular) hypersurface Σ. This hypersurface has to be interpreted as the history of a thin shell or surface layer if the extrinsic curvatures, K^-_{ab} and K^+_{ab}, of Σ^- in V^- and Σ^+ in V^+ are unequal*. The surface energy tensor S_{ab} of the layer is given by the Lanczos jump conditions[76], which can be written in the form[57]

$$- 8\pi S_{ab} = \gamma_{ab} - g_{ab}\,\gamma \tag{18}$$

where $\gamma_{ab} \equiv K^+_{ab} - K^-_{ab}$, $\gamma \equiv g^{ab}\,\gamma_{ab}$.

We shall take V^+ to be Kerr's vacuum manifold with metric (17). For the hypersurface Σ^+ we could choose quite generally the equation $r = (\text{arb.const.}) + a^2 x$ (arb. function of θ with equatorial symmetry). For our

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* Latin indices range from 2 to 4 and distinguish 3-tensors defined on Σ. Greek indices range from 1 to 4.

present purposes (explicit calculations not carried beyond a^3) it will be sufficient and convenient to assume more specifically

$$\Sigma : \quad r = R\,(1+\varepsilon k\, V^2 \cos^2\theta) \qquad (19)$$

where R and k are arbitrary constants, and

$$V^2 \equiv 1 - 2m/R + a^2/R^2 \;, \qquad \varepsilon \equiv a^2/R^2 \;. \qquad (20)$$

If $V = 0$, it can be shown(73) that Σ^+ is a regular null hypersurface of V^+ (real for $|a| \leq m$), the most natural analogue† of the critical hypercylinder $r = 2m$ in Schwarzschild's space-time, to which it in fact reduces when $a = 0$.

For the interior space V^-, we write its metric in the canonical form for axi-symmetric stationary vacuum fields, eq.(1):

$$(ds^2)_- = e^{2(\nu-\lambda)}(d\rho^2+dz^2) + \rho^2 e^{-2\lambda} d\Phi^2 - $$

$$- e^{2\lambda}(dT-\psi d\Phi)^2 \;, \qquad (21)$$

in which the coefficients are functions of ρ and z. The vacuum field equations for this line-element, essentially (2)-(5), are (subscripts indicate partial differentiation)

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† But a generalized "Schwarzschild surface" can also be defined in other ways. Compare C. V. Vishveshwara, ref. 50.

$$\nu_\rho = \rho\ (\lambda_\rho^2-\lambda_z^2) - \tfrac{1}{4}\ \rho^{-1}e^{4\lambda}(\psi_\rho{}^2-\psi_z{}^2)\ , \tag{22a}$$

$$\nu_z = 2\rho\ \lambda_\rho\lambda_z - \tfrac{1}{2}\ \rho^{-1}e^{4\lambda}\psi_\rho\psi_z\ , \tag{22b}$$

$$\lambda_{\rho\rho} + \rho^{-1}\lambda_\rho + \lambda_{zz} = -\tfrac{1}{2}\ \rho^{-2}e^{4\lambda}(\psi_\rho{}^2+\psi_z^2)\ , \tag{23a}$$

$$\psi_{\rho\rho} - \rho^{-1}\psi_\rho + \psi_{zz} = -4(\lambda_\rho\psi_\rho+\lambda_z\psi_z)\ . \tag{23b}$$

Equations (23a,b) are the integrability conditions of (22a,b).

Adopting $\xi^i = (\theta,\phi,t)$ as intrinsic co-ordinates of Σ, we postulate that the parametric equations $x_-^\alpha = x_-^\alpha\ (\theta,\phi,t)$ of Σ^- have the form

$$\Sigma\ :\qquad \rho = f_1(\theta)\ ,\qquad z = f_2(\theta)\ , \tag{24}$$

$$\Phi = \phi + a\Omega t\ ,\qquad T = At\ , \tag{25}$$

where A and Ω are constants.

Since we expect the complete solution to mirror the symmetries of the exterior Kerr metric —— i.e. invariance under reflection in the equatorial plane ($\theta \to \pi - \theta$) and under reversal of the sense of rotation ($\phi\to-\phi$, $a\to-a$)—— it follows that all quantities entering (21), (24), (25) have equatorial symmetry (except, obviously, for $f_2(\theta)$) and are even functions of a (except for ψ, which is odd).

We have now to impose the compatibility requirement that the intrinsic metric of Σ^-, computed from

(21), (24) and (25) should agree with that of Σ^+ , computed from (17) and (19). Equating the coefficients of $d\theta^2, d\theta^2, dt^2$ and $d\theta dt$ furnishes four conditions for four functions of θ , namely $f_1(\theta)$, $f_2(\theta)$ and the values of λ and ψ on Σ , and also fixes the constants A and Ω . Equations (23a,b) then in principle determine λ and ψ throughout V^- as solutions of an elliptic boundary-value problem. The function ν is obtainable by a quadrature from (22a,b) together with the requirement of "elementary flatness"(77) ($\nu=0$ on the axis).

In practice, the solution of (23a,b) of course presents difficulties. It can be reduced to a linear boundary-value problem of standard type if we proceed by successive approximations, treating the nonlinear right sides of (22) and (23) as perturbations whose values are assumed known from the previous approximation. Since all quantities in (22) and (23) are small for small a ($a=0$ corresponds to a static spherical shell with a flat interior), our procedure may be regarded as an expansion in powers of $\varepsilon = a^2/R^2$.

