University of Alberta

A REPRESENTATION THEOREM FOR COMPLETELY CONTRACTIVE DUAL BANACH ALGEBRAS AND CONNES-AMENABILITY

by

©

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics

Department of Mathematical and Statistical Sciences

Edmonton, Alberta
Fall, 2007

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ISBN: 978-0-494-33083-8
Our file Notre référence
ISBN: 978-0-494-33083-8

To my family

ABSTRACT

In the first part of the thesis, we prove a representation theorem for completely contractive dual Banach algebras. More explicitly, we prove that if $\mathfrak{A}$ is a completely contractive dual Banach algebra, then there exists a w^*-continuous complete isometry from $\mathfrak{A}$ into $\mathcal{CB}(E)$, the operator space of completely bounded operators on E, for some reflexive operator space E.

In the second part of the thesis, we study the Connes-amenability of dual Banach algebras. We first prove some hereditary properties for the Connes-amenability. We present a necessary and sufficient condition for the Connes (and strongly Connes) amenability of w^*-closed ideals of Connes (and strongly Connes) amenable dual Banach algebras. Every dual Banach algebra induces some short exact sequences. We characterize the Connes (and strongly Connes) amenability of dual Banach algebras in terms of certain homological properties of those short exact sequences.

Finally, we prove that for an arbitrary discrete group G, $B(G)$, the Fourier-Stieltjes algebra of G is Connes-amenable if and only if G has an abelian subgroup of finite index.

ACKNOWLEDGEMENTS

First of all, I would like to express my gratitude to my supervisor Professor Volker Runde for his help and guidance over the years and for sharing his experience and valuable comments during the preparation of this thesis.

I would like to thank Professor Anthony Lau for his kind support and allowing me to participate in his abstract harmonic analysis seminars. His dedication to mathematics is truly inspiring. I also would like to thank Professor Vladimir Troitsky for his friendly attitude and advice.

My sincere love goes to my parents, my sister, and of course my better half Magdalena for her patience and support in the preparation of this thesis.

My friend and mentor Professor Bünyamin Sarı is not forgotten either, nor are my fellow graduate students Shawn Desaulniers and Mehmet Eğilmez, for the numerous talks and discussions on mathematics and other matters.

Furthermore, I would like to thank my instructors from my years as an undergraduate student in Turkey: Professor Refail Alizade and Professor Oktay Veliev for introducing me to pure mathematics, and Professor Şennur Somalı for instilling confidence in me when she appointed me as the teaching assistant of her linear algebra course.

Finally, I would like to thank University of Alberta for the financial support during my graduate studies.

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Chapter 1

Introduction

A locally compact group is a group equipped with a locally compact Hausdorff topology such that the group operations and the topology are compatible with each other; that is the multiplication and the inversion maps are continuous. The main objects of abstract harmonic analysis are the locally compact groups and the algebras related to them. An operator space is a Banach space with a norm on each matrix level satisfying some compatibility conditions. In his well-known theorem ([Rua 2]), Ruan proved that every operator space is a closed subspace of bounded linear operators on some Hilbert space. Many of the fundamental objects of study in abstract harmonic analysis, such as the Fourier algebras, have natural operator space structures on them. With the advent of abstract operator spaces (which was initiated by Ruan's representation theorem [Rua 2]), we have a better understanding of objects that we study in abstract harmonic analysis. Often, investigating properties of algebras (defined over locally compact groups) as operator spaces gives us more information about the algebras and their underlying groups. Hence, generally it is more fruitful to study algebras in the category of operator spaces rather than studying their properties only on the Banach space level.

One of the most important objects of study in the category of operator spaces is completely contractive Banach algebras. Many of the central objects of study in abstract harmonic analysis, such as the Fourier algebras, are in this class. A completely contractive Banach algebra, which is a dual operator space ([E-R]), is called a completely contractive dual Banach algebra. It is known that every w^*-closed subalgebra of $\mathcal{CB}(E)$, operator space of completely bounded operators on a reflexive operator space E, is a completely contractive dual Banach algebra. In Chapter 3, we prove that every completely contractive dual Banach algebra arises in this way (see Theorem 3.4.4).

Amenability, which is a very distinctive property for locally compact groups, was defined firstly for discrete groups by von-Neumann ([vNeu]). For these groups, amenability has strong ties with the well-known Banach-Tarski paradox ([Wag]). The definition of amenability for arbitrary locally compact groups was later given by Day ([Day]). A locally compact group G is called amenable if there is a mean on $L^\infty(G)$, which is left translation invariant. All finite, abelian, and compact groups are amenable; however, the free group on two generators is not ([Run 2]).

In his famous paper ([Joh 1]), B.E. Johnson defined the concept of amenability for Banach algebras and he proved that the group algebra of a locally compact group is amenable whenever the underlying group is amenable.

A Banach algebra is called a dual Banach algebra if it is a dual Banach space and the multiplication is w^*-continuous. There is a variant of amenability called Connes-amenability, which was firstly defined for von-Neumann algebras in [J-K-R], that is better suited for dual Banach algebras. Connes-amenability was defined for arbitrary dual Banach algebras in [Run 1].

In Chapter 4, we will first prove some hereditary properties for Connes-amenability. We will present a necessary and sufficient condition for the Connes (and strongly Connes) amenability of w^*-closed ideals of Connes (and strongly Connes) amenable dual Banach algebras. Then we shall characterize the Connes (and strongly Connes) amenability of dual Banach algebras in terms of certain homological properties of some short exact sequences.

In [Run 4], Runde has proven that for amenable discrete groups, the Connes-amenability of the Fourier-Stieltjes algebra of a locally compact group G is equivalent to G's being finite by abelian. We will finish Chapter 4 by extending this result to arbitrary discrete, not necessarily amenable, groups.

Chapter 2

Preliminaries

2.1 Abstract Harmonic Analysis

The main objects of abstract harmonic analysis are locally compact groups and algebras related to them. We will start with the definition of locally compact groups and some of their important properties. For more information we refer the reader to the standard reference books [Fol] and [H-R].

Definition 2.1.1. A *topological group* is a group equipped with a topology such that the group operations and the topology are compatible. That is the maps

$$G \times G \to G,\ (g,h) \mapsto gh \qquad \text{and} \qquad G \to G,\ g \mapsto g^{-1}$$

are continuous. If the topology on G is a locally compact Hausdorff topology (that is, there is a neighborhood base for the identity element consisting of compact sets), then G is called a *locally compact group.*

Clearly, every group equipped with the discrete topology is locally compact. Also, the set of real numbers with addition and the unit circle in the complex plane with multiplication are locally compact groups. However, if E is an infinite dimensional Banach space, then $(E, +)$ is a (abelian) topological group that is not locally compact.

For a subset E of a locally compact group G and $g \in G$, gE will denote the set of all elements of the form gx, for some $x \in E$.

Definition 2.1.2. Let G be a locally compact group. A positive, regular, Borel measure m on G is called a (left) *Haar measure* if it is left invariant, that is $m(gE) = m(E)$ for each g in G and for every Borel measurable set E in G.

A. Weil ([Wei]) proved the following:

Theorem 2.1.3. *Every locally compact group has a (left) Haar measure that is unique up to a (positive) multiplicative constant.*

For discrete groups, Haar measure is counting measure; for real numbers, it is Lebesgue measure; and for the unit circle in the complex plane, it is arc length measure.

Theorem 2.1.3 is of special importance because thanks to it, we can define the L^p-spaces of an arbitrary locally compact group G. An algebra $\mathcal{A}$, which at the same time is also a Banach space, is called a *Banach algebra* if

$$\|ab\| \leq \|a\| \, \|b\|$$

for all a, b in $\mathcal{A}$.

If there is an element $e \in \mathcal{A}$ such that $ae = ea = a$, for all a in $\mathcal{A}$, then $\mathcal{A}$ is called *unital* and e is called the identity element. Not all (Banach) algebras are unital (such as $2\mathbb{Z}$, the set of all even integers, with usual addition and multiplication) but some algebras contain a net which behaves like an identity:

Definition 2.1.4. Let $\mathfrak{A}$ be a Banach algebra. A bounded net $(m_\alpha)_\alpha$ in $\mathfrak{A}$ is called a *bounded approximate identity* if it satisfies following property:

$$\lim_\alpha \; m_\alpha a = \lim_\alpha \; a m_\alpha = a \qquad (a \in \mathfrak{A}).$$

We define $L^1(G)$, the *group algebra* of G, to be the equivalence classes of Borel measurable, integrable complex valued functions of G that turns into a Banach algebra with the following convolution product

$$f * g(s) := \int_G f(t) g(t^{-1}s) dt \qquad (f, g \in L^1(G)).$$

G. Wendel ([Wen]) proved the following:

Theorem 2.1.5. *Let G and H be locally compact groups. Then $L^1(G)$ and $L^1(H)$ are isometrically isomorphic if and only if G and H are topologically isomorphic.*

Therefore, as a locally compact group, every structural property of G can be expressed in terms of $L^1(G)$. For instance, for a locally compact group G, the group algebra $L^1(G)$ is unital if and only if G is discrete ([Fol]).

Note that for any locally compact group G, the group algebra $L^1(G)$ has always a bounded approximate identity ([Fol]).

If G is a locally compact group, then $M(G)$, the measure algebra, will denote the space of all regular, complex Borel measures on G and $C_0(G)$ will denote the space of (complex valued) continuous functions on G vanishing at infinity. Then $C_0(G)$ becomes a commutative C^*-algebra with pointwise multiplication and the sup norm, that is

$$\|f\|_{C_0(G)} := \sup_{x \in G} |f(x)|.$$

The Riesz representation theorem ([H-R]) shows that $M(G)$ can be identified with the dual space of $C_0(G)$. The duality is given explicitly as follows:

$$\langle \mu, f \rangle := \int_G f(x) d\mu(x) \qquad (\mu \in M(G), f \in C_0(G)).$$

The measure algebra becomes a Banach algebra through convolution product defined by

$$\langle \mu * \nu, f \rangle := \int_G \int_G f(xy) d\mu(x) d\nu(y) \qquad (\mu, v \in M(G), f \in C_0(G)).$$

Note that the group algebra $L^1(G)$ can be viewed as a closed ideal of the measure algebra $M(G)$ by the map $T : L^1(G) \to M(G)$ where

$$\langle T(f), g\rangle = \int_G f(x)g(x)dx \qquad (f \in L^1(G), g \in C_0(G)).$$

Then the convolution product on $M(G)$ extends that on $L^1(G)$. The measure algebra $M(G)$ is a complete invariant for G. Note that $M(G)$ is always unital. The point mass δ_e at e is the identity of $M(G)$.

Definition 2.1.6. Let $\mathfrak{A}$ be a Banach algebra. A Banach space X, which is also a left $\mathfrak{A}$-module, is called a left *Banach $\mathfrak{A}$-module* if there is $C \geq 0$ such that

$$\|a.x\| \leq C\|a\| \; \|x\| \qquad (a \in \mathfrak{A},\; x \in X).$$

We similarly define *right Banach $\mathfrak{A}$-modules* and *Banach $\mathfrak{A}$-bimodules.* Clearly, every Banach algebra is a Banach bimodule over itself (with $C = 1$).

Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. Then X^* turns into a Banach $\mathfrak{A}$-bimodule via

$$\langle x, \phi.a\rangle := \langle a.x, \phi\rangle, \quad \langle x, a.\phi\rangle := \langle x.a, \phi\rangle \qquad (a \in \mathfrak{A},\; x \in X,\; \phi \in X^*).$$

Definition 2.1.7. Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D : \mathfrak{A} \to X$ is called a *derivation* if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathfrak{A}).$$

Each $x \in X$ defines a map

$$\mathrm{ad}_x : \mathfrak{A} \to X, \quad a \mapsto a.x - x.a.$$

It is easy to verify that ad_x is a derivation. Derivations of this type are called *inner derivations.*

Amenability for Banach algebras was defined first by Johnson in [Joh 1].

Definition 2.1.8. A Banach algebra $\mathfrak{A}$ is said to be *amenable* if every derivation from $\mathfrak{A}$ into X^* is inner for each Banach $\mathfrak{A}$-bimodule X.

Note that every amenable Banach algebra has a bounded approximate identity ([Joh 1]).

Definition 2.1.9. Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule X is called *pseudo-unital* if

$$X = \{a.x.b: \ a, b \in \mathfrak{A}, x \in X\}.$$

Let G be a locally compact group. The C^*-algebra of all essentially bounded complex valued Borel measurable functions on G equipped with the essential supremum norm is denoted by $L^\infty(G)$.

A bounded linear functional, $m : L^\infty(G) \to \mathbb{C}$, is called a *mean* if

$$\|m\| = \langle 1, m\rangle = 1.$$

For a function $f : G \to \mathbb{C}$, we define its left translate $L_g f$ by $g \in G$ through

$$(L_g f)(h) := f(gh) \quad (h \in G).$$

A mean m on $L^\infty(G)$ is called *left invariant* if

$$\langle L_g f, m\rangle = \langle f, m\rangle \qquad (g \in G, f \in L^\infty(G)).$$

The existence of left invariant means on (discrete) groups was first investigated by J. von Neumann ([vNeu]). We have the following definition due to M. Day ([Day]).

Definition 2.1.10. A locally compact group G is *amenable* if there is a left invariant mean on $L^\infty(G)$.

All finite, abelian, and compact groups are amenable; however, the free group on two generators is not ([vNeu]). The first connection between group amenability and Banach algebra amenability is given by Barry Johnson's theorem ([Joh 1]):

Theorem 2.1.11. *Let G be a locally compact group. Then G is amenable if and only if $L^1(G)$ is an amenable Banach algebra.*

Let G be a locally compact group. A *unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$, unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. The collection of all equivalence classes of unitary representations of G with respect to unitary equivalence is denoted by Σ_G.

We define the *left regular representation* $\lambda : G \to \mathcal{B}(L^2(G))$ of G through

$$(\lambda(g)\xi)(h) := \xi(g^{-1}h) \qquad (g, h \in G,\ \xi \in L^2(G)).$$

We call f a *coefficient function* of a representation π of G on some Hilbert space $\mathcal{H}$ if there are $\xi, \zeta \in \mathcal{H}$ such that

$$f(x) = \langle \pi(x)\xi, \zeta \rangle \qquad (x \in G).$$

We define the *Fourier-Stieltjes* algebra, $B(G)$ of G, as

$$B(G) := \{f : G \to \mathbb{C} : \ \exists\, \pi \in \Sigma_G \ \text{and} \ \xi, \zeta \in \mathcal{H}_\pi \ \text{such that} \ f(x) = \langle \pi(x)\xi, \zeta \rangle, \ \forall x \in G\}. \tag{2.1.1}$$

We define the *Fourier algebra*, $A(G)$ of G by

$$A(G) := \{f : G \to \mathbb{C} : \ \exists\, \xi, \zeta \in L^2(G) \ \text{such that} \ f(x) = \langle \lambda(x)\xi, \zeta \rangle, \ \forall x \in G\}.$$

The norm on $B(G)$ is defined by

$$\|f\|_{B(G)} := \inf \{ \|\xi\| \|\zeta\| : f \text{ is represented as in } (2.1.1) \}.$$

Fourier and Fourier-Stieltjes algebras are defined by Eymard ([Eym]) in their full generality.

Theorem 2.1.12. *Let G be a locally compact group. Then (with pointwise multiplication) $B(G)$ is a commutative unital Banach algebra that contains $A(G)$ as a norm-closed ideal.*

The first characterization of the amenability of G in terms of $A(G)$ was given by Leptin ([Lep]).