As soon as an expression for the interior metric is known, it is straightforward to compute the extrinsic curvatures from the defining relations (see eq.(11) of Chap. IV)

$$K^{\pm}_{ab} = - n_\alpha \{ \partial^2 x^\alpha / \partial \xi^a \partial \xi^b + \Gamma^\alpha_{\lambda\mu} (\partial x^\lambda / \partial \xi^a)(\partial x^\mu / \partial \xi^b) \} \Big|^{\pm} \quad (26)$$

in which n_α is the outward unit normal to Σ . The surface energy tensor of the shell is then read off from (18). The velocity u^a and the proper surface density $\sigma(\theta)$ are defined by the eigenvalue equation†

$$S^a_b\, u^b = -\sigma u^a \,, \qquad u_a u^a = -1 \,. \tag{27}$$

It can be shown easily by symmetry argument that $u^\theta = 0$, and we have $u^\phi = \omega u^t$, where $\omega(\theta) \equiv d\phi/dt$ is the angular velocity of a shell zone as measured by a stationary observer at infinity. According to (18) and (27) , ω is determinable from

$$\omega = \gamma^\phi_t \,/\, (\omega\gamma^t_\phi + \gamma^t_t - \gamma^\phi_\phi) \,. \tag{28}$$

4. First Approximation.

In lowest approximation, we neglect a^2 . To this order, Kerr's metric (17) induces the intrinsic metric

$$(ds^2)_\Sigma = R^2(d\theta^2 + \sin^2\theta d\phi^2) + (4ma/R)\sin^2\theta d\phi dt - $$
$$- (1-2m/R)dt^2 \tag{29}$$

on the shell $r = R + 0(\varepsilon)$. One verifies immediately that this agrees with the metric induced on the hypersurface

† Cf. eq. (18) of Chap. II.

$$\Sigma : \quad \rho = R \sin\theta + O(\varepsilon) , \quad z = R\cos\theta + O(\varepsilon) \tag{30}$$

by a <u>flat</u> interior line-element $d\rho^2 + dz^2 + \rho^2 d\Phi^2 - dT^2$ provided we choose the values

$$\Omega = 2m/R^3 + O(\varepsilon) , \quad A = (1-2m/R)^{\frac{1}{2}} + O(\varepsilon) \tag{31}$$

for the constants in (25). Thus, in lowest order, the boundary conditions on Σ are satisfied by the trivial null solution of (22) and (23); we have

$$\lambda,\nu = O(\varepsilon) , \quad \psi = aO(\varepsilon) . \tag{32}$$

Computation of the extrinsic curvatures and substitution in (18) yields the following nonvanishing components of the surface energy tensor (neglecting ε):*

$$S^{\theta}_{\theta} = S^{\phi}_{\phi} = [R(1-V)-m]/8\pi R^2 V , \tag{33}$$

$$S^{t}_{t} = -(1-V)/4\pi R , \tag{34}$$

$$S^{\phi}_{t} = ma(1+2V)/8\pi R^4 V , \quad S^{t}_{\phi} = -3ma \sin^2\theta/8\pi R^2 V \tag{35}$$

The derived surface density and angular velocity*

$$\sigma = (1-V)/4\pi R , \tag{36}$$

$$\omega = -\frac{2am}{R^3}\,\frac{(1+2V)}{(1-V)(1+3V)} , \tag{37}$$

are independent of θ in lowest approximation. According

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* See Sect. 4 of Appendix for details.

to (33), the shell is held in equilibrium by a uniform surface pressure. (Effects of the hoop stresses and departures from sphericity depend on the square of ω and do not enter until the next approximation.) Equations (36) and (37) agree with results obtained, using a different approach, by Brill and Cohen.[19,17]

The inertial frames in the flat interior rotate with angular speed (from (25), (31))

$$(d\phi/dt)_{\Phi=\text{const.}} = -\,a\Omega = -\,2a\dot{m}/R^3\ , \qquad (38)$$

as measured by a stationary observer at infinity. This is numerically smaller than the angular velocity ω of the shell itself, as measured by the same observer, but becomes equal to it in the limit $R \to 2m$ (i.e. $V \to 0$).

From the point of view of an inertial observer inside the shell the inertial frames at infinity spin at the faster rate

$$(d\Phi/dT)_{\phi=\text{const.}} = a\Omega/V\ , \qquad (39)$$

which tends to infinity when $R \to 2m$. The asymmetry between (38) and (39) is, of course, due to the effects of the gravitational redshift. The angular velocity of the shell relative to this interior observer is

$$\omega' \equiv (d\Phi/dT)_{\phi-\omega t=\text{const.}} = V^{-1}(\omega+a\Omega)$$

$$= - \frac{6ma}{R^3} \frac{V}{(1-V)(1+3V)} . \tag{40}$$

From (40) we see that $\omega' \to 0$ when $R \to 2m$, which means that a rigid Machian dragging of the interior inertial frame occurs in this limit.

5. Second Approximation.

For the next approximation, which neglects terms beyond order a^3 , we see from (32) that the right sides of (22) and (23) can be replaced by zero. Thus, ν has to be a constant —— in fact zero, to ensure "elementary flatness"(77) on the axis of symmetry —— and the general solutions for λ and ψ in the interior have the form

$$\begin{aligned} \lambda &= a^2 \, [c_1+c_2(2z^2-\rho^2) + \ldots] , \\ \psi &= a^3 \, [c'_1+c'_2\rho^2+c'_3(\rho^4-4\rho^2z^2) + \ldots] . \end{aligned} \tag{41}$$

The line-element (21) is invariant in form under a constant scale transformation of ρ and z , and we can use this freedom to arrange $c_1 = 0$. Similarly, the transformation

$$\Phi \to \Phi + a^3c'_2T , \qquad \psi \to \psi - a^3c'_2\rho^2$$

leaves (21) invariant to order a^3 , and effectively makes $c'_2 = 0$.

The remaining constants in (41) and the functions of order ε involved in (30) and (31) have now to be determined from the isometry of Σ^- and Σ^+ to order† a^3. Equating coefficients of dt^2 in the two line-elements and employing (30) and (31) where this entails only errors of order ε^2, we find

$$\lambda = \frac{1}{3}(ma^2/R^5)(1+k)(2z^2-\rho^2) , \qquad (42)$$

$$A^2 = V^2 [1-\varepsilon(1+\mu) + \frac{1}{3}\varepsilon\mu(1+k)] , \qquad (43)$$

where $\mu \equiv 2m/R$. From the coefficients of $d\phi^2$ and $d\theta^2$ we derive in turn

$$\Sigma : \quad \rho = R \sin\theta[1+\tfrac{1}{2}\varepsilon + \frac{1}{6}\varepsilon\mu(2-k)+\varepsilon k(1-\tfrac{1}{2}\mu)\cos^2\theta] , \qquad (44)$$

$$z = R\cos\theta[1-\frac{1}{6}\varepsilon\mu(1+k)+\varepsilon k(1-\tfrac{1}{2}\mu)\cos^2\theta] . \qquad (45)$$

Finally, equating coefficients of $d\phi dt$ yields

$$\psi = \frac{2}{5}(ma^3/R^7)(1+3k)V(\rho^4-4\rho^2z^2) , \qquad (46)$$

$$\Omega = (2m/R^3)[1-\frac{3}{5}\varepsilon(2+k)+\frac{1}{5}\varepsilon\mu(3k-4)] . \qquad (47)$$

The interior solution is thus determined in terms of four disposable parameters m, a, R and k. The surface energy tensor and physical properties of the shell can now be read off from the extrinsic curvatures $K^{\pm}_{ab}$ obtained from (26) and listed in the Appendix. Since the general

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† See Sect. 1 of Appendix for details.

formulae are somewhat unwieldy , we shall here quote explicit results only for the two extreme limiting cases.

In the weak-field limit $m/R \to 0$ we find

$$4\pi R^2\sigma/m \to 1 + (1/12)\varepsilon(8k-13) - \varepsilon(4k+13/4)\cos^2\theta \ , \quad (48)$$

$$\omega \to -(3a/2R^2)[1 + \frac{1}{60}\varepsilon(3-16k) + \varepsilon(5/12-2k)\cos^2\theta] \ , \quad (49)$$

showing that the distribution of proper mass and the angular velocity are, in general, latitude-dependent.

At the opposite extreme, letting R approach the "gravitational radius" $m + (m^2-a^2)^{\frac{1}{2}}$ (i.e. letting $V \to 0$) we find

$$4\pi R\sigma = 1 + \frac{1}{6}\varepsilon(2-k) + \tfrac{1}{2}\varepsilon(k-5)\cos^2\theta + O(V^2) \ , \quad (50)$$

$$\omega = -a/(R^2+a^2) + O(V^2) \ . \quad (51)$$

The rotation is rigid in this limit. (The result (51) actually holds exactly to all orders in a. This is easily proved by noting that the time-like vector u^a tangent to Σ must become parallel to the generators of the null hypersurface $V = 0$ of the Kerr manifold, since every other direction in this hypersurface is space-like.) We observe also that the surface density σ remains finite and, in general, is latitude-dependent. However, in this limit, it is not the density that plays the dominant role, but the surface pressure and the hoop stresses, which become infinite (see (33) and the formulae of the Appendix), and which contribute all of the gravitational mass m. This is immediately evident from the formula*

$$m = - \int_{\Sigma_2} (-g^{44})^{-\frac{1}{2}} (S^4_4 - S^2_2 - S^3_3) d\Sigma_2 , \qquad (52)$$

which holds generally for the stationary field of any shell in a condition of steady motion in vacuo.

6. Discussion.

We have not entered into the question of the convergence of our approximation procedure. But if convergence is taken for granted, we may infer the existence of an infinite variety of exact vacuum solutions, which represent the interior fields of spinning shells of differing degrees of ellipticity, and which all join continuously to the Kerr metric. At the same time our results indicate that, if any such exact solution is found in closed form, it is unlikely to be particularly simple or attractive,

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* Intrinsic coordinates ξ^2, ξ^3, $\xi^4 = t$ are presupposed in which the time-like killing vector of the field (which is also an intrinsic killing vector of Σ) has components $\eta^a = (0, 0, 1)$. The integration is over a two-dimensional slice $t =$ const of Σ ($d\Sigma_2$ is an invariant element of area). For a general proof of (52) and of a corresponding formula for the angular momentum, see V. De la Cruz and W. Israel, ref. 64, Appendix 2.

since for no choice of the ellipticity constant k is the shell both rigid and uniform (except in the limit $V \to 0$). In this sense, a spinning shell does not appear to be a "natural" source for the Kerr metric.

We note from (42) that the special choice $k = -1$ makes $\lambda = 0$. It follows that the interior space-time tends to flatness when $V \to 0$, since ψ goes to zero with V according to (46). From (47) and (51) we find again that $\omega' \to 0$ (cf.(40)). Thus the "Machian" effect obtained in Sect.4 persists at least to third order in the angular velocity. But it should be remarked that the credibility of our limiting result as an armchair version of Newton's bucket experiment is somewhat impaired by the circumstance already noted that, as $V \to 0$, the surface pressure becomes infinite and enormously exceeds the density. Of course, a model for Newton's bucket less extremely idealized than an infinitely thin shell should at least ameliorate this difficulty.

We have throughout confined attention to shells whose exterior field is of the Kerr type. The Kerr solutions actually form a subset of measure zero in the class of all stationary, asymptotically flat vacuum fields. But they are probably the only fields in this class which possess non-singular stationary null hypersurfaces[†] $(g^{44})^{-1} = 0$. If, therefore, one is primarily interested in the limit of compression to the gravitational radius, then it is worth noting that our discussion has probably been exhaustive in this respect.

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† From the formulas for small stationary vacuum perturbations of the Schwarzschild manifold given by Ernst (ref. 71), it can be seen that the only such perturbations which preserve asymptotic flatness and a regular event horizon are members of the Kerr family. See also W. Israel, ref. 62.

APPENDIX

This appendix pertains to Part B of Chapter V. The following quantities and expressions (correct to $O(a^3)$ unless otherwise stated) are here derived in some detail: interior metric of the spinning shell (Sect. 1); extrinsic curvatures $K^{\pm}_{ab}$ of the shell with respect to V^+ (Sect. 2) and V^- (Sect. 3); shell stress-energy tensor S_{ab}, proper energy density σ, and angular velocity ω, correct to order a (Sect. 4); and shell energy density and angular velocity to order a^3 (Sect. 5).