Theorem 2.1.13. *Let G be a locally compact group. Then G is amenable if and only if $A(G)$ has a bounded approximate identity.*

Since amenable Banach algebras have a bounded approximate identity, the amenability of $A(G)$ implies (group) amenability of G. However, the converse is not true ([Joh 2]). Recently Forrest and Runde ([F-R]) proved the following theorem:

Theorem 2.1.14. *Let G be a locally compact group. Then $A(G)$ is amenable if and only if G has an abelian subgroup of finite index.*

Definition 2.1.15. Let E and F be two linear spaces. The (algebraic) *tensor product* of E and F is a linear space $E \otimes F$ with a bilinear map $\tau : E \times F \to E \otimes F$ with the following universal property: For each linear space G, and for each bilinear map $T : E \times F \to G$, there exists a unique linear map $\overline{T} : E \otimes F \to G$ such that $T = \overline{T}\tau$.

Remark 2.1.1. *1.* The tensor product of linear spaces exists and it is unique up to isomorphism.

2. For an arbitrary element (x, y) in $E \otimes F$, the image $\tau(x, y)$ is denoted by $x \otimes y$ and is called an elementary tensor.

3. Any element in $E \otimes F$ can be written as a finite linear combination of elementary tensors.

Suppose that E and F are Banach spaces. Then for $u \in E \otimes F$ we define

$$\|u\| := \inf\left\{\sum_{k=1}^{m} \|x_k\|\|y_k\| : u = \sum_{k=1}^{m} x_k \otimes y_k\right\} \tag{2.1.2}$$

Then the formula 2.1.2 defines a norm on $X \otimes Y$ and the completion of $X \otimes Y$ with respect to this norm is called *the projective tensor product of Banach spaces X and Y*, denoted by $X \otimes_\gamma Y$. For more information on tensor products of Banach spaces, we refer the reader to [Rya].

Let $\mathfrak{A}$ be a Banach algebra. We define the corresponding *diagonal operator* Δ by

$$\Delta : \mathfrak{A} \otimes_\gamma \mathfrak{A} \to \mathfrak{A}, \quad a \otimes b \mapsto ab$$

where $\otimes_\gamma$ denotes the projective tensor product of Banach spaces.

$\mathfrak{A}\otimes_\gamma\mathfrak{A}$ becomes a Banach $\mathfrak{A}$-bimodule via

$$a.(b \otimes c) := ab \otimes c \quad \text{and} \quad (b \otimes c).a := b \otimes ca \qquad (a, b, c \in \mathfrak{A}).$$

It is easy to see that, the diagonal operator is an $\mathfrak{A}$-bimodule homomorphism.

2.2 Operator Spaces

If E is a linear space, then for each $m, n \in \mathbb{N}$, $M_{m,n}(E)$ will denote the space of all $m \times n$ matrices with entries in E. If $m = n$, then $M_{m,n}(E)$ will be denoted by $M_n(E)$ and in particular, $M_n = M_n(\mathbb{C})$ will denote the space of all scalar $n \times n$ matrices.

Definition 2.2.1. Let E be a linear space with a norm $\|\cdot\|_n$ on $M_n(E)$ for each $n \in \mathbb{N}$ such that

$$\left\| \begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \qquad (n, m \in \mathbb{N},\ x \in M_n(E),\ y \in M_m(E)) \quad \text{(R 1)}$$

and

$$\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \qquad (n \in \mathbb{N},\ x \in M_n(E),\ \alpha, \beta \in M_n). \qquad \text{(R 2)}$$

Then $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is called a matricial norm for E. Moreover, if each $\|\cdot\|_n$ is complete, then E is called an *abstract operator space.*

A *concrete operator space* is a closed subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is a Hilbert space, then for each $n \in \mathbb{N}$, we identify

$$M_n(\mathcal{B}(\mathcal{H})) = B(l_2^n(\mathcal{H})).$$

With this identification, $B(\mathcal{H})$ and hence all of its closed subspaces turn into concrete operator spaces. In particular, by GNS-construction ([Sak]), every C^*-algebra is a concrete operator space.

A linear operator $T : E \to F$ between two abstract operator spaces E and F induces a linear operator

$$T^{(n)} : M_n(E) \to M_n(F), \quad (x_{i,j}) \mapsto (T(x_{i,j}))$$

for each $n \in \mathbb{N}$.

Definition 2.2.2. Let E and F be two abstract operator spaces, and let $T \in \mathcal{B}(E, F)$. Then:

1. T is *completely bounded* if

$$\|T\|_{\mathrm{cb}} := \sup \left\{ \|T^{(n)}\|_{\mathcal{B}(M_n(E), M_n(F))} : \ n \in \mathbb{N} \right\} < \infty.$$

2. T is a *complete contraction* if $\|T\|_{\mathrm{cb}} \leq 1$.

3. T is a *complete isometry* if $T^{(n)}$ is an isometry for each $n \in \mathbb{N}$.

The set of completely bounded operators from E to F is denoted by $\mathcal{CB}(E, F)$.

Indeed, $\mathcal{CB}(E,F)$ with $\|\cdot\|_{\text{cb}}$ is a Banach space. Not every linear, bounded operator between operator spaces is completely bounded:

Example 2.2.3. Let $\mathcal{H} = l_2$. Then the (Banach space) adjoint operator

$$\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad T \mapsto T^*$$

is an isometry but is not completely bounded ([E-R]).

If E and F are two operator spaces, then $\mathcal{CB}(E,F)$ turns into an abstract operator space with the identification

$$M_n(\mathcal{CB}(E,F)) = \mathcal{CB}(E, M_n(F)), \qquad (n \in \mathbb{N}).$$

The following result is known as Smith's Lemma ([Smi]):

Lemma 2.2.1. *Let E be an abstract operator space and $\mathcal{A}$ be a commutative C^*-algebra. Then for every $T \in \mathcal{B}(E, \mathcal{A})$ we have*

$$\|T\|_{cb} = \|T\|.$$

In particular, if $\mathcal{A} = \mathbb{C}$, then we have:

Corollary 2.2.1. *Let E be an abstract operator space. Then for every $T \in E^*$ we have $\|T\|_{cb} = \|T\|$.*

Due to the duality theorem, the dual and the predual (if it exists) of an operator space have natural operator space structures. More explicitly, if X is an operator space and $\phi = (\phi_{i,j}) \in M_n(X^*)$ for some $n \in \mathbb{N}$, then

$$\|\phi\|_n = \sup \Big\{ |\langle \phi_{i,j}, x_{k,l} \rangle|_{n^2} : \ x = (x_{k,l}) \in M_n(X), \|x\|_n \leq 1 \Big\}.$$

Now by using the duality theorem we can give more examples of (abstract) operator spaces.

Examples 2.2.4. 1. If G is a locally compact group, then as a dual of a commutative C^*-algebra, the measure algebra $M(G)$ is an operator space.

2. If G is a locally compact group, then as a predual of a von-Neumann algebra, the Fourier algebra $A(G)$ is an operator space.

3. If G is a locally compact group, then as a dual of a C^*-algebra algebra, the Fourier-Stieltjes algebra $B(G)$ is an operator space.

The following theorem is known as Ruan's representation theorem ([Rua 2]):

Theorem 2.2.5. *Let X be an abstract operator space. Then there is a complete isometry from X into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Thanks to Ruan's representation theorem, we do not have to distinguish abstract and concrete operator spaces.

In [Rua 2], Ruan defined a new form of amenability called *operator amenability* for completely contractive Banach algebras and in the same paper he proved that:

Theorem 2.2.6. *Let G be a locally compact group. Then $A(G)$ is operator amenable if and only if G is amenable.*

Let X be a Banach space. Then X can be isometrically embedded into $C(\Omega)$ where Ω is the closed unit ball of X^* equipped with w^*-topology. By the Banach-Alaoglu theorem Ω is a compact, Hausdorff space. This embedding yields an operator space structure on X which is denoted by $\min X$.

Let X be a Banach space. Then for $x = (x_{i,j}) \in M_n(X)$ for some $n \in \mathbb{N}$, we define

$$\|x\|_n := \sup \left\{ \|(T(x_{i,j}))\|_n : \ T \in B(X, B(l_2)) \text{ is a contraction} \right\}.$$

Then X becomes an operator space with this matricial norm, and the resulting operator space is denoted by $\max X$.

The min and the max operator space structures have the following universal properties ([E-R]):

Proposition 2.2.1. *Let V be an operator space.*

1. *If $\mathcal{CB}(W, V) = \mathcal{B}(W, V)$, isometrically, for any operator space W, then $V = \min V$,*

2. *If $\mathcal{CB}(V, W) = \mathcal{B}(V, W)$, isometrically, for any operator space W, then $V = \max V$.*

When a Banach space X is equipped with the min (or max) operator space structure, the norm on the Banach space level does not change.

It is also known that if X is an infinite dimensional Banach space, then the identity map

$$I : \min X \rightarrow \max X$$

is not completely bounded ([Pau], Theorem 14.3).

This shows us two things. First, the open mapping theorem does not hold on the category of operator spaces. Second, for an infinite dimensional Banach space X, we have at least two distinct operator space structures.

Let X be an operator space. Define a new norm $\|\cdot\|'_n$ on $M_n(X)$ by

$$\|\cdot\|'_n : M_n(X) \rightarrow [0, \infty)$$

$$\|x\|'_n = \inf \left\{ |\alpha| \|\tilde{x}\| |\beta| : \; x = \alpha \tilde{x} \beta \right\}$$

where $\alpha \in HS_{n,r}$, $\tilde{x} \in M_r(V)$ and $\beta \in HS_{r,n}$ $(r \in \mathbb{N})$, and with $HS_{m,n}$ being the scalar $m \times n$ matrices with the Hilbert-Schmidt norm $|\cdot|$.

If $T_n(X)$ denotes the space $M_n(X)$ equipped with $\|\cdot\|'_n$, then we have

$$M_n(X)^* = T_n(X^*),$$

and

$$T_n(X)^* = M_n(X^*).$$

If X is a Banach space, then we have the following complete isometry

$$(\min X)^* = \max X^*$$

and

$$(\max X)^* = \min X^*.$$

In particular, as the dual of a commutative C^*-algebra, the measure algebra $M(G)$ has the max operator space structure.

Definition 2.2.7. Let E_1, E_2, and F be operator spaces. A bilinear map $T\colon E_1 \times E_2 \to F$ is called *completely contractive* if

$$\|T\|_{cb} := \sup_{n_1,n_2\in\mathbb{N}} \left\|T^{(n_1,n_2)}\right\| \leq 1,$$

where

$$T^{(n_1,n_2)} : M_{n_1}(E_1) \times M_{n_2}(E_2) \to M_{n_1 n_2}(F), \quad \big((x_{i,j}), (y_{k,l})\big) \mapsto \big(T(x_{i,j}, y_{k,l})\big).$$

Definition 2.2.8. A *completely contractive Banach algebra* is an algebra which is also an operator space such that multiplication is a completely contractive bilinear map.

Example 2.2.9. Every closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is a completely contractive Banach algebra.

Let X and Y be two operator spaces and let $X \otimes Y$ denote their algebraic tensor product. For each $n \in \mathbb{N}$, given an element u in $M_n(X \otimes Y)$ we define

$$\|u\| := \inf\left\{\|\alpha\|\|x\|\|y\|\|\beta\| : \; u = \alpha(x \otimes y)\beta\right\} \tag{2.2.1}$$

where the infimum is taken over all possible decompositions where $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$, $x \in M_p(X)$ and $y \in M_q(Y)$, with $p, q \in \mathbb{N}$ arbitrary.

Then the formula 2.2.1 defines a matricial norm on $X \otimes Y$ and the the completion of $X \otimes Y$ with respect to this norm is called *the projective tensor product of operator spaces* X and Y and is denoted by $X \widehat{\otimes} Y$. For more information on the tensor products of operator spaces, we refer the reader to [E-R].

A *Hopf–von Neumann algebra* is a pair $(\mathfrak{M}, \nabla)$ where $\mathfrak{M}$ is a von Neumann algebra and ∇ is a *co-multiplication*: a unital, injective, w^*-continuous *-homomorphism $\nabla\colon \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ (where $\bar{\otimes}$ represents the W^*-tensor product) which is co-associative: that is the diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & \mathfrak{M}\bar{\otimes}\mathfrak{M} \arrow[d, "\nabla\otimes I_{\mathfrak{M}}"] \\
\mathfrak{M}\bar{\otimes}\mathfrak{M} \arrow[r, "I_{\mathfrak{M}}\otimes\nabla"'] & \mathfrak{M}\bar{\otimes}\mathfrak{M}\bar{\otimes}\mathfrak{M}
\end{tikzcd}$$

commutes, that is $\nabla(I_{\mathfrak{M}} \otimes \nabla) = (I_{\mathfrak{M}} \otimes \nabla)\nabla$.

It is known ([E-R]) that $\mathfrak{M}\bar{\otimes}\mathfrak{M} \cong (\mathfrak{M}_*\widehat{\otimes}\mathfrak{M}_*)^*$ where $\mathfrak{M}_*$ is the unique predual of $\mathfrak{M}$ and $\widehat{\otimes}$ is the projective tensor product of operator spaces. Since ∇ is w^*-continuous, it induces a complete contraction $\nabla_* : \mathfrak{M}_*\widehat{\otimes}\mathfrak{M}_* \to \mathfrak{M}_*$, turning $\mathfrak{M}_*$ into a completely contractive Banach algebra.

Example 2.2.10. If G is a locally compact group, then the Fourier-Stieltjes algebra $B(G)$ is a Hopf–von Neumann algebra ([R-S]) and hence, it is a completely contractive Banach algebra. Since $A(G)$ is a closed ideal of $B(G)$ ([Eym]), it is also a completely contractive Banach algebra.

2.3 Amenability and Short Exact Sequences

Amenability is one of the most important notions in abstract harmonic analysis. In [Joh 1], Barry Johnson described the amenability of a locally compact group G through Hochschild cohomology groups, $\mathcal{H}(L^1(G), E)$, where E is a Banach $L^1(G)$-bimodule) of $L^1(G)$. This led us to define amenability for arbitrary Banach algebras. The notion of amenability for Banach algebras, using homological algebra, was later studied by Helemskiĭ ([Hel]).

We will start this chapter by giving some basic definitions from homological algebra that we will need later.

Definition 2.3.1. Let $\mathfrak{A}$ be a Banach algebra and X, Y, Z be (left, right or bi-) $\mathfrak{A}$-modules. Then the sequence

$$\Gamma : \qquad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{2.3.1}$$

is called a *short exact sequence* if f is injective, g is surjective, and $\mathrm{Im} f = \ker g$. The short exact sequence Γ

a) is called admissible if $\ker g$ has a Banach space complement in Y,

b) splits if $\ker g$ has a Banach space complement in Y which is a Banach $\mathfrak{A}$-module.

The proof of the next proposition is straightforward and we will omit it.

Proposition 2.3.1. *Consider the short exact sequence* Γ *in (2.3.1). Then* Γ *is admissible if and only if there is a bounded linear map*

$$F : Y \to X \quad \text{such that} \quad Ff = I_X, \text{ the identity operator on } X.$$

Moreover, Γ *splits if and only if there is a bounded* $\mathfrak{A}$*-module homomorphism*

$$F : Y \to X \quad \text{such that} \quad Ff = I_X, \text{ the identity operator on } X.$$

The following proposition will be useful in the sequel.