We introduce the following notations:

$$\alpha \equiv a/R ,$$

$$O_n \equiv O(a^n) .$$

1. Interior Line-element of the Spinning Shell, Correct to O_3 .

The interior line-element is given by (V.21) with* $\nu = 0$ and λ , ψ of the forms (V.41). Thus,

$$(ds^2)_- = (1-2\lambda)(d\rho^2+dz^2+\rho^2 d\Phi^2) - $$

$$- (1+2\lambda)dT^2 + 2\psi \, d\Phi dT , \qquad (1A)$$

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* Unless otherwise stated, all equations in the Appendix are correct to O_3 .

$$\lambda = a^2[c_1+c_2(2z^2-\rho^2)+\cdots] \ , \qquad (1B)$$

$$\psi = a^3[c_1'+c_2'\rho^2+c_3'(\rho^4-4\rho^2z^2)+\cdots] \ . \qquad (1C)$$

From (V.24,25) and the first order results (V.30,31), we see that the parametric equations of Σ in V^- have the forms

$$\begin{aligned}\Sigma : \quad & \rho = R \sin\theta \, [1+a^2f(\theta)] \ , \\ & z = R \cos\theta \, [1+a^2g(\theta)] \ , \\ & \Phi = \phi + a\Omega t \ , \\ & T = At \ ,\end{aligned} \qquad (2)$$

with

$$\Omega = 2m/R^3 + a^2\ell/R^2 \ , \qquad (3A)$$

$$A^2 = V^2(1+a^2h/R^2) \ , \qquad (3B)$$

for some functions $f(\theta)$, $g(\theta)$ and constants ℓ , h.

Using (2), we find that the interior metric (1A) induces the following intrinsic metric tensor on Σ :

$$\begin{aligned} {}^3g^-_{\theta\theta} = R^2[1-2\lambda+2a^2(f\cos^2\theta + f'\sin\theta\cos\theta + \\ + g\sin^2\theta - g'\sin\theta\cos\theta)] \ ,\end{aligned} \qquad (4A)$$

$${}^3g^-_{\phi\phi} = R^2\sin^2\theta(1+2a^2f) - 2\lambda R^2\sin^2\theta \ , \qquad (4B)$$

$$^{3}g^{-}_{\phi t} = a\Omega \; ^{3}g^{-}_{\phi\phi} + A\psi \; , \qquad (4C)$$

$$^{3}g^{-}_{tt} = - \; [A^{2}(1+2\lambda) - a^{2}R^{2} \sin^{2} \theta \; (\mu/R^{2})^{2}] \; , \qquad (4D)$$

(other components vanish) where prime stands for $d/d\theta$ and λ , ψ are evaluated on the shell.

From (V.19) ,

$$dr^{2} = 0_{4} d\theta^{2} \quad \text{on} \quad \Sigma \; .$$

The intrinsic metric induced by the exterior (Kerr) line-element (V.17) is ($\mu \equiv 2m/R$)

$$^{3}g^{+}_{\theta\theta} = R^{2}[1+\alpha^{2}\cos^{2}\theta(2kV^{2}+1)] \; , \qquad (5A)$$

$$^{3}g^{+}_{\phi\phi} = R^{2} \sin^{2} \theta \; [1+\alpha^{2}(1+2kV^{2}\cos^{2}\theta+\mu \sin^{2} \theta)] \; , \quad (5B)$$

$$^{3}g^{+}_{\phi t} = R \; \mu\alpha \sin^{2}\theta[1-\alpha^{2}\cos^{2}\theta(k \; V^{2}+1)] \; , \qquad (5C)$$

$$^{3}g^{+}_{tt} = - \; [1-\mu+\alpha^{2}\mu(1+kV^{2})\cos^{2}\theta] \; , \qquad (5D)$$

(other components zero). For later use, we list here the corresponding expressions for the non-vanishing contra-variant components:

$$^{3}g^{\theta\theta}_{+} = R^{-2}[1-\alpha^{2}(1+2kV^{2})\cos^{2}\theta] \; , \qquad (6A)$$

$$^{3}g^{\phi\phi}_{+} = (RV \sin \theta)^{-2}[1-\mu+\alpha^{2}\cos^{2}\theta(\mu-2kV^{4})] \; , \qquad (6B)$$

$$^{3}g^{\phi t}_{+} = \alpha\mu(RV^{2})^{-1}[1-\alpha^{2}\cos^{2}\theta(1+3k-2k\mu)] \; , \qquad (6C)$$

$$^{3}g^{tt}_{+} = - \; V^{-2}[1+\alpha^{2}(1+\mu \sin^{2}\theta-k\mu \cos^{2}\theta)] \; . \qquad (6D)$$

We now impose the condition $^3g^+_{ab} = {}^3g^-_{ab}$. For (a,b) = (4,4), eqs.(4D) and (5D) yield, with the help of (3B),

$$\lambda = -\tfrac{1}{2}\,\alpha^2[(1+h-k\mu)+\mu(1+k)\sin^2\theta] \quad \text{on} \quad \Sigma ,$$

or, defining a constant β by

$$-\mu\beta \equiv 1 + h - k\mu , \tag{7}$$

$$\lambda = \tfrac{1}{2}\,\alpha^2\mu[\beta-(1+k)\sin^2\theta] \quad \text{on} \quad \Sigma . \tag{8}$$

For (a,b) = (3,3), we find, when (8) is taken into account,

$$f(\theta) = \tfrac{1}{2}\,R^{-2}[1+\mu(\beta-k)+k(2-\mu)\cos^2\theta] + O_2 . \tag{9}$$

Equating the (3,4)-components yields

$$\psi = \alpha^3\mu\, VR\, \sin^2\theta[\delta-3k+(1+3k)\sin^2\theta] \quad \text{on} \quad \Sigma , \tag{10}$$

where the constant δ is defined by

$$\ell \equiv -\,\mu\, R^{-2}(2+V^2\delta) , \tag{11}$$

ℓ being the unspecified constant in (3A).

Finally, the condition $^3g^+_{\theta\theta} = {}^3g^-_{\theta\theta}$ gives rise to a differential equation for $g(\theta)$:

$$O_2 = \mu(1+k-\beta)\sin^2\theta-3k(2-\mu)\sin^2\theta\,\cos^2\theta +$$

$$+ 2R^2g\,\sin^2\theta - 2R^2g'\,\sin\theta\,\cos\theta ,$$

whose solution is easily found to be

$$2R^2 g(\theta) = k(2-\mu)\cos^2\theta - \mu(1+k-\beta) + O_2 \ , \tag{12}$$

in which the constant of integration has been set to zero.