Proposition 2.3.2. *Consider the short exact sequence* Γ *in (2.3.1). There exists a bounded linear operator*

$$F : Y \to X \quad \text{satisfying} \quad Ff = I_X$$

if and only if there exists a unique bounded linear operator

$$G : Z \to Y \quad \text{satisfying} \quad gG = I_Z.$$

Let f *and* g *be* $\mathfrak{A}$*-module homomorphisms. Then* F *is an* $\mathfrak{A}$*-module homomorphism if and only if* G *is. In either case, we have* $fF + Gg = I_Y$*, the identity operator on* Y*.*

Proof. This is ([C-L], Proposition 1.1). □

The following results were proven firstly by Helemskiĭ ([Hel]) by using homological algebra machinery, in particular Ext and Tor functors. Later, these theorems were reproved by Curtis and Loy ([C-L]) without using homological algebra machinery.

In chapter 3, we will prove analogous theorems for dual Banach algebras.

Let $\mathfrak{A}$ be a Banach algebra, and let $\Delta : \mathfrak{A}\otimes_\gamma\mathfrak{A} \to \mathfrak{A}$ be the diagonal operator. Then we may consider the following short exact sequence:

$$\Omega : \qquad 0 \longrightarrow \ker\Delta \xrightarrow{\iota} \mathfrak{A}\otimes_\gamma\mathfrak{A} \xrightarrow{\Delta} \mathfrak{A} \to 0$$

where ι denotes the inclusion map. Then by taking the adjoints, we obtain the following short exact sequence:

$$\Omega^* : \qquad 0 \longrightarrow \mathfrak{A}^* \xrightarrow{\Delta^*} (\mathfrak{A}\otimes_\gamma\mathfrak{A})^* \xrightarrow{\iota^*} \mathfrak{A} \to 0.$$

As we shall see, the amenability of $\mathfrak{A}$ and some homological properties of the short exact sequences Ω and Ω^* are closely related.

Let $\mathfrak{A}$ be a Banach algebra. Then each closed ideal J of $\mathfrak{A}$ induces the following short exact sequence:

$$\Upsilon : \qquad 0 \longrightarrow J \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\sigma} \mathfrak{A}/J \to 0$$

where ι denotes the inclusion map. The following theorem was proved in [Hel] and [C-L].

Theorem 2.3.2. *Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is unital, then Ω is admissible. Furthermore, $\mathfrak{A}$ is amenable if and only if the following conditions are satisfied:*

1. *$\mathfrak{A}$ has a bounded approximate identity,*

2. *Ω^* splits as an exact sequence of Banach $\mathfrak{A}$-bimodules.*

In their paper ([C-L]), Curtis and Loy prove that every finite codimensional subalgebra of an amenable Banach algebra is amenable. We will prove an analogous theorem for a different type of amenability called Connes amenability in Chapter 3.

2.4 Connes-Amenability

Definition 2.4.1. A Banach algebra $\mathfrak{A}$ which is a dual Banach space is called a *dual Banach algebra* if multiplication on $\mathfrak{A}$ is separately w^*-continuous, or equivalently, if $\mathfrak{A} = (\mathfrak{A}_*)^*$ for some closed submodule $\mathfrak{A}_*$ of $\mathfrak{A}^*$.

Note that $\mathfrak{A}_*$ need not be unique.

Examples 2.4.2.

1. Every von-Neumann algebra $\mathfrak{M}$ is a dual Banach algebra with the (unique) predual $\mathfrak{M}_*$, the space generated by all positive normal linear functionals on $\mathfrak{M}$.

2. The measure algebra $M(G)$ of a locally compact group G is a dual Banach algebra with the predual $C_0(G)$.

3. The Fourier-Stieltjes algebra $B(G)$ of a locally compact group G is a dual Banach algebra with the predual $C^*(G)$, the group C^*-algebra of G as defined in [Eym].

4. The bidual of every Arens regular Banach algebra is a dual Banach algebra.

5. $\mathcal{B}(E)$, the space of bounded linear operators on E, is a dual Banach algebra if E is a reflexive Banach space. In this case, the predual is $E^*\otimes_\gamma E$.

Definition 2.4.3. Let $\mathfrak{A}$ be a dual Banach algebra and let X be a dual Banach $\mathfrak{A}$-bimodule. An element $x \in X$ is called *normal* if the maps

$$\mathfrak{A} \to X, \qquad x \mapsto \begin{cases} a.x, \\ x.a \end{cases}$$

are w^*-w^*-continuous.

We say that X is normal if every element of X is normal. A dual Banach algebra $\mathfrak{A}$ is called *Connes-amenable* if every w^*-continuous derivation from $\mathfrak{A}$ into a normal, dual Banach $\mathfrak{A}$-bimodule is inner.

Let $\mathfrak{A}$ be a Connes-amenable dual Banach algebra and let E be a Banach $\mathfrak{A}$-bimodule where $E = \mathfrak{A}$ (as sets) and the module actions are defined as follows

$$a.x = ax, \quad x.a = 0 \qquad (a \in \mathfrak{A}, x \in E).$$

If $D : \mathfrak{A} \to E$ denotes the inclusion map, then it is easy to see that D is a (w^*-continuous) derivation. Hence, it is inner. This shows that there exists an $e \in E$ such that $a = D(a) = a.e - e.a = ae$ for all $a \in \mathfrak{A}$. This means that e is a left identity for $\mathfrak{A}$. Similarly, we can prove that $\mathfrak{A}$ has a left identity too. Then the associativity of the product on $\mathfrak{A}$ implies that $\mathfrak{A}$ is unital.

The Connes amenability of a dual Banach algebra (associated with a locally compact group) reflects the properties of the underlying group. For instance, as proved in [Run 2], $M(G)$ is Connes-amenable if and only if G is an amenable group.

Let $\mathfrak{A}$ be a completely contractive dual Banach algebra. An operator $\mathfrak{A}$-bimodule is a Banach $\mathfrak{A}$-bimodule which is at the same time an operator space such that the module actions are completely bounded maps. Then $\mathfrak{A}$ is said to be *operator Connes-amenable* if every w^*-continuous completely bounded derivation from $\mathfrak{A}$ into a normal, dual operator Banach $\mathfrak{A}$-bimodule is inner. It is known ([R-S]) that $B(\mathbb{F}_2)$ is operator Connes-amenable. On the other hand, $B_r(G)$ (as defined in [Eym]) is operator Connes-amenable if and only if G is an amenable group.

Definition 2.4.4. A dual Banach algebra with identity $\mathfrak{A}$ is called *strongly Connes-amenable* if for each unital Banach $\mathfrak{A}$-bimodule X, every w^*-continuous derivation from $\mathfrak{A}$ into X^* is inner.

In [Run 4], Runde gave an example of a Connes-amenable Banach algebra that is not strongly Connes-amenable.

Definition 2.4.5. *Let $\mathfrak{A}$ be a Banach algebra. A* virtual diagonal *for $\mathfrak{A}$ is an element $M \in (\mathfrak{A}\otimes_\gamma\mathfrak{A})^{**}$ such that*

$$a.M = M.a \quad \text{and} \quad a\Delta^{**}M = a \qquad (a \in \mathfrak{A}).$$

In his paper ([Joh 1]), Johnson proved that a Banach algebra $\mathfrak{A}$ is amenable if and only if $\mathfrak{A}$ has a virtual diagonal. Definition 2.4.5 has a variant suitable for dual Banach algebras.

Let $\mathfrak{A}$ be a dual Banach algebra with the predual $\mathfrak{A}_*$ and let

$$\Delta : \mathfrak{A}\otimes_\gamma\mathfrak{A} \to \mathfrak{A}, \quad a \otimes b \mapsto ab,$$

denote the diagonal operator. It is known that $\mathcal{B}^2(\mathfrak{A}, \mathbb{C})$, the space of bounded bilinear functionals on $\mathfrak{A}\times\mathfrak{A}$, can be identified with $(\mathfrak{A}\otimes_\gamma\mathfrak{A})^*$, where $\otimes_\gamma$ denotes the projective tensor product of Banach spaces. We denote by $\mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})$, the space of separately w^*-continuous bilinear functions in $\mathcal{B}^2(\mathfrak{A}, \mathbb{C})$. Clearly $\mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})$ is a Banach $\mathfrak{A}$-module. Since

$$\Delta^* : \mathfrak{A}^* \to (\mathfrak{A}\otimes_\gamma\mathfrak{A})^* = \mathcal{B}^2(\mathfrak{A}, \mathbb{C}),$$

we have the restriction map

$$\Delta^*_{|\mathfrak{A}_*} : \mathfrak{A}_* \to \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C}).$$

Finally we obtain

$$\Delta_{w^*} = (\Delta^*_{|\mathfrak{A}_*})^* : \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^* \to \mathfrak{A}.$$

As a restriction of Δ^{**}, Δ_{w^*} is an $\mathfrak{A}$-bimodule homomorphism.

Definition 2.4.6. A *normal, virtual diagonal* for a dual Banach algebra $\mathfrak{A}$ is an element $M \in B^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$ such that

$$a.M = M.a \quad \text{and} \quad a\Delta_{w^*}M = a \qquad (a \in \mathfrak{A}).$$

The following theorem was proved in [Run 1].

Theorem 2.4.7. *Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathfrak{A}$ is strongly Connes-amenable if and only if there is a normal, virtual diagonal for $\mathfrak{A}$.*

In order to characterize the Connes-amenability of a dual Banach algebra in terms of the existence of virtual diagonals, we will firstly introduce some terminology.

Definition 2.4.8. *Let $\mathfrak{A}$ be a dual Banach algebra with predual $\mathfrak{A}_*$ and let X be a Banach $\mathfrak{A}$-bimodule. An element $x \in X$ is called* w^*-weakly continuous *if the module maps*

$$\mathfrak{A} \to X, \qquad x \mapsto \begin{cases} a.x, \\ x.a \end{cases}$$

are $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-$\sigma(X, X^*)$ continuous.

The collection of all w^*-weakly continuous elements of X is denoted by $\sigma WC(X)$.

Remark 2.4.1.

1. *$\sigma WC(E)$ is a closed submodule of E.*
2. *If F is another Banach $\mathfrak{A}$-bimodule and if $\theta : E \to F$ is a bounded $\mathfrak{A}$-bimodule homomorphism, then $\sigma WC(E) \subseteq \sigma WC(F)$.*

The following result was proved in [Run 3]:

Corollary 2.4.1. *Let $\mathfrak{A}$ be a dual Banach algebra with predual bimodule $\mathfrak{A}_*$. Then $\mathfrak{A}_* \subset \sigma WC(\mathfrak{A}^*)$ holds.*

Let $\mathfrak{A}$ be a dual Banach algebra. Consider again the diagonal map $\Delta : \mathfrak{A}\hat{\otimes}_\gamma\mathfrak{A} \to \mathfrak{A}$. Since

$$\Delta^* : \mathfrak{A}^* \to (\mathfrak{A}\hat{\otimes}_\gamma\mathfrak{A})^*,$$

we have the restriction map

$$\Delta^*|_{\mathfrak{A}_*} : \mathfrak{A}_* \to \sigma WC\big((\mathfrak{A}\hat{\otimes}_\gamma\mathfrak{A})^*\big).$$

Then we obtain the following map

$$\Delta_{\sigma WC} := (\Delta^*|_{\mathfrak{A}_*})^* : WC\big((\mathfrak{A}\hat{\otimes}_\gamma\mathfrak{A})^*\big)^* \to \mathfrak{A}.$$

As a restriction of Δ^{**}, $\Delta_{\sigma WC}$ is an $\mathfrak{A}$-bimodule homomorphism too.

Definition 2.4.9. Let $\mathfrak{A}$ be a dual Banach algebra. A *σWC-virtual diagonal* for $\mathfrak{A}$ is an element $M \in \sigma WC((\mathfrak{A}\hat{\otimes}_\gamma\mathfrak{A})^*)^*$ such that

$$a.M = M.a \quad \textit{and} \quad a\Delta_{\sigma WC}M = a \qquad (a \in \mathfrak{A}).$$

Now, we can characterize Connes-amenability in terms of the existence of σWC-virtual diagonals. The following theorem was proved in [Run 3].

Theorem 2.4.10. *Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathfrak{A}$ is Connes-amenable if and only if there is a σWC-virtual diagonal for $\mathfrak{A}$.*

Chapter 3

A Representation Theorem

3.1 Introduction

If E and F are Banach spaces, then $\mathcal{B}(E, F^*)$ can be identified with the dual space of $F \otimes_\gamma E$. In particular if E is a reflexive Banach space, then $\mathcal{B}(E)$ is a dual Banach algebra with predual $E^* \otimes_\gamma E$. Hence it is clear that every w^*-closed subalgebra of $\mathcal{B}(E)$ is then also a dual Banach algebra. Surprisingly, as proven recently by Daws ([Daw]), every dual Banach algebra arises in this fashion. In this chapter, we prove an operator space analog of Daws' representation theorem: if $\mathfrak{A}$ is a completely contractive dual Banach algebra, then there is a reflexive operator space E and a w^*-w^*-continuous, completely isometric algebra homomorphism from $\mathfrak{A}$ to $\mathcal{CB}(E)$. Note that $\mathcal{CB}(E) = (E^* \hat{\otimes} E)^*$. We would like to stress that even where $\mathfrak{A}$ is of the form $\max \mathfrak{A}$ for some dual Banach algebra $\mathfrak{A}$, our result is not just a straightforward consequence of Daws' result but requires a careful adaptation of his techniques to the operator space setting. The construction of such a reflexive operator space heavily relies on the theory of real and complex interpolation of operator spaces defined by Xu ([Xu]) and Pisier ([Pis 1] and [Pis 2]) respectively.

This representation theorem is somewhat related in spirit to results by Ghahramani ([Gha]) and Neufang, Ruan, and Spronk ([N-R-S]). In [Gha], $M(G)$ is (completely) isometrically represented on $\mathcal{B}(L^2(G))$, and in [N-R-S], a similar representation is constructed for the completely contractive dual Banach algebra $M_{cb}(A(G))$. We would like to emphasize, however, that our representation theorem neither implies nor is implied by those results: $\mathcal{B}(L^2(G))$ is a dual operator space, but not reflexive.

3.2 Preliminaries

3.2.1 Interpolation Spaces

Let X_0, X_1 be two topological vector spaces. The couple (X_0, X_1) is called *compatible* if there is a Hausdorff topological vector space $\mathcal{X}$ and $\mathbb{C}$-linear continuous inclusions $X_0 \hookrightarrow \mathcal{X}$ and $X_1 \hookrightarrow \mathcal{X}$. Then one can talk about the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ (which is the set of all elements $x \in \mathcal{X}$ where $x = x_0 + x_1$ for some $x_0 \in X_0$ and $x_1 \in X_1$).

If (E_0, E_1) is a couple of compatible normed spaces, then we equip $E_0 \cap E_1$ and $E_0 + E_1$ with the following norms respectively:

$$\|x\|_{E_0 \cap E_1} = \max\left\{\|x\|_{E_0}, \|x\|_{E_1}\right\},$$

and

$$\|x\|_{E_0+E_1} = \inf\left\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : \quad x = x_0 + x_1,\ x_0 \in E_0,\ x_1 \in E_1\right\}.$$

$E_0 + E_1$ with the norm above is denoted by $E_0 +_1 E_1$.

Theorem 3.2.1. *Let (E_0, E_1) be a couple of compatible normed spaces. Then $E_0 \cap E_1$ and $E_0 +_1 E_1$ become normed spaces with the norms defined above. Furthermore, if E_0 and E_1 are Banach spaces then so are $E_0 \cap E_1$ and $E_0 +_1 E_1$.*

Proof. This is ([B-L], Lemma 2.3.1). □

Definition 3.2.2. Let E_0, E_1 be a couple of compatible normed spaces. Then a normed space E is called an *intermediate space* between E_0 and E_1 if

$$E_0 \cap E_1 \subseteq E \subseteq E_0 +_1 E_1, \quad \text{with continuous inclusions.}$$

In addition, if $T \in \mathcal{B}(E_0 +_1 E_1)$ such that the restrictions T_{E_0} and T_{E_1} are in $\mathcal{B}(E_0)$ and $\mathcal{B}(E_1)$ respectively imply $T_E \in \mathcal{B}(E)$, then E is called an *interpolation space* between E_0 and E_1.