We now determine the constant coefficients in (1B,C) for λ and ψ. From these equations and the substitutions $\rho = R\sin\theta + O_2$, $z = R\cos\theta + O_2$, we have that, on Σ,

$$\lambda = a^2[c_1 + 2R^2 c_2 - 3R^2 c_2 \sin^2\theta + \cdots] \ , \tag{13}$$

$$\psi = a^3[c_1' + R^2(c_2' - 4c_3' R^2)\sin^2\theta + 5c_3' R^4 \sin^4\theta + \cdots] \ . \tag{14}$$

Comparison of (13) with (8) yields

$$c_1 = (1/6)\mu R^{-2}[3\beta - 2(1+k)] \ ,$$
$$c_2 = (1/6)\mu R^{-4}(1+k) \ , \tag{15}$$
$$c_3 = c_4 = \cdots = 0 \ .$$

Similarly, from (14) and (10) we obtain

$$c_1' = 0 \ ,$$
$$c_2' = \mu V R^{-4}[\delta + (4-3k)/5] \ ,$$
$$c_3' = R^{-6}\mu V(1+3k)/5 \ , \tag{16}$$
$$c_4' = c_5' = \cdots = 0 \ .$$

By choosing

$$\beta = 2(1+k)/3$$
$$\delta = -(4-3k)/5 \tag{17}$$

we make both c_1 and c'_2 vanish. Correspondingly, we obtain from (7), (11),

$$-h = 1 + \mu(2-k)/3 \ ,$$
$$\ell = -\mu R^{-2}[10+(3k-4)V^2]/5 \ . \tag{18}$$

To summarize: The parametric equations of the shell in V^- in terms of intrinsic coordinates (θ,ϕ,t) are

$$\rho = R \sin\theta\{1 + \tfrac{1}{2}\alpha^2[1 + \mu(2-k)/3 + k(2-\mu)\cos^2\theta]\} \ , \tag{19}$$

$$z = R \cos\theta\{1 + \tfrac{1}{2}\alpha^2[k(2-\mu)\cos^2\theta - \mu(1+k)/3]\} \ , \tag{20}$$

$$\Phi = \phi + a\Omega t$$
$$= \phi + \alpha\mu R^{-1}\{1-\alpha^2[2+(3k-4)V^2/5]\}t \ , \tag{21}$$

$$T = At$$
$$= V\{1 - \tfrac{1}{2}\alpha^2[1+\mu(2-k)/3]\}t \ . \tag{22}$$

The interior metric is given by (1A), with

$$\lambda(\rho, z) = (1/6)\alpha^2\mu R^{-2}(1+k)(2z^2-\rho^2) \ , \tag{23}$$

$$\psi(\rho,z) = (1/5)\alpha^3\mu VR^{-3}(1+3k)\rho^2(\rho^2-4z^2) \ . \qquad (24)$$

2. <u>Exterior Curvature 3-tensor</u> K^+_{ab} <u>of</u> Σ <u>With Respect to</u> V^+ .

The coordinates in the exterior manifold V^+ are $x^\alpha = (r,\theta,\phi,t)$; the intrinsic coordinates of Σ are $\xi^a = (\theta,\phi,t)$. The equation of Σ is given by (V.19):

$$r = r(\theta) \equiv R(1+\alpha^2 kV^2\cos^2\theta) \ . \qquad (25)$$

The exterior metric is given by (V.17) or, discarding terms of O_4 ,

$$\begin{aligned}(ds^2)_+ &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= (r^2+a^2\cos^2\theta)(r^2-2mr+a^2)^{-1}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + \\ &\quad + (r^2+a^2+2ma^2\sin^2\theta/r)\sin^2\theta d\phi^2 + \\ &\quad + [4\ ma\ \sin^2\theta(1-a^2\cos^2\theta/r^2)/r]d\phi dt - \\ &\quad - (1-2m/r + 2ma^2\cos^2\theta/r^3)dt^2 \ . \qquad (26)\end{aligned}$$

The unit outward normal n_α to Σ is determined by

$$\begin{aligned}n_\alpha &= b\partial[r-r(\theta)]/\partial x^\alpha \\ &= b(1, 2\alpha^2 RkV^2\sin\theta\ \cos\theta,\ 0,\ 0) \ , \qquad (27)\end{aligned}$$

where b is a normalizing factor,

$$1 = n_\alpha n^\alpha = g^{11}(n_1)^2 + g^{22}(n_2)^2$$

$$= b^2 g^{11} \ ,$$

i.e.,

$$b = (g^{11})^{-\frac{1}{2}} \Big|_\Sigma$$

$$= V^{-1}[1+\tfrac{1}{2}\alpha^2(1-k\mu)\cos^2\theta] \ . \qquad (28)$$

From the defining equation (V.26),

$$K^+_{ab} = - n_\alpha[\partial^2 x^\alpha/\partial\xi^a\partial\xi^b + \Gamma^\alpha_{\beta\gamma}(\partial x^\beta/\partial\xi^a)(\partial x^\gamma/\partial\xi^b)] \quad (29)$$

and (25), (26), we find

$$K^+_{22} = - n_1 \ r''(\theta) - n^a[\Gamma_{11,a}(r')^2+2\Gamma_{12,a}r'+\Gamma_{22,a}] \qquad a=1,2$$

$$= - n_1 \ r'' - n^1[\tfrac{1}{2}(r')^2 g_{11,1} + r' \ g_{11,2} - \tfrac{1}{2} \ g_{22,1}] -$$

$$- n^2[-\tfrac{1}{2}(r')^2 g_{11,2} + r' g_{22,1} + \tfrac{1}{2} \ g_{22,2}] \ .$$

Now

$$n_1 = V^{-1}[1 + \tfrac{1}{2}\ \alpha^2(1-k\mu)\cos^2\theta] \ ,$$

$$n^1 = V[1 - \tfrac{1}{2}\ \alpha^2(1-k\mu)\cos^2\theta] \ ,$$

$$n^2 = 2\alpha^2 kV \sin\theta \ \cos\theta/R \ ,$$

$$r' = - 2\alpha^2 kV^2 R \sin\theta \ \cos\theta \ ,$$

$$(r')^2 = 0_4$$

$$r'' = -\ 2\alpha^2 kV^2 R(\cos^2\theta - \sin^2\theta)\ ,$$

$$g_{11,2} = -\ 2\alpha^2\ \sin\theta\ \cos\theta/V^2 \quad \text{on} \quad \Sigma\ ,$$

$$g_{22,1} = 2R(1+\alpha^2 kV^2\ \cos^2\theta) \quad \text{on} \quad \Sigma\ ,$$

and

$$g_{22,2} = -\ 2a^2\ \sin\theta\ \cos\theta \quad \text{on} \quad \Sigma\ .$$

Hence

$$K^{+}_{22} = VR\{1 - \tfrac{1}{2}\ \alpha^2[4k+(1+k\mu-10k)\cos^2\theta]\}\ . \qquad (30A)$$

Similarly,

$$K^{+}_{23} = -\ \tfrac{1}{2}\ n^a(\partial x^b/\partial\theta)(g_{ab,3}+g_{a3,b}-g_{b3,a}) \qquad a,b=1,2$$

$$= 0\ , \qquad (30B)$$

$$K^{+}_{24} = 0\ , \qquad (30C)$$

$$K^{+}_{33} = \tfrac{1}{2}\ n^1 g_{33,1} + \tfrac{1}{2}\ n^2 g_{33,2}$$

$$= VR\ \sin^2\theta\{1 - \tfrac{1}{2}\ \alpha^2[(1-6k+k\mu)\cos^2\theta+\mu\ \sin^2\theta]\}$$

$$(30D)$$

$$K^{+}_{34} = -\ \tfrac{1}{2}\ \alpha\mu\ V\ \sin^2\theta\{1 - \tfrac{1}{2}\ \alpha^2(7+12k-5k\mu)\cos^2\theta\}\ ,$$

$$(30E)$$

$$K^{+}_{44} = -\ \tfrac{1}{2}\ (\mu V/R)\{1 - \tfrac{1}{2}\ \alpha^2(7+4k-5k\mu)\cos^2\theta\}\ . \qquad (30F)$$

The trace is

$$K^+ \equiv (K^2_2)^+ + (K^3_3 + K^4_4)^+$$

$$= (V/R)\{1 - \tfrac{1}{2}\,\alpha^2[3(1-2k-k\mu)\cos^2\theta + 4k]\} +$$

$$+ (2RV)^{-1}\{1+V^2 - \tfrac{1}{2}\,\alpha^2[2+(2-\mu)\cos^2\theta +$$

$$+ k(-4+2\mu+3\mu^2)\cos^2\theta]\} \,. \qquad (31)$$

3. Extrinsic Curvature 3-tensor K^-_{ab} of Σ With Respect to V^- .

The coordinates for the interior region V^- are $x^\alpha = (\rho,z,\Phi,T)$. Representing the shell Σ by an equation of the form $\rho = \rho(z)$, we may use

$$\eta^a = (z,\Phi,T) \qquad (32)$$

as intrinsic coordinates. We first calculate K^-_{zz} , $K^-_{z\Phi}$, etc., and then make an intrinsic-coordinate transformation $(z,\Phi,T) \to (\theta,\phi,t)$ to obtain $K^-_{\theta\theta}$, etc.

The equation of Σ , $\rho = \rho(z)$, is derived from (19), (20) by eliminating θ . The result is

$$\rho^2 = R^2 + \alpha^2 R^2[1+\mu(2-k)/3] -$$

$$- [1+\alpha^2(1+\mu+\mu k-2k)]z^2 \,. \qquad (33)$$

The unit outward normal n_α to Σ is given by

$$n_\alpha = b\partial[\rho-\rho(z)]/\partial x^\alpha$$

$$= b(1,-\rho'(z),0,0) \ , \qquad (34)$$

and

$$1 = n_\alpha n^\alpha = b^2 g^{11}[1+(\rho')^2] \qquad (35)$$

where use has been made of the form of the interior metric (1A), and g^{11} is to be evaluated on Σ .

From the defining equation for K^-_{ab} [V.(26)] and the expressions for the 4-dimensional Christoffel symbols calculated from (1A), we obtain

$$K^-_{zz} = -\rho''\{g^{11}[1+(\rho')^2]\}^{-\frac{1}{2}} + \{g^{11}[1+(\rho')^2]\}^{\frac{1}{2}}(\rho'\lambda_z-\lambda_\rho) \ ,$$

$$K^-_{z\Phi} = K^-_{zT} = 0 \ ,$$

$$K^-_{\Phi\Phi} = \rho\{g_{11}[1+(\rho')^2]\}^{-\frac{1}{2}}\{1-2\lambda-\rho\lambda_\rho+\rho\rho'\lambda_z\} \ ,$$

$$K^-_{\Phi T} = \tfrac{1}{2}\ \{g_{11}[1+(\rho')^2]\}^{-\frac{1}{2}}(\psi_\rho-\rho'\psi_z) \ ,$$

$$K^-_{TT} = \{g_{11}[1+(\rho')^2]\}^{-\frac{1}{2}}\ (\rho'\lambda_z-\lambda_\rho) \ ,$$

where g^{11}, λ, $\lambda_\rho \equiv \partial\lambda/\partial\rho$, etc., are to be evaluated on Σ . With the help of (23), (24), (33), and

$$g^{11} = (g_{11})^{-1} = 1 + 2\lambda \ ,$$

it is a straightforward matter to calculate K^-_{zz}, etc., as functions of z , or of θ through (20). In this way, we find

$$K^{-}_{zz} = (R\ \sin^2\theta)^{-1}\{1 + \tfrac{1}{2}\ \alpha^2[(1+4\mu+2k+2k\mu)\sin^2\theta -$$