Example 3.2.3. The Riesz-Thorin interpolation theorem ([B-L], Theorem 1.1.1) shows that L_p is an interpolation space between L_{p_0} and L_{p_1} if $p_0 < p < p_1$.

There are different ways of obtaining interpolation spaces between two compatible normed spaces. In the subsequent sections, we will introduce two of those, namely real and complex interpolations.

3.2.2 Complex Interpolation of Banach Spaces

For $0 < \theta < 1$, let $E_{[\theta]} = (E_0, E_1)_\theta$ be the space of all elements $x \in E_0 +_1 E_1$ such that $x = f(\theta)$ for some function $f : \mathbb{C} \to E_0 +_1 E_1$ which satisfies the following conditions:

(i) f is bounded and continuous on the strip $S := \{z \in \mathbb{C} : \ 0 \le Re(z) \le 1\}$,

(ii) f is analytic on S_0, the interior of S,

(iii) $f(it) \in E_0$ and $f(1+it) \in E_1 \quad (t \in \mathbb{R})$.

Then we equip $E_{[\theta]}$ with the following norm:

$$\|x\|_{[\theta]} := \inf \Big\{ \|f\| : \ x = f(\theta), \ f \text{ satisfying the conditions listed above} \Big\}$$

where the norm of f is defined to be

$$\|f\| := \max\Big\{ \ \sup\big\{\|f(it)\|_{E_0}\big\}, \ \sup\big\{\|f(1+it)\|_{E_1}\big\} : \ t \in \mathbb{R}\Big\}.$$

By this construction, $E_{[\theta]}$ becomes an interpolation space between E_0 and E_1. For more information on interpolation of Banach spaces, we refer the reader to [B-L].

3.3 Real Interpolation of Banach Spaces

Consider a compatible couple of Banach spaces $E = (E_0, E_1)$. Then there are different ways to construct interpolation spaces between them by using the real method such as the K-method, $E_{\theta,p;K}$; the discrete K-method, $E_{\theta,p;\underline{K}}$; the J-method, $E_{\theta,p;J}$; and the discrete J-method, $E_{\theta,p;\underline{J}}$. However, for $0 < \theta < 1$ and $1 \leq p \leq \infty$, we have

$$E_{\theta,p;K} = E_{\theta,p;\underline{K}} = E_{\theta,p;J} = E_{\theta,p;\underline{J}},$$

with equivalent norms ([B-L], Prop. 3.3.1).

In this chapter, we will only investigate the discrete K-method since it will be sufficient for us to prove our representation theorem at the end of the chapter.

3.3.1 Real Interpolation of Banach Spaces by the Discrete K-Method

In this section we will introduce the real interpolation of Banach spaces by the discrete K-method. Now let E_0 and E_1 be two compatible normed spaces. For a fixed $t > 0$, we define the K-functional on the set $E_0 + E_1$ by

$$K(t,x) := \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1} : \ x = x_0 + x_1, \ x_0 \in E_0, \ x_1 \in E_1\}.$$

It is easy to see that for each $t > 0$, $K(t, \cdot)$ defines a norm (which is equivalent to $E_0 +_1 E_1$) on $E_0 + E_1$ ([B-L], Page 38).

Lemma 3.3.1. *For each fixed $x \in E_0 + E_1$, $K(t,x)$ is a positive valued, increasing function of t. Furthermore,*

$$K(t,x) \leq \max\left(1, \frac{t}{s}\right) K(s,x).$$

Proof. This is ([B-L], Lemmma 3.1.1). □

Now for $0 < \theta < 1$ and $1 \leq p \leq \infty$ or $0 \leq \theta \leq 1$ and $p = \infty$, we can define

$$K_{\theta,p} := \left\{x \in E_0 + E_1 : \ \ \|x\|_{\theta,p;K} := \Big(\sum_{v=-\infty}^{v=\infty} 2^{-v\theta}|K(2^v,x)|^p\Big)^{\frac{1}{p}} < \infty\right\}.$$

Then the set

$$E_{\theta,p;\underline{K}} := \left\{x \in E_0 + E_1 : \ \|x\|_{\theta,p;K} < \infty\right\}$$

is called the *real interpolation* of E_0 and E_1 by the discrete K-method.

The interpolation space $E_{\theta,p;\underline{K}}$ has many interesting properties. We now summarize some of them.

Theorem 3.3.1. *Let $E = (E_0, E_1)$ be a couple of compatible normed spaces. Then*

1. $(E_0, E_1)_{\theta,p;\underline{K}} = (E_1, E_0)_{1-\theta,p;\underline{K}}$ *with equal norms;*

2. $(E_0, E_1)_{\theta,p;\underline{K}} \subseteq (E_0, E_1)_{\theta,q;\underline{K}}$ *if* $p \leq q$;

3. *If E_0 and E_1 are Banach spaces, then so is $(E_0, E_1)_{\theta,p;\underline{K}}$;*

4. *If $p < \infty$, then $E_0 \cap E_1$ is dense in $(E_0, E_1)_{\theta,p;\underline{K}}$.*

Proof. This is ([B-L], Theorem 3.4.1) and ([B-L], Theorem 3.4.2). □

Theorem 3.3.2. *Suppose that $E = (E_0, E_1)$ is a couple of compatible Banach spaces such that $E_0 \cap E_1$ is dense in both E_0 and E_1. If $1 \leq p < \infty$ and $0 < \theta < 1$, then*

$$(E_0, E_1)^*_{\theta,p;\underline{K}} = (E_0^*, E_1^*)_{\theta,q;\underline{K}} \quad \textit{(with equivalent norms)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. This is ([B-L], Theorem 3.7.1). □

3.3.2 Interpolation of Operator Spaces

So far, we have shown that the real (and the complex) interpolation of two compatible Banach spaces yields another Banach space. What about the interpolation of operator spaces? This was done firstly by Pisier in [Pis 1] where he defined the complex interpolation of operator spaces. However, the theory of interpolation of operator spaces was studied systematically firstly by Xu in [Xu] where he defined the interpolation of operator spaces by the real method.

Suppose that (E_0, E_1) is a compatible couple of operator spaces. That is, E_0 and E_1 are operator spaces, and (E_0, E_1) is compatible as a couple of Banach spaces. Note that for each $n \in \mathbb{N}$, we have the continuous inclusions

$$M_n(E_0) \hookrightarrow M_n(\mathcal{X}) \quad \text{and} \quad M_n(E_1) \hookrightarrow M_n(\mathcal{X})$$

where $M_n(\mathcal{X})$ is identified with $\mathcal{X}^{n^2}$. Thus, $\big(M_n(E_0), M_n(E_1)\big)$ is a compatible couple of Banach spaces.

Clearly $E_0 \oplus_\infty E_1$ is an operator space by setting

$$M_n(E_0 \oplus_\infty E_1) = M_n(E_0) \oplus_\infty M_n(E_1).$$

Then $E_0 \oplus_1 E_1$ becomes an operator space with the embedding

$$E_0 \oplus_1 E_1 \hookrightarrow (E_0^* \oplus_\infty E_1^*)^*.$$

Now for each $1 < p < \infty$, $E_0 \oplus_p E_1$ becomes an operator space via

$$M_n(E_0 \oplus_p E_1) = \big(M_n(E_0 \oplus_1 E_1), M_n(E_0 \oplus_\infty E_1)\big)_\theta, \qquad 1/p = 1 - \theta.$$

The operator space structure on the L_p-spaces will be investigated in the proceeding sections.

3.3.3 Complex Interpolation of Operator Spaces

Let $0 < \theta < 1$ and E_0, E_1 and let (E_0, E_1) be a compatible couple of operator spaces. Then as we mentioned above, for each $n \in \mathbb{N}$, the couple $\big(M_n(E_0), M_n(E_1)\big)$ is also compatible. Now define

$$M_n(E_\theta) := (M_n(E_0), M_n(E_1))_\theta \tag{3.3.1}$$

in the sense of complex interpolation ([B-L]). By this definition, $E_\theta = (E_0, E_1)_\theta$ becomes an operator space. This is called the *complex interpolation of operator spaces* E_0 and E_1 (see [Pis 1] and [Pis 2] for more information).

3.3.4 Operator Space Structure on L_p-Spaces

Let (Ω, μ) be a measure space. As a commutative C^*-algebra $L^\infty(\mu)$ has an operator space structure. Then $L^1(\mu)$ is an operator space with the isometric inclusion $L^1(\mu) \hookrightarrow L^\infty(\mu)^*$. For $1 < p < \infty$, we define

$$L^p(\mu) = (L^1(\mu), L^\infty(\mu))_\theta \quad \text{where } \frac{1}{p} = 1 - \theta.$$

Then $L^p(\mu)$ becomes an operator space via

$$M_n(L^p(\mu)) = \big(M_n(L^1(\mu)), M_n(L^\infty(\mu))\big)_\theta \quad \text{where } \frac{1}{p} = 1 - \theta.$$

3.3.5 Vector Valued L_p-Spaces

Let (Ω, μ) be a measure space and $E \subset B(\mathcal{H})$ be an operator space, for some Hilbert space $\mathcal{H}$. Then we define $L^p(E; \mu)$ as the usual L^p-space of strongly functionals on (Ω, μ) with values in E. Then $L^\infty(E; \mu)$ has an operator space structure with $L^\infty(E; \mu) \hookrightarrow L^\infty(B(\mathcal{H}); \mu)$, since $L^\infty(B(\mathcal{H}); \mu)$ is a C^*-algebra. Again $L^1(E; \mu)$ is an operator space via $L^1(E; \mu) \hookrightarrow (L^\infty(E^*; \mu))^*$. For $1 < p < \infty$, we define

$$L^p(E; \mu) = \big(L^1(E; \mu), L^\infty(E; \mu)\big)_\theta.$$

Remark 3.3.1. Suppose that (E_0, E_1) is a compatible couple of operator spaces, and $1 \leq p_0, p_1 \leq \infty$ (at least one of them is finite). Then

$$(L^{p_0}(E; \mu), L^{p_1}(E; \mu))_\theta = L^p((E_0, E_1)_\theta; \mu), \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

with completely equal norms ([Pis 2]).

3.3.6 Vector Valued L_p-Spaces in the Discrete Case

Let (Ω, μ) be a the set of integers with counting measure and $\{E_k\}_{k=1}^{k=\infty}$ be a sequence of operator spaces. Let

$$l_p(\{E_k\}_k, \mu) = \big\{(x_k)_k : \ x_k \in E_k, \ (\|x_k\|)_k \in l_p(\mu)\big\}.$$

Then $l_p(\{E_k\}_k, \mu)$ is an operator space as before. Note that

$$\big(l_p(\{E_k\}_k, \mu)\big)^* = l_q(\{E_k^*\}_k, \mu), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

3.3.7 Real Interpolation of Operator Spaces

The construction of the interpolation of operator spaces by the real method is more complicated than by the complex method. This is because definition (3.3.1) does not work for the real interpolation $(E_0, E_1)_{\theta,p}$ if $p < \infty$. Now we introduce real interpolation of operator spaces by the discrete K-method as defined by Xu in [Xu]. This construction, as we shall see, heavily uses the complex interpolation of operator spaces.

Note that if E is an operator space and $t > 0$, then tE denotes the operator space obtained by multiplying the norm on each matrix level by t. Now let μ denote a weighted counting measure on $\mathbb{Z}$ (that is, let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of non-negative reals and for $E \subseteq \mathbb{Z}$, we define $\mu(E) := \sum_{n\in E} a_n$) and $\{E_k\}_{k\in\mathbb{Z}}$ a sequence of operator spaces. Then for $1 \leq p \leq \infty$, we define

$$l_p(\{E_k\}_{k\in\mathbb{Z}};\mu) := \big\{(x_k)_{k\in\mathbb{Z}} : \quad x_k \in E_k \text{ and } (\|x_k\|)_{k\in\mathbb{Z}} \in l_p(\mu)\big\}.$$

Clearly $l_\infty(\{E_k\}_{k\in\mathbb{Z}};\mu)$ is an operator space with its natural operator space structure. Then $l_1(\{E_k\}_{k\in\mathbb{Z}};\mu)$ becomes an operator space when it is considered as a subspace of $(l_\infty(\{E_k^*\}_{k\in\mathbb{Z}};\mu))^*$. Finally $l_p(\{E_k\}_{k\in\mathbb{Z}};\mu)$ becomes an operator space by complex interpolation:

$$l_p(\{E_k\}_{k\in\mathbb{Z}};\mu) = (l_1(\{E_k\}_{k\in\mathbb{Z}};\mu), l_\infty(\{E_k\}_{k\in\mathbb{Z}};\mu))_\theta, \quad \frac{1}{p} = 1-\theta.$$

Now for a compatible couple of operator spaces (E_0, E_1) and $1 \leq p \leq \infty$ we define $N_p(E_0, E_1) := \{(x, -x) : \ x \in E_0 \cap E_1\}$ regarded as a subspace of $E_0 \oplus_p E_1$. Then we define

$$E_0 +_p E_1 := (E_0 \oplus_p E_1)/N_p(E_0, E_1).$$

$K_p(t; E_0, E_1)$ denotes the operator space $E_0 +_p tE_1$; for any $x \in E_0 + E_1$, we let $K_p(x, t; E_0, E_1) := \|x\|_{E_0 +_p tE_1}$. Now we may give the definition of $E_{\theta,p;\underline{K}}$, *the real interpolation of the compatible couple* (E_0, E_1) *with the discrete* K*-method*, as follows:

$$\begin{aligned} E_{\theta,p;\underline{K}} &:= (E_0, E_1)_{\theta,p;\underline{K}} \\ &= \Big\{x \in E_0 + E_1 : \quad \|x\|_{\theta,p;\underline{K}} := \Big[\sum_{k\in\mathbb{Z}} (2^{-k\theta} K_p(x, 2^k; E_0, E_1))^p\Big]^{1/p} < \infty\Big\}. \end{aligned}$$

Then $E_{\theta,p;\underline{K}}$ is a Banach space ([B-L]).

If $\alpha \in \mathbb{R}$, then $l_p(2^{k\alpha})$ is the weighted space

$$l_p(2^{k\alpha}) := \Big\{x = (x_k)_{k\in\mathbb{Z}} : \quad \|x\|_{l_p(2^{k\alpha})} = \Big(\sum_{k\in\mathbb{Z}} |2^{k\alpha} x_k|^p\Big)^{1/p} < \infty\Big\}.$$

If E is an operator space, then we similarly define $l_p(E)$ and $l_p(E; 2^{k\alpha})$ of sequences with values in E. Then $l_p(E)$ and $l_p(E; 2^{k\alpha})$ are operator spaces.

For each $k \in \mathbb{Z}$, let $F_k := K_p(2^k; E_0, E_1)$. Then we define $E_{\theta,p;\underline{K}}$, *the operator space interpolation of the couple* (E_0, E_1) *by the discrete* K*-method,* as the subspace of $l_p(\{F_k\}_{k\in\mathbb{Z}}; 2^{-k\theta})$ consisting of the constant sequences.