$$- 6k - \mu(5-4k)/3]\}\ , \tag{36}$$

$$K^{-}_{\Phi\Phi} = R\ \sin^2\theta\{1 + \tfrac{1}{2}\ \alpha^2[1+\mu(5+2k)/3 -$$

$$- (1+4\mu-6k+6k\mu)\cos^2\theta]\}\ , \tag{37}$$

$$K^{-}_{\Phi T} = 2\alpha^3\mu V(1+3k)(1/5-\cos^2\theta)\sin^2\theta\ , \tag{38}$$

$$K^{-}_{TT} = \alpha^2 R^{-1}\mu(1+k)(\sin^2\theta-2/3)\ . \tag{39}$$

We now make the intrinsic-coordinate transformation $\eta^a = (z,\Phi,T) \rightarrow \xi^a = (\theta,\phi,t)$, defined by (20, (21) and (22), and use the rule of tensor transformation

$$K^{-}_{ab} = K^{-}_{a'b'}(\partial\eta^{a'}/\partial\xi^a)(\partial\eta^{b'}/\partial\xi^b)$$

which now reads:

$$K^{-}_{\theta\theta} = (\partial z/\partial\theta)^2 K^{-}_{zz}\ , \tag{40}$$

$$K^{-}_{\theta\phi} = K^{-}_{\theta t} = 0\ , \tag{41}$$

$$K^{-}_{\phi\phi} = K^{-}_{\Phi\Phi}\ , \tag{42}$$

$$K^{-}_{\phi t} = a\Omega K^{-}_{\Phi\Phi} + AK^{-}_{\Phi T}\ , \tag{43}$$

$$K^{-}_{tt} = a^2\Omega^2 K^{-}_{\Phi\Phi} + A^2 K^{-}_{TT}\ . \tag{44}$$

Since

$$(\partial z/\partial\theta) = - R\ \sin\theta\{1 + \tfrac{1}{2}\ \alpha^2[3k(2-\mu)\cos^2\theta-\mu(1+k)/3]\}\ ,$$

$$\Omega = \mu R^{-2}\{1-\alpha^2[2+(3k-4)V^2/5]\} ,$$

$$A = V + O_2 ,$$

eqs.(40)-(44) yield, when (36)-(39) are substituted into them, the following non-vanishing components of K^-_{ab} , <u>referred to the intrinsic coordinates</u> (θ,ϕ,t):

$$K^-_{22} = R\{1 + \tfrac{1}{2}\alpha^2[(1+4\mu-10k+8k\mu)\sin^2\theta + 6k - \mu(7+16k)/3]\} , \tag{45A}$$

$$K^-_{33} = R\sin^2\theta\{1 + \tfrac{1}{2}\alpha^2[1+\mu(5+2k)/3 - (1+4\mu-6k+6k\mu)\cos^2\theta]\} , \tag{45B}$$

$$K^-_{34} = \alpha\mu\sin^2\theta\{1 - \tfrac{1}{2}\alpha^2[3(1-2k)/5+(11\mu+8k\mu)/15 + (5+6k-6k\mu)\cos^2\theta]\} , \tag{45C}$$

$$K^-_{44} = \alpha^2\mu R^{-1}[(1+k-k\mu)\sin^2\theta-2(1-\mu)(1+k)/3] . \tag{45D}$$

Using (6) for the contravariant components of the intrinsic metric g^{ab} $(\equiv {}^3g^{ab}_+)$, we obtain the trace of K^-_{ab} :

$$K^- \equiv (K^a_a)^- = \frac{2}{R}\{1 - \tfrac{1}{2}\alpha^2[-\mu(1+4k)/3 + 2k + 2(1+\mu-2k+k\mu)\cos^2\theta]\} . \tag{46}$$

4. <u>Shell Energy-Stress Tensor</u> $S^a{}_b$, <u>Energy Density</u> σ , <u>and Angular Velocity</u> ω : <u>Correct to</u> 0_1 .

From Chap. V, we have

$$8\pi S_{ij} = g_{ij}\,\gamma - \gamma_{ij} \quad , \quad \gamma_{ij} = K^{+}_{ij} - K^{-}_{ij} \ , \tag{47}$$

$$S_{ij}\,u^j = -\,\sigma\,u_i \ , \tag{48}$$

with

$$u^2 = 0 \ , \tag{49}$$

$$u^3 = \omega\,u^4 \ . \tag{50}$$

Eqs. (47), (48) imply

$$\gamma_{ij}\,u^j = (\gamma+8\pi\sigma)u_i \ ,$$

i.e.,

$$\gamma_{33}\,\omega + \gamma_{34} = (\gamma+8\pi\sigma)(g_{33}\omega+g_{34}) \ , \tag{51}$$

$$\gamma_{34}\,\omega + \gamma_{44} = (\gamma+8\pi\sigma)(g_{34}\omega+g_{44}) \ , \tag{52}$$

so that, upon dividing,

$$0 = \omega^2[g_{34}\gamma_{33} - g_{33}\gamma_{34}] + \omega[g_{44}\gamma_{33} - g_{33}\gamma_{44}] +$$

$$+ [g_{44}\gamma_{34} - g_{34}\gamma_{44}] \ ,$$

which is equivalent to

$$\omega = \gamma^3_4 \ / \ (\omega\gamma^4_3 + \gamma^4_4 - \gamma^3_3) \ . \tag{53}$$

All the above equations are exact. In the lowest approximation, where 0_2 - terms are neglected, the intrinsic metric of Σ, referred to coordinates θ, ϕ, t is given by (V.29), and γ, $\gamma^a{}_b$, calculated from $K^{\pm}_{ab}$ (neglecting 0_2 - terms) of Sects. 2 and 3, are very simple. Thus,

$$\begin{aligned} \gamma^2_2 &= \gamma^3_3 = (1/R)(V-1) , \\ \gamma^3_4 &= - \tfrac{1}{2} (\alpha\mu/R^2V)[V(2+V)+\mu] , \\ \gamma^4_3 &= 3\,\alpha\mu \sin^2\theta/2V , \\ \gamma^4_4 &= \mu/2RV , \end{aligned} \tag{54}$$