More explicitly, let $x = (x_{i,j}) \in M_n\big((E_0, E_1)_{\theta,p;\underline{K}}\big)$ for some $n \in \mathbb{N}$. Then

$$\|x\|_{M_n((E_0,E_1)_{\theta,p;\underline{K}})} := \inf\Big\{\big\|(u,v)\big\|_{M_n\left(l_p(E_0;2^{-k\theta})\oplus_p l_p(E_1;2^{k(1-\theta)})\right)}\Big\}.$$

The infimum is taken over all

$$u = (u_{i,j}) \in M_n\big(l_p(E_0; 2^{-k\theta})\big),\ v = (v_{i,j}) \in M_n\big(l_p(E_1; 2^{k(1-\theta)})\big)$$

where each

$$u_{i,j} = \big(u_{i,j}^k\big)_{k\in\mathbb{Z}} \text{ and } v_{i,j} = \big(v_{i,j}^k\big)_{k\in\mathbb{Z}}$$

such that

$$x_{i,j} = u_{i,j}^k + v_{i,j}^k, \quad \text{for each} \quad k \in \mathbb{Z}, \quad i,j = 1, \dots n.$$

3.4 A Representation Theorem for Completely Contractive Dual Banach Algebras

Throughout this section, we will use the following notations:

1. Let $(E_\alpha)_{\alpha \in I}$ be a family of operator spaces where I is some index set. Then l^2-$\bigoplus_\alpha E_\alpha$ and $l^2(I, E_\alpha)$ will denote the l^2- direct sum of E_α's and the complex operator space interpolation $(l^\infty(I, E_\alpha), l^1(I, E_\alpha))_{1/2}$ respectively.

2. Let $\mathfrak{A}$ be a completely contractive Banach algebra and X be an operator (bi- or) left $\mathfrak{A}$-module. For $a = (a_{i,j})$ and $x = (x_{i,j})$ in $M_n(\mathfrak{A})$ and $M_m(X)$ respectively, for some $n, m \in \mathbb{N}$, $a \star x$ will represent the matrix

$$a \star x = (a_{i,j}.x_{k,l}) \tag{3.4.1}$$

where "." represents the module action of $\mathfrak{A}$ on X.

Definition 3.4.1. Let $\mathfrak{A}$ be a completely contractive dual Banach algebra and let $\phi \in M_n(\mathfrak{A}_*)$ for some $n \geq 1$. Suppose that for each $m \geq 1$, there is a matricial norm $\|\cdot\|_{\phi,m}$ on $M_m(\mathfrak{A}.\phi)$. Let E_ϕ denote the completion of $(\mathfrak{A}.\phi, \|\cdot\|_{\phi,1})$. Suppose that

$$\|a \star b\|_{\phi,mk} \;\leq\; \|a\|_m \|b\|_{\phi,k} \tag{3.4.2}$$

and

$$\|a \star \phi\|_{mn} \;\leq\; \|a \star \phi\|_{\phi,m} \leq \|a\|_m \|\phi\|_n \tag{3.4.3}$$

for all $a \in M_m(\mathfrak{A})$ and $b \in M_k(E_\phi), m, k \in \mathbb{N}$.

Furthermore, suppose that E_ϕ is reflexive and the inclusion $\iota_\phi : E_\phi \to M_n(\mathfrak{A}_*)$ injective. Then $\left(\|\cdot\|_{\phi,m}\right)_{m=1}^\infty$ is called an *admissible operator norm* for ϕ.

Note that in the previous definition, the inequality (3.4.2) means that E_ϕ is an operator left $\mathfrak{A}$-module.

Example 3.4.2. Let G_d be a locally compact discrete group and $\phi = (\phi_{i,j})$ be in $M_n(C^*(G_d))$ with $\|\phi\|_n = 1$ for some $n \geq 1$ where $C^*(G_d)$ is the group C^*-algebra of G_d. Then each $\phi_{i,j}$ is a finite sum of the form

$$\phi_{i,j} = \sum_{g \in G_d} \lambda_g^{i,j} \delta_g$$

where each $\lambda_g^{i,j} \in \mathbb{C}$ and δ_g is the Dirac function. Consider E_ϕ with the usual norm on $M_n(C^*(G_d))$. Clearly E_ϕ is a closed subspace of $M_n(C^*(G_d))$. Hence, it is an operator space. E_ϕ is reflexive since it is finite dimensional. Since E_ϕ is an operator $\mathfrak{A}$-module, (3.4.2) is satisfied. Since $B(G_d)$ is a completely contractive Banach algebra, (3.4.3) is also satisfied. Therefore, the (usual) norm on $M_n(C^*(G_d))$ defines an admissible operator norm for ϕ.

Theorem 3.4.3. *Let $\mathfrak{A}$ be a completely contractive dual Banach algebra and suppose that $\phi = (\phi_{i,j}) \in M_n(\mathfrak{A}_*)$ has an admissible operator norm for some $n \geq 1$. Then there is a w^*-continuous, completely contractive representation of $\mathfrak{A}$ on $\mathcal{CB}(E_\phi)$.*

Proof. It is easy to see that E_ϕ is a left $\mathfrak{A}$-module. Moreover, ι_ϕ^* has a dense range if and only if $\iota_\phi^{**} : E_\phi^{**} \to M_n(\mathfrak{A}^*)$ is injective. Since E_ϕ is reflexive, $\iota_\phi^{**} = \iota_\phi$. Hence, ι_ϕ^* has a dense range. Note that

$$\iota_\phi^* : T_n(\mathfrak{A}) \to E_\phi^*.$$

Now we define

$$S_\phi : E_\phi^* \hat{\otimes} E_\phi \longrightarrow M_{n^2}(\mathfrak{A}_*), \quad \iota_\phi^*(b) \otimes a.\phi \mapsto a.\phi \star b$$

where $\hat{\otimes}$ represents the projective tensor product of operator spaces.

Due to Definition 3.4.1, this map is well-defined. Then the map defined by

$$T_\phi := S_\phi^* : \; T_{n^2}(\mathfrak{A}) \rightarrow \mathcal{CB}(E_\phi)$$

is w^*-continuous. Since $\mathfrak{A}$ is completely isometrically isomorphic to a closed subspace of $T_{n^2}(\mathfrak{A})$ by the map

$$\mathfrak{A} \rightarrow T_{n^2}(\mathfrak{A}), \quad a \mapsto (a_{i,j}) \tag{3.4.4}$$

where

$$a_{i,j} = \begin{cases} a, & \text{if } \; (i,j) = (1,1); \\ 0, & \text{otherwise.} \end{cases}$$

T_ϕ induces a multiplicative representation from $\mathfrak{A}$ into $CB(E_\phi)$. For simplicity, we will denote this representation again by T_ϕ. In order to see that T_ϕ is multiplicative on $\mathfrak{A}$, take $a, b \in \mathfrak{A}$, $c = (c_{i,j}) \in T_n(\mathfrak{A})$ and consider a as an element of $T_{n^2}(\mathfrak{A})$ via (3.4.4). Then

$$\begin{aligned} \big\langle \langle T_\phi(a), b.\phi \rangle, \iota_\phi^*(c) \big\rangle &= \langle T_\phi(a), \iota_\phi^*(c) \otimes b.\phi \rangle \\ &= \big\langle a, S_\phi(\iota_\phi^*(c) \otimes b.\phi) \big\rangle \\ &= \langle a, b.\phi \star c \rangle \\ &= \langle a, b.\phi_{1,1}.c_{1,1} \rangle. \end{aligned}$$

Hence $\langle T_\phi(a), b.\phi\rangle = (x_{i,j}) \in M_n(\mathfrak{A}_*)$ holds where

$$x_{i,j} = \begin{cases} ab.\phi_{1,1}, & \text{if } (i,j) = (1,1); \\ 0, & \text{otherwise.} \end{cases}$$

Now let $a, b, d \in \mathfrak{A}$. Then

$$\begin{aligned} \Big\langle T_\phi(d).T_\phi(a), b.\phi \Big\rangle &= \Big\langle T_\phi(d), \langle T_\phi(a), b.\phi\rangle \Big\rangle \\ &= \Big\langle T_\phi(d), (x_{i,j}) \Big\rangle \\ &= (y_{i,j}) \\ &= \Big\langle T_\phi(da), b.\phi \Big\rangle \in E_\phi \end{aligned}$$

where

$$y_{i,j} = \begin{cases} dab.\phi_{1,1}, & \text{if } (i,j) = (1,1); \\ 0, & \text{otherwise.} \end{cases}$$

Hence, T_ϕ is multiplicative on $\mathfrak{A}$.

By using Effros and Ruan ([E-R], Proposition 7.1.2), S_ϕ is a complete contraction if and only if the induced map $\widetilde{S}_\phi \in \mathcal{B}^2(E_\phi^* \times E_\phi, M_{n^2}(\mathfrak{A}_*))$ is a complete contraction. Now

$$\begin{aligned} \|\widetilde{S}_\phi\|_{cb} &= \sup\Big\{ \big\|\widetilde{S}_\phi^{(m,m)}(x,y)\big\| : x = (x_{i,j}) \in M_m(E_\phi^*),\ y = (y_{i,j}) \in M_m(E_\phi),\ \|x\|_{\phi,m}, \\ &\qquad \|y\|_{\phi,m} \le 1,\ m \in \mathbb{N} \Big\} \\ &= \sup\Big\{ \big|\langle\langle \widetilde{S}_\phi^{(m,m)}(x,y), z\rangle\rangle\big| : x = (x_{i,j}) \in M_m(E_\phi^*),\ y = (y_{i,j}) \in M_m(E_\phi), \\ &\qquad z = (z_{i,j}) \in M_{m^2}(M_{n^2}(\mathfrak{A})),\ \|x\|_{\phi,m},\ \|y\|_{\phi,m},\ \|z\|_{m^2n^2} \le 1,\ m \in \mathbb{N} \Big\}. \end{aligned}$$

By the density of the range of ι_ϕ^*, for each $i, j = 1, \dots m$, without loss of generality, we may suppose that

$$x_{i,j} = \iota_\phi^*(A_{i,j}), \quad y_{i,j} = b_{i,j}.\phi$$

where

$$A_{i,j} = (a_{i,j}^{k,l}) \in T_n(\mathfrak{A}) \text{ and } b_{i,j} \in \mathfrak{A}.$$

Then we have

$$\left\langle\left\langle \widetilde{S}_\phi^{(m,m)}(x,y), z \right\rangle\right\rangle = \left(\left\langle \widetilde{S}_\phi(x_{i,j}, y_{k,l}), z_{s,t} \right\rangle\right) = \left(\left\langle b_{k,l}.\phi \star A_{i,j}, z_{s,t} \right\rangle\right).$$

Since

$$\left\langle b_{k,l}.\phi_{o,p}.a_{i,j}^{q,r}, z_{s,t} \right\rangle = \left\langle \phi_{o,p}.a_{i,j}^{q,r}, z_{s,t} b_{k,l} \right\rangle = \left\langle a_{i,j}^{q,r}, z_{s,t} b_{k,l}.\phi_{o,p} \right\rangle$$

for all indices $i, j, k, l, m, n, o, p, q$ and r where

$$i, j, k, l = 1, \dots m, \quad o, p, q, r = 1, \dots n, \quad \text{and} \quad s, t = 1, \dots m^2 n^2,$$

we conclude that

$$\left|\left\langle\left\langle \widetilde{S}_\phi^{(m,m)}(x,y), z \right\rangle\right\rangle\right| = \left|\left\langle \iota_\phi^*(A_{i,j}), z_{m,n} b_{k,l}.\phi \right\rangle\right| = \left|\left\langle\left\langle (\iota_\phi^*(A_{i,j})), z \star y \right\rangle\right\rangle\right|.$$

On the other hand,

$$\|z \star y\|_{\phi, m^3 n^2} \leq \|z\|_{m^2 n^2} \|y\|_{\phi, m} \leq 1$$

holds. Therefore,

$$\left|\left\langle\left\langle \widetilde{S}^{m,m}(x,y), z \right\rangle\right\rangle\right| \leq \left\| \left(\iota_\phi^*(A_{i,j}) \right) \right\|_{\phi, m} = \|x\|_{\phi, m} \leq 1$$

holds. Thus, $\widetilde{S}_\phi$ is a complete contraction. $\square$

Let $(E_\alpha)_{\alpha \in I}$ be a family of operator spaces and let $E = l_2(I, E_\alpha)$. We will need an approximation for the norm of an arbitrary element on each matrix level of E. To manage this, we will need the following two propositions.

Proposition 3.4.1. *Suppose that $(E_\alpha)_{\alpha \in I}$ is a family of operator spaces. Then*

1. $l^\infty(I, M_n(E_\alpha)) \cong M_n(l^\infty(I, E_\alpha))$ *holds for each $n \in \mathbb{N}$.*

2. *If $A = (a_{i,j}) \in M_n(l^1(I, E_\alpha))$ where $a_{i,j} = (a^\alpha_{i,j})_\alpha$, $a^\alpha_{i,j} \in E_\alpha$ for each i and j, then*

$$\|A\|_{M_n(l^1(I,E_\alpha))} \leq \sum_\alpha \|(a^\alpha_{i,j})\|_{M_n(E_\alpha)} \quad \textit{for every } n \in \mathbb{N}.$$

Proof. The first identity is obvious. Hence, we will prove only the second one. Let A be as in the claim. Then we have

$$\begin{aligned}
\|A\|_{M_n(l^1(I,E_\alpha))} &= \sup\Big\{|\langle\langle A, F\rangle\rangle| : F \in M_n\big(l^\infty(I, E^*_\alpha)\big), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leq 1\Big\} \\
&= \sup\Big\{|(\langle a_{i,j}, f_{k,l}\rangle)| : F \in M_n\big(l^\infty(I, E^*_\alpha)\big), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leq 1\Big\} \\
&= \sup\Big\{|(\sum_\alpha \langle f^\alpha_{k,l}, a^\alpha_{i,j}\rangle)| : F \in M_n\big(l^\infty(I, E^*_\alpha)\big), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leq 1\Big\} \\
&\leq \sup\Big\{\sum_\alpha |(\langle f^\alpha_{k,l}, a^\alpha_{i,j}\rangle)| : F \in M_n(l^\infty(I, E^*_\alpha)), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leq 1\Big\} \\
&\leq \sum_\alpha \|(a^\alpha_{i,j})\|_{M_n(E_\alpha)} < \infty
\end{aligned}$$

since $M_n\big(l^1(I, E_\alpha)\big)$ and $l^1\big(I, M_n(E_\alpha)\big)$ are topologically isomorphic. □

The next proposition will be needed to prove Lemma 3.4.1 and Theorem 3.4.4.

Proposition 3.4.2. *Let (X, Z) be a compatible couple of Banach spaces in the sense of Banach space interpolation. Suppose that there is a contractive embedding from a Banach space Y into Z. Let $E_1 := (X, Y)_\theta$ and $E_2 := (X, Z)_\theta$ for some $0 < \theta < 1$. Then for every $a \in E_1$, we have $\|a\|_{E_1} \geq \|a\|_{E_2}$.*

Proof. Clearly any function $f : \mathbb{C} \to X +_1 Y$ satisfying the properties given in the definition of complex interpolation of Banach spaces can be viewed as a function from $\mathbb{C}$ to $X +_1 Z$, and it will satisfy analogous properties. To distinguish these two functions, we will denote them by $f_{X,Y}$ and $f_{X,Z}$ respectively.

Recall that

$$\|f_{X,Y}\| = \max\Big\{ \sup\{\|f(it)\|_X\},\ \sup\{\|f(1+it)\|_Y\} : t \in \mathbb{R}\Big\}$$

and

$$\|f_{X,Z}\| = \max\Big\{ \sup\{\|f(it)\|_X\},\ \sup\{\|f(1+it)\|_Z\} : t \in \mathbb{R}\Big\}.$$

This shows that $E_1 \subseteq E_2$. Now let $a \in E_1$. Recall that

$$\|a\|_{E_1} = \inf\Big\{\|f_{X,Y}\| : a = f(\theta)\Big\} \text{ and } \|a\|_{E_2} = \inf\Big\{\|f_{X,Z}\| : a = f(\theta)\Big\}.$$

Since for each f we have $\|f_{X,Y}\| \geq \|f_{X,Z}\|$, we conclude that $\|a\|_{E_1} \geq \|a\|_{E_2}$. □

Proposition 3.4.1 and Proposition 3.4.2 show the following:

Lemma 3.4.1. *Let $(E_\alpha)_{\alpha \in I}$ be a family of operator spaces and let $A = (a_{i,j})$ be in $M_n(E)$ where $E = l_2(I, E_\alpha)$ for some $n \in \mathbb{N}$. If each $a_{i,j} = (a^\alpha_{i,j})_{\alpha \in I} \in E$,* *then* $\|A\|_{M_n(E)} \leq \sqrt{\sum_\alpha \|A_\alpha\|^2_{M_n(E_\alpha)}}$ *where each* $A_\alpha = (a^\alpha_{i,j}) \in M_n(E_\alpha)$. *That is, the canonical inclusion from* $l_2(I, M_n(E_\alpha))$ *into* $M_n(l_2(I, E_\alpha))$ *is a contraction.*

Proof. Note that by using Proposition 3.4.1 for each $n \in \mathbb{N}$, we have

$$M_n\big(l^\infty(I, E_\alpha), l^1(I, E_\alpha)\big) = \big(l^\infty(I, M_n(E_\alpha)), M_n(l^1(I, E_\alpha))\big).$$

Then we apply Proposition 3.4.2 where $X = l^\infty(I, M_n(E_\alpha)), Y = l^1(I, M_n(E_\alpha))$ and $Z = M_n(l^1(I, E_\alpha))$. ☐

Lemma 3.4.2. *Let* $\mathfrak{A}$ *be a completely contractive dual Banach algebra. Then for each* $n \in \mathbb{N}$, *every non-zero element in the unit ball of* $M_n(\mathfrak{A}_*)$ *has an admissible operator norm.*

Proof. Let $\mathfrak{A}$ be a completely contractive dual Banach algebra and $\phi \in M_n(\mathfrak{A}_*)$, $\phi \neq 0$, $\|\phi\|_n \leq 1$ for some $n \geq 1$. The map

$$R_\phi : \mathfrak{A} \to \mathfrak{A}.\phi, \quad a \mapsto a.\phi$$

is a complete contraction. Then the induced map $\pi : \mathfrak{A}/\ker R_\phi \to \mathfrak{A}.\phi$ is a complete isometry. For each $m \geq 1$, define a norm $\|\cdot\|_{\mathfrak{A}.\phi,m}$ on $M_m(\mathfrak{A}.\phi)$ by

$$\|x\|_{\mathfrak{A}.\phi,m} := \inf\big\{\|a\|_m : \ x = a \star \phi, \ a \in M_m(\mathfrak{A})\big\}.$$

Then $\mathfrak{A}.\phi$ becomes an operator space with this matricial norm.

Clearly $(\mathfrak{A}.\phi, M_n(\mathfrak{A}_*))$ is a compatible couple of operator spaces. Now define E_ϕ to be the space of constant sequences in $l_2(\{F_k\}_{k\in\mathbb{N}}; 2^{-k/2})$ where $F_k = K_2(2^k; \mathfrak{A}.\phi, M_n(\mathfrak{A}_*))$

for each $k \in \mathbb{N}$. By ([Bea], Section 2.3, Proposition 1) we know the following:

$$
\begin{aligned}
(\mathfrak{A}.\phi, M_n(\mathfrak{A}_*))_{\frac{1}{2},2;\underline{K}} \text{ is reflexive} &\iff \text{the inclusion } \mathfrak{A}.\phi \to M_n(\mathfrak{A}_*) \text{ is weakly compact} \\
&\iff \text{the map } R_\phi : \mathfrak{A} \to M_n(\mathfrak{A}^*), \quad a \mapsto a.\phi \text{ is weakly compact.}
\end{aligned}
$$

However, $\mathrm{Im}(R_\phi) \subseteq M_n(\mathfrak{A}_*)$ and $M_n(\mathfrak{A}_*) \subseteq WAP(M_n(\mathfrak{A}^*))$, by [Run 4]. Hence, R_ϕ is weakly compact. Therefore, as a closed subspace of $E = (\mathfrak{A}.\phi, M_n(\mathfrak{A}_*))_{\frac{1}{2},2;\underline{K}}$, the space E_ϕ is reflexive too.

Let $\|\cdot\|_{\phi,m}$ denote the norm on $M_m(E_\phi)$ for every $m \in \mathbb{N}$. If $f \in E_\phi$, then

$$
\begin{aligned}
\|f\|_{\phi,1} &= \Big[\sum_{k\in\mathbb{N}} 2^{-k}\|f\|_{F_k}^2\Big]^{1/2} \\
&= \Big[\sum_{k\in\mathbb{N}} 2^{-k} \inf_{b\in\mathfrak{A}} \big\{\sqrt{\|b.\phi\|_{\mathfrak{A}.\phi,1}^2 + 2^{2k}\|f - b.\phi\|_n^2}\big\}^2\Big]^{1/2} \\
&= \Big[\sum_{k\in\mathbb{N}} 2^{-k} \inf_{b\in\mathfrak{A}} \big\{\|b.\phi\|_{\mathfrak{A}.\phi,1}^2 + 2^{2k}\|f - b.\phi\|_n^2\big\}\Big]^{1/2} \\
&= \Big[\sum_{k\in\mathbb{N}} \inf_{b\in\mathfrak{A}} \big\{2^{-k}\|b.\phi\|_{\mathfrak{A}.\phi,1}^2 + 2^{k}\|f - b.\phi\|_n^2\big\}\Big]^{1/2}.
\end{aligned}
$$

Hence,

$$
f \in E_\phi \iff \sum_{k\in\mathbb{N}} \inf\big\{2^{-k}\|b.\phi\|_{\mathfrak{A}.\phi,1}^2 + 2^{k}\|f - b.\phi\|_n^2 : \ b \in \mathfrak{A}\big\} < \infty.
$$

Thus, there exists a sequence $(b_k)_k$ in $\mathfrak{A}$ such that $2^{2k}\|f - b_k.\phi\|_n^2 \to 0$. Hence, $\mathfrak{A}.\phi$ is dense in E_ϕ. This shows that $\|a.\phi\|_n \leq \|a.\phi\|_{\phi,1}$. Now to prove (3.4.3), we will use the following claim:

Claim 3.4.1. *If $\mathfrak{A}.\phi$ is dense in E_ϕ, then $M_m(\mathfrak{A}.\phi)$ is dense in $M_m(E_\phi)$ for every $m \geq 1$.*

Proof. Let $\epsilon > 0$ and $F = (f_{i,j}) \in M_m(E_\phi)$ for some $m \geq 1$. Then for each $i, j = 1, \dots m$, there exists a sequence $(b^k_{i,j})_k$ in $\mathfrak{A}$ such that $b^k_{i,j}.\phi \to f^k_{i,j}$ in $\|\cdot\|_n$. Consider the sequence $(F_k)_k$ in $M_m(\mathfrak{A}.\phi)$ where each $F_k = (b^k_{i,j})_k$. Then we have

$$\|F - f_k\|_{mn} \leq \sum_{i,j=1}^{m} \|f_{i,j} - b^k_{i,j}.\phi\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

$\square$

Hence, by Claim 3.4.1, we have $\|a \star \phi\|_{mn} \leq \|a \star \phi\|_{\phi,m}$ for all $a \in M_m(\mathfrak{A})$ and $m \in \mathbb{N}$.

Let $a \in M_m(\mathfrak{A})$ for some $m \geq 1$. Then by the definition of $\|\cdot\|_{\mathfrak{A}.\phi,m}$, it is clear that $\|a \star \phi\|_{\mathfrak{A}.\phi,m} \leq \|a\|_m$. Since $\mathfrak{A}$ is a completely contractive dual Banach algebra, we also have $\|a \star \phi\|_{mn} \leq \|a\|_m \|\phi\|_n \leq \|a\|_m$. By Lemma 3.4.1, we have

$$\|a \star \phi\|_{F_k} \leq \inf\left\{\left[\|b \star \phi\|^2_{\mathfrak{A}.\phi,m} + 2^{2k}\|a \star \phi - b \star \phi\|^2_{mn}\right]^{1/2} : b \in M_m(\mathfrak{A})\right\}.$$

By choosing $b = a$, we see that

$$\|a \star \phi\|_{F_k} \leq \|a \star \phi\|_{\mathfrak{A}.\phi,m} \quad \text{for each } k \in \mathbb{N}.$$

Then we have

$$\begin{aligned}
\|a \star \phi\|_{\phi,m} &\leq \left[\sum_{k\in\mathbb{N}} 2^{-k}\|a \star \phi\|^2_{F_k}\right]^{1/2} \\
&\leq \|a \star \phi\|_{\mathfrak{A}.\phi,m}\left[\sum_{k\in\mathbb{N}} 2^{-k}\right]^{1/2} \\
&= \|a \star \phi\|_{\mathfrak{A}.\phi,m} \\
&\leq \|a\|_m.
\end{aligned}$$

Now let $a \in M_m(\mathfrak{A}), b \in M_t(\mathfrak{A})$ for some $m, t \geq 1$. Since the map π is a complete isometry, we have

$$\begin{aligned}
\left\| b \star (a \star \phi) \right\|_{\mathfrak{A}.\phi, mt} &= \left\| b \star a + M_{mt}(\ker R_\phi) \right\| \\
&= \inf \left\{ \|b \star a + x\|_{mt} : \ x \in M_{mt}(\ker R_\phi) \right\} \\
&\leq \inf \left\{ \|b \star a + b \star x\|_{mt} : \ x \in M_m(\ker R_\phi) \right\} \\
&\leq \|b\|_t \inf \left\{ \|a + x\|_m : \ x \in M_m(\ker R_\phi) \right\} \\
&= \|b\|_t \|a + \ker R_\phi\| \\
&= \|b\|_t \|a \star \phi\|_{\mathfrak{A}.\phi, m}.
\end{aligned}$$

This shows that $\mathfrak{A}.\phi$ is an operator left $\mathfrak{A}$-module. Since $M_n(\mathfrak{A}_*)$ is also an operator left $\mathfrak{A}$-module, so is E_ϕ. Therefore,

$$\|b \star d\|_{\phi, mt} \leq \|b\|_t \|d\|_{\phi, m} \quad \text{for every} \ \ d \in M_m(E_\phi), \ b \in M_m(\mathfrak{A}), \ m, t \in \mathbb{N}$$

holds. □

Theorem 3.4.4. *Let $\mathfrak{A}$ be a completely contractive dual Banach algebra. Then there is a w^*-continuous complete isometry map from $\mathfrak{A}$ into $\mathcal{CB}(E)$ for some reflexive operator space E.*

Proof. Let $\dot{E} := l^2\text{-}\bigoplus_{\phi \in \mathfrak{J}} E_\phi$. For each $n \geq 1$, we equip $M_n(\dot{E})$ by the norm $\|A\|_{M_n(\dot{E})} = \sqrt{\sum_\phi \|(a^\phi_{i,j})\|^2_{\phi,n}}$ where $A = (a_{i,j}) \in M_n(\dot{E})$, $a_{i,j} = (a^\phi_{i,j})_\phi$, $a^\phi_{i,j} \in E_\phi$. Note that $\dot{E}$ is *not* an operator space. There is a natural map

$$S : \mathfrak{A} \to \mathcal{B}(\dot{E}) \quad \text{defined by} \quad \Big\langle S(a), (x_\phi) \Big\rangle := \Big(\langle T_\phi(a), x_\phi \rangle \Big)$$

where $T_\phi : \mathfrak{A} \to \mathcal{CB}(E_\phi)$ is the w^*-continuous complete contraction as defined in Theorem 3.4.3.

Note that Daws ([Daw], Theorem 4.5) proved that $S : \mathfrak{A} \to \mathcal{B}(\dot{E})$ is an isometry.

For an arbitrary $n \geq 1$, any element $(a_{i,j})$ of $M_n(\mathfrak{A})$ can be viewed as a map

$$S_n : \dot{E} \to M_n(\dot{E}).$$

We claim that S_n is a contraction. Let $\|(a_{i,j})\|_n \leq 1$ and $(x_\phi) \in \dot{E}$. Then

$$\begin{aligned}
\|S_n(x_\phi)\|_{M_n(\dot{E})} &= \Big\|\big(\langle S(a_{i,j}), (x_\phi)\rangle\big)\Big\|_{M_n(\dot{E})} \\
&= \Big\|\big((\langle T_\phi(a_{i,j}), x_\phi\rangle)\big)\Big\|_{M_n(\dot{E})} \\
&= \Big[\sum_\phi \|((\langle T_\phi(a_{i,j}), x_\phi\rangle))\|_{\phi,n}^2\Big]^{1/2} \\
&\leq \Big[\sum_\phi \|(a_{i,j})\|_n^2 \|x_\phi\|_{\phi,1}^2\Big]^{1/2} \\
&= \big\|(a_{i,j})\big\|_n \Big[\sum_\phi \|x_\phi\|_{\phi,1}^2\Big]^{1/2} \\
&= \big\|(a_{i,j})\big\|_n \big\|(x_\phi)\big\|_{\dot{E}} \\
&\leq \big\|(x_\phi)\big\|_{\dot{E}}.
\end{aligned}$$

Thus, S_n is a contraction.

Now let $E = l^2(\mathfrak{I}, E_\phi)$. We define

$$T : \mathfrak{A} \to \mathcal{CB}(E) \quad \text{by} \quad \big\langle T(a), (x_\phi)\big\rangle = \big(\langle T_\phi(a), x_\phi\rangle\big). \tag{3.4.5}$$

Then for each $n \in \mathbb{N}$, T induces the map $T^{(n)} : M_n(\mathfrak{A}) \to \mathcal{CB}(E, M_n(E))$. On the other hand, by its definition

$$M_n(E) = (M_n(l^\infty(\mathfrak{I}, E_\phi)), M_n(l^1(\mathfrak{I}, E_\phi)))_{1/2} = (l^\infty(\mathfrak{I}, M_n(E_\phi)), M_n(l^1(\mathfrak{I}, E_\phi)))_{1/2}.$$

Each $(a_{i,j}) \in M_n(\mathfrak{A})$ defines a map from E into

$$\Big(l^\infty(\mathfrak{I}, M_n(E_\phi)), l^1(\mathfrak{I}, M_n(E_\phi))\Big)_{1/2} = l^2\text{-}\bigoplus_{\phi\in\mathfrak{I}} M_n(E_\phi) \quad \text{(on the Banach space level)}.$$

Hence, we have a natural map (which we will denote by $\widetilde{T}^n$)

$$\widetilde{T}^n : M_n(\mathfrak{A}) \to \mathcal{B}\big(E, l^2\text{-}\bigoplus_{\phi\in\mathfrak{I}} M_n(E_\phi)\big).$$

However, this map is a contraction for every $n \geq 1$. On the other hand, by Proposition 3.4.2 we have $\|\widetilde{T}^n\| \geq \|T^{(n)}\|$. Hence, $T^{(n)}$ is a contraction for every $n \geq 1$. Thus, T is a complete contraction.