(other components zero) and

$$\gamma = (1/2RV)(1+3V^2-4V) . \tag{55}$$

From (47) we now get the following non-vanishing components of the shell energy-stress tensor (to 0_1) :

$$\begin{aligned} 8\pi\, S^2_2 &= 8\pi\, S^3_3 \\ &= (1/2RV)(1-V)^2 , \end{aligned} \tag{56A}$$

$$8\pi\, S^3_4 = \tfrac{1}{2} (\alpha\mu/R^2V)[V(2+V)+\mu] , \tag{56B}$$

$$8\pi\, S^4_3 = - 3\,\alpha\mu \sin^2\theta/2V , \tag{56C}$$

$$8\pi\, S^4_4 = - 2(1-V)/R . \tag{56D}$$

From (53) and (54), we obtain for the angular velocity of the shell (to 0_1) ,

$$\omega = -\alpha\mu(1+2V)/R(1+2V-3V^2) \ . \qquad (57)$$

The proper energy density of the shell is most easily calculated from (52), with the result (to 0_1)

$$\sigma = (1-V)/4\pi R \ . \qquad (58)$$

5. Shell Energy Density σ and Angular Velocity ω, Correct to 0_3 .

From (48) and (47), one gets

$$8\pi\sigma = 8\pi \ S_{ab}u^a u^b$$
$$= -\gamma - \gamma_{ab}u^a u^b$$
$$= -(K^+ - K^-) - (K^+_{ab}u^a u^b) + K^-_{ab}u^a u^b \ . \qquad (59)$$

We have from (49), (50), and the condition $u_a u^a = -1$,

$$K_{ab}u^a u^b\Big|^{\pm} = (\omega^2 K_{33} + 2\omega K_{34} + K_{44})\Big|^{\pm} (u^4)^2 \ , \qquad (60)$$

$$(u^4)^2 = -(g_{44} + 2\omega g_{34} + \omega^2 g_{33})^{-1} \ . \qquad (61)$$

Now we note that, to evaluate (60) to 0_3 , one needs ω only to 0_1 , which is already given in (57). With the help of (5) for the intrinsic metric of Σ referred to θ,ϕ,t, and of the results of Sects. 2,3 for $K^{\pm}_{ab}$, we find

$$(u^4)^2 = V^{-2}\{1+\alpha^2[1+\mu\ \sin^2\theta-\mu k\ \cos^2\theta+9\mu^2\sin^2\theta\ V^2/D^2]\}\ , \tag{62}$$

$$\omega^2K^{+}_{33} + 2\omega\ K^{+}_{34} + K^{+}_{44}$$

$$= \tfrac{1}{2}\ (\mu V/R)\{-1+\alpha^2[2\mu\ \sin^2\theta(2+9V^2/D+9V^4/D^2) + $$

$$+ \tfrac{1}{2}\ (7+4k-5k\mu)\cos^2\theta]\}\ , \tag{63}$$

$$\omega^2K^{-}_{33} + 2\omega\ K^{-}_{34} + K^{-}_{44}$$

$$= \alpha^2\mu V^2R^{-1}\{(1+9\mu V^2D^{-2})\sin^2\theta + [k\ \sin^2\theta-2(1+k)/3]\}\ , \tag{64}$$

where

$$D \equiv 1 + 2V - 3V^2\ . \tag{65}$$

Using these and (60), we obtain

$$K^{+}_{ab}u^a u^b = -\ (\mu/2VR)\{1 - \tfrac{1}{2}\ \alpha^2(5+4k-3k\mu)\cos^2\theta -$$

$$- 2\alpha^2(1+5V+6V^2+6V^3)/D(1+3V)\}\ , \tag{66}$$

$$K^{-}_{ab}u^a u^b = \alpha^2\mu R^{-1}\{9\mu V^2\sin^2\theta/D^2+(1+k)(1-3\ \cos^2\theta)/3\} \tag{67}$$

Substituting (66), (67), (31), (46) into (59), one then finds for the energy density of the shell (to 0_3)

$$8\pi\sigma = 2(1-V)/R - \tfrac{1}{2}\alpha^2 VR^{-1}[(3\mu-4+6k\mu+8k)\cos^2\theta + 9\mu^2(1+2V-3V^2)^{-1}\sin^2\theta - (2+3\mu+4k)] - \alpha^2 R^{-1}[(2+3\mu-4k+3k\mu)\cos^2\theta+2k - \mu(2+5k)/3]. \tag{68}$$

We now use (51) to calculate the shell angular velocity ω to O_3, writing

$$\omega = \alpha\omega_1 + \omega_3 \tag{69A}$$

where from (57)

$$\omega_1 = -\mu(1+2V)R^{-1}(1+2V-3V^2)^{-1}, \tag{69B}$$

and ω_3 is a higher order term to be determined. Eqs. (51) and (69A) yield

$$\omega_3[\gamma_{33}-(8\pi\sigma+\gamma)g_{33}] = (8\pi\sigma+\gamma)(\alpha\omega_1 g_{33}+ g_{34}) - (\alpha\omega_1\gamma_{33} + \gamma_{34}), \tag{70}$$

in which g_{ab} are given by (5), σ by (68), and γ_{ab} and γ can be obtained from the results of Sects. 2, 3 for the extrinsic curvatures. After some calculations, one arrives at

$$2R(1+2V-3V^2)^2\omega_3$$

$$= \alpha^3\mu(1+2V-3V^2)\{4\mu + 3V^2(3+k\mu)\cos^2\theta +$$

$$+ 4V[4/5+6\mu/5-3kV^2/5+2V^2(1+3k)\cos^2\theta]\} +$$

$$+ 3\alpha^3\mu V^2\{2+6\mu V^2-18\mu^2V^2(1+2V-3V^2)^{-1} +$$

$$+ (2-11\mu-12k+14k\mu+6\mu^2-3k\mu^2)\cos^2\theta +$$

$$+ 18\mu^2V^2(1+2V-3V^2)^{-1}\cos^2\theta +$$

$$+ V[2+2\mu(7+4k)/3-2(1+6\mu-6k+8k\mu)\cos^2\theta]\} \qquad (69C)$$

Eqs. (69A,B,C) determine ω to 0_3 .

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