Note that without loss of generality, we may suppose that $\mathfrak{A}$ is unital and let e denote its identity. For each $n \geq 1$, we have

$$T^{(n)}: \; M_n(\mathfrak{A}) \to M_n(\mathcal{CB}(E)) = M_n((E^*\hat{\otimes}E)^*) = CB((E^*\hat{\otimes}E), M_n).$$

Let $a = (a_{i,j}) \in M_n(\mathfrak{A})$. Then for every $\epsilon > 0$, there is $\phi = (\phi_{i,j}) \in M_n(\mathfrak{A}_*)$ such that

$$\|\phi\|_n \leq 1 \;\text{ and }\; |\langle\langle a, \phi\rangle\rangle| \geq (1-\epsilon)\|a\|_n.$$

For simplicity, set $\overline{T} := T^{(n)}(a) \in \mathcal{CB}((E^*\hat{\otimes}E), M_n)$. Define $x = (x_{i,j}) \in M_n(E^*\hat{\otimes}E)$ by

$$x_{i,j} = \begin{cases} (\ldots, \iota_\phi^*(B), \ldots) \otimes (\ldots, e.\phi, \ldots), & \text{if } (i,j) = (1,1); \\ 0, & \text{otherwise,} \end{cases}$$

where

$$B = (b_{i,j}) \in T_n(\mathfrak{A}) \text{ is defined by } b_{i,j} = \begin{cases} e, & \text{if } (i,j) = (1,1); \\ 0, & \text{otherwise.} \end{cases}$$

Now we have

$$\|x\|_{M_n(E^*\hat{\otimes}E)} = \|x_1\|_{E^*\hat{\otimes}E} \leq \|(\ldots, \iota_\phi^*(B), \ldots)\|_{E^*} \|(\ldots, e.\phi, \ldots)\|_E.$$

On the other hand, $\|(\ldots, \iota_\phi^*(B), \ldots)\|_{E^*} \leq 1$ since $\|B\|_{T_n(\mathfrak{A})} \leq 1$ and ι_ϕ^* is a contraction.

Clearly

$$\|(\ldots, e.\phi, \ldots)\|_E \leq \|\phi\|_n \leq 1.$$

holds. Then we have

$$\left|\overline{T}^{(n)}(x)\right| = \left|\left(\overline{T}(x_{1,1})\right)\right| = \left|\left(\langle T(a_{i,j}), x_{k,l}\rangle\right)\right| = \left|\langle\langle a, \phi\rangle\rangle\right| \geq (1-\epsilon)\|a\|_n.$$

$\square$

Chapter 4

Connes-Amenability

4.1 Connes-Amenability and Short Exact Sequences

Let $\mathfrak{A}$ be a dual Banach algebra with predual $\mathfrak{A}_*$ and

$$\Delta_{w^*} = (\Delta^*_{|\mathfrak{A}_*})^* : \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^* \to \mathfrak{A}.$$

be the $\mathfrak{A}$-bimodule homomorphism as defined before.

If $\mathfrak{A}$ is unital, then the diagonal operator and hence Δ_{w^*}, are surjective. Since the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{A}$ lies canonically in $\mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$, we obtain the following short exact sequence of Banach $\mathfrak{A}$-bimodules:

$$\Xi : \qquad 0 \longrightarrow \ker \Delta_{w^*} \xrightarrow{\iota} \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^* \xrightarrow{\Delta_{w^*}} \mathfrak{A} \to 0$$

where ι is the inclusion map.

We will start this chapter by characterizing the strong Connes-amenability of $\mathfrak{A}$ in terms of some properties of the short exact sequence Ξ.

Theorem 4.1.1. *Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathfrak{A}$ is strongly Connes-amenable if and only if $\mathfrak{A}$ is unital and the exact sequence Ξ splits as a sequence of Banach $\mathfrak{A}$-bimodules.*

Proof. ($\Rightarrow$) Let $\mathfrak{A}$ be strongly Connes-amenable. Then it is unital by [Run 1]. Thus, the sequence Ξ is exact. Let $M \in \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$ be a normal virtual diagonal for $\mathfrak{A}$. Define

$$\sigma : \mathfrak{A} \longrightarrow \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*, \qquad a \mapsto M.a$$

where "." is the module action of $\mathfrak{A}$ on $B^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$. Since Δ_{w^*} is an $\mathfrak{A}$-bimodule homomorphism, for each $a \in \mathfrak{A}$ we have

$$\Delta_{w^*}\sigma(a) = \Delta_{w^*}(M.a) = \Delta_{w^*}M.a = e.a = a$$

where e is the identity of $\mathfrak{A}$. Hence σ is a right inverse of Δ_{w^*}. Furthermore, for each $a, b \in \mathfrak{A}$, we have

$$a.\sigma(b) = a.(M.b) = a.(b.M) = (ab)M = M(ab) = (M.a).b = \sigma(a).b = \sigma(ab).$$

Hence, σ is an $\mathfrak{A}$-bimodule homomorphism, so Ξ splits.

($\Leftarrow$) If Ξ splits, then there is an $\mathfrak{A}$-bimodule homomorphism

$$\sigma : \mathfrak{A} \longrightarrow \mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^* \quad \text{via} \quad \Delta_{w^*}\sigma = I_{\mathfrak{A}}$$

where $I_{\mathfrak{A}}$ the identity operator on $\mathfrak{A}$. If e is the identity of $\mathfrak{A}$, then we have

$$a\sigma(e) = \sigma(a.e) = \sigma(e.a) = \sigma(e).a \qquad (a \in \mathfrak{A})$$

and $\Delta_{w^*}\sigma(e) = e$. Hence, $\sigma(e)$ is a normal virtual diagonal for $\mathfrak{A}$. Then by [Run 1], $\mathfrak{A}$ is strongly Connes-amenable. $\square$

Consider again the $\mathfrak{A}$-bimodule homomorphism

$$\Delta_{\sigma WC} : \sigma WC((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^* \to \mathfrak{A}.$$

Clearly the algebraic tensor product $\mathfrak{A}\otimes\mathfrak{A}$ lies canonically in $\sigma WC((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^*$. Then when $\mathfrak{A}$ is unital, we have the following exact sequence of Banach $\mathfrak{A}$-bimodules:

$$\Xi' : \qquad 0 \to \ker \Delta_{\sigma WC} \xrightarrow{\iota} \sigma WC((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^* \xrightarrow{\Delta_{\sigma WC}} \mathfrak{A} \to 0.$$

Theorem 4.1.2. *Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathfrak{A}$ is Connes-amenable if and only if $\mathfrak{A}$ is unital and the exact sequence Ξ' splits as a sequence of Banach $\mathfrak{A}$-bimodules.*

Proof. Replace the role of the normal virtual diagonal in the proof of Theorem 4.1.1 by a σWC-virtual diagonal for $\mathfrak{A}$. $\square$

Theorem 4.1.3. *Let $\mathfrak{A}$ be a unital dual Banach algebra. Then the exact sequences Ξ and Ξ' are both admissible.*

Proof. The map θ from $\mathfrak{A}$ into $\mathcal{B}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$ (resp. $\sigma WC((\mathfrak{A}\widehat{\otimes}\mathfrak{A})^*)^*$), which sends a to $e \otimes a$ where e is the identity of $\mathfrak{A}$, is the right inverse of Δ_{w^*} (resp. $\Delta_{\sigma WC}$) in Ξ (resp. Ξ'). $\square$

4.1.1 Ideals in Connes-Amenable Dual Banach Algebras

For a normed closed ideal I of an amenable Banach algebra $\mathfrak{A}$, the amenability of I is equivalent to the existence of a bounded approximate identity in I ([Run 2], Proposition 2.2.3). In this section, we will firstly prove an analogous result for dual Banach algebras and Connes-amenability.

Let J be a unital w^*-closed ideal of a dual Banach algebra $\mathfrak{A}$ and X be a Banach J-bimodule. If e is the identity of J, then for each $a \in \mathfrak{A}$ we have

$$eae = ae = ea.$$

Hence, e is in the algebraic center of $\mathfrak{A}$. Then X is a Banach $\mathfrak{A}$-bimodule as follows:

$$a.x := aex, \quad x.a := xea \qquad (a \in \mathfrak{A},\ x \in X).$$

If $D : J \to X$ is a derivation, then D can be extended to a derivation

$$\widetilde{D} : \mathfrak{A} \to X \ \text{ by } \ \widetilde{D}(a) := D(ae).$$

Now for every $a, b \in \mathfrak{A}$, we have

$$\widetilde{D}(ab) = D(abe) = D(aebe) = aeD(be) + D(ae)be = a.\widetilde{D}(b) + \widetilde{D}(a).b.$$

Thus, $\widetilde{D}$ is a derivation. Clearly $\widetilde{D}$ extends D.

Let J be a unital w^*-closed ideal of a dual Banach algebra $\mathfrak{A}$ and X be a Banach J-bimodule. Then we have the following four claims:

Claim 4.1.1. *If X is a normal, dual J-bimodule, then it is a normal $\mathfrak{A}$-bimodule.*

Proof. Let $(a_\alpha)_\alpha$ be a net in $\mathfrak{A}$ such that $a_\alpha \xrightarrow{w^*} a$, for some $a \in \mathfrak{A}$. Since multiplication on $\mathfrak{A}$ is w^*-continuous, $a_\alpha e \xrightarrow{w^*} ae$ in J. Then we have, $a_\alpha ex \xrightarrow{w^*} aex$, or equivalently $a_\alpha.x \xrightarrow{w^*} a.x$ in X. □

Claim 4.1.2. *If X is a normal, dual J-bimodule and if D is w^*-continuous, then $\widetilde{D}$ is w^*-continuous as well.*

Proof. Let $(a_\alpha)_\alpha$ be a net in $\mathfrak{A}$ such that $a_\alpha \xrightarrow{w^*} a$, for some $a \in \mathfrak{A}$. Since multiplication on $\mathfrak{A}$ is w^*-continuous, $a_\alpha e \xrightarrow{w^*} ae$, in J. Hence, $D(a_\alpha e) \xrightarrow{w^*} D(ae)$, equivalently $\widetilde{D}(a_\alpha) \xrightarrow{w^*} \widetilde{D}(a)$ in X. □

Claim 4.1.3. *Let X be a Banach J-bimodule such that X^* is a normal J-bimodule. Then X^* is also normal as an $\mathfrak{A}$-bimodule.*

Proof. Let $(a_\alpha)_\alpha$ be a net in $\mathfrak{A}$ such that $a_\alpha \xrightarrow{w^*} a$ for some $a \in \mathfrak{A}$. Since multiplication on $\mathfrak{A}$ is w^*-continuous, $a_\alpha e \xrightarrow{w^*} ae$, in J. Then for every $\phi \in X^*$ and $x \in X$ we have, $\phi(a_\alpha ex) \to \phi(aex)$, that is $\phi(a_\alpha.x) \to \phi(a.x)$. Hence $\phi.a_\alpha \xrightarrow{w^*} \phi.a$. Similarly, we can prove that $a_\alpha.\phi \xrightarrow{w^*} a.\phi$. □

Claim 4.1.4. *Let X be a Banach J-bimodule such that X^* is a normal J-bimodule and $D : J \to X^*$ be w^*-continuous. Then $\widetilde{D} : \mathfrak{A} \to X^*$ is w^*-continuous too.*

Proof. Let $(a_\alpha)_\alpha$ be a net in $\mathfrak{A}$ such that $a_\alpha \xrightarrow{w^*} a$ for some $a \in \mathfrak{A}$. Since multiplication on $\mathfrak{A}$ is w^*-continuous, $a_\alpha e \xrightarrow{w^*} ae$, in J. Then $D(a_\alpha e)(x) \to D(ae)(x) \quad (x \in X)$, equivalently $\widetilde{D}(a_\alpha) \xrightarrow{w^*} \widetilde{D}(a)$. □

Proposition 4.1.1. *Let J be a w^*-closed ideal of a Connes-amenable dual Banach algebra $\mathfrak{A}$. Then J is Connes-amenable if and only it is unital.*

Proof. ($\Rightarrow$) This is trivial.

($\Leftarrow$) Let J be unital and X^* be a dual, normal J-bimodule. Then X^* is a dual, normal $\mathfrak{A}$-bimodule. Let $D : J \to X^*$ be a w^*-continuous derivation. Then $\widetilde{D} : \mathfrak{A} \to X^*$ is w^*-continuous. Since $\mathfrak{A}$ is Connes-amenable, $\widetilde{D}$ is inner. Hence, D is inner. □

Remark 4.1.1. Let $\mathfrak{A}$ be a dual Banach algebra and J be a w^*-closed ideal of it. Let X be a dual Banach J-bimodule (not necessarily normal) and $D : J \to X$ be a derivation. Then $\mathrm{Im}(\widetilde{D})$ consists of normal elements of X when X is considered as a Banach $\mathfrak{A}$-bimodule. To see this, let $(a_\alpha)_\alpha$ be a net in $\mathfrak{A}$ such that $a_\alpha \xrightarrow{w^*} a$ for some $a \in \mathfrak{A}$. Since multiplication on $\mathfrak{A}$ is w^*-continuous, we have $a_\alpha e \xrightarrow{w^*} ae$ in J. Then

$$a_\alpha e D(be) \xrightarrow{w^*} aeD(be) \qquad (b \in \mathfrak{A}),$$

equivalently, $a_\alpha \widetilde{D}(b) \xrightarrow{w^*} a\widetilde{D}(b)$. Similarly we prove that $\widetilde{D}(b)a_\alpha \xrightarrow{w^*} \widetilde{D}(b)a$.

Hence, we have proven the following:

Corollary 4.1.1. *Let J be a w^*-closed ideal of a strongly Connes-amenable dual Banach algebra $\mathfrak{A}$. Then J is Connes-amenable if and only it is unital.*

Now let J be a w^*-closed ideal of a unital, dual Banach algebra $\mathfrak{A}$. Then we consider the following exact sequence:

$$\Gamma : \qquad 0 \longrightarrow J \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\theta} \mathfrak{A}/J \longrightarrow 0$$

where $\iota : J \to \mathfrak{A}$ is the inclusion and $\theta : \mathfrak{A} \to \mathfrak{A}/J$ is the quotient maps. The following is proved in ([C-L], page 96).

Proposition 4.1.2. *The exact sequence Γ splits as sequence of $\mathfrak{A}$-bimodules if and only if J is unital.*

Proof. ($\Rightarrow$) Assume that Γ splits as sequence of $\mathfrak{A}$-bimodules and $\sigma : \mathfrak{A} \to J$ is a left inverse of the inclusion map $\iota : J \to \mathfrak{A}$. Then for every $a \in J$, we have

$$a\sigma(e) = \sigma(ae) = \sigma(ea) = \sigma(e)a = \sigma(a) = a$$

where e is the identity of $\mathfrak{A}$. Hence, $\sigma(e)$ is the identity of J.

($\Leftarrow$) Let J be unital. Then the map from $\mathfrak{A}$ to J, $a \mapsto ae$ (where e is the identity of J) is a left inverse of ι, which is an $\mathfrak{A}$-bimodule homomorphism. $\square$

Clearly if Γ splits as sequence of $\mathfrak{A}$-bimodules then so is Γ^*: The short exact sequence obtained from Γ by taking Banach space adjoints:

$$\Gamma^* : \qquad 0 \longrightarrow J \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\theta} \mathfrak{A}/J \longrightarrow 0.$$

It is proven in ([C-L], page 97) that if Γ^* splits as sequence of $\mathfrak{A}$-bimodules, then J has a bounded approximate identity. Since J itself is a dual Banach algebra, it is unital. Hence, we conclude the following:

Corollary 4.1.2. *The exact sequence Γ splits as sequence of $\mathfrak{A}$-bimodules if and only if Γ^* splits as sequence of $\mathfrak{A}$-bimodules if and only if J is unital.*

Proposition 4.1.3. *Let J be a unital w^*-closed ideal of a dual Banach algebra $\mathfrak{A}$. Then $J^\perp$ is complemented in $\mathfrak{A}^*$ as an $\mathfrak{A}$-bimodule.*

Proof. Let e be the identity of J. Define an operator F on $\mathfrak{A}^*$ by

$$F(\phi) := \phi - e\phi \quad (\phi \in \mathfrak{A}^*).$$

Then for every $\phi \in \mathfrak{A}^*$ and $j \in J$, we have

$$(\phi - e\phi)(j) = \phi(j) - \phi(je) = \phi(j) - \phi(j) = 0.$$

Thus, $Im(F)$ is in $J^\perp$. For every $\phi \in J^\perp$ we have

$$F^2(\phi) = F(\phi - e\phi) = \phi - e\phi - e(\phi - e\phi) = \phi - e\phi - e\phi + e^2\phi = F(\phi).$$

Moreover, for every $\phi \in J^\perp$, we have

$$F(\phi)(a) = \phi(a) - \phi(ae) = \phi(a)$$

for every $a \in \mathfrak{A}$. That is $F(\phi) = \phi \quad (\phi \in J^\perp)$.

Now it remains to show that F is an $\mathfrak{A}$-bimodule homomorphism.

For every $\phi \in \mathfrak{A}^*, a, b \in \mathfrak{A}$, we have the following:

$$\begin{aligned} F(a.\phi)(b) &= (a\phi - ea\phi)(b) = \phi(ba) - \phi(bea) = \phi(ba) - \phi(bae) \\ &= (\phi - e\phi)(ba) = F(\phi)(ba) = (a.F(\phi))(b). \end{aligned}$$

Thus, $F(a.\phi) = a.F(\phi)$. Similarly, one can show that $F(\phi.a) = F(\phi).a$. □

Proposition 4.1.4. *Let J be a w^*-closed ideal of a (strongly) Connes-amenable dual Banach algebra $\mathfrak{A}$. If $J^\perp$ is complemented in $\mathfrak{A}^*$ as an $\mathfrak{A}$-bimodule, then J is (strongly) Connes-amenable.*

Proof. Let $J^\perp$ be complemented in $\mathfrak{A}^*$ as an $\mathfrak{A}$-bimodule. Then the exact sequence Γ^* splits as $\mathfrak{A}$-bimodules. Then by Corollary 4.1.2, J is unital. The result follows by Proposition 4.1.1 and Corollary 4.1.1. □

Proposition 4.1.5. *Let $\mathfrak{A}$ be a Connes-amenable dual Banach algebra and let I be a w^*-closed ideal of it such that $I_\perp := \{\phi \in \mathfrak{A}_* : \phi(x) = 0$ for all $x \in I\}$ is complemented in $\mathfrak{A}_*$ as an $\mathfrak{A}$-submodule. Then I is unital.*

Proof. Suppose that $\mathfrak{A}$ and I are as in the statement of the Proposition and $P \in \mathcal{B}(\mathfrak{A}_*)$ is the projection onto $I_\perp$. Now we define $Q := I - P \in \mathcal{B}(\mathfrak{A}_*)$, where I is the identity operator on $\mathfrak{A}_*$. Then the adjoint operator $Q^* \in \mathcal{B}(\mathfrak{A})$ is a projection onto I.

For an arbitrary $\phi \in \sigma WC(\mathcal{B}^2(\mathfrak{A}, \mathbb{C}))$ we define $\tilde{\phi} \in \mathcal{B}^2(\mathfrak{A}, \mathbb{C})$ by letting

$$\tilde{\phi}(a, b) := \phi(a, Q^*(b)) \qquad (a, b \in \mathfrak{A}).$$

It is a routine calculation to check that $\tilde{\phi}$ is in $\sigma WC(\mathcal{B}^2(\mathfrak{A},\mathbb{C}))$. Since $\mathfrak{A}$ is Connes-amenable, it has a σWC-virtual diagonal, say $M \in \big(\sigma WC(\mathcal{B}^2(\mathfrak{A},\mathbb{C}))\big)^*$. Define $\tilde{M} \in \big(\sigma WC(\mathcal{B}^2(\mathfrak{A},\mathbb{C}))\big)^*$ via

$$\langle \tilde{M}, \phi\rangle := \langle M, \tilde{\phi}\rangle, \qquad (\phi \in \sigma WC(\mathcal{B}^2(\mathfrak{A},\mathbb{C}))).$$

Now we claim that $e := \Delta_{\sigma WC}(\tilde{M})$ is an identity for I. In order to prove the claim we first show that $e \in I$. For an arbitrary $\phi \in I_\perp$ we have

$$\langle \phi, e\rangle = \langle \phi, \Delta_{\sigma WC}(\tilde{M})\rangle = \langle \Delta^*_{\sigma WC}(\phi), \tilde{M}\rangle = \langle \widetilde{\Delta^*_{\sigma WC}(\phi)}, M\rangle = 0.$$

This is because, as $\phi \in I_\perp \subset \mathfrak{A}_*$ we have $\widetilde{\Delta^*_{\sigma WC}(\phi)} = \widetilde{\Delta^*(\phi)}$ and for any $a, b \in \mathfrak{A}$ we have

$$\widetilde{\Delta^*(\phi)}(a,b) = \Delta^*(\phi)(a, Q^*(b)) = \phi(a.Q^*(b)) = 0 \quad \text{because} \quad a.Q^*(b) \in I.$$

As a result, we conclude that (since I is w^*-closed) $e \in (I_\perp)^\perp = I$.

Observe that for an arbitrary $\phi \in \sigma WC(\mathcal{B}^2(\mathfrak{A},\mathbb{C}))$ and $a \in I$ we have

$$(\widetilde{\phi.a})(x,y) = (\phi.a)(x, Q^*(y)) = \phi(a.x, Q^*(y)) = (\tilde{\phi}.a)(x,y) \quad (x, y \in \mathfrak{A}).$$

Hence $\widetilde{\phi.a} = \tilde{\phi}.a$. Similarly we have $a.\tilde{\phi} = a.\phi$ because

$$\begin{aligned}(a.\tilde{\phi})(x,y) &= \tilde{\phi}(x, y.a) = \phi(x, Q^*(y.a)) \\ &= \phi(x, y.a) = (a.\phi)(x,y) \qquad (x, y \in \mathfrak{A}).\end{aligned}$$

Let $\phi \in \sigma WC(\mathcal{B}^2(\mathfrak{A}, \mathbb{C}))$. Then we have

$$\begin{aligned} \langle a.\tilde{M}, \phi \rangle &= \langle \tilde{M}, \phi.a \rangle = \langle M, \widetilde{\phi.a} \rangle = \langle M, \tilde{\phi}.a \rangle = \langle a.M, \tilde{\phi} \rangle \\ &= \langle M.a, \tilde{\phi} \rangle = \langle M, a.\tilde{\phi} \rangle = \langle M, a.\phi \rangle = \langle M.a, \phi \rangle. \end{aligned}$$

Therefore, $a.\tilde{M} = M.a = a.M$. Hence, for every $a \in I$ we have

$$a.e = a.\Delta_{\sigma WC}(\tilde{M}) = \Delta_{\sigma WC}(a.\tilde{M}) = \Delta_{\sigma WC}(a.M) = a.\Delta_{\sigma WC}(M) = a.$$

Similarly, we can prove that $e.a = a$. □

By replacing the role of $\sigma WC(\mathcal{B}^2(\mathfrak{A}, \mathbb{C}))$ by $B^2_\sigma(\mathfrak{A}, \mathbb{C})$, we prove the following Proposition.

Proposition 4.1.6. *Let $\mathfrak{A}$ be a strongly Connes-amenable dual Banach algebra and let I be a w^*-closed ideal of it such that $I_\perp := \{\phi \in \mathfrak{A}_* : \ \phi(x) = 0$ for all $x \in I\}$ is complemented in $\mathfrak{A}_*$ as an $\mathfrak{A}$-submodule. Then I is unital.*

Remark 4.1.2. Proposition 4.1.3 and Proposition 4.1.4 together show the following: If J is a w^*-closed ideal of a (strongly) Connes-amenable dual Banach algebra $\mathfrak{A}$, then J is (strongly) Connes-amenable if and only if $J^\perp$ is complemented in $\mathfrak{A}^*$ as an $\mathfrak{A}$-bimodule.

In brief, we have the following:

Let J be a w^*-closed ideal of a (strongly) Connes-amenable dual Banach algebra $\mathfrak{A}$. Then the following are equivalent:

(i) J is (strongly) Connes-amenable,

(ii) $J^{\perp}$ is complemented in $\mathfrak{A}^*$ as an $\mathfrak{A}$-bimodule,

(iii) Γ splits as an exact sequence of $\mathfrak{A}$-bimodules,

(iv) J is unital,

(vi) $J_{\perp}$ is complemented in $\mathfrak{A}_*$ as an $\mathfrak{A}$-bimodule.

4.2 Connes-Amenability of Fourier-Stieltjes Algebra

Let G be a locally compact group. If G has an abelian subgroup of finite index, then $A(G)$ is an amenable Banach algebra ([F-R]) that is w^*-dense in $B(G)$. Hence, by (Proposition 4.2, [Run 1]), $B(G)$ is Connes-amenable. Moreover, by [Run 3], it is strongly Connes-amenable. Runde conjectured the converse of this result; that is, if $B(G)$ is Connes-amenable, then G has an abelian subgroup of finite index. In [Run 3], he proved that this is true when G is the direct product of a family of finite groups or when G is an amenable discrete group. Now we extend this result to general discrete, not necessarily amenable, groups.

Theorem 4.2.1. *Let G be a discrete group. If $B(G)$ is Connes-amenable, then G has an abelian subgroup of finite index.*

Proof. Let $B(G)$ be Connes-amenable. Then by ([Run 3]), there is a σWC-virtual diagonal $M \in \sigma WC((B(G)\otimes_\gamma B(G))^*)^*$ for $B(G)$ where $\otimes_\gamma$ denotes the projective tensor product of Banach spaces. Using the Hahn-Banach theorem, we extend M to $(B(G)\otimes_\gamma B(G))^{**}$ and let $\widetilde{M}$ be such an extension.

Let $(m_\alpha)_\alpha$ be a net in $B(G)\otimes_\gamma B(G)$, bounded by the norm of $\widetilde{M}$ such that $m_\alpha \xrightarrow{w^*} \widetilde{M}$ and let $\Gamma := \{(x, x^{-1}) : x \in G\}$ be the anti-diagonal of $G \times G$. We want to show that χ_Γ is a cluster point of $((I\otimes^\vee)m_\alpha)$.

Clearly Γ is not a subgroup of $G \times G$ (unless G is abelian). We will firstly show that the net $((I\otimes^\vee)m_\alpha)$ behaves like an approximate indicator (as defined in [A-N-R] for closed subgroups of locally compact groups) for Γ. That is,

A $\quad m_\alpha(x,x) \to 1 \qquad (x \in G)$

and

B $\quad m_\alpha(x,y) \to 0 \qquad (x \neq y \in G)$

under the (contractive) inclusion $B(G)\otimes_\gamma B(G) \hookrightarrow B(G \times G)$.

Let $(m_\alpha)_\alpha$ be as above. Then the restriction of m_α onto $\sigma WC((B(G)\otimes_\gamma B(G))^*)$ converges to M in the induced w^*-topology on $\sigma WC((B(G)\otimes_\gamma B(G))^*)^*$. We shall denote this restriction again by m_α.

Let $\Delta : B(G)\otimes_\gamma B(G) \to B(G),\ a \otimes b \mapsto ab$ be the diagonal operator. Then its second dual $\Delta^{**} : (B(G)\otimes_\gamma B(G))^{**} \to B(G)^{**}$ is w^*-w^* continuous. Then we have

$$m_\alpha \xrightarrow{w^*} M \quad \Rightarrow \quad \Delta^{**}m_\alpha \xrightarrow{w^*} \Delta^{**}M \quad \Rightarrow \quad \Delta^{**}m_\alpha \xrightarrow{w} \Delta^{**}M.$$

It is a standard theorem in functional analysis that in any normed space, the weak and the norm closures of a convex set coincide ([Con]). Hence, if $(x_\alpha)_\alpha$ is a net in a normed space E such that $x_\alpha \xrightarrow{w} x$ for some $x \in E$, then as x is in the norm closure of the convex hull of $(x_\alpha)_\alpha$ there exists a sequence in the convex hull of $(x_\alpha)_\alpha$ which converges to x in the norm topology.

Hence, passing to convex combinations without loss of generality, we may assume that

$$\Delta^{**}m_\alpha \xrightarrow{\|\cdot\|} \Delta^{**}M, \quad \text{that is } \Delta m_\alpha \xrightarrow{\|\cdot\|} \Delta_{\sigma WC}M.$$

Since $\Delta_{\sigma WC}M = 1$, we conclude that $\Delta m_\alpha \longrightarrow 1$ pointwise. Hence, A is satisfied.

Now, $m_\alpha \xrightarrow{w^*} M$ implies that

$$m_\alpha.a \xrightarrow{w^*} M.a \quad \text{and} \quad a.m_\alpha \xrightarrow{w^*} a.M \qquad (a \in B(G)).$$

Since $a.M = M.a$ for each $a \in B(G)$, we conclude that

$$\widetilde{m}_{\alpha,a} := a.m_\alpha - m_\alpha.a \xrightarrow{w^*} 0 \qquad (a \in B(G)).$$

Now we claim that the w^*-topology of $\sigma WC((B(G)\otimes_\gamma B(G))^*)^*$ is stronger than the w^*-topology of $B(G \times G)$. To see this, firstly note that

$$C^*(G \times G) \subset \sigma WC(B(G)\otimes_\gamma B(G))^*$$

since $B(G \times G)$ is a dual Banach algebra (Corrollary 4.6, [Run 3]).

On the other hand, we have the inclusion

$$B(G)\otimes_\gamma B(G) \hookrightarrow B(G \times G).$$

Hence, there is a $B(G)$-module homomorphism from $B(G \times G)^*$ to $(B(G) \otimes_\gamma B(G))^*$. Then by ([Run 3]), we have

$$\sigma WC(B(G \times G))^* \subset \sigma WC(B(G) \otimes_\gamma B(G))^*.$$

Finally, we conclude that

$$C^*(G \times G) \subset \sigma WC(B(G) \otimes_\gamma B(G))^*$$

holds. This proves the claim. Hence, $\widetilde{m}_{\alpha,a} \longrightarrow 0$ in the w^*-topology of $B(G \times G)$. When G is discrete, on norm bounded subsets of $B(G \times G)$, w^*-convergence and pointwise convergence are the same. Hence $\widetilde{m}_{\alpha,a} \longrightarrow 0$ pointwise. This shows that (B) is satisfied.

Therefore, the net $((I \otimes^\vee) m_\alpha)$ behaves like an approximate indicator for Γ. Hence, it converges to χ_Γ pointwise. Since $((I \otimes^\vee) m_\alpha)$ is a bounded net, by Alaoglu's theorem, it has an accumulation point. Under the inclusion $B(G) \otimes_\gamma B(G) \subset B(G \times G) \subset B(G_d \times G_d)$ on norm bounded subsets, w^*- and pointwise convergence topologies coincide. Hence, χ_Γ is a w^*-accumulation point of this net. Then by (Proposition 2.2, [F-R]), we are done. $\square$

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